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Graph reconstruction from subgraphs

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Abstract

The Reconstruction Conjecture asserts that every finite simple undirected graph on 3 or more vertices is determined, up to isomorphism, by its collection of (unlabeled) one-vertex-deleted subgraphs. A more general problem can be investigated if the collection consists of all (unlabeled) subgraphs with a restricted number of vertices. Kelly (Pacific J. Math. 7 (1957) 961–968) first raised the possibility of deleting several points from a graph and Manvel (Discrete Math. 8 (1974) 181–185) offered some basic observations on the problem. Here, we propose a review on the progress made in the last 25 years. Also, discussing the class of all finite trees, we go back to the original Kelly's interest. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

All graphs considered are finite, simple and undirected. More precisely: $P_2(X)$ denotes the set of all 2-point subsets of the set X. A graph is a couple $G=(X,E)$, where $E \subseteq P_2(X)$; $X = v(G)$ is the set of vertices of G and $E = e(G)$ is the set of edges of G. The size of graph G is the number of its vertices, i.e. $|G| = n = |X| = |v(G)|$; we also say that G is a graph on n vertices. If $Y \subseteq X$ then we define the *induced graph* G/Y as $G/Y = (Y, E \cap P_2(Y))$. We also use usual concepts of connectivity and connectivity components in a graph, i.e. the maximal connected induced subgraphs of a graph.

For graphs G_1, G_2 a mapping $f : v(G_1) \to v(G_2)$ is called a *homomorphism* if for every edge $\{x, y\} \in e(G_1)$ its image $\{f(x), f(y)\}$ is an edge in $e(G_2)$. In this paper, we deal with special homomorphisms f from G_1 to G_2:

- if f is a bijection and if both f and f^{-1} are homomorphisms we call f an *isomorphism*; G_1, G_2 are called isomorphic and denoted by $G_1 \simeq G_2$. Especially, an isomorphism from G to G is called an *automorphism* and the number of all automorphisms

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of G is denoted by $\mathrm{aut}(G)$; $\mathrm{aut}(G)$ also expreses the number of isomorphisms from G to any other graph isomorphic to G.

- if f is an injection and its restriction $f : G_1 \to G_2/f(v(G_1))$ is an isomorphism we call f a *monomorphism* (in a strong sense),
- if for every connectivity component C of G the restriction $f : C \to G_2/f(v(C))$ is an isomorphism, we call f a *semimonomorphism*,
- if f is a semimonomorhpism and moreover $f(v(G_1)) = v(G_2)$ we call f a *covering semimonomorphism*.

In the following series of lemmas we use special counting functions defined for any two graphs H, G:

$\mathrm{frq}(H,G)$	(the *frequency* of H in G) the number of induced subgraphs of G isomorphic to H,
$\mathrm{mon}(H,G)$	the number of monomorphisms from H to G,
$\mathrm{semi}(H,G)$	the number of semimonomorphisms from H to G,
$\mathrm{cov}(H,G)$	the number of covering semimonomorphisms from H to G.

Lemma 1. *If two graphs R, S have the same number of connectivity components then*

(1) *if R and S are not isomorphic then* $\mathrm{cov}(R,S) = 0$,
(2) *if R and S are isomorphic then* $\mathrm{cov}(R,S) = \mathrm{aut}(R) = \mathrm{aut}(S)$.

Proof (outline). Let $f : v(R) \to v(S)$ be a covering monomorphism. Since f is covering and R and S have the same number of components, f establishes a natural bijection between components of R and components of S and, moreover, the restrictions of f onto these individual components must be isomorphisms. Thus, f is an isomorphism between R and S. $\square$

Lemma 2. *Let R, S be two graphs and let $C_1, \ldots, C_q$ be the connectivity components of graph R. Then*
(1) $\mathrm{mono}(R,S) = \mathrm{aut}(R)\,\mathrm{frq}(R,S)$, (2) $\mathrm{semi}(R,S) = \prod_{i=1}^{q} \mathrm{mono}(C_i,S)$.

Proof (outline). (1) For every $Y \subseteq v(S)$ such that $S/Y \simeq R$ there are exactly $\mathrm{aut}(R)$ monomorphisms f from R to S for which $f(v(R)) = Y$.

(2) For a semimonomorphism f from R to S its restriction to each component $f : C_j \to S$ is a monomorphism which can be denoted by f_j. This establishes a natural one-to-one correspondence between the set of all semihomomorphisms from R to S and the set of all q-tuples $[f_1, \ldots, f_q]$ of monomorphisms $f_j : C_j \to S$. $\square$

Lemma 3. *Let $I = I_1 \cup I_2 \cup \cdots \cup I_m \cup \cdots$ be a set and let $\{R_i; i \in I\}$ be a family of graphs satisfying*

(A) *if $i \in I_m$ then the graph R_i has exactly m components,*
(B) *for every graph G there exists exactly one $i \in I$ such that $G \simeq R_i$.*

Then for every two graphs H, G there is

$$\mathrm{semi}(H,G) = \sum_{i \in I} \mathrm{cov}(H,R_i)\,\mathrm{frq}(R_i,G)$$

Proof (outline). First, let us remark that the above sum has only finite number of non-zero summands. Especially, if H has less components than R_i then $\mathrm{cov}(H,R_i)=0$.

