第15周高数互助课堂

16.4-16.8

线积分

DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \leq t \leq b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k, \tag{1}$$

provided this limit exists.

How to Evaluate a Line Integral

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \leq t \leq b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\,|\mathbf{v}(t)|\, dt.$$

EXAMPLE 4 A slender metal arch, denser at the bottom than top, lies along the semicircle $y^2 + z^2 = 1$, $z \geq 0$, in the yz-plane (Figure 16.4). Find the center of the arch's mass if the density at the point (x, y, z) on the arch is $\delta(x, y, z) = 2 - z$.

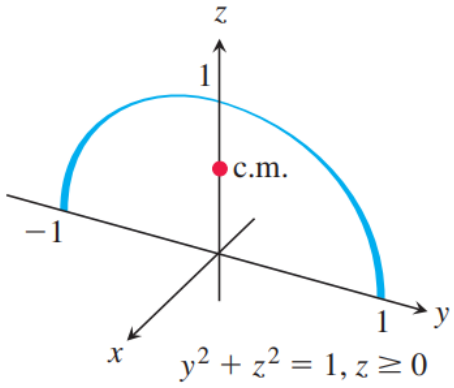


FIGURE 16.4 Example 4 shows how to find the center of mass of a circular arch of variable density.

Solution We know that $\bar{x} = 0$ and $\bar{y} = 0$ because the arch lies in the yz-plane with its mass distributed symmetrically about the z-axis. To find $\bar{z}$, we parametrize the circle as

$$\mathbf{r}(t) = (\cos t)\mathbf{j} + (\sin t)\mathbf{k}, \qquad 0 \leq t \leq \pi.$$

For this parametrization,

$$|\mathbf{v}(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(0)^2 + (-\sin t)^2 + (\cos t)^2} = 1,$$

so $ds = |\mathbf{v}|\, dt = dt$.

The formulas in Table 16.1 then give

$$M = \int_C \delta \, ds = \int_C (2 - z) \, ds = \int_0^{\pi} (2 - \sin t) \, dt = 2\pi - 2$$

$$M_{xy} = \int_C z\delta \, ds = \int_C z(2 - z) \, ds = \int_0^{\pi} (\sin t)(2 - \sin t) \, dt$$

$$= \int_0^{\pi} (2 \sin t - \sin^2 t) \, dt = \frac{8 - \pi}{2} \qquad \text{Routine integration}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{8 - \pi}{2} \cdot \frac{1}{2\pi - 2} = \frac{8 - \pi}{4\pi - 4} \approx 0.57.$$

With $\bar{z}$ to the nearest hundredth, the center of mass is (0, 0, 0.57).

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{4}$$

EXAMPLE 4 Find the work done by the force field $\mathbf{F} = (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k}$ along the curve $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, $0 \le t \le 1$, from $(0, 0, 0)$ to $(1, 1, 1)$ (Figure 16.18).

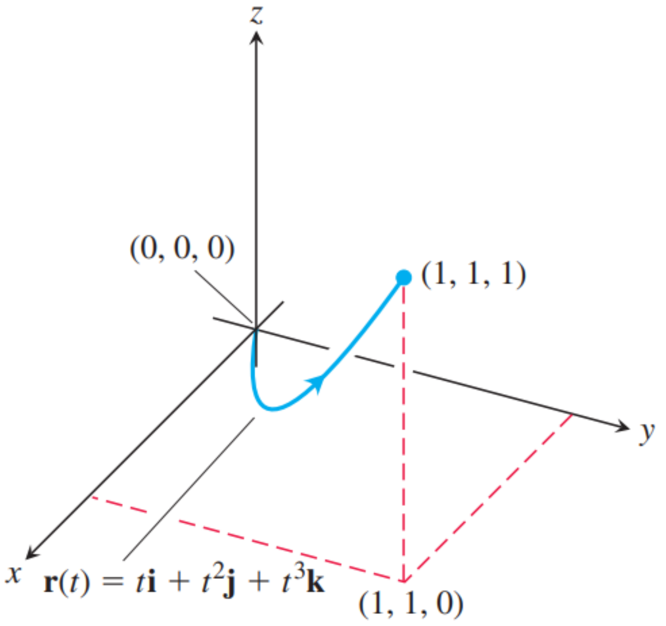


FIGURE 16.18 The curve in Example 4.

Solution First we evaluate $\mathbf{F}$ on the curve $\mathbf{r}(t)$:

$$\begin{aligned}\mathbf{F} &= (y - x^2)\mathbf{i} + (z - y^2)\mathbf{j} + (x - z^2)\mathbf{k} \\ &= \underbrace{(t^2 - t^2)}_{0}\mathbf{i} + (t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k}.\end{aligned}$$

Substitute $x = t$, $y = t^2$, $z = t^3$.

