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POINCARÉ SERIES AND THE DIVISORS OF MODULAR FORMS

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ABSTRACT. Recently, Bruinier, Kohnen and Ono obtained an explicit description of the action of the theta-operator on meromorphic modular forms f on $SL_2(\mathbb{Z})$ in terms of the values of modular functions at points in the divisor of f. Using this result, they studied the exponents in the infinite product expansion of a modular form and recurrence relations for Fourier coefficients of a modular form. In this paper, we extend these results to meromorphic modular forms on $\Gamma_0(N)$ for an arbitrary positive integer $N > 1$.

1. INTRODUCTION

Let N be a positive integer and $q := e^{2\pi i z}$, $z = x + iy$. If $f(z) = \sum_{n=h}^{\infty} a(n)q^n$ is a meromorphic modular form on $\Gamma_0(N)$, then we define the theta-operator by

$$\theta f(z) := \frac{1}{2\pi i}\frac{d}{dz}f(z) = \sum_{n=h}^{\infty} na(n)q^n.$$

Recently, Bruinier, Kohnen and Ono obtained in [6] an explicit description of the action of the theta-operator on meromorphic modular forms on $SL_2(\mathbb{Z})$ in terms of the values of modular functions at points in the divisor of f. Using this result, they also studied the exponents in the infinite product expansion of a modular form and recurrence relations for the Fourier coefficients of modular forms (see Theorems 3 and 5 in [6]). Ahlgren gave analogues of these results for meromorphic modular forms on $\Gamma_0(p)$ for $p \in \{2, 3, 5, 7, 13\}$ (see [2]). For general primes p, using eta-quotients, the author studied in [8] the action of the theta-operator on meromorphic modular forms on $\Gamma_0(p)$ and the exponents in their infinite product expansion. But, recurrence relations for the Fourier coefficients of the modular forms in [6] and [2] were not generalized in [8]. In this paper, we extend the results [2] to meromorphic modular forms on $\Gamma_0(N)$ for an arbitrary positive integer $N > 1$ and obtain recurrence relations for their Fourier coefficients.

To obtain our main theorems, we consider Poincaré series of weight 0 instead of eta-quotients. Since in general these Poincaré series are not meromorphic functions on the complex upper half plane, we cannot use the valence formula or the residue theorem as in [6], [2] and [8]. Thus, following the argument of [5], we use the regularized integral and Stokes' Theorem.

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To state our results, we introduce some notation. The group $\Gamma_0(N)$ is the congruence subgroup of $SL_2(\mathbb{Z})$ defined as

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}.$$

Let $\mathcal{C}_N$ be the set of inequivalent cusps of $\Gamma_0(N)$ and $\mathcal{C}_N^* = \mathcal{C}_N \setminus \{\infty\}$. For a cusp u of $\Gamma_0(N)$ let

$$\Gamma_0(N)_u := \{\sigma \in \Gamma_0(N) \mid \sigma u = u\}.$$

Here, $\sigma u := \frac{au+b}{cu+d}$ for $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. We denote by $\mathcal{F}_N$ a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$. The modular curve $X_0(N)$ is defined as the quotient space of orbits under $\Gamma_0(N)$, $X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. Suppose that $f(z)$ is a meromorphic modular form of weight k on $\Gamma_0(N)$. We consider the following functions: a meromorphic modular form $f_\theta(z)$ and a Poincaré series $j_{N,m}(z)$ of weight 0 and index m. For $N > 1$ let

$$f_\theta(z) := \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N-1} \cdot NE_2(Nz) + \frac{h - Nk/12}{N-1} \cdot E_2(z).$$

Here, $E_2(z)$ is the usual normalized Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n,$$

where $\sigma_k(n) := \sum_{d|n} d^k$ if $n \in \mathbb{N}$ and $\sigma_k(n) = 0$ if $n \notin \mathbb{N}$.

