Morava E-homology of Bousfield-Kuhn functors on odd-dimensional spheres

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As an application of Behrens and Rezk's spectral algebra model for unstable v_n-periodic homotopy theory, we give explicit presentations for the completed E-homology of the Bousfield-Kuhn functor on odd-dimensional spheres at chromatic level 2, and compare them to the level 1 case. The latter reflects earlier work in the literature on K-theory localizations.

1 Introduction

The rational homotopy theory of Quillen and Sullivan studies unstable homotopy types of topological spaces modulo torsion, or equivalently, after inverting primes. Such homotopy types are computable by means of their *algebraic models*. In particular, Quillen showed that there are equivalences of homotopy categories

$$\mathrm{Ho}_{\mathbb{Q}}(\mathrm{Top}_*)_2 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGL})_1 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGC})_2$$

between simply-connected pointed topological spaces localized with respect to rational homotopy equivalences, connected differential graded Lie algebras over $\mathbb{Q}$, and simply-connected differential graded cocommutative coalgebras over $\mathbb{Q}$ [Quillen1969, Theorem I].

Let p be a prime, $\mathbb{F}_p$ be the field with p elements, and $\overline{\mathbb{F}}_p$ be its algebraic closure. Working prime by prime, one has p-adic analogues where equivalences detected through $H_*(-;\mathbb{Q})$ are replaced by those through $H_*(-;\mathbb{F}_p)$. Various algebraic models for p-adic homotopy types of spaces were developed [Kříž1993, Goerss1995, Mandell2001]. In the modern language of homotopy theory, these models are often formulated in terms of "spectral" algebra. For example, Mandell's model is given by the functor that takes a connected p-complete nilpotent space X of finite p-type to the $\overline{\mathbb{F}}_p$-cochains $H\overline{\mathbb{F}}_p^X$, where $H\overline{\mathbb{F}}_p^X$ denotes the function spectrum $F(\Sigma^\infty X, H\overline{\mathbb{F}}_p)$. This spectrum is a commutative algebra over $H\overline{\mathbb{F}}_p$.

More generally, through the prism of chromatic homotopy theory, Behrens and Rezk have established spectral algebra models for unstable v_n*-periodic* homotopy types [Behrens-Rezk2015] (cf. [Arone-Ching2015, Heuts2016, Behrens-Rezk2016]). Here, instead of inverting primes, they work p-locally for a fixed prime p and invert classes of maps called "v_n-self maps" (the case of $n=0$ recovers rational homotopy). Correspondingly, there is the n'th unstable monochromatic category $M_n^f\mathrm{Top}_*$ in the sense of [Bousfield2001]. They study the functor

$$\mathbf{S}_{T(n)}^{(-)}\colon\ \mathrm{Ho}(M_n^f\mathrm{Top}_*)^{\mathrm{op}} \to \mathrm{Ho}\big(\mathrm{Alg}_{\mathrm{Comm}}(\mathrm{Sp}_{T(n)})\big) \tag{1.1}$$

that sends a space X to the $\mathbf{S}_{T(n)}$-valued cochains $\mathbf{S}_{T(n)}^X$. This last spectrum is an algebra for the reduced commutative operad Comm in modules over $\mathbf{S}_{T(n)}$, the localization of the sphere spectrum with respect to the telescope of a v_n-self map.

Considering a variant of localization with respect to the Morava K-theory $K(n)$, Behrens and Rezk have obtained an equivalence

$$\Phi_{K(n)}(X) \xrightarrow{\sim} \mathrm{TAQ}_{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^X)$$

of $K(n)$-local spectra, on a class of spaces X including spheres [Behrens-Rezk2015, Theorem 8.1] (cf. [Behrens-Rezk2016, Section 8]). In more detail, the left-hand side arises from computing homotopy groups in the source category of (1.1), where $\Phi_{K(n)} = L_{K(n)}\Phi_n$ is a version of the Bousfield-Kuhn functor (cf. [Kuhn2008]). This side is a derived realization of morphisms in the source. The right-hand side is the topological André-Quillen cohomology of $\mathbf{S}_{K(n)}^X$ as an algebra over the operad Comm in $\mathbf{S}_{K(n)}$-modules. It is a derived realization of images of morphisms under the functor (1.1) in the target category. Via a suitable Koszul duality between Comm and the Lie operad, we may view the spectrum $\mathrm{TAQ}_{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^X)$ as a Lie algebra model for the unstable v_n-periodic homotopy type of X.

