Tate normal form and level resolutions of the $K(2)$-local sphere

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Current work with Mark Behrens, Nat Stapleton, and Vesna Stojanoska

Outline

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Chromatic homotopy theory

Fix a rational prime p and an integer $n \geq 0$.

$$\text{Morava } E\text{-theory } E(n) \text{ with } \pi_* E(n) = \mathbb{Z}_{(p)}[v_1, v_2, \ldots, v_n^{\pm}],$$
$$\text{Morava } K\text{-theory } K(n) \text{ with } \pi_* K(n) = \mathbb{F}_p[v_n^{\pm}]$$

with $|v_i| = 2(p^i - 1)$.
The *chromatic tower*

$$X_{E(0)} \leftarrow X_{E(1)} \leftarrow X_{E(2)} \leftarrow \cdots$$

converges to X (when X is p-local finite).
Via the *chromatic fracture square*, we can build $E(n)$-localizations from $K(n)$-localizations.

The $K(1)$-local sphere

Lubin-Tate theory E_n with $\pi_* E_n = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^\pm]$.

At the prime p, the first (extended) Morava stabilizer group is $\mathbb{G}_1 = \mathbb{Z}_p^\times$ and

$$S_{K(1)} = E_1^{h\mathbb{Z}_p^\times}.$$

If $p > 2$, $\mathbb{Z}_p^\times$ is topologically cyclic; choose a prime ℓ such that $\langle \ell \rangle$ is dense in $\mathbb{Z}_p^\times$. Then $S_{K(1)}$ fits in the fiber sequence

$$S_{K(1)} \to E_1 \xrightarrow{\psi^\ell - 1} E_1.$$

We want to mimic this resolution for $S_{K(2)}$.

Behrens-Lawson $Q(\ell)$ via homotopy fixed points

Fix p and let C be a supersingular elliptic curve defined over $\mathbb{F}_p$.

Theorem (Behrens-Lawson)

For $p > 2$ the group

$$\Gamma := \{\ell\text{-power quasi-isogenies of } C\} \rtimes Gal$$

is dense in $\mathbb{G}_2$. If $p = 2$, Γ is dense in an index 2 subgroup of $\mathbb{G}_2$.

We expect a close connection between $S_{K(2)} = E_2^{h\mathbb{G}_2}$ and $E_2^{h\Gamma}$.

$Q(\ell)$ via level structures

Definition (Behrens-Lawson)

$$Q(\ell) := \operatorname{holim}\left(TMF \to \begin{matrix} TMF_0(\ell) \\ \times \\ TMF \end{matrix} \to TMF_0(\ell) \right)$$

By analyzing the building for $GL_2(\mathbb{Q}_\ell)$ we can prove the following.

Theorem (Behrens)

In the $K(2)$-local category

$$Q(\ell) = E_2^{h\Gamma}$$

$Q(\ell)$ and resolutions

Conjecture (Behrens-Lawson)

For $p > 2$, $Q(\ell)$ resolves "half" of the $K(2)$-local sphere in the sense that there is a $K(2)$-local fiber sequence

$$DQ(\ell) \to S \to Q(\ell).$$

For $p = 2$, let $\tilde{S} = E_2^{h\tilde{\mathbb{G}}_2}$; then there is a $K(2)$-local fiber sequence

$$DQ(\ell) \to \tilde{S} \to Q(\ell).$$

For $p = 2$, $\tilde{\mathbb{G}}_2 = \bar{\Gamma}$ with index 2 in $\mathbb{G}_2$.

Topological modular forms

There is an étale sheaf of E_∞ rings $\mathcal{O}^{\text{top}}$ on the stack of generalized elliptic curves $\mathcal{M}$ due to Goerss-Hopkins-Lurie-Miller.

$$\left(E/R \leftrightarrow \begin{array}{c} \operatorname{Spec} R \\ \downarrow \\ \mathcal{M} \end{array} \right) \mapsto \text{c.o.c.t. for } \hat{E}$$

The 576-periodic cohomology theory TMF is defined as

$$TMF := \mathcal{O}^{\text{top}}(\mathcal{M}).$$

The "level structure" TMFs are constructed by applying $\mathcal{O}^{\text{top}}$ to interesting étale covers of $\mathcal{M}$.

