Math. Z. 218, 503–518 (1995)

Mathematische Zeitschrift

$K(2)$-homology of some infinite loop spaces

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Received 15 February 1993; in final form: 8 October 1993

1 Introduction

A result of [11] states that Morava K-homology of infinite loop spaces surject to the Morava K-homology of the spectra associated to them. As it turned out, in some cases we can get a stronger conclusion. Namely, in [10] the author showed the following.

Theorem 1.1 *Let X be a spectrum with stable cells in non-negative even degrees only. Then $K_*(\underline{X}_0, Z/p) \cong \otimes K_*(QS^{2n}, Z/p)$ corresponding to the stable cell decomposition of X. Here $Q = \Omega^\infty \Sigma^\infty$.*

This result asserts that in these cases, not only mod p K-homology of the infinite loop spaces surjects to the mod p K-homology of the spectra, but also the stable mod p K-homology classes generate the mod p K-homology of the infinite loop spaces, regarded as algebras over McClure's Dyer-Lashof operation. This was proved based on the fact that $K_*(QS^{2n}, Z/p)$'s are polynomial algebras [13]. The analysis of $K(n)_*(\underline{MU}_*)$ led the author to conjecture that $K(n)_*(QS^{2m})$'s are polynomial algebras. In this paper, we resolve this conjecture affirmatively when $n = 2$ for odd primes. Namely, we prove

Theorem 1.2 *Let $p > 2$. Then*

$$\begin{aligned} K(2)_*(CS^0) &\cong K(2)_*[Q_I \iota_0 : I \text{ is an allowable sequence}] \\ K(2)_*(QS^0) &\cong K(2)_*[Q_I \iota_0 : I \text{ is an allowable sequence}] \otimes_{K(2)_*} K(2)_*[\iota_0^{-1}] \\ K(2)_*(QS^{2n}) &\cong K(2)_*[Q_I \iota_{2n} : I \text{ is an allowable sequence}] \end{aligned}$$

* Partially supported by Alfred P. Sloan doctral dissertation fellowship
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(Here $n > 0$.) For the definition of allowable sequences, see Definition 2.6, and for the definition of CX, see Definition 2.1.

Corollary 1.3 *Let X be a spectrum with stable cells in non-negative even degrees only. Then $K(2)_*(\underline{X}_0) \cong \otimes K(2)_*(QS^{2n})$ corresponding to the stable cell decomposition of X.*

The first computation of $K_*(QS^0, Z/p)$ in [5] depended on the functorial isomorphism $K^*(BG) \cong R(G)^\wedge$, which, in turn, depended on the geometric interpretentation of the complex K-theory. The computation in [13] also relies on the geometric interpretentation of the complex K-theory to define generalized Dyer-Lashof operation. As such geometric interpretentation of higher Morava K-theory doesn't exist, their methods fail to generalize. Fortunately, some ingredients in the computation in [13] do not depend on his particular construction of generalized Dyer -Lashof operation. Thus we can modify his computation, so that it works without his sophisticate machinary. That is, we can easily generalize the definition of the generalized Dyer-Lashof operation in [6], using the result of [8], and use the combinatorial method employed in [13]. Of course, one wouldn't expect such a naive method to work, without new ingredients. The new ingredients are two results in [7]. One is the rank formula for $K(n)^{even}(BG) - K(n)^{odd}(BG)$ for finite groups G, that gives lower bound for the number of generators in $K(n)_*(QS^{2m})$. The other is the fact that the dual basis for the elements in the image of our generalized Dyer-Lashof operations is given by transfer of Euler classes, which gives a more or less concrete description of the elements in the image of Dyer-Lashof operations (fortunately, as far as $K(n)_*(QS^0)$ is concerned, we don't need transfer image of Euler classes of representations of proper subgroups).

The organization of the paper is as follows. In section 2, we recall some basic facts about infinite loop space, and outline the proof of Theorem 1.2. In section 3, we essentially calculate the number of the elements in $\mathrm{Rep}((Z_p^\wedge)^2, \Sigma_m)$, which is same as the rank of $K(2)_*(B\Sigma_m)$. In section 4, we recall the relevant result in [8], and define generalized Dyer-Lashof operations. In section 5, we compute the map $K(2)_*(B(\Sigma_p \wr \Sigma_p)) \to K(2)_*(B\Sigma_{p^2})$. In section 6, we prove the Adem relations (in a very weak form, but sufficient for our purpose) based on the result of section 5, and prove the Cartan formulas as well. In section 7 we use the bar spectral sequence to prove Corollary 1.3. Section 7 and Proposition 3.1 are taken from author's doctral dissertation written under the supervison of W. S. Wilson and J. M. Boardman. He woulld like to express his gratitude to G. Nishida, H. Sadofsky, W. S. Wilson, C. -F. Bödigheimer, L. Langsetmo, J. Morava, D. Tamaki, N. Yagita, F. R. Cohen, M. Tanabe, J. -P. Meyer, J. M. Boardman, and A. Kono for helpful conversations at various stages of the work, and to D. C. Ravenel for many useful suggestions for typesetting.

2 Proof of the main theorem

Proof of Theorem 1.2. Throughout this paper we fix an odd prime p so that $K(n)^*(\_)$ has a multiplication which is associative and commutative [14] [19] [15]. We recall from the theory of infinite loop spaces the following.

Definition 2.1 For a topological space X, $CX = \cup_n E\Sigma_n \times_{\Sigma_n} X^n / \sim$, where $\sim$ is the equivalence defined in [12]. Define a filtration F on CX by $F_i(CX) = \cup_{n \leq i} \mathrm{Im}(E\Sigma_n \times_{\Sigma_n} X^n \to CX)$. Define $C'X$ to be $\vee_i F_i(CX)/F_{i-1}(CX)$.

Note that CX has a monoid structure arrising from the maps $E\Sigma_n \times X^n \times E\Sigma_m \times X^m \to E\Sigma_{m+n} \times X^{m+n}$. It is well-known and easy to verify that $F_i(CX)/F_{i-1}(CX) = E\Sigma_i^+ \wedge_{\Sigma_i} X^{\wedge i}$. Here are some standard facts.

Theorem 2.2 (e.g. [1, 12]) *The group completion of CX is homotopy equivalent to QX. Furthermore they are equivalent as H-spaces. (Here the H-space structure on QX comes from its loop sum map.)*

Theorem 2.3 (e.g. [16]) *$C'X$ is stable homotopy equivalent to CX.*

In particular, if $X = S^0$, $CX = C'X = \amalg_n B\Sigma_n$, and its monoid structure is induced component-wise by the inclusions $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$.

