MATH140A: Foundations of Real Analysis

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Abstract

Warning: This is only a piece of lecture notes written by a careless scribe. So just be careful with and tolerant of any possible typos or misunderstandings when you read [0.1]. The scribe does not intend to make anyone to be driven by his stupidity! Also, the professor's explanation is extremely helpful as he discusses a lot about the interpretable ideas behind the dull scripts. So watch the lecture before reading this. If you have any suggestions (e.g. typos, typography, logistics), please do not hesitate contacting the scribe!

Here are some resources explaining Rudin

- Supplements to the Exercises, Comments by Prof.Bergman from UCB.
- The Real Analysis Lifesaver: kind of companion for Rudin to explain ideas of a few smart proof.

List of Notations:

- $\mathbb{Z}$ integers; $\mathbb{Q}$ rational numbers; $\mathbb{Z}^+, \mathbb{N}$ positive integers; $\mathbb{Z}^{\geq 0}$ non-negative integers.
- $\bigcup_{i=1}^n A_i = A_1 \cup \cdots \cup A_n$, $\bigcap_{i=1}^n A_i = A_1 \cap \cdots \cap A_n$.
- : such that; $\Rightarrow$ so that, implies; $\Longrightarrow$ implies (if ... then ...). [a]

[a] In this note, $\Longrightarrow$ is used as "if ... then ..." often after : (e.g. in definition of function limits); but $\Rightarrow$ is used as "therefore" in derivation. The scribe realizes this point after lecture 16, when it becomes late to eradicate. So hopefully this will not influence readers' reading or raise the OCD.

[0.1] Especially '$\cap$' and '$\cup$' are often mistaken because of typos.

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Lecture 1: Issues with $\mathbb{Q}$

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 1.1

$x^2 = 2$ does not have any rational solutions.

Proof: Proof by contradiction: suppose $\exists m, n \in \mathbb{Z}, n \neq 0$ such that $x = \frac{m}{n}$ is a solution to $x^2 = 2$ and $\gcd(m, n) = 1$. Thus $(\frac{m}{n})^2 = 2$ so $m^2 = 2n^2$ then $m = 2k$ for $k \in \mathbb{Z}$. $(2k)^2 = 2n^2$ gives $n^2 = 2k^2$. Then $n = 2\ell$ for $\ell \in \mathbb{Z}$. Then $2 \mid m$ and $2 \mid n$. ■

Proof: *Question:* What is the problem with $\mathbb{Q}$?

Let $A = \{x \in \mathbb{Q} : x^2 < 2\}$. A has properties that

- A is nonempty.
- $\forall x \in A : x < 2$ (2 is an upper bound).

Question: Can we find a "best" upper bound for A?

i.e. Is there $y \in \mathbb{Q}$:
- $\forall x \in A : x \leq y$
- y is smallest with properties of A

We want to show the answer is "NO" by contradiction. Suppose $y \in \mathbb{Q}$ satisfies above properties. Then

- $1 \leq y \leq 2$

  Proof: If $y < 1$, then since $1 \in A$, we get a contradiction with the fact that y is an upper bound. If $y > 2$ then since 2 is an upper bound for A, we get a contradiction with the fact that y is the smallest upper bound. ■

- $y^2 > 2$ or $y^2 < 2$

Assume $y^2 < 2$, we find that $\exists r \in \mathbb{Q}, r > 0 : z = y + r$ such that $z \in A$. We want $(y + r)^2 = y^2 + 2yr + r^2 < 2$. The goal is to choose r: we will find $0 < r < 1$ so that the properties of A hold. Notice that $2yr \leq 4r$ and $r^2 \leq r$, then $2yr + r^2 \leq 5r$. So $(y + r)^2 = y^2 + 2yr + r^2 \leq y^2 + 5r$. So if $y^2 + 5r < 2$, then $(y + r)^2 < 2$. Recall $a = 2 - y^2 > 0$. Note that $y^2 + \frac{a}{2} < 2$ so if we let $5r = \frac{a}{2}$ that $r = \frac{a}{10} \in \mathbb{Q}$ for $r > 0$ then $y^2 + 5r = y^2 + \frac{a}{2} < 2$ gives $(y + r)^2 < 2$ so $(y + r) \in A$. Then $y + r > y$ is an upper bound, contradiction.

In view of this, $y^2 > 2$. We now show that this also leads to a contradiction. By a similar argument to show that $\exists r \in \mathbb{Q}, r > 0$ so that $y - r$ is an upper bound for A.

Claim: Suffices to find r as above such that $y - r > 0$ and $(y - r)^2 > 2$.

To find this r, recall that $1 \leq y \leq 2$. So if $0 < r < 1$ then $y - r > 0$. $(y - r)^2 = y^2 - 2yr + r^2 > 2$ and as $r^2 \geq 0$, $y^2 - 2yr + r^2 \geq y^2 - 2yr \geq y^2 - 4r$. So if $0 < r < 1$, then $y^2 - 4r > 2$. Hence $(y - r)^2 > 2$. Since $y - r < y$ and $y - r$ is an upper bound of A, we get a contradiction. To choose r, let $b = y^2 - 2 > 0$, then $2 < 2 + \frac{b}{2} = y^2 - \frac{b}{2} < y^2$. Let $r = \frac{b}{8}$ then $(y - r)^2 \geq y^2 - 4r = y^2 - \frac{b}{2} > 2$ as we want to show. ■

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Lecture 2: Infimum, Supremum

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Recall: $A = \{x \in \mathbb{Q} : x^2 < 2\}$ does not have a least upper bound in $\mathbb{Q}$. i.e. $\forall y \in \mathbb{Q}$, $y \in A$ and $\exists z \in A : y < z$ or $y \notin A$ and $\exists z \in \mathbb{Q}, z \notin A : z < y$.

Definition 2.1 (*Ordered Set*)

Let A be a set and an ***order*** *on A is a relation denoted by $<$ with the following properties* [2.1]

1. $\forall x, y \in A : x < y$ or $x = y$ or $y < x$
2. $\forall x, y, z \in A : x < y, y < z \Rightarrow x < z$

A set equipped with an order $<$ is called an ***ordered set****.*

e.g.1.

- $(Q, <)$ is an ordered set
- $(\mathbb{Z}^2, <)$ defined $(m, n) < (m', n')$ if either $m < m'$ or $m = m'$ and $n < n'$

Definition 2.2 (*Bound Above, Upper Bound*)

Let A be an ordered set, and $B \subset A$. Then B is said to be ***bounded above*** *if*

$$\exists a \in A, \forall b \in B : b \leq a$$

If B is bounded above, any element a satisfying the property is called an ***upper bound*** for B.

Bounded below and **lower bound** are defined similarly.

Definition 2.3 (*Least Upper Bound / Supremum*)

Let A be an ordered set and $B \subset A$ is bounded above. If $\exists a \in A$,

1. *a is an upper bound of B*
2. $\forall c < a, \exists b \in B : c < b \leq a$ [2.2]

Then we say B has the ***least upper bound*** *and a is called the* ***least upper bound / supremum*** *of B. This is denoted: $a = \sup B$.*

Lemma 2.1

If $B \subset A$ has a supremum, then it is unique.

Proof: Suppose $\exists a, a' \in A$ so that $a = \sup B$ and $a' = \sup B$. If $a = a'$, done. WLOG, suppose $a < a'$, since $a' = \sup B$ and $a < a'$, $\exists b \in B : a < b \leq a'$. But this implies a is not an upper bound for B which contradicts $a = \sup B$. ■

[2.1] $x \leq y$ means $x < y$ or $x = y$. $x > y$ means $y < x$.
[2.2] c is not an upper bound for B.

e.g.2. Let $A = \mathbb{Q}$.

- $B = \{x \in \mathbb{Q} : x \leq 1\}$.

  B is bounded above. $\sup B$ exists in A. $\sup B = 1$.

- $C = \{x \in \mathbb{Q} : x < 1\}$.

  C is bounded above. $\sup C$ exists in A. $\sup C = 1$. [2.3]

- $D = \{x \in \mathbb{Q} : x^2 < 2\}$.

  D is bounded above. $\sup D$ does not exist in $\mathbb{Q}$.

Definition 2.4 (*Greatest Lower Bound / Infimum*)

If $B \subset A$ is bounded below and element $a \in A$ is called the ***greatest lower bound*** *of B if*

1. $\forall b \in B : a \leq b$
2. *if $c \in A$ and $c > a$, then $\exists b \in B : c > b \geq a$*

Denoted $a = \inf B$.

Definition 2.5 (*Least Upper Bound Property*)

An ordered set A is said to have the ***least upper bound property*** *(LUB property) if $\forall B \subset A$ is nonempty bounded above, $\sup B$ exists in A.*

Greatest lower bound property is defined similarly: if $B \subset A$ is nonempty and bounded below then $\inf B$ exists in A.

e.g.3. $\mathbb{Q}$ does not have the LUB property because for $B = \{x : x^2 < 2\}$ is nonempty and bounded above but has no supremum.

Question: Does $(\mathbb{Z}, <)$ satisfies LUB property?

Proposition 2.2

If an ordered set A has the least upper bound property, then it has the greatest lower bound property.

Proof: we will show ($\Rightarrow$) and the other direction can be proved similarly.

($\Longrightarrow$): Let $B \subset A$ be nonempty and bounded below. We want to find the greatest lower bound for B. Let $C = \{a \in A : a \text{ is a lower bound for } B\}$. Then C is nonempty since B is bounded below, and C is bounded below (indeed let $b \in B$), then every $c \in C$ is a lower bound of B s.t. $\forall c \in C : c \leq b$). Since A satisfies LUB property, then $\exists d \in A : d = \sup C$. We claim $d = \inf B$. We need to show:

1. d is a lower bound for B
2. $\forall c > d, \exists b \in B : c > b \geq d$

(1) Suppose d is not a lower bound, then $\exists b \in B : d > b$. Since $d = \sup C$, $d > b$ implies that $\exists c \in C : d \geq c > b$. But c is the set of lower bounds for B so $\forall c \in C$ we have $c \leq b$. Contradiction.

(2) Let $h > d$. We want to show $\exists b \in B : h > b \geq d$. Toward a contradiction, suppose not. Then $\forall b \in B : b \geq h$. $h \in C = \{a \in A : a \text{ is a lower bound to } B\}$. However, $h > d = \sup C$, which contradicts the fact that d is an upper bound of C. ■

[2.3] This exemplifies that supremum does not need to be a maximum.

Definition 2.6 (*Field*)

*A **field** is a set $\mathbb{F}$ with two operations: addition and multiplication*

$+ - \mathbb{F}+\mathbb{F} \to \mathbb{F} \qquad (a,b) \rightarrowtail a+b$

$\cdot - \mathbb{F}\cdot\mathbb{F} \to \mathbb{F} \qquad (a,b) \rightarrowtail a\cdot b$ [2.4]

Addition Axioms:
- $\forall a,b \in \mathbb{F} : a+b = b+a$
- $\forall a,b,c \in \mathbb{F} : (a+b)+c = a+(b+c)$
- $\exists 0 \in \mathbb{F}, \forall a \in \mathbb{F} : a+0 = 0+a = a$
- $\forall a \in \mathbb{F}, \exists (-a) \in \mathbb{F} : a+(-a) = (-a)+a = 0$

Multiplication Axioms:
- $\forall a,b \in \mathbb{F} : a\cdot b = b\cdot a$
- $\forall a,b,c \in \mathbb{F} : (a\cdot b)\cdot c = a\cdot(b\cdot c)$
- $\exists 1 \neq 0 : a\cdot 1 = 1\cdot a = a$
- $\forall 0 \neq a \in \mathbb{F}, \exists \frac{1}{a} \in F : a\cdot\frac{1}{a} = \frac{1}{a}\cdot a = 1$

e.g.4.
- $(\mathbb{Q},+,\cdot)$ is a field.
- $(\mathbb{Z},+,\cdot)$ is not a field because multiplicative inverse does not exist in $\mathbb{Z}$.
- Let p be a prime number, then $\{0,1,2,\dots,p-1\}$ with addition and multiplication mod p is a field.

  To check the inverse, $\forall 0 \neq a \in F_p$, then $\gcd(a,p) = 1$. So by Bezout's Theorem $\exists m,n \in \mathbb{Z} : am+pn = 1$. Then $am \equiv 1 \mod p$, which means remainder of m divided p is the multiplicative inverse of a.
- $\left\{\dfrac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} : a_i, b_i \in \mathbb{Q} \text{ and } b_m x^m + \cdots + b_0 \neq 0\right\}$ is a field [2.5].

Definition 2.7 (*Ordered Field*)

*An **ordered field** $\mathbb{F}$ is a field with an order satisfying:*
1. $\forall a,b,c \in \mathbb{F}$ *and* $a<b$, *then* $a+c<b+c$
2. $\forall a,b \in \mathbb{F}, a>0, b>0$, *then* $ab>0$

e.g.5.
- $(\mathbb{Q},+,\cdot,<)$ is an ordered field.
- $F_p = \{0,1,2,\dots,p-1\}$ is not an ordered field.

[2.4] We denote $a\cdot b$ as ab.
[2.5] Check this in homework.

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Lecture 3: Properties of Fields, Real Numbers

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Proposition 3.1

For any field, these properties hold Addition

1. $x + y = x + z \Rightarrow y = z$
2. $-(-x) = x$

Multiplication

1. *If* $x \neq 0$ *and* $xy = xz$*, then* $y = z$.
2. *If* $x \neq 0$*, then* $\frac{1}{1/x} = x$.

Distributive Law

1. $0 \cdot x = x \cdot 0 = 0$
2. $(-x)y = -xy$

Ordered Field

1. *If* $x > 0$ *and* $y > z$*, then* $xy > xz$.
2. *If* $x < 0$ *and* $y > z$*, then* $xy < xz$.
3. $\forall x \neq 0 : x^2 > 0$. *In particular,* $1 > 0$.
4. $0 < x < y \Rightarrow \frac{1}{x} > \frac{1}{y} > 0$.

Proof: Addition

1. $x+y = x+z \Rightarrow (x+y)+(-x) = (x+z)+(-x)$. By commutativity, $(-x)+(x+y) = (-x) + (x + z)$. Then by associativity, $(-x + x) + y = (-x + x) + z \Rightarrow 0 + y = 0 + z \Rightarrow y = z$.
2. Recall by definition, $-(-x)$ is the unique element in $\mathbb{F}$ such that $-x+(-(-x)) = 0$. However by its definition, we can also write $-x+x = 0$. So we have $-x+(-(-x)) = -x + x$. By previous property, $-(-x) = x$.

Multiplication

1. All properties can be shown similarly.

Distributive Law

1. To show $0 \cdot x = x$, we use the fact that $0 + a = a$. $0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x$. Using addition property, $0 \cdot x = 0$.
2. To see $(-x)y = -xy$, note that $x+(-x) = 0$. $0 = 0 \cdot y = (x+(-x))y = xy+(-x)y$. So $-xy$ is the additive inverse of xy. i.e. $(-x)y = -xy$.

Ordered Field

1. $x > 0, y > z \Rightarrow y + (-z) > 0$. Then $x(y + (-z)) > 0 \Rightarrow xy + x(-z) \Rightarrow xy - xz > 0 \Rightarrow xy > xz$.
2. Can be shown similarly.

3. Let $x \in \mathbb{F}, x \neq 0$, then either $x > 0$ or $x < 0$. If $x > 0$, then $x^2 > 0$ by property 1. If $x < 0$, then by property 2, $x^2 > 0$.
4. Suppose not and multiply by xy. ■

Definition 3.1 (*Cuts*)

*A **cut** α is a subset of $\mathbb{Q}$ satisfying:*

- $\alpha \neq \emptyset$ and $\alpha \neq \mathbb{Q}$.
- *If $p \in \alpha$ and $q \in \mathbb{Q}, q < p$, then $q \in \alpha$.*
- *If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.*

e.g.1.

- $\alpha = \{x \in \mathbb{Q} : x < 1\}$ is a cut. Indeed, $\forall r \in \mathbb{Q} : \alpha_r = \{x \in \mathbb{Q} : x < r\}$ is a cut.
- $\beta = \{x \in \mathbb{Q} : x^2 < 2\}$.

Theorem 3.2 (*Real Numbers*)

There exists a unique ordered field which has the least upper bound property [3.1]. *We denote this field by $\mathbb{R}$. Moreover, $\mathbb{Q} \subset \mathbb{R}$.*

Theorem 3.3 (*Archimedean Property*)

Let $x, y \in \mathbb{R}, x > 0$. Then

$$\exists n \in \mathbb{Z} : nx > y$$

In particular, $\forall a \in \mathbb{R}$

$$\exists n, m \in \mathbb{Z} : m < a < n$$

Proof: Note that the particular part follows from the first claim. Indeed, apply the claim with $y = a$ and $x = 1$. Then $\exists n \in \mathbb{Z} : n = n \cdot 1 > a$. To see the lower bound, apply this with $y = -a$ and $x = 1$. Then $\exists k \in \mathbb{Z} : k = k \cdot 1 > -a \Rightarrow -k < a$. Hence the claim follows with $m = -k$.

We now show the first claim. Let $x, y \in \mathbb{R}, x > 0$. We want to find $n \in \mathbb{Z} : nx > y$. Let $A = \{kx : k \in \mathbb{Z}\}$. We need to show that $\exists n \in \mathbb{Z} : nx > y$. Toward a contradiction, assume that $\forall k \in \mathbb{Z} : kx \leq y$. Note that $0 \in A$ (A is nonempty) and y is an upper bound for A. Since $\mathbb{R}$ satisfies the least upper bound property, A has a supremum in $\mathbb{R}$, say $\alpha = \sup A$. We want to find some $k \in \mathbb{Z}$ so that $kx \leq \alpha$ but $(k+1)x > \alpha$. Since $\alpha = \sup A$ and $x > 0$ we have $\alpha - x < \alpha$ so $\exists kx \in A : \alpha - x < kx \leq \alpha$. Adding an x we conclude that $\alpha < (k+1)x$. This shows that α is not an upper bound for A and contradicts the fact that $\alpha = \sup A$. ■

Proposition 3.4

Let $x, y \in \mathbb{R}$ and $x < y$. Then [3.2]

$$\exists \frac{m}{n} \in \mathbb{Q} : x < \frac{m}{n} < y$$

[3.1] *Idea of the proof:* $\mathbb{R}$ is defined the set of cuts of $\mathbb{Q}$.

Proof: We want to find some $\frac{m}{n} \in \mathbb{Q}$ such that $x < \frac{m}{n} < y$. We always assume $n > 0$. So $x < \frac{m}{n} < y \Rightarrow nx < m < ny$. Since $y - x > 0$, $\exists n \in \mathbb{Z}^{+} : n(y-x) > 1$ [3.3]. We claim $\exists m \in \mathbb{Z} : nx < m < ny$. By the second part of the Archimedean Property $\exists k_1, k_2 : k_1 < nx < k_2$. Now the set of integers between k_1 and k_2 is a finite set. Hence $\exists m \in \mathbb{Z} : m - 1 \leq nx < m \leq k_2$. ■

[3.2] This is to say $\mathbb{Q}$ is dense in $\mathbb{R}$.

[3.3] Apply Archimedean Property with $y - x > 0$ and 1.

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Lecture 4: Properties of $\mathbb{R}$, Complex Numbers

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Theorem 4.1

For every $x \in \mathbb{R}^+$ and any $n \in \mathbb{N}$, there exists a unique $y > 0$ in $\mathbb{R}$ such that

$$y^n = x$$

Proof: (Uniqueness) By induction, if $0 < a < b$ and $n \in \mathbb{N}$, then $a^n < b^n$. Suppose $y_1, y_2 \in \mathbb{R}$ and $y_1 < y_2$ so that $y_1^n = x$ and $y_2^n = x$. This implies that $0 < y_1 < y_2 \Rightarrow x = y_1^n < y_2^n = x$. Contradiction. So $y_1 = y_2$. ■

(Existence) We want to show that $\exists y \in \mathbb{R}^+ : y^n = x$. Let $A = \{z \in \mathbb{R} : z^n < x\}$ [4.1]. Since $0^n = 0 < x$, $0 \in A$. So A is nonempty. To show A is bounded above, consider two cases: (1) $0 < x \leq 1$: then 1 is an upper bound for A. (2) $x > 1$: then $\forall n \in \mathbb{N} : x^n \geq x$. Hence $z^n < x \leq x^n \Rightarrow z \leq x$. So x is an upper bound. So x is an upper bound. Hence by the least upper bound property of $\mathbb{R}$, $\sup A$ exists in $\mathbb{R}$.

