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# The Burnside Ring of Profinite Groups and the Witt Vector Construction

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The classical construction of Witt vectors can be understood as a special case of a construction which can be defined relative to any given profinite group $G$ in terms of its (completed) Burnside ring $\hat{\Omega}(G)$. It specializes to the classical situation in case $G$ is infinite cyclic. 

## 1. INTRODUCTION

According to P. Cartier (cf. [C]), the classical Witt vector construction as contrived by E. Witt (cf. [W1]) can be described as resulting in a covariant functor $\mathbb{W}$ (or $\mathbb{W}_p$) from the category of commutative rings into itself such that for any commutative ring $A$, the ring $\mathbb{W}(A)$ (or $\mathbb{W}_p(A)$) is—as a set—canonically isomorphic to the set of all infinite sequences $(\alpha_1, \alpha_2, \ldots) \in A^{\mathbb{N}}$ (resp. $(\alpha_0, \alpha_1, \ldots) \in A^{\mathbb{N}_0}$) of elements from $A$ (in particular, for any ring homomorphism $h: A \to B$ the map $\mathbb{W}(h): \mathbb{W}(A) \to \mathbb{W}(B)$ is given by $\mathbb{W}(h)(\alpha_1, \alpha_2, \ldots) = (h(\alpha_1), h(\alpha_2), \ldots)$), while for any $i \in \mathbb{N}$ (resp. $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$) the family of maps

(1.1)
$$\phi_i^A : \mathbb{W}(A) = A^{\mathbb{N}} \to A$$
$$(\alpha_1, \alpha_2, \ldots) \mapsto \sum_{j \mid i} j a_j^{i/j}$$

(or

(1.2)
$$\phi_i^A : \mathbb{W}_p(A) = A^{\mathbb{N}_0} \to A$$
$$(\alpha_0, \alpha_1, \ldots) \mapsto \sum_{j \leqslant i} p^j a_j^{p^{i-j}})$$

is a natural transformation from $\mathbb{W}$ (or $\mathbb{W}_p$, respectively) into the identity.

In this paper we want to show that these facts can be understood as special instances of a far more general theorem. In fact, let $G$ be an



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arbitrary profinite group. For any $G$-space $X$ and any subgroup $U$ of $G$ define $\varphi_U(X)$ to be the cardinality of the set $X^U$ of $U$-invariant elements of $X$ and let $G/U$ denote the $G$-space of left cosets of $U$ in $G$. Then the following holds:

MAIN THEOREM. *There exists a unique covariant functor $\mathbb{W}_G$ from the category of commutative rings into itself such hat for any commutative ring $A$ the ring $\mathbb{W}_G(A)$ coincides—as a set—with the set $A^{\underline{\mathcal{O}(G)}}$ of all maps from the set $\mathcal{O}(G)$ of open subgroups of $G$ into the ring $A$ which are constant on conjugacy classes, in such a way that for every ring homomorphism $h: A \to B$ and every $\alpha \in \mathbb{W}_G(A)$ one has $\mathbb{W}_G(h)(\alpha) = h \circ \alpha$, while for any open subgroup $U$ of $G$ the family of maps*

$$\phi_U^A: \mathbb{W}_G(A) \to A \tag{1.3}$$

*defined by*

$$\alpha \mapsto \sum_{U \lesssim V \leqslant G}' \varphi_U(G/V) \cdot \alpha(V)^{(V:U)} \tag{1.4}$$

*provides a natural transformation from the functor $\mathbb{W}_G$ into the identity. Here $U \lesssim V$ means that the open subgroup $U$ of $G$ is subconjugate to $V$, i.e., there exists some $g \in G$ with $U \leqslant gVg^{-1}$, $(V: U)$ means the index of $U$ in $gVg^{-1}$ which coincides with $(G: U)/(G: V)$ and therefore is independent of $g$, and the symbol "$\sum'$" is meant to indicate that for each conjugacy class of subgroups $V$ with $U \lesssim V$ exactly one summand has to be taken.*

Moreover, $\mathbb{W}_G(\mathbb{Z}) \cong \hat{\Omega}(G)$, where $\hat{\Omega}(G)$ is the "completed Burnside ring" of $G$, i.e., the Grothendieck ring of those discrete $G$-spaces $X$ for which $\varphi_U(x)$ is finite for every open subgroup $U$ of $G$.

In case $G = \hat{C}$, the profinite completion of the infinite cyclic group $C$, the (conjugacy classes of) open subgroups are parametrized naturally by their index in $\hat{C}$, i.e., there is precisely one open subgroup $\hat{C}^i$ for every positive integer $i$ with $(\hat{C}: \hat{C}^i) = i$, and one has

$$\varphi_{\hat{C}^i}(\hat{C}/\hat{C}^j) = \begin{cases} j & \text{if } j \text{ divides } i, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

Similarly, if $G = \hat{C}_p$, the pro-$p$-completion of the infinite cyclic group $C$, the open subgroups have $p$-power indices in $\hat{C}_p$ and are therefore parametrized by the $p$-exponent of their index, i.e., there is precisely one open subgroup $\hat{C}_p^i$ for every non-negative integer $i$ with $(\hat{C}_p: \hat{C}_p^i) = p^i$, and one has

$$\varphi_{\hat{C}_p^i}(\hat{C}_p/\hat{C}_p^j) = \begin{cases} p^j & \text{if } j \leqslant i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the above theorem implies Witt's result as stated by P. Cartier and in addition one has the following corollaries:

COROLLARY 1. $\mathbb{W}(\mathbb{Z}) \cong \hat{\Omega}(\hat{C})$.

COROLLARY 2. $\mathbb{W}_p(\mathbb{Z}) \cong \hat{\Omega}(\hat{C}_p)$.

We suppose that the study of these functors for arbitrary profinite groups $G$ may lead to interesting relations between the arithmetic structure of $G$ and the ring theoretic invariants of the associated rings $\mathbb{W}_G(A)$. As a first result in this direction we shall prove

THEOREM 2. *If the order $|G|_p$ of a p-Sylow subgroup of G is infinite, then*

$$p^n \cdot \mathbb{W}_G(\mathbb{F}_p) \neq 0 \tag{1.6}$$

*for all n.*

*If $|G|_p$ is finite, one has*

$$|G|_p \cdot \mathbb{W}_G(\mathbb{F}_p) \neq 0 \tag{1.7}$$

*but*

$$p \cdot |G|_p \cdot \mathbb{W}_G(\mathbb{F}_p) = 0. \tag{1.8}$$

Theorem 2 is an almost immediate consequence of the characterization of the functor $\mathbb{W}_G$ given in the main theorem.

The paper is organized as follows: In Section 2 we start by considering the category of $G$-spaces associated to the profinite group $G$, i.e., of Hausdorff (topological) spaces on which $G$ is acting continuously from the left. We discuss standard constructions defined for such spaces including the "exponential" construction, symmetric powers, restriction, and induction. We introduce the notion of almost finite $G$-spaces, which are discrete $G$-spaces $X$ for which the number $\varphi_U(X)$ of $U$-invariant elements in $X$ is finite for every open subgroup $U$ of $G$ and, we define the completed Burnside ring $\hat{\Omega}(G)$ to be the Grothendieck ring of almost finite $G$-spaces. Equivalently, this ring can be constructed as the projective limit $\varprojlim_N \Omega(G/N)$ of the Burnside rings $\Omega(G/N)$ of all finite factor groups $G/N$ ($N$ an open normal subgroup of $G$), where for two open normal subgroups $M$ and $N$ of $G$ with $M \subseteq N$ the projection map $\Omega(G/M) \to \Omega(G/N)$ is given by associating to a finite $G/M$-set $X$ the $G/N$-set $X^N$ of all $N$-invariant elements of $X$.

Often we shall identify an almost finite $G$-space $X$ with the element $[X]$ in $\hat{\Omega}(G)$, represented by it.

For every open subgroup $U$ of $G$, restriction and induction define maps

$$\text{(1.9)} \qquad \operatorname{res}^G_U: \hat{\Omega}(G) \to \hat{\Omega}(U)$$

and

$$\text{(1.10)} \qquad \operatorname{ind}^G_U: \hat{\Omega}(U) \to \hat{\Omega}(G)$$

which satisfy the usual identities (cf. (2.10.4) and (2.10.4)″).

Moreover, for any open subgroup $U$ of $G$, we have an induced ring homomorphism

$$\text{(1.11)} \qquad \begin{aligned} \varphi_U: \hat{\Omega}(G) &\to \mathbb{Z} \\ [X] &\mapsto \varphi_U(X) \end{aligned}$$

from $\hat{\Omega}(G)$ to the integers, defined by associating to an almost finite $G$-space $X$ the number of its $U$-invariant elements. Together these ring homomorphisms induce an embedding

$$\text{(1.12)} \qquad \varphi := \prod_{U \in \mathcal{O}(G)} \varphi_U: \Omega(G) \to \mathbb{Z}^{\underline{\mathcal{O}(G)}}$$

of $\hat{\Omega}(G)$ into the ring of integer-valued functions, defined on the set $\mathcal{O}(G)$ of open subgroups of $G$ and constant on conjugacy classes. The image $\varphi(\hat{\Omega}(G)) \subseteq \mathbb{Z}^{\underline{\mathcal{O}(G)}}$ can be described by canonical families of congruences (cf. (2.7.3)).

We use these congruences to show that for every $G$-space $S$ with only finitely many orbits there exists a well-defined "exponential" map $x \mapsto x^S$ from the Burnside ring $\Omega(G)$ of finite $G$-spaces to the completed Burnside ring $\hat{\Omega}(G)$ of almost finite $G$-spaces such that

$$\text{(1.13)} \qquad (x \cdot y)^S = x^S \cdot y^S$$

$$\text{(1.14)} \qquad x^{(S_1 \cup S_2)} = x^{S_1} \cdot x^{S_2}$$

and

$$\text{(1.15)} \qquad [X]^S = [X^S]$$

for any finite $G$-space $X$, where $X^S$ denotes the almost finite (!) "exponential" $G$-space of all continuous functions from $S$ to $X$. This provides us in particular with a multiplicative map

$$\text{(1.16)} \qquad \tau^G: \mathbb{Z} \to \hat{\Omega}(G)$$

which assigns to a positive integer $\alpha$ the almost finite $G$-space of all continuous mappings from the transitive $G$-space $G/1$ to a discrete $G$-space with trivial $G$-action containing exactly $\alpha$ elements. This map can be considered as a combinatorial version of the Teichmüller map, well known in the context of Witt vectors. Using induction and all of the maps $\tau^U: \mathbb{Z} \to \hat{\Omega}(U)$ for all open subgroups of $G$ simultaneously we can define a mapping

$$(1.17) \qquad \tau: \mathbb{Z}^{\underline{\mathcal{O}(G)}} \to \hat{\Omega}(G)$$
$$\alpha = (\alpha(U))_U \mapsto \sum_{U \leqslant G}{}' \operatorname{ind}_U^G(\tau^U(\alpha(U)))$$

which can be shown to be a bijection (cf. (2.12.7)). This bijection will turn out to be essential in the proof of the Main Theorem. By composing $\tau$ with the embedding $\varphi: \hat{\Omega}(G) \to \mathbb{Z}^{\underline{\mathcal{O}(G)}}$ of the Burnside ring in the product ring $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$, we get the commutative diagram (cf. (1.3))

$$(1.18) \qquad \begin{tikzcd} & \hat{\Omega}(G) \arrow[d, "\varphi"] \\ \mathbb{Z}^{\underline{\mathcal{O}(G)}} \arrow[ur, "\tau"] \arrow[r, "\phi^{\mathbb{Z}} = \prod_U \phi_U^{\mathbb{Z}}"'] & \mathbb{Z}^{\underline{\mathcal{O}(G)}} \subseteq \prod_U \mathbb{Z}. \end{tikzcd}$$

The components $\phi_U^{\mathbb{Z}}$ of the mapping $\phi^{\mathbb{Z}}$ have the property stated in the Main Theorem.

Finally, we discuss symmetric powers of almost finite $G$-spaces and the $\lambda$-ring structure on $\hat{\Omega}(G)$ induced by them (cf. [DS1, DS2, S]).

In Section 3 we prove the Main Theorem. We show that the ring structure on $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ which may be defined by transport of structure relative to the bijection $\tau: \mathbb{Z}^{\underline{\mathcal{O}(G)}} \xrightarrow{\sim} \hat{\Omega}(G)$ has the property that the components of the sum and the product of two elements in $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ are integer polynomials in the components of these elements. This allows one to define a ring structure on $\mathbb{W}_G(A) := A^{\underline{\mathcal{O}(G)}}$ for any commutative ring $A$.

In the same manner we show that the transport of restriction and induction provides us with mappings

$$(1.19) \qquad f_U: \mathbb{W}_G(\mathbb{Z}) \to \mathbb{W}_U(\mathbb{Z})$$

and

$$(1.20) \qquad v_U: \mathbb{W}_U(\mathbb{Z}) \to \mathbb{W}_G(\mathbb{Z}),$$

defined for every open subgroup $U$ of $G$, for which the components of the image of an element are integer polynomials in the components of the given element. These mappings therefore extend for arbitrary commutative rings $A$ to maps

(1.21) $$f_U: \mathbb{W}_G(A) \to \mathbb{W}_U(A)$$

and

(1.22) $$v_U: \mathbb{W}_U(A) \to \mathbb{W}_G(A).$$

The maps $f_U$ are ring homomorphisms and generalize the Frobenius maps for the classical Witt vectors, whereas the maps $v_U$ are only additive homomorphisms and can be viewed as generalizations of the Verschiebung-morphisms. Frobenius reciprocity and the Mackey subgroup formula for restriction and induction generalize in an obvious way and specialize to well-known identities between Frobenius and Verschiebung in the infinite cyclic case.

In Section 4 we shall prove Theorem 2 by studying the behaviour of the ring of Witt vectors $\mathbb{W}_G(\mathbb{F}_p)$ with respect to multiplication by $p$ and relating this to the (pro-)$p$-subgroups of $G$.

Finally, in Section 5, we shall discuss the functorial properties of the functor $G \mapsto \hat{\Omega}(G)$. These become more transparent by "relativizing" this functor to a functor, defined on the category of $G$-spaces, thereby giving rise to a "Mackey functor" (cf. [Dr2] for the finite case). In this context we also define and study the concept of an "algebraic map." More precisely, a map $\eta: A \to B$ from an (additive) abelian semigroup into an (also additive) abelian group $B$ is defined to be algebraic of degree $\leqslant n$ if for all $a_0, a_1, ..., a_n \in A$ its "formal partial derivative" $D_{(a_0, a_1, ..., a_n)}\eta: A \to B: a \mapsto \sum_{k=0}^{n+1} \sum_{0 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} (-1)^{n+1-k}$ $(x + a_{i_1} + \cdots a_{i_k})$ vanishes. Using the "Taylor expansion formula"

$$\eta(x + a_1 + \cdots + a_n) = \sum_{k=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (D_{(a_{i_1}, ..., a_{i_k})}\eta)(x)$$

it is shown that the composition $\eta_2 \circ \eta_1: A \to C$ of two algebraic maps $\eta_1: A \to B$ of degree $\leqslant n_1$ and $\eta_2: B \to C$ of degree $\leqslant n_2$ is algebraic of degree $\leqslant n_1 \cdot n_2$, that for an algebraic map $\eta: A \to B$ of degree $\leqslant n$ one has a well-defined multi-additive map $\eta^n: A^n \to A$: $(a_1, ..., a_n) \to (D_{(a_1, ..., a_n)}\eta)(x)$ ($x \in A$ an arbitrary, but fixed element) and that for any $r \in \mathbb{N}$ the map $\eta': A \to B: a \to \eta(r \cdot a) - r^n \cdot \eta(a)$ is algebraic of degree $\leqslant n-1$.

We have quoted these results in detail for the following reasons: (i) they should be folklore, but seem difficult to trace, (ii) their proof, given in Section 5, is completely independent from the rest of the paper, and (iii) they might be useful in other contexts, too. In our context they allow one to give another interpretation of the Teichmüller map (on the Burnside ring level!) which may be helpful in more detailed studies of its properties.

## 2. The Burnside Ring $\hat{\Omega}(G)$

**2.1.** Let $G$ be a profinite group with neutral element $e$. A $G$-space $S$ is understood to be a Hausdorff topological space $S$ together with a continuous operation of $G$ on $S$, i.e., a continuous map

$$G \times S \to S$$
$$(g, s) \mapsto gs$$

such that

$$es = s \quad \text{and} \quad g(hs) = (gh)s \quad \text{for all} \quad s \in S \quad \text{and} \quad g, h \in G.$$

Hence, for any $s \in S$, the *stabilizer group* $G_s := \{g \in G \mid gs = s\}$ is a closed subgroup of $G$ and its *orbit* $Gs := \{gs \mid g \in G\} \subseteq S$ is a closed, compact subset of $S$. The $G$-spaces form a category with respect to the continuous $G$-maps between $G$-spaces as morphisms, i.e., continuous maps $f: S \to T$ with $f(gs) = gf(s)$ for all $g \in G$ and $s \in S$.

**2.2.** A $G$-space $S$ is defined to be *essentially finite* if for any open subgroup $U \leqslant G$ of $G$ the cardinality $\varphi_U(S)$ of the set $S^U := \{s \in S \mid us = s$ for all $u \in U\}$ of $U$-invariant elements of $S$ is finite. A discrete and essentially finite $G$-space is called *almost finite.* In an almost finite $G$-space every element lies in a finite orbit.

