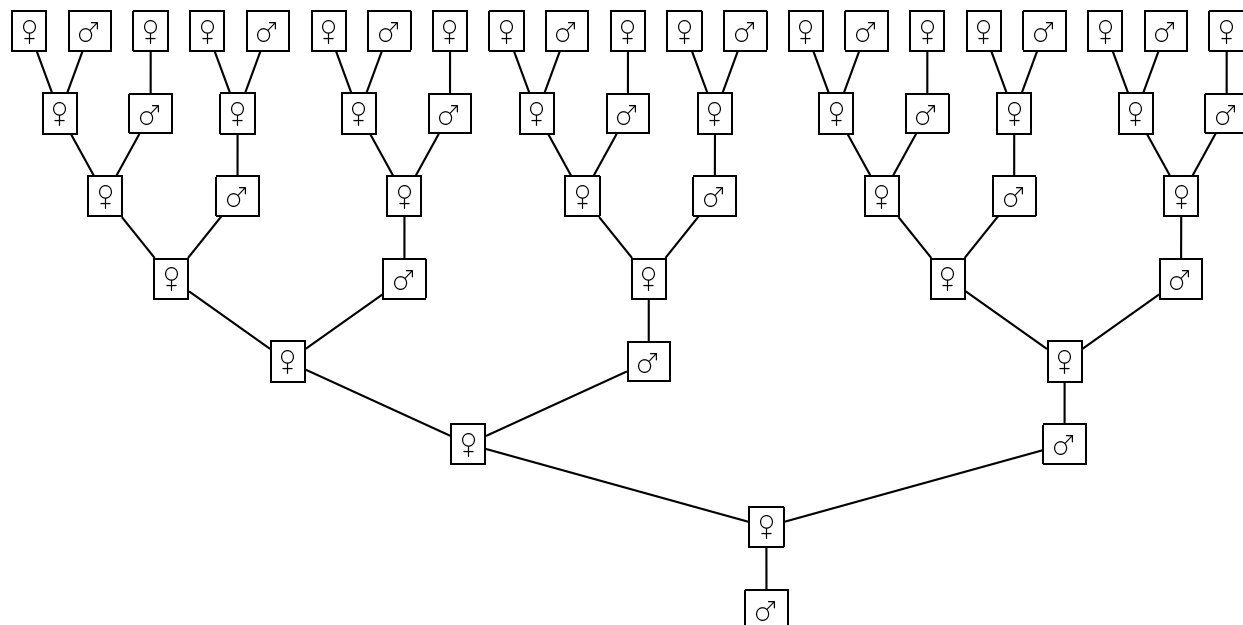# 1   Fun with Fibonacci numbers

Consider the reproductive cycle of bees. Each male bee has a mother but no father; each female bee has both a mother and a father. If we examine the generations we see the following family tree:

We easily see that the number of ancestors in each generation is the sum of the two numbers before it. For example, our male bee has three great-grandparents, two grandparents, and one parent, and $3 = 2 + 1$. The number of ancestors a bee has in generation n is defined by the Fibonacci sequence; we can also see this by applying the rule of sum.

As a second example, consider light entering two adjacent planes of glass:
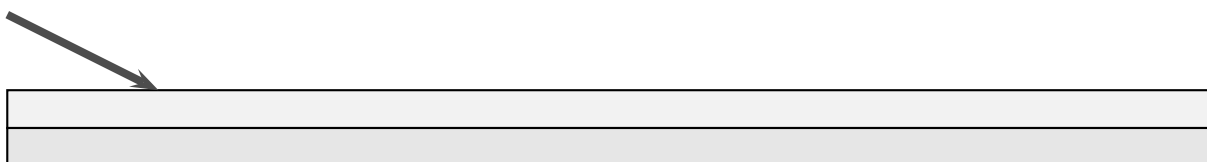
At any meeting surface (between the two panes of glass, or between the glass and air), the light may either reflect or continue straight through (refract). For example, here is the light bouncing seven times before it leaves the glass.
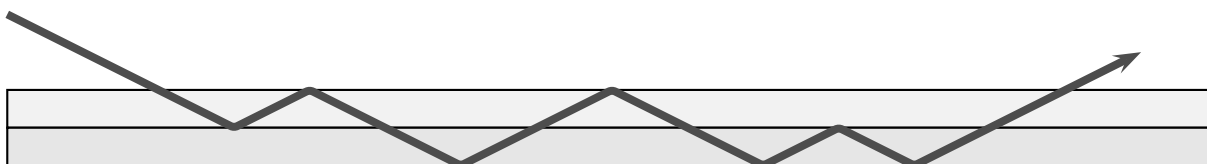
In general, how many different paths can the light take if we are told that it bounces n times before leaving the glass?

