第12周高数互助课堂

15.1-15.4

多元函数判断极限存在性的方法

- 换元法
- 夹逼定理
- 举反例

判断极限 $\boldsymbol{lim_{(x,y)\to(0,0)}\frac{x^5+x^6}{x^3+y}}$ 是否存在

令 $y=rsin\theta, x=rcos\theta$ 则

$$\lim_{(x,y)\to(0,0)}\frac{x^5+y^6}{x^3+y}$$

$$=\lim_{r\to 0}\frac{r^5\cos^5\theta+r^6\sin^6\theta}{r^3\cos^3\theta+r\,sin\,\theta}=\lim_{r\to 0}\frac{r^4\cos^5\theta+r^5\sin^6\theta}{r^2\cos^3\theta+sin\,\theta}$$

当 $\theta=0$,原极限 $=\lim_{x\to 0}\frac{x^5}{x^3}=\lim_{x\to 0}x^2=0$

当 $\theta\neq 0$,原极限 $=\lim_{r\to 0}\frac{0}{sin\theta}=0$

原极限=0?

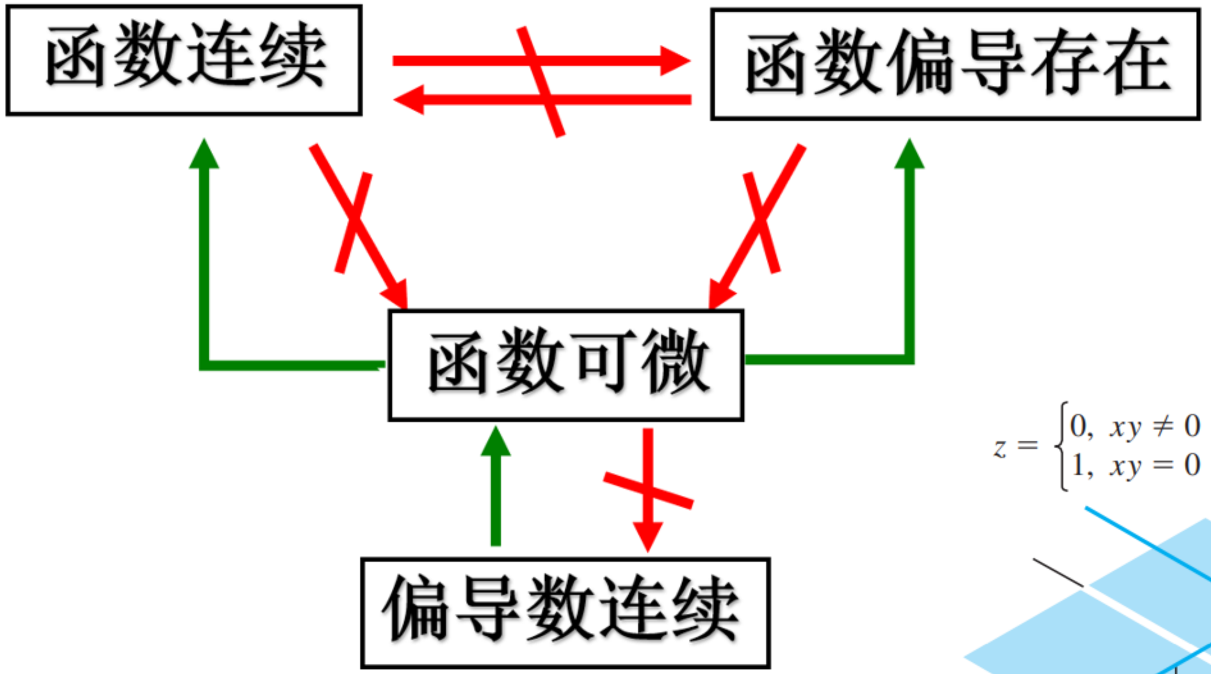
判断极限 $\boldsymbol{lim_{(x,y)\to(0,0)} \frac{x^5+y^6}{x^3+y}}$ 是否存在