1.1 Main results

The purpose of this paper is to make available calculations that apply Behrens and Rezk's theory to obtain quantitative information about unstable v_n-periodic homotopy types, in the case of $n=2$. These are based on our computation of power operations for Morava E-theory in [Zhu2015b].

Let E be a Morava E-theory spectrum of height 2 with $E_* \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!][u^{\pm 1}]$, where $|a|=0$ and $|u|=-2$. Recall that the *completed E-homology* functor is defined as $E_*^\wedge(-) \coloneqq \pi_*(E\wedge -)_{K(2)}$. It is E_0-linear dual to $E^*(-)$ with more convenient properties than $E_*(-)$ (see [Rezk2009, Section 3]).

Building on and strengthening Rezk's results in [Rezk2013, §2.13], we obtain the following.

Theorem 1.2 *Given any non-negative integer m, denote by $E_*^\wedge(\Phi_2 S^{2m+1})$ the completed E-homology groups of the Bousfield-Kuhn functor applied to the $(2m+1)$-dimensional sphere.*

(i) *The group $E_1^\wedge(\Phi_2 S^{2m+1}) \cong 0$ if $m = 0$. As an E_0-module, it equals $(E_0/p)^{\oplus p-1}$ if $m = 1$. It is a quotient of $(E_0/p^m)^{\oplus p-1} \oplus E_0/p^{m-1}$ if $m > 1$.*

(ii) *More explicitly,*

$$E_1^\wedge(\Phi_2 S^{2m+1}) \cong \begin{cases} \dfrac{\bigoplus_{i=1}^{p-1}(E_0/p^m)\cdot x_i \oplus (E_0/p^{m-1})\cdot x_p}{(r_1,\ldots,r_{m-1})} & \text{if } 2 \le m \le p+2 \\[2ex] \dfrac{\bigoplus_{i=1}^{p-1}(E_0/p^m)\cdot x_i \oplus (E_0/p^{m-1})\cdot x_p}{(r_{m-p-1},\ldots,r_{m-1})} & \text{if } m > p+2 \end{cases}$$

where $r_j = r_j(x_1,\ldots,x_p) = w_0^{m-1-j}\sum_{i=1}^p d_{i,j+1}\,x_i$. Here, as in [Zhu2015b, Theorem 1.6],

$$d_{i,\tau} = \sum_{n=0}^{\tau-1}(-1)^{\tau-n}\,w_0^n \sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau+i\\ 1\le m_s\le p+1\\ m_{\tau-n}\ge i+1}} w_{m_1}\cdots w_{m_{\tau-n}}$$

where the coefficients $w_i \in E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!]$ are defined by the identity

$$\sum_{i=0}^{p+1} w_i\,b^i = (b-p)\big(b+(-1)^p\big)^p - \big(a-p^2+(-1)^p\big)b$$

in the variable b, so that $w_{p+1} = 1$, $w_1 = -a$, $w_0 = (-1)^{p+1}p$, and the remaining coefficients

$$w_i = (-1)^{p\,(p-i+1)}\left[\binom{p}{i-1} + (-1)^{p+1}\,p\binom{p}{i}\right]$$

In particular, each relation r_j contains a term $(-1)^{j+1}w_0^{m-1-j}w_1^j\,x_p$.

(iii) *The group $E_0^\wedge(\Phi_2 S^{2m+1}) \cong 0$ for any $m \ge 0$.*

Since E is 2-periodic, the above determines the completed E-homology in all degrees.

Remark 1.3 There is a ring structure on $E_1^\wedge(\Phi_2 S^{2m+1})$. Indeed, each generator x_i is a power b^i of a certain element b. See Section 3.1 for details.

