CS762: Graph-Theoretic Algorithms
Lecture 12: Intersection Graphs
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Abstract

This lecture concerns intersection graphs and some of its variations like line graphs, circular-arc graphs, t-interval graphs, and boxicity graphs. Besides the basic properties of these graph classes, results of some common problems like recognition, coloring, independent set, relationship to other graph classes etc. (which are sometimes known to be NP-complete for some graph classes) are discussed.

1 Introduction

We have seen many examples of intersection graphs so far, for example interval graphs, chordal graphs and permutation graphs. Informally, a graph is called an intersection graph if the vertices correspond to sets of objects (of any type) and edges correspond to the similarity (or intersection) among the sets. Many real life problems can be formulated as problems on intersection graphs. Due to its large and wide range of use in problem solving, it has several subclasses like line graphs, t-interval graphs, circular-arc graphs, boxicity graphs etc. [BLS99].

Recognition of a particular type of graphs, independent set, coloring, clique, relationship to other graph classes etc. are some classical problems in graph theory. These problems are sometimes (in fact, most of the times) NP-complete for a class of graphs [GJ79]. For some intersection graphs these problems can be solved efficiently.

In Section 2 we formally define intersection graphs and discuss them briefly. Then in Section 3, 4, 5 and 6 we discuss line graphs, circular-arc graphs, t-interval graphs and boxicity graphs respectively.

2 Intersection Graphs

Definition 1 *A graph G is an intersection graph if for each vertex v there is a set S_v of objects and there is an edge (v, w) if and only if $S_v \cap S_w \neq \emptyset$.*

Theorem 1 [Mar45] *Every graph is an intersection graph.*

Proof: Set $S_v = \{e_i : v \text{ is an end point of } e_i\}$. Then $(v, w) = e_i$ if and only if $e_i \in S_v$ **and** $e_i \in S_w$, which implies $S_v \cap S_w \neq \emptyset$. □

The following is an open problem about intersection graphs.

Open Problem 1 *Every graph is an intersection graph if we represent each vertex by $S_v = \{$all edges incident to $v\}$. With this formulation, we need m different elements (one for each edge) to* *create the sets for the vertices. Now the problem is: can we do this with n elements where n is the number of vertices? Or may be even less? Or can we show that m elements are necessary for some graphs?*

3 Line Graphs

Line graphs are a variation of intersection graphs.

Definition 2 *A line graph H of a graph G is the graph where $V(H) = E(G)$ and $E(H)$ contains (e_1, e_2) if and only if e_1 and e_2 have common endpoint in G.*

Figure 1 shows a graph G (solid lines) and its corresponding line graph H (dashed lines).

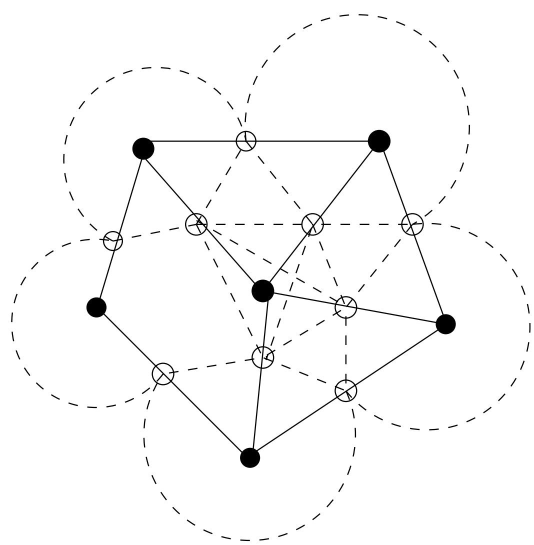


Figure 1: A graph and its corresponding line graph.

The line graph of the Figure 1 contains an induced 5-cycle (the outer dashed cycle). So, line graphs are not perfect.

