MATH140B: Foundations of Real Analysis

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Abstract

Warning: This is only a piece of lecture notes written by a careless scribe. So just be careful with and tolerant of any possible typos or misunderstandings when you read [0.1]. The scribe does not intend to make anyone to be driven by his stupidity! Also, the professor's explanation is extremely helpful as he discusses a lot about the interpretable ideas behind the dull scripts. So watch the lecture before reading this. If you have any suggestions (e.g. typos, typography, logistics), please do not hesitate contacting the scribe!

Here are some resources explaining Rudin

- Supplements to the Exercises, Comments by Prof.Bergman from UCB.
- The Real Analysis Lifesaver: kind of companion for Rudin to explain ideas of a few smart proof.

[0.1] Especially '$\cap$' and '$\cup$' are often mistaken because of typos.

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Lecture 1: Differentiation

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1.1 Differentiable

Definition 1.1 (*Differentiable*)

Let $f : [a,b] \to \mathbb{R}, x \in [a,b]$. Define

$$\varphi(t) = \frac{f(t) - f(x)}{t - x}$$

*for $a \le t \le b, t \ne x$. We say f is **differentiable** at x if and only if $f'(x) = \lim_{t \to x} \varphi(t)$ exists, and we denote the **derivative** of f at x by $f'(x)$.*

If f is differentiable at x, then $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. Note that if φ is not defined at x and f is differentiable at x, we can define $\Phi(t) = \begin{cases} \varphi(t) & t \ne x \\ f'(x) & t = x \end{cases}$, then Φ is continuous at x.

e.g.1.

- Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$. Compute $f'(0)$ if exists.

  Proof: By the definition, we get $\lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} = \frac{t^2}{t} = 0$. ■

- $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is differentiable at 0.

  Proof: We need to compute (or show DNE) $\lim_{t\to 0} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0} \frac{f(t)}{t} = \lim_{t\to 0} \begin{cases} \frac{t^2}{t} & t \in \mathbb{Q} \\ 0 & t \notin \mathbb{Q} \end{cases} = 0$. ■

- $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at 0 but not differentiable at 0.

  Proof: Proof of continuity is left as exercise. To show it is non-differentiable, we need to show the limit $\lim_{t\to 0} \frac{f(t)}{t}$ DNE. We want to find two sequences $t_n \to 0, s_n \to 0$ that $t_n \ne 0, s_n \ne 0$ such that $\frac{f(t_n)}{t_n} = 1$ and $\frac{f(s_n)}{s_n} \to 0$. So let $t_n = \frac{1}{n}$, then $\forall n : t_n \ne 0$ and $t_n \to 0$. Then $\lim_{t\to 0} \frac{f(t_n)}{t_n} = \frac{t_n}{t_n} = 1$. Let $s_n = \frac{\sqrt{2}}{n} \notin \mathbb{Q}$, then $\forall n : s_n \ne 0$ and $s_n \to 0$. Then $\lim_{t\to 0} \frac{f(s_n)}{s_n} = \frac{0}{s_n} = 0$. So $\lim_{t\to 0} \frac{f(t)}{t}$ DNE. ■

1.2 Properties of Differentiable Functions

Lemma 1.1

Let f be defined on $[a,b]$ and $x \in [a,b]$. If f is differentiable at x then f is continuous at x [1.1].

Proof: We need to show that $\lim_{t\to x} f(t) = f(x)$. We have $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x)$. Note that $f(t)-f(x) = \frac{f(t)-f(x)}{t-x}\cdot(t-x)$ since $t \neq x$. Then $f(t) = f(x) + \frac{f(t)-f(x)}{t-x}\cdot(t-x)$. Now $\lim_{t\to x} f(t) = f(x)$ and $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot(t-x) = \left(\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\right)\cdot(\lim_{t\to x}(t-x)) = f'(x)\cdot 0 = 0$. Hence $\lim_{t\to x} f(t) = \lim_{t\to x} f(x) + \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot(t-x) = f(x)$. So f is continuous at x. ■

Remark: If f is differentiable at x, then $f(t) = f(x) + (f'(x)+E(t))(t-x)$ where $\lim_{t\to x} E(t) = 0$. Indeed, $f(t) = f(x) + \frac{f(t)-f(x)}{t-x}\cdot(t-x)$. And we write $\frac{f(t)-f(x)}{t-x} = f'(x)+E(t)$. Then since $f'(x)$ exists, $\lim_{t\to x} E(t) = 0$.

Proposition 1.2

Let $f:[a,b]\to\mathbb{R}, g:[a,b]\to\mathbb{R}$ be two functions which are differentiable at $x\in[a,b]$. Then $f+g$ and $f\cdot g$ are differentiable at x. If $g(x)\neq 0$, then $\frac{f}{g}$ is differentiable at x.

1. $(f+g)'(x) = f'(x)+g'(x)$
2. $(f\cdot g)'(x) = f'(x)g(x)+f(x)g'(x)$
3. $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$

Proof:

