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ARITHMETIC MODULI OF GENERALIZED ELLIPTIC CURVES

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Abstract The theory of generalized elliptic curves gives a moduli-theoretic compactification for modular curves when the level is a unit on the base, and the theory of Drinfeld structures on elliptic curves provides moduli schemes over the integers without a modular interpretation of the cusps. To unify these viewpoints it is natural to consider Drinfeld structures on generalized elliptic curves, but some of these resulting moduli problems have non-étale automorphism groups and so cannot be Deligne–Mumford stacks. Artin's method as used in the work of Deligne and Rapoport rests on a technique of passage to irreducible fibers (where the geometry determines the group theory), and this does not work in the presence of non-étale level structures and non-étale automorphism groups. By making more efficient use of the group theory to bypass these difficulties, we prove that the standard moduli problems for Drinfeld structures on generalized elliptic curves are proper Artin stacks. We also analyze the local structure on these stacks and give some applications to Hecke correspondences.

Keywords: generalized elliptic curve; Drinfeld structure; moduli stack; Tate curve

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Contents

1. Introduction

1.1. Motivation

In [**DR**], Deligne and Rapoport developed the theory of generalized elliptic curves over arbitrary schemes and they proved that various moduli stacks for (ample) 'level-N' structures on generalized elliptic curves over $\mathbf{Z}[1/N]$-schemes are Deligne–Mumford stacks over $\mathbf{Z}[1/N]$. These stacks were proved to be $\mathbf{Z}[1/N]$-proper, and also finite flat over the $\mathbf{Z}[1/N]$-localization of the proper $\mathbf{Z}$-smooth moduli stack $\mathcal{M}_1 = \overline{\mathcal{M}}_{1,1}$ of stable marked curves of genus 1 with one marked point. Hence, by normalization over $\mathcal{M}_1$ one gets proper normal flat stacks over $\mathbf{Z}$ but the method gives no moduli interpretation in 'bad' characteristics.

In [**KM**], Katz and Mazur developed the theory of *Drinfeld* level structures on elliptic curves over arbitrary schemes, thereby removing the étaleness restriction on the level when working away from the cusps. When there is 'enough' level (to remove non-trivial isotropy groups), the work in [**KM**] constructs affine moduli schemes over $\mathbf{Z}$ for Drinfeld level structures on elliptic curves. These schemes were proved to be normal and finite flat over the 'j-line' $\mathbf{A}^1_{\mathbf{Z}}$, so they extend to proper flat $\mathbf{Z}$-schemes by normalization over $\mathbf{P}^1_{\mathbf{Z}}$. If there is 'enough' level then the $\mathbf{Z}$-proper constructions of Deligne–Rapoport and Katz–Mazur coincide.

The approach in [**KM**] does not give a moduli interpretation at the cusps (in the sense of Deligne and Rapoport), and [**DR**] uses methods in deformation theory that often do not work at the cusps in bad characteristics. One expects that the theory of Drinfeld structures on generalized elliptic curves should provide a moduli-theoretic explanation for the proper $\mathbf{Z}$-structures made in [**DR**] and [**KM**].

In unpublished work, Edixhoven [**Ed**] carried out an analysis of the situation for level structures of the types $\Gamma(N)$, $\Gamma_1(N)$, and $\Gamma_0(n)$ for squarefree n, as well as some mixtures of these level structures. He proved that the moduli stacks in all of these cases are proper and flat Deligne–Mumford stacks over $\mathbf{Z}$ that are moreover regular. Edixhoven used a method resting on considerations with an étale cover by a scheme, and he proved that the moduli stacks are isomorphic to normalizations that were constructed in [**DR**].

