CS762: Graph-Theoretic Algorithms
Lecture 24: Applications of the canonical ordering
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Abstract

In this lecture, we will look at the construction of planar triangulated graphs and the canonical order. Then, we will show, using the canonical order, that planar graphs have arboricity ≤ 3 and that every planar triangulated graph has a strong visibility representation and every planar graph has a weak visibility representation.

1 Introduction

In 1990, de Fraysseix, Pach, and Pollack [dFPP90] used the canonical ordering to create the first efficient algorithm for drawing a planar graph using straight line edges on a grid. Since then, others such as Kant [Kan96] have made improvements to this algorithm.

In these lecture notes, we will begin by looking at the construction of planar triangulated graphs and the canonical order. Then, we will look at some cases where the canonical ordering is used. Namely, showing planar triangulated graphs and planar graphs have arboricity ≤ 3 and that planar triangulated graphs have strong visibility representations while planar graphs have weak visibility representations.

2 Definitions

A graph G is *planar* if it can be drawn such that no two edges cross. Such a drawing is a *planar embedding* of G.

The *faces* of a planar graph are the maximal regions of the plane that are disjoint from the drawing. The region that is the infinite region is called the *outer-face*.

A graph G is a *planar triangulated graph* if G is both planar and every face of G is a cycle on 3 vertices or a *triangle*.

The *canonical ordering* is a vertex order $\{v_1, v_2, \ldots, v_n\}$ of a planar triangulated graph such that $\{v_1, v_2, v_n\}$ is the outer-face and for all $3 \leq k \leq n-1$:

1) the graph G_k induced by $v_1, v_2, \ldots, v_k$ is 2-connected and internally triangulated

2) the outer-face of G_k contains the edge (v_1, v_2)

3) v_{k+1} is in the outer-face of G_k

4) the neighbours of v_{k+1} in G_k are an interval on the outer-face of G_k.

3 Canonical ordering

In order to build the canonical order, we will first look at how we can build up a planar triangulated graph. First, we start with a single edge (v_1, v_2) and then at each step we add one more vertex as in Figure 1:

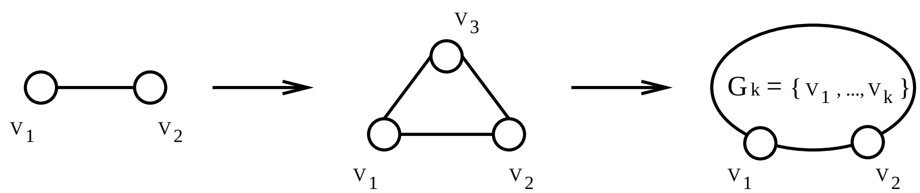

Figure 1: Build a planar triangulated graph by adding one vertex at a time.

Then, add vertex v_{k+1} in the outer-face of G_k as in Figure 2:

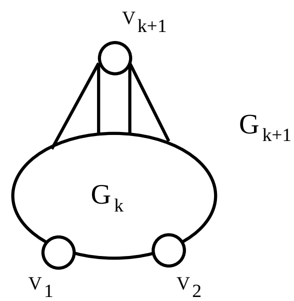

Figure 2: Add the new vertex to the outer-face.

We can guarantee that we have this structure for all choices of $\{v_1, v_2, v_n\}$ as the outer-face and use induction to show this.

First, arbitrarily fix the outer-face. Then arbitrarily pick v_n on the outer-face and let v_1, v_2 be the other two vertices on the outer-face. Then, $G_{n-1} = G - v_n$ where G_{n-1} is 2-connected since G is triangulated and therefore 3-connected and we are subtracting a vertex.

Assume we have chosen $v_{k+1}, \ldots, v_n$ such that $G_k = G - \{v_{k+1}, \ldots, v_n\}$ is 2-connected and has (v_1, v_2) on the outer-face. Now, we want to pick v_k on the outer face of G_k such that $v_k \neq v_1, v_2$ and $G_k - v_k$ is 2-connected. Note that G_k is internally triangulated and not necessarily triangulated so we have to be careful when choosing v_k to assure that $G_k - v_k$ will be 2-connected. For example, in Figure 3, making a bad choice for v_k would result in $G_k - v_k$ not being 2-connected.

In order to ensure $G_k - v_k$ is 2-connected, we need to choose a vertex on the outer-face that is not v_1 or v_2 and also is not incident to a chord. In Figure 3, edge e_1 is the problem chord. Now, we need to show that we can always find such a vertex. If there is no chord, then we can choose any vertex on the outer-face other than v_1 and v_2. If there are chords, let the outer-face be $c_1 = v_1, c_2, c_3, \ldots, c_p = v_2$. Let (c_i, c_j) be the chord that minimizes $j - i$ (see Figure 4).