Let $I_v(z)$ be the usual modified Bessel functions as in [1] and $e(x) := e^{2\pi i x}$. For a positive integer m we define the Poincaré series of weight 0 and index m by

(1.1)
$$F_{N,m}(z,s) := \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \pi |m \operatorname{Im}(\gamma z)|^{1/2} I_{s-\frac{1}{2}}(|2\pi m \operatorname{Im}(\gamma z)|) e(-m \operatorname{Re}(\gamma z)),$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$. Let $j_{N,m}(z)$ be the continuation of $F_{N,m}(z,s)$ as $s \to 1$ from the right. The function $j_{N,m}(z)$ is a weak Maass form on $\Gamma_0(N)$ (see Section 2 for details). For convenience, if t is a cusp of $\Gamma_0(N)$, then $j_{N,m}(t)$ denotes the constant term of the Fourier expansion of $F_{N,m}(z,1)$ at t. We define a differential operator ξ_0 by

$$\xi_0(j_{N,m})(z) := 2i \overline{\frac{\partial}{\partial \bar{z}} j_{N,m}(z)} \tag{1.2}$$

and consider the integral

$$\int_{\mathcal{F}_N} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy. \tag{1.3}$$

Note that in general $f_\theta(z)$ is not holomorphic on $\mathbb{H}$. Thus, we have to regularize the integral (1.3) and denote it by

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy$$

(see (3.1) in Section 3 for the exact definition). With this notation, we state our main theorem.

Theorem 1.1. *Suppose that $f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ with a positive integer N, $N > 1$. Let $\{c(n)\}_{n=1}^{\infty}$ be the complex numbers for which*

$$f(z) = q^h \prod_{n=1}^{\infty} (1-q^n)^{c(n)}. \tag{1.4}$$

Then we have

$$\sum_{d|m} d \cdot c(d) = \sum_{\tau \in \mathcal{F}_N \cup \mathcal{C}_N^*} \nu_\tau^{(N)}(f(z)) j_{N,m}(\tau) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy$$
$$+ \frac{2Nk - 24h}{N-1}\sigma_1(m) + \frac{24h-2k}{N-1} N\sigma_1(m/N).$$

Here, $\nu_\tau^{(N)}(f(z))$ denotes the order of zero of $f(z)$ at τ on $X_0(N)$.

Remark 1.2. Suppose that $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$. From the definition of the differential operator (1.2) we have

$$\xi_0(j_{N,m}(z)) = 0.$$

This implies that in Theorem 1.1,

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy = 0.$$

Remark 1.3. From Remark 1.2 it may be an interesting question when $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$. Since $\xi_0(j_{N,m}(z))$ is a cusp form, the results of [5] imply that $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$ if and only if for every cusp form $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ of weight 2 on $\Gamma_0(N)$ we have

$$b(m) = 0.$$

(1) Suppose that the genus of $\Gamma_0(N)$ is zero. Then $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$. Thus, when $N \in \{2, 3, 5, 7, 13\}$, Theorem 1.1 recovers the result of Ahlgren [2].
(2) Suppose that the genus of $\Gamma_0(N)$ is one and that $g(z) = \sum_{n=1}^{\infty} b(n)q^n$ is the unique normalized cusp form of weight 2 on $\Gamma_0(N)$. For a prime p let $\mathbf{F}_p$ denote a finite field $\mathbb{Z}/p\mathbb{Z}$. There exists an elliptic curve E_g of conductor N', $N'|N$, defined over $\mathbb{Q}$ such that for all $p \nmid N$,

$$1 - a(p) + p = \sharp E_g(\mathbf{F}_p).$$

Note that $|a(p)| \leq 2\sqrt{p}$ and $g(z)$ is a Hecke eigenform. Thus, for an odd integer m, $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$ if and only if E_g is supersingular at p for some prime p, $p|m$. In [10] Elkies proved the existence of infinitely many supersingular primes for every elliptic curve defined over $\mathbb{Q}$. This implies that there are infinitely many primes p such that $j_{N,m}(z)$ is holomorphic on $\mathbb{H}$ for every positive integer m, $p|m$.

Using Theorem 1.1, we obtain a description of the action of the theta-operator on meromorphic modular forms on $\Gamma_0(N)$.

Theorem 1.4. *Suppose that $f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ with a positive integer N, $N > 1$. Then*

$$f_\theta(z) + \sum_{m=1}^{\infty} \left(\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy \right) q^m = \sum_{m=1}^{\infty} \left(\sum_{\tau \in \mathcal{F}_N \cup\, \mathcal{C}_N^*} \nu_\tau^{(N)}(f(z)) j_{N,m}(\tau) \right) q^m.$$

From Remark 1.3(2) and Theorem 1.4 we have immediately the following corollary.

