12 Shortest Paths (March 1)

12.1 Introduction

Given a weighted *directed* graph $G = (V, E, w)$ with two special vertices, a *source* s and a *target* t, we want to find the shortest directed path from s to t. In other words, we want to find the path p starting at s and ending at t minimizing the function

$$w(p) = \sum_{e \in p} w(e).$$

For example, if we want to answer the question 'What's the fastest way to drive from my apartment in Champaign, Illinois to my wife's apartment in Columbus, Ohio?', we might use a graph whose vertices are cities, edges are roads, weights are driving times, s is Champaign, and t is Columbus.[1] The graph is directed since the driving times along the same road might be different in different directions.[2]

Perhaps counter to intuition, we will allow the weights on the edges to be negative. Negative edges make our lives complicated, since the presence of a negative cycle might mean that there is no shortest path. In general, a shortest path from s to t exists if and only if there is *at least one* path from s to t, but there is no path from s to t that touches a negative cycle. If there is a negative cycle between s and t, then se can always find a shorter path by going around the cycle one more time.