CS 373: Combinatorial Algorithms, Summer IMCS 2000

`http://www-courses.cs.uiuc.edu/~cs373`

Homework 0 (due May 25, 2000)

Instructions

- The homework is due **before** class on the day it is due. Late homeworks are not accepted. To insure that they are accepted, make sure it is submitted early.
- To submit solutions, **attach** a **postscript** file to an email sent to `maharri@cs.uiuc.edu` with the subject `cs373hw submit`. You will then get an automatic email acknowledgment.
- Label your homework with your name (family name first, with comma) and your IMCS network ID
- On the last page of your homework, please specify an alias (pseudonym) of your choice. Grades will be listed anonymously on the course web site by alias, so if you don't want anybody else to know your grades, your alias should not resemble your name (or your Net ID).
- You **must** use Postscript format; you can use any software which produces these documents, such as MSWord or LaTeX, both of which are good for typesetting math.
- No collaboration (working together) on solving or writing up the homework is allowed. You may consult others about definitions or notation. Any form of cheating will be

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This homework tests your familiarity with the prerequisite material (and *their* prerequisites)—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** CLR chapters 1-6 should be sufficient review but you may want to consult other texts, too.

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1. (5 pts total) Combinatorics
   (a) (2 pts) Prove by induction that there are $n!$ bijective functions from $\{1..n\}$ to $\{1..n\}$.
   (b) (3 pts) Prove the identity:
   $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$
   by showing that the two sides of the equation count the same sets in different ways. To get any credit, do not use induction or generating functions. What happens when $n = 0$? How is that related to the base case in the first question?
2. (5 pts total) Asymptotics
   (a) (2 pts) Prove that if $f(n)$ is $O(g(n))$, then $2^{f(n)}$ is $O(2^{g(n)})$.

(b) (3 pts) Prove that

$$\log^k n \prec n^{\frac{1}{k}}$$

(i.e. for any fixed k unboundedly large, $\log^k n$ is $O(n^{\frac{1}{k}})$, but $n^{\frac{1}{k}}$ is not $O(\log^k n)$.

3. (10 pts total, 2 each) Recurrences
   Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. Give an exact answer when possible (usually with exact base cases. You **must** give some justification for every answer, but it need not be a detailed proof).

   (a) $a(n) = 8a(n-1) - 15a(n-2) + 1, , b(0) = 0, b(1) = 1$
   (b) $b(n) = b(\lfloor n/2 \rfloor) + 1, b(1) = 1$
   (c) $c(n) = 16c(n/2) + n^3$
   (d) $d(n) = 16d(n/2) + n^4 \log n$
   (e) $e(n) = (n-1)(e(n-1) + e(n-2)), e(0) = e(1) = 1$

4. (5 pts) Probability
   You are given a coin. It may be biased, or it may not, you don't know. Show how to simulate an unbiasedy coin toss. Prove your answer. Hint: Flip the coin twice.

5. (5 pts) Binary Trees
   Prove that a full binary tree with n leaves has $n-1$ internal nodes.

6. (5 pts) Graphs
   Prove that a graph is bipartite if and only if it has no odd cycle.