MATH140B: Foundations of Real Analysis

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Abstract

Warning: This is only a piece of lecture notes written by a careless scribe. So just be careful with and tolerant of any possible typos or misunderstandings when you read [0.1]. The scribe does not intend to make anyone to be driven by his stupidity! Also, the professor's explanation is extremely helpful as he discusses a lot about the interpretable ideas behind the dull scripts. So watch the lecture before reading this. If you have any suggestions (e.g. typos, typography, logistics), please do not hesitate contacting the scribe!

Without specifications, the notation use is as the following

- $\mathbb{R}, \mathbb{C}, \mathbb{Q}, \ldots$: real, complex, quadratic, and so on
- $\mathcal{R}$: integrability
- $\mathbb{1}$: characteristic function
- $\boldsymbol{s}$: simple function
- $\mathcal{F}$: family
- $\mathcal{A}$: algebra

[0.1] Especially '∩' and '∪' are often mistaken because of typos.

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Lecture 1: Differentiation

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1.1 Differentiable

Definition 1.1 (*Differentiable*)

Let $f:[a,b]\to\mathbb{R}, x\in[a,b]$. Define

$$\varphi(t)=\frac{f(t)-f(x)}{t-x}$$

*for $a\le t\le b, t\ne x$. We say f is **differentiable** at x if and only if $f'(x)=\lim_{t\to x}\varphi(t)$ exists, and we denote the **derivative** of f at x by $f'(x)$.*

If f is differentiable at x, then $f'(x)=\lim_{t\to x}\frac{f(t)-f(x)}{t-x}$. Note that if φ is not defined at x and f is differentiable at x, we can define $\Phi(t)=\begin{cases}\varphi(t) & t\ne x\\ f'(x) & t=x\end{cases}$, then Φ is continuous at x.

e.g.1.

- Let $f:\mathbb{R}\to\mathbb{R}$ be $f(x)=x^2$. Compute $f'(0)$ if exists.

  Proof: By the definition, we get $\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}=\lim_{t\to 0}=\frac{t^2}{t}=0$. ■

- $f(x)=\begin{cases}x^2 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$ is differentiable at 0.

  Proof: We need to compute (or show DNE) $\lim_{t\to 0}\frac{f(t)-f(0)}{t-0}=\lim_{t\to 0}\frac{f(t)}{t}=\lim_{t\to 0}\begin{cases}\frac{t^2}{t} & t\in\mathbb{Q}\\ 0 & t\notin\mathbb{Q}\end{cases}=0.$ ■

- $f(x)=\begin{cases}x & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$ is continuous at 0 but not differentiable at 0.

  Proof: Proof of continuity is left as exercise. To show it is non-differentiable, we need to show the limit $\lim_{t\to 0}\frac{f(t)}{t}$ DNE. We want to find two sequences $t_n\to 0, s_n\to 0$ that $t_n\ne 0, s_n\ne 0$ such that $\frac{f(t_n)}{t_n}=1$ and $\frac{f(s_n)}{s_n}\to 0$. So let $t_n=\frac{1}{n}$, then $\forall n: t_n\ne 0$ and $t_n\to 0$. Then $\lim_{t\to 0}\frac{f(t_n)}{t_n}=\frac{t_n}{t_n}=1$. Let $s_n=\frac{\sqrt{2}}{n}\notin\mathbb{Q}$, then $\forall n: s_n\ne 0$ and $s_n\to 0$. Then $\lim_{t\to 0}\frac{f(s_n)}{s_n}=\frac{0}{s_n}=0$. So $\lim_{t\to 0}\frac{f(t)}{t}$ DNE. ■

1.2 Properties of Differentiable Functions

Lemma 1.1

Let f be defined on $[a,b]$ and $x \in [a,b]$. If f is differentiable at x then f is continuous at x [1.1].

Proof: We need to show that $\lim_{t\to x} f(t) = f(x)$. We have $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x)$. Note that $f(t)-f(x) = \frac{f(t)-f(x)}{t-x}\cdot(t-x)$ since $t \neq x$. Then $f(t) = f(x) + \frac{f(t)-f(x)}{t-x}\cdot(t-x)$. Now $\lim_{t\to x} f(t) = f(x)$ and $\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot(t-x) = \left(\lim_{t\to x}\frac{f(t)-f(x)}{t-x}\right)\cdot(\lim_{t\to x}(t-x)) = f'(x)\cdot 0 = 0$. Hence $\lim_{t\to x} f(t) = \lim_{t\to x} f(x) + \lim_{t\to x}\frac{f(t)-f(x)}{t-x}\cdot(t-x) = f(x)$. So f is continuous at x. ■

Remark: If f is differentiable at x, then $f(t) = f(x) + (f'(x)+E(t))(t-x)$ where $\lim_{t\to x} E(t) = 0$. Indeed, $f(t) = f(x) + \frac{f(t)-f(x)}{t-x}\cdot(t-x)$. And we write $\frac{f(t)-f(x)}{t-x} = f'(x)+E(t)$. Then since $f'(x)$ exists, $\lim_{t\to x} E(t) = 0$.

Proposition 1.2

Let $f:[a,b]\to\mathbb{R}, g:[a,b]\to\mathbb{R}$ be two functions which are differentiable at $x\in[a,b]$. Then $f+g$ and $f\cdot g$ are differentiable at x. If $g(x)\neq 0$, then $\frac{f}{g}$ is differentiable at x.

1. $(f+g)'(x) = f'(x)+g'(x)$
2. $(f\cdot g)'(x) = f'(x)g(x)+f(x)g'(x)$
3. $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$

Proof:

1. $\lim_{t\to x}\frac{f(t)+g(t)-(f(x)+g(x))}{t-x} = \lim_{t\to x}\left(\frac{f(t)-f(x)}{t-x}+\frac{g(t)-g(x)}{t-x}\right) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} + \lim_{t\to x}\frac{g(t)-g(x)}{t-x} = f'(x)+g'(x)$.
2. $\lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)}{t-x} = \lim_{t\to x}\frac{f(t)g(t)+f(t)g(x)-f(t)g(x)-f(x)g(x)}{t-x} = \lim_{t\to x}(\frac{g(t)-g(x)}{t-x} f(t) + \frac{f(t)-f(x)}{t-x}g(t))$. Now since f is differentiable at x, it is continuous at x. So $\lim_{t\to x} f(t) = f(x)$. $\lim_{t\to x} f(t)\frac{g(t)-g(x)}{t-x} = (\lim_{t\to x} f(t))\cdot\left(\lim_{t\to x}\frac{g(t)-g(x)}{t-x}\right) = f(x)\cdot g'(x)$. Moreover, $\lim_{t\to x} g(x)\frac{f(t)-f(x)}{t-x} = g(x)\cdot f'(x)$. So $\lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)}{t-x} = f(x)\cdot g'(x)+f'(x)g(x)$.
3. First note that since g is differentiable at x. It is continuous at x. Then $\exists\delta>0, \forall t\in(x-\delta,x+\delta)\cap[a,b] : g(t)\neq 0$. So we always assume $t\in(x-\delta,x+\delta)\cap[a,b]$ and hence $g(t)\neq 0$ and $\frac{f(t)}{g(t)}$ is defined. Now $\lim_{t\to x}\frac{1}{t-x}\cdot\left(\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}\right) = \lim_{t\to x}\frac{1}{t-x}\cdot\frac{f(t)g(x)-f(x)g(t)}{g(t)g(x)} = \lim_{t\to x}\frac{1}{g(t)g(x)}\cdot\frac{f(t)g(x)-f(x)g(t)}{t-x}$. We now consider $\lim_{t\to x}\frac{f(t)g(x)-f(x)g(t)}{t-x} = \lim_{t\to x}\frac{f(t)g(t)-f(x)g(x)+f(x)g(x)-f(x)g(t)}{t-x} = \lim_{t\to x}(\frac{f(t)-f(x)}{t-x} g(x) - \frac{g(t)-g(x)}{t-x}f(x)) = g(x)f'(x)-f(x)g'(x)$. Moreover, since g is continuous at x, $\lim_{t\to x} g(t)g(x) = (g(x))^2\neq 0$. Then $\lim_{t\to x}\frac{1}{t-x}\cdot\left(\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$. ■

[1.1] Note that the converse is not true: $f(x) = \begin{cases} x & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$ is continuous at 0 but not differentiable.

Proposition 1.3 (*Chain Rule*)

Let $f : [a, b] \to \mathbb{R}$ and g be defined on an interval containing the range of f. Let $x \in [a, b]$. Assume f is differentiable at x and g is differentiable at $f(x)$. Let $h(t) = g(f(x))$ for $t \in [a, b]$. Then h is differentiable at x and we have

$$h'(x) = g'(f(x)) \cdot f'(x)$$

Proof: Let $y = f(x), s = f(t)$. Now since f is differentiable at x and g is differentiable at y, we have $f(t) = f(x) + (f'(x) + E_f(t))(t - x)$ and $g(s) = g(y) + (g'(y) + E_g(s))(s - y)$ where $\lim_{t\to x} E_f(t) = 0$ and $\lim_{s\to y} E_g(t) = 0$. Now $h(t) - h(x) = g(f(t)) - g(f(x)) = g(s) - g(y) = (g'(y) + E_g(s))(s - y) = (g'(y) + F(s)) \cdot (f(t) - f(x)) = (g'(y) + E_g(s)) \cdot (f'(x) + E_f(t)) \cdot (t - x)$. Then $\lim_{t\to x} f'(x) + E_f(t) = f'(x)$ and $\lim_{t\to x} g'(y) + E_g(s) = g'(x)$. In order to compute $\lim_{t\to x} g'(y) + E_g(s)$. We first note that $y = f(x), s = f(t)$. Since f is differentiable at x, it is continuous at x. So $\lim_{t\to x} s = \lim_{t\to x} f(t) = f(x) = y$. Thus $\lim_{t\to x} g'(y) + E_g(s) = \lim_{s\to y} g'(y) + E_g(s) = g'(y)$. Altogether, $\lim_{t\to x} \frac{h(t)-h(x)}{t-x} = g'(f(x)) \cdot f'(x)$. ■

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Lecture 2: Rolle's Theorem, Mean Value Theorem

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e.g.1. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at 0 but f' is not continuous at 0. $f(x) = \begin{cases} x \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is continuous at 0 but not differentiable at 0 [2.1].

Lemma 2.1

Let $f : (a,b) \to \mathbb{R}$ be a convex and differentiable function. Then f' is increasing.

Proof: Recall that f is convex if $\forall a < s < u < b, 0 \leq \lambda \leq 1 : f(\lambda s + (1-\lambda)u) \leq \lambda f(s) + (1-\lambda)f(u)$. We showed in a homework in 140A that if f is convex, then $\forall a \leq s < t < u \leq b : \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$. Now taking $\lim_{t\to s^+}$ we get $\lim_{t\to s^+} \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. Similarly, we have $\lim_{t\to u^-} \frac{f(u)-f(t)}{u-t} \geq \frac{f(u)-f(s)}{u-s}$. Note that f is differentiable on (a,b) so $f'(s) = \lim_{t\to s} \frac{f(t)-f(s)}{t-s} = \lim_{t\to s^+} \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$. Similarly, we have $f'(u) \geq \frac{f(u)-f(s)}{u-s}$. Altogether, we have $f'(s) \leq f'(u)$. ■

2.1 Mean Value Theorem

Definition 2.1 (*Local Maximum*)

*Let f be a real valued function on a metric space X. Let $p \in X$, we say f has **local maximum** at p if $\exists \delta > 0, \forall x \in N_\delta(p) : f(p) \geq f(x)$. **Local minimum** is defined similarly.*

Remark: Given $f : [a,b] \to \mathbb{R}$, we can draw the graph of f and local maximum and local minimum have the usual picture.

Lemma 2.2

Let $f : [a,b] \to \mathbb{R}$ be a function. Let $x \in (a,b)$ ve a local maximum for f. Assume further that f is differentiable at x. Then $f'(x) = 0$. Similar statement holds for local minimum.

Proof: Since x is a local maximum, $\exists \delta > 0 : (x-\delta, x+\delta) \subset [a,b]$ and $\forall t \in (x-\delta, x+\delta) : f(t) \leq f(x)$. Now $f'(x)$ exists. So $f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. We compute this limit by taking $\lim_{t\to x^-}$ and $\lim_{t\to x^+}$. Note that we can always assume $t \in (x-\delta, x+\delta)$ since we are computing $\lim_{t\to x}$. If $x-\delta < t < x$, then $\frac{f(t)-f(x)}{t-x} \geq 0$ indicates $\lim_{t\to x^-} \frac{f(t)-f(x)}{t-x} \geq 0$; if $x < t < x+\delta$, then $\frac{f(t)-f(x)}{t-x} \leq 0$ indicates $\lim_{t\to x^+} \frac{f(t)-f(x)}{t-x} \leq 0$. Then $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x) = 0$.

[2.1] Although we have not yet defined trigonometric functions in a rigorous way, we know what it is.

∎

e.g.2. $f(x) = |x|$ has local minimum at 0 but not differentiable at 0. Indeed, $\lim_{t\to 0^-} \frac{f(t)-f(x)}{t-x} = -1$ and $\lim_{t\to 0^+} \frac{f(t)-f(x)}{t-x} = 1$.

Theorem 2.3 (*Rolle's Theorem*)

Let f be continuous on $[a,b]$ and differentiable on (a,b). Suppose that $f(a) = f(b)$. Then $\exists c \in (a,b) : f'(c) = 0$.

Proof: Recall that $[a,b]$ is compact. Since f is continuous, f has (absolute) maximum and (absolute) minimum on $[a,b]$. i.e. $\exists t, s \in [a,b], \forall x \in [a,b] : f(s) \le f(x) \le f(t)$.

Case1: at least one of s and t is an interior point i.e. belongs to (a,b). Let's say $t \in (a,b)$. Then by previous lemma, since $\forall x \in [a,b], t \in (a,b) : f(t) \ge f(x)$, f is differentiable at t. We conclude that $f'(t) = 0$ so $c = t$ solves this. Similarly if $s \in (a,b)$, the previous implies $f'(s) = 0$ so $c = s$.

Case2: Both t and s are end points. In this case, since $f(a) = f(b)$, f is constant. So $\forall c \in (a,b) : f'(c) = 0$. ∎

Corollary 2.4 (*Mean Value Theorem*)

Let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Then $\exists c \in (a,b)$:

$$f(b) - f(a) = f'(c)(b-a)$$

Proof: Let L be a line through $(a, f(a))$ and $(b, f(b))$. Then the equation of L can be written as $y(x) - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$ or $y(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$. Define $h(x) = f(x) - y(x)$. Then h is continuous on $[a,b]$; differentiable on (a,b); $h(a) = f(a) - y(a) = 0$; $h(b) = f(b) - y(b) = 0$. So h satisfies conditions of Rolle's Theorem. Then $\exists c \in (a,b) : h'(c) = 0$. Note that $h'(x) = f'(x) - y'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$. $\exists c \in (a,b) : h'(c) = 0$. So $f'(c) = \frac{f(b)-f(a)}{b-a}$. ∎

Corollary 2.5 (*Generalization of Mean Value Theorem*)

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). Then $\exists c \in (a,b)$: [2.2]

$$(f(b) - f(a))\, g'(c) = (g(b) - g(a))\, f'(c)$$

Proof: Define $h(x) = (f(b) - f(a))g(x) - (g(b) - f(a))f(x)$. To see this explicitly, we want to find λ and μ such that $\lambda f(a) + \mu g(a) = 0$ and $\lambda f(b) + \mu g(b) = 0$. Subtracting these two equations, we have $\lambda(f(a) - f(b)) + \mu(g(a) - g(b)) = 0$. Then $\lambda = g(a) - g(b), \mu = -(f(a) - f(b))$. Now since h is continuous on $[a,b]$ and differentiable on (a,b), $h(a) = 0$ and $h(b) = 0$. Therefore by Rolle's Theorem, $\exists c \in (a,b) : h'(c) = 0$. Note that $h'(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$. Hence $h'(c) = 0$ indicates $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$. ∎

[2.2] If $g(x) = x$, then we get the MVT Theorem.

Corollary 2.6

Suppose f is differentiable on (a, b)

1. *If $\forall x \in (a, b) : f'(x) \geq 0$, then f is increasing.*
2. *If $\forall x \in (a, b) : f'(x) \leq 0$, then f is decreasing.*
3. *If $\forall x \in (a, b) : f'(x) = 0$, then f is constant.*

Proof: Let $a < s < t < b$ then $\exists c \in (s, t) : f(t) - f(s) = f'(c)(t - s)$. Now if $\forall x \in (a, b) : f'(x) \geq 0$ holds, then this implies that $f'(c) \geq 0$. Then $f(t) \geq f(s)$. Since t and s are arbitrary, (1) holds true. The proof of (2) and (3) are similar. ■

e.g.3. Let y be twice differentiable on $\mathbb{R}$. Suppose $y = -y'', y(0) = 0, y'(0) = 0$. Prove that y is the constant function 0 [2.3].

[2.3] Hint: consider $y + y'' = 0$, then $2y'y + 2y'y'' = 0$. Note that $\left(y^2 + (y')^2\right)' = 2yy' + 2y'y''$.

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Lecture 3: Intermediate Value Theorem, L'Hospital's Rule, Higher Order Derivatives

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e.g.1. $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. $f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. But $f'(x)$ is not continuous at 0.

e.g.2. $h(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. Does there exist some f such that $f'(x) = h(x)$? [3.1]

3.1 Intermediate Value Theorem

Theorem 3.1 (*Intermediate Value Property for f'* [3.2])

Let f be differentiable on $[a, b]$. Suppose $f'(a) < c < f'(b)$ (or $f'(a) > c > f'(b)$). Then $\exists x \in (a, b) : f'(x) = c$.

Proof: Note that $f'(x) = c \implies f'(x) - c = 0$. So if we let $h(t) = f(t) - ct$ for $t \in [a, b]$. Then we are looking for some x such that $h'(x) = 0$. We will show this holds by showing that h has a local minimum in (a, b). Since h is differentiable on $[a, b]$, h is continuous. Then h has absolute maximum and minimum on $[a, b]$. We claim that the minimum cannot happen at a or b. Note $h'(a) = f'(a) - c < 0$ and $h'(b) = f'(b) - c > 0$. The first equality implies $\lim_{t \to a} \frac{h(t)-h(a)}{t-a} < 0$ for $t \in (a, b]$. So $t - a > 0$ since the limit is positive and $h(t) - h(a) < 0$ for t close to a. Thus the minimum is not at a. Similarly, the minimum is not at b. So $\exists x \in (a, b), \forall t \in [a, b] : h(x) \leq h(t)$. Then by a lemma, $h(x) = 0$. ■

Corollary 3.2

- *If f' is discontinuous at x, then it is a second-kind discontinuity.*
- *If f' is increasing, then it is continuous.*

Proof: The first part follows from the previous theorem. The second claim follows from the first and the fact that discontinuity of monotonic function is of the first kind. ■

[3.1] We will show the answer is NO.

[3.2] *By the scribe*: This is also known as **Darboux's Theorem**.

3.2 L'Hospital's Rule

Theorem 3.3 (*L'Hospital's Rule*)

Let f and g be differentiable on (a,b). Assume $\forall x \in (a,b) : g'(x) \neq 0$. Suppose:

1. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$
2. *Either* $\lim_{x\to a} f = \lim_{x\to a} g = 0$ *or* $\lim_{x\to a} g = \infty$.

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = L$.

Proof: Note that $\exists \delta_0 > 0, \forall x \in (a, a+\delta_0) : g(x) \neq 0$. Indeed, if $\exists a < t < s < b : g(t) = g(s) = 0$, then by MVT, $\exists t < x < s : g'(x) = 0$, contradiction. So g has at most one zero in (a,b). Then $\exists \delta_0, \forall x \in (a, a+\delta_0) \cup (b-\delta_0, b) : g(x) \neq 0$. So replacing (a,b) by $(a, a+\delta_0)$, we assume $g(x) \neq 0$ on (a,b).