Remark 1.4 In [Wang2015, Sections 5.3–5.4], Wang obtained an equivalent presentation of $E_*^\wedge(\Phi_2 S^{2m+1})$ for $m = 1$ and any prime p. He then used it as the input of a spectral sequence and computed $\pi_*(\Phi_{K(2)} S^3)$ at $p \geq 5$.

Example 1.5 We apply Theorem 1.2 and compute $E_1^\wedge(\Phi_2 S^{2m+1})$ at $p = 2$ for small values of m. We have $w(a, b) = b^3 - ab - 2$ so that $w_3 = 1$, $w_2 = 0$, $w_1 = -a$, and $w_0 = -2$.

- When $m = 2$, since $r_1 = 2x_1 - ax_2$, $E_1^\wedge(\Phi_2 S^5)$ is the quotient of $(E_0/4) \cdot x_1 \oplus (E_0/2) \cdot x_2$ subject to the relation $ax_2 = 2x_1$.
- When $m = 3$, we have $r_1 = -4x_1 + 2ax_2$ and $r_2 = 2ax_1 - a^2 x_2$ so that the relations are
$$\begin{aligned} a^2 x_2 &= 2ax_1 \\ 2ax_2 &= 4x_1 \end{aligned}$$
- When $m = 4$, we have $r_1 = 8x_1 - 4ax_2$, $r_2 = -4ax_1 + 2a^2 x_2$, and $r_3 = 2a^2 x_1 + (-a^3 + 4)x_2$. Thus the relations are
$$\begin{aligned} a^3 x_2 &= 2a^2 x_1 + 4x_2 \\ 2a^2 x_2 &= 4ax_1 \\ 4ax_2 &= 8x_1 \end{aligned}$$
- When $m = 5 > p + 2$, we have $r_2 = 8ax_1 - 4a^2 x_2$, $r_3 = -4a^2 x_1 + (2a^3 - 8)x_2$, and $r_4 = (2a^3 - 8)x_1 + (-a^4 + 8a)x_2$. Thus the relations are
$$\begin{aligned} a^4 x_2 &= (2a^3 - 8)x_1 + 8ax_2 \\ 2a^3 x_2 &= 4a^2 x_1 + 8x_2 \\ 4a^2 x_2 &= 8ax_1 \end{aligned}$$

The relations above show that the bounds for p-power torsion in Theorem 1.2 are sharp (see [Bousfield2005, § 2.5] and [Selick1988]). Also, as in part (ii) of the theorem, each r_j contains a term $2^{m-1-j} a^j x_2$. Unfortunately, it is impossible to simplify the relations for $E_1^\wedge(\Phi_2 S^{2m+1})$ into $2^{m-1-j} a^j x_2 = 0$ by an E_0-linear change of variables with x_i. See Remark 2.3 below.

1.2 A comparison to the case of $n = 1$

As an application of Behrens and Rezk's theory, Theorem 1.2 is a step toward the program initiated in [Arone-Mahowald1999] to compute the unstable v_n-periodic homotopy groups of spheres using stable v_n-periodic homotopy groups and Goodwillie calculus. See also [Wang2014, Wang2015]. Given the computations of Davis and Mahowald in the 1980s for the case of $n=1$, we discuss a version of Theorem 1.2 at height 1 according to this program.

Davis and Mahowald showed that, $K(1)$-locally at a fixed prime p, the Moore spectrum $\mathbf{S}^{-1}/p^m$ with i'th space $S^{i-1}\cup_{p^m} e^i$ is equivalent to the suspension spectrum of a stunted $B\Sigma_p$. Via the Goodwillie tower of the identity functor on the category of pointed spaces, the latter can be identified with $\Phi_1(S^{2m+1})$, again $K(1)$-locally (or $T(1)$-locally, due to the validity of the Telescope Conjecture at height 1). We thus obtain a variant of Theorem 1.2.

