# 第12周高数互助课堂

15.1-15.4

## 多元函数判断极限存在性的方法

- 换元法
- 夹逼定理
- 举反例

**判断极限 $\boldsymbol{lim_{(x,y)\to(0,0)}\frac{x^5+x^6}{x^3+y}}$ 是否存在**

令 $y = rsin\theta, x = rcos\theta$ 则

$$\lim_{(x,y)\to(0,0)}\frac{x^5+y^6}{x^3+y}$$

$$= \lim_{r\to 0}\frac{r^5\cos^5\theta + r^6\sin^6\theta}{r^3\cos^3\theta + r\,sin\,\theta} = \lim_{r\to 0}\frac{r^4\cos^5\theta + r^5\sin^6\theta}{r^2\cos^3\theta + sin\,\theta}$$

当 $\theta = 0$，原极限 $= \lim_{x\to 0}\frac{x^5}{x^3} = \lim_{x\to 0}x^2 = 0$

当 $\theta \neq 0$，原极限 $= \lim_{r\to 0}\frac{0}{sin\theta} = 0$

原极限=0?

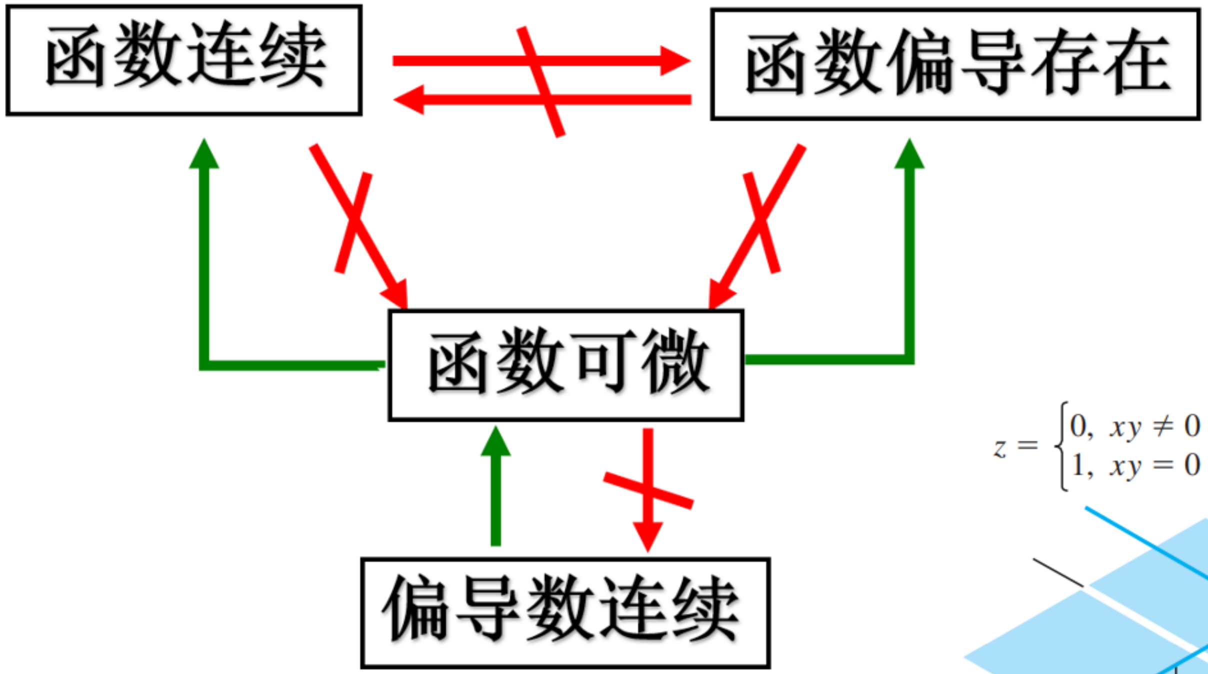
**判断极限 $\boldsymbol{lim_{(x,y)\to(0,0)}\frac{x^5+y^6}{x^3+y}}$ 是否存在**