Then we find $d\mathbf{r}/dt$,

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

Finally, we find $\mathbf{F} \cdot d\mathbf{r}/dt$ and integrate from $t = 0$ to $t = 1$:

$$\begin{aligned}
\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} &= \left[(t^3 - t^4)\mathbf{j} + (t - t^6)\mathbf{k} \right] \cdot (\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}) \\
&= (t^3 - t^4)(2t) + (t - t^6)(3t^2) = 2t^4 - 2t^5 + 3t^3 - 3t^8.
\end{aligned}$$

So,

$$\begin{aligned}
\text{Work} &= \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\, dt \\
&= \left[\frac{2}{5}t^5 - \frac{2}{6}t^6 + \frac{3}{4}t^4 - \frac{3}{9}t^9 \right]_0^1 = \frac{29}{60}.
\end{aligned}$$

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

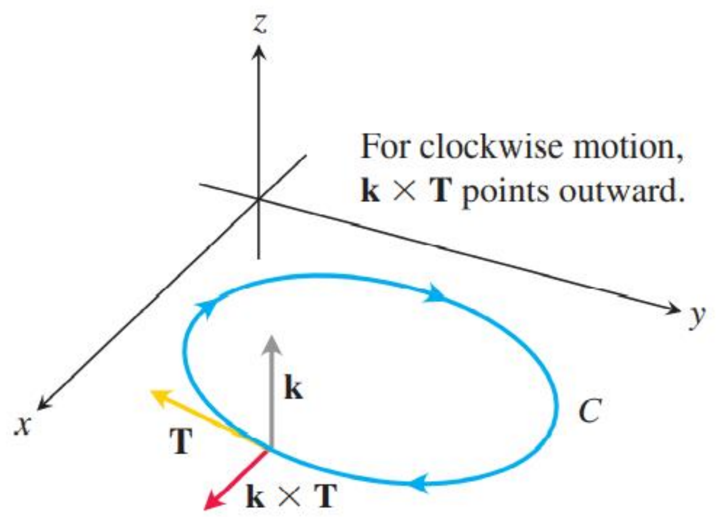
$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{5}$$

The integral is called a **flow integral**. If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across C} = \int_C \mathbf{F} \cdot \mathbf{n}\, ds. \tag{6}$$

$$\mathbf{n} = \mathbf{T} \times \mathbf{k} = \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}\right) \times \mathbf{k} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

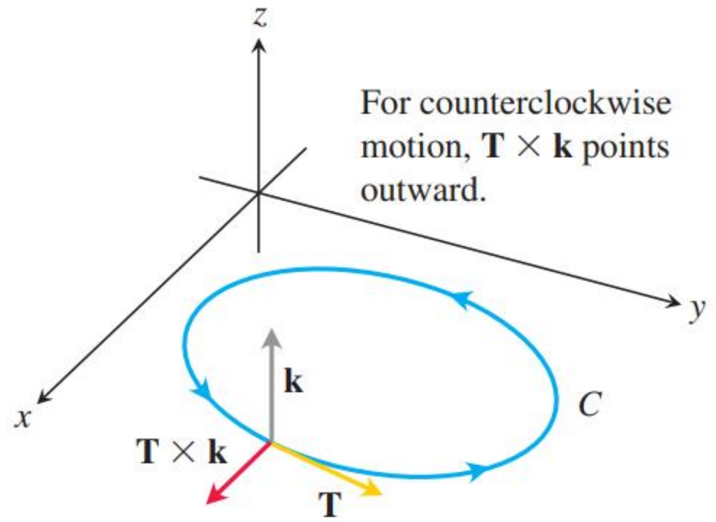


If $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$, then

$$\mathbf{F} \cdot \mathbf{n} = M(x, y)\frac{dy}{ds} - N(x, y)\frac{dx}{ds}.$$

Hence,

$$\int_C \mathbf{F} \cdot \mathbf{n}\, ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M\, dy - N\, dx.$$



Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\, dy - N\, dx \qquad (7)$$

The integral can be evaluated from any smooth parametrization $x = g(t)$, $y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for $\mathbf{F}$**.

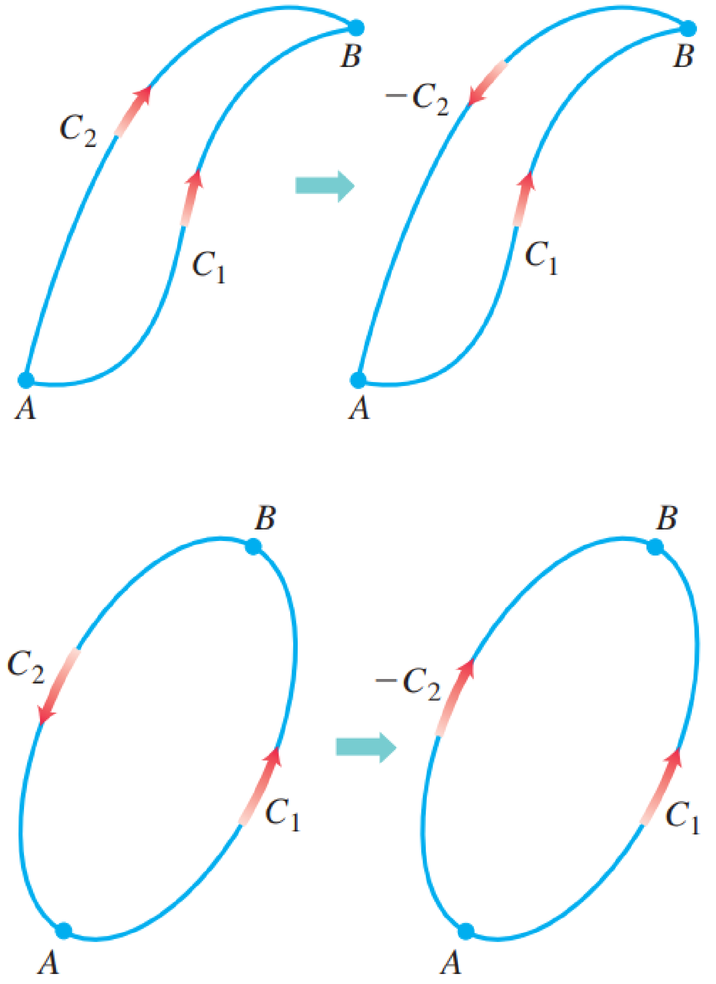
THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