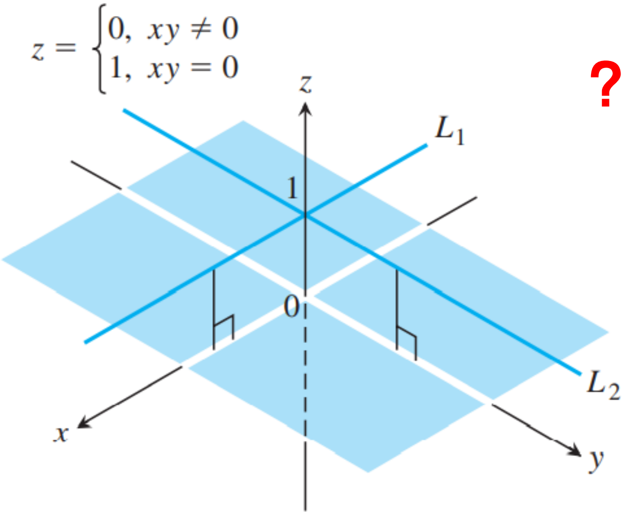
$let\ y = -x^3 + x^5,$

$$\lim_{(x,y)\to(0,0)} \frac{x^5+y^6}{x^3+y} = \lim_{(x,y)\to(0,0)} \frac{x^5+\left(-x^3+x^5\right)^6}{x^5} = 1$$

$$let\ y = 0, \lim_{(x,y)\to(0,0)} \frac{x^5+y^6}{x^3+y} = 0$$

The limit does not exist.


函数连续
函数偏导存在
函数可微
偏导数连续


$z = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$
z
L_1
1
0
L_2
x
y

?

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连续

定义 3 设二元函数 $f(P)=f(x,y)$ 的定义域为 D，$P_0(x_0,y_0)$ 为 D 的聚点，且 $P_0\in D$. 如果

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0),$$

则称函数 $f(x,y)$ 在点 $P_0(x_0,y_0)$ 连续.

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

偏导存在

定义 $\boxed{\text{设函数 } z=f(x,y) \text{ 在点}(x_0,y_0)\text{的某一邻域内有定义}}$，当 y 固定在 y_0 而 x 在 x_0 处有增量 Δx 时，相应的函数有增量

$$f(x_0+\Delta x,y_0)-f(x_0,y_0),$$

如果

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \tag{1}$$

存在，则称此极限为函数 $z=f(x,y)$ 在点 (x_0,y_0) 处对 x 的偏导数，记作

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\left.z_x\right|_{\substack{x=x_0\\y=y_0}} \text{ 或 } f_x(x_0,y_0).$$①

DEFINITION The **partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0)** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h, y_0)-f(x_0, y_0)}{h},$$

provided the limit exists.

可微

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定义 设函数 $z=f(x,y)$ 在点 (x,y) 的某邻域内有定义，如果函数在点 (x,y) 的全增量

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$$

可表示为

$$\Delta z=A\Delta x+B\Delta y+o(\rho), \tag{2}$$

其中 A、B 不依赖于 Δx、Δy 而仅与 x、y 有关，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称函数 $z=f(x,y)$ 在点 (x,y) 可微分，而 $A\Delta x+B\Delta y$ 称为函数 $z=f(x,y)$ 在点 (x,y) 的全微分，记作 $\mathrm{d}z$，即

$$\mathrm{d}z=A\Delta x+B\Delta y.$$

如果函数在区域 D 内各点处都可微分，那么称这函数在 D 内可微分.

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DEFINITION A function $z=f(x,y)$ is **differentiable at** (x_0,y_0) if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z=f_x(x_0,y_0)\Delta x+f_y(x_0,y_0)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$$

in which each of $\epsilon_1,\epsilon_2\to 0$ as both $\Delta x,\Delta y\to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

