CS762: Graph-Theoretic Algorithms
Lecture 4: Interval Graphs and their Complements
January 14, 2002

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Abstract

In this lecture, we will discuss the problem of finding a maximum independent set in an interval graph. We will also investigate the relationship between the complement of interval graphs and comparability graphs. Finally, we will give an introduction to the class of chordal graphs.

1 Introduction

In previous classes, we discussed interval graphs and the fact that several typically difficult problems can be solved in a reasonable amount of time when restricted to interval graphs. Problems such as graph colouring, finding a largest clique or independent set, determining if the graph contains a Hamiltonian cycle, and testing if two graphs are isomorphic are generally NP-hard problems. However, when restricted to the class of interval graphs, we can significantly reduce the time required to solve these problems with class specific algorithms that run in polynomial (often linear) time. One such problem that we will discuss in further detail is that of finding a maximum independent set. We will give a greedy algorithm commonly used for finding an independent set and state that the algorithm generates a maximum independent set when a perfect elimination order is given. A comprehensive list of problems and their complexities can be found at Lorna Stewart's homepage *http://web.cs.ualberta.ca/ stewart/GRAPH/index.html.*

Another class of graphs that we will look at is the complement of interval graphs. We will look at a natural way of orienting the edges of these graphs and show that the complement of interval graphs are in fact *comparability graphs.*

Finally, we will introduce *chordal graphs* and prove that a graph that has a perfect elimination order is chordal.

2 Definitions

A graph G is an *interval graph* if it is the intersection graph of some intervals.

An *independent set* or stable set in a graph $G = (V, E)$ is a subset I of the vertices V such that there are no edges in the induced subgraph $G[I]$. A k-independent set is an independent set of size at least k. A *maximal* independent set is an independent set in which no other vertex can be added to I without having an edge in $G[I]$. A *maximum* independent set is the largest independent set possible in G, denoted $\alpha(G)$.

A vertex order $\{v_1, v_2, \ldots, v_n\}$ is called a *perfect elimination order* if $Pred(v_i)$ is a clique for $1 \leq i \leq n$. $Pred(v_i)$ is the set of vertices adjacent to v_i that preceed v_i in the vertex order and $Succ(v_i)$ is the set of vertices adjacent to v_i that follow v_i in the vertex order.

3 An algorithm for finding an Independent Set

The following algorithm is a greedy algorithm used for creating an independent set. It does not guarantee that the independent set created is maximal or the maximum possible. The input for the algorithm is a graph G and a vertex order $\{v_1, v_2, \ldots, v_n\}$.

