CS 373: Combinatorial Algorithms, Spring 1999

http://www-courses.cs.uiuc.edu/~cs373

Homework 6 (due Tue. May 4, 1999 by noon)

Name:	
Net ID:	Alias:

Everyone must do the problems marked ▶. Problems marked ▷ are for 1-unit grad students and others who want extra credit. (There's no such thing as "partial extra credit"!) Unmarked problems are extra practice problems for your benefit, which will not be graded. Think of them as potential exam questions.

Hard problems are marked with a star; the bigger the star, the harder the problem.

Note: You will be held accountable for the appropriate responses for answers (e.g. give models, proofs, analyses, etc). For NP-complete problems you should prove everything rigorously, i.e. for showing that it is in NP, give a description of a certificate and a poly time algorithm to verify it, and for showing NP-hardness, you must show that your reduction is polytime (by similarly proving something about the algorithm that does the transformation) and proving both directions of the 'if and only if' (a solution of one is a solution of the other) of the many-one reduction.

Undergrad/.75U Grad/1U Grad Problems

▶1. (5 pts) Complexity

(a) (2 pts) Prove that $\mathrm{P} \subseteq \mathrm{coNP}$.

(b) (3 pts) Prove that if $\mathrm{NP} \neq \mathrm{co}-\mathrm{NP}$ then $\mathrm{P} \neq \mathrm{NP}$.

▶2. (5 pts) 2-CNF-SAT
Prove that deciding staisfiability when all clauses have at most 2 literals is in P.

▶3. (5 pts) SUBGRAPH-ISOMORPHISM
Show that the problem of deciding whether one graph is a subgraph of another is NP-complete.

▶4. (5 pts) RECTANGLE-COVER
Consider the following problem A: given a set of axis-aligned rectangles in a plane, decide whether there is a point in the plane that is covered by k planes (or less). Now also consider the CLIQUE problem. Describe and analyze a reduction of one to the other.

Only 1U Grad Problems

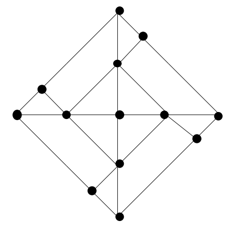

Figure 1. Gadget for PLANAR-3-COLOR.

▷1. (6 pts) PARTITION, SUBSET-SUM
PARTITION is the problem of deciding, given a set of numbers, whether there exists a subset whose sum equals the sum of the complement, i.e. given $S = s_1, s_2 \ldots, s_n$, does there exist a subset S' such that $\sum_{s \in S'} s = \sum_{t \in S - S'} t$. SUBSET-SUM is the problem of deciding, given a set of numbers and a target sum, whether there exists a subset whose sum equals the target, i.e. given $S = s_1, s_2 \ldots, s_n$ and k, does there exist a subset S' such that $\sum_{s \in S'} s = k$. Give two reduction, one in both directions.

Practice Problems

1. Show, given a set of numbers, that you can decide whether it has a subset of size 3 that adds to zero in polytime.

2. Consider finding the median of 5 numbers by using only comparisons. What is the exact worst case number of comparisons needed to find the median. Justify (exhibit a set that cannot be done in one less comparisons). Do the same for 6 numbers.

3. LONGEST-PATH
Show that the problem of deciding whether an unweighted undirected graph has a path of length greater than k is NP-complete.

4. PLANAR-3-COLOR
Using 3-COLOR, and the 'gadget' in figure 4, prove that the problem of deciding whether a planar graph can be 3-colored is NP-complete. Hint: show that the gadget can be 3-colored, and then replace any crossings in a planar embedding with the gadget appropriately.

5. DEGREE-4-PLANAR-3-COLOR
Using the previous result, and the 'gadget' in figure 5, prove that the problem of deciding whether a planar graph with no vertex of degree greater than four can be 3-colored is NP-complete. Hint: show that you can replace any vertex with degree greater than 4 with a collection of gadgets connected in such a way that no degree is greater than four.

6. Poly time subroutines can lead to exponential algorithms
Show that an algorithm that makes at most a constant number of calls to polynomial-time subroutines runs in polynomial time, but that a polynomial number of calls to polynomial-time subroutines may result in an exponential-time algorithm.

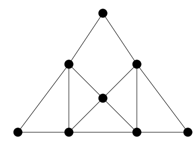

Figure 2. Gadget for DEGREE-4-PLANAR-3-COLOR.

7. (a) Prove that if G is an undirected bipartite graph with an odd number of vertices, then G is nonhamiltonian. Give a polynomial time algorithm algorithm for finding a **hamiltonian cycle** in an undirected bipartite graph or establishing that it does not exist.
   (b) Show that the **hamiltonian-path** problem can be solved in polynomial time on directed acyclic graphs by giving an efficient algorithm for the problem.
   (c) Explain why the results in previous questions do not contradict the facts that both HAM-CYCLE and HAM-PATH are NP-complete problems.

8. Consider the following pairs of problems:
   (a) MIN SPANNING TREE and MAX SPANNING TREE
   (b) SHORTEST PATH and LONGEST PATH
   (c) TRAVELING SALESMAN PROBLEM and VACATION TOUR PROBLEM (the longest tour is sought).
   (d) MIN CUT and MAX CUT (between s and t)
   (e) EDGE COVER and VERTEX COVER
   (f) TRANSITIVE REDUCTION and MIN EQUIVALENT DIGRAPH

   (all of these seem dual or opposites, except the last, which are just two versions of minimal representation of a graph).

   Which of these pairs are polytime equivalent and which are not? Why?

★9. GRAPH-ISOMORPHISM
   Consider the problem of deciding whether one graph is isomorphic to another.
   (a) Give a brute force algorithm to decide this.
   (b) Give a dynamic programming algorithm to decide this.
   (c) Give an efficient probabilistic algorithm to decide this.
   (d) Either prove that this problem is NP-complete, give a poly time algorithm for it, or prove that neither case occurs.

10. Prove that PRIMALITY (Given n, is n prime?) is in NP $\cap$ coNP. Hint: coNP is easy (what's a certificate for showing that a number is composite?). For NP, consider a certificate involving primitive roots and recursively their primitive roots. Show that knowing this tree of primitive roots can be checked to be correct and used to show that n is prime, and that this check takes poly time.

11. How much wood would a woodchuck chuck if a woodchuck could chuck wood?