Note that for any essentially finite $G$-space $S$ and any open subgroup $U$ of $G$ the number $\mu_U(S)$ of orbits $G \cdot s \subseteq S$ which are isomorphic to the finite $G$-space $G/U$ of cosets of $U$ in $G$ is also finite. One has

$$\varphi_U(S) = \sum_V{}' \mu_V(S) \cdot \varphi_U(G/V), \tag{2.2.1}$$

where the sum $\sum_V'$ is taken over all conjugacy classes of open subgroups $V \leqslant G$ of $G$. Expression (2.2.1) is summable since $\varphi_U(G/V)$ is non-zero if and only if $U$ is contained in a conjugate of $V$.

Moreover, if $S$ is essentially finite, then

$$S^0 := \{s \in S \mid (G : G_s) < \infty\}, \tag{2.2.2}$$

considered as a $G$-space with respect to the discrete topology, is almost finite and one has

$$\varphi_U(S) = \varphi_U(S^0) \tag{2.2.3}$$

for all open subgroups $U \leqslant G$. More precisely, if we define two essentially finite $G$-spaces $S$ and $T$ to be equivalent if $\varphi_U(S) = \varphi_U(T)$ for all open subgroups $U \leqslant G$, then one has

(2.2.4) LEMMA. (i) *Any essentially finite $G$-space $S$ is equivalent to the $G$-space $S^0$.*

(ii) *Two essentially finite $G$-spaces $S$ and $T$ are equivalent if and only if the associated almost finite $G$-spaces $S^0$ and $T^0$ are isomorphic.*

(iii) *An essentially finite $G$-space $S$ is almost finite if and only if $S^0 = S$ and the identity $S^0 \xrightarrow{\sim} S$ is a homeomorphism.*

In particular, two almost finite $G$-spaces are equivalent if and only if they are isomorphic.

**2.3.** Since the disjoint union and the cartesian product of two almost finite $G$-spaces are again almost finite, we can form the Burnside–Grothendieck ring $\hat{\Omega}(G)$ of the (virtual) isomorphism classes of almost finite $G$-spaces. Using Lemma (2.2.4), it is easily verified that $\hat{\Omega}(G)$ is also the Burnside–Grothendieck ring of (virtual) equivalence classes of essentially finite $G$-spaces and that two almost finite (essentially finite) $G$-spaces $S$ and $T$ represent the same element in $\hat{\Omega}(G)$ if and only if they are isomorphic (equivalent).

**2.4.** For any open subgroup $U \leqslant G$ the map $S \mapsto \varphi_U(S)$ induces a homomorphism $\hat{\Omega}(G) \to \mathbb{Z}$, also denoted by $\varphi_U$. As usual (cf. [Dr2]), $\varphi_U = \varphi_V$ if and only if $U$ and $V$ are conjugate in $G$ if and only if the $G$-spaces $G/U$ and $G/V$ are isomorphic. $\hat{\Omega}(G)$ carries a natural topology, the coarsest topology for which all the maps $\varphi_U$ from $\hat{\Omega}(G)$ into the discrete ring $\mathbb{Z}$ are continuous, or, equivalently, the topology $\hat{\Omega}(G)$ inherits from the product topology on the "ghost-ring of $G$," $gh(G) = gh(G, \mathbb{Z}) := \mathbb{Z}^{\underline{\mathscr{O}(G)}}$, through its embedding $\varphi := \prod_U \varphi_U$: $\hat{\Omega}(G) \hookrightarrow gh(G) = \mathbb{Z}^{\underline{\mathscr{O}(G)}} \subseteq \prod_U \mathbb{Z}$.

**2.5.** Note also that the (non-isomorphic) transitive, finite $G$-spaces form a "topological basis" of $\hat{\Omega}(G)$, i.e., for any element $x \in \hat{\Omega}(G)$ there exist unique coefficients $\mu_V(x) \in \mathbb{Z}$ such that

$$x = \sum_V{}' \mu_V(x) \cdot G/V. \tag{2.5.1}$$

Here the sum runs over all conjugacy classes of open subgroups $V$ of $G$ and is meant to represent the unique element in $\hat{\Omega}(G)$ such that for each neighbourhood $B$ of this element there exist a finite set $\mathfrak{B}$ of conjugacy classes with

$$\sum_{V \in \mathfrak{B}'}{}' \mu_V(x)\, G/V \in B \tag{2.5.2}$$

for all finite subsets $\mathfrak{B}'$ of conjugacy classes containing $\mathfrak{B}$.

Vice versa, for every coefficient system $(\mu_V)_V \in \prod'_V \mathbb{Z}$ there exists such a limit $x = \sum'_V \mu_V \cdot G/V$. Hence, there exists an additive homeomorphism

$$\mu : \hat{\Omega}(G) \xrightarrow{\sim} \prod_V{}' \mathbb{Z}. \tag{2.5.3}$$

(Here, "$\prod'$" denotes the product over all conjugacy classes of open subgroups $V$ of $G$.)

**2.6.** To describe the product structure on $\prod'_V \mathbb{Z}$ which makes this homeomorphism a ring isomorphism it is enough to recall (cf. [M]) that for any two open subgrous $U$, $V \leqslant G$ one has

$$G/V \times G/U \cong \dot{\bigcup_{VgU \subseteq G}} G/(V \cap gUg^{-1}). \tag{2.6.1}$$

Hence, the products of the elements $e_U := (\delta_{U,V})_V \in \prod'_V \mathbb{Z}$, representing the canonical topological basis of $\prod'_V \mathbb{Z}$, have to be defined accordingly as

$$e_V \cdot e_U = \sum_{VgU \subseteq G} e_{V \cap gUg^{-1}} \tag{2.6.2}$$

and this product has to be extended in the obvious way.

**2.7.** As in the finite case (see, for instance, [Dr3]) the image $\varphi(\hat{\Omega}(G))$ of $\hat{\Omega}(G)$ in the ghost-ring $gh(G) = \prod'_U \mathbb{Z}$ can be characterized through congruences. The main tool is

(2.7.0) BURNSIDE'S LEMMA. *If $H$ is a finite group and $M$ a finite $H$-set, then the following identity holds*:

$$\sum_{h \in H} \varphi_{\langle h \rangle}(M) = |H| \cdot \textit{number of } H\textit{-orbits of } M,$$

*where $\langle h \rangle$ denotes the cyclic subgroup of $H$ generated by $h$.*

The well-known proof of Burnside's lemma is based on the fact that the set $\Gamma = \{(h, m) \in H \times M \mid h \cdot m = m\}$ admits two different decompositions into disjoint subsets resulting from the projections of $H \times M$ onto its factors.

Now, for each pair $U \leqslant V \leqslant G$ of open subgroups $U$ and $V$ of $G$ such that $U$ is normal in $V$ and for any essentially finite $G$-set $S$, the finite set $S^U$ of $U$-invariant elements in $S$ is in a natural way a $V/U$-set, say $\pi_U^V(S)$. Obviously, $\pi_U^V$ induces a ring homomorphism $\hat{\Omega}(G) \to \Omega(V/U)$, also denoted by $\pi_U^V$.

By Burnside's lemma, the number of $V/U$-orbits in $\pi_U^V(S)$ coincides with $(1/(V\colon U)) \cdot \sum_{vU \in V/U} \#(\pi_U^V(S)^{vU})$, where $\pi_U^V(S)^{vU} = \{s \in \pi_U^V(S) = S^U \mid (vU)\, s = vs = s\}$. Note that $\#(\pi_U^V(S)^{vU})$ equals $\varphi_{\langle vU \rangle}(S)$, where $\langle vU \rangle$ is the subgroup of $G$ generated by the coset $vU$. Hence the above formula implies

$$\text{(2.7.1)} \qquad \sum_{vU \in V/U} \varphi_{\langle vU \rangle}(S) \equiv 0 \qquad (\mathrm{mod}(V\colon U)).$$

In other words, for any $x = (x(U))_U \in gh(G)$ in the image $\varphi(\hat{\Omega}(G))$ of $\hat{\Omega}(G)$ one has necessarily

$$\text{(2.7.2)} \qquad \sum_{vU \in V/U} x(\langle vU \rangle) \equiv 0 \qquad (\mathrm{mod}(V\colon U)).$$

It is easy to see (cf. [Dr3]) that, vice versa, any element $x \in gh(G)$, which satisfies all of these congruences (or only those congruences for which $V$ coincides with the normalizer $N_G(U)$ of $U$ in $G$ or only those for which $(V\colon U)$ is a prime power or only those for which $V/U$ is a Sylow-$p$-subgroup in $N_G(U)/U$ for some prime $p$), is necessarily also in the image of $\hat{\Omega}(G)$, i.e., one has the following

(2.7.3) THEOREM. *With the above notations and with* $x = (x(U))_U \in gh(G)$ *the following statements are equivalent*:

(i) *$x$ is in the image $\varphi(\hat{\Omega}(G))$ of $\hat{\Omega}(G)$ in $gh(G)$;*

(ii) *$x$ satisfies* (2.7.2) *for all pairs $U \trianglelefteq V \leqslant G$ of open subgroups $U$ and $V$ of $G$;*

(iii) *$x$ satisfies* (2.7.2) *for all pairs $U \trianglelefteq V \leqslant G$ of open subgroups $U$ and $V$ of $G$ for which the index $(V\colon U)$ is a power of some prime $p$;*

(iv) *$x$ satisfies* (2.7.2) *for all open subgroups $U \leqslant G$ with $V = N_G(U)$;*

(v) *$x$ satisfies* (2.7.2) *for all pairs $U \trianglelefteq V \leqslant G$ of open subgroups of $G$ with $V/U$ a Sylow-$p$-group of $N_G(U)/U$ for some prime $p$.*

In particular, $\hat{\Omega}(G)$ is mapped by $\varphi = \prod'_U \varphi_U$ homeomorphically onto a closed subring of the product ring $\prod_U \mathbb{Z}$. Hence, $\hat{\Omega}(G)$ is itself a complete topological ring.

**2.8.** Obviously, the Burnside ring $\Omega(G)$ of (virtual isomorphism classes of) finite $G$-spaces, studied in [Dr2, Appendix A], is a dense subring of $\hat{\Omega}(G)$. So $\hat{\Omega}(G)$ can also be viewed as the completion of $\Omega(G)$ with respect to the topology it induces on $\Omega(G)$.

**2.9.** Let us now show that $\hat{\Omega}(G)$—also as a topological ring—coincides with the projective limit $\varprojlim_N \Omega(G/N)$ considered in the Introduction. Here $N$ runs through all open normal subgroups of $G$ and the projective limit is defined relative to the maps $\pi^{G/M}_{N/M}\colon \Omega(G/M) \to \Omega((G/M)/(N/M)) = \Omega(G/N)$ which are defined for any two normal subgroups $N$, $M$ of $G$ with $M \subseteq N$. The map $\hat{\Omega}(G) \to \prod_N \Omega(G/N)$: $x \mapsto (\pi^G_N(x))_N$ induces a homomorphism from $\hat{\Omega}(G)$ into $\varprojlim_N \Omega(G/N)$, since for any essentially finite $G$-set $S$ and open normal subgroups $M \subseteq N$ of $G$ one has

$$\pi^G_N(S) = S^N = \pi^{G/M}_{N/M}(S^M) = \pi^{G/M}_{N/M}(\pi^G_N(S)). \tag{2.9.1}$$

Moreover, this homomorphism is injective since any $\varphi_U\colon \hat{\Omega}(G) \to \mathbb{Z}$ ($U$ an open subgroup of $G$) factors through $\pi^G_N\colon \hat{\Omega}(G) \to \Omega(G/N)$ whenever $N \subseteq U$. It is also surjective, e.g., by Theorem (2.7.3). One can also construct directly an inverse $\lambda\colon \varprojlim_N \Omega(G/N) \to \hat{\Omega}(G)$ in the following way. For an element $(x_N) \in \varprojlim_N \Omega(G/N) \subseteq \prod_N \Omega(G/N)$ write

$$x_N = \sum_{N \leqslant U \leqslant G}{}' a^N_U \cdot G/U, \tag{2.9.2}$$

where the sum runs through all conjugacy classes of subgroups $U$ between $N$ and $G$ and $a^N_U \in \mathbb{Z}$. Since the various $G/U$ are well known to form a $\mathbb{Z}$-basis of $\Omega(G/N)$, the coefficients $a^N_U$ are uniquely determined by $x_N$. Moreover, for $M \trianglelefteq N$ the relation $\pi^{G/M}_{N/M}(x_M) = x_N$ together with

$$\pi^{G/M}_{N/M}(G/U) = \begin{cases} G/U & \text{if } N \subseteq U \\ 0 & \text{if } N \nsubseteq U \end{cases} \tag{2.9.3}$$

implies $a^N_U = a^M_U$ whenever $M \subseteq N \subseteq U$, i.e., $a^N_U$ does not depend on $N$ (as long as $N \subseteq U$).

Hence we can define a unique $a_U \in \mathbb{Z}$ for every open subgroup $U \leqslant G$ such that

$$x_N = \sum_{N \leqslant U \leqslant G}{}' a_U \cdot G/U. \tag{2.9.4}$$

So $\lambda((x_N)) := \sum'_U a_U G/U \in \hat{\Omega}(G)$ is well defined and it is easily verified that indeed $\pi_N^G(\lambda((x_N))) = x_N$. So we have proved

$$\hat{\Omega}(G) = \varprojlim_N \Omega(G/N). \tag{2.9.5}$$

**2.10.** If $U$ is an open subgroup of $G$, the restriction of the $G$-action on an almost/essentially finite $G$-space $S$ to $U$ leads to an almost/essentially finite $U$-space, denoted by $S|_U$ or $\mathrm{res}_U^G(S)$. Since $\mathrm{res}_U^G(S \overset{\times}{\cup} T) = \mathrm{res}_U^G(S) \overset{\times}{\cup} \mathrm{res}_U^G(T)$, $\mathrm{res}_U^G$ defines a ring homomorphism $\hat{\Omega}(G) \to \hat{\Omega}(U)$, also denoted by $\mathrm{res}_U^G$.

Vice versa, if $S$ is an almost/essentially finite $U$-space, then the space $G \times_U S$, consisting of the $U$-orbits in $G \times S$ with respect to the $U$-action $U \times (G \times S) \to G \times S$: $(u, (g, s)) \mapsto (gu^{-1}, us)$, is an almost/essentially finite $G$-space with respect to the obvious $G$-action $G \times (G \times_U S) \to G \times_U S$: $(h, U \cdot (g, s)) \mapsto U \cdot (hg, s)$.

Obviously,

$$G \times_U (U/V) \cong G/V \tag{2.10.1}$$

and

$$G \times_U (S \cup T) = (G \times_U S) \cup (G \times_U T). \tag{2.10.2}$$

Hence we have a well-defined additive map $\mathrm{ind}_U^G: \hat{\Omega}(U) \to \hat{\Omega}(G)$ such that $\mathrm{ind}_U^G(S) = G \times_U S$. As usual, Frobenius reciprocity holds, i.e., for $x \in \hat{\Omega}(G)$ and $y \in \hat{\Omega}(U)$ one has

$$\mathrm{ind}_U^G(y) \cdot x = \mathrm{ind}_U^G(y \cdot \mathrm{res}_U^G(x)). \tag{2.10.3}$$

More generally, if $\sigma: U \hookrightarrow G$ is an embedding of the profinite group $U$ onto an open subgroup $\sigma(U)$ of $G$, one has well-defined induced additive maps $\mathrm{res}(\sigma) = \sigma_*: \hat{\Omega}(G) \to \hat{\Omega}(U)$ and $\mathrm{ind}(\sigma) = \sigma^*: \hat{\Omega}(U) \to \hat{\Omega}(G)$ such that

$$\sigma_*(x \cdot y) = \sigma_*(x) \cdot \sigma_*(y) \qquad (x, y \in \hat{\Omega}(G)) \tag{2.10.3$'$}$$

and

$$\sigma^*(y) \cdot x = \sigma^*(y \cdot \sigma_*(x)) \qquad (y \in \hat{\Omega}(U), x \in \hat{\Omega}(G)). \tag{2.10.3$''$}$$

In particular, if $U, V \leqslant G$ are open subgroups of $G$ and if $g \in G$ conjugates $U$ into $V$, we have corresponding restriction and induction maps $\mathrm{res}_U^V(g)$ and $\mathrm{ind}_U^V(g)$, induced from the embedding $U \hookrightarrow V$: $u \mapsto g^{-1}ug$. These maps are particularly useful to compute for any two open subgroups $U, V \leqslant G$

the composite map $\mathrm{res}_U^G \circ \mathrm{ind}_V^G \colon \hat{\Omega}(V) \to \hat{\Omega}(U)$ according to the Mackey subgroup theorem (cf. [M]) as follows:

$$\mathrm{res}_U^G \circ \mathrm{ind}_V^G = \sum_{UgV \subseteq G} \mathrm{ind}_{U \cap gVg^{-1}}^U \circ \mathrm{res}_{U \cap gVg^{-1}}^V(g). \tag{2.10.4}$$

Here one summand is taken into consideration for each double coset $UgV \subseteq G$, the resulting maps $\mathrm{ind}_{U\cap gVg^{-1}}^U \circ \mathrm{res}_{U\cap gVg^{-1}}^V(g)\colon \hat{\Omega}(V) \to \hat{\Omega}(U)$ being independent of the chosen representative $g \in UgV$.

Note that $\varphi_U = \varphi_U \circ \mathrm{res}_U^G$ and $\varphi_U \circ \mathrm{ind}_{U\cap gVg^{-1}}^U = 0$ unless $U = U \cap gVg^{-1}$ or—equivalently—$gV \in (G/V)^U$. Hence

$$\varphi_U \circ \mathrm{ind}_V^G = \sum_{gV \in (G/V)^U} \varphi_U \circ \mathrm{res}_U^V(g). \tag{2.10.4)$'$}$$

Equation (2.10.4) can also be combined with (2.10.3) to derive:

(2.10.4)″

If $U, V \leqslant G$ are two open subgroups of $G$, then one has

$$\mathrm{ind}_U^G(x) \cdot \mathrm{ind}_V^G(y) = \sum_{UgV} \mathrm{ind}_{U\cap gVg^{-1}}^G(\mathrm{res}_{U\cap gVg^{-1}}^U(x) \cdot \mathrm{res}_{U\cap gVg^{-1}}^V(g)(y))$$

for all $x \in \hat{\Omega}(U)$ and $y \in \hat{\Omega}(V)$.