1. $\lim_{t\to x}\frac{f(t)+g(t)-(f(x)+g(x))}{t-x} = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}+\frac{g(t)-g(x)}{t-x}\right) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} + \lim_{t\to x}\frac{g(t)-g(x)}{t-x} = f'(x)+g'(x)$.
2. $\lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)}{t-x} = \lim_{t\to x}\frac{f(t)g(t)+f(t)g(x)-f(t)g(x)-f(x)g(x)}{t-x} = \lim_{t\to x}(\frac{g(t)-g(x)}{t-x}f(t)+\frac{f(t)-f(x)}{t-x}g(t))$. Now since f is differentiable at x, it is continuous at x. So $\lim_{t\to x} f(t) = f(x)$. $\lim_{t\to x} f(t)\frac{g(t)-g(x)}{t-x} = (\lim_{t\to x} f(t))\cdot\left(\lim_{t\to x}\frac{g(t)-g(x)}{t-x}\right) = f(x)\cdot g'(x)$. Moreover, $\lim_{t\to x} g(x)\frac{f(t)-f(x)}{t-x} = g(x)\cdot f'(x)$. So $\lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)}{t-x} = f(x)\cdot g'(x)+f'(x)g(x)$.
3. First note that since g is differentiable at x. It is continuous at x. Then $\exists\delta>0, \forall t\in(x-\delta,x+\delta)\cap[a,b]: g(t)\neq 0$. So we always assume $t\in(x-\delta,x+\delta)\cap[a,b]$ and hence $g(t)\neq 0$ and $\frac{f(t)}{g(t)}$ is defined. Now $\lim_{t\to x}\frac{1}{t-x}\cdot\left(\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}\right) = \lim_{t\to x}\frac{1}{t-x}\cdot\frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)} = \lim_{t\to x}\frac{1}{g(t)g(x)}\cdot\frac{f(t)g(x)-f(x)g(t)}{t-x}$. We now consider $\lim_{t\to x}\frac{f(t)g(x)-f(x)g(t)}{t-x} = \lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x} = \lim_{t\to x}(\frac{f(t)-f(x)}{t-x}g(x)-\frac{g(t)-g(x)}{t-x}f(x)) = g(x)f'(x)-f(x)g'(x)$. Moreover, since g is continuous at x, $\lim_{t\to x} g(t)g(x) = (g(x))^2\neq 0$. Then $\lim_{t\to x}\frac{1}{t-x}\cdot\left(\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$. ■

[1.1] Note that the converse is not true: $f(x) = \begin{cases} x & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$ is continuous at 0 but not differentiable.

Proposition 1.3 (*Chain Rule*)

Let $f : [a, b] \to \mathbb{R}$ and g be defined on an interval containing the range of f. Let $x \in [a, b]$. Assume f is differentiable at x and g is differentiable at $f(x)$. Let $h(t) = g(f(x))$ for $t \in [a, b]$. Then h is differentiable at x and we have

$$h'(x) = g'(f(x)) \cdot f'(x)$$

Proof: Let $y = f(x), s = f(t)$. Now since f is differentiable at x and g is differentiable at y, we have $f(t) = f(x) + (f'(x) + E_f(t))(t - x)$ and $g(s) = g(y) + (g'(y) + E_g(s))(s - y)$ where $\lim_{t \to x} E_f(t) = 0$ and $\lim_{s \to y} E_g(t) = 0$. Now $h(t) - h(x) = g(f(t)) - g(f(x)) = g(s) - g(y) = (g'(y) + E_g(s))(s - y) = (g'(y) + F(s)) \cdot (f(t) - f(x)) = (g'(y) + E_g(s)) \cdot (f'(x) + E_f(t)) \cdot (t - x)$. Then $\lim_{t \to x} f'(x) + E_f(t) = f'(x)$ and $\lim_{t \to x} g'(y) + E_g(s) = g'(x)$. In order to compute $\lim_{t \to x} g'(y) + E_g(s)$. We first note that $y = f(x), s = f(t)$. Since f is differentiable at x, it is continuous at x. So $\lim_{t \to x} s = \lim_{t \to x} f(t) = f(x) = y$. Thus $\lim_{t \to x} g'(y) + E_g(s) = \lim_{s \to y} g'(y) + E_g(s) = g'(y)$. Altogether, $\lim_{t \to x} \frac{h(t) - h(x)}{t - x} = g'(f(x)) \cdot f'(x)$. ■

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Lecture 2: Rolle's Theorem, Mean Value Theorem

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e.g.1. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at 0 but f' is not continuous at 0. $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0 but not differentiable at 0 [2.1].

Lemma 2.1

Let $f : (a, b) \to \mathbb{R}$ be a convex and differentiable function. Then f' is increasing.

Proof: Recall that f is convex if $\forall a < s < u < b, 0 \leq \lambda \leq 1 : f(\lambda s + (1 - \lambda)u) \leq \lambda f(s) + (1 - \lambda)f(u)$. We showed in a homework in 140A that if f is convex, then $\forall a \leq s < t < u \leq b : \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$. Now taking $\lim_{t\to s^+}$ we get $\lim_{t\to s^+} \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. Similarly, we have $\lim_{t\to u^-} \frac{f(u)-f(t)}{u-t} \geq \frac{f(u)-f(s)}{u-s}$. Note that f is differentiable on (a, b) so $f'(s) = \lim_{t\to s} \frac{f(t)-f(s)}{t-s} = \lim_{t\to s^+} \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. Similarly, we have $f'(u) \geq \frac{f(u)-f(s)}{u-s}$. Altogether, we have $f'(s) \leq f'(u)$. ■

2.1 Mean Value Theorem

Definition 2.1 (*Local Maximum*)

*Let f be a real valued function on a metric space X. Let $p \in X$, we say f has **local maximum** at p if $\exists \delta > 0, \forall x \in N_\delta(p) : f(p) \geq f(x)$. **Local minimum** is defined similarly.*

Remark: Given $f : [a, b] \to \mathbb{R}$, we can draw the graph of f and local maximum and local minimum have the usual picture.

Lemma 2.2

Let $f : [a, b] \to \mathbb{R}$ be a function. Let $x \in (a, b)$ ve a local maximum for f. Assume further that f is differentiable at x. Then $f'(x) = 0$. Similar statement holds for local minimum.