Corollary 1.5. *Suppose that the genus of $\Gamma_0(N)$ is one and that $f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ with a positive integer N, $N > 1$. Then there are infinitely many primes p such that*

$$f_\theta(z)|U_p = \sum_{m=1}^{\infty} \left(\sum_{\tau \in \mathcal{F}_N \cup\, \mathcal{C}_N^*} \nu_\tau^{(N)}(f(z)) j_{N,pm}(\tau) \right) q^m.$$

Remark 1.6. The formula in Corollary 1.5 does not contain the regularized integral. Thus, if $f_\theta(z)|U_p \cdot E_{\ell-1}$ is a modular form for some prime ℓ, then we can study congruence for the values of $j_{N,pm}$ by using the argument of [6]. But, it seems difficult to check whether $f_\theta(z)|U_p \cdot E_{\ell-1}$ is a modular form.

As another application of Theorem 1.1, we obtain universal recursion formulas for coefficients of meromorphic modular forms on $\Gamma_0(N)$. For each $n \geq 1$, we define the polynomial

(1.5)
$$F_n^{(N)}(K, H, x_1, \cdots, x_n) := \sum_{\substack{m_1+2m_2+\cdots+(n-1)m_{n-1}=n \\ m_1,\cdots,m_{n-1}\geq 0}} (-1)^{m_1+\cdots+m_{n-1}} \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!m_2!\cdots m_{n-1}!} x_1^{m_1}\cdots x_{n-1}^{m_{n-1}} - \frac{1}{n}\left(\frac{2NK-24H}{N-1}\right)\sigma_1(m) - \frac{1}{n}\left(\frac{24H-2K}{N-1}\right) N\sigma_1(m/N).$$

Theorem 1.7. *Suppose that*

$$f(z) := q^h + \sum_{n=h+1}^{\infty} a(n)q^n$$

is a meromorphic modular form of weight k on $\Gamma_0(N)$ with a positive integer N, $N > 1$. For each $n \geq 1$, the polynomial $F_n^{(N)}(K, H, x_1, \cdots, x_n)$ is defined as in (1.5). Then we have

$$a(h+n) = F_n^{(N)}(k, h, a(h+1), \cdots, a(h+n-1)) - \frac{1}{n} \sum_{\tau \in \mathcal{F}_N \cup\, \mathcal{C}_N^*} \nu_\tau^{(N)}(f(z)) j_{N,m}(\tau) + \frac{1}{n}\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy.$$

Example 1.8. Let E be an elliptic curve of conductor N defined over $\mathbb{Q}$ and let $\mathbf{F}_p$ denote a finite field $\mathbb{Z}/p\mathbb{Z}$ for a prime p. It is known that there is a normalized

Hecke eigenform $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ of weight 2 on $\Gamma_0(N)$ such that for all $p \nmid N$,

$$1 - a(p) + p = \sharp E(\mathbf{F}_p).$$

From Theorem 1.7 we have

(1.6)

$$\begin{aligned} a(1+n) &= F_n^{(N)}(2, 1, a(2), \cdots, a(n)) \\ &\quad - \frac{1}{n} \sum_{\tau \in \mathcal{F}_N \cup\, \mathcal{C}_N^*} \nu_\tau^{(N)}(f(z)) j_{N,m}(\tau) + \frac{1}{n} \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy. \end{aligned}$$

By the definition of the Hecke operator, we obtain from (1.6) recursive relations for $\sharp E(\mathbf{F}_p)$. For example, if $N = 11$ and $m = 1$, then

$$\int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy = 0.$$

Remark 1.9. Our results (Theorems 1.1, 1.4 and 1.7) can be immediately extended to congruence subgroups $\Gamma \subset \Gamma_0(N)$ for $N > 1$.

This paper is organized as follows. In section 2 we recall the definition of Poincaré series of weight zero on $\Gamma_0(N)$. In section 3 we define the regularized integral of a meromorphic modular form of weight 2. In sections 4 and 5 we give the proofs of the main theorems.

2. Poincaré series of weight 0 on $\Gamma_0(N)$

In this section we consider non-holomorphic Poincaré series of weight 0. For details we refer to [11], [13], [4], [7].

