Research statement

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Algebraic topologists attach algebraic structures, such as groups, rings, and categories, to geometric objects, such as manifolds, simplicial complexes, and even big data sets. In homotopy theory, the main goal is to study *invariants* of geometric objects under "homotopy" transformations. This type of transformations usually turns out to provide the "right" criterion, neither too rigid nor too loose, for gaining useful insights into how these objects look and behave.

Effective *computational* tools with the associated algebraic structures give homotopy theory its unique flavor. The main players in realizing this process are the various "cohomology theories," each a systematic way of attaching specific algebraic structures to geometric objects, and each successfully capturing some aspects of the objects in question, while being blind to some others [Greenlees 1988]. A local-to-global property, manifested by Mayer–Vietoris sequences, makes cohomology theories particularly amenable to computations.

The study of Morava E-theories is like a raindrop in which all of modern homotopy theory is reflected. These cohomology theories are prominent players promoted by the "chromatic viewpoint," a deep and fruitful relationship between homotopy theory and the theory of one-dimensional formal groups. It has been steadily developing and pervading the field since Quillen's work on complex cobordism [Quillen 1969]. This approach brings homotopy theorists a "chromatic" view of the stable homotopy category, by filtering cohomology theories through *heights* and *primes* according to their corresponding formal groups. The family of Morava E-theories determines this chromatic filtration via Bousfield localizations [Ravenel 1984].

Hendrik Lenstra and Peter Stevenhagen wrote that "nothing can match the clarity of a formula when it comes to conveying a mathematical truth."[1] To understand E-theories in the specific filtration level of height 2, our research has been centering on calculations with their "power operations." These can be viewed as *algebraic structures of the algebraic structures* attached to geometric objects. They impose restrictions and refinements to the algebraic structures carried by cohomology theories, so as to enhance their ability to recognize and distinguish the subtleties between geometric objects, making those players clear-sighted.

For E-theories at height 2, contacts with algebraic geometry and number theory, particularly through the arithmetic moduli of elliptic curves, avail homotopy theorists effective tools to carry out explicit calculations. In terms of formulas for power operations, this approach imposes vastly intricate data from arithmetic. The

Date: August 9, 2019.

[1] From their book review of *Solving the Pell equation*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 345–351.

goal is to make the cohomology theories more sensitive and powerful in studying geometric questions, with a potential to witness the deep interplay between numbers and spaces.

Below we give an outline of our current research and future plans, with an emphasis on some specific aspects where algebraic topology, algebraic geometry, and number theory interact.

1 Elliptic curves: their local moduli spaces and power operations, with applications to unstable periodic homotopy theory

Cohomology operations have been a computational tool central to algebraic topology. A classical example which has been extensively studied and widely applied is the Steenrod operations in ordinary cohomology with $\mathbb{Z}/p$-coefficients. When $p = 2$, for all integers $i \geq 0$ and $n \geq 0$, each Steenrod square takes the form $\mathrm{Sq}^i\colon H^n(X;\mathbb{Z}/2) \to H^{n+i}(X;\mathbb{Z}/2)$, natural in spaces X. Together they generate the mod-2 Steenrod algebra subject to a set of axioms, among which, notably, the Adem relations require

$$\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\,\mathrm{Sq}^k \qquad\qquad 0 < i < 2j$$

In-depth study of the structure of the Steenrod algebra, and of analogous structures for other cohomology theories such as K-theory and motivic cohomology, has led to spectacular applications: Adams's solution to the problem of counting vector fields on spheres [Adams 1962] and Voevodsky's proof of the Milnor conjecture [Voevodsky 2003a, Voevodsky 2003b], just to name two.

For a Morava E-theory E, Rezk computed the first example of an explicit presentation for its algebra of power operations, in the case of height 2 at the prime 2 [Rezk 2008]. This algebra is generated over $E^0(\text{point}) \cong W(\overline{\mathbb{F}}_2)[\![h]\!]$, a ring of formal power series with coefficients in a Witt ring, by operations $Q_i\colon E^0(X) \to E^0(X)$, $0 \leq i \leq 2$. In particular, they satisfy "Adem relations"

$$Q_1Q_0 = 2Q_2Q_1 - 2Q_0Q_2 \qquad \text{and} \qquad Q_2Q_0 = Q_0Q_1 + hQ_0Q_2 - 2Q_1Q_2$$

Unlike their classical analogue, these formulas are computed from a specific moduli space of formal deformations of a supersingular elliptic curve over $\overline{\mathbb{F}}_2$. Here, bridging algebraic topology and algebraic geometry are the work of Ando, Hopkins, Strickland, and Rezk [Ando–Hopkins–Strickland 2004, Rezk 2009] together with the theorem of Serre and Tate [Lubin–Serre–Tate 1964]. Roughly speaking, power operations correspond to cyclic isogenies (a particular sort of maps) between elliptic curves (see [Rezk 2014]).