Note that (2.10.1) implies the following formula for the image of the induction maps

(2.10.5)

$$\begin{aligned}\mathrm{Im}(\mathrm{ind}_U^G) &= \mathrm{ind}_U^G(\hat{\Omega}(U)) \\ &= \{x \in \hat{\Omega}(G) \mid \varphi_V(x) = 0 \text{ for all open subgroups } V \leqslant G \text{ with } V \not\lesssim U\},\end{aligned}$$

while for the kernel of the restriction maps one has

(2.10.6)

$$\mathrm{Ker}(\mathrm{res}_U^G) = \{x \in \hat{\Omega}(G) \mid \varphi_V(x) = 0 \text{ for all subgroups } V \leqslant G \text{ with } V \lesssim U\}.$$

Hence $\mathrm{Im}(\mathrm{ind}_U^G) \cap \mathrm{Ker}(\mathrm{res}_U^G) = 0$, while—as in the finite case—one has

$$n \cdot \hat{\Omega}(G) \subseteq \mathrm{Im}(\mathrm{ind}_U^G) \oplus \mathrm{Ker}(\mathrm{res}_U^G) \tag{2.10.7}$$

for some $n \in \mathbb{N}$, e.g., the index of $\bigcap_{g\in G} gUg^{-1}$ in $G$.

**2.11.** Next consider for any two $G$-spaces $S$, $T$ the set $T^S$ of all continuous maps from $S$ into $T$, made into a $G$-space by supplying it with the compact-open topology (cf. [Sch]) and the standard $G$-action, defined by

$$(2.11.1)\qquad (g\cdot f)(s) := g\cdot f(g^{-1}s) \qquad (g\in G, f\in T^S, s\in S).$$

Note that the set $(T^S)^G$ of $G$-invariant elements in $T^S$ coincides with the set $\mathrm{Hom}_G(S, T)$ of continuous $G$-maps from $S$ into $T$. Note also that for any $G$-spaces $S$, $S_1$, $S_2$, $T$, $T_1$, $T_2$ one has canonical isomorphisms

$$(2.11.2)\qquad (T_1\times T_2)^S \cong T_1^S \times T_2^S,$$

$$(2.11.3)\qquad T^{S_1 \cup S_2} \cong T^{S_1}\times T^{S_2}$$

and, for $S_1$ being locally compact,

$$(2.11.4)\qquad (T^{S_1})^{S_2} \cong T^{S_1\times S_2}.$$

We claim

(2.11.5) LEMMA. *If $T$ is finite and if the number $\#(G\backslash S)$ of $G$-orbits in $S$ is also finite, then $T^S$ is almost finite.*

*Proof.* Since $T$ is finite and $S$ is compact, the compact-open topology on $T^S$ is necessarily discrete. Hence it is enough to show that $T^S$ is essentially finite. To this end let us observe at first that for any closed subgroup $H$ of $G$ we have

$$(2.11.6)\qquad \varphi_G(T^{G/H}) = \#\ \mathrm{Hom}_G(G/H, T) = \#(T^H) = \varphi_H(T) \leqslant \#\, T < \infty.$$

Hence, to compute $\varphi_U(T^S)$ for some open subgroup $U$ of $G$, we may decompose the restriction $S|_U$ into $U$-orbits

$$(2.11.7)\qquad S|_U = Us_1 \cup Us_2 \cdots \cup Us_k$$

of which there are only finitely many in view of

$$(2.11.8)\qquad \#(U\backslash S) \leqslant (G\colon U)\cdot \#(G\backslash S) < \infty$$

and use (2.11.3) to derive

$$(2.11.9)\ \varphi_U(T^S) = \varphi_U((T|_U)^{S|_U}) = \varphi_U\left(\prod_{i=1}^{k} (T|_U)^{Us_i}\right) = \prod_{i=1}^{k} \varphi_U((T|_U)^{U/U_{s_i}})$$

$$= \prod_{i=1}^{k} \varphi_{U_{s_i}}(T) < \infty. \quad \blacksquare$$

**2.12.** The construction $T \mapsto T^S$ extends to the level of Grothendieck rings, i.e., we have

(2.12.1) THEOREM. *Let $S$ be a $G$-space with only finitely many orbits. Then there exists a well-defined map $\tau^S$ from the Burnside ring $\Omega(G)$ of* finite *$G$-spaces into the Burnside ring $\hat{\Omega}(G)$ of* almost finite *$G$-spaces,*

$$\tau^S: \Omega(G) \to \hat{\Omega}(G): x \mapsto x^S,$$

*such that for any open subgroup $U \leqslant G$ and any $x \in \Omega(G)$ the value $\varphi_U(x^S)$ coincides with the product $\prod_{i=1}^{k} \varphi_{U_i}(x)$, where $k = \#(U\backslash S)$ is the number of $U$-orbits in $S$ and $U_1, ..., U_k$ are closed subgroups of $U$ such that $S|_U \cong \dot{\bigcup}_{i=1}^{k} U/U_i$.*

*Remark* 1. (i) Note that for a finite $G$-space $T$ and hence for any element $x = T_1 - T_2 \in \Omega(G)$ and for any closed subgroup $H$ of $G$—whether of finite index or not—the number $\varphi_H(x) = \varphi_H(T_1) - \varphi_H(T_2) = \# T_1^H - \# T_2^H \in \mathbb{Z}$ is well defined (cf. [Dr2, Appendix A]).

(ii) Note also that for a finite $G$-space $T$, considered as an element in $\Omega(G)$, one has $\tau^S(T) = T^S$ in view of (2.11.9).

*Proof.* For $x \in \Omega(G)$ one has indeed a well-defined element $y = (y(U))_U \in gh(G) = \mathbb{Z}^{\underline{\mathcal{O}(G)}}$ with $y(U) = \prod_{i=1}^{k} \varphi_{U_i}(x)$ whenever $S|_U \cong U/U_1 \dot{\cup} \cdots \dot{\cup} U/U_k$. Moreover, for actual and not only virtual $G$-sets $x = T$ the element $y$ is contained in the image of $\hat{\Omega}(G)$ in $gh(G)$ by (2.11.9). Hence, if $x = \sum_{j=1}^{h} a_j G/V_j$ for some $a_j \in \mathbb{Z}$ and open subgroups $V_j \leqslant G$ $(j = 1, ..., h)$, then for any open subgroup $W \leqslant G$ one may put $N = N_G(W)$ and one gets

$$\sum_{nW \in N/W} y(\langle nW \rangle) \equiv 0 \pmod{(N: W)}, \tag{2.12.2}$$

for

$$y(\langle nW \rangle) := \prod_{i=1}^{\#(\langle nW \rangle \backslash S)} \left( \sum_{j=1}^{h} a_j \varphi_{\langle nW \rangle_i}(G/V_j) \right) \tag{2.12.3}$$

whenever $a_1, ..., a_h \in \mathbb{N}$, where as above (cf. (2.11.7)–(2.11.9))

$$S|_{\langle nW \rangle} \cong \dot{\bigcup_{i=1}^{\#(\langle nW \rangle \backslash S)}} \langle nW \rangle / \langle nW \rangle_i.$$

But $y(\langle nW \rangle) = y_{\langle nW \rangle}(a_1, ..., a_h)$ is an integral polynomial in $a_1, ..., a_h$ by (2.12.3), hence (2.12.2) must hold for all $a_1, ..., a_h \in \mathbb{Z}$.

It follows from Theorem (2.7.3) that $y$ is contained in the image of $\hat{\Omega}(G)$ in $gh(G)$, whether $x$ is an actual or a virtual finite $G$-set.

So we have indeed a unique element $\tau^S(x)$ or $x^S$ in $\hat{\Omega}(G)$ satisfying the conditions of the theorem. ∎

*Remark* 2. If one wants to avoid the usage of Theorem (2.7.3), one could also apply an appropriate version of the theory of algebraic maps between abelian (semi-)groups, developed in [Dr1, see also Section 5].

*Remark* 3. If $T^S$ were defined to contain not only the continuous maps from $S$ into $T$, but just all maps from $S$ into $T$, then it would still be essentially finite, but not necessarily almost finite, while the associated almost finite $G$-space would coincide with the subset of continuous maps from $S$ into $T$.

(2.12.4) THEOREM. *With the notations of Theorem* (2.12.1) *one has*

$$\tau^S(x \cdot y) = \tau^S(x) \cdot \tau^S(y)$$

*for* $x, y \in \Omega(G)$. *Similarly, if* $S = S_1 \cup S_2$ is *the disjoint union of the two* $G$-*subspaces* $S_1$ *and* $S_2$, *then*

$$\tau^S(x) = \tau^{S_1}(x) \cdot \tau^{S_2}(x)$$

*for all* $x \in \Omega(G)$.

*Moreover, if* $S$ *is finite, then* $\tau^S(x)$ *is already contained in* $\Omega(G) \subseteq \hat{\Omega}(G)$, *so for any further* $G$-*space* $S_1$ *wih* $\#(G\backslash S_1) < \infty$ *one may consider* $\tau^{S_1}(\tau^S(x))$ *and one has* $\tau^{S_1}(\tau^S(x)) = \tau^{S_1 \times S}(x)$.

*Proof.* This is trivial, if $x = T$ and $y = T'$ are actual $G$-sets, and can be carried over to virtual $G$-sets in a routine manner.

Note that in particular, if $x = \alpha \cdot 1_{\Omega(G)}$ for some $\alpha \in \mathbb{Z}$ represents a (virtual) finite $G$-space with trivial $G$-action, then

$$\tau^{G/1}(x) = \tau^{G/1}(\alpha \cdot 1_{\Omega(G)}) = \alpha \cdot 1_{\Omega(G)} + \sum_{U \subsetneqq G}' x_U \cdot G/U \tag{2.12.5}$$

since

$$\mu_G(\tau^{G/1}(x)) = \varphi_G(\tau^{G/1}(x)) = \varphi_1(\alpha \cdot 1_{\Omega(G)}) = \alpha. \quad ∎ \tag{2.12.6}$$

*Remark* 4. If $S = G/1$ the map $\tau^S$ is closely related to the Teichmüller map, well known in the context of Witt vectors.

Hence, an easy recursive argument can be used to deduce

(2.12.7) THEOREM. *For any* $x \in \hat{\Omega}(G)$ *there exists a unique map* $\alpha \in \mathbb{Z}^{\underline{\mathcal{O}(G)}}$ *such that* $x = \tau(\alpha) := \sum'_{U \leqslant G} \operatorname{ind}_U^G(\alpha(U)^{U/1})$, *i.e., the "extended Teichmüller map"* $\tau : \mathbb{Z}^{\underline{\mathcal{O}(G)}} \to \hat{\Omega}(G) : \alpha \mapsto \tau(\alpha)$ *is bijective.*

*Proof.* It is clear that $\tau(\alpha)=\sum'_{U\leqslant G}\mathrm{ind}_U^G(\alpha(U)^{U/1}$ represents a well-defined element in $\hat{\Omega}(G)$ since $\varphi_V(\mathrm{ind}_U^G(x))=0$ for all $x\in\hat{\Omega}(U)$ unless $U$ contains a conjugate of $V$. To show bijectivity, it is therefore enough to observe that for a given $x\in\hat{\Omega}(G)$ one can determine the corresponding $\alpha(U)=\alpha_x(U)\in\mathbb{Z}$ in a unique way recursively, starting with $\alpha_x(G):=\mu_G(x)=\varphi_G(x)$ and continuing with

$$\alpha_x(U):=\mu_U\left(x-\sum_{U\lneqq V\leqslant G}{}'\ \mathrm{ind}_V^G(\alpha_x(V)^{V/1})\right) \tag{2.12.8}$$

assuming that the $\alpha_x(V)$ have been determined already for $V\gneqq U$ so that

$$\mu_V\left(x-\sum_{V\leqslant W\leqslant G}{}'\ \mathrm{ind}_W^G(\alpha_x(W)^{W/1})\right)=0 \tag{2.12.9}$$

for all such $V$ or—equivalently—such that

$$\varphi_V\left(x-\sum_{V\leqslant W\leqslant G}{}'\ \mathrm{ind}_W^G(\alpha_x(W)^{W/1})\right)=0 \tag{2.12.10}$$

for all such $V$. ∎

Note that in this case one has necessarily

$$\begin{aligned}\alpha_U(x)&=\mu_U\left(x-\sum_{U\lneqq V\leqslant G}{}'\ \mathrm{ind}_V^G(\alpha_x(V)^{V/1})\right)\\&=\frac{1}{\varphi_U(G/U)}\cdot\varphi_U\left(x-\sum_{U\lneqq V\leqslant G}{}'\ \mathrm{ind}_V^G(\alpha_x(V)^{V/1})\right)\\&=\frac{1}{(N_G(U):U)}\cdot\varphi_U\left(x-\sum_{U\lneqq V\leqslant G}{}'\ \mathrm{ind}_V^G(\alpha_x(V)^{V/1})\right)\\&=\frac{1}{(N_G(U):U)}\left(\varphi_U(x)-\sum_{U\lneqq V\leqslant G}{}'\ \varphi_U(G/V)\,\alpha_x(V)^{(V:U)}\right),\end{aligned} \tag{2.12.11}$$

the last equality being a consequence of

(2.12.12) LEMMA. *For any two open subgroups $U, V\leqslant G$ and $\alpha\in\mathbb{Z}$ one has $\varphi_U(\mathrm{ind}_V^G(\alpha^{V/1}))=\varphi_U(G/V)\cdot\alpha^{(V:U)}$.*

*Proof.* In view of (2.10.4)′ and (2.11.9) one has

$$\varphi_U(\operatorname{ind}_V^G(\alpha^{V/1})) = \sum_{gV \in (G/V)^U} \varphi_U(\operatorname{res}_U^V(g)(\alpha^{V/1})) = \sum_{gV \in (G/V)^U} \varphi_U(\alpha^{\operatorname{res}_U^V(g)(V/1)})$$

$$= \sum_{gV \in (G/V)^U} \varphi_U(\alpha^{\overbrace{U/1 \cup \cdots \cup U/1}^{(V:U)\text{ times}}})$$

$$= \sum_{gV \in (G/V)^U} \varphi_U(\alpha^{U/1})^{(V:U)} = \varphi_U(G/V) \cdot \alpha^{(V:U)}. \quad \blacksquare$$

*Remark* 5. Putting $V = G$ and applying (2.12.12) for all open subgroups $W$ of some open subgroup $U$ of $G$ one gets as a corollary

$$\operatorname{res}_U^G(\alpha^{G/1}) = (\alpha^{(G:U)})^{U/1}. \tag{2.12.13}$$

*Remark* 6. Note that the Teichmüller map $\tau: \mathbb{Z}^{\underline{\mathcal{O}(G)}} \to \hat{\Omega}(G)$ is by no means a ring isomorphism if $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ is supplied with the standard ghost-ring structure, defined by adding and multiplying componentwise. We will study in Section 3 what happens, if $\tau$—or rather $\tau^{-1}$—is used to transort the ring structure from $\hat{\Omega}(G)$ to $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$.

But note that the ring structure to be defined on $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ is unique and fits into a commutative diagram of ring homomorphisms (cf. (1.3)):

$$\begin{tikzcd}
\mathbb{Z}^{\underline{\mathcal{O}(G)}} \arrow[r, "\tau"] \arrow[dr, "\prod \phi_U^{\mathbb{Z}}"'] & \hat{\Omega}(G) \arrow[d, "\varphi"] \\
 & \mathbb{Z}^{\underline{\mathcal{O}(G)}}.
\end{tikzcd}$$

**2.13.** Finally, let us study symmetric powers of almost finite $G$-spaces on the level of Burnside rings. For a $G$-space $X$ its symmetric "algebra" $\mathfrak{S}(X)$ is defined to be the set of all maps $f$ from $X$ into $\mathbb{N}_0$ such that $\operatorname{supp}(f) = \{x \in X \mid f(x) \neq 0\}$ is finite. $G$ acts on $\mathfrak{S}(X)$ by

$$G \times \mathfrak{S}(X) \to \mathfrak{S}(X): (g, f) \mapsto gf, \tag{2.13.1}$$

where, as usual,

$$(gf)(x) := f(g^{-1}x). \tag{2.13.2}$$

Note that

$$\mathfrak{S}(X_1 \cup X_2) \cong \mathfrak{S}(X_1) \times \mathfrak{S}(X_2) \tag{2.13.3}$$

for any two $G$-spaces $X_1, X_2$.

It is easy to see that for an almost finite $G$-space $X$, the $G$-orbits $G \cdot f \subseteq \mathfrak{S}(X)$ are always finite. Moreover, if $G$ contains only finitely many open subgroups $U \leqslant G$ of given index $k$ (and, hence, in any almost finite $G$-set $X$ there are only finitely many $G$-orbits $G \cdot t$ of cardinality $k$) for all $k \in \mathbb{N}_0$, then for every $n \in \mathbb{N}_0$ the subset

$$\mathfrak{S}^n(X) := \left\{ f \in \mathfrak{S}(X) \,\middle|\, \sum_{x \in X} f(x) = n \right\} \subseteq \mathfrak{S}(X), \tag{2.13.4}$$

the $n$th *symmetric power* of $X$, provided with the discrete topology, is an almost finite $G$-space. By restriction, it is enough to show that $\mathfrak{S}^n(X)^G$ is finite. But $f \in \mathfrak{S}(X)$ is $G$-invariant if and only if it is constant on the $G$-orbits of $X$. Hence $f \in \mathfrak{S}^n(X)^G$ can be non-zero only on those $G$-orbits $G \cdot x$ of $X$ with $\# Gx \leqslant n$, and by our assumption on $G$ together with the fact that $X$ is assumed to be almost finite, there are only finitely many such orbits in $X$. Since in addition $f(x) \leqslant n$ for $f \in \mathfrak{S}^n(X)$, the set $\mathfrak{S}^n(X)^G$ must be finite.