Let $y = \sup A$. We try to show that $y^n \neq x$ leads to a contradiction. First observe that since y is an upper bound for A and $0 \in A$, $y \geq 0$.

Assume $y^n < x$: given our discussion above, we want to find $0 < r < 1$ so that $(y + r)^n < x$. So $y^n < (y + r)^n < x \Rightarrow (y + r)^n - y^n < x - y^n$. Recall the identity $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$. We have $(y + r)^n - y^n = (y + r - y)((y+r)^{n-1} + (y+r)^{n-2}y + \cdots + y^{n-1}) = r((y+r)^{n-1} + (y+r)^{n-2}y + \cdots + y^{n-1}) < x - y^n$. Since $0 < r < 1$, we have $(y+r)^{n-1} \leq (y+1)^{n-1}$, $(y+r)^{n-2}y \leq (y+1)^{n-2}y \leq (y+1)^{n-1}$, $\ldots$, $y^{n-1} \leq (y+1)^{n-1}$. So $((y+r)^{n-1} + \cdots + y^{n-1}) \leq n(y+1)^{n-1}$. This gives $(y+r)^n - y^n \leq r(n(y+1)^{n-1})$. So if we choose $0 < r < 1$ so that $r(n(y+1)^{n-1}) < x - y^n$. For example, if we let $r = \min\{\frac{1}{2}, \frac{x-y^n}{n(y+1)^{n-1}}\}$, then $(y + r)^n - y^n \leq r(n(y + 1)^{n-1}) \leq \frac{x-y^n}{2} < x - y^n$, hence $y < (y + r)^n < x$. Therefore, $y + r \in A$, so y is not an upper bound for A. Contradiction.

Assume $y^n > x$ [4.2]: first note that since $y^n > x$ we have $y > 0$. We will find $0 < r < \min\{1, y\}$ so that $x < (y-r)^n < y^n$ then $y - r > 0$ and $\forall z \in A : (y-r)^n > x > z^n$. Apply the identity, we have $y^n - (y-r)^n = (y-(y-n))(y^{n-1} + y^{n-2}(y-r) + \cdots + (y-r)^{n-1}) = r(y^{n-1} + \cdots + (y - r)^{n-1}) < r(ny^{n-1})$. Let $r = \{\frac{1}{2}, \frac{y}{2}, \frac{y^n-x}{2ny^{n-1}}\}$, then $x < (y - r)^n < y^n$. So $y - r$ is an upper bound for A which contradicts the fact that $y = \sup A$.

All together, we get $y^n > x$ and $y^n < x$ are impossible. Hence $y^n = x$. ■

Corollary 4.2

If $a, b \in \mathbb{R}^+$, then [4.3]

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$$

[4.1] We need to first check that A is nonempty and bounded above, then show $\sup A = x$.

[4.2] The idea is to find $0 < r < 1$ so that $y - r$ is still an upper bound for A which contradicts $y = \sup A$.

Theorem 4.3 (*Complex Numbers*)

> ***Complex numbers*** *denoted by $\mathbb{C}$ form a field.*

Proof: $\mathbb{C}$ is constructed using $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. We want to introduce addition and multiplication [4.4]on $\mathbb{R}^2$ which turns this into a field — "+": $(a, b) + (a', b') = (a + a', b + b')$ and $(0, 0) + (a, b) = (a, b)$; "$\cdot$": $(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b)$.

Note that

- $\forall(a, b) \in \mathbb{R}^2$ we have $(a, b) \cdot (1, 0) = (a - 0, 0 + b) = (a, b)$.
- Check of associativity is left as exercise.
- $(1, 0)$ is the identity element.
- $(a, b) \cdot (a', b') = (aa' - bb', ab' + a'b) = (a', b') \cdot (a, b)$.
- Let $(a, b) \in \mathbb{R}^2$ be nonzero.
- To find multiplicative inverse, $(a, b) \cdot (a', b') = (1, 0)$. We solve $(a', b') = (\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2})$.
- Check of distributive laws are left as exercise. ■

Definition 4.1 (*i*)

> *Let $i = (0, 1) \in \mathbb{R}^2$, then $i^2 = (0, 1) \cdot (0, 1) = (-1, 0)$.*

Lemma 4.4

> *$\mathbb{C}$ can be written in form of*
>
> $$\mathbb{C} = \{a + bi : a, b \in \mathbb{R}, i^2 = -1\}$$

Proof: Note that $\{(r, 0) : r \in \mathbb{R}\}$ is actually a copy of $\mathbb{R}$ in $(\mathbb{R}^2, +, \cdot)$. Indeed $(r, 0) + (r', 0) = (r + r', 0)$ and $(r, 0) \cdot (r', 0) = (rr', 0)$. Then $\forall(a, b) : (a, b) = (a, 0) + (b, 0) \cdot (0, 1) = a + bi$. ■

Definition 4.2 (*Conjugate*)

> *Let $z \in \mathbb{C}$, $z = a + bi$, then the* ***complex conjugate*** *of z is defined as*
>
> $$\overline{z} = a - bi$$
>
> *The* ***absolute value*** *of z is defined as*
>
> $$|z| = \sqrt{a^2 + b^2}$$
>
> *$a = Re(z)$ is called the real part, $b = Im(z)$ is called the* ***imaginary*** *part.*

[4.3]*Idea of the proof:* Note that $(ab)^{\frac{1}{n}}$ is the unique real number such that $((ab)^{\frac{1}{n}})^n = ab$. Check that $a^{\frac{1}{n}} b^{\frac{1}{n}}$ also satisfies the property. i.e. use induction to show that $\forall x, y \in \mathbb{R}^+ : (xy)^n = x^n y^n$.

[4.4]Problems with "common" multiplication: define $(a, b) \cdot (a', b') = (aa', bb')$. Notice that $(1, 0) \cdot (0, 1) = (0, 0)$ does not turn $\mathbb{R}^2$ into a field.

Proposition 4.5

Proof left as exercise.

- $z \cdot \overline{z} = |z|^2$
- $\overline{z+w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$
- $z + \overline{z} = 2Re(z)$
- $z - \overline{z} = 2Im(z) \cdot i$
- $z\overline{z} = |z|^2 = (Re(z))^2 + (Im(z))^2$

Proposition 4.6

1. $|z| > 0$ *unless* $z = 0$
2. $|\overline{z}| = |z|$
3. $|zw| = |z||w|$
4. $|Re(z)| \leq |z|$
5. $|z + w| \leq |z| + |w|$. *This is called the* ***Triangle Inequality***.

Proof:

1. If $z = a + bi$, then $|z|^2 = a^2 + b^2 > 0$ whereas $z = 0$. Now by definition, $|z| = \sqrt{a^2 + b^2}$ hence $|z| > 0$ unless $z = 0$.
2. $|\overline{z}| = \sqrt{a^2 + b^2} = |z|$.
3. Write $z = a + bi, w = c + di$. Check $|zw|^2 = |z|^2|w|^2$. The claim follows.
4. Write $z = a + bi$, $Re(z) = |a| = \sqrt{a^2} =\leq \sqrt{a^2 + b^2} = |z|$.
5. Since both sides are nonnegative, it is sufficed to show that $|z+w|^2 \leq (|z|+|w|)^2$. $|z + w|^2 = (z + w)(\overline{z + w}) = (z + w)(\overline{z} + \overline{w}) = z\overline{z} + z\overline{w} + w\overline{z} + w\overline{w} = |z|^2 + |w|^2 + 2Re(z\overline{w})$. Now by property 4, $|Re(z\overline{w})| \leq |z\overline{w}| = |z||w|$. Hence, $|z + w|^2 = |z|^2+|w|^2+2Re(z\overline{w}) \leq |z|^2+|w|^2+2|z||w| = (|z|+|w|)^2$, which implies the triangle inequality. ■

Lemma 4.7

$\mathbb{C}$ *cannot be turned into an ordered field. Indeed,* $i^2 = -1 < 0$.

MATH140A **Winter 2021**

Lecture 5: Cauchy-Schwartz Inequality

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Theorem 5.1 (*Cauchy-Schwartz Inequality*)

Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be elements in $\mathbb{C}$. Then

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof: Let $C^2 = \left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2, A = \sum_{j=1}^{n} |a_j|^2, B = \sum_{j=1}^{n} |b_j|^2$. We want to show $AB - C^2 \geq 0$.

For $t \in \mathbb{R}$, define $f(t) = \sum_{j=1}^{n}(a_j + tb_j)^2$. Then we get $\forall t \in \mathbb{R} : f(t) \geq 0$. Also note that $f(t) = \sum_{j=1}^{n}(a_j + tb_j)^2 = \sum_{j=1}^{n} \left(a_j^2 + 2ta_jb_j + t^2b_j^2\right) = Bt^2 + 2Ct + A$. So $Bt^2 + 2Ct + A \geq 0$. By computing the discriminant and $f(t)$ has at most one root, $C^2 - AB \leq 0$. ■

Definition 5.1 (*Euclidean Space*)

Let $n \in \mathbb{N}$, define $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}$ [5.1] *with following operations:*

- *Addition: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n)$. Then $\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.*
- *Scalar Multiplication: Let $\lambda \in \mathbb{R}, \vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Then $\lambda\vec{x} = (\lambda x_1, \ldots, \lambda x_n)$.*
- *Inner product: Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. $\vec{x} = (x_1, \ldots, x_n), \vec{y} = (y_1, \ldots, y_n)$. Then $\vec{x} \cdot \vec{y} = x_1y_1 + \cdots + x_ny_n = \sum_{i=1}^{n} x_iy_i$. In particular, $\vec{x} \cdot \vec{x} = \sum_{i=1}^{n} x_i^2$.*

Definition 5.2 (*Norm*)

*The **norm** of vector is defined $\|\vec{x}\| = \sqrt{\sum_{i=1}^{n} x_i^2} = \sqrt{\vec{x} \cdot \vec{x}}$.*

Theorem 5.2 (*Cauchy-Schwartz Inequality*)

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, then

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\|\|\vec{y}\|$$

[5.1] $\mathbb{R}^n$ is a vector space over $\mathbb{R}$.

Proposition 5.3

Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$.

1. $\|\vec{x}\| > 0$ *unless* $\vec{x} = \vec{0}$.
2. $\|\lambda\vec{x}\| = |\lambda|\|\vec{x}\|$
3. $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$
4. $\|\vec{x} - \vec{y}\| \leq \|\vec{x} - \vec{z}\| + \|\vec{y} - \vec{z}\|$

Proof: 1, 2 are easy to check.

3: $|\vec{x} + \vec{y}|^2 = (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) = \vec{x}^2 + \vec{y} \cdot \vec{x} + \vec{x} \cdot \vec{y} + \vec{y}^2 = |\vec{x}^2| + 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 \leq (|\vec{x}| + \vec{y})^2$.

4: Replace with $\vec{x} = \vec{x} - \vec{y}, \vec{y} = \vec{y} - \vec{z}$ in 3. ■

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Lecture 6: Metric Spaces

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 6.1

Let A, B be two sets and $f : A \to B$ be a function.

- *f is **one-to-one** if $f(a_1) = f(a_2) \Longrightarrow a_1 = a_2$.*
- *f is **onto** if $\forall b, \exists a \in A : f(a) = b$.*
- *f is a **one-to-one correspondence** if it is one-to-one and onto [6.1].*
- *A **finite set** is a set $A : A \sim \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. Otherwise it is **infinite**.*
- *If A is a set $A \sim \mathbb{N}$, then we say A is **countable**.*
- *If A is a set which is not finite or countable, then we say A is **uncountable** [6.2].*
- *A **sequence** is a function $f : \mathbb{N} \to X$ [6.3].*

e.g.1. $\mathbb{N}, \mathbb{Z}$ are countable.

Lemma 6.1

Let A be an infinite set, then followings are equivalent

- *$A \sim \mathbb{N}$*
- *$\exists$ a one-to-one map $f : A \to \mathbb{N}$*
- *$\exists$ an onto map $g : \mathbb{N} \to A$*

Corollary 6.2

Any subset of a countable set is either finite or countable.

e.g.2. ($\mathbb{Q}$ is countable)

Proof: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n > 0, \gcd(m, n) = 1\}$. We want $h : \mathbb{Q} \to \mathbb{Z}$ to be one-to-one. Let $h(\frac{m}{n}) = \begin{cases} 2^m 3^n & m > 0 \\ -2^{-m} 3^n & m < 0 \end{cases}$. We will show $h(\frac{m}{n}) = h(\frac{m'}{n'}) \Rightarrow \frac{m}{n} = \frac{m'}{n'}$: recall $\exists g : \mathbb{Z} \to \mathbb{N}$ that is one-to-one and onto so $g \circ h : \mathbb{Q} \to \mathbb{N}$ is one-to-one. Hence by the lemma, $\mathbb{Q}$ is countable. ■

[6.1] In this case, we denote $A \sim B$.
[6.2] The term "at most countable" is equivalent to finite or countable.
[6.3] We write x_n or a_n for $f(n)$.

Definition 6.2

If $A_1, A_2, \dots$ are subsets of X, then

$$\bigcup_{i=1}^{\infty} A_i = \{x \in X : x \in A_i \text{ for some } i\}, \qquad \bigcap_{i=1}^{\infty} A_i = \{x \in X : x \in A_i \text{ for all } i\}$$

If I is a set and $\forall \alpha \in I : A_\alpha \subset X$, then

$$\bigcup_{\alpha \in I} A_\alpha = \{x \in X : x \in A_\alpha \text{ for some } \alpha \in I\}, \qquad \bigcap_{\alpha \in I} = \{x \in X : x \in A_\alpha \text{ for all } \alpha \in I\}$$

e.g.3. $\forall 0 < y < 1$. Let $A_y = \{x \in \mathbb{R} : 0 < x < y\}$. Then $\bigcup_{0<y<1, y \in \mathbb{R}} A_y = \{x \in \mathbb{R} : x \in A_y \text{ for some } y\} = \{x \in \mathbb{R} : 0 < x < y \text{ for some } 0 < y < 1\} = \{x \in \mathbb{R} : 0 < x < 1\}$.

Lemma 6.3

Countable (or finite) union of countable sets is countable. i.e. Let $A_1, A_2, \dots$ be countable subsets of X. Then $\bigcup_{i=1}^{\infty} A_i$ is countable.

Lemma 6.4

Let A be countable and $n \in \mathbb{N}$. Then $A^n = \{(a_1, \dots, a_n) : a_i \in A\}$ is countable [6.4].

Lemma 6.5

- *$\{0,1\}^{\mathbb{N}} = \{(a_n) : a_n = 0, 1\}$ is uncountable.*
- *$P(\mathbb{N})$ which is the set of subsets of $\mathbb{N}$ is uncountable.*
- *For any set X, X and $P(X)$ are not equivalent.*

Definition 6.3 (*Metric Space*)

*Let X be a set. A **metric** on X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties:*

1. *$d(p,q) > 0$ if $p \neq q$ and $d(p,p) = 0$.*
2. *$d(p,q) = d(q,p)$.*
3. *$\forall p, q, r \in X : d(p,q) \leq d(p,r) + d(r,q)$.*

*A set X with a metric d is called a **metric space**. Elements in the **space** X are called **points**.*

e.g.4.

- $\mathbb{R}$ with d define by $d(x,y) = |x - y|$ is a metric space. We call $(\mathbb{R}, d)$ the **standard metric** on $\mathbb{R}$.
- $\mathbb{R}^n$ with $d(\vec{x}, \vec{y}) = \|\vec{x} - \vec{y}\|$ is a metric space. We call $(\mathbb{R}^n, d)$ the **standard metric** on $\mathbb{R}^n$.

[6.4] Exercise: use lemma6.3 to show this.

- Let X be a set. Define $d : X \times X \to \mathbb{R}$, $d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$. Then (X, d) is a metric space.

Definition 6.4 (***Cell***)

Cells *(or boxes) in* $\mathbb{R}^n$ *is a Cartesian product of intervals. An* ***open cell*** *is* $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$. *A* ***closed cell*** *is* $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$.

Definition 6.5 (***Ball***)

Let $\vec{a} \in \mathbb{R}^n, r > 0$. *An* ***open ball*** *is*

$$B(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| < r\}$$

A ***closed ball*** *is*

$$\overline{B}(\vec{a}, r) = \{\vec{x} \in \mathbb{R}^n : \|\vec{x} - \vec{a}\| \leq r\}$$

Definition 6.6 (***Convex Set***)

Let $E \subset \mathbb{R}^n$. *We say* E *is* ***convex*** *if for all* $0 < t < 1$:

$$\forall x, y \in E : \quad (1 - t)x + ty \in E$$

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Lecture 7: Neighborhood, Limit Points

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

e.g.1. The open ball $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is convex.

Proof: We want to show that $(1-t)y + tz \in B_r(x)$ i.e. $\|(1-t)y + tz - x\| < r$. Since $y, z \in B_r(x)$, $\|y - x\| < r$ and $\|z - x\| \leq r$. Note that $x = (1 - t + t)x$, so we have $(1-t)y + tz - (1-t)x - tx = (1-t)(y-x) + t(z-x)$. Hence by the Triangle Inequality, we have $\|(1-t)y + tz - x\| = \|(1-t)(y-x) + t(z-x)\| \leq \|(1-t)(y-x)\| + \|t(z-x)\| < \|(1-t)r\| + \|tr\| = r$. ■

Definition 7.1 (*Open Neighborhood*)

Let (X, d) be a metric space. Let $r > 0$ and $p \in X$. Then

$$N_r(p) = \{q \in X : d(p, q) < r\}$$

*is called the **open neighborhood** around p with radius r.*

e.g.2.

- Let (R, d) be the standard metric. Then $N_1(0) = \{x \in \mathbb{R} : d(x, 0) < 1\} = \{x \in \mathbb{R} : |x| < 1\} = (-1, 1)$, $N_1(0) = \{x \in \mathbb{R} : d(x, 10) < \frac{1}{2}\} = \{x \in \mathbb{R} : |x - 10| < \frac{1}{2}\} = (-9.5, 10.5)$.
- $\forall r > 0$ and $x \in \mathbb{R}^n : B_r(x) = N_r(x)$.
- Let d be the discrete metric on X. If $0 < r < 1$, then $N_r(p) \subset \{q \in X : d(p, q) < 1\} = \{p\}$. If $r = 1$, then $N_r(p) = \{q \in X : d(p, q) < 1\} = \{p\}$. If $r > 1$, then $d(p, q) \leq 1 < r\}$ gives $q \in N_r(p)$. However, by the definition of discrete metric, we have $\forall q \in X : d(p, q) \leq 1$ so $N_r(p) = X$. Then $N_r(p) = \begin{cases} \{p\} & 0 < r \leq 1 \\ X & r > 1 \end{cases}$.

Definition 7.2 (*Closed Neighborhood*)

Let (X, d) be a metric space. Let $r > 0$ and $p \in X$. Then

$$\overline{N}_r(p) = \{q \in X : d(p, q) \leq r\}$$

*is called the **closed neighborhood** around p with radius r.*

e.g.3. Let $(\mathbb{R}^n, d)$ be the standard metric. $\overline{N}_r(x) = \{y \in \mathbb{R}^n : d(x, y) \leq r\} = \{y \in \mathbb{R}^n : \|x - y\| \leq r\}$. In $\mathbb{R}$, $\overline{N}_r(x) = [x - r, x + r]$. In $\mathbb{R}^2$, $\overline{N}_r(x)$ is a circle included.

Question: Describe $\overline{N}_r(p)$ if (X, d) is X equipped with the discrete metric.

Definition 7.3 (*Limit Points*)

*Let $A \subset X$. Then $p \in X$ is called a **limit point** of A if* [7.1]

$$\forall r > 0, \exists q \in A \text{ and } q \neq p : q \in N_r(p) \cap A$$

e.g.4. Let $\mathbb{R}$ be equipped with the standard metric. Find all the limit points of the following:

1. $A = (0, 1)$

   Proof: We claim the set of limit points of A is $x = [0, 1]$. We want to show $\forall x \in [0, 1]$, x is a limit point of A and that $\forall y \notin [0, 1]$, y is not a limit point. Let $x \in [0, 1]$ and let $r > 0$, then $N_r(x) = (x - r, x + r)$.

