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## Multicurves and Equivariant Cohomology

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# Abstract

Let  $A$  be a finite abelian group. We set up an algebraic framework for studying  $A$ -equivariant complex-orientable cohomology theories in terms of a suitable kind of equivariant formal group. We compute the equivariant cohomology of many spaces in these terms, including projective bundles (and associated Gysin maps), Thom spaces, and infinite Grassmannians.

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## CHAPTER 1

# Introduction

Let  $A$  be a finite abelian group. In this memoir, we set up an algebraic framework for studying  $A$ -equivariant complex-orientable cohomology theories in terms of a suitable kind of formal groups. In part, this is a geometric reformulation of earlier work of Cole, Greenlees, Kriz and others on equivariant formal group laws [6–8, 19]. However, the theory of divisors, residues and duality for multicurves is new, and forms a substantial part of the present memoir. Although we focus on the finite case, many results can be generalised to compact abelian Lie groups. On the other hand, we have evidence that nonabelian groups will need a completely different theory. (See Section 25 for a little more discussion.)

We now briefly recall the nonequivariant theory, using the language of formal schemes. We will follow the conventions and terminology developed in [29]. Let  $E$  be an even periodic cohomology theory, and put  $S = \operatorname{spec}(E^0)$  and  $C = \operatorname{spf}(E^0 \mathbb{C}P^\infty) = \varinjlim_n \operatorname{spec}(E^0 \mathbb{C}P^n)$ . Some basic facts are as follows.

- (a)  $C$  is a formal group scheme over  $S$ .
- (b) If we forget the group structure, then  $C$  is isomorphic to the formal affine line  $\widehat{\mathbb{A}}_S^1$  as a formal scheme over  $S$ ; in other words,  $C$  is a formal curve over  $S$ .
- (c) For many interesting spaces  $X$ , the formal scheme  $\operatorname{spf}(E^0 X)$  has a natural description as a functor of  $C$ ; for example, we have  $\operatorname{spf}(E^0 BU(d)) = C^d / \Sigma_d = \operatorname{Div}_d^+(C)$ , the formal scheme of effective divisors of degree  $d$  on  $C$ . Similar descriptions are known for  $\Omega U(d)$ ,  $\mathbb{C}P^d$ ,  $BSU$ , Grassmannians, flag varieties, toric varieties and so on.
- (d) If  $M$  is a compact complex manifold then the ring  $E^0 M$  has Poincaré duality: there is a map  $\theta: E^0 M \rightarrow E^0$  such that the pairing  $(a, b) \mapsto \theta(ab)$  is perfect. When formulated this way, the map  $\theta$  is not quite canonical; we need to build in a twist by a certain line bundle to cure this. There is a formula for  $\theta$  (due to Quillen) in terms of residues of differential forms on  $C$ .

Now let  $\mathcal{U} = \mathcal{U}_A$  be a complete  $A$ -universe, and let  $\mathcal{S}_A$  be the category of  $A$ -spectra indexed on  $\mathcal{U}$  (as in [20]). Consider an  $A$ -equivariant commutative ring spectrum  $E \in \mathcal{S}_A$  that is periodic and orientable in a sense to be made precise later. In this context, the right analogue of  $\mathbb{C}P^\infty$  is the projective space  $P\mathcal{U}$ . This has an evident  $A$ -action. We put  $S = \operatorname{spec}(E^0)$  and  $C = \operatorname{spf}(E^0 P\mathcal{U})$ . This is again a formal group scheme over  $S$ , but it is no longer a formal curve. This appears to create difficulties with (c) above, because we no longer have a good hold on  $C^d / \Sigma_d$  or a good theory of divisors on  $C$ .

Our first task is to define the notion of a *formal multicurve* over  $S$ , and to show that  $C$  is an example of this notion. Later we will develop an extensive

theory of formal multicurves and their divisors, and show that many statements about generalized cohomology can be made equivariant by replacing curves with multicurves.

EXAMPLE 1.1. The simplest nontrivial example of a formal multcurve is the formal scheme  $C = \mathrm{spf}(R)$ , where  $R$  is the completion of  $\mathbb{Z}[x]$  at the ideal generated by the polynomial  $f(x) = x^2 - 2x$ , or equivalently  $R = \mathbb{Z}[[y]][x]/(x^2 - 2x - y)$ . If we consider formal schemes as functors from rings to sets, then

$$C(A) = \{a \in A \mid f(a) \text{ is nilpotent}\}.$$

One can check that  $R/2 = (\mathbb{Z}/2)[[x]]$  but  $R/3 \simeq (\mathbb{Z}/3)[[y]] \times (\mathbb{Z}/3)[[y]]$ .

EXAMPLE 1.2. Let  $A$  be a finite abelian group as before, let  $R(A) = \mathbb{Z}[A^*]$  be the complex representation ring, and put

$$f(u) = \prod_{\alpha \in A^*} (u - \alpha) \in R(A)[u].$$

Let  $R_1$  be the completion of  $R(A)[u]$  at the ideal generated by  $f(u)$ , and put  $C_1 = \mathrm{spf}(R_1)$ . Then  $C_1$  is again a formal multcurve. Moreover, it is known that  $R_1$  is the equivariant complex  $K$ -theory of  $PU_A$ , so this is an instance of the topological situation mentioned above. This is a theorem of Cole, Greenlees and Kriz, which will be discussed further in Section 7.1.

REMARK 1.3. It is not true that every formal multcurve has the form  $\mathrm{spf}(k[x]_{\hat{f}})$ , but for many purposes one can reduce to that special case. This will be discussed further in Section 16.

### 1.1. Outline of the memoir.

- In Section 2 we give the most basic definitions about multicurves.
- In Section 3 we set up a theory of differential forms on multicurves.
- In Section 4 we discuss the topology and geometry of projective spaces, being careful to use explicit and natural methods that transfer automatically to the equivariant setting.
- In Section 5 we set up the basic theory of equivariantly complex orientable ring spectra, and we show that a suitable class of such spectra give rise to equivariant formal groups.
- A nonequivariant even periodic ring spectrum  $\widehat{E}$  gives two different cohomology theories on  $A$ -spaces, namely the forgetful theory  $X \mapsto \widehat{E}^0(X)$  and the Borel theory  $X \mapsto \widehat{E}^0(X_{hA})$ . In Section 6 we discuss the (very simple) corresponding ways to convert nonequivariant formal groups to equivariant formal groups.
- In Section 7 we show how suitable algebraic groups give rise to equivariant formal groups, and we discuss the relevance to equivariant  $K$ -theory and elliptic cohomology.
- In Section 8 we introduce a new class of EFGs (those of “product type”) and show that all EFGs over fields are in this class. This makes it easy to classify EFGs over algebraically closed fields, which leads to a discussion of equivariant Morava  $K$ -theories.
- It is well-known that the category of rational even periodic spectra is contravariantly equivalent to that of formal groups over rational rings.



In Section 9 we explain and prove the corresponding fact for equivariant formal groups.

- In Section 10 we introduce another algebraic construction on EFGs. This is used in Section 11 to construct universal deformations of EFGs over fields, and to relate them to equivariant Morava  $E$ -theory.
- Various theorems are known to the effect that  $E^0EG$  is a completion of  $E^0(\text{point})$ , for suitable groups  $G$  and equivariant ring spectra  $E$ . The earliest such result is the Atiyah-Segal theorem for equivariant  $K$ -theory [5]; for some more recent results, see [15] and the papers referenced there. In Section 12 we prove a theorem of this type, following the argument of Greenlees [12] but interpreting everything in terms of EFGs.
- In Section 13 we give a brief discussion of equivariant formal group laws (as opposed to formal groups) and their relationship with complex cobordism. In particular, we discuss a conjectural generalisation of a fundamental theorem of Quillen.
- In Section 14 we discuss a salutary counterexample.
- In Section 15 we return to the algebraic theory of divisors on multicurves. We give two different definitions of divisors, and prove the crucial fact that the weaker definition is in fact equivalent to the stronger one.
- In Section 16 we define what it means for a multicurve to be *embeddable*, and show how to reduce various questions to this special case.
- In Sections 17 and 18 we discuss classification of divisors and the isomorphism  $\text{Div}_d^+(C) = C^d/\Sigma_d$ . Many results are the same as in the nonequivariant case, but the proofs are much more technical (and involve the reductions established in Section 16).
- In Section 19 we study the local structure of the formal scheme  $\text{Div}_d^+(C) = C^d/\Sigma_d$ , showing that in a suitable sense it is a formal manifold of dimension  $d$ .
- In Section 20 we give a reorganised proof of a theorem of Cole, Greenlees and Kriz [8] showing that the formal scheme associated to the Grassmannian  $G_d\mathcal{U}$  is precisely  $\text{Div}_d^+(C)$ . This allows us to build a dictionary between vector bundles and divisors, just as in the nonequivariant case.
- In Section 21 we use and extend these ideas to give an analogue of the Projective Bundle Theorem, and descriptions of the cohomology of flag bundles and Grassmann bundles.
- In Section 22 we develop an algebraic theory for certain kinds of rings with Poincaré duality, and apply it to understand residues, Gysin maps and duality for projective bundles.
- In Section 23 we return to the study of Grassmannians. Greenlees's definition of equivariant connective  $K$ -theory [14] gives canonical equivariant analogues of the spaces  $\mathbb{Z} \times BU$ ,  $BU$  and  $BSU$ , and we describe the equivariant cohomology of these.
- Finally, in Section 24 we show how to construct a Mackey functor from an arbitrary  $A$ -equivariant formal group  $C$  over  $S$ , and thus a map from the Burnside ring of  $A$  to  $\mathcal{O}_S$ . In the case where  $C$  arises from a ring spectrum, this map will agree with the usual one defined using equivariant topology.



## CHAPTER 2

# Multicurves

DEFINITION 2.1. Let  $X = \operatorname{spf}(R)$  be a formal scheme, and let  $Y$  be a subscheme of  $X$ . We say that  $Y$  is a *regular hypersurface* if  $Y = \operatorname{spf}(R/J)$  for some ideal  $J = I_Y \leq R$  that is a free module of rank one over  $R$ . Equivalently, there should be a regular element  $f \in R$  such that the vanishing locus  $V(f) = \operatorname{spf}(R/f)$  is precisely  $Y$ .

Let  $S = \operatorname{spec}(k)$  be an affine scheme.

DEFINITION 2.2. A *formal multicurve* over  $S$  is a formal scheme  $C$  over  $S$  such that

- (a)  $C = \operatorname{spf}(R)$  for some formal ring  $R$
- (b) There exists a regular element  $y \in R$  such that for all  $k \geq 0$ , the ideal  $Ry^k$  is open in  $R$ , and  $R/y^k$  is a finitely generated free module over  $\mathcal{O}_S$ , and  $R = \varprojlim_k R/y^k$ .
- (c) The diagonal subscheme  $\Delta \subset C \times_S C$  is a regular hypersurface.

A generator  $d$  for the ideal  $I_\Delta$  will be called a *difference function* for  $C$  (because  $d(a, b) = 0$  iff  $(a, b) \in \Delta$  iff  $a = b$ ). We will choose a difference function  $d$ , but as far as possible we will express our results in a form independent of this choice. An element  $y$  as in (b) will be called a *good parameter* on  $C$ .

REMARK 2.3. If  $S$  is a formal scheme, then we can write  $S = \varinjlim_\alpha S_\alpha$  for some filtered system of affine schemes, and formal schemes over  $S$  are the same as compatible systems of formal schemes over the  $S_\alpha$  by [29, Proposition 4.27]. In the rest of this memoir, we will generally work over an affine base but will silently use this result to transfer definitions and theorems to the case of a formal base where necessary.

The formal affine line  $\widehat{\mathbb{A}}_S^1 = \operatorname{spf}(k[[x]])$  is a formal multicurve, and the category of formal multicurves is closed under disjoint union. Conversely, condition (c) implies that the module  $\Omega_{C/S}^1 = I_\Delta/I_\Delta^2$  is free of rank one over  $R = \mathcal{O}_C$ , so formal multicurves may be thought of as being smooth and one-dimensional. Similarly, if  $y$  is a good parameter then  $R$  is a finitely generated projective module over  $k[[y]]$ , which means that  $C$  admits a finite flat map to  $\widehat{\mathbb{A}}_S^1$ , again indicating a one-dimensional situation. If  $k$  is an algebraically closed field, we shall see later that every formal multicurve over  $S$  is a finite disjoint union of copies of  $\widehat{\mathbb{A}}_S^1$ .

REMARK 2.4. Note that  $I_\Delta$  is the kernel of the multiplication map  $\mu: R \widehat{\otimes} R \rightarrow R$ , which is split by the map  $a \mapsto a \otimes 1$ . It follows that  $I_\Delta$  is topologically generated by elements of the form  $a \otimes b - ab \otimes 1$ . We also see by similar arguments that for any ideal  $J \leq R$ , the kernel of the multiplication map  $(R/J) \otimes (R/J) \rightarrow R/J$  is just the image of  $I_\Delta$  and thus is generated by  $d$ .

CONSTRUCTION 2.5. Let  $C = \text{spf}(R)$  be a formal scheme over  $S = \text{spec}(k)$ , and let  $y \in R$  be a good parameter. (We are not assuming that  $C$  is a formal multcurve, but we assume that  $y$  satisfies part (b) of Definition 2.2.) Choose elements  $e_0, \dots, e_{n-1} \in R$  whose residue classes give a basis for the free module  $R/y$  over  $k$ . Define elements  $e_i \in R$  for all  $i \in \mathbb{N}$  by  $e_{nj+k} = y^j e_k$ . Define  $\mu_r: \prod_{i=0}^{nr-1} k \rightarrow R/y^r$  by  $\mu_r(\underline{a}) = \sum_{i=0}^{nr-1} a_i e_i$ . Define  $\mu_\infty: \prod_{i=0}^\infty k \rightarrow R$  by  $\mu_\infty(\underline{a}) = \sum_i a_i e_i$ . (This is meaningful because  $R = \varprojlim_i R/y^i$ .)

LEMMA 2.6. *The maps  $\mu_r$  and  $\mu_\infty$  are isomorphisms. (In other words,  $\{e_i\}_{i \in \mathbb{N}}$  is a topological basis for  $R$ .)*

PROOF. Define  $\mu'_r: \left(\prod_{i=0}^{nr-1} k\right) \times R \rightarrow R$  by  $\mu'_r(\underline{a}, b) = (\sum_i a_i e_i) + y^r b$ . As  $y$  is a regular element and the elements  $e_0, \dots, e_{n-1}$  form a basis for  $R/y$  we see that  $\mu'_1$  is an isomorphism, so  $R = k^n \oplus R$ . We can substitute this into itself to see that  $R = k^{2n} \oplus R$ ; by a slightly more precise version of the same argument, we see that  $\mu'_2$  is an isomorphism. We can extend this inductively to see that  $\mu'_r$  is an isomorphism for all  $r$ . We then reduce mod  $y^r$  to see that  $\mu_r$  is an isomorphism, and pass to the limit to see that  $\mu_\infty$  is an isomorphism.  $\square$

DEFINITION 2.7. A *formal multcurve group* over  $S$  is a formal multcurve over  $S$  with a commutative group structure.

In the presence of a group structure, axiom (c) can be modified.

DEFINITION 2.8. Let  $C$  be a commutative formal group scheme over  $S$ . A *coordinate* on  $C$  is a regular element  $x \in \mathcal{O}_C$  whose vanishing locus is the zero-section. Clearly, such an  $x$  exists iff the zero-section is a regular hypersurface.

REMARK 2.9. If  $x$  is a coordinate, then so is the function  $\bar{x}$  defined by  $\bar{x}(a) = x(-a)$ .

PROPOSITION 2.10. *Let  $C$  be a formal group scheme over  $S$  satisfying axioms (a) and (b) in Definition 2.2. Then  $C$  is a formal multcurve iff the zero-section  $S \rightarrow C$  is a regular hypersurface. More precisely, if  $x$  is a coordinate on  $C$ , then the function  $d(a, b) = x(b - a)$  defines a difference function, and if  $d$  is a difference function, then the function  $x(b) = d(0, b)$  is a coordinate.*

The proof relies on the following basic lemma.

LEMMA 2.11. *Let  $C$  be a formal multcurve, and let  $f: X \rightarrow C$  be any map of schemes. Then the function  $d'(x, b) = d(f(x), b)$  on  $X \times_S C$  is regular in  $\mathcal{O}_{X \times_S C}$ .*

PROOF. We have a short exact sequence as follows:

$$R \widehat{\otimes} R \xrightarrow{\times d} R \widehat{\otimes} R \xrightarrow{\mu} R.$$

We regard  $R \widehat{\otimes} R$  as a module over  $R$  via the map  $t \mapsto t \otimes 1$ . The map  $\mu$  is then  $R$ -linearly split by the map  $t \mapsto t \otimes 1$ , so the sequence remains exact after applying the functor  $\mathcal{O}_{X \times_S C} \widehat{\otimes}_R (-)$ . The resulting sequence is just

$$\mathcal{O}_{X \times_S C} \xrightarrow{\times d'} \mathcal{O}_{X \times_S C} \rightarrow \mathcal{O}_X,$$

which proves the lemma.  $\square$

COROLLARY 2.12. *Let  $C \xrightarrow{q} S$  be a formal multicurve, and let  $S \xrightarrow{u} C$  be a section. Then the subscheme  $uS \subset C$  is a regular hypersurface, and the ideal  $I_{uS}$  is generated by the function  $d'(c) = d(u(q(c)), c)$ , or equivalently*

$$d' = (C = S \times_S C \xrightarrow{u \times 1} C \times_S C \xrightarrow{d} \mathbb{A}^1).$$

PROOF. Take  $X = S$  in the lemma.  $\square$

PROOF OF PROPOSITION 2.10. First suppose that the zero section is a regular hypersurface, so we can choose a coordinate  $x$ . It follows easily from axiom (b) that  $R \simeq \prod_{k=0}^{\infty} \mathcal{O}_S$  as topological  $\mathcal{O}_S$ -modules, so  $R \widehat{\otimes} R = \prod_{k=0}^{\infty} R$  as  $R$ -modules, so  $1 \otimes x$  is a regular element in  $R \otimes R$ . If we regard  $x$  as a function on  $C$ , this says that the function  $x_1: (a, b) \mapsto x(b)$  is a regular element of  $\mathcal{O}_{C \times_S C}$ , whose vanishing locus is precisely the closed subscheme where  $b = 0$ . The map  $s: (a, b) \mapsto (a, b - a)$  is an automorphism of  $C \times_S C$ , and  $s^*x_1$  is the function  $d(a, b) = x(b - a)$ . As  $s$  is an automorphism, we see that  $d$  is regular and its vanishing locus is the subscheme where  $a = b$ , or in other words the diagonal.  $\square$

The converse is the case  $u = 0$  of Corollary 2.12.  $\square$

To formulate the definition of an equivariant formal group, we need some basic notions about divisors.

DEFINITION 2.13. Let  $S$  be a scheme, and let  $X$  be a formal scheme over  $S$ . We say that  $X$  is *finite* over  $S$  if it is an ordinary informal scheme and  $\mathcal{O}_X$  is finitely generated as an  $\mathcal{O}_S$ -module. We say that  $X$  is *very flat* over  $S$  if it is an ordinary informal scheme and  $\mathcal{O}_X$  is projective as an  $\mathcal{O}_S$ -module. More generally, if  $X$  and  $S$  are both formal schemes we say that  $X$  is finite (resp. very flat) over  $S$  if for all informal schemes  $S'$  and all maps  $S' \rightarrow S$ , the resulting map  $X \times_S S' \rightarrow S'$  is finite (resp. very flat).

DEFINITION 2.14. A *divisor* on  $C$  is a scheme of the form  $D = \text{spec}(\mathcal{O}_C/J)$ , where  $J$  is an open ideal generated by a single regular element, and  $\mathcal{O}_C/J$  is a finitely generated projective module over  $\mathcal{O}_S$ . Thus  $D$  is a regular hypersurface in  $C$  and is finite and very flat over  $S$ . Strictly speaking, we should refer to such subschemes as *effective divisors*, but we will have little need for more general divisors in this memoir.

If  $D_i = \text{spf}(R/J_i)$  is a divisor for  $i = 0, 1$  then we put  $D_0 + D_1 := \text{spf}(R/(J_0 J_1))$ , which is easily seen to be another divisor.

The *degree* of  $D$  is the rank of  $\mathcal{O}_D$  over  $k$ . Note that this need not be constant, but that  $S$  can be split as a finite disjoint union of pieces over which  $D$  has constant degree.

If  $T$  is a scheme over  $S$ , then a divisor on  $C$  over  $T$  means a divisor on the formal multicurve  $T \times_S C$  over  $T$ .

Note that if  $D$  is an effective divisor of degree one, then the projection  $D \xrightarrow{\pi} S$  is an isomorphism, so the map  $S \xrightarrow{\pi^{-1}} D \subset C$  is a section of  $C$ . Conversely, if  $u: S \rightarrow C$  is a section, then (by Corollary 2.12) the image  $uS$  is a divisor of degree one, which is conventionally denoted by  $[u]$ .

In the case of ordinary formal curves, it is well-known that there is a moduli scheme  $\text{Div}_d^+(C)$  for effective divisors of degree  $d$  on  $C$ , and that it can be identified with the symmetric power  $C^d/\Sigma_d$ . Analogous facts are true for multicurves, but much more difficult to prove. We will return to this in Section 15.

Let  $A$  be a finite abelian group (with the group operation written additively). We write  $A^*$  for the dual group  $\text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ . This gives us a group scheme

$$A^* \times S = \coprod_{\alpha \in A^*} S = \text{spec}(\coprod_{\alpha \in A^*} \mathcal{O}_S)$$

over  $S$ .

DEFINITION 2.15. Let  $X = \text{spf}(R)$  be a formal scheme, and let  $Y = \text{spf}(R/J)$  be a closed formal subscheme. We say that  $X$  is a *formal neighbourhood* of  $Y$  if  $R$  is isomorphic to  $\varprojlim_m R/J^m$  as a topological ring, or equivalently if  $X = \varinjlim_m \text{spf}(R/J^m)$ , which essentially means that every point in  $X$  is infinitesimally close to  $Y$ .

DEFINITION 2.16. An  $A$ -equivariant formal group or  $A$ -efg over a scheme  $S$  is a formal multicurve group  $C$  over  $S$ , together with a homomorphism  $\phi: A^* \times S \rightarrow C$ , such that  $C$  is the formal neighbourhood of the divisor

$$[\phi(A^*)] := \sum_{\alpha \in A^*} [\phi(\alpha)] \subset C.$$

REMARK 2.17. Choose a coordinate  $x$  on  $C$ , and put  $d(a, b) = x(b - a)$ . For any  $\alpha \in A^*$  we have a function  $x_\alpha$  on  $C$  defined by  $x_\alpha(a) = x(a - \phi(\alpha)) = d(\phi(\alpha), a)$ . More precisely,  $x_\alpha$  is the composite

$$C = S \times_S C \xrightarrow{\phi(\alpha) \times_S 1} C \times_S C \xrightarrow{\text{subtract}} C \xrightarrow{x} \mathbb{A}^1.$$

The vanishing locus of  $x_\alpha$  is the divisor  $[\phi(\alpha)]$ , so the vanishing locus of the product  $y := \prod_\alpha x_\alpha$  is the divisor  $[\phi(A^*)]$ . We see using Corollary 2.12 that  $y$  is a regular element in  $\mathcal{O}_C$ . The final condition in Definition 2.16 says that  $y$  is topologically nilpotent. It is not hard to deduce that  $y$  is a good parameter on  $C$ .

PROPOSITION 2.18. Let  $f$  be a monic polynomial of degree  $d > 0$  over  $\mathcal{O}_S$ , and let  $R$  be the completion of  $\mathcal{O}_S[x]$  at  $f$ . Then the scheme  $C = \text{spf}(R) = \varinjlim_k V(f^k) \subseteq \mathbb{A}_S^1$  is a formal multicurve.

PROOF. Condition (b) is clear, because  $\{x^i \mid i < dj\}$  is a basis for  $R/(f^j)$  over  $\mathcal{O}_S$ . Next, observe that

$$R \simeq \mathcal{O}_S[[y]][x]/(f(x) - y) = \mathcal{O}_S[[y]]\{x^i \mid i < d\},$$

so

$$R \hat{\otimes} R \simeq \mathcal{O}_S[[y_0, y_1]][x_0, x_1]/(f(x_0) - y_0, f(x_1) - y_1) = \mathcal{O}_S[[y_0, y_1]]\{x_0^i x_1^j \mid i, j < d\}.$$

It is clear that  $y_1 - y_0$  is not a zero-divisor in this ring, and  $y_1 - y_0 = f(x_1) - f(x_0)$  which is divisible by  $x_1 - x_0$ , so  $x_1 - x_0$  is also not a zero-divisor. The multiplication map  $\mu: R \hat{\otimes} R \rightarrow R$  is surjective, split by the map  $\sigma: R \rightarrow R \hat{\otimes} R$  given by  $\sigma(f) = f \otimes 1$ . The kernel  $I_\Delta$  is the same as the image of  $1 - \sigma\mu$ , or in other words the set of functions of the form  $f(y_0, y_1, x_0, x_1) - f(y_0, y_0, x_0, x_0)$ . All such functions lie in the ideal generated by  $(y_1 - y_0, x_1 - x_0)$  and  $x_1 - x_0$  divides  $y_1 - y_0$  so  $I_\Delta = (x_1 - x_0)$ , so (c) holds.  $\square$

DEFINITION 2.19. We say that a formal multicurve  $C$  over  $S$  is *embeddable* if it has the form  $\varinjlim_k V(f^k)$  as above for some monic polynomial  $f$ .

We will see in Section 16 that any formal multicurve can be made embeddable by a faithfully flat base change; this allows us to reduce many questions to the embeddable case.

LEMMA 2.20. *Suppose that  $k = \mathcal{O}_S$  is an algebraically closed field, and that  $C$  is a formal multicurve over  $S$ . Then  $C$  is a finite disjoint union of copies of  $\widehat{\mathbb{A}}_S^1$ , and is embeddable.*

PROOF. Let  $y \in R$  be a good parameter. Then the ring  $\overline{R} := R/y$  is a finite-dimensional algebra over the field  $k$ , so it splits as a finite product of local algebras. As  $R$  is complete at  $(y)$  we can lift this splitting to  $R$ , which splits  $C$  as a disjoint union, say  $C = C_1 \amalg \dots \amalg C_r$ . One can check that each  $C_i$  is a formal multicurve. Put  $R_i = \mathcal{O}_{C_i}$ , so  $R = R_1 \times \dots \times R_r$ . Let  $y_i$  be the component of  $y$  in  $R_i$  and put  $\overline{R}_i = R_i/y_i$ , so  $\overline{R} = \overline{R}_1 \times \dots \times \overline{R}_r$ . Moreover,  $\overline{R}_i$  is local, with maximal ideal  $\mathfrak{m}_i$  say. As  $k$  is algebraically closed we see that  $\overline{R}_i/\mathfrak{m}_i = k$ . This gives an augmentation  $u_i^*: R_i \rightarrow k$ , or equivalently a section  $u_i: S \rightarrow C_i$ . It follows from Corollary 2.12 that the kernel of  $u_i^*$  is generated by a single regular element, say  $x_i$ . This means that the image of  $x_i$  in  $\overline{R}_i$  generates  $\mathfrak{m}_i$ , so  $\overline{R}_i = k \oplus \overline{R}_i x_i$ . Next, the descending chain of finite-dimensional  $k$ -spaces  $(x_i^n)$  must eventually stabilise, say with  $(x_i^n) = (x_i^{n+1})$ . This means that  $x_i^n = x_i^{n+1}y$  for some  $y$ , so  $(1 - x_i y)x_i^n = 0$  but  $1 - x_i y \in \overline{R}_i \setminus (x_i) = \overline{R}_i^\times$  so  $x_i^n = 0$ . We can combine this with the fact that  $\overline{R}_i = k \oplus \overline{R}_i x_i$  to see that  $\overline{R}_i \simeq k[x_i]/x_i^m$  for some  $m \leq n$ , and thus  $y_i$  divides  $x_i^m$ . On the other hand, we clearly have  $u^*(y_i) = 0$  so  $x_i$  divides  $y_i$ . One can now check that the obvious map  $k[x_i] \rightarrow R_i$  extends to give a map  $k[[x_i]] \rightarrow R_i$  which is an isomorphism, so  $C_i \simeq \widehat{\mathbb{A}}_S^1$ .

Finally, as  $k$  is algebraically closed, it is certainly infinite, so we can choose distinct elements  $\lambda_1, \dots, \lambda_r \in k$  say. If we put  $f(x) = \prod_i (x - \lambda_i)$  we find that the completion of  $k[x]$  at  $(f)$  is isomorphic to  $\prod_{i=1}^r k[[x]]$  and thus to  $\mathcal{O}_C$ . This proves that  $C$  is embeddable.  $\square$

We next explain how to find a topological basis for the ring  $\mathcal{O}_C$ , when  $(C, \phi)$  is an  $A$ -equivariant formal group. Put  $n = |A| = |A^*|$  and choose an enumeration of the elements of  $A^*$ , say  $A^* = \{\alpha_0, \dots, \alpha_{n-1}\}$ . More generally, for  $i \in \mathbb{N}$  we can write  $i = nj + k$  with  $0 \leq k < n$  and we define  $\alpha_i = \alpha_k$ . We then define  $e_i \in \mathcal{O}_C$  by

$$e_i(a) = \prod_{j < i} x(a - \phi(\alpha_j)) = \prod_{j < i} x_{\alpha_j}(a).$$

Note that  $e_n$  is just the good parameter  $y = \prod_{\alpha \in A^*} x_\alpha$  and more generally  $e_{nj+k} = y^j e_k$ , so  $e_i \rightarrow 0$  in the  $y$ -adic topology on  $\mathcal{O}_C$ . We can thus define a map  $\mu: \prod_{i \in \mathbb{N}} \mathcal{O}_S \rightarrow \mathcal{O}_C$  by  $\mu(t) = \sum_i t_i e_i$ . This also induces a map  $\mu_i: \prod_{j < i} \mathcal{O}_S \rightarrow \mathcal{O}_C/e_i$ .

PROPOSITION 2.21. *The maps  $\mu_i$  and  $\mu$  are isomorphisms.*

PROOF. We can define a map  $\nu_0: \mathcal{O}_S \times \mathcal{O}_C \rightarrow \mathcal{O}_C$  by  $\nu_0(t, u) = t + ux$ , and it follows from the definition of a coordinate that this is a bijection. We can twist this by the translation action of  $\phi(\alpha)$  to see that the map  $\nu_\alpha(t, u) = t + ux_\alpha$  is also a bijection.

Now define  $\mu'_i: (\prod_{j < i} \mathcal{O}_S) \times \mathcal{O}_C \rightarrow \mathcal{O}_C$  by  $\mu'_i(t, u) = (\sum_{j < i} t_j e_j) + ue_i$ . We see that  $\mu'_0 = 1$  and  $\mu'_{i+1} = \mu'_i \circ (1 \times \nu_{\alpha_i})$ ; it follows that  $\mu'_i$  is an isomorphism for all  $i$ . We can reduce mod  $e_i$  to see that  $\mu_i$  is an isomorphism, and then pass to the limit to see that  $\mu$  is an isomorphism.  $\square$

REMARK 2.22. If we knew in advance that  $\mu_n$  was an isomorphism, we could use Lemma 2.6 to see that  $\mu_i$  is an isomorphism for all  $i$  including  $i = \infty$ . However, that approach does not save us any effort, as the proof for  $\mu_n$  is no easier than the proof for  $\mu_i$  for all  $i$ .



## CHAPTER 3

### Differential forms

We next recall some basic ideas about differential forms, and record some formulae that will be useful later in our study of residues.

Given a formal multicurve  $C$  over  $S$ , we put

$$\Omega = \Omega_{C/S}^1 = I_{\Delta}/I_{\Delta}^2,$$

and call this the module of differential forms on  $C$ .

We also put  $\Delta_2 = \text{spf}(\mathcal{O}_{C \times_S C}/I_{\Delta}^2)$ , and regard this as the second-order infinitesimal neighbourhood of  $\Delta$  in  $C \times_S C$ . In these terms,  $\Omega$  is the module of functions on  $\Delta_2$  that vanish on  $\Delta$ .

Given a difference function  $d \in I_{\Delta}$ , we let  $\alpha$  be the image of  $d$  in  $\Omega$ ; this generates  $\Omega$  freely as a module over  $\mathcal{O}_C$ , so we can regard  $\Omega$  as a trivialisable line bundle on  $C$ .

For any function  $f \in \mathcal{O}_C$ , we write  $\mathbf{d}f$  for the image of  $1 \otimes f - f \otimes 1$  in  $\Omega$ ; this equivalently the function  $(a, b) \mapsto f(b) - f(a)$  on  $\Delta_2$ . As usual, we have the Leibniz rule

$$\mathbf{d}(fg) = f\mathbf{d}(g) + g\mathbf{d}(f).$$

Now suppose that  $C$  has a commutative group structure. In particular, this gives a zero-section  $Z \subset C$ , and we write  $Z_2 = \text{spec}(\mathcal{O}_C/I_Z^2)$  and

$$\omega = I_Z/I_Z^2 = \{ \text{functions on } Z_2 \text{ that vanish on } Z \}.$$

The map  $b \mapsto (0, b)$  gives an inclusion  $Z_2 \rightarrow \Delta_2$  and thus a map  $\Omega \rightarrow \omega$ , which in turn gives an isomorphism  $\Omega|_Z = \omega$  of line bundles on  $S$ . The image of  $\mathbf{d}f$  under this map is the element  $\mathbf{d}_0f$  corresponding to the function  $b \mapsto f(b) - f(0)$  on  $Z_2$ . If  $x$  is a coordinate on  $C$ , then  $\mathbf{d}_0x$  generates  $\omega$  freely as a module over  $\mathcal{O}_S$ .

Next, for any function  $f \in \mathcal{O}_C$  we define a function  $\mathbf{D}f$  on  $\Delta_2$  by

$$(\mathbf{D}f)(a, b) = f(b - a) - f(0).$$

This construction gives a map  $\mathbf{D}: \mathcal{O}_C \rightarrow \Omega$ . If  $x$  is a coordinate then  $\mathbf{D}x$  is the restriction of the usual difference function  $d(a, b) = x(b - a)$  to  $\Delta_2$ , so it is a generator of  $\Omega$ .

It is easy to see that  $\mathbf{D}f$  depends only on  $\mathbf{d}_0f$ , and thus that  $\mathbf{D}$  induces an  $\mathcal{O}_S$ -linear inclusion  $\omega \rightarrow \Omega$ , right inverse to the restriction map  $\Omega \rightarrow \Omega|_Z = \omega$ . A differential form is said to be *invariant* if it lies in the image of this map.

By extension of scalars, we obtain an  $\mathcal{O}_C$ -linear map  $\mathcal{O}_C \otimes_{\mathcal{O}_S} \omega \rightarrow \Omega$ , sending  $f \otimes \mathbf{d}_0g$  to  $f\mathbf{D}g$ . In particular, it sends  $f \otimes \mathbf{d}_0x$  to  $f\mathbf{D}x$ , and so is an isomorphism.



## CHAPTER 4

# Equivariant projective spaces

We now start to build a connection between multicurves and  $A$ -equivariant topology (where  $A$  is a finite abelian group). Naturally, this involves the generalised cohomology of the projective spaces of representations of  $A$ . In this section, we assemble some facts about the homotopy theory of such projective spaces.

For  $\alpha \in A^* = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$  we write  $L_\alpha$  for  $\mathbb{C}$  with  $A$  acting by  $a \cdot z = e^{2\pi i \alpha(a)} z$ . In particular,  $L_0$  has trivial action, and  $L_\alpha \otimes L_\beta = L_{\alpha+\beta}$ . For any finite-dimensional representation  $V$ , we put

$$V[\alpha] = \{v \in V \mid av = e^{2\pi i \alpha(a)} v \text{ for all } a \in A\} \simeq \text{Hom}_A(L_\alpha, V) \otimes L_\alpha.$$

It is well-known that  $V = \bigoplus_\alpha V[\alpha]$  and  $\text{Hom}_{\mathbb{C}[A]}(V, W) = \bigoplus_\alpha \text{Hom}_{\mathbb{C}}(V[\alpha], W[\alpha])$ . It follows that if there exists an equivariant linear embedding  $V \rightarrow W$ , then the space of such embeddings is connected, giving a canonical homotopy class of maps  $PV \rightarrow PW$  of projective spaces.

We write  $\mathcal{U}[\alpha] = L_\alpha \otimes \mathbb{C}^\infty$ , and  $\mathcal{U} = \mathcal{U}_A = \bigoplus_\alpha \mathcal{U}[\alpha]$ , so  $\mathcal{U}$  is a complete  $A$ -universe. We write  $P\mathcal{U}$  for the projective space associated to  $\mathcal{U}$ , which has a natural  $A$ -action. By the previous paragraph, for any finite-dimensional representation  $V$ , there is a canonical map  $PV \rightarrow P\mathcal{U}$  up to homotopy. Similarly, the space of equivariant linear isometries  $\mathcal{U} \otimes \mathcal{U} \rightarrow \mathcal{U}$  is contractible, which gives a canonical homotopy class of maps  $P\mathcal{U} \times P\mathcal{U} \rightarrow P\mathcal{U}$ , making  $P\mathcal{U}$  an abelian group up to equivariant homotopy. We can choose a conjugate-linear equivariant automorphism  $\chi: \mathcal{U} \rightarrow \mathcal{U}$ , and the resulting map  $P\mathcal{U} \rightarrow P\mathcal{U}$  is the negation map for our group structure.

It is well-known that  $P\mathcal{U}$  is the classifying space for equivariant complex line bundles. More precisely, for any  $A$ -space  $X$ , we write  $\text{Pic}_A(X)$  for the group of isomorphism classes of equivariant complex line bundles over  $X$ . Let  $T$  denote the tautological line bundle over  $P\mathcal{U}$ , so  $T \in \text{Pic}(P\mathcal{U})$ . Then for any  $A$ -space  $X$ , the construction  $[f] \mapsto [f^*T]$  gives a group isomorphism  $[X, P\mathcal{U}]^A \simeq \text{Pic}_A(X)$ . Note that we regard  $T$  as the universal example; some other treatments in the literature use the dual bundle  $T^* = \mathcal{O}(1)$  instead.

Note that  $A$  acts by scalars on  $\mathcal{U}[\alpha]$ , and thus acts as the identity on  $P\mathcal{U}[\alpha] \subset P\mathcal{U}$ . Moreover, the map  $L \mapsto L_\alpha \otimes L$  gives a homeomorphism  $\mathbb{C}P^\infty = P(\mathbb{C}^\infty) \rightarrow P\mathcal{U}[\alpha]$ . Using this, we have a homeomorphism  $(P\mathcal{U})^A = \coprod_\alpha P\mathcal{U}[\alpha] = A^* \times \mathbb{C}P^\infty$ , and thus a bijection  $\pi_0((P\mathcal{U})^A) = A^*$ , which is easily seen to respect the natural group structures. Thus, the group structure on  $P\mathcal{U}$  gives a translation action (up to homotopy) of  $A^*$  on  $P\mathcal{U}$ .

**DEFINITION 4.1.** We write  $\tau_\alpha: P\mathcal{U} \rightarrow P\mathcal{U}$  for translation by an element  $\alpha \in A^*$ .

For various purposes we will need to use an  $A$ -fixed basepoint in  $P\mathcal{U}$ . We have embeddings  $L_\alpha \rightarrow \mathcal{U}[\alpha] \rightarrow \mathcal{U}$ , and  $PL_\alpha$  is an  $A$ -fixed point. Any other fixed point

lies in the same component of  $(PU)^A$  as  $PL_\alpha$  for some  $\alpha$ , so it can be replaced by  $PL_\alpha$  for most purposes. Moreover, the map  $\tau_\alpha$  gives a homotopy equivalence of pairs  $(PU, PL_\beta) \rightarrow (PU, PL_{\alpha+\beta})$ . Where not otherwise stated, we use  $PL_0$  as the basepoint.

**PROPOSITION 4.2.** *Let  $V$ ,  $W$  and  $X$  be unitary representations of  $A$ , where  $V$  and  $W$  have finite dimension and  $X$  is a colimit of finite-dimensional subrepresentations. Put  $U = V \oplus W \oplus X$ . Then there is a homotopy-commutative diagram as follows, in which the maps marked  $q$  are the obvious collapses, the maps marked  $j$  are the obvious inclusions, and  $\delta$  is the diagonal map.*

$$\begin{array}{ccc} PU & \xrightarrow{\delta} & PU \times PU \\ q_{V \oplus W} \downarrow & & \downarrow q_V \wedge q_W \\ PU/P(V \oplus W) & \xrightarrow[\bar{\delta}]{} & P(V \oplus X)/PV \wedge P(W \oplus X)/PW \xrightarrow[j \wedge j]{} PU/PV \wedge PU/PW \end{array}$$

Moreover, if  $\dim(X) = 1$  then  $\bar{\delta}$  is just the standard homeomorphism

$$S^{\text{Hom}(X, V \oplus W)} = S^{\text{Hom}(X, V)} \wedge S^{\text{Hom}(X, W)}.$$

All maps and homotopies are natural for isometric embeddings of  $V$ ,  $W$  and  $X$ .

**REMARK 4.3.** For any ring spectrum  $E$ , the above diagram gives a map

$$\bar{\delta}^*: E^*(P(V \oplus X), PV) \otimes E^*(P(W \oplus X), PW) \rightarrow E^*(PU, P(V \oplus W)).$$

In his unpublished thesis [6], Cole writes  $a * b$  for  $\bar{\delta}^*(a \otimes b)$ . The idea of using this construction seems to be original to that thesis; our approach differs only in being somewhat more geometric.

**PROOF.** Assume for the moment that  $X$  is finite-dimensional. We start by defining a map

$$\bar{\gamma}: PU/P(V \oplus W) \rightarrow PU/PV \wedge PU/PW,$$

which will be homotopic to  $(j \wedge j) \circ \bar{\delta}$ . For  $u = (v, w, x) \in U^\times := U \setminus \{0\}$  we put

$$s = s(u) = (\|w\| - \|v\|)/(\|v\| + \|w\| + \|x\|).$$

Note that  $s(u) \in [-1, 1]$ , and  $s(\lambda u) = s(u)$  for all  $\lambda \in \mathbb{C}^\times$ , and  $s(u) \geq 0$  iff  $\|w\| \geq \|v\|$ . We next define  $\alpha, \beta: U^\times \rightarrow U$  by

$$\alpha(v, w, x) = \begin{cases} ((1-s)v, sw, x) & \text{if } s \geq 0 \\ (v, 0, x) & \text{if } s \leq 0 \end{cases}$$

$$\beta(v, w, x) = \begin{cases} (0, w, x) & \text{if } s \geq 0 \\ (-sv, (1+s)w, x) & \text{if } s \leq 0. \end{cases}$$

Note that  $\alpha(\lambda u) = \lambda \alpha(u)$  and similarly for  $\beta$ .

We claim that when  $u \neq 0$ , the line joining  $u$  to  $\alpha(u)$  never passes through 0 (so in particular  $\alpha(u) \neq 0$ ). Indeed, if  $s \leq 0$ , then the points on the line have the form  $(v, tw, x)$  for  $0 \leq t \leq 1$ . Thus, the line can only pass through zero if  $v = x = 0$ . The relation  $s \leq 0$  means that  $\|w\| \leq \|v\| = 0$ , so  $w = 0$  as well, contradicting the assumption that  $u \neq 0$ . In the case  $s > 0$ , the points on the line have the form  $((1-ts)v, (1-t+ts)w, x)$ . As  $s > 0$  and  $0 \leq t \leq 1$  we have  $1-t+ts > 0$ . For the line to pass through zero we must thus have  $x = w = 0$ , and the relation  $s \geq 0$

means that  $\|v\| \leq \|w\| = 0$ , again giving a contradiction. Similarly, the line from  $u$  to  $\beta(u)$  never passes through 0.

It follows that  $\alpha$  and  $\beta$  induce self-maps of  $PU$  that are homotopic to the identity, so the map  $\gamma = (\alpha, \beta): PU \rightarrow PU \times PU$  is homotopic to the diagonal map  $\delta$ .

Next, note that if  $u \in V \oplus W$ , then for  $s \geq 0$  we have  $\gamma(u) \in U \times W$ , and for  $s \leq 0$  we have  $\gamma(u) \in V \times U$ . It follows that the induced map on projective spaces has

$$\gamma(P(V \oplus W)) \subseteq (PU \times PW) \cup (PV \times PU),$$

so there is an induced map

$$\bar{\gamma}: PU/P(V \oplus W) \rightarrow PU/PV \wedge PU/PW.$$

As  $\gamma$  is homotopic to  $\delta$ , we see that  $\bar{\gamma} \circ q_{V \oplus W} \simeq (q_V \wedge q_W) \circ \delta$ .

To construct the map  $\bar{\delta}$ , we need a slightly different model. Clearly

$$PU \setminus P(V \oplus W) = (V \times W \times X^\times)/\mathbb{C}^\times = (V \times W \times S(X))/S^1,$$

and  $PU/P(V \oplus W)$  is the one-point compactification of this. Similarly,  $P(V \oplus X)/PV \wedge P(W \oplus X)/PW$  is the one-point compactification of the space  $(V \times S(X))/S^1 \times (W \times S(X))/S^1$ . We can thus define  $\bar{\delta}$  by giving a proper map

$$V \times W \times S(X) \rightarrow V \times S(X) \times W \times S(X)$$

with appropriate equivariance. The map in question just sends  $(v, w, x)$  to  $(v, x, w, x)$ .

If  $X$  is one-dimensional and  $(v, x) \in V \times S(X)$  then we have a linear map  $\alpha: X \rightarrow V$  given by  $\alpha(x) = v$ , which does not change if we multiply  $(v, x)$  by an element of  $S^1$ . This gives a homeomorphism  $(V \times S(X))/S^1 = \text{Hom}(X, V)$ , and thus  $P(V \oplus X)/PV = S^{\text{Hom}(X, V)}$ . It is easy to see that with this identification,  $\bar{\delta}$  is just the standard homeomorphism

$$S^{\text{Hom}(X, V \oplus W)} = S^{\text{Hom}(X, V)} \wedge S^{\text{Hom}(X, W)}.$$

We now show that  $(j \wedge j) \circ \bar{\delta} \simeq \bar{\gamma}$ . Put

$$T = \{((v_0, w_0, x_0), (v_1, w_1, x_1)) \in U^2 \mid \|(w_0, x_0)\| = \|(v_1, x_1)\| = 1\},$$

so that  $PU/PV \wedge PU/PW$  is the one-point compactification of  $T/(S^1 \times S^1)$ . Define maps

$$\theta_t: V \times W \times S(X) \rightarrow T$$

for  $0 \leq t \leq 1$  by

$$\theta_t(v, w, x) = \begin{cases} \left( \frac{((1-st)v, stw, x)}{\|(stw, x)\|}, (0, w, x) \right) & \text{if } s \geq 0 \\ \left( (v, 0, x), \frac{(-stv, (1+st)w, x)}{\|(-stv, x)\|} \right) & \text{if } s \leq 0, \end{cases}$$

where  $s = (\|w\| - \|v\|)/(\|v\| + \|w\| + \|x\|)$  as before. (Note that both clauses give  $\theta_t(v, w, x) = ((v, 0, x), (0, w, x))$  if  $s = 0$ , so the two clauses are consistent.)

We claim that the maps  $\theta_t$  are proper. To see this, put

$$\nu((v_0, w_0, x_0), (v_1, w_1, x_1)) = \max(\|v_0\|, \|w_1\|),$$

and  $T_k = \{t \in T \mid \nu(t) \leq k\}$ . Every compact subset of  $T$  is contained in some  $T_k$ , so it will be enough to show that  $\theta_t^{-1}T_k$  is compact. In the case  $s \geq 0$  we have  $0 \leq 1 - st \leq 1$  and  $\|(stw, x)\| \geq \|x\| = 1$  so  $\|((1-st)v/\|(stw, x)\|)\| \leq \|v\| \leq \|w\|$ , so  $\nu(\theta_t(v, w, x)) = \|w\|$ . Similarly, when  $s \leq 0$  we have  $\nu(\theta_t(v, w, x)) = \|v\|$ , so in

general  $\nu(\theta_t(v, w, x)) = \max(\|v\|, \|w\|)$ . It follows immediately that  $\theta_t$  is proper, and we get an induced family of maps

$$\theta_t: PU/P(V \oplus W) \rightarrow PU/PV \wedge PU/PW.$$

We see from the definitions  $\theta_0 = (j \wedge j) \circ \bar{\delta}$  and  $\theta_1 = \bar{\gamma}$ . The proposition follows easily (for the case where  $X$  has finite dimension).

If  $X$  has infinite dimension, we apply the above to all finite dimensional subrepresentations of  $X$ . We see by inspection that all constructions pass to the colimit, so the conclusion is valid for  $X$  itself.  $\square$

By an evident inductive extension, we obtain the following:

**COROLLARY 4.4.** *Let  $L_1, \dots, L_d$  be one-dimensional representations of  $A$ , and let  $X$  be as above. Put  $Y = \bigoplus_i L_i$  and  $U = Y \oplus X$ . Then there is a homotopy-commutative diagram as follows:*

$$\begin{array}{ccc} PU & \xrightarrow{\delta} & PU^r \\ q \downarrow & & \downarrow q \\ PU/PY & \xrightarrow{\bar{\delta}} \bigwedge_i P(L_i \oplus X)/PL_i \xrightarrow{j} & \bigwedge_i PU/PL_i \end{array}$$

Moreover, if  $\dim(X) = 1$  then  $\bar{\delta}$  is just the standard homeomorphism

$$S^{\mathrm{Hom}(X,Y)} = \bigwedge_i S^{\mathrm{Hom}(X,L_i)}. \quad \square$$

We mention one more useful special case.