Proof: Since x is a local maximum, $\exists \delta > 0 : (x-\delta, x+\delta) \subset [a, b]$ and $\forall t \in (x-\delta, x+\delta) : f(t) \leq f(x)$. Now $f'(x)$ exists. So $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. We compute this limit by taking $\lim_{t\to x^-}$ and $\lim_{t\to x^+}$. Note that we can always assume $t \in (x - \delta, x + \delta)$ since we are computing $\lim_{t\to x}$. If $x - \delta < t < x$, then $\frac{f(t)-f(x)}{t-x} \geq 0$ indicates $\lim_{t\to x^-} \frac{f(t)-f(x)}{t-x} \geq 0$; if $x < t < x+\delta$, then $\frac{f(t)-f(x)}{t-x} \leq 0$ indicates $\lim_{t\to x^+} \frac{f(t)-f(x)}{t-x} \leq 0$. Then $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x) = 0$.

[2.1] Although we have not yet defined trigonometric functions in a rigorous way, we know what it is.

■

e.g.2. $f(x) = |x|$ has local minimum at 0 but not differentiable at 0. Indeed, $\lim_{t\to 0^-} \frac{f(t)-f(x)}{t-x} = -1$ and $\lim_{t\to 0^+} \frac{f(t)-f(x)}{t-x} = 1$.

Theorem 2.3 (*Rolle's Theorem*)

Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b)$. Then $\exists c \in (a,b) : f'(c) = 0$.

Proof: Recall that $[a,b]$ is compact. Since f is continuous, f has (absolute) maximum and (absolute) minimum on $[a,b]$. i.e. $\exists t, s \in [a,b], \forall x \in [a,b] : f(s) \leq f(x) \leq f(t)$.

Case1: at least one of s and t is an interior point i.e. belongs to (a,b). Let's say $t \in (a,b)$. Then by previous lemma, since $\forall x \in [a,b], t \in (a,b) : f(t) \geq f(x)$, f is differentiable at t. We conclude that $f'(t) = 0$ so $c = t$ solves this. Similarly if $s \in (a,b)$, the previous implies $f'(s) = 0$ so $c = s$.

Case2: Both t and s are end points. In this case, since $f(a) = f(b)$, f is constant. So $\forall c \in (a,b) : f'(c) = 0$. ■

Corollary 2.4 (*Mean Value Theorem*)

Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Then $\exists c \in (a,b)$:

$$f(b) - f(a) = f'(c)(b-a)$$

Proof: Let L be a line through $(a, f(a))$ and $(b, f(b))$. Then the equation of L can be written as $y(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$ or $y(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Define $h(x) = f(x) - y(x)$. Then h is continuous on $[a,b]$; differentiable on (a,b); $h(a) = f(a) - y(a) = 0$; $h(b) = f(b) - y(b) = 0$. So h satisfies conditions of Rolle's Theorem. Then $\exists c \in (a,b) : h'(c) = 0$. Note that $h'(x) = f'(x) - y'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. $\exists c \in (a,b) : h'(c) = 0$. So $f'(c) = \frac{f(b)-f(a)}{b-a}$. ■

Corollary 2.5 (*Generalization of Mean Value Theorem*)

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$: [2.2]

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$$

Proof: Define $h(x) = (f(b) - f(a))g(x) - (g(b) - f(a))f(x)$. To see this explicitly, we want to find λ and μ such that $\lambda f(a) + \mu g(a) = 0$ and $\lambda f(b) + \mu g(b) = 0$. Subtracting these two equations, we have $\lambda(f(a) - f(b)) + \mu(g(a) - g(b)) = 0$. Then $\lambda = g(a) - g(b), \mu = -(f(a) - f(b))$. Now since h is continuous on $[a,b]$ and differentiable on (a,b), $h(a) = 0$ and $h(b) = 0$. Therefore by Rolle's Theorem, $\exists c \in (a,b) : h'(c) = 0$. Note that $h'(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$. Hence $h'(c) = 0$ indicates $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. ■

[2.2] If $g(x) = x$, then we get the MVT Theorem.

Corollary 2.6

Suppose f is differentiable on (a, b)

1. *If $\forall x \in (a, b) : f'(x) \geq 0$, then f is increasing.*
2. *If $\forall x \in (a, b) : f'(x) \leq 0$, then f is decreasing.*
3. *If $\forall x \in (a, b) : f'(x) = 0$, then f is constant.*

Proof: Let $a < s < t < b$ then $\exists c \in (s, t) : f(t) - f(s) = f'(c)(t - s)$. Now if $\forall x \in (a, b) : f'(x) \geq 0$ holds, then this implies that $f'(c) \geq 0$. Then $f(t) \geq f(s)$. Since t and s are arbitrary, (1) holds true. The proof of (2) and (3) are similar. ■

e.g.3. Let y be twice differentiable on $\mathbb{R}$. Suppose $y = -y''$, $y(0) = 0, y'(0) = 0$. Prove that y is the constant function 0 [2.3].

[2.3] Hint: consider $y + y'' = 0$, then $2y'y + 2y'y'' = 0$. Note that $\left(y^2 + (y')^2\right)' = 2yy' + 2y'y''$.

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Lecture 3: Intermediate Value Theorem, L'Hospital's Rule, Higher Order Derivatives

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e.g.1. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. $f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. But $f'(x)$ is not continuous at 0.

e.g.2. $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Does there exist some f such that $f'(x) = h(x)$? [3.1]

3.1 Intermediate Value Theorem

Theorem 3.1 (*Intermediate Value Property for f'*)

Let f be differentiable on $[a, b]$. Suppose $f'(a) < c < f'(b)$ (or $f'(a) > c > f'(b)$). Then $\exists x \in (a, b) : f'(x) = c$.