The answer to the question (in case you haven't guessed) rests with the Fibonacci sequence. We can apply the rule of sum to the event E constituting all paths through the glass in n bounces. We generate two separate sub-events, E_1 and E_2, illustrated in the following picture.
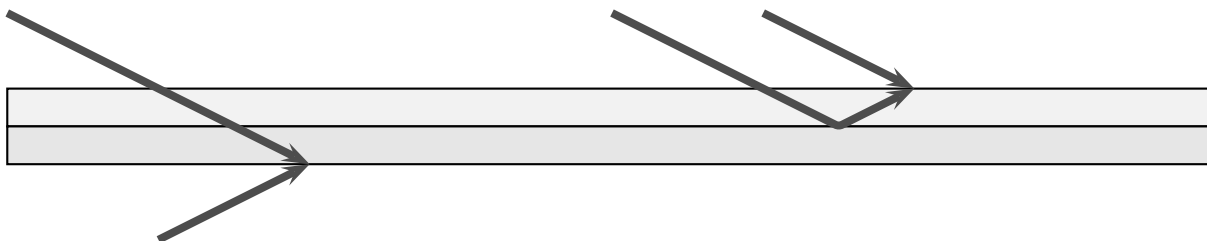
- **Sub-event E_1**: Let E_1 be the event that the first bounce is *not* at the boundary between the two panes. In this case, the light must first bounce off the bottom pane, or else we are dealing with the case of having zero bounces (there is only one way to have zero bounces). However, the number of remaining paths after bouncing off the bottom pane is the same as the number of paths entering through the bottom pane and bouncing $n - 1$ bounces more. Entering through the bottom pane is the same as entering through the top pane (but flipped over), so E_1 is the number of paths of light bouncing $n - 1$ times.

- **Sub-event E_2**: Let E_2 be the event that the first bounce *is* on the boundary between the two panes. In this case, we consider the two options for the light after the first bounce: it can either leave the glass (in which case we are dealing with the case of having one bounce, and there is only one way for the light to bounce once) or it can bounce yet again on the top of the upper pane, in which case it is equivalent to the light entering from the top with $n - 2$ bounces to take along its path.

By the rule of sum, we thus get the following recurrence relation for F_n, the number of paths in which the light can travel with exactly n bounces. There is exactly one way for the light to travel with no bounces—straight through—and exactly two ways for the light to travel with only one bounce—off the bottom and off the middle. For any $n > 1$, there are F_{n-1} paths where the light bounces off the bottom of the glass, and F_{n-2} paths where the light bounces off the middle and then off the top.

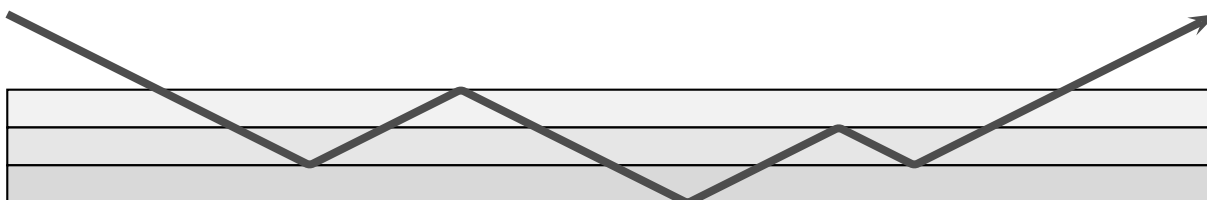$$F_0 = 1$$
$$F_1 = 2$$
$$F_n = F_{n-1} + F_{n-2}$$

Stump a professor

What is the recurrence relation for *three* panes of glass? This question once stumped an anonymous professor[1] in a science discipline, but now you should be able to solve it with a bit of effort. Aren't you proud of your knowledge?

---

[1] Not me! —*Jeff*

2   Sequences, sequence operators, and annihilators

We have shown that several different problems can be expressed in terms of Fibonacci sequences, but we don't yet know how to explicitly compute the nth Fibonacci number, or even (and more importantly) roughly how big it is. We can easily write a program to compute the n th Fibonacci number, but that doesn't help us much here. What we really want is a *closed form solution* for the Fibonacci recurrence—an explicit algebraic formula without conditionals, loops, or recursion.

In order to solve recurrences like the Fibonacci recurrence, we first need to understand *operations* on infinite sequences of numbers. Although these sequences are formally defined as *functions* of the form $A : \mathbb{N} \to \mathbb{R}$, we will write them either as $A = \langle a_0, a_1, a_2, a_3, a_4, \ldots \rangle$ when we want to emphasize the entire sequence[2], or as $A = \langle a_i \rangle$ when we want to emphasize a generic element. For example, the Fibonacci sequence is $\langle 0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots \rangle$.