Furthermore, we recall that an H- space X is called H_p^∞-space if there exists a map $E\Sigma_p \times_{\Sigma_p} X^p \to X$ that extends the p-fold multiplication map $X^p \to X$ (here we are following the terminology of [4]). Infinite loop spaces, (in particular spaces of form QY) spaces of form CX, $C'X$ are examples of such spaces. In particular, it is well known that the H_p^∞-space structure on CS^0 that is compatible with that of QS^0 (arrising from its obvious infinite loop space structure) is component-wise given by the maps $\Sigma_p \wr \Sigma_n \to \Sigma_{np}$ (For the definition of $\wr$, see §5). Now, we start proving Theorem1.2. If $q = p^r s$ where $s > 1$ and not divisible by p, then the p-Sylow subgroup of Σ_q is contained in $\Sigma_{p^r} \times \Sigma_{(s-1)p^r}$. Therefore, using the Kahn-Priddy transfer, one sees that p-locally $B\Sigma_q$ is a stable retract of $B\Sigma_{p^r} \times \Sigma_{(s-1)p^r}$. Thus the map $B\Sigma_{p^r} \times \Sigma_{(s-1)p^r} \to B\Sigma_q$ induces a surjection in any p-local homology theory, in particular in $K(2)$-homology, which means that $K(2)_*(CS^0)$ is generated as an algebra by elements in $K(2)_*(B\Sigma_{p^r})$'s. Furthermore, since the p-Sylow subgroup of Σ_{p^r} is contained in $\Sigma_p \wr \cdots \wr \Sigma_p$, by a similar argument one sees that $K(2)_*(B\Sigma_p \wr \cdots \wr \Sigma_p)$ surjects to $K(2)_*(B\Sigma_{p^r})$. Now, to deal with the former object, we will prove later in §5,

Theorem 2.4 *Let X be a space for which the Atiyah-Hirzebruch-Serre spectal sequence for the fibration $X^p \to E\Sigma_p \times_{\Sigma_p} X^p \to B\Sigma_p$ is simple (the term is defined below in §4). Then we can define the outer Dyer-Lashof operation $Q_i (1 \leq i \leq \frac{p^{i+1}-1}{p-1})$ modulo appropriate indeterminacy (which will be precised in §4) so that if A is a basis for $\tilde{K(n)}_*(X)$, then $\tilde{K(n)}_*(E\Sigma_p \times_{\Sigma_p} X^p)$ is spanned by "products" of elements in $K(n)_*(X)$ (i.e., the image of elements of $K(n)_*(X^p)$) and the elements of the form $Q_i(a), a \in A$. Here it does* not *matter how to choose representatives for $Q_i(a)$ in the indeterminacy.*

We apply this to the case $X = B\,\overbrace{\Sigma_p \wr \cdots \wr \Sigma_p}^{r-1 \text{ factors}}$ (Theorem 4.3 will show that this is applicable to our X.) to conclude that we can obtain elements of $K(2)_*(B\Sigma_p \wr \cdots \wr \Sigma_p)$ can be obtained by iterating our Dyer-Lashof operations and taking products. However, as we will prove

Proposition 2.5 *Let x be a decomposable element in $K(2)_*(CS^0)$. Then $Q_i(x)$ is again decomposable.*

This means that the elements in $K(n)_*(CS^0)$ can be obtained as a products of iterations of Dyer-Lashof operations on the fundamenatal class, i.e., the unit for $K(n)_*(B\Sigma_1)$. For $I = (I_1, \ldots I_n)$, define Q_I to be $Q_{I_1} \ldots Q_{I_n}$.

Definition 2.6 A sequence of integers $I = (I_1, \ldots I_n)$ with $1 \leq I_j \leq p+1$ is called *allowable* whenever $I_j = 1$, $I_{j-1} = 1$.

Now our version of Adem relation, which will be proved in §6 is

Proposition 2.7 *Let $1 < i \leq p+1$, and let I be an allowable sequence. Then $Q_i Q_1 Q_I(\iota_0)$ can be written as a linear combination of the elements of the form $Q_{i_1} Q_{i_2} Q_I(\iota_0)$ where (i_1, i_2) is allowable.*

Therefore $\{Q_I \iota_0 : I \text{ is an allowable seequence}\}$ generates $K(2)_*(CS^0)$. But in §3, we will prove that

Proposition 2.8 *$K(2)_*(CS^0) \cong K(2)_*[Q_I * | I$ is an allowable sequence$]$ as graded $K(2)_*$-modules. Here the grading for $K(2)_*(CS^0)$ comes from the value of m in the decomposition $K(2)_*(CS^0) \cong \oplus_m K(2)_*(B\Sigma_m)$, and that for $K(2)_*[Q_I *]$ is defined by $|Q_I *| = p^{l(I)}$ where l is the length of I.*

Here $Q_I *$'s are, for the time being, meaningless symbols for indeterminates, although later we will define Dyer-Lashof operations in appropriate Burnside ring to give them a meaning. Now this result shows that there can't be any relation in $K(2)_*(CS^0)$, which proves the first statement of Theorem 1.2. The second statement is a formal consequence of the first and Theorem 2.2. To prove the third, notice that $E\Sigma_n^+ \wedge_{\Sigma_n} S^{2n}$ is the Thom space of the vector bundle $E\Sigma_n \times_{\Sigma_n} C^{2n} \to B\Sigma_n$. Thus we have Thom isomorphism $K(2)_*(B\Sigma_n) \cong \tilde{K}(2)_{*+2n}(\Sigma_n^+ \wedge_{\Sigma_n} S^{2n})$. The naturality and multiplicativity of Thom isomorphism implies that this gives an isomorphism of algebra (which doesn't respect the degree) $K(2)_*(CS^0) \cong K(2)_*(C'S^{2n})$. As we have proved the former to be a polynomial algebra, so is the latter. Thus there is no algebra extension problem to recover $K(2)_*(CS^{2n})$, and we have

$$K(2)_*(CS^0) \cong K(2)_*(C'S^{2n}) \cong K(2)_*(CS^{2n}).$$

It is easy to see that the Thom isomorphism commutes with our Dyer-Lashof operations, and we get the desired result. The only thing that remains to be shown is that in $K(2)_*((QS^{2n}))$ the Dyer-Lsahof operations can be defined, and it will be done in §7. The argument is not circular as in §7 we only use the fact that $K(2)_*(QS^{2n})$'s are polynomial algebras, and not the fact that the generators are obtained using the Dyer-Lashof operations. □

3 The Burnside semi-ring for $(Z_p^\wedge)^2$

The goal of the section is to prove Proposition 2.8. We will prove the following.

Proposition 3.1 *$K(n)_*(CS^0) \cong K(n)_*[\{(Z_p^\wedge)^n/H\}]$ as graded graded $K(n)_*$-modules (grading of the left hand side is as in Proposition 2.8), where $(Z_p^\wedge)^n/H$ has a degree $|Z_p^{\wedge n}/H|$.*

Then we specialize to the case when $n = 2$. We define "generalized Dyer-Lashof operations" on the Burnside semi-ring of $(Z_p^\wedge)^2$, (or rather, on its free generators as abelian semigroup).