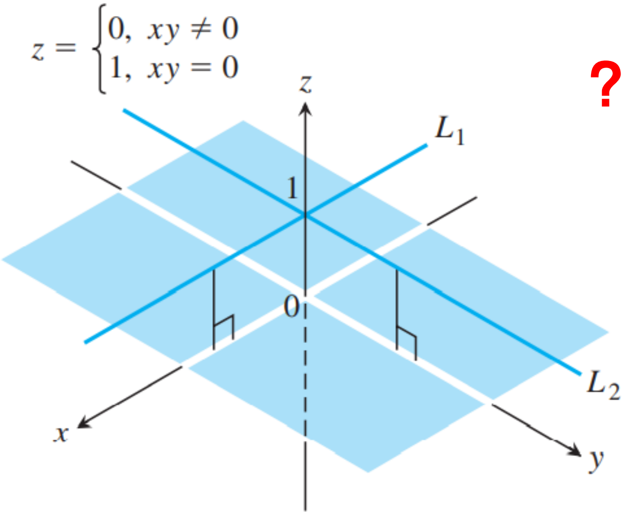
$let\ y=-x^3+x^5,$

$$\lim_{(x,y)\to(0,0)}\frac{x^5+y^6}{x^3+y}=\lim_{(x,y)\to(0,0)}\frac{x^5+\left(-x^3+x^5\right)^6}{x^5}=1$$

$let\ y=0,\ \lim_{(x,y)\to(0,0)}\frac{x^5+y^6}{x^3+y}=0$

***The limit does not exist.***


函数连续
函数偏导存在
函数可微
偏导数连续


$z = \begin{cases} 0, & xy \neq 0 \\ 1, & xy = 0 \end{cases}$
z
$L_1$
1
0
$L_2$
x
y

?

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连续

**定义 3** 设二元函数 $f(P)=f(x,y)$ 的定义域为 $D$，$P_0(x_0,y_0)$ 为 $D$ 的聚点，且 $P_0\in D$. 如果

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)=f(x_0,y_0),$$

则称函数 $f(x,y)$ 在点 $P_0(x_0,y_0)$ 连续.

**DEFINITION** A function $f(x, y)$ is **continuous at the point $(x_0, y_0)$** if

1. $f$ is defined at $(x_0, y_0)$,
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

偏导存在

**定义** 设函数 $z=f(x,y)$ 在点 $(x_0,y_0)$ 的某一邻域内有定义，当 $y$ 固定在 $y_0$ 而 $x$ 在 $x_0$ 处有增量 $\Delta x$ 时，相应的函数有增量

$$f(x_0+\Delta x,y_0)-f(x_0,y_0),$$

如果

$$\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x} \qquad (1)$$

存在，则称此极限为函数 $z=f(x,y)$ 在点 $(x_0,y_0)$ 处对 $x$ 的偏导数，记作

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\ \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}},\ \left.z_x\right|_{\substack{x=x_0\\y=y_0}} \quad 或 \quad f_x(x_0,y_0).①$$

**DEFINITION** The **partial derivative of $f(x, y)$ with respect to $x$ at the point $(x_0, y_0)$** is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h, y_0)-f(x_0, y_0)}{h},$$

provided the limit exists.

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**可微**

**定义** 设函数 $z=f(x,y)$ 在点 $(x,y)$ 的某邻域内有定义，如果函数在点 $(x,y)$ 的全增量

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$$

可表示为

$$\Delta z=A\Delta x+B\Delta y+o(\rho), \tag{2}$$

其中 $A$、$B$ 不依赖于 $\Delta x$、$\Delta y$ 而仅与 $x$、$y$ 有关，$\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$，则称函数 $z=f(x,y)$ 在点 $(x,y)$ 可微分，而 $A\Delta x+B\Delta y$ 称为函数 $z=f(x,y)$ 在点 $(x,y)$ 的全微分，记作 $\mathrm{d}z$，即

$$\mathrm{d}z=A\Delta x+B\Delta y.$$

如果函数在区域 $D$ 内各点处都可微分，那么称这函数在 $D$ 内可微分.

**DEFINITION** A function $z=f(x,y)$ is **differentiable at** $(x_0,y_0)$ if $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist and $\Delta z$ satisfies an equation of the form

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call $f$ **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

**COROLLARY OF THEOREM 3** If the partial derivatives $f_x$ and $f_y$ of a function $f(x, y)$ are continuous throughout an open region $R$, then $f$ is differentiable at every point of $R$.

