CS762: Graph-Theoretic Algorithms
Lecture 11: More on perfect graphs
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Abstract

This lecture will show some properties of perfect graphs. The strong perfect graph conjecture is presented and its consequences for perfect graphs are discussed. Some other graph classes like splits graphs and permutation graphs which are subclasses of perfect graphs are also mentioned.

1 Introduction

The class of perfect graphs was introduced by Claude Berge in the early 60s. Graphs in this class have the property that their clique number and their chromatic number have the same value for every induced subgraph. A polynomial algorithm will be presented to compute this number and its complexity is analyzed.

Interestingly, a graph seems to be perfect if and only if it has no so called odd hole or odd anti-hole. This is called the perfect graph conjecture and was first guessed by Claude Berge 1962 [Ber62]. But, nobody has been able to prove this.

Section 3 discusses recognizing perfect graphs. It turns out that not much is known about how to answer the question whether a graph is perfect or not. In the following, properties of perfect graphs will be presented and some subclasses of perfect graphs are defined, in particular split graph, threshold graphs, permutation graphs and superperfect graphs are presented.

2 Definitions

A graph G is *perfect* if $\omega(H) = \chi(H)$ for every induced subgraph H of G. Not all graphs are perfect, for example C_5 (Figure 1) is not perfect since $\omega(C_5) = 2$ and $\chi(C_5) = 3$. But, every induced smaller subgraph of C_5 is perfect. A graph with this property is called *minimally imperfect.*

For for $k \geq 2$ the C_{2k+1} is called an *odd hole* and its compliment $\overline{C_{2k+1}}$ is called an *odd anti-hole.*

Lemma 1 *The odd hole and the odd anti-hole are minimally imperfect.*

Proof: We have just seen this for C_5, the proof is similar for C_{2k+1} and $\overline{C_{2k+1}}$. $\square$

Conjecture 1 *A graph is perfect if and only if it doesn't have an odd hole or an odd anti-hole as induced subgraph.*

Conjecture 1 is called the "Strong perfect graph conjecture" (SPGC). The SPGC is not proved yet. But, it is widely believed to be true and holds for many subclasses of graphs (e.g. planar graphs [Tuc73], circular-arc graphs [Rav75]).

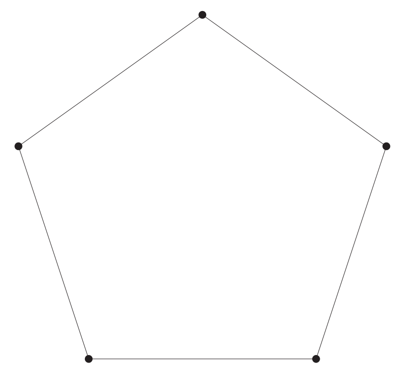

Figure 1: C_5 is not perfect: $\omega(C_5) = 2$, $\chi(C_5) = 3$

3 Properties of perfect graphs

3.1 Testing for perfect graphs

Not much is known about how to test whether a graph is perfect. It is neither known to be polynomial nor to be NP-hard. Surprisingly it is not known to be in NP but one can show that the test is Co-NP [Lov72]. This complexity for testing of perfect graphs stays unknown even if the SPGC holds [Gol80].

3.2 Stability number α, clique cover number k, clique number ω and chromatic number χ in perfect graphs

Although it is hard to find out whether a graph is perfect, we will see that we can get many properties of a graph if it is known to be perfect.

Theorem 1 *We can find $\alpha(G)$, $\omega(G)$, $\chi(G)$ of a perfect graph G in polynomial time.*

Proof: We will only sketch the proof of theorem 1 here. A full proof can be found in [BLS99]. We need a definition.

