1. (2 points) Let $f(x,y) = x^2 - xy + y^2 - y$. Find the directions $\vec{u}$ and the values of $D_{\vec{u}}f(1,-1)$ for which

a. $D_{\vec{u}}f(1,-1)$ is largest
b. $D_{\vec{u}}f(1,-1)$ is smallest
c. $D_{\vec{u}}f(1,-1) = 0$
d. $D_{u}f(1,-1) = 4$
e. $D_{\vec{u}}f(1,-1) = -3$

Solution:

$$\nabla f(x,y) = (2x-y)\vec{i} + (-x+2y-1)\vec{j}$$

(a) $\nabla f(1,-1) = 3\vec{i} - 4\vec{j} \Rightarrow |\nabla f(1,-1)| = 5 \Rightarrow \vec{u} = \frac{3}{5}i - \frac{4}{5}j$
(b) $-\nabla f(1,-1) = -3\vec{i} + 4\vec{j} \Rightarrow |-\nabla f(1,-1)| = 5 \Rightarrow \vec{u} = -\frac{3}{5}i + \frac{4}{5}j$
(c) $u = \frac{4}{5}\vec{i} + \frac{3}{5}\vec{j}$ or $u = -\frac{4}{5}\vec{i} - \frac{3}{5}\vec{j}$
(d) Assume that $\vec{u} = u_1\vec{i} + u_2\vec{j}$

We have $\begin{cases} |u| = \sqrt{u_1^2 + u_2^2} = 1 \\ D_{\vec{u}}f(1,-1) = \nabla f(1,-1)\cdot\vec{u} = 3u_1 - 4u_2 = 4 \end{cases} \Rightarrow u_1^2 + (\frac{3}{4}u_1 - 1)^2 = 1$

$\Rightarrow u_1 = \frac{24}{25},\ u_2 = \frac{7}{25}$ or $u_1 = 0,\ u_2 = -1 \Rightarrow \vec{u} = \frac{24}{25}\vec{i} - \frac{7}{25}\vec{j}$ or $\vec{u} = -\vec{j}$

(e) $\begin{cases} \vec{u}_{\nabla f} = \dfrac{3}{5}\vec{i} - \dfrac{4}{5}\vec{j} \\ \vec{u}\cdot\nabla f = |\nabla f|\cos\theta = -3 \end{cases} \Rightarrow \cos\theta = -\frac{3}{5} \Rightarrow \vec{u} = \cos(\theta_{\nabla f} \pm \theta)\vec{i} + \sin(\theta_{\nabla f} \pm \theta)\vec{j}$

$$\Rightarrow \begin{cases} \cos(\theta_{\nabla f} \pm \theta) = \cos(\theta_{\nabla f})\cos(\theta) \mp \sin(\theta_{\nabla f})\sin(\theta) = -\dfrac{3}{5}\dfrac{3}{5} \mp (-\dfrac{4}{5}\dfrac{4}{5}) = \dfrac{7}{25} \text{ or } -1 \\ \sin(\theta_{\nabla f} \pm \theta) = \sin(\theta_{\nabla f})\cos(\theta) \pm \cos(\theta_{\nabla f})\sin(\theta) = \dfrac{12}{25} \pm (\dfrac{12}{25}) = \dfrac{24}{25} \text{ or } 0 \end{cases}$$

$\Rightarrow \vec{u} = \frac{7}{25}\vec{i} + \frac{24}{25}\vec{j}$ or $\vec{u} = -\vec{i}$

2. (2 points) Find equations for the

(a) tangent plane and
(b) normal line at the point P_0 on the given surface.

$$x^2 + 2xy - y^2 + z^2 = 7, \quad P_0(1,-1,3)$$

Solution:
(a) $\nabla f = (2x+2y)\vec{i} + (2x-2y)\vec{j} + 2z\vec{k} \Rightarrow \nabla f(1,-1,3) = 4\vec{j} + 6\vec{k}$
$\Rightarrow$Tangent plane: $4(y+1) + 6(z-3) = 0 \Rightarrow 2y + 3z = 7$
(b) Normal line: $x = 1, y = -1 + 4t, z = 3 + 6t$

