Proofs and Review

CSE015 Lab 4, 2024 Fall

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Slides is posted on Canvas:Files. If you find any typos or have any concerns, please contact me ASAP!

Concepts

- **Terms**: theorem, proposition, axiom, lemma, corollary, conjecture, circular reasoning
- **Definitions**: even, odd

Goldbach's Conjecture: every even natural number greater than 2 is the sum of two prime numbers.
Collatz's Conjecture: for a number, recursively apply the following process: if it is odd, multiply it by 3 and add 1; if it is even, divide it by 2.
Fermat: "I have discovered a marvelous proof of this fact (Fermat's Last Theorem), but it was too long to fit in the margin."

1.7 Proofs

- Direct proof
- Proof by contraposition
- Proof by contradiction
- Proof by counterexample
- Proof by exhaustion

  Not covered yet (in Chapter 5): Induction

Direct Proof

1.7 Exercise 1: The sum of two odd integers is even.
proof: Let $x \in \mathbb{Z}, y \in \mathbb{Z}$ and $a = 2x + 1, b = 2y + 1$ be two odd integers. Then $a + b = (2x + 1) + (2y + 1) = 2x + 2y + 2 = 2(x + y + 1)$ is even by definition of even numbers. $\square$

1.7 Exercise 2: The sum of two even integers is even.
1.7 Exercise 3: The square of an even number is even [1].
1.7 Exercise 4: The additive inverse of an even number is even.
1.7 Exercise 5: Let $m, n, p \in \mathbb{Z}$. If $m + p$ and $n + p$ are even integers, then $m + p$ is even.

[1] A generalized proposition is: The product of two even numbers is even.

Ex.3. *proof:* We prove the generalized proposition at footnote: the product of two even numbers is even. To show that, let $a = 2x$, $b = 2y$ for $x \in \mathbb{Z}, y \in Z$ so a and b are even. Then $ab = (2x)(2y)$ $= 2(2xy)$ is even by definition of even numbers [2]. The original exercise is a special case when $a = b$. $\square$

Ex.5. *proof:* By definition of even numbers, $\exists x, y \in \mathbb{Z} : m + p = 2x, n + p = 2y$ so $m = 2x - p, n = 2y - p$. Hence $m + n = (2x - p) + (2y - p) = 2x + 2y - 2p = 2(x + y - p)$ is even. $\square$

[2] A typical mistake here is to write $ab = 4xy$ then claim it is even as the product of two even integers circular proof.

Direct Proof

1.7 Exercise 6: The product of two odd numbers is odd.
proof: Let $a = 2x + 1$, $b = 2y + 1$ be two odd integers where $x \in \mathbb{Z}, y \in \mathbb{Z}$. Now $ab = (2x+1)(2y+1) = 4xy + 2x + 2y + 1$ $= 2(2xy + x + y) + 1$ is odd as a sum of even integer and one. $\square$

1.7 Exercise 7: Every odd integer is the difference of two squares.
proof: Let $a = 2x + 1$ be an odd integer for $x \in \mathbb{Z}$. Since $(x+1)^2 - x^2 = (x^2 + 2x + 1) - x^2 = 2x + 1$, $a = (x+1)^2 - x^2$ is the difference of two squares. $\square$

1.7 Exercise 14: If x is rational and $x \neq 0$, then $\frac{1}{x}$ is rational.

Contradiction

1.7 Exercise 8: If n is a perfect square, then $n+2$ is not a perfect square.
proof: Let $n = x^2$ be a perfect square for $x \in \mathbb{Z}$. Now assume that $n+2$ is a perfect square so $\exists y \in \mathbb{Z} : n+2 = y^2$. Then $2 = (n+2) - n = y^2 - x^2 = (y+x)(y-x)$. The possible cases for $(y+x)(y-x) = 2$ are $y+x = 2, y-x = 1$ or $y+x = 1$, $y-x = 2$. There are no solutions for both cases in $\mathbb{Z}$. This contradicts to $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$. Hence $n+2$ cannot be a perfect square. $\square$

Contradiction, Some by Example

1.7 Exercise 9: The sum of an irrational number and a rational number is irrational.
1.7 Exercise 10: The product of a nonzero rational number and an irrational number is irrational.
Prove or Disapprove: Is the sum of two irrational numbers still irrational?
1.7 Exercise 11: Prove or disprove that the product of two irrational numbers is irrational.
1.7 Exercise 12: Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
1.7 Exercise 13: If x is irrational, then $\frac{1}{x}$ is irrational.
1.7 Exercise 15: If $x > 0$ is irrational, then $\sqrt{x}$ is irrational.

Contradiction

1.7 Exercise 27: There is no rational number r for which $r^3 + r + 1 = 0$.

proof. Toward a contradiction, assume $r = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ and a, b are coprime. Then $r^3 + r + 1 = 0$ implies $\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0$. So $a^3 + ab^2 + b^3 = 0$. Analyzing parity of a and b by case shows that the only possible case is a, b are both even. However, this contradicts to our assumption that a, b are coprime. In conclusion, there is no rational number r for which $r^3 + r + 1 = 0$. $\square$

Contradiction

Several kinds of contradiction we have seen

- Given or Assumed $x \in \mathbb{Z}$ Derive $x \notin \mathbb{Z}$
  Example 1.7 Exercise 8
- Given or Assumed $x \notin \mathbb{Q}$ Derive $x = \frac{p}{q} \in \mathbb{Q}$
  Example 1.7 Exercise 9,10
- Given or Assumed $x = \frac{p}{q}$ in simplest form (i.e. $\gcd(p, q) = 1$)
  Derive $x = \frac{p}{q}$ not in simplest form (e.g. p, q are both even after analyzing parity so $gcd(p, q) \geq 2$)
  Example $\sqrt{2} \notin \mathbb{Q}$; 1.7 Exercise 27

Contraposition

$$p \to q \equiv \neg q \to \neg p$$

1.7 Exercise 16: If x, y, z are integers and $x + y + z$ is odd, then at least one of x, y, z is odd.
1.7 Exercise 17: If $x \in \mathbb{R}, y \in \mathbb{R}$ and $x + y \geq 2$, then $x \geq 1$ or $y \geq 1$.
1.7 Exercise 18: If m and n are integers and mn is even, then m is even or n is even.
1.7 Exercise 19: If n is an integer and $n^3 + 5$ is odd, then n is even.
1.7 Exercise 20: If n is an integer and $3n + 2$ is even, then n is even.

Other Exercise in 1.7

- Vacuous Proof: 21, 22, 23
- Contradiction (Pigeon Hole Principle): 24, 25, 26

Prove Tautology

- Truth table: make sure to draw every columns and compute each entry correctly.
- First assuming consequence is False then showing condition is False (recall truth table of $p \to q$): can be useful if consequence is easy to analyze.
- Chain of logical equivalence: would be great if familiar with the table of logical equivalences, implication equivalence, and bi-conditional equivalences.

Truth table is usually easier to deal with, unless the problem does not allow you to use.

Other Problems

- Apply De Morgan's Law (conjunction, disjunction, and quantifier)
- Prove by giving counterexample.
- Translate logical operations (conjunction, disjunction, negation, implication, biconditional, nested quantifiers)
- ...