Let $I_v(z)$ and $K_v(z)$ be the usual modified Bessel functions as in [1]. We define for $s \in \mathbb{C}$ and $y \in \mathbb{R} \setminus \{0\}$:

$$\mathcal{I}_s(y) := \sqrt{\frac{\pi|y|}{2}} I_{s-\frac{1}{2}}(|y|),$$
$$\mathcal{K}_s(y) := \sqrt{\frac{\pi|y|}{2}} K_{s-\frac{1}{2}}(|y|).$$

Note that $\mathcal{I}_s(y)$ and $\mathcal{K}_s(y)$ are holomorphic in s. If $s = 1$, then we have

$$\mathcal{I}_1(y) = \sinh(|y|),$$
$$\mathcal{K}_1(y) = e^{-|y|},$$
$$2\mathcal{I}_1(y) + \mathcal{K}_1(y) = e^{|y|}.$$

For a cusp u let σ_u denote a matrix in $SL_2(\mathbb{R})$ such that

$$\sigma_u^{-1}\infty = u \;\text{ and }\; \sigma_u \Gamma_0(N)_u \sigma_u^{-1} = \left\{ \pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix} \right) \mid n \in \mathbb{Z} \right\}.$$

If t_1 and t_2 are equivalent cusps of $\Gamma_0(N)$, i.e., $\gamma t_1 = t_2$ for some $\gamma \in \Gamma_0(N)$, then we write $t_1 \sim t_2$. For a positive integer m we define the Poincaré series of weight 0 and index m by

$$F_{N,m}(z,s) := \sum_{\gamma \in \Gamma_0(N)_\infty \backslash \Gamma_0(N)} \mathcal{I}_s(2\pi m \operatorname{Im}(\gamma z)) e(-m \operatorname{Re}(\gamma z)), \tag{2.1}$$

where $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$. The special value $F_{N,m}(z,1)$ satisfies the following properties.

Theorem 2.1. *The function $F_{N,m}(z,1)$ is a harmonic weak Maass form of weight 0 on $\Gamma_0(N)$. Moreover, $F_{N,m}(z,1)$ has the following properties at each cusp t:*

$$F_{N,m}(z,1) = q^{-m} + \sum_{n\geq 0} b_m(n,1)q^n + \sum_{n>0} b_m(-n,1)e(n\bar{z}) \text{ if } t \sim \infty, \tag{2.2}$$

$$\lim_{\operatorname{Im} z\to\infty} F_{N,m}(\sigma_t z,1) = j_{N,m}(t) \qquad \text{if} \quad t \nsim \infty. \tag{2.3}$$

Proof. Let

$$I_\gamma(z,s) := \sum_{\beta\in\Gamma_0(N)_\infty} \mathcal{I}_s(2\pi m \operatorname{Im}(\gamma\beta z))e(-m\operatorname{Re}(\gamma\beta z)). \tag{2.4}$$

If we define

$$\delta_{t\infty} := 1 \text{ if } t\sim\infty,$$
$$\delta_{t\infty} := 0 \text{ if } t\nsim\infty,$$

then

$$F_{N,m}(\sigma_t z,s) = \delta_{t\infty}\cdot\mathcal{I}_s(2\pi m\operatorname{Im}(z))e(-m\operatorname{Re}(z)) + \sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)\sigma_t/\Gamma_0(N)_\infty} \mathcal{I}_\gamma(z,s).$$

Considering the Fourier expansion of

$$\sum_{\gamma\in\Gamma_0(N)_\infty\backslash\Gamma_0(N)\sigma_t/\Gamma_0(N)_\infty} \mathcal{I}_\gamma(z,s),$$

we can obtain (2.2) and (2.3). For the details of the proof, see §1.9 in [4] or [11], [13]. □

3. Regularized integration

Suppose that $g(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Let $S(g)$ be the set of the singular points of $g(z)$ on $\mathcal{F}_N$. We define an ε-disk $B(t,\varepsilon)$ at t by

$$B(t,\varepsilon) := \begin{cases} \{z\in\mathbb{H} \mid |z-t|<\varepsilon\} & \text{if } t\in\mathcal{F}_N, \\ \{z\in\mathcal{F}_N \mid \operatorname{Im}(\sigma_t z) > \frac{1}{\varepsilon}\} & \text{if } t\in\mathcal{C}_N. \end{cases}$$