We can naturally define several sequence operators:

- We can add or subtract any two sequences:

$$\langle a_i \rangle + \langle b_i \rangle = \langle a_0, a_1, a_2, \ldots \rangle + \langle b_0, b_1, b_2, \ldots \rangle = \langle a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots \rangle = \langle a_i + b_i \rangle$$
$$\langle a_i \rangle - \langle b_i \rangle = \langle a_0, a_1, a_2, \ldots \rangle - \langle b_0, b_1, b_2, \ldots \rangle = \langle a_0 - b_0, a_1 - b_1, a_2 - b_2, \ldots \rangle = \langle a_i - b_i \rangle$$

- We can multiply any sequence by a constant:

$$c \cdot \langle a_i \rangle = c \cdot \langle a_0, a_1, a_2, \ldots \rangle = \langle c \cdot a_0, c \cdot a_1, c \cdot a_2, \ldots \rangle = \langle c \cdot a_i \rangle$$

- We can shift any sequence to the left by removing its initial element:

$$\mathbf{E}\langle a_i \rangle = \mathbf{E}\langle a_0, a_1, a_2, a_3, \ldots \rangle = \langle a_1, a_2, a_3, a_4, \ldots \rangle = \langle a_{i+1} \rangle$$

Example: We can understand these operators better by looking at some specific examples, using the sequence T of powers of two.

$$T = \langle 2^0, 2^1, 2^2, 2^3, \ldots \rangle = \langle 2^i \rangle$$
$$\mathbf{E}T = \langle 2^1, 2^2, 2^3, 2^4, \ldots \rangle = \langle 2^{i+1} \rangle$$
$$2T = \langle 2 \cdot 2^0, 2 \cdot 2^1, 2 \cdot 2^2, 2 \cdot 2^3, \ldots \rangle = \langle 2^1, 2^2, 2^3, 2^4, \ldots \rangle = \langle 2^{i+1} \rangle$$
$$2T - \mathbf{E}T = \langle 2^1 - 2^1, 2^2 - 2^2, 2^3 - 2^3, 2^4 - 2^4, \ldots \rangle = \langle 0, 0, 0, 0, \ldots \rangle = \langle 0 \rangle$$

2.1   Properties of operators

It turns out that the distributive property holds for these operators, so we can rewrite $\mathbf{E}T - 2T$ as $(\mathbf{E} - 2)T$. Since $(\mathbf{E} - 2)T = \langle 0, 0, 0, 0, \ldots \rangle$, we say that the operator $(\mathbf{E} - 2)$ *annihilates* T, and we call $(\mathbf{E} - 2)$ an *annihilator* of T. Obviously, we can trivially annihilate any sequence by multiplying it by zero, so as a technical matter, we do not consider multiplication by 0 to be an annihilator.

What happens when we apply the operator $(E - 3)$ to our sequence T?

$$(\mathbf{E} - 3)T = \mathbf{E}T - 3T = \langle 2^{i+1} \rangle - 3\langle 2^i \rangle = \langle 2^{i+1} - 3 \cdot 2^i \rangle = \langle -2^i \rangle = -T$$

---

[2]It really doesn't matter whether we start a sequence with a_0 or a_1 or a_5 or even a_{-17}. Zero is often a convenient starting point for many recursively defined sequences, so we'll usually start there.

The operator $(\mathbf{E} - 3)$ did very little to our sequence T; it just flipped the sign of each number in the sequence. In fact, we will soon see that *only* $(\mathbf{E} - 2)$ will annihilate T, and all other simple operators will affect T in very minor ways. Thus, if we know how to annihilate the sequence, we know what the sequence must look like.

In general, $(\mathbf{E} - c)$ annihilates any geometric sequence $A = \langle a_0, a_0 c, a_0 c^2, a_0 c^3, \ldots \rangle = \langle a_0 c^i \rangle$:

$$(\mathbf{E} - c)\langle a_0 c^i \rangle = \mathbf{E}\langle a_0 c^i \rangle - c\langle a_0 c_i \rangle = \langle a_0 c^{i+1} \rangle - \langle c \cdot a_0 c_i \rangle = \langle a_0 c^{i+1} - a_0 c^{i+1} \rangle = \langle 0 \rangle$$

To see that this is the only operator of this form that annihilates A, let's see the effect of operator $(\mathbf{E} - d)$ for some $d \neq c$:

$$(\mathbf{E} - d)\langle a_0 c^i \rangle = \mathbf{E}\langle a_0 c^i \rangle - d\langle a_0 c_i \rangle = \langle a_0 c^{i+1} - d a_0 c_i \rangle = \langle (c - d) a_0 c^i \rangle = (c - d)\langle a_0 c^i \rangle$$

So we have a more rigorous confirmation that an annihilator annihilates exactly one type of sequence, but multiplies other similar sequences by a constant.