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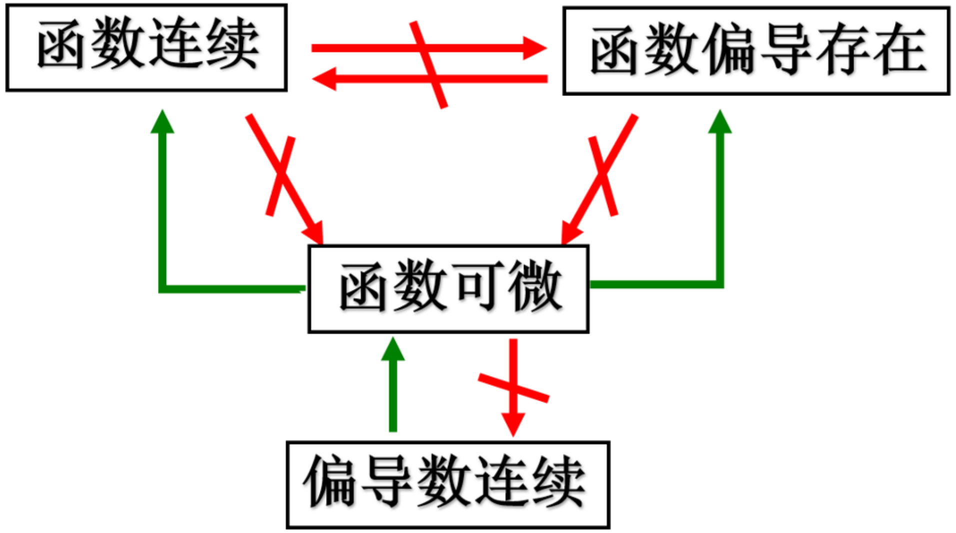


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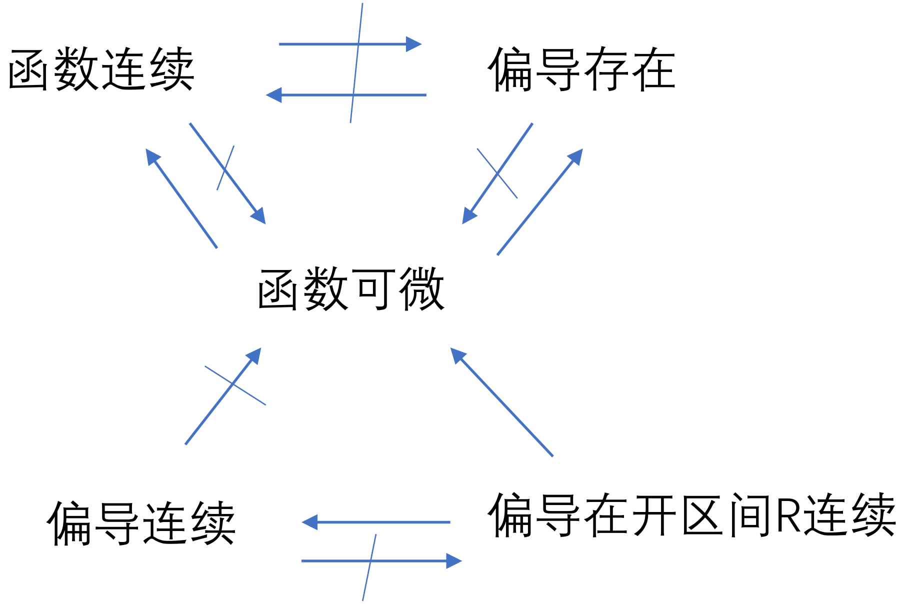