**COROLLARY 4.5.** *For any ring spectrum  $E$ , the group  $E^*(P(V \oplus W), PV)$  is naturally a module over  $E^*PW$ .*

**PROOF.** Take  $W = 0$  in Proposition 4.2 to get a map

$$\bar{\delta}: P(V \oplus X)/PV \rightarrow (P(V \oplus X)/PV) \wedge PX_+,$$

and thus a map

$$\bar{\delta}^*: E^0 PX \otimes E^0(P(V \oplus X), PV) \rightarrow E^0(P(V \oplus X), PV).$$

If we identify  $P(V \oplus X)/PV$  as the one-point compactification of  $(V \times S(X))/S^1$  then the formula is just  $\bar{\delta}([v, x]) = ([v, x], [x])$ ; this is clearly coassociative and counital, so the corresponding map in cohomology gives a module structure. A slight change of notation recovers the stated corollary.  $\square$

We conclude with some further miscellaneous observations about the space  $PU$ .

**PROPOSITION 4.6.** *The space  $PU$  is equivariantly equivalent to  $F(EA_+, \mathbb{C}P^\infty)$  (where  $\mathbb{C}P^\infty$  is the usual space with trivial  $A$ -action). Equivalently,  $PU$  is the second space in the Borel cohomology spectrum  $F(EA_+, H)$ , so  $[X, PU]^A = H^2(X_{hA})$  for any  $A$ -space  $X$ . Moreover, the space  $\Omega PU$  is equivariantly equivalent to  $S^1$  with the trivial action.*

**PROOF.** There is an evident inclusion  $\mathbb{C}P^\infty = P(\mathcal{U}^A) \rightarrow (PU)^A \rightarrow PU$ , which is a nonequivariant equivalence. It follows that the resulting map  $F(EA_+, \mathbb{C}P^\infty) \rightarrow F(EA_+, PU)$  is an equivariant equivalence (see Lemma 4.7). On the other hand, the collapse map  $EA_+ \rightarrow S^0$  gives a map  $j: PU \rightarrow F(EA_+, PU) \simeq F(EA_+, \mathbb{C}P^\infty)$ .

We claim that this is an equivalence. Indeed, if we take fixed points for a subgroup  $A_0 \leq A$  we get a map  $A_0^* \times \mathbb{C}P^\infty \rightarrow F((BA_0)_+, \mathbb{C}P^\infty)$  of commutative  $H$ -spaces. It is clear that

$$\pi_k(A_0^* \times \mathbb{C}P^\infty) = \begin{cases} A_0^* & \text{if } k = 0 \\ \mathbb{Z} & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, we have

$$\pi_k F((BA_0)_+, \mathbb{C}P^\infty) = [\Sigma^k(BA_0)_+, K(\mathbb{Z}, 2)] = H^{2-k} BA_0.$$

This clearly vanishes for  $k > 2$  and gives  $\mathbb{Z}$  for  $k = 2$ . The short exact sequence  $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$  gives long exact sequences of cohomology groups, using which we find that  $H^1 BA_0 = 0$  and  $H^2 BA_0 = A_0^*$ . This shows that  $\pi_* F((BA_0)_+, \mathbb{C}P^\infty)$  is abstractly isomorphic to  $\pi_*(A_0^* \times \mathbb{C}P^\infty)$ . With a little more work one sees that the isomorphism is induced by  $j$ , and the first part of the proposition follows.

We now see that

$$\Omega PU \simeq \Omega F(EA_+, \mathbb{C}P^\infty) = F(EA_+, \Omega \mathbb{C}P^\infty) = F(EA_+, S^1).$$

As above we find that

$$\pi_k(F(EA_+, S^1)^{A_0}) = H^{1-k} BA_0 = \begin{cases} \mathbb{Z} & \text{if } k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the obvious map  $S^1 \rightarrow F(EA_+, S^1)$  is an equivariant equivalence.  $\square$

For the convenience of the reader we record a proof of a standard lemma that was used above.

**LEMMA 4.7.** *If  $f: X \rightarrow Y$  is an  $A$ -equivariant based map and a nonequivariant homotopy equivalence, then the induced map  $F(EA_+, X) \rightarrow F(EA_+, Y)$  is an equivariant homotopy equivalence.*

**PROOF.** Let  $\mathcal{C}$  be the category of those based  $A$ -spaces  $C$  for which  $f_*: F(C, X) \rightarrow F(C, Y)$  is an equivariant equivalence. For any based  $A$ -spaces  $P, Q$  we have

$$[P, F(A_+ \wedge Q, X)]^A = [A_+ \wedge P \wedge Q, X]^A = [P \wedge Q, X].$$

Using this and the Yoneda lemma we see that  $A_+ \wedge Q \in \mathcal{C}$ . Moreover, the category  $\mathcal{C}$  is closed under homotopy pushouts, so it follows by cellular induction that it contains all finite free based  $A$ -CW-complexes. Moreover,  $\mathcal{C}$  is closed under telescopes, so one can take a colimit over skeleta to see that  $\mathcal{C}$  contains all free based  $A$ -CW-complexes. In particular we have  $EA_+ \in \mathcal{C}$ , as required.  $\square$

**PROPOSITION 4.8.** *Let  $T$  be the tautological line bundle over  $PU$ , and let  $S(T^n)$  be the unit circle bundle in the  $n$ 'th tensor power of  $T$ . Then  $S(T^n)$  is equivariantly equivalent to  $F(EA_+, B(\mathbb{Z}/n))$ .*

**PROOF.** First, we let  $\mathbb{Z}/n$  act freely on the contractible space  $S(\mathcal{U})$  by multiplication by  $n$ 'th roots of unity, so  $S(\mathcal{U})/(\mathbb{Z}/n)$  is a model for  $B(\mathbb{Z}/n)$ . Given a point  $v \in S(\mathcal{U})$  we have a line  $L = \mathbb{C}v \in P(\mathcal{U})$  and an element  $v^{\otimes n} \in L^n$ , giving a point  $(L, v^{\otimes n}) \in S(T^n)$  that depends only on the  $\mathbb{Z}/n$ -orbit of  $v$ . This construction gives a homeomorphism  $S(\mathcal{U})/(\mathbb{Z}/n) \rightarrow S(T^n)$ , so  $S(T^n)$  is also a model (nonequivariantly) for  $B(\mathbb{Z}/n)$ .

We now analyse the equivariant picture. Suppose that  $(L, u) \in S(T^n)$  is fixed by a subgroup  $A_0 \leq A$ . We see that  $A_0$  acts on  $L$  by some character  $\alpha \in A_0^*$ , so  $A_0$  acts on  $u$  by  $n\alpha$ , but  $u$  is fixed so  $n\alpha = 0$ . Given that  $n\alpha = 0$ , we see that every point in  $L^n$  is fixed by  $A_0$ . Using this, we see that  $S(T^n)^{A_0} = A_0^*[n] \times B(\mathbb{Z}/n)$ , where  $A_0^*[n]$  denotes the subgroup of points of order  $n$  in  $A_0^*$ . Using this, we find that  $\pi_* S(T^n)^{A_0} = H^{1-*}(BA_0; \mathbb{Z}/n)$ , and the claim follows by the same method as in the previous proposition.  $\square$

It is also useful to describe  $S(\mathcal{U})$  as the universal space for a family of subgroups, as in the following definition.

**DEFINITION 4.9.** Let  $G$  be a finite group, and let  $\mathcal{F}$  be a collection of subgroups such that

- (a)  $1 \in \mathcal{F}$
- (b) If  $K \leq H \in \mathcal{F}$  then  $K \in \mathcal{F}$
- (c) If  $H \in \mathcal{F}$  and  $g \in G$  then  $gHg^{-1} \in \mathcal{F}$ .

Then  $E\mathcal{F}$  is the  $G$ -space characterised up to unique equivariant homotopy equivalence by the following properties:

- (A)  $(E\mathcal{F})^H$  is contractible if  $H \in \mathcal{F}$
- (B)  $(E\mathcal{F})^H = \emptyset$  if  $H \notin \mathcal{F}$ .

We also write  $\tilde{E}\mathcal{F}$  for the unreduced suspension  $\tilde{\Sigma}E\mathcal{F}$ , or equivalently the cofibre of the collapse map  $E\mathcal{F}_+ \rightarrow S^0$ .

(For the existence and uniqueness of  $E\mathcal{F}$ , see [10]. Alternatively, uniqueness is straightforward by obstruction theory, and we will have concrete models to prove existence in all cases that we need.)

**PROPOSITION 4.10.** Put  $\mathcal{F} = \{B \leq S^1 \times A \mid B \cap S^1 = \{1\}\}$ , which is a family of subgroups of  $S^1 \times A$ . Then the unit sphere  $S(\mathcal{U})$  is a model for  $E\mathcal{F}$ , and so  $P\mathcal{U} = (E\mathcal{F})/S^1$ .

**PROOF.** First, we let  $S^1 \subset \mathbb{C}^\times$  act on  $S(\mathcal{U})$  by multiplication, and let  $A$  act in the usual way. These actions commute and so give an action of  $S^1 \times A$ . We need only check that  $S(\mathcal{U})$  has the characterizing property of  $E\mathcal{F}$ , or in other words that  $S(\mathcal{U})^B$  is contractible for  $B \in \mathcal{F}$  and empty for  $B \notin \mathcal{F}$ . If  $B \in \mathcal{F}$  then  $B \cap S^1$  is trivial so  $B$  is the graph of a homomorphism  $\phi: A_0 \rightarrow S^1$  for some subgroup  $A_0 \leq A$ . Put

$$\mathcal{V} = \{v \in \mathcal{U} \mid a.v = \phi(a)^{-1}v \text{ for all } a \in A_0\},$$

so  $S(\mathcal{U})^B = S(\mathcal{V})$ . As  $\mathcal{U}$  is a complete  $A_0$ -universe, we see that  $\mathcal{V}$  is infinite dimensional, and so  $S(\mathcal{V})$  is contractible as required. On the other hand, as  $S^1$  acts freely on  $S(\mathcal{U})$ , it is clear that  $S(\mathcal{U})^B = \emptyset$  whenever  $B \notin \mathcal{F}$ .  $\square$



## CHAPTER 5

# Equivariant orientability

Now let  $E$  be a commutative  $A$ -equivariant ring spectrum. We next need to formulate suitable notions of orientability and periodicity for  $E$ , and deduce consequences for the rings  $E^*PV$ . Our results differ from those of [6] only in minor points of technical detail. We start by introducing some notation and auxiliary ideas.

CONVENTION 5.1. All  $A$ -spectra are implicitly assumed to be indexed by a complete  $A$ -universe.

NOTATION 5.2. Given an  $A$ -equivariant spectrum  $X$ , we write  $E^n(X)$  for the group  $[X, \Sigma^n E]^A$  of equivariant homotopy classes of equivariant maps. In many parts of the literature this is written as  $E_A^n(X)$ , but we omit the subscript to avoid cluttering the notation. If we have a  $B$ -spectrum  $Y$  (for some subgroup  $B \leq A$ ) then we will write  $\text{res}_B^A(E)^n(Y)$  or  $E^n(A_+ \wedge_B Y)$  or  $[Y, \Sigma^n E]^B$  for the group that might otherwise be denoted  $E_B^n(Y)$ . For specific choices of  $E$  the subscript may reappear as part of the name of  $E$ . For example,  $A$ -equivariant complex  $K$ -theory is represented by an  $A$ -spectrum called  $KU_A$ , and we write  $KU_A^n(X)$  for  $[X, \Sigma^n KU_A]^A$ .

Similarly, we write  $\pi_n(E)$  rather than  $\pi_n^A(E)$  for the group  $[S^n, E]^A = E^0(S^n) = E^{-n}(S^0)$ . When  $X$  is a based  $A$ -space we usually write  $\Sigma^\infty X$  (rather than  $\Sigma_A^\infty X$ ) for the corresponding  $A$ -equivariant suspension spectrum.

NOTATION 5.3. Another potential source of clutter is the distinction between reduced and unreduced cohomology groups. When  $Y$  is an  $A$ -spectrum, we write  $E^*(Y) = [Y, E]_{-*}^A$  as above. When  $X$  is an  $A$ -space, we write  $E^*(X)$  for the unreduced groups, so  $E^*(X) = E^*(\Sigma_+^\infty X)$ . If  $X$  has an  $A$ -fixed basepoint then we can also define reduced groups  $\tilde{E}^*(X) = E^*(\Sigma^\infty X)$ . Sometimes we will use notation that is slightly ambiguous, in that it could refer to a based space or to its suspension spectrum. In such cases we will use either a tilde or an explicit  $\Sigma^\infty$  to resolve the ambiguity.

DEFINITION 5.4. Let  $R$  be an  $E$ -algebra spectrum, and  $M$  a module spectrum over  $R$ . We say that  $M$  is a free  $R$ -module if it is equivalent as an  $R$ -module to a wedge of (unsuspended) copies of  $R$ , or equivalently, there is a family of elements  $e_i \in \pi_0 M$  such that the resulting maps  $\bigoplus_i [\Sigma^n A/B_+, R] \rightarrow [\Sigma^n A/B_+, M]$  are isomorphisms for all  $n \in \mathbb{Z}$  and all  $B \leq A$ . We say that such elements  $e_i$  are *universal generators* for  $\pi_0 M$  over  $\pi_0 R$ . We will often leave the identification of  $R$  and  $M$  implicit. For example, if we say that an element  $e$  is a universal generator for  $E^0(X, Y)$  over  $E^0 X$ , we are referring to the case  $R = F(X_+, E)$  and  $M = F(X/Y, E)$ .

DEFINITION 5.5. Let  $E$  an  $A$ -equivariant ring spectrum, and consider a class  $x \in E^0(P\mathcal{U}, PL_0)$ . For any  $\alpha \in A^*$  we can embed  $L_\alpha \oplus L_0$  in  $\mathcal{U}$ , and thus restrict  $x$  to get a class  $u_{L_\alpha} \in E^0(P(L_\alpha \oplus L_0), PL_0) = \tilde{E}^0 S^{L_\alpha}$ . This in turn gives an  $E$ -module map  $m_\alpha: \Sigma^{L_\alpha} E \rightarrow E$ .

We say that  $x$  is a *complex coordinate* for  $E$  if for all  $\alpha$  the map  $m_\alpha$  is an equivalence, or equivalently  $u_{L_\alpha}$  generates  $\Sigma^{-L_\alpha} E$  as an  $E$ -module. We say that  $E$  is *periodically orientable* if it admits such a coordinate. We say that  $E$  is *evenly orientable* if in addition, the group  $\pi_1^B E = [\Sigma A/B_+, E]^A = E^{-1}(A/B)$  vanishes for all  $B \leq A$ .

From now on, we assume that  $E$  is periodically orientable. We choose a complex coordinate  $x$ , but as far as possible we state our results in a form independent of this choice. We write  $\bar{x} = \chi^* x$ , where  $\chi: P\mathcal{U} \rightarrow P\mathcal{U}$  is the negation map for the group structure. It is easy to see that this is again a coordinate.

Recall that for any line bundle  $L$  over  $X$ , there is an essentially unique map  $f_L: X \rightarrow P\mathcal{U}$  with  $f^* T \simeq L$  (where  $T$  is the tautological bundle over  $P\mathcal{U}$ ). We define the *Euler class* of  $L$  by

$$e(L) = f_{L*}^*(x) = f_L^*(\bar{x}).$$

Thus, the element  $x \in E^0 P\mathcal{U}$  is the Euler class of  $T^*$ , and  $\bar{x}$  is the Euler class of  $T$ .

REMARK 5.6. There is some inconsistency in the literature about whether  $e(L)$  should be  $f_L^*(x)$  or  $f_L^*(\bar{x})$ . The convention adopted here is the opposite of that used in [29], but I believe that it is more common in other work and has some technical advantages. The conventions used elsewhere in this memoir are fixed by the following requirements.

- (a) We have  $e(V \oplus W) = e(V)e(W)$ .
- (b) The Euler class of  $V$  is the restriction of the Thom class in  $\tilde{E}^0 X^V$  to the zero section  $X \subset X^V$ .

Our substitute for the nonequivariant theory of Chern classes will be more abstract, so we will not need sign conventions. The rôle normally played by the Chern polynomial  $\sum_{i+j=\dim(V)} \pm c_i x^j$  will be played by a certain element  $f_V$ ; if  $A = 0$  and  $V = \bigoplus_i L_i$  then  $f_V = \prod_i (x +_F e(L_i))$ .

Next note that we can define

$$x_\alpha := \tau_{-\alpha}^* x \in E^0(P\mathcal{U}, PL_\alpha).$$

(Here  $\tau_\alpha$  is the translation map as in Definition 4.1.) Because

$$(\tau_{-\alpha}^* T)^* = (L_{-\alpha} \otimes T)^* = L_\alpha \otimes T^* = \text{Hom}(T, L_\alpha),$$

we have  $x_\alpha = e(\text{Hom}(T, L_\alpha))$ . If  $L$  is a one-dimensional representation isomorphic to  $L_\alpha$ , we also use the notation  $x_L$  for  $x_\alpha$ . We can identify  $E^0(P(L_\beta \oplus L_\alpha), PL_\alpha)$  with  $\tilde{E}^0 S^{L_\beta - \alpha}$ , and we find that  $x_\alpha$  restricts to  $u_{\beta - \alpha}$ , which is a universal generator.

Now consider a finite-dimensional representation  $V$  of  $A$ . We have a canonical homotopy class of embeddings  $PV \rightarrow P\mathcal{U}$ , and thus a well-defined group  $E^0(P\mathcal{U}, PV)$ . We can write  $V$  as  $\bigoplus_{i=1}^d L_i$ , and Corollary 4.4 gives a map

$$P\mathcal{U}/PV \rightarrow \bigwedge_i P\mathcal{U}/PL_i$$

compatible with the diagonal. Using this, we can pull back  $x_{L_1} \wedge \dots \wedge x_{L_d}$  to get a class  $x_V \in E^0(P\mathcal{U}, PV)$  that maps to  $\prod_i x_{\alpha_i}$  in  $E^0 P\mathcal{U}$ . Note that for any

representation  $W$  containing  $V$  we can choose an embedding  $W \rightarrow \mathcal{U}$  and pull back  $x_V$  along the resulting map  $PW \rightarrow PU$  to get a class in  $E^0(PW, PV)$ , which we again denote by  $x_V$ .

LEMMA 5.7. *Let  $V \leq W$  be complex representations of  $A$ , with  $\dim(W/V) = 1$ . Then  $x_V$  is a universal generator for  $E^0(PW, PV)$ .*

PROOF. Write  $V = L_1 \oplus \dots \oplus L_d$  as before, and  $X = W \ominus V$ , so  $W = V \oplus X$  and  $PW/PV = S^{\text{Hom}(X, V)} = \bigwedge_i S^{\text{Hom}(X, L_i)}$ . Because  $x$  is a complex coordinate, we know that  $x_{L_i} \in E^0(PW, PL_i)$  restricts to a universal generator  $v_i$  of  $S^{\text{Hom}(X, L_i)}$ . It follows from Corollary 4.4 that  $x_V = \prod_i v_i \in \tilde{E}^0 S^{\text{Hom}(X, V)} = E^0(PW, PV)$ , and this is easily seen to be a universal generator.  $\square$

COROLLARY 5.8. *Let  $0 = U_0 < U_1 < \dots < U_d = U$  be representations of  $A$  with  $\dim(U_i) = i$ . Then  $\{x_{U_i} \mid i < d\}$  is a universal basis for  $E^0 PU$  over  $E^0$ .*

PROOF. This follows by an evident induction from the lemma.  $\square$

REMARK 5.9. As  $\bar{x}$  is another coordinate, it gives rise to another universal basis  $\{\bar{x}_{U_i} \mid i < d\}$  for  $E^0 PU$ , which is sometimes more convenient.

We record separately some easy consequences that are independent of the choice of flag  $\{U_i\}$ :

PROPOSITION 5.10. *Let  $U$  be a  $d$ -dimensional representation of  $A$ . Then*

- (a)  *$F(PU_+, E)$  is a free module of rank  $d$  over  $E$ .*
- (b) *If  $U = V \oplus W$  then the restriction map  $F(PU_+, E) \rightarrow F(PV_+, E)$  is split surjective. The kernel is a free module of rank one over  $F(PW_+, E)$ , generated by  $x_V$ .*  $\square$

We now put  $S = \text{spec}(E^0)$  and  $R = E^0 PU$  and  $C = \text{spf}(R)$ . We must show that  $C$  is an equivariant formal group over  $S$ .

We first exhibit a topological basis for  $R$ . This will be essentially the same as in Proposition 2.21, but we cannot appeal to that result because we do not yet know that we have a multicurve. We can list the elements of  $A^*$  as

$$A^* = \{\alpha_0 = 0, \alpha_1, \dots, \alpha_{n-1}\}$$

(where  $n = |A|$ ), and then define  $\alpha_k$  for all  $k \geq 0$  by  $\alpha_{ni+j} = \alpha_j$ . We then have an evident filtration

$$0 = V_0 < V_1 < V_2 < \dots < \mathcal{U} = \varinjlim_k V_k$$

where  $V_k = \bigoplus_{j < k} L_{\alpha_j}$ . If we put  $e_k = x_{V_k}$  we find that  $\{e_i \mid 0 \leq i < k\}$  is a universal basis for  $E^0 PV_k$ , and it follows by an evident limiting argument that  $\{e_i \mid i \geq 0\}$  is a universal topological basis for  $E^0 PU$ , giving an isomorphism  $F(PU_+, E) = \prod_k E$ . If we put  $y = x_{\mathbb{C}[A]} = x_{V_n} = e_n$ , it is easy to see that  $e_{ni+j} = y^i e_j$ , and it follows that  $E^0 PU$  is a free module over  $E^0 \llbracket y \rrbracket$  with basis  $\{e_i \mid i < n\}$ . Thus, conditions (a) and (b) in Definition 2.2 are satisfied.

Next, we have

$$F(PU_+^2, E) = F(PU_+, F(PU_+, E)) = F(PU_+, \prod_j E) = \prod_{i,j} E.$$

By working through the definitions, we deduce that the elements  $e_i \otimes e_j$  form a universal topological basis for  $E^0(PU \times PU)$ , so  $E^0(PU \times PU) = R \hat{\otimes} R$ , so

$\mathrm{spf}(E^0(PU \times PU)) = C \times_S C$ . As  $PU$  is a commutative group up to equivariant homotopy, we now see that  $C$  is a commutative formal group scheme over  $S$ .

Now note that  $e_1$  is just the coordinate  $x$ , and this divides  $e_k$  for all  $k > 0$ . In particular it divides  $y$ , which is a regular element in  $R$ , so  $x$  is also a regular element. It is also now easy to see  $x$  generates the ideal  $E^0(PU, PL_0)$ , which is just the augmentation ideal in the Hopf algebra  $R$ , so the vanishing locus of  $x$  is the zero-section in  $C$ . Thus  $x$  is a coordinate on  $C$ , showing (via Proposition 2.10) that  $C$  is in fact a formal multicurve group.

Next, recall that  $\pi_0((PU)^A) = A^*$ , which gives a map  $A^* \rightarrow PU$  of groups up to homotopy, and thus a map  $\phi: A^* \times S \rightarrow C$  of formal group schemes over  $S$ . By working through the definitions, we see that the image of the section  $\phi(\alpha)$  is the closed subscheme  $\mathrm{spec}(E^0 PL_\alpha) = \mathrm{spec}(R/x_\alpha)$ , so the divisor  $D := \sum_\alpha [\phi(\alpha)]$  is

$$\mathrm{spec}(R/\prod_\alpha x_\alpha) = \mathrm{spec}(R/y) = E^0 PC[A].$$

As  $y$  is topologically nilpotent, we see that any function on  $C$  that vanishes on  $D$  is topologically nilpotent, so  $C$  is a formal neighbourhood of  $D$ . We have thus proved the following result:

**THEOREM 5.11.** *Let  $E$  be a periodically orientable  $A$ -equivariant ring spectrum. Then the scheme  $C := \mathrm{spf}(E^0 PU)$  is an  $A$ -equivariant formal group over  $S := \mathrm{spec}(E^0)$ .  $\square$*

**REMARK 5.12.** We have  $I_0 = \{f \in \mathcal{O}_C \mid f(0) = 0\} = E^0(PU, PC)$ , and thus  $I_0^2 = E^0(PU, P(\mathbb{C} \oplus \mathbb{C}))$ , and thus

$$\omega = I_0/I_0^2 = E^0(P(\mathbb{C} \oplus \mathbb{C}), PC) = \tilde{E}^0 S^2 = \pi_2 E.$$

It will be helpful to record the naturality properties of the above construction.

**DEFINITION 5.13.** We write  $\mathcal{PO}_A$  for the category whose objects are periodically orientable  $A$ -equivariant ring spectra, and whose morphisms are homotopy classes of  $A$ -equivariant ring maps. We also write  $\mathcal{EO}_A$  for the evenly orientable subcategory.

Next, we write  $\mathcal{G}_A$  for the category of triples  $(S, C, \phi)$ , where  $S$  is an affine scheme and  $(C, \phi)$  is an  $A$ -EFG over  $S$ . The morphisms from  $(S, C, \phi)$  to  $(S', C', \phi')$  are the pairs  $(f, \tilde{f})$  where

- $f$  is a map  $S \rightarrow S'$  of schemes;
- $\tilde{f}$  is an isomorphism  $C \rightarrow f^* C'$  of formal group schemes over  $S$ ;
- $\phi' = \tilde{f} \circ \phi$ .

The construction  $E \mapsto (\mathrm{spec}(E^0), \mathrm{spf}(E^0(PU_A)), \phi)$  then defines a contravariant functor  $\Gamma: \mathcal{PO}_A \rightarrow \mathcal{G}_A$ .

## CHAPTER 6

### Simple examples

Let  $\widehat{C}$  be a nonequivariant formal group over a scheme  $S$ , so  $\widehat{C}$  is the formal neighbourhood of its zero section. For any finite abelian group  $A$ , we can of course let  $\phi: A^* \times S \rightarrow \widehat{C}$  be the zero map, and this gives us an  $A$ -equivariant formal group. More generally, *any* homomorphism  $A^* \times S \rightarrow \widehat{C}$  will give an  $A$ -efg, although often there will not be any homomorphisms other than zero.

Now suppose that  $\widehat{C}$  is the formal group associated to a nonequivariant even periodic ring spectrum  $\widehat{E}$ . We then have an  $A$ -equivariant ring spectrum  $E = F(A_+, \widehat{E})$  (which the Wirthmüller isomorphism also identifies with  $A_+ \wedge \widehat{E}$ ). This satisfies  $E^*X = \widehat{E}^* \text{res}(X)$ , where  $\text{res}: \mathcal{S}_A \rightarrow \mathcal{S}_0$  is the restriction functor. It follows easily that  $E$  is periodically orientable, and that the associated equivariant formal group is just  $\widehat{C}$ , equipped with the zero map  $\phi: A^* \times S \rightarrow \widehat{C}$  as above.

For a slightly more subtle construction, suppose we allow  $S$  to be a formal scheme, and assume that some prime  $p$  is topologically nilpotent in  $\mathcal{O}_S$ . Suppose also that the formal group  $\widehat{C}$  has finite height  $n$ . Put  $S' = \text{Hom}(A^*, \widehat{C})$ ; it is well-known that  $\mathcal{O}_{S'}$  is a free module of rank  $|A_{(p)}|^n$  over  $\mathcal{O}_S$ , so  $S'$  is finite and flat over  $S$ . By definition,  $S'$  is the universal example of a formal scheme  $T$  over  $S$  equipped with a homomorphism from  $A^*$  to the group of maps  $T \rightarrow \widehat{C}$  of formal schemes over  $S$ , or equivalently the group of sections of  $T \times_S C$  over  $T$ . If we put  $C' = S' \times_S \widehat{C}$ , there is thus a tautological map  $\phi: A^* \times S' \rightarrow C'$ . Here  $C'$  is an ordinary formal group over  $S'$  and thus is the formal neighbourhood of its zero section. It follows that  $(C', \phi)$  is automatically an  $A$ -equivariant formal group over  $S'$ .

Now suppose we have a  $K(n)$ -local even periodic ring spectrum  $\widehat{E}$ . We give the ring  $\pi_0 \widehat{E}$  the natural topology as in [17, Section 11] — in most cases of interest, this is the same as the  $I_n$ -adic topology. We then put  $S = \text{spf}(\pi_0 \widehat{E})$  and  $\widehat{C} = \text{spf}(\widehat{E}^0 \mathbb{C}P^\infty)$ , which gives an ordinary formal group of height  $n$  over  $S$ . Let  $EA$  denote a contractible space with free  $A$ -action, and put  $E = F(EA_+, \widehat{E})$ . This is a commutative  $A$ -equivariant ring spectrum, with  $E^*X = \widehat{E}^*X_{hA}$ , where  $X_{hA}$  denotes the homotopy orbit space or Borel construction. In particular, we have  $E^0(\text{point}) = \widehat{E}^0 BA$ . A character  $\alpha \in A^*$  gives a line bundle  $L_\alpha$  over  $BA$ , classified by a map  $f_\alpha: BA \rightarrow \mathbb{C}P^\infty$ . This gives a map  $\text{spf}(f_{\alpha!}): \text{spf}(\widehat{E}^0 BA) \rightarrow \widehat{C}$ , and by letting  $\alpha$  vary we get a map  $\text{spf}(\widehat{E}^0 BA) \rightarrow \text{Hom}(A^*, C)$ . By reduction to the cyclic case one can check that this is an isomorphism; see [16, Proposition 5.12] for details.

Next, observe that we have an  $A$ -equivariant inclusion  $PU[0] \rightarrow PU$ , which is nonequivariantly a homotopy equivalence, so the map  $EA \times PU[0] \rightarrow EA \times PU$  is an equivariant homotopy equivalence. It follows that  $E^*PU = E^*PU[0] = \widehat{E}^*(BA \times \mathbb{C}P^\infty) = \widehat{E}^*BA \otimes_{\widehat{E}^*} \widehat{E}^*\mathbb{C}P^\infty$ , and thus that  $\text{spf}(E^0PU) = \text{Hom}(A^*, \widehat{C}) \times_S \widehat{C}$ .

This shows that the equivariant formal group associated to  $E$  is just the pullback  $C' = S' \times_S \widehat{C}$  as discussed above.

## Formal groups from algebraic groups

We now show how to pass from algebraic groups (in particular, elliptic curves or the multiplicative group) to equivariant formal groups.

**7.1. The multiplicative group.** Let  $S = \operatorname{spec}(k)$  be a scheme, and consider the group scheme  $\mathbb{G}_m \times S = \operatorname{spec}(k[u, u^{-1}])$  over  $S$ . Suppose we are given a homomorphism  $\phi$  from  $A^* \times S$  to  $\mathbb{G}_m \times S$  of group schemes over  $S$ , or equivalently a homomorphism  $\phi: A^* \rightarrow k^\times$  of abstract groups. We can then form the divisor

$$D = \sum_{\alpha} [\phi(\alpha)] = \operatorname{spec}(k[u^{\pm 1}]/y),$$

where  $y = \prod_{\alpha} (1 - u/\phi(\alpha))$ . It is convenient to observe that  $u$  is invertible in  $k[u]/y$  and thus in  $k[u]/y^m$  for all  $m$ , so  $D$  can also be described as  $\operatorname{spec}(k[u]/y)$ . We then define  $C$  to be the formal neighbourhood of  $D$  in  $\mathbb{G}_m \times S$ , so

$$C = \varinjlim_m \operatorname{spec}(k[u]/y^m) = \operatorname{spf}(k[u]_y^{\wedge}),$$

which is an embeddable formal multicurve. It is easy to see that this is a subgroup of  $\mathbb{G}_m \times S$  and is an equivariant formal group, with coordinate  $x = 1 - u$ .

The universal example of a ring with a map  $A^* \rightarrow k^\times$  is  $k = \mathbb{Z}[A^*]$ , which can be identified with the representation ring  $R(A)$ . Thus, the universal example of a scheme  $S$  with a map  $A^* \times S \rightarrow \mathbb{G}_m \times S$  as above is  $S = \operatorname{Hom}(A^*, \mathbb{G}_m) = \operatorname{spec}(R(A))$ . We can apply the above construction in this tautological case to get an equivariant formal group  $C$  over  $\operatorname{Hom}(A^*, \mathbb{G}_m)$ . Explicitly, if we let  $v_{\alpha} \in \mathbb{Z}[A^*]$  be the basis element corresponding to  $\alpha \in A^*$  and put  $y = \prod_{\alpha} (1 - uv_{-\alpha}) \in \mathbb{Z}[A^*][u]$ , then  $C = \operatorname{spf}(\mathbb{Z}[A^*][u]_y^{\wedge})$ .

**THEOREM 7.1** (Cole-Greenlees-Kriz). *The  $A$ -efg associated to the equivariant complex  $K$ -theory spectrum  $K_A$  is isomorphic to the  $A$ -efg  $C$  over  $\operatorname{Hom}(A^*, \mathbb{G}_m)$  constructed above.*

**PROOF.** This is just a geometric restatement of [7, Section 6]. It is proved by identifying  $K_A^*PU$  with  $K_{A \times S^1}^*E\mathcal{F}_+$  (where  $\mathcal{F} = \{B \leq A \times S^1 \mid B \cap S^1 = \{1\}\}$ ) as in Proposition 4.10) and applying a suitable completion theorem.  $\square$

**7.2. Elliptic curves.** We now carry out the same program with the multiplicative group replaced by an elliptic curve (with some technical conditions assumed for simplicity). The resulting equivariant formal groups should be associated to equivariant versions of elliptic cohomology. See Definition 7.3 and subsequent comments for more detail.

Suppose that we are given a ring  $k$  and an element  $\lambda \in k$ , and that  $2$ ,  $\lambda$  and  $1 - \lambda$  are invertible in  $k$ . Let  $\tilde{C}$  be the elliptic curve given by the homogeneous cubic  $y^3 = x(x - z)(x - \lambda z)$ , so the zero element is  $O = [0 : 1 : 0]$ , and the points

$P := [0 : 0 : 1]$ ,  $Q := [1 : 0 : 1]$  and  $R := [\lambda : 0 : 1]$  are the three points of exact order two in  $\tilde{C}$ . Define rational functions  $t$  and  $r$  on  $\tilde{C}$  by  $t([x : y : z]) = x/y$  and  $r([x : y : z]) = z/y$ . One checks that the subscheme  $U = \tilde{C} \setminus \{P, Q, R\}$  is the affine curve with equation  $r = t(t - r)(t - \lambda r)$ , and that on  $U$ , the function  $t$  has a simple zero at  $O$  and no other poles or zeros.

Now let  $A$  be an abelian group of odd order  $n$ , and let  $\phi: A^* \times S \rightarrow \tilde{C}$  be a homomorphism. Define  $V = \bigcap_{\alpha} (U + \phi(\alpha))$ , which is an affine open subscheme of  $U$ .

LEMMA 7.2. *For each  $\beta \in A^*$ , the section  $\phi(\beta): S \rightarrow \tilde{C}$  actually lands in  $V$ .*

PROOF. We first show that for all  $\gamma \in A^*$ , the section  $\phi(\gamma)$  lands in  $U$ . Put  $D = [P] + [Q] + [R]$ , so  $U = \tilde{C} \setminus D$ . Let  $T$  be the closed subscheme of points  $s \in S$  where  $\phi(\gamma)(s) \in D$ ; we must show that  $T = \emptyset$ . As  $n$  is odd and  $D$  is the divisor of points of exact order 2, we see that multiplication by  $n$  is the identity on  $D$ , but of course  $n \cdot \phi(\alpha) = O$ . We conclude that over  $T$  we have  $O \in D$ . As 2 is invertible in  $k$  we know that  $O$  and  $D$  are disjoint, so  $T = \emptyset$  as required.

We now apply this to  $\gamma = \beta - \alpha$  to deduce that  $\phi(\beta) \in U + \phi(\alpha)$ . This holds for all  $\alpha$ , so  $\phi(\beta) \in V$  as claimed.  $\square$

We now define  $C$  to be the formal neighbourhood of the divisor  $D = \sum_{\alpha} [\phi(\alpha)]$  in  $V$ . If we put  $s(a) = \prod_{\alpha} t(a - \phi(\alpha))$  then  $s \in \mathcal{O}_V$  and the vanishing locus of  $s$  is just  $D$ , so we have  $\mathcal{O}_C = (\mathcal{O}_V)_s^{\wedge}$ . Using this, we see that  $C$  is an equivariant formal group, with coordinate  $t$  and good parameter  $s$ .

Now suppose instead that we are given a curve  $\tilde{C}$  over  $S$  as above, but not the map  $\phi: A^* \times S \rightarrow \tilde{C}$ . We can then consider the scheme  $S_1 = \text{Hom}(A^*, \tilde{C})$ , which is easily seen to be a closed subscheme of  $\text{Map}(A^*, U)$  and thus affine. We can thus pull back  $\tilde{C}$  to get a curve  $\tilde{C}_1$  over  $S_1$  equipped with a tautological map  $\phi: A^* \times S_1 \rightarrow \tilde{C}_1$ , and we can carry out the previous construction to get an equivariant formal group  $C_1$  over  $S_1$ .

DEFINITION 7.3. An  $A$ -equivariant elliptic spectrum consists of an evenly orientable equivariant ring spectrum  $E$  together with an elliptic curve  $\tilde{C}$  over an affine scheme  $S$  and a compatible system of isomorphisms  $\text{spec}(E^0(A/B)) = \text{Hom}(B^*, \tilde{C})$  and  $\text{spf}(E^0 P\mathcal{U}_A) = C$  (where  $C$  is constructed from  $\tilde{C}$  as above).

We will not take the trouble to make this definition more precise, as we will not use it very seriously.

In Section 9 we will construct equivariant elliptic spectra associated to elliptic curves over  $\mathbb{Q}$ -algebras, as a simple application of the general theory of rational  $A$ -spectra.

Elsewhere we have obtained partial results about the existence of integral equivariant elliptic spectra. These have unsatisfactory indeterminacy and awkward technical hypotheses, but nonetheless they are sufficient to make it clear that Definition 7.3 is the right one. Much better results (relying on a far-reaching theory of equivariant  $E_{\infty}$  ring spectra and derived algebraic geometry) have been announced by Jacob Lurie, but details have yet to appear.



## Equivariant formal groups of product type

A simple class of  $A$ -EFGs can be constructed as follows.

DEFINITION 8.1. Let  $\widehat{C}$  be an ordinary, nonequivariant formal group, and let  $B$  be a subgroup of  $A$ . We then have a formal multicurve  $C := B^* \times \widehat{C}$  and a homomorphism

$$\phi := (A^* \xrightarrow{\text{res}} B^* \xrightarrow{\text{inc}} B^* \times \widehat{C} = C),$$

giving an  $A$ -efg. Equivariant formal groups of this kind are said to be of *product type*. We call  $B$  the *core* of  $(C, S, \phi)$ .

We will show that EFGs over fields are of product type, and EFGs over  $\mathbb{Q}$ -algebras are locally of product type. Moreover, we will introduce equivariant analogues of the Morava  $K$ -theory spectra, and show that the associated EFGs are of product type.

PROPOSITION 8.2. *An  $A$ -efg  $(C, \phi)$  is of product type iff for every character  $\alpha \in A^*$  with  $\phi(\alpha) \neq 0$  in  $C$  (or equivalently,  $x(\phi(\alpha)) \neq 0$  in  $\mathcal{O}_S$ ), the element  $x(\phi(\alpha))$  is invertible in  $\mathcal{O}_S$ . (This is easily seen to be independent of the choice of coordinate.)*

PROOF. First suppose that for all  $\alpha$  with  $\phi(\alpha) \neq 0$ , the element  $x(\phi(\alpha))$  is invertible. The kernel of  $\phi$  is a subgroup of  $A^*$ , so it necessarily has the form  $\text{ann}(B)$  for some  $B \leq A$ , so  $\phi$  factors as  $A^* \xrightarrow{\text{res}} B^* \xrightarrow{\psi} C$  for some  $\psi$ . By assumption,  $x(\psi(\beta))$  is invertible for all  $\beta \in B^* \setminus \{0\}$ .

Let  $\widehat{C} = \{a \in C \mid x(a) \text{ is nilpotent}\}$  be the formal neighbourhood of 0 in  $C$ , and define  $\sigma: B^* \times \widehat{C} \rightarrow C$  by  $\sigma(\beta, a) = \psi(\beta) + a$ . We need to show that  $\sigma$  is an isomorphism. For this, we define  $x_\beta(a) = x(a - \psi(\beta))$  and  $y = \prod_{\beta \in B^*} x_\beta$  and  $R = \mathcal{O}_C$ . From the definition of an equivariant formal group, we know that  $R = R_y^\wedge$ , and it is clear that

$$\mathcal{O}_{B^* \times \widehat{C}} = \prod_{\beta} R_{x_\beta}^\wedge.$$

It will thus suffice to show that the natural map

$$R_y^\wedge \rightarrow \prod_{\beta} R_{x_\beta}^\wedge$$

is an isomorphism. This will follow from the Chinese Remainder Theorem if we can check that the ideal  $(x_\beta(a), x_\gamma(a))$  contains 1 whenever  $\beta \neq \gamma$ . This is clear because modulo that ideal, we have  $\psi(\beta) = a = \psi(\gamma)$ , so  $\psi(\beta - \gamma) = 0$ , so  $x(\psi(\beta - \gamma)) = 0$ , but  $x(\psi(\beta - \gamma))$  is invertible by assumption. Thus,  $C$  is of product type, as claimed.

Conversely, suppose that  $C$  is of product type. The vanishing locus of  $x$  is contained in  $\{0\} \times \widehat{C}$ , so  $x$  must be invertible on  $(B^* \setminus \{0\}) \times \widehat{C}$ . It follows immediately

that when  $\phi(\alpha) \neq 0$  we have  $\phi(\alpha) \in (B^* \setminus \{0\}) \times \widehat{C}$  and so  $x(\phi(\alpha))$  is invertible, as required.  $\square$

**COROLLARY 8.3.** *Every  $A$ -equivariant formal group over a field is of product type.*

**PROOF.** This is immediate from the proposition.  $\square$

We next show how groups of product type occur in topology. For this we need to use the geometric fixed point functors  $\overline{\phi}^B : \mathcal{S}_A \rightarrow \mathcal{S}_0$  for  $B \leq A$ . These preserve smash products and satisfy  $\overline{\phi}^B \Sigma^\infty X = \Sigma^\infty X^B$  for based  $A$ -spaces  $X$ . (The definition and further properties of these functors will be recalled in Section 10.)

**THEOREM 8.4.** *Let  $\widehat{K}$  be a nonequivariant even periodic cohomology theory, with associated formal group  $\widehat{C}$  over  $S$ , and let  $B$  be a subgroup of  $A$ . Define a cohomology theory  $K^*$  on  $\mathcal{S}_A$  by  $K^*X = \widehat{K}^*\overline{\phi}^B X$ . Then  $K$  is evenly orientable, and the associated equivariant formal group is just  $B^* \times \widehat{C}$  over  $S$ .*

**PROOF.** Note that  $\overline{\phi}^B S^V = S^{V^B}$  for any virtual complex representation  $V$ , and that  $\overline{\phi}^B \Sigma^\infty X = \Sigma^\infty X^B$  for any based  $A$ -space  $X$ . It follows that  $\pi_1 K = \pi_1 \widehat{K} = 0$  and that the periodicity isomorphism  $F(S^{2n}, \widehat{K}) = \widehat{K}$  gives an isomorphism

$$K^*(X_+ \wedge S^V) = \widehat{K}^*(X_+^B \wedge S^{V^B}) = \widehat{K}^*X_+^B = K^*X_+$$

of modules over  $K^*X_+$ . This implies that  $K$  is evenly periodic, with  $K^0(\text{point}) = \widehat{K}^0(\text{point})$  and thus  $\text{spec}(K^0(\text{point}))$  is the base scheme  $S$  for  $\widehat{C}$ . We also have

$$K^0 P\mathcal{U} = \widehat{K}^0(P\mathcal{U})^B = \widehat{K}^*(B^* \times \mathbb{C}P^\infty) = \mathcal{O}_{B^* \times \widehat{C}},$$

so the equivariant formal group associated to  $K$  is just  $B^* \times \widehat{C}$  as claimed.  $\square$

**EXAMPLE 8.5.** Let  $\widehat{K} = \widehat{K}(p, n)$  be the two-periodic version of Morava  $K$ -theory at a prime  $p$ , with height  $n$ . We define an equivariant theory  $K = K(p, n, B)$  as above; this is called *equivariant Morava  $K$ -theory*. In [30] we present evidence that these theories deserve this name, because they play the expected rôle in equivariant analogues of the Hopkins-Devnatz-Smith nilpotence theorems, among other things. The same paper also explains the representing object for the theory  $K$ , and shows that we have natural isomorphisms as follows:

$$\begin{aligned} K_*(X \wedge Y) &= K_*(X) \otimes_{K_*} K_*(Y) \\ K^*X &= \text{Hom}_{K_*}(K_*X, K_*). \end{aligned}$$

We now give a slight generalisation.

**DEFINITION 8.6.** We say that  $(C, S, \phi)$  is *locally of product type* if there is a splitting  $S = \coprod_{B \leq A} S_B$  such that the restricted group  $C_B = C \times_S S_B$  is of product type with core  $B$ .

**DEFINITION 8.7.** Let  $R$  be a ring, and  $r$  an element of  $R$ . We say that  $r$  is *split* if there is an idempotent  $e \in R$  with  $Rr = Re$ , or equivalently there is a splitting  $R = R_0 \times R_1$  with respect to which  $r \in \{0\} \times R_1^\times$ .

DEFINITION 8.8. Let  $(C, S, \phi)$  be an  $A$ -equivariant formal group, and let  $x$  be a coordinate on  $C$ . For each  $B \leq A$  we put

$$I_B = (x(\phi(\alpha)) \mid \alpha \in \text{ann}(B) \leq A^*) \leq \mathcal{O}_S.$$

We also put  $S[B] = \text{spec}(\mathcal{O}_S/I_B)$ , which can be described more invariantly as the largest closed subscheme  $T \subseteq S$  over which  $\phi: A^* \times T \rightarrow C \times_S T$  factors through  $B^* \times T$ .

PROPOSITION 8.9. *Let  $(C, S, \phi, x)$  be as above. Then the following are equivalent:*

- (a)  $C$  is locally of product type.
- (b) For each  $B \leq A$ , the subscheme  $S[B] \subseteq S$  is open as well as closed.
- (c) For each  $\alpha \in A^*$  the element  $e_\alpha = x(-\phi(\alpha)) \in \mathcal{O}_S$  is split.

PROOF. First suppose that (a) holds, giving a decomposition  $S = \coprod_B S_B$ . It follows that each  $S_B$  is open and closed, and  $S[B] = \coprod_{B' \leq B} S_{B'}$ , which proves (b).

Next, suppose that (b) holds. For  $\alpha \in A^*$  one checks that  $S[\ker(\alpha)]$  is just the largest closed subscheme of  $S$  where  $\phi(\alpha)$  vanishes, or equivalently where  $e_\alpha$  vanishes. As this is open as well as closed, it is the vanishing locus of an idempotent, say  $f_\alpha$ . It follows that  $e_\alpha$  and  $f_\alpha$  generate the same ideal, so  $e_\alpha$  is split. This proves (c).

Finally, suppose that (c) holds. This means that for each  $\alpha$  we can split  $S$  as  $E_\alpha \amalg F_\alpha$  with  $e_\alpha$  invertible on  $E_\alpha$  and zero on  $F_\alpha$ . Next, for any subset  $U \subseteq A^*$  we put

$$F_U = \bigcap_{\alpha \in U} F_\alpha \cap \bigcap_{\alpha \notin U} E_\alpha.$$

We find that  $F_U$  is both open and closed in  $S$ , and that  $S$  is the disjoint union of all the sets  $F_U$ .

Now suppose that  $U$  is not a subgroup of  $A^*$ , so there must exist  $\alpha, \beta \in U$  with  $\gamma = \alpha + \beta \notin U$ . Then we have  $F_U \subseteq F_\alpha \cap F_\beta \cap E_\gamma$ . This means that over  $F_U \subseteq S$  we have  $x(\phi(\alpha)) = x(\phi(\beta)) = 0$  but  $x(\phi(\gamma))$  is invertible. As  $x$  is a coordinate we have  $\phi(\alpha) = \phi(\beta) = 0$ , which implies  $\phi(\gamma) = 0$ , so  $x(\phi(\gamma))$  is zero as well as invertible. It follows that  $F_U$  is the empty scheme.

Suppose instead that  $U$  is a subgroup of  $A^*$ , so  $U = \text{ann}(B)$  for some subgroup  $B \leq A$ . We then define  $S_B$  to be  $F_U$ , and observe that  $S = \coprod_B S_B$ , and that the restriction  $C_B = C \times_S S_B$  is of product type with core  $B$ . This proves (a).  $\square$



## Equivariant formal groups over rational rings

We next prove an equivariant analogue of the well-known fact that all formal groups over a  $\mathbb{Q}$ -algebra are additive. We write  $\widehat{G}_a$  for the ordinary additive formal group over  $S$ . If we consider formal schemes over  $S$  as functors in the usual way, this sends an  $\mathcal{O}_S$ -algebra  $R$  to the set  $\text{Nil}(R)$  of nilpotents in  $R$ . Given a free module  $L$  of rank one over  $\mathcal{O}_S$  (or equivalently, a trivialisable line bundle over  $S$ ), we can instead consider the functor  $R \mapsto L \otimes_{\mathcal{O}_S} \text{Nil}(R)$ , which we denote by  $L \otimes \widehat{G}_a$ . This gives a formal group over  $S$ , noncanonically isomorphic to  $\widehat{G}_a$ . If  $C$  is a formal multicurve group over  $S$ , then the cotangent spaces to the fibres give a trivialisable line bundle  $\omega_C$  on  $S$ . This is easily seen to be the same as  $\omega_{\widehat{C}}$ , where  $\widehat{C}$  is the formal neighbourhood of zero, as usual. From now on we just write  $\omega$  for this module. If  $S$  lies over  $\text{spec}(\mathbb{Q})$  then the theory of logarithms for ordinary formal groups gives a canonical isomorphism  $\widehat{C} \rightarrow \omega^{-1} \otimes \widehat{G}_a$ .

