CS 373: Combinatorial Algorithms, Fall 2000

Homework 0, due August 31, 2000 at the beginning of class

Name:	
Net ID:	Alias:

Neatly print your name (first name first, with no comma), your network ID, and a short alias into the boxes above. **Do not *sign* your name. Do *not* write your Social Security number.** Staple this sheet of paper to the top of your homework.

Grades will be listed on the course web site by alias give us, so your alias should not resemble your name or your Net ID. If you don't give yourself an alias, we'll give you one that you won't like.

Before you do anything else, read the Homework Instructions and FAQ on the CS 373 course web page (http://www-courses.cs.uiuc.edu/~cs373/hw/faq.html), and then check the box below. This web page gives instructions on how to write and submit homeworks—staple your solutions together in order, write your name and netID on every page, don't turn in source code, analyze everything, use good English and good logic, and so forth.

☐ I have read the CS 373 Homework Instructions and FAQ.

This homework tests your familiarity with the prerequisite material from CS 173, CS 225, and CS 273—many of these problems have appeared on homeworks or exams in those classes—primarily to help you identify gaps in your knowledge. **You are responsible for filling those gaps on your own.** Parberry and Chapters 1–6 of CLR should be sufficient review, but you may want to consult other texts as well.

Required Problems

1. Sort the following 25 functions from asymptotically smallest to asymptotically largest, indicating ties if there are any:

$$\begin{array}{ccccc} 1 & n & n^2 & \lg n & \lg(n \lg n) \\ \lg^* n & \lg^* 2^n & 2^{\lg^* n} & \lg \lg^* n & \lg^* \lg n \\ n^{\lg n} & (\lg n)^n & (\lg n)^{\lg n} & n^{1/\lg n} & n^{\lg \lg n} \\ \log_{1000} n & \lg^{1000} n & \lg^{(1000)} n & \left(1+\frac{1}{n}\right)^n & n^{1/1000} \end{array}$$

   To simplify notation, write $f(n) \ll g(n)$ to mean $f(n) = o(g(n))$ and $f(n) \equiv g(n)$ to mean $f(n) = \Theta(g(n))$. For example, the functions n^2, n, $\binom{n}{2}$, n^3 could be sorted either as $n \ll n^2 \equiv \binom{n}{2} \ll n^3$ or as $n \ll \binom{n}{2} \equiv n^2 \ll n^3$.

2. (a) Prove that any positive integer can be written as the sum of distinct powers of 2. For example: $42 = 2^5 + 2^3 + 2^1$, $25 = 2^4 + 2^3 + 2^0$, $17 = 2^4 + 2^0$. [Hint: "Write the number in binary" is *not* a proof; it just restates the problem.]

   (b) Prove that any positive integer can be written as the sum of distinct *nonconsecutive* Fibonacci numbers—if F_n appears in the sum, then neither F_{n+1} nor F_{n-1} will. For example: $42 = F_9 + F_6$, $25 = F_8 + F_4 + F_2$, $17 = F_7 + F_4 + F_2$.

   (c) Prove that *any* integer (positive, negative, or zero) can be written in the form $\sum_i \pm 3^i$, where the exponents i are distinct non-negative integers. For example: $42 = 3^4 - 3^3 - 3^2 - 3^1$, $25 = 3^3 - 3^1 + 3^0$, $17 = 3^3 - 3^2 - 3^0$.

3. Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. You do not need to turn in proofs (in fact, please *don't* turn in proofs), but you should do them anyway just for practice. If no base cases are given, assume something reasonable but nontrivial. Extra credit will be given for more exact solutions.

   (a) $A(n) = 3A(n/2) + n$

   (b) $B(n) = \max_{n/3<k<2n/3} \big(B(k) + B(n-k) + n\big)$

   (c) $C(n) = 4C(\lfloor n/2 \rfloor + 5) + n^2$

   *(d) $D(n) = 2D(n/2) + n/\lg n$

   *(e) $E(n) = \dfrac{E(n-1)}{E(n-2)}$, where $E(1) = 1$ and $E(2) = 2$.

4. Penn and Teller have a special deck of fifty-two cards, with no face cards and nothing but clubs—the ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ..., 52 of clubs. (They're big cards.) Penn shuffles the deck until each each of the 52! possible orderings of the cards is equally likely. He then takes cards one at a time from the top of the deck and gives them to Teller, stopping as soon as he gives Teller the three of clubs.

   (a) On average, how many cards does Penn give Teller?

   (b) On average, what is the smallest-numbered card that Penn gives Teller?

   *(c) On average, what is the largest-numbered card that Penn gives Teller?

   [Hint: Solve for an n-card deck, and then set $n = 52$.] Prove that your answers are correct. If you have to appeal to "intuition" or "common sense", your answers are probably wrong!