CSE291D: Machine Learning for Robotics

Rabbittac

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Abstract

Course webpage: https://haosulab.github.io/ml-for-robotics/SP21/index.html

Without specific clarification, the notations are in the form below

- x, lowercase ordinary fonts: points
- $\mathbf{x}$, lowercase bf fonts: vectors
- A, upper case ordinary fonts: matrix
- $\mathscr{F}$: frame (in motion)
- $\mathbf{i}, \mathbf{j}, \mathbf{k}$: axis

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CSE291D **Spring 2021**

Lecture 1: Introduction, Differential Geometry

Lecturer: Hao Su *Scribes: Rabbittac*

1.1 Introduction

Topics in this course:

– Modeling robotics by rigid-body geometry
– Forward and inverse kinematics of robots
– Generalized force and inertia
– Friction, contact model, grasp
– Classical planning and control
– Reinforcement learning
– Deep RL framework
– Hierarchical RL
– Generalizability of RL

1.2 Differential Geometry

Not scribe.

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Lecture 2: Rigid Transformations

Lecturer: Hao Su *Scribes: Rabbittac*

An observer records the position of any point in the space using a **frame** $\mathscr{F}_s$. There is a **rigid object**, to which we bind a frame $\mathscr{F}_b$ (body frame) tightly, so that $\mathscr{F}_b$ moves along with the object.

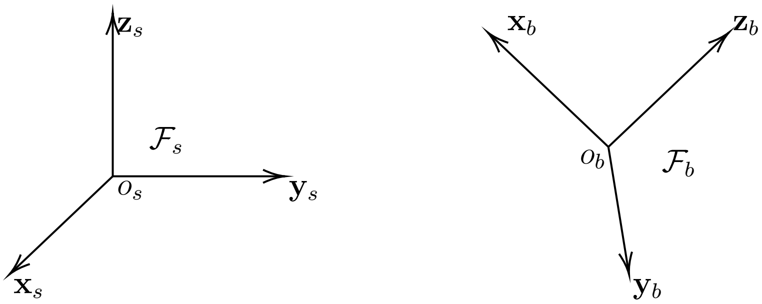


Figure 2.0.1

In order to learn the pose of the rigid object, we want to know the transformation from $\mathscr{F}_s$ to $\mathscr{F}_b$: we first **translate** $\mathscr{F}_s$ by $\mathbf{t}_{s\to b}$ to align o_s and o_b [2.1]

$$o_b^s = o_s^s + \mathbf{t}_{s\to b}^s$$

then **rotate** by $R_{s\to b}$ to align $\{\mathbf{x_i}, \mathbf{y_i}, \mathbf{z_i}\}$

$$[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = R_{s\to b}^s[\mathbf{x}_s^s, \mathbf{y}_s^s, \mathbf{z}_s^s]$$

Since the observer records everything using $\mathscr{F}_s$, then $o_s^s = 0$ and $[\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] = I_{3\times 3}$. Therefore, in this case

$$\mathbf{t}_{s\to b}^s = o_b^s \qquad R_{s\to b}^s = [\mathbf{x}_b^s, \mathbf{y}_b^s, \mathbf{z}_b^s] \in \mathbb{R}^{3\times 3}$$

2.1 Coordinate Transformation

Now assume a second observer that records coordinates by $\mathscr{F}_b$. Assume a point p on the body, since $\mathscr{F}_b$ moves along the body, its coordinate recorded in $\mathscr{F}_b$, denoted as p^b, should never change. If we view $(R_{s\to b}, \mathbf{t}_{s\to b})$ as the *coordinate transformation* from $\mathscr{F}_s$ to $\mathscr{F}_b$, the movement of p along $\mathscr{F}_b$ at state t is

$$p_t^s = R_{s\to b}^s p_t^b + \mathbf{t}_{s\to b}^s$$

The **homogeneous coordinates** for $x \in \mathbb{R}^3$ is [2.2].

$$\tilde{x} := \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$$

[2.1] *Notation*: When writing equations, we add a superscript to denote the **recording frame**.
[2.2] *Notation*: For simplicity, $\tilde{}$ will be ignored in the future.