The same argument shows that for $X = G/U$ ($U$ some open subgroup in $G$) one has (cf. [S])

$$\varphi_G(\mathfrak{S}^n(G/U)) = \begin{cases} 1 & \text{if } (G:U) \mid n \\ 0 & \text{otherwise.} \end{cases} \tag{2.13.5}$$

It follows from (2.13.3) that for a disjoint union $X_1 \,\dot{\cup}\, X_2$ of two almost finite $G$-spaces $X_1$ and $X_2$ one has always

$$\mathfrak{S}^n(X_1 \,\dot{\cup}\, X_2) \cong \dot{\bigcup_{i+j=n}} \mathfrak{S}^i(X_1) \times \mathfrak{S}^j(X_2). \tag{2.13.6}$$

Note that there is another way of describing symmetric powers of $G$-spaces. To show this consider the canonical projection

$$p: X^n \to \mathfrak{S}^n(X)$$

of the $n$th cartesian power $X^n = X \times \cdots \times X$ of the $G$-space $X$ onto its $n$th symmetric power $\mathfrak{S}^n(X)$. It is defined by mapping an element $x = (x_1, \ldots, x_n)$ of $X^n$ to the map $p(x): X \to \mathbb{N}$ which maps any element $y$ in $X$ to the number of those indices $i \in \{1, \ldots, n\}$ for which $x_i = y$, i.e., $p(x)(y) = \# \{ i \in \{1, \ldots, n\} \mid x_i = y \}$. Obviously $p$ is a surjective $G$-map, invariant under the canonical action of the symmetric group $\Sigma_n$ on $X^n$. It therefore induces a $G$-map

$$\pi: X^n/\Sigma_n \to \mathfrak{S}^n(X), \tag{2.13.7}$$

which is well known to be a $G$-isomorphism.

**2.14.** Let us now define the formal power series

$$\mathfrak{S}_t(X) := \sum_{n=0}^{\infty} \mathfrak{S}^n(X) \cdot t^n, \tag{2.14.1}$$

considered as an element in the ring $\hat{\Omega}(G)[\![t]\!]$ of formal power series over $\hat{\Omega}(G)$. Note that $\mathfrak{S}_t(X)$ has constant term 1, i.e., it is contained in $\Lambda(\hat{\Omega}(G)) := \{1 + \sum_{n=1}^{\infty} x_n t^n \mid x_n \in \hat{\Omega}(G)\} = 1 + t\hat{\Omega}(G)[\![t]\!]$. Since

$$\mathfrak{S}_t(X_1 \cup X_2) = \mathfrak{S}_t(X_1) \cdot \mathfrak{S}_t(X_2) \tag{2.14.2}$$

it follows as usual that the mapping $\mathfrak{S}_t$ defines a homomorphism, also denoted by $\mathfrak{S}_t$, from the additive group $\hat{\Omega}(G)$ into the multiplicative group $\Lambda(\hat{\Omega}(G))$:

$$\mathfrak{S}_t \colon \hat{\Omega}(G) \to \Lambda(\hat{\Omega}(G)) \colon x \mapsto 1 + \sum \mathfrak{S}^n(x)\, t^n. \tag{2.14.3}$$

Hence, taking logarithmic derivatives, we get additive homomorphisms $\psi^r = \psi_r \circ \mathfrak{S}_t$: $\hat{\Omega}(G) \to \hat{\Omega}(G)$ $(r = 1, 2, \ldots)$—the so-called Adams operations (cf. [A])—such that

$$\psi_t(x) := t \cdot \frac{d}{dt} \log \mathfrak{S}_t(x) = \sum_{r=1}^{\infty} \psi^r(x)\, t^n. \tag{2.14.4}$$

Note that (2.13.5) implies

$$\varphi_G(\mathfrak{S}_t(G/U)) = 1 + t^{(G:U)} + t^{2(G:U)} + \cdots = \frac{1}{1 - t^{(G:U)}} \tag{2.14.5}$$

as well as

$$\varphi_G(\psi^r(G/U)) = \begin{cases} (G:U) & \text{if } (G:U) \text{ divides } r \\ 0 & \text{otherwise} \end{cases} \tag{2.14.6}$$

and

$$\varphi_G \psi_t(G/U) = \frac{(G:U)\, t^{(G:U)}}{1 - t^{(G:U)}}. \tag{2.14.7}$$

As has been shown in [S], the $\psi^n$ are not only additive, but also ring homomorphisms from $\hat{\Omega}(G)$ into itself if and only if $G$ is cyclic.

These constructions have many further interesting properties which are elaborated in another paper (cf. [DS2]).

## 3. The Witt-Burnside Construction

**3.1.** Let us now introduce the notation $\mathbb{W}_G(\mathbb{Z})$ for the set $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ (i.e., the same set which suports the ring structure of $gh(G) = gh(G, \mathbb{Z})$), made into a ring by carrying the ring structure of $\hat{\Omega}(G)$ back to $\mathbb{Z}^{\underline{\mathcal{O}(G)}}$ via $\tau^{-1}$, i.e., by defining

$$\alpha + \beta = \tau^{-1}(\tau(\alpha) + \tau(\beta)) \tag{3.1.1}$$

$$\alpha \cdot \beta = \tau^{-1}(\tau(\alpha) \cdot \tau(\beta)) \tag{3.1.1$'$}$$

for all $\alpha, \beta \in \mathbb{W}_G(\mathbb{Z}) = \mathbb{Z}^{\underline{\mathcal{O}(G)}}$, or, equivalently (cf. (2.12.12)), such that

$$\phi_U := \varphi_U \circ \tau : \mathbb{W}_G(\mathbb{Z}) \to \mathbb{Z} : \alpha \mapsto \sum_{U \leqslant V \leqslant G}{}' \varphi_U(G/U)\, \alpha(V)^{V/1} \tag{3.1.2}$$

is a ring homomorphism for all open subgroups $U$ of $G$.

**3.2.** We claim

(3.2.1) Theorem. *There exist integral polynomials*

$$s_U = s_U^G, \qquad p_U = p_U^G \in \mathbb{Z}[x_V, y_V \mid U \lesssim V \leqslant G]$$

*in two times as many variables $x_V, y_V$ $(U \leqslant V \leqslant G)$ as there are conjugacy classes of subgroups $V \leqslant G$ which contain a conjugate of $U$ such that*

$$\alpha + \beta = (s_U(\alpha(V), \beta(V)) \mid U \lesssim V \leqslant G)_{U \leqslant G}$$

*and*

$$\alpha \cdot \beta = (p_U(\alpha(V), \beta(V)) \mid U \lesssim V \leqslant G))_{U \leqslant G}$$

*for $\alpha, \beta \in W_G(\mathbb{Z})$. Similarly, there exist polynomials $m_U(x_V) \in \mathbb{Z}[x_V \mid U \lesssim V \leqslant G]$ such that $-\alpha = (m_U(\alpha_V \mid U \lesssim V \leqslant G))'_{U \leqslant G}$ for every $\alpha \in \mathbb{W}_G(\mathbb{Z})$.*

*Proof.* The proof is based on the three following basic lemmata (3.2.2), (3.2.5), and (3.2.11):

(3.2.2) Lemma. *For a subset $A$ of $G$ let $U_A := \{g \in G \mid Ag = A\}$ denote its stabilizer group and let $i_A := \#(A/U_A)$ denote the number of $U_A$-orbits in $A$. Then*

(i) *$A$ is simultaneously closed and open in $G$ if and only if $U_A$ is an open subgroup in $G$.*

(ii) *If the set $\mathfrak{A}(\mathfrak{G}) := \{A \subseteq G \mid U_A$ open in $G\}$ of simultaneously* *closed and open subsets of $G$ is considered as an (almost finite!) $G$-space via $G \times \mathfrak{A}(G) \to \mathfrak{A}(G)$: $(g, A) \mapsto Ag^{-1}$, then for any $\alpha, \beta \in \mathbb{Z}$ one has*

$$(\alpha+\beta)^{G/1} = \sum_{G \cdot A \in G \backslash \mathfrak{A}(G)} \operatorname{ind}_{U_A}^{G}((\alpha^{i_A} \cdot \beta^{i_{G-A}})^{U_A/1}).$$

*Proof.* Part (i) is well known. Part (ii) follows by showing that $\varphi_U$, applied to both sides, yields the same number $(\alpha+\beta)^{(G:U)}$ for all open subgroups $U \leqslant G$. Indeed, using (2.12.12), one has

$$\begin{aligned}
&\varphi_U\Bigg(\sum_{G \cdot A \in G \backslash \mathfrak{A}(G)} \operatorname{ind}_{U_A}^{G}((\alpha^{i_A} \cdot \beta^{i_{G-A}})^{U_A/1})\Bigg) \\
&= \sum_{\substack{G \cdot A \in G \backslash \mathfrak{A}(G) \\ U \lesssim U_A}} \varphi_U(G/U_A) \cdot (\alpha^{i_A} \cdot \beta^{i_{G-A}})^{(U_A:U)} \\
&= \sum_{A \in \mathfrak{A}(G),\, U \leqslant U_A} \alpha^{\#(A/U)} \cdot \beta^{\#(G-A)/U} \\
&= \sum_{A \subseteq G/U} \alpha^{\# A} \cdot \beta^{\#(G/U) - \# A} = (\alpha+\beta)^{\#(G/U)} \\
&= \varphi_U((\alpha+\beta)^{G/1}). \quad \blacksquare
\end{aligned} \tag{3.2.3}$$

(3.2.4) *Remark.* In case $\alpha, \beta \in \mathbb{N}$ one can also give a set-theoretic proof of (ii) by establishing for any $A \in \mathfrak{A}(G)$ an (almost canonical) isomorphism between $\{f \in (\boldsymbol{\alpha} \cup \boldsymbol{\beta})^{G/1} \mid f^{-1}(\boldsymbol{\alpha}) \in G \cdot A\}$ and $\operatorname{ind}_{U_A}^{G}((\boldsymbol{\alpha}^{i_A} \times \boldsymbol{\beta}^{i_{G-A}})^{U_A/1})$ (with $\boldsymbol{\alpha} := \{0, 1, ..., \alpha-1\}$ and $\boldsymbol{\beta} := \{0, 1, ..., \beta-1\}$, as usual). See Section 5 for details.

(3.2.5) LEMMA. *For some $k \in \mathbb{N}$ let $V_1, ..., V_k \leqslant G$ be a sequence of open subgroups of $G$. Then for every open subgroup $U \leqslant G$ and every sequence $\varepsilon_1, ..., \varepsilon_k \in \{\pm 1\}$ there exists a unique polynomial $\xi_U = \xi^G_{(U; V_1, ..., V_k; \varepsilon_1, ..., \varepsilon_k)} = \xi_U(x_1, ..., x_k) \in \mathbb{Z}[x_1, ..., x_k]$ such that for all $\alpha_1, ..., \alpha_k \in \mathbb{Z}$ one has $\tau^{-1}(\sum_{i=1}^k \varepsilon_i \cdot \operatorname{ind}_{V_i}^G(\alpha_i^{V_i/1}))(U) = \xi_U(\alpha_1, ..., \alpha_k)$.*

*Proof.* Without loss of generality one may assume $U \leqslant \bigcap_{i=1}^k V_i$. If $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_k = 1$ and if $V_i$ is not conjugate to $V_j$ for $i \neq j$, then $\sum_{i=1}^k \varepsilon_i \cdot \operatorname{ind}_{V_i}^G(\alpha_i^{V_i/1}) = \tau(\alpha)$ for $\alpha = (\alpha_V) \in \mathbb{Z}^{\underline{\mathcal{O}(G)}}$ with $\alpha_V = \alpha_i$ for $V \sim V_i$ and $\alpha_V = 0$ if $V$ is not conjugate to either of the $V_1, ..., V_k$. So in this case we are done: $\xi_U(\alpha_1, ..., \alpha_k)$ equals $\alpha_U$. Hence we may use triple induction, first with respect to $m_1 = m_1(U; V_1, ..., V_k;\ \varepsilon_1, ..., \varepsilon_k) := \max((V_i : U) \mid \varepsilon_i = -1$ or there exists some $j \neq i$ with $V_j \sim V_i)$, then with respect to $m_2 := \#\{i \mid (V_i : U) = m_1$ and $\varepsilon_i = -1\}$ and then with respect to $m_3 := \#\{i \mid (V_i : U = m_1$ and there exists some $j \neq i$ with $V_j \sim V_i\}$.

In case $m_1 = 0$ the induction hypothesis holds in view of the above remark. In case $m_1 > 0$ we have either $m_2 > 0$ or $m_3 > 0$. In case $m_2 > 0$, say $(V_1 : U) = m_1$ and $\varepsilon_1 = -1$, we may use (3.2.2)(ii) with $G = V_1$, $\alpha = -\alpha_1$, $\beta = +\alpha_1$ to conclude that

$$(3.2.6) \qquad 0 = \sum_{V_1 \cdot A \in V_1 \backslash \mathfrak{A}(V_1)} \mathrm{ind}_{U_A}^{V_1}(((-1)^{i_A} \cdot \alpha_1^{(V_1 : U_A)})^{U_A/1}).$$

Therefore, considering the two special summands $A = \emptyset$ and $A = V_1$ and putting $\mathfrak{A}_0(V_1) := \{A \in \mathfrak{A}(V_1) \mid A \neq \emptyset \text{ and } A \neq V_1\}$, one gets

$$(3.2.7) \qquad -(\alpha_1^{V_1/1}) = (-\alpha_1)^{V_1/1} + \sum_{V_1 \cdot A \in V_1 \backslash \mathfrak{A}_0(V_1)} \mathrm{ind}_{U_A}^{V_1}(((-1)^{i_A} \cdot \alpha_1^{(V_1 : U_A)})^{U_A/1}).$$

Hence, if $A_{k+1}, A_{k+2}, ..., A_{k'} \in \mathfrak{A}_0(V_1)$ denote representatives of the (finitely many!) $V_1$-orbits $V_1 \cdot A \subseteq \mathfrak{A}_0(V_1)$ with $U \leqslant V_{k+1} := U_{A_{k+1}}$, $V^{k+2} := U_{A_{k+2}}, ..., V_{k'} := U_{A_{k'}} \lneqq V_1$ and if we put $\varepsilon_{k+1} = \cdots = \varepsilon_{k'} := 1$ and $\alpha_{k+1} := (-1)^{i_{A_{k+1}}} \cdot \alpha_1^{(V_1 : V_{k+1})}, ..., \alpha_{k'} := (-1)^{i_{A_{k'}}} \cdot \alpha_1^{(V_1 : V_{k'})}$, then we have

$$(3.2.8) \qquad \begin{aligned} &\xi^G_{(U; V_1, ..., V_k; -1, \varepsilon_2, ..., \varepsilon_k)}(\alpha_1, \alpha_2, ..., \alpha_k) \\ &= \xi^G_{(U; V_1, ..., V_{k'}; 1, \varepsilon_2, ..., \varepsilon_{k'})}(-\alpha_1, \alpha_2, ..., \alpha_{k'}), \end{aligned}$$

so the result follows by induction.