Elliptic curves with level structures

Definition

$$\{\Gamma_1(n)\text{-structures}\} := \left\{ (E, P) : \begin{array}{c} E \text{ elliptic curve} \\ P \in E \text{ of order } n \end{array} \right\}$$

$$\{\Gamma_0(n)\text{-structures}\} := \left\{ (E, H) : \begin{array}{c} E \text{ elliptic curve} \\ H \subset E \text{ cyclic of order } n \end{array} \right\}$$

The moduli spaces of $\Gamma_1(n)$- and $\Gamma_0(n)$-structures are denoted $\mathcal{M}_1(n)$ and $\mathcal{M}_0(n)$.
The "level structure *TMF*s" we study are

$$TMF_i(n) := \mathcal{O}^{\text{top}}(\mathcal{M}_i(n))$$

for $i = 0, 1$.

Constructing $Q(\ell)$

We get $Q(\ell)$ by applying the *TMF* sheaf $\mathcal{O}^{\mathrm{top}}$ to a semi-simplicial stack $\mathcal{M}_\bullet$:

$$
\begin{tikzcd}
 & \mathcal{M}_0(\ell) \arrow[dl, shift right, "q"'] \arrow[dl, shift left, "f"] & \\
\mathcal{M} & \amalg & \mathcal{M}_0(\ell) \arrow[ul, shift right, "t"'] \arrow[ul, shift left, "1"] \arrow[dl, "f"] \\
 & \mathcal{M} \arrow[ul, shift right, "\psi^\ell"] \arrow[ul, shift left, "1"'] &
\end{tikzcd}
$$

Maps in the simplicial stack

$$\begin{tikzcd}
 & \mathcal{M}_0(\ell) \arrow[dl, shift right, "q"'] \arrow[dl, shift left, "f"] & \\
\mathcal{M} \arrow[r, phantom, "\amalg"] & {} & \mathcal{M}_0(\ell) \arrow[ul, shift right, "t"'] \arrow[ul, shift left, "1"] \arrow[dl, "f"] \\
 & \mathcal{M} \arrow[ul, shift right, "\psi^\ell"'] \arrow[ul, shift left, "1"] &
\end{tikzcd}$$

$\mathcal{M}_0(\ell) \xrightarrow{f} \mathcal{M}$ $(E, H) \mapsto E$	$\mathcal{M}_0(\ell) \xrightarrow{q} \mathcal{M}$ $(E, H) \mapsto E/H$
$\mathcal{M} \xrightarrow{\psi^\ell} \mathcal{M}$ $E \mapsto E/E[\ell]$	$\mathcal{M}_0(\ell) \xrightarrow{t} \mathcal{M}_0(\ell)$ $\phi \mapsto \hat{\phi}$

Goals

Project goals:

- Understand $Q(\ell)$ and the Behrens-Lawson conjecture at $p = 2$ for $\ell \geq 5$.
  - Mahowald-Rezk have studied $Q(3)$ at $p = 2$.
  - We should find different "$K(2)$-local semi-hemispheres" at varying ℓ.
- Shed conceptual light on the difficult computations of Shimomura-Wang.
- Connect to the group-theoretic $K(2)$-local resolutions of Goerss-Henn-Mahowald-Rezk.

Talk goals:

- Describe models for $\mathcal{M}_0(\ell)$ that permit computations on the Hopf algebroid level.
- See how the divided β-family sits in $Q(5)$.

$TMF_0(5)$ and the Kervaire invariant

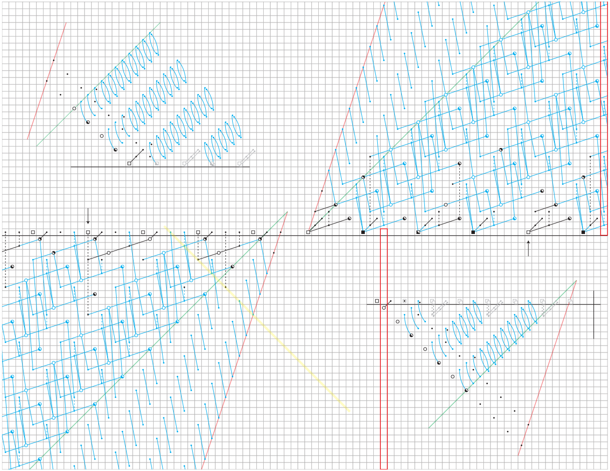


image: Hill-Hopkins-Ravenel

Outline

1. Resolutions via isogenies
2. Normal forms for level structures
3. Computations

Elliptic curves

Recall that every elliptic curve has a Weierstrass form

$$C_{\mathbf{a}} : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

where $\mathbf{a} = (a_1, a_2, a_3, a_4, a_6)$ and $\Delta(\mathbf{a})$ is invertible.
There's an associated Weierstrass Hopf algebroid

$$(A, \Gamma) = (\mathbb{Z}[\mathbf{a}, \Delta^{-1}], A[r, s, t, \lambda^{\pm}])$$