Case1: $L \in \mathbb{R}$. We need to show $\forall \epsilon > 0, \exists \delta > 0 : x \in (a, a+\delta) \implies \left|\frac{f(x)}{g(x)} - L\right| < \epsilon$. Since $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$, we have $\exists \delta_1, \forall x \in (a, a+\delta_1) : \left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\epsilon}{2}$. Fix $x \in (a, a+\delta_1)$ and let $a < t < x < a+\delta_1$. Then by MVT, $\frac{f(x)-f(a)}{g(x)-g(a)} = \frac{f'(c_t)}{g'(c_t)}$ for $t < c_t < x$. Assume $\lim_{t\to a} f(t) = \lim_{t\to a} g(t) = 0$. Then $\left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right| = \left|\frac{f'(c_t)}{g'(c_t)} - L\right| < \frac{\epsilon}{2}$. Take limit as $t \to a$. Hence we get $\left|\frac{f(x)}{g(x)} - L\right| \leq \frac{\epsilon}{2}$. Since $x \in (a, a+\delta_1)$ is arbitrary, we get $\left|\frac{f(x)}{g(x)} - L\right| < \epsilon$ in the case $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Now we consider $\lim_{t\to\infty} g(t) = \infty$. Then $\left|\frac{f(x)-f(t)}{g(x)-g(t)} - L\right| = \left|\frac{f'(c_t)}{g'(c_t)} - L\right| < \frac{\epsilon}{2}$ for $a < t < c_t < x < a+\delta_1$. Thus $L - \frac{\epsilon}{2} < \frac{f(t)-f(x)}{g(t)-g(x)} < L + \frac{\epsilon}{2}$. Since $\lim_{t\to\infty} g(t) = \infty$ and $x \in (a, a+\delta_1)$ is fixed, $\exists \delta_2 : a < t < a+\delta_2 < x < a+\delta_1 \implies g(t) > 0, g(t) - g(x) > 0$. Multiplying $L - \frac{\epsilon}{2} < \frac{f(t)-f(x)}{g(t)-g(x)} < L + \frac{\epsilon}{2}$ by $(g(t) - g(x))$, we have $(L - \frac{\epsilon}{2})(g(t) - g(x)) < f(t) - f(x) < (L + \frac{\epsilon}{2})(g(t) - g(x))$. Adding $f(x)$ to the inequality and dividing by $g(t)$ give $(L - \frac{\epsilon}{2})\frac{g(t)-g(x)}{g(t)} + \frac{f(x)}{g(t)} < \frac{f(t)}{g(t)} < (L + \frac{\epsilon}{2})\frac{g(t)-g(x)}{g(t)} + \frac{f(x)}{g(t)}$. Since x is fixed, $\exists \delta_3 < \delta_2 : a < t < a+\delta_3 < a+\delta_2 < x < a+\delta_1 \implies \left|\frac{f(x)}{g(t)}\right| < \frac{\epsilon}{10}, \left|\frac{g(x)}{g(t)}\right| < \frac{\epsilon}{10}$. Then we have $L - \epsilon < \frac{f(t)}{g(t)} < L + \epsilon$. So if $t \in (a, a+\delta_3)$, then $\left|\frac{f(t)}{g(t)} - L\right| < \epsilon$. So $\lim_{t\to a} \frac{f(t)}{g(t)} = L$ in this case as well.

Case2: $L = \pm\infty$. The proof is similar and left as exercise. ■

3.3 Higher Order Derivatives

Definition 3.1

Let f be differentiable on $[a,b]$. $\forall x \in [a,b] : f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$. So f' is a function on $[a,b]$. If f' is differentiable on $[a,b]$, then $(f')'$ will be denoted by f''. Continuing inductively, we define $f, f'', f''', f^{(n)}$ if they exist [3.3].

[3.3] Note that in order for $f^{(n)}$ to exist at $x \in [a,b]$. The $(n-1)$-derivative should exist on an interval around.

Definition 3.2 (*Taylor's Polynomial*)

Suppose f is defined on $[a,b]$. Let $c \in [a,b]$ and suppose f is n-time differentiable at c. Then we define

$$P_{n,c}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

$P_{n,c}$ is the called n-th (degree) ***Taylor's polynomial*** *at c.*

Remark: Without further restrictions, $P_{n,c}(x)$ gives only information about c. Let $f(x) = \begin{cases} e^{-\frac{1}{x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $\forall n : f^{(n)} = 0$ so $P_{n,0}(x) = 0$.

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Lecture 4: Taylor's Theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 4.1

Let f be n-times differentiable at c. Then

1. $f^{(k)}(c) = P_{n,c}^{(k)}(c)$ *for* $0 \le k \le n$
2. $P_{n,c}^{(k)}(t) = 0$ *for* $k > n$
3. $\lim_{t\to c} \frac{f(t)-P_{n,c}(t)}{(t-c)^k/k!} = 0$ *for* $0 \le k \le n$

Proof:

1. follows from the definition. Indeed, $P_{n,c}(t) = f(c) + f'(c)(t-c) + \cdots + \frac{f^{(k)}(c)}{k!}(t-c)^k + \cdots + \frac{f^{(n)}(c)}{n!}(t-c)^n$. Then $P_{n,c}^{(k)}(t) = f^{(k)}(c) + (k+1)\times\cdots\times 2\times\frac{f^{(k+1)}(c)}{(k+1)!}(t-c) + \cdots + n(n-1)\dots(n-(k-1))\frac{f^{(n)}(c)}{n!}(t-c)^{n-k}$. So if we evaluate this at c, we get $P_{n,c}^{(k)}(c) = f^{(k)}(c)$.
2. is clear since $P_{n,c}$ has degree at most n for every $0 \le k \le n$.
3. It suffices to prove this for $k = n$. We want to compute $\lim_{t\to c}\frac{f(t)-P_{n,c}(t)}{(t-c)^n/n!}$. For $n=1$, $\lim_{t\to c}\frac{f(t)-P_{1,c}(t)}{t-c} = \lim_{t\to c}\frac{f(t)-(f(c)+f'(c)(t-c))}{t-c} = \lim_{t\to c}\left(\frac{f(t)-f(c)}{t-c} - f'(c)\right)$. Note that the conditions of L'Hospital Rule are satisfied $(n-1)$-times. So by the conclusion of L'Hospital, we compute: $\lim_{t\to c}\frac{f^{(n-1)}(t)-(f^{(n-1)}(c)+f^{(n)}(c)(t-c))}{t-c} = \lim_{t\to c}(\frac{f^{(n-1)}(t)-f^{(n-1)}(c)}{t-c} - f^{(n)}(c)) = 0$. Thus, by L'Hospital, $\lim_{t\to c}\frac{f(t)-P_{1,c}(t)}{t-c} = 0$. ■

Corollary 4.2

Suppose f is n-times differentiable at c and $f'(c) = \cdots = f^{(n-1)}(c) = 0$

1. *Suppose n is even, if $f^{(n)}(c) > 0$, then c is the local minimum; if $f^{(n)}(c) < 0$, then c is the local maximum.*
2. *Suppose n is odd, $f^{(n)}(c) \ne 0$. Then c is not local maximum or minimum.*

Proof: (1) Suppose n is even, $P_{n,c}(t) = f(c) + \frac{f^{(n)}(c)}{n!}(t-c)^n$. Now by previous lemma, $\lim_{t\to c}\frac{f(t)-P_{n,c}(t)}{(t-c)^n/n!} = 0$. Suppose $f^{(n)}(c) > 0$, then $0 = \lim_{t\to c}\frac{f(t)-f(c)-\frac{f^{(n)}}{n!}(t-c)^n}{(t-c)^n/n!} = \lim_{t\to c}\left(\frac{f(t)-f(c)}{(t-c)^n/n!} - f^{(n)}(c)\right)$. So we get $\lim_{t\to c}\frac{f(t)-f(c)}{(t-c)^n/n!} = f^{(n)}(c)$. Since $f^{(n)} > 0$, $\frac{f(t)-f(c)}{(t-c)^n/n!} > 0$. If t is close to c, n is even so $\frac{(t-c)^n}{n!} > 0$. Altogether we get $\exists\delta, \forall t \in (c-\delta, c+\delta) : f(t) - f(c) \ge 0$, so c is the local minimum.

The proof for $f^{(n)}(c) < 0$ and n is even and part (2) are similar. ■

Theorem 4.3 (*Taylor's Theorem*)

Suppose f is real-valued on $[a,b]$. Assume that $f^{(n-1)}$ is continuous on $[a,b]$ and that $f^{(n)}$ exists on (a,b). Let $c,d \in [a,b], c \neq d$. Then $\exists t \in (c,d)$: [4.1]

$$f(d) = P_{n-1,c}(d) + \frac{f^{(n)}(t)}{n!}(d-c)^n$$

Proof: Recall the proof of MVT: $g(x) = f(x) - (f(c) + \frac{f(d)-f(c)}{d-c}(x-c))$, $g(c) = g(d) = 0$ with Rolle's Theorem implies $\exists t \in (c,d) : g'(t) = 0$.

Use a similar strategy. Define $h(x) = P_{n-1,c}(x) + M(x-c)^n$ where M is chosen so that $h(d) = f(d)$. i.e. we want to solve $f(d) = P_{n-1,c}(d) + M(d-c)^n$ for M. Let $g(x) = f(x) - h(x) = f(x) - P_{n-1,c}(x) - M(x-c)^n$. Then $g(c) = f(c) - P_{n-1,c}(c) - M(c-c)^n = 0$. Similarly, using the lemma 4.1, we have $g(c) = 0, g'(c) = 0, \ldots, g^{(n-1)}(c) = 0$. $g(d) = f(d) - h(d) = 0$. Now g is continuous on $[a,b]$ and differentiable on (a,b). So by MVT, $\exists t_1 \in (c,d) : g'(t_1) = 0$. Repeat this using the fact that $f^{(n-1)}$ is continuous on $[a,b]$ and $f^{(n)}$ exists on (a,b). We get $t_2 \in (c,t_1), t_3 \in (c,t_2), \ldots, t_{n-1} \in (c,t_{n-2})$ such that $g^{(n-1)}(t_{n-1}) = 0$. We now have $g^{(n-1)}(c) = 0, g^{(n-1)}(t_{n-1}) = 0$. Then $g^{(n-1)}$ is continuous on $[a,b]$ and differentiable on (a,b). So $\exists t \in (c,t_{n-1}) : g^{(n)}(t) = 0$. Recall that $g(x) = f(x) - P_{n-1,c}(x) - M(x-c)^n$, then $g^{(n)}(x) = f^{(n)}(x) - 0 - n!M$ since $P_{n-1,c}$ is a polynomial of degree $k \leq n-1$. Hence $0 = g^{(n)}(t) = f^{(n)}(t) - n!M$ implies $M = \frac{f^{(n)}(t)}{n!}$. ■

[4.1] Note that $P_{n,c}(d) = P_{n-1,c}(d) + \frac{f^{(n)}(c)}{n!}(d-c)^n$.

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Lecture 5: Riemann Integrability

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Suppose $f : [a, b] \to \mathbb{R}^n$ is a function. Then $f(x) = (f_1(x), \ldots, f_n(x))$ where $f_j : [a, b] \to \mathbb{R}$ for $1 \le j \le n$. Then we say f is differentiable at x if $\lim_{t\to x} \frac{f(t)-f(x)}{t-x}$ exists. Note that f is differentiable at x if and only if $\forall j : f_j$ is differentiable at x. Indeed,

$$\frac{f(t) - f(x)}{t - x} = \left(\frac{f_1(t) - f_1(x)}{t - x}, \ldots, \frac{f_n(t) - f_n(x)}{t - x}\right)$$

If f is differentiable at x, then $f'(x) = (f_1'(x), \ldots, f_n'(x))$.

Lemma 5.1

Let $f : [a, b] \to \mathbb{R}^n, g : [a, b] \to \mathbb{R}^n$ be differentiable. Then

1. $(f + g)'(x) = f'(x) + g'(x)$
2. $(f \cdot g)'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

Proof: Left as exercise. ■

e.g.1.

- $f(t) = (t, t^2, t^3)$, then $f'(t) = (1, 2t, 3t^2)$.
- $f : [0, 2\pi] \to \mathbb{R}^2, f(x) = (\cos x, \sin x)$, then $f'(x) = (-\sin x, \cos x)$ and $\|f'(x)\| = 1$. Note however that MVT may fail: $\nexists t : (0, 0) = f(2\pi) - f(0) = 2\pi f'(t)$.

Proposition 5.2

Suppose $f : [a, b] \to \mathbb{R}^n$ is continuous and differentiable on (a, b). Then $\exists x \in (a, b)$: [5.1]

$$\|f(b) - f(a)\| \le (b - a) \|f'(x)\|$$

Proof: Let $v \in \mathbb{R}^n$. Define $f_v : [a, b] \to \mathbb{R}, f_v(t) = v \cdot f(t)$. So for every v the function f_v satisfies conditions of MVT. Then $\forall v, \exists x_v \in (a, b) : f_v(b) - f_v(a) = (b - a) \cdot f_v'(x_v)$. Thus, $v \cdot f(b) - v \cdot f(a) = (b - a)(v \cdot f'(x_v))$ implies $v \cdot (f(b) - f(a)) = (b - a)(v \cdot f'(x_v))$. Suppose now that $v = f(b) - f(a)$. Then by MVT, $\exists x \in (a, b) : (f(b) - f(a)) \cdot (f(b) - f(a)) = (b - a)(f(b) - f(a)) \cdot f'(x)$ so $\|f(b) - f(a)\|^2 = (b - a)((f(b) - f(a)) \cdot f'(x)) \le (b - a)\|f(b) - f(a)\|\|f'(x)\|$ by applying Cauchy-Schwartz Inequality. Now if $\|f(b) - f(a)\| = 0$, then the lemma is obvious; otherwise if $\|f(b) - f(a)\| \ne 0$, then $\|f(b) - f(a)\| \le (b - a)\|f'(x)\|$. ■

Remark: L'Hospital's Rule for vector-valued functions also fails [5.2].

[5.1] Recall $\|(v_1, \ldots, v_n)\| = \sqrt{\sum v_i^2}$.
[5.2] See the example in the book.

5.1 Riemann Integrable

Definition 5.1 (*Partition*)

Let $[a, b]$ be an interval. A ***partition*** *of $[a, b]$ is a finite set of points $\{x_0 = a, \ldots, x_n = b : x_0 \le \cdots \le x_n\}$.*

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, we write

$$\Delta x_i = x_i - x_{i-1}$$

for $1 \le i \le n$. Let f be a bounded function on $[a, b]$ and $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$. Define

$$\begin{aligned} M_i &= \sup f(x) \quad x_{i-1} \le x \le x_i \\ m_i &= \inf f(x) \quad x_{i-1} \le x \le x_i \end{aligned}$$

Since f is bounded, M_i, m_i exist as real numbers.

e.g.2.

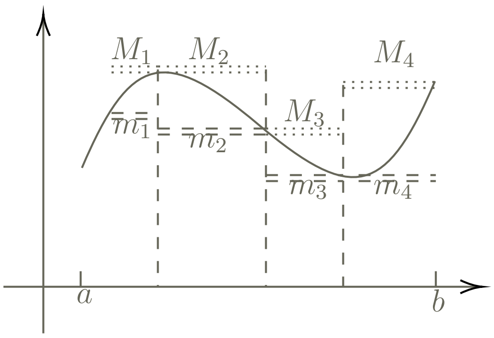

Figure 5.1: Illustrations of M_i, m_i

Definition 5.2 (*Upper Sum, Lower Sum*)

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$

Definition 5.3 (*Upper Integral, Lower Integral*)

$$\overline{\int_a^b} f \mathrm{d}x = \inf U(P, f) \qquad \underline{\int_a^b} f \mathrm{d}x = \sup L(P, f)$$

Definition 5.4 (*Riemann Integrable*)

We say f is ***Riemann integrable*** *on $[a, b]$ if*

$$\overline{\int_a^b} f \mathrm{d}x = \underline{\int_a^b} f \mathrm{d}x$$

If so, we write $f \in \mathcal{R}[a, b]$ or simply $f \in \mathcal{R}$.

Remark: Suppose $m \le f \le M$ for f is bounded. Then $m \le m_i \le M_i \le M$. So $U(P,f) = \sum_{i=1}^n M_i \Delta x_i \le \sum_{i=1}^n M\Delta x_i = M\sum_{i=1}^n \Delta x_i = M(b-a)$. Similarly, $L(P,f) \ge m(b-a)$. So we get

$$\overline{\int} f = \inf U(P,f) \le M(b-a)$$

$$\underline{\int} f = \sup L(P,f) \ge m(b-a)$$

e.g.3.

- $f(x) = c$ on $[a,b]$. Then $f \in \mathcal{R}$.

  Proof: Let $P = \{x_0, \ldots, x_n\}$ be any partition. Then $M_i = \sup f(x) = c$ and $m_i = \inf f(x) = c$. This implies $U(P,f) = \sum_{i=1}^n M_i\Delta x_i = c(b-a)$ and $L(P,f) = \sum_{i=1}^n m_i \Delta x_i = c(b-a)$. Then $\overline{\int_a^b} f = c(b-a) = \underline{\int_a^b} f$. ■

- Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ on $[0,1]$. Then f is not Riemann integrable.

  Proof: Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$. Then $U(P,f) = \sum_{i=1}^n M_i \Delta x_i$ and $L(P,f) = \sum_{i=1}^n m_i\Delta x_i = c(b-a)$. We can assume $x_0 = 0 < x_1 < x_2 \le \cdots \le x_n = 1$. Since $[x_{i-1}, x_i]$ contains more than one point, $\exists x \in [x_{i-1}, x_i] \cap \mathbb{Q}$ so $M_i = \sup f(x) = 1$ for $x \in [x_{i-1}, x_i]$. Similarly, $\exists x \in [x_{i-1}, x_i] \setminus \mathbb{Q}$ so $m_i = \inf f(x) = 0$. Therefore, $U(P,f) = \sum_{i=1}^n M_i \Delta_i = \sum_{i=1}^n \Delta x_i = 1$ and $L(P,f) = \sum_{i=1}^n m_i \Delta x_i = 0$. Then $\overline{\int} f = 1$ and $\underline{\int} f = 0$. Thus $f \notin \mathcal{R}$. ■

- Let $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \ne \frac{1}{2} \end{cases}$ on $[0,1]$. Then $f \in \mathcal{R}$.

  Proof: Note that $0 \le f \le 1$. Let P be any partition. $L(P,f) = \sum_{i=1}^n m_i \Delta_i = 0$ so $\underline{\int_0^1} f \mathrm{d}x = 0$. Note that $U(P,f) \ge 0$. We now show that $\forall \epsilon > 0, \exists P : U(P,f) < \epsilon$. Then $\inf U(P,f) = 0$ so $\overline{\int_0^1} f \mathrm{d}x = 0$. Now given $\epsilon > 0$, define $P_\epsilon = \{x_0 = 0, x_1 = \frac{1}{2} - \frac{\epsilon}{4}, x_2 = \frac{1}{2} + \frac{\epsilon}{4}, x_3 = 1\}$. Then $U(P_\epsilon, f) = M_1\Delta x_1 + M_2 \Delta x_2 + M_3 \Delta x_3$. $M_1 = \sup f = 0$ for $x_0 \le x \le x_1$, $M_3 = \sup f = 0$ for $x_2 \le x \le x_3$, and $M_2 = \sup f = 1$ for $x_1 \le x \le x_2$. Then $U(P_\epsilon, f) = M_2 \Delta x_2 = 1(x_2 - x_1) = \frac{\epsilon}{2} < \epsilon$. This implies $\overline{\int_0^1} f \mathrm{d}x = 0$ so $f \in \mathcal{R}$. ■

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Lecture 6: Riemann-Stieltjes Integrability

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

6.1 Riemann-Stieltjes Integrable

Let f be a bounded (real-valued) function on $[a,b]$. Let α be an increasing function on $[a,b]$. Given a partition $P = \{x_0 = a, x_1, \ldots, x_n = b\}$, we define

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

and with

$$\begin{aligned} M_i &= \sup f(x) \quad x_{i-1} \le x \le x_i \\ m_i &= \inf f(x) \quad x_{i-1} \le x \le x_i \end{aligned}$$

We update the notion of **upper sum** and **lower sum** by

$$U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

and define

$$\overline{\int_a^b} f \mathrm{d}\alpha = \inf U(P,f,\alpha) \qquad \underline{\int_a^b} f \mathrm{d}\alpha = \sup L(P,f,\alpha)$$

Definition 6.1 (*Riemann-Stieltjes Integrable*)

*We say f is **Riemann-Stieltjes integrable** w.r.t. α and write $f \in \mathcal{R}(\alpha)$ if*

$$\overline{\int_a^b} f \mathrm{d}\alpha = \underline{\int_a^b} f \mathrm{d}\alpha$$

If so, we define

$$\int_a^b f \mathrm{d}\alpha = \overline{\int_a^b} f \mathrm{d}\alpha = \underline{\int_a^b} f \mathrm{d}\alpha$$

e.g.1.

- Let f be a constant function and α be arbitrary. Then $U(P,f) = \sum_{i=1}^n M_i\Delta\alpha_i = \sum_{i=1}^n c\Delta\alpha_i = c\sum_{i=1}^n(\alpha(x_i) - \alpha(x_{i-1})) = c(\alpha(b) - \alpha(a))$, $L(P,f) = \sum_{i=1}^n m_i\Delta\alpha_i = c(\alpha(b) - \alpha(a))$. Therefore, $\overline{\int} f\mathrm{d}\alpha = \underline{\int} f\mathrm{d}\alpha = c(\alpha(b) - \alpha(a))$. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f\mathrm{d}\alpha = c(\alpha(b) - \alpha(a))$.
- Let f be arbitrary and α be constant. Then $\Delta x_i = 0$. Then $\forall i : \overline{\int} f\mathrm{d}\alpha = \overline{\int} f\mathrm{d}\alpha = 0$. So $f \in \mathcal{R}(\alpha)$.