Proposition 1.6 *Let E be a Morava E-theory spectrum of height 1, with $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p$. Assume that $p \neq 3$ and that, if $p=2$, $m \equiv 0,3 \mod 4$. Then*

$$E_0^{\wedge}(\Phi_1 S^{2m+1}) \cong E_0/p^m \qquad\qquad m \geq 0$$

Proof For non-negative integers k and b, let $L(k)_b := e_k\Sigma^\infty(B\mathbb{F}_p^k)^{b\bar{\rho}_k}$ be the stable summand of the Thom space $(B\mathbb{F}_p^k)^{b\bar{\rho}_k}$ associated to the Steinberg idempotent, where $b\bar{\rho}_k$ denotes the direct sum of b copies of the k-dimensional reduced real regular representation of $\mathbb{F}_p^k$ (cf. [Behrens-Rezk2015, Remark 5.4]). For $b \leq t$, let $L(k)_b^t$ be the fiber of the natural map of spectra $L(k)_b \to L(k)_{t+1}$ (see [Behrens2012, Chapter 2, esp. Section 2.3]). When $k=1$, it is a stunted $B\Sigma_p$ (cf. [Mitchell-Priddy1983]). In particular, if $p=2$, $L(1)_b^t \simeq \mathbf{P}_b^t$, the suspension spectrum of the stunted real projective space $\mathbb{RP}^t/\mathbb{RP}^{b-1}$. In this case, when $m=4n$, we have

$$\begin{aligned}
\Phi_1(S^{2m+1}) &\simeq L_{K(1)}\Sigma^{2m+1}L(1)_1^{2m} && \text{by [Kuhn2007, Theorem 7.20]}\\
&\simeq L_{K(1)}\Sigma^{2m+1}\mathbf{P}_1^{8n}\\
&\simeq L_{K(1)}\Sigma^{2m+1}\mathbf{P}_{1-8n}^{0} && \text{by [Davis-Mahowald1987, Proposition 2.1]}\\
&\simeq L_{K(1)}\Sigma^{2m+1}\mathbf{S}^{-1}/2^m && \text{by [Davis-Mahowald1987, proof of Theorem 4.2]}\\
&\simeq L_{K(1)}\mathbf{S}^{2m}/2^m
\end{aligned}$$

and thus $E_0^{\wedge}(\Phi_1 S^{2m+1}) \cong E_0/2^m$. It is similar when $m=4n+3$. For $p>3$, we apply [Davis1986, Corollary 1.7 and Theorem 1.8]. □

Remark 1.7 There has been extensive work on the case of v_1-periodic homotopy theory. See [Bousfield1999, §9.11] and [Bousfield2005, §§8.6–8.7] for an alternative approach to a more general result than the above where the assumptions on p and m are removed (cf. [Rezk2013, §2.13]). See also [Davis1995, esp. Theorem 3.1] and the references therein for related information in this case.

Remark 1.8 For any n, since Φ_n preserves fiber sequences, there is a natural map

$$\Sigma^2\Phi_n(X) \to \Phi_n(\Sigma^2 X)$$

In view of the 2-periodicity of E, this induces a map on completed E-homology in the same degree. Thus the groups $\{E^\wedge_*(\Phi_n S^{2m+1})\}_{m\geq 0}$ form a direct system. Homotopy (co)limits of generalized Moore spectra are closely related to various kinds of localizations of the sphere spectrum (see, e.g., [Arone-Mahowald1999, Proposition A.3] and [Hovey-Strickland1999, Proposition 7.10]). On the other hand, note that completed E-homology does not preserve homotopy colimits [Hovey2008]. Nevertheless, based on computational evidence from Theorem 1.2, Proposition 1.6, and further, we hope to study the relationship between the $K(n)$-local sphere and the Bousfield-Kuhn functor on odd-dimensional spheres hinted in [Rezk2016, §§3.20–3.21].

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2 Koszul complexes for modules over the Dyer-Lashof algebra of Morava E-theory

Let E be a Morava E-theory spectrum of height n at the prime p. Its formal group $\operatorname{Spf} E^0\mathbb{CP}^\infty$ over $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![u_1, \ldots, u_{n-1}]\!]$ is the Lubin-Tate universal deformation of a formal group $\mathbb{G}$ over $\overline{\mathbb{F}}_p$ of height n.