THEOREM 2—Conservative Fields are Gradient Fields Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.



DEFINITIONS Any expression $M(x, y, z)\,dx + N(x, y, z)\,dy + P(x, y, z)\,dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M\,dx + N\,dy + P\,dz$

The differential form $M\,dx + N\,dy + P\,dz$ is exact on an open simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

F *is conservative*

$\oint_C F dr$ *is path in dependent*

$\oint_C F dr = 0$ *around every loop*

there exist a f *such that* $F = \nabla f$

F *is a field on an open simple connected domain*

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$Mdx + Ndy + Pdz$ *is exact*

如何证明F是保守的?

- 找到f使得$F = \nabla f$
- Component Test

如何证明F是不保守的?

- 找到两条积分结果不一样的路径
- 环路积分不为0
- Component Test

EXAMPLE 3 Show that $\mathbf{F} = (e^x \cos y + yz)\mathbf{i} + (xz - e^x \sin y)\mathbf{j} + (xy + z)\mathbf{k}$ is conservative over its natural domain and find a potential function for it.

Solution The natural domain of $\mathbf{F}$ is all of space, which is open and simply connected. We apply the test in Equations (2) to

$$M = e^x \cos y + yz, \qquad N = xz - e^x \sin y, \qquad P = xy + z$$

and calculate

$$\frac{\partial P}{\partial y} = x = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = y = \frac{\partial P}{\partial x}, \qquad \frac{\partial N}{\partial x} = -e^x \sin y + z = \frac{\partial M}{\partial y}.$$

We find f by integrating the equations

$$\frac{\partial f}{\partial x} = e^x \cos y + yz, \qquad \frac{\partial f}{\partial y} = xz - e^x \sin y, \qquad \frac{\partial f}{\partial z} = xy + z. \tag{3}$$

We integrate the first equation with respect to x, holding y and z fixed, to get

$$f(x, y, z) = e^x \cos y + xyz + g(y, z).$$

We write the constant of integration as a function of y and z because its value may depend on y and z, though not on x. We then calculate $\partial f / \partial y$ from this equation and match it with the expression for $\partial f / \partial y$ in Equations (3). This gives

$$-e^x \sin y + xz + \frac{\partial g}{\partial y} = xz - e^x \sin y,$$

so $\partial g / \partial y = 0$. Therefore, g is a function of z alone, and

$$f(x, y, z) = e^x \cos y + xyz + h(z).$$

We now calculate $\partial f / \partial z$ from this equation and match it to the formula for $\partial f / \partial z$ in Equations (3). This gives

$$xy + \frac{dh}{dz} = xy + z, \qquad \text{or} \qquad \frac{dh}{dz} = z,$$

so

$$h(z) = \frac{z^2}{2} + C.$$

Hence,

$$f(x, y, z) = e^x \cos y + xyz + \frac{z^2}{2} + C.$$

We found infinitely many potential functions of **F**, one for each value of C.

EXAMPLE 4 Show that $\mathbf{F} = (2x - 3)\mathbf{i} - z\mathbf{j} + (\cos z)\mathbf{k}$ is not conservative.

Solution We apply the Component Test in Equations (2) and find immediately that

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(\cos z) = 0, \qquad \frac{\partial N}{\partial z} = \frac{\partial}{\partial z}(-z) = -1.$$

The two are unequal, so $\mathbf{F}$ is not conservative. No further testing is required.

$$r = (0, t, 0), t \in (0,1)$$

$$v = (0,1,0)$$

$$\int_C F dr = \int_0^1 (-3,0,1)(0,1,0) dt = 0$$

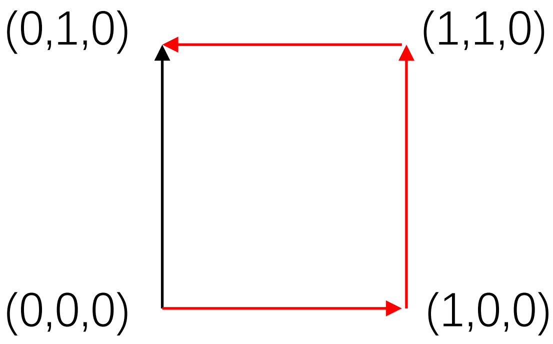
$$r_1 = (t, 0,0), r_2 = (0, t, 0), r_3 = (-t, 0,0), t \in (0,1)$$

$$v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (-1,0,0)$$

$$\int_{C_1} F dr = \int_0^1 (2t - 3,0,1)(1,0,0) dt = -2$$

$$\int_{C_2} F dr = \int_0^1 (-3,0,1)(0,1,0) dt = 0$$

$$\int_{C_3} F dr = \int_0^1 (-2t - 3,0,1)(-1,0,0) dt = 4$$



$$\int_C F dr = 2 \neq 0$$

F *is not conservative*

EXAMPLE 5 Show that the vector field

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + 0\mathbf{k}$$

satisfies the equations in the Component Test, but is not conservative over its natural domain. Explain why this is possible.