We can use this fact about annihilators of geometric sequences to solve certain recurrences. For example, consider the sequence $R = \langle r_0, r_1, r_2, \ldots \rangle$ defined recursively as follows:

$$r_0 = 3$$
$$r_{i+1} = 5r_i$$

We can easily prove that the operator $(\mathbf{E} - 5)$ annihilates R:

$$(\mathbf{E} - 5)\langle r_i \rangle = \mathbf{E}\langle r_i \rangle - 5\langle r_i \rangle = \langle r_{i+1} \rangle - \langle 5r_i \rangle = \langle r_{i+1} - 5r_i \rangle = \langle 0 \rangle$$

Since $(\mathbf{E} - 5)$ is an annihilator for R, we must have the closed form solution $r_i = r_0 5^i = 3 \cdot 5^i$. We can easily verify this by induction, as follows:

$$
\begin{aligned}
r_0 &= 3 \cdot 5^0 = 3 \quad \checkmark && \text{[definition]} \\
r_i &= 5r_{i-1} && \text{[definition]} \\
    &= 5 \cdot (3 \cdot 5^{i-1}) && \text{[induction hypothesis]} \\
    &= 5^i \cdot 3 \quad \checkmark && \text{[algebra]}
\end{aligned}
$$

2.2   Multiple operators

An operator is a function that transforms one sequence into another. Like any other function, we can apply operators one after another to the same sequence. For example, we can multiply a sequence $\langle a_i \rangle$ by a constant d and then by a constant c, resulting in the sequence $c(d\langle a_i \rangle) = \langle c \cdot d \cdot a_i \rangle = (c\,d)\langle a_i \rangle$. Alternatively, we may multiply the sequence by a constant c and then shift it to the left to get $\mathbf{E}(c\langle a_i \rangle) = \mathbf{E}\langle c \cdot a_i \rangle = \langle c \cdot a_{i+1} \rangle$. This is exactly the same as applying the operators in the reverse order: $c(\mathbf{E}\langle a_i \rangle) = c\langle a_{i+1} \rangle = \langle c \cdot a_{i+1} \rangle$. We can also shift the sequence twice to the left: $\mathbf{E}(\mathbf{E}\langle a_i \rangle) = \mathbf{E}\langle a_{i+1} \rangle = \langle a_{i+2} \rangle$. We will write this in shorthand as $\mathbf{E}^2\langle a_i \rangle$. More generally, the operator $\mathbf{E}^k$ shifts a sequence k steps to the left: $\mathbf{E}^k\langle a_i \rangle = \langle a_{i+k} \rangle$.

We now have the tools to solve a whole host of recurrence problems. For example, what annihilates $C = \langle 2^i + 3^i \rangle$? Well, we know that $(\mathbf{E} - 2)$ annihilates $\langle 2^i \rangle$ while leaving $\langle 3^i \rangle$ essentially unscathed. Similarly, $(\mathbf{E} - 3)$ annihilates $\langle 3^i \rangle$ while leaving $\langle 2^i \rangle$ essentially unscathed. Thus, if we apply both operators one after the other, we see that $(\mathbf{E} - 2)(\mathbf{E} - 3)$ annihilates our sequence C.

In general, for any integers $a \neq b$, the operator $(\mathbf{E} - a)(\mathbf{E} - b)$ annihilates any sequence of the form $\langle c_1 a^i + c_2 b^i \rangle$ but nothing else. We will often 'multiply out' the operators into the shorthand notation $\mathbf{E}^2 - (a + b)\mathbf{E} + ab$. It is left as an exhilarating exercise to the student to verify that this shorthand actually makes sense—the operators $(\mathbf{E} - a)(\mathbf{E} - b)$ and $\mathbf{E}^2 - (a + b)\mathbf{E} + ab$ have the same effect on every sequence.

We now know finally enough to solve the recurrence for Fibonacci numbers. Specifically, notice that the recurrence $F_i = F_{i-1} + F_{i-2}$ is annihilated by $\mathbf{E}^2 - \mathbf{E} - 1$:

$$(\mathbf{E}^2 - \mathbf{E} - 1)\langle F_i \rangle = \mathbf{E}^2 \langle F_i \rangle - \mathbf{E} \langle F_i \rangle - \langle F_i \rangle$$
$$= \langle F_{i+2} \rangle - \langle F_{i+1} \rangle - \langle F_i \rangle$$
$$= \langle F_{i-2} - F_{i-1} - F_i \rangle$$
$$= \langle 0 \rangle$$

Factoring $\mathbf{E}^2 - \mathbf{E} - 1$ using the quadratic formula, we obtain

$$\mathbf{E}^2 - \mathbf{E} - 1 = (\mathbf{E} - \phi)(\mathbf{E} - \hat{\phi})$$

where $\phi = (1 + \sqrt{5})/2 \approx 1.618034$ is the golden ratio and $\hat{\phi} = (1 - \sqrt{5})/2 = 1 - \phi = -1/\phi$. Thus, the operator $(E - \phi)(E - \hat{\phi})$ annihilates the Fibonacci sequence, so F_i must have the form

$$F_i = c\phi^i + \hat{c}\hat{\phi}^i$$

for some constants c and $\hat{c}$. We call this the *generic solution* to the recurrence, since it doesn't depend at all on the base cases. To compute the constants c and $\hat{c}$, we use the base cases $F_0 = 0$ and $F_1 = 1$ to obtain a pair of linear equations:

$$F_0 = 0 = c + \hat{c}$$
$$F_1 = 1 = c\phi + \hat{c}\hat{\phi}$$

Solving this system of equations gives us $c = 1/(2\phi - 1) = 1/\sqrt{5}$ and $\hat{c} = -1/\sqrt{5}$.

