Modular equations for Lubin-Tate formal groups at chromatic level 2

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Northwestern University

AMS session on algebraic geometry 2016

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \text{Spaces} \to \text{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\text{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

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Cartan formula $\mathrm{Sq}^i(xy) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k}(x)\,\mathrm{Sq}^k(y)$

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Example (complex K-theory)

Adams operations $\psi^i\colon\ K(-) \to K(-)$

Power operation $\psi^p(x) \equiv x^p \mod p$

$\psi^i\psi^j = \psi^{ij}$ $\qquad\qquad$ $\psi^i(xy) = \psi^i(x)\psi^i(y)$

J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962)

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

stable homotopy theory $\leftrightsquigarrow$ 1-dim formal group laws

complex-oriented $h^*(-)$ $\quad$ $F(x,y)$ over $h^*(\mathrm{pt})$

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big)$$

Example

$$H^*(-;\mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x,y) = x + y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x + y - xy$$

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Example (Morava 1978, Goerss-Hopkins-Miller 1990s–2004)

k = perfect field of characteristic $p > 0$

F = formal group over k of height $n < \infty$

$E_n(k, F)(-) \leftrightsquigarrow$ the Lubin-Tate universal deformation of F/k

$\pi_* E_n(k, F) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$ with $|u| = -2$

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Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2$ = form of K-theory

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Results: power operations on E-theories of height 2

Theorem (Z.)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i \colon E(-) \to E(-)$, $0 \le i \le p$, and quadratic relations

$$Q_i Q_0 = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $1 \le i \le p$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

Remark The first example, for $p = 2$, is computed by Rezk (2008).

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Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

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An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

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Recall E-theory at height n and prime p has an underlying model

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$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$(\alpha-1)^p(\alpha-p) - \big((-1)^p + (-1)^{p-1}(-p)p + h\big)\alpha = 0$

Future directions   Can we get an explicit presentation for the Dyer-Lashof algebra of Morava E-theory at a **higher** chromatic level?

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Future directions How much (local) structure of the moduli carries on to formal groups of higher height
(or higher-dimensional abelian varieties)?

Thank you.