Definition 6.2 (*Refinement*)

*Let P be a partition of $[a,b]$. We say P^* is a **refinement** of P if*

$$P \subset P^*$$

If P_1 and P_2 are two partitions of $[a,b]$, then their common refinement is defined to be $P_1 \cup P_2$.

Lemma 6.1

Let P^*, P be two partitions where $P^* \supset P$. Then

1. $U(P^*, f, \alpha) \leq U(P, f, \alpha)$
2. $L(P^*, f, \alpha) \geq L(P, f, \alpha)$

Proof: We prove the lemma for upper sums; the proof for the lower sums is similar.

We want to prove this by induction. First assume $P^* = P \cup \{y\}$. i.e. P^* has one more point than P. Let $P = \{x_0, \ldots, x_n\}$. $U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i$. Since $P^* = P \cup \{y\}$ where $x_{j-1} \leq y \leq x_j$ for some j. $U(P^*, f, \alpha) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + \cdots + A(\alpha(y) - \alpha(x_{j-1})) + B(\alpha(x_j) - \alpha(y)) + M_{j+1}\Delta\alpha_i + \cdots + M_n\Delta\alpha_n$ where $A = \sup f(x)$ for $x_{j-1} \leq x \leq y$, $B = \sup f(x)$ for $y \leq x \leq x_j$. $U(P, f, \alpha) = \sum_{i=1}^n M_i\Delta\alpha_i = M_1\Delta\alpha_1 + \cdots + M_{j-1}\Delta\alpha_{j-1} + M_j(\alpha(x_j) - \alpha(x_{j-1})) + \cdots + M_n\Delta\alpha_n$. To show $U(P^*, f, \alpha) \leq U(P, f, \alpha)$ is equivalent to prove $A(\alpha(y) - \alpha(x_{j-1})) + B(\alpha(x_j) - \alpha(y)) \leq M_j(\alpha(x_j) - \alpha(x_{j-1})) = M_j(\alpha(y) - \alpha(x_{j-1})) + M_j(\alpha(x_j) - \alpha(y))$. Indeed, this is implied by that $[x_{j-1}, y] \subset [x_{j-1}, x_j]$ and $[y, x_j] \subset [x_{j-1}, x_j]$. To complete the proof for an arbitrary $P^* \supset P$, we just repeat the above process ℓ-times where ℓ is the difference between the numbers of points of P^* and of P. ■

Proposition 6.2

Let f be bounded on $[a,b]$ and α be increasing. Let P_1 and P_2 be two partitions of $[a,b]$, then

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

Proof: Let P_1 and P_2 be two partitions. Let P^* be the common refinement. Then by the previous lemma, $U(P^*, f, \alpha) \leq U(P_2, f, \alpha)$ and $L(P_1, f, \alpha) \leq L(P^*, f, \alpha)$. Since $L(P^*, f, \alpha) \leq U(P^*, f, \alpha)$, we get that $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$. ■

Theorem 6.3

$$\underline{\int_a^b} f \mathrm{d}\alpha \leq \overline{\int_a^b} f \mathrm{d}\alpha$$

Proof: Let P be an arbitrary partition and fix a partition P'. Then $L(P, f, \alpha) \leq U(P', f, \alpha)$. Taking sup over P gives $\underline{\int_a^b} f \mathrm{d}\alpha \leq U(P', f, \alpha)$. Then taking inf over P' gives $\underline{\int_a^b} f \mathrm{d}\alpha \leq \overline{\int_a^b} f \, \mathrm{d}\alpha$. ■

Proposition 6.4

Let f be bounded on $[a,b]$ and α be increasing. Then $f \in \mathcal{R}(\alpha)$ if and only if $\forall \epsilon > 0, \exists P : U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$.

Proof:

($\Leftarrow$) Suppose $\forall \epsilon > 0, \exists P : U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. We want to show that $\underline{\int_a^b} \mathrm{d}\alpha = \overline{\int_a^b} \mathrm{d}\alpha$. Now $L(P, f, \alpha) \leq \underline{\int_a^b} \mathrm{d}\alpha \leq \overline{\int_a^b} \mathrm{d}\alpha \leq U(P, f, \alpha)$. Since P is so that $U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$, $0 \leq \overline{\int_a^b} f \mathrm{d}\alpha - \underline{\int_a^b} f \mathrm{d}\alpha < \epsilon$. As our choice of ϵ is arbitrary, $\overline{\int} f \mathrm{d}\alpha = \underline{\int} f \mathrm{d}\alpha$.

($\Rightarrow$) Suppose $f \in \mathcal{R}(\alpha)$ and $\epsilon > 0$. We want to show $\exists P : U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. Since $f \in \mathcal{R}(\alpha)$, we have $\sup_P L(P, f, \alpha) = \underline{\int} f \mathrm{d}\alpha = \overline{\int} f \mathrm{d}\alpha = \inf_{P'} U(P', f, \alpha)$. By definition of sup and inf, $\exists P, P' : 0 \leq U(P', f, \alpha) - \overline{\int} f \mathrm{d}\alpha < \frac{\epsilon}{2}, 0 \leq \underline{\int} f \, \mathrm{d}\alpha - L(P, f, \alpha) < \frac{\epsilon}{2}$. Let P^* be the common refinement of P and P'. Then by a lemma, $L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P', f, \alpha)$. So $U(P^*, f, \alpha) \leq U(P', f, \alpha) \leq \int_a^b f \mathrm{d}\alpha + \frac{\epsilon}{2} \leq L(P, f, \alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \leq L(P^*, f, \alpha) + \epsilon$ gives $0 \leq U(P, f, \alpha) - L(P, f, \alpha) < \epsilon$. This claim holds with $P_\epsilon = P^*$. ∎

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Lecture 7: Integrability and Monotonicity

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proposition 7.1

Let f be bounded on $[a,b]$ and α be increasing. Let P be a partition of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ for some $\epsilon > 0$.

1. *If $P' \supset P$, then $U(P',f,\alpha) - L(P',f,\alpha) < \epsilon$.*
2. *Suppose P as above and write $P = \{x_0, \dots, x_n\}$. Let $r_i, s_i \in [x_{i-1}, x_i]$. Then $\left|\sum_{i=1}^n f(s_i)\Delta\alpha_i - \sum_{i=1}^n f(r_i)\Delta\alpha_i\right| \le \sum_{i=1}^n |f(s_i) - f(r_i)|\,\Delta\alpha_i < \epsilon$.*
3. *Suppose $f \in \mathcal{R}(\alpha)$ and P is as above. Let $s_i \in [x_{i-1}, x_i]$ be arbitrary. Then $\left|\int_a^b f \mathrm{d}\alpha - \sum_{i=1}^n f(s_i)\Delta\alpha_i\right| < \epsilon$.*

Proof: Left as exercise. ■

e.g.1. Let $a < c < b$ and define $\alpha(x) = \begin{cases} 0 & a \le x \le c \\ 1 & c < x \le b \end{cases}$. Let f be continuous at c. Then $f \in \mathcal{R}(\alpha)$. Moreover, $\int_a^b f \mathrm{d}\alpha = f(c)$.

Proof: We will use ϵ-P condition for integrability. Let $P = \{x_0 = a, x_1 = c, x_2, x_3 = b\}$. Then $U(P,f,\alpha) = M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3$. Since $\Delta\alpha_1 = \Delta\alpha_3 = 0$ and $\Delta\alpha_2 = \alpha(x_2) - \alpha(x_1) = 1$, $U(P,f,\alpha) = M_2$. Similarly, $L(P,f,\alpha) = m_2$. Recall that $M_2 = \sup f(x)$ for $x_1 = c \le x \le x_2$ and $m_2 = \inf f(x)$ for $x_1 = c \le x \le x_2$. Since f is continuous at c, $\exists\delta > 0 : |x - c| < \delta \implies |f(x) - f(c)| < \frac{\delta}{4}$. Let $x_2 = c + \frac{\delta}{2}$. Then $\forall t, s \in [c, x_2] : |t - c| < \delta, |s - c| < \delta \implies |f(t) - f(s)| \le \frac{\epsilon}{2}$ so $|M_2 - m_2| \le \frac{\epsilon}{2}$. Altogether, we have $U(P,f,\alpha) - L(P,f,\alpha) \le \frac{\epsilon}{2} < \epsilon$. So $f \in \mathcal{R}(\alpha)$. Note that $\forall P' : L(P',f,\alpha) \le \int_a^b f \mathrm{d}\alpha \le U(P',f,\alpha)$. Now let P be as above. Then $m_2 \le \int_a^b f \mathrm{d}\alpha \le M_2$. Now as $x_2 \to c$, we have $M_2 \to f(c)$ and $m_2 \to f(c)$. Thus $\int_a^b f \mathrm{d}\alpha = f(c)$. ■

Theorem 7.2

Let α be increasing on $[a,b]$ and f be continuous on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$.

Proof: Let $\epsilon > 0$. We need to find P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. Note $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i < \epsilon$. Since f is continuous on $[a,b]$ and $[a,b]$ is compact, we have f is uniformly continuous. Thus $\forall\epsilon', \exists\delta > 0, \forall r, s \in [a,b] : |r - s| < \delta \implies |f(r) - f(s)| < \epsilon'$. Let P be a partition so that $\forall i : 0 < x_i - x_{i-1} < \delta$. Then $\forall r, s \in [x_{i-1}, x_i]$ we have $|f(r) - f(s)| < \epsilon'$. So $|M_i - m_i| \le \epsilon'$. Hence for such a P, we have $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^n (M_i - m_i)\Delta\alpha_i \le \epsilon' \sum_{i=1}^n \Delta\alpha_i = \epsilon'(\alpha(b) - \alpha(a))$. We want $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. We want $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. So if we let ϵ' be so that $\epsilon'(\alpha(b) - \alpha(a)) < \epsilon$. For example, $\epsilon' = \frac{\epsilon}{2(\alpha(b) - \alpha(a))}$ for $\alpha(b) - \alpha(a) \ne 0$ or $\epsilon' = 1$ for $\alpha(b) = \alpha(a) = 0$. ■

Theorem 7.3

> *Let f is monotone on $[a,b]$ and α be continuous (and increasing) on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$.*

Proof: Let $\epsilon > 0$. We need to find P such that $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \epsilon$. We will show this for f is increasing. The proof for that f is decreasing is similar.

Let $\epsilon' > 0$. Since α is continuous on $[a,b]$ and $[a,b]$ is compact. $\exists\delta, \forall r,s \in [a,b] : |r-s| < \delta \implies |\alpha(r) - \alpha(s)| < \epsilon'$. Let P be a partition such that $x_i - x_{i-1} < \delta$. Then $\Delta\alpha_i < \epsilon'$. Now $U(P,f,\alpha) - L(P,f,\alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \leq \epsilon' \sum_{i=1}^{n}(M_i - m_i) = \epsilon'(f(b) - f(a))$. Since f is increasing, $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. So if ϵ' is such that $\epsilon'(f(b) - f(a)) < \epsilon$, then $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$. ■

e.g.2. Let $f : [0,1] \to \mathbb{R}$

- $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Then $f \notin \mathcal{R}$ [7.1].
- $f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q}, \gcd(p,q) = 1, p, q > 0 \\ 0 & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$. Then $f \in \mathcal{R}$ [7.2].

Proposition 7.4

> *Suppose that f is bounded on $[a,b]$ and the set of discontinuity A of f is a finite subset of $[a,b]$. Further assume that α is continuous at every $x \in A$. Then $f \in \mathcal{R}(\alpha)$.*

The proof is similar to the argument we discussed for showing that $f(x) = \begin{cases} 1 & x = \frac{1}{2} \\ 0 & x \neq \frac{1}{2} \end{cases}$ on $[0,1]$ satisfies $f \in \mathcal{R}$. We will prove this next time.

[7.1] The proof is similar to non-integrability of $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$.

[7.2] Recall f is continuous on $\mathbb{Q}^C \cap [0,1]$ at 0 but discontinuous at every other rational point.

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Lecture 8: Integrability and Continuity, Integrability of Composite Functions

Lecturer: Amir Mohammadi — *Scribes: Rabbittac*

Proof to 7.4: Let $A = \{t_1, \ldots, t_\ell\}$ and $a \le t_1 < t_2 < \cdots < t_\ell \le b$. For every i, let $c_i < d_i$ be such that

1. If $t_1 = a$, then $c_1 = a$; if $t_2 = b$, then $d_\ell = b$.
2. All (c_i, d_i) are disjoint.
3. $\alpha(d_i) - \alpha(c_i) < \epsilon$.

We now construct a partition of $[a, b]$ by refining $P_0 = \{c_1, d_1, c_2, d_2, \ldots, c_\ell, d_\ell\}$. If $c_1 \ne a$ and $d_\ell \ne b$, then let $B = [a, b] \setminus \bigcup_{i=1}^{\ell}(c_i, d_i)$; otherwise, if $c_1 = a$, or $d_\ell = b$, or both, then in the definition of B, we exclude $[c_1, d_1)$, or $(c_\ell, d_\ell]$, or both [8.1]. Note that in either case, B is a closed subset of $[a, b]$ so it is compact. Moreover, since f is continuous on B, f is uniformly continuous on $[a, b]$. Let $\delta > 0$ be such that if $r, s \in B$, $|r - s| < \delta \implies |f(r) - f(s)| < \epsilon$. Let P be a refinement of P_0 constructed as following: include P_0; no point is added between c_i and d_i; if $d_i < x_{j-1} < x_j < c_{i+1}$, then $x_j - x_{j-1} < \delta \implies M_j - m_j \le \epsilon$; if $d_i < x_{j-1} < c_{i+1}$ and $c_{i+1} - x_{j-1} < \delta$, we do not add any point before d_{i+1}. Now $U(P, f, \alpha) - L(P, f, \alpha) \le \sum_{i=1}^{n}(M_i - m_i)(\alpha(d_i) - \alpha(c_i)) + \sum_{j=1}^{n}(M_i^* - m_i^*)\Delta\alpha_j$ [8.2]. Since f is bounded, we have $|f(x)| \le M$ on $[a, b]$. So we have $U(P, f, \alpha) - L(P, f, \alpha) \le \sum_{i=1}^{\ell} 2M\epsilon + \epsilon \sum_{i=1}^{n} \Delta\alpha_j \le 2M\epsilon\ell + (\alpha(b) - \alpha(a))\epsilon$. ■

Proposition 8.1

Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and assume that $m \le f(x) \le M$. Let φ be continuous on $[m, M]$. Then $\varphi \circ f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof: Since φ is continuous on $[m, M]$ and $[m, M]$ is compact, φ is uniformly continuous. So $\forall \epsilon > 0, \exists \delta, r, s \in [m, M] : |r - s| < \delta \implies |\varphi(r) - \varphi(s)| < \epsilon$. Since $f \in \mathcal{R}(\alpha)$, $\forall \eta > 0, \exists P = \{x_0, \ldots, x_n\} : U(P, f, \alpha) - L(P, f, \alpha) < \eta$. Then $\sum(M_i - m_i)\Delta\alpha_i < \eta$. Note that if $M_i - m_i < \delta$ for some i, then $\forall c, d \in [x_{i-1}, x_i] : |f(c) - f(d)| < \delta \implies |\varphi(f(c)) - \varphi(f(d))| < \epsilon$. So if we define $M_i^* = \sup \phi(f(x))$ and $m_i^* = \inf \phi(f(x))$ for $x_{i-1} \le x \le x_i$, and i is such that $M_i - m_i < \delta$. Then $M_i^* - m_i^* \le \epsilon$. Let now $B = \{i : M_i - m_i \ge \delta\}$ then $U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \eta$. Then $\delta \sum_{i \in B} \Delta\alpha_i \le \sum_{i \in B}(M_i - m_i)\Delta\alpha_i \le U(P, f, \alpha) - L(P, f, \alpha) < \eta$. So $\sum_{i \in B} \Delta\alpha_i < \frac{\eta}{\delta}$. So if $\eta < \epsilon\delta$, then we get $\sum_{i \in B} \Delta\alpha_i < \epsilon$. Since φ is continuous on $[m, M]$ and $[m, M]$ is compact, $\exists L, \forall y \in [m, M] : |\varphi(y)| \le L$. So $\forall x \in [a, b] : -L \le \varphi(f(x)) \le L$. Altogether we have $U(P, \varphi \circ f, \alpha) - L(P, \varphi \circ f, \alpha) = \sum_{i=1}^{n}(M_i^* - m_i^*)\Delta\alpha_i = \sum_{i \in B}(M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \notin B}(M_i^* - m_i^*)\Delta\alpha_i \le 2L \sum_{i \in B} \Delta\alpha_i + \epsilon \sum_{i \notin B} \Delta\alpha_i < 2L\epsilon + (\alpha(b) - \alpha(a))\epsilon$. ■

Corollary 8.2

Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Then $\forall p \ge 0 : |f|^p \in \mathcal{R}(\alpha)$ on $[a, b]$.

[8.1] For example, if $c_1 = a$ but $d_\ell \ne b$, then $B = [a, b] \setminus \Big([c_1, d_1) \cup \bigcup_{i=2}^{\ell}(c_i, d_i)\Big)$.

[8.2] The first term is for P_0 and the second term is for new intervals which are not part of P_0.

Proof: Since $x \to |x|^p$ is continuous on $\mathbb{R}$, applying the proposition gives the result. ■

e.g.1. $f(x) = \begin{cases} \frac{\sqrt{2}}{q} & x = \frac{p}{q}, \gcd(p,q) = 1, p, q > 0 \\ 0 & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$ and $\varphi = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Then $f \in \mathcal{R}$ on $[0,1]$ and φ is integrable. But $\varphi \circ f(x) = \begin{cases} \varphi(\frac{\sqrt{2}}{q}) = 0 & x = \frac{p}{q}, x \neq 0 \\ \varphi(0) = 1 & x \notin \mathbb{Q} \text{ or } x = 0 \end{cases}$ is not integrable on $[\frac{1}{2}, 1]$.