Generalizing the Lubin-Tate deformation theory, Strickland shows that for each $k \geq 0$ there is a ring $A_k \cong E^0B\Sigma_{p^k}/I_k$ classifying subgroups of degree p^k in the universal deformation, where I_k is the ideal generated by the image of all transfer maps from inclusions of the form $\Sigma_i \times \Sigma_{p^k-i} \subset \Sigma_{p^k}$ with $0 < i < p^k$ [Strickland1998, Theorem 1.1]. In particular, $A_0 \cong E_0$ and there are ring homomorphisms

$$s = s_k,\ t = t_k\colon\ A_0 \to A_k \quad \text{and} \quad \mu = \mu_{k,m}\colon\ A_{k+m} \to A_k\,{}^s\!\otimes^t_{A_0} A_m$$

classifying the source and target of an isogeny of degree p^k on the universal deformation and the composition of two isogenies.

As E is an E_∞-ring spectrum, there are (additive) power operations acting on the homotopy of $K(n)$-local commutative E-algebra spectra. A *Γ-module* is an A_0-module M equipped with structure maps (the power operations)

$$P_k\colon\ M \to {}^tA_k\,{}^s\!\otimes_{A_0} M \qquad k \geq 0$$

which are a compatible family of A_0-module homomorphisms. These power operations form the *Dyer-Lashof algebra* Γ for the E-theory, with graded pieces $\Gamma[k] := \mathrm{Hom}_{A_0}({}^sA_k, A_0)$, $k \geq 0$. There is a tensor product $\otimes$ for Γ-modules [Rezk2013, §4.1].

The structure of a Γ-module is determined by P_1, subject to a condition involving A_2, i.e. the existence of the dashed arrow in the diagram

$$\begin{tikzcd} M \arrow[r, "P_1"] \arrow[d, dashed] & {}^tA_1\,{}^s\!\otimes_{A_0} M \arrow[d, "\mathrm{id}\otimes P_1"] \\ {}^tA_2\,{}^s\!\otimes_{A_0} M \arrow[r, "\mu\otimes\mathrm{id}"] & {}^tA_1\,{}^s\!\otimes^t_{A_0} A_1\,{}^s\!\otimes_{A_0} M \end{tikzcd} \tag{2.1}$$

[Rezk2013, Proposition 7.2]. This manifests the fact that the ring Γ is *Koszul* and, in particular, *quadratic* [Rezk2012].

Let $D_0 := A_0$, $D_1 := A_1$, and

$$D_k := \mathrm{coker}\left(\bigoplus_{i=0}^{k-2} A_1^{\otimes i} \otimes A_2 \otimes A_1^{k-i-2} \xrightarrow{\mathrm{id}\otimes\mu\otimes\mathrm{id}} A_1^{\otimes k} \right) \qquad k \geq 2$$

Given Γ-modules M and N, Rezk defines the *Koszul complex* $\mathcal{C}^\bullet(M,N)$ by

$$\mathcal{C}^k(M,N) := \mathrm{Hom}_{A_0}(M, D_k \otimes_{A_0} N)$$

with appropriate coboundary maps [Rezk2013, §7.3].

Proposition 2.2 *If M is projective as an A_0-module, then*

$$\mathrm{Ext}^k_\Gamma(M,N) \cong H^k\mathcal{C}^\bullet(M,N)$$

In particular, if $k > n$, $D_k \cong 0$ and so $\mathrm{Ext}^k_\Gamma(M,N) \cong 0$.

Proof This is [Rezk2013, Proposition 7.4]. $\square$

2.1 The case of $n = 2$

Choose a *preferred $\mathcal{P}_N$-model* for E in the sense of [Zhu2015a, Definition 3.29] so that the formal group of E is isomorphic to the formal group of a universal deformation of a supersingular elliptic curve satisfying a list of properties.

