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Note

On the reconstruction of the degree sequence

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Abstract

Harary's edge reconstruction conjecture states that a graph $G=(V,E)$ with at least four edges is uniquely determined by the multiset of its edge-deleted subgraphs, i.e. the graphs of the form $G-e$ for $e\in E$. It is well-known that this multiset uniquely determines the degree sequence of a graph with at least four edges. In this note we generalize this result by showing that the degree sequence of a graph with at least four edges is uniquely determined by the *set* of the degree sequences of its edge-deleted subgraphs with one well-described class of exceptions. Moreover, the *multiset* of the degree sequences of the edge-deleted subgraphs always allows one to reconstruct the degree sequence of the graph. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs will be finite, simple and undirected. For a graph $G=(V,E)$, the deletion of an edge $e\in E$ produces an *edge-deleted subgraph* of G and the multiset of the edge-deleted subgraphs of G is the *edge deck* of G. The *vertex-deleted subgraphs* and the *vertex deck* of a graph are defined similarly.

The decks of a graph play a central role in the theory of reconstruction which is motivated by two famous open conjectures: Kelly [4,5] and Ulam's [7] *vertex reconstruction conjecture* which states that a graph of order at least three is uniquely

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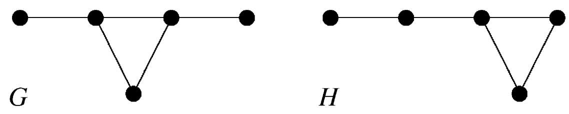

Fig. 1. A small exceptional pair.

determined (up to isomorphism) by its vertex deck and Harary's [2] *edge reconstruction conjecture* which states that a graph with at least four edges is uniquely determined by its edge deck. For detailed information on these conjectures we refer the reader to Bondy's survey [1].

It has been shown that two graphs with the same edge (vertex) deck share many properties. The edge version of a fundamental lemma due to Kelly [5] implies, for example, that two graphs with at least four edges and the same edge deck have the same degree sequence. Manvel [6] generalized this by proving that already the *set* of the edge-deleted subgraphs is sufficient to determine the degree sequence of a graph with at least four edges.

In the present note we will further generalize this result by showing that the degree sequence of a graph with at least four edges is uniquely determined by the *set* of degree sequences of its edge-deleted subgraphs with one well-described class of exceptions. Moreover, the *multiset* of degree sequences of the edge-deleted subgraphs always allows one to reconstruct the degree sequence of the graph.

We need some notation and terminology. Let $G=(V,E)$ be a graph. The degree of a vertex u in G will be denoted by $d(u,G)$. The set of edge-deleted subgraphs of G will be denoted by $\mathscr{E}(G)$, i.e. $\mathscr{E}(G)=\{G-e \mid e\in E\}$.

It is convenient for our purposes to define the *degree sequence* of a graph G as the mapping $\mathbf{d}_G : \mathbf{N}_0=\{0,1,2,3,\ldots\}\to\mathbf{N}_0$ with $\mathbf{d}_G(i)=|\{v\in V(G)|d(v,G)=i\}|$ for $i\geqslant 0$. This definition slightly differs from the one given in [3], but carries the same information. To wit, if $\mathbf{d}_G(\Delta)>0$ and $\mathbf{d}_G(i)=0$ for all $i>\Delta$, then G has maximum degree Δ, order $\sum_{i=0}^{\Delta}\mathbf{d}_G(i)$, and size $\frac{1}{2}\sum_{i=0}^{\Delta} i\mathbf{d}_G(i)$.

Now, the set of degree sequences of the elements in $\mathscr{E}(G)$ is the set of mappings $\{\mathbf{d}_H : \mathbf{N}_0\to\mathbf{N}_0|H\in\mathscr{E}(G)\}$ and will be denoted by $\mathscr{D}(G)$. Whenever convenient we will write a mapping $m : \mathbf{N}_0\to\mathbf{N}_0$ as the sequence $[m(0),m(1),m(2),\ldots]$.