Lemma 8.3 (*Properties of Integrals*)

1. *If $f_1, f_2 \in \mathcal{R}(\alpha)$, then $f_1 + f_2 \in \mathcal{R}(\alpha)$ and $\int (f_1 + f_2) \mathrm{d}\alpha = \int f_1 \mathrm{d}\alpha + \int f_2 \mathrm{d}\alpha$.*
2. *If $f \in \mathcal{R}$ and $c \in \mathbb{R}$, then $cf \in \mathcal{R}(\alpha)$ and $\int cf \mathrm{d}\alpha = c \int f \mathrm{d}\alpha$.*
3. *If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and $[c,b]$, and $\int_a^b f \mathrm{d}\alpha = \int_a^c f \mathrm{d}\alpha + \int_c^b f \mathrm{d}\alpha$.*
4. *If $f_1 \leq f_2$ on $[a,b]$ and $f_1, f_2 \in \mathcal{R}(\alpha)$, then $\int f_1 \mathrm{d}\alpha \leq \int f_2 \mathrm{d}\alpha$. In particular, if $f \in \mathcal{R}(\alpha)$ and $f \geq 0$, then $\int f \mathrm{d}\alpha \geq 0$.*
5. *If $f \in \mathcal{R}(\alpha_1), f \in \mathcal{R}(\alpha_2)$ on $[a,b]$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and $\int_a^b f \,\mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f \,\mathrm{d}\alpha_1 + \int_a^b f \,\mathrm{d}\alpha_2$.*
6. *If $f \in \mathcal{R}(\alpha), c > 0$, then $f \in \mathcal{R}(c\alpha)$ and $\int f \,\mathrm{d}(c\alpha) = c \int f \mathrm{d}\alpha$.*
7. *If $|f| \leq M$ on $[a,b]$ and $f \in \mathcal{R}(\alpha)$, then $\int f \mathrm{d}\alpha \leq M(\alpha(b) - \alpha(a))$.*

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Lecture 9: Properties of Integrals

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proof to 8.3:

3. Since $f \in \mathcal{R}(\alpha)$, $\forall \epsilon, \exists P : \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i < \epsilon$. Let $P' = P \cup \{c\}$. Since P' is a refinement of P, $\sum_{i=1}^{n}(M_i' - m_i')\Delta\alpha_i < \epsilon$ where M_i' and m_i' are w.r.t. P'. Note that $\sum_{i=1}^{n}(M_i' - m_i')\Delta\alpha_i = \sum_{x_i' \leq c}(M_i' - m_i')\Delta\alpha_i + \sum_{c \leq x_i'}(M_i' - m_i')\Delta\alpha_i < \epsilon$. Since both terms are non-negative, we get $\sum_{x_i' \leq c}(M_i' - m_i')\Delta\alpha_i < \epsilon$ and $\sum_{c \leq x_i'}(M_i' - m_i')\Delta\alpha_i < \epsilon$, which gives the desired partition for $[a, c]$ and $[c, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $[c, b]$.
4. Since $f \geq 0$, $\forall P : L(P, f, \alpha) \geq 0$. Hence $\int_a^b f \mathrm{d}\alpha = \sup L(P, f, \alpha) \geq 0$. To see the second part, let $f(x) = \begin{cases} 1 & x = 1 \\ 0 & 0 \leq x \leq 2, x \neq 1 \end{cases}$. Then $f \in \mathbb{R}$ on $[0, 2]$ and $\int_0^2 f \mathrm{d}\alpha = 0$. However $f \neq 0$. ∎

Proposition 9.1

Let $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$. Then

1. $\forall p \geq 0 : |f|^p \in \mathcal{R}(\alpha)$. $\left|\int_a^b f \mathrm{d}\alpha\right| \leq \int_a^b |f| \mathrm{d}\alpha$.
2. $fg \in \mathcal{R}(\alpha)$

Proof:

1. Recall that if $f \in \mathcal{R}(\alpha)$ and φ is continuous on an interval which contains the image of f. Then $\varphi \circ f \in \mathbb{R}(\alpha)$ on $[a, b]$. So $|f|^p \in \mathcal{R}(\alpha)$. To see the inequality, $\left|\int_a^b f \mathrm{d}\alpha\right| \leq \int_a^b |f| \mathrm{d}\alpha$. Note that $-|f(x)| \leq f(x) \leq |f(x)|$. Then by properties of integration, $-\int_a^b |f| \mathrm{d}\alpha \leq \int_a^b f \mathrm{d}\alpha \leq \int_a^b |f| \mathrm{d}\alpha$. Then $\left|\int_a^b f \mathrm{d}\alpha\right| \leq \int_a^b |f| \mathrm{d}\alpha$.
2. Recall that $\forall c, d \in \mathbb{R} : (c + d)^2 - (c - d)^2 = 4cd$. Hence $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$. Since $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$, $f + g \in \mathcal{R}(\alpha)$ and $f - g \in \mathcal{R}(\alpha)$. By (1), $|f \pm g|^2 = (f \pm g)^2 \in \mathcal{R}(\alpha)$. So $fg = \frac{1}{4}((f + g)^2 - (f - g)^2) \in \mathcal{R}(\alpha)$. ∎

Recall that we showed: let $\alpha(x) = \begin{cases} 0 & x < c \\ 1 & x \geq c \end{cases}$ and let f be continuous at c. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f \mathrm{d}\alpha = f(c)$. Define

$$\mathbb{I}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Proposition 9.2

Suppose $\lambda_n \geq 0$ and $\sum \lambda_n$ converges. Let $\{c_n\}$ be a sequence of different points in an interval $[a,b]$. Let f be continuous on $[a,b]$. Define $\alpha(x) = \sum \lambda_n \mathbb{I}(x - c_n)$. Then $f \in \mathcal{R}(\alpha)$. Moreover,

$$\int_a^b f \mathrm{d}\alpha = \sum_{n=1}^{\infty} \lambda_n f(c_n)$$

Proof: Before proving the statement, we first rewrite $\alpha(x)$ more explicitly: $\alpha(x) = \sum_{n=1}^{\infty} \lambda_n \mathbb{I}(x - c_n)$. Given x, we have $\mathbb{I}(x - c_n) = \begin{cases} 0 & x \leq c_n \\ 1 & x > c_n \end{cases}$. Hence $\alpha(x) = \sum_{c_n < x} \lambda_n$. Clearly, α is increasing.

Since α is increasing and f is continuous on $[a,b]$, $f \in \mathcal{R}(\alpha)$. We want to compute the integral to show $\int_a^b f \mathrm{d}\alpha = \sum_{n=1}^{\infty} \lambda_n f(c_n)$. This is equivalent to: $\forall \epsilon > 0, \exists N$: $\left| \int_a^b f \mathrm{d}\alpha - \sum_{i=1}^{N} \lambda_n f(c_n) \right| < \epsilon$. Recall $\alpha(x) = \sum_{n=1}^{\infty} \lambda_n \mathbb{I}(x - c_n) = \sum_{c_n < x} \lambda_n$. Let $\epsilon > 0$, $\exists N : \sum_{n=N+1}^{\infty} \lambda_n \mathbb{I}(x - c_n) < \epsilon$. Define $\alpha_1(x) = \sum_{i=1}^{N} \lambda_n \mathbb{I}(x - c_n)$ and $\alpha_2 = \sum_{n=N+1}^{\infty} \lambda_n \mathbb{I}(x - c_n)$. Then $\alpha = \alpha_1 + \alpha_2$; $0 \leq \alpha(x) \leq \sum_{n=N+1}^{\infty} \lambda_n < \epsilon$; α_1 is a finite combination of $\mathbb{I}(x - c_n)$. Hence $\int_a^b f \mathrm{d}\alpha_1 = \sum_{i=1}^{N} \lambda_n f(c_n)$. Then $\int_a^b f \, \mathrm{d}(\sum_{i=1}^{N} \lambda_n \mathbb{I}(x - c_n)) = \sum_{i=1}^{N} \int_a^b f \, \mathrm{d}(\lambda_n \mathbb{I}(x - c_n)) = \sum_{i=1}^{N} \lambda_n \int_a^b f \, \mathrm{d}\mathbb{I}(x - c_n) = \sum_{i=1}^{N} \lambda_n f(c_n)$. Now since $\forall a, b$: $0 \leq \alpha_2(b) - \alpha_2(a) < \epsilon$, we have $|\int_a^b f \mathrm{d}\alpha_2| \leq M\epsilon$ where $|f| \leq M$. Then $\int_a^b f \mathrm{d}\alpha = \int_a^b f \, \mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f \mathrm{d}\alpha_1 + \int_a^b f \mathrm{d}\alpha_2 = \sum_{i=1}^{N} \lambda_n f(c_n) + \int_a^b f \mathrm{d}\alpha_2$. Therefore, $|\int_a^b f \mathrm{d}\alpha - \sum_{i=1}^{N} \lambda_n f_{c_n}| = |\int_a^b f \mathrm{d}\alpha_2| < M\epsilon$. ■

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Lecture 10: Simple Function, Change of Variable, Convolution

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 10.1 (*Characteristic Function*)

Given an interval J (closed, open, half open), the ***characteristic function*** *of J is defined as* [10.1]

$$\mathbb{1}_J(x) = \begin{cases} 1 & x \in J \\ 0 & x \notin J \end{cases}$$

Definition 10.2 (*Simple Function*)

A ***simple function*** *s is a function defined as*

$$\boldsymbol{s} = \sum_{i=1}^{n} c_i \mathbb{1}_{J_i}$$

where $n \in \mathbb{N}, c_i \in \mathbb{R}$, and J_i's are disjoint intervals.

e.g.1. Let $J_1 = [0,1]$, $J_2 = [1,2]$, and $\boldsymbol{s}(x) = 10\mathbb{1}_{J_1}(x) - 2\mathbb{1}_{J_2}(x)$. Then $\boldsymbol{s}(x) = \begin{cases} 10 & x \in [0,1] \\ -2 & x \in [-1,2] \\ 0 & \text{otherwise} \end{cases}$.

Lemma 10.1

Let f be bounded on $[a,b]$. Assume that $f \in \mathcal{R}$. Then

$$\exists \boldsymbol{s} : \left| \int_a^b f \mathrm{d}x - \int_a^b \boldsymbol{s} \mathrm{d}x \right| < \epsilon$$

Proof: Let $\epsilon > 0$. Since $f \in \mathcal{R}$, $\exists P = \{x_0, x_1, \ldots, x_n\} : \left| U(P,f) - \int_a^b f \mathrm{d}x \right| < \epsilon$. Note that $U(P,f) = \sum_{i=1}^n M_i \Delta x_i = \sum_{i=1}^n M_i(x_i - x_{i-1})$. Define $\boldsymbol{s}(t) = M_1 \mathbb{1}_{[x_0,x_1]} + M_2 \mathbb{1}_{[x_1,x_2]} + \cdots + M_n \mathbb{1}_{[x_{n-1},x_n]}$. Then $\boldsymbol{s}$ is a simple function. We know $\boldsymbol{s} \in \mathbb{R}$. Moreover, $\int_a^b \boldsymbol{s} \mathrm{d}x = \sum_{i=1}^n M_i \int_a^b \mathbb{1}_{[x_{i-1},x_i]}(t) \mathrm{d}t = \sum_{i=1}^n M_i(x_i - x_{i-1}) = \sum_{i=1}^n M_i \Delta x_i = U(P,f)$. Hence, we have $\left| \int_a^b \boldsymbol{s} \mathrm{d}x - \int_a^b f \mathrm{d}x \right| < \epsilon$. ■

[10.1] χ_J is another commonly used notation for this.

Theorem 10.2

Suppose α is increasing and $\alpha' \in \mathcal{R}$ on $[a,b]$. Let f be bounded on $[a,b]$. Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In that case, we have

$$\int_a^b f\mathrm{d}\alpha = \int_a^b f\alpha'\mathrm{d}x$$

Proof: $\forall t_i \in [x_{i-1}, x_i]$: LHS $= \sum_{i=1}^n f(t_i)\Delta\alpha_i = \sum_{i=1}^n f(t_i)(\alpha(x_i) - \alpha(x_{i-1}))$; RHS $= \sum_{i=1}^n f(t_i)\alpha'(t_i)(x_i - x_{i-1})$. By MVT, $\exists c_i \in (x_{i-1}, x_i) : \alpha(x_i) - \alpha(x_{i-1}) = \alpha'(c_i)(x_i - x_{i-1})$ so LHS $= \sum_{i=1}^n f(t_i)\alpha'(c_i)(x_i - x_{i-1})$. Since $\alpha' \in \mathcal{R}$, $\exists P = \{x_0, x_1, \dots, x_n\}, \forall P \supset P_0 : U(P, \alpha') - U(P, \alpha') < \epsilon$. Hence, $\forall p_i, q_i \in [x_{i-1}, x_i] : \sum_{i=1}^n |\alpha'(p_i) - \alpha'(q_i)| \Delta x_i < \epsilon$ because $|\alpha'(p_i) - \alpha'(q_i)| \le M_i - m_i$. Then $\forall t_i \in [x_{i-1}, x_i] : \sum_{i=1}^n f(t_i)\Delta\alpha_i = \sum_{i=1}^n f(t_i)\alpha'(c_i)(x_i - x_{i-1})$. Moreover, since we have a Riemann sum for $f\alpha'$ of the form $\sum_{i=1}^n f(t_i)\alpha'(t_i)(x_i - x_{i-1})$. Since f is bounded, $|f| \le M$. Now $|\sum_{i=1}^n f(t_i)\Delta\alpha_i - \sum_{i=1}^n f(t_i)\alpha'(t_i)\Delta x_i| = |\sum_{i=1}^n f(t_i)\alpha'(c_i)\Delta x_i - \sum_{i=1}^n f(t_i)\alpha'(t_i)\Delta x_i| = |\sum_{i=1}^n f(t_i)(\alpha'(c_i) - \alpha'(t_i))\Delta x_i| \le \sum_{i=1}^n |f(t_i)||\alpha'(c_i) - \alpha'(t_i)|\Delta x_i \le M\sum_{i=1}^n |\alpha'(c_i) - \alpha'(t_i)|\Delta x_i < M\epsilon$. Then we can conclude that $\sum_{i=1}^n f(t_i)\Delta\alpha_i \le \sum_{i=1}^n f(t_i)\alpha'(t_i)\Delta x_i \le U(P, f\alpha') + M\epsilon$. Similarly, $U(f\alpha', P) \le U(P, f, \alpha)$. So $|\overline{\int_a^b} f\mathrm{d}\alpha - \overline{\int_a^b} f\alpha'\mathrm{d}x| \le M\epsilon$ implies $\overline{\int_a^b} f\mathrm{d}\alpha = \overline{\int_a^b} f\alpha'\mathrm{d}x$. Similarly, $\underline{\int_a^b} f\mathrm{d}\alpha = \underline{\int_a^b} f\alpha'\mathrm{d}x$. So $f \in \mathcal{R}(\alpha) \iff \overline{\int_a^b} f\mathrm{d}\alpha = \underline{\int_a^b} f\mathrm{d}\alpha = \int_a^b f\mathrm{d}\alpha \iff \overline{\int_a^b} f\alpha'\mathrm{d}x = \underline{\int_a^b} f\alpha'\mathrm{d}x = \int_a^b f\alpha'\mathrm{d}x$. ■

Theorem 10.3 (*Change of Variable*)

1. *Let $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and α be increasing on $[a,b]$. Let $\varphi : [A,B] \to [a,b]$ be continuous and strictly increasing. Define $\beta(t) = \alpha(\varphi(t))$ and $g(t) = f(\varphi(t))$. Then β is increasing on $[A,B]$. Moreover, $g \in \mathcal{R}(\beta)$ and*

$$\int_a^b f\mathrm{d}\alpha = \int_A^B g\mathrm{d}\beta$$

2. *Let f and α be as in (1). Let $\varphi : [A,B] \to [a,b]$ to be continuous and strictly decreasing. Define $g(t) = f(\varphi(t))$ and $\beta(t) = -\alpha(\varphi(t))$. Then β is increasing. Moreover, $g \in \mathcal{R}(\beta)$ and*

$$\int_a^b f\mathrm{d}\alpha = \int_A^B g\mathrm{d}\beta$$

Proof will be in the next lecture.

Definition 10.3 (*Support*)

Let (X,d) be a metric space and $f : X \to \mathbb{R}$. Then the ***support*** *of f is defined*

$$\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$$

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, we say f has ***compact support*** *if $\operatorname{supp} f$ is compact. The set of all continuous compactly supported functions on $\mathbb{R}$ is denoted by $C_c(\mathbb{R})$* [10.2].

Definition 10.4 (*Convolution*)

Let $f, g \in C_c(\mathbb{R})$. The ***convolution*** *$f * g : \mathbb{R} \to \mathbb{R}$ is defined as*

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)\mathrm{d}y$$

Lemma 10.4

Let $f, g \in C_c(\mathbb{R})$. Then $h(y) = f(x-y)g(y) \in C_c(\mathbb{R})$.

Proof: If y satisfies $g(y) = 0$, then $h(y) = 0$. So $\operatorname{supp} h \subset \operatorname{supp} g$. Note if $h \in C_c(\mathbb{R})$, then $\operatorname{supp} h \subset [a, b]$ and $\int_{\mathbb{R}} h\mathrm{d}x = \int_a^b h\mathrm{d}x$ [10.3]. ■

Corollary 10.5

Suppose $\alpha(x) = x$ and $f \in \mathcal{R}$. Assume further that $\varphi' \in \mathcal{R}$.

1. *If $\varphi : [A, B] \to [a, b]$ is continuous and strictly increasing, then*

$$\int_a^b f(x)\mathrm{d}x = \int_A^B f(\varphi(t))\varphi'(t)\mathrm{d}t$$

2. *If $\varphi : [A, B] \to [a, b]$ is continuous and strictly decreasing, then*

$$\int_a^b f(x)\mathrm{d}x = -\int_A^B f(\varphi(t))\varphi'(t)\mathrm{d}t$$

Proof: We will prove the first statement, and the second can be shown proved similarly. To see this, note that by the previous theorem: $g(t) = f(\varphi(t)) \in \mathcal{R}(\beta)$ where $\beta(t) = \alpha(\varphi(t)) = \varphi(t)$ in (1) and $\beta(t) = -\alpha(\varphi(t)) = -\varphi(t)$ in (2). Now by the previous theorem, $\int_a^b f(x)\mathrm{d}x = \int_A^B g(t)\mathrm{d}\beta = \int_A^B f(\varphi(t))\mathrm{d}\varphi = \int_A^B f(\varphi(t))\varphi'(t)\mathrm{d}t$. ■

[10.2] More generally, $C_c(X) = \{f : X \to \mathbb{R}, f$ is continuous and supported compactly.$\}$

[10.3] Note that if $\operatorname{supp} h \subset [c, d] \subset [a, b]$, then $\int_a^b h\mathrm{d}x = \int_c^d h\mathrm{d}x$.

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Lecture 11: Fundamental Theorem of Calculus, Integrate by Parts

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proof to 10.3: We will prove 2. The proof of 1 is similar. The main observation is the fact that φ induces a bijection between partitions of $[A, B]$ and $[a, b]$. More explicitly, let $Q = \{y_1, y_2, \dots, y_n\}$ be a partition of $[A, B]$. Then $\varphi(y_0) = b$ and $\varphi(y_n) = a$ since φ is decreasing. Let $\varphi(y_0) = x_n, \varphi(y_1) = x_{n-1}, \dots, \varphi(y_n) = x_0$ so $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$. Since φ is a continuous bijection, this is a bijection. Let now Q and $\varphi(Q) = P$ be partitions of $[A, B]$ and $[a, b]$. Note that $M_i = \sup_{[y_{i-1}, y_i]} g = \sup_{y \in [y_{i-1}, y_i]} f(\varphi(y)) = \sup_{x \in [\varphi(y_i), \varphi(y_{i-1})]} f(x) = \sup_{x \in [x_{n-1}, x_{n-i+1}]} f(x)$. Moreover, $\beta(y_i) - \beta(y_{i-1}) = ((-\alpha(\varphi(y_i))) - (-\alpha(\varphi(y_{i-1}))) = -\alpha(x_{n-1}) + \alpha(x_{n-i+1})$. Hence $U(Q, g, \beta) = \sum_{i=1}^n M_i(\beta(y_i) - \beta(y_{i-1})) = \sum_{i=1}^n (\sup_{[x_{n-1}, x_{n-i+1}]} f(x))(\alpha(x_{n-i+1}) - \alpha(x_{n-1})) = U(P, f, \alpha)$. Similarly, $L(Q, g, \beta) = L(P, f, \alpha)$. Hence $f \in \mathcal{R}(\alpha) \implies g \in \mathcal{R}(\beta)$ and $\int_a^b f \mathrm{d}\alpha = \int_A^B g \mathrm{d}\beta$. ■

Proposition 11.1

Let $f \in \mathcal{R}$ on $[a, b]$ and define

$$F(x) = \int_a^x f(t) \mathrm{d}t$$

for $a \le x \le b$. Then F is continuous on $[a, b]$. Moreover, if f is continuous at $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof: Since $f \in \mathcal{R}$, we have f is bounded. Let M be such that $\forall x \in [a, b] : |f(x)| < M$. Let $s \in [a, b]$ and $\epsilon > 0$, we want to find $\delta > 0 : |s - r| < \delta \implies |F(s) - F(r)| < \epsilon$. Let $a \le s \le r \le b$, then $F(r) - F(s) = \int_a^r f(t)\mathrm{d}t - \int_a^s f(t)\mathrm{d}t = \int_a^s f\mathrm{d}t + \int_s^r f(t)\mathrm{d}t - \int_a^s f(t)\mathrm{d}t = \int_s^r f(t)\mathrm{d}t$. Similarly, if $a \le u \le s \le b$, then $F(s) - F(u) = \int_u^s f(t)\mathrm{d}t$. Hence if $a \le s \le r \le b$, then $|F(r) - F(s)| = \left|\int_s^r f(t)\mathrm{d}t\right| \le \int_s^r |f(t)|\mathrm{d}t \le M(r - s)$. If $a \le u \le s \le b$, then $|F(s) - F(u)| \le M(s - u)$. Altogether, $|s - y| < \delta \implies |F(s) - F(y)| \le M\delta$. Hence letting $\delta = \frac{\epsilon}{2M}$, we get $|F(s) - F(y)| \le \epsilon$. This proves F is continuous at s.