For sufficiently small positive ε, let $\mathcal{F}_N(g,\varepsilon)$ denote a punctured fundamental domain for $\Gamma_0(N)$ defined as

$$\mathcal{F}_N(g,\varepsilon) = \mathcal{F}_N - \bigcup_{t\in S(g)\cup\mathcal{C}_N} B(t,\varepsilon).$$

We define the regularized integral of $g(z)\cdot\xi_0(j_{N,m}(z))dxdy$ on $\mathcal{F}_N$ by

$$\int_{\mathcal{F}_N}^{reg} g(z)\cdot\xi_0(j_{N,m}(z))dxdy := \lim_{\varepsilon\to 0}\int_{\mathcal{F}_N(g,\varepsilon)} g(z)\cdot\xi_0(j_{N,m}(z))dxdy. \tag{3.1}$$

Note that $g(z)\cdot\overline{j_{N,m}(z)}dz$ is a $\Gamma_0(N)$-invariant 1-form on $\mathbb{H}$. We define

$$\gamma(t,\varepsilon) := \begin{cases} \{z\in\mathbb{H} \mid |z-t|=\varepsilon\} & \text{if } t\in\mathbb{H}, \\ \{z\in\mathcal{F}_N \mid \operatorname{Im}(\sigma_t z)=\frac{1}{\varepsilon}\} & \text{if } t \text{ is a cusp of } \Gamma_0(N). \end{cases}$$

Then

$$\partial\mathcal{F}_N(g,\varepsilon) = \bigcup_{t\in S(g)\cup\mathcal{C}_N} \gamma(t,\varepsilon).$$

For $t \in \mathbb{H}$ let $\mathrm{Res}_t(g)$ be the residue of g at t on $\mathbb{H}$. Using the argument of Lemma 3.1 and Proposition 3.5 in [5], we prove the following lemma.

Lemma 3.1. *Let $g(z) := \sum_{n=0}^{\infty} a(n)q^n$ be a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Suppose that $g(z)$ is holomorphic at each cusp and that every pole of $g(z)$ is a simple pole. Then we have*

$$\lim_{\varepsilon\to 0}\int_{\mathcal{F}_N(g,\varepsilon)} g(z)\cdot \xi_0(j_{N,m}(z))dxdy$$
$$= b_m(1,0)a(0) + a(m) + \sum_{t\in\mathcal{C}_N^*} \alpha_t g(t) j_{N,m}(t) + \sum_{t\in S(f)} \frac{2\pi i}{l_t}\operatorname*{Res}_t(g) j_{N,m}(t).$$

Proof. Note that

$$d(g\cdot \overline{j_{N,m}}dz) = \overline{\partial}(g\cdot \overline{j_{N,m}}dz) = g\cdot \overline{\frac{\partial}{\partial \overline{z}} j_{N,m}} d\bar{z}dz = g(z)\cdot \xi_0(j_{N,m})(z)dxdy. \tag{3.2}$$

Thus, by Stokes' Theorem, we have

$$\int_{\mathcal{F}_N(g,\varepsilon)} g(z)\cdot \xi_0(j_{N,m}(z))dxdy$$
$$= \sum_{t\in S(g)} \frac{1}{l_\tau}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz + \sum_{t\in C_N}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz.$$

Note that if $t \in \mathcal{C}_N$ and ε is sufficiently small, then

$$\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz = \alpha_t \int_{-\frac{1}{2}}^{\frac{1}{2}} g\left(\sigma_t^{-1}\left(x + i\frac{1}{\varepsilon}\right)\right) j_{N,m}\left(\sigma_t^{-1}\left(x+i\frac{1}{\varepsilon}\right)\right)dx.$$

Here, α_t is a non-zero constant, and $\alpha_t = 1$ for a cusp $t\sim\infty$.