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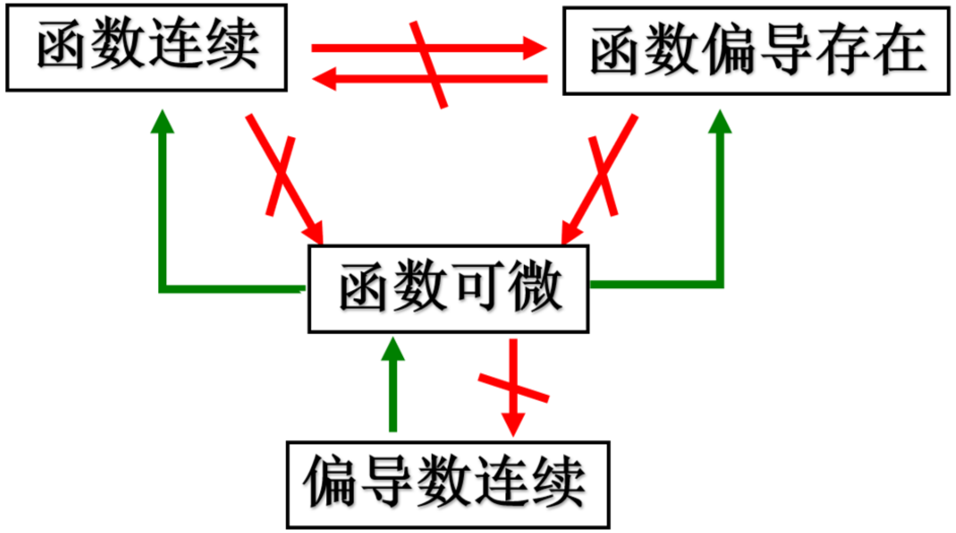

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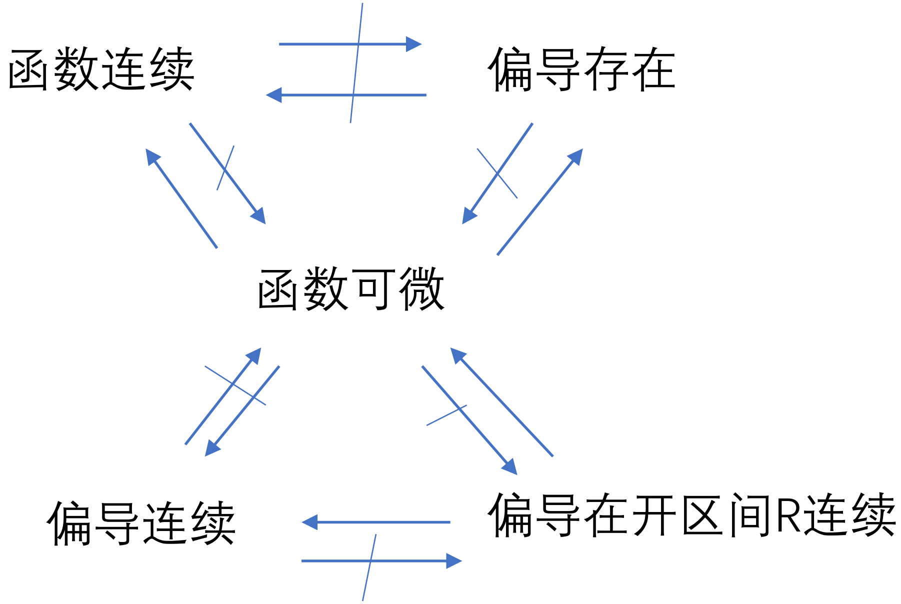


方向导数（标量）

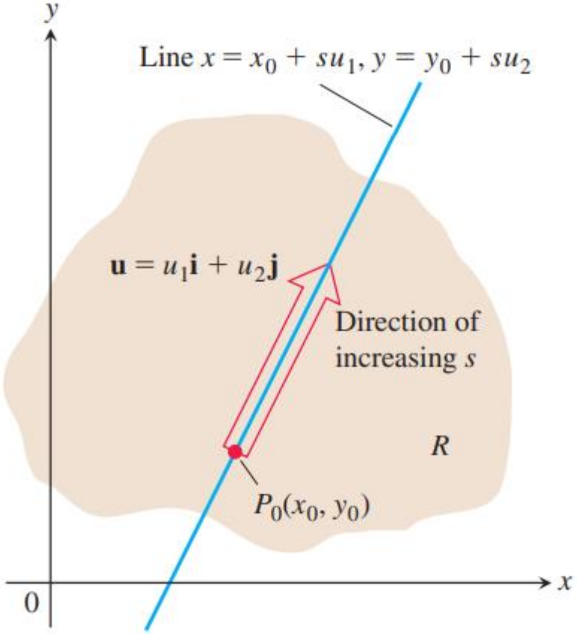
大小：函数在 P_0 点沿 $\vec{u}$ 方向的的变化率

定义：

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0}\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.



