第16周高数互助课堂

Chapter 10 复习课

内容

- 判断数列敛散性
- 判断级数敛散性
- 求收敛半径和收敛区间
- 泰勒级数的应用

数列敛散性判断

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).

EXAMPLE 1 Show that

(a) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

(a) Let $\epsilon > 0$ be given. We must show that there exists an integer N such that for all n,

$$n > N \quad \Rightarrow \quad \left|\frac{1}{n} - 0\right| < \epsilon.$$

This implication will hold if $(1/n) < \epsilon$ or $n > 1/\epsilon$. If N is any integer greater than $1/\epsilon$, the implication will hold for all $n > N$. This proves that $\lim_{n\to\infty}(1/n) = 0$.

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

EXAMPLE 8 Does the sequence whose nth term is

$$a_n = \left(\frac{n+1}{n-1}\right)^n$$

converge? If so, find $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} n \ln\left(\frac{n+1}{n-1}\right) \qquad \infty \cdot 0 \text{ form}$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{n+1}{n-1}\right)}{1/n} \qquad \frac{0}{0} \text{ form}$$

$$= \lim_{n\to\infty} \frac{-2/(n^2-1)}{-1/n^2} \qquad \text{L'Hôpital's Rule: differentiate numerator and denominator.}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2-1} = 2.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1 + \frac{2}{n-1}\right)^n = n^2$$

Since $\ln a_n \to 2$ and $f(x) = e^x$ is continuous, Theorem 4 tells us that

$$a_n = e^{\ln a_n} \to e^2.$$

The sequence $\{a_n\}$ converges to e^2.

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

$x_{n+1} = \frac{x_n}{2} + \frac{2}{x_n}$，$x_1 = 1$，判断$x_n$的是否收敛，若收敛求$\lim\limits_{n\to\infty} x_n$

$x_1 > 2$，$x_k > 2 \to x_{k+1} > 2$，由数学归纳法易得，$x_n > 2$，故

$$x_{n+1} - x_n = \frac{x_n}{2} + \frac{2}{x_n} - x_n = \frac{4 - x_n^2}{2x_n} < 0$$

x_n单调递减，x_n极限存在

令$\lim\limits_{n\to\infty} x_n = L$，

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{x_n}{2} + \frac{2}{x_n}$$

$$L = \frac{L}{2} + \frac{2}{L} \quad \Longrightarrow \quad L = 2$$

判断数列$\boldsymbol{a_n}$极限存在的方法

- 找到一个N，使得$|a_n - L| < \epsilon$，对任意ϵ成立
- $b_n \le a_n \le c_n$且$b_n \to L$，$c_n \to L$
- $\lim\limits_{x\to\infty} f(x) = L$，其中$a_n = f(n)$
- a_n单调有界

级数敛散性判断

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

Caution Remember that $\sum(a_n + b_n)$ can converge when $\sum a_n$ and $\sum b_n$ both diverge. For example, $\sum a_n = 1 + 1 + 1 + \cdots$ and $\sum b_n = (-1) + (-1) + (-1) + \cdots$ diverge, whereas $\sum(a_n + b_n) = 0 + 0 + 0 + \cdots$ converges to 0.

THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a continuous, positive, decreasing function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

EXAMPLE 3 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \le 1$.

Solution If $p > 1$, then $f(x) = 1/x^p$ is a positive decreasing function of x. Since

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^p}\,dx &= \int_1^{\infty} x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b \\ &= \frac{1}{1-p}\lim_{b\to\infty}\left(\frac{1}{b^{p-1}} - 1\right) \\ &= \frac{1}{1-p}(0-1) = \frac{1}{p-1}, \end{aligned}$$

$b^{p-1} \to \infty$ as $b \to \infty$ because $p - 1 > 0$.

the series converges by the Integral Test. We emphasize that the sum of the p-series is *not* $1/(p-1)$. The series converges, but we don't know the value it converges to.

If $p \le 0$, the series diverges by the nth-term test. If $0 < p < 1$, then $1 - p > 0$ and

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p}\lim_{b\to\infty}(b^{1-p} - 1) = \infty.$$

The series diverges by the Integral Test.

If $p = 1$, we have the (divergent) harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We have convergence for $p > 1$ but divergence for all other values of p.