   (1). If $x = 0$, then we want to show that $(-r, r) \cap (0, 1)$ has a point $p \neq 0$. Either $0 < r < \frac{1}{2}$ then $p = \frac{r}{4} \in (-r, r) \cap (0, 1)$ or $r > \frac{1}{2}$ then $p = \frac{1}{2} \in (-r, r) \cap (0, 1)$.

   (2). If $x = 1$, note that since $\forall r > \frac{1}{2}$ we have $N_r(x) \supset N_{\frac{1}{2}}(x)$. It suffices to show that $N_r(x) \cap (0, 1)$ contains $p \neq 1$ for $r \leq \frac{1}{2}$. Now let $0 < r \leq \frac{1}{2}$ we have $N_r(1) = (1 - r, 1 + r)$. Since $0 < r \leq \frac{1}{2}$, we have $1 - \frac{r}{2} \in (0, 1) \cap (1 - r, 1 + r)$. $1 \neq 1 - \frac{r}{2}$. So 1 is the limit points.

   (3). If $0 < x < 1$, let $r_0 = \min\{1 - x, x\}$. It suffices to consider $0 < r < r_0$ (why?). Let $0 < r < r_0$ then $(x - r, x + r) \subset (0, 1)$. Let $p = x + \frac{r}{2} \in (0, 1)$ and $p \neq x$. Therefore x is a limit point of $(0, 1)$.

   So far we have shown $[0, 1]$ is contained in the set of limit points of $(0, 1)$. We now show that $\forall y \notin [0, 1]$, y is not a limit point. Let $y \in \mathbb{R}$ but not in $[0, 1]$. In order to show y is not a limit point, we need to find some $r > 0$ such that $N_r(y) \cap (0, 1) \subset \{y\}$. If $y > 1$, let $r = \frac{y-1}{2} > 0$, we claim that $(y - r, y + r) \cap (0, 1) = \emptyset$. Indeed let $p \in (y - r, y + r)$. Then $p > y - r = y - \frac{y-1}{2} = \frac{2y-y+1}{2} > 1$. So $p \notin (0, 1)$. If $y < 0$, let $r = -\frac{y}{2}$. Then $(y - r, y + r) \cap (0, 1) = \emptyset$. We can show that if $p \in (y - r, y + r)$ then $p < 0$. Therefore, if $y \notin [0, 1]$ then y is not a limit point of $(0, 1)$. ■

2. $A = [0, 1]$

   As above show that the limit points of A is $[0, 1]$.

3. $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

   Will be shown in the next lecture.

[7.1] This can also be written as: $\forall r > 0 : (N_r(p) \setminus p) \cap E \neq \emptyset$.

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Lecture 8: Limit Points

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

e.g.1. Find all the limit points of $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ as a subset of $\mathbb{R}$.

Proof: We claim that the set of limit points of A is $\{0\}$. Let $r > 0$ be arbitrary. We need to find points in $(-r, r) \cap A$. Note that points in A are of the form $\frac{1}{n}$ so we need to find some $n \in \mathbb{N} : \frac{1}{n} \in (-r, r) \Rightarrow \frac{1}{n} < r \Rightarrow n > \frac{1}{r}$. Note that by the Archimedean Property, $\exists n \in \mathbb{N} : n > \frac{1}{r} \Rightarrow 0 \neq \frac{1}{n} \in (-r, r) \cap A$. Hence 0 is a limit point of A.

We now need to show that if $p \in \mathbb{R}$ is nonzero then p is not a limit point of A. That means $\forall p \neq 0$, we want to find some $r > 0 : (p - r, p + r) \cap A \subset \{p\}$.

(1). $p > 1$ or $p < 0$: If $p > 1$, let $r = p - 1 > 0$, then $(p - r, p + r) \cap A = \emptyset$. If $p < 0$, let $r = -p$, then $(p - r, p + r) \cap A = \emptyset$.

(2). $0 < p \leq 1$:

- $p \notin A$ (i.e. $p \neq \frac{1}{n}$ for any $n \in \mathbb{N}$). Then $\exists n : \frac{1}{n+1} < p < \frac{1}{n}$. Let $r = \min\{p - \frac{1}{n+1}, \frac{1}{n} - p\}$. Then $(p - r, p + r) \cap A = \emptyset$. To see this check that if $x \in (p - r, p + r)$ then $\frac{1}{n+1} < x < \frac{1}{n} \Rightarrow x \notin A$.
- $p = \frac{1}{n}$ for some $n \in \mathbb{N}$. Then let $r = \frac{1}{n} - \frac{1}{n+1}$. Note that $\frac{1}{n} - \frac{1}{n+1} < \frac{1}{n+1} - \frac{1}{n}$ for $n \neq 1$. Then $(p - r, p + r) = (\frac{1}{n} - r, \frac{1}{n} + r) = (\frac{1}{n+1}, \frac{2}{n} - \frac{1}{n+1}) = (\frac{1}{n+1}, \frac{n+2}{n(n+1)})$. Thus $\frac{n+2}{n(n+1)} < \frac{1}{n-1}$ and $(p - r, p + r) \cap A = \{p\}$. So $p = \frac{1}{n}$ is also not a limit point.

Hence the set of limit points is $\{0\}$. ■

Definition 8.1 (*Closed Set*)

> *Let $A \subset X$. We say A is **closed** if A contains all its limit points.*

e.g.2. Consider $\mathbb{R}$ with the standard metric

- $A = (0, 1)$ is not closed.
- $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not closed.
- $C = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is a closed set.

Definition 8.2 (*Closure*)

> *Let $A \subset X$. Then the **closure** of A is*
>
> $$\overline{A} = A \cup A'$$
>
> *where A' is the set of limit points.*

e.g.3. Consider $\mathbb{R}$ with the standard metric

- $\overline{(0, 1)} = [0, 1]$
- $\overline{\{\frac{1}{n} : n \in \mathbb{N}\}} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$
- $\overline{[0, 1]} = [0, 1]$

Proposition 8.1

A is closed $\Leftrightarrow$ $A' \subset A$ $\Leftrightarrow$ $\overline{A} = A$.

Definition 8.3 (*Dense Set*)

*Let $A \subset X$. We say A is **dense** in X if*

$$X = \overline{A} = A \cup A'$$

i.e. $\forall x \in X$: either $x \in A$ or x is a limit point of A.

e.g.4. $\boxed{\textit{Question:}}$ Find some dense subsets of $\mathbb{R}$

- $\mathbb{R}$ is dense in $\mathbb{R}$. More generally, X is dense in X.
- $A = \mathbb{R} \setminus \{0, 1\}$ is dense in $\mathbb{R}$.

  If $x \in \mathbb{R}, x \neq 0$ or 1, then $x \in A$; if $x = 0$ or 1, then $x \in A'$.

Lemma 8.2

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Proof: We will show that every point $x \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. Let $r > 0$, we need to show $\exists p \in \mathbb{Q} : p \neq x$ and $p \in (x-r, x+r)$. Recall from chapter 1, this is the corollary to the Archimedean Property that $\forall a < b \in \mathbb{R}, \exists q \in \mathbb{Q} : a < q < b$. Hence $\exists p \in \mathbb{Q}$, $x < p < x + r \Rightarrow p \neq x, p \in (x - r, x + r)$. ■

Definition 8.4 (*Interior Points*)

*Let $A \subset X$, and let $p \in A$. We say p is an **interior point** of A if*

$$\exists r > 0 : N_r(p) \subset A$$

We denote the set of interior points of A by $\mathring{A}$.

Definition 8.5 (*Open Set*)

*A subset $A \subset X$ is called an **open set** if $A = \mathring{A}$. i,e. every point in A is an interior point.*

e.g.5. Consider $\mathbb{R}$ with the standard metric. Find $\mathring{A}$ for $A = [0, 1]$.

Proof: We will show that $\mathring{A} = (0, 1)$. To see this we need to show (1). $\forall x \in (0, 1), \exists r$: $(x - r, x + r) \subset [0, 1]$; (2). if $x \in \{0, 1\}$, then $\nexists r : (x - r, x + r) \subset [0, 1]$.

(1). Let $x \in (0, 1)$. Let $r = \min\{1 - x, x\}$. Then since $x \in (0, 1)$, we have $r > 0$. Moreover, $(x - r, x + r) \subset (0, 1) \subset [0, 1] = A$. Thus x is an interior point.

(2). $x = 0$ or 1. We want to show that $\forall r > 0 : (x - r, x + r) \not\subset [0, 1]$. We will show this for $x = 0$. Note that $-\frac{r}{2} < 0$ and $-\frac{r}{2} \in (-r, r)$ so $(-r, r) \not\subset [0, 1]$. This shows 0 is not an interior point. Similarly we can show that 1 is not an interior point. ■

Lemma 8.3

Let (X, d) be a metric space. Let $p \in X, r > 0$. Then $N_r(p)$ is an open set.

Proof: Recall that $N_r(p) = \{q \in X : d(p,q) < r\}$. Let $x \in N_r(p)$. We need to find some $s > 0 : N_s(x) \subset N_r(p)$ [8.1]. Let $a = d(x,p)$. Since $x \in N_r(p)$ we have $a = d(x,p) < r$. Define $s = r - a$. We claim $N_s(x) \subset N_r(p)$. To see this let $y \in N_s(x)$. Then by the triangle inequality, $d(y,p) \le d(y,x) + d(x,p) < s + a = r$. So $y \in N_r(p)$. So $N_s(x) \subset N_r(p)$. ■

[8.1] Thoughts: we want $N_s(x) = \{y \in X : d(x,y) < s\} \subset N_r(p) = \{q \in X : d(p,q) < r\}$. Note that by the triangle inequality, we have $d(y,p) \le d(x,y) + d(x,p) < r$. Since $d(x,p) = a$, we want to set $d(y,x) = s < r - a$.

MATH140A **Winter 2021**

Lecture 9: Open Set, Closed Set

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 9.1 (*Complement*)

*If $A \subset X$, then the **complement** of A is*

$$A^C = X \setminus A = \{x \in X : x \notin A\}$$

Definition 9.2 (*Bounded Sets*)

*A subset $A \subset X$ is said to be **bounded** if*

$$\exists x \in X \text{ and } R > 0 : A \subset N_R(x)$$

Lemma 9.1

Let (X, d) be a metric space. If $A \subset X$ is bounded, then $\forall p \in X, \exists L > 0 : A \subset N_L(p)$.

Proof: Since $A \subset X$ is bounded, $\exists x \in X, R > 0 : A \subset N_R(x)$. Let $p \in X$ and $L = R + d(x, p)$. We want to show $A \subset N_L(p)$. To show this, let $y \in A$, then we need to show $y \in N_L(p)$ i.e. $d(y, p) < L$. Note that $d(y, p) \leq d(y, x) + d(x, p) < R + d(x, p) = L$. Thus $y \in N_L(p)$. Then $A \subset N_L(p)$. ■

Definition 9.3 (*Perfect Set*)

*A subset $A \subset X$ is said to be a **perfect set** if every point in A is a limit point of A. i.e. $A = A'$* [9.1].

e.g.1.

- $[0, 1]$ is a perfect set.
- $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is closed but not perfect.

Lemma 9.2

Let (X, d) be a metric space. Then $A \subset X$ is open if and only if A^C is closed.

Proof: $(\Rightarrow)$ Suppose A is open. We want to show that A^C is closed i.e. A^C contains all the limit points. That is to say we need to show $\forall x \in A : x$ is not a limit point of A^C. Therefore, given $a \in A$, we need to find some $r > 0 : N_r(x) \cap A^C = \emptyset$. Since A is open, $\exists r > 0 : N_r(x) \subset A$. Hence $N_r(x) \cap A^C = \emptyset$.

$(\Leftarrow)$ Can be proved similarly. ■

[9.1] A perfect set is always closed.

Lemma 9.3

Let $A \subset X$ be a subset. Let $p \in X$ be a limit point of A. Then $\forall r > 0$, $N_r(x) \cap A$ is an infinite set.

Proof: Toward a contradiction. Suppose $\exists r > 0 : N_r(p) \cap A$ is finite. Write $N_r(p) \cap A = \{a_1, \dots, a_n\}$. Let $s = \min\{d(p, a_i) : 1 \le i \le n, p \ne a_i\}$. Then $s > 0$. We claim $N_s(p) \cap A \subset \{p\}$ [9.2]. This contradicts the fact that p is a limit point of A. ■

Corollary 9.4

Let $A \subset X$ be a finite set. Then A has no limit points. In particular, any finite set of X is closed.

Proof: The first claim is immediate from the lemma.

To see the second claim, note that A is closed if and only if $A' \subset A$. Since A is finite, $A' = \emptyset$ so A is closed. ■

[9.2] To see this, note that if $\exists x \in N_s(p) \cap A : x \ne p$. Then $x = a_i$ for some $1 \le i \le n$. Hence we get $\exists 1 \le i \le n, a_i = x \ne p : d(a_i, p) = d(x, p) < s$. This contradicts the definition of s. (Indeed $s = \min\{d(p, a_i) : 1 \le i \le n, p \ne a_i\} < r$ so $N_s(p) \subset N_r(p)$).

MATH140A **Winter 2021**

Lecture 10: Induced Metric, Compact

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 10.1

Let (X, d) be a metric space

1. *$\emptyset$ and X are open and closed.*
2. *An arbitrary union of open sets is open.*
3. *An arbitrary intersection of closed sets is closed.*
4. *A finite intersection of open sets is open.*
5. *A finite union of closed sets is closed.*

Proof: 1. is clear from the definition.

2. Let O_α be a collection of open sets. Put $O = \bigcup_\alpha O_\alpha$. We want to show that O is open. Let $x \in O$. Then by the definition $\exists \alpha : x \in O_\alpha$. Since O_α is open, $\exists r > 0 : N_r(x) \subset O_\alpha$. Then $N_r(x) \subset \bigcup_\alpha O_\alpha = O \Rightarrow O$ is open.
3. Let $\{F_\alpha\}$ be a collection of closed sets in X, we want to show $F = \bigcap_\alpha F_\alpha$ is closed by showing F^C is open. Recall De Morgan's Law $(\bigcup_\alpha B_\alpha)^C = \bigcap_\alpha B_\alpha^C$ and $(\bigcap_\alpha B_\alpha)^C = \bigcup_\alpha B_\alpha^C$. Note that $F^C = (\bigcap_\alpha F_\alpha)^C = \bigcup_\alpha F_\alpha^C$. Since F_α is closed, F_α^C is open and by (2) we have $F^C = \bigcup_\alpha F_\alpha^C$ is open. Therefore F is closed.
4. Let $O_1, \dots, On$ be a finite collection of open sets and let $O = O_1 \cap O_2 \cap \cdots \cap O_n$. We want to show that O is open. Let $x \in O$, we need to find $r > 0$ such that $N_r(x) \subset O$. Since $x \in O_i$ for $i \leq i \leq n$. So $\exists r_i > 0 : N_{r_i}(x) \subset O_i$. Let $r = \min\{r_i : 1 \leq i \leq n\}$. Then $r > 0$ since $r_i > 0$ and there are only finitely many r_i. Then $N_r(x) \subset N_{r_i}(x) \subset O_i \Rightarrow N_r(x) \subset \bigcap_{i=1}^n O_i = O$. So O is open.
5. follows from (4) and De Morgan's Law. ■

e.g.1. Let $I_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap_{i=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = \{0\}$ and $\{0\}$ is not open.
Let $F_n = \{\frac{1}{n}\}$ then F_n is closed. However, $\bigcup_{i=1}^\infty F_n$ is not closed.

Lemma 10.2

Let (X, d) be a metric space, and $A \subset X$.

1. *$\overline{A}$ is closed.*
2. *$A = \overline{A}$ if and only if A is closed.*
3. *Suppose A is closed and $B \subset A$, then $\overline{B} \subset A$.*

Proof: We have already seen (1) and (2). To see (3), note that $B \subset A$, hence any limit points of B is a limit point of A. i.e. $B' \subset A'$. But A is closed. Then $A' \subset A$. Altogether $\overline{B} = B \cap B' \subset A$. ■

Definition 10.1 (***Open Relative***)

> *Let (X, d) be a metric space, and let $Y \subset X$, then (Y, d) is a metric space as well. A subset $A \subset Y$ is said to be* ***open relative*** *to Y if A is open in the metric space (Y, d).*

e.g.2. Let $X = \mathbb{R}$ with the standard metric. Let $Y = (0, 1]$. Then $(\frac{1}{2}, 1]$ is not open in X but it is open relative to Y.

Proof: Let $x \in (\frac{1}{2}, 1]$. We consider two cases. If $x \in (\frac{1}{2}, 1)$, then $\exists r : (x - r, x + r) \subset (\frac{1}{2}, 1)$. If $x = 1$, then $(\frac{1}{2}, 1] = N_{\frac{1}{2}}(1) = \{y \in Y : |y - 1| < \frac{1}{2}\} = Y \cap \{p \in \mathbb{R} : |p - 1| < \frac{1}{2}\} = (0, \frac{1}{2}]$ by applying [10.1]. ■

Lemma 10.3

> *Let $Y \subset X$. Then $A \subset Y$ is open relative to Y if and only if $A = O \cap Y$ for an open subset $O \subset X$. Similarly, $B \subset Y$ is closed relative to Y if and only if $B = Y \cap F$ for a closed subset $F \subset X$.*

Proof: We prove this for open sets; the proof for closed sets follows by taking complement.

$(\Rightarrow)$ We want to find an open subset $O \subset X$ such that $A = O \cap Y$. Indeed $\forall p \in A, \exists r > 0 : N_r^Y(p) \subset A$. Recall that $N_r^Y(p) = \{y \in Y : d(y, p) < r_p\} = N_{r_p}^X(p) \cap Y$. Let $O = \bigcup_{p \in A} N_{r_p}^X(p)$. Then O is open and $O \cap Y = A$.

$(\Leftarrow)$ Suppose $O \subset X$ is open and let $A = O \cap Y$. We need to show A is open relative to Y. Let $p \in A$. Since O is open in X, $\exists r > 0 : N_r(p) \subset O$. Then $N_r^Y(p) = N_r(p) \cap Y \subset O \cap Y = A$. Thus A is open relative to Y. ■

Lemma 10.4

> *Let $A \subset \mathbb{R}$ be a bounded subset. Then $\sup A \in \overline{A}$ and $\inf A \in \overline{A}$.*

Proof: We will show $\sup A \in \overline{A}$; the proof for inf is similar.

First note that by the LUB Property, $\sup A$ exists.

Case 1: $\sup A \in A$, then $\sup A \in A \subset \overline{A}$.

Case 2: $\sup A \notin A$. We want to show $\alpha = \sup A \in A'$. We need to show that $\forall r > 0, \exists b \in A, b \neq \alpha$ and $b \in (\alpha - r, \alpha + r)$. Note that $\alpha - r < \alpha$. Since $\alpha = \sup A$, $\exists b \in A : \alpha - r < b \leq \alpha$. Since $\alpha \notin A$ and $b \in A$, we get $\alpha - r < b < \alpha$. Then $b \neq \alpha, b \in A, b \in (\alpha - r, \alpha + r)$. Therefore, $\alpha \in A' \Rightarrow A' \subset A \Rightarrow \alpha \in \overline{A}$. ■

Definition 10.2 (***Open Cover***)

> *Let (X, d) be a metric space and $A \subset X$. An* ***open covering*** *of A is a collection $\{O_\alpha : \alpha \in \Lambda\}$ of open sets so that $A \subset \bigcup_{\alpha \in \Lambda} O_\alpha$.*

[10.1] Let (X, d) be a metric space and $Y \subset X$. For $p \in Y, r > 0$, $N_r^Y(p) = \{y \in Y : d(y, p) < r\} = \{x \in X : d(x, p) < r\} \cap Y = N_r^X(p) \cap Y$.

Definition 10.3 (*Compact*)

A subset $K \subset X$ is called ***compact*** *if for all open covering $\{O_\alpha\}$ of K, there exists a finite collection $O_{\alpha_1}, O_{\alpha_2}, \ldots, O_{\alpha_n}$ such that* [10.2]

$$K \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$$

e.g.3.

1. Any finite subset of X is compact.

   Proof: Let $K = \{X_1, \ldots, X_n\}$. Suppose $\{O_\alpha\}$ is an open covering of K. Then $K = \{x_1, \ldots, x_n\} \subset \bigcup_{\alpha \in \Lambda} O_\alpha$. Now by definition of union, $\forall x_i \in K, \exists \alpha_i : x_i \in O_{\alpha_i}$. Since K is finite, we get that $\{x_1, \ldots, x_n\} \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Thus K is compact. ■

2. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ is not compact.