Now assume that f is continuous at c. We want to show that F is differentiable at c and $F'(c) = f(c)$. Since f is continuous at c, we have $\forall \epsilon, \exists \delta > 0 : |t - c| < \delta \implies |f(t) - f(c)| < \epsilon$. Let now $a \le c \le x \le c + \delta$ and $x \in [a, b]$. Then $\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| = \left|\frac{1}{x-c}\int_c^x f(t)\mathrm{d}t - f(c)\right| = \left|\frac{1}{x-c}\int_c^x f(t)\mathrm{d}t - \frac{1}{x-c}\int_c^x f(c)\mathrm{d}t\right| = \left|\frac{1}{x-c}\int_c^x (f(t) - f(c))\mathrm{d}t\right| \le \frac{1}{x-c}\int_c^x |f(t) - f(c)| \, \mathrm{d}t < \frac{1}{x-c}\epsilon(x - c) = \epsilon$. Similarly, if $x \in [a, b]$ and $c - \delta \le x \le c \le b$, then $\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| < \epsilon$. Therefore, F is differentiable at c and $F'(c) - f(c)$. ■

Theorem 11.2 (*Fundamental Theorem of Calculus*)

Let $f \in \mathcal{R}$ on $[a,b]$ and $\exists F : F' = f$. Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Proof: Let $\epsilon > 0$ be arbitrary. Then $\exists P = \{x_0, \ldots, x_n\} : \left|\sum_{i=1}^n f(t_i)\Delta x_i - \int_a^b f(x)\mathrm{d}x\right| < \epsilon$ for arbitrary $t_i \in [x_{i-1}, x_i]$. Fix i, by MVT, $\exists s_i : F'(s_i)\Delta x_i = F(x_i) - F(x_{i-1})$. Since $F' = f$, $\forall i, \exists s_i \in [x_{i-1}, x_i] : F(x_i) - F(x_{i-1}) = F'(s_i)\Delta x_i = f(s_i)\Delta x_i$. Therefore, $\sum_{i=1}^n F(x_i) - F(x_{i-1}) = \sum_{i=1}^n f(s_i)\Delta x_i$. Note that $\sum_{i=1}^n F(x_i) - F(x_{i-1}) = F(b) - F(a)$. Then $F(b) - F(a) = \sum_{i=1}^n f(s_i)\Delta x_i$. Hence, $|F(b) - F(a) - \int_a^b f(x)\mathrm{d}x| < \epsilon$. Since ϵ is arbitrary, we get $F(b) - F(a) = \int_a^b f(x)\mathrm{d}x$. ■

Corollary 11.3 (*Integration by Parts*)

Let F and G be differentiable on $[a,b]$. Assume that $F' = f$ and $G' = g$ are both integrable. Then

$$\int_a^b F(x)g(x)\mathrm{d}x = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)\mathrm{d}x$$

Proof: Let $H(x) = F(x)G(x)$. Then H is differentiable and $H'(x) = F'(x)G(x) + F(x)G'(x) = f(x)G(x) + F(x)g(x)$. Since $f \in \mathcal{R}$ and $g \in \mathcal{R}$, F and G are differentiable, by FTC, $\int_a^b (fG + Fg) = \int_a^b H' = H(b) - H(a) = F(b)G(b) - F(a)G(a)$. ■

11.1 Vector-valued Functions

Let $f : [a,b] \to \mathbb{R}^n$, $f = (f_1, \ldots, f_n)$. Then we say $f \in \mathcal{R}$ if $\forall i : f_i \in \mathcal{R}$. If $f \in \mathcal{R}(\alpha)$, then $\int_a^b f\mathrm{d}\alpha = \left(\int_a^b f_1\mathrm{d}\alpha, \ldots, \int_a^b f_n\mathrm{d}\alpha\right)$.

Theorem 11.4 (*Fundamental Theorem of Calculus*)

Let $f : [a,b] \to \mathbb{R}^n$ be in $\mathbb{R}$. Suppose $\exists F : [a,b] \to \mathbb{R}^n : F' = f$. Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

Proposition 11.5

Let $f : [a,b] \to \mathbb{R}^n$. Suppose $f \in \mathcal{R}(\alpha)$. Then $\|f\| \in \mathcal{R}$ and

$$\left\|\int_a^b f\mathrm{d}\alpha\right\| \leq \int_a^b \|f\|\,\mathrm{d}\alpha$$

e.g.1. **(Riemann-Lebesgue Theorem)** Let $f \in \mathcal{R}$ on $[a, b]$. Then

$$\lim_{n\to\infty} \int_0^1 f(x)\sin(nx)\,\mathrm{d}x = 0$$

Proof: The proof will be completed in some steps. A complete proof will be in the next lecture.

1. Prove for constant functions $f(x) = c$.
2. Prove for simple functions $f(x) = \sum_{i=1}^{n} c_i \mathbb{1}_{I_i}$.
3. Given $f \in \mathcal{R}$ and $\epsilon > 0$, $\exists \boldsymbol{s} : f \leq \boldsymbol{s}$ and $0 \leq \int_a^b s - \int_a^b f < \epsilon$. ■

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Lecture 12: Riemann-Lebesgue Lemma, Curve, Sequence of Functions

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proof to *e.g.*11.1:

1. Let $f(x) = c$ be a constant function where $c \in \mathbb{R}$.
   We want to compute $\int_0^1 c\sin(nx)\mathrm{d}x = c\int_0^1 \sin(nx)\mathrm{d}x = c\int_0^n \sin u\frac{\mathrm{d}u}{n} = \frac{c}{n}\int_0^n \sin u\mathrm{d}u$. Let k be the largest integer so that $2\pi k \le n$. Then $0 \le n - 2\pi k < 2\pi$. Hence $\int_0^1 c\sin(nx)\mathrm{d}x = \frac{c}{n}\int_0^n \sin u\mathrm{d}u = \frac{c}{n}(\int_0^{2k\pi} \sin u\mathrm{d}u + \int_{2k\pi}^n \sin u\mathrm{d}u) = \frac{c}{n}\int_{2k\pi}^n \sin u\mathrm{d}u$. Then $\left|c\int_0^1 \sin(nx)\mathrm{d}x\right| = |\frac{c}{n}\int_{2k\pi}^n \sin u\mathrm{d}u| \le \frac{c}{n}\int_{2k\pi}^n |\sin u|\mathrm{d}u \le \frac{c}{n}\cdot 1\cdot(n-2k\pi) \le \frac{2\pi c}{n} \xrightarrow[n\to\infty]{} 0$.
2. Let $f(x) = \sum_{i=1}^{\ell} c_i\mathbb{1}_{I_i}$ be a simple function where I_i are disjoint intervals.
   Since ℓ is fixed, it suffices to show that $\int_0^1 c_i\mathbb{1}_{I_i}\sin(nx)\mathrm{d}x \to 0$. Note that this can be rewritten as $\int_0^1 c_i\mathbb{1}_{I_i}\sin(nx)\mathrm{d}x = c_i\int_{a_i}^{b_i}\sin(nx)\mathrm{d}x$. We need to show that $c_i\int_{a_i}^{b_i}\sin(nx)\mathrm{d}x \to 0$. By Change of Variable, $c_i\int_{a_i}^{b_i}\sin(nx)\mathrm{d}x = c_i\int_{na_i}^{nb_i}\sin u\frac{\mathrm{d}u}{n}$. Let k_1 be the smallest integer such that $2k_1\pi \ge na_i$ and k_2 be the largest integer such that $2k_2\pi \le nb_i$. Then $\frac{c_i}{n}\int_{na_i}^{nb_i}\sin u\mathrm{d}u = \frac{c_i}{n}(\int_{na_i}^{2k_1\pi}\sin u\mathrm{d}u + \int_{2k_1\pi}^{2k_2\pi}\sin u\mathrm{d}u + \int_{2k_2\pi}^{nb_i}\sin u\mathrm{d}u) = \frac{c_i}{n}(\int_{na_i}^{2k_1\pi}\sin u\mathrm{d}u + \int_{2k_2\pi}^{nb_i}\sin u\mathrm{d}u)$. Therefore, $|c_i\int_{a_i}^{b_i}\sin(nx)\mathrm{d}x| \le \frac{c_i}{n}(2\pi + 2\pi) \xrightarrow[n\to\infty]{} 0$.
3. Let $f \in \mathcal{R}$.
   Let $\epsilon > 0$. $\exists \boldsymbol{s}, \forall x \in [0,1] : f(x) \le \boldsymbol{s}(x), 0 \le \int(s-f) < \epsilon$. Write $\boldsymbol{s} = \sum_{i=1}^{\ell} c_i\mathbb{1}_{I_i}$. Now by step 2, we have $\exists N : n > N \implies |\int_0^1 \boldsymbol{s}\sin(nx)\mathrm{d}x| < \epsilon$. Let $n > N$, then $\int_0^1 f(x)\sin(nx)\mathrm{d}x = \int_0^1 (f(x) - \boldsymbol{s}(x) + \boldsymbol{s}(x))\sin(nx)\mathrm{d}x = \int_0^1 (f(x)-\boldsymbol{s}(x))\sin(nx)\mathrm{d}x + \int_0^1 \boldsymbol{s}(x)\sin(nx)\mathrm{d}x$. Hence, $|\int_0^1 f(x)\sin(nx)\mathrm{d}x| \le |\int_0^1 (f(x)-\boldsymbol{s}(x))\sin(nx)\mathrm{d}x| + |\int_0^1 \boldsymbol{s}(x)\sin(nx)\mathrm{d}x| \le |\int_0^1 (f(x)-\boldsymbol{s}(x))\sin(nx)\mathrm{d}x| + \epsilon \le \int_0^1 |f(x)-\boldsymbol{s}(x)||\sin(nx)|\mathrm{d}x + \epsilon \le \int_0^1 (\boldsymbol{s}(x) - f(x))\mathrm{d}x + \epsilon = 2\epsilon$. ■

Remark: Similarly, one can show $\forall f \in \mathcal{R} : \lim_{n\to\infty}\int_0^1 f(x)\cos(nx)\mathrm{d}x = 0$.

12.1 Curve

Definition 12.1 (*Curve*)

*A **curve** on $[a,b]$ is a continuous function $\gamma : [a,b] \to \mathbb{R}^n$.*

e.g.1.

- If $f : [a,b] \to \mathbb{R}$ is continuous, then $\gamma(t) = (t, f(t))$ is a curve.
- $\gamma : [0,2\pi] \to \mathbb{R}^2, \gamma(t) = (\cos t, \sin t)$ is a curve.

- $\gamma : [0, 2\pi] \to \mathbb{R}^2, \gamma(t) = (\cos 2t, \sin 2t)$ is a curve [12.1].

Definition 12.2 (*Rectifiable*)

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve. Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. Define

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} \|\gamma(x_i) - \gamma(x_{i-1})\|$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^n$. Define

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

A curve is called ***rectifiable*** *if $\Lambda(\gamma) < \infty$.*

e.g.2. We now construct a curve on $[0, 1]$ which has infinite length.

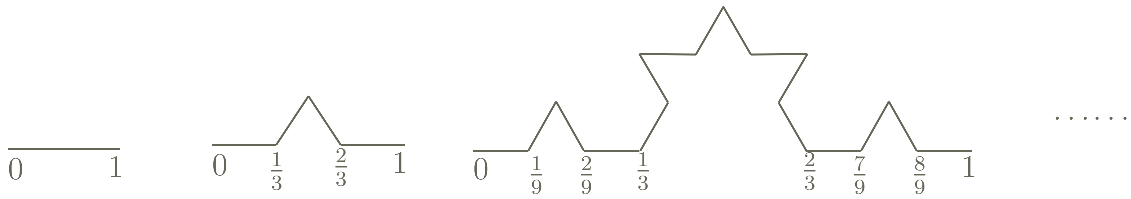

Figure 12.1: Curve of infinite length: $1, \frac{4}{3}, \frac{16}{9}, \ldots$.

Theorem 12.1

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a curve and γ' be continuous. Then γ is rectifiable and

$$\Lambda(\gamma) = \int_a^b \|\gamma'(t)\| \mathrm{d}t$$

Proof: We want to show $\Lambda(\gamma) \le \int_a^b \|\gamma'(t)\|\mathrm{d}t$ and $\Lambda(\gamma) \ge \int_a^b \|\gamma'(t)\|\mathrm{d}t$. Let $P = \{x_0, \ldots, x_n\}$ be a partition. Then $\|\gamma(x_i) - \gamma(x_{i-1})\| = \|\int_{x_{i-1}}^{x_i} \gamma'(t)\mathrm{d}t\| \le \int_{x_{i-1}}^{x_i} \|\gamma'(t)\|\mathrm{d}t$. Then $\Lambda(P, \gamma) \le \int_a^b \|\gamma'(t)\|\mathrm{d}t$. So γ is rectifiable and $\Lambda(\gamma) \le \int_a^b \|\gamma'(t)\|\mathrm{d}t$. The other side of the inequality $\int_a^b \|\gamma'(t)\|\mathrm{d}t \le \Lambda(\gamma)$ follows from the uniform continuity of γ' on $[a, b]$. Indeed, given $\epsilon > 0$, $\exists\delta : |r - s| < \delta \implies \|\gamma'(r) - \gamma'(s)\| < \epsilon$. So if P is so that $\Delta x_i < \delta$, then $\|\gamma'(t) - \gamma'(x_{i-1})\| < \epsilon$. Hence, $\int_{x_{i-1}}^{x_i} \|\gamma'(t)\|\mathrm{d}t \le \|\gamma'(x_{i-1})\|\Delta x_i + \epsilon\Delta x_i$. Note $\|\gamma'(x_{i-1})\|\Delta x_i = \int_{x_{i-1}}^{x_i} \|\gamma'(x_{i-1})\|\mathrm{d}t = \|\int_{x_{i-1}}^{x_i} \gamma'(x_{i-1})\mathrm{d}t\| = \|\int_{x_{i-1}}^{x_i}(\gamma'(x_{i-1}) - \gamma'(t) + \gamma'(t))\mathrm{d}t\| \le \|\int_{x_{i-1}}^{x_i} \gamma'(t)\mathrm{d}t\| + \|\int_{x_{i-1}}^{x_i}(\gamma'(x_{i-1}) - \gamma'(t))\mathrm{d}t\| = \|\gamma(x_i) - \gamma(x_{i-1})\| + \epsilon\Delta x_i$. Hence, $\int_{x_{i-1}}^{x_i} \|\gamma'(t)\|\mathrm{d}t \le \|\gamma(x_i) - \gamma(x_{i-1})\| + 2\Delta x_i$. Summing this for $i = 1, \ldots, n$, we get $\int_a^b \|\gamma'(t)\|\mathrm{d}t \le \Lambda(P, \gamma) + 2\epsilon(b - a) \le \Lambda(P, \gamma)$. ■

[12.1] Two curves may have the same image in $\mathbb{R}^n$ even though they are different continuous function.

12.2 Sequence of Functions

Let (X, d) be a metric space, and $\forall n \in \mathbb{N} : f_n : X \to \mathbb{R}$. Then $\{f_n\}$ is a sequence of functions

Definition 12.3 (*Pointwise Converge*)

*Let $f : X \to \mathbb{R}$ be a function. We say $\{f_n\}$ converges to f **pointwise** if*

$$\forall x \in X : f_n(x) \to f(x)$$

Definition 12.4 (*Uniformly Converge*)

*Let $f : X \to \mathbb{R}$ be a function. We say $\{f_n\}$ converges to f **uniformly** on X if*

$$\forall \epsilon > 0, \exists N, \forall x \in X, \forall n > N : |f_n(x) - f(x)| < \epsilon$$

Remark: in definition 1, $\forall x \in X, \forall \epsilon > 0, \exists N = N(x, \epsilon), \forall n > N : |f_n(x) - f(x)| < \epsilon$; in definition 2, $\forall x \in X, \forall \epsilon > 0, \exists N = N(\epsilon), \forall n > N : |f_n(x) - f(x)| < \epsilon$. Similarly, we define convergence of $\sum f_n$ using sequence $s_n = \sum_{i=1}^{n} f_i$ of partial sums.

e.g.3. $f_n : [0, 1] \to \mathbb{R}, f_n(x) = \begin{cases} 1 & \frac{1}{n} < x < \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$. Then $\forall x \in [0, 1]$, we have

1. $x = 0$. $\forall n : f_n(0) = 0$ so $f_n(0) \to 0$.
2. $0 < x \leq 1$. Then $\exists N, \forall n > N : x > \frac{2}{n}$. So $\forall n > N : f_n(x) = 0$. Then $f_n(0) \to 0$.

Then we can conclude that $\{f_n\}$ converges pointwise to 0 [12.2].

[12.2] Note that this sequence is not uniformly convergent to 0. Indeed, suppose it does. Let $\epsilon = \frac{1}{2}$. Then $\exists N, \forall n > N, \forall x : |f_n(x) - 0| < \epsilon = \frac{1}{2}$. However, if $\frac{1}{n} < x < \frac{2}{n}$, then $f_n(x) = 1 < \frac{1}{2}$, contradiction.

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Lecture 13: Cauchy's Criterion for Uniform Convergence

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

e.g.1. Consider $f_n(x) = x^n$ on $[0,1]$. $\forall x \in [0,1) : x^n \to 0$ and $\forall n : x = 1 \implies x^n = 1$. So $f_n(x) \xrightarrow{\text{pointwise}} f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & x = 1 \end{cases}$.

Remark: Continuity of f at 1 means $\lim_{x\to 1} f(x) = f(1)$. Note that $\forall x : f(x) = \lim_{n\to\infty} f_n(x)$. So we are asking whether $\lim_{x\to 1}(\lim_{n\to\infty} f_n(x)) = \lim_{n\to\infty} f_n(1)$ [13.1]? *e.g.*2 shows that $\lim_{x\to 1}(\lim_{n\to\infty} f_n(x)) \neq \lim_{n\to\infty} f_n(1)$.

e.g.2. Let $a_{n,m} = \frac{m}{m+n}$. Note that $\lim_{m\to\infty} a_{n,m} = 1$ and $\lim_{n\to\infty} a_{n,m} = 0$. So $\lim_{n\to\infty}(\lim_{m\to\infty} a_{n,m}) = 1$. However, $\lim_{m\to\infty}(\lim_{n\to\infty} a_{n,m}) = 0$.

e.g.3. Let $T = \{t_1, t_2, \dots\} \subset [0,1]$ be a countable set. $f_n(x) = \begin{cases} 1 & x \in \{t_1, \dots, t_n\} \\ 0 & x \notin \{t_1, \dots, t_n\} \end{cases}$.
f_n is continuous except at $\{t_1, \dots, t_n\}$. Note that $f_n(x) \to f(x) \begin{cases} 1 & x \in T \\ 0 & x \notin T \end{cases}$. So if T is dense in $[0,1]$, then f is discontinuous at every point in $[0,1]$ [13.2].

Lemma 13.1

Let $\{f_n\}$ be a sequence of functions on X, and let $f : X \to \mathbb{R}$ be another function so that $\forall x \in X : \lim_{n\to\infty} f_n(x) = f(x)$. Let

$$M_n = \sup_{x\in X} |f_n(x) - f(x)|$$

Then $f_n \xrightarrow{\text{uniformly}} f$ on X if and only if $\lim_{n\to\infty} M_n = 0$.

Proof:

$\implies$ Suppose $f_n \xrightarrow{\text{uniformly}} f$. Then $\forall \epsilon > 0, \exists N, \forall n > N, \forall x \in X : |f_n(x) - f(x)| < \epsilon$. Then $\forall n > N : \sup_{x\in X} |f_n(x) - f(x)| \le \epsilon$ implies $M_n \to 0$.

$\impliedby$ Can be shown similarly. ■

Lemma 13.2 (*Cauchy's Criterion for Uniform Convergence*)

Let $\{f_n\}$ be a sequence of functions on X. Then $\{f_n\}$ be uniformly convergent on X if and only if

$$\forall \epsilon > 0, \exists N, \forall x \in X, \forall m, n > N : |f_n(x) - f_m(x)| < \epsilon$$

[13.1] In this example, switching the order of taking limit is not allowed as in order to have $f_n(x) \to f(x)$ for $x \to 1$, n needs to be large. And similarly, in order to see continuity of f_n at 1 for n large, x needs to be close to 1.
[13.2] **Remark**: If $T = [0,1] \cap \mathbb{Q}$, then $f \in \mathcal{R}$ on $[0,1]$ but $f \notin \mathcal{R}$.

Proof:

$\Longrightarrow$ Since $\{f_n\}$ is uniformly convergent, $\exists f : X \to \mathbb{R} : \forall \epsilon, \exists N, \forall x \in X, \forall n > N : |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Let $m, n > N$ and $x \in X$. Then $|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

$\Longleftarrow$ Since $\{f_n\}$ satisfied the Cauchy Criterion, we have $\forall x \in X : \{f_n(x)\}$ is a Cauchy in $\mathbb{R}$. Hence, $f_n(x) \to \ell_x$ where ℓ_x is the unique limit. Define $f(x) = \ell_x$. Then $f : X \to \mathbb{R}$. We claim $f_n \xrightarrow{\text{uniformly}} f$. Let $\epsilon > 0, \exists N, \forall x \in X, \forall m, n > N : |f_n(x) - f_m(x)| < \frac{\epsilon}{2}$. Now $\forall x \in X : |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$. So $f_n \xrightarrow{\text{uniformly}} f$. ■

Corollary 13.3

Suppose $\{f_n\}$ is a sequence of functions on X and assume $\forall x \in X : |f_n(x)| \leq M_n$. Assume that $\sum_{n=1}^{\infty} M_n$ converges. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly.

Proof: By the previous lemma, it suffices to show that $\sum f_n$ satisfies the Cauchy's Criterion. Let $\epsilon > 0$. Since $\sum M_n$ converges, $\exists N, \forall m \geq n > N : 0 \leq \sum_{i=n}^{m} M_i < \epsilon$. Now $|\sum_{i=n}^{m} f_i(x)| \leq \sum_{i=n}^{m} |f_i(x)| \leq \sum_{i=n}^{m} M_i < \epsilon$. Hence the sequence $s_n(x) = \sum_{i=1}^{n} f_i(x)$ is Cauchy and the previous lemma implies uniform convergent. ■

Corollary 13.4

Let f be ∞-differentiable on $\mathbb{R}$. Suppose $\forall M > 0, \exists T(M), \forall n :$ [13.3]

$$\sup_{x \in [-M, M]} |f^{(n)}(x)| < T$$

Then $P_{n,0}(x) \xrightarrow{\text{uniformly}} f$ on $[-M, M]$.