Similarly, if $m_2 = 0$, but $m_3 > 0$, say $V_1 = V_2$, then we may use (3.2.3)(ii) once more with $G = V_1$, $\alpha = \alpha_1$, and $\beta = \alpha_2$ to conclude that

$$(3.2.9) \qquad \begin{aligned} (\alpha_1 + \alpha_2)^{V_1/1} &= \sum_{V_1 \cdot A \in V_1 \backslash \mathfrak{A}(V_1)} \mathrm{ind}_{U_A}^{V_1}((\alpha_1^{i_A} \cdot \alpha_2^{i_{V_1 - A}})^{U_A/1}) \\ &= \alpha_1^{V_1/1} + \alpha_2^{V_1/1} + \sum_{V_1 \cdot A \in V_1 \backslash \mathfrak{A}_0(V_1)} \mathrm{ind}_{U_A}^{V_1}((\alpha_1^{i_A} \cdot \alpha_2^{i_{V_1 - A}})^{U_A/1}), \end{aligned}$$

so with $V_{k+1}, ..., V_{k'}$ as above, but $\varepsilon_{k+1} = \cdots = \varepsilon_{k'} = -1$ and $\alpha_{k+1} := \alpha_1^{i_{A_{k+1}}} \alpha_2^{i_{V_1 - A_{k+1}}}, ..., \alpha_{k'} := = \alpha_1^{i_{A_{k'}}} \cdot \alpha_2^{i_{V_1 - A_{k'}}}$, we get

$$(3.2.10) \qquad \begin{aligned} &\xi^G_{(U; V_1, ..., V_k; \varepsilon_1, ..., \varepsilon_k)}(\alpha_1, ..., \alpha_k) \\ &= \xi^G_{(U; V_2, V_3, ..., V_{k'}; \varepsilon_2, ..., \varepsilon_{k'})}(\alpha_1 + \alpha_2, \alpha_3, ..., \alpha_{k'}), \end{aligned}$$

so again our result follows by induction, since $A \in \mathfrak{A}_0(V_1)$ implies $U_A \lneqq V_1$. ∎

(3.2.11) LEMMA. *For every two open subgroups $V, W \leqslant G$ and for every $\alpha, \beta \in \mathbb{Z}$ one has the modified Mackey formula*

$$\begin{aligned} &\mathrm{ind}_V^G(\alpha^{V/1}) \cdot \mathrm{ind}_W^G(\beta^{W/1}) \\ &= \sum_{VgW \subseteq G} \mathrm{ind}^G_{V \cap gWg^{-1}}((\alpha^{(V : V \cap gWg^{-1})} \cdot \beta^{(W : g^{-1}Vg \cap W)})^{(V \cap gWg^{-1})/1}). \end{aligned}$$

*Proof.* We may use either (2.10.4)″ or (2.12.13). Alternatively, we may show that $\varphi_U$, applied to both sides of the above equation, yields the same number for all open subgroups $U$ in $G$. Indeed, using the convention

$$(V:U) = (G:U)/(G:V) \tag{3.2.12}$$

for any two open subgroups $U, V \leqslant G$ (which is obviously consistent with the definition of $(V:U)$ in case $U \subseteq V$) we get

$$\begin{aligned}
&\varphi_U(\operatorname{ind}_V^G(\alpha^{V/1}) \cdot \operatorname{ind}_W^G(\beta^{W/1})) \\
&= \varphi_U(\operatorname{ind}_V^G(\alpha^{V/1})) \cdot \varphi_U(\operatorname{ind}_W^G(\beta^{W/1})) \\
&= \varphi_U(G/V) \cdot \alpha^{(V:U)} \cdot \varphi_U(G/W)\, \beta^{(W:U)} \\
&= \varphi_U(G/V \times G/W)\, \alpha^{(V:U)} \cdot \beta^{(W:U)} \\
&= \sum_{VgW \subseteq G} \varphi_U(G/(V \cap gWg^{-1}))(\alpha^{(V:V \cap gWg^{-1})} \\
&\qquad \cdot \beta^{(W:g^{-1}Vg \cap W)})^{(V \cap gWg^{-1}:U)} \\
&= \varphi_U\Bigg(\sum_{VgW \subseteq G} \operatorname{ind}_{V \cap gWg^{-1}}^G((\alpha^{(V:V \cap gWg^{-1})} \\
&\qquad \cdot \beta^{(W:g^{-1}Vg \cap W)})^{(V \cap gWg^{-1}/1)})\Bigg). \quad \blacksquare
\end{aligned} \tag{3.2.13}$$

We are now ready to return to the proof of (3.2.1). So let $G = V_1, V_2, ..., V_k = U$ be a system of representatives of subgroups of $G$ containing a conjugate of $U$. Obviously

$$\begin{aligned}
&s_U^G(x_{V_1}, y_{V_1}, ..., x_{V_k}, y_{V_k}) \\
&= \xi^G_{(U;V_1,V_1,...,V_k,V_k;1,...,1)}(x_{V_1}, y_{V_1}, ..., x_{V_k}, y_{V_k})
\end{aligned} \tag{3.2.14}$$

and

$$m_U^G(x_{V_1}, ..., x_{V_k}) = \xi^G_{(U;V_1,...,V_k;-1,...,-1)}(x_{V_1}, ..., x_{V_k}). \tag{3.2.15}$$

So $s_U = s_U^G$ and $m_U = m_U^G$ are indeed integral polynomials by (3.2.5).

To compute $p_U = p_U^G$ we use (3.2.11). First we choose a system $s_1, s_2, ..., s_h$ of representatives of the $G$-orbits in

$$S := \dot{\bigcup_{i,j=1}^{k}} G/V_i \times G/V_j. \tag{3.2.16}$$

Next we put $W_r := G_{s_r}$ and

$$p_r(x_{V_1}, y_{V_1}, ..., x_{V_k}, y_{V_k}) := x_i^{(V_i:W_r)} \cdot y_j^{(V_j:W_r)} \tag{3.2.17}$$

in case $s_r = (g_r V_i, g'_r V_j) \in G/V_i \times G/V_j \subseteq S$. Using these conventions and (3.2.11), we get the equation

$$(3.2.18) \quad p_U^G(x_{V_1}, y_{V_1}, ..., x_{V_k}, y_{V_k}) \\ = \xi^G_{(U;W_1,...,W_h;1,...,1)}(p_1(x_{V_1}, ..., y_{V_k}), ..., p_h(x_{V_1}, ..., y_{V_k})).$$

So $p_U$ is also in $\mathbb{Z}[x_V, y_V \mid U \lneq V \leqslant G]$. ▮

**3.3.** Theorem (3.2.1) has the amazing consequence that we can define a commutative ring structure $\mathbb{W}_G(A)$ on $\prod'_{U \leqslant G} A$ for any commutative ring $A$ by defining

$$(3.3.1) \qquad (\alpha_U)'_{U \leqslant G} \overset{+}{\cdot} (\beta_U)'_{U \leqslant G} = \begin{cases} (s_U(\alpha_V, \beta_V \mid U \lneq V \leqslant G))'_{U \leqslant G} \\ (p_U(\alpha_V, \beta_V \mid U \lneq V \leqslant G))'_{U \leqslant G} \end{cases}$$

for all $\alpha = (\alpha_U)$, $\beta = (\beta_U) \in \prod'_{U \leqslant G} A$.

More precisely, as stated in the Introduction, we can reformulate Theorem (3.2.1) in the following form, suggested by P. Cartier (cf. [C]), to get our main theorem:

(3.3.2) THEOREM. *For any profinite group $G$ there exists one and only one functor $\mathbb{W}_G$ from the category of commutative rings into itself such that the following two conditions are satisfied*:

(i) $\mathbb{W}_G(A) = A^{\underline{\mathcal{O}(G)}}$ *and* $W_G(h)\,\alpha = h \circ \alpha$ *for every commutative ring $A$, every ring homomorphism $h: A \to B$, and every $\alpha \in \mathbb{W}_G(A)$*;

(ii) *for every open subgroup $V \leqslant G$ the map*

$$\phi_V^A : A^{\underline{\mathcal{O}(G)}} \to A : (\alpha_U)'_{U \leqslant G} \mapsto \sum_{V \lesssim U \leqslant G} \varphi_V(G/U)\, \alpha_U^{(U:V)}$$

*is a ring homomorphism from $\mathbb{W}_G(A)$ into $A$.*

Obviously, the Witt–Burnside functor $\mathbb{W}_G$ satisfies in addition

$$(3.3.3) \qquad \mathbb{W}_G(\mathbb{Z}) \cong \hat{\Omega}(G),$$

$$(3.3.4) \qquad \mathbb{W}_{\hat{C}}(A) = \mathbb{W}(A),$$

and

$$(3.3.5) \qquad \mathbb{W}_{\hat{C}_p}(A) = \mathbb{W}_p(A),$$

where $\mathbb{W}(A)$ and $\mathbb{W}_p(A)$ denote the universal and the classical ring of Witt vectors with coefficients in $A$ (cf. [W2, L, C]), respectively, while $\hat{C}_p :=$

$\varprojlim_n C/C^{p^n}$ denotes the pro-$p$-completion of $C$, i.e., the free pro-$p$-group with one generator.

Using (3.2.5) and (2.12.13) one can also construct natural transformations

$$\mathbb{W}\,\mathrm{res}_U^G\colon \mathbb{W}_G \to \mathbb{W}_U \tag{3.3.6}$$

and

$$\mathbb{W}\,\mathrm{ind}_U^G\colon \mathbb{W}_U \to \mathbb{W}_G \tag{3.3.7}$$

for any open subgroup $U$ of $G$.

Indeed, for $\alpha \in \mathbb{W}_G(\mathbb{Z})$ one has

$$\begin{aligned}(3.3.8)\quad \mathrm{res}_U^G(\tau(\alpha)) &= \sum_V{}' \mathrm{res}_U^G \cdot \mathrm{ind}_V^G(\alpha(V)^{V/1})\\ &= \sum_V{}' \sum_{UgV \subseteq G} \mathrm{ind}_{U\cap gVg^{-1}}^U \cdot \mathrm{res}_{U\cap gVg^{-1}}^V(g)(\alpha(V)^{V/1})\\ &= \sum_V{}' \sum_{UgV \leqslant G} \mathrm{ind}_{U\cap gVg^{-1}}^U((\alpha(V)^{V:U\cap gVg^{-1}})^{U\cap gVg^{-1}/1}).\end{aligned}$$

Hence, for any open subgroup $W$ of $U$ one may enumerate (with repetitions, if necessary) all subgroups of the form $U\cap gVg^{-1}$ to which $W$ is subconjugate in $U$, where $V$ runs through a system of representatives of conjugacy classes of open subgroups of $G$ and $g$ runs through a system of representations of the double cosets $UgV \subseteq G$. Assume we get $k = k_W$ such subgroups, say $W_1 = U\cap g_1V_1g_1^{-1}$, $W_2 = U\cap g_2V_2g_2^{-1}, \ldots,$ $W_k = U\cap g_kV_kg_k^{-1}$. Then one gets from (3.2.5) for any $a \in \mathbb{W}_G(\mathbb{Z})$ the formula

$$\begin{aligned}(3.3.9)\quad &\tau^{-1}(\mathrm{res}_U^G(\tau(\alpha)))(W)\\ &\quad = \xi^U_{(W;W_1,\ldots,W_k;1,\ldots,1)}(\alpha(V_1)^{(V_1:W_1)}, \ldots, \alpha(V_k)^{(V_k:W_k)}).\end{aligned}$$

Similarly, one shows that $\tau^{-1}(\mathrm{ind}_U^G(\tau(\alpha)))(W)$ is a polynomial with integral coefficients in those $\alpha(V)$ ($V$ an open subgroup of $U$) to which $W$ is subconjugate in $G$. Hence one has indeed well-defined natural transformations $f_U\colon \mathbb{W}_G(-) \to \mathbb{W}_U(-)$ and $v_U\colon \mathbb{W}_U(-) \to \mathbb{W}_G(-)$, satisfying all relations which are known to hold generally for restriction and induction.

Finally, if $N$ is a closed normal subgroup of $G$ and if $\bar{G} := G/N$ denotes the factor group of $G$ modulo $N$, then the obvious canonical isomorphisms

$$(3.3.10)\quad \mathrm{ind}_U^G(\alpha^{U/1})^N \cong \begin{cases}\mathrm{ind}_{\bar{U}}^{\bar{G}}(\alpha^{\bar{U}/1}) & \text{if } N \subseteq U \text{ and } \bar{U} := U/N\\ 0 & \text{otherwise}\end{cases}$$

($U$ an open subgroup of $G$) show that the map $\pi_N^G\colon \hat{\Omega}(G)\to\hat{\Omega}(\bar{G})\colon X\mapsto X^N$ induces a natural surjective transformation

(3.3.11) $$\mathbb{W}\pi_N^G\colon \mathbb{W}_G\to\mathbb{W}_{\bar{G}}.$$

Note that the right inverse $\rho\colon\hat{\Omega}(\bar{G})\to\hat{\Omega}(G)$ to $\pi_N^G$, defined by considering an almost finite $\bar{G}$-set as an almost finite $G$-set in the obvious way, does not carry over in any natural way to the level of Witt-vectors. This follows, for instance, from the results proved in the next section, stating that $p^{n+1}\cdot\mathbb{W}_{\bar{G}}(\mathbb{F}_p)=0$ but $p^{n+1}\cdot\mathbb{W}_G(\mathbb{F}_p)\neq 0$ if $\bar{G}$ has a finite $p$-Sylow-subgroup of order $p^n$ and $N$ has a non-trivial $p$-Sylow subgroup.

## 4. The Proof of Theorem 2

**4.1.** In this section we shall use the canonical isomorphism

$$\tau\colon W_G(\mathbb{Z})\to\hat{\Omega}(G)$$

and the commutative diagram

$$\begin{tikzcd} & \hat{\Omega}(G) \arrow[dd, "\varphi"] \\ W_G(\mathbb{Z}) \arrow[ur, "\tau"] \arrow[dr, "\Pi\phi_U^{\mathbb{Z}}"'] & \\ & gh(G) \end{tikzcd}$$

to derive ring-theoretic properties of the ring of generalized Witt vectors from properties of the group $G$. This will lead to well-known results in the classical cases $G=\hat{C}$ and $G=\hat{C}_p$.

Recall (cf. 2.12 Remark) that by definition of the map

$$\phi^{\mathbb{Z}}\colon W_G(\mathbb{Z})\to gh(G)$$

one has

(4.1.1) $$\phi_U^{\mathbb{Z}}(\alpha)=\sum_{U\lesssim V\leqslant G}{}' \varphi_U(G/V)\,\alpha(V)^{(V:U)}.$$

Since the group $\mathrm{Aut}(G/V)$ is acting freely on the set of $G$-morphisms from $G/U$ to $G/V$ and the number of elements of this set equals $\varphi_U(G/V)$, it follows that $\varphi_V(G/V)$ divides $\varphi_U(G/V)$ for every $U\lesssim V$. Therefore, if some integer $k$ divides the number $\varphi_U(G/U)\cdot\alpha(U)$ for every open subgroup $U$ of

$G$ then $k$ divides the number $\phi_U^{\mathbb{Z}}(\alpha)$ for every such $U$. We shall prove that the converse is also true.

(4.2) THEOREM. *Let $\alpha = (\alpha(U))_{U \leqslant G}$ be an element of $W_G(\mathbb{Z})$. Then*

$$\phi_U^{\mathbb{Z}}(\alpha) \equiv 0 \qquad (\operatorname{mod} k) \text{ for all open subgroups } U \text{ in } G$$

*if and only if*

$$\varphi_U(G/U) \cdot \alpha(U) \equiv 0 \qquad (\operatorname{mod} k) \text{ for all open subgroups } U \text{ in } G.$$

*Proof.* We only have to show one implication of the theorem. So let $\phi_U^{\mathbb{Z}}(\alpha) \equiv 0 \pmod k$ for all open subgroups $U$ of $G$. Since by (4.1.1) we have

$$\phi_G^{\mathbb{Z}}(\alpha) = \varphi_G(G/G) \cdot \alpha(G),$$

the assertion is true for the group $G$ and we can use induction with respect to the index $(G: U)$ of the subgroup $U$ of $G$ to prove the theorem. By (4.1.1) one has

$$\varphi_U(G/U) \cdot \alpha(U) = \phi_U^{\mathbb{Z}}(\alpha) - \sum_{\substack{U \lesssim V \\ U \not\sim V}}{}' \varphi_U(G/V) \cdot \alpha(V)^{(V:U)},$$

i.e., $\Phi_U^{\mathbb{Z}}(\alpha) \equiv \varphi_U(G/U) \cdot \alpha(U)$ modulo $(\text{l.c.d.}(\varphi_V(G/V) \cdot \alpha(V)^2 \mid U \precnsim V \leqslant G))$.

Now, since the group $\operatorname{Aut}(G/V)$ is acting (by composition) freely on the set of $G$-mappings from $G/U$ to $G/V$, the number $\varphi_U(G/V)$ is a multiple of $\#\operatorname{Aut}(G/V) = \varphi_V(G/V)$. Therefore by induction one knows that the right-hand side is congruent to 0 modulo $k$. This proves our theorem. ∎

One has the following immediate consequence:

(4.2.1) COROLLARY 1. *Let $|G|_p$ denote the order of a $p$-Sylow group of the group $G$, whereby $|G|_p$ may be finite or infinite.*

(a) *If $|G|_p$ is finite and if $\alpha = p \cdot |G|_p \cdot 1_{W_G(\mathbb{Z})}$ then*

$$\alpha(U) \equiv 0 \qquad (\operatorname{mod} p)$$

*for all $U$. If $\beta = |G|_p \cdot 1_{W_G(\mathbb{Z})}$, then*

$$\beta(V) \not\equiv 0 \qquad (\operatorname{mod} p)$$

*for some open subgroup $V \leqslant G$.*

(b) *If $|G|_p$ is infinite and if*

$$\gamma = p^n \cdot 1_{W_G(\mathbb{Z})}, \qquad n \in \mathbb{N}$$

*then*

$$\gamma(U) \not\equiv 0$$

*for some open subgroup* $U \leqslant G$.

*Proof.* (a) Since $\phi^{\mathbb{Z}}(\alpha) = (p|G|_p, p|G|_p, ...)$ one has

$$\phi^{\mathbb{Z}}_U(\alpha) \equiv 0 \qquad (\mathrm{mod}\ p|G|_p)$$

for all $U$. By 4.2 one has $\varphi_U(G/U) \cdot \alpha(U) \equiv 0$ (mod $p|G|_p$) for all $U$ and since the $p$-part $(\varphi_U(G/U))_p$ of $\varphi_U(G/U) = (N_G(U) \colon U)$ is at most $|G|_p$ one has $\alpha(U) \equiv 0$ mod $p$, proving the first part of (a).

To prove the second part one has to show that $\beta(U) \not\equiv 0$ (mod $p$) for some open subgroup $U$ of $G$. Again $|G|_p$ divides $\varphi_U(G/U) \cdot \beta(U)$ for all open subgroups $U$ of $G$. Choose a maximal open subgroup $V$ of $G$ for which

$$(\varphi_V(G/V))_p = |G|_p.$$

Consider

$$\Phi^{\mathbb{Z}}_V(\beta) = |G|_p = \sum_{V \lneqq U \leqslant G} \varphi_V(G/U) \cdot (\beta(U))^{(U:V)}.$$

If $U$ is not conjugate to $V$ then it follows from $|G|_p \mid \varphi_U(G/U)\, \beta(U)$ that $p$ divides $\beta(U)$. Therefore

$$p \cdot |G|_p \text{ divides } \varphi_V(G/U) \cdot (\beta(U))^{(U:V)}.$$

Consequently one has

$$|G|_p \equiv \varphi_V(G/V) \cdot \beta(V) \qquad (\mathrm{mod}(p|G|_p)).$$

This implies that $\beta(V) \not\equiv 0$ (mod $p$) and therefore proves the second part of (a).