In view of the prominent role of the modular curve $X_0(n)$ in the study of elliptic curves over $\mathbf{Q}$, it is natural to ask if the restriction to squarefree n is really necessary in order that the moduli stack for $\Gamma_0(n)$-structures be a reasonable algebro-geometric object over $\mathbf{Z}$. Let us recall why non-squarefree n may seem to present a difficulty. Suppose n has a prime factor p with $\mathrm{ord}_p(n) \geqslant 2$. Choose any $d \mid n$ and let C_d denote the standard

d-gon over an algebraically closed field of characteristic p, considered as a generalized elliptic curve in the usual manner. Assume $p \mid d$, so $\underline{\mathrm{Aut}}(C_d) = \mu_d \rtimes \langle \mathrm{inv} \rangle$ contains μ_p (with 'inv' denoting the unique involution of C_d that restricts to inversion on the smooth locus $C_d^{\mathrm{sm}} = \mathbf{G}_m \times (\mathbf{Z}/d\mathbf{Z})$). If, moreover, $p \mid (n/d)$, then there exist cyclic subgroup schemes G in the smooth locus C_d^{sm} such that G is ample on C_d and has order n with p-part that is non-étale and disconnected. Such a subgroup contains the p-torsion μ_p in the identity component of C_d^{sm}, and so the infinitesimal subgroup μ_p in the automorphism scheme of C_d preserves G. In particular, the finite automorphism scheme of the $\Gamma_0(n)$-structure (C_d, G) contains μ_p and thus is not étale. The moduli stack for $\Gamma_0(n)$-structures therefore cannot be a Deligne–Mumford stack in characteristic p if $p^2 \mid n$. However, failure of automorphism groups to be étale does not prevent the possibility that such stacks can be Artin stacks.

1.2. Results

The first main result of this paper is that $\Gamma_0(n)$-structures form a proper flat Artin stack over $\mathbf{Z}$ for arbitrary n. This is a special case of a more general result, as follows. For any positive integers N and n such that $\mathrm{ord}_p(n) \leqslant \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$ (e.g., $N = 1$ and arbitrary n), define a *$\Gamma_1(N; n)$-structure* on a generalized elliptic curve E over a scheme S to be a pair (P, G) consisting of a $\mathbf{Z}/N\mathbf{Z}$-structure P on E^{sm} and a cyclic subgroup G of order n on E^{sm} such that the Cartier divisor $\sum_{j \in \mathbf{Z}/N\mathbf{Z}} (jP + G)$ in E is ample and

$$\sum_{j \in \mathbf{Z}/d\mathbf{Z}} (j(N/d)P + G_d) = E^{\mathrm{sm}}[d],$$

where $d = \gcd(N, n)$ and $G_d \subseteq G$ is the standard cyclic subgroup of order d in the sense of Definition 2.3.6. (In Theorem 2.4.5 we prove that $\sum_{j \in \mathbf{Z}/N\mathbf{Z}} (jP + G)$ must be a subgroup of E^{sm}.)

Theorem 1.2.1. *The moduli stack $\mathcal{M}_{\Gamma_1(N;n)}$ classifying $\Gamma_1(N; n)$-structures on generalized elliptic curves is a proper flat Artin stack over $\mathbf{Z}$, and it is Deligne–Mumford away from the 0-dimensional closed cuspidal substacks in characteristics p with $p^2 \mid n$. This stack is regular and has geometrically connected fibers with pure dimension 1 over $\mathrm{Spec}\, \mathbf{Z}$.*

We establish this theorem in § 3 by methods that are necessarily rather different from those of Edixhoven (since Artin stacks do not generally admit an étale cover by a scheme and do not generally admit universal deformation rings at geometric points). Our arguments require the use of fine structural properties of auxiliary moduli stacks for $\Gamma(M)$-structures and $\Gamma_1(M)$-structures (even for the treatment of $\Gamma_0(n)$-structures). Thus, for unity in the exposition we treat the moduli stack $\mathcal{M}_\Gamma$ of Γ-structures for all of these Γ *ab ovo* by using the viewpoint of Artin stacks; the main issue is to incorporate cusps in bad characteristics (especially for $\Gamma_1(N; n)$-structures in characteristic p with $p^2 \mid n$). Our proof that every $\mathcal{M}_\Gamma$ is an Artin stack is modelled on the method of tri-canonical embeddings that is used for moduli stacks of stable curves. However, whereas universal tri-canonically embedded stable curves are easy to construct via Hilbert schemes, we need to do additional work in the case of generalized elliptic curves because the group law involves the possibly non-proper smooth locus. The only 'messy calculation' in the proof of Theorem 1.2.1 is in the proof of regularity for $\mathcal{M}_{\Gamma_1(N)}$ along its cusps, but this step is not too unpleasant (we build on calculations in [**KM**, Chapter 10]).