Solution We have $M = -y/(x^2 + y^2)$, $N = x/(x^2 + y^2)$, and $P = 0$. If we apply the Component Test, we find

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \qquad \frac{\partial P}{\partial x} = 0 = \frac{\partial M}{\partial z}, \qquad \text{and} \qquad \frac{\partial M}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial N}{\partial x}.$$

$x^2 + y^2 \neq 0!$

To show that $\mathbf{F}$ is not conservative, we compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ around the loop C. First we write the field in terms of the parameter t:

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} = \frac{-\sin t}{\sin^2 t + \cos^2 t}\mathbf{i} + \frac{\cos t}{\sin^2 t + \cos^2 t}\mathbf{j} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}.$$

Next we find $d\mathbf{r}/dt = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$, and then calculate the line integral as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{2\pi} \left(\sin^2 t + \cos^2 t\right) dt = 2\pi.$$

Since the line integral of $\mathbf{F}$ around the loop C is not zero, the field $\mathbf{F}$ is not conservative, by Theorem 3. The field $\mathbf{F}$ is displayed in Figure 16.28d in the next section. ■

16.4-16.8

DEFINITION The **circul**... point (x, y) is the scalar e...

... at the

(1)

This expression is also ...d by (curl **F**) • **k**.

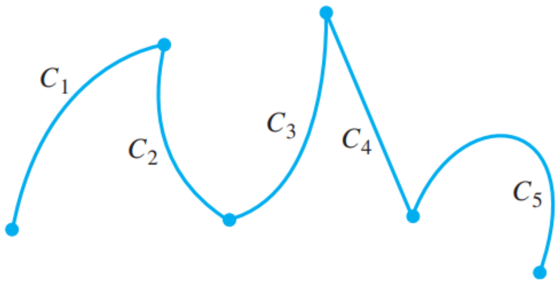


FIGURE 13.6 A piecewise smooth curve made up of five smooth curves connected end to end in a continuous fashion. The curve here is not smooth at the points joining the five smooth curves.

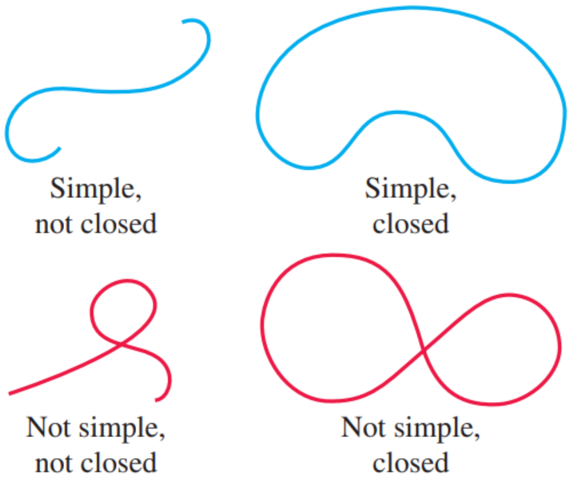
DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\operatorname{div} \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{2}$$

THEOREM 4—Green's Theorem (Circulation-Curl or Tangen...

be a piecewise smooth, simple closed curve enclosing a regi... Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having cont... derivatives in an open region containing R. Then the counter... tion of $\mathbf{F}$ around C equals the double integral of $(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}$ ov...

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{T}\, ds}_{\text{Counterclockwise circulation}} = \oint_C M\, dx + N\, dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{Curl integral}}$$



5—Green's Theorem (Flux-Divergence or Normal Form) Let C ...wise smooth, simple closed curve enclosing a region R in the plane. ...$M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial ...s in an open region containing R. Then the outward flux of $\mathbf{F}$ across C ... double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F} \cdot \mathbf{n}\, ds}_{\text{Outward flux}} = \oint_C M\, dy - N\, dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\, dy}_{\text{Divergence integral}} \tag{4}$$

EXAMPLE 4 Evaluate the line integral

$$\oint_C xy\,dy - y^2\,dx,$$

where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

Solution We can use either form of Green's Theorem to change the line integral into a double integral over the square, where C is the square's boundary and R is its interior.