**THEOREM 9.1.** *Let  $(C, \phi)$  be an  $A$ -equivariant formal group over a scheme  $S$ , such that the integer  $n = |A|$  is invertible in  $\mathcal{O}_S$ . Then  $C$  is of local product type. Moreover, if  $\mathcal{O}_S$  is a  $\mathbb{Q}$ -algebra then  $\widehat{C} \simeq \omega_C^{-1} \otimes \widehat{G}_a$  and so*

$$C \simeq \coprod S_B \times_S B^* \times (\omega_C^{-1} \otimes \widehat{G}_a).$$

**PROOF.** Put  $n = |A|$ , and choose a coordinate  $x$  on  $C$ . For formal reasons we have  $x(a+b) = x(a) + x(b) \pmod{x(a)x(b)}$  as functions on  $C^2$ , and it follows that  $x(na) = v_n(a)x(a)$  for some function  $v_n$  on  $C$  with  $v_n(0) = n$ . (These functions will be studied in greater detail in Section 24.) Put  $Q = \mathcal{O}_C/(x.v_n)$ , so  $\text{spf}(Q)$  is just the closed subscheme  $C[n]$  of points of order  $n$  in  $C$ . In  $Q$  we have  $xv_n = 0$ , and for any function  $f \in \mathcal{O}_C$  we have  $f - f(0) \in (x)$  so  $f.v_n = f(0).v_n$ . In particular, we can take  $f = v_n$  to see that  $v_n^2 = nv_n$ , so the function  $u = 1 - v_n/n$  is idempotent in  $Q$ . We have  $u(0) = 0$  so  $u \in Qx$ , but also  $v_nx = 0$  so  $ux = x$  so  $x \in Qu$ . For any  $\alpha \in A^*$  we know that  $n\phi(\alpha) = 0$  so  $\phi(\alpha): S \rightarrow C[n]$ , and we find that  $x(\phi(\alpha))$  generates the same ideal in  $\mathcal{O}_S$  as  $u(\phi(\alpha))$ , so  $x(\phi(\alpha))$  is split (in the sense of Definition 8.7). It follows by Proposition 8.9 that  $C$  is of local product type.

The rational statement now follows from the nonequivariant theory.  $\square$

The following slight extension can easily be proved in the same way.

**PROPOSITION 9.2.** *Let  $(C, \phi)$  be an  $A$ -equivariant formal group over a scheme  $S$ , such that  $\mathcal{O}_S$  is an algebra over  $\mathbb{Z}_{(p)}$ . There is of course a unique splitting  $A = A_0 \times A_1$ , where  $A_0$  is a  $p$ -group and  $p$  does not divide  $|A_1|$ . Let  $C_0 \subseteq C$  be the formal neighbourhood of  $[\phi(A_0^*)]$ , and let  $\phi_0: A_0^* \rightarrow C_0$  be the restriction of  $\phi$ . Then there is a canonical decomposition  $S = \coprod_{B \leq A_1} S_B$ , and a corresponding decomposition*

$$C \simeq \coprod S_B \times_S C_0 \times B^*,$$

such that over  $S_B$ , the map  $\phi$  is the product of  $\phi_0$  and the restriction map  $A_1^* \rightarrow B^*$ .  $\square$

We would like to understand how the splitting in Theorem 9.1 works out when the equivariant formal group comes from a ring spectrum. For simplicity we will treat only the rational case, although many parts of our analysis can also be made to work assuming only that  $|A|$  is invertible. We start by recalling an algebraic description of the category  $\mathbb{Q}\mathcal{S}_A$  of rational  $A$ -spectra. Let  $\mathcal{V}_*[A]$  denote the category of graded modules over the group ring  $\mathbb{Q}[A]$ . If  $X \in \mathbb{Q}\mathcal{S}_A$  and  $B \leq A$  then the nonequivariant spectrum  $\bar{\phi}^B(X)$  has a homotopical action of the Weyl group  $A/B$ , so  $\pi_*(\bar{\phi}^B(X)) \in \mathcal{V}_*[A/B]$ . Put  $\mathcal{A}_A = \prod_{B \leq A} \mathcal{V}_*[A/B]$ , and define  $\Phi: \mathbb{Q}\mathcal{S}_A \rightarrow \mathcal{A}_A$  by

$$\Phi(X)_B = \pi_*(\bar{\phi}^B(X)).$$

We can make  $\mathcal{A}_A$  into a symmetric monoidal category by the rule  $(M \otimes N)_B = M_B \otimes N_B$ . For nonequivariant rational spectra  $X$  and  $Y$  we have

$$\pi_*(X \wedge Y) = H_*(X \wedge Y; \mathbb{Q}) = H_*(X; \mathbb{Q}) \otimes H_*(Y; \mathbb{Q}) = \pi_*(X) \otimes \pi_*(Y).$$

Using this, we see that  $\Phi$  takes smash products to tensor products.

**THEOREM 9.3.** *The functor  $\Phi: \mathbb{Q}\mathcal{S}_A \rightarrow \mathcal{A}$  is an equivalence.*

As far as we know, the literature only contains a rather indirect proof of this fact, going via the theory of Mackey functors. Here we give a slightly more direct argument.

**PROOF.** All spectra in this proof are implicitly rationalised; we will not indicate this explicitly in the notation.

We have a natural map

$$\Phi_{XY}: \mathbb{Q}\mathcal{S}_A(X, Y)_* \rightarrow \mathcal{A}_A(\Phi(X), \Phi(Y))_*,$$

which we claim is an isomorphism.

Using Maaschke's theorem we see that all objects in  $\mathcal{A}$  are both injective and projective. It is also clear that  $\Phi$  preserves all coproducts. It follows that both  $\mathbb{Q}\mathcal{S}_A(X, Y)_*$  and  $\mathcal{A}(\Phi(X), \Phi(Y))_*$  are cohomology theories of  $X$ . If  $X$  is finite then they are also homology theories in  $Y$ . We can thus reduce easily to the case where  $X$  and  $Y$  are both finite.

Now let  $DY$  be the Spanier-Whitehead dual of  $Y$ . The duality between  $Y$  and  $DY$  is encoded by the unit map  $\eta: S \rightarrow DY \wedge Y$  and the counit  $\epsilon: Y \wedge DY \rightarrow S$ , which make the following diagrams commute:

$$\begin{array}{ccc} DY & \xrightarrow{\eta \wedge 1} & DY \wedge Y \wedge DY \\ & \searrow 1 & \downarrow 1 \wedge \epsilon \\ & & DY \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{1 \wedge \eta} & Y \wedge DY \wedge Y \\ & \searrow 1 & \downarrow \epsilon \wedge 1 \\ & & Y \end{array}$$

We can apply  $\Phi$  to this to get a perfect duality between  $\Phi(Y)$  and  $\Phi(DY)$ , which in turn gives a commutative square

$$\begin{array}{ccc} \mathbb{Q}\mathcal{S}_A(X, Y) & \xrightarrow{\Phi_{XY}} & \mathcal{A}_A(\Phi(X), \Phi(Y)) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathbb{Q}\mathcal{S}_A(X \wedge DY, S^0) & \xrightarrow{\Phi_{X \wedge DY, S^0}} & \mathcal{A}_A(\Phi(X \wedge DY), \Phi(S^0)). \end{array}$$

Using this we reduce to the case  $Y = S^0$ . We can then reduce further by cellular induction to the case  $X = A/B_+$ . Here  $\mathbb{Q}\mathcal{S}_A(A/B_+, S^0)_0$  is the rationalised Burnside ring of  $B$ , which we will call  $\mathbb{Q}\Omega(B)$ . Using the tom Dieck splitting  $(\Sigma_A^\infty S^0)^B = \bigvee_{C \leq B} \Sigma^\infty B(B/C)_+$  we also see that  $\mathbb{Q}\mathcal{S}_A(A/B_+, S^0)_n = H_n((\Sigma_A^\infty S^0)^B; \mathbb{Q}) = 0$  for  $n \neq 0$ . We also find that  $\mathcal{A}_A(\Phi(A/B_+), \Phi(S^0)) = \prod_{C \leq B} \mathbb{Q}$ , and the map  $\Phi_{A/B_+, S^0}$  is just the usual fixed-point map, which is well-known to be an isomorphism.

We now know that the functor  $\Phi$  is full and faithful. As  $\mathbb{Q}\mathcal{S}_A$  has all coproducts, and splittings for idempotents, we see that the essential image of  $\Phi$  is closed under coproducts and retracts. Moreover, all monomorphisms and epimorphisms in  $\mathcal{A}_A$  are split, so  $\text{image}(\Phi)$  is closed under taking subobjects and quotient objects. If  $M \in \mathcal{A}_A$  and  $m \in M_{B,d}$  then there is a map  $f_m: \Phi(\Sigma^d A/B_+) \rightarrow M$  sending an evident generator to  $m$ . By taking a large direct sum of maps like this, we can express  $M$  as a quotient of an object in the image of  $\Phi$ . It follows that  $\Phi$  is essentially surjective, as claimed.  $\square$

REMARK 9.4. The group  $Y^m X = [X, Y]_{-m} = \mathcal{A}_A(\Phi(X), \Phi(Y))_{-m}$  can also be described as  $\prod_B \prod_i H^{m+i}(\overline{\phi}^B(X); \pi_i(\overline{\phi}^B(Y)))^{A/B}$ .

REMARK 9.5. The action of  $A/B$  on  $\pi_*(\overline{\phi}^B(X))$  plays a central rôle here, so it is useful to know when this action is trivial. Choose an embedding of  $A$  in a torus  $T$ , and suppose that  $X$  arises by applying the restriction functor  $\text{res}_A^T$  to a  $T$ -spectrum which we also call  $X$ . Then the action map  $A/B \rightarrow [\overline{\phi}^B(X), \overline{\phi}^B(X)]$  factors through  $\pi_0(T/B) = 1$ , so  $A/B$  acts trivially on all homotopical invariants of  $\overline{\phi}^B(X)$ .

In particular, elementary character theory tells us that any complex representation  $V$  of  $A$  admits a compatible  $T$ -action, so the above remarks apply to all spaces constructed functorially from  $V$  such as the projective space  $PV$ , the sphere  $S(V)$  and so on.

PROPOSITION 9.6. *Let  $E$  be an evenly orientable rational  $A$ -spectrum, let  $(C, S, \phi)$  be the corresponding formal group, and let  $S_B \subseteq S$  be as in Theorem 9.1. Put  $j_B = \pi_0(\overline{\phi}^B E)$  and  $k_B = j_B^{A/B}$  and  $\omega = \pi_2 E$ . Then  $S_B = \text{spec}(k_B)$ , and for all  $X$  we have*

$$E^m(X) = \prod_{B \leq A} \prod_{d \in \mathbb{Z}} H^{m+2d}(\overline{\phi}^B(X); j_B \otimes \omega^d)^{A/B}.$$

PROOF. Using Theorem 9.3 and Remark 9.4 we have

$$E^m(X) = \prod_B \prod_i H^{m+i}(\overline{\phi}^B(X); \pi_i(\overline{\phi}^B(E)))^{A/B}.$$

We now take  $m = 1$  and  $X = (A/B')_+$ . As  $E$  is evenly periodic, the left hand side vanishes. On the right, the factor where  $B = B'$  is

$$H^0(A/B'; \pi_{-1}(\overline{\phi}^{B'}(E)))^{A/B'} = \pi_{-1}(\overline{\phi}^{B'}(E));$$

this must therefore vanish as well. As  $E$  is periodically orientable we find that  $\pi_{i+2}(\overline{\phi}^B(E)) \simeq \pi_i(\overline{\phi}^B(E)) \otimes_{E^0} \omega$ . It follows that  $\pi_{2i}(\overline{\phi}^B(E)) = j_B \otimes_{E^0} \omega^i$  and  $\pi_{2i+1}(\overline{\phi}^B(E)) = 0$ . Using this we get

$$E^m(X) = \prod_{B \leq A} \prod_{d \in \mathbb{Z}} H^{m+2d}(\overline{\phi}^B(X); j_B \otimes \omega^d)^{A/B}$$

as claimed. We apply this to the spectrum  $X = \Sigma_+^\infty P\mathcal{U}$ , noting that the action of  $A/B$  on  $\overline{\phi}^B(X) = \Sigma_+^\infty(B^* \times \mathbb{C}P^\infty)$  is homotopically trivial by Remark 9.5. This gives

$$E^0 P\mathcal{U} = \prod_B \prod_d \text{Map}(B^*, H^{2d}(\mathbb{C}P^\infty; k_B \otimes \omega^d))$$

and thus

$$C \simeq \prod_B B^* \times \text{spec}(k_B) \times \widehat{G}_a.$$

We also find that the map  $\phi: A^* \times S \rightarrow C$  is given over  $\text{spec}(k_B)$  by the restriction  $A^* \rightarrow B^*$  and the obvious inclusion  $B^* \rightarrow B^* \times \widehat{G}_a$ , so  $S_B = \text{spec}(k_B)$ .  $\square$

CONSTRUCTION 9.7. Let  $(C, S, \phi)$  be an  $A$ -EFG as in Theorem 9.1. Put  $k_B = \mathcal{O}_{S_B}$ , which is a  $\mathbb{Q}$ -algebra. Let  $\omega$  be the cotangent space to  $C$  at zero, and put  $\omega_B = \omega|_{S_B}$ , which is a free module of rank one over  $k_B$ . Define a multiplicative cohomology theory  $E^*$  on  $A$ -spectra by

$$E^m(X) = \prod_B \prod_d H^{m+2d}(\overline{\phi}^B(X); \omega_B^d)^{A/B}.$$

(Here we use the trivial action of  $A/B$  on  $\omega_B^d$ , so the action on the cohomology group comes solely from the action on  $\overline{\phi}^B(X)$ .) Write  $E = \Delta(C, S, \phi)$  for the spectrum that represents this cohomology theory.

PROPOSITION 9.8.  $\Delta(C, S, \phi)$  is evenly orientable, with  $\Gamma \Delta(C, S, \phi) \simeq (C, S, \phi)$ .

PROOF. Put  $E = \Delta(C, S, \phi)$ . From the definitions we have

$$E^0 = \prod_B H^0(1; k_B)^{A/B} = \prod_B k_B = \mathcal{O}_S$$

and

$$E^0 P\mathcal{U} = \prod_B \prod_d H^{2d}(B^* \times \mathbb{C}P^\infty; \omega_B^d) = \prod_B \prod_d \prod_{\beta \in B^*} H^{2d}(\mathbb{C}P^\infty; \omega^d).$$

Choose a generator  $u$  of  $\omega$ , and let  $y$  be the standard generator of  $H^2(\mathbb{C}P^\infty)$ . Put

$$x_{B,d,\beta} = \begin{cases} 1 & \text{if } d = 0, \beta \neq 0 \\ y \otimes u & \text{if } d = 1, \beta = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This defines a class  $x \in E^0 P\mathcal{U}$ . If we quotient out the ideal generated by  $x$  then all factors where  $\beta \neq 0$  or  $d > 0$  are killed, and this leaves  $(E^0 P\mathcal{U})/x = \prod_B k_B = E^0$ .



Now consider a character  $\alpha$  of  $A$  and the subspace  $P(L_0 \oplus L_\alpha) \subset PU$ . If  $B \leq \ker(\alpha)$  we find that

$$\overline{\phi}^B P(L_0 \oplus L_\alpha) = \{0\} \times \mathbb{C}P^1 \subset B^* \times \mathbb{C}P^\infty = \overline{\phi}^B PU.$$

On the other hand, if  $B \not\leq \ker(\alpha)$  we find that

$$\overline{\phi}^B P(L_0 \oplus L_\alpha) = \{0, \alpha|_B\} \subseteq B^* \subset B^* \times \mathbb{C}P^\infty = \overline{\phi}^B PU.$$

Either way, we see that the reduced cohomology of this space with coefficients in  $k_B^* = \bigoplus_d \omega_B^d$  is freely generated over  $k_B^*$  by the image of  $x$ . By taking the product over all  $B$ , we see that the image of  $x$  generates  $E^*(P(L_0 \oplus L_\alpha), PL_0)$  as a module over  $E^*$ . We leave it to the reader to check in the same way that  $E^*((A/A') \times P(L_0 \oplus L_\alpha), A/A' \times PL_0)$  is freely generated by  $x$  over  $E^*(A/A')$ , so  $x$  has the required universal generating property and is a coordinate for  $E$ . This shows that  $E$  is periodically orientable, and it is immediate from the definitions that  $E^{-1}(A/A') = 0$ , so in fact it is evenly orientable.

It is clear by construction that

$$\mathrm{spf}(E^0 PU) = \coprod_B B^* \times S_B \times (\omega_B^{-1} \otimes \widehat{G}_a) \simeq C,$$

so  $\Gamma\Delta$  is the identity functor.  $\square$

**PROPOSITION 9.9.** *There is a natural map  $\xi_E: \Delta\Gamma(E) \rightarrow E$  for  $E \in \mathbb{Q}\mathcal{EO}_A$ . Moreover, the following are equivalent:*

- (a)  $E^0(A/B) = E^0/(x(\phi(\alpha)) \mid \alpha \in (A/B)^* \leq A^*)$  (for any coordinate  $x$  and any subgroup  $B \leq A$ ).
- (b) The unit map  $E^0 \rightarrow E^0(A/B)$  is surjective (for any subgroup  $B \leq A$ ).
- (c)  $A/B$  acts trivially on  $E^0(A/B)$  (for any subgroup  $B \leq A$ ).
- (d)  $\xi_E$  is an isomorphism.

**REMARK 9.10.** If  $E$  is the restriction of a  $T$ -spectrum for some torus  $T \supseteq A$  then we see as in Remark 9.5 that (c) holds, and so the other conditions hold as well. In particular, this will hold automatically for the rationalisations of popular examples such as  $K$ -theory and cobordism, which are constructed in a uniform way for all compact Lie groups.

**PROOF.** Let  $E$  be a rational, evenly periodic  $A$ -equivariant ring spectrum. Put  $j_B = \pi_0(\overline{\phi}^B E)$  and  $k_B = j_B^{A/B}$  and  $\omega = \pi_2 E$  as in Proposition 9.6. The inclusions  $k_B \rightarrow j_B$  give an inclusion

$$\xi_{E,X}: \coprod_B H^*(\overline{\phi}^B(X); k_B \otimes \omega^*)^{A/B} \rightarrow \coprod_B H^*(\overline{\phi}^B(X); j_B \otimes \omega^*)^{A/B},$$

or equivalently  $\Delta\Gamma(E)^*(X) \rightarrow E^*(X)$ . By the Yoneda lemma, this is induced by a map  $\Delta\Gamma(E) \rightarrow E$  of equivariant ring spectra.

Next, it is trivial that (a) implies (b), and a naturality argument shows that (b) implies (c). Now note that

$$E^0(A/A') = \coprod_{B \leq A'} H^0(A/A'; j_B)^{A/B} = \coprod_{B \leq A'} j_B^{A'/B},$$

and the factor indexed by  $B = A'$  is just  $j_{A'}$ . It follows that  $A$  acts trivially on  $E^0(A/A')$  for all  $A'$  iff it acts trivially on  $j_{A'}$  for all  $A'$  iff  $k_{A'} = j_{A'}$  for all  $A'$ . If this holds, it is clear that  $\xi_E$  is an isomorphism. Thus (c) implies (d). Finally, suppose

that (d) holds. Note that  $\Delta\Gamma(E)^0(A/A') = \prod_{B \leq A'} k_B$ , so if  $\xi_E$  is an isomorphism we must have  $k_B = j_B^{A'/B}$  for all  $B \leq A' \leq A$ . By considering the case  $A' = B$  we see that  $j_B = k_B$ , so the action of  $A$  on  $j_B$  is trivial. This gives  $E^0 = \prod_B k_B$  and  $E^0(A/A') = \prod_{B \leq A'} k_B$ . We also know that the map  $\phi: A^* \times S \rightarrow C$  is given over  $S_B = \text{spec}(k_B)$  by the restriction  $A^* \rightarrow B^*$  and the inclusion  $B^* \rightarrow B^* \times \widehat{C}$ , so the largest closed subscheme where  $\phi((A')^*) = 0$  is just  $\prod_{B \leq A'} S_B$ , which is the spectrum of the ring  $\prod_{B \leq A'} k_B = E^0(A/A')$ . On the other hand, this closed subscheme can also be described as the spectrum of the ring

$$k'_{A'} = E^0/(x(\phi(\alpha)) \mid \alpha \in (A/A')^* \leq A^*).$$

We must therefore have  $E^0(A/A') = k'_{A'}$ , so (a) holds.  $\square$

The following corollary follows directly.

**COROLLARY 9.11.** *Let  $\mathbb{Q}\mathcal{EO}'_A$  be the full subcategory of  $\mathbb{Q}\mathcal{EO}_A$  generated by objects for which the equivalent conditions of Proposition 9.9 are satisfied. Then the functors  $\Gamma$  and  $\Delta$  give an equivalence between  $\mathbb{Q}\mathcal{EO}'_A$  and  $\mathbb{Q}\mathcal{G}_A$ .*  $\square$

**EXAMPLE 9.12.** Suppose we have an elliptic curve  $\widetilde{C}$  over a scheme  $S = \text{spec}(k)$  as in Section 7.2, and suppose that  $k$  is a  $\mathbb{Q}$ -algebra. We then define an equivariant formal group  $C$  over  $S_1 = \text{Hom}(A^*, \widetilde{C})$  as described in Section 7.2, and use this to construct an evenly periodic  $A$ -equivariant ring spectrum  $E = \Delta(C)$ . Using Proposition 9.9 we see that  $E^0(A/B) = \mathcal{O}_{\text{Hom}(B^*, \widetilde{C})}$  for all  $B \leq A$ , so we have an elliptic spectrum as defined in Definition 7.3.

## Equivariant formal groups of pushout type

We next consider a slightly different generalization of the notion of a group of product type.

DEFINITION 10.1. Suppose we have a subgroup  $B \leq A$  and a formal multicurve group  $C'$ , with a map  $\phi': (A/B)^* \rightarrow C'$  making it an  $A/B$ -equivariant formal group. There is an evident embedding  $(A/B)^* \rightarrow A^*$ , which we can use to form a pushout

$$\begin{array}{ccc} (A/B)^* & \xrightarrow{\phi'} & C' \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\phi} & C. \end{array}$$

If we choose a transversal  $T$  to  $(A/B)^*$  in  $A^*$ , then the underlying scheme of  $C$  is just  $\coprod_{\alpha \in T} C'$ . This implies that the formation of the pushout is compatible with base change, and that  $C$  is an  $A$ -equivariant formal group. Formal groups constructed in this way are said to be of *pushout type*. (The case where  $\phi' = 0$  evidently gives groups of product type.)

We next examine how formal groups of this kind can arise in equivariant topology. For this, we need to recall in more detail the various different change of group functors and fixed-point functors for  $A$ -spectra.

Given a homomorphism  $\zeta: B \rightarrow A$ , there is a pullback functor  $\zeta^*: \mathcal{S}_A \rightarrow \mathcal{S}_B$ , which preserves smash products and function spectra. (Note that if  $\zeta$  is not injective, then  $\zeta^* \mathcal{U}_A$  is not a complete  $B$ -universe, so the definition of  $\zeta^*$  contains an implicit change of universe.) If  $\zeta$  is the inclusion of a subgroup then  $\zeta^*$  is called *restriction* and written  $\text{res}_B^A$ . This functor has a left adjoint written  $X \mapsto A_+ \wedge_B X$ , and a right adjoint written  $X \mapsto F_B(A_+, X)$ . These two adjoints are actually isomorphic, by the generalized Wirthmüller isomorphism [20, Theorem II.6.2].

If  $\zeta$  is the projection  $A \rightarrow A/B$  then  $\zeta^*$  is called *inflation*. This has a right adjoint functor  $\lambda^B: \mathcal{S}_A \rightarrow \mathcal{S}_{A/B}$ , which we call the Lewis-May fixed point functor. The adjunction is discussed in [20, Section II.7]; there  $\lambda^B X$  is written  $X^B$ . One can check that the following square commutes up to natural isomorphism:

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\lambda^C} & \mathcal{S}_{A/C} \\ \text{res}_B^A \downarrow & & \downarrow \text{res}_{B/C}^{A/C} \\ \mathcal{S}_B & \xrightarrow{\lambda^C} & \mathcal{S}_{B/C}. \end{array}$$

It will be convenient to write

$$\bar{\lambda}^B = \text{res}_0^{A/B} \lambda^B = \lambda^B \text{res}_B^A: \mathcal{S}_A \rightarrow \mathcal{S}_0.$$

The usual equivariant homotopy groups of  $X$  are defined by  $\pi_*^B X = \pi_* \overline{\lambda}^B X$ . The functors  $\lambda^B$  and  $\overline{\lambda}^B$  do not preserve smash products, and there is no sense in which  $\lambda^B$  acts as the identity on  $B$ -fixed objects.

Lewis and May also introduce another functor  $\phi^B: \mathcal{S}_A \rightarrow \mathcal{S}_{A/B}$ , called the *geometric fixed point functor*. To explain the definition, let  $V$  be a complex representation of  $A$ . We write  $\chi_V$  for the usual inclusion  $S^0 \rightarrow S^V$ , which can be regarded as an element of the  $R(A)$ -graded homotopy ring  $\pi_* S^0$  in dimension  $-V$ . It is easily seen to be zero if  $V^A \neq 0$ , but it turns out to be nonzero otherwise. It is also clear that  $\chi_{V \oplus W} = \chi_V \chi_W$ .

By dualizing the standard cofibration  $S(V)_+ \rightarrow S^0 \xrightarrow{\chi_V} S^V$ , we see that  $D(S(V)_+)$  deserves to be called  $S^0/\chi_V$ . On the other hand, we have

$$S^0[\chi_V^{-1}] = \varinjlim (S^0 \xrightarrow{\chi_V} S^V \xrightarrow{\chi_V} S^{2V} \rightarrow \dots) = S^{\infty V}.$$

It follows that for any  $X \in \mathcal{S}_A$ , the spectrum  $X[\chi_V^{-1}] = X \wedge S^{\infty V}$  is a Bousfield localization of  $X$ , or more specifically, the finite localization away from the thick ideal generated by  $S^0/\chi_V$ . There is another characterization as follows. Let  $\mathcal{F}$  be the family of those subgroups  $A' \leq A$  such that  $V^{A'} \neq 0$ , and let  $\mathcal{C}$  be the thick ideal generated by  $\{A/A'_+ \mid A' \in \mathcal{F}\}$ . It is not hard to see that  $S(\infty V)^{A'}$  is contractible for  $A' \in \mathcal{F}$  and empty for  $A' \notin \mathcal{F}$ . Thus  $S(\infty V)$  is a model for the space  $E\mathcal{F}$  (as in Definition 4.9) and  $S^{\infty V} = \widetilde{\Sigma} S(\infty V) = \widetilde{\Sigma} E\mathcal{F} = \widetilde{E}\mathcal{F}$ . It follows that  $X[\chi_V^{-1}]$  is the finite localization of  $X$  away from  $\mathcal{C}$ . It also follows that  $\mathcal{C}$  is the same as the thick ideal generated by  $S^0/\chi_V$ .

Now fix a subgroup  $B \leq A$ , and take

$$V = V_{A,B} := \mathbb{C}[A] \ominus (\mathbb{C}[A]^B) = \bigoplus_{\alpha \in A^* \setminus \text{ann}(B)} L_\alpha.$$

In this context, we write  $\chi_{A,B}$  for  $\chi_V$ , and we also write  $\chi_A$  for  $\chi_{A,A}$ . We also put  $\mathcal{F}[\not\geq B] = \{C \leq A \mid C \not\geq B\}$ , and note that  $\widetilde{E}\mathcal{F}[\not\geq B] = S^{\infty V}$ . The geometric fixed-point functor  $\phi^B: \mathcal{S}_A \rightarrow \mathcal{S}_{A/B}$  is defined by

$$\phi^B X = \lambda^B(X[\chi_{A,B}^{-1}]) = \lambda^B(X \wedge \widetilde{E}\mathcal{F}[\not\geq B]).$$

(In [20] the functor  $\phi^B$  is actually defined in a different way, but the above description is proved as Theorem II.9.8). Let  $\pi: A \rightarrow A/B$  be the projection. One can check that  $\phi^B$  preserves smash products [20, Proposition 9.12], the composite

$$\mathcal{S}_{A/B} \xrightarrow{\pi^*} \mathcal{S}_A \xrightarrow{\phi^B} \mathcal{S}_{A/B}$$

is the identity [20, Proposition 9.10], and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}_A & \xrightarrow{\phi^C} & \mathcal{S}_{A/C} \\ \text{res}_B^A \downarrow & & \downarrow \text{res}_{B/C}^{A/C} \\ \mathcal{S}_B & \xrightarrow{\phi^C} & \mathcal{S}_{B/C}. \end{array}$$

Moreover, for any  $A$ -space  $X$  we have  $\phi^B \Sigma^\infty X = \Sigma^\infty X^B$  [20, Corollary 9.9]. It will be convenient to write

$$\overline{\phi}^B = \text{res}_0^{A/B} \phi^B = \phi^B \text{res}_B^A: \mathcal{S}_A \rightarrow \mathcal{S}_0.$$

This again preserves smash products, and it is known that a spectrum  $X \in \mathcal{S}_A$  satisfies  $X = 0$  iff  $\overline{\phi}^B X = 0$  in  $\mathcal{S}_0$  for all  $B \leq A$ . We will also need the following property:

LEMMA 10.2. *Suppose that  $B \leq A$ , and write  $\chi = \chi_{A,B}$ . Then for  $X, Y \in \mathcal{S}_A$  there are natural equivalences*

$$\lambda^B F(X, Y[\chi^{-1}]) = \lambda^B F(X[\chi^{-1}], Y[\chi^{-1}]) = F(\phi^B X, \phi^B Y).$$

PROOF. First note that the map  $W \rightarrow W[\chi^{-1}]$  is an equivalence iff  $W$  is concentrated over  $B$  as defined in [20, page 109]. Let  $\mathcal{C}$  be the category of such  $W$ , so we have functors  $\phi^B = \lambda^B: \mathcal{C} \rightarrow \mathcal{S}_{A/B}$  and  $\psi: \mathcal{S}_{A/B} \rightarrow \mathcal{C}$  given by  $\psi(Z) = (\pi^* Z)[\chi^{-1}]$ . We see from [20, Corollary II.9.6] that  $\phi^B$  and  $\psi$  are mutually inverse equivalences, and it follows that

$$[X, Y[\chi^{-1}]]_*^B = [X[\chi^{-1}], Y[\chi^{-1}]]_*^B = [\phi^B X, \phi^B Y]_*^{A/B}.$$

Now consider  $W \in \mathcal{S}_{A/B}$  and replace  $X$  by  $(\pi^* W) \wedge X$  in the above. We deduce that

$$\begin{aligned} [W, \lambda^B F(X, Y[\chi^{-1}])]_*^{A/B} &= [W, \lambda^B F(X[\chi^{-1}], Y[\chi^{-1}])]_*^{A/B} \\ &= [W, F(\phi^B X, \phi^B Y)]_*^{A/B}. \end{aligned}$$

The claim now follows by the Yoneda lemma.  $\square$

THEOREM 10.3. *Let  $E'$  be an  $A/B$ -equivariant periodically orientable ring spectrum, with associated equivariant formal group  $(A/B)^* \xrightarrow{\phi'} C'$ . Let  $\pi: A \rightarrow A/B$  be the projection, and put  $E = (\pi^* E')[\chi_{A,B}^{-1}]$ . Then  $E$  is an  $A$ -equivariant periodically orientable ring spectrum, and for all  $X \in \mathcal{S}_A$  we have*

$$\begin{aligned} E_* X &= E'_* \phi^B X \\ E^* X &= (E')^* \phi^B X. \end{aligned}$$

Moreover, the formal group associated to  $E$  is the pushout of  $C'$  along the inclusion  $(A/B)^* \rightarrow A^*$ .

PROOF. Because  $\pi^*$  preserves smash products, it is clear that  $\pi^* E'$  is a commutative  $A$ -equivariant ring spectrum, and so the same is true of  $E$ . We saw earlier that  $\phi^B \pi^* = 1$ , so  $\phi^B E = E'$ . Also, we have  $E \wedge X = E \wedge X[\chi_{A,B}^{-1}]$ , so

$$\lambda^B(E \wedge X) = \phi^B(E \wedge X) = \phi^B(E) \wedge \phi^B(X) = E' \wedge \phi^B(X).$$

We can apply  $\lambda^{A/B}$  to this to see that  $\lambda^A(E \wedge X) = \lambda^{A/B}(E' \wedge \phi^B(X))$ , and by applying  $\pi_*$  we deduce that  $E_* X = E'_* \phi^B X$ .

For the corresponding statement in cohomology, we see using Lemma 10.2 that  $\lambda^B F(X, E) = F(\phi^B X, \phi^B E) = F(\phi^B X, E')$ . We again apply the functor  $\pi_* \lambda^{A/B}(-)$  to see that  $E^* X = E'^* \phi^B X$ , as claimed.

In particular, if  $X$  is an  $A$ -space we have  $\phi^B \Sigma^\infty X = \Sigma^\infty X^B$  and so  $E^* X = E'^* X^B$ . Thus, if we put  $S = \text{spec}(E'^0(\text{point}))$ , then  $S$  is also the same as  $\text{spec}(E^0(\text{point}))$ . We next consider the space  $PV$ , where  $V$  is a representation of  $A$ . We can split  $V$  into isotypical parts for the action of  $B$ , say  $V = \bigoplus_\beta V[\beta]$ , where  $V[\beta]$  is a sum of representations  $L_\alpha$  with  $\alpha|_B = \beta$ . We then have  $(PV)^B = \prod_\beta PV[\beta]$ , and so  $E^* PV = \prod_\beta E'^* PV[\beta]$ . Using this, it is easy to see that  $E$  is periodically orientable. Next, consider the space  $PU_A$ , so  $(PU_A)^B = \prod_\beta P(U_A[\beta])$ .

The space  $P(\mathcal{U}_A[0])$  is canonically identified with  $P\mathcal{U}_{A/B}$ , so  $\mathrm{spf}(E'^0 P\mathcal{U}_A[0]) = C'$ . For  $\beta \neq 0$ , we can choose  $\tilde{\beta} \in A^*$  extending  $\beta$ , and then tensoring with  $L_{-\tilde{\beta}}$  gives an equivalence  $\theta: P\mathcal{U}_A[\beta] \simeq P\mathcal{U}_{A/B}$ . If we change  $\tilde{\beta}$  by an element  $\gamma \in (A/B)^*$ , then  $\theta$  changes by the automorphism  $\tau_{-\gamma}$  of  $P\mathcal{U}_{A/B}$ . Using this, it is not hard to identify the curve  $C = \mathrm{spf}(E^0 P\mathcal{U}_A) = \coprod \mathrm{spf}(E'^0 P\mathcal{U}_A[\beta])$  with the pushout of  $C'$  along the map  $(A/B)^* \rightarrow A^*$ .  $\square$

## Equivariant Morava $E$ -theory

There is a standard  $p$ -typical formal group law of height  $n$  over  $\mathbb{Z}_p$ , with logarithm  $\sum_{k \geq 0} x^{p^{n_k}}/p^k$ . We can reduce this mod  $p$  and use it to define a formal group  $\widehat{C}_0$  over  $S_0 = \text{spec}(\mathbb{F}_p)$ . We write  $\widehat{K}$  for the two-periodic Morava  $K$ -theory spectrum whose associated formal group is  $\widehat{C}_0$ , so  $\widehat{K}_* = \mathbb{F}_p[u^{\pm 1}]$  with  $|u| = 2$ . This formal group has a universal deformation  $\widehat{C}_1$  over  $S_1 := \text{spf}(\mathbb{Z}_p[[u_1, \dots, u_{n-1}]])$ . We write  $\widehat{E}$  for the corresponding Landweber-exact cohomology theory, and refer to it as Morava  $E$ -theory. Now suppose we have a finite abelian group  $A$  and a subgroup  $B$ . We define  $C_0 = B^* \times \widehat{C}_0$ , which is an  $A$ -efg of product type over  $S_0$  (via the restriction  $A^* \rightarrow B^*$ ), associated to the equivariant Morava  $K$ -theory  $K^*X := \widehat{K}^*\overline{\phi}^B X$ . We can also define an  $A/B$ -equivariant cohomology theory by  $X \mapsto \widehat{E}^*X_{h(A/B)}$ , as in Section 6. The associated equivariant formal group is  $\widehat{C}_2 = \widehat{C}_1 \times_{S_1} S$  over  $S$ , where  $S = \text{Hom}((A/B)^*, \widehat{C}_1)$ . We then perform the construction in Section 10. This gives an  $A$ -equivariant theory  $E = E(p, n, B)$ , defined by

$$E^*X = \widehat{E}^*((\phi^B X)_{h(A/B)}),$$

whose associated equivariant formal group is the pushout of  $\widehat{C}_2$  along the inclusion  $(A/B)^* \rightarrow A^*$ . We write  $C$  for this pushout, and we refer to  $E$  as *equivariant Morava  $E$ -theory*. In [30] we give some evidence that this name is reasonable, related to the theory of Bousfield classes and nilpotence. Here we give a further piece of evidence, based on formal group theory.

We first note that  $S_0$  is a closed subscheme of  $S_1$ , which is in turn a closed subscheme of  $S = \text{Hom}((A/B)^*, \widehat{C}_1)$  (corresponding to the zero homomorphism). The restriction of  $C$  to  $S_1$  is just  $B^* \times \widehat{C}_1$ , and the restriction of this to  $S_0$  is just  $C_0$ . The inclusion  $C_0 \rightarrow C$  corresponds to a ring map  $\mathcal{O}_C \rightarrow \mathcal{O}_{C_0}$ , or equivalently  $E^0PU \rightarrow K^0PU$ . It can be shown that this comes from a natural map  $E^*X \rightarrow K^*X$  of cohomology theories. Indeed, there is certainly a nonequivariant map  $q: \widehat{E} \rightarrow \widehat{K}$ . Moreover, up to homotopy there is a unique map  $A/B \rightarrow E(A/B)$  of  $A/B$ -spaces, which gives a natural map

$$\text{res}(Y) = (A/B_+ \wedge Y)/(A/B) \rightarrow (E(A/B)_+ \wedge Y)/(A/B) = Y_{h(A/B)}$$

for  $A/B$ -spectra  $Y$ . If  $Y = \phi^B X$  then  $\text{res}(Y) = \overline{\phi}^B X$  and so we get a map

$$E^*X = \widehat{E}^*(\phi^B X)_{h(A/B)} \rightarrow \widehat{E}^*\overline{\phi}^B X \xrightarrow{q_*} \widehat{K}^*\overline{\phi}^B X = K^*X,$$

as required.

DEFINITION 11.1. A *deformation* of the  $A$ -efg  $C_0$  over  $S_0$  consists of an  $A$ -efg  $C'$  over a base  $S'$  together with a commutative square

$$\begin{array}{ccc} C_0 & \xrightarrow{\tilde{f}} & C' \\ \downarrow & & \downarrow \\ S_0 & \xrightarrow{f} & S' \end{array}$$

such that

- (a)  $f$  is a closed inclusion, and  $S'$  is a formal neighbourhood of  $f(S_0)$
- (b)  $\tilde{f}$  induces an isomorphism  $C_0 \rightarrow f^*C'$  of  $A$ -efg's over  $S_0$ .

If  $C'$  and  $C''$  are deformations, a morphism between them means a commutative square

$$\begin{array}{ccc} C' & \xrightarrow{\tilde{g}} & C'' \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S'' \end{array}$$

such that  $\tilde{g}$  induces an isomorphism  $C' \rightarrow g^*C''$  of  $A$ -efg's over  $S'$ . A *universal deformation* means a terminal object in the category of deformations.

As mentioned previously, the formal group  $\widehat{C}_1$  associated to  $\widehat{E}$  is the universal deformation of the formal group  $\widehat{C}_0$  associated to  $\widehat{K}$ . Equivariantly, we have the following analogue.

THEOREM 11.2. *The  $A$ -equivariant formal group  $C$  (associated to equivariant Morava  $E$ -theory) is the universal deformation of  $C_0$  (associated to equivariant Morava  $K$ -theory).*

PROOF. Suppose we have an  $A$ -efg  $(C', \phi')$  over  $S'$  equipped with maps  $(f, \tilde{f})$  making it a deformation of  $C_0$ . We will identify  $S_0$  with  $f(S_0)$  and thus regard it as a closed subscheme of  $S'$ . Similarly, we regard  $C_0$  as the closed subscheme  $C'|_{S_0}$  of  $C'$ . Note that  $S'$  is a formal neighbourhood of  $S_0$ , and it follows that  $C'$  is a formal neighbourhood of  $C_0$ . We choose a coordinate  $x'$  on  $C'$ , and note that it restricts to give a coordinate on  $C_0$ .

Now let  $\widehat{C}'$  denote the formal neighbourhood of the zero section in  $C'$ . We have  $(\widehat{C}')|_{S_0} = \widehat{C}_0$ , so we can regard  $\widehat{C}'$  as a deformation of the ordinary formal group  $\widehat{C}_0$ . As  $\widehat{C}_1$  is the universal deformation of  $\widehat{C}_0$ , this gives us a pullback square

$$\begin{array}{ccc} \widehat{C}' & \xrightarrow{\tilde{g}} & \widehat{C}_1 \\ \downarrow & & \downarrow \\ S' & \xrightarrow{g} & S_1. \end{array}$$

Next, suppose we have  $\alpha \in (A/B)^* \subset A^*$ , giving a section  $\phi'(\alpha)$  of  $C'$  and an element  $x'(\phi'(\alpha)) \in \mathcal{O}_{S'}$ . As  $C'|_{S_0} = C_0 = B^* \times \widehat{C}_0$  and  $\alpha|_B = 0$  we have  $\phi'(\alpha)|_{S_0} = 0$ , so  $x'(\phi'(\alpha))$  maps to 0 in  $\mathcal{O}_{S_0}$ . As  $S'$  is a formal neighbourhood of  $S_0$ , it follows that  $x'(\phi'(\alpha))$  is topologically nilpotent in  $\mathcal{O}_{S'}$ , and thus that  $\phi'(\alpha)$  is actually a



section of  $\widehat{C}'$ . Thus,  $\tilde{g} \circ \phi'$  gives a map  $(A/B)^* \rightarrow \widehat{C}_1$ , which is classified by a map  $h: S' \rightarrow \text{Hom}((A/B)^*, \widehat{C}_1) = S$ . The maps  $\tilde{g}$  and  $h$  combine to give a map

$$\theta: \widehat{C}' \rightarrow h^* \widehat{C} = h^*(\widehat{C}_1 \times_{S_1} S) = g^* \widehat{C}_1.$$

This can be regarded as an isomorphism of  $A/B$ -equivariant formal groups.

Next, the decomposition  $C_0 = B^* \times \widehat{C}_0 = \coprod_{\beta \in B^*} \widehat{C}_0$  gives orthogonal idempotents  $e_\beta \in \mathcal{O}_{C_0}$  with  $\sum_\beta e_\beta = 1$ . As  $C'$  is a formal neighbourhood of  $C_0$ , these can be lifted to orthogonal idempotents in  $\mathcal{O}_{C'}$ , giving a decomposition  $C' = \coprod_\beta C'_\beta$  say. One can check that  $C'_\beta = \phi'(\alpha) + \widehat{C}'$  for any  $\alpha \in A^*$  with  $\alpha|_B = \beta$ , and it follows that  $C'$  is just the pushout of the map  $\phi': (A/B)^* \rightarrow \widehat{C}'$  and the inclusion  $(A/B)^* \rightarrow A^*$ . It follows in turn that  $\tilde{h}$  extends to give an isomorphism  $C' \rightarrow g^* C$ , and thus a morphism  $C' \rightarrow C$  of deformations. All steps in this construction are forced, so one can check that this morphism is unique. This means that  $C$  is the universal deformation of  $C_0$ , as claimed.  $\square$



## A completion theorem

Suppose we have an  $A$ -equivariant formal group  $(C, \phi)$ , and a subgroup  $B \leq A$ , giving a subgroup  $(A/B)^* \leq A^*$ . Let  $S_0 = S[B]$  be the closed subscheme of  $S$  where  $\phi((A/B)^*) = 0$ . Equivalently, if we put  $e_\alpha = x(\phi(-\alpha))$  and  $I_B = (e_\alpha \mid \alpha \in (A/B)^*)$ , then  $S_0 = V(I_B) = \text{spec}(\mathcal{O}_S/I_B)$ . If we put  $C_0 = S_0 \times_S C$  then  $\phi$  induces a map  $\psi: B^* \times S_0 \rightarrow C_0$  making  $C_0$  into a  $B$ -equivariant formal group over  $S_0$ . Next, we put  $S_1 = \lim_{\rightarrow m} \text{spec}(\mathcal{O}_S/J^m) = \text{spf}((\mathcal{O}_S)^\wedge_J)$ , the formal neighbourhood of  $S_1$  in  $S$ , and  $C_1 = S_1 \times_S C$ . This is an  $A$ -equivariant formal group over  $S_1$  for which  $\phi((A/B)^*)$  is infinitesimally close to 0.

Now suppose that  $C$  comes from an  $A$ -equivariant periodically orientable theory  $E$ . We would like to interpret  $C_0$  and  $C_1$  topologically.

**PROPOSITION 12.1.** *Let  $E_0$  be the  $B$ -spectrum  $\text{res}_B^A(E)$ , representing the theory  $E^*(A \times_B Y)$  for  $B$ -spaces  $Y$ . Let  $C'_0/S'_0$  be the associated  $B$ -equivariant formal group. Then there is a map  $S'_0 \rightarrow S_0$  (which may or may not be an isomorphism) and an isomorphism  $C'_0 = C_0 \times_{S_0} S'_0$ .*

**PROOF.** We have  $S'_0 = \text{spec}(\pi_0 E_0) = \text{spec}(E^0 A/B)$ , so there is a natural map  $S'_0 \rightarrow S$ . Moreover, we have  $PU_B \simeq \text{res}_B^A PU_A$ , which gives an isomorphism  $A \times_B PU_B \simeq A/B \times PU_A$  and thus

$$E_0^0 PU_B \simeq E^0(A/B \times PU_A) = E^0(A/B) \otimes_{E^0} E^0 PU_A.$$

This shows that the formal group for  $E_0$  is just  $C'_0 := C \times_S S'_0$ . All that is left is to check that the map  $S'_0 \rightarrow S$  factors through  $S_0$ , so  $C'_0$  can also be described as  $C_0 \times_{S_0} S'_0$ . To see this, note that  $\phi$  comes from the inclusion  $j: A^* = \pi_0^A PU \rightarrow PU$ , so the corresponding map  $\phi'_0$  over  $S'_0$  comes from the map  $1 \times j: (A/B) \times A^* \rightarrow (A/B) \times PU$ . Using the isomorphism

$$\begin{aligned} [(A/B) \times A^*, (A/B) \times PU]^A &= [A^*, (A/B) \times PU]^B \\ &= \text{Map}(A^*, \pi_0^B((A/B) \times PU)) \\ &= \text{Map}(A^*, (A/B) \times B^*) \end{aligned}$$

we see that the restriction of  $(1 \times j)$  to  $(A/B) \times (A/B)^*$  is null, so that  $\phi'_0((A/B)^*) = 0$  as claimed.  $\square$

If  $E$  is the complex  $K$ -theory spectrum  $KU_A$ , then we saw earlier that  $S = \text{Hom}(A^*, \mathbb{G}_m)$  and so

$$S_0 = \{\phi \in \text{Hom}(A^*, \mathbb{G}_m) \mid \phi((A/B)^*) = 0\} = \text{Hom}(B^*, \mathbb{G}_m).$$

On the other hand, it is well-known that  $KU_A^*(A \times_B Y) = KU_B^* Y$  so  $E_0 = KU_B$  so  $S'_0 = \text{Hom}(B^*, \mathbb{G}_m) = S_0$ . A similar argument works for theories of the form  $E^* X = \widehat{E}^* X_{hA}$  where  $\widehat{E}$  is  $K(n)$ -local as in Section 6, in which case we have

$S = \text{Hom}(A^*, \widehat{C})$  and  $S_0 = S'_0 = \text{Hom}(B^*, \widehat{C})$ . At the other extreme, for theories of the form  $E^*X = \widehat{E}^*(\text{res}_0^A(X))$ , we have  $S_0 = S$  and  $S'_0 = (A/B) \times S$ .

We next consider  $C_1$ . Recall that there is an  $A$ -space  $E[\leq B]$  characterised by the property that  $E[\leq B]^C$  is contractible for  $C \leq B$  and empty for  $C \not\leq B$ . The first approximation would be to consider the ring spectrum  $F(E[\leq B]_+, E)$ . However, as  $S_1$  is a formal scheme rather than an affine scheme, we need a pro-spectrum rather than a spectrum. The solution is to define  $F_\bullet(X_+, E)$  to be the pro-system of ring spectra  $F(X_{\alpha+}, E)$ , where  $X_\alpha$  runs over finite subcomplexes of  $X$ , and to put  $E_1 = F_\bullet(E[\leq B]_+, E)$ . The desired description of  $E_1^*PU$  is a kind of completion theorem in the style of Atiyah-Segal, so we expect to need finiteness hypotheses. However, with these hypotheses, we have an exact result rather than an approximate one as in the previous proposition.