For an arbitrary semimonomorphism $f: H \to G$ find $i_f \in I$ such that $G/f(v(H)) \simeq R_{i_f}$. Then the restriction $f: H \to G/f(v(H))$ is a covering semimonomorphism. The rest of the proof is a matter of grouping of semimonomorphisms over the indices i_f. □

2. Function rec

Definition 1. Let k be a natural number. Two graphs G_1, G_2 are called k-congruent ($G_1 \sim^k G_2$) if for every graph H on k vertices the equality $\mathrm{frq}(H,G_1) = \mathrm{frq}(H,G_2)$ holds.

Remark. The well-known Reconstruction Conjecture [38] asserts that any two graphs on $n \geqslant 3$ vertices that are $(n-1)$-congruent must be isomorphic. The conjecture was verified for many important classes of graphs the progress in results reached and techniques employed can be viewed in papers [1–4,8–11,14,19,21,23,28,36,37]. But the question of its validity for the class of all graphs still remains open.

We are interested in a more general problem. Namely, for which values of k any two graphs on n vertices being k-congruent must be isomorphic.

Definition 2. Let $\mathscr{A}$ be a class of graphs. We define the function $\mathrm{rec}_{\mathscr{A}}$ as follows:

$$\begin{aligned} \mathrm{rec}_{\mathscr{A}}(n) &= \min\{k;\ \forall G_1, G_2 \in \mathscr{A}((|G_1| = |G_2| = n \wedge G_1 \sim^k G_2) \Rightarrow G_1 \simeq G_2)\} \\ &= 0 \text{ if the minimum above does not exist.} \end{aligned}$$

We will use the symbol rec instead of $\mathrm{rec}_{\mathscr{A}}$ if it is clear what class $\mathscr{A}$ we are talking about.

Example. Let us consider the class of all graphs. The following table indicates some known results from a computer research by McKay (see [25]) and the author

n	1	2	3	4	5	6	7	8
$\mathrm{rec}(n)$	1	1	2	3	4	4	$\leqslant 6$	$\leqslant 7$

As we already mentioned, it is not clear if for all large values of n the inequality $\mathrm{rec}(n) \leqslant n-1$ holds. On the other hand, Manvel (1974) in [22] gave the first lower bound for $\mathrm{rec}(n)$. Our best estimate (see [32]) is that for each real number $\varepsilon > 0$ the inequality $n(1-\varepsilon) < \mathrm{rec}(n)$ holds for all sufficiently large values of n.

Let us also remind a result by Müller (see [27]). He has shown that, given $\varepsilon > 0$, there exists a class $\mathscr{A}$ containing asymptotically the most graphs such that $\mathrm{rec}_{\mathscr{A}}(n) \leqslant (n/2)(1+\varepsilon)$ for all large values of n.

The investigation of function rec has two aspects. If for some class of graphs $\mathscr{A}$ the Reconstruction Conjecture is rejected, i.e. it is proved that there exists an arbitrarily large n such that $\mathrm{rec}_{\mathscr{A}}(n) = n$, then one can try to find a subclass $\mathscr{B} \subseteq \mathscr{A}$ such that for all large values of n the inequality $\mathrm{rec}_{\mathscr{B}}(n) \leqslant n-1$ holds.

On the other hand, if for some class of graphs $\mathscr{A}$ the Reconstruction Conjecture is proved then this fact can initiate seeking lower and upper bounds for rec, i.e. some estimations of the form $l_n < \mathrm{rec}_{\mathscr{A}}(n) \leqslant u_n$. Especially, to find an integral lower bound l_n means to construct in $\mathscr{A}$ a family of couples of non-isomorphic graphs G_1, G_2 (on arbitrarily large number of vertices n) such that $G_1 \sim^{l_n} G_2$. In these constructions we exploit the following lemma first proved in [29].

Main lemma. *Let G_1, G_2 be two graphs on n vertices and let $k \leqslant n$ be a natural number. The following three statements are equivalent*

(i) $G_1 \sim^k G_2$,
(ii) $\mathrm{frq}(H, G_1) = \mathrm{frq}(H, G_2)$ *for every graph* $H, |H| \leqslant k$,
(iii) $\mathrm{frq}(H, G_1) = \mathrm{frq}(H, G_2)$ *for every connected graph* $H, |H| \leqslant k$.

