CS762: Graph-Theoretic Algorithms
Lecture 22: Triangulated Graphs
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Abstract

In this lecture we start discussing planar graphs in detail, and focus on triangulated graphs. We discuss some of its properties like simplicity, maximality, connectivity, and uniqueness in embedding. We prove that every triangulated graph is 3-connected. We also give the idea of how to make a planar graph, which is not already triangulated, triangulated.

1 Introduction

In this section we review some basic definitions, and properties of planar graphs related to our lecture from the previous lecture and from [NC88].

A graph is *planar* if it can be drawn in the plane so that no two edges cross each other except at their end point (which is a vertex), and such a drawing is called a *planar embedding* of that planar graph. A planar graph does not necessarily have a unique planar embedding. For example, Figure 1(a) and 1(b) show two different planar embeddings of same planar graph. A planar graph with a fixed planar embedding is called a *plane graph*.

Figure 1: Two different planar embeddings of a planar graph.

Faces in a planar embedding are the connected pieces of the plane after the drawing is removed. The *degree of a face* F is the number of edges forming the boundary of F. We know the following famous theorem which is from Euler.

Theorem 1 [Eul1758] *Let G be a connected plane graph, and let n, m, and f denote the number of vertices, edges, and faces of G respectively. Then $n - m + f = 2$.*

In the following section (Section 2) we discuss triangulated graphs in detail.

2 Triangulated Graphs

Definition 1 *A graph is a planar triangulated graph or simply triangulated graph if it is planar and in a fixed planar embedding every face is a triangle (i.e. a 3-cycle).*

Figure 2(a) shows a triangulated graph. It is important to observe that by the definition the outer face of the triangulated graph is also a triangle. For example, the planar graph in Figure 2(b) seems to be a triangulated graph but it is not, because its outer face is not a triangle. It is actually an *internally triangulated graph.*

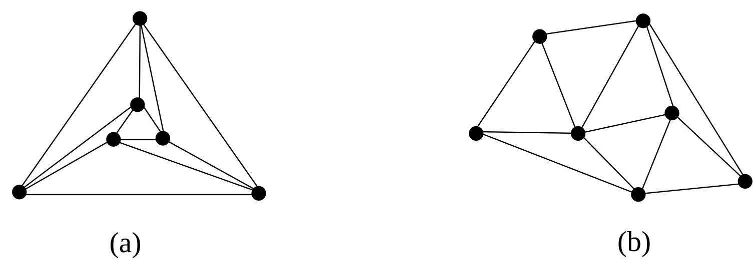

Figure 2: (a) A triangulated graph, (b) an internally triangulated graph.

Chordal graphs are sometimes also called triangulated graphs in the literature, so it may be interesting to compare the two classes and to observe that they are unrelated. For example, the graph in Figure 3(a) is triangulated but not chordal (as the vertices a, b, c, d, e, f form an induced 6-cycle); on the other hand, the graphs in Figure 3(b) and 3(c) are chordal and planar but not triangulated (the outer face, (a, b, c, d, b, a), is not a triangle for both cases). The graph in Figure 3(c) is not even internally triangulated.

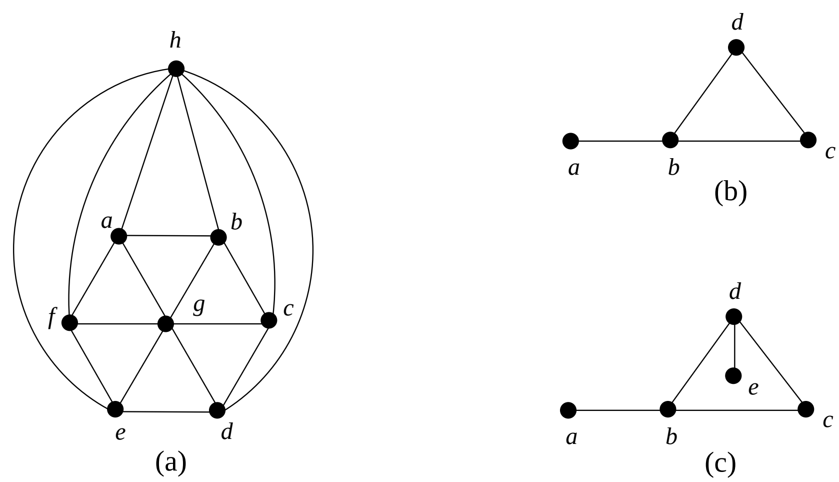

Figure 3: (a) A triangulated graph which is not chordal, (b,c) planar chordal graphs which are not triangulated.