梯度（矢量）

方向：函数增长最快的方向

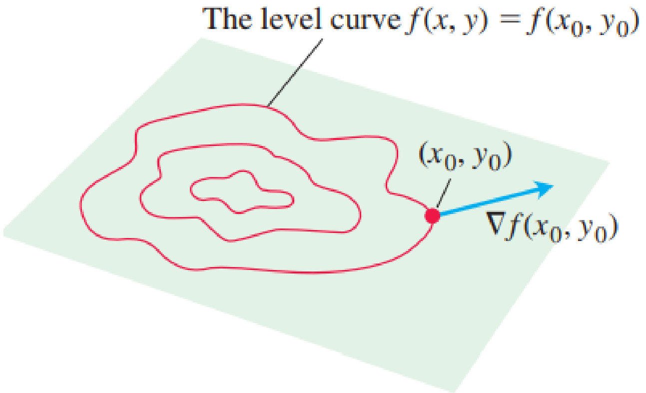
大小：方向导数的最大值

定义：

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.



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THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

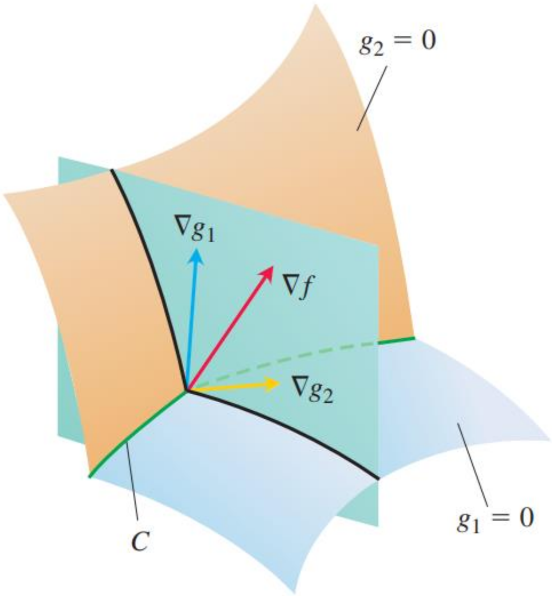
the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

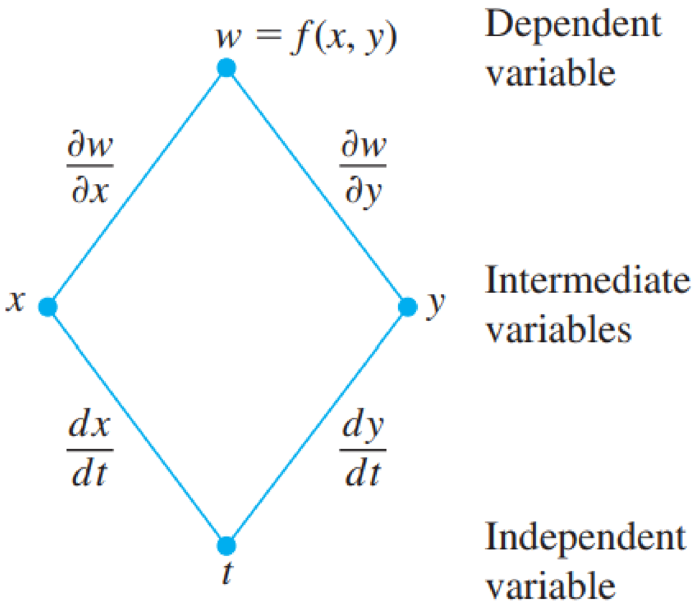
$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.



