A Mathematical Model of Guarantee Mechanism in Gacha Systems

Jacky Wang
https://jackywang2001.github.io
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TOC

What is a Gacha System?

- Originated from Japanese collectible trading card game (TCG) where the player needs to open a pack (black box) to get random cards and build his own deck to battle.
- Standing for "a pool of cards".
- Used in current video games, players draw for characters, dress-up, etc.
- Similar to **loot box** used by Overwatch and Dota2, but a number of repeating items are still useful in Gacha!

A Few Examples

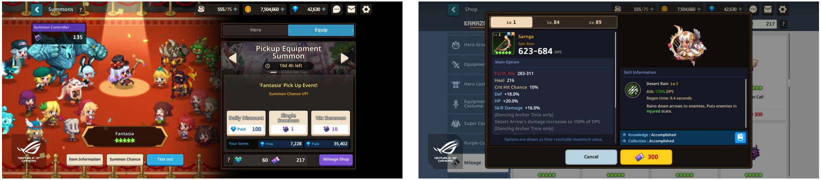


Figure: Summons in Guardian Tales

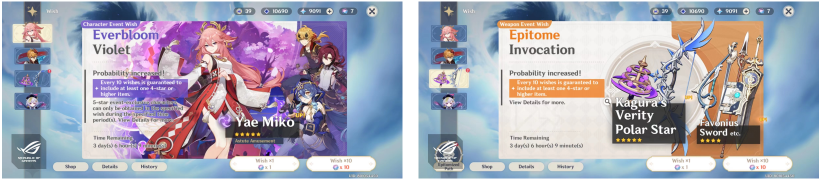


Figure: Wishes in Genshin Impact

Problems of Interest

- The expected number of steps to obtain K repeating target items is

$$\mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=K}$$

- For players, this helps estimate the budget since

$$\mathcal{C} = C\mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=K}$$

- For industries, the total revenue can be estimated after knowing the population of their user groups

$$\mathcal{R} = C\sum_{k} p_k \mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=k}$$

Models

- **Draw-Relevant Guarantee Mechanism**
- **Failure-Relevant Guarantee Mechanism**

Draw-Relevant Guarantee Mechanism

for $\tau = 1, 2, \ldots$ **do**
  if $\tau \mid G$ **then**
    Guarantee
  else
    Perform a random draw [1]
  end if
end for

[1] In some games, the procedure "perform a random draw" may be out of the else.

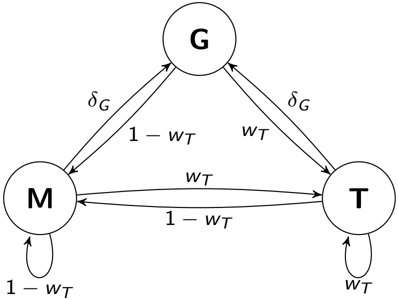


Figure: **M** is the state that the player misses to draw a target item; **T** is the state that the player successfully draws a target item; and **G** is the guarantee. The initial distribution $\mathbf{I} = \{\mathbf{M} : 1 - w_T, \mathbf{T} : w_T\}$.

Definition (Draw-Relevant Guarantee Mechanism)

A guarantee mechanism is said to be deterministic guarantee after G draws if its transition matrix for the random walk $\{\mathbf{M}, \mathbf{T}, \mathbf{G}\}$ at any time step τ is in the form

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{M} & \mathbf{T} & \mathbf{G} \end{array} \\ \begin{array}{c} \mathbf{M} \\ \mathbf{T} \\ \mathbf{G} \end{array} & \begin{bmatrix} (1-w_T)(1-\delta_G) & w_T(1-\delta_G) & \delta_G \\ (1-w_T)(1-\delta_G) & w_T(1-\delta_G) & \delta_G \\ 1-w_T & w_T & 0 \end{bmatrix} \end{array}$$

where $\delta_G = \delta(\tau - G) := \begin{cases} 1 & \tau \mid G \\ 0 & \tau \nmid G \end{cases}$ is the delta function.

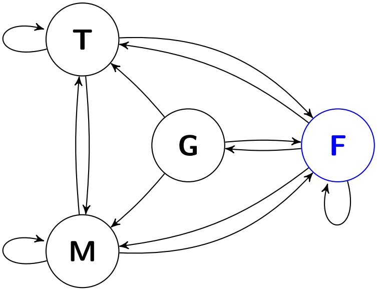
It should be possible for one to derive a form for $\mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=K}$.

Failure-Relevant Guarantee Mechanism

- Rare items $\mathcal{S}$ and $t_T \in \mathcal{S}$
- Rare items have higher chance to be drawn at step τ_S.
- If the player fails to draw t_T for a consecutive F times, the next rare item must be t_T.

for $\tau = 1, 2, \ldots$ **do**
  if $|\mathbf{F}| = F$ **then**
    if $\tau - \tau_{\mathbf{F}} = S$ **then**
      Deterministic Guarantee
    else
      Perform a random draw in $(\mathcal{P} \setminus \mathcal{S}) \cup \{t_T\}$ [a]
    end if
  else
    if $\tau - \tau_{\mathbf{G}} = S$ **then**
      Perform a random draw in $\mathcal{S}$
    else
      Perform a random draw in $\mathcal{P}$
    end if
    if the draw is t_T **then**
      Clear the accumulation of $|\mathbf{F}|$
    end if
  end if
end for

[a] i.e. $\mathcal{S} = \{t_T\}$ in this case.



- It is hard to write down the transition matrix.
- Need to embed a delta function that has condition $|\mathbf{F}| = F$.
- This is a hypergraph with **F** itself being a graph.