Definition 1 *A symmetric matrix $F \in \mathbb{R}^{n\times n}$ is feasible for a graph G if*

$$F_{i,j} = \begin{cases} 1, & \text{if } i = j \\ 1, & \text{if } (i,j) \text{ is an edge of } G \\ \in \mathbb{R}, & \text{otherwise} \end{cases} \tag{1}$$

For a symmetric matrix $M \in R^{n\times n}$, let $\lambda_{max}(M)$ be the largest eigenvalue of M. We define *Lavász's* ϑ-function of a graph as follows:

$$\vartheta(G) = \min\{\lambda_{max}(F) : F \text{ is feasible for } G\}.$$

With this function ϑ, one can show that for every graph G,

$$\omega(G) \leq \vartheta(G) \leq \chi(G).$$

We know that for a perfect graph $\omega(G) = \chi(G)$ holds, so $\vartheta(G) = \omega(G) = \chi(G)$.

The next thing to show is that the ϑ-function can be computed in polynomial time: One can show that $\vartheta(G)$ is the solution of a semidefinite programming problem, which can be solved with the ellipsoid method. More precisely, we can compute for every $\epsilon > 0$ in polynomial time a rational number r such that $|r - \vartheta(G)| < \epsilon$. Since G is perfect, we know that $\vartheta(G)$ is integer. Therefore with $\epsilon < \frac{1}{2}$, this will compute $\vartheta(G)$ exactly.

Thus we get $\omega(G)$, $\chi(G)$ directly and we can get $\alpha(G) = \omega(\overline{G})$ and $k(G) = \chi(\overline{G})$ since $\overline{G}$ is also perfect. $\square$

4 Other perfect graph classes

In the remainder, we will introduce some other graph classes that are perfect. We will not be giving many details; see Golumbic's book [Gol80].

4.1 Split graphs

A graph $G = (V, E)$ is a *split graph* if V can be partitioned in $V = I \cup C$ such that I forms an independent set and C forms a clique. See Figure 2 for an example. One can show the following:

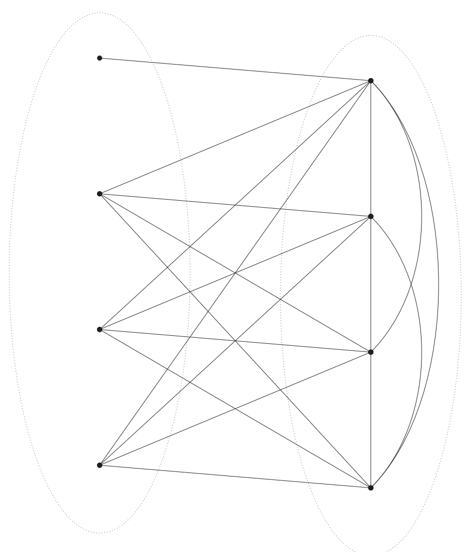

Figure 2: A split graph with one of its four partitions indicated.

Theorem 2 *G is a split graph if and only if G and $\overline{G}$ are chordal graphs.*

4.2 Permutation graphs

Suppose that π is a permutation of the numbers $1, 2, \ldots, n$. We can define an undirected graph $G = (V, E)$ from π by numbering the vertices from 1 to n. Two vertices (i, j) have an edge if and only if the order between them is reversed in π.

A definition in a more formal way is the following: With $\pi^{-1}(i)$ giving the position in the sequence where the number i can be found, then (i, j) is an edge in G if and only if the lines $(i, \pi^{-1}(i))$ and $(j, \pi^{-1}(j))$ cross. That means for $i < j$

$$(i,j) \in E \Leftrightarrow \pi^{-1}(i) > \pi^{-1}(j)$$

For an example, see figure 3. One can show (this was left as an exercise):

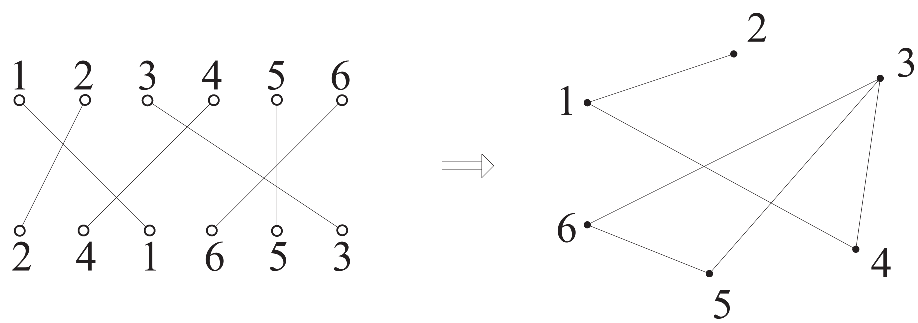

Figure 3: Example of a permutation graph.