Define $f_i : Z_p^\wedge \times Z_p^\wedge \to Z_p^\wedge \times Z_p^\wedge$ $(1 \le i \le p+1)$ as follows.

$$\begin{aligned} f_1(1,0) &= (1,0) \\ f_1(0,1) &= (0,p) \end{aligned}$$

and for $2 \le i$

$$\begin{aligned} f_i(1,0) &= (p,0) \\ f_i(0,1) &= (i-1,1) \end{aligned}$$

Now, let A be any transitive $Z_p^\wedge \times Z_p^\wedge$-set. Thus $A = Z_p^\wedge \times Z_p^\wedge/H$. Define $Q_i(A)$ to be $Z_p^\wedge \times Z_p^\wedge/f_i(H)$.

Lemma 3.2 *Any non-empty transitive $(Z_p^\wedge)^2$-set can be written uniquely as $Q_I *$, where $*$ is the $(Z_p^\wedge)^2$-set with single element, and I is an allowable sequence.*

Now Proposition 2.8 can be obtained just by combining this with Proposition 3.1.

To prove the Proposition 3.1, we recall the following.

Theorem 3.3 ([7]) Rank$K(n)^{even}(BG)$ − Rank$K(n)^{odd}(BG)$ = #Rep$[(Z_p^\wedge)^n, G]$. *Here $(Z_p^\wedge)$ denotes the additive group of p-adic integers, and* Rep$[H, G]$ *denotes the set* Hom$[H, G]/G$, *where G acts on* Hom$[H, G]$ *by conjugation.*

Proof of Proposition 3.1. By [8], we know that $K(n)^{odd}(B\Sigma_m) = 0$. Therefore, by Theorem 3.3 we have, Rank$K(n)^*(B\Sigma_m)$ = #Rep$[(Z_p^\wedge)^n, \Sigma_m]$. Thus we have

$$K(n)_*(CS^0) \cong \oplus_m K(n)_*(B\Sigma_m) \cong K(n)_*\langle \cup_m \mathrm{Rep}[(Z_p^\wedge)^n, \Sigma_m]\rangle.$$

Here $F\langle I\rangle$ denotes the free module over F whose basis is I, and the isomorphism is an isomorphism of graded $K(n)_*$-modules, grading coming from the value of m. Now, Rep$[(Z_p^\wedge)^n, \Sigma_m]$ can be identified with the set of isomorphism classes of $(Z_p^\wedge)^n$-sets with m elements, so one can define an addition in $\cup_m$Rep$[(Z_p^\wedge)^n, \Sigma_m]$ by disjoint union of sets. This makes $\cup_m$Rep$[(Z_p^\wedge)^n, \Sigma_m]$ into an abelian semigroup. It is easy to see it is a free abelian semigroup generted bu $(Z_p^\wedge)^n/H$'s, where H is a subgroup of $Z_p^{\wedge n}$. This makes $K(n)_*\langle\cup_m$Rep$[(Z_p^\wedge)^n, \Sigma_m]\rangle$ into an algebra, namely, semigroup-ring. But as semigroup-ring of a free semigroup is nothing but a polynomial algebra, we have established the Proposition. □

Proof of Lemma 3.2. Let $\#A = p^r$. Suppose $(p^{r-1}, 0) \in H$. Then it is easy to see that $(a, i) \notin H$ unless $p|i$. Thus $H \subset Im(f_1)$. By repeating, one obtains that $H = f_1^s(K)$, $(p^{r-s-1}, 0) \notin K$. Furthermore, s and K are uniquely determined by H. Next task is to classify subgroups $H \subset Z_p^\wedge \times Z_p^\wedge$ with the property that Index$(H) = p^r$, $p^{r-1}(1,0) \notin H$. Such a H is uniquely determined by a number d such that $0 \le d < p^{r-1}$, $(d, 1) \in H$. Call such H H_d^r. Notice that $f_i(H_d^r) = H_{pd+i-1}^{r+1}$. Therefore $H_d^r \in Imf_i \Leftrightarrow d \cong i-1 \bmod p$. Now the statement follows.

Remark. The number of allowable sequences of length l is $\frac{p^{l+1}-1}{p-1}$. Thus the rank of $K(2)_*(B\Sigma_m)$ is the coefficient of t^m in the formal power series $\Pi(1 - t^{p^l})^{-(p^{l+1}-1)/(p-1)}$. This can be proved by simply counting elements of given order in groups $Z/(p^r) \times Z/(p^r)$, etc., but we presented the proof above, as our method s more likely to be generalized to the case for arbitrary n.

4 $K(n)$-homology of wreath products of spaces after Hunton and generalized Dyer-Lashof operations

In this section, we prove Theorem 2.4. First we recall a definition.

Definition 4.1 ([2]) *The Atiyah-Hirzebruch-Serre spectral sequence (AHSSS in short) for the fibration $X^p \to EZ/p \times_{Z/p} X^p \to BZ/p$*

$$H^*(Z/p, K(n)^*(X)^{\otimes p}) \Rightarrow K(n)^*(EZ/p \times_{Z/p} X^p)$$

is called simple if it has just one non-zero differential, namely d_{2p^n-1}, forced by the action of the Atiyah-Hirzebruch spectral sequence for $K(n)^*(BZ/p)$.

At the end of the section, we will prove,

Proposition 4.2 *Let $\{a_i | i \in I\}$ be a basis for $K(n)_*(X)$, and give a total order on I. The Atiyah-Hirzebruch-Serre spectral sequence for the fibration $X^p \to E\Sigma_p \times_{\Sigma_p} X^p \to B\Sigma_p$*

$$H_*(\Sigma_p, K(n)_*(X)^{\otimes p}) \Rightarrow K(n)_*(E\Sigma_p \times_{\Sigma_p} X^p)$$

has just one non-zero differential, namely d^{2p^n-1}. Thus, $E^\infty \cong .A \otimes_{Z/p} K(n)_*(B\Sigma_p) \oplus B$, where A is as above, and B has representatives in E_2 of the form

$$\{a_{i_1} \otimes \ldots \otimes a_{i_p} | i_j \le i_{j+1}, \exists j, i_j < i_{j+1}\}.$$

Proof of Theorem 2.4 modulo Proposition 4.2. For $1 \le i \le p^{n-1} + \ldots + 1$, denote by $e_{2i(p-1)}$ the dual basis elements for the basis $x^{i(p-1)}$ of $K(n)^*(B\Sigma_p)$. (Here we identify $K(n)^*(B\Sigma_p)$ with a subalgebra of $K(n)^*(BZ/p)$ via the inclusion $Z/p \subset \Sigma_p$.) Now we can define outer homology operation Q_i's from $K(n)_*(X)$ to $K(n)_*(E\Sigma_p \times_{\Sigma_p} X^p)$ for spaces X satisfying our hypothesis by $Q_i(y) = e_{2i(p-1)} \otimes y^{\otimes p}$ in the E^∞-term of the AHSSS. Thus they are well-defined modulo the indeterminacy generated by elements in the image of the elements of $K(n)_*(X^p)$ and image of Q_j's with $j < i$. If X has an H_p^∞ structure, then send them by the structure map $E\Sigma_p \times_{\Sigma_p} X^p \to X$ to define (modulo appropriate indeterminacy) inner homology operation, which will be denoted by Q_i as well. This agrees, of course, with the definition given by [6] when $n = 1$. □

Remark. In the case of ordinary mod p homology, the corresponding spectral sequence always collapses at E_2, and there is a canonical choice of the elements $e_{2i(p-1)} \otimes y^{\otimes p}$, namely, the chain which has the same form. Thanks to these facts, the theory of Dyer-Lashof operations for mod p homology is much more straightforward.

