Morava E-homology of Bousfield-Kuhn functors on odd-dimensional spheres

YIFEI ZHU

As an application of Behrens and Rezk's spectral algebra model for unstable v_n-periodic homotopy theory, we give explicit presentations for the completed E-homology of the Bousfield-Kuhn functor on odd-dimensional spheres at chromatic level 2, and compare them to the level 1 case. The latter reflects work of Davis and Mahowald in the 1980s on K-theory localizations.

1 Introduction

The rational homotopy theory of Quillen and Sullivan studies unstable homotopy types of topological spaces modulo torsion, or equivalently, after inverting primes. Such homotopy types are computable by means of their algebraic models. In particular, Quillen showed that there are equivalences of homotopy categories

$$\mathrm{Ho}_{\mathbb{Q}}(\mathrm{Top})_2 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGL})_1 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGC})_2$$

between simply-connected rational spaces, connected differential graded Lie algebras over $\mathbb{Q}$, and simply-connected differential graded coalgebras over $\mathbb{Q}$ [Quillen1969, Theorem I].

Working integrally, one has p-adic analogues where equivalences detected through rational homotopy $\pi_*(-)\otimes\mathbb{Q}$ are replaced by those through $\pi_*(-)\otimes\mathbb{Z}_p^\wedge$. Various algebraic models for p-adic homotopy types of spaces were developed [Kříž1993, Goerss1995, Mandell2001]. In the modern language of homotopy theory, these models are often formulated in terms of "spectral" algebra. For example, Mandell's model is given by the functor that takes a pointed nilpotent p-complete space X of finite p-type, to the $\overline{\mathbb{F}}_p$-cochains $H\overline{\mathbb{F}}_p^X$ (i.e. the function spectrum $F(\Sigma^\infty X, H\overline{\mathbb{F}}_p)$), which is a commutative $H\overline{\mathbb{F}}_p$-algebra.

More generally, through the prism of chromatic homotopy theory, Behrens and Rezk have established spectral algebra models for unstable v_n-periodic homotopy types

[Behrens-Rezk2015, Behrens-Rezk2016] (cf. [Arone-Ching2015, Heuts2016]). Here, instead of inverting primes, they invert classes of maps called "v_n-self maps" (the case of $n=0$ recovers rational homotopy). They work with the n'th unstable monochromatic category $M_n^f\mathrm{Top}_*$, in the sense of [Bousfield2001], and study the functor

$$\mathrm{Ho}(M_n^f\mathrm{Top}_*) \to \mathrm{Ho}\big(\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Sp}_{T(n)})\big) \tag{1.1}$$

that sends X to the $S_{T(n)}$-valued cochains $S_{T(n)}^X$. This last spectrum is an algebra for the (reduced) commutative operad Comm, in modules over the localization $S_{T(n)}$ of the sphere spectrum with respect to the telescope of v_n-self maps.

Considering a variant of localization with respect to the Morava K-theory $K(n)$, Behrens and Rezk have obtained an equivalence

$$\Phi_{K(n)}(X) \xrightarrow{\sim} \mathrm{TAQ}_{S_{K(n)}}(S_{K(n)}^X)$$

of $K(n)$-local spectra, on a class of spaces X including spheres [Behrens-Rezk2015, Theorem 8.1] (cf. [Behrens-Rezk2016]). The left-hand side arises from computing homotopy groups in the source category of (1.1), where $\Phi_{K(n)} = L_{K(n)}\Phi_n$ is a version of the Bousfield-Kuhn functor (cf. [Kuhn2008]). This side is a derived realization of morphisms in the source. The right-hand side is the topological André-Quillen cohomology of $S_{K(n)}^X$ as an algebra over the operad Comm in $S_{K(n)}$-modules. It is a derived realization of images of morphisms under the functor (1.1) in the target category. Via a suitable Koszul duality between Comm and the Lie operad, we may view the spectrum $\mathrm{TAQ}_{S_{K(n)}}(S_{K(n)}^X)$ as a Lie algebra model for the unstable v_n-periodic homotopy type of X.

The purpose of this paper is to make available calculations that apply Behrens and Rezk's theory to obtain quantitative information about unstable v_n-periodic homotopy types, in the case of $n=2$.