Using the theory of dual isogenies of elliptic curves, Rezk identifies that $D_2 \cong A_1/s(A_0)$ [Rezk2013, Proposition 9.3]. He also classifies Γ-modules of rank 1 in this case [Rezk2013, Proposition 9.7]. In particular, each of them takes the form L_β with structure map

$$\begin{aligned} P\colon\ L_\beta &\to {}^{t}A_1{}^{s}\!\otimes_{A_0} L_\beta \\ x &\mapsto \beta \otimes x \end{aligned}$$

where x is a generator for the underlying A_0-module, and $\beta \in A_1$ is such that $\iota(\beta)\cdot\beta \in s(A_0)$ with $\iota(-)$ the Atkin-Lehner involution (this condition on β corresponds to the condition in (2.1)). Moreover, L_1 is the unit object in the symmetric monoidal category of Γ-modules with respect to $\otimes$ and $L_{\beta_1} \otimes L_{\beta_2} \cong L_{\beta_1\beta_2}$. Thus L_β is $\otimes$-invertible as a Γ-module if and only if $\beta \in A_1^\times$.

Now let $M = L_\alpha$ and $N = L_\beta$. We have identifications

$$\begin{aligned} &A_0 \xrightarrow{\sim} \mathcal{C}^0(M,N) = \mathrm{Hom}_{A_0}(M,N) && f \mapsto (x \mapsto f\,y) \\ &A_1 \xrightarrow{\sim} \mathcal{C}^1(M,N) = \mathrm{Hom}_{A_0}(M,\ {}^{t}A_1{}^{s}\!\otimes_{A_0} N) && g \mapsto (x \mapsto g \otimes y) \\ &A_1/s(A_0) \xrightarrow{\sim} \mathcal{C}^2(M,N) = \mathrm{Hom}_{A_0}\Big(M,\ {}^{\iota^2 s}\big(A_1/s(A_0)\big){}^{s}\!\otimes_{A_0} N\Big) && h \mapsto (x \mapsto h \otimes y) \end{aligned}$$

Thus the Koszul complex in this case is

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_1/s(A_0)$$

with $d_0 f = \iota(f)\beta - f\alpha$ and $d_1 g = \iota(g)\beta + g\iota(\alpha)$ [Rezk2013, §9.18].

More explicitly, we have identifications

$$A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!] \quad \text{and} \quad A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a,b]\!]/\big(w(a,b)\big)$$

where

$$w(a,b) = \sum_{i=0}^{p+1} w_i\, b^i = (b-p)\big(b+(-1)^p\big)^p - \big(a - p^2 + (-1)^p\big)b$$

[Zhu2015b, Theorem 1.2]. Note that the parameters a and b are chosen as in [Rezk2013, § 9.15] and they correspond precisely to h and α in [Zhu2015a, Zhu2015b]. In particular, the Γ-module of invariant 1-forms is $\omega = L_b$.

Remark 2.3 As we will see in Section 4, the generators x_i in Theorem 1.2 (ii) depend on the choice of the parameter b for A_1. We do not know if a different choice would make the presentations simpler.

The ring homomorphism $s\colon A_0 \to A_1$ is simply the inclusion of scalars, as A_1 is a free left module over A_0 of rank $p+1$. We will thus abbreviate $s(A_0)$ as A_0. Following [Rezk2013], we will also abbreviate $\iota(x)$ as x', which is written as $\widetilde{x}$ in [Zhu2015a, Zhu2015b]. Note that $w_{p+1} = 1$, $p|w_i$ for $2 \le i \le p$, $w_1 = -a$, and

$$w_0 = (-1)^{p+1} p = bb' \tag{2.4}$$

[Zhu2015a, (3.30)].