Proof: Note that $f'(x) = c \implies f'(x) - c = 0$. So if we let $h(t) = f(t) - ct$ for $t \in [a, b]$. Then we are looking for some x such that $h'(x) = 0$. We will show this holds by showing that h has a local minimum in (a, b). Since h is differentiable on $[a, b]$, h is continuous. Then h has absolute maximum and minimum on $[a, b]$. We claim that the minimum cannot happen at a or b. Note $h'(a) = f'(a) - c < 0$ and $h'(b) = f'(b) - c > 0$. The first equality implies $\lim_{t\to a} \frac{h(t)-h(a)}{t-a} < 0$ for $t \in (a, b]$. So $t - a > 0$ since the limit is positive and $h(t) - h(a) < 0$ for t close to a. Thus the minimum is not at a. Similarly, the minimum is not at b. So $\exists x \in (a, b), \forall t \in [a, b] : h(x) \leq h(t)$. Then by a lemma, $h(x) = 0$. ■

Corollary 3.2

- *If f' is discontinuous at x, then it is a second-kind discontinuity.*
- *If f' is increasing, then it is continuous.*

Proof: The first part follows from the previous theorem. The second claim follows from the first and the fact that discontinuity of monotonic function is of the first kind. ■

[3.1] We will show the answer is NO.

3.2 L'Hospital's Rule

Theorem 3.3 (*L'Hospital's Rule*)

Let f and g be differentiable on (a,b). Assume $\forall x \in (a,b) : g'(x) \neq 0$. Suppose:

1. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$
2. *Either* $\lim_{x\to a} f = \lim_{x\to a} g = 0$ *or* $\lim_{x\to a} g = \infty$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Proof: Note that $\exists \delta_0 > 0, \forall x \in (a, a+\delta_0) : g(x) \neq 0$. Indeed, if $\exists a < t < s < b : g(t) = g(s) = 0$, then by MVT, $\exists t < x < s : g'(x) = 0$, contradiction. So g has at most one zero in (a,b). Then $\exists \delta_0, \forall x \in (a, a+\delta_0) \cup (b-\delta_0, b) : g(x) \neq 0$. So replacing (a,b) by $(a, a+\delta_0)$, we assume $g(x) \neq 0$ on (a,b).

Case1: $L \in \mathbb{R}$. We need to show $\forall \epsilon > 0, \exists \delta > 0 : x \in (a, a+\delta) \implies \left|\frac{f(x)}{g(x)} - L\right| < \epsilon$. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, we have $\exists \delta_1, \forall x \in (a, a+\delta_1) : \left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$. Fix $x \in (a, a+\delta_1)$ and let $a < t < x < a+\delta_1$. Then by MVT, $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_t)}{g'(c_t)}$ for $t < c_t < x$. Assume $\lim_{t\to a} f(t) = \lim_{t\to a} g(t) = 0$. Then $\left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right| = \left|\frac{f'(c_t)}{g'(c_t)} - L\right| < \frac{\epsilon}{2}$. Take limit as $t \to a$. Hence we get $\left|\frac{f(x)}{g(x)} - L\right| \leq \frac{\epsilon}{2}$. Since $x \in (a, a+\delta_1)$ is arbitrary, we get $\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$ in the case $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Now we consider $\lim_{t\to\infty} g(t) = \infty$. Then $\left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right| = \left|\frac{f'(c_t)}{g'(c_t)} - L\right| < \frac{\epsilon}{2}$ for $a < t < c_t < x < a+\delta_1$. Thus $L - \frac{\epsilon}{2} < \frac{f(t)-f(x)}{g(t)-g(x)} < L + \frac{\epsilon}{2}$. Since $\lim_{t\to\infty} g(t) = \infty$ and $x \in (a, a+\delta_1)$ is fixed, $\exists \delta_2 : a < t < a+\delta_2 < x < a+\delta_1 \implies g(t) > 0, g(t)-g(x) > 0$. Multiplying $L - \frac{\epsilon}{2} < \frac{f(t)-f(x)}{g(t)-g(x)} < L + \frac{\epsilon}{2}$ by $(g(t)-g(x))$, we have $(L-\frac{\epsilon}{2})(g(t)-g(x)) < f(t) - f(x) < (L+\frac{\epsilon}{2})(g(t)-g(x))$. Adding $f(x)$ to the inequality and dividing by $g(t)$ give $(L-\frac{\epsilon}{2})\frac{g(t)-g(x)}{g(t)} + \frac{f(x)}{g(t)} < \frac{f(t)}{g(t)} < (L+\frac{\epsilon}{2})\frac{g(t)-g(x)}{g(t)} + \frac{f(x)}{g(t)}$. Since x is fixed, $\exists \delta_3 < \delta_2 : a < t < a+\delta_3 < a+\delta_2 < x < a+\delta_1 \implies \left|\frac{f(x)}{g(t)}\right| < \frac{\epsilon}{10}, \left|\frac{g(x)}{g(t)}\right| < \frac{\epsilon}{10}$. Then we have $L - \epsilon < \frac{f(t)}{g(t)} < L + \epsilon$. So if $t \in (a, a+\delta_3)$, then $\left|\frac{f(t)}{g(t)} - L\right| < \epsilon$. So $\lim_{t\to a} \frac{f(t)}{g(t)} = L$ in this case as well.

Case2: $L = \pm\infty$. The proof is similar and left as exercise. ■

3.3 Higher Order Derivatives

Definition 3.1

Let f be differentiable on $[a,b]$. $\forall x \in [a,b] : f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. So f' is a function on $[a,b]$. If f' is differentiable on $[a,b]$, then $(f')'$ will be denoted by f''. Continuing inductively, we define $f, f'', f''', f^{(n)}$ if they exist [3.2].

[3.2] Note that in order for $f^{(n)}$ to exist at $x \in [a,b]$. The $(n-1)$-derivative should exist on an interval around.

Definition 3.2 (*Taylor's Polynomial*)

Suppose f is defined on $[a,b]$. Let $c \in [a,b]$ and suppose f is n-time differentiable at c. Then we define

$$P_{n,c}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

$P_{n,c}$ is the called n-th (degree) ***Taylor's polynomial*** *at c.*

Remark: Without further restrictions, $P_{n,c}(x)$ gives only information about c. Let $f(x) = \begin{cases} e^{-\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $\forall n : f^{(n)} = 0$ so $P_{n,0}(x) = 0$.