1. *With the Tangential Form* Equation (3): Taking $M = -y^2$ and $N = xy$ gives the result:

$$\oint_C -y^2\,dx + xy\,dy = \iint_R (y - (-2y))\,dx\,dy = \int_0^1 \int_0^1 3y\,dx\,dy$$

$$= \int_0^1 \Big[3xy\Big]_{x=0}^{x=1} dy = \int_0^1 3y\,dy = \frac{3}{2}y^2\Big]_0^1 = \frac{3}{2}.$$

2. *With the Normal Form* Equation (4): Taking $M = xy$, $N = y^2$, gives the same result:

$$\oint_C xy\,dy - y^2\,dx = \iint_R (y + 2y)\,dx\,dy = \frac{3}{2}.$$

■

EXAMPLE 5 Calculate the outward flux of the vector field $\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = 2e^{xy}$, $N = y^3$, C the square, and R the square's interior, we have

$$\begin{aligned}
\text{Flux} &= \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx \\
&= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy \qquad \text{Green's Theorem, Eq. (4)} \\
&= \int_{-1}^{1} \int_{-1}^{1} (2ye^{xy} + 3y^2)\, dx\, dy = \int_{-1}^{1} \Big[2e^{xy} + 3xy^2 \Big]_{x=-1}^{x=1} dy \\
&= \int_{-1}^{1} (2e^{y} + 6y^2 - 2e^{-y})\, dy = \Big[2e^{y} + 2y^3 + 2e^{-y} \Big]_{-1}^{1} = 4.
\end{aligned}$$

■

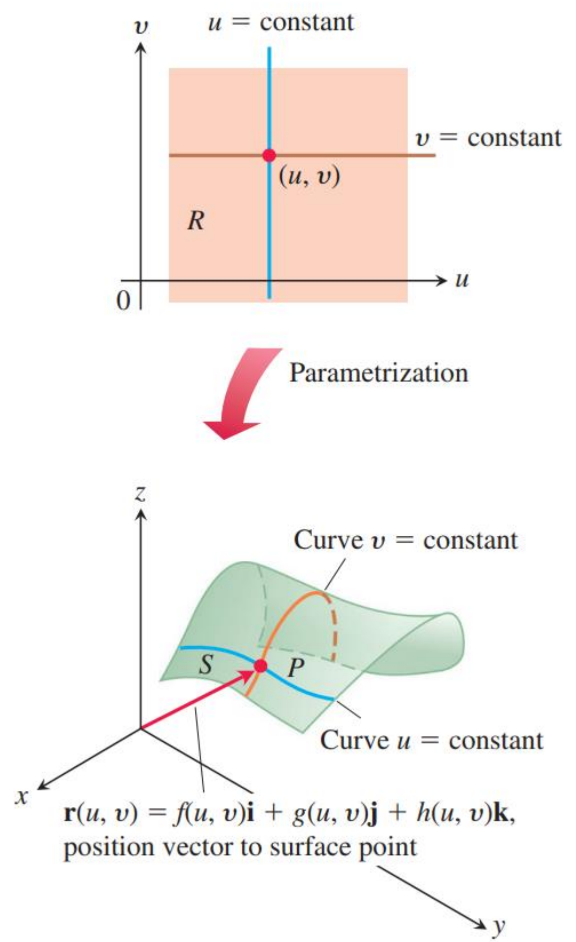
DEFINITION The **area** of the smooth surface

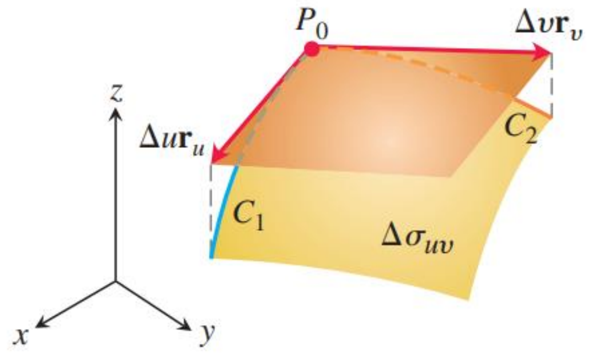
$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

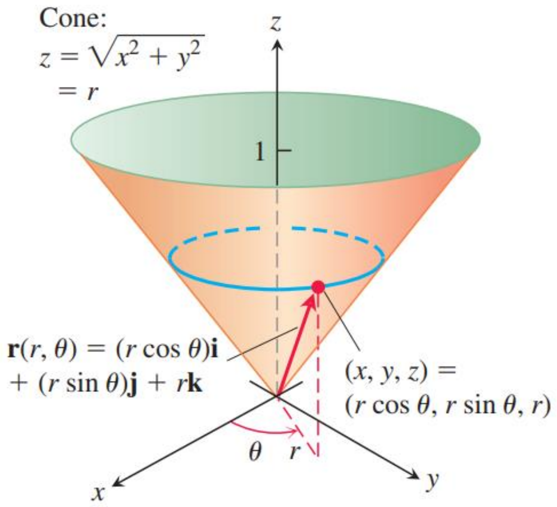
$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv. \qquad (4)$$

DEFINITION A parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is **smooth** if $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is never zero on the interior of the parameter domain.





EXAMPLE 4 Find the surface area of the cone in Example 1 (Figure 16.37).



Solution In Example 1, we found the parametrization

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

To apply Equation (4), we first find $\mathbf{r}_r \times \mathbf{r}_\theta$:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= -(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + \underbrace{(r\cos^2\theta + r\sin^2\theta)}_{r}\mathbf{k}.$$

Thus, $|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}r$. The area of the cone is

$$A = \int_0^{2\pi}\int_0^1 |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta \qquad \text{Eq. (4) with } u = r, v = \theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{2}\, r\, dr\, d\theta = \int_0^{2\pi} \frac{\sqrt{2}}{2}\, d\theta = \frac{\sqrt{2}}{2}(2\pi) = \pi\sqrt{2} \text{ units squared.}$$ ■

EXAMPLE 5 Find the surface area of a sphere of radius a.