3.1 Some Results on Line Graphs

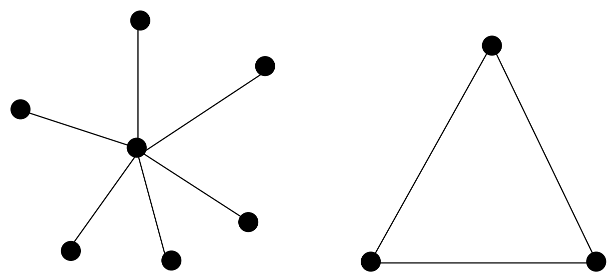
In this subsection we give some results for line graphs and compare some familiar problems for a line graph (H) and its corresponding general graph (G).

- *recognition*: Line graphs can be recognized in $O(n+m)$ time where n and m are number of vertices and number of edges respectively [Rou73, Leh74].
- *independent set*: Maximum independent set is NP-hard for general graphs. But this problem for a line graph H turns to maximum matching for the corresponding general graph G. Since maximum matching problem can be solved in polynomial time for any graph, independent set for line graphs can be solved in polynomial time.
- *covering*: Clique cover is NP-hard for general graphs. This problem for a line graph H turns to vertex cover, which is also known to be NP-hard, for the corresponding general graph G.

So, clique cover remains NP-hard for line graphs (except for a triangle). If G is bipartite then the vertex cover can be solved in polynomial time and so the clique cover can be solved in polynomial time for the corresponding line graph H.

- *coloring*: The (vertex-)coloring for a line graph H is equivalent to the edge coloring for the corresponding general graph G which is know to be NP-hard. So, coloring problem remains NP-hard for line graphs. It becomes polynomial if G is bipartite.

- *clique*: Finding a maximum clique is NP-hard in general. A clique in a line graph H is a triangle or a collection of edges with only one end point (see Figure 2). So, finding a maximum clique in a line graph H is equivalent to testing the corresponding general graph G to be a triangle or an object of Figure 2(a).



(a) (b)

Figure 2: Graphs whose corresponding line graphs are clique.

Although line graphs are not necessarily perfect, one can show that line graphs of bipartite graphs are perfect. We have the following theorem which is a consequence of König Lemma [Kön16].

Theorem 2 *If G is bipartite, then the line graph H of G is perfect.*

4 Circular-arc Graphs

Circular-arc graphs are another type of intersection graphs. They are also a generalization of interval graphs.

Definition 3 *G is a circular-arc graph if there is a finite collection of arcs on a circle such that each vertex v in G corresponds to an arc and there is an edge (v, w) in G if and only if the arcs corresponding to v and w intersect.*

Figure 3 shows a collection of arcs (solid lines) on a circle (dotted line) and its corresponding circular-arc graph (dashed lines).

From the example of Figure 3, we see that the circular-arc graph can contain an induced 5-cycle. So, circular-arc graphs are not perfect.

There are many applications of circular-arc graphs. Problems related to traffic scheduling, DNA sequencing (prokaryote DNA is circular) etc. can be formulated as problems on circular-arc graphs.

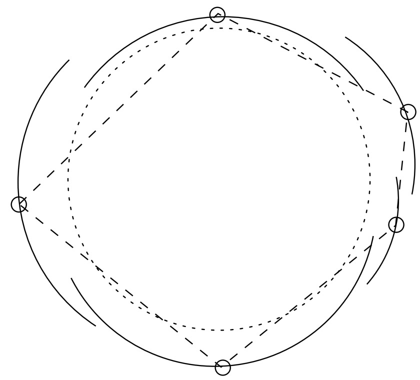


Figure 3: A circular-arc graph.

4.1 Some Results on Circular-arc Graphs

Now we give some results for circular-arc graphs.

- *recognition*: Circular-arc graphs can be recognized in $O(n+m)$ time where n and m are the number of vertices and edges respectively [HBH90, DHH96].
- *coloring*: Coloring is NP-hard for circular-arc graphs (as for general graphs) [GJMP78].
- *clique*: Although clique is NP-hard for general graphs, it is polynomial-time solvable for circular-arc graphs [Gav74].

5 t-interval Graphs

t-interval graphs are another type of intersection graphs.