Based on computational evidence from Rezk's work and our further, significantly more involved calculations at the primes 3 and 5 [Zhu 2014, Zhu 2015], we obtained a presentation, *uniform for all primes*, of the power operation algebra of Morava E-theories at height 2. This presentation was deduced from our new semistable integral model for the modular curve $X_0(p)$ that parametrizes degree-p subgroup schemes of elliptic curves.

Theorem 1.1 ([Zhu 2018c, Theorem A]) *The modular curve $X_0(p)$, completed at a mod-p supersingular point, is given by the equation*

$$(\alpha - p)\big(\alpha + (-1)^p\big)^p - \big(h - p^2 + (-1)^p\big)\alpha = 0 \tag{1.2}$$

over the Lubin–Tate ring $W(\overline{\mathbb{F}}_p)[\![h]\!]$. In particular, this equation reduces to $\alpha(\alpha^p - h) \equiv 0 \mod p$ so that the singularity of the special fiber of $X_0(p)$ is a normal crossing, making the local model semistable.

Theorem 1.3 ([Zhu 2018c, Theorem C]) *Let E be a Morava E-theory of height 2 at the prime p. Its algebra of power operations admits a presentation as the associative ring generated over $E^0(\text{point}) \cong W(\overline{\mathbb{F}}_p)[\![h]\!]$ by elements Q_i, $0 \le i \le p$, subject to Adem relations and commutation relations. In particular, the Adem relations require*

$$Q_k Q_0 = -\sum_{j=1}^{p-k} w_0^j\, Q_{k+j} Q_j - \sum_{j=1}^{p} \sum_{i=0}^{j-1} w_0^i\, d_{k,j-i}\, Q_i Q_j \qquad\qquad 1 \le k \le p$$

where $w_0, d_{k,j-i} \in W(\overline{\mathbb{F}}_p)[\![h]\!]$ are determined explicitly from the equation (1.2).

To prove the key Theorem 1.1, we found a concrete modular form to model the Hasse invariant, and then used explicit q-expansion computations to see how this form transformed under a certain operation. This transported formulas at the cusps to the supersingular point.

Recent progress in unstable v_n-periodic homotopy theory gave immediate computational applications to unstable v_2-periodic homotopy [Behrens–Rezk 2017, Rezk 2013, Zhu 2018a]. The power opreation algebra structure of K-theory has proven to be a central ingredient for computations at height 1. Our presentation in Theorem 1.3 should have a similarly important impact.

In joint work in progress with Guozhen Wang, we have further studied the homological algebra involved in [Zhu 2018a]. In particular, we obtained an equivariantly commutative infinite tower of Koszul complexes, which we conjecture to generalize to *all* heights in relation to the EHP sequence and the Lubin–Tate tower [Zhu 2018b].

2 Generalized modular forms: Hecke algebras and logarithmic operations, with potential applications to multiplicative bordism invariants

In [Rezk 2006], using Bousfield–Kuhn functors, Rezk constructed "logarithmic" cohomology operations that naturally act on the units of any strictly commutative ring spectrum. In particular, given a Morava E-theory spectrum E of height n at the prime p, he wrote down a formula for its "logarithm" $\ell_{n,p}\colon E^0(X)^\times \to E^0(X)$ in terms of its power operations ψ_A [Rezk 2006, Theorem 1.11] and he interpreted this formula in terms of certain "Hecke operators" $T_{j,p}$ as follows:

$$\ell_{n,p}(x) = \frac{1}{p} \log \left(\prod_{j=0}^{n} \prod_{\substack{A \subset (\mathbb{Q}_p/\mathbb{Z}_p)^n[p] \\ |A| = p^j}} \psi_A(x)^{(-1)^j p^{(j-1)(j-2)/2}} \right) = \sum_{j=0}^{n} (-1)^j p^{j(j-1)/2}\, T_{j,p}(\log x) \tag{2.1}$$

These Hecke operators are cohomology operations constructed from power operations which were known to act on the E-cohomology of a space [Ando 1995].

Based on explicit calculations and a particular choice of parameters in the case of height 2, we revisited these operators to make a precise connection with Hecke operators acting on modular forms. In particular, we obtained the following, again via q-expansion arguments (cf. Theorem 1.1).

Theorem 2.2 ([Zhu 2015, Theorem 1.1]) *Let E be a Morava E-theory of height 2 at the prime p, and let $N > 3$ be any integer prime to p.*

(i) *There is a ring homomorphism $\beta\colon \mathrm{MF}[\Gamma_1(N)] \to E^0$ from the graded ring of weakly holomorphic modular forms for $\Gamma_1(N)$ to the coefficient ring of E in degree zero.*

(ii) *Given $f \in \big(\mathrm{MF}[\Gamma_1(N)]\big)^\times$ with trivial Nebentypus, if its Serre derivative equals zero, then the element $\beta(f)$ is contained in the kernel of Rezk's logarithmic cohomology operation $\ell_{2,p}\colon (E^0)^\times \to E^0$.*

The logarithms in E-theories at height 2 are critical in the work of Ando, Hopkins, and Rezk on rigidification of the string-bordism elliptic genus [Ando–Hopkins–Rezk 2010, Theorem 12.3]. Roughly speaking, the kernel of a logarithm contains the desired genera, which they identified with certain Eisenstein series.