Solution We use the parametrization from Example 2:

$$\mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta)\mathbf{i} + (a \sin \phi \sin \theta)\mathbf{j} + (a \cos \phi)\mathbf{k},$$
$$0 \le \phi \le \pi, \quad 0 \le \theta \le 2\pi.$$

For $\mathbf{r}_\phi \times \mathbf{r}_\theta$, we get

$$\begin{aligned}
\mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\
&= (a^2 \sin^2 \phi \cos \theta)\mathbf{i} + (a^2 \sin^2 \phi \sin \theta)\mathbf{j} + (a^2 \sin \phi \cos \phi)\mathbf{k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} \\
&= \sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi} = \sqrt{a^4 \sin^2 \phi (\sin^2 \phi + \cos^2 \phi)} \\
&= a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi,
\end{aligned}$$

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + h(u, v)\mathbf{k} \tag{6}$$

gives a parametrization of the surface S. We use Equation (4) to find the area of S.

Calculating the partial derivatives of $\mathbf{r}$, we find

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}.$$

Applying the Chain Rule for implicit differentiation (see Equation (2) in Section 14.4) to $F(x, y, z) = c$, where $x = u$, $y = v$, and $z = h(u, v)$, we obtain the partial derivatives

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}.$$

Substitution of these derivatives into the derivatives of $\mathbf{r}$ gives

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \quad \text{and} \quad \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}.$$

From a routine calculation of the cross product we find

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} && F_z \neq 0 \\
&= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}} \\
&= \frac{\nabla F}{\nabla F \cdot \mathbf{p}}. && \mathbf{p} = \mathbf{k}
\end{aligned}$$

$$d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dx\, dy. \qquad u = x \text{ and } v = y$$

$$\nabla F = f_x i + f_y j - k$$

$$|\nabla F \cdot p| = 1$$

$$d\sigma = |\nabla F| dxdy = \sqrt{f_x^2 + f_y^2 + 1} dxdy$$

Formulas for a Surface Integral of a Scalar Function

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

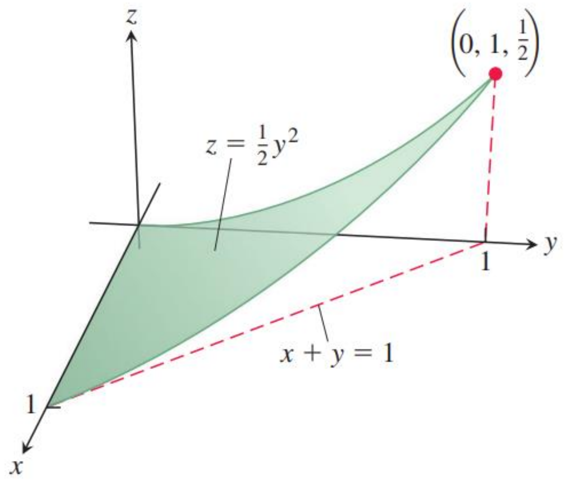
2. For a surface S given **implicitly** by $F(x, y, z) = c$, where F is a continuously differentiable function, with S lying above its closed and bounded shadow region R in the coordinate plane beneath it, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, z)\, \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|}\, dA, \qquad (3)$$

   where $\mathbf{p}$ is a unit vector normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

3. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y))\, \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy. \qquad (4)$$



EXAMPLE 4 Evaluate $\iint_S \sqrt{x(1 + 2z)}\, d\sigma$ on the portion of the cylinder $z = y^2/2$ over the triangular region R: $x \ge 0$, $y \ge 0$, $x + y \le 1$ in the xy-plane (Figure 16.48).

Solution The function G on the surface S is given by

$$G(x, y, z) = \sqrt{x(1 + 2z)} = \sqrt{x}\sqrt{1 + y^2}.$$

With $z = f(x, y) = y^2/2$, we use Equation (4) to evaluate the surface integral:

$$d\sigma = \sqrt{f_x{}^2 + f_y{}^2 + 1}\, dx\, dy = \sqrt{0 + y^2 + 1}\, dx\, dy$$

$$\begin{aligned}
\iint_S G(x, y, z)\, d\sigma &= \iint_R \left(\sqrt{x}\sqrt{1 + y^2}\right)\sqrt{1 + y^2}\, dx\, dy \\
&= \int_0^1 \int_0^{1-x} \sqrt{x}\,(1 + y^2)\, dy\, dx \\
&= \int_0^1 \sqrt{x}\left[(1 - x) + \frac{1}{3}(1 - x)^3\right] dx && \text{Integrate and evaluate.} \\
&= \int_0^1 \left(\frac{4}{3}x^{1/2} - 2x^{3/2} + x^{5/2} - \frac{1}{3}x^{7/2}\right) dx && \text{Routine algebra} \\
&= \left[\frac{8}{9}x^{3/2} - \frac{4}{5}x^{5/2} + \frac{2}{7}x^{7/2} - \frac{2}{27}x^{9/2}\right]_0^1 \\
&= \frac{8}{9} - \frac{4}{5} + \frac{2}{7} - \frac{2}{27} = \frac{284}{945} \approx 0.30.
\end{aligned}$$