Theorem 1.2 *Let E be a Morava E-theory spectrum of height 2, with $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!]$. Then the first completed E-homology of the Bousfield-Kuhn functor applied to an odd-dimensional sphere has a presentation*

$$E_1^\wedge(\Phi_2 S^{2m+1}) \cong \begin{cases} 0 & m=0 \\ (E_0/p^m)^{\oplus p-m} \oplus E_0/(p^{m-1}, p^{m-2}a, \ldots, a^{m-1}) \oplus E_0^{\oplus m-1} & 1 \le m \le p \\ E_0/(p^p a^{m-p-1}, p^{p-1}a^{m-p}, \ldots, a^{m-1}) \oplus E_0^{\oplus p-1} & m > p \end{cases}$$

We also have $E_0^\wedge(\Phi_2 S^{2m+1}) \cong 0$ for any m. Since E is 2-periodic, these determine the completed E-homology in all degrees.

This builds on and strengthens Rezk's results in [Rezk2013, 2.13] and is a step toward the program initiated in [Arone-Mahowald1999] to compute the unstable v_n-periodic homotopy groups of spheres using stable v_n-periodic homotopy groups and Goodwillie calculus.

1.1 Comparison to the case of $n = 1$

In the 1980s, Davis and Mahowald showed that, $K(1)$-locally at a prime p, the mod-p^m Moore spectrum with i'th space $S^{i-1} \cup_m e^i$ is equivalent to the suspension spectrum of a certain stunted real projective space [Davis1986, Corollary 1.7 and Theorem 1.8], [Davis-Mahowald1987, proof of Theorem 4.2]. Via the Goodwillie tower of the identity functor on the category of pointed spaces, the latter can be identified with $\Phi_1(S^{2m+1})$, again $K(1)$-locally (cf. [Kuhn2007, Theorem 7.20] and [Behrens-Rezk2015, Remark 5.4]). We thus obtain a version of Theorem 1.2, for E of height 1 with $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p$, that identifies

$$E_0^\wedge(\Phi_1 S^{2m+1}) \cong E_0/p^m \qquad\qquad m \geq 0$$

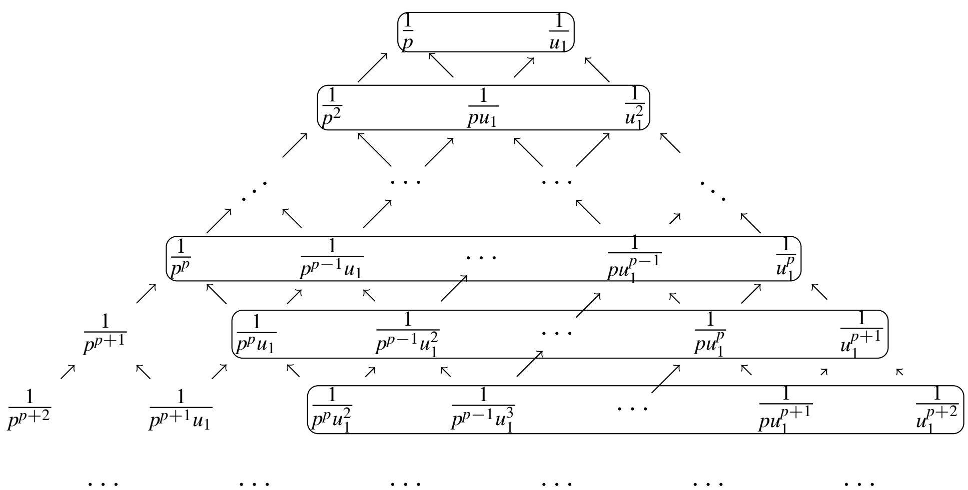
For any n, since Φ_n preserves fiber sequences, there is a natural map $\Sigma^2\Phi_n(X) \to \Phi_n(\Sigma^2 X)$. In view of the 2-periodicity of E, this induces a map on completed E-homology in the same degree. Thus the groups $\{E_*^\wedge(\Phi_n S^{2m+1})\}_{m\geq 0}$ form a direct system.