Following the argument of Proposition 3.5 in [5], for a cusp $t\sim\infty$ we have

$$\lim_{\varepsilon\to 0}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz = b_m(1,0)a(0) + a(m),$$

and for $t \not\sim \infty$,

$$\lim_{\varepsilon\to 0}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz = \alpha_t g(t) j_{N,m}(t).$$

From now on, we consider

$$\lim_{\varepsilon\to 0}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz$$

for $t \in S(g)$. Suppose that $g(z)$ has the Laurent series at t:

$$g(z) = \sum_{n=-1}^{\infty} a_n(t)(z-t)^n.$$

Then we have

$$\begin{aligned}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz &= \frac{1}{l_\tau}\int_0^1 g(t+\varepsilon e^{2\pi i u}) j_{N,m}(t+\varepsilon e^{2\pi i u}) 2\pi i \varepsilon e^{2\pi i u} du \\ &= \frac{2\pi i}{l_\tau}\int_0^1 \left(\sum_{n=0}^{\infty} a_{n-1}(t)(\varepsilon e^{2\pi i u})^n\right) j_{N,m}(t+\varepsilon e^{2\pi i u}) du \\ &= \frac{2\pi i}{l_\tau}\sum_{n=0}^{\infty} a_{n-1}(t)\varepsilon^n \int_0^1 e^{2\pi i n u} j_{N,m}(t+\varepsilon e^{2\pi i u}) du.\end{aligned}$$

Thus, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\gamma(t,\varepsilon)} g(z) j_{N,m}(z)dz &= \lim_{\varepsilon\to 0}\frac{2\pi i}{l_\tau}\sum_{n=0}^{\infty} a_{n-1}(t)\varepsilon^n \int_0^1 e^{2\pi i n u} j_{N,m}(t+\varepsilon e^{2\pi i u}) du \\ &= \frac{2\pi i}{l_\tau} a_{-1}(t)\int_0^1 j_{N,m}(t) du = \frac{1}{l_\tau} 2\pi i a_{-1}(t) j_{N,m}(t) \\ &= \frac{2\pi i}{l_\tau}\operatorname*{Res}_t(g) j_{N,m}(t).\end{aligned}$$

This completes the proof. □

4. Proof of Theorem 1.1

Suppose that g is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. For $\tau \in \mathbb{H}$, let Q_τ be the image of τ under the canonical map from $\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ to $X_0(N)$. The residue of g at Q_τ on $X_0(N)$, denoted by $\mathrm{Res}_{Q_\tau} gdz$, is well defined since we have the canonical correspondence between a meromorphic modular form of weight 2 on $\Gamma_0(N)$ and a meromorphic 1-form on $X_0(N)$. Suppose that t is a cusp of $\Gamma_0(N)$. Let σ_t^* be a matrix in $SL_2(\mathbb{Z})$ such that $\sigma_t^*\infty = t$. Then there exists α_t such that

$$\sigma_t^{*-1}\Gamma_0(N)_t\sigma_t^* = \left\{\pm\begin{pmatrix}1 & k\alpha_t\\ 0 & 1\end{pmatrix}\ \Big|\ k\in\mathbb{Z}\right\},$$

where $\Gamma_0(N)_t$ denotes the stabilizer of the cusp t in $SL_2(\mathbb{Z})$. For convenience we define $g(t)$ by the constant term of the Fourier expansion of $g(z)$ at t. If $\mathrm{Res}_\tau g$ denotes the residue of g at τ on $\mathbb{H}$, then we obtain

$$\begin{aligned}&\mathrm{Res}_{Q_\tau} gdz = \tfrac{1}{l_\tau}\mathrm{Res}_\tau g \text{ if } \tau\in\mathbb{H},\\ &\mathrm{Res}_{Q_t} gdz = \alpha_t g(t) \text{ if } t\in\mathcal{C}_N.\end{aligned}$$

Here, l_τ is the order of the isotropy group at τ. In particular, if f is a meromorphic modular form of weight k on $\Gamma_0(N)$ and $g = \frac{\theta f}{f}$, then the residue of g at each point on $X_0(N)$ is determined by the order of its zero or pole. If we denote by $\mathrm{ord}_\tau(f)$ the order of the zero or pole of f at τ as a complex function on $\mathbb{H}$, then

$$\nu_\tau^{(N)}(f) = \tfrac{1}{l_\tau}\mathrm{ord}_\tau(f).$$

Note that the constant term of the Fourier expansion of $\frac{\theta f}{f}$ at a cusp t is equal to the order of its zero or pole at the cusp. Thus, for each $t\in\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ we have

$$2\pi i\cdot\mathrm{Res}_{Q_t}\tfrac{\theta f}{f} = \nu_t^{(N)}(f). \tag{4.1}$$

Proof of Theorem 1.1. We begin by stating a lemma which was proved by Eholzer and Skoruppa in [9].