We now have a closed-form expression for the ith Fibonacci number:

$$\boxed{F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right)^i - \frac{1}{\sqrt{5}}\left(\frac{1 - \sqrt{5}}{2}\right)^i}$$

With all the square roots in this formula, it's quite amazing that Fibonacci numbers are integers. However, if we do all the math correctly, all the square roots cancel out when i is an integer. (In fact, this is pretty easy to prove using the binomial theorem.)

2.3 Degenerate cases

We can't quite solve *every* recurrence yet. In our above formulation of $(\mathbf{E} - a)(\mathbf{E} - b)$, we assumed that $a \neq b$. What about the operator $(\mathbf{E} - a)(\mathbf{E} - a) = (\mathbf{E} - a)^2$? It turns out that this operator annihilates sequences such as $\langle ia^i \rangle$:

$$(\mathbf{E} - a)\langle ia^i \rangle = \langle (i + 1)a^{i+1} - (a)ia^i \rangle$$
$$= \langle (i + 1)a^{i+1} - ia^{i+1} \rangle$$
$$= \langle a^{i+1} \rangle$$
$$(\mathbf{E} - a)^2 \langle ia^i \rangle = (\mathbf{E} - a)\langle a^{i+1} \rangle = \langle 0 \rangle$$

More generally, the operator $(\mathbf{E} - a)^k$ annihilates any sequence $\langle p(i) \cdot a^i \rangle$, where $p(i)$ is any polynomial in i of degree $k - 1$. As an example, $(\mathbf{E} - 1)^3$ annihilates the sequence $\langle i^2 \cdot 1^i \rangle = \langle i^2 \rangle = \langle 1, 4, 9, 16, 25, \ldots \rangle$, since $p(i) = i^2$ is a polynomial of degree $n - 1 = 2$.

As a review, try to explain the following statements:

- $(\mathbf{E} - 1)$ annihilates any constant sequence $\langle \alpha \rangle$.

- $(\mathbf{E} - 1)^2$ annihilates any arithmetic sequence $\langle \alpha + \beta i \rangle$.

- $(\mathbf{E} - 1)^3$ annihilates any quadratic sequence $\langle \alpha + \beta i + \gamma i^2 \rangle$.

- $(\mathbf{E} - 3)(\mathbf{E} - 2)(\mathbf{E} - 1)$ annihilates any sequence $\langle \alpha + \beta 2^i + \gamma 3^i \rangle$.

- $(\mathbf{E} - 3)^2(\mathbf{E} - 2)(\mathbf{E} - 1)$ annihilates any sequence $\langle \alpha + \beta 2^i + \gamma 3^i + \delta i 3^i \rangle$.

2.4   Summary

In summary, we have learned several operators that act on sequences, as well as a few ways of combining operators.

Operator	Definition
Addition	$\langle a_i \rangle + \langle b_i \rangle = \langle a_i + b_i \rangle$
Subtraction	$\langle a_i \rangle + \langle b_i \rangle = \langle a_i + b_i \rangle$
Scalar multiplication	$c\langle a_i \rangle = \langle c\,a_i \rangle$
Shift	$\mathbf{E}\langle a_i \rangle = \langle a_{i+1} \rangle$
Composition of operators	$(\mathbf{X} + \mathbf{Y})\langle a_i \rangle = \mathbf{X}\langle a_i \rangle + \mathbf{Y}\langle a_i \rangle$
	$(\mathbf{X} - \mathbf{Y})\langle a_i \rangle = \mathbf{X}\langle a_i \rangle - \mathbf{Y}\langle a_i \rangle$
	$\mathbf{X}\mathbf{Y}\langle a_i \rangle = \mathbf{X}(\mathbf{Y}\langle a_i \rangle) = \mathbf{Y}(\mathbf{X}\langle a_i \rangle)$
k-fold shift	$\mathbf{E}^k\langle a_i \rangle = \langle a_{i+k} \rangle$

Notice that we have not defined a multiplication operator for two sequences. This is usually accomplished by *convolution*:

$$\langle a_i \rangle * \langle b_i \rangle = \left\langle \sum_{j=0}^{i} a_j\, b_{i-j} \right\rangle.$$

Fortunately, convolution is unnecessary for solving the recurrences we will see in this course.