## 方向导数（标量）

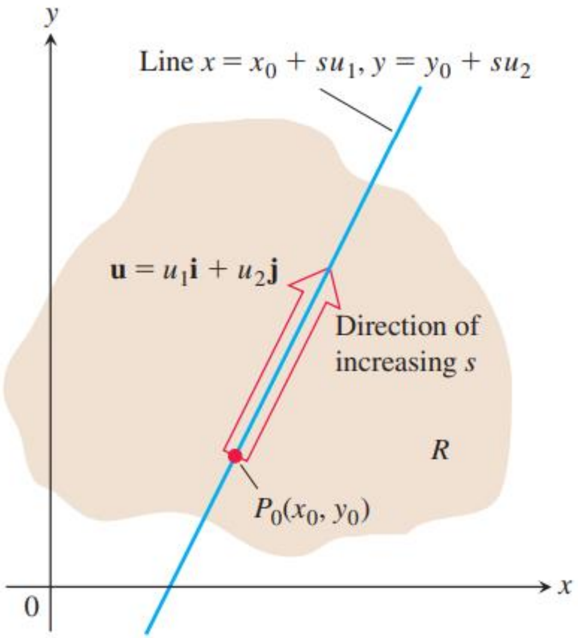
大小：函数在 $P_0$ 点沿 $\vec{u}$ 方向的的变化率

定义：

**DEFINITION** The **derivative of $f$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \tag{1}$$

provided the limit exists.




## 梯度（矢量）

方向：函数增长最快的方向

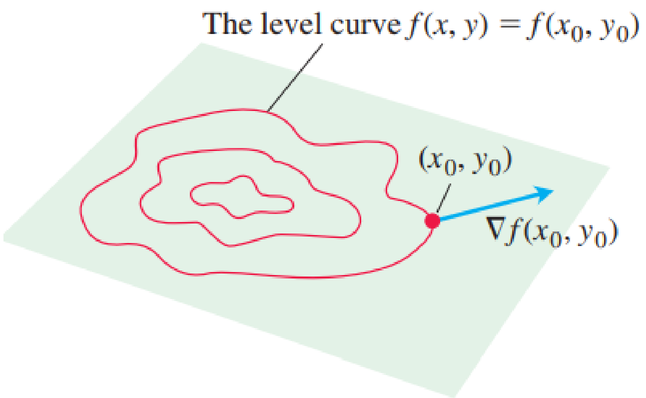
大小：方向导数的最大值

定义：

**DEFINITION** The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.



**DEFINITION** The **derivative of $f$ at $P_0(x_0, y_0)$ in the direction of the unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

**DEFINITION** The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of $f$ at $P_0$.

**THEOREM 9—The Directional Derivative Is a Dot Product** If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u}, P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \qquad (4)$$

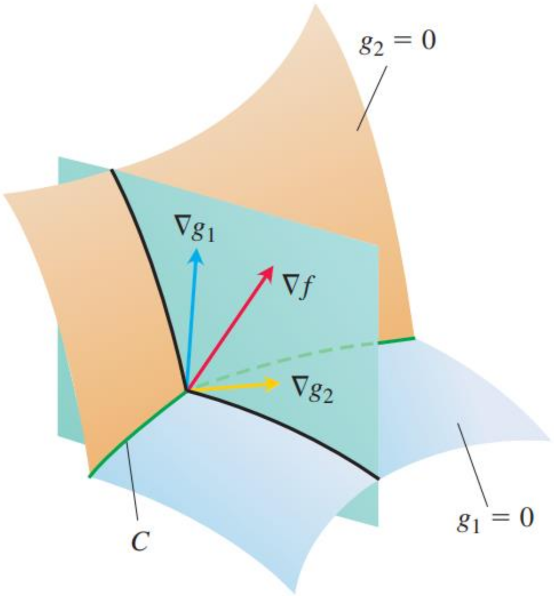
the dot product of the gradient $\nabla f$ at $P_0$ and $\mathbf{u}$. In brief, $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$.