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Lecture 4: Taylor's Theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 4.1

Let f be n-times differentiable at c. Then

1. $f^{(k)}(c) = P_{n,c}^{(k)}(c)$ *for* $0 \le k \le n$
2. $P_{n,c}^{(k)}(t) = 0$ *for* $k > n$
3. $\lim_{t\to c} \frac{f(t)-P_{n,c}(t)}{(t-c)^k/k!} = 0$ *for* $0 \le k \le n$

Proof:

1. follows from the definition. Indeed, $P_{n,c}(t) = f(c) + f'(c)(t-c) + \cdots + \frac{f^{(k)}(c)}{k!}(t-c)^k + \cdots + \frac{f^{(n)}(c)}{n!}(t-c)^n$. Then $P_{n,c}^{(k)}(t) = f^{(k)}(c) + (k+1) \times \cdots \times 2 \times \frac{f^{(k+1)}(c)}{(k+1)!}(t-c) + \cdots + n(n-1)\dots(n-(k-1))\frac{f^{(n)}(c)}{n!}(t-c)^{n-k}$. So if we evaluate this at c, we get $P_{n,c}^{(k)}(c) = f^{(k)}(c)$.
2. is clear since $P_{n,c}$ has degree at most n for every $0 \le k \le n$.
3. It suffices to prove this for $k = n$. We want to compute $\lim_{t\to c}\frac{f(t)-P_{n,c}(t)}{(t-c)^n/n!}$. For $n=1$, $\lim_{t\to c}\frac{f(t)-P_{1,c}(t)}{t-c} = \lim_{t\to c}\frac{f(t)-(f(c)+f'(c)(t-c))}{t-c} = \lim_{t\to c}\left(\frac{f(t)-f(c)}{t-c} - f'(c)\right)$. Note that the conditions of L'Hospital Rule are satisfied $(n-1)$-times. So by the conclusion of L'Hospital, we compute: $\lim_{t\to c}\frac{f^{(n-1)}(t)-(f^{(n-1)}(c)+f^{(n)}(c)(t-c))}{t-c} = \lim_{t\to c}(\frac{f^{(n-1)}(t)-f^{(n-1)}(c)}{t-c} - f^{(n)}(c)) = 0$. Thus, by L'Hospital, $\lim_{t\to c}\frac{f(t)-P_{1,c}(t)}{t-c} = 0$. ■

Corollary 4.2

Suppose f is n-times differentiable at c and $f'(c) = \cdots = f^{(n-1)}(c) = 0$

1. *Suppose n is even, if $f^{(n)}(c) > 0$, then c is the local minimum; if $f^{(n)}(c) < 0$, then c is the local maximum.*
2. *Suppose n is odd, $f^{(n)}(c) \ne 0$. Then c is not local maximum or minimum.*

Proof: (1) Suppose n is even, $P_{n,c}(t) = f(c) + \frac{f^{(n)}(c)}{n!}(t-c)^n$. Now by previous lemma, $\lim_{t\to c}\frac{f(t)-P_{n,c}(t)}{(t-c)^n/n!} = 0$. Suppose $f^{(n)}(c) > 0$, then $0 = \lim_{t\to c}\frac{f(t)-f(c)-\frac{f^{(n)}}{n!}(t-c)^n}{(t-c)^n/n!} = \lim_{t\to c}\left(\frac{f(t)-f(c)}{(t-c)^n/n!} - f^{(n)}(c)\right)$. So we get $\lim_{t\to c}\frac{f(t)-f(c)}{(t-c)^n/n!} = f^{(n)}(c)$. Since $f^{(n)} > 0$, $\frac{f(t)-f(c)}{(t-c)^n/n!} > 0$. If t is close to c, n is even so $\frac{(t-c)^n}{n!} > 0$. Altogether we get $\exists\delta, \forall t \in (c-\delta, c+\delta) : f(t) - f(c) \ge 0$, so c is the local minimum.

The proof for $f^{(n)}(c) < 0$ and n is even and part (2) are similar. ■

Theorem 4.3 (*Taylor's Theorem*)

Suppose f is real-valued on $[a,b]$. Assume that $f^{(n-1)}$ is continuous on $[a,b]$ and that $f^{(n)}$ exists on (a,b). Let $c,d \in [a,b], c \neq d$. Then $\exists t \in (c,d)$: [4.1]

$$f(d) = P_{n-1,c}(d) + \frac{f^{(n)}(t)}{n!}(d-c)^n$$

Proof: Recall the proof of MVT: $g(x) = f(x) - (f(c) + \frac{f(d)-f(c)}{d-c}(x-c))$, $g(c) = g(d) = 0$ with Rolle's Theorem implies $\exists t \in (c,d) : g'(t) = 0$.

Use a similar strategy. Define $h(x) = P_{n-1,c}(x) + M(x-c)^n$ where M is chosen so that $h(d) = f(d)$. i.e. we want to solve $f(d) = P_{n-1,c}(d) + M(d-c)^n$ for M. Let $g(x) = f(x) - h(x) = f(x) - P_{n-1,c}(x) - M(x-c)^n$. Then $g(c) = f(c) - P_{n-1,c}(c) - M(c-c)^n = 0$. Similarly, using the lemma 4.1, we have $g(c) = 0, g'(c) = 0, \ldots, g^{(n-1)}(c) = 0$. $g(d) = f(d) - h(d) = 0$. Now g is continuous on $[a,b]$ and differentiable on (a,b). So by MVT, $\exists t_1 \in (c,d) : g'(t_1) = 0$. Repeat this using the fact that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). We get $t_2 \in (c,t_1), t_3 \in (c,t_2), \ldots, t_{n-1} \in (c,t_{n-2})$ such that $g^{(n-1)}(t_{n-1}) = 0$. We now have $g^{(n-1)}(c) = 0, g^{(n-1)}(t_{n-1}) = 0$. Then $g^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b). So $\exists t \in (c,t_{n-1}) : g^{(n)}(t) = 0$. Recall that $g(x) = f(x) - P_{n-1,c}(x) - M(x-c)^n$, then $g^{(n)}(x) = f^{(n)}(x) - 0 - n!M$ since $P_{n-1,c}$ is a polynomial of degree $k \leq n-1$. Hence $0 = g^{(n)}(t) = f^{(n)}(t) - n!M$ implies $M = \frac{f^{(n)}(t)}{n!}$. ■

[4.1] Note that $P_{n,c}(d) = P_{n-1,c}(d) + \frac{f^{(n)}(c)}{n!}(d-c)^n$.