   Proof: We want to find a collection of open sets, i.e. open intervals in this case $I_1, I_2, \ldots$ so that $A \subset \bigcup_{n=1}^{\infty} I_n$, but no finite number of them suffices. Let $r_1 = \frac{1}{4}$. For every $n > 1$ we have $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$. Let $r_n = \frac{1}{2}\min\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\} = \frac{1}{2}\frac{1}{n(n+1)}$. Put $I_n = (\frac{1}{n} - r_n, \frac{1}{n} + r_n)$ then $I_n \cap I_m = \emptyset$ for $n \neq m$, and $\forall n : \frac{1}{n} \in I_n$. We get that $A \subset \bigcup_{n=1}^{\infty} I_n$. However, if I_{n_0} is omitted for some n_0 then $A \not\subset I_1 \cup \cdots \cup I_{n_0-1} \cup I_{n_0+1} \cup \ldots$. So in particular, since no finite subcover exists, A is not compact. ■

Lemma 10.5

Let $K \subset Y \subset X$. Then K is compact relative to Y if and only if K is compact.

Proof: $(\Rightarrow)$ Suppose K is compact relative to Y. We want to show it is compact in X. Let $\{O_\alpha : \alpha \in \Lambda\}$ be a covering of K with open sets in X. Then $K \subset \bigcup_{\alpha \in \Lambda} O_\alpha$, O_α is open in X. We want to show $\exists \alpha_1, \ldots, \alpha_n : K \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Put $A_\alpha = O_\alpha \cap Y$. Then A_α is open relative to Y and $K \subset Y$. Since $K \subset \bigcup_{\alpha \in \Lambda} O_\alpha$, $K \subset \bigcup_{\alpha \in \Lambda} A_\alpha$. Then K is compact in Y so $\exists \alpha_1, \ldots, \alpha_n : K \subset A_{\alpha_1} \cup \cdots \cup A_{\alpha_n} \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Then K is compact in X.

$(\Leftarrow)$ This direction is similar and left as exercise. ■

e.g.4. If $A \subset \mathbb{R}$ is compact, then $A \subset \mathbb{R}^2$ is compact. Note however that $(0, 1) \subset \mathbb{R}$ is open but $(0, 1) \subset \mathbb{R}^2$ is not open in $\mathbb{R}^2$ [10.3].

e.g.5. Show that $\mathbb{R}$ is not compact.

Lemma 10.6

Let $K \subset X$ be compact. Then K is closed [10.4].

[10.2] i.e. Finite subcover can be chosen to cover K.
[10.3] This exemplifies that compactness is intrinsic but openness is sensitive to the space.
[10.4] As we see the above example, the converse is not necessarily true.

Proof: Let $K \subset X$ be compact. We want to show K is closed by showing K^C is open. We need to find some $r > 0$ such that $N_r(p) \subset K^C \Leftrightarrow N_r(p) \cap K = \emptyset$. Let $q \in K$ be arbitrary, then $p \neq q \Rightarrow d(p,q) > 0$. Let $r_q = \frac{1}{2}d(p,q)$ then $N_{r_q}(p) \cap N_{r_q}(q) = \emptyset$ by the Triangle Inequality. Now $\{N_{r_q}(q) : q \in K\}$ is an open covering of K. They are open and every $q \in K$ belongs to $N_{r_q}(q)$. Since K is compact, we get that $\exists q_1, \ldots, q_n : K \subset N_{r_{q_1}}(q_1) \cup \cdots \cup N_{r_{q_n}}(q_n)$. Now note that $N_{r_{q_i}}(p) \cap N_{r_{q_i}}(q_i) = \emptyset$. Let $r = \min\{r_{q_i} : 1 \leq i \leq n\}$ then $r > 0$. Then $N_r(p) \cap N_{r_{q_i}}(q_i) = \emptyset \Rightarrow N_r(p) \cap (\bigcup N_{r_{q_i}}(q_i)) = \emptyset \Rightarrow N_r(p) \cap K = \emptyset$. Then K^C is open. ■

Lemma 10.7

Let $K \subset X$ be compact and $F \subset K$ be closed. Then F is compact. In particular, if K is compact and $A \subset X$ is closed. Then $A \cap K$ is compact.

Proof: For the second part, note that since K is compact, K is closed. So $A \cap K$ is closed and $A \cap K \subset K$. Then $A \cap K$ is compact by the first claim.

So we need to show the first claim. Let $\{O_\alpha : \alpha \in \Lambda\}$ be an open covering of F. That is $F \subset \bigcup_{\alpha \in \Lambda} O_\alpha$. We want to show that $\exists \alpha_1, \ldots, \alpha_n : F \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Since $F \subset \bigcup_{\alpha \in \Lambda} O_\alpha$ we have $F^C \cup (\bigcup_{\alpha \in \Lambda} O_\alpha) = X$. In particular, $K \subset F^C \cup (\bigcup_{\alpha \in \Lambda} O_\alpha)$. Since K is compact, $\exists \alpha_1, \ldots, \alpha_n : K \subset F^C \cup O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. Then $F \subset K \subset F^C \cup O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. However, since $F \cap F^C = \emptyset$, $F \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. This shows F is compact. ■

MATH140A **Winter 2021**

Lecture 11: Heine–Borel theorem

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 11.1

Let $K \subset X$ be compact and $A \subset K$ be an infinite subset. Then A has a limit point in K.

Proof: Toward a contradiction. Let $A \subset K$ be an infinite set. Suppose the claim fails: no point in K is a limit point of A. Therefore, $\forall p \in K, \exists r > 0 : N_r(p) \cap A = \begin{cases} \emptyset \\ \{p\} \end{cases}$. So $\{N_{r_p}(p) : p \in K\}$ is an open covering of K. However, since A is infinite, $N_r(p) \cap A$ implies that infinitely-many of coverings are needed to cover $A \subset K$. So there does not exist finite subcover of this covering. ■

Theorem 11.2

Let (X, d) be a metric space. Then (X, d) is compact if and only if every infinite subset of X has a limit point in X.

Proof: We will prove $(\Leftarrow)$. $(\Rightarrow)$ is left as a Hw problem. ■

Lemma 11.3

Suppose $\{K_\alpha\}$ is a collection of compact subsets of X. Further assume that $\bigcap_\alpha K_\alpha = \emptyset$, then $\exists \alpha_1, \ldots, \alpha_n : K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$ [11.1].

Proof: We will use compactness of one of these and the fact that any compact set is closed. Fix some K_{α_0} from this collection. Now by the assumption, $K_{\alpha_0} \cap (\bigcap_{\alpha \neq \alpha_0} K_\alpha) = \bigcap_\alpha K_\alpha = \emptyset$. Then $K_{\alpha_0} \subset (\bigcap_{\alpha \neq \alpha_0} K_\alpha)^C = \bigcup_{\alpha \neq \alpha_0} K_\alpha$. Since K_α is compact for all α, K_α is closed. Then K_α^C is open. So $\{K_\alpha^C : \alpha \neq \alpha_0\}$ is an open covering of the compact set K_{α_0}. Hence $\exists \alpha_1, \ldots, \alpha_n : K_{\alpha_0} \subset K_{\alpha_1}^C \cap \cdots \cap K_{\alpha_n}^C$. Then $K_{\alpha_0} \cap (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}) = \emptyset \Rightarrow K_{\alpha_0} \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \emptyset$. ■

Corollary 11.4

Let $K_1 \supset K_2 \supset \ldots$ be a collection of nonempty compact sets. Then $\bigcap_{i=1}^\infty K_i$ is nonempty.

Proof: Toward a contradiction, suppose not. Then by previous lemma 11.3, $\exists i_1 < i_2 < \cdots < i_n : K_{i_1} \cap K_{i_2} \cap \cdots \cap K_{i_n} = \emptyset$. Then $K_{i_1} \subset K_{i_2} \subset \cdots \subset K_{i_n}$ and $K_{i_1} \cap K_{i_2} \cap \cdots \cap K_{i_n} = K_{i_n} = \emptyset$. Contradiction. ■

e.g.1. Let $n \in \mathbb{N}$ and $I_n = (1 - \frac{1}{n}, 1)$. Then $I_1 \subset I_2 \subset \ldots$. But $\bigcap_{i=1}^\infty I_n = \emptyset$ while $\forall n : I_n \neq \emptyset$. So I_n is not compact.

[11.1] i.e. There exists a finite subcollection whose intersection is empty.

Theorem 11.5 (*Heine-Borel Theorem*)

Let $A \subset \mathbb{R}^d$. Then the followings are equivalent.

1. *A is compact.*
2. *A is bounded and closed.*

e.g.2.

1. $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact in $\mathbb{R}$ since it is closed and bounded.
2. $[0, 1]$ is compact in $\mathbb{R}$ since it is closed and bounded.
3. $(0, 1]$ is not compact because it is bounded but not closed.

Lemma 11.6

Let $I_m = [a_m, b_m]$ be closed interval for $m \in \mathbb{N}$. Assume $I_1 \supset I2 \supset \dots$, then $\bigcap_{i=1}^{\infty} I_m$ is nonempty.

Proof: Note that since $I_1 \supset I2 \supset \dots$, $a_1 \le a_2 \le \dots \le b_2 \le b_1$. Let $A = \{a_1, a_2, \dots\}$ then A is a bounded subset of $\mathbb{R}$. So by the LUB Property, $\sup A = \alpha$ exists in $\mathbb{R}$. We claim $\alpha \in \bigcap_{i=1}^{\infty} I_m$. To see this, we need to show $\alpha \in I_m \Rightarrow a_m \le \alpha \le b_m$. Since $\alpha = \sup A = \sup\{a_1, a_2, \dots\}$, α is an upper bound of A so $\forall m : \alpha \ge a_m$. To see the second inequality, we want to show b_m is an upper bound for A. Note $I_m = [a_m, b_m] \supset I_{m+k} = [a_{m+k}, b_{m+k}]$, then $\sup A = \alpha \le b_m$. Altogether, we showed $a_m \le \alpha \le b_m$ i.e. $\alpha \in I_m \Rightarrow \alpha \in \bigcap_{i=1}^{\infty} I_m$. So $\bigcap_{i=1}^{\infty} I_m \neq \emptyset$. ■

e.g.3.

1. $I_m = [-\sqrt{2} - \frac{1}{m}, \sqrt{2} + \frac{1}{m}]$, $\bigcap_{i=1}^{\infty} I_m = \{\sqrt{2}\}$
2. $I'_m = I_m \cap \mathbb{Q} = [\sqrt{2} - \frac{1}{m}, \sqrt{2} + \frac{1}{m}] \cap \mathbb{Q}$. Then I'_m is closed relative in $\mathbb{Q}$. $\bigcap_{i=1}^{\infty} I'_m = \emptyset$.

Lemma 11.7

Let $B_1 \supset B_2 \supset \dots$ be closed nonempty boxes in $\mathbb{R}^n$. Then $\bigcap_{i=1}^{\infty} B_i$ is nonempty.

Proof: Note by the definition, we have $B_i = [a_{1,i}, b_{1,i}] \times \dots \times [a_{n,i}, b_{n,i}]$. Since $B_1 \supset B_2 \supset \dots$, then $[a_{1,1}, b_{1,1}] \supset [a_{1,2}, b_{1,2}] \supset \dots, \dots, [a_{n,1}, b_{n,1}] \supset [a_{n,2}, b_{n,2}] \supset \dots$. By the previous lemma 11.6: $\bigcap_{i=1}^{\infty}[a_{1,i}, b_{1,j}] \neq \emptyset, \dots, \bigcap_{i=1}^{\infty}[a_{n,i}, b_{n,j}] \neq \emptyset$. Let $x_j \in \bigcap_{i=1}^{\infty}[a_{j,1}, b_{j,1}]$ then $(x_1, x_2, \dots, x_n) \in \bigcap_{i=1}^{\infty} B_i$. ■

Theorem 11.8

Let B be a closed box in $\mathbb{R}^n$ then B is compact.

Proof: Let $B = [a_1, b_1] \times \dots \times [a_n, b_n]$ and $d = \sqrt{(b_1 - a_1)^2 + \dots + (b_n - a_n)^2} > 0$. Toward a contradiction, suppose there exists some open covering $\{O_\alpha : \alpha \in \Lambda\}$ which covers B but no finite subcollection covers B. We will construct $B = B_1 \supset B_2 \supset \dots$ as follows: consider 2^n sub-boxes of B which are a result of taking half point of intervals in B so $c_1 = \frac{a_1+b_1}{2}, \dots, c_n = \frac{a_n+b_n}{2}$. Then we have $I_{1,1} = [a_1, c_1], I_{1,2} = [c_1, b_1], \dots, I_{n,1} = [a_n, c_n], I_{n,2} = [c_n, b_n]$. A sub-box is $I_{1,i_1} \times \dots \times I_{n,i_n}$. We get 2^n sub-boxes since B is not covered by finitely many $\{O_\alpha\}$. Hence there exists a sub-box which is not covered by finitely many, call this box B_2. Then (1). $B_1 \supset B_2$; (2). any two point in B_2 are at most $\frac{1}{2}d$ far from each other; (3). B_2 cannot be covered with finitely many $\{O_\alpha\}$.

Continue inductively, we get (1). $B_1 \supset B_2 \supset \ldots$; (2). any two point in B_m are at most $\frac{1}{2^{m-1}}d$ far from each other; (3). B_m cannot be covered with finitely many $\{O_\alpha\}$. By the previous lemma, $\bigcap_{m=1}^{\infty} B_m \neq \emptyset$. Let $x \in \bigcap_{m=1}^{\infty} B_m$. Then since $x \in B_1$ and $\{O_\alpha\}$ covers B_1, $\exists \alpha : x \in O_\alpha$. Hence O_α is open. Then $\exists r > 0 : B_r(x) \subset O_\alpha$. However, $\forall m, \forall y \in B_m : d(x,y) \leq \frac{d}{2^{m-1}}$. So if m is large enough then $\frac{d}{2^{m-1}} < r$. Thus $B_m \subset O_\alpha$. This contradicts property (3) of $\{B_i\}$. Indeed, B_m is covered with O_α. ■

MATH140A **Winter 2021**

Lecture 12: Connected Sets, Perfect Sets, Cantor Sets, Sequences

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Theorem 12.1 (*Heine-Borel Theorem*)

Let $A \subset \mathbb{R}^n$. Then the followings are equivalent.

1. *A is compact.*
2. *A is bounded and closed.*

Proof: (2) $\Rightarrow$ (1) Suppose A is bounded and closed. Since A is bounded, $\exists M > 0 : A \subset [-M, M] \times \ldots [-M, M] = B$ where B is a closed box in $\mathbb{R}^n$. Now by theorem 11.8, B is compact. Since $A \subset B$ and A is closed. Thus A is compact.

(1) $\Rightarrow$ (2) Since $A \subset \mathbb{R}^n$ is compact, A is closed. We want to show that A is bounded. Toward a contradiction, suppose A is unbounded. Then $\forall k \in \mathbb{N}, \exists x_k \in A : \|x_k\| > k$ [12.1]. We now claim that $B = \{x_k\} \subset A$ is infinite and has no limit points. This will contradict the fact that A is compact. Note that B is an infinite set since $\|x_k\| > k$ and B has no limit points in $\mathbb{R}^n$. Suppose $y \in \mathbb{R}^n$ is a limit point of B. Then $B_1(y) \cap B$ should be infinite. Hence for $k_1 < k_2 < \ldots$ we have $x_{k_i} \in B_1(y) \cap B$. Then $\forall i : \|x_{k_i} - y\| < 1$. By Triangle Inequality, $\|x_{k_i}\| < 1 + \|y\|$. Recall $\|x_{k_i}\| > k_i$ so that $k_i < 1 + \|y\|$.This contradicts the Archimedean Property. So y cannot be a limit point of B. ■

Corollary 12.2 (*Weierstrass Theorem*)

Any infinite bounded subset of $\mathbb{R}^n$ has a limit point.

Proof: Let $A \subset \mathbb{R}^n$ be infinite and bounded. Then $A \subset B$ for some box $B \subset \mathbb{R}^n$. Since B is compact, A has a limit point. ■

Definition 12.1 (*Separated Sets*)

*Let (X, d) be a metric space. Two subsets $A, B \subset X$ are called **separated** if $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.*

e.g.1.

1. $(0, 1)$ and $[0, 2]$ are disjoint but not separated. Indeed, $1 \in \overline{(0, 1]} \cap [1, 2]$.
2. Any two disjoint open intervals in $\mathbb{R}$ are separated.

Question: :

- How about two disjoint open subsets of $\mathbb{R}$?
- How about two disjoint open neighborhoods in (X, d)?

Definition 12.2 (*Connected Sets*)

*Let $C \subset X$, we say C is **connected** if C cannot be written as a union of two nonempty separated subsets.*

[12.1] Otherwise $A \subset B_{k_0}(0)$ for some $k_0 \in \mathbb{N}$ so A is bounded.

e.g.2. $C = (0,1) \cup (1,2)$ is disconnected.

Proposition 12.3

Let $C \subset \mathbb{R}$. Then C is connected if and only if $\forall x, y \in C, x < z < y : z \in C$.

Proof: ($\Rightarrow$) Given C is connected. Toward a contradiction, suppose the statement fails. Then $\exists x, y \in C, x < z < y : z \notin C$. Then denote $A = C \cap (-\infty, z)$ and $B = C \cap (z, \infty)$, $C = A \cup B = (C \cap (-\infty, z)) \cup (C \cap (z, \infty))$. Since $x \in A, y \in B$, A, B are nonempty. Since $\overline{A} \subset (-\infty, z]$ and $\overline{B} \subset [z, \infty)$, $\overline{A} \cap B = \overline{B} \cap A = \emptyset$. Thus A, B are separated, Then C is not connected. Contradiction.

($\Leftarrow$) Given $\forall x, y \in C, x < z < y : z \in C$ holds, we want to show C is connected. Toward a contradiction, suppose $C = A \cup B$ where A and B are separated and nonempty. Let $x \in A, y \in B, x \neq y$ and $(A \cap B = \emptyset)$. Assume $x < y$. Let $z = \sup(A \cap [x, y])$. Now since $z \in \overline{A}$ and $\overline{A} \cap B = \emptyset$, $z \notin B$. Then $z \neq y \Rightarrow x \leq z < y$. Two cases: (1). if $z \in A$: then since $A \cap \overline{B} = \emptyset$ so $z \notin \overline{B}$. Then $\exists r > 0 : (z, z+r) \subset (z, y)$ and $(z, z+r) \cap B = \emptyset$. Hence $z < z' < y$ and $z' \notin B$. Since $z = \sup(A \cap [x, y])$ and $z' > z$, we have $z' \notin A, x \leq z < z' < y, z' \notin B$. Therefore $z' \notin A \cup B = C$. Contradiction to $x < z' < y$. (2). if $z \notin A, x < z < y$: then $z \in C = A \cup B \Rightarrow z \in B$. Hence $z \in \overline{A} \cap B$, which contradicts the fact that A and B are separated. ■

Definition 12.3 (*Perfect Sets*)

*A subset $P \subset X$ is called **perfect** if P is closed and every point in P is a limit point.*

e.g.3. In $\mathbb{R}$ with the usual metric

- $[0, 1]$ is a perfect set.
- $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ is not perfect.

Theorem 12.4

Let $P \subset \mathbb{R}^n$ be a nonempty perfect set, then P is uncountable.

Proof: See theorem 2.43 in the book. ■

Definition 12.4 (*Cantor Sets*)

***Cantor sets** are constructed with $K_1 = [0,1] \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, $K_2 = K_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})) = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1], \ldots, K_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup \cdots \cup [\frac{3^n - 1}{3^n}, \frac{3^n}{3^n}]$ [12.2].*

Proposition 12.5

$K_1 \supset K_2 \supset \ldots$ and K_i is closed and bounded. So K_i is compact subset of $\mathbb{R}$. Since $K_1 \supset K_2 \supset \ldots$ are nonempty, $C = \bigcap_{i=1}^{\infty} K_n$ is nonempty.

Theorem 12.6

$C = \bigcap_{i=1}^{\infty} K_n$ is a perfect set. Thus uncountable.

[12.2] K_n is union of 2^n intervals where each has length 3^{-n}. Intervals of the form $(\frac{3^{k+1}}{3^n}, \frac{3^{k+2}}{3^n})$ are cut out.