The deformation-theoretic technique at the cusps in [**DR**] rests on an étale quotient argument that usually does not work when the level structure is not étale (also see Remark 2.1.13), and we have to prove that some results in [**KM**] for Drinfeld structures on elliptic curves are valid for Drinfeld structures on generalized elliptic curves $E \to S$ (especially the cyclicity criterion [**KM**, 6.1.1]; see Theorem 2.3.7). The intervention of Artin stacks contributes additional complications, since geometric points on Artin stacks do not generally admit universal deformation rings. To circumvent these problems, we must make more effective use of the group theory. For example, we use group-theoretic structures, and *not* the deformation-theoretic method of Artin, to prove that various moduli stacks $\mathcal{M}_\Gamma$ are Artin stacks. The 'unramified diagonal' criterion then implies that these stacks are often Deligne–Mumford, and in §§ 3.3–4.3 we study the fine structure by exploiting the *a posteriori* existence of universal deformation rings at geometric points on Deligne–Mumford stacks. Such deformation rings are useful in the study of the Artin stack $\mathcal{M}_{\Gamma_1(N;n)}$ because $\mathcal{M}_{\Gamma_1(N;n)}$ admits a canonical finite flat covering by a (not necessarily regular) Deligne–Mumford stack.

The good structure exhibited over $\mathbf{Z}$ in Theorem 1.2.1 suggests considering the associated $\mathbf{Z}$-structure on spaces of classical modular forms. For $\Gamma = \Gamma_1(N;n)$, let $\mathcal{E}_\Gamma \to \mathcal{M}_\Gamma$ be the universal generalized elliptic curve and let ω_Γ be the invertible sheaf on $\mathcal{M}_\Gamma$ that is the pushforward of the relative dualizing sheaf. Let $\mathcal{M}_\Gamma^\infty \subseteq \mathcal{M}_\Gamma$ be the closed substack classifying Γ-structures on degenerate objects, and let $\mathcal{M}_\Gamma^0$ be the complementary open substack. By coherence of higher direct images [**O**] and Theorem 1.2.1, the $\mathbf{Z}$-module $\mathrm{H}^0(\mathcal{M}_\Gamma, \omega_\Gamma^{\otimes k})$ is finite and free, and so it provides a natural $\mathbf{Z}$-structure on the space $\mathrm{M}_k(\Gamma, \mathbf{C})$ of weight-k classical modular forms for Γ; see Remark 4.4.2 for a description via q-expansions when $\Gamma = \Gamma_1(N)$. Upon inverting Nn this recovers the familiar $\mathbf{Z}[1/Nn]$-structure defined by means of q-expansions at a single cusp. For $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$, an extremely tedious calculation with q-expansions at all cusps shows that this $\mathbf{Z}$-structure is preserved under all Hecke operators T_p (even allowing $p \mid N$). However, there is a much better and more useful way to understand Hecke-stability of the $\mathbf{Z}$-structure without using q-series: the underlying correspondence is well-posed over $\mathbf{Z}$. More precisely, we have the following theorem.

Theorem 1.2.2. *For any $N \geqslant 1$ and prime p, the pth Hecke correspondence $\pi_1^0, \pi_2^0 : \mathcal{M}^0_{\Gamma_1(N;p)} \rightrightarrows \mathcal{M}^0_{\Gamma_1(N)}$ uniquely extends to a finite flat correspondence $\pi_1, \pi_2 : \mathcal{M}_{\Gamma_1(N;p)} \rightrightarrows \mathcal{M}_{\Gamma_1(N)}$ on proper stacks over $\mathbf{Z}$. Moreover, the pullback map*

$$\xi^0 : (\pi_2^0)^*(\omega_{\Gamma_1(N)}|_{\mathcal{M}^0_{\Gamma_1(N)}}) \to \omega_{\Gamma_1(N;p)}|_{\mathcal{M}^0_{\Gamma_1(N;p)}} = (\pi_1^0)^*(\omega_{\Gamma_1(N)}|_{\mathcal{M}^0_{\Gamma_1(N)}})$$

along the universal p-isogeny over $\mathcal{M}^0_{\Gamma_1(N;p)}$ uniquely extends to a map $\xi : \pi_2^*\omega_{\Gamma_1(N)} \to \pi_1^*\omega_{\Gamma_1(N)}$ over $\mathcal{M}_{\Gamma_1(N;p)}$.