For two positive integers i and j an edge uv is called a *i-edge* of G, if $i\in\{d(u,G), d(v,G)\}$, and it is called a *i,j-edge* of G, if $\{i,j\}=\{d(u,G),d(v,G)\}$. A graph is said to *be of type i*, if all of its edges are i-edges. A graph is said to *be of some (no) type*, if there is some (no) integer i such that the graph is of type i.

Let e be an i,j-edge of the graph G. If $|i-j|\geqslant 2$, then $\mathbf{d}_{G-e}(i-1)=\mathbf{d}_G(i-1)+1$, $\mathbf{d}_{G-e}(j-1)=\mathbf{d}_G(j-1)+1$, $\mathbf{d}_{G-e}(i)=\mathbf{d}_G(i)-1$, $\mathbf{d}_{G-e}(j)=\mathbf{d}_G(j)-1$ and $\mathbf{d}_{G-e}(k)=\mathbf{d}_G(k)$ for all $k\in\mathbf{N}_0\backslash\{i-1,j-1,i,j\}$. Similar relations hold if $|i-j|\leqslant 1$.

Hence $\mathbf{d}_{G-e}=\mathbf{d}_G+\delta_i+\delta_j$ where for $k\geqslant 1$, $\delta_k : \mathbf{N}_0\to\mathbf{Z}=\{0,\pm 1,\pm 2,\ldots\}$ is the mapping defined by $\delta_k(k-1)=1$, $\delta_k(k)=-1$ and $\delta_k(l)=0$ for $l\in\mathbf{N}_0\backslash\{k-1,k\}$. Note that every mapping $\mathbf{f} : \mathbf{N}_0\to\mathbf{Z}$ has a unique linear decomposition in terms of the δ_i's.

To illustrate these notions we consider the pair of graphs G and H in Fig. 1 which is a small member of the above mentioned class of exceptions. In this example

$\mathbf{d}_G=[0,2,1,2,0,0,\ldots]$, $\mathbf{d}_H=[0,1,3,1,0,0,\ldots]$ and

$$\mathscr{D}(G)=\mathscr{D}(H)=\{[1,1,2,1,0,0,\ldots],[0,3,1,1,0,0,\ldots],[0,2,3,0,0,\ldots]\}.$$

The exposition of our results naturally splits into three parts. In Section 2 we will consider the degenerate case of graphs G for which $|\mathscr{D}(G)|=1$. Then, in Section 3, we consider graphs that are of no type. If G is of no type, then $\mathscr{D}(G)$ has enough structure to determine $\mathbf{d}_G$. Finally, in Section 4, we consider graphs G of some type with $|\mathscr{D}(G)|\geqslant 2$. Our results entirely settle the question when $\mathscr{D}(G)$ uniquely determines $\mathbf{d}_G$ for some graph G.

2. Graphs G with $|\mathscr{D}(G)|=1$

It is obvious that $|\mathscr{D}(G)|=1$ if and only if all edges of G are d_1,d_2-edges for some $d_1,d_2\in\mathbf{N}$. The next theorem characterizes the possible unique elements of $\mathscr{D}(G)$ in this case.

Theorem 1. *Let G be a graph with at least four edges and let $d_1,d_2\in\mathbf{N}$ with $d_1\leqslant d_2$. Then all edges of G are d_1,d_2-edges if and only if $\mathscr{D}(G)=\{\mathbf{f}\}$ for some $\mathbf{f}:\mathbf{N}_0\to\mathbf{N}_0$ and there are integers $n(0)\in\mathbf{N}_0$ and $n(d_1),n(d_2)\in\mathbf{N}$ such that if $d_1=d_2=1$, then $n(1)\geqslant 8$ and*

$$\mathbf{f}:\mathbf{N}_0\to\mathbf{N}_0:\mathbf{f}(i)=\begin{cases} n(0)+2 & \text{if } i=0,\\ n(1)-2 & \text{if } i=1,\\ 0 & \text{else},\end{cases}$$

if $d_1=d_2\geqslant 2$, then $n(d_1)\geqslant 4$ and

$$\mathbf{f}:\mathbf{N}_0\to\mathbf{N}_0:\mathbf{f}(i)=\begin{cases} n(0) & \text{if } i=0,\\ 2 & \text{if } i=d_1-1,\\ n(d_1)-2 & \text{if } i=d_1,\\ 0 & \text{else}\end{cases}$$