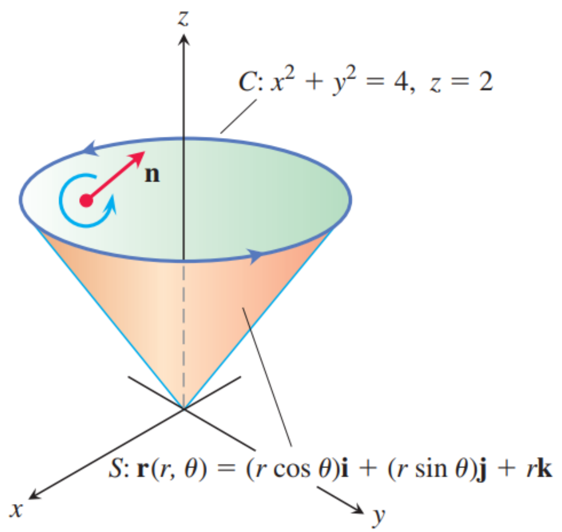
THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of the curl vector field $\nabla \times \mathbf{F}$ over S:

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}} \tag{4}$$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}.$$



EXAMPLE 4 Find the circulation of the field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above (Figure 16.59).

Solution Stokes' Theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the counterclockwise direction viewed from above corresponds to taking the *inner* normal $\mathbf{n}$ to the cone, the normal with a positive $\mathbf{k}$-component.

We parametrize the cone as

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \qquad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

We then have

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} = \frac{-(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}}{r\sqrt{2}} \qquad \text{Section 16.5, Example 4}$$

$$= \frac{1}{\sqrt{2}}\left(-(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} + \mathbf{k}\right)$$

$$d\sigma = r\sqrt{2}\,dr\,d\theta \qquad \text{Section 16.5, Example 4}$$

$$\nabla \times \mathbf{F} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k} \qquad \text{Routine calculation}$$

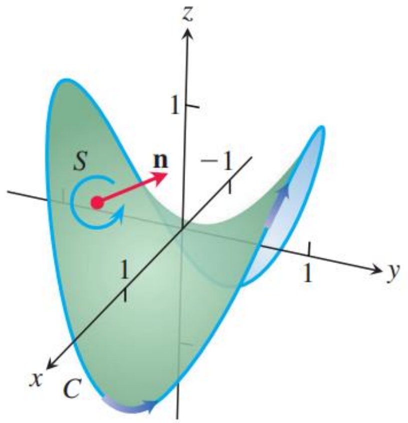
$$= -4\mathbf{i} - 2r\cos\theta\mathbf{j} + \mathbf{k}. \qquad x = r\cos\theta$$

Accordingly,

$$\begin{aligned}\nabla \times \mathbf{F} \cdot \mathbf{n} &= \frac{1}{\sqrt{2}}\left(4 \cos \theta + 2r \cos \theta \sin \theta + 1\right) \\ &= \frac{1}{\sqrt{2}}\left(4 \cos \theta + r \sin 2\theta + 1\right)\end{aligned}$$

and the circulation is

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma \qquad \text{Stokes' Theorem, Eq. (4)} \\ &= \int_0^{2\pi} \int_0^{2} \frac{1}{\sqrt{2}}(4 \cos \theta + r \sin 2\theta + 1)\left(r\sqrt{2}\, dr\, d\theta\right) = 4\pi.\end{aligned}$$



EXAMPLE 6 Find a parametrization for the surface S formed by the part of the hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around the z-axis and for the boundary curve C of S. (See Figure 16.60.) Then verify Stokes' Theorem for S using the normal having positive $\mathbf{k}$-component and the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$.

Solution As the unit circle is traversed counterclockwise in the xy-plane, the z-coordinate of the surface with the curve C as boundary is given by $y^2 - x^2$. A parametrization of C is given by

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin^2 t - \cos^2 t)\mathbf{k}, \quad 0 \le t \le 2\pi$$

with

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (4 \sin t \cos t)\mathbf{k}, \ 0 \le t \le 2\pi.$$

Along the curve $\mathbf{r}(t)$ the formula for the vector field $\mathbf{F}$ is

$$\mathbf{F} = (\sin t)\mathbf{i} - (\cos t)\mathbf{j} + (\cos^2 t)\mathbf{k}.$$

The counterclockwise circulation along C is the value of the line integral

$$\begin{aligned}\int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt &= \int_0^{2\pi} \left(-\sin^2 t - \cos^2 t + 4 \sin t \cos^3 t\right) dt \\ &= \int_0^{2\pi} \left(4 \sin t \cos^3 t - 1\right) dt \\ &= \Big[-\cos^4 t - t\Big]_0^{2\pi} = -2\pi.\end{aligned}$$

We now compute the same quantity by integrating $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over the surface S. We use polar coordinates and parametrize S by noting that above the point (r, θ) in the plane, the z–coordinate of S is $y^2 - x^2 = r^2 \sin^2\theta - r^2 \cos^2\theta$. A parametrization of S is