Definition 4 *A graph G is a t-interval graph if each vertex is a set of at most t intervals and there is an edge (v, w) if and only if v and w intersect in at least one interval.*

Figure 4 shows a 2-interval graph (dashed lines) and the corresponding intervals (solid lines), and we see that it contains a 5-cycle. So, t-interval graphs are not perfect (even for $t = 2$).

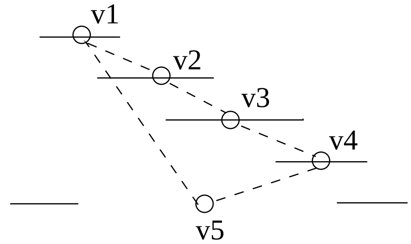


Figure 4: A 2-interval graph.

5.1 Some Results on t-interval Graphs

- *relationship to other graph classes*:
  - From the definition of t-interval graphs it is clear that every interval graph is 1-interval graph.
  - Griggs and West [GW79] have shown that every graph G is a t-interval graph for some $t \leq \lceil (\Delta + 1)/2 \rceil$, where Δ is the maximum degree of G.
  - Trees are 2-interval graphs [GK76, TH79].
  - Planar graphs are 3-interval graphs [SW83].
  - Line graphs are 2-interval graphs but the reverse is not true.
- *recognition*: Recognition is NP-hard for 2-interval graphs [WS84], but for 1-interval graphs it is polynomial as we have seen earlier.

6 Boxicity Graphs

This is another type of intersection graphs.

Definition 5 *A graph is a boxicity d-graph if every vertex is an axis-aligned d-dimensional box and there is an edge (v, w) if and only if the box for v and the box for w intersect.*

Figure 5 shows a boxicity 2-graph (dashed lines), and we see that the graph contains a 5-cycle. So, boxicity graphs are not perfect.

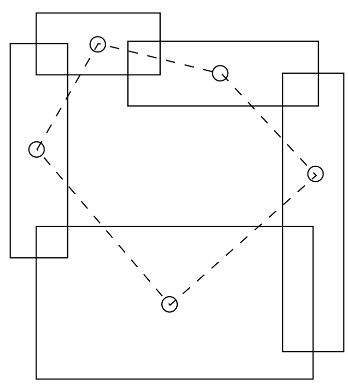


Figure 5: A boxicity 2-graph.

6.1 Some Results on Boxicity Graphs

- *recognition*: Recognition of boxicity d-graph with $d \geq 2$ is NP-complete [Coz82, Yan82, Kra94].
- *relationship to interval graphs*: It may be interesting to compare boxicity d-graphs with d-interval graphs. Although these two classes seem to be similar, there are differences. Boxicity 1-graphs are 1-interval graphs (as well as interval graphs), but for $d \geq 2$ there is no clear relationship between boxicity d-graphs and d-interval graphs. We summarize the differences with $d = 2$ as follows:

– Each vertex v in a boxicity 2-graph is a *cross product* of two intervals I_v^x and I_v^y in x and y dimension respectively, that is $v = I_v^x \times I_v^y$. But for a 2-interval graph each vertex v is the *union* of two intervals I_v^1 and I_v^2, that is $v = I_v^1 \cup I_v^2$.
– For an edge (v, w) in a boxicity 2-graph both I_v^x, I_w^x intersect and I_v^y, I_w^y intersect, that is $I_v^x \cap I_w^x \neq \emptyset$ **and** $I_v^y \cap I_w^y \neq \emptyset$. But in a 2-interval graph any pair of (I_v^1, I_w^1), (I_v^1, I_w^2), (I_v^2, I_w^1) or (I_v^2, I_w^2) intersects, that is $I_v^1 \cap I_w^1 \neq \emptyset$ **or** $I_v^1 \cap I_w^2 \neq \emptyset$ **or** $I_v^2 \cap I_w^1 \neq \emptyset$ **or** $I_v^2 \cap I_w^2 \neq \emptyset$

- *relationship to other graph classes*: Every graph can be represented as an boxicity d-graph of suitable dimension d [Rob69].
- *independent set*: Maximum independent set is NP-hard for boxicity graphs (as for general graphs).

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