and if $d_1<d_2$, then $d_1n(d_1)=d_2n(d_2)\geqslant 4$ and $\mathbf{f}=\mathbf{d}+\delta_{d_1}+\delta_{d_2}$ for

$$\mathbf{d}:\mathbf{N}_0\to\mathbf{N}_0:\mathbf{d}(i)=\begin{cases} n(0) & \text{if } i=0,\\ n(d_1) & \text{if } i=d_1,\\ n(d_2) & \text{if } i=d_2,\\ 0 & \text{else}.\end{cases}$$

Proof. If all edges of G are d_1,d_2-edges, then trivially $\mathscr{D}(G)=\{\mathbf{f}\}$ for some $\mathbf{f}$ as in the statement of the theorem.

Conversely, let $\mathscr{D}(G)=\{\mathbf{f}\}$ for some $\mathbf{f}$ as in the statement of the theorem. It is straightforward (but tedious) to verify, that d_1, d_2, $n(0)$, $n(d_1)$ and $n(d_2)$ are uniquely determined. Since $|\mathscr{D}(G)|=1$, all edges of G are d_1', d_2'-edges for some $d_1', d_2' \in \mathbf{N}$ with $d_1' \leqslant d_2'$.

As d_1 and d_2 are uniquely determined, we obtain that $d_1' = d_1$, $d_2' = d_2$ and $\mathbf{d}_G = \mathbf{f} - \delta_{d_1} - \delta_{d_2}$. This completes the proof. $\square$

Note that the following corollary contains the case of regular graphs. The straightforward proof is left to the reader.

Corollary 2. *Let G be a graph with at least four edges.*

(i) *Given $\mathscr{D}(G)$, it is possible to decide whether there are integers $d_1, d_2 \in \mathbf{N}$ such that all edges of G are d_1, d_2-edges and to determine d_1 and d_2, if they exist.*
(ii) *Given $\mathscr{D}(G)$, it is possible to determine $\mathbf{d}_G$, if all edges of G are d_1, d_2-edges for some $d_1, d_2 \in \mathbf{N}$.*
(iii) *Given $\mathscr{D}(G)$ and one graph in $\mathscr{E}(G)$, it is possible to determine G, if all edges of G are d_1, d_2-edges for some $d_1, d_2 \in \mathbf{N}$.*

3. Graphs of no type

Theorem 3. *Let G be a graph with at least four edges.*

(i) *Given $\mathscr{D}(G)$, it is possible to decide whether there is an integer $d \in \mathbf{N}$ such that G is of type d.*
(ii) *Given $\mathscr{D}(G)$, it is possible to determine the degree sequence $\mathbf{d}_G$ of G, if there is no integer $d \in \mathbf{N}$ such that G is of type d.*

Proof. In view of Corollary 2, we can assume that $|\mathscr{D}(G)| \geqslant 2$. We fix an arbitrary element $\mathbf{f}_1 = \mathbf{d}_G + \delta_{i_1} + \delta_{i_2} \in \mathscr{D}(G)$ and consider the set $\mathscr{D}' = \{\mathbf{f}_1 - \mathbf{f} \mid \mathbf{f} \in \mathscr{D}(G), \mathbf{f} \neq \mathbf{f}_1\}$. All elements of $\mathscr{D}'$ have a unique minimal linear decomposition using either two or four δ_i's.

If $\mathbf{f}' = \delta_{i_1} + \delta_{i_2} - \delta_{i_3} - \delta_{i_4}$ for some $\mathbf{f}' \in \mathscr{D}'$, then there exist edges e_1 and e_2 in G such that e_1 is incident with vertices of degree i_1 and i_2, respectively, and e_2 is incident with vertices of degree i_3 and i_4, respectively, with $\{i_1, i_2\} \cap \{i_3, i_4\} = \emptyset$. Hence, G is of no type, $\mathbf{f}'$ determines $\{i_1, i_2\}$ and $\mathbf{f}_1$ and $\{i_1, i_2\}$ determine the degree sequence $\mathbf{d}_G$ of G as $\mathbf{d}_G = \mathbf{f}_1 - \delta_{i_1} - \delta_{i_2}$.