3 Computing with Koszul complexes

Recall that $\omega = L_b$ is the Γ-module of invariant 1-forms defined in Section 2.1. Write $\mathrm{nul} := L_0$, the Γ-module annihilated by Γ. In this section, we compute $\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{nul})$ for $m \ge 0$. By Proposition 2.2,

$$\mathrm{Ext}^*_\Gamma(\omega^m, \mathrm{nul}) \cong H^* \mathcal{C}^\bullet(L_{b^m}, L_0)$$

where

$$\mathcal{C}^\bullet(L_{b^m}, L_0)\colon\ A_0 \xrightarrow{-b^m} A_1 \xrightarrow{b'^m} A_1/A_0$$

Proposition 3.1 *For all $m \ge 0$, $H^0 \mathcal{C}^\bullet(L_{b^m}, L_0) \cong 0$.*

Proof We need to show that $A_0 \xrightarrow{-b^m} A_1$ is injective. Given $f(a) \in A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a]\!]$, suppose $-b^m \cdot f(a) = 0 \in A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[\![a,b]\!]/\big(w(a,b)\big)$.

If $0 \le m \le p$, since $w(a,b)$ is a polynomial in b of degree $p+1$ with coefficients in A_0, clearly $f(a)$ must be 0.

If $m > p$, we need only show that $b^m \not\equiv 0 \mod w(a,b)$. Since $w(a,b) \equiv b(b^p - a) \mod p$, we have $b^{p+1} \equiv ab \mod (p,w)$, and thus $b^m \not\equiv 0 \mod (p,w)$. □

Proposition 3.2 *For all $m \ge 0$, $H^1 \mathcal{C}^\bullet(L_{b^m}, L_0) \cong 0$.*

Proof Let $g(a,b)$ be a polynomial in b of degree at most p with coefficients in A_0 that represents an element in A_1. Suppose $b'^m \cdot g(a,b) = 0 \in A_1/A_0$. We need to show that $g(a,b) \equiv -b^m \cdot f(a) \mod w(a,b)$ for some $f(a) \in A_0$.

We do this by induction on m. The case of $m = 0$ is clear. Let $m \geq 1$. By the induction hypothesis, since $b'^{m-1} \cdot b'g(a,b) = 0 \in A_1/A_0$, we have $b'g(a,b) \equiv -b^{m-1} \cdot f(a) \mod w(a,b)$. Multiplying both sides by b, in view of (2.4), we get

$$w_0 g(a,b) \equiv -b^m f(a) \mod w \tag{3.3}$$

and thus

$$0 \equiv -b^m f(a) \mod (p,w) \tag{3.4}$$

Since $b^{p+1} \equiv ab \mod (p,w)$, (3.4) implies that $p \mid f(a)$ in A_1. As p is not a zero-divisor in A_1, (3.3) implies that $g(a,b) \equiv -b^m \tilde{f}(a) \mod w$ for some $\tilde{f}(a) \in A_0$. □

3.1 The second cohomology

Finally, we compute $H^2\mathcal{C}^\bullet(L_{b^m}, L_0)$. Write $B_m := H^2\mathcal{C}^\bullet(L_{b^m}, L_0) \cong A_1/(A_0 + b'^m A_1)$. Clearly, $B_0 \cong 0$. Let $m > 0$ for the rest of this section.

As a free module over A_0, the ring A_1 has a basis consisting of

$$1,\ b,\ b^2,\ \ldots,\ b^p \tag{3.5}$$

Proposition 3.6 *In the A_0-module B_m, $p^m b^i = 0$ for $1 \leq i \leq p-1$ and $p^{m-1}b^p = 0$.*

Proof In view of (2.4), we have $w_0^m b^i = b'^m b^m b^i = b'^m b^{m+i} = 0$ and

$$\begin{aligned} w_0^{m-1} b^p &= w_0^{m-1}(-b' - w_p b^{p-1} - \cdots - w_2 b) \\ &= -b^{m-1}b'^m - w_0^{m-1} w_p b^{p-1} - \cdots - w_0^{m-1} w_2 b \\ &= -w_0^{m-1} w_p b^{p-1} - \cdots - w_0^{m-1} w_2 b \end{aligned} \tag{3.7}$$

Since $p | w_i$ for $2 \leq i \leq p$, the last expression has a factor of w_0^m and so vanishes as we have just shown. □

Let $1 \leq m \leq p$. Under the map of multiplication by b'^m, the elements in (3.5) become

$$b'^m,\ w_0 b'^{m-1},\ w_0^2 b'^{m-2},\ \ldots,\ w_0^{m-1} b',\ w_0^m,\ w_0^m b,\ \ldots,\ w_0^m b^{p-m} \tag{3.8}$$

Note that $w_0^{m-1}b' = 0$ in B_m is equivalent to (3.7). Thus, as a quotient of $(A_0/p^m)^{\oplus p-1} \oplus A_0/p^{m-1}$ from the above proposition, B_m has relations given precisely by the vanishing of the first $(m-1)$ terms in (3.8).