The essential content in this theorem is that the morphism $\pi_2^0 : (E; P, C) \rightsquigarrow (E/C, P \bmod C)$ over $\mathcal{M}^0_{\Gamma_1(N;p)}$ may be (uniquely up to unique isomorphism) extended to the entire moduli stack $\mathcal{M}_{\Gamma_1(N;p)}$ (on which the traditional quotient E/C does not make sense as a generalized elliptic curve). The proof rests on two ingredients: a general formal criterion for extending maps between suitable Deligne–Mumford stacks, and a study of descent theory over deformation rings on $\mathcal{M}_{\Gamma_1(N)}$ at geometric points in $\mathcal{M}^{\infty}_{\Gamma_1(N)}$. In § 4.4 we prove Theorem 1.2.2 (see Theorem 4.4.3). The analogue of Theorem 1.2.2 for $\Gamma_0(N)$ is true, but this requires additional arguments because $\mathcal{M}_{\Gamma_0(N)}$ is generally not Deligne–Mumford (and so does not admit universal deformation rings at its geometric points); we give some brief indications on this variant, using the result for $\Gamma_1(N)$, at the end of § 4.4.

Recall that the definition of T_p on meromorphic modular forms over $\mathbf{C}$ involves division by p upon the operations in coherent cohomology:

$$T_p = (1/p)\,\mathrm{Tr}_{\pi_1^0} \circ (\xi^0)^{\otimes k} \circ (\pi_2^0)^*.$$

Hence, to work with T_p in coherent cohomology over $\mathbf{Z}$ we need to analyze divisibility by p, which is to say that we have to prove a vanishing property in characteristic p. In § 4.5 we address this vanishing via conceptual local considerations on $\mathcal{M}^0_{\Gamma_1(N)/\mathbf{F}_p}$, thereby constructing all operators T_p over $\mathbf{Z}$ without requiring any explicit q-series calculations to verify holomorphicity along the cusps. (This gives a purely arithmetic proof, without topological cohomology or the artifice of q-expansions, that eigenvalues of classical Hecke operators are algebraic integers.) An interesting application of these local arguments is that they lead to a new uniform construction of the Hecke operator T_p on Katz modular forms for $\Gamma_1(N)$ in characteristic $p \nmid N$ and any weight, especially weight 1. In contrast with the approach in [**G**, § 4] that uses the q-expansion formula for T_p in characteristic 0 to 'define' the operator T_p on Katz forms in weight 1, we deduce this formula *a posteriori* from our uniform definition in all weights.

1.3. Notation and terminology

If S is a scheme and X is an S-scheme, then $X_{/S'}$ and $X_{S'}$ denote $X \times_S S'$ for an S-scheme S'. If $X \to S$ is flat and locally of finite presentation, then X^{sm} denotes the (open) S-smooth locus. If $G \to S$ is a finite locally free commutative S-group of order N, then $G^{\times}$ denotes the scheme of $\mathbf{Z}/N\mathbf{Z}$-generators of G [**KM**, 1.10.13]; this is finite and finitely presented over S. For example, we have $\mu_N^{\times} = \mathrm{Spec}(\mathbf{Z}[T]/\Phi_N(T))$ with universal generator T, where Φ_N is the Nth cyclotomic polynomial [**KM**, 1.12.9]. We define $\phi(N) = \deg \Phi_N$ for $N \geqslant 1$.

