Deformation structures and norm coherence

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We generalize Ando's construction of H_∞ complex orientations on Morava E-theories associated to Honda formal group laws over $\mathbb{F}_p$. We show the existence and uniqueness of such an orientation on any Morava E-theory associated to a formal group law over an algebraic extension of $\mathbb{F}_p$.

1 Introduction

fg vs fgl, coords, notations F_x and F_x/H (induced by Lubin isog, def, defo of Frob)

Lubin-Tate E_n [Ando95, thm 2.3.1], Strickland A_r [iph, thm 4.4], univ ex of quot $F^{(p^r)}$

contravariant, omitting Spec

1.1 Acknowledgements

I thank

2 Deformation structures

2.1 Wide categories of formal groups

2.2 Deformation of isogenies and pushforward of deformation structures

bivariant: pullback and pushforward of defo strs and coords

3 Norm coherence

3.1 Norm-coherent coordinates

Let k be an algebraic extension of $\mathbb{F}_p$, and G be a formal group over k of finite height n. Let F/E_n be the universal deformation of G/k, and $F[p]$ be its subgroup of p-torsions. The latter is defined over an extension $\overline{E}_n$ of E_n obtained by adjoining the roots of the p-series of F. Note that $F[p](\overline{E}_n)$ is stable under the action of $\mathrm{Aut}(\overline{E}_n/E_n)$. Thus, given a coordinate x on F lifting any preferred coordinate on G, the Lubin isogeny

$$f_p^x\colon\ F_x \to F_x/F[p]$$

can be defined over E_n (cf. [**?**, Theorem 1.4] and see [**?**, proof of Corollary 3.2] for an explicit example).

Remark 3.1 On the special fiber $\mathcal{O}_{G^{(p^n)}} = k[\![y]\!]$ over k, this Lubin isogeny is k-linear, sending $g(y)$ to $g(x^{p^n})$. It is the relative p^n-power Frobenius and is not an endomorphism unless $k \subset \mathbb{F}_{p^n}$ (cf. [**?**, proof of Proposition 2.5.1]).

By Theorem **??**, in view of the above remark, we have a unique local homomorphism $\alpha_p\colon\ A_n \to E_n$ and a unique $\star$-isomorphism $g_p^x\colon\ F_x/F[p] \to \alpha_p^* F_x^{(p^n)}$ that classify f_p^x as a deformation of a degree-p^n isogeny. Define

$$l_p^x\colon\ F_x \to \alpha_p^* F_x^{(p^n)}$$

to be the composite $g_p^x \circ f_p^x$. It is uniquely characterized by the following properties (cf. [**?**, Proposition 2.5.4]).

(i) The isogeny l_p^x of formal group laws has source F_x and target of the form $\alpha^* F_x^{(p^n)}$ for some ring homomorphism $\alpha\colon\ A_n \to E_n$.

(ii) The kernel of l_p^x applied to $F_x(\overline{E}_n)$ is $x(F[p])$.

(iii) Reducing coefficients to the residue field transforms l_p^x to the relative p^n-power Frobenius.

Explicitly, f_p^x and l_p^x fit into the following commutative diagram, where their restrictions over the residue field are highlighted in corresponding colors and thick arrows indicate homomorphisms.

$$\begin{tikzcd}
 & & F \arrow[dl, red, "f_p"'] \arrow[ddll, blue, bend right, "l_p"'] & & \pi^*F \arrow[ll] \arrow[rr, "\eta"] \arrow[dl, "\pi^*f_p"'] & & i^*G \arrow[rr] \arrow[dl, "i^*\mathrm{Frob}^n"'] & & G \arrow[dl, red, "\mathrm{Frob}^n"'] \arrow[dl, blue, bend left] \\
 & F/F[p] \arrow[dl, "g_p"'] & & \pi^*(F/F[p]) \arrow[ll] \arrow[rr, "\eta"] \arrow[dl, equal, "\pi^*g_p"'] & & i^*G^{(p^n)} \arrow[rr] \arrow[dl, equal] & & G^{(p^n)} \arrow[dl] & \\
\alpha_p^*F^{(p^n)} & & \pi^*\alpha_p^*F^{(p^n)} \arrow[ll] \arrow[rr, "\eta"] & & i'^*G \arrow[rr] & & G & &
\end{tikzcd}$$

More generally, let H be a subgroup of F of order p^r, and $\psi_H\colon F \to F/H$ be an isogeny with kernel H. For comparison, there are classifying maps α_H and g_H giving rise to the diagram below, where by definition $l'_H := g_H \circ \psi_H$ (the prime distinguishes ψ_H from the particular Lubin isogeny f_H).

$$\begin{tikzcd}
 & & F \arrow[dl, red, "\psi_H"'] \arrow[ddll, blue, bend right, "l'_H"'] & & \pi^*F \arrow[ll] \arrow[rr, "\eta"] \arrow[dl, "\pi^*\psi_H"'] & & i^*G \arrow[rr] \arrow[dl, "i^*\mathrm{Frob}^r"'] & & G \arrow[dl, "\mathrm{Frob}^r"'] \\
 & F/H \arrow[dl, "g_H"'] & & \pi^*(F/H) \arrow[ll] \arrow[rr, "\eta'"] \arrow[dl, equal, "\pi^*g_H"'] & & i^*G^{(p^r)} \arrow[rr] \arrow[dl, equal] & & G^{(p^r)} \arrow[dl] & \\
\alpha_H^*F^{(p^r)} & & \pi^*\alpha_H^*F^{(p^r)} \arrow[ll] \arrow[rr, "\eta'"] & & i'^*G \arrow[rr] & & G & &
\end{tikzcd}$$

3.2 Ando's construction of norm-coherent coordinates in greater generality

References