### The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of $f$ subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of $x$, $y$, $z$, and $\lambda$ that simultaneously satisfy the equations

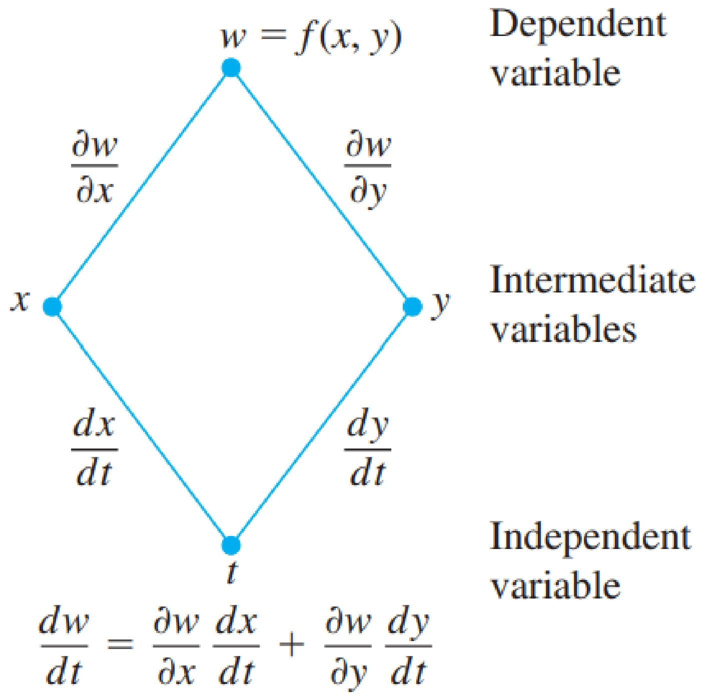
$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable $z$.




$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$

## Chain Rule

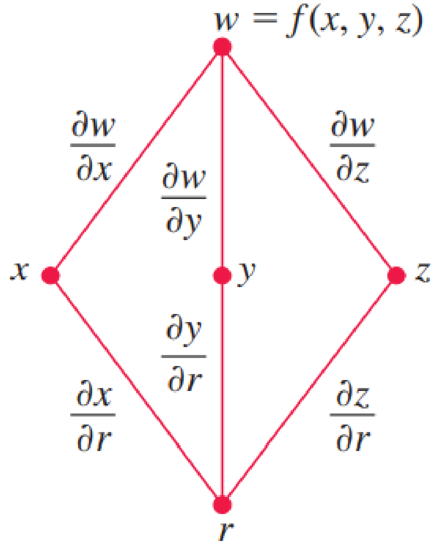



**THEOREM 5—Chain Rule For Functions of One Independent Variable and Two Intermediate Variables** If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of $t$, then the composite $w = f(x(t), y(t))$ is a differentiable function of $t$ and

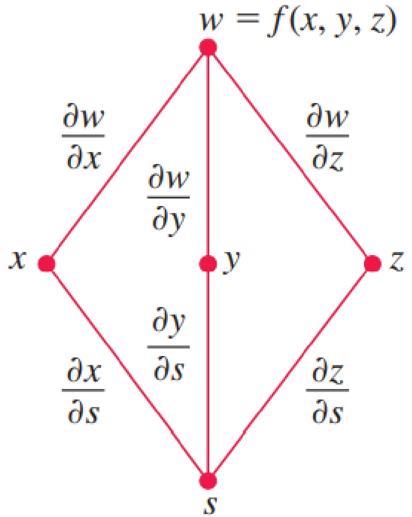
$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$




$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\delta y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$




$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

**THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables** Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then $w$ has partial derivatives with respect to $r$ and $s$, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$

## Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i) **boundary points** of the domain of $f$

ii) **critical points** (interior points where $f_x = f_y = 0$ or points where $f_x$ or $f_y$ fails to exist).

If the first- and second-order partial derivatives of $f$ are continuous throughout a disk centered at a point $(a, b)$ and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i) $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local maximum**

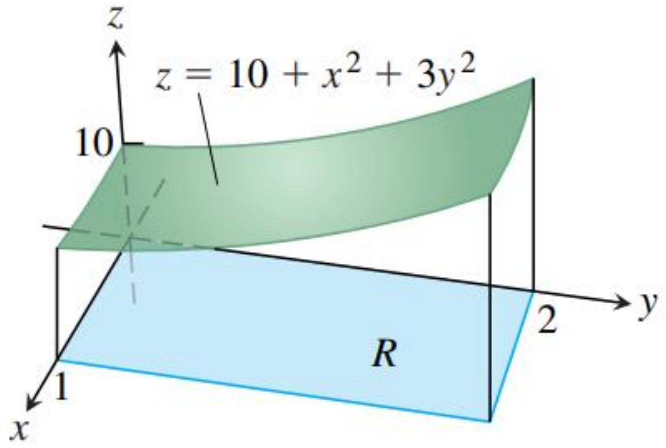
ii) $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}{}^2 > 0$ at $(a, b) \Rightarrow$ **local minimum**

iii) $f_{xx}f_{yy} - f_{xy}{}^2 < 0$ at $(a, b) \Rightarrow$ **saddle point**

iv) $f_{xx}f_{yy} - f_{xy}{}^2 = 0$ at $(a, b) \Rightarrow$ **test is inconclusive**