**THEOREM 12.2.** *Suppose that  $E^*(\text{point})$  is a Noetherian ring and that  $E^*(A/C)$  is finitely generated over it for all  $C \leq A$ . Then the  $A$ -equivariant formal group associated to  $E_1$  is  $C_1$ .*

**PROOF.** This is essentially taken from [12]. Choose generators  $\alpha_1, \dots, \alpha_r$  for  $(A/B)^*$ , let  $L_i$  be the one-dimensional representation corresponding to  $\alpha_i$  and let  $\chi_i$  denote the inclusion  $S^0 \rightarrow S^{L_i}$ . There is a canonical Thom class  $u_i$  in  $E^0 S^{L_i}$ , and  $\chi_i^*(u_i)$  is the Euler class  $e_i = x(\phi(-\alpha_i))$ . One checks easily that the space  $P := \prod_i S(\infty L_i)$  is a model for  $E[\leq B]$ , and the spaces  $T(m) := \prod_i S(mL_i)$  form a cofinal system of finite subcomplexes, so  $E_1$  is equivalent to the tower of ring spectra  $F(T(m)_+, E) = D(T(m)_+) \wedge E$ . Next, by taking the Spanier-Whitehead dual of the cofibration  $S(mL_i)_+ \rightarrow S^0 \xrightarrow{\chi_i^m} S^{mL_i}$ , we see that  $D(S(mL_i)_+)$  deserves to be called  $S/\chi_i^m$ , and so  $D(T(m)_+)$  deserves to be called  $S/(\chi_1^m, \dots, \chi_r^m)$ . This suggests that  $\pi_*(E \wedge D(T(m)_+))$  should be  $E_*/J_m$ , where  $J_m = (u_1^m, \dots, u_r^m) \leq E_*$ . Unfortunately, there are correction terms. More precisely, the cofibration displayed above gives a two-stage filtration of  $D(S(mL_i)_+)$  for each  $i$ , and by smashing these together we get a  $(r+1)$ -stage filtration of  $D(T(m)_+)$ , and thus a spectral sequence converging to  $\pi_*(E \wedge D(T(m)_+))$ . The first page is easily seen to be the Koszul complex for the sequence  $e_1^m, \dots, e_r^m$ , so the bottom line of the second page is  $E_*/J_m$ , and the remaining lines are higher Koszul homology groups. The filtrations are compatible as  $m$  varies, so we get a spectral sequence in the abelian category of pro-groups converging to  $\pi_* E_1$ . In the second page, the bottom line is the tower  $\{E_*/J_m\}_{m \geq 0}$ , and the remaining lines are pro-trivial by [12, Lemma 3.7]. It follows that  $\pi_* E_1 \simeq \{E_*/J_m\}$  as pro-groups, and so the formal scheme corresponding to  $\pi_0 E_1$  is  $\lim_{\rightarrow m} \text{spec}(E^0/J_m) = \lim_{\rightarrow m} \text{spec}(E^0/J^m) = S_1$ . We now replace  $E$  by  $F(P(n, \mathbb{C}[A])_+, E)$  and then take the limit as  $n$  tends to infinity to conclude that  $\text{spf}(E_1^*PU) = C \times_S S_1 = C_1$  as claimed.  $\square$

**REMARK 12.3.** Using the same circle of ideas one proves that the kernel of the map  $E^0/J \rightarrow E^0(A/B)$  is nilpotent, so the map  $S'_0 \rightarrow S_0$  is dominant; compare [12, Theorem 1.4].

## Equivariant formal group laws and complex cobordism

We can define an *A-equivariant formal group law* over  $S$  to be an isomorphism class of triples  $(C, \phi, x)$ , where  $(C, \phi)$  is an  $A$ -efg over  $S$  and  $x$  is a coordinate on  $C$ . We write  $\mathrm{FGL}_A(S)$  for the set of such things. In the case  $A = 0$ , there is a power series  $F \in \mathcal{O}_S[[s, t]]$  such that  $x(a+b) = F(x(a), x(b))$  for all  $a, b \in C$ . Using this we see that  $\mathrm{FGL}_0(S)$  bijects with the set of formal group laws over  $\mathcal{O}_S$  as traditionally defined. Cole, Greenlees and Kriz have shown [7] that there is a ring  $L_A$  (called the *equivariant Lazard ring*) and a natural bijection  $\mathrm{FGL}_A(S) \simeq \mathrm{Hom}(L_A, \mathcal{O}_S)$ . Next, we can define an  $A$ -equivariant complex cobordism spectrum  $MU_A$  as explained in [23, Chapter XV] for example, and then put  $MP_A = \bigvee_{n \in \mathbb{Z}} \Sigma^{2n} MU_A$ . It is shown in [23, Chapter XXVIII] that  $MU_A^*$  is a free module over the nonequivariant complex cobordism ring  $MU^*$ , and is concentrated in even degrees. It follows that  $MP_A$  is an evenly orientable  $A$ -equivariant ring spectrum. There is a map  $P\mathcal{U} \rightarrow MP_A$  built into the definition of  $MP_A$ , and this gives a canonical coordinate  $x$  on the resulting  $A$ -efg. Using this we obtain a map  $\lambda_A: L_A \rightarrow \pi_0(MP_A)$ . In the case  $A = 0$  it is a fundamental theorem of Quillen that the map  $\lambda_0: L_0 \rightarrow \pi_0(MP)$  is an isomorphism, and that  $L_0$  is a polynomial algebra over  $\mathbb{Z}$  on a countable sequence of generators. When  $A \neq 0$  it is known that  $\lambda_A$  is surjective; let  $J_A$  denote the kernel. It is known that  $J_A$  is torsion and nilpotent and has various other exotic properties that force it to map to zero in any Noetherian quotient of  $L_A$ ; but it is not known whether  $J_A = 0$ . This is discussed in [13]. In the case  $|A| = 2$  an explicit calculation of  $MU_A^*$  is given in [27], and using this one can check that there is a ring map  $\mu_A: MP_A^0 \rightarrow L_A$  with  $\lambda_A \mu_A = 1$ . A very similar argument also works when  $|A|$  is an odd prime.



## A counterexample

Here we exhibit a  $\mathbb{Z}/2$ -equivariant formal group  $C$  with a number of unusual properties, which are only possible because the base scheme  $S$  is not Noetherian. The possibility of such behaviour makes it difficult to understand the equivariant Lazard ring  $L_A$ , which is one reason why we have not yet succeeded in proving that  $L_A = \pi_0(MP_A)$ .

For any  $A$ -equivariant formal group  $(C, \phi)$ , there is a natural map  $\psi: A^* \times \widehat{C} \rightarrow C$  given by  $\psi(\alpha, a) = \phi(\alpha) + a$ . As  $C$  is a formal neighbourhood of  $[\phi(A^*)]$ , it is natural to expect that  $\psi$  should be an epimorphism, or equivalently that the map  $\psi^*: \mathcal{O}_C \rightarrow \prod_{\alpha} \mathcal{O}_{\widehat{C}}$  should be injective. The key feature of the example to be constructed here is that  $\psi^*$  is not in fact injective.

Start with  $k_0 = \mathbb{F}_2[e]$ , let  $M$  be the module  $\mathbb{F}_2[e^{\pm 1}]/\mathbb{F}_2[e]$ , and let  $k$  be the square-zero extension  $k_0 \oplus M$ . More explicitly,  $k$  is generated over  $k_0$  by elements  $u_1, u_2, \dots$  subject to  $eu_{i+1} = u_i$  (with  $u_0$  interpreted as 0) and  $u_i u_j = 0$ . Put  $S = \text{spec}(k)$ .

Next, let  $R$  be the completion of  $k[x]$  at the element  $y = x^2 + ex$ , so  $R = k[[y]][1, x]$ , and put

$$C = \text{spf}(R) = \{x \in \mathbb{A}_S^1 \mid x^2 + ex \text{ is nilpotent}\}.$$

This is a subgroup of  $\mathbb{A}_S^1$  under addition. In the corresponding Hopf algebra structure on  $R$ , the elements  $x$  and  $y$  are both primitive. There is a homomorphism  $\phi: \mathbb{Z}/2 \rightarrow C$  sending 0 to 0 and 1 to  $e$ . The corresponding divisor is just  $R/y$ , and as  $R$  is complete at  $y$ , we deduce that  $(C, \phi)$  is an equivariant formal group.

Next, we can define maps  $\lambda_0, \lambda_a: R \rightarrow k[[t]]$  by

$$\begin{aligned}\lambda_0(x) &= t \\ \lambda_a(x) &= t + e \\ \lambda_0(y) &= \lambda_a(y) = t^2 + te.\end{aligned}$$

The map  $\psi^*: \mathcal{O}_C \rightarrow \prod_{\alpha} \mathcal{O}_{\widehat{C}}$  is just the map  $(\lambda_0, \lambda_a): R \rightarrow k[[t]] \times k[[t]]$ . Now consider the element

$$f = \sum_{k \geq 0} u_{1-2^{k+1}} y^{2^k} \in R.$$

We then have

$$\begin{aligned}
 \lambda_0(f) &= \sum_{k \geq 0} u_{1-2^{k+1}} (t^2 + et)^{2^k} \\
 &= \sum_{k \geq 0} u_{1-2^{k+1}} t^{2^{k+1}} + \sum_{k \geq 0} u_{1-2^{k+1}} e^{2^k} t^{2^k} \\
 &= \sum_{k \geq 0} u_{1-2^{k+1}} t^{2^{k+1}} + \sum_{k \geq 0} u_{1-2^k} t^{2^k} \\
 &= u_{1-2^0} t^{2^0} = u_0 t = 0.
 \end{aligned}$$

We also have  $\lambda_a(f) = 0$  by the same argument, so  $\psi^*(f) = 0$ .



## CHAPTER 15

# Divisors

We now return to the purely algebraic theory of formal multicurves and their divisors.

Recall that a divisor on  $C$  is a regular hypersurface  $D \subseteq C$  such that  $\mathcal{O}_D$  is a finitely generated projective module over  $\mathcal{O}_S$ , which is discrete in the quotient topology. We also make the following temporary definition; one of our main tasks in this section is to show (in Proposition 15.15) that it is equivalent to the preceding one. (For divisors of degree one, this follows from Corollary 2.12.)

**DEFINITION 15.1.** A *weak divisor* on  $C$  is a closed subscheme  $D \subset C$  that is finite and very flat over  $S$  (so  $\mathcal{O}_D$  is a discrete finitely generated projective module over  $\mathcal{O}_S$ ). Thus, a weak divisor  $D = \text{spf}(R/J)$  is a divisor iff the ideal  $J$  is open and generated by a regular element. If  $y$  is a good parameter on  $C$ , we note that  $J$  is open iff  $y^N \in J$  for  $N \gg 0$ .

If  $D_0 = \text{spf}(R/J_0)$  is a divisor and  $D_1 = \text{spf}(R/J_1)$  is a weak divisor then one checks that the scheme  $D_0 + D_1 := \text{spf}(R/(J_0 J_1))$  is again a weak divisor.

**DEFINITION 15.2.** Now suppose we have a map  $q: T \rightarrow S$  of schemes which is finite and very flat, so that  $\mathcal{O}_T$  is a discrete finitely generated projective module over  $\mathcal{O}_S$ . If  $g \in \mathcal{O}_T$  then multiplication by  $g$  gives an  $\mathcal{O}_S$ -linear endomorphism  $\mu_g$  of  $\mathcal{O}_T$ , whose determinant we denote by  $N_q(g)$  or  $N_{T/S}(g)$ .

**DEFINITION 15.3.** Fix a difference function  $d$  on  $C$ . For any weak divisor  $D$  on  $C$  over, we can regard  $d$  by restriction as a function on  $D \times_S C$ . We also have a projection  $q: D \times_S C \rightarrow C$ , and we put

$$f_D = N_q(d) = N_{(D \times_S C)/C}(d) \in \mathcal{O}_C.$$

We will eventually show that  $D = \text{spf}(\mathcal{O}_C/f_D)$ .

**REMARK 15.4.** Consider the case where  $C$  is an ordinary formal group, with coordinate  $x$  and associated formal group law  $F$ . We then have  $\mathcal{O}_C = \mathcal{O}_S[[x]]$  and  $\mathcal{O}_{C \times C} = \mathcal{O}_S[[x_0, x_1]]$ , and we can take  $d = x_1 -_F x_0$ . If  $D$  has the form  $\sum_i [u_i]$  for some family of sections  $u_i$ , then we have elements  $a_i = x(u_i) \in \mathcal{O}_S$  and we will see that  $f_D = \prod_i (x -_F a_i)$ . This is a unit multiple of the Chern polynomial  $g_D = \prod_i (x - a_i)$ , and it is familiar that  $D = \text{spf}(\mathcal{O}_C/g_D)$ , so  $D = \text{spf}(\mathcal{O}_C/f_D)$  also. In the multicurve case, one can still define  $g_D$  (as the norm of the function  $(a, b) \mapsto x(b) - x(a)$ ) and we find that it is divisible by  $f_D$ , but  $g_D/f_D$  need not be invertible so  $\mathcal{O}_C/g_D \neq \mathcal{O}_D$ .

**LEMMA 15.5.** *Let  $R$  be a ring,  $P$  a finitely generated projective  $R$ -module, and  $\alpha$  an endomorphism of  $P$ . Then  $\alpha$  is injective iff  $\det(\alpha)$  is a regular element.*

PROOF. After localising we may assume that  $P = R^d$  for some  $d$ , and  $\alpha$  is represented by a  $d \times d$  matrix  $A$ . If  $\det(A)$  is regular, the equation  $\operatorname{adj}(A)A = \det(A)I_d$  implies immediately that  $\alpha$  is injective. Conversely, suppose that  $\alpha$  is injective. As  $P$  is flat, it follows that  $\alpha^{\otimes d}: P^{\otimes d} \rightarrow P^{\otimes d}$  is also injective. It is easy to check with bases that  $\lambda^d P$  is naturally isomorphic to the image of the antisymmetrisation map  $P^{\otimes d} \rightarrow P^{\otimes d}$ . In particular, it embeds naturally in  $P^{\otimes d}$ , and it therefore follows that  $\lambda^d \alpha$  is injective. On the other hand,  $\lambda^d P$  is an invertible  $R$ -module, so  $\operatorname{End}(\lambda^d P) = R$ , and  $\lambda^d(\alpha) = \det(\alpha)$  under this isomorphism. It follows that  $\det(\alpha)$  is regular as claimed.  $\square$

COROLLARY 15.6. *For any weak divisor  $D$  on  $C$ , the element  $f_D \in \mathcal{O}_C$  is regular.*

PROOF. Take  $R = \mathcal{O}_C$  and  $P = \mathcal{O}_{D \times_S C}$  and  $\alpha = \mu_d$ . We know from Lemma 2.11 that  $\alpha$  is injective, and the claim follows.  $\square$

LEMMA 15.7. *Let  $q: T \rightarrow S$  be finite and very flat, and let  $g$  be a function on  $T$ . If there is a section  $u: S \rightarrow T$  such that  $g \circ u = 0$  then  $N_q(g) = 0$ .*

PROOF. Put  $J = \ker(u^*: \mathcal{O}_T \rightarrow \mathcal{O}_S)$ , so  $g \in J$ . We have a short exact sequence of  $\mathcal{O}_S$  modules  $J \rightarrow \mathcal{O}_T \xrightarrow{u^*} \mathcal{O}_S$ , which is split by the map  $q^*: \mathcal{O}_S \rightarrow \mathcal{O}_T$ . The sequence is preserved by  $\mu_g$ , and  $\mu_g(\mathcal{O}_T) = \mathcal{O}_T \cdot g \leq J$  so the induced map on the cokernel is zero. Zariski-locally on  $S$  we can choose bases adapted to the short exact sequence and it follows easily that  $\det(\mu_g) = 0$  as claimed.  $\square$

COROLLARY 15.8. *The function  $f_D \in \mathcal{O}_C$  vanishes on  $D$ .*

PROOF. We have  $f_D|_D = N_{q'}(d)$ , where  $q': D \times_S D \rightarrow D$  is the projection on the second factor. The diagonal map  $\delta: D \rightarrow D \times_S D$  is a section of  $q'$  with  $d \circ \delta = 0$ , so  $N_{q'}(d) = 0$ .  $\square$

LEMMA 15.9. *If  $D = D_0 + D_1$  (where  $D_0, D_1$  are divisors) and  $g \in \mathcal{O}_D$  then*

$$N_{D/S}(g) = N_{D_0/S}(g)N_{D_1/S}(g).$$

PROOF. Put  $R = \mathcal{O}_C$ , and let the ideals corresponding to  $D_i$  be  $J_i = (f_i)$  for  $i = 0, 1$ . We then have a short exact sequence of  $\mathcal{O}_D$ -modules as follows:

$$\mathcal{O}_{D_0} = R/f_0 \xrightarrow{\times f_1} \mathcal{O}_D = R/(f_0 f_1) \rightarrow \mathcal{O}_{D_1} = R/f_1.$$

This is splittable, because  $\mathcal{O}_{D_1}$  is projective over  $\mathcal{O}_S$ . The map  $\mu_g$  preserves the sequence, and it follows easily that  $\det(\mu_g) = \det(\mu_g|_{\mathcal{O}_{D_0}}) \det(\mu_g|_{\mathcal{O}_{D_1}})$ , as required.  $\square$

COROLLARY 15.10. *If  $D = D_0 + D_1$  as above then  $f_D = f_{D_0} f_{D_1}$ .*

PROOF. Just change base to  $C$  and take  $g = d$ .  $\square$

LEMMA 15.11. *Suppose that  $D$  is a weak divisor of degree  $r$ , that  $D'$  is a divisor of degree  $r'$ , and that  $D' \subseteq D$ . Then  $D = D' + D''$  for some weak divisor  $D''$  of degree  $r - r'$ .*

PROOF. Put  $J = I_D$  and  $J' = I_{D'}$ . As  $D'$  is a genuine divisor, we have  $J' = Rf'$  for some regular element  $f' \in R$ . As  $D' \subseteq D$ , we have  $J \leq J'$ . Put  $J'' = \{g \in R \mid f'g \in J\} \geq J$ . We then have a short exact sequence

$$R/J'' \xrightarrow{\times f'} R/J \rightarrow R/J'.$$

As  $R/J$  and  $R/J'$  are projective modules of ranks  $r$  and  $r'$  over  $k$ , it follows that  $R/J''$  is a projective module of rank  $r - r'$ . Thus, the scheme  $D'' := \text{spf}(R/J'')$  is a weak divisor. From the definition of  $J''$  we have  $J'J'' \leq J$ . Conversely, if  $h \in J$  then certainly  $h \in J' = Rf'$  so  $h = gf'$  for some  $g \in R$ . From the definitions we have  $g \in J''$ , so  $h \in J''J'$ . This shows that  $J = J'J''$  and so  $D = D' + D''$ .  $\square$

DEFINITION 15.12. Let  $D$  be a weak divisor of constant degree  $r$ . A *full set of points* for  $D$  is a list  $u_1, \dots, u_r$  of sections of  $S$  such that  $D = \sum_i [u_i]$ . If there exists a full set of points, it is clear that  $D$  is actually a genuine divisor. (This concept is due to Drinfeld, and is explained and used extensively in [18].)

PROPOSITION 15.13. *If  $u_1, \dots, u_r$  is a full set of points for  $D$ , then  $N_{D/S}(g) = \prod_i g(u_i)$  for any function  $g$  on  $D$ . Moreover, we have  $f_D(a) = \prod_i d(a, u_i)$ , and so  $\mathcal{O}_D = \mathcal{O}_C/f_D$ .*

PROOF. As the projection  $[u_i] \rightarrow S$  is an isomorphism, we see that  $N_{[u_i]/S}(g) = g(u_i)$ . The first claim follows easily using from Lemma 15.9 by induction on  $r$ . It follows similarly from Corollary 15.10 that  $f_D(a) = \prod_i d(a, u_i)$ . As  $d$  is a difference function we have  $\mathcal{O}_{[u_i]} = \mathcal{O}_C/d(a, u_i)$  and so  $\mathcal{O}_D = \mathcal{O}_C/f_D$  as claimed.  $\square$

LEMMA 15.14. *Let  $D$  be a weak divisor of constant degree  $r$ . Then there is a finite, very flat scheme  $T$  over  $S$  such that the weak divisor  $T \times_S D$  on  $T \times_S C$  has a full set of points (and so is genuine).*

PROOF. By an evident induction, it suffices to show that after very flat base change we can split  $D$  as  $[u] + D''$  for some section  $u$  and some weak divisor  $D''$ . It is enough to find a section  $u: S \rightarrow D$ , for then  $[u] \subseteq D$  and we can apply the previous lemma. For this we can simply pull back along the projection map  $D \rightarrow S$  (which is very flat by assumption) and then the diagonal map  $D \rightarrow D \times_S D$  gives the required “tautological” section.  $\square$

PROPOSITION 15.15. *Every weak divisor is a genuine divisor.*

PROOF. Let  $D = \text{spf}(R/J)$  be a weak divisor. We may assume without loss that it has constant degree  $r$ . We know from Corollary 15.6 and Corollary 15.8 that  $f_D$  is regular in  $R$  and lies in  $J$ ; we need only show that it generates  $J$ . It is enough to do this after faithfully flat base change, so by Lemma 15.14 we may assume that we have a full set of points. Proposition 15.13 completes the proof.  $\square$



## Embeddings

Let  $C$  be a nonempty formal multicurve over a scheme  $S$ . In this section we study embeddings of  $S$  in the affine line  $\mathbb{A}_S^1 = \mathbb{A}^1 \times S$ . If  $q$  is the given map  $C \rightarrow S$ , then any map  $C \rightarrow \mathbb{A}_S^1$  of schemes over  $S$  has the form  $(x, q)$  for some  $x: C \rightarrow \mathbb{A}^1$ , or equivalently  $x \in \mathcal{O}_C$ .

Now choose a difference function  $d$  on  $C$ . Given  $x \in \mathcal{O}_C$ , we can define  $x': C \times_S C \rightarrow \mathbb{A}^1$  by  $x'(a, b) = x(b) - x(a)$ . Equivalently,  $x'$  is the element  $1 \otimes x - x \otimes 1$  in  $\mathcal{O}_{C \times_S C} = \mathcal{O}_C \widehat{\otimes}_{\mathcal{O}_S} \mathcal{O}_C$ . It is clear that  $x'$  vanishes on the diagonal, and thus is divisible by  $d$ , say  $x' = \theta(x)d$  for some  $\theta(x) \in \mathcal{O}_{C \times_S C}$ . This element  $\theta(x)$  is unique, because  $d$  is not a zero-divisor.

**PROPOSITION 16.1.** *Let  $C \xrightarrow{q} S$  be a nonempty formal multicurve. A map  $(x, q): C \rightarrow \mathbb{A}_S^1$  is injective if and only if  $\theta(x)$  is invertible. If so, then  $(x, q)$  induces an isomorphism  $C \rightarrow \varinjlim_k V(f^k) \subset \mathbb{A}_S^1$  for some monic polynomial  $f \in \mathcal{O}_S[t]$ , showing that  $C$  is embeddable.*

**PROOF.** Put  $X = \{(a, b) \in C \times_S C \mid x(a) = x(b)\} = V(x') = V(\theta(x)d)$ . We see that  $x$  is injective if and only if  $V(x') = \Delta = V(d)$ , if and only if  $d = ux' = u\theta(x)d$  for some  $u \in \mathcal{O}_{C \times_S C}$ . As  $d$  is not a zero divisor, this holds if and only if  $\theta(x)$  is invertible.

If so, we may assume without loss that  $d = x'$ . Choose a good parameter  $y$ , so  $\mathcal{O}_C/y$  has constant rank  $r$  over  $\mathcal{O}_S$  for some  $r$ . Put  $D = \text{spec}(R/y)$ , let  $p: C \times_S D \rightarrow C$  be the projection, and put  $z = N_p(x')$ . The proof of Proposition 15.15 shows that  $z$  is a unit multiple of  $y$ .

We next claim that  $\{1, x, \dots, x^{r-1}\}$  is a basis for  $R/y = R/z$  over  $k$ , and that  $z = f(x)$  for a unique monic polynomial  $f$  of degree  $r$ . It is enough to check this after faithfully flat base change, so we may assume that  $D = [u_0] + \dots + [u_{r-1}]$  for some list of sections  $u_i$  of  $D$ . If we put  $a_i = x(u_i)$  we see that  $z = \prod_i (x - a_i)$ . If we put  $e_i = \prod_{j < i} (x - a_j)$  we also find that  $\{e_0, \dots, e_{n-1}\}$  is a basis for  $R/z$ . As  $e_i = x^i + \text{lower terms}$ , we also find that  $\{1, \dots, x^{n-1}\}$  is a basis as claimed.

The rest of the proposition follows easily from this.  $\square$

Now suppose we have an arbitrary element  $x \in \mathcal{O}_C$ . Given a map  $u: S' \rightarrow S$  we get a multicurve  $C' := S' \times_S C$  over  $S'$  and a function  $x' = (C' \rightarrow C \xrightarrow{x} \mathbb{A}^1) \in \mathcal{O}_{C'}$ .

**LEMMA 16.2.** *There is a basic open subscheme  $U \subset S$  such that  $(x', q'): C' \rightarrow \mathbb{A}_{S'}^1$  is an embedding if and only if  $u: S' \rightarrow S$  factors through  $U$ .*

**PROOF.** Choose a good parameter  $y$  on  $C$  and put  $D = \text{spec}(\mathcal{O}_C/y)$ . Put  $w = N_{D \times_S D/S}(\theta(x)) \in \mathcal{O}_S$ . We see that  $w$  is invertible in  $\mathcal{O}_S$  if and only if  $\theta(x)$  is invertible in  $\mathcal{O}_{D \times_S D}$ . As  $\mathcal{O}_C$  is complete at  $y$ , we see that  $\theta(x)$  is invertible in  $\mathcal{O}_{C \times_S C}$  if and only if it is invertible in  $\mathcal{O}_{D \times_S D}$ . Given this, it is clear that the scheme  $U = \text{spec}(\mathcal{O}_S[1/w])$  has the stated property.  $\square$

COROLLARY 16.3. *Let  $C$  be a formal multicurve over  $S$ . Then there is a faithfully flat map  $S' \rightarrow S$  such that the pullback  $C' := S' \times_S C$  is embeddable.*

PROOF. Put  $R = \mathcal{O}_C$  and  $k = \mathcal{O}_S$ , and let  $y$  be a very good parameter on  $C$ . Let  $P$  be the continuous dual of  $R$ , which is a projective module of countable rank over  $k$ . We have  $P \widehat{\otimes}_k R \simeq \text{Hom}_k^{\text{cts}}(R, R)$ , so there is an element  $x \in P \widehat{\otimes}_k R$  corresponding to the identity map  $1_R$ . The scheme  $M := \text{Map}_S(C, \mathbb{A}^1)$  is the spectrum of the symmetric algebra  $k[P]$ , with the tautological map  $M \times_S C \rightarrow \mathbb{A}^1$  corresponding to the element  $x \in P \widehat{\otimes}_k R \subset k[P] \widehat{\otimes}_k R = \mathcal{O}_{M \times_S C}$ . As in the lemma, there is a largest open subscheme  $S' \subseteq M$  where this tautological map gives an embedding  $S' \times_S C \rightarrow \mathbb{A}_{S'}^1$ . Note that  $M = \text{spec}(k[P])$  is flat over  $S$  and  $S'$  is open in  $M$ , it is again flat over  $S$ . It is clear by construction that  $S' \times_S C$  has a canonical embedding in  $\mathbb{A}_{S'}^1$ . All that is left is to check that the map  $u: S' \rightarrow S$  is *faithfully* flat. It will suffice to show that  $u$  is surjective on geometric points, and this follows easily from Lemma 2.20.  $\square$

## Symmetric powers of multicurves

In this section, we study the formal schemes  $C^r/\Sigma_r$ , or in other words the symmetric powers of  $C$ . As usual, we write  $R = \mathcal{O}_C$  and  $k = \mathcal{O}_S$ . We choose a good parameter  $y$  on  $C$ , and a basis  $\{e_0, \dots, e_{n-1}\}$  for  $R/y$ . We then put  $e_{ni+j} = y^i e_j$ , which gives a topological basis  $\{e_i \mid i \geq 0\}$  for  $R$  over  $k$  and thus an isomorphism  $R \simeq \prod_i k$  of topological  $k$ -modules. We write

$$\begin{aligned} R_r &= R \hat{\otimes}_k \dots \hat{\otimes}_k R \\ S_r &= R_r^{\Sigma_r} \\ \overline{R} &= k[[y]] \\ \overline{R}_r &= \overline{R} \hat{\otimes}_k \dots \hat{\otimes}_k \overline{R} = k[[y_1, \dots, y_r]] \\ u_i &= i\text{'th elementary symmetric function of } y_1, \dots, y_r \\ \overline{S}_r &= \overline{R}_r^{\Sigma_r} = k[[u_1, \dots, u_r]] \\ C^r &= C \times_S \dots \times_S C = \text{spf}(R_r) \\ C^r/\Sigma_r &= \text{spf}(S_r) \\ \overline{C} &= \text{spf}(\overline{R}) \\ \overline{C}^r &= \overline{C} \times_S \dots \times_S \overline{C} = \text{spf}(\overline{R}_r) \\ \overline{C}^r/\Sigma_r &= \text{spf}(\overline{S}_r). \end{aligned}$$

Here we have topologized  $\overline{R}_r$ ,  $S_r$  and  $\overline{S}_r$  as closed subrings of  $R_r$ . We clearly have a commutative square of topological rings as shown on the left below, and thus a commutative square of formal schemes as shown on the right.

$$\begin{array}{ccc} R_r & \longleftarrow & S_r \\ \uparrow & & \uparrow \\ \overline{R}_r & \longleftarrow & \overline{S}_r \end{array} \quad \begin{array}{ccc} C^r & \longrightarrow & C^r/\Sigma_r \\ \downarrow & & \downarrow \\ \overline{C}^r & \longrightarrow & \overline{C}^r/\Sigma_r \end{array}$$

We next exhibit topological bases for the above rings. Put

$$\begin{aligned} A &= \mathbb{N}^r \\ \overline{A} &= (n\mathbb{N})^r = \{\alpha \in A \mid \alpha_i = 0 \pmod{n} \text{ for all } i\} \\ B &= \{\beta \in \mathbb{N}^\infty \mid \sum_{i=0}^\infty \beta_i = r\} \\ \overline{B} &= \{\beta \in B \mid \beta_i = 0 \text{ whenever } i \not\equiv 0 \pmod{n}\}. \end{aligned}$$

Next, for  $\alpha \in A$  we put

$$e_\alpha = e_{\alpha_1} \otimes \dots \otimes e_{\alpha_r} \in R_r.$$

Note that  $e_{n\alpha} = \prod_{i=1}^r y_i^{\alpha_i} \in \overline{R}_r$ , and  $e_{n\alpha+\alpha'} = e_{n\alpha} e_{\alpha'}$ .

Now define  $\tau: A \rightarrow B$  by  $\tau(\alpha)_j = |\{i \mid \alpha_i = j\}|$ . This gives bijections  $A/\Sigma_r = B$  and  $\overline{A}/\Sigma_r = \overline{B}$ . For  $\beta \in B$ , we put

$$e'_\beta = \sum_{\tau(\alpha)=\beta} e_\alpha \in S_r.$$

It is clear that

- $\{e_\alpha \mid \alpha \in A\}$  is a topological basis for  $R_r$  over  $k$ , giving an isomorphism  $R_r = \prod_A k$ .
- $\{e_\alpha \mid \alpha \in \overline{A}\}$  is a topological basis for  $\overline{R}_r$ .
- $\{e'_\beta \mid \beta \in B\}$  is a topological basis for  $S_r$ .
- $\{e'_\beta \mid \beta \in \overline{B}\}$  is a topological basis for  $\overline{S}_r$ .

Of course, the monomials in the symmetric functions  $u_i$  give another topological basis for  $\overline{S}_r$  over  $k$ .

**PROPOSITION 17.1.** *If  $S' = \text{spec}(k')$  is any scheme over  $S$ , and  $C' = S' \times_S C$  (considered as a multicurve over  $S'$ ) then  $(C')^r/\Sigma_r = S' \times_S (C^r/\Sigma_r)$ . The schemes  $\overline{C}$ ,  $\overline{C}'$  and  $\overline{C}'/\Sigma_r$  are also compatible with base change in the same sense.*

**PROOF.** Put  $R' = \mathcal{O}_{C'} = k' \widehat{\otimes}_k R = \prod_{i \in \mathbb{N}} k'$ ,  $R'_r = \mathcal{O}_{(C')^r} = R' \widehat{\otimes}_{k'} \dots \widehat{\otimes}_{k'} R' = \prod_{\alpha \in A} k'$ , and  $S'_r = \mathcal{O}_{(C')^r/\Sigma_r} = \prod_{\beta \in B} k'$ . This is clearly the same as  $k' \widehat{\otimes}_k S_r$ , so  $(C')^r/\Sigma_r = (C^r/\Sigma_r) \times_S S'$ . The same argument works for the other claims.  $\square$

We next need to formulate and prove various compatibility statements for the topologies on the rings considered above.

**DEFINITION 17.2.** Let  $A$  be a linearly topologised ring, and let  $M$  be a topological module over  $A$ . We say that  $M$  is *topologically free of rank  $r$*  if it is isomorphic to  $A^r$  (with the product topology) as a topological module.

**DEFINITION 17.3.** Let  $A$  be a linearly topologised ring, and let  $B$  be a closed subring (with the subspace topology). We write  $I \leq_O A$  to indicate that  $I$  is an open ideal in  $A$ . We say that  $B$  is *neat* if for every open ideal  $J \leq_O B$ , the ideal  $JA$  is open in  $A$ .

**REMARK 17.4.** As  $B$  has the subspace topology, we see that  $\{I \cap B \mid I \leq_O A\}$  is a basis of open ideals in  $B$ . It follows that  $B$  is neat iff  $(I \cap B)A$  is open in  $A$  whenever  $I \leq_O A$ . If so, then (using the inclusion  $(I \cap B)A \leq I$ ) we see that  $\{(I \cap B)A \mid I \leq_O A\}$  is a basis of open ideals in  $A$ .

**REMARK 17.5.** Suppose that we start with a linear topology on  $B$ . We can then give  $A$  a linear topology by declaring  $\{AJ \mid J \leq_O B\}$  to be a basis of open ideals in  $A$ . By regarding  $B$  as a subspace of  $A$ , we obtain a new linear topology on  $B$ , which may or may not be the same as the old one. Now suppose that  $A$  is faithfully flat over  $B$ . It follows that  $A/JA = A \otimes_B B/J$  is faithfully flat over  $B/J$ , and in particular that the map  $B/J \rightarrow A/JA$  is injective, so  $J = (JA) \cap B$ . Using this we see that the two topologies on  $B$  are the same, and that  $B$  is neat in  $A$ .

In particular, if  $A$  is topologically free of finite rank over  $B$ , then  $B$  is neat in  $A$ . Conversely, if  $A$  is free of finite rank over  $B$  and  $B$  is neat, it is easy to see that  $A$  is topologically free.



PROPOSITION 17.6. (a)  $R_r$  is topologically free of rank  $n^r$  over  $\overline{R}_r$

(b)  $\overline{R}_r$  is topologically free of rank  $r!$  over  $\overline{S}_r$

(c)  $R_r$  is topologically free of rank  $n^r r!$  over  $\overline{S}_r$

(d)  $S_r$  is topologically free of rank  $n^r$  over  $\overline{S}_r$

(e)  $R_r$  is a finitely generated module over  $S_r$ , and  $S_r$  is neat in  $R_r$ .

Moreover, in each of the four rings there is a finitely generated ideal  $J$  such that  $\{J^m \mid m \geq 0\}$  is a basis of open ideals.

The proof will follow after a number of lemmas. In Corollary 17.14, we will extend part (e) by proving that  $R_r$  is a projective module of rank  $r!$  over  $R_r$ .

LEMMA 17.7. Suppose we have a ring  $A$  and elements  $a_1, \dots, a_n \in A$ , and we put  $I_m = (a_1^m, \dots, a_n^m)$ . Then  $I_1^{n(m-1)+1} \leq I_m \leq I_1^m$ , so the topology defined by the ideals  $I_m$  is the same as that defined by the ideals  $I_1^m$ .

PROOF. The ideal  $I_1^m$  is generated by the monomials  $\prod_i a_i^{v_i}$  for which  $\sum_i v_i = m$ . It is clear from this that  $I_m \leq I_1^m$ , and thus that  $I_1^m$  is open in the topology defined by the ideals  $I_k$ . Now suppose we have a monomial  $\prod_i a_i^{v_i}$  that is not contained in  $I_m$ . This means that  $v_i \leq m-1$  for all  $i$ , and thus  $\sum_{i=1}^n v_i < n(m-1)+1$ . By the contrapositive, we see that  $I_1^{n(m-1)+1} \leq I_m$ , so  $I_m$  is open in the topology defined by the ideals  $I_1^k$ .  $\square$

LEMMA 17.8. Let  $A$  be a linearly topologised ring with a continuous action of a finite group  $G$ . Suppose that there exists a finitely generated  $G$ -invariant ideal  $I = (a_1, \dots, a_n)$  such that  $\{I^m \mid m \geq 0\}$  is a basis of open ideals. Then  $A^G$  is neat in  $A$ . Moreover, if  $A$  is faithfully flat over  $A^G$  then  $\{(I^G)^m \mid m \geq 0\}$  is a basis of open ideals in  $A^G$ .

PROOF. Put  $r = |G|$ . For any  $a \in A$ , put  $\phi_a(t) = \prod_{g \in G} (t - g.a)$ , so  $\phi_a(a) = 0$ . If  $J$  is any  $G$ -invariant ideal containing  $a$  we see that  $\phi_a(t) \in t^r + J^G[t]$ , so the equation  $\phi_a(a) = 0$  gives  $a^r \in A.J^G$ . Thus, all elements of  $J$  are nilpotent modulo  $A.J^G$ . If  $J$  is finitely generated we deduce that there exists  $s > 0$  with  $J^s \leq A.J^G$ .

Now apply this with  $J = I^m$ ; we see that  $A.(I^m)^G$  contains  $I^{ms}$  for some  $s$ , and thus is open. This shows that  $A^G$  is neat in  $A$ .

Now suppose that  $A$  is faithfully flat over  $A^G$ . We claim that  $(I^G)^m$  is open in  $A^G$ . Indeed, the above shows that for large  $j$  we have  $I^j \leq A.I^G$ . It follows that  $I^{jm} \leq (I^G A)^m = A.(I^G)^m$ . It is also clear that  $A.(I^{jm})^G \leq I^{jm}$ , so  $A.(I^{jm})^G \leq A.(I^G)^m$ . By faithful flatness, for any ideals  $J, J' \leq A^G$  we have  $A.J \leq A.J'$  iff  $J \leq J'$ . We deduce that  $(I^{jm})^G \leq (I^G)^m$ . The ideal  $(I^{jm})^G = I^{jm} \cap A^G$  is open in the subspace topology, so the same is true of  $(I^G)^m$ . We also have  $(I^G)^m \leq (I^m)^G$  and the ideals  $(I^m)^G$  form a basis of neighbourhoods of 0; it follows that the same is true of the ideals  $(I^G)^m$ .  $\square$

COROLLARY 17.9. Let  $A, I$  and  $G$  be as in the lemma, and let  $H$  be a subgroup of  $G$ . Suppose that the inclusion  $A^H \rightarrow A$  is faithfully flat, and that  $I^H$  is finitely generated. Then  $A^G$  is neat in  $A^H$ .

PROOF. The lemma (with  $G$  replaced by  $H$ ) tells us that  $\{(I^H)^m \mid m \geq 0\}$  is a basis of open ideals in  $A^H$ . As  $I^H$  is finitely generated, the same is true of  $(I^H)^m$ , say  $(I^H)^m = (b_1, \dots, b_n)$ . Consider the polynomial  $\phi_{b_i}(t) = \prod_{g \in H} (t - g.b_i)$  as in the proof of the lemma. As  $b_i \in (I^H)^m \subseteq I^m$ , we see that  $\phi_{b_i}(t) \in t^r + (I^m)^G[t]$ . Using

the relation  $\phi_{b_i}(b_i) = 0$  we see that  $b_i^r \in (I^m)^G A^H$ , so

$$(I^H)^{m(n(r-1)+1)} \leq (b_1^r, \dots, b_n^r) \leq (I^m)^G A^H,$$

so  $(I^m)^G A^H$  is open in  $A^H$ . As the ideals  $(I^m)^G$  are a basis of open ideals in  $A^G$ , we deduce that  $A^G$  is neat as claimed.  $\square$

LEMMA 17.10. *Suppose that  $A = k[y_1, \dots, y_r]$ , with the evident action of  $G = \Sigma_r$ , and with topology determined by the powers of the ideal  $I = (y_1, \dots, y_r)$ . Let  $H$  be a subgroup of  $G$  of the form  $\Sigma_{r_1} \times \dots \times \Sigma_{r_k}$ , with  $r = r_1 + \dots + r_k$ . Then*

- (a)  *$A$  is topologically free of rank  $|G| = r!$  over  $A^G$*
- (b)  *$A$  is topologically free of rank  $|H| = \prod_i r_i!$  over  $A^H$*
- (c)  *$A^H$  is topologically free of rank  $|G/H|$  over  $A^G$*
- (d) *The topology on  $A^H$  (resp.  $A^G$ ) is determined by powers of the ideal  $I^H$  (resp.  $I^G$ ), which is finitely generated.*

PROOF. It is well-known that  $A^G = k[u_1, \dots, u_r]$ , where  $u_i$  is the  $i$ 'th elementary symmetric function in the variables  $y_i$ . Similarly, we have  $A^H = k[v_1, \dots, v_r]$ , where the elements  $v_1, \dots, v_{r_1}$  are the elementary symmetric functions of  $y_1, \dots, y_{r_1}$ , and  $v_{r_1+1}, \dots, v_{r_1+r_2}$  are the elementary symmetric functions of  $y_{r_1+1}, \dots, y_{r_1+r_2}$  and so on. By considering the maps

$$A^G \rightarrow A^H \rightarrow A \rightarrow A/I = k,$$

we see that  $I^G = (u_1, \dots, u_r)$  and  $I^H = (v_1, \dots, v_r)$ , so in particular these ideals are finitely generated.

We next claim that  $A$  is algebraically free of rank  $|H|$  over  $A^H$ . Everything is compatible with base change, so it will be enough to prove this when  $k = \mathbb{Z}$ . In this case, all the rings involved are Noetherian domains with unique factorisation and the claim is a standard piece of invariant theory. Similarly, we see that  $A$  and  $A^H$  are algebraically free of the indicated ranks over  $A^G$ , and so the inclusions  $A^G \rightarrow A^H \rightarrow A$  are faithfully flat.

Using Lemma 17.8 and Corollary 17.9 we deduce that the inclusions  $A^G \rightarrow A^H \rightarrow A$  are neat. A neat extension that is an algebraically free module is always topologically free, which proves (a), (b) and (c). We have seen that  $I^G$  and  $I^H$  are finitely generated, and the rest of (d) follows from Lemma 17.8.  $\square$

PROOF OF PROPOSITION 17.6. Claim (a) is clear. Claim (b) follows from part (a) of Lemma 17.10 by passing to completions, and part (c) follows immediately from (a) and (b).

For claim (d), put

$$\begin{aligned} A' &= \{\alpha \in A \mid \alpha_i < n \text{ for all } i\} \\ B' &= A'/\Sigma_r = \{\beta \in B \mid \beta_j = 0 \text{ for all } j \geq n\}. \end{aligned}$$

For  $\beta \in B'$  we put  $H_\beta = \prod_i \Sigma_{\beta_i} \leq \Sigma_r$ , so  $\tau^{-1}\{\beta\} \simeq \Sigma_r/H_\beta$ . As  $A'$  is a basis for  $R_r$  over  $\overline{R}_r$ , we deduce that  $S_r = R_r^{\Sigma_r}$  is isomorphic to  $\bigoplus_\beta \overline{R}_r^{H_\beta}$  as a module over  $\overline{S}_r$ . It will thus suffice to show that  $k[[y_1, \dots, y_r]]^{H_\beta}$  is topologically free of rank  $|\Sigma_r/H_\beta|$  over  $k[[y_1, \dots, y_r]]^{\Sigma_r}$ , and this follows from part (c) of Lemma 17.10 by passing to completions.

For part (e), note that  $R_r$  is finitely generated over  $\overline{S}_r$  and thus is certainly finitely generated over the larger ring  $S_r$ . Neatness follows from Lemma 17.8.

Finally, we must show that for each of our rings there is a finitely generated ideal  $J$  whose powers determine the topology. For  $\overline{R}_r$ , we can obviously take  $J$  to be the ideal  $\overline{I}_r := (y_1, \dots, y_r)$ . Lemma 17.8 tells us that for  $\overline{S}_r$  we can use the ideal  $\overline{J}_r := \overline{I}_r^{\Sigma_r} = (u_1, \dots, u_r)$ . For  $S_r$  (which is topologically free over  $\overline{S}_r$ ) we can therefore use the ideal  $J_r := \overline{J}_r S_r$ . Similarly, for  $R_r$  we can use the ideal  $I_r = \overline{I}_r R_r$ .  $\square$

LEMMA 17.11. *If the curve  $C$  is embeddable, then  $R_r$  is topologically free of rank  $r!$  over  $S_r$ .*

PROOF. We may assume that

$$C = \operatorname{spf}(k[x]_{f(x)}^\wedge) = \varinjlim_m \operatorname{spec}(k[x]/f(x)^m)$$

for some monic polynomial  $f(x)$ . Put  $A = k[x_1, \dots, x_r]$ , and give this the topology determined by the powers of the ideal  $I = (f(x_1), \dots, f(x_r))$ , so  $C^r = \operatorname{spf}(A_I^\wedge)$ . The evident action of  $G := \Sigma_r$  on  $A$  is continuous, and  $A$  is free of rank  $r!$  over  $A^G$ . We see from Lemma 17.8 that  $A^G$  is neat in  $A$ , so  $A$  is topologically free over  $A^G$  of rank  $r!$ , and the claim follows by passing to completions.  $\square$

LEMMA 17.12. *Let  $A$  be a ring,  $M$  a finitely generated  $A$ -module, and  $B$  a faithfully flat  $A$ -algebra. Suppose that  $B \otimes_A M$  is a free  $B$ -module of rank  $s$ . Then  $M$  is a projective  $A$ -module of the same rank.*

PROOF. First, we claim that if  $\mathfrak{m}$  is a maximal ideal in  $A$  with residue field  $K = A/\mathfrak{m}$ , then  $\dim_K(K \otimes_A M) = s$ . Indeed, by faithful flatness there exists a prime ideal  $\mathfrak{n} \leq B$  with  $\mathfrak{n} \cap A = \mathfrak{m}$ . Using Zorn's lemma we can find a maximal element of the set of all such ideals  $\mathfrak{n}$ , and this is easily seen to be a maximal ideal in  $B$ . It follows that the residue field  $L = B/\mathfrak{n}$  is a field extension of  $K$ , so

$$\dim_K(K \otimes_A M) = \dim_L(L \otimes_K K \otimes_A M) = \dim_L(L \otimes_B (B \otimes_A M)),$$

which is evidently equal to  $s$ .

We now choose a finite generating set  $\{m_1, \dots, m_t\}$  for  $M$ . For each subset  $S \subseteq \{1, \dots, t\}$  with  $|S| = s$ , we let  $f_S: A^S \rightarrow M$  be the map  $\underline{a} \mapsto \sum_s a_s m_s$ , and we let  $P_S$  and  $Q_S$  be the kernel and cokernel of  $f_S$ .

Next, we put  $I_S = \operatorname{ann}(Q_S) \leq A$ . If  $\mathfrak{m}$  is maximal as before, we claim that there exists  $S$  such that  $I_S \not\leq \mathfrak{m}$ . Indeed, as  $\dim_K(K \otimes M) = s$ , we can certainly choose  $S$  such that  $\{m_i \mid i \in S\}$  gives a basis for  $K \otimes_A M$ . It follows that  $K \otimes_A Q_S = 0$ , or equivalently that  $\mathfrak{m}Q_S = Q_S$ . The module  $Q_S$  is generated by the elements  $m_j$  for  $j \notin S$ , so we can find elements  $u_{jk} \in \mathfrak{m}$  for each  $j, k \notin S$  such that  $m_j = \sum_k u_{jk} m_k$ . Let  $U$  be the square matrix with entries  $u_{jk}$  and put  $u = \det(I - U)$ . As in [21, Theorem 2.1], we see that  $u = 1 \pmod{\mathfrak{m}}$  and  $u \in I_S$ , so  $I_S \not\leq \mathfrak{m}$  as claimed.

It follows from this claim that  $\sum_S I_S$  is not contained in any maximal ideal, so  $\sum_S I_S = A$ . We can thus choose  $a_S \in I_S$  with  $\sum_S a_S = 1$ . It follows that  $\operatorname{spec}(A)$  is the union of the basic open subschemes  $D(a_S) = \operatorname{spec}(A[a_S^{-1}])$ .

We have  $a_S Q_S = 0$  and so  $Q_S[a_S^{-1}] = 0$ , so the map  $f_S$  becomes surjective after inverting  $a_S$ . It follows that the resulting map  $1 \otimes f_S: B[a_S^{-1}]^S \rightarrow B[a_S^{-1}] \otimes_A M$  is also surjective. Here both source and target are free modules of the same finite rank over  $B[a_S^{-1}]$ , so our map must in fact be an isomorphism. As  $B[a_S^{-1}]$  is faithfully flat

over  $A[a_S^{-1}]$ , we deduce that  $f_S$  actually gives an isomorphism  $A[a_S^{-1}]^s \rightarrow M[a_S^{-1}]$ . This shows that  $M$  is locally free of rank  $s$ , and thus is projective.  $\square$

**COROLLARY 17.13.** *Let  $k$  be a ring, and let  $A$  be a formal  $k$ -algebra whose topology is defined by the powers of a single open ideal  $J$  (so  $A = \varprojlim_m A/J^m$ ). Let  $M$  be a finitely generated  $A$ -module such that  $M = \varprojlim_m M/J^m M$ . Let  $k'$  be a faithfully flat  $k$ -algebra, and put  $A' = k' \hat{\otimes}_k A$  and  $M' = k' \hat{\otimes}_k M = A' \hat{\otimes}_A M$ . Suppose that  $M'$  is a free module of rank  $s$  over  $A'$ ; then  $M$  is a projective module of rank  $s$  over  $A$ .*

**PROOF.** First, note that the map  $A/J^m \rightarrow A'/J^m A' = k' \otimes_k A/J^m$  is a faithfully flat extension of discrete rings. We can thus apply the lemma and deduce that  $M/J^m M$  is a finitely generated projective module of rank  $s$  over  $A/J^m$ .