THEOREM 10—The Comparison Test Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \leq a_n \leq c_n \qquad \text{for all} \qquad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

THEOREM 11—Limit Comparison Test Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

EXAMPLE 3 Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

Solution Because $\ln n$ grows more slowly than n^c for any positive constant c (Section 10.1, Exercise 105), we can compare the series to a convergent p-series. To get the p-series, we see that

$$\frac{\ln n}{n^{3/2}} < \frac{n^{1/4}}{n^{3/2}} = \frac{1}{n^{5/4}}$$

for n sufficiently large. Then taking $a_n = (\ln n)/n^{3/2}$ and $b_n = 1/n^{5/4}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\ln n}{n^{1/4}} \\
&= \lim_{n\to\infty} \frac{1/n}{(1/4)n^{-3/4}} && \text{l'Hôpital's Rule} \\
&= \lim_{n\to\infty} \frac{4}{n^{1/4}} = 0.
\end{aligned}$$

Since $\sum b_n = \sum (1/n^{5/4})$ is a p-series with $p > 1$, it converges, so $\sum a_n$ converges by Part 2 of the Limit Comparison Test. ■

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

THEOREM 13—The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

THEOREM 14—The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n! n!}{(2n)!}$

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$

Does $\sum a_n$ converge?

EXAMPLE 2 Investigate the convergence of the following series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n + 5}{3^n}$ **(b)** $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{n!n!}$ **(c)** $\displaystyle\sum_{n=1}^{\infty} \frac{4^n n!n!}{(2n)!}$

(a) For the series $\sum_{n=0}^{\infty} (2^n + 5)/3^n$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2^{n+1} + 5)/3^{n+1}}{(2^n + 5)/3^n} = \frac{1}{3} \cdot \frac{2^{n+1} + 5}{2^n + 5} = \frac{1}{3} \cdot \left(\frac{2 + 5 \cdot 2^{-n}}{1 + 5 \cdot 2^{-n}}\right) \rightarrow \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

The series converges absolutely (and thus converges) because $\rho = 2/3$ is less than 1.

(b) If $a_n = \dfrac{(2n)!}{n!n!}$, then $a_{n+1} = \dfrac{(2n + 2)!}{(n + 1)!(n + 1)!}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n!n!(2n + 2)(2n + 1)(2n)!}{(n + 1)!(n + 1)!(2n)!}$$

$$= \frac{(2n + 2)(2n + 1)}{(n + 1)(n + 1)} = \frac{4n + 2}{n + 1} \rightarrow 4.$$

The series diverges because $\rho = 4$ is greater than 1.

(c) If $a_n = 4^n n! n!/(2n)!$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{4^{n+1}(n+1)!(n+1)!}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{4^n n! n!}$$

$$= \frac{4(n+1)(n+1)}{(2n+2)(2n+1)} = \frac{2(n+1)}{2n+1} \to 1.$$

Because the limit is $\rho = 1$, we cannot decide from the Ratio Test whether the series converges. When we notice that $a_{n+1}/a_n = (2n+2)/(2n+1)$, we conclude that a_{n+1} is always greater than a_n because $(2n+2)/(2n+1)$ is always greater than 1. Therefore, all terms are greater than or equal to $a_1 = 2$, and the nth term does not approach zero as $n \to \infty$. The series diverges. ■

EXAMPLE 3 Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$

Does $\sum a_n$ converge?

Solution We apply the Root Test, finding that

$$\sqrt[n]{|a_n|} = \begin{cases} \sqrt[n]{n}/2, & n \text{ odd} \\ 1/2, & n \text{ even}. \end{cases}$$

Therefore,

$$\frac{1}{2} \le \sqrt[n]{|a_n|} \le \frac{\sqrt[n]{n}}{2}.$$

Since $\sqrt[n]{n} \to 1$ (Section 10.1, Theorem 5), we have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1/2$ by the Sandwich Theorem. The limit is less than 1, so the series converges absolutely by the Root Test. ■

THEOREM 15—The Alternating Series Test The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.

DEFINITION A convergent series that is not absolutely convergent is **conditionally convergent**.

