CS 373: Combinatorial Algorithms, Spring 2001
Homework 0, due January 23, 2001 at the beginning of class

Name:	
Net ID:	Alias:

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not sign your name. Do not write your Social Security number.** Staple this sheet of paper to the top of your homework.

Grades will be listed on the course web site by alias give us, so your alias should not resemble your name or your Net ID. If you don't give yourself an alias, we'll give you one that you won't like.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 273—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Parberry and Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Before you do anything else, read the Homework Instructions and FAQ on the CS 373 course web page (http://www-courses.cs.uiuc.edu/~cs373/hw/faq.html), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don't turn in source code, analyze everything, use good English and good logic, and so forth.

☐   I have read the CS 373 Homework Instructions and FAQ.

Required Problems

1. (a) Prove that any positive integer can be written as the sum of distinct powers of 2. For example: $42 = 2^5 + 2^3 + 2^1$, $25 = 2^4 + 2^3 + 2^0$, $17 = 2^4 + 2^0$. *[Hint: 'Write the number in binary' is not a proof; it just restates the problem.]*

   (b) Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers—if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, $17 = F_7 + F_4 + F_2$.

   (c) Prove that *any* integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example: $42 = 3^4 - 3^3 - 3^2 - 3^1$, $25 = 3^3 - 3^1 + 3^0$, $17 = 3^3 - 3^2 - 3^0$.

2. Sort the following 20 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice.

$$1 \qquad n \qquad n^2 \qquad \lg n \qquad \lg^* n$$

$$2^{2^{\lg \lg n + 1}} \qquad \lg^* 2^n \qquad 2^{\lg^* n} \qquad \lfloor \lg(n!) \rfloor \qquad \lfloor \lg n \rfloor!$$

$$n^{\lg n} \qquad (\lg n)^n \qquad (\lg n)^{\lg n} \qquad n^{1/\lg n} \qquad n^{\lg \lg n}$$

$$\log_{1000} n \qquad \lg^{1000} n \qquad \lg^{(1000)} n \qquad \left(1 + \frac{1}{1000}\right)^n \qquad n^{1/1000}$$

To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

3. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. Assume reasonable but nontrivial base cases if none are supplied. Extra credit will be given for more exact solutions.

(a) $A(n) = 5A(n/3) + n \log n$

(b) $B(n) = \min_{0 < k < n} \left(B(k) + B(n - k) + 1 \right)$.

(c) $C(n) = 4C(\lfloor n/2 \rfloor + 5) + n^2$

(d) $D(n) = D(n - 1) + 1/n$

*(e) $E(n) = n + 2\sqrt{n} \cdot E(\sqrt{n})$

4. This problem asks you to simplify some recursively defined boolean formulas as much as possible. In each case, prove that your answer is correct. Each proof can be just a few sentences long, but it must be a *proof*.

(a) Suppose $\alpha_0 = p$, $\alpha_1 = q$, and $\alpha_n = (\alpha_{n-2} \wedge \alpha_{n-1})$ for all $n \geq 2$. Simplify α_n as much as possible. *[Hint: What is α_5?]*

(b) Suppose $\beta_0 = p$, $\beta_1 = q$, and $\beta_n = (\beta_{n-2} \Leftrightarrow \beta_{n-1})$ for all $n \geq 2$. Simplify β_n as much as possible. *[Hint: What is β_5?]*
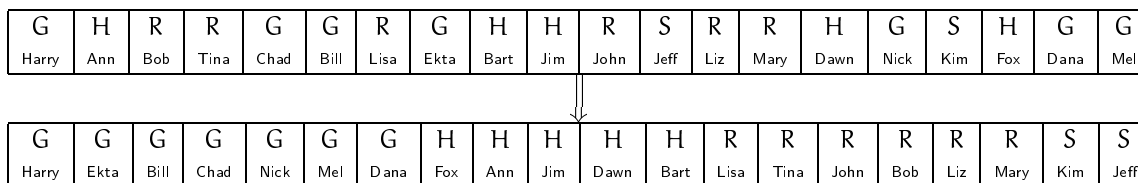
(c) Suppose $\gamma_0 = p$, $\gamma_1 = q$, and $\gamma_n = (\gamma_{n-2} \Rightarrow \gamma_{n-1})$ for all $n \geq 2$. Simplify γ_n as much as possible. *[Hint: What is γ_5?]*

(d) Suppose $\delta_0 = p$, $\delta_1 = q$, and $\delta_n = (\delta_{n-2} \bowtie \delta_{n-1})$ for all $n \geq 2$, where $\bowtie$ is some boolean function with two arguments. Find a boolean function $\bowtie$ such that $\delta_n = \delta_m$ if and only if $n - m$ is a multiple of 4. *[Hint: There is only one such function.]*

5. Every year, upon their arrival at Hogwarts School of Witchcraft and Wizardry, new students are sorted into one of four houses (Gryffindor, Hufflepuff, Ravenclaw, or Slytherin) by the Hogwarts Sorting Hat. The student puts the Hat on their head, and the Hat tells the student which house they will join. This year, a failed experiment by Fred and George Weasley filled almost all of Hogwarts with sticky brown goo, mere moments before the annual Sorting. As a result, the Sorting had to take place in the basement hallways, where there was so little room to move that the students had to stand in a long line.

   After everyone learned what house they were in, the students tried to group together by house, but there was too little room in the hallway for more than one student to move at a time. Fortunately, the Sorting Hat took CS 373 many years ago, so it knew how to group the students as quickly as possible. What method did the Sorting Hat use?

   More formally, you are given an array of n items, where each item has one of four possible values, possibly with a pointer to some additional data. Describe an algorithm[1] that rearranges the items into four clusters in $O(n)$ time using only $O(1)$ extra space.

G	H	R	R	G	G	R	G	H	H	R	S	R	R	H	G	S	H	G	G
Harry	Ann	Bob	Tina	Chad	Bill	Lisa	Ekta	Bart	Jim	John	Jeff	Liz	Mary	Dawn	Nick	Kim	Fox	Dana	Mel

G	G	G	G	G	G	G	H	H	H	H	H	R	R	R	R	R	R	S	S
Harry	Ekta	Bill	Chad	Nick	Mel	Dana	Fox	Ann	Jim	Dawn	Bart	Lisa	Tina	John	Bob	Liz	Mary	Kim	Jeff

6. *[This problem is required only for graduate students taking CS 373 for a full unit; anyone else can submit a solution for extra credit.]*

   Penn and Teller have a special deck of fifty-two cards, with no face cards and nothing but clubs—the ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 52 of clubs. (They're big cards.) Penn shuffles the deck until each each of the 52! possible orderings of the cards is equally likely. He then takes cards one at a time from the top of the deck and gives them to Teller, stopping as soon as he gives Teller the three of clubs.

   (a) On average, how many cards does Penn give Teller?

   (b) On average, what is the smallest-numbered card that Penn gives Teller?

   *(c) On average, what is the largest-numbered card that Penn gives Teller?

   [Hint: Solve for an n-card deck and then set $n = 52$.] In each case, give *exact* answers and prove that they are correct. If you have to appeal to "intuition" or "common sense", your answers are probably wrong!

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[1]Since you've read the Homework Instructions, you know what the phrase 'describe an algorithm' means. Right?