Proof of Proposition 4.2. Let X be a space for which the AHSSS for the fibration $X^p \to EZ/p \times_{Z/p} X^p \to BZ/p$

$$H^*(Z/p, K(n)^*(X)^{\otimes p}) \Rightarrow K(n)^*(EZ/p \times_{Z/p} X^p)$$

is simple. Then, it follows almost immediately from the definition that if $\{a_i | i \in I\}$ is a basis for $K(n)^*(X)$, then the E^∞ term has the form $E_\infty \cong A \otimes_{Z/p} K(n)^*(BZ/p) \oplus B$, where $A \subset E_\infty^{0,*}$ has representatives in E_2 of the form $\{a_i^{\otimes p} | i \in I\}$, and $B \subset E_\infty^{0,*}$ has representatives in E_2 of the form

$$\{\Sigma_{\sigma \in Z/p} \sigma(a_{i_1} \otimes \ldots \otimes a_{i_p}) | \text{not all } i_j\text{'s are equal}\}.$$

Remark. Notice that if $X = BG$, and $y \in K(n)^*(BG) = e(\phi)$ for some Unitary representation ϕ, then $y^{\otimes p} \in K(n)^*(BZ/p \wr G)$ can be represented by $e(Z/p \wr \phi)$, where e denotes the Euler class. Furthermore, if $x \in K(n)^2(BZ/p)$ is the image of the orientation class, then $x^i \otimes y^{\otimes p} \in K(n)^*(BZ/p \wr G)$ can be represented by some Euler classes for $0 \leq i \leq 2(p^n - 1)$. A generalization of this fact was used in [7] to prove the collapsing of the Atiyah-Hirzebruch-Serre spectral sequence when $X = BG$ and G is good.

As the Atiyah-Hirzebruch-Serre spectral sequence for the fibration $X^p \to E\Sigma_p \times_{\Sigma_p} X^p \to B\Sigma_p$ splits off from the one we have just discussed, we see that, the Atiyah-Hirzebruch-Serre spectral sequence for the fibration $X^p \to E\Sigma_p \times_{\Sigma_p} X^p \to B\Sigma_p$

$$H^*(\Sigma_p, K(n)^*(X)^{\otimes p}) \Rightarrow K(n)^*(E\Sigma_p \times_{\Sigma_p} X^p)$$

also has just one non-zero differential, namely d_{2p^n-1}. Thus, $E_\infty \cong A \otimes_{Z/p} K(n)^*(B\Sigma_p) \oplus B$, where A is as above, and B has representatives in E_2 of the form

$$\{\Sigma_{\sigma \in \Sigma_p} \sigma(a_{i_1} \otimes \ldots \otimes a_{i_p}) | \text{not all } i_j\text{'s are equal}\}.$$

By dualizing we obtain the desired result. □

Following results show that our arguments can be applied to a number of spaces.

Theorem 4.3 ([2]) *If $K(n)^*(X)$ has finite rank over $K(n)^*$ and $K(n)^{odd}(X) = 0$, then The Atiyah-Hirzebruch-Serre spectral sequence (AHSSS in short) for the fibration $X^p \to EZ/p \times_{Z/p} X^p \to BZ/p$ is simple.*

Definition 4.4 ([8]) *A space Y is called to be* Unitary-Like *if in $K(n)^*(Y)$ the p-th power map is injective. A space X is called to have a Unitary-Like embedding if there is a map $f : X \to Y$ with Y unitary-like and $K(n)_*(f)$ is injective.*

Theorem 4.5 ([8]) *If X has a ULE then the AHSSS for the fibration $X^p \to EZ/p \times_{Z/p} X^p \to BZ/p$ is simple.*

In particular, Theorem 4.3 applies to $B\Sigma_m$'s (the vanishing of $K(n)^{odd}(B\Sigma_m)$ was shown in [8]), and therefore to CS^0.

Remark. It was shown in [8] that if X has a ULE, then so does $EZ/p \times_{Z/p} X^p$. Although our work doesn't depend on this fact, it was this fact which motivated our work.

5 $K(2)$-cohomology of Σ_{p^2}

In this section, we describe the map $K(2)^*(B\Sigma_{p^2}) \to K(2)^*(B(\Sigma_p \wr \Sigma_p))$. First we fix some notations and recall necessary preliminaries.

Definition 5.1
- if G is a subgroup of Σ_r and H is any group, by $G \wr H$ we mean the semi-direct product of G and H^r with G acting on H^r by permutation of factors.
- $$\pi_i^j : \underbrace{\overbrace{\Sigma_p \wr \ldots \wr \Sigma_p}^{i \text{ factors}} \wr \ldots \wr \Sigma_p}_{j \text{ factors}} \to \overbrace{\Sigma_p \wr \ldots \wr \Sigma_p}^{i \text{ factors}}$$
  will denote the projection.
- $\rho_{p^n-1} : \Sigma_{p^n} \to U(p^n - 1)$ will denote the representation of the symmetric group on $C^{p^n-1} \subset C^{p^n}$ given by the equation $z_1 + \ldots + z_{p^n} = 0$ via the permutation of the basis of C^{p^n}.
- ι is the inclusion $\Sigma_p \wr \Sigma_p \subset \Sigma_{p^2}$.
- $Z/(p^2)$ and $(Z/p)^2$ are considered as their subgroups via the regular representations.
- $(\Sigma_p)^{\times p}$ is considered as their subgroup in the natural way.
- $(Z/p)^{\times p}$ is considered as their subgroup via the inclusion $Z/p \subset \Sigma_p$. This can't cause a confusion, as $p \neq 2$.
- We fix inclusions $\Sigma_n \wr U(m) \subset U(nm)$.
- If ϕ is any representation $G \to U(m)$, by abuse of notation, we use $\Sigma_n \wr \phi$ to denote the representation obtained by the composition $\Sigma_n \wr \phi : \Sigma_n \wr G \to \Sigma_n \wr U(m) \subset U(nm)$.
- $\theta_{p^i} : Z/(p^i) \subset U(1)$ is a fixed inclusion.
- $\mu : U(1) \times U(1) \to U(1)$ denotes the multiplication.