Lemma 4.1 ([9]). *Suppose that $f = \sum_{n=h}^{\infty} a(n)q^n$ is a meromorphic modular function in a neighborhood of $q = 0$ and that $a(h) = 1$. Then there are uniquely determined complex numbers $c(n)$ such that*

$$f = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. Moreover, the following identity is true:

$$\frac{\theta f}{f} = h - \sum_{n \geq 1} \sum_{d|n} c(d) d q^n.$$

Recall that

$$f_\theta(z) = \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N-1} \cdot N E_2(Nz) + \frac{h - Nk/12}{N-1} \cdot E_2(z).$$

The function $f_\theta(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Note that $f_\theta(z)$ is holomorphic at each cusp of $\Gamma_0(N)$. Moreover, the function

$$\frac{k/12 - h}{N-1} \cdot N E_2(Nz) + \frac{h - Nk/12}{N-1} \cdot E_2(z)$$

is holomorphic on $\mathbb{H}$. Thus, we have

$$\mathrm{Res}_\tau\, f_\theta = \mathrm{Res}_\tau\, \tfrac{\theta f}{f}$$

for $\tau \in \mathbb{H}$. Using Lemma 3.1 and (4.1), we complete the proof. $\square$

5. Proofs of Theorems 1.4 and 1.7

Proof of Theorem 1.4. Note that

$$\begin{aligned} f_\theta(z) &= \frac{\theta f(z)}{f(z)} + \frac{k/12 - h}{N-1} \cdot N E_2(Nz) + \frac{h - Nk/12}{N-1} \cdot E_2(z) \\ &= \sum_{n=1}^{\infty} \left(- \sum_{d|n} c(d) d + \frac{2Nk - 24h}{N-1} \sigma_1(n) + \frac{24h - 2k}{N-1} N \sigma_1(n/N) \right) q^n. \end{aligned}$$

This completes the proof by Theorem 1.1. $\square$

Proof of Theorem 1.7. We follow the argument of Theorem 3 in [6] (or Theorem 4 in [2]). Suppose that $f(z)$ has the q-expansion of the form

$$f(z) := q^h + \sum_{n=h+1}^{\infty} a(n) q^n = q^h \prod_{n=1}^{\infty} (1 - q^n)^{c(n)}$$

as in Lemma 4.1. Let

$$b(n) := \sum_{d|n} c(d) d.$$

Lemma 4.1 implies that

$$\left(q^h + \sum_{n=h+1}^{\infty} a(n) q^n \right) \left(h - \sum_{n=1}^{\infty} b(n) q^n \right) = h q^h + \sum_{n=h+1}^{\infty} n a(n) q^n.$$

Thus, for $n \geq 1$ we have

$$na(h+n) = -b(1)a(h+n-1) - b(2)a(h+n-2) - \cdots - b(n).$$

From this recurrence we obtain

$$b(n) = -na(h+n)$$

$$n \cdot \sum_{\substack{m_1+2m_2+\cdots+(n-1)m_{n-1}=n \\ m_1,\cdots,m_{n-1}\geq 0}} (-1)^{m_1+\cdots+m_{n-1}} \frac{(m_1+\cdots+m_{n-1}-1)!}{m_1!m_2!\cdots m_{n-1}!} a(h+1)^{m_1}\cdots$$
$$a(h+n-1)^{m_{n-1}}$$

(see (2.11) and Example 20 in §1.2 of [12]). Note that by Theorem 1.1,

$$b(n) = \sum_{\tau \in \mathcal{F}_N \cup \mathcal{C}_N^*} \nu_\tau^{(p)}(f(z)) j_{N,m}(\tau) - \int_{\mathcal{F}_N}^{reg} f_\theta(z) \cdot \xi_0(j_{N,m}(z)) dxdy$$
$$+ \frac{2Nk-24h}{N-1}\sigma_1(m) + \frac{24h-2k}{N-1} N\sigma_1(m/N).$$

This completes the proof. □

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