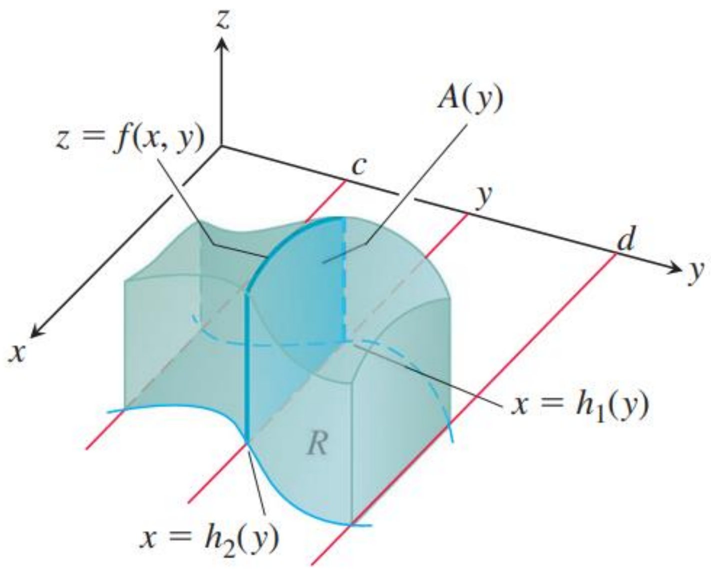
$$f_{xx}f_{yy} - f_{xy}{}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

# 15.1-15.4




**THEOREM 1—Fubini's Theorem (First Form)** If $f(x, y)$ is continuous throughout the rectangular region $R$: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$



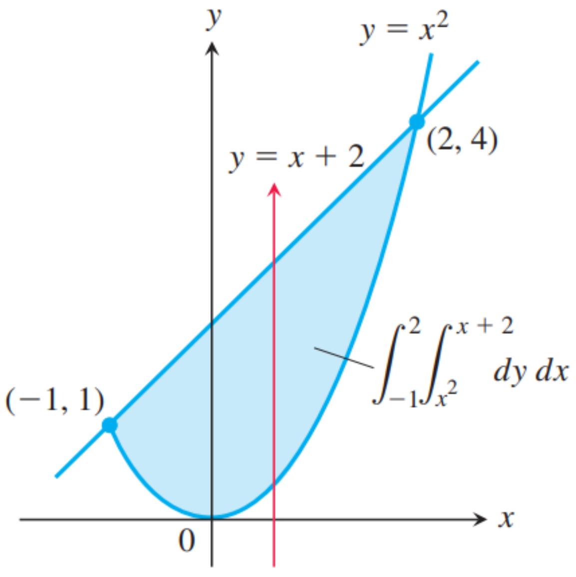

**THEOREM 2—Fubini's Theorem (Stronger Form)** Let $f(x, y)$ be continuous on a region $R$.

1. If $R$ is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with $g_1$ and $g_2$ continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If $R$ is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with $h_1$ and $h_2$ continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



(b)