Finally, set $v_k = c_{i+1}$. Since (c_i, c_j) is the chord that minimizes $j - i$ we know there is no other chord between the vertices $\{c_i, c_{i+1}, \ldots, c_j\}$ and since the graph is planar there cannot be a chord between c_{i+1} and a vertex outside of interval c_i, c_j. Therefore, we can always choose a next vertex, and by induction a canonical ordering exists.

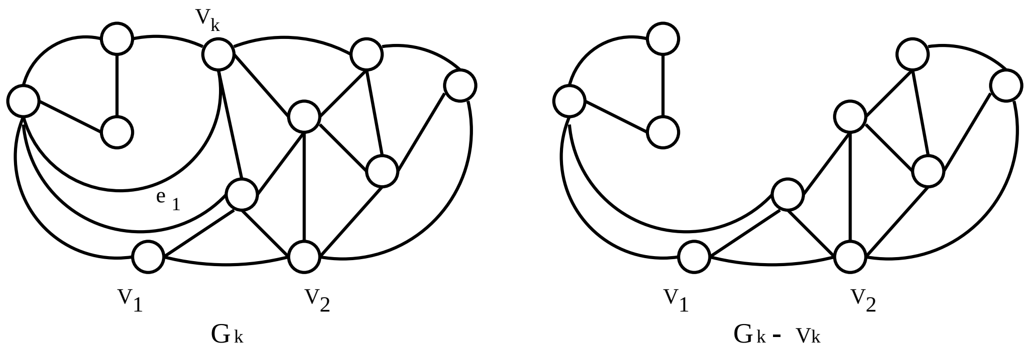

Figure 3: This choice of v_k causes $G_k - v_k$ not to be 2-connected.

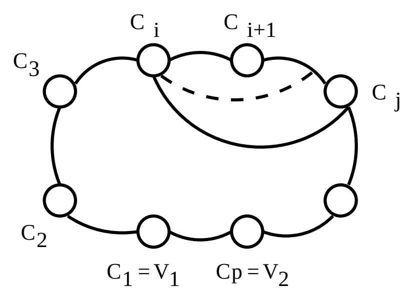

Figure 4: The chord minimizes $j - i$.

4 Applications of the canonical ordering

4.1 Arboricity

Recall that we needed the following definition in the previous lecture.

Definition 1 *Let G be a graph. Assume there are k forests $T_1, \ldots, T_k$ on the vertices of G such that $E(G) \subset E(T_1) \cup \ldots \cup E(T_k)$. We say that G has arboricity $a(G) \leq k$.*

Assume that we have the canonical ordering $\{v_1, v_2, \ldots, v_n\}$ on a planar triangulated graph. Then, we can create an acyclic orientation of the graph by directing all edges $v_i \rightarrow v_j$ where i is less than j. Next, label all of the edges with $\{1, 2, 3\}$ as in Figure 5. For edges directed into vertex v_{k+1}, assign "1" to the leftmost edge, "2" to the rightmost edge and "3" to the all of the edges in the middle.

Now, every vertex has at most one incoming edge labelled "1" and edges labelled "1" form a forest. Similarly, every vertex has at most one incoming edge labelled "2" and edges labelled "2" form a forest. Also, every vertex has at most one outgoing edge labelled "3" because it gets one only if it disappears from the outer-face. All edges labelled "3" also form a forest.

Since we have 3 forests that together cover all edges in the graph, we can conclude that planar triangulated graphs have arboricity less than or equal to 3. In fact, planar graphs have arboricity less than or equal to 3.

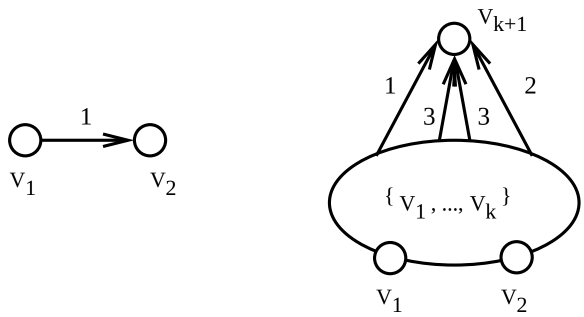

Figure 5: Label the directed edges with 1, 2 or 3.

