Algebraic geometry of ring spectra and multiplicative invariants for families of manifolds

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Algebraic and geometric topology workshop 2017

The Witten genus and its refinements

Definition

A *genus* is a function which assigns to each closed manifold M of some type an element $g(M) \in R$ of a commutative ring R, satisfying

- $g(M_1 \amalg M_2) = g(M_1) + g(M_2)$
- $g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$

This is the same as giving a ring homomorphism from a suitable cobordism ring, e.g., $g\colon MSO_* \to R, \quad g\colon MU_* \to R.$

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The Witten genus (Witten '87): $MString_* \to MF_*$

$MString_* = MO\langle 8\rangle_* :=$ the cobordism ring of *string manifolds*

$0 = w_1(M) = w_2(M) = p_1(M)/2$

$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$ $\qquad MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$

$MO\langle 8\rangle^{-*}(X) \rightsquigarrow$ genus of a *family* parametrized by X (orientation)

$$\begin{tikzcd}
MO\langle 8\rangle \arrow[rr, "\text{Ando-Hopkins-Rezk '10}", "\text{an } E_\infty \text{ map}"'] & & TMF \arrow[d] & \text{topological modular forms} \\
MU\langle 6\rangle \arrow[u] \arrow[rr, "\text{Ando-Hopkins-Strickland '04}", "\text{an } H_\infty \text{ map}"'] & & E & \text{an elliptic cohomology theory}
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Multiplicative structures and symmetries of the geometry

$$\begin{tikzcd}
D_n MU\langle 6\rangle \arrow[r, "D_n g"] \arrow[d] & D_n E \arrow[d] & D_n(-) := (-)^{\wedge n}/\Sigma_n, \quad n \geq 0 \\
MU\langle 6\rangle \arrow[r, "g"] & E & \text{H}_\infty \text{ means commutative up to homotopy}
\end{tikzcd}$$

The vertical maps correspond to *power operations*; e.g., given

$$E^0 X \cong \pi_0 E^{\Sigma_+^\infty X} =: \pi_0 A = [S, A]_S \cong [E, A]_E \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation $Q_\alpha \colon E^0 X \to E^0 X$. When $E = KU$, this is the n-fold tensor product of $\mathbb{C}$-vector bundles over X.

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D_n MU\langle 6\rangle \arrow[r, "D_n g"] \arrow[d] & D_n E \arrow[d] \\
MU\langle 6\rangle \arrow[r, "g"] & E
\end{tikzcd}$$

$D_n(-) \coloneqq (-)^{\wedge n}/\Sigma_n, \quad n \geq 0$

H_∞ means commutative up to homotopy

The vertical maps correspond to *power operations*; e.g., given

$$E^0 X \cong \pi_0 E^{\Sigma^\infty_+ X} =: \pi_0 A = [S, A]_S \cong [E, A]_E \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation $Q_\alpha \colon E^0 X \to E^0 X$. When $E = KU$, this is the n-fold tensor product of $\mathbb{C}$-vector bundles over X.

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Definition (Ando-Hopkins-Strickland '01, Lurie '09)

elliptic cohomology theory $= \left\{ \begin{array}{lll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} & \end{array} \right\}$

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elliptic cohomology theory $= \left\{ \begin{array}{ll} R, \quad C/R, & E, \\ E^0(*) \cong R, & \mathrm{Spf}\, E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$

- A *formal group* G is a group object in formal schemes.
- A *formal group law* F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F\big(t(P_1), t(P_2)\big) = t(P_1 \underset{G}{+} P_2)$$

$$F\big(c_1(\mathcal{L}_{\mathrm{univ}}), c_1(\mathcal{L}_{\mathrm{univ}})\big) = c_1(\mathcal{L}_{\mathrm{univ}} \otimes \mathcal{L}_{\mathrm{univ}})$$

- Coordinates on $G_E := \mathrm{Spf}\, E^0(\mathbb{CP}^\infty) \leftrightarrow$ ortns $MU\langle 0\rangle \to E$.
  Question   Which coordinates correspond to H_∞ orientations?

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Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

$\mathcal{E}\colon$ {formal groups over perfect fields, isos} $\to$ {E_∞-ring spectra}

- $\mathrm{Spf}\, E^0(\mathbb{CP}^\infty) =$ universal deformation of a fg Γ of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}], \quad |u| = 2$

A *deformation* (G, i, η) of Γ/k to R (Lubin-Tate '66):

$$\begin{tikzcd}
G \arrow[d] & \pi^*G \arrow[l] \arrow[r, "\eta"] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] & i^*\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma \arrow[d] \\
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H_∞ $MU\langle 0\rangle$-orientations for Morava E-theories

Theorem (Z.)

Let k be an algebraic extension of $\mathbb{F}_p$, Γ be a formal group over k of height n, and E be the Morava E-theory associated to Γ/k. Given any coordinate x_Γ on Γ, there exists a unique coordinate x on G_E lifting x_Γ such that $MU\langle 0\rangle \xrightarrow{x} E$ is an H_∞ map.

Remark

- (Ando '95) $k = \mathbb{F}_p$, $\Gamma =$ Honda fg, $E \neq$ elliptic cohomology
- When $n = 2$, the composite $MU\langle 6\rangle \to MU\langle 0\rangle \to E$ does not factor through the Witten genus.
  $E =$ local TMF $\rightsquigarrow$ $E_* =$ completion of MF_* $\rightsquigarrow$? genus

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- (Ando '95) $k = \mathbb{F}_p$, $\Gamma =$ Honda fg, $E \neq$ elliptic cohomology
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Norm-coherent coordinates

Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$\begin{tikzcd}
E(\Gamma/k) \arrow[loop below] & \leftrightsquigarrow & \Gamma/k \arrow[r, "\text{univ defo}"] & G_E/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \arrow[loop below] \\
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Fix Γ/k. Then deformations of Frobenius $(G, i, \eta) \to (G', i', \eta')$ are classified by rings A_r, $r \geq 0$, with p^r the order of the subgroup scheme $\ker(G \to G') \subset G$.

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Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left(x \underset{G}{+} x(Q)\right)$$

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Let E be the Morava E-theory associated to Γ/k as before. Then the orientation $MU\langle 0\rangle \to E$ is an H_∞ map if and only if its corresponding coordinate on G_E is norm-coherent.

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