We will use following facts in the subsequent sections of the paper without explicit reference. The proof is straightforward and omitted.

Proposition 5.2 *(i)* $\rho_{p^2-1}|\Sigma_p \wr \Sigma_p = \rho_{p-1} \circ \pi_1^2 \oplus \Sigma_p \wr \rho_{p-1}$.
(ii)

$$\begin{aligned}
\rho_{p-1} \circ \pi_1^2 | Z/(p^2) &= \oplus_{p|j \in Z/(p^2)-\{0\}} \theta_{p^2} \circ (j \times _) \\
\Sigma_p \wr \rho_{p-1} | Z/(p^2) &= \oplus_{p \nmid j \in Z/(p^2)-\{0\}} \theta_{p^2} \circ (j \times _) \\
\rho_{p-1} \circ \pi_1^2 | Z/p \times Z/p &= \oplus_{j \in Z/p-\{0\}} \mu(\theta_p \times \theta_p(j \times _, j \times _)) \\
\Sigma_p \wr \rho_{p-1} | Z/p \times Z/p &= \oplus_{(i,j) \in Z/p \times Z/p - \mathrm{Im}\Delta} \mu(\theta_p \times \theta_p(i \times _, j \times _))
\end{aligned}$$

(iii) $\pi_i^j \circ \pi_j^k = \pi_i^k$
(iv) $\Sigma_p \wr \pi_i^j = \pi_{i+1}^{j+1}$
(v) $\Sigma_p \wr \phi_{|G^p} = \phi^{\oplus p}$
Here Δ is the diagonal, μ is the addition, and $\phi : G \to U(n)$ is any representation. An equality means an isomorphism of representations.

Furthermore, we will need the following. $K(n)^*(BU(1)) \cong K(n)^*[[x]]$, $K(n)^*(BU(1) \times BU(1)) \cong K(n)^*[[x_1, x_2]]$, $K(n)^*(B\mu)(x) = x_1 +_F x_2$, where $+_F$ is the formal group law for $K(n)$-cohomology, $[1]x = x$, $[i]x = [i-1]x +_F x$, $K(n)^*(BZ/(p^i)) \cong K(n)^*[x]/([p^i]x)$, where x is the image of $x \in K(n)^*(BU(1))$ by $\theta_{p^i}^*$. As to the formal group law for $K(2)$, all we need is the following. $x_1 +_F x_2 \equiv x_1 + x_2$ mod higher power of (x_1, x_2), $[p]x = v_2 x^{p^2}$. First we have to deal with the "less interesting part" of $K(2)_*(B\Sigma_{p^2})$, namely the decomposables.

Lemma 5.3 *The polynomials on basis elements of $K(2)_*(B\Sigma_1)$ and $K(2)_*(B\Sigma_p)$ remains linearly independent in $K(2)_*(B\Sigma_{p^2})$.*

Proof The images of the basis elements in $K(2)_*(BU \times Z)$ by the unit map for complex K-theory spectrum, (which is just regular representations at space level) are $[1], b_1^{p-1}, \ldots, b_{p+1}^{p-1}$ which are algebraically independent. □

Now we can start discussing "more interesting" part. Our goal is

Theorem 5.4 $c_{p^2-1}(\rho_{p^2-1}) \times c^i_{p(p-1)}(\rho_{p^2-1})$ $(0 \le i \le p^2+p)$ *form a basis for the kernel of the map $K(2)^*(B\Sigma_{p^2}) \to K(2)^*((BZ/p)^{\times p})$.*

We prove it in three steps. In *step 1* we will analyze the restriction maps from $K(2)^*(B(\Sigma_p \wr \Sigma_p))$ to its abelian subgroups, in *step 2*, we will analize the restriction maps from $K(2)_*(B\Sigma_{p^2})$ to its abelian subgroups, and in *step 3* we complete the proof. However, before getting into the details of these steps, we state following two Lemmas for the sake of memory. It will be used without explicit reference in both of the steps. They are more or less immediate consequences of Proposition 5.2 and the remark made in the course of the proof of Proposition 4.2 respectively and proofs are omitted.

Lemma 5.5 *(i)* $c_{p-1}(\rho_{p-1} \circ \pi_1^2)_{|Z/(p^2)} \equiv (-v_n) x^{p^2(p-1)}$ *mod* $x^{p^2(p-1)+p^2-1}$.
(ii) $c_{p-1}(\rho_{p-1} \circ \pi_1^2)_{|(Z/p)^2} \equiv -(x_1 + x_2)^{p-1}$ *mod higher powers of* (x_1, x_2)
(iii) $c_{p^2-p}(\Sigma_p \wr \rho_{p-1})_{|Z/(p^2)} \equiv x^{p^2-p}$ *mod* $x^{p^2-p+p^2-1}$.
(iv) $c_{p^2-p}(\Sigma_p \wr \rho_{p-1})_{|(Z/p)^2} = x_1^{p-1} x_2^{p-1} \Pi_{i=2}^{p-1} (x_1 + i x_2)^{p-1}$.

Lemma 5.6 *A basis for* Ker : $K(2)^*(B(\Sigma_p \wr \Sigma_p)) \to K(2)^*((B\Sigma_p)^{\times p})$ *is given by* $c_{p-1}^i(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}^j(\Sigma_p \wr \rho_{p-1})$*'s with* $1 \leq i \leq p+1$, $0 \leq j \leq p+1$.

Results in step 1. We will prove the following results. To avoid interrupting the flow of ideas, we postpone the proof until the end of the section.

Proposition 5.7 *The kernel of the map*

$$\begin{aligned} K(2)^*(B\Sigma_p \wr \Sigma_p) \to K(2)^*((BZ/p)^{\times p}) \quad &\oplus \quad K(2)^*(BZ/(p^2)) \\ &\oplus \quad K(2)^*(B(Z/p \times Z/p)) \end{aligned}$$

is spanned by $c_{p-1}(\rho_{p-1} \circ \pi_1^2)^{p+1} c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^r$*'s (* $2 \leq r \leq p+1$*).*

Lemma 5.8 Ker$K(2)^*(B\Sigma_p \wr \Sigma_p) \to K(2)^*((BZ/p)^{\times p}) \oplus K(2)^*(B(Z/p \times Z/p))$ *has a basis consisting of elements* $c_{p-1}(\rho_{p-1} \circ \pi_1^2)^i c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^j$ *with* $i \geq 1$, $j \geq 2$, *and* $(i,j) \neq (1,2)$ *and possibly an element of the form*

$$c_{p-1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^2 + ac_{p-1}(\rho_{p-1} \circ \pi_1^2)^{p+1}c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$$