Next, as  $M$  is finitely generated, we can choose an epimorphism  $f: A^t \rightarrow M$  for some  $t$ . Let  $X_m$  be the set of  $A$ -module maps  $g: M/J^m M \rightarrow (A/J^m)^t$  such that the induced map

$$M/J^m M \xrightarrow{g} (A/J^m)^t \xrightarrow{f} M/J^m M$$

is the identity. As  $M/J^m M$  is projective over  $A/J^m$ , we see that  $X_m$  is nonempty. There is an evident projection  $\pi_m: X_m \rightarrow X_{m-1}$ , which we claim is surjective. Indeed, given  $g \in X_{m-1}$  we can use the projectivity of  $M/J^m M$  again to see that there exists a map  $h: M/J^m M \rightarrow (A/J^m)^t$  lifting  $g$ . Let  $\delta$  be the determinant of the resulting map  $fh: M/J^m M \rightarrow M/J^m M$ , so  $\delta \in A/J^m$ . Because  $g \in X_{m-1}$ , we see that  $\delta$  maps to 1 in  $A/J^{m-1}$ . As the kernel of the projection  $A/J^{m-1} \rightarrow A/J^m$  is nilpotent, it follows that  $\delta$  is a unit, so  $fh$  is an isomorphism. After replacing  $h$  by  $h(fh)^{-1}$  we may assume that  $fh = 1$ , so  $h \in X_m$  and  $\pi(h) = g$ . It follows that  $\varprojlim_m X_m \neq \emptyset$ , and this gives a map  $g: M \rightarrow A^t$  with  $fg = 1$ . Thus,  $M$  is a retract of a free module, and hence is projective.  $\square$

**COROLLARY 17.14.**  *$R_r$  is a projective module of rank  $r!$  over  $S_r$ , so the projection  $C^r \rightarrow C^r/\Sigma_r$  is a finite, faithfully flat map of degree  $r!$ .*

**PROOF.** In Corollary 17.13, we take  $A = S_r$  and  $M = R_r$ . We know from Proposition 17.6 that the topology on  $S_r$  is determined by powers of the ideal  $J_r = (u_1, \dots, u_r)$ , and that  $S_r$  is neat in  $R_r$ . This means that the given topology on  $R_r$  is determined by the ideals  $J_r^m R_r$ . As  $R_r$  is complete, we deduce that  $R_r = \varprojlim_r R_r/J_r^m R_r$ . We next take  $k' = \mathcal{O}_{S'}$  to be a faithfully flat extension of  $k$  such that the curve  $C' = S' \times_S C$  is embeddable; this is possible by Corollary 16.3. Using Lemma 17.11, we see that  $M'$  is topologically free of rank  $r!$  over  $A'$ , so we can apply Corollary 17.13 and deduce that  $R_r$  is projective over  $S_r$ .  $\square$

## Classification of divisors

Our main task in this section is to prove the following result.

**THEOREM 18.1.** *Let  $C$  be a formal multicurve over a scheme  $S$ . Then for formal schemes  $S'$  over  $S$ , there is a natural bijection between divisors of degree  $r$  on  $S' \times_S C$  and maps  $S' \rightarrow C^r/\Sigma_r$  over  $S$ .*

**CONSTRUCTION 18.2.** We must first construct a universal example. We start by putting  $\Delta_i = \{(a_1, \dots, a_r, b) \in C^{r+1} \mid b = a_i\}$ , which is a divisor of degree one on  $C$  over  $C^r$ . If we define  $d_i(\underline{a}, b) = d(a_i, b)$  then  $\mathcal{O}_{\Delta_i} = R_{r+1}/d_i$ . Now put  $\delta_r = \prod_i d_i \in R_{r+1}$  and  $\tilde{D}_r = \sum_i \Delta_i = \text{spf}(R_{r+1}/\delta_r)$ , which is a divisor of degree  $r$  on  $C$  over  $C^r$ . On the other hand, we note that  $\delta_r \in R_{r+1}^{\Sigma_r} = S_r \hat{\otimes} R$ , so we can define  $D_r = \text{spf}((S_r \hat{\otimes} R)/\delta_r)$ , which is a closed formal subscheme of  $C^r/\Sigma_r \times_S C$ . It is clear that  $R_r \otimes_{S_r} \mathcal{O}_{D_r} = \mathcal{O}_{\tilde{D}_r}$ , which is free of rank  $r$  over  $R_r$ . We know from Corollary 17.14 that  $R_r$  is faithfully flat over  $S_r$ , and it follows from Lemma 17.12 that  $\mathcal{O}_{D_r}$  is a projective module of rank  $r$  over  $S_r$ . Moreover, the relevant ideal is generated by the regular element  $\delta_r$ , so  $D_r$  is a divisor on  $C$  over  $C^r/\Sigma_r$ .

Now put  $Q_r = C^r/\Sigma_r$  for brevity. As in Section 17, we choose a topological basis  $\{e_i\}$  for  $\mathcal{O}_C$ , and use it to construct a topological basis  $\{e'_\beta \mid \beta \in B\}$  for  $\mathcal{O}_{Q_r}$ . We then put  $M = \text{spec}(\mathbb{Z}[t_0, t_1, \dots])$ , and put  $g = \sum_i t_i e_i$ , regarded as a function on  $M \times Q_r \times C$ . We then put

$$h = N_{M \times D_r / M \times Q_r}(g) \in \mathcal{O}_{M \times Q_r} = \prod_{\beta} \mathcal{O}_S[t_i \mid i \geq 0] e'_\beta.$$

We claim that  $h$  is actually equal to  $\sum_{\beta} t^\beta e'_\beta$ . Indeed, although this is an equation in  $\mathcal{O}_{M \times Q_r}$ , it will suffice to prove it in the larger ring  $\mathcal{O}_{M \times C^r}$ . In that context, we can describe  $h$  as  $N_{M \times \tilde{D}_r / M \times C^r}(g)$ . Now let  $\pi_j: C^r \rightarrow C$  be the  $j$ 'th projection. Using Proposition 15.13 we see that  $h = \prod_j \pi_j^* g = \prod_j \sum_i t_i \pi_j^* e_i$ . Expanding this out gives

$$h = \sum_{\alpha \in A} t^{\tau(\alpha)} e_\alpha = \sum_{\beta \in B} \left( t^\beta \sum_{\tau(\alpha)=\beta} e_\alpha \right) = \sum_{\beta \in B} t^\beta e'_\beta$$

as claimed.

Now suppose we have a map  $c: S' \rightarrow Q_r$  over  $S$ , and  $D = c^* D_r$  over  $S'$ . We deduce easily that

$$N_{M \times D / M \times S'}(g) = \sum_{\beta} t^\beta c^*(e'_\beta).$$

This shows that  $c^*(e'_\beta)$  depends only on  $D$ , and  $\{e'_\beta \mid \beta \in B\}$  is a topological basis for  $S_r$ , so the ring map  $c^*: S_r \rightarrow \mathcal{O}_{S'}$  depends only on  $D$ , so the map  $c: S' \rightarrow C^r/\Sigma_r$  depends only on  $D$ . We record this formally as follows:

PROPOSITION 18.3. *Suppose we have two maps  $c_0, c_1: S' \rightarrow C^r/\Sigma_r$  over  $S$ , and that  $c_0^*D_r = c_1^*D_r$  as divisors over  $S'$ . Then  $c_0 = c_1$ .*  $\square$

PROOF OF THEOREM 18.1. Let  $S'$  be a scheme over  $S$ , and let  $A$  be the set of maps  $S' \rightarrow C^r/\Sigma_r$  over  $S$ , and let  $B$  be the set of divisors of degree  $r$  on  $C$  over  $S'$ . The construction  $c \mapsto c^*D_r$  gives a map  $\phi: A \rightarrow B$ , which is injective by Proposition 18.3. To show that  $\phi$  is surjective, suppose we have a divisor  $D \in B$ . We can choose a faithfully flat map  $q: T \rightarrow S'$  such that  $q^*D$  has a full set of points, say  $\underline{u} = (u_1, \dots, u_r)$ . We deduce that  $q^*D$  is the pullback of  $\tilde{D}_r$  along the map  $T \xrightarrow{\underline{u}} C^r$ , and thus is the pullback of  $D_r$  along the composite  $c = (T \xrightarrow{\underline{u}} C^r \rightarrow C^r/\Sigma_r)$ . Now let  $q_0, q_1: T \times_{S'} T \rightarrow T$  be the two projections, so  $qq_0 = qq_1$ . Note that

$$(cq_0)^*D_r = q_0^*c^*D_r = q_0^*q^*D = (qq_0)^*D,$$

and similarly

$$(cq_1)^*D_r = q_1^*c^*D_r = q_1^*q^*D = (qq_1)^*D.$$

As  $qq_0 = qq_1$  we see that  $(cq_0)^*D = (cq_1)^*D$ , and so (by Proposition 18.3) we have  $cq_0 = cq_1$ . By faithfully flat descent, we have  $c = \bar{c}q$  for a unique map  $\bar{c}: S' \rightarrow C^r/\Sigma_r$ . We then have  $q^*\bar{c}^*D_r = c^*D_r = q^*D$ , and using the faithful flatness of  $q$ , we deduce that  $D = \bar{c}^*D_r = \phi(\bar{c})$ . This shows that  $\phi$  is also surjective, and thus a natural bijection.  $\square$

DEFINITION 18.4. In the light of Theorem 18.1, it makes sense to write  $\text{Div}_r^+(C)$  for  $C^r/\Sigma_r$ . The evident projection

$$C^r/\Sigma_r \times_S C^s/\Sigma_s = C^{r+s}/(\Sigma_r \times \Sigma_s) \rightarrow C^{r+s}/\Sigma_{r+s}$$

gives a map

$$\sigma_{r,s}: \text{Div}_r^+(C) \times_S \text{Div}_s^+(C) \rightarrow \text{Div}_{r+s}^+(C).$$

It is easy to check that this classifies addition of divisors, in the following sense: if we have divisors  $D = u^*D_r$  and  $D' = v^*D_s$  on  $C$  over  $S'$ , then  $D + D' = w^*D_{r+s}$ , where

$$w = (S' \xrightarrow{(u,v)} \text{Div}_r^+(C) \times_S \text{Div}_s^+(C) \xrightarrow{\sigma} \text{Div}_{r+s}^+(C)).$$

We put  $\text{Div}^+(C) = \coprod_r \text{Div}_r^+(C)$ . As one would expect, this is the free abelian monoid scheme generated by  $C$ ; see [29, Section 6.2] for technical details.

DEFINITION 18.5. Now suppose that  $C$  has an abelian group structure, written additively. We can then define  $\tilde{\mu}: C^r \times_S C^s \rightarrow C^{rs}$  by

$$\tilde{\mu}(a_0, \dots, a_{r-1}; b_0, \dots, b_{s-1})_{i+rj} = a_i + b_j$$

(for  $0 \leq i < r$  and  $0 \leq j < s$ ). The composite

$$C^r \times_S C^s \xrightarrow{\tilde{\mu}} C^{rs} \xrightarrow{q} C^{rs}/\Sigma_{rs}$$

is invariant under  $\Sigma_r \times \Sigma_s$ , so we get an induced map

$$\mu_{r,s}: \text{Div}_r(C) \times_S \text{Div}_s(C) \rightarrow \text{Div}_{rs}(C).$$

If we have divisors  $D = u^*D_r$  and  $D' = v^*D_s$  on  $C$  over  $S'$ , then we define  $D * D'$  to be the divisor  $w^*D_{rs}$ , where

$$w = (S' \xrightarrow{(u,v)} \text{Div}_r^+(C) \times_S \text{Div}_s^+(C) \xrightarrow{\mu} \text{Div}_{rs}^+(C)).$$

We call this the *convolution* of  $D$  and  $D'$ . This operation makes  $\text{Div}^+(C)$  into a semiring. If we have full sets of points, say  $D = \sum_i [a_i]$  and  $D' = \sum_j [b_j]$  then  $D * D'$  is just  $\sum_{i,j} [a_i + b_j]$ .

**PROPOSITION 18.6.** *Let  $D$  and  $D'$  be divisors on  $C$  over  $S$ . Then there exists a closed subscheme  $T \subseteq S$  such that for any scheme  $S'$  over  $S$ , we have  $S' \times_S D \leq S' \times_S D'$  iff the map  $S' \rightarrow S$  factors through  $T$ .*

**PROOF.** As  $\mathcal{O}_D$  is finitely generated and projective over  $\mathcal{O}_S$ , we can choose an embedding  $i: \mathcal{O}_D \rightarrow \mathcal{O}_S^N$  of  $\mathcal{O}_S$ -modules, and a retraction  $r: \mathcal{O}_S^N \rightarrow \mathcal{O}_D$ . We then have  $i(f_{D'}) = (a_1, \dots, a_N)$  for some elements  $a_j \in \mathcal{O}_S$ , and we put  $J = (a_1, \dots, a_N)$  and  $T = \text{spec}(\mathcal{O}_S/J)$ . We find that a map  $S' \rightarrow S$  factors through  $T$  iff  $J$  maps to 0 in  $\mathcal{O}_{S'}$ , iff  $f_{D'}$  maps to 0 in  $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_D$ , iff  $S' \times_S D \subseteq S' \times_S D'$ .  $\square$

**PROPOSITION 18.7.** *Let  $D$  be a divisor on  $C$  over  $S$ , and suppose that  $r \geq 0$ . Then there is a scheme  $\text{Sub}_r(D)$  over  $S$  such that maps  $S' \rightarrow \text{Sub}_r(D)$  over  $S$  biject with divisors  $D' \leq S' \times_S D$  of degree  $r$ .*

**PROOF.** Over the formal scheme  $\text{Div}_r^+(C)$  we have both the originally given divisor  $D$  and the universal divisor  $D_r$ . We let  $\text{Sub}_r(D)$  denote the largest closed subscheme of  $\text{Div}_r^+(C)$  where  $D_r$  is contained in  $D$  (which makes sense by Proposition 18.6). It is formal to check that this has the required property.  $\square$

**PROPOSITION 18.8.** *Let  $D$  be a divisor on  $C$  over  $S$ , and suppose that  $r \geq 0$ . Then there is a scheme  $P_r(D)$  over  $S$  such that maps  $S' \rightarrow P_r(D)$  over  $S$  biject with lists  $(u_1, \dots, u_r)$  of sections of  $C$  over  $S'$  such that  $\sum_i [u_i] \leq S' \times_S D$ .*

**PROOF.** Over the formal scheme  $C^r$  we have both the originally given divisor  $D$  and the divisor  $\tilde{D}^r$ . We let  $P_r(D)$  denote the largest closed subscheme of  $C^r$  where  $\tilde{D}^r$  is contained in  $D$  (which makes sense by Proposition 18.6). It is formal to check that this has the required property.  $\square$

**REMARK 18.9.** Suppose that  $D$  has degree  $r$ . Then  $P_r(D)$  classifies  $r$ -tuples for which  $\sum_i [u_i] \leq D$ , but by comparing degrees we see that this means that  $\sum_i [u_i] = D$ . Thus,  $P_r(D)$  classifies full sets of points for  $D$ .

**LEMMA 18.10.** *Suppose we have ring maps  $A \rightarrow B \rightarrow C$ , and  $C$  is a projective module of degree  $m > 0$  over  $B$ , and also a projective module of degree  $nm > 0$  over  $A$ . Then  $B$  is a projective module of degree  $n$  over  $A$ .*

**PROOF.** We can use the second copy of  $B$  to make  $\text{Hom}_A(B, B)$  into a  $B$ -module. For any  $B$ -module  $N$  there is an evident map  $\text{Hom}_A(B, B) \otimes_B N \rightarrow \text{Hom}_A(B, N)$ . This is evidently an isomorphism if  $N$  is a free module of finite rank, and thus (by taking retracts) also when  $N$  is projective of finite rank over  $B$ . In particular, we have  $\text{Hom}_A(B, B) \otimes_B C = \text{Hom}_A(B, C)$ . As  $C$  is also projective over  $A$ , the same kind of argument shows that

$$\text{Hom}_A(B, C) = \text{Hom}_A(B, A) \otimes_A C = (\text{Hom}_A(B, A) \otimes_A B) \otimes_B C.$$

It follows that  $(\text{Hom}_A(B, A) \otimes_A B) \otimes_B C = \text{Hom}_A(B, B) \otimes_B C$ . More precisely, there is a natural map

$$\alpha: \text{Hom}_A(B, A) \otimes_A B \rightarrow \text{Hom}_A(B, B),$$

given by  $\alpha(\phi \otimes b)(b') = \phi(b')b$ . By working through the above argument more carefully, we see that  $\alpha \otimes_B 1_C$  is an isomorphism. However,  $C$  is faithfully flat over

$B$  so  $\alpha$  itself must be an isomorphism. In particular, we see that  $1_B$  lies in the image of  $\alpha$ , so  $1_B = \sum_{i=1}^N \alpha(\phi_i \otimes b_i)$  for some maps  $\phi_i: B \rightarrow A$  and elements  $b_i \in B$ . This means that for all  $b \in B$  we have  $b = \sum_i \phi_i(b) b_i$ . We can use the elements  $\phi_i$  to give a map  $\phi: B \rightarrow A^N$ , and the elements  $b_i$  to give a map  $\beta: A^N \rightarrow B$ . We find that  $\beta\phi = 1$ , which proves that  $B$  is projective. It is now clear that the rank must be  $n$ .  $\square$

PROPOSITION 18.11. *Let  $D$  be a divisor of degree  $s$  on  $C$  over  $S$ , and suppose that  $0 \leq r \leq s$ . Then there are natural maps  $P_r(D) \xrightarrow{p} \text{Sub}_r(D) \xrightarrow{q} S$  which are finite and very flat, with  $\deg(p) = r!$  and  $\deg(q) = s!/(r!(s-r)!)$  (so  $\deg(qp) = s!/(s-r)!$ ).*

PROOF. Over  $P_r(D)$  we have tautological sections  $u_1, \dots, u_r$  of  $C$  giving a divisor  $D'_r := \sum_i [u_i]$  on  $C$ . This is contained in  $(qp)^*D$ , so we can form the divisor  $D''_r := (qp)^*D - D'_r$ , which has degree  $s - r$  over  $P_r(D)$ . It is easy to identify  $P_{r+1}(D)$  with  $D''_r$ , so  $\deg(P_{r+1}(D) \rightarrow S) = (s-r) \deg(P_r(D) \rightarrow S)$ . By an evident induction, we see that the map  $pq$  is finite and very flat, with degree  $s!/(s-r)!$ , as claimed.

Next, let  $\overline{D}$  be the tautological divisor of degree  $r$  on  $C$  over  $\text{Sub}_r(D)$ . We can then form the scheme  $P_r(\overline{D})$ , which classifies full sets of points on  $\overline{D}$ . As above, we see that the map  $P_r(\overline{D}) \rightarrow \text{Sub}_r(D)$  is finite and very flat, with degree  $r!$ . We claim that  $P_r(\overline{D}) = P_r(D)$ . Indeed, a map  $S' \rightarrow P_r(\overline{D})$  over  $S$  corresponds to a map  $S' \rightarrow \text{Sub}_r(D)$ , together with a lifting to  $P_r(\overline{D})$ . Equivalently, it corresponds to a divisor of degree  $r$  contained in  $S' \times_S D$ , together with sections  $u_1, \dots, u_r: S' \rightarrow C$  giving a full set of points for that divisor. The full set of points determines the divisor, so it is equivalent to just give sections  $u_i$  with  $\sum_i [u_i] \leq S' \times_S D$ , or equivalently, a map  $S' \rightarrow P_r D$  over  $S$ . The claim follows by Yoneda's lemma. It follows that the map  $p$  is finite and very flat, with degree  $r!$ . We can now apply Lemma 18.10 to see that  $q$  is finite and very flat, with degree  $s!/(r!(s-r)!)$ .  $\square$

PROPOSITION 18.12. *For the universal divisor  $D_s$  over  $\text{Div}_s^+(C)$  we have*

$$\begin{aligned} \text{Sub}_r(D_s) &= \text{Div}_r^+(C) \times_S \text{Div}_{s-r}^+(C) \\ P_r(D_s) &= C^r \times_S \text{Div}_{s-r}^+(C). \end{aligned}$$

PROOF. Let  $S'$  be a scheme over  $S$ . Then a map  $S' \rightarrow \text{Sub}_r(D_s)$  over  $S$  corresponds to a map  $S' \rightarrow \text{Div}_r^+(C)$ , together with a lifting to  $\text{Sub}_r(D_s)$ . Equivalently, it corresponds to a divisor  $D$  of degree  $s$  on  $C$  over  $S'$ , together with a subdivisor  $D' \leq D$  of degree  $r$ . Given such a pair  $(D, D')$ , we have another divisor  $D'' = D - D'$ , which has degree  $s - r$ . There is evidently a bijection between pairs  $(D, D')$  as above, and pairs  $(D', D'')$  where  $D'$  and  $D''$  are arbitrary divisors of degrees  $r$  and  $s - r$ . These pairs correspond in turn to maps  $S' \rightarrow \text{Div}_r^+(C) \times_S \text{Div}_{s-r}^+(C)$  over  $S$ . The first claim follows by Yoneda's lemma, and the second claim can be proved in the same way.  $\square$

COROLLARY 18.13. *We have  $D_s = \text{Div}_{s-1}^+(C) \times_S C = C^s / \Sigma_{s-1}$ .*

PROOF. Take  $r = 1$ , and observe that  $P_1(D_s) = \text{Sub}_1(D_s) = D_s$  and  $\text{Div}_1^+(C) = C$ .  $\square$



## Local structure of the scheme of divisors

Let  $C$  be a formal multicurve over a base  $S$ . In the nonequivariant case, we know that  $\mathrm{Div}_n^+(C) \simeq \mathrm{spf}(\mathcal{O}_S[[c_1, \dots, c_n]]) = \widehat{\mathbb{A}}_S^n$ , so  $\mathrm{Div}_n^+(C)$  is a formal affine space of dimension  $n$  over  $S$ . Equivariantly, this is not even true when  $n = 1$ . However, we will show in this section that  $\mathrm{Div}_n^+(C)$  is still a “formal manifold”, in the sense that the formal neighbourhood of any point is isomorphic to  $\widehat{\mathbb{A}}_S^n$ , at least up to a slight twisting. Later we will apply this to calculate  $E^0BU(V)$ , where  $BU(V)$  is the simplicial classifying space of the unitary group of a representation  $V$  of  $A$ .

We state the result more formally as follows.

**THEOREM 19.1.** *Let  $C = \mathrm{spf}(R)$  be a formal multicurve over  $S = \mathrm{spec}(k)$ , with a difference function  $d$ . Let  $s: S \rightarrow \mathrm{Div}_n^+(C)$  be a section, classifying a divisor  $D = \mathrm{spf}(R/J) \subset C$ . Then the formal neighbourhood of  $sS$  in  $\mathrm{Div}_n^+(C)$  is isomorphic to the formal neighbourhood of zero in  $\mathrm{Map}_S(D, \mathbb{A}_S^1)$  (by an isomorphism that depends on the choice of  $d$ ).*

The rest of this section constitutes a more detailed explanation and a proof.

We first examine the two formal schemes that are claimed to be isomorphic. We put  $A_0 = (R^{\widehat{\otimes} n})^{\Sigma_n}$  and  $X_0 = \mathrm{spf}(A_0) = \mathrm{Div}_n^+(C)$ . The section  $s$  corresponds to a  $k$ -algebra map  $A_0 \rightarrow k$ , with kernel  $K$  say. We put  $A = (A_0)_K^\wedge$  and  $X = \mathrm{spf}(A)$ . This is the formal neighbourhood of  $sS$  in  $\mathrm{Div}_n^+(C)$ .

Now consider the scheme  $Y_0 = \mathrm{Map}_S(D, \mathbb{A}_S^1)$ . For any scheme  $T$  over  $S$ , the maps  $T \rightarrow Y_0$  over  $S$  are (essentially by definition) the maps  $D \times_S T \rightarrow \mathbb{A}^1$  of schemes, or equivalently the elements in the ring  $\mathcal{O}_D \otimes_k \mathcal{O}_T$ . These biject with the maps  $\mathcal{O}_D^\vee = \mathrm{Hom}_k(\mathcal{O}_D, k) \rightarrow \mathcal{O}_T$  of  $k$ -modules, or with the maps  $B_0 = \mathrm{Sym}_k[\mathcal{O}_D^\vee] \rightarrow \mathcal{O}_T$  of  $k$ -algebras. Thus, we have  $Y_0 = \mathrm{spec}(B_0)$ . We let  $B$  be the completion of  $B_0$  at the augmentation ideal, and put  $Y = \mathrm{spf}(B)$ , which is the formal neighbourhood of the zero section in  $Y_0$ . Of course  $B_0$  is just the direct sum of all the symmetric tensor powers of  $\mathcal{O}_D^\vee$ , and  $B$  is the direct product of the same terms. If  $\mathcal{O}_D$  is free over  $k$  (rather than just projective) then  $B$  is isomorphic to  $k[[c_1, \dots, c_n]]$ ; in the general case, it should be regarded as a slight twist of this. Note that maps  $T \rightarrow Y$  over  $S$  biject with  $k$ -linear maps  $\mathcal{O}_D^\vee \rightarrow \mathrm{Nil}(\mathcal{O}_T)$ , or equivalently elements of  $\mathcal{O}_D \otimes_k \mathrm{Nil}(\mathcal{O}_T)$ . Note also that a choice of generators  $x_1, \dots, x_r$  for  $\mathcal{O}_D^\vee$  gives a split surjection  $k[[x_1, \dots, x_r]] \rightarrow B$ .

There is an evident map

$$\alpha: \mathrm{Div}_n^+(C) = X_0 \rightarrow Y_0 = \mathrm{Map}_S(D, \mathbb{A}_S^1),$$

sending the section  $s'$  classifying a divisor  $D'$  to the function  $(f_{D'})|_D: D \rightarrow \mathbb{A}^1$ . This clearly sends  $s$  itself to zero, so it sends the formal neighbourhood of  $s$  to the

formal neighbourhood of zero, so it gives a map  $\alpha: X \rightarrow Y$ . We shall show that this is an isomorphism.

Note that because  $D$  and  $D'$  have the same degree, we have  $s' = s$  if and only if  $f_{D'}$  is divisible by  $f_D$ , if and only if  $\alpha(s') = 0$ . This shows that the kernel  $K$  of the map  $s^*: A_0 \rightarrow k$  is generated by the image under  $\alpha^*$  of the augmentation ideal in  $B_0$ . In particular, we see that  $K$  is finitely generated.

Because  $\mathcal{O}_D = R/J$  is projective over  $k$ , we can choose a  $k$ -submodule  $P \leq R$  such that  $R = P \oplus J$ . It follows that the map  $P \rightarrow R \rightarrow \mathcal{O}_D$  is an isomorphism, with inverse  $\xi$  say.

**LEMMA 19.2.** *Let  $I \leq k$  be a finitely generated ideal with  $I^m = 0$ , and let  $g \in R$  be such that  $g = f_D \pmod{IR}$ . Then  $g$  is a regular element, the ideal  $Rg$  is open, and we have  $R = Rg \oplus P$ .*

**PROOF.** A standard topological basis for  $R$  gives an isomorphism  $R = \prod_i k$ , and using the fact that  $I$  is finitely generated we see that  $I^j R = (IR)^j = \prod_i I^j$ . We thus have a finite filtration of  $R$  with quotients  $\prod_i I^j/I^{j+1}$ .

Now consider the  $k$ -linear self-map of  $R$  given by  $\lambda(qf_D + r) = qg + r$  for  $q \in R$  and  $r \in P$ . This is easily seen to induce the identity map on the quotients of the above filtration, so it is an isomorphism. It follows easily that  $g$  is regular and  $R = Rg \oplus P$ .

As  $D$  is a divisor, we know that  $Rf_D$  is open. Thus, for any good parameter  $y$  we have  $y^l \in Rf_D$  for large  $l$ , say  $y^l = uf_D$ . We also know that  $f_D = g + h$  for some  $h \in IR$ , so  $y^l = uh \pmod{g}$ . As  $I^m = 0$  we have  $y^{lm} = u^m h^m = 0 \pmod{g}$ , so  $Rg$  is also open.  $\square$

We now define a map  $\beta$  from sections of  $Y$  to sections of  $X$ . A section of  $Y$  is an element  $r \in \text{Nil}(k)\mathcal{O}_D$ . As  $\mathcal{O}_D$  is finitely generated we have  $r \in I\mathcal{O}_D$  for some finitely generated ideal  $I \leq \text{Nil}(k)$ , and by finite generation this satisfies  $I^m = 0$  for some  $m$ . We can thus apply the lemma to the function  $g = f_D + \xi(r) \in R$  and conclude that the subscheme  $D' = \text{spec}(R/g)$  is a divisor of degree  $n$ , classified by a section  $s'$  of  $X_0$  say. Over the subscheme  $\text{spec}(k/I) \subset S$  it clearly coincides with  $s$ , so  $(s')^*(K) \leq I$ , so  $(s')^*(K^m) = 0$ . This shows that  $s'$  is actually a section of  $X$ , as required. We can thus define  $\beta(r) = s'$ .

In order to define a map  $\beta: Y \rightarrow X$  of formal schemes over  $S$ , we need to define maps  $\beta_T$  from sections of  $Y$  over  $T$  to sections of  $X$  over  $T$ , naturally for all schemes  $T$  over  $S$ . For this we just replace  $C$  by  $T \times_S C$ ,  $P$  by  $\mathcal{O}_T \otimes_k P$  and follow the same procedure.

We now define another map  $\alpha': X \rightarrow Y$ . It will again be sufficient to do this for sections defined over  $S$ . Let  $s'$  be a section of  $X$ , classifying a divisor  $D'$ . Put  $I = (s')^*K \leq k$ ; this is finitely generated because  $K$  is, and nilpotent because  $s'$  lands in  $X$ . Over  $\text{spec}(k/I)$  we have  $D' = D$ , so  $f_{D'} = f_D \pmod{IR}$ . The lemma tells us that  $R = Rf_{D'} \oplus P$ , so there are unique elements  $h \in R$  and  $p \in P$  such that  $f_D = hf_{D'} - p$ . By reducing modulo  $I$  we see that  $h = 1 \pmod{IR}$  and  $p \in IP$ . We let  $r$  be the image of  $p$  in  $R/f_D = \mathcal{O}_D$ , so  $r \in I\mathcal{O}_D$  and  $\xi(r) = p$ . The map  $\alpha': X \rightarrow Y$  is defined by  $\alpha'(s') = r$ . Note that  $h$  is invertible so  $f_{D'}$  is a unit multiple of  $f_D + p = f_D + \xi(r)$ , so  $D' = \text{spec}(R/(f_D + \xi(r))) = \beta(r)$ . This shows that  $\beta\alpha' = 1$ .

In the other direction, suppose we start with  $r \in I\mathcal{O}_D$  and put  $D' = \operatorname{spec}(R/(f_D + \xi(r)))$  (corresponding to  $\beta(r)$ ). There is then a unique element  $p \in P$  congruent to  $-f_D$  modulo  $f_{D'}$ , and  $\alpha'\beta(r)$  is by definition the image of  $p$  in  $\mathcal{O}_D$ . It is clear that  $-f_D$  is congruent to  $\xi(r)$  modulo  $f_D + \xi(r)$ , which is a unit multiple of  $f_{D'}$ , so  $p = \xi(r)$  and  $\alpha'\beta(r) = r$ . This shows that  $\alpha'\beta = 1$ , so  $\alpha'$  and  $\beta$  are isomorphisms.

We actually started by claiming that the (slightly more canonical) map  $\alpha$  is an isomorphism. As  $\beta$  is an isomorphism, it suffices to check that the map  $\alpha\beta: Y \rightarrow Y$  is an isomorphism, or that  $(\alpha\beta)^*$  is an automorphism of  $\mathcal{O}_Y = B$ . As  $B$  is the completed symmetric algebra of a finitely generated projective module, it will suffice to show that  $(\alpha\beta)^*$  is the identity modulo the square of the augmentation ideal. By base-change to the universal case, it will suffice to show that  $\alpha\beta(r) = r$  whenever  $r \in I\mathcal{O}_D$  with  $I^2 = 0$ . Given such an  $r$ , we form the divisor  $D' = \operatorname{spec}(R/(f_D + \xi(r)))$  corresponding to  $\beta(r)$ , and observe that  $f_{D'} = u(f_D + \xi(r))$  for some  $u \in R^\times$ . As  $f_{D'} = f_D \pmod{IR}$  we must have  $u = 1 \pmod{IR}$ . As  $\xi(r) \in IR$  and  $I^2 = 0$  we have  $u\xi(r) = \xi(r)$  and so  $f_{D'} = \xi(r) \pmod{f_D}$ , so  $\alpha\beta(r) = r$  as claimed.



## Generalised homology of Grassmannians

Consider a periodically orientable theory  $E$  with associated equivariant formal group  $C = \mathrm{spf}(E^0 P\mathcal{U})$  over  $S = \mathrm{spec}(E^0)$ . Let  $G_r \mathcal{U}$  be the space of  $r$ -dimensional subspaces of  $\mathcal{U}$ , and put  $G\mathcal{U} = \coprod_{r=0}^{\infty} G_r \mathcal{U}$ . Here we reprove the following result from [8].

THEOREM 20.1 (Cole, Greenlees, Kriz). *There are natural isomorphisms*

$$\begin{aligned} E_* G_r \mathcal{U} &= (E_* P\mathcal{U})_{\Sigma_r}^{\otimes r} \\ E^* G_r \mathcal{U} &= ((E^* P\mathcal{U})^{\widehat{\otimes} r})^{\Sigma_r} \\ \mathrm{spf}(E^0 G_r \mathcal{U}) &= C^r / \Sigma_r. \end{aligned}$$

We first introduce some additional structure. Rather than working with spaces, it will be convenient to use (pre)spectra that happen to be homotopy equivalent to suspension spectra of spaces. Recall from [22] that an *orthogonal prespectrum*  $X$  consists of a space  $X(V)$  for each finite-dimensional inner product space  $V$ , together with maps  $\alpha_* : X(V) \rightarrow X(W)$  for each isometric isomorphism  $\alpha : V \rightarrow W$  and maps  $\sigma : S^U \wedge X(V) \rightarrow X(U \oplus V)$  satisfying certain axioms. The category of orthogonal prespectra has a smash product, defined so that pairings  $X(U) \wedge Y(V) \rightarrow Z(U \oplus V)$  satisfying some obvious axioms biject with maps  $X \wedge Y \rightarrow Z$ . There is a parallel theory of equivariant unitary prespectra, where we use unitary representations of  $A$  rather than inner product spaces. Our work in this section will be based on that theory.

For any complex inner product space  $V$ , we put

$$R_0(V) = \Sigma^V G V_+ = \bigvee_r \Sigma^V G_r V_+.$$

Using the evident maps  $G_r U \times G_s V \rightarrow G_{r+s}(U \oplus V)$  we get maps  $\mu_{U,V} : R_0(U) \wedge R_0(V) \rightarrow R_0(U \oplus V)$ . We also have inclusions  $\eta_U : S^U = \Sigma^U G_0 U_+ \rightarrow R_0(U)$ . These maps make  $R_0$  into a commutative and associative ring in the category of unitary prespectra. All this works equivariantly in an obvious way. The weak homotopy type of  $R_0$  is

$$R_0 \simeq \lim_{\substack{\longrightarrow \\ U \ll \mathcal{U}}} \Sigma^{-U} \Sigma^{\infty} R_0(U) = \lim_{\substack{\longrightarrow \\ U \ll \mathcal{U}}} \Sigma_+^{\infty} G U = \Sigma_+^{\infty} G \mathcal{U} = \Sigma_+^{\infty} \coprod_r G_r \mathcal{U}.$$

We write  $Q_r R_0$  for the subfunctor  $V \mapsto \Sigma^V G_r V_+$ , so that  $R_0 = \bigvee_r Q_r R_0$  and  $Q_r R_0 \simeq \Sigma_+^{\infty} G_r \mathcal{U}$ . In particular, we have  $Q_0 R_0 = S^0$  and  $Q_1 R_0 = \Sigma_+^{\infty} P\mathcal{U}$ . This gives a map  $E_* P\mathcal{U} \rightarrow E_* R_0$  and thus a ring map  $\mathrm{Sym}_{E_*} E_* P\mathcal{U} \rightarrow E_* R_0$ . The theorem says that this is an isomorphism. For the proof, we need some intermediate spectra. Let  $T$  denote the tautological bundle over  $G_r V$ . For any representation

$W$ , we put

$$Q_r R_W(V) = \Sigma^V (G_r V)^{\text{Hom}(T, W)}$$

$$R_W(V) = \bigvee_r Q_r R_W(V) = \Sigma^V G V^{\text{Hom}(T, W)}.$$

This again gives a commutative unitary ring spectrum, with weak homotopy type  $R_W \simeq G\mathcal{U}^{\text{Hom}(T, W)}$ . In the case  $W = 0$  we recover  $R_0$  as before. An inclusion  $W \rightarrow W'$  gives a ring map  $i: R_W \rightarrow R_{W'}$ . In particular, we have a ring map  $R_0 \rightarrow R_W$ , whose fibre we denote by  $J_W$ . This is weakly equivalent to the stable fibre of the zero section  $G\mathcal{U}_+ \rightarrow G\mathcal{U}^{\text{Hom}(T, W)}$ , and thus is the sphere bundle of the bundle  $\text{Hom}(T, W)$  over  $G\mathcal{U}$ .

Next, recall that there is an isometric embedding  $\mathcal{U} \oplus W \rightarrow \mathcal{U}$ , and that the space of such embeddings is connected. We have

$$Q_1 R_W = P\mathcal{U}^{\text{Hom}(T, W)} = P(\mathcal{U} \oplus W)/PW \simeq P\mathcal{U}/PW.$$

Using this, we have a diagram as follows, in which the rows are cofibrations:

$$\begin{array}{ccccc} PW_+ & \longrightarrow & P\mathcal{U}_+ & \longrightarrow & P\mathcal{U}/PW \\ \downarrow & & \downarrow & & \downarrow \\ J_W & \longrightarrow & R_0 & \longrightarrow & R_W. \end{array}$$

The map  $P\mathcal{U}/PW \rightarrow R_W$  gives a ring map

$$\theta_W: \text{Sym}_{E_*} E_*(P\mathcal{U}, PW) \rightarrow E_* R_W.$$

**THEOREM 20.2.** *The above maps  $\theta_W$  are isomorphisms.*

The proof will be given after some preparatory results.

First, suppose we have representations  $W$  and  $L$  with  $\dim(L) = 1$ . We put  $W' = W \oplus L$  and investigate the fibre of the map  $R_W \rightarrow R_{W'}$ . We may assume that  $W' \leq \mathcal{U}$ , and then we have a map

$$S^{\text{Hom}(L, W)} = PL^{\text{Hom}(T, W)} \subseteq P\mathcal{U}^{\text{Hom}(T, W)} \rightarrow R_W,$$

which we denote by  $b_{W, L}$ . Multiplication by  $b_{W, L}$  gives a map  $\Sigma^{\text{Hom}(L, W)} R_W \rightarrow R_W$ , which we again denote by  $b_{W, L}$ . (Note that this sends  $\Sigma^{\text{Hom}(L, W)} Q_r R_W$  into  $Q_{r+1} R_W$ , or in other words, it increases internal degrees by one.)

**PROPOSITION 20.3.** *The sequence*

$$\Sigma^{\text{Hom}(L, W)} R_W \xrightarrow{b_{W, L}} R_W \rightarrow R_{W \oplus L} = R_{W'}$$

*is a cofibration.*

**PROOF.** This is a special case of the following fact. Suppose we have a space  $X$  with vector bundles  $U$  and  $V$ . Let  $S(U)$  for the unit sphere bundle in  $U$ , and  $D(U)$  for the unit disc bundle, so  $X^U$  is homeomorphic to  $D(U)/S(U)$ . We can pull back  $V$  along the projection  $q: S(U) \rightarrow X$  and thus form the Thom space  $S(U)^{q^* V} = (S(U) \times_X D(V))/(S(U) \times_X S(V))$ . It is not hard to see that there is a cofibration

$$S(U)^{q^* V} \rightarrow D(U)^{q^* V} \simeq X^V \rightarrow X^{U \oplus V}.$$

We will apply this with  $X = G\mathcal{U}$  and  $V = \text{Hom}(T, W)$  and  $U = \text{Hom}(T, L)$ , so that  $X^V = R_W$  and  $X^{U \oplus V} = R_{W'}$ . To prove the proposition, we need to identify  $S(U)^{q^* V}$  with  $\Sigma^{\text{Hom}(L, W)} R_W$ .

To do this, observe that  $S(U)$  is the space of pairs  $(M, \alpha)$  where  $M$  is a finite-dimensional subspace of  $\mathcal{U}$  and  $\alpha: M \rightarrow L$  is a linear map of norm one. As  $L$  has dimension one, we find that  $\alpha$  can be written as the composite of the orthogonal projection  $M \rightarrow M \ominus \ker(\alpha)$  with an isometric isomorphism  $M \ominus \ker(\alpha) \rightarrow L$ . Using this, we identify  $S(U)$  with the space of pairs  $(N, \beta)$ , where  $N$  is a finite-dimensional subspace of  $\mathcal{U}$  and  $\beta: L \rightarrow \mathcal{U} \ominus N$  is an isometric embedding; the correspondence is that  $M = N \oplus \beta(L)$  and

$$\alpha = (N \oplus \beta(L) \xrightarrow{\text{proj}} \beta(L) \xrightarrow{\beta^{-1}} L).$$

We can thus define a map  $k: S(U) \rightarrow G\mathcal{U}$  by  $k(N, \beta) = N$  (or equivalently,  $k(M, \alpha) = \ker(\alpha)$ ). This makes  $S(U)$  into an equivariant fibre bundle over  $G\mathcal{U}$ . The fibre over a point  $N \in G\mathcal{U}$  is the space  $\mathcal{L}(L, \mathcal{U} \ominus N)$ , which is well-known to be contractible, and the contraction can be chosen to be equivariant with respect to the stabilizer of  $N$  in  $A$ . It follows that  $k$  is an equivariant equivalence. We next analyse the inverse of  $k$  (which will help to make the above argument more explicit).

Recall that  $L \leq W' \leq \mathcal{U}$ , so we can put

$$Y = \{N \in G\mathcal{U} \mid N \text{ is orthogonal to } L\} = G(\mathcal{U} \ominus L).$$

Define  $j: Y \rightarrow X$  by  $j(N) = N \oplus L$ , and then define  $\tilde{j}: Y \rightarrow S(U)$  by

$$\tilde{j}(N) = (N \oplus L, \text{proj}: N \oplus L \rightarrow L),$$

so  $q\tilde{j} = j$ . Clearly  $k\tilde{j}: Y = G(\mathcal{U} \ominus L) \rightarrow G\mathcal{U} = X$  is just the map induced by the inclusion  $\mathcal{U} \ominus L \rightarrow \mathcal{U}$ . As the space of linear isometries between any two complete  $A$ -universes is equivariantly contractible, we see that this inclusion is an equivariant equivalence. As the same is true of  $k$ , we deduce that  $\tilde{j}$  is also an equivariant equivalence. We can thus identify  $S(U)$  with  $Y$  and  $q: S(U) \rightarrow X$  with  $j: Y \rightarrow X$ . It follows that we can identify  $q^*V$  with  $j^*V$ , but the fibre of  $j^*V$  over a point  $N \in Y$  is  $\text{Hom}(j(N), W) = \text{Hom}(L, W) \oplus \text{Hom}(N, W)$ , so

$$Y^{j^*V} = \Sigma^{\text{Hom}(L, W)} Y^{\text{Hom}(N, W)} \simeq \Sigma^{\text{Hom}(L, W)} R_W.$$

This gives a cofibration  $\Sigma^{\text{Hom}(L, W)} R_W \rightarrow R_W \rightarrow R_{W'}$ , and one can check from the definitions that the first map is just multiplication by  $b_{W, L}$ .  $\square$

Now choose a complete flag

$$0 = W_0 < W_1 < \dots < \mathcal{U},$$

where  $\dim_{\mathbb{C}} W_i = i$  and  $\mathcal{U} = \lim_{\rightarrow i} W_i$ . Put  $R(i) = R_{W_i}$ , so we have maps

$$R_0 = R(0) \rightarrow R(1) \rightarrow R(2) \rightarrow \dots$$

Put  $L_i = W_{i+1} \ominus W_i$  and  $U_i = \text{Hom}(L_i, W_i)$  and  $b_i = b_{W_i, L_i}$ , so we have a cofibration

$$\Sigma^{U_i} R(i) \xrightarrow{b_i} R(i) \rightarrow R(i+1).$$

LEMMA 20.4. *Suppose that  $B \leq A$ , and split  $W$  as  $\bigoplus_{\beta \in B^*} W[\beta]$  in the usual way. Then*

$$\overline{\phi}^B R_W = \bigwedge_{\beta \in B^*} R_{W[\beta]},$$

where as before

$$W[\beta] = \{w \in W \mid bw = e^{2\pi i \beta(b)} w \text{ for all } b \in B\},$$

and so the connectivity of  $(\overline{\phi}^B R_W)/S^0$  is at least  $\min_{\beta}(2 \dim_{\mathbb{C}}(W[\beta]) - 1)$ .

PROOF. We have

$$\overline{\phi}^B G\mathcal{U} = (G\mathcal{U})^B = \{ B\text{-invariant subspaces of } \mathcal{U} \}.$$

Any complex representation  $U$  of  $B$  splits as  $\bigoplus_{\beta} U[\beta]$ , so a subspace  $U \leq \mathcal{U}$  is invariant iff it is the direct sum of its intersections with the subspaces  $\mathcal{U}[\beta]$ . It follows that

$$(G\mathcal{U})^B = \prod_{\beta} G\mathcal{U}[\beta] \simeq \prod_{\beta} G\mathbb{C}^{\infty}.$$

We have a tautological bundle  $T[\beta]$  over  $G\mathcal{U}[\beta]$ , and the bundle  $\mathrm{Hom}_{\mathbb{C}B}(T, W)$  over  $(G\mathcal{U})^B$  is the external direct sum of the bundles  $\mathrm{Hom}_{\mathbb{C}}(T[\beta], W[\beta])$ . The Thom complex  $G\mathcal{U}[\beta]^{\mathrm{Hom}(T[\beta], W[\beta])}$  is just  $R_{W[\beta]}$ , and it follows that  $[(G\mathcal{U})^{\mathrm{Hom}(T, W)}]^B$  is just the smash product of these factors, as claimed.

For the last statement, note that if  $X$  is a space and  $U$  is a vector bundle of real dimension  $d$  over  $X$ , then  $X^U$  is always  $(d-1)$ -connected. Now let  $\mathcal{V}$  be a complex universe, and  $V$  a complex vector space of finite dimension  $d$ . The bundle  $\mathrm{Hom}(T, V)$  over  $G_r \mathcal{V}$  has real dimension  $2rd$ , so  $\mathrm{conn}(Q_r R_V) \geq 2rd - 1$ , and

$$\mathrm{conn}(R_V/S^0) = \mathrm{conn}\left(\bigvee_{r>0} Q_r R_V\right) \geq 2d - 1.$$

The claim follows easily.  $\square$

COROLLARY 20.5.  $\lim_{\rightarrow i} R(i) = S^0$ .

PROOF. The unit map  $S^0 \rightarrow Q_0 R(i)$  is an isomorphism for all  $i$ , so  $\lim_{\rightarrow i} Q_0 R(i) = S^0$ . It will thus suffice to show that  $\lim_{\rightarrow i} R(i)/S^0 = 0$ , or equivalently that the spectrum  $\overline{\phi}^B(\lim_{\rightarrow i} R(i)/S^0) = \lim_{\rightarrow i} ((\overline{\phi}^B R(i))/S^0)$  is nonequivariantly contractible for all  $B$ . As  $\mathcal{U}$  is a complete universe, we have  $\dim W_i[\beta] \rightarrow \infty$  as  $i \rightarrow \infty$  for all  $\beta$ , so  $\mathrm{conn}(\overline{\phi}^B R(i)/S^0) \rightarrow \infty$ , and the claim follows.  $\square$

We now let  $E$  be a periodically oriented theory, with orientation  $x$  say. This gives a universal generator  $u_i$  for  $\widetilde{E}_0 S^{U_i}$ , and a basis  $\{c_i \mid i \geq 0\}$  for  $\widetilde{E}_0 P\mathcal{U}$ . Put

$$ER(i) = \mathrm{Sym}_{E_*} E_*(P\mathcal{U}, PW_i) = E_*[c_j \mid j \geq i] = ER(0)/(c_k \mid k < i),$$

and let  $Q_r ER(i)$  be the submodule generated by monomials of weight  $r$  (where each generator  $c_j$  is considered to have weight one). We then have maps

$$\theta_i = \theta_{W_i}: ER(i) \rightarrow E_* R(i),$$

which restrict to give maps

$$\theta_{ir}: Q_r ER(i) \rightarrow E_* Q_r R(i).$$

The elements  $u_i$  and  $c_i$  are related as follows: the inclusion  $PL_i \rightarrow P\mathcal{U}$  gives an inclusion  $S^{U_i} \rightarrow P\mathcal{U}^{\mathrm{Hom}(T, W_i)} \simeq P\mathcal{U}/PW_i$ , and the image of  $u_i$  under this map is the same as the image of  $c_i$  under the evident quotient map  $P\mathcal{U} \rightarrow P\mathcal{U}/PW_i$ . It follows that the cofibration  $\Sigma^{U_i} R(i) \xrightarrow{b_i} R(i) \rightarrow R(i+1)$  gives rise to a cofibration

$$E \wedge R(i) \xrightarrow{c_i} E \wedge R(i) \rightarrow E \wedge R(i+1),$$



which restricts to give a cofibration

$$E \wedge Q_{r-1}R(i) \xrightarrow{c_i} E \wedge Q_rR(i) \rightarrow E \wedge Q_rR(i+1).$$

PROPOSITION 20.6. *The maps  $\theta_{ir}$  are isomorphisms for all  $i$  and  $r$ .*

PROOF. The maps  $\theta_{j0}$  and  $\theta_{j1}$  are visibly isomorphisms, so we may assume inductively that  $\theta_{j,r-1}$  is an isomorphism for all  $j$ . The cofibration displayed above gives a diagram  $D(i)$  as follows, with exact rows:

$$\begin{array}{ccccc} Q_{r-1}ER(i) & \xrightarrow{c_i} & Q_rER(i) & \longrightarrow & Q_rER(i+1) \\ \theta_{i,r-1} \downarrow \simeq & & \theta_{i,r} \downarrow & & \downarrow \theta_{i+1,r} \\ E_*Q_{r-1}R(i) & \xrightarrow{c_i} & E_*Q_rR(i) & \xrightarrow{q_{ir}} & E_*Q_rR(i+1) \end{array}$$

We first prove that  $\theta_{ir}$  is surjective for all  $i$ . Let  $\Theta(i)$  be the image of  $\theta_{ir}$ , so the claim is that  $\Theta(i) = E_*Q_rR(i)$ . For  $j \geq i$  we write  $K(j)$  for the kernel of the map  $E_*Q_rR(i) \rightarrow E_*Q_rR(j)$ . Clearly  $K(i) = 0 \leq \Theta(i)$ . Suppose that  $K(j) \leq \Theta(i)$ , and consider the diagram  $D(j)$ . Suppose that  $a \in K(j+1)$ . Let  $b$  be the image of  $a$  in  $E_*Q_rR(j)$ , so  $q_{jr}(b) = 0$ . As the bottom row of  $D(j)$  is exact and  $\theta_{j,r-1}$  is an isomorphism we see that there exists  $c \in Q_rER(j)$  with  $\theta_{jr}(c) = b$ . Moreover, the map  $Q_rER(i) \rightarrow Q_rER(j)$  is epi, so we can lift  $c$  to get an element  $d \in Q_rER(i)$ . Now  $a - \theta_{ir}(d) \in K(j) \leq \Theta(i)$  and visibly  $\theta_{ir}(d) \in \Theta(i)$  so  $a \in \Theta(i)$ . It follows by induction that  $K(j) \leq \Theta(i)$  for all  $j$ . Moreover, Corollary 20.5 implies that  $E_*Q_rR(i)$  is the union of the subgroups  $K(j)$ , so  $\Theta(i) = E_*Q_rR(i)$  as required.