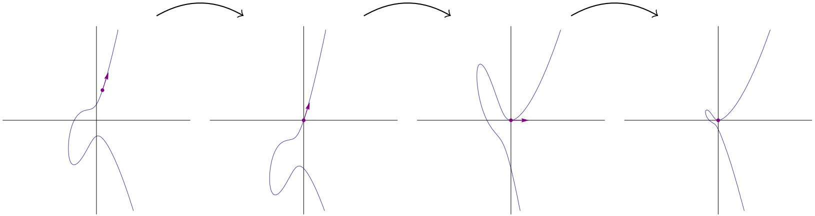
where the automorphisms are

$$\begin{aligned} C_{\mathbf{a}'} &\xrightarrow{\varphi_{r,s,t,\lambda}} C_{\mathbf{a}} \\ x &\mapsto \lambda^{-2} x + r \\ y &\mapsto \lambda^{-3} y + \lambda^{-2} s x + t. \end{aligned}$$

Tate normal form for $\Gamma_1(\ell)$-structures

Apply coordinate transforms to $(C_{\mathbf{a}''}, P) \in \mathcal{M}_1(\ell)$, $\ell \geq 5$:

	$P \mapsto (0,0)$	Set $a_4 = 0$	Set $a_2 = a_3$



$C_{\mathbf{a}''}$	$C_{\mathbf{a}'}$	$T : y^2 + a_1xy + a_3y = x^3 + a_2x^2$	$E : y^2 + (1-c)xy - by = x^3 - bx^2$
$(-2, 4, 0, 3, 7)$	$(-2, 7, 8, 24, 0)$	$(4, 4, 8, 0, 0)$	$(2, 1, 1, 0, 0)$

Relations from torsion

Since $(0,0)$ has order ℓ,

$$\left[\frac{\ell+1}{2}\right](0,0) = \left[\frac{1-\ell}{2}\right](0,0).$$

We get polynomials encoding the fact that $(0,0)$ is ℓ-torsion.

ℓ	$g_\ell(a_1, a_2, a_3)$	$f_\ell(b, c)$
5	$-a_1 a_2 a_3 + a_2^3 + a_3^2$	$-b + c$
7	$a_1^3 a_2^3 a_3 - 3a_1^2 a_2^2 a_3^2 + \cdots$	$b^2 - bc - c^3$
11	$-a_1^7 a_2^{10} a_3 + 10 a_1^6 a_2^9 a_3^2 + \cdots$	$b^5 - 3b^4 c - 4b^3 c^3 + \cdots$
13	$-a_1^{10} a_2^{10} a_3^4 - a_1^9 a_2^{15} a_3 + \cdots$	$-b^7 + 6b^6 c - 4b^5 c^3 + \cdots$
17	$a_1^{18} a_2^{21} a_3^4 - 21 a_1^{17} a_2^{20} a_3^5 + \cdots$	$-b^{12} + 10b^{11} c - 10b^{10} c^3 + \cdots$

Associated Hopf algebroids

The moduli space $\mathcal{M}_1^{tan}(\ell)$ of $\Gamma_1(\ell)$-structures $(E, P) +$ a tangent vector at P is represented by

$$\mathbb{Z}[a_1, a_2, a_3, \Delta^{-1}]/g_\ell(a_1, a_2, a_3)$$

[To get a Hopf algebroid stackifying to $\mathcal{M}_1(\ell)$ we just need to add in a $\mathbb{G}_m$ worth of automorphisms (they scale the tangent vector but retain homogeneous Tate normal form).]

Passing to non-homogeneous Tate normal form we also have that $\mathcal{M}_1(\ell)$ is represented by

$$\mathbb{Z}[b, c, \Delta^{-1}]/f_\ell(b, c).$$

Immediate corollaries of Tate normal form

The 2-series for $E(b, c)$ is

$$[2]_T(z) = 2z - (1 - c)z^2 + 2bz^3 + \cdots .$$

Thus we see the supersingular locus at $p = 2$ of $\mathcal{M}_1(\ell)$ by computing $f_\ell(b, 1) \pmod 2$.