We use [**LMB**] as the basic reference on stacks; in particular, we require Artin stacks to have diagonal morphisms that are represented by separated algebraic spaces of finite type. We adopt one abuse of terminology that we hope will not cause confusion: rather than speak of 1-morphisms (of stacks) and 2-isomorphisms between 1-morphisms, we will use the words 'morphism' and 'isomorphism', respectively. One deviation we make from [**LMB**] concerns the role of the base scheme: in [**LMB**] the general theory is developed over a fixed quasi-separated base scheme S, and we consider the general theory to have $\mathrm{Spec}\,\mathbf{Z}$ as the base scheme. This allows us to discuss Artin stacks over arbitrary schemes, as follows. An *Artin stack over a scheme S* is an Artin stack $\mathcal{S}'$ over $\mathrm{Spec}\,\mathbf{Z}$ (in the sense of [**LMB**]) equipped with a morphism to S; the diagonal of the structural morphism

$\mathcal{S}' \to S$ is automatically representable in algebraic spaces and is both separated and of finite type. For quasi-separated S, this definition recovers the notion of 'Artin stack over S' as in [**LMB**]. To be precise, if S is quasi-separated and $\mathcal{S}$ is a stack over S in the sense of [**LMB**] then let $\mathcal{S}'$ be the stack over $\operatorname{Spec} \mathbf{Z}$ whose fiber category over a ring A is the groupoid of pairs (f, x) where $f : \operatorname{Spec} A \to S$ is a map of schemes and x is an object in the fiber category $\mathcal{S}_{(A,f)}$ over the affine scheme $\operatorname{Spec} A$ over S. There is an evident morphism $\mathcal{S}' \to S$ that 'is' $\mathcal{S}$, and $\Delta_{\mathcal{S}/S}$ is representable in algebraic spaces if and only if $\Delta_{\mathcal{S}'/\operatorname{Spec}\mathbf{Z}}$ is so, in which case the quasi-separatedness of S ensures that $\Delta_{\mathcal{S}/S}$ is quasi-compact if and only if $\Delta_{\mathcal{S}'/\operatorname{Spec}\mathbf{Z}}$ is quasi-compact, and the same holds for the properties of the diagonal being separated or of finite type. It follows that $\mathcal{S}$ is an Artin stack over S in the sense of [**LMB**] if and only if $\mathcal{S}'$ is an Artin stack over $\operatorname{Spec} \mathbf{Z}$. Hence, all theorems in [**LMB**] for Artin stacks over a quasi-separated S are valid for arbitrary S via our general definition because any scheme S is covered by quasi-separated (e.g., affine) opens.

2. The fundamental definitions and examples

2.1. Basic definitions

For the convenience of the reader, let us first review some standard notions.

Definition 2.1.1. A morphism of schemes $X \to S$ is *Cohen–Macaulay* (or CM) if it is flat and locally of finite presentation with Cohen–Macaulay fibers.

Definition 2.1.2. A *curve* over a scheme S is a morphism $C \to S$ that is separated, flat, and finitely presented with all fibers non-empty of pure dimension 1. A *Deligne–Rapoport (DR) semistable genus-*1 *curve* over S is a proper curve $f : C \to S$ such that the geometric fibers are connected and semistable with trivial dualizing sheaf.

In [**DR**], DR semistable genus-1 curves are called 'stable genus-1' curves. A basic fact that is useful in universal constructions is the following lemma.

Lemma 2.1.3 ([DR, II, 1.5]). *Let $f : C \to S$ be a proper flat map of finite presentation. The set of $s \in S$ such that C_s is a DR semistable genus-*1 *curve is open.*

In [**DR**, II, 1.2, 1.3] it is shown that the DR semistable genus-1 curves over an algebraically closed field are exactly the smooth curves with genus 1 and the so-called Néron n-gons (for $n \geqslant 1$) whose definition we now recall. For any $n > 1$, the *standard n-gon* (or *Néron n-gon*) C_n over a scheme S is the S-proper curve obtained from $\mathbf{P}^1_S \times \mathbf{Z}/n\mathbf{Z}$ by 'gluing' the ∞-section on $\mathbf{P}^1_S \times \{i\}$ to the 0-section on $\mathbf{P}^1_S \times \{i+1\}$ for all $i \in \mathbf{Z}/n\mathbf{Z}$. The formation of this gluing naturally commutes with base change. The tautological action of $\mathbf{Z}/n\mathbf{Z}$ on $\mathbf{P}^1_S \times \mathbf{Z}/n\mathbf{Z}$ uniquely factors through an action of $\mathbf{Z}/n\mathbf{Z}$ on each C_n. This action on C_n is free and for any $d \mid n$ with $d > 1$ there is a unique map