We now see that in  $D(i)$ , the vertical maps are surjective, so  $q_{ir}$  is surjective. As the bottom row is part of a long exact sequence and the right hand map is surjective, we conclude that the bottom row is actually a short exact sequence. Using the snake lemma, we conclude that the induced map  $\ker(\theta_{ir}) \rightarrow \ker(\theta_{i+1,r})$  is an isomorphism. It follows that for any  $m > i$ , the map  $\ker(\theta_{ir}) \rightarrow \ker(\theta_{mr})$  is an isomorphism. However, we have  $\ker(\theta_{ir}) \leq Q_rER(i)$ , and it is also clear that when  $r > 0$ , any element of  $Q_rER(i)$  maps to zero in  $Q_rER(m)$  for  $m \gg 0$ . It follows that  $\ker(\theta_{ir})$  must be zero, so  $\theta_{ir}$  is an isomorphism as claimed.  $\square$

PROOF OF THEOREM 20.2. Given any subrepresentation  $W < \mathcal{U}$ , we can choose our flag  $\{W_i\}$  such that  $W = W_i$  for some  $i$ . The theorem then follows from Proposition 20.6.  $\square$



## Thom isomorphisms and the projective bundle theorem

Let  $E$  be a periodically orientable  $A$ -equivariant cohomology theory, with associated equivariant formal group  $(C, \phi)$  over  $S$ . For any  $A$ -space  $X$ , we will write  $X_E = \text{spf}(E^0 X)$ .

Now let  $V$  be an equivariant complex vector bundle over  $X$ . We write  $PV$  for the associated bundle of projective spaces, and  $X^V$  for the Thom space (so  $X^V = P(V \oplus \mathbb{C})/PV$ ). In this section, we will give a Thom isomorphism and a projective bundle theorem to calculate  $\tilde{E}^* X^V$  and  $E^* PV$ .

First, it is well-known that equivariant bundles of dimension  $r$  over  $X$  are classified by homotopy classes of  $A$ -maps  $X \rightarrow G_r \mathcal{U}$ . (See for example [4, Section 1.6].) We saw above that  $E^0 G_r \mathcal{U} = S_r$ , and moreover the standard topological basis  $\{e'_\beta\}$  for  $S_r$  is dual to a universal basis for  $E_0 G_r \mathcal{U}$ . It follows that  $(G_r \mathcal{U})_E = C^r / \Sigma_r = \text{Div}_r^+(C)$ .

Now let  $T$  denote the tautological bundle over  $G_r \mathcal{U}$ . It is not hard to identify the projective bundle  $PT \rightarrow G_r \mathcal{U}$  with the addition map

$$G_{r-1} \mathcal{U} \times P\mathcal{U} = G_{r-1} \mathcal{U} \times G_1 \mathcal{U} \rightarrow G_r \mathcal{U},$$

and thus to identify  $E^0 PT$  with  $S_{r-1} \hat{\otimes} R = \mathcal{O}_{C^r / \Sigma_{r-1}}$ , so  $PT_E = C^r / \Sigma_{r-1}$ . On the other hand, we can use Corollary 18.13 to identify  $C^r / \Sigma_{r-1}$  with the universal divisor  $D_r$  over  $C^r / \Sigma_r$ .

Now suppose we have a vector bundle  $V$  over  $X$ , classified by a map  $c: X \rightarrow G_r \mathcal{U}$ , so  $c^* T \simeq V$ . The map  $c$  is then covered by a map  $\tilde{c}: PV \rightarrow PT$ , which gives a map  $\tilde{c}^*: \mathcal{O}_{D_r} = E^0 PT \rightarrow E^0 PV$ . We can combine this with the evident map  $E^* X \rightarrow E^* PV$  to get a map

$$\theta_{X,V}: \mathcal{O}_{D_r} \otimes_{S_r} E^* X \rightarrow E^* PV.$$

**THEOREM 21.1.** *For any  $X$  and  $V$  as above, the map  $\theta_{X,V}$  is an isomorphism (and so  $E^* PV$  is a projective module of rank  $r$  over  $E^* X$ ).*

**PROOF.** We first examine the simplifications that occur when  $V$  admits a splitting  $V = L_1 \oplus \dots \oplus L_r$ , where each  $L_i$  is a line bundle. In this case, the classifying map  $X \rightarrow G_r \mathcal{U}$  factors through  $PU^r$ , so the map  $S_r \rightarrow E^0 X$  factors through  $R_r$ . As  $\mathcal{O}_{D_r} \otimes_{S_r} R_r = \mathcal{O}_{\tilde{D}_r}$ , we see that  $\theta_{X,V}$  is the composite of an isomorphism with a map

$$\theta'_{X,V}: \mathcal{O}_{\tilde{D}_r} \otimes_{R_r} E^* X \rightarrow E^* PV.$$

Next, choose a coordinate  $x$  on  $C$  and define a difference function  $d(a, b) = x(b - a)$  as usual. Define a function  $d_i$  on  $C^{r+1}$  by

$$d_i(a_1, \dots, a_r, b) = d(a_i, b) = x(b - a_i),$$

as in Construction 18.2. We then put  $c_i = \prod_{j < i} d_j$ . By the method of Proposition 2.21 we see that  $\{c_1, \dots, c_r\}$  is a basis for  $\mathcal{O}_{\bar{D}}$  over  $R_r$ , so  $\theta'_{X,V}$  is just the map  $(E^*X)^r \rightarrow E^*PV$  given by  $(t_1, \dots, t_r) \mapsto \sum_i t_i c_i$ .

Now consider the case where  $X$  is a point, so  $V$  is just a representation of  $A$ . In this case there is always a splitting  $V = L_1 \oplus \dots \oplus L_r$  as above, where  $L_i = L_{\alpha_i}$  for some  $\alpha_i \in A^*$ . In this case the image of  $c_i$  in  $E^0PV$  is just the element  $x_{U_{i-1}}$  from Corollary 5.8, so the map  $\theta_{X,V}$  is an isomorphism.

More generally, suppose that  $X$  is arbitrary but  $V$  is a constant bundle, with fibre given by a representation  $W = \sum_i L_{\alpha_i}$  say. As the elements  $x_{U_i}$  form a universal basis for  $E^*PW$ , we see that  $E^*PV = E^*X \otimes_{E^*} E^*PW = \bigoplus_i E^*.x_{U_i}$  and it follows easily that  $\theta_{X,V}$  is again an isomorphism.

Now consider the case  $X = A/B$  for some  $B \leq A$ . It is easy to see that any bundle over  $X$  has the form  $A \times_B W_0$  for some representation  $W_0$  of  $B$ . However, as  $A$  is a finite abelian group, we can find a representation  $W$  of  $A$  such that  $W|_B = W_0$ , and it follows that  $A \times_B W_0$  is isomorphic to the constant bundle  $A/B \times W$ . It follows that  $\theta_{X,V}$  is again an isomorphism.

Now let  $X$  and  $V$  be arbitrary, and suppose we can decompose  $X$  as the union of two open sets  $X_0$  and  $X_1$ , with intersection  $X_2$ . Suppose we know that the maps  $\theta_{X_i,V}$  are isomorphisms for  $i = 0, 1$  and  $2$ ; we claim that  $\theta_{X,V}$  is also an isomorphism. Indeed, the decomposition gives a Mayer-Vietoris sequence involving  $E^*X$ . We can tensor this by the projective module  $\mathcal{O}_{D_r}$  over  $S_r$ , and it will remain exact. Alternatively, we can pull back the decomposition to get a decomposition of  $PV$ , and obtain another Mayer-Vietoris sequence. The  $\theta$  maps are easily seen to be compatible with these sequences, so the claim follows by the five lemma.

We can now now prove that  $\theta_{X,V}$  is an isomorphism for all  $X$  and  $V$ , by induction on the number of cells and passage to colimits.  $\square$

**COROLLARY 21.2.** *If  $V$  is an equivariant vector bundle over  $X$ , then the formal scheme  $D(V) := PV_E$  is a divisor on  $C$  over  $X_E$ , of degree equal to the dimension of  $V$ . Moreover, we have  $D(V \oplus W) = D(V) + D(W)$  and  $D(V \otimes W) = D(V) * D(W)$ .*

**PROOF.** The first statement is clear from the theorem. We need only check the equation  $D(V \oplus W) = D(V) + D(W)$  in the universal case, where  $X = G_r\mathcal{U} \times G_s\mathcal{U}$ . As the map  $PU^{r+s} \rightarrow G_r\mathcal{U} \times G_s\mathcal{U}$  induces a faithfully flat map  $C^{r+s} \rightarrow C^r/\Sigma_r \times_S C^s/\Sigma_s$ , it suffices to check that equation for the obvious bundles  $V = L_1 \oplus \dots \oplus L_r$  and  $W = L_{r+1} \oplus \dots \oplus L_{r+s}$  over  $PU^{r+s}$ , in which case it is clear. A similar approach works for convolution of divisors.  $\square$

We next consider the Thom isomorphism.

**DEFINITION 21.3.** Let  $C$  be a formal multicurve group over  $S$ , with zero section  $\zeta: S \rightarrow C$ . Given a divisor  $D$  on  $C$  over  $S$ , we let  $J_D$  denote the kernel of the restriction map  $\mathcal{O}_C \rightarrow \mathcal{O}_D$ , which is a free module of rank one over  $\mathcal{O}_C$ . We also use the map  $\zeta^*: \mathcal{O}_C \rightarrow \mathcal{O}_S$  to make  $\mathcal{O}_S$  into a module over  $\mathcal{O}_C$ , and we define  $L(D) = \mathcal{O}_S \otimes_{\mathcal{O}_C} J_D$ , which is a free module of rank one over  $\mathcal{O}_S$ , or equivalently a trivialisable line bundle over  $S$ . We call this the *Thom module* for  $D$ . More generally, given a scheme  $S'$  over  $S$  and a divisor  $D$  on  $C$  over  $S'$ , we obtain a trivialisable line bundle  $L(D)$  over  $S'$ .

REMARK 21.4. Note that  $\ker(\zeta^*) = J_{[0]}$  and that  $J_D J_{[0]} = J_{D+[0]}$ . It follows that

$$L(D) = J_D / J_{D+[0]} = \ker(\mathcal{O}_{D+[0]} \rightarrow \mathcal{O}_D).$$

REMARK 21.5. If we fix a coordinate  $x$  and put  $d(a, b) = x(b - a)$ , we get a generator  $f_D$  for  $J_D$  as in Definition 15.3, and thus a generator  $u_D = 1 \otimes f_D$  for  $L(D)$ , which we call the *Thom class*. However, these generators are not completely canonical because of the choice of coordinate.

We also define the *Euler class*  $e_D$  to be the element  $f_D(0) = \zeta^* f_D \in \mathcal{O}_S$ . Note that if  $D = [u]$  for some section  $u$ , then  $f_D(a) = x(a - u)$  and so  $e_D = x(-u) = \bar{x}(u)$ .

REMARK 21.6. For any two divisors  $D$  and  $D'$ , we have  $J_{D+D'} = J_D J_{D'}$ , which can be identified with  $J_D \otimes_{\mathcal{O}_C} J_{D'}$  (because  $J_D$  and  $J_{D'}$  are each generated by a single regular element). It follows that  $L(D + D') = L(D) \otimes_{\mathcal{O}_S} L(D')$ . In terms of a coordinate, we have  $u_{D+D'} = u_D \otimes u_{D'}$  and  $e_{D+D'} = e_D e_{D'}$ .

THEOREM 21.7. *Let  $V$  be an equivariant complex bundle over a space  $X$ , giving a divisor  $D(V) = PV_E$  on  $C$  over  $X_E$  as in Corollary 21.2 and thus a free rank one module  $L(D(V))$  over  $E^0 X$ . Then there is a natural isomorphism  $\tilde{E}^0 X^V = L(D(V))$  (and  $\tilde{E}^* X^V = L(D(V)) \otimes_{E^0 X} E^* X$ ). Moreover, if we choose a coordinate and thus obtain a Thom class  $u_{D(V)}$  as in Remark 21.5, then this gives a universal generator for  $\tilde{E}^0 X$ .*

PROOF. Consider the cofibration  $P(V) \rightarrow P(V \oplus \mathbb{C}) \rightarrow X^V$ . Using Theorem 21.1 we see that

$$E^* P(V) = E^* X \otimes_{E^0 X} \mathcal{O}_{D(V)}$$

$$E^* P(V \oplus \mathbb{C}) = E^* X \otimes_{E^0 X} \mathcal{O}_{D(V \oplus \mathbb{C})} = E^* X \otimes_{E^0 X} \mathcal{O}_{D(V) + [0]}.$$

As the map  $\rho: \mathcal{O}_{D(V) + [0]} \rightarrow \mathcal{O}_{D(V)}$  is a split surjection of  $E^0 X$ -modules, we see that the long exact sequence of the cofibration splits into short exact sequences. As  $\ker(\rho) = L(D(V))$ , we see that  $\tilde{E}^* X^V = L(D(V)) \otimes_{E^0 X} E^* X$ . By looking in degree zero, we see that  $\tilde{E}^0 X^V = L(D(V))$ . As this isomorphism is natural in  $X$ , it is easy to see that the generator is universal.  $\square$

REMARK 21.8. If we have two bundles  $V$  and  $V'$ , the above results give

$$\begin{aligned} \tilde{E}^0 X^{V \oplus V'} &= L(D(V \oplus V')) = L(D(V) + D(V')) \\ &= L(D(V)) \otimes_{E^0 X} L(D(V')) = \tilde{E}^0 X^V \otimes_{E^0 X} \tilde{E}^0 X^{V'}. \end{aligned}$$

One can check that this isomorphism  $\tilde{E}^0 X^V \otimes \tilde{E}^0 X^{V'} = \tilde{E}^0 X^{V \oplus V'}$  is induced by the usual diagonal map  $X^{V \oplus V'} \rightarrow X^V \wedge X^{V'}$ .

DEFINITION 21.9. We write  $u_V$  for  $u_{D(V)}$ , and call this the *Thom class* of  $V$ . We also write  $e_V$  for  $e_{D(V)}$ , and call this the *Euler class* of  $V$ . (Using Remark 21.5, we see that this is consistent with the definition for line bundles given in Section 5.)

It is easy to see that the Euler class is the pullback of the Thom class along the zero section  $X \rightarrow X^V$ , and that  $e_{V \oplus W} = e_V e_W$ .

Now suppose that  $r \leq \dim(V)$ , and consider the space  $P_r(V)$  consisting of all tuples  $(x; L_1, \dots, L_r)$  where  $x \in X$  and the  $L_i$  are orthogonal lines in  $V_x$ . Recall also that  $P_r(D(V))$  is the classifying scheme for  $r$ -tuples  $(u_1, \dots, u_r)$  of sections of  $C$  such that  $\sum_i [u_i] \leq D(V)$ , as in Proposition 18.8.

PROPOSITION 21.10. *There is a natural isomorphism  $P_r(V)_E = P_r(D(V))$ .*

PROOF. For each  $i$  we have a line bundle over  $P_r(V)$  whose fibre over  $(x, L_1, \dots, L_r)$  is  $L_i$ . This is classified by a map  $P_r(V) \rightarrow \mathcal{PU}$ , which gives rise to a map  $u_i: P_r(V)_E \rightarrow C$ . The direct sum of these line bundles corresponds to the divisor  $[u_1] + \dots + [u_r]$ . This direct sum is a subbundle of  $V$ , so  $[u_1] + \dots + [u_r] \leq D(V)$ . This construction therefore gives us a map  $P_r(V)_E \rightarrow P_r(D(V))$ .

In the case  $r = 1$  we have  $P_1(V) = PV$  and  $P_1(D(V)) = D(V)$  so the claim is that  $(PV)_E = D(V)$ , which is true by definition. In general, suppose we know that  $P_{r-1}(V)_E = P_{r-1}(D(V))$ . We can regard  $P_r(V)$  as the projective space of the bundle over  $P_{r-1}(V)$  whose fibre over a point  $(x, L_1, \dots, L_{r-1})$  is the space  $V_x \ominus (L_1 \oplus \dots \oplus L_{r-1})$ . It follows that  $P_r(V)_E$  is just the divisor  $D(V) - ([u_1] + \dots + [u_{r-1}])$  over  $P_{r-1}(D(V))$ , which is easily identified with  $P_r(D(V))$ . The proposition follows by induction.  $\square$

We next consider the Grassmannian bundle

$$G_r(V) = \{(x, W) \mid x \in X, W \leq V_x \text{ and } \dim(W) = r\}.$$

PROPOSITION 21.11. *There is a natural isomorphism  $G_r(V)_E = \text{Sub}_r(D(V))$ .*

PROOF. Let  $T$  denote the tautological bundle over  $G_r(V)$ . This is a rank  $r$  subbundle of the pullback of  $V$  so we have a degree  $r$  subdivisor  $D(T)$  of the pullback of  $D(V)$  over  $G_r(V)_E$ . This gives rise to a map  $f: G_r(V)_E \rightarrow \text{Sub}_r(D(V))$ , so if we put  $A = \mathcal{O}_{\text{Sub}_r(D(V))}$  we get a map  $f^*: A \rightarrow E^0 G_r(V)$ , and we must show that this is an isomorphism. Now consider the tautological divisor  $\overline{D}$  of degree  $r$  over  $\text{Sub}_r(D(V))$ . As the module  $B = \mathcal{O}_{P_r \overline{D}}$  is faithfully flat over  $A$ , it will suffice to show that the map  $f^*: B \rightarrow B \otimes_A E^0 G_r(V)$  is an isomorphism. However, we saw in the proof of Proposition 18.11 that  $P_r \overline{D} = P_r D(V) = (P_r V)_E$ , so  $B = E^0 P_r V$ . If we let  $T$  be the tautological bundle over  $G_r V$ , it is easy to see that  $P_r T = P_r V$  and so  $B = E^0 P_r T = \mathcal{O}_{P_r D(T)}$ . It is also easy to see that  $D(T) = f^* \overline{D}$ , so  $P_r D(T) = f^* P_r \overline{D}$ , and so

$$B = \mathcal{O}_{P_r D(T)} = E^0 G_r(V) \otimes_A \mathcal{O}_{P_r \overline{D}} = E^0 G_r(V) \otimes_A B,$$

as required.  $\square$

We conclude this section with a consistency check that will be useful later.

DEFINITION 21.12. Given a one-dimensional complex vector space  $L$  and an arbitrary complex vector space  $V$ , we define  $\rho: PV \rightarrow P(L \otimes V)$  by  $\rho(M) = L \otimes M$ . This is evidently a homeomorphism. If  $V$  has the form  $V = \text{Hom}(L, W) = L^* \otimes W$  then we identify  $L \otimes V$  with  $W$  in the obvious way, and thus obtain a homeomorphism  $\rho: P(\text{Hom}(L, W)) \rightarrow PW$ . All this clearly works equivariantly, and fibrewise for vector bundles.

PROPOSITION 21.13. *Let  $X$  be a space equipped with two complex vector bundles  $V$  and  $W$ . Let  $p: PV \rightarrow X$  be the projection, and let  $T$  be the tautological bundle over  $PV$ , so we have a bundle  $\text{Hom}(T, p^* W)$  over  $PV$ . Then there is a natural homeomorphism*

$$P(V \oplus W)/PW = PV^{\text{Hom}(T, p^* W)}.$$

PROOF. Put  $U = \text{Hom}(T, p^*W)$ . We will construct a diagram as follows:

$$\begin{array}{ccccccc}
 PU & \xrightarrow[\simeq]{\rho} & P(p^*W) & \xrightarrow{\text{proj}} & PV \\
 \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 \\
 P(\mathbb{C} \oplus U) & \xrightarrow[\rho]{\simeq} & P(T \oplus p^*W) & \xrightarrow{\quad} & P(p^*(V \oplus W)) & \xrightarrow[\text{proj}]{\quad} & P(V \oplus W)
 \end{array}$$

First note that the obvious map  $\mathbb{C} \rightarrow \text{Hom}(T, T)$  is an isomorphism (because  $T$  has dimension one), so

$$\mathbb{C} \oplus U = \text{Hom}(T, T) \oplus \text{Hom}(T, p^*W) = \text{Hom}(T, T \oplus p^*W).$$

Given this, it is clear that we have homeomorphisms  $\rho$  as indicated; this gives the left hand half of the diagram, and shows that the cofibre of  $i_0$  is homeomorphic to that of  $i_1$ .

Next, observe that  $T$  is a subbundle of  $p^*V$  so  $T \oplus p^*W \leq p^*(V \oplus W)$ , so  $P(T \oplus p^*W) \subseteq P(p^*(V \oplus W))$  as indicated. There is also an obvious projection  $P(p^*(V \oplus W)) \rightarrow P(V \oplus W)$ , giving the right hand rectangle in the diagram. Note also that  $P(p^*W) = p^*PV = PV \times_X PW$ .

We next consider in more detail the map  $P(T \oplus p^*W) \rightarrow P(V \oplus W)$ , which we shall call  $\tau$ . A point in  $P(T \oplus p^*W)$  consists of a triple  $(x, L, M)$ , where  $x \in X$  and  $L$  is a one-dimensional subspace of  $V_x$  and  $M$  is a one-dimensional subspace of  $L \oplus W_x$ . We have  $\tau(x, L, M) = (x, M) \in P(V \oplus W)$ . Suppose we start with a point  $(x, M) \in P(V \oplus W)$ . If  $M \in PW_x$  then it is clear that  $\tau^{-1}\{(x, M)\} = PV_x \times \{M\}$ . On the other hand, if  $M \notin PW_x$  then the image of  $M$  under the projection  $V_x \oplus W_x \rightarrow V_x$  is a one-dimensional subspace  $L \leq V_x$ , and the point  $(x, L, M)$  is the unique preimage of  $(x, M)$  under  $\tau$ . This means that the rectangle is a pullback, in which the horizontal maps are surjective. Using this, we see that  $\tau$  induces a homeomorphism from the cofibre of  $i_1$  to that of  $i_2$ .

The cofibre of  $i_0$  is  $PV^U$ , and the cofibre of  $i_2$  is  $P(V \oplus W)/PW$ , so these are homeomorphic as claimed.  $\square$

As a corollary of the above, we have  $E^0(P(V \oplus W), PW) = \tilde{E}^0 PV^{\text{Hom}(T, p^*W)}$ . We can use the projective bundle theorem and the Thom isomorphism to calculate both sides in terms of divisors, and they are not obviously the same. Nonetheless, there is an isomorphism between them that can be constructed by pure algebra, as explained in the following result.

PROPOSITION 21.14. *Let  $C = \text{spf}(R)$  be a formal multicurve group over  $S = \text{spec}(k)$ , equipped with two divisors  $P = \text{spec}(R/K)$  and  $Q = \text{spec}(R/L)$ . Define an automorphism  $\rho$  of  $P \times_S C$  by  $\rho(a, b) = (a, b + a)$  (so  $\rho^{-1}(a, b) = (a, b - a)$ ) and let  $p: P \rightarrow S$  be the projection. Then there is a natural isomorphism*

$$L(\rho^{-1}(p^*Q)) = L/KL = \ker(\mathcal{O}_{P+Q} \rightarrow \mathcal{O}_Q).$$

Moreover, if we have a coordinate  $x$  and use it to define a difference function and Thom classes, then the above isomorphism sends the Thom class in  $L(\rho^{-1}(p^*Q))$  to the element  $f_Q \in L/KL$ .

REMARK 21.15. In the last part of the statement, it is important that we are using the generator  $f_Q$  defined as a norm as in Definition 15.3. As explained in Remark 15.4, in the nonequivariant case, this is different from the Chern polynomial which is more usually used as a generator.

PROOF. Write  $Z$  for the scheme  $P \times 0$  and  $\Delta$  for the image of the diagonal map  $P \rightarrow P \times_S P$ . Both of these can be regarded as divisors on the multcurve  $P \times_S C$  over  $P$ , and it is clear that  $\rho^{-1}(\Delta) = Z$ , and so  $\rho^{-1}(\Delta + p^*Q) = Z + \rho^{-1}(p^*Q)$ . It is also clear that  $\Delta \leq p^*P$  and so  $\Delta + p^*Q \leq p^*(P + Q)$ . There are evident projection maps  $p^*Q \rightarrow Q$  and  $p^*(P + Q) \rightarrow P + Q$ . All this fits together into the following diagram.

$$\begin{array}{ccccccc}
 \rho^{-1}(p^*Q) & \xrightarrow[\simeq]{\rho} & p^*Q & \xrightarrow{\hspace{2cm}} & Q \\
 \downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 \\
 Z + \rho^{-1}(p^*Q) & \xrightarrow[\rho]{\simeq} & \Delta + p^*Q & \twoheadrightarrow & p^*(P + Q) & \twoheadrightarrow & P + Q
 \end{array}$$

The kernel of the ring map  $i_0^*$  is (essentially by definition) the Thom module  $L(\rho^{-1}(p^*Q))$ . As  $\rho$  is an isomorphism, it induces an isomorphism  $\ker(i_1^*) \simeq \ker(i_0^*)$ . It will thus be enough to show that the map  $\ker(i_2^*) \rightarrow \ker(i_1^*)$  is also an isomorphism. To be more explicit, write  $A = \mathcal{O}_P = R/K$  and  $B = \mathcal{O}_Q = R/L$ . Let  $I$  be the ideal in  $A \otimes R = \mathcal{O}_{P \times_S C}$  defining the closed subscheme  $\Delta$ , so  $I$  is generated by the images of elements  $1 \otimes a - a \otimes 1$  for  $a \in R$ . The right hand half of the above diagram then gives the following diagram of rings and ideals:

$$\begin{array}{ccc}
 \frac{A \otimes R}{A \otimes L} & \xleftarrow{f_0} & \frac{R}{L} \\
 \uparrow i_1^* & & \uparrow i_2^* \\
 \frac{A \otimes R}{I.(A \otimes L)} & \xleftarrow{f_1} & \frac{R}{KL} \\
 \uparrow & & \uparrow \\
 \frac{A \otimes L}{I.(A \otimes L)} & \xleftarrow{f_2} & \frac{L}{KL}
 \end{array}$$

The maps  $f_i$  have the form  $a \mapsto 1 \otimes a$ , and we must show that  $f_2$  is an isomorphism. Now choose a generator  $g$  for the ideal  $L$ , giving an isomorphism  $L \simeq R$  of  $R$ -modules. This gives isomorphisms  $(A \otimes L)/I.(A \otimes L) \simeq (A \otimes R)/I$  and  $L/KL \simeq R/K = A$ , in terms of which  $f_2$  becomes the ring map  $A \rightarrow (A \otimes R)/I$  given by  $a \mapsto (a \otimes 1 + I) = (1 \otimes a + I)$ . This corresponds to the projection  $\Delta \rightarrow B$ , which is evidently an isomorphism as required.  $\square$

REMARK 21.16. In the context of Proposition 21.13, we can take  $P = D(V)$  and  $Q = D(W)$ . We find that  $D(\text{Hom}(T, q^*V)) = \rho^{-1}(q^*P)$ , and the diagram in the proof of Proposition 21.14 can be identified with that in the proof of Proposition 21.13. It follows that the isomorphism  $L(\rho^{-1}(q^*P)) = \ker(\mathcal{O}_{P+Q} \rightarrow \mathcal{O}_P)$  obtained by applying  $E^0(-)$  to Proposition 21.13 is the same as that given by Proposition 21.14.



## Duality

Let  $D = \text{spf}(R/I)$  be a divisor of degree  $r$  on  $C$ . In this section we will prove that  $\text{Hom}_S(\mathcal{O}_D, \mathcal{O}_S)$  is a free module of rank one over  $\mathcal{O}_D$ , which means that  $\mathcal{O}_D$  is a Poincaré duality algebra over  $\mathcal{O}_S$ . More precisely, we will identify  $\text{Hom}_S(\mathcal{O}_D, \mathcal{O}_S)$  with a subquotient of the module of meromorphic differential forms on  $C$ . In the case where  $C$  is embeddable, the duality is given by a kind of residue. It is therefore reasonable to define the residue map in the general case so that this continues to hold.

**22.1. Abstract duality.** It will be convenient to start by considering a more abstract situation. Fix a ground ring  $k$ , and write  $\text{Hom}(M, N)$  for  $\text{Hom}_k(M, N)$  and  $M \otimes N$  for  $M \otimes_k N$ . Let  $A$  be a  $k$ -algebra that is a finitely generated projective module of rank  $r$  over  $k$ , and write

$$\begin{aligned} M^\vee &= \text{Hom}(M, k) = \text{Hom}_k(M, k) \\ N^* &= \text{Hom}_A(N, A). \end{aligned}$$

If  $M$  is an  $A$ -module, then we make  $M^\vee$  an  $A$ -module by the usual rule  $(a\phi)(m) = \phi(am)$ .

Now let  $I$  be the kernel of the multiplication map  $\mu: A \otimes A \rightarrow A$ , and let  $J$  be the annihilator of  $I$  in  $A \otimes A$ , and put  $B = (A \otimes A)/J$ . Assume that  $I$  and  $J$  are both principal.

Given a  $k$ -linear map  $\phi: A \rightarrow k$ , we get an  $A$ -linear map  $1 \otimes \phi: A \otimes A \rightarrow A$ , so we can define

$$\tilde{\phi} = \theta_0(\phi) = (1 \otimes \phi)|_J: J \rightarrow A.$$

This construction gives a map  $\theta_0: A^\vee \rightarrow J^*$ .

**THEOREM 22.1.** *The  $A$ -modules  $A^\vee$  and  $J$  are both free of rank one (but without canonical generator) and the map  $\theta_0: A^\vee \rightarrow J^*$  is an  $A$ -linear isomorphism.*

The rest of this section constitutes the proof.

**LEMMA 22.2.** *The map  $\theta_0$  is  $A$ -linear, and the adjoint map  $\theta_1: J \rightarrow A^{\vee*}$  is an isomorphism.*

**PROOF.** First suppose that  $a \in A$  and  $\phi \in A^\vee$  and  $u \in J$ ; we must show that  $(1 \otimes a\phi)(u) = a((1 \otimes \phi)(u))$ . From the definitions we have  $(1 \otimes a\phi)(u) = (1 \otimes \phi)((1 \otimes a)u)$ , and  $(1 \otimes \phi)((a \otimes 1)u) = a((1 \otimes \phi)(u))$ , so it will suffice to show that  $(1 \otimes a)u = (a \otimes 1)u$ . This holds because  $1 \otimes a - a \otimes 1 \in I$  and  $IJ = 0$ . We now see that  $\theta_0$  is  $A$ -linear, which allows us to define the adjoint map  $\theta_1: J \rightarrow A^{\vee*}$  by  $\theta_1(u)(\phi) = \theta_0(\phi)(u) = (1 \otimes \phi)(u)$ .

Next, as  $A$  is  $k$ -projective, we have  $A \otimes A = \text{Hom}(A^\vee, A)$ . If an element  $u \in A \otimes A$  corresponds to a map  $\alpha: A^\vee \rightarrow A$ , then  $(1 \otimes a)u$  corresponds to the

map  $x \mapsto a\alpha(x)$ , and  $(a \otimes 1)u$  corresponds to the map  $x \mapsto \alpha(ax)$ . It follows that  $(1 \otimes a - a \otimes 1)u = 0$  iff  $\alpha(ax) = a\alpha(x)$  for all  $x \in A$ . As  $I$  is generated by elements of the form  $1 \otimes a - a \otimes 1$ , we find that

$$A^{\vee*} = \text{Hom}_A(A^\vee, A) = \text{ann}(I, \text{Hom}(A^\vee, A)) = \text{ann}(I, A \otimes A) = J.$$

One can check that the isomorphism arising from this argument is just  $\theta_1$ .  $\square$

REMARK 22.3. It follows immediately that if  $A^\vee$  has an inverse as an  $A$ -module, then that inverse must be  $J$ , and  $\theta_0$  must be an isomorphism.

We now define  $\eta_0, \eta_1: A \rightarrow A \otimes A$  by

$$\begin{aligned}\eta_0(a) &= a \otimes 1 \\ \eta_1(a) &= 1 \otimes a.\end{aligned}$$

We regard  $A \otimes A$  as an  $A$ -algebra (and thus  $I$ ,  $J$  and  $B$  as  $A$ -modules) via the map  $\eta_0$ .

LEMMA 22.4. *The  $A$ -modules  $I$  and  $B$  are projective, both with rank  $r - 1$ . Moreover,  $J$  is free of rank one as an  $A$ -module.*

PROOF. As  $A$  is projective over  $k$  with rank  $r$ , we see that  $A \otimes A$  is projective over  $A$  with rank  $r$ . There is a short exact sequence  $I \rightarrow A \otimes A \xrightarrow{\mu} A$ , that is  $A$ -linearly split by  $\eta_0$ . It follows that  $I$  is projective over  $A$ , with rank  $r - 1$ . As  $I$  is principal with annihilator  $J$ , we see that  $I \simeq (A \otimes A)/J = B$  as  $A$ -modules, so  $B$  is also projective of rank  $r - 1$ . It follows that the short exact sequence  $J \rightarrow A \otimes A \rightarrow B$  is  $A$ -linearly split, and so  $J$  is projective of rank one. It is also a principal ideal and thus a cyclic module, so it must in fact be free of rank one.  $\square$

Next, write  $\lambda^t$  for the  $t$ 'th exterior power functor, and observe that  $\eta_1$  induces a  $k$ -linear map

$$\hat{\eta}_1: \lambda^{r-1}A \rightarrow A \otimes \lambda^{r-1}A = \lambda_A^{r-1}(A \otimes A),$$

and the projection  $q: A \otimes A \rightarrow B$  induces a map  $\lambda^{r-1}(q): \lambda_A^{r-1}(A \otimes A) \rightarrow \lambda_A^{r-1}B$ . We define  $\psi: \lambda^{r-1}A \rightarrow \lambda_A^{r-1}B$  to be the composite of these two maps, so

$$\psi(a_1 \wedge \cdots \wedge a_{r-1}) = q(1 \otimes a_1) \wedge_A \cdots \wedge_A q(1 \otimes a_{r-1}).$$

LEMMA 22.5. *The map  $\psi: \lambda^{r-1}A \rightarrow \lambda_A^{r-1}B$  is an isomorphism.*

PROOF. One can see from the definitions that the image of  $\psi$  generates  $\lambda_A^{r-1}B$  as an  $A$ -module. We will start by showing that the image is itself an  $A$ -submodule, so  $\psi$  must be surjective.

First, we show that  $\lambda^{r-1}A$  has a natural structure as an  $A$ -module. Indeed, there is an evident multiplication  $\nu: A \otimes \lambda^{r-1}A \rightarrow \lambda^r A$ , which induces an isomorphism  $\nu^\#: \lambda^{r-1}A \simeq \text{Hom}(A, \lambda^r A)$ . The  $A$ -module structure on  $A$  gives an  $A$ -module structure on  $\text{Hom}(A, \lambda^r A)$  which can thus be transported to  $\lambda^{r-1}A$ . More explicitly, there is a unique bilinear operation  $*$ :  $A \otimes \lambda^{r-1}A \rightarrow \lambda^{r-1}A$  such that  $a \wedge (b * u) = (ab) \wedge u$  for all  $a, b \in A$  and  $u \in \lambda^{r-1}A$ .

This in turn gives an  $A \otimes A$ -module structure on the group  $A \otimes \lambda^{r-1}A = \lambda_A^{r-1}(A \otimes A)$ , by the formula  $(a \otimes b) * (c \otimes u) = (ac) \otimes (b * u)$ . It follows that

$$\hat{\eta}_1(b * u) = \eta_1(b) * \hat{\eta}_1(u).$$

We next claim that  $\lambda_A^{r-1}B$  is a quotient  $A \otimes A$ -module of  $\lambda_A^{r-1}(A \otimes A)$ , and is annihilated by  $I$ . Indeed, we certainly have an  $A$ -module structure and  $(A \otimes A)/I =$

$A$  so it will suffice to show that the map  $\lambda^{r-1}(q): \lambda_A^{r-1}(A \otimes A) \rightarrow \lambda_A^{r-1}B$  annihilates  $I * \lambda_A^{r-1}(A \otimes A)$ . To see this, choose an element  $e$  that generates  $J$  (so  $Ie = 0$ ). Using the splittable exact sequence  $J \rightarrow A \otimes A \xrightarrow{q} B$ , we see that there is a commutative diagram as follows, in which  $\chi$  is an isomorphism.

$$\begin{array}{ccc} \lambda_A^{r-1}(A \otimes A) & \xrightarrow{e \wedge (-)} & \lambda_A^r(A \otimes A) \\ \lambda^{r-1}(q) \downarrow & \nearrow \chi & \\ \lambda_A^{r-1}B & \simeq & \end{array}$$

Now for  $u \in \lambda_A^{r-1}(A \otimes A)$  we have  $e \wedge (I * u) = (Ie) \wedge u = 0$ , so  $\lambda^{r-1}(q)(I * u) = 0$  as required.

We can now apply the map  $\lambda^{r-1}(q)$  to the equation  $\hat{\eta}_1(b * u) = \eta_1(b) * \hat{\eta}_1(u)$  to see that our map  $\psi: \lambda^{r-1}A \rightarrow \lambda_A^{r-1}B$  is  $A$ -linear. In particular, the image of  $\psi$  is an  $A$ -submodule, and thus  $\psi$  is surjective as explained previously.

Next, note that  $A$  is a projective  $k$ -module of rank  $r$ , so the same is true of  $\lambda^{r-1}A$ . On the other hand,  $B$  is a projective  $A$ -module of rank  $r - 1$ , so  $\lambda_A^{r-1}B$  is a projective  $A$ -module of rank one, and thus also a projective  $k$ -module of rank  $r$ . Thus  $\psi$  is a surjective map between projective  $k$ -modules of the same finite rank, so it is necessarily an isomorphism.  $\square$

EXAMPLE 22.6. It is illuminating to see how this works out in the case where  $A = k[x]/f(x)$ , where  $f(x) = \sum_{i=0}^r a_i x^{r-i}$  is a monic polynomial of degree  $r$ . We write  $x_0$  for  $x \otimes 1$  and  $x_1$  for  $1 \otimes x$ , so  $A \otimes A = k[x_0, x_1]/(f(x_0), f(x_1))$ . Put

$$\begin{aligned} d(x_0, x_1) &= x_1 - x_0 \\ e(x_0, x_1) &= (f(x_1) - f(x_0))/(x_1 - x_0) = \sum_{i+j < r} a_{r-i-j-1} x_0^i x_1^j. \end{aligned}$$

One checks that  $I$  is generated by  $d$  and  $J$  is generated by  $e$ . Put

$$u_i = (-1)^i x^0 \wedge \dots \wedge \hat{x}^i \wedge \dots \wedge x^{r-1},$$

so  $\{u_0, \dots, u_{r-1}\}$  is a basis for  $\lambda^{r-1}A$  over  $k$ . If we put  $v = x^0 \wedge \dots \wedge x^{r-1} \in \lambda^r A$ , then  $x^i \wedge u_j = \delta_{ij}v$ , so  $x_1^i \wedge \hat{\eta}_1(u_j) = \delta_{ij}\hat{\eta}_1(v)$ . Using this, we find that

$$\chi\psi(u_j) = e \wedge \hat{\eta}_1(u_j) = \left( \sum_{i=k}^{r-1} a_{i-k} x^{r-1-i} \right) \hat{\eta}_1(v).$$

This means that the matrix of the map  $\chi\psi: \lambda^{r-1}A \rightarrow \lambda_A^r(A \otimes A)$  (with respect to the obvious bases) is triangular, with ones on the diagonal. The map  $\chi\psi$  is thus an isomorphism, and the same is true of  $\chi$ , so  $\psi$  is an isomorphism as expected.

PROOF OF THEOREM 22.1. First, suppose we have a map  $\phi: A \rightarrow k$ . It is well-known that there is a unique derivation  $i_\phi$  of the exterior algebra  $\lambda^*A$  whose effect on  $\lambda^1A$  is just the map  $\lambda^1A = A \xrightarrow{\phi} k = \lambda^0A$  (this is called *interior multiplication* by  $\phi$ ). We write  $\zeta(\phi)$  for the map  $i_\phi: \lambda^r A \rightarrow \lambda^{r-1}A$ . This construction gives a map  $\zeta: A^\vee \rightarrow \text{Hom}(\lambda^r A, \lambda^{r-1}A)$ . If we have a basis for  $A$  then we find that  $\zeta$  sends the obvious basis of  $A^\vee$  to the obvious basis for  $\text{Hom}(\lambda^r A, \lambda^{r-1}A)$  (up to sign), so  $\zeta$  is an isomorphism. In general,  $A$  need only be projective over  $k$  but we can still choose a basis Zariski-locally on  $\text{spec}(k)$  and the argument goes through. It follows that  $\zeta$  is always an isomorphism.

Next, as mentioned above, the short exact sequence  $J \rightarrow A \otimes A \rightarrow B$  gives  $J \otimes_A \lambda_A^{r-1} B = \lambda_A^r(A \otimes A) = A \otimes \lambda^r A$ . As the modules  $J$ ,  $\lambda_A^{r-1} B$  and  $A \otimes \lambda^r A$  are all dualisable, we deduce that

$$\begin{aligned} J^* &= \operatorname{Hom}_A(A \otimes \lambda^r A, \lambda_A^{r-1} B) \\ &= \operatorname{Hom}(\lambda^r A, \lambda_A^{r-1} B) \\ &= \operatorname{Hom}(\lambda^r A, \lambda^{r-1} A) \\ &= A^\vee. \end{aligned}$$

In particular, we see that  $A^\vee$  is an invertible  $A$ -module, so Remark 22.3 tells us that the map  $\theta_0$  must be an isomorphism. (In fact, the above chain of identifications implicitly constructs an isomorphism  $\theta'_0: A^\vee \rightarrow J^*$ , and one could presumably check directly that  $\theta'_0 = \theta_0$ , but we have not done so.)  $\square$

DEFINITION 22.7. The isomorphism  $J^* = A^\vee$  gives  $J^{*\vee} = A^{\vee\vee} = A$ ; we let  $\epsilon: J^* \rightarrow k$  be the element of  $J^{*\vee}$  corresponding to  $1 \in A$  under this isomorphism. Equivalently,  $\epsilon$  is the unique map such that  $\epsilon(\theta_0(\phi)) = \epsilon((1 \otimes \phi)|_J) = \phi(1)$  for all  $\phi \in A^\vee$ .

We will prove later that in cases arising from topology, the map  $\epsilon$  can be identified with a Gysin map. We conclude this section with an algebraic characterisation of  $\epsilon$  that will be the basis of the proof.

CONSTRUCTION 22.8. Given  $\lambda \in J^*$  and  $a \in A$  we can define a map  $m(a \otimes \lambda): J \rightarrow A \otimes A$  by  $e \mapsto a \otimes \lambda(e)$ . This map is  $A$ -linear if we use the second copy of  $A$  to make  $A \otimes A$  into an  $A$ -module. Thus, this construction gives a map  $m: A \otimes J^* \rightarrow \operatorname{Hom}_A(J, A \otimes A)$ , and as  $A$  is projective over  $k$ , this is easily seen to be an isomorphism. Under the inverse of this isomorphism, the inclusion  $J \rightarrow A \otimes A$  corresponds to an element  $u \in A \otimes J^*$ . Lemma 22.11 will give a more concrete description of this element.

PROPOSITION 22.9. *The map  $\epsilon: J^* \rightarrow k$  is such that  $(1 \otimes \epsilon)(u) = 1$  (where  $u$  is as constructed above). Moreover,  $\epsilon$  is the unique map with this property.*

The proof will follow after some discussion.

It is convenient to choose a generator  $e = \sum_i a_i \otimes b_i$  for  $J$ , and a dual generator  $\eta$  for  $J^*$ , so  $\eta(e) = 1$ . We then put  $\psi = \theta_0^{-1}(\eta) \in A^\vee$ ; this is the unique map  $\psi: A \rightarrow k$  such that  $(1 \otimes \psi)(e) = 1$ .

LEMMA 22.10. *We have  $\psi(a) = \epsilon(a\eta)$  for all  $a \in A$ .*

PROOF. Using the  $A$ -linearity of  $\theta_0$  and the fact that  $\epsilon\theta_0(\phi) = \phi(1)$ , we see that

$$\epsilon(a\eta) = \epsilon\theta_0(a\psi) = (a\psi)(1) = \psi(a).$$

$\square$

LEMMA 22.11. *The element  $u$  in Construction 22.8 is given by*

$$u = e.(1 \otimes \eta) = \sum_i a_i \otimes (b_i \eta).$$

PROOF. The element  $v := e.(1 \otimes \eta)$  corresponds to the map  $i = m(v): J \rightarrow A \otimes A$  given by  $i(x) = \sum_i a_i \otimes (b_i \eta(x))$ . In particular, we have  $i(e) = \sum_i a_i \otimes b_i = e$ , so  $i$  is the inclusion, so  $v = u$ .  $\square$

PROOF OF PROPOSITION 22.9. Recall that  $\epsilon$  is the image of 1 under an isomorphism  $A \simeq A^{\vee\vee} \simeq J^{*\vee}$ , so it certainly generates  $J^{*\vee}$ . It will thus suffice to check that  $(1 \otimes (t\epsilon))(u) = t$  for all  $t \in A$ . The calculation is as follows:

$$\begin{aligned} (1 \otimes (t\epsilon))(u) &= \sum_i a_i \epsilon(tb_i \eta) \\ &= \sum_i a_i \psi(tb_i) \\ &= (1 \otimes \psi)((1 \otimes t)e) \\ &= (1 \otimes \psi)((t \otimes 1)e) \\ &= t \cdot (1 \otimes \psi)(e) \\ &= t. \end{aligned}$$

The first equality is Lemma 22.11, and the second is Lemma 22.10. The fourth equality holds because  $(1 \otimes t)e = (t \otimes 1)e$ , and the last equality is essentially the definition of  $\psi$ .  $\square$

**22.2. Duality for divisors.** Now consider a divisor  $D = \text{spec}(A)$  on a multicurve  $C = \text{spf}(R)$  over a scheme  $S = \text{spec}(k)$ . In this section, we explain and prove the following theorem.

**THEOREM 22.12.** *For any divisor  $D = \text{spec}(\mathcal{O}_C/I_D)$ , there is a natural isomorphism*

$$\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S) = (I_D^{-1}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega.$$

(The right hand side consists of meromorphic differential forms whose polar divisor is less than or equal to  $D$ , modulo holomorphic differential forms; it is easily seen to be free of rank one over  $\mathcal{O}_D$ .) Moreover, if a map  $\phi: \mathcal{O}_D \rightarrow \mathcal{O}_S$  corresponds to a meromorphic form  $\mu$ , then  $\phi(1) = \text{res}(\mu)$ .

The proof is postponed to the end of the section. The last part of the theorem is not yet meaningful, as we have not defined residues. The definition will be such as to make the claim trivial, but we will also check that the definition is compatible with the usual one in the case of embeddable multicurves.

The first step in proving the theorem is to show that the theory in the previous section is applicable.

**DEFINITION 22.13.** Let  $X$  be a scheme over  $S$ , with closed subschemes  $Y$  and  $Z$ . Suppose that  $\mathcal{O}_X$ ,  $\mathcal{O}_Y$  and  $\mathcal{O}_Z$  are all finitely generated projective modules over  $\mathcal{O}_S$ . We say that  $Y$  and  $Z$  are *perfectly complementary* if

- (a) the ideals  $I_Y$  and  $I_Z$  are principal.
- (b)  $\text{ann}(I_Y) = I_Z$  and  $\text{ann}(I_Z) = I_Y$ .

**LEMMA 22.14.** *If  $D_0$  and  $D_1$  are divisors on a multicurve  $C$ , then  $D_0$  and  $D_1$  are perfectly complementary in  $D_0 + D_1$ .*

**PROOF.** Put  $A_i = \mathcal{O}_{D_i}$ ; this is a finitely generated projective module over  $\mathcal{O}_S$ , and has the form  $\mathcal{O}_C/f_i$  for some regular element  $f_i \in \mathcal{O}_C$ . We also put  $B = \mathcal{O}_C/(f_0 f_1) = \mathcal{O}_{D_0 + D_1}$ , so  $A_i = B/f_i$ .