Proof: $\boxed{\text{Sketch:}}$ $x \in [0,1]$ can be represented by $x = \sum_{i=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0,2\}$. Then $C = \{x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0,2\}\}$. ■

Definition 12.5 (*Sequence*)

*Let (X,d) be a metric space. A **sequence** is a function*

$$p : \mathbb{N} \to X$$

*We usually write $p_1, p_2, \ldots$ for $p(1), p(2), \ldots$. The **range** of a sequence is $\{p_n : n \in \mathbb{N}\} \subset X$. We say the sequence $\{p_n\}$ is bounded if the range is bounded.*

e.g.4.

- $p_n = \frac{1}{n}$ for $n \in \mathbb{N}$ is a sequence and its range $\{\frac{1}{n}, n \in \mathbb{N}\}$.
- $p_n = (-1)^n$ for $n \in \mathbb{N}$ is a sequence and its range is $\{-1,1\}$.

Definition 12.6 (*Convergent*)

*A sequence $\{p_n\}$ in X is called **convergent** if $\exists p \in X$ satisfying the following* [12.3]

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, n \geq N : d(p_n, p) < \epsilon$$

e.g.5. Let $X = \mathbb{R}$ with Standard metric

- $p_n = 1$. Then p_n converges to 1.
- $p_n = \frac{1}{n}$. Then p_n converges to 0.

  Proof: We want to show $\forall \epsilon > 0, \exists N \in \mathbb{N}$: if $n \geq N$, then $d(\frac{1}{n}, 0) < \epsilon \Rightarrow |\frac{1}{n}| < \epsilon$. Now $|\frac{1}{n}| < \epsilon \Leftrightarrow \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$. By the Archimedean Property of $\mathbb{R}$, $\exists N \in \mathbb{N} : N > \frac{1}{\epsilon}$. So if $n \geq N$ then $n \geq N > \frac{1}{\epsilon} \Rightarrow n > \frac{1}{\epsilon}$. ■

Lemma 12.7

Let (X,d) be a metric space and $\{p_n\}$ be a sequence in X

1. $\lim_{n\to\infty} p_n = p$ *if and only if $\forall N_r(p)$: for all but finitely many n's, $p_n \in N_r(p)$.*
2. *Suppose $p_n \to p$ and $p_n \to p'$. Then $p = p'$.*
3. *If $p_n \to p$, then $\{p_n\}$ is bounded.*
4. *Let $A \subset X$. Suppose $p \in X$ is a limit point of A, then $\exists \{p_n\} \in A : p_n \to p$.*

Proof:

1. ($\Rightarrow$) Suppose $p_n \to p$ and let $N_r(p)$ be a neighborhood. Apply definition of convergence with $\epsilon = r$. Then $\exists N, \forall n \geq N : d(p_n, p) < \epsilon \Rightarrow p_n \in N_r(p)$. ($\Leftarrow$) is similar and left as exercise.
2. Toward a contradiction. Suppose $p \neq p'$, $p_n \to p$ and $p_n \to p'$. Since $p \neq p'$, $d(p,p') > 0$. Set $\epsilon = \frac{1}{2}d(p,p')$. Now by the definition of convergence $\exists N, \forall n \geq N : d(p_n, p) < \epsilon$. Similarly, $\exists N' : \forall n \geq N', d(p_n, p') < \epsilon$. Let $n \geq \max\{N, N'\}$. We

[12.3] *Notation:* If $\{p_n\}$ converges to p, we write $\lim_{n\to\infty} p_n = p$ or $p_n \to p$

have $d(p_n, p) < \epsilon$ and $d(p_n, p') < \epsilon$. Therefore, $d(p, p') \leq d(p, p_n) + d(p_n, p') < 2\epsilon = d(p, p')$. Contradiction.

3. Suppose $p_n \to p$. Then $\forall \epsilon > 0, \exists N, \forall n \geq N : d(p_n, p) < \epsilon$. Let $\epsilon = 1$ then $\exists N_1$ such that if $n \geq N_1$, then $d(p_n, p) < 1$. Let $R' = \max\{d(p_n, p) : n = 1, \ldots, N_1 - 1\}$. Then $R' \geq 0$. Put $R = \max\{1, R'\}$. Then $\forall n : d(p_n, p) \leq R$. Thus $\{p_n\}$ is bounded.

4. Let $A \subset X$ and let $p \in X$ be a limit point of A. $\forall n > 0, \exists p_n \in A$ and $p_n \in N_{\frac{1}{n}}(p)$. We claim $p_n \to p$. Indeed, $\forall \epsilon > 0$, we need to find N such that if $n \geq N$, then $d(p_n, p) < \epsilon$. Given $\epsilon > 0, \exists N_0 : \frac{1}{N_0} < \epsilon$. Then $\forall n \geq N_0$, we have $p_n \in N_{\frac{1}{n}}(p)$. Thus $d(p, p_n) < \frac{1}{n} \leq \frac{1}{N_0} < \epsilon \Rightarrow p_n \to p$. ■

MATH140A **Winter 2021**

Lecture 13: Sequences, Subsequence

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 13.1

Let $\{a_n\}$ and $\{b_n\}$ be two sequences in $\mathbb{C}$. Assume $a_n \to a$ and $b_n \to b$. Then

1. $a_n + b_n \to a + b$
2. $a_n b_n \to ab$
3. *Suppose $b \neq 0$, then $\exists N, \forall n \geq N : b_n \neq 0$ and $\{\frac{a_n}{b_n} : n \geq N\}$ is defined and $\frac{a_n}{b_n} \to \frac{a}{b}$.*

Proof: 1. We want to show $\forall \epsilon, \exists N \in \mathbb{N}, \forall n \geq N : |(a_n + b_n) - (a + b)| < \epsilon$. By definition of convergence, $\forall \epsilon' > 0, \exists N_1, \forall n \geq N_1 : d(a_n, a) = |a_n - a| < \epsilon'$ and $\exists N_2, \forall n \geq N_2 : d(b_n, b) = |b_n - b| < \epsilon'$. Let $N = \max\{N_1, N_2\}$ and $n \geq N$. Then $|a_n + b_n - (a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < 2\epsilon'$. So if we let $\epsilon' = \frac{\epsilon}{2}$, we get the claim for this N.

2. We want to show $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |a_n b_n - ab| < \epsilon$. We know $\forall \epsilon' > 0 \exists N_1, \forall n \geq N_1 : |a_n - a| < \epsilon'$ and $\exists N_2, \forall n \geq N_2 : |b_n - b| < \epsilon'$. $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| \leq |a_n b_n - a_n b| + |a_n b - ab| = |a_n||b_n - b| + |a_n - a||b|$. Note that $a_n \to a$ so by the lemma we proved earlier $\exists A, \forall n : |a_n| \leq A$. Let now $N = \max\{N_1, N_2\}$ and let $n \geq N$ then we have $|a_n b_n - ab| \leq |a_n||b_n - b| + |a_n - a||b| \leq A|b_n - b| + |a_n - a||b| < A\epsilon' + |b|\epsilon' = (A + |b|)\epsilon'$. So if we let $\epsilon' = \frac{\epsilon}{A+|b|}$, then $\forall n \geq N : |a_n b_n - ab| < \epsilon$.

3. We first show that $\exists N \in \mathbb{N}, \forall n \geq N : b_n \neq 0$. We know $b_n \to b$ and $b \neq 0$. Let $\epsilon = \frac{|b|}{2}$ then $\exists n \in \mathbb{N}, \forall n \geq N_0 : |b_n - b| < \epsilon = \frac{|b|}{2}$. Then $|b_n| > \frac{|b|}{2} \Rightarrow \forall n \geq N_0 : b_n \neq 0$. Hence $\frac{a_b}{b_n}$ is defined for $n \geq N_0$.

   Note $\frac{a_n}{b_n} = a_n \cdot \frac{1}{b_n}$. So the second claim in (3) follows from (2) if we show that $\frac{1}{b_n} \to \frac{1}{b}$. Then we need to show $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : |\frac{1}{b_n} - \frac{1}{b}| < \epsilon$. Note $|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{b_n b}| = \frac{|b_n - b|}{|b_n b|}$. Note $b_n \to b$ so $\exists N_0, \forall n \geq N_0 : |b_n - b| < \frac{|b|}{2} \Rightarrow \frac{|b|}{2} < |b_n| < \frac{3|b|}{2} \Rightarrow \forall n \geq N_0 : \frac{|b|^2}{2} < |bb_n| < \frac{3|b|^2}{2} \Rightarrow \frac{2}{3|b|^2} < \frac{1}{|bb_n|} < \frac{2}{|b|^2} \Rightarrow \forall n \geq N_0 : |\frac{1}{b_n} - \frac{1}{b}| = \frac{|b - b_n|}{|b_n b|} < \frac{2}{|b|^2}|b - b_n|$. Let $\epsilon' > 0, \exists N_1 \in \mathbb{N}, \forall n \geq N_1 : |b - b_n| < \epsilon'$. Altogether if $N = \max\{N_0, N_1\}$ and $\forall n \geq N : |\frac{1}{b_n} - \frac{1}{b}| < \frac{2}{|b|^2}|b_n - b| < \frac{2}{|b|^2}\epsilon' = 2$. Let $\epsilon' = \frac{|b|^2}{2}\epsilon$. Then we get $\forall n \geq N : |\frac{1}{b_n} - \frac{1}{b}| < \epsilon$. ■

Lemma 13.2

let $\{\vec{x_n} = (x_{1,1}, \ldots, x_{k,n})\}$ be a sequence in $\mathbb{R}^k$. Then $\{\vec{x_n}\}$ converges to $\vec{x} = (x_1, \ldots, x_k)$ if and only if $\forall 1 \leq j \leq k : x_{j,n} \to x_j$.

Proof: Repeat the argument we used for Hw that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Left as exercise. ■

Definition 13.1 (*Subsequence*)

Let $\{p_n\}$ be a sequence, and let $\{n_i\}$ be a sequence of natural numbers such that $n_1 < n_2 < n_3$. Then $p_{n_1}, p_{n_2}, \dots$ is a ***subsequence*** *of $\{p_n\}$. We write this as $\{p_{n_i}\}$.*

Definition 13.2 (*Subsequential Limit*)

Let $\{p_n\}$ be a sequence. A point p is called a ***subsequential limit*** *of p_n if $\exists \{p_{n_i}\} : p_{n_i} \to p$.*

e.g.1. Let $p_n = (-1)^n$ then $\{p_n\}$ does not converge. However, $\{p_{2k+1}\}$ converges and $\{p_{2k}\}$ converges. So -1 and 1 are the only subsequential limits of $\{p_n\}$.

e.g.2. Consider $\mathbb{Q} \cap [0,1]$ which is a countable set. So we can write $\mathbb{Q} \cap [0,1] = \{q_1, \dots\}$. What is the set of subsequential limits of $\mathbb{Q} \cap [0,1]$? [13.1]

Lemma 13.3

Let $\{p_n\}$ be a sequence. Then $p_n \to p$ if and only if every subsequence converges to p.

Proof: $(\Rightarrow)$ Suppose $p_n \to p$ and let $\{p_{n_i}\}$ be a subsequence. We need to show that $p_{n_i} \to p$. Let $\epsilon > 0$, we need to find $I \in \mathbb{N}$ such that $\forall i \geq N : d(p_{n_i}, p) < \epsilon$. Since $p_n \to p$, $\exists N \in \mathbb{N}, \forall n \geq N : d(p_n, p) < \epsilon$. Let I be such $\forall i \geq I : n_i \geq N$ [13.2]. Then $d(p_{n_i}, p) < \epsilon \Rightarrow p_{n_i} \to p$.

$(\Leftarrow)$ Suppose all subsequences converge to p. We want to show $p_n \to p$. Toward a contradiction, suppose $p_n \not\to p$. So $\exists \epsilon > 0, \forall N \in \mathbb{N}, \exists n \geq N : d(p_n, p) \geq \epsilon$. Then there exists $\epsilon > 0$ and $n_1 < n_2 < \dots$ such that $d(p_{n_i}, p) \geq \epsilon$. Thus $\{p_{n_i}\}$ does not converge to p. ■

Lemma 13.4

1. *Let $\{p_n\}$ be a sequence in a compact set $K \subset X$. Then $\{p_n\}$ has a converging subsequence in K.*
2. *Every bounded sequence in $\mathbb{R}^k$ has a converging subsequence* [13.3].

Proof:
1. If $\{p_n\}$ has a finite range, then $\exists n_1 < n_2 < \cdots : p_{n_1} = p_{n_2} = \cdots = p$. Then $\{p_{n_i}\}$ converges to p. Now if $\{p_n\}$ has infinite range, then the set $A = \{p_n : n \in \mathbb{N}\} \subset K$ is an infinite set. By a lemma, A has a limit point in K. By another lemma, there exists a sequence in A which converges to this limit. i.e. a subsequence of $\{p_n\}$ converges to a point in K.
2. Let $\{x_n\}$ be a bounded sequence of $\mathbb{R}^k$. Then $\{x^n\} \subset B$ where B is a closed (and bounded) box in $\mathbb{R}^k$. By Heine-Borel Theorem, B is compact. So by (1), $\{x_n\}$ has a converging subsequence in B. ■

Theorem 13.5

A metric space X is compact if and only if every sequence in X has a converging subsequence.

[13.1] Show that $[0,1]$ is the set of subsequential limits of $\{q_n\}$.

[13.2] i.e. $n_1 < n_2 < \cdots < n_{I-1} \leq N \leq n_I$.

[13.3] *By the scribe*: this is referred to **Bolzano-Weierstrass Theorem**.

Proof: Follows from the lemma + the Hw problem. ■

Lemma 13.6

Let $\{p_n\}$ be a sequence in X. Let $A \subset X$ be the set of subsequential limits of $\{p_n\}$. Then A is closed.

Proof: We want to show that $A' \subset A$. i.e. if $p \in X$ is a limit point of A, then p is a subsequential limit. Since p is a limit point of A, $\forall i \geq 1, \exists q_i \in A : q_i \in N_{\frac{1}{i}}(p)$. Now for every i, we have $q_i \in A$. So a subsequence of $\{p_n\}$ converges to q_i. Applying the definition of convergence with $\epsilon = \frac{1}{i}, \exists p_{n_i} : d(p_{n_i}, q_i) < \frac{1}{i}$. So $\exists J, \forall j \geq J : d(p_{n_{i,j}}, q_i) < \epsilon = \frac{1}{i}$. Then $\forall i : d(p_{n_i}, p) \leq d(p_{n_i}, q_i) + d(q_i, p) < \frac{2}{i}$. Then $p_{n_i} \to p$. ■

MATH140A **Winter 2021**

Lecture 14: Cauchy Sequence, Divergence, Series

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 14.1 (*Cauchy Sequence*)

*Let (X, d) be a metric space. A sequence $\{p_n\}$ in X is called a **Cauchy Sequence** if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N : d(p_n, p_m) < \epsilon$$

Lemma 14.1

Let $\{p_n\}$ be a convergent sequence, then $\{p_n\}$ is Cauchy.

Proof: Suppose $p_n \to p$. Then $\forall \epsilon > 0, \exists n \in \mathbb{N}, \forall n \geq N : d(p_n, p) < \frac{\epsilon}{2}$. Let now $n, m \geq N$ then $d(p_n, p_m) \leq d(p_n, p) + d(p_m, p) < \epsilon$. ■

Definition 14.2 (*Complete*)

*A metric space (X, d) is **complete** if every Cauchy sequence in X converges.*

e.g.1. $\mathbb{Q}$ with the standard metric is not complete.

Proof: Let $p_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2}) \cap \mathbb{Q}$ then $\{p_n\}$ is Cauchy. But $\{p_n\}$ is not convergent in $\mathbb{Q}$. ■

Lemma 14.2

Let (X, d) be a metric space. A Cauchy sequence is bounded.

Proof: Indeed $\exists N \in \mathbb{N}, \forall m, n \geq N : d(p_m, p_n) < 1$. Then $\forall m \geq N : d(p_m, p_N) < 1$. Let $R' = \max\{d(p_n, p_N) : n = 1, \ldots, N-1\}$, $R' \in \mathbb{R}$. Let $R = \max\{1, R'\}$. Then $\forall n : d(p_n, p_N) \leq R$. ■

Lemma 14.3

Let $\{p_n\}$ be a Cauchy sequence. Suppose there exists a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that $p_{n_i} \to p$, then $p_n \to p$.

Proof: Since p_{n_i} converges to p, $\forall \epsilon > 0, \exists I, \forall i \geq I : d(p_{n_i}, p) < \frac{\epsilon}{2}$. Since $\{p_n\}$ is Cauchy $\exists N', \forall n, m \geq N' : d(p_n, p_m) < \frac{\epsilon}{2}$. Now let i be that $i \geq I$ and $n_i \geq N'$. Then $\forall n \geq N' : d(p_{n_i}, p) < \frac{\epsilon}{2}, d(p_n, p_{n_i}) < \frac{\epsilon}{2}$. Thus $d(p_n, p) \leq d(p_n, p_{n_i}) + d(p_{n_i}, p) < \epsilon$. So if we let $N = N'$ then $\forall n \geq N : d(p_n, p) < \epsilon \Rightarrow p_n \to p$. ■

Theorem 14.4 (*Cauchy Criterion for Convergence of Sequence*)

1. *Let (X, d) be a compact metric space. Then X is complete.*
2. *$\mathbb{R}^k$ is complete.*

Proof:

1. Let $\{p_n\}$ be a Cauchy sequence in X. Since X is compact, $\exists\{p_{n_i}\}$ and $\exists p \in X : p_{n_i} \to p$. Since $\{p_n\}$ is Cauchy, by lemma 14.3, X is complete.
2. Let $\{p_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Then by lemma 14.2, $\{p_n\}$ is bounded. So there exists a closed bounded box B such that $\{p_n\}$ is a sequence in B. By Heine-Borel Theorem, B is compact. So by part (1), $\{p_n\}$ converges in B. Thus $\{p_n\}$ converges in $\mathbb{R}^k$. ■

Definition 14.3 (*Diameter*)

Let $A \subset X$, define

$$\operatorname{diam}(A) = \sup\{d(x,y) : x, y \in A\}$$

Here $d(x,y) \in \mathbb{R}$. So if A is bounded, then $\operatorname{diam}(A) \in \mathbb{R}$; otherwise, $\operatorname{diam}(A) = \infty$.

Lemma 14.5

Let $\{p_n\}$ be a Cauchy sequence and $A_N = \{p_N, p_{N+1}, \dots\}$. Note that $A_N \supset A_{N+1} \supset \dots$ we get $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N : \operatorname{diam}(A_N) \leq \epsilon$ then $\inf_N \operatorname{diam}(A_N) = 0$[14.1].

.

Lemma 14.6

Let $A \subset X$

1. $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.
2. *If $K_1 \supset K_2 \supset \dots$ are compact and nonempty, and $\operatorname{diam}(K_n) \to 0$, then $\bigcap K_n = \{p\}$.*

Proof:

1. $\forall p, q \in \overline{A}$ and $\forall \epsilon > 0, \exists p', q' \in A : d(p,p') < \epsilon$ and $d(q',q) < \epsilon$. This indicates $d(p,q) \leq d(p,p') + d(p',q') + d(q',q) < 2\epsilon + d(p',q')$. Since ϵ is arbitrary, $\operatorname{diam}(\overline{A}) \leq \operatorname{diam}(A)$. Since $A \subset \overline{A}$, $\operatorname{diam}(A) \leq \operatorname{diam}(\overline{A})$. Thus $\operatorname{diam}(A) = \operatorname{diam}(\overline{A})$.
2. Since K_n is nonempty compact, $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$. Now if $p \neq q \in \bigcap_{n=1}^{\infty} K_n$, then $\forall n : p, q \in K_n$. So $\operatorname{diam}(K_n) \geq d(p,q) > 0$. But $\operatorname{diam}(K_n) \to 0$, contradiction. ■

e.g.2. Not all bounded sequences in $\mathbb{R}^k$ converge. For example, $(-1)^n$ is bounded but not convergent.

Lemma 14.7

[14.2]

1. *Let $p_1 \leq p_2 \leq \dots$ be a sequence in $\mathbb{R}$. Assume $\{p_n\}$ is bounded, then $p_n \to p$.*
2. *Let $p_1 \geq p_2 \geq \dots$ be a sequence in $\mathbb{R}$. Assume $\{p_n\}$ is bounded, then $p_n \to p$.*

[14.1] This is another way of defining the Cauchy sequence using diam.
[14.2] Any bounded monotonically increasing / decreasing sequence converges.