The **homogeneous transformation matrix** is in the form

$$T^s_{s\to b} = \begin{bmatrix} R^s_{s\to b} & \mathbf{t}^s_{s\to b} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

Homogeneous coordinates allow us to write transformation in linear form

$$x^1 = T^1_{1\to 2}x^2$$

By $x^1 = T^1_{1\to 2}x^2$, we have $x^2 = T^2_{2\to 1}x^1$ and $x^3 = T^3_{3\to 2}x^2$. Then $T^3_{3\to 1}x^1 = x^3 = T^3_{3\to 2}T^2_{2\to 1}x^1$ gives the **composition rule**

$$T^3_{3\to 1} = T^3_{3\to 2}T^2_{2\to 1}$$

Since $x^1 = T^1_{1\to 2}x^2$, $x^2 = (T^1_{1\to 2})^{-1}x^1$. This derives the **change of observer's frame**

$$T^2_{2\to 1} = \left(T^1_{1\to 2}\right)^{-1}$$

e.g.1. Consider a simple robot arm with 2 degree of freedom shown in figure 2.1.2, which revolute θ_1 and slide θ_2. In this example, $T^0_{0\to 1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -l_2\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & l_2\cos\theta_1 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

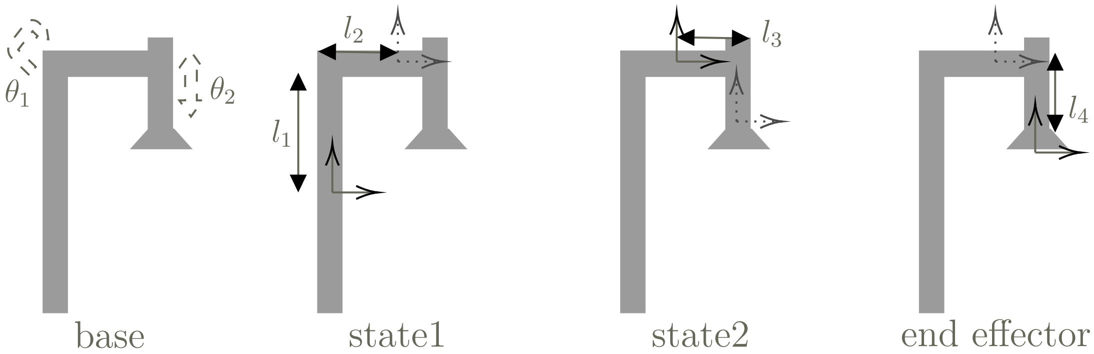


Figure 2.1.2: a robot arm with DoF=2

$T^1_{1\to 2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $T^2_{2\to 3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -l_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. By the composition rule,

$$T^0_{0\to 3} = T^0_{0\to 1}T^1_{1\to 2}T^2_{2\to 3} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & -\sin\theta_1\,(l_2+l_3) \\ \sin\theta_1 & \cos\theta_1 & 0 & \cos\theta_1\,(l_2+l_3) \\ 0 & 0 & 1 & l_1 - l_4 + \theta_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

2.2 Linear Transformation

In another perspective, we can view $(R_{s\to b}, \mathbf{t}_{s\to b})$ as a *linear transformation* that transforms any point in the whole space by

$$x'^s = R^s_{s\to b}x^s + \mathbf{t}^s_{s\to b}$$

Suppose $\mathscr{F}_p^s = \{p^s, [\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$ is a frame at an arbitrary point p^s. Then the new origin becomes

$$p'^s = R_{s\to b}^s p^s + \mathbf{t}_{s\to b}^s$$

Now to transform the base vectors of the frame, assume three curves $\gamma_x, \gamma_y, \gamma_z$ pass p^s at $t = 0$ with tangents $\mathbf{x}_p, \mathbf{y}_p, \mathbf{z}_p$. The new tangents after the transformations are $\frac{\mathrm{d}}{\mathrm{d}t} R_{s\to b}^s \gamma_x^s(0)$, $\frac{\mathrm{d}}{\mathrm{d}t} R_{s\to b}^s \gamma_y^s(0)$, $\frac{\mathrm{d}}{\mathrm{d}t} R_{s\to b}^s \gamma_z^s(0)$. So the new frame is

$$\mathscr{F}_{p'}^s = \{p'^s, R_{s\to b}^s[\mathbf{x}_p^s, \mathbf{y}_p^s, \mathbf{z}_p^s]\}$$