($a \in K(2)_*$*).*

Result in step 2. We will prove later, assuming the result of previous step,

Proposition 5.9 *The kernel of the map*

$$\begin{aligned} K(2)^*(B\Sigma_p \wr \Sigma_p) \quad &\to \quad K(2)^*((BZ/p)^{\times p}) \oplus K(2)^*(BZ/(p^2)) \\ &\oplus \quad K(2)^*(B(Z/p \times Z/p)) \end{aligned}$$

lies in the image of $K(2)^*(B\Sigma_{p^2})$*. Furthermore the restriction of the elements* $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}^{p^2+l}(\rho_{p^2-1})$ *(*$1 \leq l \leq p$*) gives a basis for the kernel.*

The proof is postponed until we deduce its consequences in the next step.
Step 3. Here we deduce Theorem 5.4 and a corollary from the Proposition 5.9.
Proof of Theorem 5.4. Thanks to Proposition 5.9, we have only to consider the part of $K(2)^*(B\Sigma_{p^2})$ that is detected by its abelian subgroups. Denote by K the kernel in the statement. Now consider the restriction to (Z/p^2) of the elements $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^i$ $(0 \leq i \leq p^2)$. Just by looking at the leading term, which is $x^{p^2(p-1)+(i+1)p(p-1)}$ up to multiplication by invertible elements, one can conclude that they are linearly independent. Since we have just proved that $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^{p^2+l}$ $(1 \leq l \leq p)$ form a basis of K, we obtain that $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^i$ $(0 \leq i \leq p^2+p)$'s are linearly independent. But our counting argument Proposition 2.8 and Lemma 5.3 shows that the dimension of the kernel of the map $K(2)^*(B\Sigma_{p^2}) \to K(2)^*((B\Sigma_p)^{\times p})$ can't be larger. This finishes the proof of Theorem 5.4. □

Corollary 5.10

$$\begin{aligned}\mathrm{Im}(K(2)^*(B\Sigma_{p^2}) &\to K(2)^*(B\Sigma_p \wr \Sigma_p)) \cap \mathrm{Ker}(K(2)^*(B\Sigma_p \wr \Sigma_p)\\ &\to K(2)^*((B\Sigma_p)^{\times p})\end{aligned}$$

is spanned by the elements of the form $c_{p-1}^i(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}^j(\Sigma_p \wr \rho_{p-1})$ *with* $2 \leq i \leq p+1$, $1 \leq j \leq p+1, (i,j) \neq (1,2)$, $c_{p-1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$, *and* $c_{p-1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}^2(\Sigma_p \wr \rho_{p-1}) + ac_{p-1}^{p+1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$, *where* $a \in K(2)^*$.

Proof It is easy to see that $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^i$ restricts to 0 in $Z/p \times Z/p$ if $i > 1$. Thus using Lemma 5.8, Proposition 5.9, and the fact that $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^j$ restricts in $Z/(p^2)$ to $x^{(p^2-p)(j+1)+(p-1)p^2}$ up to multiplication by invertible elements, one sees that the vector space spanned by $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})^j$'s with $j > 1$ is same as the vector space spanned by $c_{p-1}^i(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}^j(\Sigma_p \wr \rho_{p-1})$ with $2 \leq i \leq p+1$, $1 \leq j \leq p+1, (i,j) \neq (1,2)$. Obviously $c_{p^2-1}(\rho_{p^2-1})$ restricts to $c_{p-1}^i(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$. As to $c_{p^2-1}(\rho_{p^2-1})c_{p^2-p}(\rho_{p^2-1})$, by looking at its restriction to $(Z/p)^2$, one sees that up to linear combination of elements that are already taken care of, it must be of the form

$$a_1c_{p-1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}^2(\Sigma_p \wr \rho_{p-1}) + a_2c_{p-1}^{p+1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1}),$$

but by looking at its restriction to $Z/(p^2)$, a_1 must be invertible. This finishes the proof. □

Proofs in step 2. Now we prove Proposition 5.9. In view of Lemma 5.8, we can define a total ordering of the basis elements specified in the statement of the Lemma by the value of $p^2(p-1)i + p(p-1)j$ for $c_{p-1}(\rho_{p-1} \circ \pi_1^2)^i c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^j$'s, and if an element of the form $c_{p-1}(\rho_{p-1} \circ \pi_1^2)c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^2 + ac_{p-1}(\rho_{p-1} \circ \pi_1^2)^{p+1}c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$ is in the kernel we are considering, then by defining it to be smaller than any other elements in this basis. (If $x^{p(p-1)(p+1)^2}$ were non-zero in $K(2)^*(BZ/(p^2))$, then we could simply call it the ordering by the power of x in the leading term of the restriction to $Z/(p^2)$.) Now, note that $c_{p^2-p}(\rho_{p^2-1})^{p^2+l}$ $(1 \leq l \leq p)$ restricts to 0 in $(Z/p)^p$ and $(Z/p)^2$. Therefore we can express them as a linear combination of the basis elements we have constructed. Now by restricting them to $Z/(p^2)$, we see that

$$\begin{aligned}c_{p^2-p}(\rho_{p^2-1})^{p^2+1} &\equiv ac_{p-1}(\rho_{p-1} \circ \pi_1^2)^{p-1}c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^{p+1},\\ c_{p^2-p}(\rho_{p^2-1})^{p^2+l} &\equiv ac_{p-1}(\rho_{p-1} \circ \pi_1^2)^p c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^l (2 \leq l \leq p)\end{aligned}$$

modulo the vector space spanned by larger (in the ordering just defined) basis elements. Furthermore one has

$$c_{p-1}(\rho_{p-1} \circ \pi_1^2)^{p-1}c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^{p+2} \equiv c_{p-1}(\rho_{p-1} \circ \pi_1^2)^p c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^2.$$

Since $c_{p^2-1}(\rho_{p^2-1})$ restricts to $c_{p-1}(\rho_{p-1}\circ\pi_1^2)c_{p^2-p}(\Sigma_p\wr\rho_{p-1})$, the desired result follows from Proposition 5.7. □

Proofs in step 1. As promised, we come back to the proof of the Proposition 5.7, and Lemma 5.8. First we prove following Lemmas.