(b) This is proved similarly as the second part of (a). ∎

Corollary 1 implies

(4.2.2) Corollary 2. *If the order* $|G|_p$ *of a p-Sylow subgroup of the group G is finite then*

$$p \cdot |G|_p \cdot \mathbb{W}_G(\mathbb{F}_p) = 0,$$

*but*

$$|G|_p \cdot \mathbb{W}_G(\mathbb{F}_p) \neq 0.$$

*In particular one has*

$$|G|_p \cdot 1_{\mathbb{W}_G(\mathbb{F}_p)} \neq 0.$$

*If the order* $|G|_p$ *of a p-Sylow subgroup of the group G is infinite, then*

$$p^n \cdot \mathbb{W}_G(\mathbb{F}_p) \neq 0$$

*for all* $n \in \mathbb{N}$ *and one has an injective ring homomorphism*

$$\hat{\mathbb{Z}}_p \hookrightarrow \mathbb{W}_G(\mathbb{F}_p).$$

*Proof.* If $h: \mathbb{Z} \to \mathbb{F}_p: x \mapsto \bar{x}$ denotes the canonical surjection of $\mathbb{Z}$ onto $\mathbb{F}_p$, then $W_G(h)$: $W_G(\mathbb{Z}) \to W_G(\mathbb{F}_p)$ is given by $\alpha \mapsto h \circ \alpha$. Hence $\alpha_n := p^n \cdot 1_{W_G(\mathbb{Z})}$ is in the kernel of $W_G(h)$, i.e., one has $p^n \cdot 1_{W_G(\mathbb{F}_p)} = 0$, if and only if $\alpha_n(U) \equiv 0$ $(p)$ for all open subgroups $U$ of $G$. In particular, if $G$ is infinite, then the maps $\mathbb{Z} \to W_{G/N}(\mathbb{F}_p)$, where $N$ runs through all open normal subgroups of $G$, together define an injective ring homomorphism $\hat{\mathbb{Z}}_p \hookrightarrow W_G(\mathbb{F}_p) = \varprojlim W_{G/N}(\mathbb{F}_p)$. ∎

## 5. The Burnside Ring as a Mackey Functor

**5.1.** Some of the functorial properties of the construction $G \to \hat{\Omega}(G)$ become more transparent if this construction is "relativized," this way giving rise to a "Mackey functor," defined on the category of $G$-spaces (cf. [Dr2] for the finite case). To do this, one observes at first that for any $G$-map $\sigma: S \to T$ between two $G$-spaces $S$ and $T$ the fibers $\sigma^{-1}(t) \subseteq S$ are $G_t$-spaces for every $t \in T$. Hence one may define such a $G$-map $\sigma: S \to T$ to be (almost/essentially) finite if $\sigma^{-1}(t)$ is (almost/essentially) finite as a $G_t$-set for every $t \in T$. In other words, $\sigma$ is essentially finite if and only if for every $t \in T$ and for every open subgroup $U$ of $G_t$ the number

$$(5.1.1) \qquad \varphi_{(t,U)}(\sigma) := \#(\sigma^{-1}(t)^U)$$

is finite and $\sigma$ is almost finite if in addition all of its fibers are discrete.

The following statements are more or less immediate consequences of these definitions and their proof will be left to the reader:

(5.1.2) Lemma. (i) *A G-space S is (almost/essentially) finite if and only if the trivial G-map*

$$\sigma_S: S \to * := G/G$$

*from $S$ onto the trivial "one-point $G$-space" $* = G/G$ is (almost/essentially) finite.*

(ii) *The composition $\sigma \circ \rho : R \to^{\rho} S \to^{\sigma} T$ of two (almost/essentially) finite $G$-maps $\rho : R \to S$ and $\sigma : S \to T$ is also (almost/essentially) finite.*

(iii) *For two (almost/essentially) finite $G$-maps $\sigma_1 : S_1 \to T$ and $\sigma_2 : S_2 \to T$ the disjoint union $\sigma_1 \cup \sigma_2 : S_1 \cup S_2 \to T$ and the fiber product $\sigma_1 \times_T \sigma_2 : S_1 \times_T S_2 \to T$ are also (almost/essentially) finite and for $t \in T$ and $U$ an open subgroup of $G_t$ one has*

$$\varphi_{(t,U)}(\sigma_1 \cup \sigma_2) = \varphi_{(t,U)}(\sigma_1) + \varphi_{(t,U)}(\sigma_2)$$

*and*

$$\varphi_{(t,U)}(\sigma_1 \times_T \sigma_2) = \varphi_{(t,U)}(\sigma_1) \cdot \varphi_{(t,U)}(\sigma_2).$$

(iv) *For any two $G$-maps $\sigma : S \to T$ and $\xi : G/U \to T$ one has a canonical bijection*

$$\{\tau : G/U \to S \mid \tau \text{ a } G\text{-map with } \xi = \sigma \circ \tau\} \xrightarrow{\sim} \sigma^{-1}(\xi(U))^U : \tau \mapsto \tau(U).$$

*In particular, if $\xi$ is finite (i.e., if $U$ is an open subgroup of $G_{\xi(U)}$), then $\varphi_{(\xi(U),U)}(\sigma)$ coincides with the number of commutative triangles*

$$\begin{tikzcd} G/U \arrow[rr, "\tau"] \arrow[dr, "\xi"'] & & S \arrow[dl, "\sigma"] \\ & T & \end{tikzcd}$$

*and therefore with $\varphi_{(\xi(gU), gUg^{-1})}(\sigma)$ for any $g \in G$.*

*Remark.* Hence for any two $G$-spaces over $T$, i.e., for any two $G$-maps $\xi : X \to T$ and $\sigma : S \to T$, we may define the set

(5.1.3) $$\operatorname{Hom}(\xi, \sigma) := \{\tau : X \to S \mid \tau \text{ a } G\text{-map with } \xi = \sigma \circ \tau\}$$

and for any finite simple $G$-map $\xi : X \to T$ (i.e., for any finite $G$-map $\xi : X \to T$ from a transitive $G$-space $X \cong G/H$ into $T$) and for any essentially finite $G$-map $\sigma : S \to T$ we may define

(5.1.4) $$\varphi_\xi(\sigma) := \# \operatorname{Hom}(\xi, \sigma)$$

to reformulate (5.1.2)(iv) into

(5.1.5) $$\varphi_\xi = \varphi_{(\xi(x), G_x)}$$

for any finite, simple map $\xi : X \to T$ and any $x \in X$.

**5.2.** We may now define two essentially finite $G$-maps $\sigma_1: S_1 \to T_1$ and $\sigma_2: S_2 \to T_1$ to be equivalent if $T_1 = T_2$ and if for all $t \in T_1 = T_2$ the fibers $\sigma_1^{-1}(t)$ and $\sigma_2^{-1}(t)$ are equivalent as essentially finite $G_t$-spaces, i.e., if $\varphi_\rho(\sigma_1) = \varphi_\rho(\sigma_2)$ for all simple, finite $G$-maps $\rho: X \to T$ (cf. 2.2), and we may define the Burnside ring $\Omega(T)$ (or $\hat{\Omega}(T)$) of equivalence classes of (essentially) finite "$G$-spaces over $T$," i.e., of (essentially) finite $G$-maps $\sigma: S \to T$, as the associated Grothendieck ring, consisting of all formal differences $[\sigma_1] - [\sigma_2]$ of equivalence classes $[\sigma_1]$ and $[\sigma_2]$ of (essentially) finite $G$-spaces $\sigma_i: S_i \to T$ $(i = 1, 2)$ over $T$. As usual, one has $[\sigma_1] - [\sigma_2] = [\sigma_1'] - [\sigma_2']$ for two additional (essentially) finite $G$-spaces $\sigma_i': S_i' \to T$ over $T$ if and only if $\sigma_1 \cup \sigma_2'$ is equivalent to $\sigma_1' \cup \sigma_2$, and one has $([\sigma_1] - [\sigma_2]) + ([\sigma_1'] - [\sigma_2']) = ([\sigma_1 \cup \sigma_1'] - [\sigma_2 \cup \sigma_2'])$ and $([\sigma_1] - [\sigma_2]) \cdot ([\sigma_1'] - [\sigma_2]) = [(\sigma_1 \times_T \sigma_1') \cup (\sigma_2 \times_T \sigma_2')] - [(\sigma_1 \times_T \sigma_2') \cup (\sigma_1' \times_T \sigma_2)]$.

In other words, $\Omega(T)$ and $\hat{\Omega}(T)$ are (canonically isomorphic to) the subrings of the product ring

$$(5.2.1) \qquad gh(T) := \prod_{\substack{\xi: X \to T \\ \xi \text{ finite and simple}}} \mathbb{Z}$$

(one factor for each isomorphism class of finite, simple $G$-spaces over $T$), generated by the elements

$$(5.2.2) \qquad \Phi(\sigma) := (\varphi_\xi(\sigma))_\xi \in gh(T),$$

where $\sigma$ runs over all (essentially) finite $G$-maps $\sigma: S \to T$. Note that as in Section 2, $\hat{\Omega}(T)$ is a closed subring of $gh(T)$ which can be characterized by congruences, that $\Omega(T)$ is dense in $\hat{\Omega}(T)$, and that the (isomorphism classes of) simple, finite $G$-spaces over $T$ form a (topological) $\mathbb{Z}$-basis of $\Omega(T)$ ($\hat{\Omega}(T)$, respectively).

Note also that the congruences characterizing $\hat{\Omega}(T)$ as a closed subring of $gh(T)$ can be derived most easily and directly by considering for any simple finite $G$-map $\xi: X \to T$ and for any essentially finite $G$-map $\sigma: S \to T$ the finite set $\operatorname{Hom}(\xi, \sigma)$ as a finite $\operatorname{Aut}(\xi) = \operatorname{Hom}(\xi, \xi)$-space, noting that $\operatorname{Aut}(\xi) \cong N_{G_{\xi(t)}}(G_x)/G_x$ for any $x \in X$.

More precisely (cf. 2.9), the various maps

$$(5.2.3) \qquad h_\xi: \Omega(T) \to \Omega(\operatorname{Aut}(\xi)): \sigma \mapsto \operatorname{Hom}(\xi, \sigma)$$

combine into an isomorphism

$$(5.2.4) \qquad h: \hat{\Omega}(T) \to \varprojlim \Omega(\operatorname{Aut}(\xi)),$$

where the projective limit is taken over the category of all finite, simple $G$-spaces $\xi: X \to T$ over $T$ for which $\mathrm{Aut}(\xi)$ acts transitively on the fibers of $\xi$ (i.e., those of "Galois type" for which $G_x$ is a normal subgroup of $G_{\xi(x)}$ for one or—equivalently— for all $x \in X$) with respect to the $G$-maps $\tau \in \mathrm{Hom}(\xi, \xi')$ over $T$, noting that any such $G$-map $\tau$ induces an epimorphism $\tilde{\tau}: \mathrm{Aut}(\xi) \twoheadrightarrow \mathrm{Aut}(\xi')$, defined by associating to each $\rho \in \mathrm{Aut}(\xi)$ the unique $\rho' \in \mathrm{Aut}(\xi')$ for which the diagram

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\tau"'] & X \arrow[d, "\tau"] \\ X' \arrow[r, "\rho'"] & X' \end{tikzcd}$$

commutes, and hence it induces a surjective ring homomorphism

$$\Omega(\tilde{\tau}) = \pi^{\mathrm{Aut}(\xi)}_{\mathrm{Ker}(\tilde{\tau})}: \Omega(\mathrm{Aut}(\xi)) \twoheadrightarrow \Omega(\mathrm{Aut}(\xi')): S \mapsto S^{\mathrm{Ker}(\tilde{\tau})}.$$

**5.3.** If every $G$-orbit $G \cdot t \subseteq T$ is an open subset of $T$, then two almost finite $G$-spaces $\sigma_i: S_i \to T$ $(i = 1, 2)$ over $T$ are isomorphic over $T$ (i.e., there exists some $G$-isomorphism $\rho: S_1 \xrightarrow{\sim} S_2$ with $\sigma_1 = \sigma_2 \circ \rho$) if and only if they are equivalent. Hence, in this case we can identify $\Omega(T)$ (or $\hat{\Omega}(T)$) with the Grothendieck ring of isomorphism classes of (almost) finite $G$-spaces over $T$ (cf. 2.3).

In particular, if $T = G/H$ for some subgroup $H$ of $G$ we have canonical isomorphisms

(5.3.1) $$\omega_H: \Omega(G/H) \xrightarrow{\sim} \Omega(H)$$

and

(5.3.2) $$\hat{\omega}_H: \hat{\Omega}(G/H) \xrightarrow{\sim} \hat{\Omega}(H)$$

defined by associating to each (almost/essentially) finite $G$-space $\sigma: S \to G/H$ over $G/H$ the $H$-space which one gets as the fiber $\sigma^{-1}(H \cdot 1)$ of the particular coset $H \cdot 1 \in G/H$.

Note that the diagram

(5.3.3) $$\begin{tikzcd} \Omega(G/H) \arrow[r, hook] \arrow[d, "\omega_H"] & \hat{\Omega}(G/H) \arrow[d, "\hat{\omega}_H"] \\ \Omega(H) \arrow[r, hook] & \hat{\Omega}(H) \end{tikzcd}$$

commutes and that moreover for any simple finite $G$-map $\rho: X \to G/H$ and any $x \in X$ with $\rho(x) = H \cdot 1 \in G/H$ the diagram

(5.3.4)
$$\begin{tikzcd}
\Omega(G/H) \arrow[rr, hook] \arrow[dd, "\omega_H"'] \arrow[dr, "\varphi_\rho"] & & \hat{\Omega}(G/H) \arrow[dd, "\hat{\omega}_H"] \arrow[dl, "\varphi_\rho"'] \\
& \mathbb{Z} & \\
\Omega(H) \arrow[rr, hook] \arrow[ur, "\varphi_{G_x}"'] & & \hat{\Omega}(H) \arrow[ul, "\varphi_{G_x}"]
\end{tikzcd}$$

commutes, too.

**5.4.** Any $G$-map $\sigma: S \to T$ induces a restriction map

(5.4.1) $$\Omega_*(\sigma): \Omega(T) \to \Omega(S)$$

defined by associating to each finite $G$-space $\rho: R \to T$ over $T$ the finite pull-back $\rho \times_T S$: $R \times_T S \to S$. If $\sigma$ has discrete fibers, the same construction yields as well a restriction map

(5.4.2) $$\hat{\Omega}_*(\sigma): \hat{\Omega}(T) \to \hat{\Omega}(S).$$

Vice versa, if $\sigma: S \to T$ is (essentially) finite, it induces an "induction map"

(5.4.3) $$\Omega^*(\sigma): \Omega(S) \to \Omega(T)$$

(or

(5.4.4) $$\hat{\Omega}^*(\sigma): \Omega(S) \to \Omega(T)),$$

defined by associating to each (essentially) finite $G$-space $\rho: R \to S$ over $S$ the (essentially) finite $G$-space $\sigma \circ \rho$: $R \to T$ over $T$.

The following results follow immediately or by standard arguments from these definitions:

(5.4.5) PROPOSITION. (i) *In case* $\sigma = \sigma_{GH}: G/H \to G/G$ *we have commutative diagrams*

$$\begin{tikzcd} \Omega(G/G) \arrow[r, "\sim", "\omega_G"'] \arrow[d, "\Omega_*(\sigma)"] & \Omega(G) \arrow[d, "\mathrm{res}_H^G"] \\ \Omega(G/H) \arrow[r, "\sim", "\omega_H"'] & \Omega(H) \end{tikzcd}$$

$$\begin{tikzcd} \hat{\Omega}(G/G) \arrow[r, "\sim", "\hat{\omega}_G"'] \arrow[d, "\Omega_*(\sigma)"] & \hat{\Omega}(G) \arrow[d, "\mathrm{res}_H^G"] \\ \hat{\Omega}(G/H) \arrow[r, "\sim", "\hat{\omega}_H"'] & \hat{\Omega}(H) \end{tikzcd}$$

$$\begin{tikzcd} \Omega(G/G) \arrow[r, "\sim", "\omega_G"'] & \Omega(G) \\ \Omega(G/H) \arrow[r, "\sim", "\omega_H"'] \arrow[u, "\Omega^*(\sigma)"] & \Omega(H) \arrow[u, "\mathrm{ind}_H^G"'] \end{tikzcd}$$

*and*

$$\begin{tikzcd} \hat{\Omega}(G/G) \arrow[r, "\sim", "\hat{\omega}_G"'] & \hat{\Omega}(G) \\ \hat{\Omega}(G/H) \arrow[r, "\sim", "\hat{\omega}_H"'] \arrow[u, "\Omega^*(\sigma)"] & \hat{\Omega}(H) \arrow[u, "\mathrm{ind}_H^G"'] \end{tikzcd}$$

*whenever the maps in these diagrams are well defined.*

(ii) *$\Omega_*$ and $\hat{\Omega}_*$ are contravariantly functorial on the category of G-spaces with respect to arbitrary G-maps or G-maps with discrete fibers, respectively. $\Omega^*$ and $\hat{\Omega}^*$ are covariantly functorial on the category of G-spaces with respect to (essentially) finite G-maps.*

(iii) *If $\sigma: S \to T$ is an (almost) finite G-map, then we have*

$$\Omega^*(\sigma)(x) \cdot y = \Omega^*(\sigma)(x \cdot \Omega_*(\sigma)(y))$$

*(or*

$$\hat{\Omega}^*(\sigma)(x) \cdot y = \hat{\Omega}^*(\sigma)(x \cdot \hat{\Omega}_*(\sigma)(y)))$$

*for all $x \in \Omega(S)$ and $y \in \Omega(T)$ (or $x \in \hat{\Omega}(S)$ and $y \in \hat{\Omega}(T)$).*