Suppose that  $g f_1 = 0 \pmod{f_0 f_1}$ . Then  $(g - h f_0) f_1 = 0$  for some  $h \in \mathcal{O}_C$ , but  $f_1$  is regular, so  $g = h f_0 = 0 \pmod{f_0}$ . This shows that the annihilator of  $f_1$  in  $B$  is generated by  $f_0$ , and by symmetry, the annihilator of  $f_0$  is generated by  $f_1$ , and this proves the lemma.  $\square$

COROLLARY 22.15. *Let  $D$  be a divisor on a multicurve  $C$ . Then the diagonal subscheme  $\Delta \subseteq D \times_S D$  and the subscheme  $P_2 D \subseteq D \times_S D$  are perfectly complementary.*

PROOF. Let  $q: D \rightarrow S$  be the projection. We can regard  $\Delta$  and  $P_2 D$  as divisors on the multicurve  $q^* C$  over  $D$ , with  $D \times_S D = q^* D = \Delta + P_2 D$ , so the claim follows from the lemma.  $\square$

COROLLARY 22.16. *Let  $D$  be a divisor on a multicurve  $C$ , and put  $J = \ker(\mathcal{O}_{D^2} \rightarrow \mathcal{O}_{P_2 D})$ . Then  $J$  is a free  $\mathcal{O}_D$ -module of rank one, and there is a natural isomorphism*

$$\theta_0: \operatorname{Hom}_{\mathcal{O}_D}(J, \mathcal{O}_D) \rightarrow \operatorname{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S)$$

*given by  $\theta_0(\phi) = (1 \otimes \phi)|_J$ .*

PROOF. Let  $I$  be the kernel of the multiplication map  $\mathcal{O}_{D^2} = \mathcal{O}_D^{\otimes 2} \rightarrow \mathcal{O}_D$ , so that  $\mathcal{O}_\Delta = \mathcal{O}_{D^2}/I$ . Note that  $I$  is principal, generated by any difference function on  $C$ . We see from Corollary 22.15 that  $J$  is the annihilator of  $I$ , and that  $J$  is also principal. We can thus apply Theorem 22.1 to get the claimed isomorphism.  $\square$

To proceed further, we need a better understanding of the ideal  $J$ .

DEFINITION 22.17. For the rest of this section, we will use the following notation.

$$\begin{aligned} k &= \mathcal{O}_S \\ R &= \mathcal{O}_C \\ A &= \mathcal{O}_D \\ R_2 &= R \widehat{\otimes} R \\ A_2 &= A \otimes A \\ \tilde{I} &= \ker(R_2 \xrightarrow{\mu} R) \\ I &= \ker(A_2 \xrightarrow{\mu} A) \\ J &= \operatorname{ann}_{A_2}(I) \\ K &= I_D = \ker(R \rightarrow A) \\ K_2 &= K \widehat{\otimes} R + R \widehat{\otimes} K = \ker(R_2 \rightarrow A_2) \\ \tilde{J} &= \{u \in R_2 \mid u\tilde{I} \subseteq K_2\}. \end{aligned}$$

We will also choose a difference function  $d$  on  $C$  (so  $d \in \tilde{I}$ ), and let  $\alpha$  denote the image of  $d$  in  $\Omega$ . We choose a generator  $f$  of  $K$ , and we let  $e$  denote the unique element of  $R_2$  such that  $1 \otimes f - f \otimes 1 = de$ . We will check later that the image of  $e$  in  $A_2$  is a generator of  $J$ . After that, we will write  $\eta$  for the dual generator of  $J^*$ , and  $\psi$  for the corresponding element of  $A^\vee$ .

REMARK 22.18. It is clear that  $\tilde{J}$  is the preimage of  $J$  in  $R_2$ , so  $J = \tilde{J}/K_2$ . As  $\tilde{I}$  is generated by a single regular element, one can check that  $\tilde{I} \cap K_2 = \tilde{I}\tilde{J}$ .

DEFINITION 22.19. We define  $\chi_0: R \rightarrow \tilde{I}$  by  $\chi_0(a) = 1 \otimes a - a \otimes 1$ , and observe that

$$\chi_0(ab) = (a \otimes 1)\chi_0(b) + (1 \otimes b)\chi_0(a).$$

Given a difference function  $d$  on  $C$ , we let  $\xi(a) \in R_2$  denote the unique element such that  $\chi_0(a) = \xi(a)d$ . We again have

$$\xi(ab) = (a \otimes 1)\xi(b) + (1 \otimes b)\xi(a).$$

LEMMA 22.20. *There is a unique map  $\nu: K_2 \rightarrow K/K^2$  such that*

$$\nu(a \otimes b) = ab \text{ whenever } a \in K \text{ and } b \in R$$

$$\nu(a \otimes b) = 0 \text{ whenever } a \in R \text{ and } b \in K.$$

Moreover, we have  $\nu(K_2\tilde{I}) = 0$ .

PROOF. Using a  $k$ -linear splitting of the sequence  $K \rightarrow R \rightarrow A$ , we see that  $(K\hat{\otimes}R) \cap (R\hat{\otimes}K) = K\hat{\otimes}K$ , so we have

$$\frac{K_2}{R\hat{\otimes}K} = \frac{K\hat{\otimes}R}{K\hat{\otimes}K}.$$

The multiplication map  $\mu: R\hat{\otimes}R \rightarrow R$  evidently induces a map  $(K\hat{\otimes}R)/(K\hat{\otimes}K) \rightarrow K/K^2$ . Putting this together, we get a map

$$\nu = \left( K_2 \rightarrow \frac{K_2}{R\hat{\otimes}K} = \frac{K\hat{\otimes}R}{K\hat{\otimes}K} \xrightarrow{\mu} \frac{K}{K^2} \right).$$

It is clear that this is uniquely characterised by the stated properties. As  $\nu$  is essentially given by  $\mu$  on  $K\hat{\otimes}R$  and  $\mu(\tilde{I}) = 0$ , we see that  $\nu(\tilde{I}(K\hat{\otimes}R)) = 0$ . We also have  $\nu(R\hat{\otimes}K) = 0$  and so  $\nu(\tilde{I}(R\hat{\otimes}K)) = 0$ , so  $\nu(\tilde{I}K_2) = 0$  as claimed.  $\square$

PROPOSITION 22.21. *The map  $\xi$  induces a  $A$ -linear isomorphism  $K/K^2 \rightarrow J$ , so  $J$  is freely generated over  $A$  by the element  $e = \xi(f)$ . (Note however that this isomorphism is not canonical, because it depends on the choice of  $d$ .)*

PROOF. Suppose that  $a \in K$ , so  $\chi_0(a) \in K_2$ . As  $\tilde{I} = R_2d$  we see that  $\xi(a)\tilde{I} = R_2\chi_0(a) \subseteq K_2$ , so  $\xi(K) \subseteq \tilde{J}$ , so we get an induced map  $K \rightarrow \tilde{J}/K_2 = J$ . Using the product formula for  $\xi$ , we deduce that this map is  $R$ -linear and induces a map  $K/K^2 \rightarrow J$ . In the opposite direction, we define  $\zeta: \tilde{J} \rightarrow K/K^2$  by  $\zeta(u) = \nu(ud)$ . If  $u \in K_2$  then  $ud \in \tilde{I}K_2$  so  $\nu(ud) = 0$ . Thus,  $\zeta$  induces a map  $J = \tilde{J}/K_2 \rightarrow K/K^2$ . It is easy to see that  $\zeta\xi = 1: K/K^2 \rightarrow K/K^2$ , and both  $J$  and  $K/K^2$  are invertible  $A$ -modules, so  $\zeta$  and  $\xi$  must be mutually inverse isomorphisms. It follows immediately that  $J$  is freely generated by  $e$ .  $\square$

Our next task is to reformulate the above isomorphism in a way that is independent of any choices.

PROPOSITION 22.22. *Put*

$$\overline{\Omega} = \Omega|_D = \Omega \otimes_R A = \tilde{I} \otimes_{R_2} A.$$

*There is a natural isomorphism  $\chi: K/K^2 \rightarrow J \otimes_R \Omega = J \otimes_A \overline{\Omega}$  of  $A$ -modules, satisfying  $\chi(a) = \xi(a) \otimes \alpha$ , where  $\alpha$  is the image of  $d$  in  $\Omega$ . By adjunction, there is also a natural isomorphism  $\chi': J^* \rightarrow (K/K^2)^* \otimes_A \overline{\Omega} = (K/K^2)^* \otimes_R \Omega$ .*

PROOF. We have

$$J \otimes_A \overline{\Omega} = J \otimes_A (A \otimes_{R_2} \tilde{I}) = (\tilde{J}/K_2) \otimes_{R_2} \tilde{I} = (\tilde{I}\tilde{J})/(\tilde{I}K_2) = (\tilde{I} \cap K_2)/(\tilde{I}K_2).$$

We have seen that the map  $\chi$  sends  $K$  to  $\tilde{I} \cap K_2$  and  $K^2$  to  $\tilde{I}K_2$ , so it induces a map  $\chi: K/K^2 \rightarrow J \otimes_A \overline{\Omega}$ , which is obviously canonical. One checks from the

definitions that  $\chi(a) = \xi(a) \otimes \alpha$ , where  $\xi$  and  $\alpha$  are defined in terms of a difference function  $d$  as above. As  $\xi$  is an isomorphism and  $\overline{\Omega}$  is freely generated by  $\alpha$  over  $A$ , we conclude that  $\chi$  is an isomorphism.  $\square$

Our next task is to interpret the module  $(K/K^2)^*$ .

**DEFINITION 22.23.** Let  $C$  be a formal multicurve over  $S$ . We say that an element  $f \in \mathcal{O}_C$  is *divisorial* if it is not a zero-divisor, and  $\mathcal{O}_C/f$  is a projective  $\mathcal{O}_S$ -module of finite rank. One can check that the set of divisorial elements is closed under multiplication, so we can invert it to get a new ring  $\mathcal{M}_C$ , whose elements we call *meromorphic functions*. We say that a meromorphic function is *divisorial* if it can be written as  $f/g$ , where  $f$  and  $g$  are divisorial elements of  $\mathcal{O}_C$  (this can be seen to be compatible with the previous definition). A *fractional ideal* is an  $\mathcal{O}_C$ -submodule  $I \leq \mathcal{M}_C$  that can be generated by a divisorial meromorphic function. The set of fractional ideals forms a group under multiplication, with  $I^{-1} = \{f \in \mathcal{M}_C \mid fI \subseteq \mathcal{O}_C\}$ .

**LEMMA 22.24.** *There is a natural isomorphism  $(K/K^2)^* = (K^{-1}/R)$ . With respect to this, the generator  $f$  of  $K/K^2$  is dual to the generator  $1/f$  of  $K^{-1}/R$ .*

**PROOF.** The multiplication map  $K \widehat{\otimes}_R K^{-1} \rightarrow R$  induces a map  $(K/K^2) \otimes_A (K^{-1}/R) \rightarrow A$ , and thus a map  $K^{-1}/R \rightarrow (K/K^2)^*$ . This is easily seen to be an isomorphism. The statement about generators is clear.  $\square$

**PROOF OF THEOREM 22.12.** Everything except the last part now follows immediately from Corollary 22.16, Proposition 22.22 and Lemma 22.24. The residue map will be defined in Definition 22.31, and then the last part of the theorem will be true by definition.  $\square$

We can make the theorem more explicit as follows.

**PROPOSITION 22.25.** *The natural isomorphism*

$$A^\vee \xrightarrow{\theta_0} J^* \xrightarrow{\chi'} K^{-1}/R \otimes_A \Omega$$

*sends  $\phi$  to the element  $((1 \otimes \phi)(e)/f) \otimes \alpha$ .*

**PROOF.** Using Proposition 22.21, we see that  $e$  generates  $J$ , so there is a unique element  $\eta \in J^*$  with  $\eta(e) = 1$ . The natural isomorphism  $K/K^2 \rightarrow J \otimes_A \Omega$  sends  $f$  to  $e \otimes \alpha$ , so the adjoint map  $\chi': J^* \rightarrow K^{-1}/R \otimes \Omega$  sends  $\eta$  to  $(1/f) \otimes \alpha$ , and thus sends  $a\eta$  to  $(a/f) \otimes \alpha$ . Next, we certainly have  $\theta_0(\phi) = a\eta$  for some  $a \in A$ , and by evaluating this equation on  $e$  we find that  $a = (1 \otimes \phi)(e)$ . It follows that

$$\chi' \theta_0(\phi) = (a/f) \otimes \alpha = ((1 \otimes \phi)(e)/f) \otimes \alpha$$

as claimed.  $\square$

We next examine how this works in the case where  $C$  is embeddable, say  $C = \text{spf}(k[x]_{(g)}^\wedge)$  for some monic polynomial  $g$ . We then have  $K = Rf$  for some monic polynomial  $f$  that divides some power of  $g$ , and  $A = R/K = k[x]/f$ .

**DEFINITION 22.26.** Suppose we have a ring  $k$  and an expression  $\alpha = f(x)\mathbf{d}x = p(x)\mathbf{d}x/q(x)$ , where  $p$  and  $q$  are polynomials with  $q$  monic; we then define the residue  $\text{res}(\alpha)$  as follows. Let  $R'$  denote the ring of series of the form  $u(x) = \sum_{n=-\infty}^N a_n x^n$  for some finite  $N$ . Clearly  $k[x] \subseteq R'$ . Moreover, if  $q(x)$  is a monic polynomial then we can write  $q(x) = x^n r(1/x)$  for some polynomial  $r(t)$  with  $r(0) = 1$ . It follows



that  $r(1/x)$  is invertible in  $R'$ , and thus the same is true of  $q(x)$ , so we can expand out  $f(x) = p(x)/q(x)$  as an element of  $R'$ , say  $f(x) = \sum_{n=-\infty}^N a_n x^n$ . We then put

$$\text{res}(\alpha) = a_{-1}.$$

REMARK 22.27. In the case  $k = \mathbb{C}$ , one can check that  $\text{res}(\alpha)$  is the sum of the residues of  $\alpha$  at all its poles, so the integral of  $\alpha$  around any sufficiently large circle is  $2\pi i \text{res}(\alpha)$ . By standard arguments, most formulae that hold when  $k = \mathbb{C}$  will be valid for all  $k$ . In particular, we have

- $\text{res}(f(x)\mathbf{d}x) = 0$  if  $f$  is a polynomial
- $\text{res}(f'(x)\mathbf{d}x) = 0$  for any  $f = p/q$  as above
- $\text{res}(f'(x)\mathbf{d}x/f(x)) = \deg(p) - \deg(q)$  if  $f = p/q$  for some monic polynomials  $p$  and  $q$ .

LEMMA 22.28. Suppose that  $f(x) = p(x)/q(x)$  where  $q$  is monic of degree  $n$ , and

$$p(x) = \sum_{i=0}^{n-1} b_i x^i \pmod{q(x)}.$$

Then  $\text{res}(f(x)\mathbf{d}x) = b_{n-1}$ .

PROOF. First, put  $m(x) = \sum_{i=0}^{n-1} b_i x^i$ , so  $p(x) = m(x) + l(x)q(x)$  for some polynomial  $l(x)$ , so  $f(x) = m(x)/q(x) + l(x)$ . We have  $\text{res}(l(x)\mathbf{d}x) = 0$ , so it will suffice to show that  $\text{res}(m(x)/q(x)\mathbf{d}x) = b_{n-1}$ .

Next, write  $q(x) = x^n r(1/x)$  as in the definition, and put  $u(x) = 1/r(1/x)$ , so  $u(x) = \sum_{i=0}^{\infty} a_i x^{-i}$  for some coefficients  $a_i \in k$  with  $a_0 = 1$ . We have

$$x^j/q(x) = x^{j-n}u(x) = \sum_{i \geq 0} a_i x^{j-n-i},$$

so

$$\text{res}(x^j \mathbf{d}x/q(x)) = \begin{cases} 0 & \text{if } 0 \leq j < n-1 \\ 1 & \text{if } j = n-1. \end{cases}$$

The claim follows immediately. □

PROPOSITION 22.29. If  $A = k[x]/f(x)$ , then the map

$$(K^{-1}/R) \otimes \Omega \xrightarrow{(\chi')^{-1}} J^* \xrightarrow{\epsilon} k$$

is just the residue.

PROOF. As in example 22.6, we put

$$\begin{aligned} f(x) &= \sum_{i=0}^r a_i x^{r-i} \\ d(x_0, x_1) &= x_1 - x_0 \\ \alpha &= \mathbf{d}x \in \Omega \\ e(x_0, x_1) &= (f(x_1) - f(x_0))/(x_1 - x_0) = \sum_{i+j < r} a_{r-i-j-1} x_0^i x_1^j. \end{aligned}$$

This is of course compatible with the notation in Proposition 22.25, so  $\chi'\theta_0(\phi) = (1 \otimes \phi)(e)\mathbf{d}x/f(x)$ . The map  $\epsilon$  is characterised by the fact that  $\epsilon(\theta_0(\phi)) = \phi(1)$ , so it will suffice to check that  $\text{res}((1 \otimes \phi)(e)\mathbf{d}x/f(x)) = \phi(1)$  for all  $\phi \in A^\vee$ .

Now let  $\{\zeta_0, \dots, \zeta_{r-1}\}$  be the basis for  $A^\vee$  dual to the basis  $\{x^0, \dots, x^{r-1}\}$  for  $A$ . We then have  $(1 \otimes \zeta_j)(x_1^k) = \delta_{jk}$ , and it follows that

$$(1 \otimes \zeta_j)(e) = \sum_{i=0}^{r-1-j} a_{r-i-j-1} x^i.$$

Using Lemma 22.28 we deduce that  $\text{res}((1 \otimes \zeta_j)(e) \mathbf{d}x / f(x)) = 0$  for  $j > 0$ , whereas for  $j = 0$  we get  $a_0$ , which is 1 because the polynomial  $f(x) = \sum_{i+j=r} a_j x^j$  is monic. On the other hand, we also have  $\zeta_j(1) = \delta_{0j}$  by definition, so  $\text{res}((1 \otimes \zeta_j)(e) \mathbf{d}x / f(x)) = \zeta_j(1)$  as required.  $\square$

Given this, it would be reasonable to define residues on multicurves using the maps  $\epsilon$ . To make this work properly, we need to check that these maps are compatible for different divisors.

**PROPOSITION 22.30.** *Suppose we have divisors  $D_0 \subseteq D_1$ , corresponding to ideals  $K_1 \leq K_0 \leq R$ . Let  $j: K_0^{-1}/R \rightarrow K_1^{-1}/R$  be the evident inclusion, and let  $q: A_1 = R/K_1 \rightarrow R/K_0 = A_0$  be the projection. Define  $\delta_i: A_i^\vee \rightarrow k$  by  $\delta_i(\phi) = \phi(1)$ . Then the following diagram commutes:*

$$\begin{array}{ccccc} k & \xleftarrow{\delta_0} & A_0^\vee & \xrightarrow[\simeq]{\chi' \theta_0} & K_0^{-1}/R \otimes \Omega \\ \parallel & & \downarrow q^\vee & & \downarrow j \\ k & \xleftarrow[\delta_1]{} & A_1^\vee & \xrightarrow[\chi' \theta_0]{\simeq} & K_1^{-1}/R \otimes \Omega \end{array}$$

**PROOF.** As  $q(1) = 1$ , it is clear that the left hand square commutes. For the right hand square, choose generators  $f_i$  for  $K_i$  and put  $e_i = \xi(f_i)$ , so that

$$\chi' \theta_0(\phi) = ((1 \otimes \phi)(e_i) / f_i) \otimes \alpha$$

for  $\phi \in A_i^\vee$ .

As  $D_0 \subseteq D_1$ , we have  $f_1 = g f_0$  for some  $g$ , and so  $\xi(f_1) = (g \otimes 1)\xi(f_0) + (1 \otimes f_0)\xi(g)$ , or in other words  $e_1 = (g \otimes 1)e_0 + (1 \otimes f_0)\xi(g)$ .

Now suppose we have  $\phi \in A_0^\vee$ , so  $(q^* \phi)(f_0) = 0$ , so  $(1 \otimes q^* \phi)((1 \otimes f_0)\xi(g)) = 0$ . We then have

$$\begin{aligned} \chi' \theta_0 q^*(\phi) &= ((1 \otimes q^* \phi)(e_1) / f_1) \otimes \alpha \\ &= (g(1 \otimes \phi)(e_0)) / (g f_0) \otimes \alpha \\ &= (1 \otimes \phi)(e_0) / f_0 \otimes \alpha \\ &= j \chi' \theta_0(\phi) \end{aligned}$$

as claimed.  $\square$

**DEFINITION 22.31.** Define  $\delta: \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S) \rightarrow \mathcal{O}_S$  by  $\delta(\phi) = \phi(1)$ . We let  $\text{res}: \mathcal{M}_C \otimes_{\mathcal{O}_C} \Omega \rightarrow \mathcal{O}_S$  be the unique map whose restriction to  $I_D^{-1} \otimes_{\mathcal{O}_C} \Omega$  is the composite

$$I_D^{-1} \otimes_{\mathcal{O}_C} \Omega \rightarrow (I_D^{-1} / \mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega \simeq \text{Hom}_{\mathcal{O}_S}(\mathcal{O}_D, \mathcal{O}_S) \xrightarrow{\delta} \mathcal{O}_S.$$

(This is well-defined, by Proposition 22.30, and compatible with the classical definition, by Proposition 22.29.)

PROPOSITION 22.32. *For any  $g, f \in \mathcal{O}_C$ , if  $f$  is divisorial then*

$$\operatorname{res}((g/f)\mathbf{d}f) = \operatorname{trace}_{(\mathcal{O}_C/f)/\mathcal{O}_S}(g).$$

*In particular, we have  $\operatorname{res}((1/f)\mathbf{d}f) = \dim_{\mathcal{O}_S}(\mathcal{O}_C/f)$ . Moreover, we also have*

$$\operatorname{res}(\mathbf{d}(g/f)) = \operatorname{res}\left(\frac{f\mathbf{d}(g) - g\mathbf{d}(f)}{f^2}\right) = 0.$$

PROOF. Both facts are well-known for residues in the classical sense, so they hold whenever  $C$  is embeddable. Using Corollary 16.3, we deduce that they hold for a general multcurve  $C$ . We will give a more direct and illuminating proof for the first fact; we have not been able to find one for the second fact.

We use abbreviated notation as before, with  $K = R/f$  so that  $A = R/f$ . The multiplication map  $\mu: A_2 \rightarrow A$  restricts to give an  $A$ -linear map  $\mu: J \rightarrow A$ , or in other words an element of  $J^*$ . The trace map  $\tau: A \rightarrow k$  can be regarded as an element of  $A^\vee$ . We claim that the elements  $\tau$ ,  $\mu$  and  $(\mathbf{d}f)/f$  correspond to each other under our standard isomorphisms

$$A^\vee \simeq J^* \simeq (K^{-1}/R) \otimes \Omega.$$

To see this, note that  $(1 \otimes \tau)(u) = \operatorname{trace}_{A_2/A}(u)$  for all  $u \in A_2$ . Using the splittable short exact sequence

$$I \rightarrow A_2 \xrightarrow{\mu} A,$$

we see that

$$(1 \otimes \tau)(u) = \operatorname{trace}(I \xrightarrow{\times u} I) + \mu(u).$$

If  $u \in J$  then multiplication by  $u$  is zero on  $I$  and we deduce that  $(1 \otimes \tau)(u) = \mu(u)$ . This shows that  $\theta_0(\tau) = \mu$  as claimed.

Next, let  $e = \xi(f)$  be the standard generator of  $J$ , and let  $\eta$  be the dual generator of  $J^*$ , so  $\eta(e) = 1$ . Using Proposition 22.25, we see that  $\mu$  corresponds to the element  $(\mu(e)/f) \otimes \alpha = (1/f) \otimes (\mu(e)\alpha)$  in  $(K^{-1}/R) \otimes \Omega$ . Now, the module  $\Omega = \tilde{I}/\tilde{I}^2$  is originally a module over  $R_2$  that happens to be annihilated by  $\ker(\mu) = \tilde{I}$ , and so is regarded as a module over  $R$  via  $\mu$ . Thus,  $\mu(e)\alpha$  is just the same as  $e\alpha$ . Moreover,  $\alpha$  is just the image of  $d$  in  $\Omega$ , so  $e\alpha$  is the image of  $ed = \xi(f)d = 1 \otimes f - f \otimes 1$ , and this image is by definition just  $\mathbf{d}f$ . Thus,  $\mu \in J^*$  corresponds to  $(1/f) \otimes \mathbf{d}f$  as claimed.

As the isomorphism  $A^* \rightarrow (K^{-1}/R) \otimes \Omega$  is  $A$ -linear, we see that  $g\tau$  maps to  $(g/f)\mathbf{d}f$ , so

$$\operatorname{res}((g/f)\mathbf{d}f) = (g\tau)(1) = \tau(g),$$

as claimed.  $\square$

REMARK 22.33. It is useful and interesting to reconcile this result with [28, Proposition 9.2]. There we have a  $p$ -divisible formal group  $\widehat{C} = \operatorname{spf}(R)$  of height  $n$  over a formal scheme  $S = \operatorname{spf}(k)$ , where  $k$  is a complete local Noetherian ring of residue characteristic  $p$ , and we will assume for simplicity that  $k$  is torsion-free. Fix  $m \geq 1$  and let  $\psi: \widehat{C} \rightarrow \widehat{C}$  be  $p^m$  times the identity map. In this context the subgroup scheme  $D := \widehat{C}[p^m] = \ker(\psi)$  is a divisor of degree  $p^{nm}$ , so the ring  $\mathcal{O}_D$  is self-dual (with a twist) as before. Given any coordinate  $x$ , we note that  $\mathcal{O}_D = R/\psi^*x$ , so the meromorphic form  $\alpha = \mathbf{D}(x)/(\psi^*x)$  is a generator of the twisting module  $(K^{-1}/R) \otimes \Omega$ . We claim that  $\alpha$  is actually independent of  $x$ . Indeed, any other coordinate  $x'$  has the form  $x' = (x + x^2q)u$  for some  $u \in k^\times$  and  $q \in R$ . It follows that  $\mathbf{d}_0(x') = u\mathbf{d}_0(x)$ , so that  $\mathbf{D}(x') = u\mathbf{D}(x)$ . We also have  $\psi^*(x') =$

$u\psi^*(x) \pmod{\psi^*(x)^2}$ , and it follows that  $\psi^*(x')^{-1} = u^{-1}\psi^*(x)^{-1} \pmod{R}$ , so  $\mathbf{D}(x')/\psi^*(x') = \mathbf{D}(x)/\psi^*(x) \pmod{\text{holomorphic forms}}$ , as claimed. Thus, we have a canonical generator for  $(K^{-1}/R) \otimes \Omega$  and thus a canonical generator for  $A^\vee$ , giving a Frobenius form on  $\mathcal{O}_D$ . The cited proposition says that this is the same as the Frobenius form coming from a transfer construction. As discussed in the preamble to that proposition,  $p^m$  times the transfer form is the same as the trace form. In view of Proposition 22.32, this means that  $p^m\alpha = \mathbf{d}(\psi^*x)/(\psi^*x)$ . In fact, this is easy to see directly. We know that  $\mathbf{D}(x)$  generates  $\Omega$  and agrees with  $\mathbf{d}(x)$  at zero, so  $\mathbf{d}(x) = (1+xr)\mathbf{D}(x)$  for some function  $r \in R$ . It follows that

$$\mathbf{d}(\psi^*x) = \psi^*(\mathbf{d}(x)) = (1 + \psi^*(x)\psi^*(r))\psi^*(\mathbf{D}(x)).$$

As  $\mathbf{D}(x)$  is invariant we have  $\psi^*\mathbf{D}(x) = p^m\mathbf{D}(x)$ . It follows that  $\mathbf{d}(\psi^*x)/(\psi^*x) = p^m\mathbf{D}(x)/(\psi^*x) = p^m\alpha$  in  $(K^{-1}/R) \otimes \Omega$ , as claimed.

REMARK 22.34. It should be possible to connect our treatment of residues with that of Tate [31]. However, Tate assumes that the ground ring  $k$  is a field, and it seems technically awkward to remove this hypothesis. In any case, Proposition 22.32 should be sufficient to ensure that our residues coincide with Tate's in cases where both are defined, although we have not formulated or checked a rigorous statement.

**22.3. Topological duality.** Consider a periodically orientable theory  $E$ , an  $A$ -space  $X$ , and an equivariant complex bundle  $V$  over  $X$ . To avoid some minor technicalities, we will assume that  $X$  is a finite  $A$ -CW complex; everything can be generalised to the infinite case by passage to (co)limits. Let  $C$  be the multicurve  $\text{spf}(E^0(PU \times X))$  over  $S := \text{spec}(E^0X)$ . We then have a divisor  $D = D(V)$  on  $C$ , to which we can apply all the machinery in the previous section. In particular, this gives us a residue map

$$\text{res}: (I_D^{-1}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega \rightarrow \mathcal{O}_S.$$

On the other hand, if we let  $\tau$  denote the tangent bundle along the fibres of  $PV$ , then there is a stable Pontrjagin-Thom collapse map  $X_+ \rightarrow PV^{-\tau}$ , giving a Gysin map

$$p_!: \tilde{E}^0 PV^{-\tau} \rightarrow E^0 X = \mathcal{O}_S.$$

THEOREM 22.35. *There is a natural isomorphism  $\tilde{E}^0 PV^{-\tau} = (I_D^{-1}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} \Omega$ , which identifies the Gysin map with the residue map.*

This is an equivariant generalisation of a result stated by Quillen in [25]. Even in the nonequivariant case, we believe that there is no published proof. The rest of this section constitutes the proof of our generalisation. (The case of nonequivariant ordinary cohomology is easy, and is a special case of the result proved in [9].)

We retain our previous notation for rings, and write  $P^2V = PV \times_X PV$ , so

$$\begin{aligned} k &= \mathcal{O}_S = E^0 X \\ R &= \mathcal{O}_C = E^0(PU \times X) \\ A &= \mathcal{O}_D = E^0(PV) \\ R_2 &= R \hat{\otimes} R = E^0(PU \times PU \times X) \\ A_2 &= A \otimes A = E^0(P^2V). \end{aligned}$$

Next, observe that  $P_2V$  is a subspace of  $P^2V$ , and Proposition 21.10 tells us that  $E^0P_2V = \mathcal{O}_{P_2D} = A_2/J$ , so  $J = E^0(P^2V, P_2V)$ . On the other hand, there is another natural description of  $E^0(P^2V, P_2V)$ , which we now discuss. Let  $T$  be the tautological bundle on  $PV$ , consider the vector bundles  $T^\perp = V \ominus T$  and  $U = \text{Hom}(T, T^\perp)$ , and let  $B^\circ U$  denote the open unit ball bundle in  $U$ . A point in  $B^\circ U$  is a triple  $(x, L, \alpha)$  where  $x \in X$  and  $L \in PV_x$  and  $\alpha: L \rightarrow V_x \ominus L$ , such that  $\|\alpha(u)\| < \|u\|$  for all  $u \in L \setminus \{0\}$ . We can thus consider  $\text{graph}(\alpha)$  and  $\text{graph}(-\alpha)$  as one-dimensional subspaces of  $L \times (V_x \ominus L) = V_x$ , or in other words points of  $PV_x$ , so we have a map  $\delta': B^\circ U \rightarrow P^2V$  given by

$$\delta'(x, L, \alpha) = (\text{graph}(\alpha), \text{graph}(-\alpha)).$$

PROPOSITION 22.36. *The map  $\delta'$  is a diffeomorphism  $B^\circ U \rightarrow P^2V \setminus P_2V$ .*

PROOF. First, we must show that  $\delta'(x, L, \alpha) \notin P_2V$ , or in other words that  $\text{graph}(\alpha)$  is not perpendicular to  $\text{graph}(-\alpha)$ . For this, we choose a nonzero element  $u \in L$ , so  $v_0 = u + \alpha(u) \in \text{graph}(\alpha)$  and  $v_1 = u - \alpha(u) \in \text{graph}(-\alpha)$ . It follows that  $\langle v_0, v_1 \rangle = \|u\|^2 - \|\alpha(u)\|^2$ ; this is strictly positive because  $\|\alpha\| < 1$ , so the lines are not orthogonal, as required. We therefore have a map  $\delta': B^\circ U \rightarrow P^2V \setminus P_2V$ .

Any element of  $P^2V \setminus P_2V$  has the form  $(x, M_0, M_1)$  where  $x \in X$  and  $M_0, M_1 \in PV_x$  and  $M_0$  is not orthogonal to  $M_1$ . This means that we can choose elements  $u_i \in M_i$  with  $\|u_i\| = 1$  and such that  $t := \langle u_0, u_1 \rangle$  is a positive real number. One checks that the pair  $(u_0, u_1)$  is unique up to the diagonal action of  $S^1$ . Put  $v = u_0 + u_1$  and  $w = u_0 - u_1$ . By Cauchy-Schwartz we have  $t \leq 1$ , and by direct expansion we have

$$\begin{aligned} \langle v, w \rangle &= 0 \\ \langle v, v \rangle &= 2(1 + t) > 0 \\ \langle w, w \rangle &= 2(1 - t) < \langle v, v \rangle. \end{aligned}$$

We can thus put  $L = \mathbb{C}v \in PV_x$  and define  $\alpha: L \rightarrow L^\perp$  by  $\alpha(zv) = zw$ ; these are clearly independent of the choice of pair  $(u_0, u_1)$ . As  $\|w\| < \|v\|$  we have  $\|\alpha\| < 1$ . As  $v + \alpha(v) = 2u_0$  we have  $\text{graph}(\alpha) = M_0$ , and as  $v - \alpha(v) = 2u_1$  we have  $\text{graph}(-\alpha) = M_1$ . It follows that the construction  $(x, M_0, M_1) \mapsto (x, L, \alpha)$  gives a well-defined map  $\zeta: P^2V \setminus P_2V \rightarrow B^\circ U$ , with  $\delta'\zeta = 1$ . One can check directly that  $\zeta\delta'$  is also the identity, so  $\delta'$  is a diffeomorphism as claimed.  $\square$

COROLLARY 22.37. *The bundle  $U$  is the normal bundle to the diagonal embedding  $\delta: PV \rightarrow P^2V$ , there is a homeomorphism  $P^2V/P_2V = PV^U$ , and the quotient map  $P^2V \rightarrow P^2V/P_2V$  can be thought of as the Pontrjagin-Thom collapse associated to  $\delta$ .*  $\square$

REMARK 22.38. There are easier proofs of this corollary if one is willing to be less symmetrical.

Now, the above corollary together with Proposition 22.22 and Section 3 gives

$$\tilde{E}^0PV^U = E^0(P^2V, P_2V) = J = K/K^2 \otimes_A \overline{\Omega}^* = K/K^2 \otimes_k \omega^\vee.$$

On the other hand, we have

$$U = \text{Hom}(T, T^\perp) = \text{Hom}(T, p^*V) \ominus \mathbb{C},$$

so (using the case  $W = V$  of Proposition 21.13)

$$PV^U = \Sigma^{-2}PV^{\text{Hom}(T, p^*V)} = \Sigma^{-2}P(V \oplus V)/PV.$$

Remark 5.12 tells us that  $\tilde{E}^0(S^{-2}) = \omega^\vee$ , and it is clear that  $E^0(P(V \oplus V), PV) = K/K^2$ . We thus obtain

$$\tilde{E}^0 PV^U = K/K^2 \otimes_k \omega^\vee$$

again. These two arguments apparently give two different isomorphisms  $\tilde{E}^0 PV^U \rightarrow K/K^2 \otimes_k \omega^\vee$ , but one can show (using Remark 21.16) that they are actually the same.

We next recall some ideas about Gysin maps. We discuss the situation for manifolds, and leave it to the reader to check that everything works fibrewise for bundles of manifolds, at least in sufficient generality for the arguments below. Let  $f: M \rightarrow N$  be an analytic map between compact complex manifolds. (It is possible to work with much less rigid data, but we shall not need to do so.) Let  $\tau_M$  and  $\tau_N$  be the tangent bundles of  $M$  and  $N$ , and let  $\nu_f$  be the virtual bundle  $f^*\tau_N - \tau_M$  over  $M$ . Then for any virtual bundle  $U$  over  $N$ , a well-known variant of the Pontrjagin-Thom construction gives a stable map  $T(f, U): N^U \rightarrow M^{f^*U + \nu_f}$ , and thus a map  $f_! = T(f, U)^*: \tilde{E}^0 M^{f^*U + \nu_f} \rightarrow \tilde{E}^0 N^U$ . Using the ring map  $f^*: E^0 N \rightarrow E^0 M$  we regard the source and target of  $T(f, U)^*$  as  $E^0 N$ -modules, and we find that  $T(f, U)^*$  is  $E^0 N$ -linear. We also find that  $T(f, U)^*$  can be obtained from  $T(f, 0)^*$  by tensoring over  $E^0 N$  with  $\tilde{E}^0 N^U$ . Finally, we have a composition formula: given maps  $M \xrightarrow{f} N \xrightarrow{g} P$ , we have  $\nu_{gf} = \nu_f + f^*\nu_g$  and

$$T(f, \nu_g) \circ T(g, 0) = T(gf, 0): P^V \rightarrow M^{(gf)^*V + \nu_{gf}}.$$

Now consider the maps  $M \xrightarrow{\delta} M^2 \xrightarrow{1 \times \pi} M$ , where  $\pi$  is the constant map from  $M$  to a point. We have  $\nu_\delta = \tau_M$  and  $\nu_\pi = -\tau_M$ , so the transitivity formula says that the composite

$$M_+ \xrightarrow{1 \wedge T(\pi, 0)} M_+ \wedge M^{-\tau} \xrightarrow{T(\delta, \nu_{1 \times \pi})} M_+$$

is the identity. Assuming a Künneth isomorphism, we get maps

$$E^0 M \xrightarrow{\delta_!} E^0 M \otimes \tilde{E}^0 M^{-\tau} \xrightarrow{1 \otimes \pi_!} E^0 M,$$

whose composite is again the identity.

Now specialise to the case  $M = PV$ . As before we put  $A = E^0 PV$  and identify  $\tilde{E}^0 M^\tau$  with  $J$ , and the map  $\delta_! = T(\delta, 0)^*: \tilde{E}^0 M^\tau \rightarrow E^0(M^2)$  with the inclusion  $J \rightarrow A \otimes A$ . We know that the map

$$\delta_! = T(\delta, \nu_{1 \times \pi})^*: A = E^0 M \rightarrow E^0 M \otimes \tilde{E}^0 M^{-\tau} = A \otimes J^*$$

is obtained from  $T(\delta, 0)^*$  by tensoring over  $A \otimes A$  with  $A \otimes J^*$ . It follows easily that  $\delta_!(1) = u \in A \otimes J^*$ , where  $u$  is as in Construction 22.8. The equation  $(1 \otimes \pi_!)\delta_! = 1$  now tells us that  $(1 \otimes \pi_!)(u) = 1$ . Proposition 22.9 now tells us that  $\pi_! = \epsilon: J^* \rightarrow k$ . This proves Theorem 22.35.

## Further theory of infinite Grassmannians

Recall from Section 20 that  $G\mathcal{U}$  denotes the space of finite-dimensional subspaces of  $\mathcal{U}$ , which is the natural equivariant generalization of the space  $G\mathbb{C}^\infty = \coprod_{d \geq 0} BU(d)$ . We know from Theorem 20.1 that  $E_0 G\mathcal{U}$  is the symmetric algebra over  $E_0$  generated by  $E_0 P\mathcal{U} = \mathcal{O}_C^\vee$ . It follows that

$$\begin{aligned}\mathrm{spec}(E_0 G\mathcal{U}) &= \mathrm{Map}(C, \mathbb{A}^1) \\ \mathrm{spf}(E^0 G_d \mathcal{U}) &= \mathrm{Div}_d^+(C) \\ \mathrm{spf}(E^0 G\mathcal{U}) &= \mathrm{Div}^+(C).\end{aligned}$$

In this section, we obtain similar results for spaces analogous to  $\mathbb{Z} \times BU$ ,  $BU$  and  $BSU$ .

**DEFINITION 23.1.** For any finite-dimensional  $A$ -universe  $U$ , we put  $\mathbf{2}U = U \oplus U$ . We write  $U_+$  for  $U \oplus 0$  and  $U_-$  for  $0 \oplus U$  so  $\mathbf{2}U = U_+ + U_-$ . We put

$$\tilde{G}(U) = G(\mathbf{2}U) = \prod_{d=0}^{2 \dim(U)} G_d(\mathbf{2}U);$$

a point  $X \in \tilde{G}(U)$  should be thought of as a representative of the virtual vector space  $X - U_-$ . We embed  $G(U)$  in  $\tilde{G}(U)$  by  $X \mapsto X \oplus U = X_+ + U_-$ . We define  $\widetilde{\dim}: \tilde{G}(U) \rightarrow \mathbb{Z}$  by  $\widetilde{\dim}(X) = \dim(X) - \dim(U)$ , and  $\widetilde{G}_d(U) = \{X \mid \widetilde{\dim}(X) = d\}$ . Given an isometric embedding  $j: U \rightarrow V$ , we define  $j_*: \tilde{G}(U) \rightarrow \tilde{G}(V)$  by  $j_*(X) = (j \oplus j)(X) + W_-$ , where  $W = V \ominus j(U)$ . There is an evident map  $\sigma: \tilde{G}(U) \times \tilde{G}(V) \rightarrow \tilde{G}(U \oplus V)$  sending  $(X, Y)$  to  $X \oplus Y$ ; one checks that  $\widetilde{\dim}(X \oplus Y) = \widetilde{\dim}(X) + \widetilde{\dim}(Y)$  and that the map  $\sigma$  is compatible in an obvious sense with the maps  $j_*$ .

If  $\mathcal{U}$  is an infinite  $A$ -universe, we define  $\mathbf{2}\mathcal{U} = \mathcal{U} \oplus \mathcal{U}$  as before, and put  $\tilde{G}(\mathcal{U}) = \varinjlim_U \tilde{G}(U)$ , where  $U$  runs over finite-dimensional subspaces. Equivalently,  $\tilde{G}(\mathcal{U})$  is the space of subuniverses  $\mathcal{V} < \mathbf{2}\mathcal{U}$  such that the space  $\mathcal{V} \cap \mathcal{U}_-$  has finite codimension in  $\mathcal{V}$  and also has finite codimension in  $\mathcal{U}_-$ . This is a natural analogue of the space  $\mathbb{Z} \times BU$ .

**PROPOSITION 23.2.** *For any  $B \leq A$  we have*

$$\begin{aligned}(G\mathcal{U})^B &= \prod_{\beta \in B^*} G(\mathcal{U}[\beta]) = \mathrm{Map}(B^*, \prod_d BU(d)) \\ (\tilde{G}\mathcal{U})^B &= \prod_{\beta \in B^*} \tilde{G}(\mathcal{U}[\beta]) = \mathrm{Map}(B^*, \mathbb{Z} \times BU)\end{aligned}$$

where

$$\mathcal{U}[\beta] = \{u \mid b.u = \exp(2\pi i \beta(b))u \text{ for all } b \in B\}$$

is the  $\beta$ -isotypical part of  $\mathcal{U}$ . In each case, the first equivalence is  $A/B$ -equivariant, but the second is not.

PROOF. For the first isomorphism, just note that  $\mathcal{U}$  splits  $A$ -equivariantly as  $\bigoplus_{\beta} \mathcal{U}[\beta]$ , and a subspace  $V < \mathcal{U}$  is  $B$ -invariant iff it is the direct sum of its intersections with the subspaces  $\mathcal{U}[\beta]$ . This gives an  $A/B$ -equivariant isomorphism  $(GU)^B = \prod_{\beta} G(\mathcal{U}[\beta])$ , and it is clear that  $G(\mathcal{U}[\beta])$  is nonequivariantly a copy of  $\coprod_d BU(d)$  so  $(GU)^B = \text{Map}(B^*, \coprod_d BU(d))$ . The argument for  $(\tilde{GU})^B$  is essentially the same.  $\square$

We next write  $R^+A = \mathbb{N}[A^*] = \pi_0^A(GU)$  for the additive semigroup of honest representations of  $A$ , and  $RA = \mathbb{Z}[A^*] = \pi_0^A(\tilde{GU})$  for the additive group of virtual representations. It is clear that the semigroup ring  $E_0[R^+A]$  is a polynomial algebra over  $E_0$  with one generator  $u_{\alpha}$  for each character  $\alpha$ , and the group ring  $E_0[RA]$  is the Laurent series ring with the same generators. In other words, we have

$$\begin{aligned} E_0[R^+A] &= E_0[u_{\alpha} \mid \alpha \in A^*] \\ E_0[RA] &= E_0[u_{\alpha}, u_{\alpha}^{-1} \mid \alpha \in A^*] = E_0[R^+A][v^{-1}] \end{aligned}$$

where  $v = \prod_{\alpha} u_{\alpha}$ . Note that

$$\begin{aligned} \text{spec}(E_0[R^+A]) &= \text{Map}(A^*, \mathbb{A}^1) \\ \text{spec}(E_0[RA]) &= \text{Map}(A^*, \mathbb{G}_m), \end{aligned}$$

and the isomorphisms  $R^+A = \pi_0^A GU$  and  $RA = \pi_0^A \tilde{GU}$  give maps  $E_0[R^+A] \rightarrow E_0 GU$  and  $E_0[RA] \rightarrow E_0 \tilde{GU}$ .

Now let  $\phi$  be the obvious isomorphism

$$\mathbb{C}[A] \oplus \mathcal{U} = \mathbb{C}[A] \oplus \mathbb{C}[A]^{\infty} \rightarrow \mathbb{C}[A]^{\infty} = \mathcal{U},$$

and define  $\phi': G_d \mathcal{U} \rightarrow G_{d+|A|} \mathcal{U}$  by  $\phi'(X) = \phi(\mathbb{C}[A] \oplus X)$ .

PROPOSITION 23.3. *The space  $\tilde{GU}$  is the telescope of the self-map  $\phi'$  of  $GU$ . We thus have*

$$E_0 \tilde{GU} = v^{-1} E_0 GU = E_0[RA] \otimes_{E_0[R^+A]} E_0 GU,$$

and so  $\text{spec}(E_0 \tilde{GU}) = \text{Map}(C, \mathbb{G}_m)$ .

PROOF. Put  $\mathcal{U}' = \mathbb{C}[A][z, z^{-1}]$ , and identify this with  $2\mathcal{U}$  by sending  $(e_k, 0)$  to  $z^k$  and  $(0, e_k)$  to  $z^{-k-1}$ . The standard embedding  $GU \rightarrow \tilde{GU}$  now sends  $X$  to  $X \oplus \mathcal{U}_-$ . It is easy to check that  $\tilde{GU} = \lim_{\rightarrow k} z^{-k} GU$  on the nose, and that the inclusion  $z^{-k} GU \rightarrow z^{-k-1} GU$  is isomorphic to the map  $\phi'$ . The first claim follows, and the second claim is just the obvious consequence in homology. The tensor product description of  $E_0 \tilde{GU}$  gives us a pullback square

$$\begin{array}{ccc} \text{spec}(E_0 \tilde{GU}) & \longrightarrow & \text{spec}(E_0 GU) = \text{Map}(C, \mathbb{A}^1) \\ \downarrow & & \downarrow \\ \text{Map}(A^*, \mathbb{G}_m) = \text{spec}(E_0[RA]) & \longrightarrow & \text{spec}(E_0[R^+A]) = \text{Map}(A^*, \mathbb{A}^1). \end{array}$$



As  $C$  is a formal neighbourhood of the image of  $\phi$ , we see that a map  $C \xrightarrow{f} \mathbb{A}^1$  lands in  $\mathbb{G}_m$  if and only if the composite  $A^* \times S \xrightarrow{\phi} C \xrightarrow{f} \mathbb{A}^1$  lands in  $\mathbb{G}_m$ . Given this, we see that the pullback is just  $\text{Map}(C, \mathbb{G}_m)$  as required.  $\square$

We next introduce the analogue of  $BU$ .

**PROPOSITION 23.4.** *There is a natural splitting  $\widetilde{GU} = \mathbb{Z} \times \widetilde{G}_0\mathcal{U}$ , and we have  $\text{spec}(E_0\widetilde{G}_0\mathcal{U}) = \text{Map}_0(C, \mathbb{G}_m)$  (the scheme of maps  $f: C \rightarrow \mathbb{G}_m$  with  $f(0) = 1$ ).*

**PROOF.** We have already described an equivariant map  $\widetilde{\dim}: \widetilde{GU} \rightarrow \mathbb{Z}$ , and defined  $\widetilde{G}_0\mathcal{U} = \ker(\widetilde{\dim})$ . We also have  $(\widetilde{GU})^A = \text{Map}(A^*, \mathbb{Z} \times BU)$  so  $\pi_0^A(\widetilde{GU}) = \text{Map}(A^*, \mathbb{Z}) = RA$ , which gives an equivariant map  $i: RA \rightarrow \widetilde{GU}$  (where  $RA$  has trivial action). The composite  $\widetilde{\dim} \circ i: RA \rightarrow \mathbb{Z}$  is just the usual augmentation map  $\epsilon$  sending a virtual representation to its dimension. Thus, if we let  $\eta: \mathbb{Z} \rightarrow RA$  be the unit map, then  $i \circ \eta$  is a section of  $\widetilde{\dim}$ . As  $\widetilde{GU}$  is a commutative equivariant  $H$ -space, we can define a map  $\delta: \widetilde{GU} \rightarrow \widetilde{G}_0\mathcal{U}$  by  $x \mapsto (i(\eta(\dim(x))) - x)$ , and we find that the resulting map  $(\widetilde{\dim}, \delta): \widetilde{GU} \rightarrow \mathbb{Z} \times \widetilde{G}_0\mathcal{U}$  is an equivalence. This is easily seen to be parallel to the splitting  $\text{Map}(C, \mathbb{G}_m) = \mathbb{G}_m \times \text{Map}_0(C, \mathbb{G}_m)$  given by  $f \mapsto (f(0), f(0)/f)$ , which gives the claimed description of  $\text{spec}(E_0\widetilde{GU})$ .  $\square$

**REMARK 23.5.** There are two other possible analogues of  $BU$ . Firstly, one could take the colimit of the spaces  $G_d\mathcal{U}$  using the maps  $V \mapsto V \oplus \mathbb{C}$ ; the scheme associated to the corresponding space is then  $\text{Map}_0(C, \mathbb{A}^1)$ , which classifies maps  $f: C \rightarrow \mathbb{A}^1$  with  $f(0) = 1$ . Alternatively, one could take the colimit of the spaces  $G_{d|A|}\mathcal{U}$  using the maps  $V \mapsto V \oplus \mathbb{C}[A]$ . This gives the scheme of maps  $f: C \rightarrow \mathbb{G}_m$  for which  $\prod_{\alpha \in A^*} f(\phi(\alpha)) = 1$ . However, the space  $\widetilde{G}_0\mathcal{U}$  described above is the one that occurs in Greenlees's definition of the spectrum  $kU_A$ , and is also the one whose Thom spectrum is  $MU_A$ .

