Logical Equivalence

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TOC

Slides is posted on Canvas:Files. If you find any typos or have any concerns, please contact me ASAP!

My proofs may not be the most beautiful ones. If you find a better one, please share it via Canvas:Discussion.

Abstract of 1.1 $\sim$ 1.3

- **Terms**: propositions, tautology, contradiction, contingency, logically equivalent
- **Operations**: negation, conjunction, disjunction, exclusive, conditional statement, biconditional statement
- **Apply**: truth table, logical precedence, Boolean variable, De Morgan laws, conditional-disjunction equivalence, distributive law of disjunction over conjunction
- More contents: refer to Prof's slides.

Table 6

TABLE 6 **Logical Equivalences.**

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Proof of Table 6

Sketch of Proof.

By truth table. Can be found on Textbook p.26-28.

Table 7

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \vee q \equiv \neg p \to q$$

$$p \wedge q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \wedge \neg q$$

$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

Proof of Table 7

Theorem (Implication Equivalence)

$$p \to q \equiv \neg p \vee q$$

proof. via truth table.

Corollary

$$p \to q \equiv \neg q \to \neg p$$

proof.

$$\begin{aligned} p \to q &\equiv \neg p \vee q && \text{apply IE} \\ &\equiv q \vee \neg p && \text{commutativity} \\ &\equiv \neg(\neg q) \vee \neg p && \text{double negation} \\ &\equiv \neg q \to \neg p && \text{apply IE (simplify triple-neg to single-neg)} \end{aligned}$$

Corollary

$$p \vee q \equiv \neg p \rightarrow q$$

proof. $p \vee q \equiv \neg(\neg p) \vee q \equiv \neg p \rightarrow q$

Corollary

$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$

Look at Table 6. Which law is most relevant to such pattern?

proof.

$$
\begin{aligned}
p \wedge q &\equiv \neg(\neg(p \wedge q)) && \text{double negation} \\
&\equiv \neg(\neg p \vee \neg q) && \text{De Morgan in inner parenthesis} \\
&\equiv \neg(p \rightarrow \neg q) && \text{apply IE in inner parenthesis}
\end{aligned}
$$

For the rest, I will simply write the sketch of the proof for convenience.

Corollary

$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$

proof. $\neg(p \rightarrow q) \equiv \neg(\neg p \vee q) \equiv p \wedge \neg q$ where we applied IE and De Morgan's Laws.

Corollary

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

proof. $(\neg p \lor q) \land (\neg p \lor r) \equiv \neg p \lor (q \land r) \equiv p \to (q \land r)$ where we applied distributive laws and IE.

Corollary

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

proof. $(p \to r) \lor (q \to r) \equiv (\neg p \lor r) \lor (\neg q \lor r) \equiv$
$\neg p \lor r \lor \neg q \lor r \equiv (\neg p \lor \neg q) \lor (r \lor r) \equiv \neg(p \land q) \lor r \equiv (p \lor q) \to r$

Corollary

$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$

proof.
$(p \to q) \wedge (p \to r) \equiv (\neg p \vee q) \wedge (\neg p \vee r) \equiv \neg p \vee (q \wedge r) \equiv p \to (q \vee r)$

Corollary

$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

proof. $(\neg p \vee r) \vee (\neg q \vee r) \equiv \neg p \vee r \vee \neg q \vee r \equiv$
$(\neg p \vee \neg q) \vee (r \vee r) \equiv \neg (p \wedge q) \vee r \equiv (p \wedge q) \to r$

Table 8

Theorem

1. $p \leftrightarrow q \equiv (p \to q) \land (q \to p)$
2. $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
3. $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
4. $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

You can prove one via truth table, then construct others by applying laws. But truth tables can be easier for these.

Proof of Table 8

proof. (sketch)

$1 \to 3$ $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\neg p \vee q) \wedge (\neg q \vee p) \equiv$
$((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p) \equiv$
$((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee ((\neg p \wedge p) \vee (q \wedge p)) \equiv$
$((\neg p \wedge \neg q) \vee F) \vee (F \vee (q \wedge p)) \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$.

$1 \to 2$ $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\neg p \vee q) \wedge (\neg q \vee p) \equiv$
$(q \vee \neg p) \wedge (p \vee \neg q) \equiv (\neg q \to \neg p) \wedge (\neg p \to \neg q) \equiv \neg p \leftrightarrow \neg q$.

$1 \to 4$ $\neg(p \leftrightarrow q) \equiv \neg((p \wedge q) \vee (\neg p \wedge \neg q)) \equiv \neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q) \equiv$
$(\neg p \vee \neg q) \wedge (p \vee q) \equiv (p \to \neg q) \wedge (\neg q \to p) \equiv p \leftrightarrow \neg q$.

Tips

- If you are uncertain, use **truth table**.
- When you construct logical equivalence:
  1. Make sure the rules applied are allowed i.e. in our scope.
  2. Comment the rule explicitly at each step.
  3. Break down complex expressions into simpler components.
  4. Look for patterns and common equivalences.
  5. Do NOT skip any procedure.

Patterns

1. The atomic rules are Table 6 + IE + 1 of Table 8
2. $\neg(p \wedge / \vee q)$: De Morgan.
3. $\wedge$, $\vee$ alternatively appears inside/outside parenthesis: distributive law.
4. If the term includes biconditional, first convert it to simpler form (conj, disj, cond). Even simplifying the RHS helps you find the correct direction.
5. There can be more ... Come to lab to learn.

Practice