证明$\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{nlnn}$为条件收敛

$\sum_{n=2}^{\infty}|a_n| = \sum_{n=2}^{\infty}\frac{1}{nlnn}$， $f(x) = \frac{1}{xlnx}$ is positive continuous and decreasing for n ≥ 2

$$\int_2^{\infty}\frac{1}{xlnx}dx = \lim_{b\to\infty}\frac{d(lnx)}{lnx} = \lim_{b\to\infty} lnlnb - lnln2 = \infty$$

$$\sum_{n=2}^{\infty}|a_n| \text{ diverges}$$

$$u_n = \frac{1}{nlnn} > 0 \text{ for n} \geq 2$$

$$u_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{nlnn} = u_n \text{ for n} \geq 2$$

$$\lim_{n\to\infty} \frac{1}{nlnn} = 0$$

by the alternating series test, $\sum_{n=2}^{\infty} a_n$ converges conditionally

小结

1. **The *n*th-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.
2. **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
3. ***p*-series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
4. **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
5. **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
6. **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

求收敛半径和收敛区间

1. 令$\rho = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$或$\rho = \lim_{n \to \infty} \sqrt[n]{a_n} < 1$
2. 测试边界点

(1) Find the radius and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^n x^n}{\sqrt{n^2+n+1}}$$

(2) For what values of x does the series converge absolutely, or conditionally?

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{2^{n+1}\sqrt{n^2+n+1}}{2^n\sqrt{(n+1)^2+n+1+1}}|\mathrm{x}|$$

$$|x| < \frac{1}{2}$$

当 $x = \frac{1}{2}$，原式 $= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^2+n+1}}$，条件收敛

当 $x = -\frac{1}{2}$，原式 $= \sum_{n=0}^{\infty} \frac{-1}{\sqrt{n^2+n+1}}$，发散

当 $x \in (-\frac{1}{2}, \frac{1}{2})$，绝对收敛

$$\left(-\frac{1}{2}, \frac{1}{2}\right], R = \frac{1}{2}$$

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series of f** is the Taylor series generated by f at $x = 0$, or

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots.$$

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

TABLE 10.1 Frequently used Taylor series

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \leq 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \leq 1$$

求泰勒展开式

- 化为已知的泰勒级数
- 先求导后积分

求$\frac{1}{(2-x)^2}$的在$x=0$处的泰勒展开

$$\frac{1}{(2-x)^2}=\frac{1}{4\left(1-\frac{x}{2}\right)^2}$$

$$=\frac{1}{4}\times(1+\sum_{k=1}^{\infty}\binom{-2}{k}\left(-\frac{x}{2}\right)^k)$$

$$=\frac{1}{4}\times\left(1+\frac{-2}{1}\left(\frac{-x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{-x}{2}\right)^2+\cdots\right)$$

因为$\left|-\frac{x}{2}\right|<1$

$$-2<x<2$$

求$\ln\left(x+\sqrt{x^2+1}\right)$在$x=0$处的泰勒级数

令$f(x)=\ln\left(x+\sqrt{x^2+1}\right)$，$f'=\frac{1}{x+\sqrt{x^2+1}}\times\left(1+\frac{2x}{2\sqrt{x^2+1}}\right)=(x^2+1)^{\frac{1}{2}}$

$f(x)=\int_0^x f'(x)=\int_0^x 1+\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{k}x^{2k}dx=x+\sum_{k=1}^{\infty}\frac{\binom{\frac{1}{2}}{k}x^{2k+1}}{2k+1}$

求$\frac{x^2}{2-4x}$在$x=2$的泰勒级数

$$\frac{x^2}{-6-4(x-2)}=-\frac{1}{6}\times\frac{(x-2)^2+4x-8+8}{1+\frac{2}{3}(x-2)}=-\frac{1}{6}\times\frac{(x-2)^2+4(x-2)+8}{1+\frac{2}{3}(x-2)}$$

$$=-\frac{1}{6}\times\left(\frac{x-2}{1+\frac{2}{3}(x-2)}+\frac{4(x-2)}{1+\frac{2}{3}(x-2)}+\frac{8}{1+\frac{2}{3}(x-2)}\right)$$

利用泰勒展开求极限

1. 通分
2. 泰勒展开

EXAMPLE 7 Find $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$.