We next introduce the analogue of  $BSU$ . For this, we need an analogue of the map  $B \det: BU \rightarrow \mathbb{C}P^\infty$ .

**DEFINITION 23.6.** Given a universe  $U$  of finite dimension  $d$ , we put  $\widetilde{FU} = \text{Hom}(\lambda^d U_-, \lambda^d(2U))$ . We make this a functor as follows. Given an isometric embedding  $j: U \rightarrow V$ , we put  $W = V \ominus jU$  and  $e = \dim(W)$ . As  $j: U \rightarrow jU$  is an isomorphism and  $\lambda^e W$  is one-dimensional, we have an evident isomorphism

$$\widetilde{FU} = \text{Hom}(\lambda^d jU_- \otimes \lambda^e W_-, \lambda^d(2jU) \otimes \lambda^e W_-).$$

The isomorphism  $V = jU \oplus W$  gives an isomorphism  $\lambda^d jU_- \otimes \lambda^e W_- = \lambda^{d+e} V$  and an embedding  $\lambda^d(2jU) \otimes \lambda^e W_- \rightarrow \lambda^{d+e}(2V)$ . Putting this together gives the required map  $j_*: \widetilde{FU} \rightarrow \widetilde{FV}$ .

There are also obvious maps  $\widetilde{F}(U) \otimes \widetilde{F}(V) \rightarrow \widetilde{F}(U \oplus V)$ , compatible with the above functoriality. This gives maps  $P\widetilde{F}(U) \times P\widetilde{F}(V) \rightarrow P\widetilde{F}(U \oplus V)$  of the associated projective spaces.

Next, recall that a point of  $\widetilde{G}_0\mathcal{U}$  is a  $d$ -dimensional subspace  $X \leq 2U$ . We define

$$\widetilde{\det}(X) = \text{Hom}(\lambda^d U_-, \lambda^d X) \in P\widetilde{FU}.$$

One can check that this gives a natural map  $\widetilde{\det}: \widetilde{G}_0 \rightarrow P\widetilde{F}$ , with  $\widetilde{\det}(X \oplus Y) = \widetilde{\det}(X) \otimes \widetilde{\det}(Y)$ .

Finally, for our complete universe  $\mathcal{U}$  we put  $\widetilde{F}\mathcal{U} = \lim_{\rightarrow U} \widetilde{F}U$ , where  $U$  runs over the finite-dimensional subuniverses. It is easy to check that this is again a complete  $A$ -universe, and thus is unnaturally isomorphic to  $\mathcal{U}$ . The maps  $\widetilde{\det}$  pass to the colimit to give an  $H$ -map  $\widetilde{\det}: \widetilde{G}_0\mathcal{U} \rightarrow P\widetilde{F}\mathcal{U} \simeq P\mathcal{U}$ . We write  $S\widetilde{G}_0\mathcal{U}$  for the pullback of the projection  $S(\widetilde{F}\mathcal{U}) \rightarrow P\widetilde{F}\mathcal{U}$  along the map  $\widetilde{\det}$ , or equivalently the space of pairs  $(\mathcal{V}, u)$  where  $\mathcal{V} \in \widetilde{G}_0\mathcal{U}$  and  $u$  is a unit vector in the one-dimensional space  $\widetilde{\det}(\mathcal{V})$ . As  $S(\widetilde{F}\mathcal{U})$  is equivariantly contractible, this is just the homotopy fibre of  $\widetilde{\det}$ .

PROPOSITION 23.7. *There is a natural splitting  $\widetilde{G}_0\mathcal{U} = S\widetilde{G}_0\mathcal{U} \times P\mathcal{U}$  (which does not respect the  $H$ -space structure).*

PROOF. It is enough to give a section of the  $H$ -map  $\widetilde{\det}: \widetilde{G}_0\mathcal{U} \rightarrow P\mathcal{U}$ . We can include  $P\mathcal{U} = G_1\mathcal{U}$  in  $\widetilde{G}_1\mathcal{U} \subset \widetilde{G}\mathcal{U}$  in the usual way, then use the projection  $\widetilde{G}\mathcal{U} \rightarrow \widetilde{G}_0\mathcal{U}$  from Proposition 23.4. We find that the resulting composite  $P\mathcal{U} \rightarrow P\mathcal{U}$  is actually minus the identity, but we can precompose by minus the identity to get the required section.  $\square$

REMARK 23.8. Cartier duality identifies  $\mathrm{spec}(E_0P\mathcal{U})$  with  $\mathrm{Hom}(C, \mathbb{G}_m)$ , and the proposition suggests that  $\mathrm{spec}(E_0S\widetilde{G}_0\mathcal{U})$  should be identified with the quotient scheme  $\mathrm{Map}_0(C, \mathbb{G}_m)/\mathrm{Hom}(C, \mathbb{G}_m)$ . However, there are difficulties in interpreting this quotient, and it is in fact more useful to take a slightly different approach as in [1, 2]. We will not give details here.

Next, recall that Greenlees has defined an equivariant analogue of connective  $K$ -theory (denoted by  $kU_A$ ) by the homotopy pullback square

$$\begin{array}{ccc} kU_A & \longrightarrow & F(EA_+, kU) \\ \downarrow & & \downarrow \\ KU_A & \longrightarrow & F(EA_+, KU). \end{array}$$

If  $v \in \pi_2 kU$  is the Bott element then  $kU[v^{-1}] = KU$  and  $kU/v = H$ . It is not hard to see that there is a corresponding element in  $\pi_2^A kU_A$  with  $kU_A[v^{-1}] = KU_A$  and  $kU_A/v = F(EA_+, H)$ .

PROPOSITION 23.9. *The zeroth, second and fourth spaces of  $kU_A$  are  $\widetilde{G}\mathcal{U}$ ,  $\widetilde{G}_0\mathcal{U}$  and  $S\widetilde{G}_0\mathcal{U}$  respectively.*

PROOF. We take it as well-known that the zeroth space of  $KU_A$  is  $\widetilde{G}\mathcal{U}$ , and  $KU_A$  is two-periodic so this is also the  $2k$ 'th space for all  $k$ . Let  $X_k$  denote the  $2k$ 'th space of  $kU_A$ , so we have a homotopy pullback square

$$\begin{array}{ccc} X_k & \longrightarrow & F(EA_+, BU\langle 2k \rangle) \\ \downarrow i & & \downarrow \\ \widetilde{G}\mathcal{U} & \xrightarrow{j} & F(EA_+, \mathbb{Z} \times BU) \end{array}$$

(where  $BU\langle 0 \rangle$  is interpreted as  $\mathbb{Z} \times BU$ ). In the case  $k = 0$ , the map  $i$  is the identity and so  $X_0 = \widetilde{G}\mathcal{U}$ . In the case  $k = 1$ , the map  $i$  is just the inclusion

$$F(EA_+, BU) \rightarrow \mathbb{Z} \times F(EA_+, BU) = F(EA_+, \mathbb{Z} \times BU)$$

and the map  $j$  sends  $\tilde{G}_k\mathcal{U}$  into  $\{k\} \times F(EA_+, BU)$ . It follows easily that  $X_1 = \tilde{G}_0\mathcal{U}$ . In the case  $k = 2$ , we note that the cofibration  $\Sigma^2 kU_A \xrightarrow{v} kU_A \rightarrow F(EA_+, H)$  gives a fibration  $X_2 \rightarrow X_1 \rightarrow F(EA_+, K(\mathbb{Z}, 2))$ . We know that  $X_1 = \tilde{G}_0\mathcal{U}$  and Proposition 4.6 that  $F(EA_+, K(\mathbb{Z}, 2)) = P\mathcal{U}$ . One can check that the resulting map  $\tilde{G}_0\mathcal{U} \rightarrow P\mathcal{U}$  is just  $\pm\widehat{\det}$ , and so  $X_2 = S\tilde{G}_0\mathcal{U}$  as claimed.  $\square$



## Transfers and the Burnside ring

There is a well-known relationship between transfers, the Burnside ring, and equivariant stable homotopy. In this section we will show how this relationship is encoded in the theory of equivariant formal groups.

Let  $\Omega = \Omega(A)$  be the Burnside ring of  $A$ , which is the group completion of the monoid of isomorphism classes of finite  $A$ -sets. Additively this is freely generated over  $\mathbb{Z}$  by the elements  $[A/B]$  for subgroups  $B \leq A$ , with the product rule

$$[A/B] \cdot [A/B'] = \left| \frac{A}{BB'} \right| [A/B \cap B'].$$

There is a well-known isomorphism  $\Omega \rightarrow \pi_0^G$ , sending  $[A/B]$  to  $\mathrm{tr}_B^A(1)$ .

Now let  $E$  be an  $E$ -equivariant evenly periodic ring spectrum, with associated equivariant formal group  $(C, \phi)$  over the scheme  $S = \mathrm{spec}(E_0)$ . The unit map  $\eta: S_G \rightarrow E$  gives us a ring map  $\Omega \rightarrow \mathcal{O}_S = E_0$ . We will show that this map is determined by the EFG structure.

**24.1. The functions  $v_n$ .** Let  $C$  be a formal multicurve group over a scheme  $S$ , and let  $x$  be a coordinate on  $C$ . (Later we will show that our constructions are independent of  $x$ , but we have not been able to make this visible from the outset.) It will be convenient to write  $x_n(a) = x(na)$ .

LEMMA 24.1. *There is a unique function  $f: C \times_S C \rightarrow \mathbb{A}^1$  such that*

$$x(a+b) = x(a) + x(b) + x(a)x(b)f(a,b)$$

*for all  $(a, b) \in C \times_S C$ .*

PROOF. From the axioms for formal multicurves, we see that  $\mathcal{O}_C = \mathcal{O}_S \oplus \mathcal{O}_C \cdot x$ . After tensoring this decomposition with itself, we see that and function  $p$  on  $C \times_S C$  can be written uniquely in the form

$$p(a,b) = q + x(a)r(b) + s(a)x(b) + f(a,b)x(a)x(b),$$

for some  $q \in \mathcal{O}_S$  and some functions  $r, s$  and  $f$ . If  $p(a,0) = p(0,b) = 0$  then it follows that  $q, r$  and  $s$  vanish, so  $p(a,b) = f(a,b)x(a)x(b)$ . We apply this to the function  $p(a,b) = x(a+b) - x(a) - x(b)$  to prove the lemma.  $\square$

We can apply the lemma inductively to understand  $x_n(a)$ . For any finite set  $I$  of integers we put  $y_I(a) = \prod_{i \in I} f(a, ia)$ . We also write  $\min_n(I)$  for the smallest element of  $I$ , with the convention  $\min_n(\emptyset) = n$ .

LEMMA 24.2.  $x_n = \sum_I \min_n(I) x^{|I|+1} y_I$ , where  $I$  runs over subsets of  $\{1, 2, \dots, n-1\}$ .

PROOF. Induction, using the identity  $x_{n+1} = x + x_n + x x_n y_n = x + (1 + x y_n) x_n$ .  $\square$

DEFINITION 24.3. We put

$$v_n = \sum_I \min_n(I) x^{|I|} y_I$$

$$w_n = \sum_{I \neq \emptyset} \min_n(I) x^{|I|-1} y_I,$$

so that

$$x_n = x v_n = n x + x^2 w_n.$$

We also write  $v_{n,k}(a) = v_n(ka)$  and  $w_{n,k}(a) = w_n(ka)$  and  $y_{I,k}(a) = y_I(ka)$ .

LEMMA 24.4.  $v_{nm} = v_n v_{m,n} = v_m v_{n,m}$ .

PROOF. By evaluating the identity  $x(ma) = x(a)v_m(a)$  at  $a = nb$ , we get  $x_{nm} = x_n v_{m,n} = x v_n v_{m,n}$ . On the other hand, we have  $x_{nm} = x v_{nm}$  and  $x$  is not a zero divisor so  $v_{nm} = v_n v_{m,n}$ . The other identity follows symmetrically.  $\square$

COROLLARY 24.5.  $v_n^2 = n v_n \pmod{x_n}$ .

PROOF. We have  $v_n^2 - n v_n = x w_n v_n = w_n x_n$ .  $\square$

We can push the lemma one step further, as follows.

LEMMA 24.6.  $v_n^2 w_{m,n} - v_m^2 w_{n,m} = n w_m - m w_n = v_n w_m - v_m w_n$ .

PROOF. We have

$$x_{nm} = n x_m + x_m^2 w_{n,m}$$

$$= n m x + n x^2 w_m + x^2 v_m^2 w_{n,m}$$

By exchanging the roles of  $n$  and  $m$ , we also have

$$x_{nm} = n m x + m x^2 w_n + x^2 v_n^2 w_{m,n}.$$

Subtracting these expressions gives

$$x^2(n w_m - m w_n + v_m^2 w_{n,m} - v_n^2 w_{m,n}) = 0.$$

As  $x$  is not a zero-divisor in  $\mathcal{O}_C$ , it follows that

$$v_n^2 w_{m,n} - v_m^2 w_{n,m} = n w_m - m w_n.$$

Now write  $v_n = n + x w_n$  and  $v_m = m + x w_m$  to see that

$$n w_m - m w_n = v_n w_m - v_m w_n.$$

$\square$

LEMMA 24.7. *Suppose that  $n$  and  $m$  are coprime. Then  $v_n = v_{n,m} \pmod{x_n}$ .*

PROOF. Put

$$s_{n,m} = v_m w_{n,m} - w_n.$$

Note that

$$v_{n,m} - v_n = (n + x_m w_{n,m}) - (n + x w_n) = x_m w_{n,m} - x w_n = x(v_m w_{n,m} - w_n) = x s_{n,m}.$$

On the other hand, a simple rearrangement of Lemma 24.6 tells us that  $v_m s_{n,m} = v_n s_{m,n}$ , so  $x_m s_{n,m} = x_n s_{m,n}$ .

Put  $I = \{t \in \mathcal{O}_C \mid t s_{n,m} \in (x_n)\}$ . As  $x s_{n,m} = v_{n,m} - v_n$ , it will suffice to show that  $x \in I$ .

As  $n$  and  $m$  are coprime, an element  $a \in C$  has  $a = 0$  iff  $na = ma = 0$ . This means that the vanishing locus of  $x$  is the intersection of the vanishing loci of  $x_n$  and  $x_m$ , or in other words that  $(x) = (x_n) + (x_m)$ . Visibly  $x_n \in I$ , so it will suffice to show that  $x_m \in I$ . This follows from the identity  $x_m s_{n,m} = x_n s_{m,n}$ .  $\square$

**COROLLARY 24.8.** *If  $n$  and  $m$  are coprime then  $v_{nm} = v_{n,m}v_{m,n} \pmod{x_{nm}}$ .*

**PROOF.** Lemma 24.4 tells us that  $v_{nm} = v_n v_{m,n}$ , so it will suffice to show that  $(v_n - v_{n,m})v_{m,n}$  is divisible by the element  $x_{nm} = x_n v_{m,n}$ , or to show that  $v_n - v_{n,m}$  is divisible by  $x_n$ . This is just Lemma 24.7.  $\square$

**PROPOSITION 24.9.** *Let  $x'$  be another coordinate on  $C$ , and let  $v'_n$  be the unique function such that  $x'_n(a) = x'(na) = x'(a)v'_n(a)$  for all  $a$ . Then  $v'_n(a) = v_n(a)$  whenever  $na = 0$ .*

**PROOF.** Both  $x$  and  $x'$  generate the ideal of functions that vanish on the zero section, so we must have  $x = px'$  and  $x' = qx$  for some functions  $p$  and  $q$ . This implies that  $(1 - pq)x = 0$  but  $x$  is regular, so  $pq = 1$ . Now

$$v'_n(a)x'(a) = x'(na) = q(na)x(na) = q(na)v_n(a)x(a) = q(na)p(a)v_n(a)x'(a),$$

and  $x'$  is regular so  $v'_n(a) = q(na)p(a)v_n(a)$  for all  $a$ . If  $na = 0$  this gives  $v'_n(a) = q(0)p(a)v_n(a)$ . Moreover, we have  $p(a) = p(0) \pmod{x(a)}$  and  $x(a)v_n(a) = x(na) = 0$  so  $p(a)v_n(a) = p(0)v_n(a)$ . We also have  $q(0)p(0) = 1$  so  $v'_n(a) = v_n(a)$  as claimed.  $\square$

Given any coordinate  $x$  we could apply the above to the coordinate  $x'(a) = x(-a)$ . For that case, however, some additional things can be said, as we now explain.

**PROPOSITION 24.10.** *If  $na = 0$  then  $v_{-n}(a) = -v_n(-a)$ .*

**PROOF.** Take  $m = -1$   $\square$

## 24.2. Transfer elements.

**DEFINITION 24.11.** Let  $U$  be a finite abelian group, and let  $\phi: U \times S \rightarrow C$  be a map of group schemes over  $S$ . We write  $I(U)$  or  $I(\phi)$  or  $I(U, \phi)$  for the ideal in  $\mathcal{O}_S$  generated by the elements  $x(\phi(u))$  for  $u \in U$ , so  $\text{spec}(\mathcal{O}_S/I(U))$  is the largest closed subscheme of  $S$  over which  $\phi$  vanishes.

**THEOREM 24.12.** *There is a unique way to define elements  $t(U) = t(U, \phi) \in \mathcal{O}_S$  with properties as follows.*

- (a) *If  $\phi = 0$  then  $t(U, 0) = |U|$ ; in particular,  $t(0, 0) = 1$ .*
- (b) *If  $U = U_0 \oplus U_1$  then  $t(U) = t(U_0)t(U_1)$ .*
- (c) *If  $U$  is cyclic of order  $n$ , generated by  $\alpha$ , then  $t(U) = v_n(\phi(\alpha))$ .*
- (d) *If  $\lambda: U' \rightarrow U$  is an isomorphism then  $t(U', \phi\lambda) = t(U, \phi)$ .*
- (e)  *$t(U)I(U) = 0$ .*
- (f) *Suppose that  $V \leq U$ , and that  $\bar{\phi}$  is the induced map  $U/V \rightarrow C$  defined over  $\text{spec}(\mathcal{O}_S/I(V))$ . Then  $t(U, \phi) = t(V, \phi)t(U/V, \bar{\phi})$ . (Note that (e) makes the right hand side well-defined.)*
- (g) *If  $V, W \leq U$  then  $t(V)t(W) = |V \cap W|t(V + W)$ .*

The proof will be given at the end of this section, after some preliminary results. Note that the elements  $v_n(\phi(\alpha))$  in clause (c) are independent of the coordinate, by Proposition 24.9.

It is clear that there is at most one way to satisfy (a) to (d); the problem is to show that this gives a well-defined answer which also has properties (e) to (g). The next definition makes this more formal.

DEFINITION 24.13. A *presentation* of a subgroup  $U$  is a set  $P$  of nonzero elements of  $U$  such that the resulting map  $\bigoplus_{\alpha \in P} \mathbb{Z}/\text{ord}(\alpha) \rightarrow U$  is an isomorphism. Given a presentation  $P$ , we put

$$t(P) = t(P, \phi) = \prod_{\alpha \in P} v_{\text{ord}(\alpha)}(\phi(\alpha)).$$

We say that  $U$  is *canonical* if  $t(P)$  is independent of  $P$ ; if so, we write  $t(U)$  instead of  $t(P)$ .

LEMMA 24.14. *For any presentation  $P$  of  $U$  we have  $t(P)I(U) = 0$ .*

PROOF. Suppose that  $\alpha \in P$  has order  $n$ , and put  $a = \phi(\alpha)$ . Then  $v_n(a)$  is a factor in  $t(P)$  and  $x(a)v_n(a) = x(na) = x(0) = 0$ , so  $x(a)t(P) = 0$ . These elements  $x(a)$  generate  $I(U)$ , so  $t(P)I(U) = 0$ .  $\square$

LEMMA 24.15. *If  $U$  is cyclic of order  $n$  and  $\alpha$  is a generator, then  $t(\{\alpha\})$  is independent of  $\alpha$ . (We can thus write  $t_1(U)$  for  $t(\{\alpha\})$ .)*

PROOF. If  $\beta$  is another generator then  $\beta = m\alpha$  for some  $m$  that is coprime to  $n$ . If  $a = \phi(\alpha)$  then  $t(\{\alpha\}) = v_n(a)$  and  $t(\{\beta\}) = v_{n,m}(a)$ , but these are the same by Lemma 24.7.  $\square$

LEMMA 24.16. *If  $U$  is cyclic and  $U = \prod_{i=1}^r U_i$  then  $t_1(U) = \prod_i t_1(U_i)$ .*

PROOF. By induction we can reduce to the case  $r = 2$ . Put  $n = |U_1|$  and  $m = |U_2|$ . As  $U$  is cyclic, these must be coprime. Choose a generator  $\alpha$  for  $U$  and put  $a = \phi(\alpha)$ . Note that  $m\alpha$  generates  $U_1$  and  $n\alpha$  generates  $U_2$ , so  $t_1(U_1) = v_{n,m}(a)$  and  $t_1(U_2) = v_{m,n}(a)$ , whereas  $t_1(U) = v_{nm}(a)$ . The claim now follows from Corollary 24.8.  $\square$

COROLLARY 24.17. *Cyclic groups are canonical.*

PROOF. Let  $U$  be cyclic, and let  $P = \{\alpha_1, \dots, \alpha_r\}$  be a presentation, and let  $U_i$  be the subgroup generated by  $\alpha_i$ . Directly from the definitions we see that  $t(P) = \prod_i t_1(U_i)$ , but the lemma tells us that this is the same as  $t_1(U)$  and thus is independent of  $P$ .  $\square$

LEMMA 24.18. *Suppose that  $U = \prod_p U(p)$ , where  $U(p)$  is a  $p$ -group, and that each  $U(p)$  is canonical. Then  $U$  is canonical, with  $t(U) = \prod_p t(U(p))$ .*

PROOF. Let  $P = \{\alpha_1, \dots, \alpha_d\}$  be a presentation of  $U$ . Let  $U_i$  be the cyclic subgroup generated by  $\alpha_i$ , and put  $n_i = |U_i|$  and  $t_i = t(U_i) = v_{n_i}(\phi(\alpha_i))$ . Clearly  $t(P) = \prod_i t_i$ , so it will suffice to prove that this is equal to  $\prod_p t(U(p))$ .

Now let  $U_i(p)$  be the  $p$ -torsion part of  $U_i$ . Put  $n_{i,p} = |U_i(p)|$  and choose an element  $\beta_{i,p}$  generating  $U_i(p)$ . Put  $t_{i,p} = t(U_i(p)) = v_{n_{i,p}}(\phi(\beta_{i,p}))$ . If we fix  $i$  then the elements  $\beta_{i,p}$  give a presentation of  $U(i)$ , so  $\prod_p t_{i,p} = t(U_i) = t_i$ , so  $\prod_{i,p} t_{i,p} = \prod_i t_i$ . On the other hand, if we fix  $p$  then the elements  $\beta_{i,p}$  present



$U(p)$ , so  $\prod_i t_{i,p} = t(U(p))$ , so  $\prod_{i,p} t_{i,p} = \prod_p t(U(p))$ . Thus  $\prod_i t_i = \prod_p t(U(p))$  as required.  $\square$

LEMMA 24.19. *Elementary abelian  $p$ -groups are canonical.*

PROOF. Suppose  $U = (\mathbb{Z}/p)^d$ . The element  $t(\{\alpha_1, \dots, \alpha_d\})$  is clearly unchanged by permuting the elements  $\alpha_i$ , and Lemma 24.15 tells us that it is also unchanged if we replace each  $\alpha_i$  by  $m_i \alpha_i$  for some  $m_i \in (\mathbb{Z}/p)^\times$ . Next, we claim that if  $a, b \in C$  and  $pa = pb = 0$  then  $v_p(a+b)v_p(b) = v_p(a)v_p(b)$ . Indeed, we have  $x(b)v_p(b) = x_p(b) = 0$ , so it will suffice to show that  $v_p(a+b) = v_p(a) \pmod{x(b)}$ . This is clear because the vanishing locus of  $x$  is the zero section. It now follows that  $t(\{\alpha_1, \dots, \alpha_d\})$  is unchanged by substitutions of the form  $\alpha_i \mapsto \alpha_i + \alpha_j$  with  $j \neq i$ . These substitutions, together with permutations and diagonal matrices, generate  $GL_d(\mathbb{Z}/p)$ , and the claim follows.  $\square$

LEMMA 24.20. *If  $U$  is an abelian  $p$ -group then  $U$  is canonical. Moreover, if we put  $V = \{\alpha \in U \mid p\alpha = 0\}$ , then  $t(U, \phi) = t(V, \phi)t(U/V, \bar{\phi})$ .*

PROOF. By induction on the exponent (starting with Lemma 24.19), we may assume that  $V$  and  $U/V$  are canonical. We therefore have well-defined elements  $t(V, \phi) \in \mathcal{O}_S$  and  $t(U/V, \bar{\phi}) \in \mathcal{O}_S/I(V, \phi)$ , with  $t(V, \phi)I(V, \phi) = 0$  by Lemma 24.14. This means that there is a well-defined product  $t_1 = t(V, \phi)t(U/V, \bar{\phi}) \in \mathcal{O}_S$ . It will suffice to show that  $t(P) = t_1$  for any presentation  $P = \{\alpha_1, \dots, \alpha_d\}$  of  $U$ . To see this, let the order of  $\alpha_i$  be  $p^{r_i}$ , so  $r_i > 0$ . We will assume for simplicity that in fact  $r_i > 1$  for all  $i$ ; only minor notational adjustments are needed for the remaining cases. Put  $\beta_i = p^{r_i-1}\alpha_i$ , so that  $\{\beta_1, \dots, \beta_d\}$  is a presentation of  $V$ . Let  $\bar{\alpha}_i$  be the image of  $\alpha_i$  in  $U/V$ , so  $\{\bar{\alpha}_1, \dots, \bar{\alpha}_d\}$  is a presentation for  $U/V$ . Put  $a_i = \phi(\alpha_i) \in C$ . Now  $t(P)$  is the product of the terms  $v_{p^{r_i}}(a_i)$ , whereas  $t(U/V, \bar{\phi})$  is the product of the terms  $v_{p^{r_i-1}}(a_i)$ , and  $t(V, \phi)$  is the product of the terms  $v_p(p^{r_i-1}a_i)$ . Lemma 24.4 tells us that  $v_{p^{r_i}}(a_i) = v_{p^{r_i-1}}(a_i)v_p(p^{r_i-1}a_i)$ , so  $t(P) = t(U/V, \bar{\phi})t(V, \phi)$  as required.  $\square$

COROLLARY 24.21. *All finite abelian groups are canonical.*

PROOF. This is immediate from Lemmas 24.20 and 24.18.  $\square$

DEFINITION 24.22. Let  $U$  be a finite abelian group, and let  $V$  be a subgroup. We say that the pair  $(U, V)$  is *good* if for all  $C$  and  $\phi$  as above, we have  $t(U, \phi) = t(V, \phi)t(U/V, \bar{\phi})$ .

REMARK 24.23. From now on we will just write  $t(U)$  for  $t(U, \phi)$  and so on, and not mention explicitly the quotient rings in which these elements lie.

LEMMA 24.24. (a) *If  $W \leq V \leq U$  and the pairs  $(V, W)$  and  $(U, W)$  and  $(U/W, V/W)$  are good then so is  $(U, V)$ .*  
 (b) *If the pairs  $(U_i, V_i)$  are good then so is  $(\bigoplus_i U_i, \bigoplus_i V_i)$ .*  
 (c) *If the pairs  $(U(p), V(p))$  are all good (where  $U(p) = \{\alpha \in U \mid p^N \alpha = 0 \text{ for } N \gg 0\}$ ) then so is  $(U, V)$ .*

PROOF. For (a), we can use the goodness of  $(U, W)$ ,  $(U/W, V/W)$  and  $(V, W)$  in turn to see that

$$t(U) = t(W)t(U/W) = t(W)t(V/W)t(U/V) = t(V)t(U/V).$$

Part (b) is clear, and part (c) is a special case.  $\square$

LEMMA 24.25. *If  $|V| = p$  then  $(U, V)$  is good.*

PROOF. We can use Lemma 24.24(c) to reduce to the case where  $U$  is a  $p$ -group. If  $U$  is cyclic, the claim then follows from Lemma 24.20.

If  $U$  is not cyclic, let the exponent of  $U$  be  $p^u$ , and choose a cyclic summand  $W \leq U$  of order  $p^u$ . If  $V \leq W$  then we choose a complementary group  $X$  with  $U = W \oplus X$ , note that the pairs  $(W, V)$  and  $(X, 0)$  are good, and apply Lemma 24.24(b) to these pairs.

Suppose instead that  $V \not\leq W$ , so  $V \oplus W \leq U$ . Note that all groups mentioned are modules over the ring  $R = \mathbb{Z}/p^u$ . Now  $R \simeq \text{Hom}(R, \mathbb{Q}/\mathbb{Z})$ , so  $R$  is self-injective, so  $W$  is an injective  $R$ -module. This means that the projection  $\pi: V \oplus W \rightarrow W$  can be extended to give a map  $\pi: U \rightarrow W$ . If we put  $X = \ker(\pi)$  we see that  $U = W \oplus X$  with  $V \leq X$ . The pair  $(X, V)$  can be assumed to be good by induction on the order, and the pair  $(W, 0)$  is good, so the pair  $(U, V) = (W, 0) \oplus (X, V)$  is good.  $\square$

COROLLARY 24.26. *All pairs  $(U, V)$  are good.*

PROOF. We can use Lemma 24.24(c) to reduce to the case where  $U$  is a  $p$ -group. We then argue by induction on the order of  $V$ , starting with Lemma 24.25. Choose a subgroup  $W \leq V$  of order  $p$ . The pairs  $(U, W)$  and  $(V, W)$  are good by the lemma, and  $(U/W, V/W)$  is good by induction. Part (a) of Lemma 24.24 therefore tells us that  $(U, V)$  is good.  $\square$

PROPOSITION 24.27. *If  $V, W \leq U$  then  $t(V)t(W) = |V \cap W|t(V + W)$ .*

PROOF. Put  $X = V \cap W$ . We have  $t(V) = t(V/X)t(X)$ ,  $t(W) = t(W/X)t(X)$  and  $t(X)^2 = |X|t(X)$  so  $t(V)t(W) = |X|t(V/X)t(W/X)t(X) = |X|t(V/X)t(W)$ . Now  $V/X$  can be identified with  $(V + W)/W$  and  $t((V + W)/W)t(W) = t(V + W)$ , so  $t(V)t(W) = |X|t(V + W)$  as claimed.  $\square$

PROOF OF THEOREM 24.12. Part (a) follows from the definitions if we remember that  $v_n(0) = n$ . Parts (b) to (d) are built in to the definition of  $t$ . Part (e) follows from Lemma 24.14, and parts (f) and (g) are Corollary 24.26 and Proposition 24.27.  $\square$

We now change notation, and consider an  $A$ -equivariant formal group  $(C, \phi)$ , so  $\phi: A^* \times S \rightarrow C$ .

DEFINITION 24.28. We define a map

$$\eta: \Omega = \Omega(A) = \mathbb{Z}\{[A/B] \mid B \leq A\} \rightarrow \mathcal{O}_S$$

by  $\eta([A/B]) = t(\text{ann}(B), \phi)$ .

PROPOSITION 24.29.  *$\eta$  is a ring map.*

PROOF. For every point in  $(A/B) \times (A/C)$ , the isotropy group is  $B \cap C$ . This means that in  $\Omega$  we have  $[A/B].[A/C] = n[A/(B \cap C)]$ , where  $n$  is characterised by the fact that  $|A/B|.|A/C| = n|A/(B \cap C)|$ . Now put  $V = \text{ann}(B) = (A/B)^*$ , so  $|V| = |A/B|$ . Similarly, put  $W = \text{ann}(C) = (A/C)^*$ , so  $|W| = |A/C|$ . We see that  $\text{ann}(B \cap C) = V + W$  and  $|V + W| = |A/(B \cap C)|$ , so  $n$  is defined by the equation  $|V||W| = n|V + W|$ , so  $n = |V \cap W|$ . We have  $\eta([A/B]) = t(V)$  and  $\eta([A/C]) = t(W)$  and

$$\eta([A/B][A/C]) = n\eta([A/(B \cap C)]) = |V \cap W|t(V + W).$$

The claim now follows from Proposition 24.27.  $\square$

**24.3. Comparison with topology.** Now consider the case where  $C$  arises from an evenly orientable  $A$ -equivariant ring spectrum  $E$ . We then have a map  $\eta: \Omega(A) \rightarrow \mathcal{O}_S = E_0$  as above, and also a topologically defined map  $\eta': \Omega(A) = \pi_0^A(S) \rightarrow E_0$ , given by  $\eta'([A/B]) = \text{tr}_B^A(1)$ .

PROPOSITION 24.30. *We have  $\eta([A/B]) = \text{tr}_B^A(1)$ , so  $\eta = \eta'$ .*

We will do the most basic case as a separate lemma, and then do the general case.

LEMMA 24.31. *If  $\alpha \in A^*$  has order  $m$  and  $B = \ker(\alpha)$  then  $\text{tr}_B^A(1) = v_m(\phi(\alpha))$ .*

This was originally proved by Quillen [24], but we will give an argument here for convenience.

PROOF. First, we need to deal with a sign issue. Recall that any coordinate  $x$  gives a natural system of Thom classes  $u_V$  for all bundles  $V$ . We also have another coordinate  $\bar{x}(a) = x(-a)$ , and we write  $\bar{u}_V$  for the Thom class defined using  $\bar{x}$  instead of  $x$ .

Now let  $T$  be the tautological bundle over  $PU$ , and let  $PU^T$  be the corresponding Thom space. Restriction to the zero section sends  $u_T$  to  $\bar{x}$  and  $\bar{u}_T$  to  $x$ .

Define  $\theta_0: PU^T \rightarrow PU^{T^m}$  by  $\theta_0(L, x) = (L, x^{\otimes m})$ . We claim that  $\theta_0^*(\bar{u}_{T^m}) = v_m u_T$ . (where  $v_m$  is defined by  $x_m = v_m x$  as previously). Indeed, we know that  $\widehat{E}^0(PU^T)$  is freely generated over  $E^0 PU$  by  $\bar{u}_T$ , so certainly  $\theta_0^*(\bar{u}_{T^m}) = f \bar{u}_T$  for some  $f$ . Next, note that  $\theta$  is the identity on the zero section, and that restriction to the zero section converts Thom classes to Euler classes, and that the Euler class of  $T^m$  is just  $x_m$ . We deduce that  $x_m = f x$ , so  $f = v_m$  as claimed. We now restrict to the point  $L_\alpha \in PU$ , noting that  $L_\alpha^{\otimes m} \simeq L_0$  and that the restriction  $\mathcal{O}_C = E^0 PU \rightarrow E^0 PL_\alpha = \mathcal{O}_S$  is just evaluation at  $\phi(\alpha)$ . This gives us a map  $\theta_0: S^{L_\alpha} \rightarrow S^{L_0}$  with  $\theta_0(z) = z^m$  and  $\theta_0^*(u_{L_0}) = v_m(\phi(\alpha))u_{L_\alpha}$ . In this context it is meaningful to define  $\theta: S^{L_\alpha} \rightarrow S^{L_0}$  by  $\theta(z) = (z^m - 1)/m$ ; this is equivariantly homotopic to  $\theta_0$  and so also satisfies  $\theta^*(\bar{u}_{L_0}) = v_m(\phi(\alpha))\bar{u}_{L_\alpha}$ .

We next need to relate this to the transfer. Recall that  $\alpha \in A^* = \text{Hom}(A, \mathbb{Q}/\mathbb{Z})$ ; we define  $\alpha'(a) = \exp(2\pi i \alpha(a))$  so that  $\alpha': A \rightarrow S^1$ . Let  $q: A \rightarrow A/B$  be the projection. We have an equivariant embedding  $j_0: A/B \rightarrow L_\alpha$  given by  $j_0(q(a)) = \alpha'(a)$ . To construct the transfer, we need an equivariant embedding  $j: A/B \times L_\alpha \rightarrow L_\alpha$  with  $j(q(a), z) \simeq j_0(q(a)) + z = \alpha'(a) + z$  to first order in  $z$ . There is a unique such  $j$  up to isotopy, so we can choose any  $j$  that is convenient for our purposes. We will define a diffeomorphism

$$i: \mathbb{C} \rightarrow \{z \in \mathbb{C} \mid |z| < 1/m\}$$

by  $i(z) = z/(1 + m|z|)$ , and then define  $j$  by

$$j(q(a), z) = \alpha'(a)(1 + m i(z/\alpha'(a)))^{1/m}.$$

(Here we note that  $1 + m i(z/\alpha'(a))$  lies in the right half plane, and we implicitly use the principal branch of the  $m$ 'th root giving an answer whose argument lies in  $(-\pi/(2m), \pi/(2m))$ ; the injectivity of  $j$  follows easily.)

Now define  $\delta: A/B \times L_\alpha \rightarrow L_0$  by  $\delta(q(a), z) = z/\alpha'(a)$ . We then have a diagram as shown on the left below:

$$\begin{array}{ccccc}
 A/B \times L_\alpha & \xrightarrow{j} & L_\alpha & & S^{L_\alpha} \xrightarrow{j^!} (A/B)_+ \wedge S^{L_\alpha \epsilon \wedge 1} \rightarrow S^{L_\alpha} \\
 \delta \downarrow & & \downarrow \theta & & \downarrow \theta & & \downarrow \delta & & \downarrow \zeta_\alpha \\
 L_0 & \xrightarrow{i} & L_0 & & S^{L_0} \xrightarrow{i^! \simeq 1} S^{L_0} \xrightarrow{\zeta_0} PU^T.
 \end{array}$$

One can check directly that the left hand square is a pullback, the horizontal maps are open inclusions, and the vertical maps are proper. We can thus apply the Pontrjagin-Thom construction to the horizontal maps, and one-point compactification to the vertical maps, to give a commutative square, which is the left-hand half of the right-hand diagram. Now let  $\epsilon: A/B \rightarrow 1$  be the projection, and let  $\zeta_0$  and  $\zeta_\alpha$  be the evident inclusions, to give the remaining square. We claim that this commutes up to equivariant homotopy. To see this, note that  $\mathcal{U}^B$  is a complete universe for the group  $A/B$ , so  $S(\mathcal{U}^B)$  is connected, and it contains  $S(L_0)$  and  $S(L_\alpha)$ . Choose a path  $\lambda: [0, 1] \rightarrow S(\mathcal{U}^B)$  with  $\lambda(0) \in S(L_0)$  and  $\lambda(1) \in S(L_\alpha)$ . Define  $A$ -equivariant maps  $\pi_t: (A/B)_+ \wedge S^{L_\alpha} \rightarrow PU^T$  by

$$\pi_t(q(a) \wedge z) = (\mathbb{C}.a.\lambda(t), z\alpha'(a)^{-1}a.\lambda(t)) = a.(\mathbb{C}.\lambda(t), z\alpha'(a)^{-1}\lambda(t)).$$

We find that  $\epsilon_0 = \zeta_0\delta$  and  $\epsilon_1 = \zeta_\alpha(\epsilon \wedge 1)$ , so these maps are homotopic as claimed.

We now chase the Thom class  $\bar{u}_T \in \tilde{E}^0(PU^T)$  around the diagram. We have  $(\zeta_0 i^!)*\bar{u}_T = \zeta_0^*\bar{u}_T = \bar{u}_{L_0}$  so

$$(\zeta_0 i^! \theta)^*(\bar{u}_T) = \theta^*(\bar{u}_{L_0}) = v_m(\phi(\alpha))\bar{u}_{L_\alpha}.$$

On the other hand, we have  $\zeta_\alpha^*(\bar{u}_T) = \bar{u}_{L_\alpha}$ . Moreover,  $j^!$  is  $1_{S^{L_\alpha}}$  smashed with the transfer map  $S^0 \rightarrow (A/B)_+$ , so  $((\epsilon \wedge 1)j^!)*(\bar{u}_{L_\alpha}) = \text{tr}_B^A(1)\bar{u}_{L_\alpha}$ . Putting this together we find that  $\text{tr}_B^A(1)\bar{u}_{L_\alpha} = v_m(\phi(\alpha))\bar{u}_{L_\alpha}$ , but  $\tilde{E}^0(S^{L_\alpha})$  is freely generated over  $E^0$  by  $\bar{u}_{L_\alpha}$  so  $\text{tr}_B^A(1) = v_m(\phi(\alpha))$  as claimed.  $\square$

**PROOF OF PROPOSITION 24.30.** Write  $A/B$  as a product of cyclic groups. Each cyclic factor can itself be regarded as a quotient of  $A$ , so we get a decomposition  $A/B = \prod_i A/B_i$  say. Now  $[A/B] = \prod_i [A/B_i]$  in  $\Omega(A)$ , and both  $\eta$  and  $\eta'$  are ring maps, so it will suffice to prove that  $\eta[A/B_i] = \eta'[A/B_i]$ . This is clear from Lemma 24.31.  $\square$

**24.4. Mackey structure.** We now want to take this further, by constructing a Mackey functor.

**CONSTRUCTION 24.32.** Put  $k = \mathcal{O}_S$  for convenience. Given a subgroup  $B \leq A$ , put  $k_B = k/I(\text{ann}(B), \phi)$ , so that the subscheme  $S[B] = \text{spec}(k_B)$  is the largest closed subscheme of  $S$  over which  $\phi: A^* \times S \rightarrow C$  factors through  $B^*$ . We will also write  $C[B] = C \times_S S[B]$ , so we have an induced map  $\phi_B: B^* \times S[B] \rightarrow C[B]$ , and thus a ring map  $\eta_B: \Omega(B) \rightarrow k_B$ .

If  $C \leq B$  then  $\text{ann}(C) \geq \text{ann}(B)$  so  $I(\text{ann}(C), \phi) \geq I(\text{ann}(B), \phi)$ , so we have a quotient map  $k_B \rightarrow k_C$ . We write  $\text{res}_C^B$  for this map, and note that the kernel is  $I(\text{ann}(C)/\text{ann}(B), \phi_B)$ .

Next, note that the element  $\tau_C^B = \eta_B([B/C]) = t(\text{ann}(C)/\text{ann}(B), \phi_B)$  annihilates the ideal  $I(\text{ann}(C)/\text{ann}(B), \phi_B)$ , so multiplication by  $\tau_C^B$  induces a well-defined map  $\text{tr}_C^B: k_C \rightarrow k_B$ .

As the third ingredient for a Mackey functor we need to specify conjugation maps  $\gamma_a: k_B \rightarrow k_B$  for all  $a \in A$ . (Here the target is morally  $k_{B'}$ , where  $B'$  is conjugate to  $B$  by  $a$ , but of course  $B' = B$  because  $A$  is abelian.) We take all the maps  $\gamma_a$  to be the identity.

**PROPOSITION 24.33.** *The above definitions make the system  $\{k_B\}_{B \leq A}$  into a Mackey functor.*

**PROOF.** The axioms to be checked are as follows.

- (a)  $\text{res}_B^B = 1$  and  $\text{res}_D^B = \text{res}_D^C \text{res}_C^B$  whenever  $D \leq C \leq B \leq A$ .
- (b)  $\text{tr}_B^B = 1$  and  $\text{tr}_D^B = \text{tr}_C^B \text{tr}_D^C$  whenever  $D \leq C \leq B \leq A$ .
- (c) If  $C, D \leq B \leq A$  then

$$\text{res}_C^B \text{tr}_D^B(r) = |B/(C + D)| \text{tr}_{C \cap D}^C \text{res}_{C \cap D}^D(r).$$

(In (c) we have silently made the obvious simplifications to the double coset formula arising from the commutativity of  $A$ . We have also not bothered to list the compatibility conditions between the conjugation maps  $\gamma_a$  and the other structure; these hold for trivial reasons, given that  $\gamma_a = 1$ .)

Here axiom (a) follows directly from the definitions, and axiom (b) is a translation of Theorem 24.12(f). Similarly, axiom (c) can be derived from part (g) of Theorem 24.12.  $\square$

**REMARK 24.34.** Recall that a *Green ring* is a Mackey functor  $R$  with a ring structure on each group  $R(B)$  such that the restriction maps  $\text{res}_C^B: R(B) \rightarrow R(C)$  and the conjugation maps  $\gamma_a$  are ring maps, and the Frobenius reciprocity formula holds: for  $C \leq B$  and  $r \in R(B)$  and  $s \in R(C)$  we have  $\text{tr}_C^B(\text{res}_C^B(r)s) = r \text{tr}_C^B(s)$ . The Mackey functor  $\{k_B\}$  clearly has these properties.



## Generalisations

We now offer some brief remarks about possible generalisations of the theory discussed in the main body of this memoir.

Firstly, one might want to work  $A$ -equivariantly when  $A$  is an infinite abelian group. For this one would need a good category of  $A$ -equivariant spectra. There is a well-established theory for the case where  $A$  is a compact Lie group. More recently a theory has been developed that covers all compact Hausdorff topological groups [11]. We do not know of any significant work on equivariant stable homotopy for noncompact groups. For our purposes it is probably most natural to focus on the compact Lie case. In that case one could define an  $A$ -equivariant formal group to consist of a commutative formal group scheme  $C$  over a base  $S$  together with a map  $\phi: A^* \times S \rightarrow C$  with certain properties. To formulate these, put  $I_\alpha = \{f \in \mathcal{O}_C \mid f(\phi(\alpha)) = 0\}$ , and for each representation  $V = \bigoplus_i L_{\alpha_i}$  put  $I_V = \prod_i I_{\alpha_i}$ . The conditions are:

- (a)  $C = \text{spf}(R)$  for some formal ring  $R = \mathcal{O}_C$ .
- (b)  $I_0$  is generated by a regular element (which implies the same for all the ideals  $I_V$ ).
- (c)  $I_V$  is open in  $R$ , and the map  $R \rightarrow \varprojlim_V R/I_V$  is an isomorphism of formal rings.

The formal scheme  $C$  here will not be a multicurve according to our definitions, because there is no single element  $y$  as in Definition 2.2. Nonetheless, we are not aware of any major obstructions to extending most of our work to this context; it should just be a case of making technical adjustments and extensions in a large number of places.

In a different direction, one might want to cover nonabelian groups. We believe that it is not possible to do this without a very different theory. The issues involved are clearest if we consider nonabelian compact Lie groups of positive dimension. Take  $G = SU(2)$ , and let  $T$  be the subgroup of diagonal matrices, which is isomorphic to  $S^1$  and is a maximal torus. Let  $V$  be the standard 2-dimensional complex representation of  $G$ , so  $S(V) \simeq G$  and  $PV \simeq G/T$ .

LEMMA 25.1.  $MP_G^*(PV)$  is not a free module over  $MP_G^*$ .

PROOF. As  $PV = G/T$  we have  $MP_G^*(PV) = MP_T^*$ . Let  $L$  be a generator of  $T^*$ , so that  $\text{res}_T^G(V) = L + L^{-1}$ . Let  $V_k$  be the  $k$ 'th symmetric tensor power of  $V = V_1$ ; it is well known that these give all the irreducible complex representations of  $G$ . Let  $e_k \in MP_G^0$  be the Euler class of  $V_k$ . One checks that  $\text{res}_T^G(V_2) = L^2 + 1 + L^{-2}$ . As this contains a trivial summand, we see that  $e_2$  maps to zero in  $MP_T^0$ . The claim will therefore follow if we can prove that  $e_2 \neq 0$  in  $MP_G^0$ . Note that if we invert  $e_k$  for all  $k > 0$ , what we get is  $MP_0^G(S^{\mathcal{U} \oplus \mathcal{U}^G})$  or equivalently  $\pi_0(\phi^G(MP_G))$ . One

can check that  $\phi^G(MP)$  is just the Thom spectrum of a certain virtual bundle over  $(\tilde{GU})^G$ , where  $\tilde{GU}$  is the semiinfinite grassmannian as in Section 23. An analogue of Proposition 23.2 gives

$$(\tilde{GU})^G = \lim_{\substack{\longrightarrow \\ n}} \prod_{k < n} (\mathbb{Z} \times BU),$$

with the  $k$ 'th piece corresponding to the representation  $V_k$ . The relevant virtual bundle is the  $G$ -fixed part of the tautological bundle over  $\tilde{GU}$ , which is the standard thing over the  $k = 0$  factor and zero over the other factors. This gives

$$\phi^G MP_G = MP \wedge \lim_{\substack{\longrightarrow \\ n}} ((\mathbb{Z} \times BU)_+)^{(n)}.$$

This is certainly nonzero, so none of the Euler classes  $e_k$  (with  $k > 0$ ) can be zero in  $MP_G^0$ .  $\square$

Although we have proved the lemma for  $SU(2)$ , we believe, based on various partial calculations, that the same phenomenon is common for finite nonabelian groups. The assumption that  $E_A^*(PV)$  is free over  $E_A^*$  is very deeply embedded in our work, so the above lemma is a serious obstruction to extending it.

It is worth noting, however, that  $KU_G^*(PV)$  is always free over  $KU_G^* = R(G)[u^{\pm 1}]$ , and in fact if  $\dim(V) = n$  then

$$KU_G^0(PV) = R(G)[L] / \sum_{i=0}^n (-L)^{n-i} \lambda^i(V).$$

(This is stated as Proposition 3.9 in Segal's paper [26]. The proof relies on a result that Segal states as Proposition 3.8, but does not prove; for that, see Proposition 4.9 of [3].) Given this, one might hope for a theory of  $G$ -equivariant formal groups applicable to a more limited range of cohomology theories, not including  $MP_G$ . In particular, one might hope for a version of equivariant elliptic cohomology that is similarly well-behaved. This may arise from some work of Jacob Lurie, but details have not yet appeared. We do not have many ideas on this front.



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