We have also learned some things about annihilators, which can be summarized as follows:

Sequence	Annihilator
$\langle \alpha \rangle$	$\mathbf{E} - 1$
$\langle \alpha a^i \rangle$	$\mathbf{E} - a$
$\langle \alpha a^i + \beta b^i \rangle$	$(\mathbf{E} - a)(\mathbf{E} - b)$
$\langle \alpha_0 a_0^i + \alpha_1 a_1^i + \cdots + \alpha_n a_n^i \rangle$	$(\mathbf{E} - a_0)(\mathbf{E} - a_1) \cdots (\mathbf{E} - a_n)$
$\langle \alpha i + \beta \rangle$	$(\mathbf{E} - 1)^2$
$\langle (\alpha i + \beta) a^i \rangle$	$(\mathbf{E} - a)^2$
$\langle (\alpha i + \beta) a^i + \gamma b^i \rangle$	$(\mathbf{E} - a)^2(\mathbf{E} - b)$
$\langle (\alpha_0 + \alpha_1 i + \cdots \alpha_{n-1} i^{n-1}) a^i \rangle$	$(\mathbf{E} - a)^n$
If $\mathbf{X}$ annihilates $\langle a_i \rangle$, then $\mathbf{X}$ also annihilates $c\langle a_i \rangle$ for any constant c.	
If $\mathbf{X}$ annihilates $\langle a_i \rangle$ and $\mathbf{Y}$ annihilates $\langle b_i \rangle$, then $\mathbf{X}\mathbf{Y}$ annihilates $\langle a_i \rangle \pm \langle b_i \rangle$.	

3   Solving Linear Recurrences

3.1   Homogeneous Recurrences

The general expressions in the annihilator box above are really the most important things to remember about annihilators because they help you to solve any recurrence for which you can write down an annihilator. The general method is:

> 1. Write down the annihilator for the recurrence
> 2. Factor the annihilator
> 3. Determine the sequence annihilated by each factor
> 4. Add these sequences together to form the generic solution
> 5. Solve for constants of the solution by using initial conditions

Example: Let's show the steps required to solve the following recurrence:

$$r_0 = 1$$
$$r_1 = 5$$
$$r_2 = 17$$
$$r_i = 7r_{i-1} - 16r_{i-2} + 12r_{i-3}$$

1. *Write down the annihilator.* Since $r_{i+3} - 7r_{i+2} + 16r_{i+1} - 12r_i = 0$, the annihilator is $\mathbf{E}^3 - 7\mathbf{E}^2 + 16\mathbf{E} - 12$.

2. *Factor the annihilator.* $\mathbf{E}^3 - 7\mathbf{E}^2 + 16\mathbf{E} - 12 = (\mathbf{E} - 2)^2(\mathbf{E} - 3)$.

3. *Determine sequences annihilated by each factor.* $(\mathbf{E} - 2)^2$ annihilates $\langle (\alpha i + \beta)2^i \rangle$ for any constants α and β, and $(\mathbf{E} - 3)$ annihilates $\langle \gamma 3^i \rangle$ for any constant γ.

4. *Combine the sequences.* $(\mathbf{E}-2)^2(\mathbf{E}-3)$ annihilates $\langle (\alpha i + \beta)2^i + \gamma 3^i \rangle$ for any constants α, β, γ.

5. *Solve for the constants.* The base cases give us three equations in the three unknowns α, β, γ:

$$r_0 = 1 = (\alpha \cdot 0 + \beta)2^0 + \gamma \cdot 3^0 = \beta + \gamma$$
$$r_1 = 5 = (\alpha \cdot 1 + \beta)2^1 + \gamma \cdot 3^1 = 2\alpha + 2\beta + 3\gamma$$
$$r_2 = 17 = (\alpha \cdot 2 + \beta)2^2 + \gamma \cdot 3^2 = 8\alpha + 4\beta + 9\gamma$$

We can solve these equations to get $\alpha = 1$, $\beta = 0$, $\gamma = 1$. Thus, our final solution is $\boxed{r_i = i2^i + 3^i}$, which we can verify by induction.

3.2   Non-homogeneous Recurrences

A *height balanced tree* is a binary tree, where the heights of the two subtrees of the root differ by at most one, and both subtrees are also height balanced. To ground the recursive definition, the empty set is considered a height balanced tree of height -1, and a single node is a height balanced tree of height 0.

Let T_n be the smallest height-balanced tree of height n—how many nodes does T_n have? Well, one of the subtrees of T_n has height $n-1$ (since T_n has height n) and the other has height either $n-1$ or $n-2$ (since T_n is height-balanced and as small as possible). Since both subtrees are themselves height-balanced, the two subtrees must be T_{n-1} and T_{n-2}.

We have just derived the following recurrence for t_n, the number of nodes in the tree T_n:

$$
\begin{aligned}
t_{-1} &= 0 & &\text{[the empty set]} \\
t_0 &= 1 & &\text{[a single node]} \\
t_n &= t_{n-1} + t_{n-2} + 1
\end{aligned}
$$

The final '+1' is for the root of T_n.