To record an arbitrary transformation from $\mathscr{F}_1$ to $\mathscr{F}_2$ in $\mathscr{F}_s$, we denote [2.3]

$$T_{1\to 2}^s := T_{s\to 2}^s T_{1\to s}^1$$

With this definition, we can derive the **composition rule** as

$$T_{1\to 2}^s = T_{3\to 2}^s T_{1\to 3}^s$$

Now we ask the question: if we change the observer, how can we describe the same motion? So given $T_{1\to 2}^s$, we want to compute $T_{1\to 2}^b$. By composition rule as linear transformation, $T_{1\to 2}^s T_{s\to 1}^s = T_{s\to 2}^s$. By composition rule as coordinate transformation, $T_{1\to 2}^s T_{s\to b}^s T_{b\to 1}^b = T_{s\to b}^s T_{b\to 2}^b$. Apply composition rule as linear transformation again, we obtain $T_{1\to 2}^s T_{s\to b}^s T_{b\to 1}^b = T_{s\to b}^s T_{1\to 2}^b T_{b\to 1}^b$. So we get the **similarity transformation**

$$T_{1\to 2}^b = (T_{s\to b}^s)^{-1} T_{1\to 2}^s T_{s\to b}^s$$

In a special case, if $\mathscr{F}_1 = \mathscr{F}_s$ and $\mathscr{F}_2 = \mathscr{F}_b$, then $T_{s\to b}^s = T_{s\to b}^b$ [2.4].

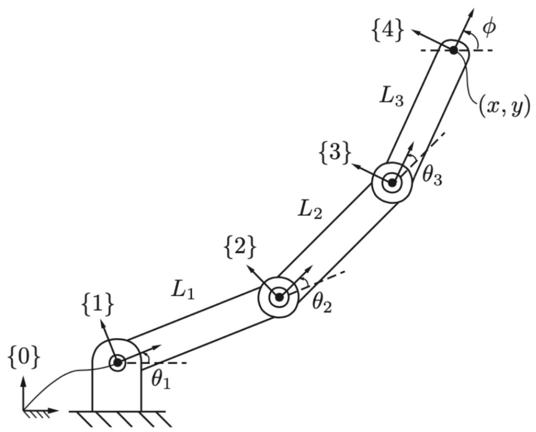
e.g.2. Consider a camera with frame $\mathscr{F}_c$ observing a red car with the current frame $\mathscr{F}_1$. Then the red car move to a new frame $\mathscr{F}_2$. By the composition rule of linear transformation $T_{c\to 2}^c = T_{c\to 1}^c T_{1\to 2}^1$. Then we compute $T_{1\to 2}^1 = (T_{c\to 1}^c)^{-1} T_{c\to 2}^c = \left(\begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & -l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} \cos\pi & -\sin\pi & 0 & l \\ \sin\pi & \cos\pi & 0 & l \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} & 0 & 2l \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.
We can also achieve this by computing $T_{1\to 2}^1$ with the composition rule of coordinate transformation and then applying the similarity transformation.

[2.3] This is actually hard to describe in perspectives of the coordinate transformation.
[2.4] *Notation*: So we often use the abbreviated notations $T^b \equiv T_b^s \equiv T_{s\to b}^s$.

CSE291D	**Spring 2021**
Lecture 3: $\mathbb{SO}(3)$	
Lecturer: Hao Su	*Scribes: Rabbittac*

3.1 Multi-Link Rigid-Body Geometry

Links are the rigid-body connected in sequence. **Joints** are the connector between links, which determine the DoF of motion between adjacent links. The 0-th link of the robot is called **base link**/root link, which is often static and attached by the spatial frame $\mathscr{F}_s$. The last link is called the **end-effector link**, which is attached by the frame $\mathscr{F}_e$. Two common joint types are **revolute**/hinge/rotational and **prismatic**/translational joint.