We can now assume that $\mathbf{f}' = \delta_i - \delta_j$ with $i \in \{i_1, i_2\}$ for every $\mathbf{f}' \in \mathscr{D}'$. This implies that each edge of G is incident with a vertex of degree i_1 or a vertex of degree i_2.

Therefore, either G is of type i_1 or i_2 or G is of no type and there exist edges e_1, e_2 and e_3 in G such that e_1 is incident with vertices of degree i_1 and i_2, respectively, e_2 is incident with vertices of degree i_1 and i_3, respectively, and e_3 is incident with vertices of degree i_2 and i_3, respectively, with $|\{i_1, i_2, i_3\}| = 3$.

If G is of type i_1 or i_2, say i_1, then $\mathbf{f}' = \delta_{i_2} - \delta_j$ for every $\mathbf{f}' \in \mathscr{D}'$. If G is of no type, then $\mathbf{f}_1' = \delta_{i_1} - \delta_j$ and $\mathbf{f}_2' = \delta_{i_2} - \delta_{j'}$ for some $\mathbf{f}_1', \mathbf{f}_2' \in \mathscr{D}'$. Therefore, we can differentiate between these two possibilities. Moreover, if G is of no type, then $\mathscr{D}'$ determines $\{i_1, i_2\}$ and $\mathbf{f}_1$ and $\{i_1, i_2\}$ determine the degree sequence $\mathbf{d}_G$ of G as above. $\square$

4. Graphs G of some type with $|\mathscr{D}(G)|\geqslant 2$

The following theorem gives a complete description of the pairs of degree sequences of graphs G and H with $\mathbf{d}_G\neq\mathbf{d}_H$ and $\mathscr{D}(G)=\mathscr{D}(H)$. By Theorem 3, these graphs are necessarily of some type.

Theorem 4. *Let G and H be graphs with at least 4 edges such that $\mathbf{d}_G\neq\mathbf{d}_H$ and $\mathscr{D}(G)=\mathscr{D}(H)$. Then G is of type i and H is of type j, for some $i,j\in\mathbf{N}$ with $i>j$, $|\mathscr{D}(G)|=|\mathscr{D}(H)|\geqslant 2$, and*

either (i) $i\geqslant 3$, $j=i-1$, *and there is some* $k\in\mathbf{N}_0$ *such that*

$$\mathbf{d}_G:\mathbf{N}_0\to\mathbf{N}_0:\mathbf{d}_G(l)=\begin{cases}(i-1)+k(i-1) & \textit{if } l=i,\\ (i-2)+ki & \textit{if } l=i-1,\\ 2 & \textit{if } l=i-2,\\ \mathbf{d}_G(0) & \textit{if } l=0,\\ 0 & \textit{else}\end{cases}$$

and

$$\mathbf{d}_H:\mathbf{N}_0\to\mathbf{N}_0:\mathbf{d}_H(l)=\begin{cases}(i-2)+k(i-1) & \textit{if } l=i,\\ i+ki & \textit{if } l=i-1,\\ 1 & \textit{if } l=i-2,\\ \mathbf{d}_G(0) & \textit{if } l=0,\\ 0 & \textit{else}\end{cases}$$

and G has exactly one i,i-edge and H has exactly one $(i-1),(i-1)$-edge

or (ii) $i=2$, *the connected components of G are one path on* 4 *vertices*, $\ell\geqslant 1$ *paths on* 3 *vertices, and* $\mathbf{d}_G(0)\geqslant 1$ *isolated vertices, and the connected components of H are one path on two vertices*, $\ell+1$ *paths on* 3 *vertices, and* $\mathbf{d}_G(0)-1$ *isolated vertices.*