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Lecture 5: Vector-valued Functions, Riemann Integrable

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

5.1 Vector-valued Functions

Suppose $f : [a,b] \to \mathbb{R}^n$ is a function. Then $f(x) = (f_1(x), \dots, f_n(x))$ where $f_j : [a,b] \to \mathbb{R}$ for $1 \le j \le n$. Then we say f is differentiable at x if $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ exists. Note that f is differentiable at x if and only if $\forall j : f_j$ is differentiable at x. Indeed,

$$\frac{f(t)-f(x)}{t-x} = \left(\frac{f_1(t)-f_1(x)}{t-x}, \dots, \frac{f_n(t)-f_n(x)}{t-x}\right)$$

If f is differentiable at x, then $f'(x) = (f_1'(x), \dots, f_n'(x))$.

Lemma 5.1

Let $f : [a,b] \to \mathbb{R}^n, g : [a,b] \to \mathbb{R}^n$ be differentiable. Then

1. $(f+g)'(x) = f'(x) + g'(x)$
2. $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Proof: Left as exercise. ∎

e.g.1.

- $f(t) = (t, t^2, t^3)$, then $f'(t) = (1, 2t, 3t^2)$.
- $f : [0, 2\pi] \to \mathbb{R}^2, f(x) = (\cos x, \sin x)$, then $f'(x) = (-\sin x, \cos x)$ and $\|f'(x)\| = 1$. Note however that MVT may fail: $\nexists t : (0,0) = f(2\pi) - f(0) = 2\pi f'(t)$.

Proposition 5.2

Suppose $f : [a,b] \to \mathbb{R}^n$ is continuous and differentiable on (a,b). Then $\exists x \in (a,b)$: [5.1]

$$\|f(b) - f(a)\| \le (b-a)\,\|f'(x)\|$$

Proof: Let $v \in \mathbb{R}^n$. Define $f_v : [a,b] \to \mathbb{R}, f_v(t) = v \cdot f(t)$. So for every v the function f_v satisfies conditions of MVT. Then $\forall v, \exists x_v \in (a,b) : f_v(b) - f_v(a) = (b-a) \cdot f_v'(x_v)$. Thus, $v \cdot f(b) - v \cdot f(a) = (b-a)(v \cdot f'(x_v))$ implies $v \cdot (f(b) - f(a)) = (b-a)(v \cdot f'(x_v))$. Suppose now that $v = f(b) - f(a)$. Then by MVT, $\exists x \in (a,b) : (f(b)-f(a)) \cdot (f(b)-f(a)) = (b-a)(f(b)-f(a)) \cdot f'(x)$ so $\|f(b)-f(a)\|^2 = (b-a)((f(b)-f(a)) \cdot f'(x)) \le (b-a)\|f(b) - f(a)\|\|f'(x)\|$ by applying Cauchy-Schwartz Inequality. Now if $\|f(b) - f(a)\| = 0$, then the lemma is obvious; otherwise if $\|f(b)-f(a)\| \ne 0$, then $\|f(b)-f(a)\| \le (b-a)\|f'(x)\|$. ∎

[5.1] Recall $\|(v_1, \dots, v_n)\| = \sqrt{\sum v_i^2}$.

Remark: L'Hospital's Rule for vector-valued functions also fails [5.2].

5.2 Riemann Integrable

Definition 5.1 (*Partition*)

*Let $[a,b]$ be an interval. A **partition** of $[a,b]$ is a finite set of points $\{x_0 = a, \ldots, x_n = b : x_0 \le \cdots \le x_n\}$.*

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$, we denote

$$\Delta x_i = x_i - x_{i-1}$$

for $1 \le i \le n$. Let f be a bounded function on $[a,b]$ and $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$. Define

$$\begin{aligned} M_i &= \sup f(x) \quad x_{i-1} \le x \le x_i \\ m_i &= \inf f(x) \quad x_{i-1} \le x \le x_i \end{aligned}$$

Since f is bounded, M_i, m_i exist as real numbers.

e.g.2.

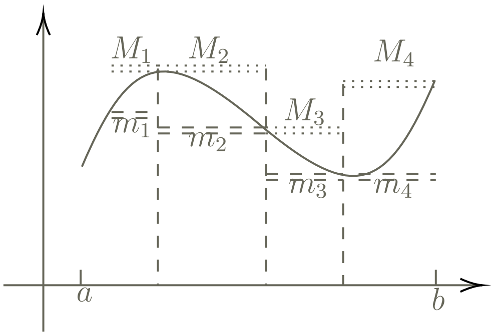

Figure 5.1: Illustrations of M_i, m_i

We denote **upper sum** and **lower sum** by

$$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$$

and **upper integral** and **lower integral** by

$$\overline{\int_a^b} f \mathrm{d}x = \inf\{U(P,f)\} \qquad \underline{\int_a^b} f \mathrm{d}x = \sup\{L(P,f)\}$$

Definition 5.2 (*Riemann Integrable*)

*We say f is **Riemann integrable** on $[a,b]$ if*

$$\overline{\int_a^b} f \mathrm{d}x = \underline{\int_a^b} f \mathrm{d}x$$

If so, we write $f \in \mathcal{R}[a,b]$ or simply $f \in \mathcal{R}$.