(iv) *If $T = T_1 \cup T_2$ is a partition of $T$ into two open and G-invariant subsets, then the embeddings $\sigma_i: T_i \to T$ $(i = 1, 2)$ define isomorphisms*

$$\Omega_*(\sigma_1) \times \Omega_*(\sigma_2): \Omega(T) \simeq \Omega(T_1) \times \Omega(T_2),$$

$$\Omega^*(\sigma_1) \oplus \Omega^*(\sigma_2): \Omega(T_1) \oplus \Omega(T_2) \simeq \Omega(T),$$

$$\hat{\Omega}_*(\sigma_1) \times \hat{\Omega}_*(\sigma_2): \hat{\Omega}(T) \simeq \hat{\Omega}(T_1) \times \hat{\Omega}(T_2),$$

*and*

$$\hat{\Omega}^*(\sigma_1) \oplus \hat{\Omega}^*(\sigma_2) : \hat{\Omega}(T_1) \oplus \hat{\Omega}(T_2) \xrightarrow{\sim} \hat{\Omega}(T).$$

(v) *Let*

$$\begin{tikzcd} S \arrow[r, "\sigma_2'"] \arrow[d, "\sigma_1'"'] \arrow[dr, "\sigma"] & S_1 \arrow[d, "\sigma_1"] \\ S_2 \arrow[r, "\sigma_2"'] & T \end{tikzcd}$$

*be a pull-back diagram. If $\sigma_2$ is finite, it induces a commutative diagram*

$$\begin{tikzcd} \Omega(S) \arrow[r, "\Omega^*(\sigma_2')"] & \Omega(S_1) \\ \Omega(S_2) \arrow[u, "\Omega_*(\sigma_1')"] \arrow[r, "\Omega^*(\sigma_2)"'] & \Omega(T). \arrow[u, "\Omega_*(\sigma_1)"'] \end{tikzcd}$$

*If $\sigma_2$ is essentially finite and if $\sigma_1$ has discrete fibers* (*in particular, if $\sigma_1$ and $\sigma_2$ are almost finite*), *then it induces a commutative diagram*

$$\begin{tikzcd} \hat{\Omega}(S) \arrow[r, "\hat{\Omega}^*(\sigma_2')"] \arrow[d, "\hat{\Omega}_*(\sigma_1')"] & \hat{\Omega}(S_1) \arrow[d, "\hat{\Omega}_*(\sigma_1)"] \\ \hat{\Omega}(S_2) \arrow[r, "\hat{\Omega}^*(\sigma_2)"'] & \hat{\Omega}(T). \end{tikzcd}$$

*In particular, for $x_i \in \hat{\Omega}(S_i)$ $(i = 1, 2)$ and if $\sigma_1$ and $\sigma_2$ are almost finite, one has*

$$\hat{\Omega}^*(\sigma_1)(x_1) \cdot \hat{\Omega}^*(\sigma_2)(x_2) = \hat{\Omega}^*(\sigma)(\hat{\Omega}_*(\sigma_2')(x_1) \cdot \hat{\Omega}_*(\sigma_1')(x_2)).$$

(vi) *For any finite, simple G-space $\xi: X \to S$ over $S$ and any G-map $\sigma: S \to T$ with discrete fibers one has*

$$\varphi_\xi \circ \Omega_*(\sigma) = \varphi_{\sigma \circ \xi} : \hat{\Omega}(T) \to \mathbb{Z}$$

*and for any finite, simple G-space $\xi: X \to T$ over $T$ and any essentially finite G-map $\sigma: S \to T$ one has* (cf. (2.10.4)′)

$$\varphi_\xi \circ \hat{\Omega}^*(\sigma) = \sum_{\tau \in \mathrm{Hom}(\xi, \sigma)} \varphi_\tau : \hat{\Omega}(S) \to \mathbb{Z}$$

*as well as* (cf. (5.2.3))

$$h_\xi \circ \hat{\Omega}^*(\sigma) = \sum_{\tau \in \mathrm{Hom}(\xi, \sigma)} h_\tau : \hat{\Omega}(S) \to \Omega(\mathrm{Aut}(\xi)).$$

*Remark.* If $U, V \leqslant G$ are open subgroups of $G$, then (i), (iv), and (v), applied with respect to the diagram

$$\begin{array}{ccc} \dot{\bigcup}\limits_{UgV \subseteq G} G/U \cap gVg^{-1} \cong G/U \times G/V & \longrightarrow & G/V \\ \downarrow & & \downarrow \\ G/U & \longrightarrow & G/G, \end{array}$$

give the Mackey subgroup formula stated in (2.10.4) and its corollary (2.10.4)″. It is essentially this rather transparent and convenient reformulation of (2.10.4) contained in (i), (iv), and (v) which makes it so attractive to work with the construction "$T \mapsto \hat{\Omega}(T)$" rather than with "$U \mapsto \hat{\Omega}(U)$."

**5.5.** Finally, it is worthwhile to observe that for any $G$-map $\sigma: S \to T$ which induces a finite map $G\backslash\sigma: G\backslash S \to G\backslash T$ from the set $G\backslash S$ of $G$-orbits in $S$ into the set $G\backslash T$ of $G$-orbits in $T$ (or, equivalently, with $\#(G_t\backslash\sigma^{-1}(t)) < \infty$ for all $t \in T$) one has a well-defined map

(5.5.1) $$\hat{\Omega}^{\otimes}(\sigma): \hat{\Omega}(S) \to \hat{\Omega}(T)$$

which is constructed as follows: for any $t \in T$ and for any almost finite $G$-space $\rho: R \to S$ consider the set $\Gamma_t(\rho) = \Gamma_t^{\sigma}(\rho)$ of continuous sections $\gamma: \sigma^{-1}(t) \to \rho^{-1}(\sigma^{-1}(t)) \subseteq R$, i.e., of continuous maps $\gamma: \sigma^{-1}(t) \to R$ with $\rho \circ \gamma = \mathrm{Id}_{\sigma^{-1}(t)}$—if $\sigma^{-1}(t) = \varnothing$, put $\Gamma_t(\rho) := \{t\}$. For $g \in G$ and $\gamma \in \Gamma_t(\rho)$ the map

(5.5.2) $$g\gamma: \sigma^{-1}(gt) \to R: s \mapsto g(\gamma(g^{-1}s))$$

is obviously in $\Gamma_{gt}(\rho)$ and one has $1 \cdot \gamma = \gamma$ as well as $g_1(g_2\gamma) = (g_1 g_2)\gamma$. Hence the set

(5.5.3) $$\Gamma_{\sigma}(\rho) := \dot{\bigcup_{t \in T}} \Gamma_t(\rho)$$

is a $G$-set and so it can be topologized (as any $G$-set) to become a $G$-space in such a way that any $G$-orbit becomes an open subset. Moreover, there is a natural $G$-map

(5.5.4) $$\sigma^{\otimes}(\rho): \Gamma_{\sigma}(\rho) \to T$$

which maps $\Gamma_t(\rho)$ onto $t$. It is easy to check that $\Gamma_t(\rho)$ is an almost finite $G_t$-space, hence $\sigma^{\otimes}(\rho)$ represents an element

(5.5.5) $$\hat{\Omega}^{\otimes}(\sigma)(\rho) := [\sigma^{\otimes}(\rho)] \in \hat{\Omega}(T).$$

Note that for $\sigma = \sigma_{G/1}\colon G/1 \to *$ and $\rho = \rho_\alpha\colon \{0, 1, ..., \alpha - 1\} \times G/1 \to G/1$ one has

(5.5.6) $$\alpha^{G/1} = \hat{\Omega}^{\otimes}(\sigma_{G/1})(\rho_\alpha).$$

More generally, if $T$ is a finite $G$-set and if $\#(G\backslash S)$ is finite, too, then

(5.5.6)′ $$\hat{\Omega}^{\otimes}(\sigma_S)(\mathrm{pr}\colon T \times S \to S) \cong (\sigma_{T^S}\colon T^S \to *).$$

Note also that analogously to (5.4.5)(v) one has

(5.5.7) LEMMA. *If*

$$\begin{tikzcd} S \arrow[r, "\sigma_2'"] \arrow[d, "\sigma_1'"] & S_1 \arrow[d, "\sigma_1"] \\ S_2 \arrow[r, "\sigma_2"'] & T \end{tikzcd}$$

*is a pull-back diagram, if $G\backslash\sigma_2\colon G\backslash S_2 \to G\backslash T$ is finite and if $\sigma_1$ has discrete fibers, then $G\backslash\sigma_2'\colon G\backslash S \to G\backslash S_1$ is also finite and $\sigma_1'\colon S \to S_2$ has discrete fibers and one has*

$$\hat{\Omega}_*(\sigma_1) \circ \hat{\Omega}^{\otimes}(\sigma_2) = \hat{\Omega}^{\otimes}(\sigma_2') \circ \hat{\Omega}_*(\sigma_1').$$

We want to show next that the construction $\rho \mapsto \hat{\Omega}^{\otimes}(\sigma)(\rho)$ can be extended in a canonical way to a map from $\hat{\Omega}(S)$ into $\hat{\Omega}(T)$, i.e., there is a canonical way to define $\hat{\Omega}^{\otimes}(\sigma)([\rho_1] - [\rho_2]) \in \hat{\Omega}(T)$ for any two almost finite $G$-spaces $\rho_1\colon R_1 \to S$ and $\rho_2\colon R_2 \to S$ over $S$.

**5.6.** To this end we may either proceed as in (2.12) or we may use a more conceptual approach based on the elementary theory of algebraic maps (cf. [Dr1]). Since the second alternative seems to give some additional valuable insight into the structure of the $\Gamma$-construction, we will follow it in this section.

To define algebraic maps, let us first define for any map $\eta\colon A \to B$ from an (additive) abelian semigroup $A$ into an (additive) abelian group $B$ and any sequence $a_1, a_2, ..., a_n \in A$ the map

(5.6.1)

$$D_{(a_1,...,a_n)}\eta\colon A \to B\colon x \mapsto \sum_{k=0}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{n-k}\, \eta(x + a_{i_1} + \cdots + a_{i_k}).$$

Obviously, $D_{(a_1,\dots,a_n)}(\eta)$ does not depend on the order of $a_1, \dots, a_n$, i.e., if $\pi\colon \{1, \dots, n\} \simeq \{1, \dots, n\}$ is a permutation of $\{1, \dots, n\}$, then one has

$$D_{(a_{\pi(1)},\dots,a_{\pi(n)})}\eta = D_{(a_1,\dots,a_n)}\eta. \tag{5.6.2}$$

Other obvious identities are

$$D_{(a_1,\dots,a_k)}(D_{(a_{k+1},\dots,a_n)}\eta) = D_{(a_1,\dots,a_n)}\eta \tag{5.6.3}$$

whenever $k \leqslant n$, and

$$D_{a+b}\eta = D_a\eta + D_b\eta + D_{(a,b)}\eta \tag{5.6.4}$$

for $a, b \in A$ (with $D_{a_1}\eta := D_{(a_1)}\eta$).

Moreover, Möbius inversion leads to the "Taylor expansion formula"

$$\eta(x + a_1 + \cdots + a_n) = \sum_{k=0}^{n} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} (D_{(a_{i_1},\dots,a_{i_k})}\eta)(x). \tag{5.6.5}$$

One can use (5.6.5) to compute $D_{(a_1,\dots,a_n)}(\eta_2 \circ \eta_1)$ for the composition of two maps $\eta_1\colon A_1 \to A_2$ and $\eta_2\colon A_2 \to A_3$ between abelian groups $A_1$, $A_2$, and $A_3$.

To this end let us introduce the following convention: for a map $\vartheta\colon I \to A$ from a finite index set $I$ into some abelian group $A$ and a subset $J$ of $I$ let us write $\vartheta(J)$ for $\sum_{j \in J} \vartheta(j)$ and $D_{(\vartheta,J)}$ for the operator $D_{(\vartheta(j_1),\dots,\vartheta(j_k))}$ if $J = \{j_1, \dots, j_k\}$ is of cardinality $k$ (which is well-defined in view of (5.6.2)). Then with

$$\vartheta\colon I := \{0, \dots, n\} \to A_1 : i \mapsto a_i,$$

$$\vartheta_x\colon \mathfrak{P}(I) := \{J \mid J \subseteq I\} \to A_2 : J \mapsto (D_{(\vartheta,J)}\eta_1)(x) \qquad \text{for any} \quad x \in A_1$$

$$|\mathfrak{K}| := \bigcup_{K \in \mathfrak{K}} K \qquad \text{for} \quad \mathfrak{K} \subseteq \mathfrak{P}(I)$$

we have the relation

$$\begin{aligned} D_{(a_0,\dots,a_n)}&(\eta_2 \circ \eta_1)(x) \\ &= \sum_{J \subseteq I} (-1)^{\#(I-J)} (\eta_2 \circ \eta_1)(x + \vartheta(J)) \\ &= \sum_{J \subseteq I} (-1)^{\#(I-J)} \eta_2\Big(\sum_{K \subseteq J} (D_{(\vartheta,K)}\eta_1)(x)\Big) \\ &= \sum_{J \subseteq I} (-1)^{\#(I-J)} \eta_2(\vartheta_x(\mathfrak{P}(J))) \end{aligned} \tag{5.6.6}$$

$$= \sum_{J \subseteq I} \sum_{\mathfrak{R} \subseteq \mathfrak{P}(J)} (-1)^{\#(I-J)} (D_{(\vartheta_x, \mathfrak{R})} \eta_2)(0)$$

$$= \sum_{\mathfrak{R} \subseteq \mathfrak{P}(I)} \left( \sum_{\substack{J \\ |\mathfrak{R}| \subseteq J \subseteq I}} (-1)^{\#(I-J)} \right) (D_{(\vartheta_x, \mathfrak{R})} \eta_2)(0)$$

$$= \sum_{\substack{\mathfrak{R} \subseteq \mathfrak{P}(I) \\ |\mathfrak{R}| = I}} (D_{(\vartheta_x, \mathfrak{R})} \eta_2)(0).$$

We define $\eta: A \to B$ to be algebraic if for some $n \geqslant 0$ and all sequences $a_1, ..., a_n \in A$ of length $n$ one has $D_{(a_1, ..., a_n)} \eta \equiv 0$, in which case $D_{(a_1, ..., a_m)} \eta \equiv 0$ for all $m \geqslant n$ and $a_1, ..., a_m \in A$ by (5.6.3). If $\eta$ is algebraic, we define the degree $\deg \eta$ of $\eta$ to be $-1$, if $\eta \equiv 0$, and we define

(5.6.7) $\deg \eta := \max(n \mid \text{there exist } a_1, ..., a_n \in A \text{ with } D_{(a_1, ..., a_n)} \eta \not\equiv 0)$

if $\eta \not\equiv 0$.

Hence

— $\eta$ is algebraic of degree $\leqslant 0$ if and only if it is constant;

— $\eta$ is algebraic of degree $\leqslant 1$ if and only if it is "affine," i.e., a sum of a constant map and a homomorhism;

— $\eta$ is algebraic of degree $\leqslant n$ if and only if for some (or all) $k \leqslant n$ the maps $D_{(a_1, ..., a_k)} \eta$ are algebraic of degree $\leqslant n-k$ for all $a_1, ..., a_k \in A$;

— $\eta: \mathbb{Z} \to \mathbb{Z}$ is algebraic of degree $\leqslant n$ if and only if there exist $r_0, r_1, ..., r_n \in \mathbb{Z}$ with $\eta(a) = r_0 + r_1\binom{a}{1} + \cdots + r_n\binom{a}{n}$ for all $a \in \mathbb{Z}$.

Note also that an algebraic map $\eta: A \to B$ of degree $\leqslant n$ defines a multi-additive map $\eta^n: A^n \to B$ by associating to each $n$-tuple $(a_1, ..., a_n) \in A^n$ the constant value of the map $D_{(a_1, ..., a_n)} \eta$,

(5.6.8) $$\eta^n(a_1, ..., a_n) := D_{(a_1, ..., a_n)} \eta,$$

since

(5.6.9) $$\begin{aligned} &\eta^n(a+b, a_2, ..., a_n) \\ &= D_{a+b}(D_{(a_2, ..., a_n)} \eta) \\ &= D_a(D_{(a_2, ..., a_n)} \eta) + D_b(D_{(a_2, ..., a_n)} \eta) + D_{(a,b)}(D_{(a_2, ..., a_n)} \eta) \\ &= \eta^n(a, a_2, ..., a_n) + \eta^n(b, a_2, ..., a_n) + 0. \end{aligned}$$

As a corollary we get:

(5.6.10) LEMMA. *If $\eta: A \to B$ is algebraic of degree $\leqslant n$, then $\eta' = \eta_{(r,n)}$: $A \to B: a \mapsto \eta(r \cdot a) - r^n \cdot \eta(a)$ is algebraic of degree $\leqslant n-1$ for any $r \in \mathbb{N}$ (or* *even for any $r \in \mathbb{Z}$ if $A$ happens to be an abelian group rather than just a semigroup*).

*Proof.* For all $x, a_1, ..., a_n \in A$ we have

$$\begin{aligned}(D_{(a_1,...,a_n)}\eta')(x) &= (D_{(ra_1,...,ra_n)}\eta)(rx) - r^n(D_{(a_1,...,a_n)}\eta)(x)\\ &= \eta^n(ra_1, ..., ra_n) - r^n\eta^n(a_1, ..., a_n) = 0. \quad \blacksquare\end{aligned}$$

*Remark.* Lemma (5.6.10) suggests to define $R$-algebraic maps $\eta: M_1 \to M_2$ of degree $\leqslant n$ between $R$-modules $M_1$ and $M_2$ for a given commutative ring $R$ recursively as (additively or $\mathbb{Z}$-) algebraic maps of degree $\leqslant n$ for which the maps $D_m\eta: M_1 \to M_2$ $(m \in M_1)$ and $\eta_{(r,n)}: M_1 \to M_2$ $(r \in R)$ are $R$-algebraic of degree $\leqslant n-1$. Then one can develop the theory of $R$-algebraic maps in perfect analogy with the theory of algebraic maps.