$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$

Chain Rule



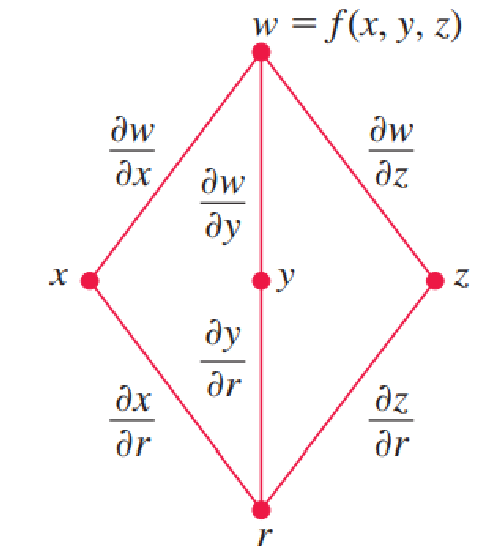
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

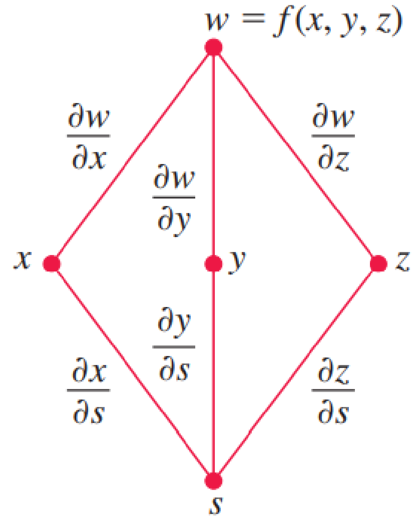
$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\delta y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$



$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of f

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fails to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

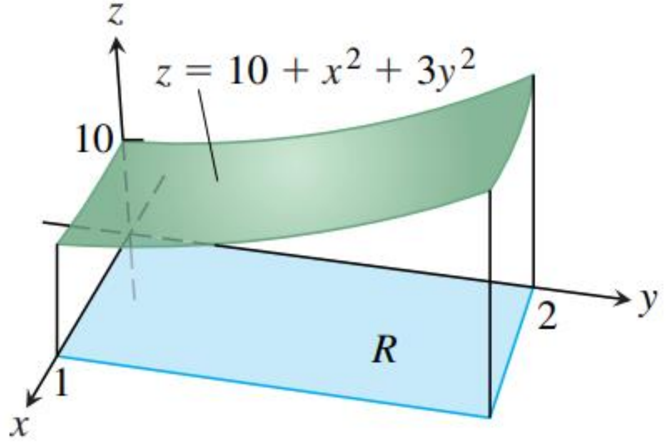
ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive**

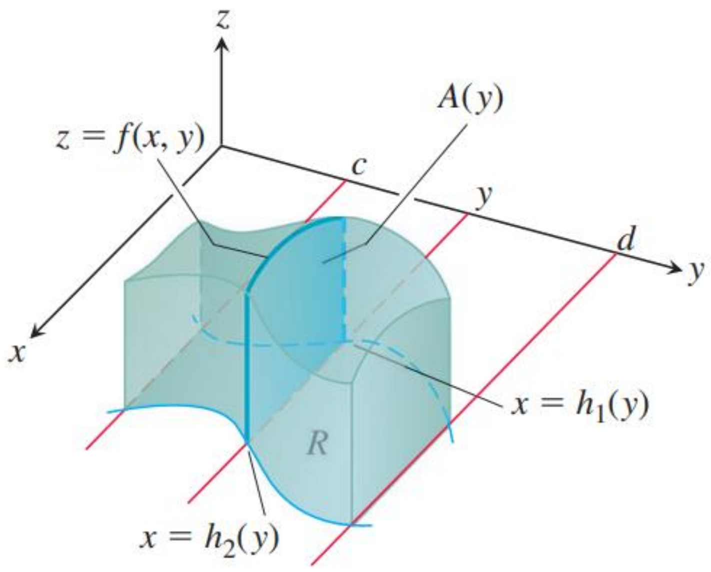
$$f_{xx}f_{yy} - f_{xy}{}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

15.1-15.4



THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$



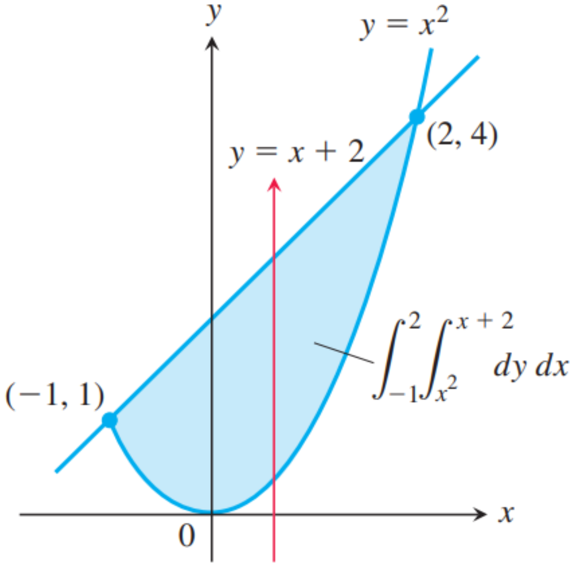
THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



(b)

EXAMPLE 2 Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Solution If we divide R into the regions R_1 and R_2 shown in Figure 15.20a, we may calculate the area as

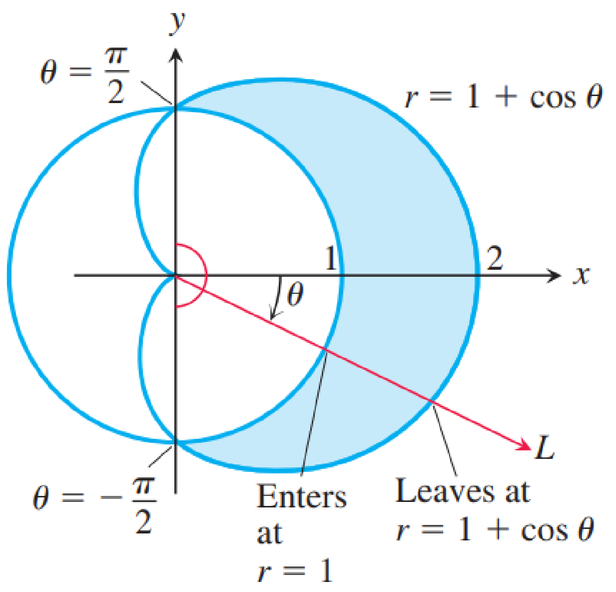
$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy.$$

On the other hand, reversing the order of integration (Figure 15.20b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx.$$

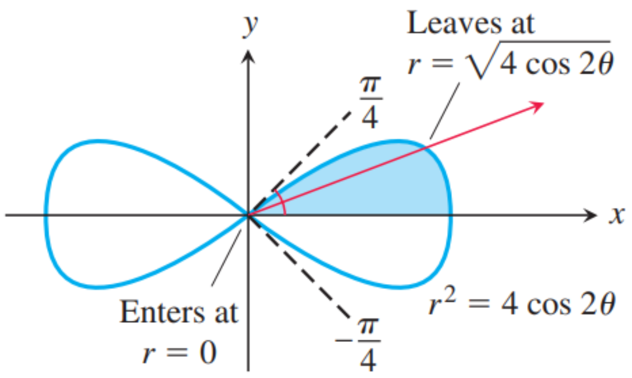
This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^{2} \Big[y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\,dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$



EXAMPLE 1 Find the limits of integration for integrating $f(r, \theta)$ over the region R that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

$$\int_{-\pi/2}^{\pi/2} \int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$



EXAMPLE 2 Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

$$A = 4\int_0^{\pi/4} \int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

EXAMPLE 4 Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

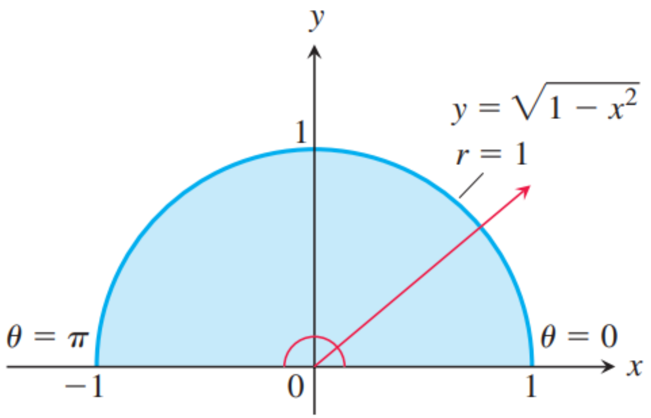
Solution Integration with respect to y gives

$$\int_0^1 \left(x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.

EXAMPLE 4 Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$



$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^4}{4}\right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$