3. (2 points) Find the linearization $L(x,y,z)$ of the function $f(x,y,z)$ at P_0. Then find an upper bound for the magnitude of the error E in the approximation $f(x,y,z) \approx L(x,y,z)$ over the region R.

$$f(x,y,z) = xz - 3yz + 2 \text{ at } P_0(1,1,2)$$
$$R:\ |x-1| \le 0.01,\ |y-1| \le 0.01,\ |z-2| \le 0.02$$

Solution:
$f(1,1,2) = -2, f_x = z, f_y = -3z, f_z = x - 3y$
$\Rightarrow L(x,y,z) = -2 + 2(x-1) - 6(y-1) - 2(z-2) = 2x - 6y - 2z + 6$
$f_{xx} = 0, f_{yy} = 0, f_{zz} = 0, f_{xy} = 0, f_{xz} = 1, f_{yz} = -3$
$\Rightarrow M = 3$, thus, $|E(x,y,z)| \le (\frac{1}{2})(3)(0.01 + 0.01 + 0.02)^2 = 0.0024$

4. (2 points) Find all the local maximam, local minima, and saddle points of the function

$$f(x,y) = x^3 + 3xy + y^3$$

Solution:

$$\begin{cases} f_x(x,y) = 3x^2 + 3y = 0 \\ f_y(x,y) = 3x + 3y^2 = 0 \end{cases} \Rightarrow \begin{cases} x = -y^2 \\ y^4 + y = 0 \end{cases} \Rightarrow \begin{cases} y = -1 \\ x = -1 \end{cases} \text{ or } \begin{cases} y = 0 \\ x = 0 \end{cases}$$

The critical points are $(0,0)$ and $(-1,-1)$.

$$f_{xx} = 6x, f_{xy} = 3, f_{yy} = 6y \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$$

For $(0,0)$, $f_{xx}f_{yy} - f_{xy}^2 = -9 < 0 \Rightarrow$ saddle point.
For $(-1,-1)$, $f_{xx}f_{yy} - f_{xy}^2 = 27 > 0, f_{xx} = -6 < 0 \Rightarrow$ local maximum.

5. (2 points) Find the absolute maxima and minima of the functions on the given domains. $f(x,y) = 48xy - 32x^3 - 24y^2$ on the rectangular plate $0 \le x \le 1, 0 \le y \le 1$.

Solution:

(i) When $x = 0, 0 \le y \le 1, f(x,y) = f(0,y) = -24y^2$
$f'(0,y) = -48y \le 0 \Rightarrow$ the critical points are $f(0,0) = 0, f(0,1) = -24$.

(ii) When $x = 1, 0 \le y \le 1, f(x,y) = f(1,y) = 48y - 32 - 24y^2$
$f'(1,y) = 48 - 48y \ge 0 \Rightarrow$ the critical points are $f(1,0) = -32, f(1,1) = -8$.

(iii) When $0 \le x \le 1, y = 0, f(x,y) = f(x,0) = -32x^3$
$f'(x,0) = -96x^2 \le 0 \Rightarrow$ the critical points are $f(0,0) = 0, f(1,0) = -32$.

(iv) When $0 \le x \le 1, y = 1, f(x,y) = f(x,1) = 48x - 32x^3 - 24$
$f'(x,1) = 48 - 96x^2 = 0 \Rightarrow x = \frac{1}{\sqrt{2}} \in (0,1)$
$\Rightarrow$ the critical points are $f(\frac{1}{\sqrt{2}}, 1) = 16\sqrt{2} - 24, f(0,1) = -24, f(1,1) = -8$.

(v) For interior points, $\begin{cases} f_x(x,y) = 48y - 96x^2 = 0 \\ f_y(x,y) = 48x - 48y = 0 \end{cases} \Rightarrow \begin{cases} x = y \\ x - 2x^2 = 0 \end{cases} \Rightarrow x = y = 0$ or $x = y = \frac{1}{2}$
the critical points is $f(\frac{1}{2}, \frac{1}{2}) = 2$
Therefore the absolute maximum is 2 at $(\frac{1}{2}, \frac{1}{2})$, the absolute minimum is -32 at $(1,0)$.