Proof: In order to show $P_{n,0}(x) \to f(x)$. We need to show: $\forall \epsilon, \exists N, \forall x \in [-M, M], \forall n > N : |P_{n,0}(x) - f(x)| < \epsilon$. Recall from Taylor's Theorem that $f(x) = P_{n,0}(x) + \frac{f^{(n+1)}(t)}{(n+1)!} x^{n+1}$ for some $t \in [-M, M]$. Hence $\forall x \in [-M, M] : |f(x) - P_{n,0}(x)| \leq \sup_{t \in [-M, M]} |f^{(n+1)}(t)| \cdot \frac{M^{n+1}}{(n+1)!} < T \cdot \frac{M^{n+1}}{(n+1)!}$. Recall now that $\lim_{n \to \infty} \frac{M^{n+1}}{(n+1)!} = 0$. So $\forall x \in [-M, M] : |f(x) - P_{n,0}(x)| < \epsilon$. ■

[13.3] $T(M)$ means that T depends on M.

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Lecture 14: Uniform Convergence and Continuity

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proposition 14.1

For $n \in \mathbb{N}$, define

$$\varphi_n(x) = \begin{cases} \frac{n}{2} & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Let $f \in C_c(\mathbb{R})$. Then $f * \varphi_n \xrightarrow{\text{uniformly}} f$.

Proof: Since $f \in C_c(\mathbb{R})$, f is uniformly continuous. So $\forall \epsilon > 0, \exists \delta, \forall s, t \in \mathbb{R} : |s - t| < \delta \implies |f(s) - f(t)| < \epsilon$. Recall the definition of $f * \varphi_n(x) = \int_{\mathbb{R}} f(x-y)\varphi_n(y)\mathrm{d}y = \int_{-1/n}^{1/n} f(x-y)\varphi_n(y)\mathrm{d}y = \frac{n}{2}\int_{-1/n}^{1/n} f(x-y)\mathrm{d}y$. Let now n be larger enough such that $\frac{1}{n} < \delta$. Then $|f * \varphi_n(x) - f(x)| = \frac{n}{2}\int_{-1/n}^{1/n} f(x-y)\mathrm{d}y - f(x) = |\frac{n}{2}\int_{-1/n}^{1/n} f(x-y)\mathrm{d}y - \frac{n}{2}\int_{-1/n}^{1/n} f(x)\mathrm{d}y| \leq \frac{n}{2}\int_{-1/n}^{1/n} |f(x-y) - f(x)|\mathrm{d}y \leq \frac{n}{2}(\epsilon(\frac{1}{n} - (-\frac{1}{n}))) = \epsilon$. Since $x \in \mathbb{R}$ is arbitrary, we get $f * \varphi_n \xrightarrow{\text{uniformly}} f$. ■

Remark: All we used in the proof is that:

1. $\varphi_n \geq 0$
2. $\int \varphi_n = 1$
3. $\forall \epsilon > 0, \exists N, \forall t \notin [-\delta, \delta], \forall n > N : \varphi_n(t) = 0$

Lemma 14.2

Let $E \subset X$ be a subset. Suppose $\{f_n\}$ and f are functions on E such that $f_n \xrightarrow{\text{uniformly}} f$ on E. Let $x \in X$ be a limit point of E, and $\forall n \in \mathbb{N}$: $\lim_{t\to x} f_n(t) = A_n$ exist. Then $\{A_n\}$ converges and

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

Proof: Let $\epsilon > 0$. Since $f_n \xrightarrow{\text{uniformly}} f$, they are uniformly Cauchy so $\exists N, \forall m, n > N, \forall t \in E : |f_m(t) - f_n(t)| < \epsilon$. Fix m, n and take the limit as $t \to x$. We get that $|A_m - A_n| \leq \epsilon$. So $\{A_n\}$ is Cauchy, and $\{A_n\} \to A$. For $t \in E$ and $n \in \mathbb{N}$, we have $|f(t) - A| = |f(t) - f_n(t) + f_n(t) - A_n + A_n - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|$. Since $f_n \xrightarrow{\text{uniformly}} f$ and $A_n \to A$, $\exists N, \forall t \in E, \forall n > N : |f(t) - f_n(t)| < \frac{\epsilon}{3}$ and $|A - A_n| < \frac{\epsilon}{3}$. Fix $n_0 > N$. Since $f_{n_0}(t) \to A_{n_0}$, $\exists \delta > 0 : 0 < d(t,x) < \delta \implies |f_{n_0}(t) - A_{n_0}| < \frac{\epsilon}{3}$. Let now $t \in E$ be such that $0 < d(t,x) < \delta$, then we have $|f(t) - A| \leq |f(t) - f_{n_0}(t)| + |f_{n_0}(t) - A_{n_0}| + |A_{n_0} - A| \leq 3 \cdot \frac{\epsilon}{3} = \epsilon$. Altogether, we conclude that $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$. ■

Remark: The lemma is asserting the following: if $f_n \xrightarrow{\text{uniformly}} f$, then

$$\lim_{t\to x}(\lim_{n\to\infty} f_n(t)) = \lim_{n\to\infty}(\lim_{t\to x} f_n(t))$$

Theorem 14.3

Let $\{f_n\}$ be a sequence of continuous functions on X. Suppose $f_n \xrightarrow{\text{uniformly}} f$ on X. Then f is continuous on X.

Proof: We will deduce this theorem from the lemma14.2. Indeed since f_n's are continuous, we have $\lim_{t\to x} f_n(t) = f_n(x) = A_n$. Since $f_n \xrightarrow{\text{uniformly}} f$, we have $\{f_n(x)\}$ converges to $f(x)$. Moreover, by the lemma, $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n = \lim_{n\to\infty} f_n(x) = f(x)$. Hence f is continuous. ■

e.g.1. For $n \in \mathbb{N}$, let $f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{n}] \cup [\frac{2}{n}, 1] \\ nx - \frac{1}{2} & x \in [\frac{1}{n}, \frac{3}{2n}] \\ -nx + \frac{5}{2} & x \in [\frac{3}{2n}, \frac{2}{n}] \end{cases}$

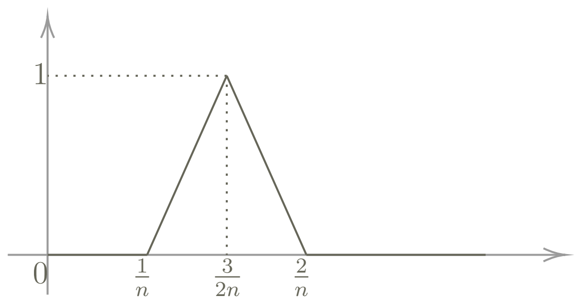

Figure 14.1: *e.g.*1

Then $f_n(x) \xrightarrow{\text{pointwise}} 0$. Indeed, $\forall x \neq 0$. If n is such that $\frac{2}{n} < x$, then $f_n(x) = 0$ so $\forall x \neq 0 : f_n(x) \to 0$. Moreover, $\forall n : f_n(x) = 0$ and f_n is continuous. However, f_n does not converge to f uniformly. Indeed, let $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)| = 1 \not\to 0$.

Proposition 14.4

Let K be compact. Assume

1. *$\{f_n\}$ is a sequence of continuous functions on K.*
2. *$\{f_n\}$ pointwise converge to a continuous function f on K.*
3. *$\forall n, \forall x \in K : f_n(x) \geq f_{n+1}(x)$.*

Then $f_n \xrightarrow{\text{uniformly}} f$.

Proof: Define $M_n = \sup_{x\in K} |f_n(x) - f(x)|$. We want to show $f_n \xrightarrow{\text{uniformly}} f$ by showing that $\lim_{n\to\infty} M_n = 0$. Let $\epsilon > 0$. We need to find N such that $\forall n > N : M_n < \epsilon$. This is to say, if $n > N$, then $\{x \in K : |f_n(x) - f(x)| < \epsilon\} = K$. i.e. $\{x \in K : |f_n(x) - f(x)| \geq \epsilon\} = \emptyset$. Define $g_n(x) = f_n(x) - f(x)$. Then g_n is continuous, $g_n(x) \xrightarrow{\text{pointwise}} 0$, and $g_n \geq g_{n+1}$. So in terms of g_n, we want to show $\exists N, \forall n > N : \{x \in K : |g_n| \geq \epsilon\} = \emptyset$. Define $K_n = \{x \in K : |g_n(x)| \geq \epsilon\}$. Then K_n is closed, and $K_n \subset K$. So K_n is compact. Since $g_n \geq g_{n+1}$, if $x \in K$ satisfies $g_{n+1}(x) \geq \epsilon$, then $g_n(x) \geq \epsilon$. Then $K_{n+1} \subset K_n$. Hence we get a nested sequence of compact sets $K_1 \subset K_2 \subset \ldots$. Note that $\bigcap_{n=1}^{\infty} K_n = \emptyset$. To see this, assume $\exists x \in \bigcap_{n=1}^{\infty} K_n$. Then $\forall x : g_n(x) \to 0$ so $\bigcap_{n=1}^{\infty} K_n = \emptyset$. We conclude that $\exists N : K_n = \emptyset$. Then $\forall n \geq N : K_n \subset K_N, K_n = \emptyset$ implies $\forall n \geq N : M_n < \epsilon$. ■

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Lecture 15: $C_b(X)$, Uniform Convergence and Integration & Differentiation

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Let X be a metric space. We denote

$$C_b(X) := \{f : X \to \mathbb{C} \quad \text{s.t.} \quad f \text{ is continuous and bounded}\}$$

If X is compact, then $C_b(X) = C(X)$ where $C(X)$ is the space of continuous functions. Also note that $C_c(X) \subset C_b(X)$. Define

$$\|f\|_\infty = \sup_{x\in X} |f(x)|$$

Proposition 15.1

$\|\cdot\|$ is a norm on $C_b(X)$.

Proof:

- $\|f\|_\infty \geq 0$. Moreover, if $\|f\|_\infty = 0$, then $\forall x \in X : f(x) = 0$ implies $f = 0$.
- $\|f + g\|_\infty = \sup_{x\in X} |f(x) + g(x)| \leq \sup_{x\in X} |f(x)| + |g(x)| \leq \sup_{x\in X} |f(x)| + \sup_{y\in X} |g(y)| = \|f\|_\infty + \|g\|_\infty$.
- $\forall \lambda \in \mathbb{C} : \|\lambda f\|_\infty = |\lambda|\|f\|_\infty$. ■

In view of this proposition, if we define

$$d(f, g) = \|f - g\|_\infty$$

Then $d(\cdot, \cdot)$ is a metric on $C_b(X)$.

e.g.1. $X = \{a, b\}$. Then $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$. $C_b(X) = \{f : \{a, b\} \to \mathbb{C}\}$. So $f \in C_b(X)$ is uniquely determined by two complex numbers namely $(f(a), f(b))$. So $C_b(X) \simeq \mathbb{C}^2$. Moreover, $d((z_1, z_2), (w_1, w_2)) = \|(z_1, z_2)-(w_1, w_2)\|_\infty = \max\{|z_1-w_1|, |z_2 - w_2|\}$.

Lemma 15.2

Let X be a metric space. A sequence $\{f_n\}$ of function in $C_b(X)$ converge in the metric d if and only if $\{f_n\}$ is uniformly convergent.

Proof: Suppose $\{f_n\}$ converge in the metric d. Then $\exists f \in C_b(X) : d(f_n, f) \to 0 \iff d(f_n, f) = \sup_{x\in X} \|f_n(x) - f(x)\|_\infty \to 0 \iff f_n \xrightarrow{\text{uniformly}} f$. ■

Proposition 15.3

$C_b(X)$ with d_∞ is a complete metric space.

Proof: Let $\{f_n\}$ be a Cauchy sequence in $C_b(X)$. Then $\forall\epsilon > 0, \exists N, \forall m, n > N$: $d(f_n, f_m) = \|f_n - f_m\|_\infty < \epsilon$. Therefore, $\{f_n\}$ is uniformly Cauchy. By a lemma, we know that uniform Cauchy implies uniform convergence. So let $f : X \to \mathbb{C}$ be the function such that $f_n \xrightarrow{\text{uniformly}} f$. We need to show that $f \in C_b(X)$: (1). Since $\{f_n\}$ are continuous and f is the uniform limit, f is continuous; (2). Since $f_n \xrightarrow{\text{uniformly}} f$ and $\{f_n\}$ are bounded, $\exists N : \forall n > N : \|f_n - f\|_\infty < 1$. Fix some $n > N$. Then $\|f\|_\infty < 1 + \|f_n\|_\infty$. Since $f_n \in C_b(X)$, $\|f_n\|_\infty < \infty$ so $\|f\|_\infty < \infty$. Then $f \in C_b(X)$. ■

15.1 Uniform Convergence and Integration

Proposition 15.4

Let α be increasing on $[a, b]$. Suppose $\forall n \in \mathbb{N} : f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, and assume that $f_n \xrightarrow{\text{uniformly}} f$. Then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f_n \mathrm{d}\alpha \to \int_a^b f \mathrm{d}\alpha$.

Proof: Since $f_n \xrightarrow{\text{uniformly}} f$, we have $\epsilon_n = \sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0$. Therefore, $\forall\epsilon > 0, \exists N, \forall n > N, \forall x \in [a, b] : |f_n(x) - f(x)| < \epsilon$. So $f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$. Hence $f_n \mathrm{d}\alpha - \epsilon(\alpha(b) - \alpha(a)) \leq \underline{\int} f \mathrm{d}\alpha \leq \int f_n \mathrm{d}\alpha + \epsilon(\alpha(b) - \alpha(a))$ and $f_n \mathrm{d}\alpha - \epsilon(\alpha(b) - \alpha(a)) \leq \overline{\int} f \mathrm{d}\alpha \leq \int f_n \mathrm{d}\alpha + \epsilon(\alpha(b) - \alpha(a))$. Then $|\underline{\int} f \mathrm{d}\alpha - \int f_n \mathrm{d}\alpha| < \epsilon(\alpha(b) - \alpha(a))$ and $|\overline{\int} f \mathrm{d}\alpha - \int f_n \mathrm{d}\alpha| < \epsilon(\alpha(b) - \alpha(a))$. This gives $|\underline{\int} f \mathrm{d}\alpha - \overline{\int} f \mathrm{d}\alpha| < 2\epsilon(\alpha(b) - \alpha(a))$. Since ϵ is arbitrary, $f \in \mathcal{R}(\alpha)$. Moreover, we have $\forall n > N : |\int f \mathrm{d}\alpha - \int f_n \mathrm{d}\alpha| < \epsilon(\alpha(b) - \alpha(a))$ so $\int_a^b f_n \mathrm{d}\alpha \to \int f \mathrm{d}\alpha$. ■

Remark: This say we can change the order of lim and $\int$ under the assumption of uniform convergence. i.e.

$$\lim_{n\to\infty} \int_a^b f_n \mathrm{d}\alpha = \int_a^b \lim_{n\to\infty} f_n \mathrm{d}\alpha$$

Remark: Uniform convergence is too strong for integration. Indeed the following holds true (do not prove it here):

Theorem 15.5 (*Lebesgue Dominated Convergence Theorem*)

Suppose $f_n \xrightarrow{\text{pointwise}} f$ and $\exists g \in \mathcal{R}, \forall n, \forall x \in [a, b] : |f_n(x)| \leq g(x)$. Then $\int_a^b f_n \mathrm{d}x = \int_a^b f \mathrm{d}x$.

Corollary 15.6

Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and that $f(x) = \sum_{n=1}^\infty f_n(x)$ uniformly converge on $[a, b]$. Then

$$\int_a^b f \mathrm{d}\alpha = \sum_{n=1}^\infty \int_a^b f_n \mathrm{d}\alpha$$

e.g.2. $f_n(x) = \frac{x^n}{n!}$, then $\sum f_n(x)$ uniformly converges and $\sum_{n=1}^\infty \int_a^b f_n \mathrm{d}x = \int_a^b (\sum f_n) \mathrm{d}x$.

15.2 Uniform Convergence and Differentiation

e.g.3. $f_n(x) = \frac{\sin nx}{\sqrt{n}}, f_n(x) \xrightarrow{\text{uniformly}} 0$. $f'_n(x) = \sqrt{n}\cos nx$. Then $\{f'_n(x)\}$ is not bounded.

Proposition 15.7

Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$. Assume

1. *$\exists x_0 \in [a, b] : \{f_n(x_0)\}$ converges.*
2. *$\{f'_n\}$ is continuous and uniformly convergent.*

Then $\{f_n\}$ uniformly converges to a differentiable function f on $[a, b]$ and $\forall x \in [a, b]$:

$$f'(x) = \lim_{n\to\infty} f'_n(x)$$

Proof: Define $\{f'_n\}$ are continuous. By FTC, $f_n(x) - f_n(x_0) = \int_{x_0}^{x} f'_n(t)\mathrm{d}t$. Now we show that $\{f_n\}$ uniformly converges. We will show that $\{f_n\}$ is uniformly Cauchy. Let $\epsilon > 0$ be arbitrary. $\exists N, \forall m, n > N, \forall x \in [a, b] : |f_n(x_0) - f_m(x_0)| < \epsilon$ and $|f'_n(x) - f'_m(x)| < \epsilon$. Let $m, n > N$. Then $|f_n(x) - f_n(x_0) - (f_m(x) - f_m(x_0))| = |f_n(x) - f_m(x) + f_m(x_0) - f_n(x_0)| = |\int_{x_0}^{x}(f'_n(t) - f'_m(t))\mathrm{d}t| \le \int_{x_0}^{x} |f'_n(t) - f'_m(t)|\mathrm{d}t < \epsilon(b - a)$. Now we have $|f_n(x) - f_m(x)| \le |f_n(x_0) - f_m(x_0)| + \epsilon(b - a) < \epsilon + \epsilon(b - a)$. Hence $\{f_n\}$ is uniformly Cauchy. Let $f_n \xrightarrow{\text{uniformly}} f$. Recall that $f'_n \xrightarrow{\text{uniformly}} g$. Since $\{f'_n\}$ are continuous, g is continuous. Let G be defined by $G(x) = f(x_0) + \int_{x_0}^{x} g(t)\mathrm{d}t$ [15.1]. Then by FTC, we have $G' = g = \lim_{n\to\infty} f'_n$. So the proof will be complete if we show $G = f$. Note that $f_n(x) - f_n(x_0) = \int_{x_0}^{x} f'_n(t)\mathrm{d}t$. By the proposition about uniform convergence and integration, we have $\int_{x_0}^{x} f'_n(t)\mathrm{d}t \to \int_{x_0}^{x} g(t)\mathrm{d}t$. So $f_n(x) - f_n(x_0) \to G(x) - f(x_0)$. Moreover, since $f_n \xrightarrow{\text{uniformly}} f$, we have $f_n(x) - f_n(x_0) \to f(x) - f(x_0)$ which implies $G = f$. ■

Remark: Condition (1) is necessary: $f_n(x) = n$ on $[a, b]$. Then $f'_n = 0$ uniformly converge but $\{f_n\}$ do not converge.

[15.1] Note that $f(x_0)$ is a constant.

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Lecture 16: Equicontinuity

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proposition 16.1

Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$. Assume

1. *$\exists x_0 \in [a, b] : \{f_n(x_0)\}$ converge.*
2. *$\{f_n'\}$ uniformly converges.*

Then $\{f_n\}$ uniformly converge to a differentiable function f on $[a, b]$ and $\forall x \in [a, b]$:

$$f'(x) = \lim_{n\to\infty} f_n'(x)$$

Proof: We will first show that $\{f_n\}$ uniformly converges. For that, we need to show $\{f_n\}$ is uniformly Cauchy. Given $\epsilon > 0, \exists N, \forall m, n > N, \forall x \in [a, b] : |f_n(x_0) - f_m(x_0)| < \epsilon$ and $|f_n'(x) - f_m'(x)| < \epsilon$. Fix $m, n > N$ and define $g(x) = f_n(x) - f_m(x)$ for $x \in [a, b]$. Let $s, t \in [a, b]$. By MVT, $\exists \xi : |g(s) - g(t)| = |g'(\xi)||s - t| \leq |g'(\xi)|(b - a)$. Rewriting this in terms of f, we have $|f_n(s) - f_m(s)(f_n(x_0) - f_m(x_0))| \leq |f_n'(\xi) - f_m'(\xi)|(b - a) \leq \epsilon(b - a)$. This implies that $\forall x \in [a, b], s = x, t = x_0 : |f_n(s) - f_m(s) - (f_n(x_0) - f_m(x_0))| \leq \epsilon(b - a)$. Then $|f_n(s) - f_m(s)| < \epsilon(b-a) + |f_n(x_0) - f_m(x_0)| < \epsilon(b-a) + \epsilon$. Hence $\{f_n\}$ is uniformly Cauchy so $f_n \xrightarrow{\text{uniformly}} f$ on $[a, b]$.