We refer to the terms in the equation involving t_i's as the *homogeneous* terms and the rest as the *non-homogeneous* terms. (If there were no non-homogeneous terms, we would say that the recurrence itself is homogeneous.) We know that $\mathbf{E}^2 - \mathbf{E} - 1$ annihilates the homogeneous part $t_n = t_{n-1} + t_{n-2}$. Let us try applying this annihilator to the entire equation:

$$
\begin{aligned}
(\mathbf{E}^2 - \mathbf{E} - 1)\langle t_i \rangle &= \mathbf{E}^2 \langle t_i \rangle - \mathbf{E}\langle a_i \rangle - 1\langle a_i \rangle \\
&= \langle t_{i+2} \rangle - \langle t_{i+1} \rangle - \langle t_i \rangle \\
&= \langle t_{i+2} - t_{i+1} - t_i \rangle \\
&= \langle 1 \rangle
\end{aligned}
$$

The leftover sequence $\langle 1, 1, 1, \ldots \rangle$ is called the *residue*. To obtain the annihilator for the entire recurrence, we compose the annihilator for its homogeneous part with the annihilator of its residue. Since $\mathbf{E} - 1$ annihilates $\langle 1 \rangle$, it follows that $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$ annihilates $\langle t_n \rangle$. We can factor the annihilator into

$$(\mathbf{E} - \phi)(\mathbf{E} - \hat{\phi})(\mathbf{E} - 1),$$

so our annihilator rules tell us that

$$t_n = \alpha \phi^n + \beta \hat{\phi}^n + \gamma$$

for some constants α, β, γ. We call this the *generic solution* to the recurrence. Different recurrences can have the same generic solution.

To solve for the unknown constants, we need three equations in three unknowns. Our base cases give us two equations, and we can get a third by examining the next nontrivial case $t_1 = 2$:

$$
\begin{aligned}
t_{-1} &= 0 = \alpha \phi^{-1} + \beta \hat{\phi}^{-1} + \gamma = \alpha/\phi + \beta/\hat{\phi} + \gamma \\
t_0 &= 1 = \alpha \phi^0 + \beta \hat{\phi}^0 + \gamma = \alpha + \beta + \gamma \\
t_1 &= 2 = \alpha \phi^1 + \beta \hat{\phi}^1 + \gamma = \alpha \phi + \beta \hat{\phi} + \gamma
\end{aligned}
$$

Solving these equations, we find that $\alpha = \frac{\sqrt{5}+2}{\sqrt{5}}$, $\beta = \frac{\sqrt{5}-2}{\sqrt{5}}$, and $\gamma = -1$. Thus,

$$
\boxed{t_n = \frac{\sqrt{5}+2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{\sqrt{5}-2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n - 1}
$$

Here is the general method for non-homogeneous recurrences:

> 1. Write down the homogeneous annihilator, directly from the recurrence
> $1\frac{1}{2}$. 'Multiply' by the annihilator for the residue
> 2. Factor the annihilator
> 3. Determine what sequence each factor annihilates
> 4. Add these sequences together to form the generic solution
> 5. Solve for constants of the solution by using initial conditions

3.3   Some more examples

In each example below, we use the base cases $a_0 = 0$ and $a_1 = 1$.

- $a_n = a_{n-1} + a_{n-2} + 2$

    - The homogeneous annihilator is $\mathbf{E}^2 - \mathbf{E} - 1$.
    - The residue is the constant sequence $\langle 2, 2, 2, \ldots \rangle$, which is annihilated by $\mathbf{E} - 1$.
    - Thus, the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$.
    - The annihilator factors into $(\mathbf{E} - \phi)(\mathbf{E} - \hat{\phi})(\mathbf{E} - 1)$.
    - Thus, the generic solution is $a_n = \alpha \phi^n + \beta \hat{\phi}^n + \gamma$.
    - The constants α, β, γ satisfy the equations

    $$a_0 = 0 = \alpha + \beta + \gamma$$
    $$a_1 = 1 = \alpha \phi + \beta \hat{\phi} + \gamma$$
    $$a_2 = 3 = \alpha \phi^2 + \beta \hat{\phi}^2 + \gamma$$

    - Solving the equations gives us $\alpha = \frac{\sqrt{5}+2}{\sqrt{5}}$, $\beta = \frac{\sqrt{5}-2}{\sqrt{5}}$, and $\gamma = -2$
    - So the final solution is $\boxed{a_n = \dfrac{\sqrt{5}+2}{\sqrt{5}} \left(\dfrac{1+\sqrt{5}}{2} \right)^n + \dfrac{\sqrt{5}-2}{\sqrt{5}} \left(\dfrac{1-\sqrt{5}}{2} \right)^n - 2}$

(In the remaining examples, I won't explicitly enumerate the steps like this.)