Theorem 3 *G is a permutation graph if and only if G and $\overline{G}$ are comparability graphs.*

4.3 Threshold graphs

G is a *threshold graph* if there exists a weight function $w : V \to \mathbb{N}$ and a threshold value $t \in \mathbb{N}$ such that $I \subseteq V$ is an independent set if and only if

$$\sum_{v \in I} w(v) \leq t.$$

An example for a threshold graph with $t = 6$ is given at figure 4.

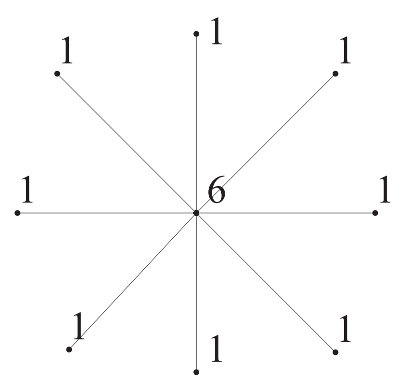

Figure 4: Example for a threshold graph with $t = 6$.

[Note from Therese: The above picture is not correct; it should be a $K_{1,6}$, not a $K_{1,8}$.]

One can show that if G is a threshold graph then G and $\overline{G}$ are interval graphs, but the reverse does not hold.

4.4 Superperfect graphs

There exist weighted versions of clique cover and coloring. One definition for weighted coloring used for solving the shipbuilding problem [Gol80] is the following:

To each vertex x of a graph $G = (V, E)$ we associate a non-negative number $w(x)$, and we define the *weight* of a subset $S \subseteq V$ to be the quantity

$$w(S) = \sum_{x \in S} w(x).$$

The pair $(G; w)$ is called *weighted graph.* The subset S will often be the vertices of a simple cycle or a clique or a stable set. An *interval coloring* of a weighted graph $(G; w)$ maps each vertex x onto an (open) interval I_x of the real line of width (or measure) $w(x)$ such that adjacent vertices are mapped to disjoint intervals, that is, $xy \in E$ implies $I_x \cap I_y = \varnothing$. The *number of hues* of a coloring (i.e. its total *width*) is defined to be $| \cup_x I_x|$. The *interval chromatic number* $\chi(G; w)$ is the least number of hues needed to color the vertices with intervals.

The *clique number* for a weighted graph $(G; w)$ is defined as

$$w(G; w) = \max\{w(K)|K \text{ is a clique of } G\}.$$

Definition 2 *A graph G is superperfect if $\omega_w(H) = \chi_w(H)$ for **every** weight-function on **every** induced subgraph H of G.*

We saw earlier that weighted clique can be solved in linear time on comparability graphs. In fact, so can weighted coloring and the two numbers are equal. So we know:

Theorem 4 *Every comparability graph is superperfect.*

References

[Ber62] Claude Berge. *Sur une conjecture relative au problème des codes optimaux.* Tokyo, 1962.

[BLS99] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes: A Survey.* SIAM Monographs on Discrete Mathematics and Applications, 1999.

[Gol80] Martin Charles Golumbic. *Algorithmic graph theory and perfect graphs.* Academic Press, New York, 1980.

[Lov72] L. Lovász. A characterization of perfect graphs. *J. of Comb. Th.*, 13:95–98, 1972.

[Rav75] G. Ravindra. On Berge's conjecture concerning perfect graphs. *Proc. Indian Nat. Sci. Acad.*, 41A:294–296, 1975.

[Tuc73] Alan C. Tucker. The strong perfect graph conjecture for planar graphs. *Canad. J. Math*, 25:103–114, 1973.