Kinematics describes the motion (position and velocities) of bodies, without considering the force that leads the motion. Two representations of the pose of the end-effector are joint space and Cartesian space. The **joint space** is the space that each coordinate is a vector of joint poses. The **Cartesian space** is the space of the rigid transformations of the end-effector by $(R_{s\to e}, \mathbf{t}_{s\to e})$. We can map the joint space coordinates $\theta \in \mathbb{R}^n$ to a transformation matrix T by composing transformations along the kinematic chain, i.e. $T_{s\to e} = f(\theta)$ [3.1]. The problem of finding this mapping is known as **forward kinematics**. The inverse problem is known as **inverse kinematics**: given the forward kinematics $T_{s\to e}(\theta)$ and the target pose $T_{\text{target}} = \mathbb{SE}(3)$, we want to know θ so that $T_{s\to e}(\theta) = T_{\text{target}}$.

[3.1] *e.g.* See example 1 in lecture 2.

3.2 Special Orthogonal Group

We define the **Special Orthogonal Group** $\mathbb{SO}(n)$ as [3.2]

$$\mathbb{SO}(n) = \{R \in \mathbb{R}^{n\times n} : \det(R) = 1, RR^T = I\}$$

e.g.1. $\mathbb{SO}(2)$ works with 2D rotations and has 1 DoF; $\mathbb{SO}(3)$ works with 3D rotations and has 3 DoF.

3.2.1 Rotation Matrix, Angle-Axis Parameterization

The **Euler's Theorem** states that any rotation is equivalent to a rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3 : \|\hat{\omega}\| = 1$ through a positive angle θ. The rotation matrix R is defined

$$R \in \mathbb{SO}(3) := \mathrm{Rot}(\hat{\omega}, \theta)$$

Given $\hat{\omega}$ and θ, we want to find $R \in \mathbb{SO}(3)$. We can derive $\mathrm{Rot}(\hat{\omega}, \theta)x = x + (\sin\theta)\hat{\omega} \times x + (1 - \cos\theta)\hat{\omega} \times (\hat{\omega} \times x) = (I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1-\cos\theta))\, x = (I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 + \frac{\theta^3}{3!}[\hat{\omega}]^3 + \cdots)x = e^{[\hat{\omega}]\theta}x$ [3.3] [3.4]

$$\mathrm{Rot}(\hat{\omega}, \theta) = e^{[\hat{\omega}]\theta}$$

where $[\cdot]$ is the **skew-symmetric matrix operator** [3.5], and $\vec{\theta} = \hat{\omega}\theta$ is called rotation vector or exponential coordinate. From the derivation, we can also get **Rodrigues formula**

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin\theta + [\hat{\omega}]^2(1-\cos\theta)$$

Now consider the inverse problem: given $R \in \mathbb{SO}(3)$, we want to find $\hat{\omega}$ and θ. First note that this parameterization will not be unique.

e.g.2. $(\hat{\omega}, \theta)$ and $(-\hat{\omega}, -\theta)$ give the same rotation; If $R = I, \theta = 0$, then $\hat{\omega}$ can be arbitrary; $(\hat{\omega}, \pi)$ and $(-\hat{\omega}, \pi)$ give the same rotation since $\mathrm{tr}(R) = -1$.

But if we restrict $\theta \in (0, \pi)$, a unique parameterization exists

$$\theta = \arccos\frac{\mathrm{tr}\left(R_2R_1^T\right) - 1}{2}, \quad [\hat{\omega}] = \frac{1}{2\sin\theta}\left(R - R^T\right)$$

This also enables us to define **distance between rotations**: the interpretation is to measure the minimal effort to rotate the body at R_1 pose to R_2 pose

$$d(R_1, R_2) = \theta(R_2R_1^T) = \arccos\frac{\mathrm{tr}\left(R_2R_1^T\right) - 1}{2}$$

[3.2] **Group** means roughly closed under matrix multiplication; **orthogonal** means $RR^T = I$; **special** means $\det(R) = 1$.