- $a_n = a_{n-1} + a_{n-2} + 3$

    The homogeneous annihilator $(\mathbf{E}^2 - \mathbf{E} - 1)$ leaves a constant residue $\langle 3, 3, 3, \ldots \rangle$, so the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)$, and the generic solution is $a_n = \alpha \phi^n + \beta \hat{\phi}^n + \gamma$. Solving the equations

    $$a_0 = 0 = \alpha + \beta + \gamma$$
    $$a_1 = 1 = \alpha \phi + \beta \hat{\phi} + \gamma$$
    $$a_2 = 4 = \alpha \phi^2 + \beta \hat{\phi}^2 + \gamma$$

    gives us the final solution $\boxed{a_n = \dfrac{\sqrt{5}+3}{\sqrt{5}} \left(\dfrac{1+\sqrt{5}}{2} \right)^n + \dfrac{\sqrt{5}-3}{\sqrt{5}} \left(\dfrac{1-\sqrt{5}}{2} \right)^n - 3}$

- $a_n = a_{n-1} + a_{n-2} + 2^n$

  The homogeneous annihilator $(\mathbf{E}^2 - \mathbf{E} - 1)$ leaves an exponential residue $\langle 4, 8, 16, 32, \ldots \rangle = \langle 2^{i+2} \rangle$, which is annihilated by $\mathbf{E} - 2$. Thus, the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 2)$, and the generic solution is $a_n = \alpha\phi^n + \beta\hat{\phi}^n + \gamma 2^n$. The constants α, β, γ satisfy the following equations:

$$a_0 = 0 = \alpha + \beta + \gamma$$
$$a_1 = 1 = \alpha\phi + \beta\hat{\phi} + 2\gamma$$
$$a_2 = 5 = \alpha\phi^2 + \beta\hat{\phi}^2 + 4\gamma$$

- $a_n = a_{n-1} + a_{n-2} + n$

  The homogeneous annihilator $(\mathbf{E}^2 - \mathbf{E} - 1)$ leaves a linear residue $\langle 2, 3, 4, 5 \ldots \rangle = \langle i+2 \rangle$, which is annihilated by $(\mathbf{E} - 1)^2$. Thus, the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^2$, and the generic solution is $a_n = \alpha\phi^n + \beta\hat{\phi}^n + \gamma + \delta n$. The constants $\alpha, \beta, \gamma, \delta$ satisfy the following equations:

$$a_0 = 0 = \alpha + \beta + \gamma$$
$$a_1 = 1 = \alpha\phi + \beta\hat{\phi} + \gamma + \delta$$
$$a_2 = 3 = \alpha\phi^2 + \beta\hat{\phi}^2 + \gamma + 2\delta$$
$$a_3 = 7 = \alpha\phi^3 + \beta\hat{\phi}^3 + \gamma + 3\delta$$

- $a_n = a_{n-1} + a_{n-2} + n^2$

  The homogeneous annihilator $(\mathbf{E}^2 - \mathbf{E} - 1)$ leaves a quadratic residue $\langle 4, 9, 16, 25 \ldots \rangle = \langle (i+2)^2 \rangle$, which is annihilated by $(\mathbf{E} - 1)^3$. Thus, the annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^3$, and the generic solution is $a_n = \alpha\phi^n + \beta\hat{\phi}^n + \gamma + \delta n + \varepsilon n^2$. The constants $\alpha, \beta, \gamma, \delta, \varepsilon$ satisfy the following equations:

$$a_0 = 0 = \alpha + \beta + \gamma$$
$$a_1 = 1 = \alpha\phi + \beta\hat{\phi} + \gamma + \delta + \varepsilon$$
$$a_2 = 5 = \alpha\phi^2 + \beta\hat{\phi}^2 + \gamma + 2\delta + 4\varepsilon$$
$$a_3 = 15 = \alpha\phi^3 + \beta\hat{\phi}^3 + \gamma + 3\delta + 9\varepsilon$$
$$a_4 = 36 = \alpha\phi^4 + \beta\hat{\phi}^4 + \gamma + 4\delta + 16\varepsilon$$

- $a_n = a_{n-1} + a_{n-2} + n^2 - 2^n$

  The homogeneous annihilator $(\mathbf{E}^2 - \mathbf{E} - 1)$ leaves the residue $\langle (i+2)^2 - 2^{i-2} \rangle$. The quadratic part of the residue is annihilated by $(\mathbf{E} - 1)^3$, and the exponential part is annihilated by $(\mathbf{E} - 2)$. Thus, the annihilator for the whole recurrence is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - 1)^3(\mathbf{E} - 2)$, and so the generic solution is $a_n = \alpha\phi^n + \beta\hat{\phi}^n + \gamma + \delta n + \varepsilon n^2 + \eta 2^i$. The constants $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ satisfy a system of six equations in six unknowns determined by $a_0, a_1, \ldots, a_5$.

- $a_n = a_{n-1} + a_{n-2} + \phi^n$

  The annihilator is $(\mathbf{E}^2 - \mathbf{E} - 1)(\mathbf{E} - \phi) = (\mathbf{E} - \phi)^2(\mathbf{E} - \hat{\phi})$, so the generic solution is $a_n = \alpha\phi^n + \beta n\phi^n + \gamma\hat{\phi}^n$. (Other recurrence solving methods will have a "interference" problem with this equation, while the operator method does not.)