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Initially, all vertices are "untouched".
for i = 1, ..., n{
    if v_i is "untouched"{
        add v_i to the independent set
        mark all of Succ(v_i) as "touched"
    }
}
```

Suppose that a perfect elimination order is used as input to the greedy algorithm. Would this then guarantee a maximum independent set is created? Consider the graph in Figure 1.

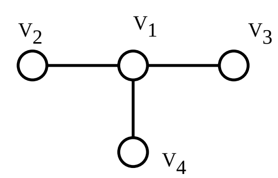

Figure 1: A graph with a perfect elimination order

It is easy to verify that this graph contains a perfect elimination order $\{v_1, v_2, v_3, v_4\}$. However, when the greedy algorithm is applied to this order, v_1 is selected to be the first vertex of the independent set. The three remaining vertices are then marked as "*touched*" and will not be considered for the independent set. Clearly, the independent set $\{v_1\}$ is not the maximum independent set since $\{v_2, v_3, v_4\}$ would be a larger independent set in this graph. So, we cannot use the perfect elimination order directly with the greedy algorithm to get a maximum independent set. However, if we use the perfect elimination order in reverse order, we will obtain a maximum independent set.

Theorem 1 *Let $\{v_1, v_2, \ldots, v_n\}$ be a perfect elimination order. Then the greedy algorithm applied with order $v_n, v_{n-1}, \ldots, v_1$ gives a maximum independent set.*

Proof: This proof is given as a homework assignment. A hint is to look for it in [Gol80].

By using algorithms such as the one above, many problems that are generally NP-hard can be solved in linear or polynomial time when restricted to interval graphs. Some such problems include colouring, maximum clique, maximum independent set, Hamiltonian cycle, and graph isomorphism. However, other problems, such as the travelling salesman problem still remain NP-hard when restricted to interval graphs. Given a set of cities and distances between each pair of cities, the travelling salesman problem is to find the shortest path that he can take to visit all of the cities. If we represent each city by a vertex and then add edges with weights corresponding to the distance between cities, the travelling salesman problem then becomes the problem of finding the shortest path that contains all vertices in the graph. Since it is possible to travel from any city to any other city, we have an edge between every pair of vertices. This means that the graph is a complete graph denote K_n where n is the number of vertices. The complete graph is an interval graph and can be represented by n intervals each with the same start and end points. Thus each interval will intersect with every other interval causing each vertex to be connected to all other vertices in the graph.

4 Complements of interval graphs

Since we have now looked at interval graphs, it only makes sense to learn more about their complement. In many cases, a graph and its complement are contained within the same class. Is this also true for interval graphs? Is the complement of an interval graph another interval graph?

To create a graph from a given set of intervals, we represent each interval as a vertex and add an edge between intervals that intersect. So, in the complement of this graph, we will still have each vertex represented by an interval but an edge is added only when two intervals do not intersect (see Figure 2). Moreover, the edges can be directed according to which interval is on the left. Given an interval graph $G = (V, E)$, the complement of G, $\overline{G} = (V, DE)$ where $(u, v) \in DE$ for $u, v \in V$ is a directed edge $u \rightarrow v$ and interval I_u is to the left of interval I_v.

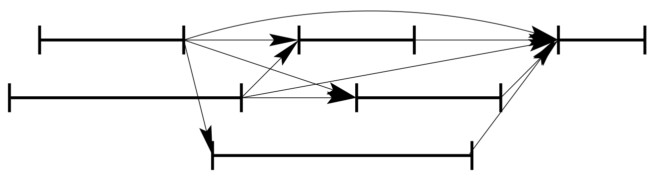

Figure 2: Given a set of intervals, directed edges can be added to nonadjacent intervals to create the complement.

Definition 1 *An orientation of a graph $G = (V, E)$ is a directed graph obtained by choosing an orientation ($u \rightarrow v$ or $v \rightarrow u$) for each $(u, v) \in E$ and $u, v \in V$. An acyclic orientation of G is an orientation that does not contain a directed cycle. A transitive orientation satisfies the property that for all $u, v, w \in V$ if $u \rightarrow v$ and $v \rightarrow w$ then $u \rightarrow w$. An orientation of the edges that is acyclic and transitive is called a transitive orientation.*

Definition 2 *A graph that has a transitive orientation is called a comparability graph. A graph whose complement is a comparability graph is called a co-comparability graph*[1].

The orientation of $\overline{G}$ is both acyclic and transitive. Thus, the complement of an interval graph is a *comparability graph* or interval graphs are *co-comparability graphs*.

We have shown that interval graphs are co-comparability graphs, but are all co-comparabilty graphs also interval graphs? Figure 3 shows a graph G that is a comparability graph while $\overline{G}$ is a co-comparability graph, but not an interval graph. The graph $\overline{G}$ is not an interval graph since it contains a cycle on 4 vertices, C_4, as an induced subgraph. In the previous lecture, it is shown that C_4 does not have an interval representation. So not all co-comparability graphs are also interval graphs.

The class of bipartite graphs, however, are comparability graphs. This is easily shown by considering the two bipartite sets A and B. Edges only exist between A and B not within each set. So to create a transitive orientation, we can simply direct all edges from A to B. Hence, bipartite graphs are comparability graphs.

[1]The prefix "*co-*" is used when discussing properties of the complement of a graph G. In the previous definition, the complement of a graph G is a comparability graph, so, we say that G is a co-comparability graph.

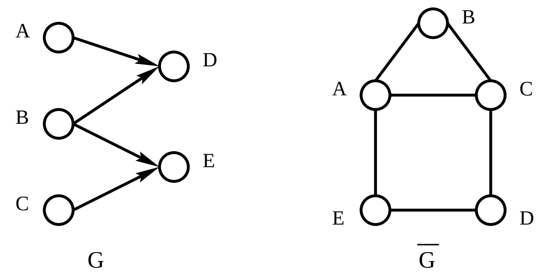

Figure 3: The complement of comparability graph G is not an interval graph.

5 Chordal graphs

Definition 3 *A graph is a chordal graph if it does not contain an induced k-cycle for $k \geq 4$ (see Figure 4). Figure 5 shows an example of a chordal graph.*

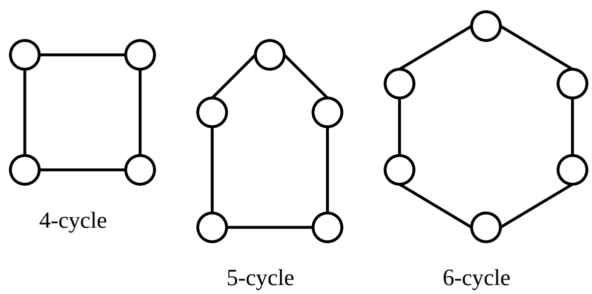

Figure 4: A 4-cycle, 5-cycle, and 6-cycle

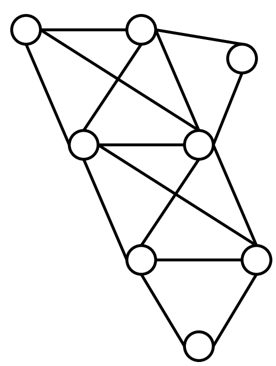

Figure 5: A chordal graph

Similar to interval graphs, we are interested in determining what properties chordal graphs have and what classes of graphs are contained within or intersect with chordal graphs. This gives rise to the next theorem.

Theorem 2 *If G has a perfect elimination order, then G is chordal.*

Proof: Assume C is a k-cycle where $k \geq 4$ and C has no chord. Let v be the vertex in the cycle that appears last in the perfect elimination order of all the cycle vertices. Then v must have at least 2 predecessors - its neighbours in the cycle. These two neighbours must form a clique since they are predecessors of v in the perfect elimination order. However, there is no edge between them since they are both adjacent to v in a chordless cycle (see Figure 6).

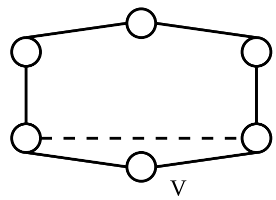

Figure 6: The neighbours of v are not adjacent.

References

[Gol80] Martin Charles Golumbic. *Algorithmic graph theory and perfect graphs.* Academic Press, New York, 1980.