We now show that f is differentiable and $f_n'(x) \to f'(x)$. Recall that we define $f_n'(x) = \lim_{t\to x} \frac{f_n(x)-f_n(t)}{x-t}$. Fix $x \in [a, b]$ and define $\varphi_n(t) = \frac{f_n(x)-f_n(t)}{x-t}$ and $\varphi(t) = \frac{f(x)-f(t)}{x-t}$. Then $\forall t \in [a, b], t \neq x : \lim_{n\to\infty} \varphi_n(t) = \varphi(t)$; $\lim_{t\to x} \varphi_n(t) = f_n'(x)$; and $\ell_x = \lim_{n\to\infty} f_n'(x)$ exists.

We claim that $\{\varphi_n\}$ is uniformly convergent on $[a, b] \setminus \{x\}$. To see this, we need to show $\{\varphi_n\}$ is uniformly Cauchy. Let N be as before so $\forall m, n > N, \forall x \in [a, b]$: $|f_n(x_0) - f_m(x_0)| < \epsilon$ and $|f_n'(x) - f_m'(x)| < \epsilon$. Let $m, n > N$, recall that $|f_n(x) - f_m(x) - (f_n(t) - f_m(t))| = |f_n'(\xi) - f_m'(\xi)| \cdot |x - t| < \epsilon|x - t|$. Then $|(f_n(x) - f_n(t)) - (f_m(x) - f_m(t))| < \epsilon|x - t|$. This implies $\left|\frac{f_n(x)-f_n(t)}{x-t} - \frac{f_m(x)-f_n(t)}{x-t}\right| < \epsilon$. In other words, $|\varphi_n(t) - \varphi_m(t)| < \epsilon$. Now we have $\ell_x = \lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}(\lim_{t\to x} \varphi_n(t)) = \lim_{t\to x}(\lim_{n\to\infty} \varphi_n(t)) = \lim_{t\to x} \varphi(t) = \lim_{t\to x} \frac{f(x)-f(t)}{t-x}$. So f is differentiable at x, and $f'(x) = \ell_x = \lim_{n\to\infty} f_n'(x)$. ■

Equicontinuous Family of Function

Definition 16.1 (*Pointwise Bounded Family*)

*Let $\{f_n\}$ be a sequence of functions on set A. We say $\{f_n\}$ is **pointwise bounded family** if $\forall a \in A, \exists M_a \in \mathbb{R}, \forall n \in \mathbb{N} : |f_n(a)| \leq M_a$.*

Remark: If $\{f_k\}$ is a pointwise bounded family of functions, then $\exists g : A \to \mathbb{R}$ so $\forall a \in A, \forall n \in \mathbb{N} : f_n(a) \leq g(a)$. Indeed $\forall a \in A$, define $g(a) = \sup\{|f_n| : n =$

$1, 2, \dots\}$. Note that $\{|f_n(a) : n \in \mathbb{N}|\}$ is a bounded nonempty subset of $\mathbb{R}$ so the sup exists and is unique.

Definition 16.2 (*Uniformly Bounded Family*)

*Let $\{f_n\}$ be a sequence of functions on A. We say $\{f_n\}$ is **uniformly bounded family** if $\exists M \in \mathbb{R}, \forall a \in A, \forall n \in \mathbb{N} : |f_n(a)| \le M$.*

e.g.1. $A = [0, 1]$

- $\{f_n(x) = n\}$ is not pointwise bounded.
- $f_n(x) = \begin{cases} \frac{1}{x} & \frac{1}{n} \le x \le 1 \\ n^2 x & 0 \le x \le \frac{1}{n} \end{cases}$. Then $\{f_n\}$ is pointwise bounded. But not uniformly bounded.
- $f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{n}] \cup [\frac{2}{n}, 1] \\ nx - \frac{1}{2} & x \in [\frac{1}{n}, \frac{3}{2n}] \\ -nx + \frac{5}{2} & x \in [\frac{3}{2n}, \frac{2}{n}] \end{cases}$. Then $\{f_n\}$ is uniformly bounded: $\forall n, x : f_n(x) \le 1$ [16.1].

Remark: Uniform boundedness does not generally imply existence of a pointwise convergent subsequence.

e.g.2. $f_n(x) = \sin nx$. See the book.

Definition 16.3 (*Equicontinuous Family of functions*)

*Let $\mathcal{F}$ be a family of functions on a metric space X. We say $\mathcal{F}$ is an **equicontinuous family** if $\forall f \in \mathcal{F}$*

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in X, d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Remark: Note that if $\mathcal{F}$ is an equicontinuous family. Then every $f \in \mathcal{F}$ is uniformly continuous. But the converse if not necessarily true.

e.g.3. $f_n(x) = x^n$ on $[0, 1]$. Then f_n is continuous and $[0, 1]$ is compact. So f_n is uniformly continuous. However, $\{f_n\}$ is not equicontinuous.
Toward a contradiction, suppose $\{f_n\}$ is equicontinuous. Then for $\epsilon = \frac{1}{2}, \exists \delta > 0, \forall n : |x - y| < \delta \implies |f_n(x) - f_n(y)| < \frac{1}{2}$. Now suppose $y = 1$ and $x = 1 - \frac{\delta}{2}$, then $|x - y| = \frac{\delta}{2} < \delta$. We have $f_n(x) = (1 - \frac{\delta}{2})^n \to 0$ and $f(y) = 1$. So $|f_n(x) - f_n(y)| < \frac{1}{2}$. Contradiction.

e.g.4. $f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{n}] \cup [\frac{2}{n}, 1] \\ nx - \frac{1}{2} & x \in [\frac{1}{n}, \frac{3}{2n}] \\ -nx + \frac{5}{2} & x \in [\frac{3}{2n}, \frac{2}{n}] \end{cases}$ is not an equicontinuous family.
Indeed, let ϵ, if $\exists \delta : |x - y| < \delta \implies |f_n(x) - f_n(y)| < \frac{1}{2}$. $\forall x, y \in [0, 1], \forall n : \frac{1}{n} < \delta \implies \left|\frac{1}{n} - \frac{3}{2n}\right| = \frac{1}{2n} < \delta$. However, $\left|f_n(\frac{1}{n}) - f_n(\frac{3}{2n})\right| = |0 - 1| = 1 > \frac{1}{2}$.

[16.1] But $\{f_n\}$ is not uniformly convergent.

Lemma 16.2

Let $\{f_n\}$ be a family of functions on $[a, b]$. Assume that $\{f_n'\}$ is uniformly bounded. Then $\{f_n\}$ is equicontinuous.

Proof: Note that by MVT, we have $\forall x, y, \exists \xi \in (a, b) : |f_n(x) - f_n(y)| = |f_n'(\xi)||x - y|$. Let M be such that $\forall t \in [a, b] : |f_n'(t)| \leq M$. Then $\forall n : |f_n(x) - f_n(y)| \leq M|x - y|$. So given $\epsilon > 0$, if we let $\delta = \frac{\epsilon}{2M}$, then $|x - y| < \delta \implies |f_n(x) - f_n(y)| \leq M|x - y| \leq M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2} < \epsilon$. ■

Proposition 16.3

Let K be a compact metric space. Let $\{f_n\}$ be a sequence of continuous functions on K which is uniformly convergent. Then $\{f_n\}$ is equicontinuous.

Will prove in the next lecture.

Remark: Compactness is needed: let $f_n(x) = x^2$ for $x \in \mathbb{R}, n \in \mathbb{N}$. Then $f_1(x) = x^2, f_2(x) = x^2, \ldots, f(x) = x^2$. So $|f_n(x) - f_m(x)| = 0$ implies $f_n(x) \xrightarrow{\text{uniformly}} f(x) = x^2$. Note however that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$ so the family $\{f_n(x)\}$ is not equicontinuous.

MATH140B **Spring 2021**

Lecture 17: Arzela-Ascoli Theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Recall a family $\mathcal{F}$ of functions on X is said to be equicontinuous if $\forall\epsilon, \exists\delta > 0, \forall f \in \mathcal{F}, \forall x, y \in X : d(x, y) < \delta \implies d(f(x) - f(y)) < \epsilon$.

Proof to 16.3: Let $\epsilon > 0$. Since $\{f_n\}$ is uniformly convergent, $\exists N, \forall m, n > N$: $\|f_n - f_m\|_\infty < \frac{\epsilon}{3}$. So in particular, $\forall m \geq N : \|f_m - f_N\|_\infty < \frac{\epsilon}{3}$. Since f_N is continuous and K is compact, f_N is uniformly continuous. Then $\exists\delta > 0, \forall x, y \in K$: $d(x, y) < \delta \implies |f_N(x) - f_N(y)| < \frac{\epsilon}{3}$. Let $m \geq N$, and suppose $|f_m(x) - f_m(y)| \leq |f_m(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_m(y)| < \epsilon$. Therefore, if $d(x, y) < \delta$, then $\forall m \geq N : |f_m(x) - f_N(y)| < \epsilon$. Now the functions $f_1, \ldots, f_{N-1}$ are continuous, and K is compact. Therefore, they are uniformly continuous. $\exists\delta', \forall n \in \mathbb{N} : d(x, y) < \delta' \implies |f_n(x) - f_n(y)| < \epsilon$. Let $\hat{\delta} = \min\{\delta, \delta'\}$. Then $\forall x, y \in K, \forall n \in \mathbb{N} : d(x, y) < \hat{\delta} \implies |f_n(x) - f_n(y)| < \epsilon$. ∎

Lemma 17.1

Let X be any metric space. If $f_n \xrightarrow{\text{pointwise}} f$, then $\{f_n\}$ is pointwise bounded.

Proof: Since every converging sequence of number is bounded, $\forall x \in X : \{f_n(x)\}$ is bounded. So $\{f_n\}$ is pointwise bounded. ∎

Lemma 17.2

Let $\{f_n\}$ be a sequence of pointwise bounded complex-valued functions on a countable set A. Then there exists a subsequence $\{f_{n_j}\}$ that is pointwise convergent.

Simpler version of the lemma: Let $A = \{a_1, a_2\}$. Then $f_1(a_1), f_2(a_1), \ldots$ is a bounded sequence. By Heine-Borel Theorem, there exists a subsequence $n_{1,k}$ such that $f_{n_{1,k}}(a_1)$ is convergent, and there exists a subsequence $n_{2,k}$ such that $f_{n_{2,k}}(a_2)$ is convergent.

Proof: Since A is countable, we will write $A = \{a_1, a_2, \ldots\}$. Since $\{f_n(a_1)\}$ is bounded, there exists a subsequence of this sequence which converges by Heine-Borel Theorem. We denote this subsequence by $f_{1,k}$. Then $f_{1,k}(a_1)$ converges. Now consider $\{f_{1,k}(a_2)\}$. This is a bounded sequence so there exists a subsequence $\{f_{2,k}\}$ so that $\{f_{2,k}(a_2)\}$ converges. Continue this inductively, we get an array of functions

$$\begin{array}{cccc} f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

so that $f_{i,k}$ is a subsequence of $f_{i-1,k}$; $\forall i : \{f_{i,k}(a_i)\}$ converges; the order at which the functions appear is the same as the order in the original sequence $\{f_n\}$. Now choose the diagonal terms $f_{1,1}, f_{2,2}, \ldots$. We claim $\{f_{n,n}(a_k)\}$ converges. Indeed, $\{f_{n,n}(a_i) : n \geq i\}$ is a subsequence of the i-th row which is $\{f_{i,k}(a_i)\}$ and converges by the definition. ∎

Theorem 17.3 (*Arzela-Ascoli*)

Let K be a compact metric space. Let $\{f_n\}$ be a sequence of continuous functions on K. Assume $\{f_n\}$ is equicontinuous and pointwise bounded. Then

1. *$\{f_n\}$ is uniformly bounded.*
2. *There exists a subsequence of $\{f_n\}$ that is uniformly convergent.*

Proof:

1. Since $\{f_n\}$ is equicontinuous, $\forall\epsilon > 0, \exists\delta > 0, \forall n : d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \epsilon$. Let $\{N_\delta(x) : x \in K\}$ be a covering of K with δ-neighborhood of points. Since K is compact, $\exists x_1, \ldots, x_\ell : K = \bigcup_{i=1}^{\ell} N_\delta(x_\ell)$. Let now $x \in K$ be arbitrary. Then $\exists 1 \leq i \leq \ell : x \in N_\delta(x_i)$. So $\forall n : d(x, x_i) < \delta \implies |f_n(x) - f_n(x_i)| < \epsilon \implies |f_n(x)| < |f_n(x_i)| + \epsilon$. Now let $\epsilon = 1$, and $M = \sup\{f_n(x_i) : 1 \leq i \leq \ell, n \in \mathbb{N}\}$. Then M is finite because $\{f_n\}$ is pointwise bounded. Hence, $|f_n(x)| < M + 1$. So $\{f_n\}$ is uniformly bounded.
2. Since K is compact, $\exists A \subset K$ which is countable and dense. By a lemma applied to $\{f_n\}$ and A, there exists a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ which is pointwise convergent on A. Since $\{f_{n_j}\}$ is equicontinuous, $\forall\epsilon > 0, \exists\delta > 0, \forall n : d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \epsilon$. Let $\{N_\delta(a) : a \in A\}$ be a covering of K. Since K is compact, $\exists a_1, \ldots, a_\ell \in A : K = \bigcup_{i=1}^{\ell} N_\delta(a_i)$. Since $\{f_{n_j}(a_k)\}$ converges., $\forall k, \exists N, \forall n_j, n_i > N : \left|f_{n_i}(a_k) - f_{n_j}(a_k)\right| < \epsilon$. Now let $x \in K$, then $\exists 1 \leq k \leq \ell : x \in N_\delta(a_k)$. Now if $n_i, n_j > N$, we have $\left|f_{n_i}(x) - f_{n_j}(x)\right| \leq |f_{n_i}(x) - f_{n_i}(a_k)| + \left|f_{n_i}(a_k) - f_{n_j}(a_k)\right| + \left|f_{n_j}(a_k) - f_{n_i}(x)\right| < 3\epsilon$. So $\{f_{n_j}\}$ is uniformly Cauchy on K. Hence $\{f_{n_j}\}$ is uniformly convergent. ■

Corollary 17.4

Let $\{f_n\}$ be a sequence of continuous functions on a compact metric space K. Let $A = \overline{\{f_n\}}$ w.r.t. d_∞. If $\{f_n\}$ is equicontinuous and pointwise bounded, then A is compact.

Proof: Let $\{g_n\} \subset A$. In order to show that A is compact, we want to show that $\exists\{g_{n_i}\} : \{g_{n_i}\}$ is uniformly convergent. By AA Theorem, we need to show that $\{g_n\}$ is pointwise bounded and continuous. Since $\{g_n\} \subset A = \overline{\{f_n\}}$, $\forall n, \exists f_{i_n} : \|g_n - f_{i_n}\| < 1$. So $g_n(x) \leq f_{i_n}(x) + 1$. Since $\{f_{i_n}\}$ is pointwise bounded, $\{g_n\}$ is pointwise bounded. Now let $\epsilon > 0$. $\forall n \in \mathbb{N}, \exists i_n : \|g_n - f_{i_n}\| < \epsilon$. Now $\{f_n\}$ is equicontinuous. So $\exists\delta > 0, \forall n : d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \epsilon$. Let $x, y \in K$ be such that $d(x, y) < \delta \implies |g_n(x) - g_n(y)| < |g_n(x) - f_{i_n}(x)| + |f_{i_n}(x) - f_{i_n}(y)| + |f_{i_n}(y) - g_n(y)| < 3\epsilon$. So $\{g_n\}$ is equicontinuous. So by AA Theorem, there exists a subsequence that is uniformly convergent. ■

MATH140B **Spring 2021**

Lecture 18: Weierstrass Theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

e.g.1. Consider $f(x) = |x|$ on $[-1, 1]$. There exists a sequence of polynomials $\{P_n\}$ on $[-1, 1]$ such that $P_n(x) \xrightarrow{\text{uniformly}} |x|$.

Lemma 18.1

Let $a_1, \dots, a_n \in \mathbb{R}$ where $\forall i \neq j : a_i \neq a_j$, and $b_1, \dots, b_n \in \mathbb{R}$. Then there exists a polynomial of degree at most $n-1$ such that

$$P(a_i) = b_i$$

Proof 1° (sketch): A polynomial of degree $n-1$ is determined by $c_0, c_1, \dots, c_{n-1} \in \mathbb{R}$. i.e. $p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1}x^{n-1}$. Then $p(a_1) = b_1, p(a_2) = b_2, \dots, p(a_n) = b_n$. Since we have n unknown variable and n equations, we construct

$$A = \begin{pmatrix} 1 & a_1 & \dots & a_1^{n-1} \\ \vdots & & & \\ 1 & a_n & \dots & a_n^{n-1} \end{pmatrix}$$

where $\det(A) \neq 0$ since $a_i \neq a_j$. ■

Proof 2°: Construct $p(x)$ as $p(x) = b_1 \frac{(x-a_2)\cdots(x-a_n)}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)} + b_1 \frac{(x-a_1)\cdots(x-a_n)}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)} + \cdots + b_n \frac{(x-a_1)(x-a_2)\cdots(x-a_{n-1})}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}$. Then $\forall i : p(a_i) = \sum_{i=1}^n b_i (\prod_{j \neq i} \frac{x-a_j}{a_i-a_j})$. ■

Theorem 18.2 (*Weierstrass Theorem*)

Let f be a continuous complex-valued function on $[a, b]$. Then there exists a sequence of polynomials $\{P_n\}$ such that

$$P_n \xrightarrow{\text{uniformly}} f \quad \text{on } [a, b]$$

If f is real-valued, then $\{P_n\}$ may be taken to be real.

Proof: The complete proof is implemented by the following steps:

Step 1: We may replace $[a, b]$ with $[0, 1]$.

Indeed, there is a bijection between $[0, 1]$ and $[a, b]$: $f(t) = a + t(b-a)$. So if $f \in C[0, 1]$ is a uniform limit of polynomial, then $\forall g \in C([a, b])$: g is a uniform limit of polynomial. Let $g \in [a, b]$, $g : [a, b] \to \mathbb{C}$. Define $f(t) = g(\varphi(t))$. Then $f : [0, 1] \to \mathbb{C}$. So $\exists \{P_n\}$ on $[0, 1]$ such that $P_n \xrightarrow{\text{uniformly}} f$. Hence, $P_n \circ \varphi^{-1} : [a, b] \to \mathbb{C}$ is polynomial and $P_n \circ \varphi^{-1} \xrightarrow{\text{uniformly}} g$.

Step 2: We may assume $f(0) = f(1) = 0$.

Let $g(x) = f(x) - L(x)$ where $L(x)$ is linear and $L(0) = f(0), L(1) = f(1)$ [18.1]. So $L(x) = f(0) + (f(1) - f(0))x$. Now $g(x) = f(x) - f(0) - (f(1) - f(0))x$ satisfies $g(0) = g(1) = 0$. So if $\exists P_n : P_n \xrightarrow{\text{uniformly}} g$, then $P_n + L \xrightarrow{\text{uniformly}} f$.

So far we have $f \in C([0,1])$, $f(0) = f(1) = 0$. We assume f is real valued. Now we extend f to $\mathbb{R}$ as follows: $f(x) = 0$ if $x \notin [0,1]$. Then we may consider $f \in C_c(\mathbb{R})$.

Step 3: If $p : \mathbb{R} \to \mathbb{R}$ is a polynomial, then $f * p$ is a polynomial [18.2].

Step 4: Let $\varphi_n(x) = c_n(1-x^2)^n$ where $c_n > 0$ is chosen so that $\int_{-1}^1 \varphi_n = 1$. Then $c_n \le \sqrt{n}$; $\forall \delta > 0, \varphi_n \xrightarrow{\text{uniformly}} 0$ on $\delta \le |x| \le 1$.

Recall c_n is defined so that $\int_{-1}^1 c_n(1-x^2)^n \mathrm{d}x = 1$. So $\frac{1}{c_n} = \int_{-1}^1 (1-x^2)^n \mathrm{d}x$. Then we need to estimate $\int_{-1}^1 (1-x^2)^n \mathrm{d}x$. Since $(1-x^2)^n \ge 1 - nx^2$ on $(0,1)$, $\int_{-1}^1 (1-x^2)^n \mathrm{d}x = 2\int_0^1 (1-x^2)^n \mathrm{d}x \ge 2\int_0^{1/\sqrt{n}} (1-x^2)^n \mathrm{d}x \ge 2\int_0^{1/\sqrt{n}} (1-nx^2)\mathrm{d}x = \frac{4}{3\sqrt{n}} > \frac{1}{\sqrt{n}}$. So $c_n < \sqrt{n}$. Now note that $\delta \le |x| \le 1$. Then $\varphi_n(x) = c_n(1-x^2)^n \le \sqrt{n}(1-\delta^2)^n$. So $\varphi_n \xrightarrow{\text{uniformly}} 0$ on $\delta \le |x| \le 1$.