2.1 Some Results and Properties of Triangulated Graphs

For the following results, we assume that we have a simple triangulated graph:

- *Maximality:* It is not so hard to observe that every triangulated graph is maximally planar, that is, no edge can be added (to any face) without destroying its planarity or simplicity.
- *Number of edges and faces:* Every face in a triangulated graph is a triangle, and therefore has degree three. In a planar embedding each edge is adjacent to two faces. So, summation of degree of all faces equals twice the number of edges in the graph, that is, $3m = 2f$. From Euler's formula we get two results in terms of number of nodes, n, (i) $m = 3n - 6$ and (ii) $f = 2n - 4$.
- *Connectivity:* We have the following theorem about the connectivity of triangulated graphs.

  Theorem 2 *Every triangulated graph is 3-connected, i.e., after deleting 2 arbitrary vertices the graph is still connected.*

  Proof: We prove this by contradiction. We prove that if a simple planar graph is not 3-connected then it cannot be triangulated. If a graph is not 3-connected then it either has a cut vertex or it has a cutting pair, i.e., a pair of vertices whose removal leaves the graph disconnected.

  Case 1 (cut vertex): Let the cut vertex be v (see Figure 4(a)). Without loss of generality, we will assume that removing v creates only two components, and that both are incident to the outer face. Since the graph is triangulated, the outer face is a triangle. As v counts twice towards the degree of the outer face, there is only one other vertex x which, together with v (twice), forms the triangle for the outer face. Now, x can be in any one of the two components. But there will be multiple edges between x and v in the component that contains x, and there will be a self loop in v in the other component (see Figure 4(b)). This contradicts the simplicity of the triangulated graph.

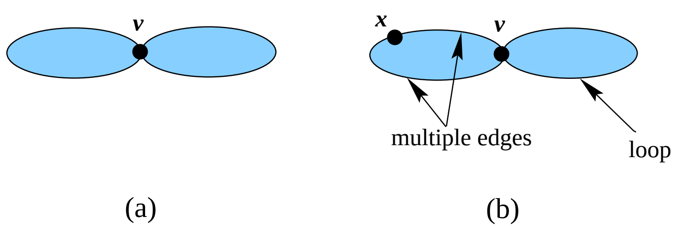

Figure 4: Proof of Theorem 2, Case 1.

  Case 2 (cutting pair): Let the cutting pair be (u, v). Again, without loss of generality we can assume that removing u and v creates only two connected components, and that u and v are on the outer-face. For our proof we are interested in two faces: the outer face and the face enclosed by the two connected components, F_1 and F_2 respectively in Figure 5(a). Now, if the graph is triangulated then each of F_1 and F_2 is a triangle. So, for F_1 there is a vertex x in the boundary which, together with u and v, forms the triangle; similarly, for F_2 there is a vertex y in the boundary which, together with u and v, forms the triangle (see Figure 5(b)). The result is multiple edges between u and v which contradicts the simplicity of the triangulated graph.

  So, every triangulated graph is 3-connected. □

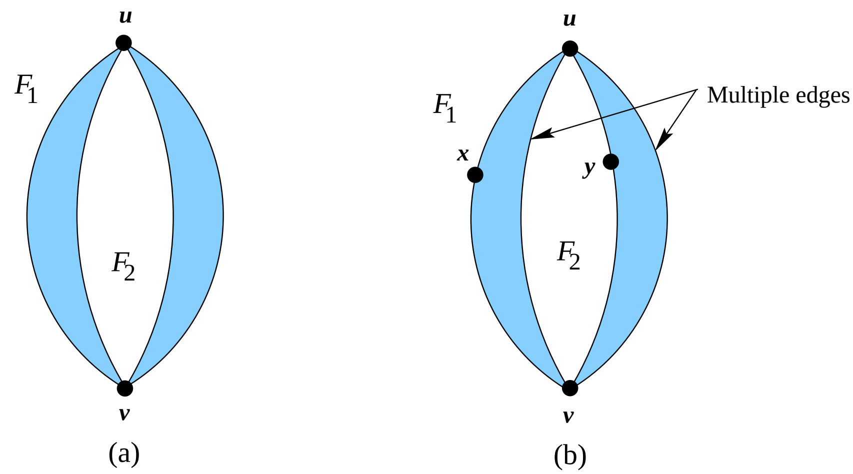

Figure 5: Proof of Theorem 2, Case 2.

