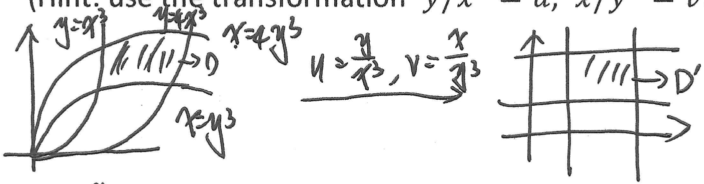
1. Find the area of the region enclosed by $y = x^3$, $y = 4x^3$, $x = y^3$, $x = 4y^3$ in the first quadrant. (Hint: use the transformation $y/x^3 = u,\ x/y^3 = v$.)



$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{3y}{x^4} & \frac{1}{x^3} \\ \frac{1}{y^3} & -\frac{3x}{y^4} \end{vmatrix} = \frac{8}{(xy)^3} = 8(uv)^{\frac{3}{2}}$$

令 $\begin{cases} u = \frac{y}{x^3} \\ v = \frac{x}{y^3} \end{cases}$，则

$$\therefore J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{8}u^{-\frac{3}{2}}v^{-\frac{3}{2}}$$

$y=x^3 \Rightarrow u=1$，$x=y^3 \Rightarrow v=1$

$y=4x^3 \Rightarrow u=4$，$x=4y^3 \Rightarrow v=4$，

$$\therefore \text{该面积 } S = \iint_D dxdy = \iint_{D'} |J|\,dudv = \frac{1}{8}\int_1^4\int_1^4 u^{-\frac{3}{2}}v^{-\frac{3}{2}}dudv$$
$$= \frac{1}{8}\int_1^4 u^{-\frac{3}{2}}du \int_1^4 v^{-\frac{3}{2}}dv = \frac{1}{8}$$

2. Evaluate $\int_C ye^{x^2}\,ds$, where C is the curve $\vec{r}(t) = 4t\vec{i} - 3t\vec{j}$, for $-1 \le t \le 2$.

解. $\vec{r}(t) = 4t\vec{i} - 3t\vec{j}$，$x = 4t$, $y = -3t$, $x'(t) = 4$, $y'(t) = -3$

$$\therefore \int_C ye^{x^2}ds = \int_{-1}^2 -3t\,e^{16t^2}\cdot\sqrt{4^2+(-3)^2}\,dt = \int_{-1}^2 -15t\,e^{16t^2}dt$$
$$= -\frac{15}{32}\int_{-1}^2 e^{16t^2}d(16t^2) = -\frac{15}{32}\left[e^{16t^2}\right]_{-1}^2 = \frac{15}{32}(e^{16}-e^{64})$$

3. Find the work done by the force field $\mathbf{F} = 2y\vec{i} + 3x\vec{j} + (x+y)\vec{k}$ in moving an object along the curve

$\vec{r}(t) = (\cos t)\vec{i} + (\sin t)\vec{j} + (\frac{t}{6})\vec{k}$, $0 \le t \le 2\pi$.

解. $\vec{F}$ 所做的功 $W = \int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot\frac{d\vec{r}}{dt}dt = \int_0^{2\pi}(-2\sin^2t + 3\cos^2t + \frac{1}{6}(\cos t+\sin t))\,dt$

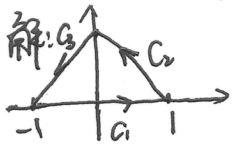
$\vec{F} = (2\sin t)\vec{i} + (3\cos t)\vec{j} + (\cos t+\sin t)\vec{k}$

$\frac{d\vec{r}}{dt} = (-\sin t)\vec{i} + (\cos t)\vec{j} + \frac{1}{6}\vec{k}$.

$$= \int_0^{2\pi}\left[5\cos^2t - 2 + \frac{1}{6}\cos t + \frac{1}{6}\sin t\right]dt$$
$$= \frac{5}{2}\int_0^{2\pi}(\cos 2t+1)\,dt - 2\cdot 2\pi$$
$$= \frac{5}{2}\cdot 2\pi - 2\cdot 2\pi = \pi$$

4. Find the circulation and flux of the field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ around and across the triangle with vertices $(1,0)$, $(0,1)$, $(-1,0)$.

解:



C_1: $\vec{r}_1(t) = t\vec{i}$, $-1 \le t \le 1$, C_2: $\vec{r}_2(t) = t\vec{i} + (1-t)\vec{j}$, $0 \le t \le 1$, (t从1到0变化)

C_3: $\vec{r}_3(t) = t\vec{i} + (t+1)\vec{j}$, t从0到-1. $\vec{r}_1'(t) = \vec{i}$, $\vec{r}_2'(t) = \vec{i} - \vec{j}$, $\vec{r}_3'(t) = \vec{i} + \vec{j}$

$\vec{F}(\vec{r}_1(t)) = t\vec{i} - t^2\vec{j}$, $\vec{F}(\vec{r}_2(t)) = \vec{i} - (2t^2-2t+1)\vec{j}$, $\vec{F}(\vec{r}_3(t)) = (2t+1)\vec{i} - (2t^2+2t+1)\vec{j}$

Circulation : $\int_{C_1}\vec{F}\cdot\frac{d\vec{r}_1}{dt}\cdot dt = \int_{-1}^1 t\,dt = \left[\frac{t^2}{2}\right]_{-1}^1 = 0$, $\int_{C_2}\vec{F}\cdot\frac{d\vec{r}_2}{dt}dt = \int_1^0(1+2t^2-2t+1)\,dt = \left[\frac{2}{3}t^3 - t^2 + 2t\right]_1^0 = -\frac{5}{3}$

$$\int_{C_3}\vec{F}\cdot\frac{d\vec{r}_3}{dt}dt = \int_0^{-1}(2t+1-2t^2-2t-1)\,dt = -\left[\frac{2t^3}{3}\right]_0^{-1} = \frac{2}{3}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1}\vec{F}\cdot d\vec{r}_1 + \int_{C_2}\vec{F}\cdot d\vec{r}_2 + \int_{C_3}\vec{F}\cdot d\vec{r}_3 = -1$$

Flux: $\oint_{C_1}\vec{F}\cdot\vec{n}_1ds = \oint_{C_1}M_1dy - N_1dx = \int_{-1}^1 t^2\,dt = \left[\frac{t^3}{3}\right]_{-1}^1 = \frac{2}{3}$, $\oint_{C_2}\vec{F}\cdot\vec{n}_2ds = \oint_{C_2}M_2dy - N_2dx = \int_1^0(-1+2t^2-2t+1)\,dt = \frac{1}{3}$

$\oint_{C_3}\vec{F}\cdot\vec{n}_3ds = \oint_{C_3}M_3dy - N_3dx = \int_0^{-1}(2t+1+2t^2+2t+1)\,dt = -\frac{2}{3}$, $\therefore \oint_C\vec{F}\cdot\vec{n}ds = \frac{2}{3}+\frac{1}{3}-\frac{2}{3} = \frac{1}{3}$