Proof: We will prove (1) and (2) can be shown similarly.

Since $\{p_n\}$ is bounded, by the LUB property of $\mathbb{R}$, $\sup\{p_n : n \in \mathbb{N}\} = \alpha$ exists. Then $p_n \le \alpha$. We claim that $p_n \to \alpha$. We need to show $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \ge N : |p_n - p| < \epsilon \Rightarrow \alpha - p_n < \epsilon$. Since $\epsilon > 0$, $\alpha - \epsilon < \alpha$. Since $\alpha = \sup\{p_n\}$, $\exists P_N : \alpha - \epsilon < P_N \le \alpha$. Now $\forall n \ge N : \alpha - \epsilon < P_N \le p_n \le \alpha \Rightarrow 0 < \alpha - p_n < \epsilon \Rightarrow p_n \to \alpha$. ∎

Definition 14.4 (*Diverge*)

Let $\{p_n\}$ be a sequence in $\mathbb{R}$. We write $p_n \to \infty$ if $\forall L \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \ge N : p_n \ge L$. Similarly, we say $p_n \to -\infty$ if $\forall L \in \mathbb{R}, \exists N \in \mathbb{N}, \forall n \ge N : p_n \le L$. We define $\sup$ and $\inf$ of subsets in $\mathbb{R} \cup \{-\infty, \infty\}$ to be usual $\sup$ and $\inf$ in the set which is bounded, If the set A is not bounded above, we say $\sup A = \infty$. If the set is not bounded below, we say $\sup A = -\infty$.

Definition 14.5 (*Upper Limits, Lower Limits*)

Let $\{p_n\}$ be a sequence in $\mathbb{R}$. Let A be the set of all subsequential limits of $\{p_n\}$ in $\mathbb{R} \cup \{-\infty, \infty\}$. Put $p^* = \sup A$ and $p_* = \inf A$. Note p^* and p_* always exist. We will write [14.3]

$$\lim_{n\to\infty} \sup p_n = p^* \text{ and } \lim_{n\to\infty} \inf p_n = p_*$$

Note by definition $\overline{\lim} p_n \le \underline{\lim} p_n$ and the equality holds if and only if A has one point.

Lemma 14.8

Let $\{p_n\}$ be a sequence in $\mathbb{R}$. Define A, p^*, p_* as above.

1. *$p^* \in A$ and $p_* \in A$.*
2. *If $x > p^*$, then $\exists N, \forall n \ge N : p_n < x$. Similarly, if $y < p^*$, then $\exists N, \forall n \ge N : y < p_n$.*

Proof:

1. When $p^*, p_* \in \mathbb{R}$, this follows $A = \overline{A}$ and $\sup A, \inf A \in \overline{A}$. The case of $\infty, -\infty$ follows similarly using the definition.
2. Suppose not, then $\exists n_1 < n_2 < \cdots : p_{n_i} > x$. This implies $\exists \alpha \in A$ which is a subsequential limit of $\{p_{n_i}\}$ so that $\alpha \ge x > p^*$. Therefore, $\exists \alpha \in A, \alpha > p^* = \sup A$. Contradiction. ∎

Lemma 14.9

Let $\{p_n\}$ and $\{q_n\}$ be two sequences. Assume $\forall n \ge N : p_n \le q_n$. Then $\overline{\lim} p_n \le \overline{\lim} q_n$ and $\underline{\lim} p_n \le \underline{\lim} q_n$.

e.g.3. Let $p_n = \begin{cases} 10 & n = 1 \\ -10 & n = 2 \\ \frac{1}{n} & n \ge 3 \end{cases}$. Then $\overline{\lim} p_n = \underline{\lim} p_n = 0$ and $\sup\{p_n\} = 10, \inf\{p_n\} = -10$.

[14.3] *Notation:* We write $\lim\sup$ as $\overline{\lim}$ and $\lim\inf$ as $\underline{\lim}$.

Definition 14.6 (*Series*)

*Let $\{a_n\}$ be a sequence in $\mathbb{C}$. $\forall n \in \mathbb{N}$, define $s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i$. Then the sequence $\{s_n\}$ is called the **sequence of partial sums**. We denote $a_1 + a_2 + \ldots$ by $\sum_{i=1}^{\infty} a_i$ or $\sum a_i$.*

Definition 14.7 (*Convergent Series*)

We say the series $\sum a_i$ converges if the sequence of partial sums $\{s_n\}$ converges. If $s_n \to s$, we write $\sum_{i=1}^{\infty} a_i = s$.

Lemma 14.10

Let $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ converges.

Proof: We need to show that the sequence of partial sums converges. $s_\ell = \sum_{n=0}^{\ell} x^n = 1 + x + x^2 + \cdots + x^\ell = \frac{1-x^{\ell+1}}{1-x}$. Now $0 \leq x < 1$, hence $x^{\ell+1} \to 0$. Therefore $s_\ell \to \frac{1}{1-x}$. So by the definition $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. ■

Theorem 14.11 (*Cauchy Criterion for Convergence of Series*)

The series $\sum a_n$ converges if and only if $\forall \epsilon > 0, \exists N, \forall m \geq n \geq N$:

$$|s_m - s_{n-1}| = \left|\sum_{i=n}^{m} a_i\right| < \epsilon$$

Proof: $\{s_n\}$ converges if and only if $\{s_n\}$ is Cauchy. Now $\{s_n\}$ is Cauchy if and only if $\forall \epsilon > 0, \exists N, \forall m, n \geq N : |s_n - s_m| < \epsilon$. Recall $s_n = \sum_{i=1}^{n} a_i$ and $s_m = \sum_{i=1}^{n} a_i$. Let $m > n \geq N$ then $s_m - s_n = a_{n+1} + \cdots + a_m = \sum_{i=n+1}^{m} a_i$. Therefore, the series $\sum a_n$ converges if and only if $\forall \epsilon > 0, \exists N, \forall m \geq n \geq N : |s_m - s_{n-1}| = |\sum_{i=n}^{m} a_i| < \epsilon$. ■

Corollary 14.12

Let $\sum a_n$ be a convergent series then $a_n \to 0$.

Proof: By Cauchy Criterion, we have $\forall \epsilon > 0, \exists N, \forall m \geq n \geq N : |\sum_{i=1}^{m} a_i| < \epsilon$. Let $m = n$ then we get $\forall n \geq N : |a_n| < \epsilon$. Then $a_n \to 0$. ■

MATH140A **Winter 2021**

Lecture 15: Absolute Convergence, Test

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Lemma 15.1

Let $a_n \geq 0$. Then $\sum a_n$ converges if and only if $\{s_n\}$ is bounded.

Proof: ($\Rightarrow$) If $\sum a_n$ converges, then by definition, $\{s_n\}$ is convergent. Hence $\{s_n\}$ is bounded.

($\Leftarrow$) Suppose $\{s_n\}$ is bounded. Note also that $s_{n+1} = a_1 + \cdots + a_n + a_{n+1} \geq a_1 + \cdots + a_n = s_n$. So $\{s_n\}$ is monotonically increasing. Then $\{s_n\}$ converges so $\sum a_n$ converges. ∎

Lemma 15.2

Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences in $\mathbb{R}$. Suppose $\forall n \geq N_0 : a_n \leq b_n \leq c_n$. If $\sum a_n$ and $\sum c_n$ converge, then $\sum b_n$ converges.

Proof: We need to show that the sequence of partial sum for $\sum b_n$ is Cauchy. Let $A_n = a_1 + \cdots + a_n, B_n = b_1 + \cdots + b_n, C_n = c_1 + \cdots + c_n$. Let $\epsilon > 0$, then $\exists N_1, \forall m \geq n \geq N_1 : |A_m - A_{n-1}| = |\sum_{i=n}^m a_i| < \epsilon$. Similarly, $\exists N_2, \forall m \geq n \geq N_2 : |C_m - C_{n-1}| = |\sum_{i=n}^m c_i| < \epsilon$. Let $N = \max\{N_0, N_1, N_2\}$. Let $m \geq n \geq N$. Then $\forall n \leq i \leq m : a_i \leq b_i \leq c_i \Rightarrow \sum_{i=1}^m a_i \leq \sum_{i=n}^m b_i \leq \sum_{i=n}^m c_i$. Hence $|\sum_{i=n}^m b_i| < \epsilon$. $\sum b_n$ converges. ∎

Definition 15.1 (*Absolutely Convergent*)

*Let $\{a_n\}$ be a sequence. We say $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ converges.*

Corollary 15.3

Let $\{a_n\}$ be a sequence. If $\sum |a_n|$ converges, then $\sum a_n$ converges [15.1].

Proof: $\forall n : -|a_n| \leq a_n \leq |a_n|$. Since $\sum a_n$ converges, $\sum -|a_n|$ converges. So by previous lemma, $\sum a_n$ converges. ∎

Lemma 15.4

Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive numbers. Assume $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$. Then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof: We will show ($\Leftarrow$). The other direction is similar and left as exercise.

Since $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$, if we let $\epsilon = \frac{\ell}{2}$, then $\exists N, \forall n \geq N : |\frac{a_n}{b_n} - \ell| < \frac{\ell}{2} \Rightarrow -\frac{\ell}{2} < \frac{a_n}{b_n} - \ell < \frac{\ell}{2} \Rightarrow \frac{\ell}{2} < \frac{a_n}{b_n} < \frac{3\ell}{2}$. Since $b_n > 0$, $\forall n \geq N : \frac{\ell}{2} b_n < a_n < \frac{3\ell}{2} b_n$. Since $\sum b_n$ converges, by lemma 15.2, $\sum a_n$ converges. ∎

[15.1] The converse is not true. Counterexample: $\sum \frac{(-1)^n}{n}$ converges but $\sum \frac{1}{n}$ does not converge. So $\sum \frac{(-1)^n}{n}$ is not absolutely convergent.

Theorem 15.5

Let $a_1 \geq a_2 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} 2^k a_{2^k}$ converges [15.2].

Proof: Since $a_n \geq 0$, we need to show that the sequence of partial sums are bounded to get convergence.

($\Rightarrow$) Assume $\sum a_n$ converges, we want to show $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. We regroup the sum as the following $\underbrace{a_1}_{\geq a_1} + \underbrace{a_2}_{\geq a_2} + \underbrace{a_3 + a_4}_{\geq 2a_4} + \underbrace{a_5 + a_6 + a_7 + a_8}_{\geq 4a_8} + \cdots + \underbrace{a_{2^{n-1}+1} + \cdots + a_{2^n}}_{\geq 2^{n-1} a_{2^n}}$.
Recall that $\{s_{2^n} = a_1 + \cdots + a_{2^n}\}$ is bounded by S. Then $a_1 + a_2 + 2a_4 + \cdots + 2^{n-1} a_{2^n} \leq S \Rightarrow a_1 + 2(a_2 + 2a_4 + \ldots 2^n a_{2^n}) \leq 2S - a_1 \leq 2S$. So $\{t_n = a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n}\}$ is bounded by $2S$. Thus $\sum 2^k a_{2^k}$ converges.

($\Leftarrow$) Assume $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, we want to show that $\sum a_n$ converges. First note that $\{s_n = a_1 + \cdots + a_n\}$ is increasing. So it suffices to show that $\{s_{2^k}\}$ is bounded. $s_{2^n} = a_1 + \cdots + a_{2^n} \leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_n) + \cdots + (a_{2^n} + \cdots + a_{2^n-1}) \leq a_1 + 2a_2 + 4a_4 + \cdots + 2^{n-1} a_{2^n-1}$. However, $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. So $\{t_n = \sum_{k=0}^{n} 2^k a_{2^k}\}$ is bounded. Then $\{s_{2^n-1}\}$ is bounded. Therefore $\{s_n\}$ is bounded. ■

Corollary 15.6

1. *$\sum \frac{1}{n^p}$ converges if and only if $p > 1$.*
2. *$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ converges if and only if $p > 1$.*

Proof: We will use the previous theorem. By that theorem, $\sum \frac{1}{n^p}$ converges if and only if $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges. Note if $p > 1$, then $2^{1-p} < 1$. So by the lemma 14.10, $\sum 2^{(1-p)k}$ converges. If $p \leq 1$, then $2^{(1-p)k} \not\to 0$. Then by the divergence test lemma, $\sum 2^{(1-p)k}$ does not converge. We conclude that $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p}$ converges if and only if $p > 1$. So by the theorem $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Part (2) can be proved similarly. ■

Proposition 15.7 (*Root Test*)

[15.3]*Let $\{a_n\}$ be a sequence. Let*

$$\alpha = \overline{\lim_{n \to \infty}} \sqrt[n]{|a_n|}$$

1. *If $\alpha < 1$, then $\sum a_n$ absolutely converges.*
2. *If $\alpha > 1$, then $\sum a_n$ diverges.*
3. *If $\alpha = 1$, no information is given.*

Proof: 1. Since $\alpha < 1$, $\exists \alpha < \beta < 1$. Now by property of $\overline{\lim}$, $\exists N, \forall n \geq N : \sqrt[n]{|a_n|} \leq \beta \Rightarrow |a_n| \leq \beta^n$. Therefore, $\sum |a_n|$ converges.

2. $\alpha > 1$. By property of $\overline{\lim}$, $\exists n_1 < n_2 < \cdots : \sqrt[n_i]{|a_{n_i}|} \to \alpha \Rightarrow a_n \not\to 0$. So by divergence test, $\sum a_n$ diverges.

[15.2]*By the scribe*: this is referred to **Cauchy Condensation Test**.
[15.3]This test only detects convergence as fast as geometric series.

3. For example, $\sum n$ diverges but the test gives no information; $\sum \frac{1}{n^2}$ converges but the test gives no information. ■

Proposition 15.8 (*Ratio Test*)

Let $\{a_n\}$ be a sequence, $a_n \neq 0$. Let

$$\alpha = \overline{\lim} \left| \frac{a_{n+1}}{a_n} \right|$$

1. *If $\alpha < 1$, then $\sum a_n$ absolutely converges.*
2. *If $\exists N_0, \forall n \geq N_0 : |\frac{a_{n+1}}{a_n}| \geq 1$, then $\sum a_n$ diverges.*

Proof: 1. Let $0 \leq \alpha < \beta < 1$. Then $\exists N, \forall n \geq N : |\frac{a_{n+1}}{a_n}| < \beta$. Let $n \geq N$. we want to estimate $|a_n|$. $|\frac{a_{N+1}}{a_N}| \leq \beta \Rightarrow |a_{N+1}| \leq \beta |a_N|$. Inductively, $|a_n| \leq \beta |a_{n-1}|$. Hence $\forall n \geq N : |a_n| \leq \beta^{n-N} |a_N|$. Then $\sum_{n=N}^{\infty} |a_n|$ converges. This tells $\sum a_n$ converges absolutely.

2. If $\exists N_0 : |\frac{a_{n+1}}{a_n}| \geq 1$, then $\forall n \geq N_0 : |a_n| \geq |a_{N_0}|$. Hence $a_n \not\to 0$ so $\sum a_n$ diverges. ■

e.g.1. $\sum \frac{1}{2^n}$ converges and $\sum \frac{1}{3^n}$ converges. Now let $a_n = \begin{cases} \frac{1}{2^n} & n \text{ is odd.} \\ \frac{1}{3^n} & n \text{ is even.} \end{cases}$. Then $\sum a_n$ converges by the Root Test, but the Ratio Test cannot give convergence since $\overline{\lim} |\frac{a_{n+1}}{a_n}| = \infty$.

Proposition 15.9

Let $\{a_n\}$ be a sequence of positive numbers. Then

$$\underline{\lim} \frac{a_{n+1}}{a_n} \leq \underline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \sqrt[n]{a_n} \leq \overline{\lim} \frac{a_{n+1}}{a_n}$$

Proof: We will prove the first inequality. The last one is similar and proved in the book.

Let $\underline{\lim} \frac{a_{n+1}}{a_n} = \alpha$. If $\alpha = 0$, then $\sqrt[n]{a_n} \geq 0 \Rightarrow \underline{\lim} \sqrt[n]{a_n} \geq 0$ since $a_n \geq 0$. Now we consider $\alpha > 0$. Let $0 < \beta < \alpha$. We will show $\underline{\lim} \sqrt[n]{a_n} \geq \beta$. Since $\beta < \alpha = \underline{\lim} \frac{a_{n+1}}{a_n}$, $\exists N, \forall n \geq N : \frac{a_{n+1}}{a_n} \geq \beta \Rightarrow a_n \geq \beta^{n-N} a_N \Rightarrow \sqrt[n]{a_n} \geq \beta^{\frac{n-N}{n}} \sqrt[n]{a_N}$. Since N is fixed, $\frac{n-N}{n} \to 1$ and $\sqrt[n]{a_N} \to 1$. Altogether we conclude that $\underline{\lim} \sqrt[n]{a_n} \geq \beta$. Since β is arbitrary, this implies $\underline{\lim} \sqrt[n]{a_n} \geq \alpha$. ■

MATH140A **Winter 2021**

Lecture 16: Power Series, Summation by Parts, Rearrangement, Function Limits

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 16.1 (*Power Series*)

Let $\{a_n\}$ be a sequence of numbers. A ***power series*** *is a formed sum*

$$\sum_{n=0}^{\infty} a_n z^n$$

The series converge at $z = 0$.

Corollary 16.1

Let $\sum a_n z^n$ be a power series and let $\alpha = \overline{\lim} \sqrt[n]{|a_n|}$ and $R = \frac{1}{\alpha}$ [16.1]*. Then if $z \in \mathbb{C}$ and $|z| < R$, then the series $\sum_{n=0}^{\infty} a_n z^n$ absolutely converge; if $|z| > R$, then the series $\sum_{n=0}^{\infty} a_n z^n$ diverges.*

Proof: Apply the Root Test to the series $\sum_{n=0}^{\infty} a_n z^n$ for $|z| < R$ and $|z| > R$ respectively. ■

e.g.1. $\sum_{n=0}^{\infty} n^n z^n$ has radius of convergence 0. $\sum \frac{z^n}{n}$ has radius of convergence 1.

e.g.2. **(Summation by Parts)**: Let $\{a_n\}$ and $\{b_n\}$ be sequences of numbers. We want to discuss convergence of the series $\sum a_n b_n$. So we introduce theorem 16.3.

- $a_n = (-1)^n, b_n = \frac{1}{n}$. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
- $a_n = z^n$ for some $z \in \mathbb{C}$, $b_n = \frac{1}{n}$. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

Lemma 16.2 (*Summation by Parts*)

Let $\{a_n\}$ and $\{b_n\}$ be sequences of numbers and let $A_n = a_1 + \cdots + a_n$. Then [16.2]

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

Proof: Recall that $a_n = A_n - A_{n-1}$. Hence $\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q}(A_n - A_{n-1})b_n = \sum_{n=p}^{q}(A_n b_n - A_{n-1} b_n) = A_p b_p + A_{p+1} b_{p+1} + \cdots + A_q b_q - A_{p-1} b_p - A_p b_{p+1} - \cdots - A_{q-1} b_q = A_p(b_p + b_{p+1}) + \cdots + A_{q-1}(b_{q-1} - b_q) + A_q b_q - A_{p-1} b_p = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$. ■

[16.1] R is called the **radius of convergence**.

[16.2] Connections to Integration by parts: $\int_a^b f'g = \int_a^b fg' +$ boundary terms.

Theorem 16.3

Let $\{a_n\}$ and $\{b_n\}$ be sequences of numbers satisfying the following:

1. *The sequence of partial sums $A_n = a_1 + \cdots + a_n$ is bounded*
2. $b_1 \geq b_2 \geq \cdots \geq 0$
3. $\lim b_n = 0$

Then $\sum a_n b_n$ converges.