- *Uniqueness in embedding:* We have already seen that a planar graph may have more that one planar embedding. This is not true for all planar graphs. We have the following theorem, whose proof is omitted.

  Theorem 3 *A 3-connected planar graph has only one planar embedding.*

  As every triangulated graph is 3-connected this theorem implies that every triangulated graph has a unique planar embedding. This theorem is subject to choice of outer face and flipping. In other words, it does not consider flipping of drawing of a subgraph or changing the outer face. More precisely, it only considers *fixed edge orientation around the vertices*[1].

2.2 Making Graphs Triangulated

If a planar graph is not triangulated then we can make it triangulated in one the following ways:

- adding edges only,
- adding edges and vertices, or
- adding vertices (and edges incident to at least one new vertex) only.

Here we give an outline for the first way, which of course implies the second way. The third way was left as an exercise. We have the following theorem.

Theorem 4 *Let G be a (simple, connected) planar graph with $n \geq 3$. Then we can find a set of $O(n)$ edges E' such that $(V, E \cup E')$ is a simple, triangulated and planar graph; and this takes $O(n)$ time.*

Proof: (Sketch) Assume we have a plane graph G. Let F be a face of degree ≥ 4 with vertices $v_1, v_2, ..., v_k$ of G (see Figure 6(a)). The simplest idea to make the face F triangulated is to add edges $(v_1, v_3), (v_1, v_4), ..., (v_1, v_{k-1})$ (dashed lines in Figure 6(b)). But it does not work, because there may be an edge (v_1, v_j) for $3 \leq j \leq k-1$ (dotted line in Figure 6(b)) in a face other than F, and in that case there will be multiple edges between v_1 and v_j. So, we have to consider the edges (v_i, v_j) for $1 \leq i, j \leq k$ in faces other than F. We solve it as follows.

[1]See the previous lecture note for the definition of fixed edge orientation around the vertices.

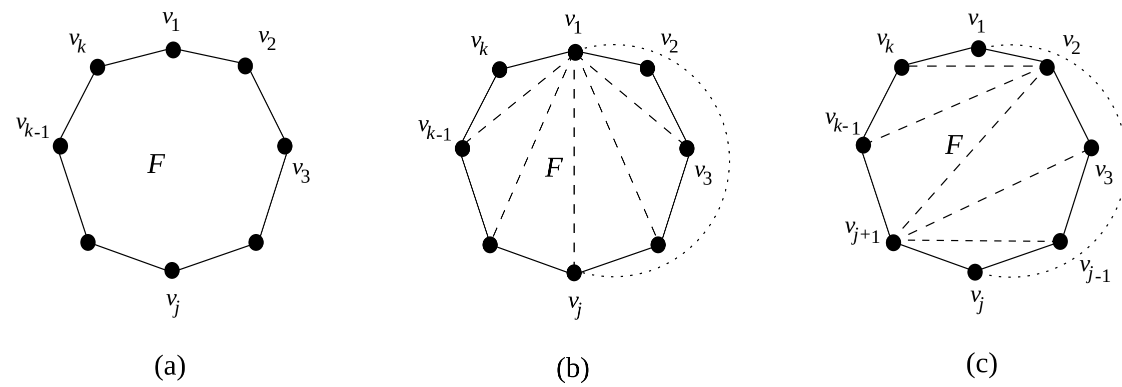

Figure 6: Making a planar graph triangulated.

Assume we have an edge (v_1, v_j) for $3 \leq j \leq k-1$ in a face other than F. It is easy to observe that F does not have the edge (v_2, v_k), otherwise it would exclude v_1. Also, there is no edge (v_2, v_k) in a face other than F, otherwise it would cross the edge (v_1, v_j) outside F. Similar arguments hold for each of the edges $(v_2, v_{k-1}), (v_2, v_{k-2}), ..., (v_2, v_{j+1})$, and $(v_{j+1}, v_{j-1}), (v_{j+1}, v_{j-2}), ..., (v_{j+1}, v_3)$. We add all these edges inside F (dashed lines in Figure 6(c)). This divides F into triangles.

We take another face with degree ≥ 4 and follow the same procedure. In this way, by induction, the whole graph becomes triangulated.

The time complexity will be discussed in the next lecture. $\square$

References

[Eul1758] L. Euler, Demonstratio nonnullarum insignium proprietatum quibus solida hedris planis inclusa sunt praedita, *Novi Comm. Acad. Sci. Omp. Petropol,* 4, (1758), 140-160.

[NC88] T. Nishizeki and N. Chiba, *Planar Graphs: Theory and Algorithms,* North-Holland, 1988.