Define $\hat{\varphi}_n(x) = \begin{cases} \varphi_n & x \in [-1,1] \\ 0 & x \notin [-1,1] \end{cases}$. Since $f * \varphi = \int_{\mathbb{R}} f(y)\varphi_n(x-y)\mathrm{d}y = \int_{-1}^1 f(y)\varphi_n(x-y)\mathrm{d}y$ and $\hat{\varphi}_n$ satisfies the properties of hw7(C) [18.3]. So $f * \hat{\varphi}_n \xrightarrow{\text{uniformly}} f$. ■

Corollary 18.3

Let $a > 0$. Then there exists a sequence of polynomials $\{P_n\}$ on $[-a,a]$, satisfying $P_n(0) = 0$, so that $P_n \xrightarrow{\text{uniformly}} |x|$ on $[-a,a]$.

Proof: By Weierstrass Theorem, $\exists \{q_n\} : q_n \xrightarrow{\text{uniformly}} |x|$ on $[-a,a]$. Let $P_n = q_n - q_n(0)$. Then $P_n(0) = 0$. Moreover, $P_n \xrightarrow{\text{uniformly}} |x|$ on $[-a,a]$ (note that $q_n(0) \xrightarrow{\text{uniformly}} 0$). ■

Definition 18.1 (*Algebra*)

*Let E be a set. A collection $\mathcal{A}$ of complex-valued functions on E is said to be an **algebra** if*

1. $\forall f, g \in \mathcal{A} : f + g \in \mathcal{A}$.
2. $\forall f, g \in \mathcal{A} : fg \in \mathcal{A}$.
3. $\forall \lambda \in \mathbb{C}, \forall f \in \mathcal{A} : \lambda f \in \mathcal{A}$.

e.g.2. $\mathcal{A} = \{p_n \text{ on } [0,1]\}$ is an algebra. Note that $\forall x \in [0,1] : p(x) = 1 \in \mathcal{A}$, and $\forall p \in \mathcal{A} : 1 \cdot p = p$.

e.g.3. $\mathcal{A}' = \{p : [0,1] \to \mathbb{C} : p(x) = xq(x) \text{ where } q(x) \text{ is a polynomial}\}$ is an algebra. Let $p, q \in \mathcal{A}'$. Then $p = c_1x + \cdots + c_nx^n$, $q = d_1x + \cdots + d_mx^m$. Then

[18.1] i.e. L is the line through $(0, f(0))$ and $(1, f(1))$.
[18.2] Done in homework.
[18.3] Let $\varphi_n : \mathbb{R} \to \mathbb{R}$ be a sequence of functions with the following properties.

1. $\int_{-\infty}^{\infty} \varphi_n(x)\mathrm{d}(x) = 1$ for all $n \in \mathbb{N}$.
2. For all $n \in \mathbb{N}$ we have $\varphi_n(x) \ge 0$ for all $x \in \mathbb{R}$ and $\varphi_n(x) = 0$ for all $|x| \ge 1$.
3. For every $\delta > 0$ we have $\lim_{n\to\infty} \int_{|x|>\delta} \varphi_n(x)\mathrm{d}(x) = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function so that $f(x) = 0$ for all $|x| \ge 1$.

Then $\{f * \varphi_n\} \xrightarrow{\text{uniformly}} f$.

$p_1 + p_2 = (c_1 + d_1)x + \cdots \in \mathcal{A}'$; $p_1 p_2 \in \mathcal{A}'$; $\forall \lambda \in \mathbb{C}, \forall p \in \mathcal{A}' : \lambda p \in \mathcal{A}'$. Note however that $p_0(x) = 1$ does not belong to $\mathcal{A}'$. Indeed, $\nexists p_1 \in \mathcal{A}', \forall p \in \mathcal{A}' : p_1 \cdot p = p$.

Remark: If we consider real-valued functions on a set E. We say $\mathcal{A}$ is an algebra

1. $\forall f, g \in \mathcal{A} : f + g \in \mathcal{A}$.
2. $\forall f, g \in \mathcal{A} : fg \in \mathcal{A}$.
3. $\forall \lambda \in \mathbb{R}, \forall f \in \mathcal{A} : \lambda f \in \mathcal{A}$.

MATH140B **Spring 2021**

Lecture 19: Stone-Weierstrass Theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 19.1 (*Closed, Closure*)

*Let E be a set and $C(E) = \{f : E \to \mathbb{C} : f \text{ is bounded and continuous}\}$. A subset $F \subset C_b(F)$ is said to be **closed** if $\forall f_n \in F : f_n \xrightarrow{\text{uniformly}} f \implies f \in F$. If $\mathcal{F} \subset C_b(E)$ is a set, we define the (uniform) **closure** of F to be the set of all uniform limits of sequences in F.*

Proposition 19.1

Let $\mathcal{A} \subset C_b(E)$ be an algebra, and B be the uniform closure of $\mathcal{A}$. Then B is an algebra.

Proof: Let $f, g \in B$ and $\lambda \in \mathbb{C}$. We need to show (1) $f + g \in B$: Since $f, g \in B$ and B is the uniform closure of $\mathcal{A}$, there exists $\{f_n\} \subset \mathcal{A}$ and $\{g_n\} \subset \mathcal{A}$ such that $f_n \xrightarrow{\text{uniformly}} f$ and $g_n \xrightarrow{\text{uniformly}} g$. Now $f_n + g_n \in \mathcal{A}$. Moreover, $f_n + g_n \xrightarrow{\text{uniformly}} f + g \in B$. Other conditions $fg \in B$ and $\lambda f \in B$ can be shown similarly. ■

e.g.1. If $K = [0, 1]$ and $A \subset C([0, 1])$ is the algebra of polynomials. Then by Weierstrass's Theorem, we know that $\overline{A} = C([0, 1])$.

e.g.2. Recall $\mathcal{A}' = \{p : [0, 1] \to \mathbb{C} \mid p(x) = xq(x) \text{ where } q(x) \text{ is a polynomial}\}$. Then $\overline{\mathcal{A}'} \neq C([0, 1])$. Indeed, $\forall p \in \mathcal{A}' : p(0) = 0$. Now if $p_n \xrightarrow{\text{uniformly}} f$, we have $0 = p_n(0) \to f(0) \implies f(0) = 0$. So $\forall f \in \overline{\mathcal{A}'} : f(0) = 0$. Then $\overline{\mathcal{A}'} \neq C([0, 1])$.

Definition 19.2

Let $\mathcal{A}$ be an algebra of functions on E.

1. *We say $\mathcal{A}$ **separates** points if $\forall x_1, x_2 \in E, \exists f \in \mathcal{A} : x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$.*
2. *We say $\mathcal{A}$ **vanishes** at $x \in E$ if $\forall f \in \mathcal{A} : f(x) = 0$. If $\mathcal{A}$ vanishes at no point in E, $\forall x \in E, \exists f \in \mathcal{A} : f(x) \neq 0$.*

e.g.3. $\mathcal{A}' = \{p : [0, 1] \to \mathbb{C} \mid p(x) = xq(x) \text{ where } q(x) \text{ is a polynomial}\}$ vanishes at 0. $\mathcal{A}'' = \{p : [-1, 1] \to \mathbb{C} \mid p \text{ is a polynomial and even}\}$. Then $\mathcal{A}''$ does not separate points. Indeed, $\forall p \in \mathcal{A}'' : p(1) = p(-1)$.

Lemma 19.2

Let $\mathcal{A}$ be an algebra of functions on E which separates points and vanishes at no point in E. Let $x_1, x_2 \in E, x_1 \neq x_2$ and $c_1, c_2 \in \mathbb{C}$ (we take $c_1, c_2 \in \mathbb{R}$ when $\mathcal{A}$ is an algebra of real-valued functions). Then $\exists f \in \mathcal{A} : f(x_1) = c_1, f(x_2) = c_2$.

Proof:

Simpler case: Assume $\mathcal{A}$ contains constant function $f_0 : E \to \mathbb{C}, f_0(x) = 1$. Recall that $\mathcal{A}$ separates points. So $\exists g \in \mathcal{A} : g(x_1) \neq g(x_2)$. Let $u(x) = g(x) - g(x_1) \cdot f_0(x)$ and $v(x) = g(x) - g(x_2) \cdot f_0(x)$. Then $u(x_1) = 0, u(x_2) = g(x_2) - g(x_1) \neq 0$ and $v(x_2) = 0, v(x_1) = g(x_1) - g(x_2) \neq 0$. Define $f(x) = \frac{c_1 v(x)}{v(x_1)} + \frac{c_2 u(x)}{u(x_2)}$. Then $f(x_1) = c_1$ and $f(x_2) = c_2$.

General case: Since $\mathcal{A}$ separates points, $\exists g \in \mathcal{A} : g(x_1) \neq g(x_2)$. Since $\mathcal{A}$ vanishes at no points, $\exists p, q \in \mathcal{A} : p(x_1) \neq 0, q(x_2) \neq 0$. Let $u(x) = g(x)p(x) - g(x_1)p(x)$ and $v(x) = g(x)q(x) - g(x_2)q(x)$. Then $u, v \in \mathcal{A}$, and we have $u(x_1) = 0, u(x_2) = p(x_2)(g(x_2) - g(x_1)) \neq 0$ and $v(x_2) = 0, u(x_1) = p(x_1)(g(x_1) - g(x_2)) \neq 0$. Now define $f(x) = \frac{c_1 v(x)}{v(x_1)} + \frac{c_2 u(x)}{u(x_2)}$. Then $f(x_1) = c_1$ and $f(x_2) = c_2$. ■

Theorem 19.3 (*Stone-Weierstrass Theorem*)

Let $\mathcal{A}$ be an algebra of real-valued continuous functions on a compact set K. Assume $\mathcal{A}$ separates points and vanishes at no points. Then the uniform closure of $\mathcal{A}$ is

$$\overline{\mathcal{A}} = \{f : K \to \mathbb{R} \quad | \quad f \text{ is continuous}\}$$

In other words, for any continuous function $f : K \to \mathbb{R}$, $\exists \{f_n\} \in \mathcal{A} : f_n \xrightarrow{\text{uniformly}} f$.

Proof: Let B be the uniform closure of $\mathcal{A}$. Then B is a uniformly closed algebra.

Step1: Let $f \in B$. Then $|f| \in B$.

Let $a = \sup\{|f(x)| : x \in K\} \in \mathbb{R}$. By the corollary to Weierstrass Theorem, $\forall \ell \in \mathbb{N}, \exists p_\ell(y) = \sum_{i=1}^n c_i y^i, p(0) = 0, \forall y \in [-a, a] : \left|\sum_{i=1}^n c_i y^i - |y|\right| < \frac{1}{\ell}$. Let now $y = f(x)$ for some $x \in K$. Then $\forall x \in K : \left|\sum_{i=1}^n c_i (f(x))^i - |f(x)|\right| < \frac{1}{\ell}$. Note that $g_\ell = \sum_{i=1}^n c_i f^i \in B$ [19.1]. Altogether, we have $\|g_\ell - |f|\|_\infty < \frac{1}{\ell}$. So $g_\ell \xrightarrow{\text{uniformly}} |f|, g_\ell \in B$. Since B is uniformly closed, we get that $|f| \in B$.

Step2: Let $f, g \in B$. Then $\max\{f, g\}, \min\{f, g\} \in B$ [19.2].

Given two functions f, g, we can compute $\max\{f, g\} = \frac{f+g}{2} + \frac{|f-g|}{2}$ and $\min\{f, g\} = \frac{f+g}{2} - \frac{|f-g|}{2}$. Since B is an algebra, the conclusion follows from step1. Note that using induction, we have $\forall f_1, \ldots, f_n \in B : \max\{f_1, \ldots, f_n\} \in B, \min\{f_1, \ldots, f_n\} \in B$.

Continue in the next lecture. ■

Remark: If $K = [a, b]$ and $\mathcal{A}$ is the algebra of polynomials, then we get the Weierstrass Theorem.

[19.1] This explains why we should not have the constant term c_0 in p_ℓ.

[19.2] $m = \max\{f, g\}$ is defined by $m(x) = \max\{f(x), g(x)\}$. $\min\{f, g\}$ is defined similarly.

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Lecture 20: Baire's Category Theorem

Lecturer: Amir Mohammadi — *Scribes: Rabbittac*

Proof continuing of 19.3:

Step3: Let $f : K \to \mathbb{R}$ be continuous and $\epsilon > 0$. $\forall x \in K, \exists g_x \in B : g_x(x) = f(x)$ and $\forall y \in K : g_x(y) > f(y) - \epsilon$.

Recall that $\mathcal{A}$ separates points and vanishes at no points. Moreover, $A \subset B$ implies B separates points and vanishes at no points. Hence, by a lemma from the last lecture, $\forall y \in K, \exists h_y \in B : h_y(x) = f(x), h_y(y) = f(y)$. Since h_y and f are continuous on K, $h_y - f$ is continuous on K. So $\exists \delta_y > 0, \forall t \in N_{\delta_y}(y) : |h_y(t) - f(t)| < \epsilon$. Now $\{N_{\delta_y}(y) : y \in K\}$ is an open covering of K. Since K is compact, $\exists y_1, \ldots, y_n : K = \bigcup_{i=1}^n N_{\delta_{y_i}}(y_i)$. Let $g_x = \max\{h_{y_1}, \ldots, h_{y_n}\}$. Then by step2, $g_x \in B$. We show that $g_x(x) = f(x)$ and $\forall s \in K : g_x(s) > f(s) - \epsilon$. Indeed, $g_x(x) = \max\{h_{y_1}(x), \ldots, h_{y_n}(x)\} = \max\{f(x), \ldots, f(x)\} = f(x)$. Let $s \in K$ be arbitrary. Since $K = \bigcup_{i=1}^n N_{\delta_{y_i}}(y_i)$, $\exists 1 \le i \le n, \forall s \in N_{\delta_{y_i}}(y_i) : h_{y_i} > f(s) - \epsilon$. Hence, $g_x(s) = \max\{h_{y_1}(s), \ldots, h_{y_n}(s)\} \ge h_{y_i}(s) > f(s) - \epsilon$.

Step4: Let $f : K \to \mathbb{R}$ be continuous and $\epsilon > 0$. Then $\exists g \in B : \|f - g\|_\infty < \epsilon$.

Note that this indeed finishes the proof. Indeed, applying step4 with $\epsilon = \frac{1}{n}$ for $n \in \mathbb{N}$, we get a sequence $\{g_n\} \subset B : g_n \xrightarrow{\text{uniformly}} f$. Since B is closed, $f \in B$. So $B = \{f : K \to \mathbb{R} \quad | \quad f \text{ is continuous}\}$.

By step3, $\forall x \in K, \exists g_x \in B : g_x(x) = x$ and $\forall s \in K : g_x(s) > f(s) - \epsilon$. Since g_x is continuous and f are continuous, $\exists \eta_x > 0, \forall t \in N_{\eta_x}(x) : |f(t) - g_x(t)| < \epsilon$. So $\forall t \in N_{\eta_x}(x) : f(t) - \epsilon < g_x(t) < f(t) + \epsilon$ [20.1]. So we write $g_x(t) < f(t) + \epsilon$ for $t \in N_{\eta_x}(x)$. Then $\{N_{\eta_x}(x) : x \in K\}$ is an open covering of K, and K is compact. So $\exists x_1, \ldots, x_\ell : K = \bigcup_{i=1}^\ell N_{\ell_{x_i}}(x_i)$. Define $g = \min\{g_{x_1}, \ldots, g_{x_\ell}\}$. We claim $g \in B$ and $\|g - f\|_\infty < \epsilon$. Indeed, $g \in B$ by step2. To see $\|g - f\|_\infty < \epsilon$, let $s \in K$ be arbitrary. Then $\forall 1 \le i \le \ell : g_{x_i}(s) > f(s) - \epsilon$. So $g(s) = \min\{g_{x_i}(x)\} > f(s) - \epsilon$. To see the upper bound, since $K = \bigcup_{i=1}^\ell N_{\eta_{x_i}}(x_i)$, $\exists 1 \le i \le \ell, \forall s \in N_{\eta_{x_i}}(x_i) : g_{x_i}(s) < f(s) + \epsilon$. So $g(x) = \min\{g_{x_1}(s), \ldots, g_{x_n}(s)\} \le g_{x_i}(s) < f(s) + \epsilon$. Altogether, $\forall s \in K : f(s) - \epsilon < g(s) < f(s) + \epsilon$ implies $\|f - g\| < \epsilon$. ∎

Definition 20.1 (*Self-adjoint*)

Let $\mathcal{A}$ be an algebra of complex valued functions on a set E. We say $\mathcal{A}$ is ***self-adjoint*** *if $\forall f \in \mathcal{A} : \overline{f} \in \mathcal{A}$ where $\forall x \in E : (\overline{f})(x) = \overline{f(x)}$.*

Theorem 20.1

Let $\mathcal{A}$ be a self-adjoint algebra of complex-valued functions on a compact set K. Assume $\mathcal{A}$ separates points and vanishes at no points. Then the uniform closure of $\mathcal{A}$ is

$$\overline{\mathcal{A}} = \{f : K \to \mathbb{C} \quad | \quad f \text{ is continuous}\}$$

[20.1] This actually holds for all $s \in K$.

Proof: Recall that $f : K \to \mathbb{C}$ can be written as $f = u + vi$ where $u : K \to \mathbb{R}$ and $v : K \to \mathbb{R}$. So it suffices to show that the uniform closure of $\mathcal{A}$ contains all real-valued continuous functions. Let $\mathcal{A}_{\mathbb{R}} = \{f \in \mathcal{A} : f \text{ is real-valued}\}$. Note that $\mathcal{A}_{\mathbb{R}}$ is an algebra of real-valued functions. We want to show that $\mathcal{A}_{\mathbb{R}}$ separates points and vanishes at no points. To see this, we will show that $\mathcal{A}_{\mathbb{R}}$ contains the real and imaginary parts of all $f \in \mathcal{A}$. Indeed, let $f = u+vi$. Since $\mathcal{A}$ is self-adjoint, $\overline{f} = u-vi \in \mathcal{A}$. Now $\frac{f+\overline{f}}{2} = u \in \mathcal{A}$. But u is real-valued so $u \in \mathcal{A}_{\mathbb{R}}$. Similarly, $fi = ui - v \in \mathcal{A}$. So $\frac{fi+\overline{fi}}{2} = -v \in \mathcal{A}_{\mathbb{R}}$, which implies $v \in \mathcal{A}_{\mathbb{R}}$. Now since $\mathcal{A}$ separates points, $\forall x_1 \neq x_2 \in K, \exists f \in \mathcal{A} : f(x_1) \neq f(x_2)$. Hence, $u(x_1) + v(x_1)i \neq u(x_2) + v(x_2)i$. Then either $u(x_1) \neq u(x_2)$ or $v(x_1) \neq v(x_2)$ holds. So $\mathcal{A}_{\mathbb{R}}$ separates points. Since $\mathcal{A}$ vanishes at no points, $\forall x \in K, \exists f \in \mathcal{A} : f(x) \neq 0$. Then $u(x) + v(x)i \neq 0$ implies either $u(x) \neq 0$ or $v(x) \neq 0$. Therefore, $\mathcal{A}_{\mathbb{R}}$ vanishes at no points. ■

Theorem 20.2 (*Baire's Theorem*)

Let X be a non-empty complete metric space. Let $\{G_n\}$ be a collection of dense open subsets of X. Then $\bigcap_{n=1}^{\infty} G_n$ is dense. Equivalently, let $\{F_n\}$ be a collection of closed subsets of X so that $X = \bigcup_{n=1}^{\infty} F_n$. Then $\exists n, O : O \subset F_n$ where O is non-empty open set.

Proof: Let $X = \bigcup_{n=1}^{\infty} F_n$ and F_n is closed. Assume F_n has no interior points for all n. We want to get a contradiction. Note that $\exists x \in X \setminus F_1$. Since F_1 is closed, $\exists N_r(x) \subset X \setminus F_1$. Since F_2 has no interior points, $\exists x_2 \in N_{r_1}(x_1) \setminus F_2$. Let $r_2 < \frac{r_1}{2}$ be such that $N_{r_2}(x_2) \subset N_{r_1}(x_1) \setminus F_2$ and $\overline{N_{r_2}(x_2)} \subset N_{r_1}(x_1)$. Repeat inductively, we find $N_{r_1}(x_1) \supset N_{r_2}(x_2) \supset \dots$ such that $r_i < \frac{r_{i-1}}{2}$, $\overline{N_{r_i}(x_i)} \subset N_{r_{i-1}}(x_{i-1})$, and $N_{r_i}(x_i) \subset N_{r_{i-1}}(x_{i-1}) \setminus F_i$. Now $\operatorname{diam}(\overline{N_{r_i}(x_i)}) \to 0, N_{r_i}(x_i) \neq \infty$. We get that $\bigcap \overline{N_{r_i}(x_i)} = \{p\}$. Note that $p \notin F_i$ for any i. However, $X = \bigcup_{n=1}^{\infty} F_n$, contradiction. ■

e.g.1.

1. $\mathbb{R}^n$ cannot be written as a countable union of closed sets without interior points.
2. $\overline{B_r(x)} = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ cannot be written as a countable union of closed sets without interior points.
3. $(C([0,1]), d_\infty)$ is a complete space so it cannot be written as a union of closed sets without interior.