[5.2] See the example in the book.

Remark: Suppose $m \le f \le M$ for f is bounded. Then $m \le m_i \le M_i \le M$. So $U(P,f) = \sum_{i=1}^n M_i \Delta x_i \le \sum_{i=1}^n M\Delta x_i = M\sum_{i=1}^n \Delta x_i = M(b-a)$. Similarly, $L(P,f) \ge m(b-a)$. So we get [5.3]

$$\overline{\int} f = \inf U(P,f) \le M(b-a)$$

$$\underline{\int} f = \sup L(P,f) \ge m(b-a)$$

e.g.3.

- $f(x) = c$ on $[a,b]$. Then $f \in \mathcal{R}$.

  Proof: Let $P = \{x_0, \ldots, x_n\}$ be any partition. Then $M_i = \sup f(x) = c$ and $m_i = \inf f(x) = c$. This implies $U(P,f) = \sum_{i=1}^n M_i \Delta x_i = c(b-a)$ and $L(P,f) = \sum_{i=1}^n m_i \Delta x_i = c(b-a)$. Then $\overline{\int_a^b} f = c(b-a) = \underline{\int_a^b}$. ■

- Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ on $[0,1]$. Then f is not Riemann integrable.

  Proof: Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$. Then $U(P,f) = \sum_{i=1}^n M_i \Delta x_i$ and $L(P,f) = \sum_{i=1}^n m_i \Delta x_i = c(b-a)$. We can assume $x_0 = 0 < x_1 < x_2 \le \cdots \le x_n = 1$. Since $[x_{i-1}, x_i]$ contains more than one point, $\exists x \in [x_{i-1}, x_i] \cap \mathbb{Q}$ so $M_i = \sup f(x) = 1$ for $x \in [x_{i-1}, x_i]$. Similarly, $\exists x \in [x_{i-1}, x_i] \setminus \mathbb{Q}$ so $m_i = \inf f(x) = 0$. Therefore, $U(P,f) = \sum_{i=1}^n M_i \Delta_i = \sum_{i=1}^n \Delta x_i = 1$ and $L(P,f) = \sum_{i=1}^n m_i \Delta x_i = 0$. Then $\overline{\int} f = 1$ and $\underline{\int} f = 0$. Thus $f \notin \mathcal{R}$. ■

- Let $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \ne \frac{1}{2} \end{cases}$ on $[0,1]$. Then $f \in \mathcal{R}$.

  Proof: Note that $0 \le f \le 1$. Let P be any partition. $L(P,f) = \sum_{i=1}^n m_i \Delta_i = 0$ so $\underline{\int_0^1} f \mathrm{d}x = 0$. Note that $U(P,f) \ge 0$. We now show that $\forall \epsilon > 0, \exists P : U(P,f) < \epsilon$. Then $\inf U(P,f) = 0$ so $\overline{\int} f \mathrm{d}x = 0$. Now given $\epsilon > 0$, define $P_\epsilon = \{x_0 = 0, x_1 = \frac{1}{2} - \frac{\epsilon}{4}, x_2 = \frac{1}{2} + \frac{\epsilon}{4}, x_3 = 1\}$. Then $U(P_\epsilon, f) = M_1 \Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3$. $M_1 = \sup f = 0$ for $x_0 \le x \le x_1$, $M_3 = \sup f = 0$ for $x_2 \le x \le x_3$, and $M_2 = \sup f = 1$ for $x_1 \le x \le x_2$. Then $U(P_\epsilon, f) = M_2 \Delta x_2 = 1(x_2 - x_1) = \frac{\epsilon}{2} < \epsilon$. This implies $\overline{\int} f \mathrm{d}x = 0$ so $f \in \mathcal{R}$. ■

[5.3] *By the scribe*: in the next lecture, we will show that this can be indeed simplified as $m(b-a) \le \underline{\int} f \le \overline{\int} f \le M(b-a)$.

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Lecture 6: Riemann-Stieltjes Integrable

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Let f be a bounded (real-valued) function on $[a, b]$. Let α be an increasing function on $[a, b]$. Given a partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$, we define

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

and with

$$\begin{aligned} M_i &= \sup f(x) \quad x_{i-1} \le x \le x_i \\ m_i &= \inf f(x) \quad x_{i-1} \le x \le x_i \end{aligned}$$

We update the notion of **upper sum** and **lower sum** by

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

and define

$$\overline{\int_a^b} f \mathrm{d}\alpha = \inf U(P, f, \alpha) \qquad \underline{\int_a^b} f \mathrm{d}\alpha = \sup L(P, f, \alpha)$$

Definition 6.1 (*Riemann-Stieltjes Integrable*)

*We say f is **Riemann-Stieltjes integrable** w.r.t. α and write $f \in \mathcal{R}(\alpha)$ if*

$$\overline{\int_a^b} f \mathrm{d}\alpha = \underline{\int_a^b} f \mathrm{d}\alpha$$

If so, we define

$$\int_a^b f \mathrm{d}\alpha = \overline{\int_a^b} f \mathrm{d}\alpha = \underline{\int_a^b} f \mathrm{d}\alpha$$

e.g.1.

- Let f be a constant function and α be arbitrary. Then $U(P, f) = \sum_{i=1}^{n} M_i \Delta\alpha_i = \sum_{i=1}^{n} c\Delta\alpha_i = c\sum_{i=1}^{n}(\alpha(x_i) - \alpha(x_{i-1})) = c(\alpha(b) - \alpha(a))$, $L(P, f) = \sum_{i=1}^{n} m_i \Delta\alpha_i = c(\alpha(b) - \alpha(a))$. Therefore, $\overline{\int} f \mathrm{d}\alpha = \underline{\int} f \mathrm{d}\alpha = c(\alpha(b) - \alpha(a))$. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f \mathrm{d}\alpha = c(\alpha(b) - \alpha(a))$.
- Let f be arbitrary and α be constant. Then $\Delta x_i = 0$. Then $\forall i : \overline{\int} f \mathrm{d}\alpha = \overline{\int} f \mathrm{d}\alpha = 0$. So $f \in \mathcal{R}(\alpha)$.