Homotopy (co)limits of generalized Moore spectra are closely related to various kinds of localizations of the sphere spectrum (see, e.g., [Arone-Mahowald1999, Proposition A.3] and [Hovey-Strickland1999, Proposition 7.10]). On the other hand, note that completed E-homology does not preserve homotopy colimits [Hovey2008]. Nevertheless, we hope to study the relationship between the $K(n)$-local sphere and the Bousfield-Kuhn functor on odd-dimensional spheres hinted in [Rezk2016, 3.20–3.21]. We observe the following.

Corollary 1.3 *Let E be a Morava E-theory spectrum of height n, with $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![u_1, \ldots, u_{n-1}]\!]$. Write $B_m := E_{n-1}^\wedge(\Phi_n S^{2m+1})$ as abstract E_0-modules. Then there are homomorphisms $B_m \to B_{m+1}$ making $\{B_m\}_{m\geq 0}$ into a direct system with*

$$\operatorname*{colim}_m B_m \cong \begin{cases} E_0/p^\infty & n = 1 \\ E_0/(p^\infty, u_1^\infty) \oplus E_0^{\oplus p-1} & n = 2 \end{cases}$$

Proof The case of $n = 1$ is clear. Let $n = 2$. Recall that $E_0/(p^\infty, u_1^\infty) \cong E_0[p^{-1}, u_1^{-1}]/E_0$ (see, e.g., [May-Ponto2012, Section 10.1]). Below we list its E_0-basis with arrows indicating the obvious multiplications. Under this identification, row by row beginning with $m = 2$, each box of elements and their "descendants" (elements able to reach the box from below along the arrows) are eliminated in the corresponding quotient of B_m as stated in Theorem 1.2. We see from the diagram that as m increases, more and more elements are preserved, until having them all in the limit.



□

1.2 Acknowledgements

I thank Mark Behrens, Paul Goerss, and Charles Rezk for helpful discussions. I am especially grateful to Mark for introducing me into the program of computing unstable periodic homotopy groups of spheres and to Charles for sharing his knowledge in power operations.

2 Koszul complexes for Γ-modules

Let E be a Morava E-theory spectrum of height n at the prime p. Its formal group $\mathrm{Spf}\, E^0\mathbb{CP}^\infty$ over $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![u_1, \ldots, u_{n-1}]\!]$ is the Lubin-Tate universal deformation of a formal group $\mathbb{G}$ over $\overline{\mathbb{F}}_p$ of height n.

Generalizing the Lubin-Tate deformation theory, Strickland shows that for each $k \geq 0$ there is a ring $A_k \cong E^0 B\Sigma_{p^k}/I_k$ classifying subgroups of degree p^k in the universal deformation, where I_k is a transfer ideal [Strickland1998, Theorem 1.1]. In particular, $A_0 \cong E_0$, and there are ring homomorphisms

$$s = s_k, t = t_k \colon A_0 \to A_k \quad \text{and} \quad \mu = \mu_{k,m} \colon A_{k+m} \to A_k^s \otimes_{A_0} {}^t A_m$$

classifying the source and target of an isogeny of degree p^k on the universal deformation and the composition of two isogenies.

As E is an E_∞-ring spectrum, there are (additive) power operations acting on the homotopy of $K(n)$-local commutative E-algebra spectra. A Γ*-module* is an A_0-module M equipped with structure maps (the power operations)

$$P_k \colon M \to {}^t A_k^s \otimes_{A_0} M, \qquad k \geq 0$$

which are a compatible family of A_0-module homomorphisms. These power operations form the *Dyer-Lashof algebra* Γ for the E-theory, with graded pieces $\Gamma[k] := \mathrm{Hom}_{A_0}({}^s A_k, A_0)$, $k \geq 0$. There is a tensor product $\otimes$ for Γ-modules [Rezk2013, Section 4].