Proof. By the results of Section 2, we have that $|\mathscr{D}(G)|=|\mathscr{D}(H)|\geqslant 2$ and, by the results of Section 3, we have that G is of type i and H is of type j for some $i,j\in\mathbf{N}$. Let $\mathbf{f}_1,\mathbf{f}_2\in\mathscr{D}(G)=\mathscr{D}(H)$ with $\mathbf{f}_1\neq\mathbf{f}_2$. Then

$$\mathbf{f}_1=\mathbf{d}_G+\delta_i+\delta_{i_1}=\mathbf{d}_H+\delta_j+\delta_{j_1}\quad\text{and}\quad\mathbf{f}_2=\mathbf{d}_G+\delta_i+\delta_{i_2}=\mathbf{d}_H+\delta_j+\delta_{j_2}.$$

Therefore, $\mathbf{f}_1-\mathbf{f}_2=\delta_{i_1}-\delta_{i_2}=\delta_{j_1}-\delta_{j_2}$ which implies that $i_1=j_1$, $i_2=j_2$, and

$$\mathbf{d}_G+\delta_i=\mathbf{d}_H+\delta_j.$$

Hence $i\neq j$ and we may assume without loss of generality that $j<i$.

Let $n_l = \mathbf{d}_G(l)$ for all $l \geqslant 0$. Since G is of type i, we have that

$$in_i \geqslant \sum_{l \neq i} ln_l. \tag{1}$$

We assume that $j \leqslant i-2$. Since H is of type j and $\mathbf{d}_H = \mathbf{d}_G + \delta_i - \delta_j$, we have that

$$j(n_j+1) \geqslant (j-1)(n_{j-1}-1) + (i-1)(n_{i-1}+1) + i(n_i-1) + \sum_{l \notin \{j-1,j,i-1,i\}} ln_l$$

which implies

$$jn_j \geqslant \sum_{l \neq j} ln_l - 2j. \tag{2}$$

By (1) and (2), we have

$$j \geqslant \sum_{l \notin \{j,i\}} ln_l \tag{3}$$

which implies that $n_l = 0$ for all $j < l < i$. Since $\mathbf{d}_H(i-1) = \mathbf{d}_G(i-1) + \delta_i(i-1) - \delta_j(i-1) = 1$, the graph H has a $j,(i-1)$-edge. Since G has no $i,(i-1)$-edge, this yields that

$$\mathbf{d}_H + \delta_j + \delta_{i-1} = \mathbf{d}_G + \delta_i + \delta_{i-1} \in \mathscr{D}(H) \backslash \mathscr{D}(G),$$

which is a contradiction. This implies that $j = i-1$.

Let m_G be the number of i,i-edges of G and let m_H be the number of $(i-1),(i-1)$-edges of H. As above, we obtain

$$in_i \geqslant (i-1)n_{i-1} + (i-2)n_{i-2} + \sum_{l \notin \{i-2,i-1,i\}} ln_l + 2m_G \tag{4}$$

and

$$(i-1)n_{i-1} \geqslant in_i + (i-2)n_{i-2} - 4i + 4 + \sum_{l \notin \{i-2,i-1,i\}} ln_l + 2m_H, \tag{5}$$

which implies

$$(i-2)n_{i-2} + 2m_G \leqslant in_i - (i-1)n_{i-1} \leqslant 4i - 4 - (i-2)n_{i-2} - 2m_H. \tag{6}$$

We have that $n_i \geqslant 2$, since otherwise G would be a star contradicting $|\mathscr{D}(G)| \geqslant 2$. This implies that $\mathbf{d}_H(i) \geqslant 1$.

If $m_G = 0$, then

$$\mathbf{d}_H + \delta_j + \delta_i = \mathbf{d}_G + 2\delta_i \in \mathscr{D}(H) \backslash \mathscr{D}(G),$$

which is a contradiction.

If $n_{i-1} = 0$, then $\mathbf{d}_H(i-1) = 0+1+1 = 2$. Since $n_i \geqslant 2$ and H is of type $i-1$, we have that $i = 2$ and $n_i = 2$. Since there is no graph with degree sequence $[\mathbf{d}_G(0), 0, 2, 0, \ldots]$, this is a contradiction. Hence $n_{i-1} \geqslant 1$.