Corollary (O.-Stapleton-Stojanoska)

ℓ	$f_\ell(b, 1) \pmod 2$	$TMF_1(\ell)_{K(2)}$
5	$b + 1$	$E_2(\mathbb{F}_2)$
7	$b^2 + b + 1$	$E_2(\mathbb{F}_{2^2})$
11	$b^5 + b^4 + b^3 + b^2 + 1$	$E_2(\mathbb{F}_{2^5})$
13	$(b^4 + b + 1)(b^3 + b + 1)$	$E_2(\mathbb{F}_{2^4}) \times E_2(\mathbb{F}_{2^3})$
17	$(b^8 + b^7 + b^6 + b^5 + b^4 + b^3 + 1)\cdot$ $(b^4 + b^3 + b^2 + b + 1)$	$E_2(\mathbb{F}_{2^8}) \times E_2(\mathbb{F}_{2^4})$

The Hopf algebroid for $\mathcal{M}_0(5)$

Specializing to the case $\ell = 5$ we can compute

$$g_5(a_1, a_2, a_3) = a_2^3 + a_3^2 - a_1 a_2 a_3.$$

This allows us to put the universal elliptic curve $T(\mathbf{a})$ in the form

$$y^2 + a_1 xy + u^2(a_1 - u)y = x^3 + u(a_1 - u)x^2$$

with u invertible. Hence

$$\mathcal{M}_1^{tan}(5) = \operatorname{Spec} \mathbb{Z}[a_1, u^{\pm}, \Delta^{-1}].$$

The Hopf algebroid for $\Gamma_0(5)$-structures is then

$$(B, \Lambda) = (\mathbb{Z}[a_1, u^{\pm}, \Delta^{-1}], \operatorname{Map}(\mathbb{Z}/5^{\times}, B)).$$

Maps between Hopf algebroids

On the Hopf algebroid level, the forgetful map *f* in the cosimplicial is represented by

$$\begin{aligned}(A,\Gamma) &\xrightarrow{f} (B,\Lambda)\\ a_i &\mapsto a_i,\ i=1,2,3\\ a_4,a_6 &\mapsto 0\end{aligned}$$

We see the quotient map *q* as

$$\begin{aligned}(A,\Gamma) &\xrightarrow{q} (B,\Lambda)\\ a_i &\mapsto a_i,\ i=1,2,3\\ a_4 &\mapsto 5a_1^2a_2-10a_1a_3-10a_2^2\\ a_6 &\mapsto a_1^4a_2-2a_1^3a_3-12a_1^2a_2^2+19a_2^3-a_3^2\end{aligned}$$

via *Vélu's formulae*.

The Atkin-Lehner dual

We can lift the map

$$\phi \mapsto \widehat{\phi}$$

to $\Gamma_1(5)$-structures via the Weil pairing

$$\langle\ ,\ \rangle_\phi : \ker\phi \times \ker\widehat{\phi} \to \mu_5$$

after adjoining a fifth root of unity ζ.
On the level of Hopf algebroids, we find

$$\begin{aligned} a_1 &\mapsto \frac{1}{5}(-8\zeta^3 - 6\zeta^2 - 14\zeta - 7)a_1 + \frac{1}{5}(14\zeta^3 - 2\zeta^2 + 12\zeta + 6)u, \\ u &\mapsto \frac{1}{5}(-\zeta^3 - 7\zeta^2 - 8\zeta - 4)a_1 + \frac{1}{5}(8\zeta^3 + 6\zeta^2 + 14\zeta + 7)u. \end{aligned}$$

The Atkin-Lehner dual on modular forms

Let $b_1 := a_1 - u$. Then

$$MF_*(\Gamma_0(5)) = \frac{\mathbb{Z}[1/5, b_2, b_4, \Delta^{\pm 1/3}, D^{\pm}]}{(b_4^2 = b_2\Delta^{1/3} - 4\Delta^{2/3}, D = 11(\Delta^{1/3})^3 + \Delta^{2/3}b_4)}$$

where

$$\begin{aligned} b_2 &:= (u^2 + b_1^2)^2, \\ b_4 &:= u^3 b_1 - u b_1^3, \\ \Delta^{1/3} &:= u^2 b_1^2. \end{aligned}$$