The structure of a Γ-module is determined by P_1, subject to a condition involving A_2, i.e. the existence of the dashed arrow in the diagram

$$\begin{tikzcd}
M \arrow[r, "P_1"] \arrow[d, dashed] & {}^t A_1^s \otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes P_1"] \\
{}^t A_2^s \otimes_{A_0} M \arrow[r, "\mu\otimes\mathrm{id}"] & {}^t A_1^s \otimes_{A_0} {}^t A_1^s \otimes_{A_0} M
\end{tikzcd} \tag{2.1}$$

[Rezk2013, Proposition 7.2]. This manifests the fact that the ring Γ is *Koszul* and, in particular, *quadratic* [Rezk2012].

Let $D_0 := A_0$, $D_1 := A_1$, and

$$D_k := \mathrm{coker}\left(\bigoplus_{i=0}^{k-2} A_1^{\otimes i} \otimes A_2 \otimes A_1^{k-i-2} \xrightarrow{\mathrm{id}\otimes\mu\otimes\mathrm{id}} A_1^{\otimes k} \right), \qquad k \geq 2$$

Given Γ-modules M and N, Rezk defines the *Koszul complex* $\mathcal{C}^\bullet(M,N)$ by

$$\mathcal{C}^k(M,N) := \mathrm{Hom}_{A_0}(M, D_k \otimes_{A_0} N)$$

with appropriate coboundary maps [Rezk2013, Section 7.3].

Proposition 2.2 *If M is projective as an A_0-module, then*

$$H^k\mathcal{C}^\bullet(M,N) \cong \mathrm{Ext}^k_\Gamma(M,N)$$

In particular, if $k > n$, $D_k \cong 0$ and so $\mathrm{Ext}^k_\Gamma(M,N) \cong 0$.

Proof This is [Rezk2013, Proposition 7.4]. □

2.1 The case of $n = 2$

Choose a preferred $\mathcal{P}_N$-model for E in the sense of [Zhu2015a, Definition 3.29] so that the formal group of E is isomorphic to the formal group of a universal deformation of a supersingular elliptic curve satisfying a set of properties.

Using the theory of dual isogenies of elliptic curves, Rezk identifies that $D_2 \cong A_1/s(A_0)$ [Rezk2013, Proposition 9.3]. He also classifies Γ-modules of rank 1 [Rezk2013, Proposition 9.7]. In particular, each takes the form L_β with structure map

$$P\colon\ L_\beta \to {}^tA_1^s \otimes_{A_0} L_\beta$$
$$x \mapsto \beta \otimes x$$

where x is a generator for the underlying A_0-module, and $\beta \in A_1$ is such that $\iota(\beta)\cdot\beta \in s(A_0)$ with $\iota(-)$ the Atkin-Lehner involution (this condition on β corresponds to the condition in (2.1)). Moreover, $L_{\beta_1} \otimes L_{\beta_2} \cong L_{\beta_1\beta_2}$, so that L_β is $\otimes$-invertible as a Γ-module if and only if $\beta \in A_1^\times$.

Now let $M = L_\alpha$ and $N = L_\beta$. We have identifications

$$A_0 \xrightarrow{\sim} \mathcal{C}^0(M,N) = \mathrm{Hom}_{A_0}(M,N) \qquad f \mapsto (x \mapsto f\,y)$$
$$A_1 \xrightarrow{\sim} \mathcal{C}^1(M,N) = \mathrm{Hom}_{A_0}(M,\ {}^tA_1^s \otimes_{A_0} N) \qquad g \mapsto (x \mapsto g \otimes y)$$
$$A_1/s(A_0) \xrightarrow{\sim} \mathcal{C}^2(M,N) = \mathrm{Hom}_{A_0}\Big(M,\ {}^{\iota^2 s}\big(A_1/s(A_0)\big)^s \otimes_{A_0} N\Big) \qquad h \mapsto (x \mapsto h \otimes y)$$

Thus the Koszul complex in this case is

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_1/s(A_0)$$

with $d_0 f = \iota(f)\beta - f\alpha$ and $d_1 g = \iota(g)\beta + g\iota(\alpha)$ [Rezk2013, Section 9.18].