Lemma (5.6.10) suggests also to define an $R$-algebraic map $\eta$ of degree $n$ to be homogeneous if $\eta_{(r,n)}$ vanishes for all $r \in \mathbb{Z}$ (or $r \in R$ in the case of $R$-algebraic maps) and to be integral if it can be written as a sum of homogeneous algebraic maps of various degrees $\leqslant n$.

If $\eta: A \to B$ is algebraic and homogeneous of degree $n$, then the relation

$$\text{(5.6.11)} \qquad \begin{aligned}\eta^n(a, ..., a) &= \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k}\eta(k\cdot a)\\ &= \left(\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\cdot k^n\right)\cdot\eta(a) = n!\cdot\eta(a)\end{aligned}$$

shows that up to the factor $n!$, $\eta$ can be derived from a symmetric multiadditive map $A^n \to B$ by combining this map with the diagonal map $\Delta: A \to A^n$: $a \mapsto (a, ..., a)$. It is clear that, more generally, for any multiadditive map $\eta': A^n \to B$, whether symmetric or not, the map $\eta = \eta' \circ \Delta$: $A \to B$: $a \mapsto \eta'(a, ..., a)$ is algebraic and homogeneous of degree $n$. Still, the examples

$$\text{(5.6.12)} \qquad \eta_p: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}: a + p\mathbb{Z} \mapsto a^p + p^2\mathbb{Z} \qquad (p \text{ a prime})$$

show that one cannot remove the factor $n!$ in (5.6.11), i.e., one cannot write any algebraic and homogeneous map $\eta$ in the form $\eta' \circ \Delta$ for some multi-additive map $\eta': A^n \to B$, even if one is willing to allow $\eta'$ not to be symmetric.

From (5.6.6) one can easily deduce:

(5.6.13) LEMMA. *If $\eta_1: A_1 \to A_2$ and $\eta_2: A_2 \to A_3$ are algebraic maps between abelian groups $A_1, A_2$, and $A_3$ of degree $n_1$ and $n_2$, respectively, then $\eta_2 \circ \eta_1: A_1 \to A_3$ is algebraic of degree $\leqslant n := n_1 \cdot n_2$.*

*Proof.* In view of (5.6.6) we have to show that for any $x \in A_1$, any map $\vartheta: \{0, 1, ..., n\} \to A_1$, and any covering $\mathfrak{K} \subseteq \mathfrak{P}(\{0, 1, ..., n\})$ of $\{0, 1, ..., n\}$ one has

$$(5.6.14) \qquad (D_{(\vartheta_x, \mathfrak{K})}\eta_2)(0) = 0.$$

But if $\# \mathfrak{K} > n_2$, (5.6.14) follows from $\deg \eta_2 < n_2$, while otherwise there exists some $K \in \mathfrak{K}$ with $\# K > n_1$ which implies $(D_{(\vartheta, K)}\eta_1)(x) = 0$ and $D_{(\vartheta_x, \mathfrak{K})}\eta = 0$ for any $\eta: A_2 \to A_3$ in view of (5.6.2), (5.6.3), and the trivial fact that $D_0\eta \equiv 0$ for any such $\eta$. ▮

In our context, the most important property of algebraic maps is the following one:

(5.6.15) LEMMA. *If $\eta: A \to B$ is an algebraic map of degree $n$ from some abelian semigroup $A$ into some abelian group $B$ and if $\iota: A \to \bar{A}: a \mapsto [a]$ denotes the canonical homomorphism from $A$ into the associated Grothendieck group $\bar{A} = \{[a_1] - [a_2] \mid a_1, a_2 \in A\}$ of $A$, then there exists a unique algebraic map $\bar{\eta}: \bar{A} \to B$ such that $\eta = \bar{\eta} \circ \iota$ and for this $\bar{\eta}$ one has $\deg \bar{\eta} = \deg \eta = n$.*

*Proof.* At first we observe by induction with respect to $\max(\deg \bar{\eta}_1, \deg \bar{\eta}_2)$ that for any two algebraic maps $\bar{\eta}_1, \bar{\eta}_2: \bar{A} \to B$ with $\eta = \bar{\eta}_1 \circ \iota = \bar{\eta}_2 \circ \iota$ one has $\bar{\eta}_1 = \bar{\eta}_2$. This is clear if $\deg \bar{\eta}_1 = \deg \bar{\eta}_2 = -1$ and it follows in case $-1 < \deg \bar{\eta}_1 \geqslant \deg \bar{\eta}_2$ from the relation

$$(5.6.16) \qquad \begin{aligned} \bar{\eta}_i([a_1] - [a_2]) &= \bar{\eta}_i([a_1]) - (D_{[a_2]}\bar{\eta}_i)([a_1] - [a_2]) \\ &= \eta(a_1) - (D_{[a_2]}\bar{\eta}_i)([a_1] - [a_2]) \end{aligned}$$

and from the induction hypothesis, applied with respect to $D_{[a_2]}\bar{\eta}_1$ and $D_{[a_2]}\bar{\eta}_2$, taking into account that

$$(5.6.17) \qquad (D_{[a_2]}\bar{\eta}_i) \circ \iota = D_{a_2}\eta.$$

To prove existence, one uses induction with respect to $n = \deg \eta$ to define

$$(5.6.18) \qquad \bar{\eta}([a_1] - [a_2]) := \eta(a_1) - \overline{(D_{a_2}\eta)}([a_1] - [a_2]).$$

That this is well-defined follows from the fact that for a given $a \in A$ the relation

$$(5.6.19) \qquad (D_{a_2 + a}\eta)(x) = (D_a\eta)(x + a_2) + (D_{a_2}\eta)(x) \qquad (x \in A)$$

implies by induction the relation

$$(5.6.20) \qquad \overline{D_{a_2 + a}\eta}(x) = \overline{D_a\eta}(x + [a_2]) + \overline{D_{a_2}\eta}(x) \qquad (x \in \bar{A})$$

and hence

$$
\begin{aligned}
(5.6.21)\qquad \eta(a_1)-\overline{D_{a_2}\eta}([a_1]-[a_2]) &= \eta(a_1)-\overline{D_{a_2+a}\eta}([a_1]-[a_2])+\overline{D_a\eta}([a_1])\\
&= \eta(a_1)-\overline{D_{a_2+a}\eta}([a_1+a]-[a_2+a])+D_a\eta(a_1)\\
&= \eta(a_1+a)-\overline{D_{a_2+a}\eta}([a_1+a]-[a_2+a]).
\end{aligned}
$$

It is also straightforward to check that for $a=[a_1]-[a_2]$ and $x\in[x_1]-[x_2]$ in $\bar{A}$ one has

(5.6.22)

$$
\begin{aligned}
(D_a\bar{\eta})(x) &= \bar{\eta}(x+a)-\bar{\eta}(x)\\
&= \eta(x_1+a_1)-\overline{D_{x_2+a_2}\eta}(x+a)-\eta(x_1)+\overline{D_{x_2}\eta}(x)\\
&= (D_{a_1}\eta)(x_1)-\overline{D_{a_2}\eta}(x+a)-(\overline{D_{x_2}\eta}(x+a+[a_2])-\overline{D_{x_2}\eta}(x))\\
&= (D_{a_1}\eta)(x_1)-(\overline{D_{a_2}\eta}(x)+D_a\overline{D_{a_2}\eta}(x))-D_{[a_1]}\overline{(D_{x_2}\eta)}(x)\\
&= (D_{a_1}\eta)(x_1)-\overline{D_{x_2}(D_{a_1}\eta)}(x)-\overline{D_{a_2}\eta}(x)-D_a\overline{D_{a_2}\eta}(x)\\
&= \overline{D_{a_1}\eta}(x)-\overline{D_{a_2}\eta}(x)-D_a\overline{(D_{a_2}\eta)}(x).
\end{aligned}
$$

So using induction once more, it follows that $\bar{\eta}$ is algebraic of degree $\leqslant n$. ∎

(5.6.23) COROLLARY. *If $B=\varprojlim_{i\in I}B_i$ is a projective limit of discrete groups $B_i$ ($i\in I$), then for any map $\eta\colon A\to B$ from some abelian semigroup $A$ into $B$ for which all of the projections $\eta_i\colon A\to^{\eta} B\to^{\mathrm{pr}_i} B_i$ are algebraic (of various degrees), then there exists a unique extension $\bar{\eta}\colon\bar{A}\to B$ such that all of the projections $\bar{\eta}_i\colon\bar{A}\to^{\bar{\eta}} B\to^{\mathrm{pr}_i} B_i$ are algebraic.*

*Remark.* Note that $\eta$ is algebraic if and only if $\sup_{i\in I}(\deg(\eta_i))<\infty$. Otherwise $\eta$ will be called quasi-algebraic.

**5.7.** It follows that to construct the canonical extension $\hat{\Omega}^{\otimes}(\sigma)\colon \hat{\Omega}(S)\to\hat{\Omega}(T)$ of the map $\sigma^{\otimes}$ from the abelian semigroup

$$
(5.7.1)\qquad \hat{\Omega}_+(S):=\{x\in\hat{\Omega}(S)\mid \text{there exist some almost finite } G\text{-map } \rho\colon R\to S \text{ with } x=[\rho]\}
$$

of actual and not only virtual almost finite $G$-spaces over $S$ into $\hat{\Omega}(T)$, defined by $[\rho]\to[\sigma^{\otimes}(\rho)]$, it is enough to show that $\sigma^{\otimes}$ is quasi-algebraic with respect to the representation of $\hat{\Omega}(T)$ as a projective limit of the rings $\Omega(\mathrm{Aut}(\xi))$ given in (5.2.4'). Hence it follows from

(5.7.2) THEOREM. *If the G-map $\sigma: S \to T$ induces a finite map $G\backslash\sigma: G\backslash S \to G\backslash T$ on the G-orbits and if $\xi: X \to T$ is a simple finite G-space over T. then the map*

$$h_\xi \circ \sigma^\otimes : \hat{\Omega}_+(S) \to \hat{\Omega}(T) \to \Omega(\mathrm{Aut}(\xi))$$

*is algebraic of degree $\leqslant n$, where n equals the number of G-orbits in $S \times_T X$ or—equivalently—the number of $G_x$-orbits in $\sigma^{-1}(\xi(x))$ for any $x \in X$.*

*Proof.* One can prove this result by showing that for any finite, simple $G$-space $\xi: X \to T$ over $T$ (whether of Galois type or not) the map $\varphi_\xi \circ \sigma^\otimes$: $\hat{\Omega}_+(S) \to \hat{\Omega}(T) \to \mathbb{Z}$ is algebraic of degree $\leqslant \# G\backslash(S \times_T X)$, which in turn follows from

$$\varphi_{(U,t)} \circ \sigma^\otimes = \prod_{Us \subseteq \sigma^{-1}(t)} \varphi_{(s, U_s)} \tag{5.7.3}$$

(cf. (2.11.9)) and

(5.7.4) LEMMA. *If $\eta_i: A \to B_i$ are algebraic maps of degree $n_i$ $(i = 1, 2)$, then $\eta_1 \otimes \eta_2: A \to B_1 \otimes B_2$ is algebraic of degree $\leqslant n_1 + n_2$.*

A perhaps more conceptual way is to observe at first that, generalizing (3.2.2)(ii), for any two almost finite $G$-spaces $\rho_1: R_1 \to S$ and $\rho_2: R_2 \to S$ one has a canonical $G$-isomorphism over $T$ between $\sigma^\otimes(\rho_1 \mathbin{\dot\cup} \rho_2)$ and the space $\sigma^\otimes(\rho_2) \mathbin{\dot\cup} (\Omega^*(\tau) \circ \sigma'^\otimes \circ \Omega_*(\zeta))(\rho_2)$, where $\zeta: S' \to S$, $\sigma': S' \to T'$, and $\tau: T' \to T$ are defined as

(5.7.5) $S' := \{(s, A, \gamma) \mid s \in S$, $A$ a non-empty, simultaneously closed and open subset of $\sigma^{-1}(\sigma(s))$, $s \notin A$, $\gamma: A \to R_1$ a continuous map with $\rho_1 \circ \gamma = \mathrm{Id}_A\}$,

$$\zeta: S' \to S: (s, A, \gamma) \mapsto s, \tag{5.7.6}$$

(5.7.7) $T' := \{(t, A, \gamma) \mid t \in T$, $A$ a non-empty, simultaneously closed and open subset of $\sigma^{-1}(t)$, $\gamma: A \to R_1$ a continuous map with $\rho_1 \circ \gamma = \mathrm{Id}_A\}$,

$$\sigma': S' \to T': (s, A, \gamma) \mapsto (\sigma(s), A, \gamma) \tag{5.7.8}$$

and

$$\tau: T' \to T: (t, A, \gamma) \mapsto t, \tag{5.7.9}$$

using the canonical $G$-actions and topologies on $S'$ and $T'$. Indeed, a canonical $G$-isomorphism

$$\sigma^\otimes(\rho_1 \mathbin{\dot\cup} \rho_2) \xrightarrow{\sim} \sigma^\otimes(\rho_2) \mathbin{\dot\cup} (\Omega^*(\tau) \circ \sigma'^\otimes \circ \Omega_*(\zeta))(\rho_2) \tag{5.7.10}$$

can be established easily by observing that for each $t \in T$ and $\gamma: \sigma^{-1}(t) \to R_1 \cup R_2$ in $\Gamma_t(\rho_1 \cup \rho_2)$ one has either $\gamma(\sigma^{-1}(t)) \subseteq R_2$, i.e., $\gamma \in \Gamma_t(\rho_2) \subseteq \Gamma_t(\rho_1 \cup \rho_2)$, or $(t, A := \gamma^{-1}(R_1), \gamma|_A) \in T'$. In the second case the set of all $\gamma \in \Gamma_t(\rho_1 \cup \rho_2)$ which lead to the same element $(t, A, \gamma')$ in $T'$ can then be identified with the fiber $\Gamma^{\sigma'}_{(t,A,\gamma')}(\rho_2 \times_S \zeta)$ of all continuous maps $\gamma'': \sigma^{-1}(t) - A \to R_2$ with $\rho_2 \circ \gamma'' = \mathrm{Id}_{\sigma^{-1}(t)-A}$, associating to each such $\gamma: \sigma^{-1}(t) \to R_1 \cup R_2$ the map $\gamma'' = \gamma|_{\sigma^{-1}(t)-A}: \sigma^{-1}(t) - A \to R_2$.

Now the theorem follows by induction with respect to $\#(G\backslash(S \times_T X))$: for any $[\rho_1] \in \hat{\Omega}_+(S)$ the map $D_{[\rho_1]}(h_\xi \circ \sigma^{\otimes})$ coincides by (5.7.10) and (5.4.5)(vi) with the map

$$(5.7.11) \qquad h_\xi \circ \Omega^*(\tau) \circ \sigma'^{\otimes} \circ \Omega_*(\zeta) = \sum_{\xi' \in \mathrm{Hom}(\xi, \tau)} h_{\xi'} \circ \sigma'^{\otimes} \circ \Omega_*(\zeta)$$

and this map is algebraic of degree $< \#(G\backslash(S \times_T X))$, since the map $\Omega_*(\zeta)$ is linear and the maps $h_{\xi'} \circ \sigma'^{\otimes}$ are algebraic of degree $\leqslant \#(G\backslash(S' \times_{T'} X))$ by induction in view of the fact that $\#(G\backslash(S' \times_{T'} X)) < \#(G\backslash(S \times_T X))$, which in turn follows from the observation that for any $\xi': X \to T'$ with $\tau \circ \xi' = \xi$ the $G$-map from $S' \times_{T'} X = \{((s, A, \gamma), x) \mid (\sigma(s), A, \gamma) = \xi'(x)\}$ into $S \times_T X = \{(s, x) \mid \sigma(s) = \xi(x)\}$ defined by $((s, A, \gamma), x) \mapsto (s, x)$ maps $S' \times_{T'} X$ bijectively onto the proper subset $\{(s, x) \in S \times_T X \mid s \notin A$ if $\xi'(x) = (t, A, \gamma)\}$ of $S \times_T X$.

The map $\tau: \prod'_{U \leqslant G} \mathbb{Z} \xrightarrow{\sim} \hat{\Omega}(G)$, considered in (2.12.7) and studied in Section 3, can now be interpreted in the following way: let $T$ denote the disjoint union of all simple finite $G$-spaces, taking one copy out of each isomorphism class. For any component $X \subseteq T$ of $T$ choose some $x \in X$ and consider the $G$-map $G/1 \to X$: $g \mapsto gx$. Let $\sigma: S := \dot{\bigcup} G/1 \to \dot{\bigcup} X = T$ denote the disjoint union of these maps and let $\sigma_T: T \to *$ denote the trivial $G$-map from $T$ onto $* = G/G$. Then $\hat{\Omega}(S)$ is obviously canonically isomorphic to $gh(G)$, even as a ring, and the diagram

$$\begin{tikzcd}
gh(G) = \prod'_{U \leqslant G} \mathbb{Z} \arrow[rr, "\tau"] \arrow[d, equal, "\wr"] & & \hat{\Omega}(G) \\
\hat{\Omega}(S) \arrow[r, "\Omega^{\otimes}(\sigma)"] & \hat{\Omega}(T) \arrow[r, "\Omega^*(\sigma_T)"] & \hat{\Omega}(*) \arrow[u, "\hat{\omega}_G"']
\end{tikzcd}$$

commutes. We believe that this interpretation of the Teichmüller map $\tau$ will be helpful in further and more detailed studies of its properties as well as for setting up appropriate generalizations of this construction, e.g., in the case of arbitrary compact transformation groups.

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