Proof: We want to show $\sum a_n b_n$ converges. We will use Cauchy's Criterion: let $\epsilon > 0$. We need to show that $\exists N, \forall m \geq n \geq N : |\sum_{i=n} m a_i b_i| < \epsilon$. Since $A_n = a_1 + \cdots + a_n$ is bounded, say $|A_n| \leq A$. Moreover, $b_n \to 0$ so $\exists N_1, \forall n \geq N_1 : |b_n| < \epsilon$. Now write $|\sum_{i=n}^{m} a_i b_i| = |\sum_{i=n}^{m-1} A_i(b_i - b_{i+1}) + A_m b_m - A_{n-1} b_n| \leq |\sum_{i=n}^{m-1} A_i(b_i - b_{i+1})| + |a_m b_m| + |A_{n-1} b_m| \leq |\sum_{i=n}^{m-1} A_i(b_i - b_{i+1})| + A\epsilon + A\epsilon$. We now control $|\sum_{i=n}^{m-1} A_i(b_i - b_{i+1})| \leq \sum_{i=n}^{m-1} |A_i(b_i - b_{i+1})| \leq A \sum_{i=n}^{m-1}(b_i - b_{i+1}) = A(b_n - b_m) \leq 2A\epsilon$. Hence if $m, n \geq N_1$, then we have $|\sum_{i=n}^{m} a_i b_i| < 4A\epsilon$. ■

Corollary 16.4

Let $\{b_n\}$ be a sequence of numbers such that $b_1 \geq b_2 \geq \ldots$ and $\lim b_n = 0$. Then $\sum_{n=1}^{\infty} b_n z_n$ converges $\forall z \in \mathbb{C}, |z| = 1, z \neq 1$. In particular, $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.

Proof: Note that if we let $z = -1$ then the second claim follows the first claim.

To see the first claim, let $a_n = z^n$. Then $|\sum_{n=1}^{N} a_n| = |z + z^2 + \cdots + z^N| = |\frac{1 - z^{N+1}}{1 - z} - 1| \leq |\frac{1 - z^{N+1}}{1 - z}| + 1 \leq \frac{2}{|1 - z|} + 1$. So this follows from the theorem 16.3. ■

Definition 16.2 (*Rearrangement*)

Suppose $r_n : \mathbb{N} \to \mathbb{N}$ is a bijection defines $a'_n = a_{r_n}$. Then $\sum a'_n$ is a ***rearrangement*** *of $\sum a_n$.*

e.g.3. $a_2, a_1, a_3, a_4, \ldots$ is a rearrangement for $\{a_1, a_2, a_3, a_4, \ldots\}$.

Theorem 16.5

Let $\sum a_n$ be absolutely convergent. Then every rearrangement $\sum a'_n$ also converges to the same limit as $\sum a_n$.

Proof: Recall that $\sum a_n = \lim A_n$. Let $A'_n = a'_1 + \cdots + a'_n$. We will show that $\forall \epsilon > 0 : |A_n - A'_n| < \epsilon$ for large n. To see this, note that since $\sum |a_n|$ converges, $\exists N, \forall k \geq \ell \geq N : \sum_{i=\ell}^{k} |a_i| < \epsilon$. Let M be large enough so that $\{1, 2, \ldots, N\} \subset \{r(1), r(2), \ldots, r(M)\}$. Now let $n \geq M$, we have $|A_n - A'_n| \leq \sum_{N=1}^{\infty} |a_i| < \epsilon$. Since $\lim A_n = A$, we get $\lim A'_n = A$. ■

Lemma 16.6

If $\sum a_n$ converge to A and $\sum b_n$ converge to B. Then

1. $\sum a_n + b_n = A + B$
2. $\forall \lambda \in \mathbb{R} : \sum \lambda a_n = \lambda A$

Proof: Convergence of series is that of the sequence of partial sums. For sequences, we have shown this.

■

e.g.4. For multiplication, we want to multiply to have $(\sum_{n=0}^{\infty} a_n z^n)(\sum_{\ell=0}^{\infty} b_\ell z^n) = (\sum_{m=0}^{\infty} c_m z^m)$. We get $c_m = \sum_{k=0}^{m} a_k b_{m-k}$. Now the question turns: suppose $\sum a_n$ and $\sum b_n$ converge and define c_n as previous, is it true that $\sum c_n$ converges? If so, dies it converge to $(\sum a_n)(\sum b_n)$? The answer is **NO**. Counterexample: $\forall n \geq 0 : a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$.

Theorem 16.7

If $\sum a_n$ converges absolutely and $\sum b_n$ converges, then $\sum c_n = \sum_{k=0}^{n} a_k b_{n-k}$ converges and $\sum c_n = (\sum a_n)(\sum b_n)$.

Proof: See the book 3.50. ■

Definition 16.3 (*Limit*)

Let (X, d_X) and (Y, d_Y) be metric spaces. Let $A \subset X$. Let $f : A \to Y$ and $p \in X$ be a limit point of A. We say $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q$$

if there is $q \in Y$ such that the following holds:

$$\forall \epsilon > 0, \exists \delta > 0 : x \in A, 0 < d_X(x,p) < \delta \implies d_Y(f(x), q) < \epsilon$$

e.g.5. Let $f : (0,1) \to \mathbb{R}, f(x) = x, g(x) = \frac{1}{x}$. Then 0 is a limit point of $(0,1)$. Indeed, $\lim_{x\to 0} f(x) = 0$. However, $\lim_{x \to 0} g(x)$ does not exist in $\mathbb{R}$.

e.g.6. Let $f(x) = \begin{cases} x & x \in \mathbb{Q} \cap (0,1) \\ 0 & x \notin \mathbb{Q} \cap (0,1) \end{cases}$. We will show later that $\lim_{x \to 0} f(x) = 0$.

Lemma 16.8

Let (X, d_X) and (Y, d_Y) be metric spaces, $A \subset X$. Let $f : A \to Y$. Then $\lim_{x \to p} f(x) = q$ if and only if $\forall \{a_n\} \in A, a_n \neq p, a_n \to p : f(a_n) \to q$.

Proof: $(\Rightarrow)$ Suppose $f(x) \to q$ as $x \to p$. Then $\forall \epsilon > 0, \exists \delta > 0 : 0 < d_X(x,p) < \delta \implies d_Y(f(x), q) < \epsilon$. Let $a_n \to p$ where $a_n \in A, a_n \neq p$. We want to show that $f(a_n) \to q$. So $\forall \epsilon > 0$, we need to find N such that $\forall n \geq N : d_Y(f(a_n), q) < \epsilon$. Apply previous definition with this ϵ, then $0 < d_X(x,p) < \delta \implies d_Y(f(x), q) < \epsilon$. Since $a_n \to p$, $\exists N, \forall n \geq N : d_X(a_n, p) < \delta$. Since $a_n \neq p$, we have $\forall n \geq N : 0 < d_X(a_n, p) < \delta$. So by the choice of δ, we have $\forall n \geq N : d_Y(f(a_n), q) < \epsilon$.

$(\Leftarrow)$ Toward a contradiction. Suppose $\forall \{a_n\} \in A, a_n \neq p, a_n \to p : f(a_n) \to q$ holds but $f(x) \not\to q$ as $x \to p$. So $\exists \epsilon > 0 : \forall \delta > 0, \exists x_\delta \in A, 0 < d_X(s_\delta, p) < \delta$ but $d_Y(f(x_\delta), q) \geq \epsilon$. Apply with $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$, we get $\{x_n\} \in A : 0 < d_X(x_n, p) < \frac{1}{n}$ and $d_Y(f(x_n), q) \geq \epsilon$. So $\{x_n\}$ violates that $x_n \to p, x_n \neq p, f(x_n) \not\to q$. ■

Corollary 16.9

If f has a limit at p, then the limit is unique.

Proof: Limit of sequences are unique. So the previous lemma implies this claim. ■

Corollary 16.10

Let $A \subset X$ and $f : A \to \mathbb{C}, g : A \to \mathbb{C}$. Suppose $\lim_{x\to p} f(x) = L$ and $\lim_{x\to p} g(x) = M$. Then

1. $\lim_{x\to p} f \pm g = L \pm M$
2. $\lim_{x\to p} fg = LM$
3. *If $M \neq 0$, then* $\lim_{x\to p} \frac{f}{g} = \frac{L}{M}$.

Proof: This follows from the lemma 16.8 and a similar statement for sequences. ■

e.g.7. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$. Then $\lim_{x\to a} f(x) = a^2$.

Proof: Left as exercise: Given $\epsilon > 0$, find δ s.t. $0 < |x-a| < \delta \Longrightarrow |f(x)-f(a)| < \epsilon$. ■

MATH140A **Winter 2021**

Lecture 17: Continuity, Homeomorphism

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Definition 17.1 (*Continuous*)

Let $A \subset X$ and $f : X \to Y$. Let $p \in A$. We say f is **continuous** *at p if*

$$\forall \epsilon > 0, \exists \delta > 0 : d(x,p) < \delta \implies d(f(x), f(p)) < \epsilon$$

If $f : A \to Y$ we say f is continuous on A if it is continuous at every point of A.

e.g. If p is an isolated point in A, then any function $f : A \to Y$ is continuous at p. Indeed, p is isolated if and only if $\exists \delta_0 > 0 : \{x \in A : d(x,p) < \delta_0\} = \{p\}$. $\forall \epsilon$, we can take $\delta = \delta_0$. Then $\forall x \in A, d(x,p) < \delta : d(f(x), f(p)) = 0 < \epsilon$

e.g.1. $A = [0,1] \cup \{2\}, f : A \to \mathbb{R}$. Let $f(x) = \begin{cases} x^3 + 1 & x \in [0,1] \\ 10^{10} & x = 2 \end{cases}$. Then f is continuous at $x = 2$.

e.g.2. $A = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}, f : A \to \mathbb{R}$. Let $f(x) = \begin{cases} n & x = \frac{1}{n} \\ 0 & x = 0 \end{cases}$. Then f is continuous at all points $\{\frac{1}{n}\}$ but not at 0.

Lemma 17.1

If $f : A \to Y$ and $p \in A$ is a limit point of A, then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.

Proof: This follows from the definition of limits and continuity. ■

Corollary 17.2

Let f and g be two functions from X to $\mathbb{C}$. If f and g are continuous at $x \in X$, then

1. *$f + g$ is continuous at x.*
2. *$f \cdot g$ is continuous at x.*
3. *$\frac{f}{g}$ is continuous at x, given $g(x) \neq 0$.*

Proof: We have this for limits of functions. ■

Corollary 17.3

Let X be a metric space. Let $f : X \to \mathbb{R}^n, f(x) = (f_1(x), f_2(x), \ldots, f_n(x))$ where $f_i : X \to \mathbb{R}$. Then f is continuous on X if and only if f_i is continuous for all i.

Proof: Let $p, q \in X$. Then $|f_i(p) - f_i(q)| \leq (\sum_{j=1}^n |f_j(p) - f_j(q)|^2)^{\frac{1}{2}} = \|f(p) - f(q)\|$. So if f is continuous, then f_i is continuous. Moreover, if f_i is continuous, then the last equality implies f is continuous. ■

e.g.3. Any polynomial $f(x) = \sum_{i=1}^n a_i x^i$ is continuous on $\mathbb{R}$.

Lemma 17.4

Let $A \subset X$ and $f : A \to Y$. Let $g : f(A) \to Z$. Define $h = g \circ f : A \to Z$. Let $p \in A$ and assume that f is continuous at p and g is continuous at $f(p)$. Then h is continuous at p.

Proof: We want to show that $\forall \epsilon > 0, \exists \delta > 0 : d(x,p) < \delta \implies d(h(x), h(p)) = d(g(f(x)), g(f(p))) < \epsilon$. Since g is continuous at $f(p)$, $\exists \beta > 0 : d(y, f(p)) < \beta \implies d(g(y), g(f(p))) < \epsilon$ for $y \in A$. Now since f is continuous at p, $\exists \delta > 0 : d(x,p) < \delta \implies d(f(x), f(p)) < \beta$. Let now $x \in A$, $d(x,p) < \delta$. Then $d(f(x), f(p)) < \beta$ indicates $d(g(f(x)), g(f(p))) < \epsilon$, which finishes the proof. ■

Proposition 17.5

Let $f : X \to Y$ be a function. Then f is continuous on X if and only if $f^{-1}(O)$ is open in X for all open subsets $O \subset Y$ [17.1].

Proof: $(\Rightarrow)$ Suppose f is continuous on X and let $O \subset Y$ be an open set. In order to show $f^{-1}(O)$ is open, we need to show that $\forall x \in f^{-1}(O), \exists r > 0 : N_r(x) \subset f^{-1}(O)$. Since $x \in f^{-1}(O)$, we have $f(x) \in O$. Since O is open, $\exists \epsilon > 0 : N_\epsilon(f(x)) \subset O$. Since f is continuous, $\exists \delta > 0 : d(x, x') < \delta \implies d(f(x), f(x')) < \epsilon$. That is to say: if $x' \in N_\delta(x)$, then $f(x') \in N_\delta(f(x))$. i.e. $f(N_\delta(x)) \subset N_\epsilon(f(x)) \subset O$. Therefore, $N_\delta(x) \subset f^{-1}(O)$ so $r = \delta$ satisfies $N_\epsilon(f(x)) \subset O$.

$(\Leftarrow)$ We know that $f^{-1}(O)$ is open for any open set $O \subset Y$. We want to show that f is continuous. Let $\epsilon > 0$, we need to find $\delta > 0$ such that $d(x, x') < \delta \implies d(f(x), f(x')) < \epsilon$. That is, we want to find $\delta > 0$ such that $f(N_\delta(x)) \subset N_\epsilon(f(x))$. Since $N_\epsilon(f(x))$ is open, we know that $f^{-1}(N_\epsilon(f(x)))$ is open and $x \in f^{-1}(N_\epsilon(f(x)))$. So $\exists \delta > 0 : N_\delta \subset f^{-1}(N_\epsilon(f(x)))$. Then $f(N_\delta(x)) \subset N_\epsilon(f(x))$. i.e. $d(x, x') < \epsilon \implies d(f(x), f(x')) < \epsilon$. ■

Corollary 17.6

A function $f : X \to Y$ is continuous if and only if $f^{-1}(F)$ is closed for all closed subsets $F \subset Y$.

Proof: Left as exercise. Hint: A set is open if and only if its complement is closed; $f^{-1}(A^C) = (f^{-1}(A))^C$. ■

e.g.4. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{1+x^2}$. Then f is continuous. So $f^{-1}(O)$ is open, where $O \subset \mathbb{R}$ is open. Note however that $f(\mathbb{R}) = (0, 1]$ is not an open set and $f((-1, 1)) = (\frac{1}{2}, 1]$ is not an open set.

Definition 17.2

Let $f : X \to Y$ be a continuous bijection and $f^{-1} : Y \to X$ be continuous. Then f is called a ***homeomorphism****.*

[17.1] Recall $f^{-1}(O) = \{x \in X : f(x) \in O\}$.

Lemma 17.7

Let $f : X \to Y$ be a continuous bijection. Then the followings are equivalent

1. *f is homeomorphism.*
2. *$f(O)$ is open for all open subsets $O \subset X$.*
3. *$f(F)$ is closed for all closed subsets $F \subset X$.*

Proof: We know f is continuous and bijection. So $f^{-1} : Y \to X$ is defined as a map between set. f is a homeomorphism $\Leftrightarrow$ f is continuous $\Leftrightarrow$ $(f^{-1})^{-1}(O)$ is open $\forall O \subset X$ open $\Leftrightarrow$ $(f^{-1})^{-1}(F)$ is closed $\forall F \subset X$ closed. ■

e.g.5. $f : [0, 1) \to \{z \in \mathbb{C} : |z| = 1\}$, $f(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$ is a continuous bijection. However, f^{-1} is not continuous at $z = 1$.

Proposition 17.8

Let $f : X \to Y$ be continuous. Let $K \subset X$ be compact. Then $f(K)$ is compact.

Proof: In order to show $f(K)$ is compact, we need to show that every open covering of $f(K)$ has a finite subcover. So let $\{O_\alpha : \alpha \in \Lambda\}$ be an open covering of $f(K)$. i.e. $f(K) \subset \bigcup_\alpha O_\alpha$. Taking f^{-1} we conclude that $f^{-1}(f(K)) \subset f^{-1}(\bigcup_\alpha O_\alpha) = \bigcup_\alpha f^{-1}(O_\alpha)$. Since $K \subset f^{-1}(f(K))$, we get $K \subset \bigcup_\alpha f^{-1}(O_\alpha)$. Since f is continuous and O_α is open, $f^{-1}(O_\alpha)$ is open. Now K is compact, so $\exists \alpha_1, \ldots, \alpha_n : K \subset f^{-1}(O_{\alpha_1}) \cup \cdots \cup f^{-1}(O_{\alpha_n})$. Hence, by applying f, we get $f(K) \subset f(f^{-1}(O_{\alpha_1})) \cup \cdots \cup f(f^{-1}(O_{\alpha_n})) \subset O_{\alpha_1} \cup \cdots \cup O_{\alpha_n}$. So $f(K)$ is compact. ■

e.g.6. $f : [0, 1) \to \{z \in \mathbb{C} : |z| = 1\}$, $f(x) = e^{2\pi i x} = \cos 2\pi x + i \sin 2\pi x$ is a continuous bijection. Then $\{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}$ is compact. But $f^{-1}(\{z \in \mathbb{C} : |z| = 1\}) = [0, 1)$ is not compact.

MATH140A **Winter 2021**

Lecture 18: Uniform Continuity, Infinite Limits

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Corollary 18.1

Let X be a compact metric space. Let $f : X \to \mathbb{R}^n$ be continuous. Then $f(X)$ is compact. Hence closed and bounded.

Proof: By the previous result, $f(X)$ is a compact subset of $\mathbb{R}^n$. By Heine-Borel Theorem, it is closed and bounded. ■

Corollary 18.2

Let X be a compact metric space. Let $f : X \to \mathbb{R}$ be continuous. Then f has maximum and minimum [18.1]. In particular, if $f : [a, b] \to \mathbb{R}$ is continuous, then f has maximum and minimum.

Proof: Note that by Heine-Borel Theorem, $[a, b]$ is compact so the second claim of a consequence of the first.

We now show the first claim. By corollary 18.1, $f(X) \subset \mathbb{R}$ is compact. So it is closed. Thus, $\sup f(X) \in f(X)$ and $\inf f(X) \in f(X)$. Then $\exists p \in X : f(p) = \sup f(X)$ and $\exists q \in X : f(q) = \inf f(X)$. ■

Proposition 18.3

Let X be a compact metric space. Let $f : X \to Y$ be a one-to-one continuous map. Then $f^{-1} : f(X) \to X$ is continuous.

Proof: It suffices to show that for all closed sets $F \subset X$, we have $(f^{-1})^{-1}(F) = f(F)$ is closed. Note however that X is compact and $F \subset X$ is closed. Then F is compact. So by a lemma $f(F)$ is compact. Then $f(F)$ is closed so f^{-1} is continuous. ■

Definition 18.1

Let $f : X \to Y$ be a map. We say f is ***uniformly continuous*** *if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall p, q \in X : d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$$

Continuity at a point is a local property. Uniform continuity is a global property of f.

e.g.1. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ be continuous. $\forall a \in \mathbb{R}, \forall \epsilon > 0, \exists \delta > 0 : |x - a| < \delta \implies |x^2 - a^2| < \epsilon$. Note however that if a is chosen large, then δ should be chosen small. More precisely, δ depends on a and tends 0 if $a \to \infty$. So $f(x) = x^x$ is not uniformly continuous on $\mathbb{R}$.

[18.1] i.e. $\exists p, q \in X : f(p) = \sup f(X), f(q) = \inf f(X)$.

e.g.2. $f : \mathbb{R} \to \mathbb{R}, f(x) = x$ is uniformly continuous.

e.g.3. Is $f : (1, \infty) \to \mathbb{R}, f(x) = \sqrt{x}$ uniformly continuous on $(1, \infty)$?

Theorem 18.4

Let X be a compact metric space. Let $f : X \to Y$ be a continuous function. Then f is uniformly continuous [18.2].