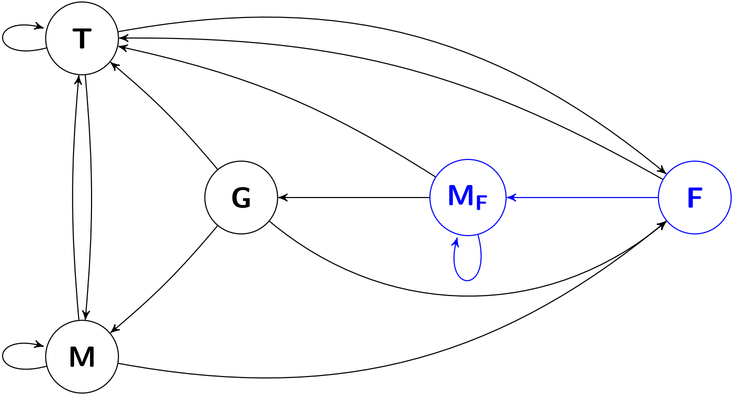


Figure: $F = 1$: **M** is the state that the player draws $t_i \in \mathcal{P} \setminus \{t_T\}$; **T** is the state where the player wins one target item; **F** is the state where the player draws a rare item $\mathcal{S} \setminus \{t_T\}$; **G** is the state that the deterministic guarantee is drawn after F failures.

Failure-Relevant Guarantee Mechanism

Definition (Deterministic Guarantee after 1 Failure)

if its transition matrix for the random walk $\{\mathbf{M}, \mathbf{T}, \mathbf{F}, \mathbf{M_F}, \mathbf{G}\}$ at any time step τ is in the form

$$
\begin{array}{c c c c c c}
 & \mathbf{M} & \mathbf{T} & \mathbf{F} & \mathbf{M_F} & \mathbf{G} \\
\begin{matrix} \mathbf{M} \\ \mathbf{T} \\ \mathbf{F} \\ \mathbf{M_F} \\ \mathbf{G} \end{matrix} &
\multicolumn{5}{c}{
\begin{bmatrix}
(1-w_S)(1-\delta_{\tau_\mathbf{G}+S}) & w_T(1-\delta_{\tau_\mathbf{G}+S}) & \Delta_\mathbf{F}\delta_{\tau_\mathbf{G}+S} & 0 & 0 \\
(1-w_S)(1-\delta_{\tau_\mathbf{G}+S}) & w_T(1-\delta_{\tau_\mathbf{G}+S}) & \Delta_\mathbf{F}\delta_{\tau_\mathbf{G}+S} & 0 & 0 \\
0 & w_S & 0 & 1-w_S & 0 \\
0 & w_S(1-\delta_{\tau_\mathbf{F}+G-S}) & 0 & (1-w_S)(1-\delta_{\tau_\mathbf{F}+G-S}) & \delta_{\tau_\mathbf{F}+G-S} \\
1-w_S & w_T & w_S-w_T & 0 & 0
\end{bmatrix}}
\end{array}
$$

where $\Delta_\mathbf{F} = \begin{cases} w_S - w_T & \tau \neq S \\ 1 & \tau = S \end{cases}$, $\tau_\mathbf{G}$ is the time step that $\mathbf{G}$ is hit and $\tau_\mathbf{F}$ is the time step that $\mathbf{F}$ is hit.

Numerical Simulations

- Draw-relevant models: Guardian Tales
- We follow the settings of pick-up weapon summon in Guardian Tales. In this case, $w_T = 0.0135$, $G = 300$, $K = 6$.

$$\mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=6} \approx 375.516$$

Numerical Simulations

- Failure-relevant models: Genshin Impact
- We follow the settings of promotional character wishes in Genshin Impact. In this case, $w_T = 0.009$, $w_T = 0.018$, $F = 1$, $S = 90$, $G = 180$, $K = 7$.

$$\mathbb{E}\mathcal{T}_{|\mathbf{T}|+|\mathbf{G}|=7} \approx 1071.491$$

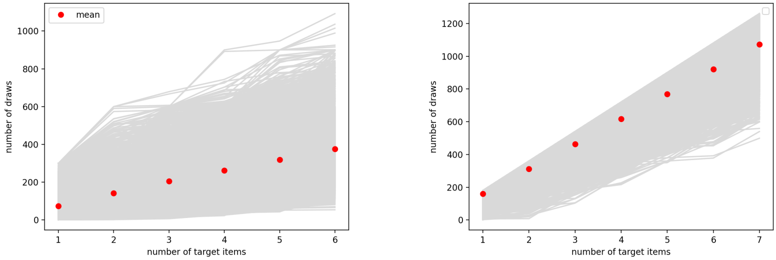


Figure: Left: for draw-relevant models; Right: for failure-relevant models.

- Failure-relevant guarantee mechanism has smaller variances.
- The upper bounds of number of draws are close.
- The mean number of draws for failure-relevant models is very large.

Limitations

- Games following draw-relevant models may promise $\mathbb{E}[T|\tau = G] = 1 + w_T$. i.e. One possibly gets two target items at that step, one for guarantee and one for random draw.
- In real world games, the draws and failures can be cumulative in different pools (long term). We only discuss the model in one pool (short term).

Future Work & Open Problems

- Theoretical results for expected number of steps to win K target items in a gacha with deterministic guarantee.
- We only consider the simple case $F = 1$ for this work. How about arbitrary K?
- Design more delicate rules to win more revenue.

Contributions

- We describe the gacha system with guarantee mechanism in a formal way.
- We perform numerical simulations for this system by applying the models to popular games like Guardian Tales and Genshin Impact.
- We found that the failure-relevant guarantee mechanism requires more draws than the draw-relevant guarantee mechanism in expectations.
- We present an open problem of value.

END
THANKS