[3.3] For derivation of the first equality, see wikipedia.

[3.4] Note that for matrix exponential, $e^{A+B} = e^Ae^B$ if and only if $AB = BA$.

[3.5] A is **skew-symmetric** if $A = -A^T$. The **skew-symmetric matrix operator** allows us to write cross product in linear form: $a \times b = [a]b$ where $a = (a_1, a_2, a_3)^T, b = (b_1)$ and $[a] := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$.

3.2.2 Quaternion

Another representation of the rotation is **quaternion**, which a more generalized complex number

$$q := w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (w, \vec{v})$$

where w is the real part, $\vec{v} = (x, y, z)$ is the imaginary part that satisfies $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ and anti-commutative law $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}, \mathbf{jk} = \mathbf{i} = -\mathbf{kj}, \mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The product rule for two quaternions q_1 and q_2 is

$$q_1 q_2 := (w_1 w_2 - \vec{v}_1^T \vec{v}_2, w_1 \vec{v}_2 + w_2 \vec{v}_1 + \vec{v}_1 \times \vec{v}_2)$$

Note that quaternion product is non-commutable. The **conjugate** of q is

$$q^* := (w, -\vec{v})$$

the **norm** is defined

$$\|q\|^2 := w^2 + \vec{v}^T \vec{v} = qq^* = q^* q$$

and the **inverse** is

$$q^{-1} := \frac{q^*}{\|q\|^2}$$

A unit quaternion (3 DoF) can represent a rotation. To map the exponential coordinate to quaternion

$$q = \left[\cos\frac{\theta}{2}, \hat{\omega}\sin\frac{\theta}{2}\right]$$

To map the quaternion to exponential coordinate

$$\theta = 2\arccos w, \quad \hat{\omega} = \begin{cases} \frac{1}{\sin\frac{\theta}{2}}\vec{v} & \theta \neq 0 \\ 0 & \theta = 0 \end{cases}$$

e.g.3. Rotate a vector $\vec{x} \in \mathbb{R}^3$ by a quaternion q: First augment $\vec{x}$ to $x = (0, \vec{x})$. Then $x' = qxq^{-1}$. If we want to compose two rotations by unit quaternions q_1 and q_2. $x' = q_2(q_1 x q_1^*)q_2^* = (q_2 q_1)x(q_1^* q_2^*)$. The computational complexity of using quaternions is often cheaper.

e.g.4. Notations may be different in various systems:

- (w, x, y, z): SAPIEN, Eigen, blender, transforms3d, MuJoCo, V-Rep
- (x, y, z, w): ROS, PhysX, PyBullet

3.2.3 Local Structure of $\mathbb{SO}(3)$

When $\theta \to 0$, $e^{[\vec{\theta}]} = I + [\vec{\theta}] + o([\vec{\theta}])$ where $[\vec{\theta}]$ is a linear subspace of $\mathbb{R}^{3\times 3}$. This means that any local movement in $\mathbb{SO}(3)$ around I can be approximated by $[\vec{\theta}]$. In other words, the set of $[\vec{\theta}]$ forms the tangent space of $\mathbb{SO}(3)$ at $R = I$. More generally, the tangent space at $R \in \mathbb{SO}(3)$ is $\{SR : S \in \mathbb{R}^{3\times 3}, S^T = -S\}$ [3.6]. We call this **Lie Algebra of** $\mathbb{SO}(3)$ and denote [3.7]

$$\mathfrak{so}(3) := \{S \in \mathbb{R}^{3\times 3} : S^T = -S\}$$

Now we want to parameterize the rotation of a body frame by time. An observer associated to $\mathscr{F}_o$ records the motion as $R^o_{s'\to b(t)}$, where the body frame is at $\mathscr{F}_{b(t)}$. We can derive $R^o_{s'\to b(t+\Delta t)} - R^o_{s'\to b(t)} = R^o_{b(t)\to b(t+\Delta t)} R^o_{s'\to b(t)} - R^o_{s'\to b(t)} = e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} R^o_{s'\to b(t)} - R^o_{s'\to b(t)} \approx [\chi^o_{b(t)\to b(t+\Delta t)}] R^o_{s'\to b(t)}$. Divide this by Δt and take the limit, we obtain

$$\dot{R}^o_{s'\to b(t)} = \left[\omega^o_{b(t)}\right] T^o_{s'\to b(t)}$$

where $\omega^o_{b(t)} := \lim_{\Delta t\to 0} \frac{\vec{\theta}^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$ is the **instant angular velocity**.