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2(\sin^2\theta - \cos^2\theta)\mathbf{k}, \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

We next compute $\nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & x^2 \end{vmatrix} = -2x\mathbf{j} - 2\mathbf{k} = -(2r\cos\theta)\mathbf{j} - 2\mathbf{k}$$

and

$$\begin{aligned}
\mathbf{r}_r &= (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r(\sin^2\theta - \cos^2\theta)\mathbf{k} \\
\mathbf{r}_\theta &= (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j} + 4r^2(\sin\theta\cos\theta)\mathbf{k} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r(\sin^2\theta - \cos^2\theta) \\ -r\sin\theta & r\cos\theta & 4r^2(\sin\theta\cos\theta) \end{vmatrix} \\
&= 2r^2(2\sin^2\theta\cos\theta - \sin^2\theta\cos\theta + \cos^3\theta)\mathbf{i} \\
&\quad -2r^2(2\sin\theta\cos^2\theta + \sin^3\theta + \sin\theta\cos^2\theta)\mathbf{j} + r\mathbf{k}.
\end{aligned}$$

We now obtain

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi}\int_0^1 \nabla \times \mathbf{F} \cdot \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|}|\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \nabla \times \mathbf{F} \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left[4r^3\left(2 \sin\theta \cos^3\theta + \sin^3\theta \cos\theta + \sin\theta \cos^3\theta\right) - 2r\right] dr\, d\theta$$

$$= \int_0^{2\pi} \left[r^4(3 \sin\theta \cos^3\theta + \sin^3\theta \cos\theta) - r^2 \right]_{r=0}^{r=1} d\theta \qquad \text{Integrate.}$$

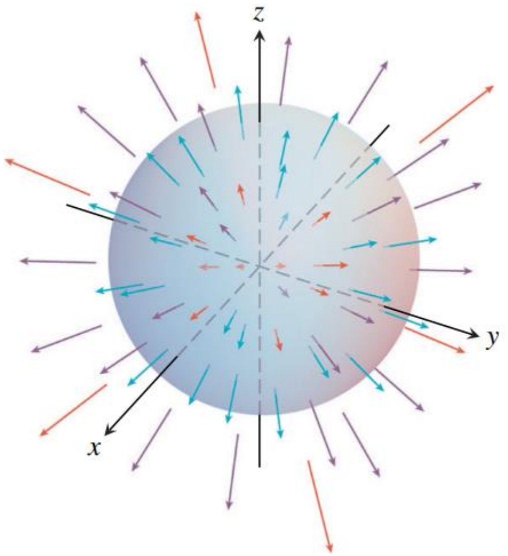
$$= \int_0^{2\pi} (3 \sin\theta \cos^3\theta + \sin^3\theta \cos\theta - 1)\, d\theta \qquad \text{Evaluate.}$$

$$= \left[-\frac{3}{4}\cos^4\theta + \frac{1}{4}\sin^4\theta - \theta \right]_0^{2\pi}$$

$$= \left(-\frac{3}{4} + 0 - 2\pi + \frac{3}{4} - 0 + 0 \right) = -2\pi.$$

THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the triple integral of the divergence $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underbrace{\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Outward flux}} = \underbrace{\iiint_D \nabla \cdot \mathbf{F}\, dV}_{\text{Divergence integral}}. \tag{2}$$



EXAMPLE 2 Evaluate both sides of Equation (2) for the expanding vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ over the sphere $x^2 + y^2 + z^2 = a^2$ (Figure 16.70).

Solution The outer unit normal to S, calculated from the gradient of $f(x, y, z) = x^2 + y^2 + z^2 - a^2$, is

$$\mathbf{n} = \frac{2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{4(x^2 + y^2 + z^2)}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}. \qquad x^2 + y^2 + z^2 = a^2 \text{ on } S$$

It follows that

$$\mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{x^2 + y^2 + z^2}{a}\, d\sigma = \frac{a^2}{a}\, d\sigma = a\, d\sigma.$$

Therefore, the outward flux is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S a\, d\sigma = a \iint_S d\sigma = a(4\pi a^2) = 4\pi a^3. \qquad \text{Area of } S \text{ is } 4\pi a^2.$$

For the right-hand side of Equation (2), the divergence of **F** is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3,$$

so we obtain the divergence integral,

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iiint_D 3\, dV = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3.$$

If we think of a two-dimensional field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ as a three-dimensional field whose $\mathbf{k}$-component is zero, then $\nabla \cdot \mathbf{F} = (\partial M/\partial x) + (\partial N/\partial y)$ and the normal form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy = \iint_R \nabla \cdot \mathbf{F}\, dA.$$

Similarly, $\nabla \times \mathbf{F} \cdot \mathbf{k} = (\partial N/\partial x) - (\partial M/\partial y)$, so the tangential form of Green's Theorem can be written as

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA.$$

Green's Theorem and Its Generalization to Three Dimensions

Tangential form of Green's Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_R \nabla \times \mathbf{F} \cdot \mathbf{k}\, dA$

Stokes' Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma$

Normal form of Green's Theorem: $\displaystyle\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \nabla \cdot \mathbf{F}\, dA$

Divergence Theorem: $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV$