1. (5 points) Use Stokes' Theorem to calculate the flux of the curl of the field $\vec{F}$ across the surface S in the direction of the outward unit normal $\vec{n}$.

$$\begin{aligned}
&\vec{F} = (y-z)\vec{i} + (z-x)\vec{j} + (x+z)\vec{k} \\
&S := \vec{r}(r,\theta) = (r\cos\theta)\vec{i} + (r\sin\theta)\vec{j} + (9-r^2)\vec{k} \\
&0 \le r \le 3, \quad 0 \le \theta \le 2\pi
\end{aligned}$$

Solution:

The boundary $C := \vec{r}(\theta) = (3\cos\theta)\vec{i} + (3\sin\theta)\vec{j}$

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot \vec{n}\mathrm{d}\sigma &= \oint_C \vec{F} \cdot \mathrm{d}\vec{r} \\
&= \int_0^{2\pi} (3\sin\theta\vec{i} - 3\cos\theta\vec{j} + 3\cos\theta\vec{k}) \cdot (-3\sin\theta\vec{i} + 3\cos\theta\vec{j})\mathrm{d}\theta \\
&= -18\pi
\end{aligned}$$

2. (5 points) Use the Divergence Theorem to find the outward flux of $\vec{F}$ across the boundary of the region D.

$$\begin{aligned}
&\vec{F} = 2xz\vec{i} - xy\vec{j} - z^2\vec{k}) \\
&D : \text{The wedge cut from the first octant by the plane } y+z=4 \\
&\quad \text{and the elliptical cylinder } 4x^2 + y^2 = 16
\end{aligned}$$

Solution:

$$\begin{aligned}
\nabla \cdot \vec{F} &= 2z - x - 2z = -x \\
\Rightarrow \iint_S \vec{F} \cdot \vec{n}\mathrm{d}\sigma &= \iiint_D \nabla \cdot \vec{F}\mathrm{d}V = \iint_D -x\mathrm{d}V \\
&= \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} -x\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \int_0^2 \int_0^{\sqrt{16-4x^2}} (xy - 4x)\mathrm{d}y\mathrm{d}x \\
&= \int_0^2 \left[\frac{1}{2}x(16-4x^2) - 4x\sqrt{16-4x^2}\right]\mathrm{d}x \\
&= -\frac{40}{3}
\end{aligned}$$