Definition 6.2 (*Refinement*)

Let P be a partition of $[a,b]$. We say P^* is a ***refinement*** *of P if*

$$P \subset P^*$$

If P_1 and P_2 are two partitions of $[a,b]$, then their common refinement is defined to be $P_1 \cup P_2$.

Lemma 6.1

Let P^*, P be two partitions where $P^* \supset P$. Then

1. *$U(P^*, f, \alpha) \leq U(P, f, \alpha)$*
2. *$L(P, f, \alpha) \leq L(P^*, f, \alpha)$*

Proof: We prove the lemma for upper sums; the proof for the lower sums is similar.

We want to prove this by induction. First assume $P^* = P \cup \{y\}$. i.e. P^* has one more point that P. Let $P = \{x_0, \ldots, x_n\}$. $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$. Since $P^* = P \cup \{y\}$ where $x_{j-1} \leq y \leq x_j$ for some j. $U(P^*, f, \alpha) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + \cdots + A(\alpha(y) - \alpha(x_{j-1})) + B(\alpha(x_j) - \alpha(y)) + M_{j+1}\Delta\alpha_i + \cdots + M_n\Delta\alpha_n$ where $A = \sup f(x)$ for $x_{j-1} \leq x \leq y$, $B = \sup f(x)$ for $y \leq x \leq x_j$. $U(P, f, \alpha) = \sum_{i=1}^n M_i\Delta\alpha_i = M_1\Delta\alpha_1 + \cdots + M_{j-1}\Delta\alpha_{j-1} + M_j(\alpha(x_j) - \alpha(x_{j-1})) + \cdots + M_n\Delta\alpha_n$. To show $U(P^*, f, \alpha) \leq U(P, f, \alpha)$ is equivalent to prove $A(\alpha(y) - \alpha(x_{j-1})) + B(\alpha(x_j) - \alpha(y)) \leq M_j(\alpha(x_j) - \alpha(x_{j-1})) = M_j(\alpha(y) - \alpha(x_{j-1})) + M_j(\alpha(x_j) - \alpha(y))$. Indeed, this is implied by that $[x_{j-1}, y] \subset [x_{j-1}, x_j]$ and $[y, x_j] \subset [x_{j-1}, x_j]$. To complete the proof for an arbitrary $P^* \supset P$, we just repeat the above process ℓ-times where ℓ is the difference between the numbers of points of P^* and of P. ■

Proposition 6.2

Let f be bounded on $[a,b]$ and α be increasing. Let P_1 and P_2 be two partitions of $[a,b]$, then

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Proof: Let P_1 and P_2 be two partitions. Let P^* be the common refinement. Then by the previous lemma, $U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$ and $L(P_1, f, \alpha) \leq L(P^*, f, \alpha)$. Since $L(P^*, f, \alpha) \leq U(P^*, f, \alpha)$, we get that $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$. ■

Theorem 6.3

$$\underline{\int_a^b} f \mathrm{d}\alpha \leq \overline{\int_a^b} f \mathrm{d}\alpha$$

Proof: Let P be an arbitrary partition and fix a partition P'. Then $L(P, f, \alpha) \leq U(P', f, \alpha)$. Taking sup over P gives $\underline{\int_a^b} f \mathrm{d}\alpha \leq U(P', f, \alpha)$. Then taking inf over P' gives $\underline{\int_a^b} f \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha$. ■

Proposition 6.4

Let f be bounded on $[a,b]$ and α be increasing. Then $f \in \mathcal{R}(\alpha)$ if and only if $\forall \epsilon > 0, \exists P : U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Proof:

($\Leftarrow$) Let $\epsilon > 0$, then $\exists P : U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ w.t.s. that $\underline{\int_a^b} \mathrm{d}\alpha = \overline{\int_a^b} \mathrm{d}\alpha$. Now $L(P,f,\alpha) \leq \underline{\int_a^b} \mathrm{d}\alpha \leq \overline{\int_a^b} \mathrm{d}\alpha \leq U(P,f,\alpha)$. Since P is so that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$, $0 \leq \overline{\int_a^b} f \mathrm{d}\alpha - \underline{\int_a^b} f \mathrm{d}\alpha < \epsilon$. As our choice of ϵ is arbitrary, $\overline{\int} f \mathrm{d}\alpha = \underline{\int} f \mathrm{d}\alpha$.

($\Rightarrow$) Suppose $f \in \mathcal{R}(\alpha)$ and let $\epsilon > 0$ w.t.s. $\exists P : U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. Since $f \in \mathcal{R}(\alpha)$, we have $\sup_P L(P,f,\alpha) = \underline{\int} f \mathrm{d}\alpha = \overline{\int} f \mathrm{d}\alpha = \inf_{P'} U(P',f,\alpha)$. By definition of sup and inf, $\exists P, P' : 0 \leq U(P',f,\alpha) - \overline{\int} f \mathrm{d}\alpha < \frac{\epsilon}{2}, 0 \leq \underline{\int} f \, \mathrm{d}\alpha - L(P,f,\alpha) < \frac{\epsilon}{2}$. Let P^* be the common refinement of P and P'. Then by a lemma, $L(P,f,\alpha) \leq L(P^*,f,\alpha) \leq U(P^*,f,\alpha) \leq U(P',f,\alpha)$. So $U(P^*,f,\alpha) \leq U(P',f,\alpha) \leq \int_a^b f \mathrm{d}\alpha + \frac{\epsilon}{2} \leq L(P,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq L(P^*,f,\alpha) + \epsilon$ gives $0 \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. ■