Solution Using algebra and the Taylor series for $\sin x$, we have

$$\frac{1}{\sin x}-\frac{1}{x}=\frac{x-\sin x}{x\sin x}=\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}{x\cdot\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots\right)}$$

$$=\frac{x^3\left(\frac{1}{3!}-\frac{x^2}{5!}+\cdots\right)}{x^2\left(1-\frac{x^2}{3!}+\cdots\right)}=x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}.$$

Therefore,

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)=\lim_{x\to 0}\left(x\cdot\frac{\frac{1}{3!}-\frac{x^2}{5!}+\cdots}{1-\frac{x^2}{3!}+\cdots}\right)=0.$$

级数求和/求和函数

求$\sum_{n=0}^{\infty}\frac{n+2}{n!}$

$$\sum_{n=0}^{\infty}\frac{n+2}{n!}=\sum_{n=0}^{\infty}\frac{nx^n}{n!}+\frac{2}{n!}x^n$$

$$=\sum_{n=0}^{\infty}x(\frac{x^n}{n!})'+\frac{2}{n!}x^n$$

$$=x(e^x)'+2e^x$$

$$=xe^x+2e^x$$

令$x=1$，原式$=3\mathrm{e}$

求$\sum_{n=0}^{\infty}(n+2)^2x^n$

$$\sum_{n=0}^{\infty}(n+2)^2x^n=\sum_{n=0}^{\infty}(n(n+1)+3(n+1)+1)x^n$$

$$=x(\sum x^{n+1})''+3(x^{n+1})'+x^n$$

$$=x\left(\frac{x}{1-x}\right)''+3\left(\frac{1}{1-x}\right)'+x^n$$

求泰勒展开前k项

求$lncosx$在$x = 0$的泰勒展开的前三项

In order to compute the 7th degree Maclaurin polynomial for the function

$$f(x) = \ln(\cos x), \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

one may first rewrite the function as

$$f(x) = \ln\left(1 + (\cos x - 1)\right).$$

The Taylor series for the natural logarithm is (using the big O notation)

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + O\left(x^4\right)$$

and for the cosine function

$$\cos x - 1 = -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^8\right).$$

https://en.wikipedia.org/wiki/Taylor_series

The latter series expansion has a zero constant term, which enables us to substitute the second series into the first one and to easily omit terms of higher order than the 7th degree by using the big O notation:

$$\begin{aligned}
f(x) &= \ln\left(1+(\cos x - 1)\right) \\
&= (\cos x - 1) - \tfrac{1}{2}(\cos x - 1)^2 + \tfrac{1}{3}(\cos x - 1)^3 + O\left((\cos x - 1)^4\right) \\
&= \left(-\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O\left(x^8\right)\right) - \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + O\left(x^6\right)\right)^2 + \frac{1}{3}\left(-\frac{x^2}{2} + O\left(x^4\right)\right)^3 + O\left(x^8\right) \\
&= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} - \frac{x^4}{8} + \frac{x^6}{48} - \frac{x^6}{24} + O\left(x^8\right) \\
&= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + O\left(x^8\right).
\end{aligned}$$

Since the cosine is an even function, the coefficients for all the odd powers $x, x^3, x^5, x^7, \ldots$ have to be zero.

https://en.wikipedia.org/wiki/Taylor_series

求$\frac{e^x}{cosx}$在$x = 0$的泰勒展开的前四项

Suppose we want the Taylor series at 0 of the function

$$g(x) = \frac{e^x}{\cos x}.$$

We have for the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and, as in the first example,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

https://en.wikipedia.org/wiki/Taylor_series

Assume the power series is

$$\frac{e^x}{\cos x} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

Then multiplication with the denominator and substitution of the series of the cosine yields

$$\begin{aligned} e^x &= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots\right) \cos x \\ &= \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots\right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) \\ &= c_0 - \frac{c_0}{2} x^2 + \frac{c_0}{4!} x^4 + c_1 x - \frac{c_1}{2} x^3 + \frac{c_1}{4!} x^5 + c_2 x^2 - \frac{c_2}{2} x^4 + \frac{c_2}{4!} x^6 + c_3 x^3 - \frac{c_3}{2} x^5 + \frac{c_3}{4!} x^7 + c_4 x^4 + \cdots \end{aligned}$$

Collecting the terms up to fourth order yields

$$e^x = c_0 + c_1 x + \left(c_2 - \frac{c_0}{2}\right) x^2 + \left(c_3 - \frac{c_1}{2}\right) x^3 + \left(c_4 - \frac{c_2}{2} + \frac{c_0}{4!}\right) x^4 + \cdots$$

The values of c_i can be found by comparison of coefficients with the top expression for e^x, yielding:

$$\frac{e^x}{\cos x} = 1 + x + x^2 + \frac{2x^3}{3} + \frac{x^4}{2} + \cdots.$$

https://en.wikipedia.org/wiki/Taylor_series