Proof: Let $\epsilon > 0$, we need to find $\delta > 0$ such that $\forall p, q \in X : d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$. Since f is continuous, $\forall p \in X, \exists \delta_p > 0 : d(p, q) < \delta_p \implies d(f(p), f(q)) < \epsilon$. Consider the covering $\{N_{\delta_{\frac{p}{2}}}(p) : p \in X\}$. Since X is compact, $\exists p_1, \ldots, p_n : X = \bigcup_{i=1}^{n} N_{\delta_{\frac{p_i}{2}}}(p_i)$. Let $\delta = \min\{\delta_{\frac{p_i}{2}} : i = 1, \ldots, n\} > 0$. Let $p, q \in X, d(p, q) < \delta$. Then $p \in X = \bigcup_{i=1}^{n} N_{\delta_{\frac{p_i}{2}}}(p_i) \Rightarrow \exists i_0 : p \in N_{\delta_{\frac{p_0}{2}}}(p_0)$ so we get $d(q, p_{i_0}) \leq d(p, q) + d(p, p_{i_0}) < \delta + \delta_{\frac{p_{i_0}}{2}} < \delta_{p_{i_0}}$. So by the choice of $\delta_{p_{i_0}}$, $d(f(p), f(p_{i_0})) < \frac{\epsilon}{2}$ and $d(f(q), f(p_{i_0})) < \frac{\epsilon}{2}$. Thus $d(f(p), f(q)) \leq d(f(p), f(p_{i_0})) + d(f(p_{i_0}), f(q)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $d(p, q) < \delta \implies d(f(p), f(q)) < \epsilon$. Then f is uniformly continuous. ■

Proposition 18.5

Let $A \subset \mathbb{R}$ be non-compact. Then

1. *There exists a continuous function on A that is unbounded.*
2. *There exists a continuous function on A that is bounded but has no maximum.*
3. *If A is bounded* [18.3], *then there exists a continuous function on A which is not uniformly continuous.*

Proof: First assume A is bounded but not compact. Since A is not compact, by Heine-Borel Theorem, A is not closed. So $\exists b \in \mathbb{R}, b \in A'$ but $b \notin A$. Let $f(x) = \frac{1}{x-b}$. Since $b \notin A$, f is defined on A and is continuous. This function is not bounded, and it is not uniformly continuous.

To see (2) for bounded non-compact set A. Let $g(x) = \frac{1}{1+(x-b)^2}$. Then $\forall x \in A : 1 + (x - b)^2 > 1$ and $\exists \{a_n\} \in A : a_n \to b$. So $g(x) < 1$ on A. However, g has no maximum. Indeed, $\sup\{g(x) : x \in A\} = 1$. If A is not bounded, then$h(x) = \frac{x^2}{1+x^2}$ is bounded on A but has no maximum on A. Indeed, $\sup\{h(x) : x \in A\} = 1$. But $\nexists x \in \mathbb{R} : h(x) = 1$. ■

Note that limit concerns neighborhoods of points: $0 < d(x, p) < \delta \implies d(f(x), \ell) < \epsilon$. i.e. $x \in N_\delta(p) \implies f(x) \in N_\epsilon(\ell)$. In $\mathbb{R}$, $x \in (p - \delta, p + \delta) \implies f(x) \in (\ell - \epsilon, \ell + \epsilon)$. In the extended real line $[-\infty, \infty]$, we can think of (M, ∞) as neighborhoods of ∞ and $(-\infty, -M)$ as neighborhoods of $-\infty$.

[18.2] *By the scribe*: this is referred to **Heine-Cantor Theorem**.

[18.3] Boundedness here is necessary. Indeed, if $A = \mathbb{N}$, then any function on A is uniformly continuous. Let $\delta = \frac{1}{2}$. Then $\forall n \in \mathbb{N}, (n - \delta, n + \delta) \cap \mathbb{N} = \{n\}$. So $\forall x, y \in \mathbb{N} : d(x, y) < \delta \Rightarrow x = y \Rightarrow d(f(x), f(y)) = 0 < \epsilon$.

Definition 18.2 (*Limit at Infinity, Infinite Limits*)

For $f : \mathbb{R} \to \mathbb{R}$, *we say*

$$\lim_{x\to\infty} f(x) = \ell$$

for $\ell \in \mathbb{R}$ *if* $\forall \epsilon > 0, \exists M \in \mathbb{R} : x > M \implies |f(x) - \ell| < \epsilon$.

$$\lim_{x\to p} f(x) = \infty$$

for $p \in \mathbb{R}$ *if* $\forall M \in \mathbb{R}, \exists \delta > 0 : 0 < |x - p| < \delta \implies f(x) > M$.

$$\lim_{x\to\infty} f(x) = \infty$$

if $\forall M \in \mathbb{R}, \exists N \in \mathbb{R}, \forall x > N : f(x) > M$. *Similarly we define the notion of limits at* $-\infty$.

e.g.4. $f(x) = \frac{1}{x}$. Then $\lim_{x\to\infty} \frac{1}{x} = 0$. $\forall \epsilon > 0$, we want to find M such that $\forall x > M : |\frac{1}{x}| < \epsilon$. Since we can take x to be positive, we need to have $\frac{1}{x} < \epsilon \Rightarrow x > \frac{1}{\epsilon}$. So suffice to let $M = \frac{1}{\epsilon}$.

Lemma 18.6

Let f *and* g *be functions on* $\mathbb{R}$. *Suppose that* $f(x) \to \ell_1, g(x) \to \ell_2$ *as* $x \to p$ *where* $\ell_1, \ell_2, p \in [-\infty, \infty]$. *Then (provided* $\ell_1 + \ell_2, \ell_1\ell_2, \frac{\ell_1}{\ell_2}$ *are defined in* $[-\infty, \infty]$*)*

1. *Limit is unique*
2. $\lim_{x\to p}(f \pm g) = \ell_1 \pm \ell_2$
3. $\lim_{x\to p}(f \cdot g) = \ell_1\ell_2$
4. $\lim_{x\to p} \frac{f}{g} = \frac{\ell_1}{\ell_2}$

MATH140A	**Winter 2021**

Lecture 19: Intermediate Value Theorem, Discontinuities

Lecturer: Amir Mohammadi *Scribes: Rabbittac*

Proposition 19.1

Let $f : X \to Y$ be continuous. Let $C \subset X$ be connected. Then $f(C)$ is connected.

Proof: Toward a contradiction, suppose $f(C)$ is not connected, and write $f(C) = A \cup B$ where $A \neq \emptyset, B \neq \emptyset, A \cap \overline{B} = \overline{A} \cap B = \emptyset$. We apply f^{-1} (set map) to $f(C) = A \cup B$ and conclude that $f^{-1}(f(C)) = f^{-1}(A \cup B)$. Note that $C \subset f^{-1}(f(C))$. Let $D = C \cap f^{-1}(A), E = C \cap f^{-1}(B)$. Since A and B are nonempty and $f(C) = A \cup B$, we have $D \neq \emptyset$ and $E \neq \emptyset$. Moreover, $C = D \cup E$. We want to show D and E are separated: $D \cap \overline{E} = \overline{D} \cap E = \emptyset$. We consider $D \cap \overline{E}$, the other case is similar. $D = C \cap f^{-1}(A) \subset f^{-1}(A), E = C \cap f^{-1}(B) \subset f^{-1}(B)$ implies $\overline{E} \subset \overline{f^{-1}(B)}$. Moreover, $f^{-1}(B) \subset f^{-1}(\overline{B})$. Since f is continuous, $f^{-1}(\overline{B})$ is closed. Then $\overline{E} \subset \overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$. Since $A \cap \overline{B} = \emptyset$, we have $f^{-1}(A) \cap f^{-1}(\overline{B}) = \emptyset$. Then $D \cap \overline{E} = \emptyset$. $\overline{D} \cap E = \emptyset$ can be shown similarly. Hence we get a contradiction to the fact that C is connected. ■

Corollary 19.2 (*Intermediate Value Theorem*)

Let f be a continuous function on $[a, b]$. Suppose $f(a) < f(b)$ and let $f(a) < r < f(b)$, then $\exists c \in [a, b] : f(c) = r$. Similarly, if $f(a) > f(b)$ and $f(a) > r > f(b)$, then $\exists c \in [a, b] : f(c) = r$.

Proof: f is continuous. $[a, b]$ is connected. So $f([a, b])$ is a connected subset of $\mathbb{R}$. Therefore $\forall x, y \in f([a, b]) : x < y$ and $x < r < y$. Then $r \in f([a, b])$. Since $x = f(a), y = f(b)$ belong to $f([a, b])$ and $f(a) < r < f(b)$, $r \in f([a, b])$. i.e. $\exists c \in [a, b] : f(c) = r$. The case $f(a) > f(b)$ can be shown similarly. ■

e.g.1. $f(x) = x^3 + 2x^2 + x + 10^6$ has a real root.

Proof: We claim $\exists M > 0 : f(M) > 0$ and $f(-M) < 0$. Assume this claim follows from the previous corollary. Indeed, $f(-M) < 0 < f(M) \Rightarrow \exists c \in [-M, M] : f(c) = 0$. We show that $\lim_{x \to -\infty} f(x) = -\infty$. Recall $f(x) = x^3 + 2x^2 + x + 10^6$ so $f(x) = x^3(1 + \frac{2}{x} + \frac{1}{x^2} + \frac{10^6}{x^3})$. Now as $x \to -\infty$, we have $\frac{2}{x}, \frac{1}{x^2}, \frac{10^6}{x^3}$ approach 0. So $\exists N, \forall x < -N :$ $|\frac{2}{x}| < 0.1, |\frac{1}{x^2}| < 0.1, |\frac{10^6}{x^3}| < 0.1$. Hence $\forall x < -N : |1 + \frac{2}{x} + \frac{1}{x^2} + \frac{10^6}{x^3}| \geq 1 - 0.3 = 0.7$ by triangle inequality. Hence $\forall x < -N : f(x) = 1 + \frac{2}{x} + \frac{1}{x^2} + \frac{10^6}{x^3} \leq -0.7N^3$. Hence given M we can choose $N' > 0$ such that $x < -N' \implies f(x) < -M$. Then $\lim_{x \to -\infty} f(x) = -\infty$ ■

Question: Is there any homeomorphism between $\mathbb{R}$ and $\mathbb{R}^2$? [19.1]

Theorem 19.3

There is no one-to-one continuous map from $[0, 1] \times [0, 1]$ to $[0, 1]$.

[19.1] **Space Filling Curves** is an onto and continuous map from $[0, 1]$ to $[0, 1] \times [0, 1]$.

Proof: Suppose $f : [0,1] \times [0,1] \to [0,1]$ is continuous. Let $I = f([0,1] \times [0,1])$. Then I is connected. Moreover, $[0,1] \times [0,1]$ is compact so I is compact. Then $I = [a,b]$ where $a \neq b$. Let $c \in (a,b)$. Since f is one-to-one, $\exists p \in [0,1] \times [0,1] : f(p) = c$. Let $E = [0,1] \times [0,1] \setminus \{p\}$ be a connected subset and $F = [a,b] \setminus \{c\} = [a,c) \cup (c,b]$ be a disconnected subset. However, $f : E \to F$ is continuous and one-to-one and onto. So $f(E) = F$ must be connected. Contradiction. ■

Definition 19.1 (*Right Limit, Left Limit*)

*Let f be a real-valued function on (a,b). Let $p \in (a,b)$. Then we say **limit from right** at p equals ℓ and write*

$$\lim_{x \to p^+} f(x) = \ell = f(p^+)$$

if $\forall \epsilon > 0, \exists \delta, \forall x \in (a,b) : 0 < x - p < \delta \implies |f(x) - \ell| < \epsilon$. Or equivalently, we say $f(p^+) = \ell$ if for every sequence $x_n \in (p,b)$ with $x_n \to p$, we have $f(x_n) \to \ell$. Similarly, we say

$$\lim_{x \to p^-} f(x) = q = f(p^-)$$

if $\forall \epsilon > 0, \exists \delta, \forall x \in (a,b) : 0 < q - x < \delta \implies |f(x) - q| < \epsilon$. One can write this using sequence as we did above.

f has limit at p if and only if $f(p^+) = f(p^-)$. In particular, both must exist.
f is continuous at p if and only if $f(p^+) = f(p^-) = f(p)$.

Definition 19.2 (*Discontinuity*)

*Let f be defined on (a,b) and discontinuous at $p \in (a,b)$. We say f has **first kind discontinuity** if $f(p^+)$ and $f(p^-)$ both exist. Otherwise, we say f has **second kind discontinuity**.*

e.g.2. $f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$. Then $f(0^+) = f(0^-) =\neq f(0)$. Note if we defined $g(x) = \begin{cases} x & x \neq 0 \\ \lim_{x \to 0} f(x) & x = 0 \end{cases} = x$, then g is continuous.

e.g.3. $f(x) = \begin{cases} 0 & x > 0 \\ 1 & x \leq 0 \end{cases}$. Then $f(0^+) = 1$ and $f(0^-) = 0$.

e.g.4. $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. This function has second kind discontinuity at every point. Indeed, $x \in \mathbb{Q}$, then $\exists p_n \to x, p_n > x, p_n \in \mathbb{Q}$. Recall every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$ so $f(p_n) = 1$. Similarly, $\exists q_n \to x, q_n > x, q_n \notin \mathbb{Q}$. Since $\mathbb{R}$ is uncountable, we can show that every point in $\mathbb{R}$ is a limit point of irrational points both from above and below. Then $f(q_n) = 0$. Thus $f(x^+)$ and $f(x^-)$ do not exists.

e.g.5. Define $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \cap [0,1] \\ \frac{1}{q} & x = \frac{p}{q} \in [0,1], \gcd(p,q) = 1 \end{cases}$. Then f is continuous at all irrational points and has first kind discontinuity at rational points.

Proof: Let $x \in [0,1]$ be irrational. Then $f(x) = 0$. We want to show $\lim_{t\to x} f(t) = 0$. Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|t - x| < \delta \implies |f(t)| < \epsilon$. Note that $\forall t \notin [0,1] \cap \mathbb{Q} : f(t) = 0$. So we only need to consider rational points. Let N be such that $\frac{1}{N} < \epsilon$. Now the set $\{x = \frac{p}{q} : \frac{p}{q} \in [0,1], \gcd(p,q) = 1, 0 < q < N\}$ is a finite set. Indeed, there are $N - 1$ choices for denominator and for every q we have $\frac{0}{q}, \frac{1}{q}, \ldots, \frac{q}{q}$ are the only possibilities of $\frac{p}{q} \in [0,1]$. Let $\delta = \min\{|x - t| : t \in A\}$. Then A is finite, $x \notin A$ so $\delta > 0$. Moreover, $(x - \delta, x + \delta) \subset [0,1]$. Let now $t \in (x - \delta, x + \delta)$ then either $t \notin \mathbb{Q}, f(t) = 0$ so $|f(t)| < \epsilon$ or $t \in \mathbb{Q}, t = \frac{p}{q}, \gcd(o,q) = 1, q \geq N$. Then $0 < f(t) = \frac{1}{q} \leq \frac{1}{N} \leq \epsilon$ so $|x - t| < \delta$ implies $|f(t)| < \epsilon$ as we wanted to show. ■

MATH140A **Winter 2021**

Lecture 20: Monotonic Functions

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Definition 20.1

*Let f be a real-valued function on (a, b). We say f is **monotonically increasing** if $\forall a < x < y < b : f(x) \leq f(y)$. Similarly, we say f is monotonically decreasing if $\forall a < x < y < b : f(x) \geq f(y)$.*

e.g.1. $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$ and $f(x) = \begin{cases} 0 & x > 0 \\ \frac{1}{2} & x = 0 \\ 0 & x < 0 \end{cases}$ are increasing and first-kind discontinuous at 0

Theorem 20.1

Let f be a monotonically increasing function on (a, b). Then $\forall x \in (a, b) : f(x^-)$ and fx^+ exist. Moreover, $f(x^-) \leq f(x) \leq f(x^+)$ and if $a < x < y < b$, then $f(x^+) \leq f(y^-)$.
Similar statement holds for monotonically decreasing function with reversed inequalities.

Proof: We will show that $f(x^+)$ exists and satisfies $f(x^+) \geq f(x)$. The proof for $f(x^-)$ is similar. Let $A = \{f(t) : x < t < b\}$. Then $A \neq \emptyset$ and A is bounded below. Since f is monotonically increasing, $a < x < t < b$ implies $f(x) \leq f(t)$. So $f(x)$ is a lower bound for A. Let $\ell = \inf A$. We claim that $\ell = f(x^+)$. To see this, let $\epsilon > 0$, we want to find $\delta > 0$ such that $0 < t - x < \delta \implies |f(t) - \ell| < \epsilon$. Since $\ell = \inf A$ and $\epsilon > 0$, we have $\ell + \epsilon$ is not a lower bound for A. i.e. $\exists x < t_0 < b : \ell \leq f(t_0) < \ell + \epsilon$. Let $\delta = t_0 - x$ and $x < t < x + \delta = t_0$ then $\ell \leq f(t) \leq f(t_0) < \ell + \epsilon$. So $\forall 0 < t - x < \delta$, we have $0 \leq f(t) - \ell < \epsilon$ then $f(x^+) = \ell$. Since $f(x)$ is a lower bound for A and $\ell = \inf A$, we get $f(x) \leq f(x^+) = \ell$. Let now $a < x < y < b$, then $f(x^+) = \inf\{f(t) : x < t < b\} = \inf\{f(t) : x < t < y\}$ and $f(y^-) = \sup\{f(s) : a < s < y\} = \sup\{f(s) : x < s < y\}$. Then $f(x^+) \leq f(y^-)$. ■

Corollary 20.2

If f is monotonic, then all its discontinuities are first-kind.

Proof: We showed $f(x^+)$ and $f(x^-)$ exist. ■

Corollary 20.3

If f is monotonic, then the set of discontinuities of f is at most countable.

Proof: Let us assume f is increasing. By the previous theorem, we have $\forall x \in (a, b) : f(x^-) \leq f(x) \leq f(x^+)$. So if f is discontinuous at x, then at least one inequality in $f(x^-) \leq f(x) \leq f(x^+)$ is strict. So $(f(x^-), f(x^+)) \neq \emptyset$. Now let $a < x < y < b$

and assume both x and y are points of discontinuities. Then $(f(x^-), f(x^+)) \neq \emptyset$ and $(f(y^-), f(y^+)) \neq \emptyset$. We also have $(f(x^0), f(x^+)) \cap ((f(y^-), f(y^+)) = \emptyset$. Let $A = \{x \in (a,b) : f \text{ is discontinuous at } x\}$. For every $x \in A$, let $r(x) \in (f(x^-), f(x^+)) \cap \mathbb{Q}$. Then the map is $g : A \to \mathbb{Q}$ is well-defined by $(f(x^-), f(x^+)) \neq \emptyset$ and one-to-one by $(f(x^0), f(x^+)) \cap ((f(y^-), f(y^+)) = \emptyset$. Since $\mathbb{Q}$ is countable, we get that A is countable. ■

Proposition 20.4

Given any countable subset $A = \{a_n\} \subset (a,b)$. There exists a monotonic function which is discontinuous exactly on A.

Proof: Fix a converging series of positive numbers. e.g. we let $b_n = \frac{1}{2^n}$. Then $b_n > 0, \sum b_n < \infty$. Define $f(x) = \sum_{a_n < x} \frac{1}{2^n}$. Note since $\frac{1}{2^n} > 0$ and $\sum \frac{1}{2^n}$ converges, rearrangements also converge. So $f(x)$ is well-defined and increasing. Note that $f(a_{n_0}) = \sum_{a_n < a_{n_0}} \frac{1}{2^n}$. However, for all $x > a_{n_0}$ we have $f(x) = \sum_{a_n < x} \frac{1}{2^n} = \sum_{a_n < a_{n_0}} \frac{1}{2^n} + \sum_{x > a_n \geq a_{n_0}} \frac{1}{2^n} = f(a_{n_0}) + \sum_{a_{n_0} \leq a_n < x} \frac{1}{2^n}$. So $f(x) - f(a_{n_0}) \geq \frac{1}{2^{n_0}}$ implies $f(a_{n_0}^+) > f(a_{n_0})$. So $\forall a_{n_0} \in A : f$ is discontinuous at a_{n_0}. We now show that f is continuous for all $x \in (a,b) \setminus A$. Indeed, given $\epsilon > 0$, we need to find $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Since $\sum \frac{1}{2^n}$ converges, $\exists N, \forall m \geq k \geq N : \sum_{n=k}^{m} \frac{1}{2^k} < \frac{\epsilon}{2}$ by Cauchy Criterion. Let δ be small that $a_1, \ldots, a_N \notin (x - \delta, x + \delta)$. Let now $y \in (x - \delta, x + \delta)$. Then $f(x) = \sum_{a_n < x} \frac{1}{2^n}$ and $f(y) = \sum_{a_n < y} \frac{1}{2^n}$. By the choice of δ, $a_1, \ldots, a_N$ either appear for both x and y or for neither. Indeed, $a_1, \ldots, a_N \notin (x - \delta, x + \delta)$. Thus $|f(x) - f(y)| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^n} \leq \frac{\epsilon}{2} < \epsilon$. ■