4.2 Visibility representations

A *visibility representation* of a graph is a drawing of the graph using axis-parallel, disjoint boxes for vertices and vertical lines for edges such that the edges do no cross any box. A graph has a *strong visibility representation* if the boxes can be drawn so that boxes that see each other (have a common horizontal-axis point) must be connected by an edge. In otherwords, a strong visibility representation must have all visibility lines as edges in the graph. If not all overlapping boxes are connected by an edge, the representation is a called a *weak visibility representation.* (see Figure 6)

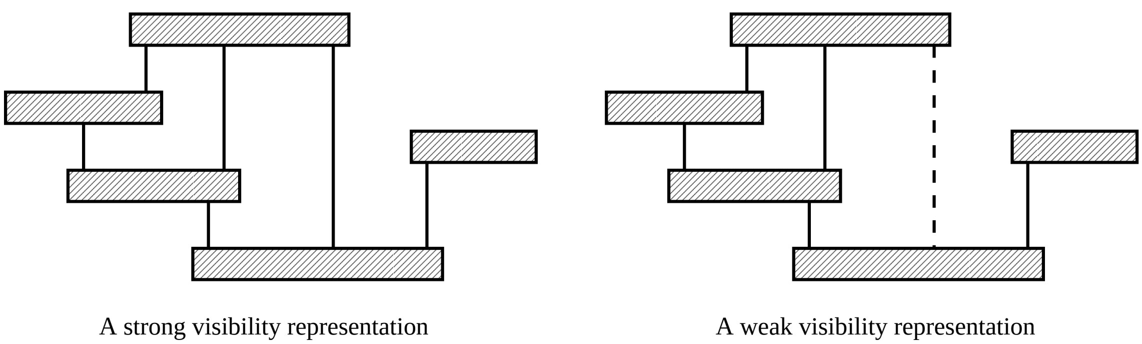

Figure 6: An example of both a strong and a weak visibility representation.

In 1978, Otten and Van Wijk [OW78] showed that every planar graph admits a visibility representation and in 1986 Rosenstiehl and Tarjan [RT86] and independently, Tamassia and Tollis [TT86] gave a linear time algorithm for constructing this representation.

Theorem 1 *Every planar triangulated graph has a strong visibility representation and every planar graph has a weak visibility representation.*

Proof: Let $\{v_1, v_2, \ldots, v_n\}$ be the canonical ordering of the graph G. We start with a single edge (v_1, v_2) and represent it as in Figure 7. Above the visibility representation, we have also labelled the visibility interval of each vertex from above. These intervals are created by scanning a vertical ray (whose source is above the graph) from left to right and determining which vertex the ray hits first.

As we add the next vertex in the canonical order to the top of the graph, we create a new box above the current visibility representation sized and located so that its adjacencies can be

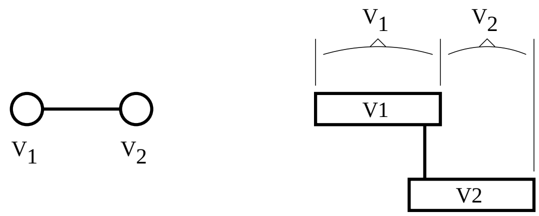

Figure 7: The base case for creating a visibility representation from a canonical order.

accommodated. Since this new box is on a higher level than the other boxes in graph, it will block part or all of the visibility intervals of boxes it is adjacent to. In Figure 8 when v_{k+1} is added, its interval takes half of c_a and half of c_b and covers all intervals in between c_a and c_b. The intervals for c_a and c_b are shaded to show that with the addition of vertex v_{k+1}, the intervals for c_a and c_b have become smaller.

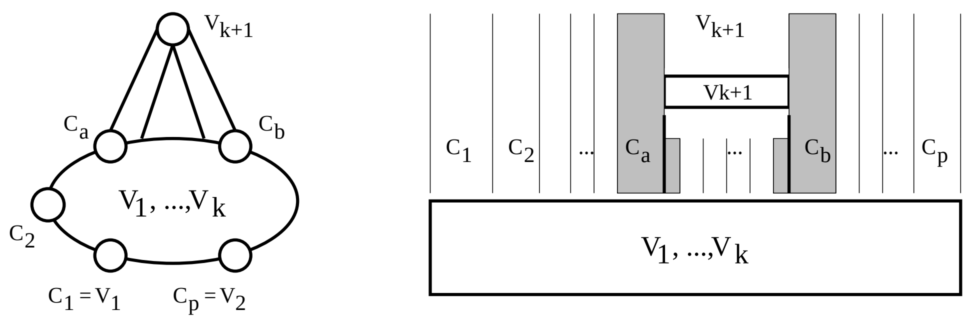

Figure 8: Update the visibility intervals as a new vertex is added.

Thus, we have shown how to build a visibility representation for planar triangulated graphs from a canonical order. This can also be done for planar graphs.

The height of this visibility representation is n since as we add each new vertex, we place it on a new level. Determining the width of the drawing if all edges must have integer coordinates is left as an exercise for the reader.

References

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