To write down these relations explicitly, with notation as in [Zhu2015b, Theorem 1.6 (ii)], we have

$$b'^k = d_{p,k}b^p + d_{p-1,k}b^{p-1} + \cdots + d_{0,k} \qquad 2 \le k \le m \le p$$

(cf. [Zhu2015b, Section 4.1] for $k > p$). In particular, the formula for the coefficient $d_{p,k}$ has a leading term $(-1)^k w_1^{k-1} w_{p+1}$. Thus setting $w_0^{m-k}b'^k$ to be zero in B_m gives an expression for $w_0^{m-k}w_1^{k-1}b^p$ in terms of an A_0-linear combination of $b^{p-1}, \ldots, b$, and, possibly, b^p itself if there are more than one term in $d_{p,k}$ not divisible by p^{m-1}.

The case of $m > p$ is similar.

4 Proof of Theorem 1.2

Recall that given a Morava E-theory E of height n, the completed E-homology functor is defined as $E_*^\wedge(-) := \pi_*(E \wedge -)_{K(n)}$. In particular,

$$E_*^\wedge(\Phi_n X) \cong E_*^\wedge(\Phi_{K(n)} X) \tag{4.1}$$

since the map $\mathrm{id} \wedge L_{K(n)}\colon E \wedge \Phi_n X \to E \wedge L_{K(n)}\Phi_n X$ induces a $K(n)$-equivalence by the Künneth isomorphism.

In [Rezk2013], Rezk sets up a composite functor spectral sequence (CFSS) followed by a mapping space spectral sequence (MSSS) to compute the homotopy groups of derived mapping spaces $\widehat{\mathcal{R}}_E(A,B)$ between $K(n)$-local augmented commutative E-algebras A and B. He identifies the E_2-term in the CFSS as Ext-groups over the Dyer-Lashof algebra Γ. The CFSS converges to the E_2-term in the MSSS.

In particular, [Rezk2013, §2.13] shows that this setup specializes to compute the E-cohomology of the topological André-Quillen homology $\mathrm{TAQ}^{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^{S_+^{2m+1}})$, and that the two spectral sequences both collapse at the E_2-term when $n=2$. Here $A = E^{S_+^{2m+1}} := F(\Sigma_+^\infty S^{2m+1}, E)$ and $B = E \rtimes E$ is a *square-zero extension* (see [Rezk2013, §5.10]).

Now, by [Behrens-Rezk2015, Theorem 8.1] and (4.1), we identify the abutment of the MSSS as

$$\pi_{t-s}\widehat{\mathcal{R}}_E(E^{S_+^{2m+1}}, E \rtimes E) \cong \pi_{t-s}F\big(\mathrm{TAQ}^{\mathbf{S}_{K(2)}}(\mathbf{S}_{K(2)}^{S_+^{2m+1}}), E\big) \cong E_{t-s}^\wedge(\Phi_2 S^{2m+1})$$

For a fixed t, Rezk identifies the possibly nonzero terms on the E_2-page of the CFSS as $\mathrm{Ext}^s_{\mathrm{Mod}^\star_\Gamma}(\omega^m, \omega^{(t-1)/2} \otimes \mathrm{nul})$, where $\mathrm{Mod}^\star_\Gamma$ is the category of $\mathbb{Z}/2$-*graded* Γ-modules in the sense of [Rezk2013, §5.6]. Thus for degree and periodicity reasons, we may set $t = 1$ and the calculations in Section 3 then complete the proof, with b^i written as x_i in Theorem 1.2. □

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