3.3 Euler Angle

This part is not included in the lecture.

Representations	Inverse?	Composing	Movements [3.8]	Usage
Rotation Matrix	✓	✓	N/A	Define concepts
Euler Angle	Complicated	Complicated	No	Visualization
Angle-axis	✓	Complicated	Part	Visualization Derivatives
Quaternion	✓	✓	✓	Write code

Table 3.3.1: Summary

[3.6] Since $e^{[\vec{\theta}]} - I = [\vec{\theta}] + o([\vec{\theta}])$, $e^{[\vec{\theta}]}R - R = [\vec{\theta}]R + o([\vec{\theta}])$

[3.7] By introducing Lie bracket $[A, B] = AB - BA$, the set of skew-symmetric matrices form an "algebra" because the set is closed and left and right distributive law are satisfied under Lie bracket.

[3.8] In complete sentence, local movement in $\mathbb{SO}(3)$ can be achieved by local movement in the domain.

CSE291D	**Spring 2021**
Lecture 4: Screw and Twist	
Lecturer: Hao Su	*Scribes: Rabbittac*

4.1 Rigid Transformation and $\mathbb{SE}(3)$

We define **Special Euclidean Group** $\mathbb{SE}(3)$ as [4.1]

$$\mathbb{SE}(3) := \left\{ T = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}, R \in \mathbb{SO}(3), t \in \mathbb{R}^3 \right\}$$

e.g.1. $\mathbb{SE}(3)$ has 6 DoF.

4.1.1 Screw Parameterization

Recall Euler's Theorem for $\mathbb{SO}(3)$, similarly in $\mathbb{SE}(3)$, we have **Screw Motion Theorem** which states that any rigid body motion is equivalent to rotating about one axis while also translating along the axis, where the axis may not pass the origin.

	$\mathbb{SO}(3)$	$\mathbb{SE}(3)$
Model Interpretation	$\hat{\omega}$: motion direction	$\hat{\xi}$: 6D motion direction
Exponential Coordinate	rot vector: $\vec{\theta} = \hat{\omega}\theta \in \mathbb{R}^3$	screw: $\chi = \hat{\xi}\theta \in \mathbb{R}^6$
Exponential Map	$R = \exp\Big([\hat{\omega}]\theta\Big) \in \mathbb{SO}(3)$	$T = \exp\Big([\hat{\xi}]\theta\Big) \in \mathbb{SE}(3)$
Tangent Space	At $R = I : [\hat{\omega}]\theta \in \mathfrak{so}(3)$	At $T = I : [\hat{\xi}]\theta \in \mathfrak{se}(3)$

Table 4.1.1: Summary of $\mathbb{SO}(3)$ and $\mathbb{SE}(3)$

In $\mathbb{SE}(3)$, we have $\mathrm{Trans}(\hat{\omega}, \theta, q, d_\omega)x = (I + A + \frac{A^2}{2!} + \dots)x = e^A x$ where $x \in \mathbb{R}^4$ and $A = \begin{bmatrix} [\hat{\omega}\theta] & -[\hat{\omega}\theta]q + d_\omega \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times4}$. To align the expression in the form of $\mathrm{Rot}(\hat{\omega}, \theta) \equiv e^{[\hat{\omega}]\theta}$, we introduce $\hat{\xi} = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \in \mathbb{R}^6$ so $[\hat{\xi}]\theta = \begin{bmatrix} [\hat{\omega}] & d \\ 0 & 0 \end{bmatrix}\theta$, where $d = \frac{-[\hat{\omega}\theta]q + d_\omega}{\theta}$. So we can rewrite

$$T = \exp\Big([\hat{\xi}]\theta\Big)$$

e.g.2. In a special case, when the motion is translation-only, we define $\hat{\omega} = 0, \theta = \|d_\omega\|$, and $d = \frac{d_\omega}{\|d_\omega\|}$.