More explicitly, we have identifications

$$A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!] \quad \text{and} \quad A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a,b]\!]/\big(w(a,b)\big)$$

where

$$w(a,b) = (b-p)\big(b+(-1)^p\big)^p - \big(a-p^2+(-1)^p\big)b$$

[Zhu2015b, Theorem 1.2]. Note that the parameters a and b are chosen as in [Rezk2013, Section 9.15], and they correspond precisely to h and α in [Zhu2015b]. In particular, the Γ-module of invariant 1-forms is $\omega = L_b$.

The ring homomorphism $s\colon A_0 \to A_1$ is simply the inclusion of scalars, viewing A_1 as a free left module over A_0 of rank $p+1$. We will thus abbreviate $s(A_0)$ as A_0. Following [Rezk2013], we will also abbreviate $\iota(x)$ as x', which is written as $\widetilde{x}$ in [Zhu2015b].

3 Calculation of $\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{nul})$

Write $\mathrm{nul} := L_0$. By Proposition 2.2,

$$\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{nul}) \cong H^*\mathcal{C}^\bullet(L_{b^m}, L_0)$$

where

$$\mathcal{C}^\bullet(L_{b^m}, L_0)\colon\ A_0 \xrightarrow{-b^m} A_1 \xrightarrow{b'^m} A_1/A_0$$

Proposition 3.1 *For all $m \geq 0$, $H^0\mathcal{C}^\bullet(L_{b^m}, L_0) \cong 0$.*

Proof We need to show that $A_0 \xrightarrow{-b^m} A_1$ is injective. Given $f(a) \in A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!]$, suppose $-b^m \cdot f(a) = 0 \in A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a,b]\!]/\big(w(a,b)\big)$.

If $0 \leq m \leq p$, since $w(a,b)$ is a polynomial in b of degree $p+1$ with coefficients in A_0, clearly $f(a)$ must be 0.

If $m > p$, we need only show that $b^m \not\equiv 0 \mod w(a,b)$. Since $w(a,b) \equiv b(b^p - a) \mod p$, we have $b^{p+1} \equiv ab \mod (p,w)$, and thus $b^m \not\equiv 0 \mod (p,w)$. □

Proposition 3.2 *For all $m \geq 0$, $H^1\mathcal{C}^\bullet(L_{b^m}, L_0) \cong 0$.*

Proof Let $g(a,b)$ be a polynomial in b of degree at most p with coefficients in A_0 that represents an element in A_1. Suppose $b'^m \cdot g(a,b) = 0 \in A_1/A_0$. We need to show that $g(a,b) \equiv -b^m \cdot f(a) \mod w(a,b)$ for some $f(a) \in A_0$.

We do this by induction on m. The case of $m=0$ is clear. Let $m \geq 1$. By the induction hypothesis, since $b'^{m-1} \cdot b' g(a,b) = 0 \in A_1/A_0$, we have $b' g(a,b) \equiv -b^{m-1} \cdot f(a) \mod w$. Multiplying both sides by b, we get

$$\pm\, p g(a,b) \equiv -b^m f(a) \mod w \tag{3.3}$$

and thus

$$0 \equiv -b^m f(a) \mod (p,w) \tag{3.4}$$

Since $b^{p+1} \equiv ab \mod (p,w)$, (3.4) implies that $p \mid f(a)$. As p is not a zero-divisor in A_1, (3.3) implies that $g(a,b) \equiv -b^m \tilde{f}(a) \mod w$ for some $\tilde{f}(a) \in A_0$. □

Proposition 3.5

$$H^2\mathcal{C}^\bullet(L_{b^m}, L_0) \cong \begin{cases} 0 & m=0 \\ (A_0/p^m)^{\oplus p-m} \oplus A_0/(p^{m-1}, p^{m-2}a, \ldots, a^{m-1}) \oplus A_0^{\oplus m-1} & 1 \leq m \leq p \\ A_0/(p^p a^{m-p-1}, p^{p-1}a^{m-p}, \ldots, a^{m-1}) \oplus A_0^{\oplus p-1} & m > p \end{cases}$$

Proof We have $H^2\mathcal{C}^\bullet(L_{b^m}, L_0) \cong A_1/(A_0 + b'^m A_1)$. The case of $m=0$ is clear.