Lemma 5.11 *$c_{p-1}(\rho_{p-1}\circ\pi_1^2)^{p+1}c_{p^2-1}(\Sigma_p\wr\rho_{p-1})\neq 0$ in $Z/p\times Z/p$.*

Proof If σ_i denotes the i-th elementary symetric polynomial, then in Z/p, $\sigma_i(1,2,\ldots,p-1)=0$ if $i\leq p-1$ and $=-1$ if $i=p$, so it follows that

$$\Pi_{i=1}^{i=p-1}(x_1+ix_2)=x_1^{p-1}-x_2^{p-1},$$

from which one deduces that

$$\begin{aligned}&c_{p-1}(\rho_{p-1}\circ\pi_1^2)^{p+1}c_{p^2-1}(\Sigma_p\wr\rho_{p-1})_{|Z/p\times Z/p}\\&=-x_1^{p-1}x_2^{p-1}(x_1+x_2)^{p(p-1)}(x_1^{p-1}-x_2^{p-1})^{p-1},\end{aligned}$$

and this latter expression shows that the coefficient of $(x_1x_2)^{(p-1)(2p+1)/2}$ is $-2\times\binom{p-1}{(p-1)/2}$, therefore, : $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^{p+1}c_{p^2-1}(\Sigma_p\wr\rho_{p-1})\neq 0$ in $Z/p\times Z/p$.
□

Lemma 5.12 *$c_{p-1}(\rho_{p-1}\circ\pi_1^2)^ic_{p^2-p}(\Sigma_p\wr\rho_{p-1})^j$ is in the kernel of the map $K(2)^*(B\Sigma_p\wr\Sigma_p)\to K(2)^*(B(Z/p\times Z/p))$ if $i\geq 1$, $j\geq 2$, and $(i,j)\neq(1,2)$.*

Proof Unless $(i,j)=(2,2)$, their restriction in $K(2)^*(BZ/p\times BZ/p)$ lies in $(x_1,x_2)^{2p^2}$ which is trivial. When $(i,j)=(2,2)$, the coefficient of $x_1^{p^2-1}x_2^{p^2-1}$ is a multiple of $\binom{2(p-1)}{p-1}$, which is zero *mod* p. □

Proof of Proposition 5.8. The elements $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^ic_{p^2-1}(\Sigma_p\wr\rho_{p-1})^j$ with $i\geq 1, j\geq 2$, and $(i,j)\neq(1,2)$ are in the kernel according to Lemma 5.12. On the other hand by Lemma 5.11 the elements $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^ic_{p^2-1}(\Sigma_p\wr\rho_{p-1})^j$'s with $i\leq p+1, j\leq 1$ restricts nontrivially to $Z/p\times Z/p$. Now we consider the restriction of linear combination of these elements and $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^2c_{p^2-1}(\Sigma_p\wr\rho_{p-1})$. Just by considering the degree of the leading term, which is $2(p-1)(i+pj)$, one can conclude that if such a linear combination actually lies in the kernel, it must have the for of either $c_{p-1}(\rho_{p-1}\circ\pi_1^2)c_{p^2-1}(\Sigma_p\wr\rho_{p-1})+ac_{p-1}(\rho_{p-1}\circ\pi_1^2)^{p+1}+$ linear combination of some other $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^ic_{p^2-1}(\Sigma_p\wr\rho_{p-1})^j$'s, where $a\neq 0\in K(2)_*$, or $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^2c_{p^2-1}(\Sigma_p\wr\rho_{p-1})+ac_{p-1}(\rho_{p-1}\circ\pi_1^2)c_{p^2-1}(\Sigma_p\wr\rho_{p-1})^{p+1}$. where $a\in K(2)_*$. However the former is impossible because the term $x_1^{p^2-1}$ appears in $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^{p+1}$ whereas it doesn't in $c_{p-1}(\rho_{p-1}\circ\pi_1^2)c_{p^2-1}(\Sigma_p\wr\rho_{p-1})$. This completes the proof. □

Proof of Proposition 5.7. First we consider the restriction of the basis elements to the subgroup $Z/(p^2)$. Since the restriction of $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^ic_{p^2-p}(\Sigma_p\wr\rho_{p-1})^j$ has the leading term of the form $ax^{(p-1)p^2i+(p^2-p)j}$, and

$$(p+1)(p-1)p^2+2(p^2-p)=p^4+p^2-2p\geq p^4,$$

the specified basis elements are actually in the kernel. Furthermore, if a linear combination of other basis elements lie in the kernel, by looking at the degree of the leading term of its restriction to $Z/(p^2)$ one concludes that if it is non-trivial, then the coefficient of some basis element of the form $c_{p-1}(\rho_{p-1}\circ\pi_1^2)^i c_{p^2-p}(\Sigma_p \wr \rho_{p-1})^j$ with $0 \leq j \leq 1$, $1 < i$ must be non-zero. Thus using Lemma 5.12, one sees that it must contain terms which are non-zero multiple of $c_{p-1}(\rho_{p-1}\circ \pi_1^2)^{p+1}c_{p^2-p}(\Sigma_p \wr \rho_{p-1})$ and $c_{p-1}(\rho_{p-1}\circ\pi_1^2)c_{p^2-p}(\Sigma_p\wr\rho_{p-1})^2$. But, again by looking at the degree of the leading term of its restriction to $Z/(p^2)$ one concludes that their is no way to cancel the contribution of the latter in Z/p^2. This proves Proposition 5.7. □

6 Adem relations and Cartan formulas

In this section, we prove Adem relations and Cartan formulas in a very weak form, which still suffices for our purpose.

Proof of Proposition 2.5. This follows from the following commutative diagram.

$$\begin{tikzcd}
\Sigma_p \wr ((\Sigma_{p^n})^p) \arrow[r] \arrow[dd] & \Sigma_p \wr (\Sigma_{p^{n+1}}) \arrow[dr] & \\
 & & \Sigma_{p^{n+2}} \\
(\Sigma_p \wr \Sigma_{p^n})^p \arrow[r] & (\Sigma_{p^{n+1}})^p \arrow[ur] &
\end{tikzcd}$$

□

Remark. To be precise, we should say, "we can choose a representative of $Q_i(x)$ to be a decomposable element." But as all we want to establish is a generation result, indeterminacy is not going to affect our arguments. Henceforward, any statements on operations are meant to be modulo indeterminacy. Of course, we could have said that "$Q_i(x) = 0$", but we chose the statement as above because it reflects the proof.

Proof of Proposition 2.7. First suppose that I is non-empty. Now, according to the arguments in the remark in the proof of Proposition 4.2, the dual basis of these elements are given by

$$\begin{aligned}&c_{p-1}^{i_1}(\rho_{p-1}\circ\pi_1^n)c_{p(p-1)}^{i_2}(\Sigma_p \wr (\rho_{p-1}\circ\pi_1^{n-1}))\dots\\ &c_{p^{n-1}(p-1)}^{i_n}(\Sigma_p\wr\dots\wr\Sigma_p\wr(\rho_{p-1}\circ\pi_1^1))\end{aligned}$$

Now for any representation $\kappa : G \to U(r)$, $\Sigma_p \wr \Sigma_p \wr \kappa : \Sigma_p \wr \Sigma_p \wr G \to U(p^2 r)$ factors through the representation $\Sigma_{p^2}\wr\kappa : \Sigma_{p^2}\wr G \to U(p^2 r)$. Therefore,

$$\begin{aligned}&c_{p^2(p-1)}^{i_3}(\Sigma_p\wr\Sigma_p\wr(\rho_{p-1}\circ\pi_1^{n-2}))\dots c_{p^{n-1}(p-1)}^{i_n}(\Sigma_p\wr\dots\wr\Sigma_p\wr(\rho_{p-1}\circ\pi_1^1))\\ \in\ &\mathrm{Im}K(2)^*(B(\Sigma_{p^2}\wr\overbrace{\Sigma_p\wr\dots\wr\Sigma_p}^{n-2\text{ factors}}))\to K(2)^*(B(\Sigma_p\wr\Sigma_p\wr\overbrace{\Sigma_p\wr\dots\wr\Sigma_p}^{n-2\text{ factors}})).\end{aligned}$$