Proof. The implications (ii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (iii) are evident. The implication (i) $\Rightarrow$ (ii) is the well-known Kelly's lemma (cf. [3], for example).

The proof of (iii) $\Rightarrow$ (ii):

Let I, I_m, R_i be the same as in Lemma 3. We prove by induction for every $m \leqslant k$ the validity of proposition

$$A(m):\ \text{if } q \in I_m \text{ and } |R_q| \leqslant k \text{ then } \mathrm{frq}(R_q, G_1) = \mathrm{frq}(R_q, G_2).$$

$\langle m = 1 \rangle$ $A(1)$ is true because of the assumption (i.e. for connected graphs which are in I_1),

$\langle m-1 \to m \rangle m \geqslant 2$ and $A(1), \ldots, A(m-1)$ are supposed to be true; let $q \in I_m$ and $|R_q| \leqslant k$; denote $M = I_1 \cup \cdots \cup I_{m-1}$ and also denote

$$s = \sum_{i \in M} \mathrm{cov}(R_q, R_i)\,\mathrm{frq}(R_i, G_1) = \sum_{i \in M} \mathrm{cov}(R_q, R_i)\,\mathrm{frq}(R_i, G_2).$$

According to Lemmas 1 and 3 we have

$$\mathrm{semi}(R_q, G_1) = \sum_{i \in I} \mathrm{cov}(R_q, R_i)\,\mathrm{frq}(R_i, G_1) = s + \mathrm{aut}(R_q)\,\mathrm{frq}(R_q, G_1),$$

$$\mathrm{semi}(R_q, G_2) = \sum_{i \in I} \mathrm{cov}(R_q, R_i)\,\mathrm{frq}(R_i, G_2) = s + \mathrm{aut}(R_q)\,\mathrm{frq}(R_q, G_2).$$

But by Lemma 2 we have $\mathrm{semi}(R_q, G_1) = \mathrm{semi}(R_q, G_2)$. Thus,

$$s + \mathrm{aut}(R_q)\,\mathrm{frq}\,(R_q, G_1) = s + \mathrm{aut}(R_q)\,\mathrm{frq}(R_q, G_2)$$

which yields $\mathrm{frq}(R_q, G_1) = \mathrm{frq}(R_q, G_2)$. $\square$

Corollary. *Let $\mathscr{G}[r]$ be the class of all graphs with at least r components. Then* $\mathrm{rec}_{\mathscr{G}[r]}(n) \leqslant n - r + 1$ *for every* $n \geqslant r$.

Proof (outline). In any graph on n vertices with at least r components every component has at most $n - r + 1$ vertices. Apply the Main lemma (cf. also [5]). □

Remark. Using the Main lemma we also found in [30] some bounds of reconstructability in the class of all finite equivalences (i.e. sums of complete graphs).

3. Trees

Let us denote by $\mathscr{T}$ the class of all finite trees. It was the first class investigated in connection with the Reconstruction Conjecture. Kelly already in 1957 [13] proved $\mathrm{rec}_{\mathscr{T}}(n) \leqslant n-1$ for every $n \geqslant 2$, and later on (in 1976) Giles gave in [6] $\mathrm{rec}_{\mathscr{T}}(n) \leqslant n-2$ for every $n \geqslant 5$. According to [4] (we have no other source of this information) Giles in his preprint to [7] he even showed that for every natural q there is $\mathrm{rec}_{\mathscr{T}}(n) \leqslant n - q$ if n is sufficiently large.

On the other hand, in [31] we described a simple family of counterexamples which implies that $[n/2]+1 \leqslant \mathrm{rec}_{\mathscr{T}}(n)$ (for every $n \geqslant 4$). We also exhibited the following table which gives the first ten values of $\mathrm{rec}_{\mathscr{T}}(n)$

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{rec}_{\mathscr{T}}(n)$	1	1	1	3	3	4	4	5	5	6

We had conjectured that $\mathrm{rec}_{\mathscr{T}}(n) = [n/2] + 1$ (for every $n \geqslant 4$).

4. Related results

There are more generalizations possible of our approach to the reconstruction problem. One is the question of determining some characteristics of a graph from the collection of all its cardinality restricted subgraphs. For example, Taylor in [35] investigates the possibility of reconstructing degree sequence of a graph from k-vertex deleted subgraphs.

Another direction is based on the idea that also for other structures than graphs the concept of induced substructure makes sense. Stockmayer [34], Ramachandran [33], Ille [12], and Lopez and Rauzy [20] investigated binary relations (especially tournaments, for example). Kocay and Lui gave some basic results on non-reconstructibility of hypergraphs [15,16] while Kratsh and Rampon exhibited in [18] a counterexample about poset reconstruction. Some bounds of reconstructibility of sequences from subsequences were given by Manvel et al. in [24] and then improved by Krasikov and Roditty (cf. [17]). Recently, Miller [26] investigates matroid reconstruction.

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