If $m_H = 0$, then

$$\mathbf{d}_G + \delta_i + \delta_{i-1} = \mathbf{d}_H + 2\delta_{i-1} \in \mathscr{D}(G) \backslash \mathscr{D}(H),$$

which is a contradiction. Hence $m_G, m_H \geqslant 1$.

If $i = 2$, then (6) yields $2n_2 - n_1 = 2$. Together with (4) this implies that $n_l = 0$ for $l \notin \{0, 1, 2\}$ and $m_G = m_H = 1$. Hence, G consists of one path on 4 vertices, $\ell \geqslant 1$ paths on three vertices, and n_0 isolated vertices and H consists of one path on 2 vertices, $\ell + 1$ paths on three vertices, and $n_0 - 1$ isolated vertices.

If $i \geqslant 3$, then (6) yields that $n_{i-2} \leqslant 2$. If $n_{i-2} \leqslant 1$, then $\mathbf{d}_H(i-2) = 0$ and $n_{i-2} = 1$ and

$$\mathbf{d}_G + \delta_i + \delta_{i-2} = \mathbf{d}_H + \delta_{i-1} + \delta_{i-2} \in \mathscr{D}(G) \backslash \mathscr{D}(H),$$

which is a contradiction. Hence $n_{i-2} = 2$. By (6), we have that $in_i - (i-1)n_{i-1} = 2i - 2$. This equality has the following integer solutions

$$\begin{aligned} n_i &= (i-1) + k(i-1) \\ n_{i-1} &= (i-2) + ki \end{aligned}$$

for some $k \geqslant 0$. By (4), we have that $n_l = 0$ for $l \notin \{0, i-2, i-1, i\}$. Furthermore, by (6), we have that $m_G = m_H = 1$ and the proof is complete. □

The graphs with the degree sequences described in Theorem 4 are not uniquely determined for $i \geqslant 4$ and $k \geqslant 1$. (If $k = 0$, then the graphs are the uniquely determined, see e.g. Fig. 1 for the case $i = 3$).

We have the following corollary.

Corollary 5. *Let G be a graph with at least four edges.*

(i) *The multiset $\mathscr{D}_m(G)$ of the degree sequences of the edge-deleted subgraphs of G uniquely determines the degree sequence of G.*
(ii) (Manvel [6]) *$\mathscr{E}(G)$ uniquely determines the degree sequence of G.*

Proof. Trivially, if $\mathscr{D}(G)$ uniquely determines $\mathbf{d}_G$, then also either $\mathscr{D}_m(G)$ or $\mathscr{E}(G)$ does. Hence we assume that $\mathscr{D}(G)$ does not uniquely determine $\mathbf{d}_G$.

By Theorems 3 and 4, G is either of type i for some $i \geqslant 2$ and has the first degree sequence $\mathbf{d}_1$ given in Theorem 4 or G is of type $i-1$ and has the second degree sequence $\mathbf{d}_2$ given in Theorem 4.

We have seen in the proof of Theorem 4 that $\mathscr{D}(G)$ uniquely determines $\mathbf{d}_1 + \delta_i = \mathbf{d}_2 + \delta_{i-1}$. If G has degree sequence $\mathbf{d}_1$, then the degree sequence $\mathbf{d}_1 + \delta_i + \delta_i = \mathbf{d}_2 + \delta_{i-1} + \delta_i$ appears exactly once (since $m_G = 1$) in $\mathscr{D}_m(G)$, and, if G has degree sequence $\mathbf{d}_2$, then it appears at least $i \geqslant 2$ times in $\mathscr{D}_m(G)$. This proves (i).

If $i \geqslant 3$, then G has degree sequence $\mathbf{d}_1$ if and only if $\mathscr{E}(G)$ contains no graph with an $(i-2),(i-2)$-edge. If $i = 2$, then G has degree sequence $\mathbf{d}_1$, if and only if one graph in $\mathscr{E}(G)$ has a path on four vertices as a connected component. This proves (ii). □

It is possible to generalize the above results to graphs with loops or multiple edges. This leads to larger classes of exceptions.

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