[4.1] **Group** means closed under matrix multiplication and other conditions of forming a group; **Euclidean** means R and t; **special** means $\det(R) = 1$.

We call $\hat{\xi}$ **unit twist** which describes the motion direction, and call $\chi = \hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix} \theta = \begin{bmatrix} -[\hat{\omega}]q\theta + d_\omega \\ \hat{\omega}\theta \end{bmatrix}$ **screw** or exponential coordinate. By $T = \exp([\chi])$, we have

$$\chi = \log(T)$$

Using Taylor expansion, we can express $e^{[\hat{\xi}]}$ as $e^{[\hat{\xi}]\theta} = I + [\hat{\xi}] + \frac{1-\cos\theta}{\theta^2}[\hat{\xi}]^2 + \frac{\theta-\sin\theta}{\theta^3}[\hat{\xi}]^3$ and further computation gives

$$e^{[\hat{\xi}\theta]} = \begin{bmatrix} e^{[\hat{\omega}\theta]} & \left(I - e^{[\hat{\omega}\theta]}\right)(\hat{\omega} \times d) + \hat{\omega}\hat{\omega}^T d\theta \\ 0 & 1 \end{bmatrix}$$

which enables us to compute T from $\hat{\xi}\theta$. Now we consider computing $T \in \mathbb{SE}(3)$ from $\hat{\xi}\theta$. We first need to determine $\hat{\omega}\theta \in \mathfrak{so}(3)$ from $\mathbb{SO}(3)$ rotation. The translation component of T is t. Then d can be computed by

$$d = \left(\frac{1}{\theta}I - \frac{1}{2}[\hat{\omega}] + (\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2})[\hat{\omega}]^2\right) t$$

e.g.3. If $t \perp \hat{\omega}$, then $\frac{1}{\theta}\left(I + [\hat{\omega}]^2\right) t = 0$. Thus $d = (-\frac{1}{2}[\hat{\omega}] - \frac{1}{2}\cot\frac{\theta}{2}[\hat{\omega}]^2)t$.

In order to extract motion parameters from $\hat{\xi}\theta$ [4.2], $\hat{\omega}$ and θ can be read directly;

$$q = [\hat{\omega}](\hat{\omega}\hat{\omega}^T - I)d$$

$$d_\omega = \hat{\omega}\hat{\omega}^T d\theta$$

where d is from $\hat{\xi}$.

e.g.4. What is the screw χ given $T(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha t) & -\sin(\alpha t) & \sin(\alpha t) \\ 0 & \sin(\alpha t) & \cos(\alpha t) & 1-\cos(\alpha t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$?
Recall from lecture 3, given $R \in \mathbb{SO}(3)$, we can compute θ and $[\hat{\omega}]$: $\theta = \alpha t, [\hat{\omega}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, thus $\hat{\omega} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. We can also compute $d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Thus, $\hat{\xi}\theta = \begin{bmatrix} d \\ \hat{\omega} \end{bmatrix}, \theta = [0, 1, 0, 1, 0, 0]^T \alpha t$. So $\chi = \hat{\xi}\theta = [0, \alpha t, 0, \alpha t, 0, 0]^T$. If $T^s_{s\to b}(\theta) \equiv T(\theta)$ describes the relative transformation of a body frame relative to spatial frame, then $\chi^s_{s\to b} = \hat{\xi}^s_{s\to b}\theta^s_{s\to b} = [0, \alpha t, 0, \alpha t, 0, 0]^T$ represents the linear transformation of rotating about a fixed axis.