Therefore, by the property of the transfer [3], we have

$$\begin{aligned}
&Tr^{\Sigma_{p^2}\wr G}_{\Sigma_p\wr\Sigma_p\wr G}\Pi^n_{m=1} c^{i_m}_{p^{m-1}(p-1)}(\overbrace{\Sigma_p\wr\ldots\wr\Sigma_p}^{m-1\text{ factors}}\wr(\rho_{p-1}\circ\pi_1^{n-m+1}))\\
=\;& Tr^{\Sigma_{p^2}\wr G}_{\Sigma_p\wr\Sigma_p\wr G}(c^{i_1}_{p-1}(\rho_{p-1}\circ\pi_1^n))c^{i_2}_{p(p-1)}(\Sigma_p\wr(\rho_{p-1}\circ\pi_1^{n-1}))\times\\
&\Pi^n_{m=3} c^{i_m}_{p^{m-1}(p-1)}(\Sigma_{p^2}\wr\overbrace{\Sigma_p\wr\ldots\wr\Sigma_p}^{m-3\text{ factors}}\wr(\rho_{p-1}\circ\pi_1^{n-m+1}))
\end{aligned}$$

Now notice that we have the following pull-back diagram of fibration

$$\begin{tikzcd}
B(\Sigma_p\wr\Sigma_p\wr G) \arrow[r] \arrow[d] & B(\Sigma_{p^2}\wr G) \arrow[d] \\
B(\Sigma_p\wr\Sigma_p) \arrow[r] & B\Sigma_{p^2}
\end{tikzcd}$$

Thus by the naturality of the transfer [3],

$$\begin{aligned}
&Tr^{\Sigma_{p^2}\wr G}_{\Sigma_p\wr\Sigma_p\wr G}(c^{i_1}_{p-1}(\rho_{p-1}\circ\pi_1^n)c^{i_2}_{p(p-1)}(\Sigma_p\wr(\rho_{p-1}\circ\pi_1^{n-1})))\\
=\;& Tr^{\Sigma_{p^2}\wr G}_{\Sigma_p\wr\Sigma_p\wr G}(c^{i_1}_{p-1}(\rho_{p-1}\circ\pi_1^2\circ\pi_2^n)c^{i_2}_{p(p-1)}(\Sigma_p\wr\rho_{p-1}\circ\pi_2^n))\\
=\;& Tr^{\Sigma_{p^2}\wr G}_{\Sigma_p\wr\Sigma_p\wr G}\circ K(2)^*(\pi_2^n)(c^{i_1}_{p-1}(\rho_{p-1}\circ\pi_1^2)c^{i_2}_{p(p-1)}(\Sigma_p\wr\rho_{p-1}))\\
=\;& K(2)^*(\pi)\circ Tr^{\Sigma_{p^2}}_{\Sigma_p\wr\Sigma_p}(c^{i_1}_{p-1}(\rho_{p-1}\circ\pi_1^2)c^{i_2}_{p(p-1)}(\Sigma_p\wr\rho_{p-1}))
\end{aligned}$$

Thus it suffices to prove the Adem relation when I is empty. But this is just dual to Corollary 1. □

7 The decomposition results

In this section we prove Corollary 1.3, and complete the proof of Theorem 1.2 by showing that QS^{2n} admits an ULE for $K(2)$.

Proposition 7.1 *Let h_* be a generalized homology theory that satisfies the Künneth isomorphism, and such that $h_*(QS^{2n+1})$ is an exterior algebra with generators in odd degrees for any non-negative integer n. Then for any spectrum X that admits a stable cell decomposition consisting only of non-negative even dimensional cells, $h_*(\underline{X}_{2m})$ $(m \geq 0)$ decomposes as a tensor product of $h^*(QS^{2n})$'s corresponding to the stable cell decomposition of X.*

Proof First assume that X has finitely many cells. Thus there exist spectra

$$X^0 = S^{2d}, X^1, \ldots, X^l = X \text{ such that}$$
$$\exists \text{ cofibrations } S^{2d_i-1} \to X^{i-1} \to X^i.$$

The bar spectral sequences for the corresponding fibrations of the 0-spaces collapse, and the desired result follows. One obtains the result for the general case by passing to the colimit. □

Proposition 7.2 *Let h be a generalized homology theory satisfying the Künneth isomorphism. If $h_*(QS^{2n})$ for all positive integers n and $h_*(QS^0_0)$ are polynomial algebras, then $h_*(QS^{2n+1})$ is an exterior algebra for all non-negative integers n. Here QS^0_0 is the component of the constant map. Furthermore, the isomorphism in the Proposition 7.1 becomes an isomorphism of algebras.*

Proof The bar spectral sequence for the fibration $QS^{2n} \to * \to QS^{2n+1}$ collapses, and there is no extension problem. This takes care of the first assertion. For the second assertion, use the map that pinches everything except the top cell to solve the algebra extension problem in the bar spectral sequence argument of the Proposition 7.1. □

Now Corollary 1.3 follows immediately. Of course, we can obtain similar decomposition for odd space by using the bar spectral sequence for the fibration $\underline{X}_{2n} \to * \to \underline{X}_{2n+1}$. Furthermore, we have,

Corollary 7.3 *Let X be S^0, $T(n)$, or $X(n)$ as defined in [17]. (They are spectra such that $H_*(X(n);Z) = Z[b_1,\ldots,b_n] \subset H_*(MU;Z) = Z[b_1,\ldots,b_i,\ldots]$, $H_*(T(n);Z_{(p)}) = Z_{(p)}[m_1,\ldots,m_n] \subset H_*(BP;Z_{(p)}) = Z_{(p)}[m_1,\ldots,m_i,\ldots]$,) Then $\underline{X}_{2n}$ admits a ULE for $K(2)$.*

Proof By the result of [18], $\underline{MU}_{2n}$ and $\underline{BP}_{2n}$ are unitary like spaces [8]. But Proposition 1.3 says that the map $\underline{X}_{2n} \to \underline{MU}_{2n}$ (or $\underline{X}_{2n} \to \underline{BP}_{2n}$) induces an injection of $K(2)$-homology. □

Thus for these infinite loop spaces we can define our Dyer-Lashof operations, and the proof of Theorem 1.2 is now complete. To put everything together, we have the following.

Theorem 7.4 *Let X be S^0, $T(n)$, or $X(n)$, and $A = \{a\}$ be a set of basis elements for $K(2)_*(X)$. Then $K(2)_*(\underline{X}_{2n})$ is a polynomial algebra on $Q^I\iota_a$'s, with I admissible and ι_a an element which suspends to a.*

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