On $MF_*(\Gamma_0(5))$, t^* takes the form

$$\begin{aligned} t^*(b_2) &= -5b_2, \\ t^*(b_4) &= \frac{1}{5}(11b_2^2 - 117b_4 - 88\Delta^{1/3}), \\ t^*(\Delta^{1/3}) &= \frac{1}{5}(b_2^2 - 22b_4 + 117\Delta^{1/3}). \end{aligned}$$

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The double complex for $H^*(Q(\ell); \omega^{\otimes *})$

There are spectral sequences

$$H^*\begin{Bmatrix} C^*(\Gamma) \\ C^*(\Lambda) \end{Bmatrix} = H^*\left(\begin{Bmatrix} \mathcal{M} \\ \mathcal{M}_0(5) \end{Bmatrix}; \omega^{\otimes *}\right) \implies \pi_*\begin{Bmatrix} TMF \\ TMF_0(5) \end{Bmatrix}.$$

A double complex computes $E_2(Q(5)) = \mathbb{H}^*(\mathcal{M}_\bullet; \omega^{\otimes *})$:

$$\begin{tikzcd}
C^0(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d, dashed, "d^i"] & C^0(\Lambda) \oplus C^0(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d, dashed, "d^i"] & C^0(\Lambda) \arrow[d, dashed, "d^i"] \\
C^1(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d, dashed, "d^i"] & C^1(\Lambda) \oplus C^1(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d, dashed, "d^i"] & C^1(\Lambda) \arrow[d, dashed, "d^i"] \\
C^2(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d] & C^2(\Lambda) \oplus C^2(\Gamma) \arrow[r, dashed, "d^e"] \arrow[d] & C^0(\Lambda) \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

Accessing the second monochromatic layer

If N is a BP_*BP-comodule, then

$$M_2^0 N := v_2^{-1} N/v_0, v_1, \quad M_1^1 N := v_2^{-1} N/v_0, v_1^\infty,$$

$$M_0^2 N := v_2^{-1} N/v_0^\infty, v_1^\infty.$$

There are Bockstein spectral sequences

$$\mathrm{Ext}^{*,*} M_2^0 N \otimes \frac{\mathbb{F}_2[v_0, v_1]}{v_0^\infty, v_1^\infty} \implies \mathrm{Ext}^{*,*} M_1^1 N \otimes \frac{\mathbb{F}_2[v_0]}{v_0^\infty} \implies \mathrm{Ext}^{*,*} M_0^2 N.$$

If $N = BP_* X$ we also have the monochromatic Adams-Novikov spectral sequence

$$\mathrm{Ext}^{*,*} M_0^2 N \implies \pi_* M_2 X$$

If $X = S$, this is the strategy of Miller-Ravenel-Wilson and $\mathrm{Ext}^{0,*} M_0^2 BP_*$ detects the β-family.

The β-family in $Q(\ell)$

Let $C^\bullet_{tot}(Q(\ell)$ denote the totalization of the double complex.

Theorem (Behrens-O)

The map

$$\mathrm{Ext}^{0,*} M_0^2 BP_* \to H^{0,*} M_0^2 C^\bullet_{tot} Q(3)$$

is an isomorphism.

Surprise (Behrens-O)

The map

$$\mathrm{Ext}^{0,*} M_0^2 BP_* \to H^{0,*} M_0^2 C^\bullet_{tot} Q(5)$$

is *not*.

S and $Q(3)$ see only up to $\beta_{4/6}$ while $Q(5)$ sees $\beta_{4/7}$.

Symmetry breaking in the $Q(\ell)$ spectra

Let $\mathbb{S}_2^{(\ell)}$ denote the preimage of $\ell^{\mathbb{Z}}$ under the reduced norm

$$\mathbb{S}_2 \xrightarrow{N} \mathbb{Z}_2^\times.$$

If $\ell^{\mathbb{Z}}$ is dense in an index 2 subgroup of $\mathbb{Z}_2^\times = \pm 1 \times (1 + 4\mathbb{Z}_2)$, then $[\mathbb{S}_2 : \mathbb{S}_2^{(\ell)}] = 2$.

Projecting out the Galois part, Γ_ℓ is dense in $\mathbb{S}_2^{(\ell)}$.

On the level of second monochromatic layers, the $\mathbb{S}_2^{(3)}$-invariants of $\pi_* E_2$ match the $\mathbb{S}_2$-invariants, but there are more $\mathbb{S}_2^{(5)}$-invariants!