e.g.5. What are motion parameters $\hat{\omega}, q, \theta, d_\omega$ for $T(\theta) = e^{[\hat{\xi}]\theta}$, where $\hat{\xi}\theta = [0, \alpha t, 0, \alpha t, 0, 0]^T$?
Since $\hat{\omega} = [1, 0, 0]^T, d = [0, 1, 0]^T$, $q = [\hat{\omega}]^\dagger \left(\hat{\omega}\hat{\omega}^T - I\right) d = q = [0, 0, 1]^T$. With $\theta = \alpha t$, $d_\omega = \hat{\omega}\hat{\omega}^T d\theta = [0, 0, 0]^T$.

e.g.6. A robot has two links (green stick and blue stick) connected by a revolute joint (green sphere). The end-effector (blue sphere) is connected to the end of the second link. The spatial frame is at the red sphere (static). What is the screw $\chi^s_{s\to e}(t)$ of the end-effector in the spatial frame?
Left as a reading (page 35 – 45).

[4.2] One reason we need to do this is that $T \in \mathbb{R}^{4\times 4}$, which is computationally expensive.

4.1.2 Local Structure of $\mathbb{SE}(3)$

When $\theta \to 0$, $e^{[\hat{\xi}]\theta} \to I + \theta[\hat{\xi}] + o(\theta[\hat{\xi}])$. So $\forall T \in \mathbb{SE}(3) : e^{[\hat{\xi}]\theta}T \approx T + \theta[\hat{\xi}]T$ when $\theta \approx 0$. This implies that $\mathbb{SE}(3)$ has a linear local structure (differentiable manifold). When $\theta \approx 0$, $e^{[\chi]} - I \to [\chi] + o([\chi])$ indicates $[\chi] \to 0 \implies e^{[\chi]} \to I$. This implies that any local movement in $\mathbb{SE}(3)$ around I can be approximated by some small $[\chi]$. Moreover, the set of $[\chi]$ forms the tangent space of $\mathbb{SE}(3)$ at $T = I$. We call this set **Lie Algebra of** $\mathbb{SE}(3)$

$$\mathfrak{se}(3) := \left\{ \begin{bmatrix} S & t \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4} : S^T = -S \right\}$$

4.2 Twist

To parameterize the motion of a body frame by time: an observer associated to $\mathscr{F}_o$ records the motion as $T^o_{s'\to b(t)}$, where the body frame is at $\mathscr{F}_{b(t)}$.
Then $T^o_{s'\to b(t+\Delta t)} - T^o_{s'\to b(t)} = T^o_{b(t)\to b(t+\Delta t)} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} = e^{[\chi^o_{b(t)\to b(t+\Delta t)}]} T^o_{s'\to b(t)} - T^o_{s'\to b(t)} \approx [\chi^o_{b(t)\to b(t+\Delta t)}] T^o_{s'\to b(t)}$. Divide by Δt and take the limit, we have

$$\dot{T}^o_{s'\to b(t)} = \left[\xi^o_{b(t)}\right] T^o_{s'\to b(t)}$$

where we define $\xi^o_{b(t)}$ as **twist**, which is 6D instant velocity [4.3]

$$\xi^o_{b(t)} := \lim_{\Delta t\to 0} \frac{\chi^o_{b(t)\to b(t+\Delta t)}}{\Delta t}$$

The **linear velocity** of p^o caused by $T^o_{s'\to b(t)}$ at time t can derived by $\mathbf{v}^o_p(t) = \lim_{\Delta t\to 0} \frac{T^o_{b(t)\to b(t+\Delta t)} p^o - p^o}{\Delta t} = \lim_{\Delta t\to 0} \frac{\exp\left([\chi^o_{b(t)\to b(t+\Delta t)}]\right) - I}{\Delta t} p^o = [\xi^o_{b(t)}] p^o$ which gives

$$\mathbf{v}^o_p(t) = \left[\xi^o_{b(t)}\right] p^o$$

e.g.7. If a motion is a pure rotation, then $\mathbf{v}^o_p(t) = \omega^o_{b(t)} \times p^o$.

[4.3] In general, $\xi^o_{b(t)} \neq \dot{\chi}^o_{s'\to b(t)}$.