Let $1 \leq m \leq p$. Without loss of generality, we may assume p is odd so that $bb' = p$. As a free module over A_0, the ring A_1 has a basis consisting of

$$1,\ b,\ b^2,\ \ldots,\ b^p \tag{3.6}$$

Under the map of multiplication by b'^m, these elements become b'^m, pb'^{m-1}, $p^2 b'^{m-2}$, $\ldots$, $p^{m-1}b'$, p^m, $p^m b$, $\ldots$, $p^m b^{p-m}$, and thus the submodule $A_0 + b'^m A_1$ of A_1 is generated by

$$1,\ p^m b,\ \ldots,\ p^m b^{p-m}, \text{ as well as } b'^m,\ pb'^{m-1},\ p^2 b'^{m-2},\ \ldots,\ p^{m-1} b' \tag{3.7}$$

Comparing these generators to (3.6), quotient by the first $(p-m+1)$ free ranks results in the summand $(A_0/p^m)^{\oplus p-m}$ as stated in the proposition. We next verify that the remaining m elements in (3.7) contribute m relations on b^p to forming the summand $A_0/(p^{m-1}, p^{m-2}a, \ldots, a^{m-1})$, so that the rest elements in (3.6), b^{p-m+1}, $\ldots$, b^{p-1}, survive and give the free summand $A_0^{\oplus m-1}$.

Indeed, with notation as in [Zhu2015b, Theorem 1.6 (ii)], we have

$$b'^k = d_{p,k} b^p + d_{p-1,k} b^{p-1} + \cdots + d_{0,k}, \qquad 1 \leq k \leq m \leq p$$

(cf. [Zhu2015b, Section 4.1]). In particular, the formula for the coefficient $d_{p,k}$ has a leading term $(-1)^k w_1^{k-1} w_{p+1} = -a^{k-1}$. Thus setting $p^{m-k} b'^k$ to be zero in the quotient

$A_1/(A_0 + b'^m A_1)$ gives an expression for $p^{m-k}a^{k-1}b^p$ in terms of an A_0-linear combination of $b^{p-1}, \ldots, 1$, and, possibly, b^p itself if there are more than one term in $d_{p,k}$. As k ranges from 1 to m, this explains the summand $A_0/(p^{m-1}, p^{m-2}a, \ldots, a^{m-1})$.

Finally, the case of $m > p$ is similar and simpler. □

4 Proof of Theorem 1.2

Recall that given a Morava E-theory E of height n, the completed E-homology functor is defined as $E^\wedge_*(-) := \pi_*(E \wedge -)_{K(n)}$, where $(-)_{K(n)} = L_{K(n)}(-)$ denotes the $K(n)$-localization. In particular,

$$
\begin{aligned}
E^\wedge_*(\Phi_{K(n)}X) &= \pi_*(E \wedge \Phi_{K(n)}X)_{K(n)} \\
&= \pi_*(L_n \Phi_{K(n)}X)_{K(n)} \\
&= \pi_*(L_n L_{K(n)} \Phi_n X)_{K(n)} \\
&= \pi_*(L_n \Phi_n X)_{K(n)} \\
&= E^\wedge_*(\Phi_n X)
\end{aligned}
\tag{4.1}
$$

In [Rezk2013], Rezk sets up a composite functor spectral sequence (abbrev. CFSS) followed by a mapping space spectral sequence (abbrev. MSSS) to compute the homotopy groups of derived mapping spaces between $K(n)$-local augmented commutative E-algebras. He identifies the E_2-term in the CFSS as Ext-groups over the Dyer-Lashof algebra Γ. The CFSS converges to the E_2-term in the MSSS.

In particular, [Rezk2013, Section 2.13] shows that this setup specializes to compute the E-cohomology of the topological André-Quillen homology $\mathrm{TAQ}^{S_{K(n)}}(S_{K(n)}^{S_+^{2m+1}})$, and that the two spectral sequences both collapse at the E_2-term when $n = 2$.

By [Behrens-Rezk2015, Theorem 8.1] and (4.1), we identify the abutment of the MSSS as $E^\wedge_*(\Phi_n S^{2m+1})$. Rezk identifies the E_2-term in the CFSS as $\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{nul})$. The calculations in Section 3 then give the claimed result. □

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