**EXAMPLE 2** Find the area of the region $R$ enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

**Solution** If we divide $R$ into the regions $R_1$ and $R_2$ shown in Figure 15.20a, we may calculate the area as

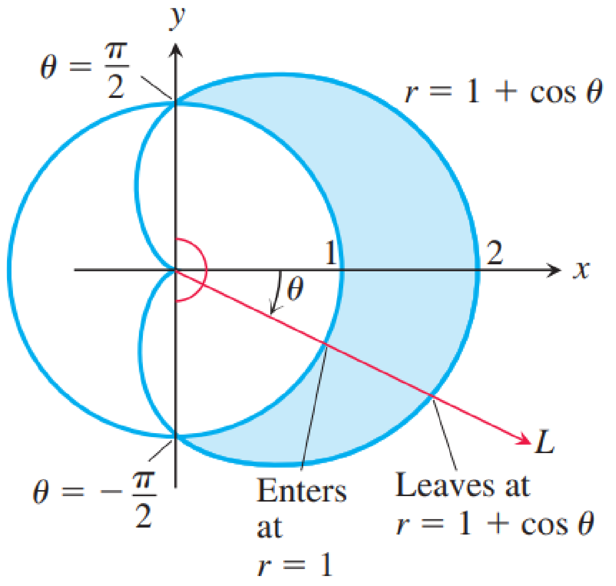
$$A = \iint_{R_1} dA + \iint_{R_2} dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy + \int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy.$$

On the other hand, reversing the order of integration (Figure 15.20b) gives

$$A = \int_{-1}^{2} \int_{x^2}^{x+2} dy\, dx.$$

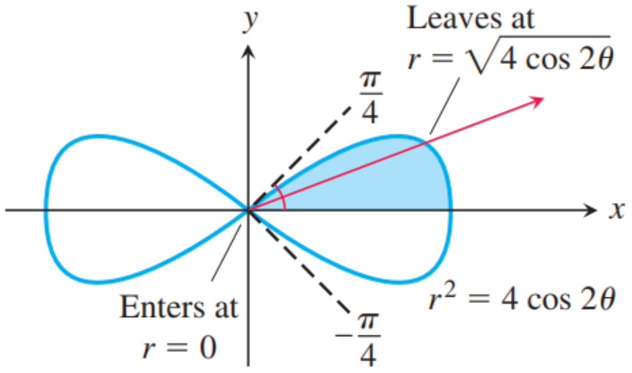
This second result, which requires only one integral, is simpler and is the only one we would bother to write down in practice. The area is

$$A = \int_{-1}^{2} \Big[ y \Big]_{x^2}^{x+2} dx = \int_{-1}^{2} (x + 2 - x^2)\, dx = \left[ \frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^{2} = \frac{9}{2}.$$ ■




**EXAMPLE 1** Find the limits of integration for integrating $f(r, \theta)$ over the region $R$ that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

$$\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} f(r, \theta)\, r\, dr\, d\theta.$$




**EXAMPLE 2** Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\, dr\, d\theta = 4\int_0^{\pi/4}\left[\frac{r^2}{2}\right]_{r=0}^{r=\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\, d\theta = 4\sin 2\theta\Big]_0^{\pi/4} = 4.$$

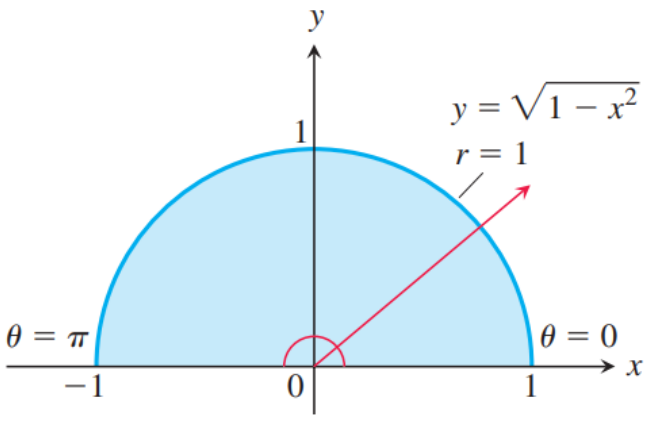
**EXAMPLE 4** Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

**Solution** Integration with respect to $y$ gives

$$\int_0^1 \left( x^2 \sqrt{1 - x^2} + \frac{(1 - x^2)^{3/2}}{3} \right) dx,$$

an integral difficult to evaluate without tables.




**EXAMPLE 4** Evaluate the integral

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx.$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dy\, dx = \int_0^{\pi/2} \int_0^1 (r^2)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_{r=0}^{r=1} d\theta = \int_0^{\pi/2} \frac{1}{4}\, d\theta = \frac{\pi}{8}.$$