Question 2.3 With our result in Theorem 2.2 about the kernel of a logarithmic operation, can we develop an analysis of E_∞-orientations analogous to the work of Ando, Hopkins, and Rezk?

The logarithm of a meromorphic modular form (on which Hecke operators act) appears in Rezk's formula (2.1). Serre's differential operator appears in Theorem 2.2 (see also [Katz 1973, Section A1.4]). In view of these, we ask the following.

Question 2.4 Do these specific pieces of number theory enter homotopy theory in a *structural* way? Do Rezk's logarithmic operations bring in a wider class of automorphic functions to homotopy theory? What is present at chromatic level higher than 2?

In [Zhu 2015, Section 3.4], we have started investigating certain aspects of the aforementioned type of elliptic functions, not totally modular, in the framework of "logarithmic q-series" originally studied by Knopp and Mason [Knopp–Mason 2011]. It has a curious relationship to mock modular forms [Zhu 2015, Remark 3.29].

3 Formal groups: norm-coherent coordinates, with applications to complex orientations and towards local class field theory in a derived setting

The various technical choices made in establishing Theorems 1.1 and 2.2 forced us to carefully study a distinguished coordinate on the one-dimensional Lubin–Tate universal deformation in question, characterized by a "norm coherence" property. Roughly speaking, such a coordinate is functorial under pushforward along any isogeny, with respect to a norm map which comes from a Galois action by the kernel of the isogeny.

In [Ando 1995], Ando constructed norm-coherent coordinates for deformations of the Honda formal groups over $\mathbb{F}_p$. Given these, he established the existence and uniqueness of H_∞ complex orientations for the corresponding Morava E-theories. However, as the underlying formal groups in Theorems 1.1 and 2.2 come from supersingular elliptic curves over algebraic extensions of $\mathbb{F}_p$, Ando's results do not apply to our cases. Placing the problem in a more flexible context, we have obtained analogous results for Morava E-theories associated to any formal group as follows.

Let $\mathrm{FG}_{\mathrm{isog}}$ be the category with objects

$$\begin{tikzcd} \mathcal{G} \arrow[d] \\ \mathrm{Spf}(k) \end{tikzcd} \qquad \begin{aligned} k &= \text{perfect field of characteristic } p \\ \mathcal{G} &= \text{formal group of height } n \end{aligned}$$

and morphisms

$$\begin{tikzcd} \mathcal{G} \arrow[r, "f"] \arrow[d] & \mathcal{G}' \times_{k'} k \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{G}' \arrow[d] \\ \mathrm{Spf}(k) \arrow[r, equal] & \mathrm{Spf}(k) \arrow[r] & \mathrm{Spf}(k') \end{tikzcd}$$

where f is an isogeny of formal groups over k. Let $\mathrm{NCoh}\colon \mathrm{FG}_{\mathrm{isog}} \to \mathrm{Set}$ be the bivariant functor

$$\mathcal{G}/k \mapsto \{\text{norm-coherent coordinates on } \mathcal{F}/E_n\}$$

where $\mathcal{F}$ is the universal deformation of $\mathcal{G}$ over the Lubin–Tate ring E_n. It respects pullback and base change contravariantly and pushforward covariantly. Let $\mathrm{Coord}\colon \mathrm{FG}_{\mathrm{isog}} \to \mathrm{Set}$ be the bivariant functor

$$\mathcal{G}/k \mapsto \{\text{coordinates on } \mathcal{G}\}$$

Theorem 3.1 ([Zhu 2017, Theorems 1.2 and 1.5]) *The natural transformation* $\mathrm{NCoh} \to \mathrm{Coord}$ *of restriction to the special fiber is an isomorphism. Moreover, it respects Galois descent: given a Galois extension* K/k*, the following diagram commutes.*

$$\begin{tikzcd} \mathrm{NCoh}(\mathcal{G} \times_k K) \arrow[r, "\sim"] \arrow[d, "(-)^{\mathrm{Gal}(K/k)}"'] & \mathrm{Coord}(\mathcal{G} \times_k K) \arrow[d, "(-)^{\mathrm{Gal}(K/k)}"] \\ \mathrm{NCoh}(\mathcal{G}) \arrow[r, "\sim"] & \mathrm{Coord}(\mathcal{G}) \end{tikzcd}$$

As a consequence, any Morava E-theory admits a unique H_∞ *complex orientation.*

Although we have solved the technical problem with norm-coherent coordinates for E-theories, we would like to have a more conceptual understanding of norm coherence, particularly in view of its relevance to an action of Coleman's norm operator in explicit local class field theory [Zhu 2017, Remark 1.3] (also cf. [Peterson 2019, Theorem A.2.34]).

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