$$C_n \to C_d \tag{2.1.1}$$

that is compatible with the projection $\mathbf{P}^1_S \times \mathbf{Z}/n\mathbf{Z} \to \mathbf{P}^1_S \times \mathbf{Z}/d\mathbf{Z}$. This map is invariant under the free action of $d\mathbf{Z}/n\mathbf{Z}$ on C_n, and it realizes C_n as a finite étale $d\mathbf{Z}/n\mathbf{Z}$-torsor over C_d.

Since the action of $\mathbf{Z}/n\mathbf{Z}$ on C_n is free and each orbit lies in an open affine, by [**SGA3**, Exposé V, Theorem 4.1] there exists an S-scheme quotient map $C_n \to C_{1,n}$ that is a finite étale $\mathbf{Z}/n\mathbf{Z}$-torsor, and its formation is compatible with arbitrary base change over S. Using the torsor maps (2.1.1) we see that $C_{1,n}$ is independent of n and hence it may be denoted C_1. As an S-scheme, C_1 is proper with a canonical section '1' that is induced by any of the sections $(1,i)$ of any C_n with $n > 1$.

We call C_1 the *standard* 1-*gon*; this is equipped with a canonical finite map $\mathbf{P}^1_S \to C_1$ realizing C_1 as the gluing of the sections 0 and ∞ in $\mathbf{P}^1_S$ (the corresponding universal property follows from a comparison with C_m for any $m > 1$). The map $t \mapsto (t^2+1, t(t^2+1))$ from $\mathbf{P}^1$ to the nodal plane curve $y^2z = x^3 - x^2z$ factorizes through C_1 and induces an isomorphism between C_1 and this nodal cubic.

The natural action $(\mathbf{G}_m \times \mathbf{Z}/n\mathbf{Z}) \times_S (\mathbf{P}^1_S \times \mathbf{Z}/n\mathbf{Z}) \to \mathbf{P}^1_S \times \mathbf{Z}/n\mathbf{Z}$ uniquely descends to a morphism

$$+ : C_n^{\mathrm{sm}} \times_S C_n \to C_n \tag{2.1.2}$$

for all $n \geqslant 1$. This is an action extending the group law on $C_n^{\mathrm{sm}} = \mathbf{G}_m \times \mathbf{Z}/n\mathbf{Z}$ and it is compatible with base change and with change in n. The structure (2.1.2) is an example of the following notion, first introduced in [**DR**, II, 1.12].

Definition 2.1.4. A *generalized elliptic curve* over S is a triple $(E, +, e)$ where E is a DR semistable genus-1 curve, $+ : E^{\mathrm{sm}} \times_S E \to E$ is an S-morphism, and $e \in E^{\mathrm{sm}}(S)$ is a section such that

- $+$ restricts to a commutative group scheme structure on E^{sm} with identity section e,
- $+$ is an action of E^{sm} on E such that on singular geometric fibers the translation action by each rational point in the smooth locus induces a rotation on the graph of irreducible components (this forces the component groups of geometric fibers E_s^{sm} to be cyclic).

A *morphism* between generalized elliptic curves E and E' over a scheme S is a map $f : E \to E'$ as S-schemes such that $f(E^{\mathrm{sm}}) \subseteq E'^{\mathrm{sm}}$ (e.g., a finite étale S-map, or the zero map) and the induced map on smooth loci is a map of S-groups. Considerations with universal schematic density [**EGA**, IV$_3$, 11.10.4, 11.10.10] ensure that a morphism in this sense is automatically equivariant with respect to the actions of E^{sm} on E and of E'^{sm} on E'.

Over an algebraically closed field, a generalized elliptic curve is either an elliptic curve or is isomorphic to a standard n-gon with the structure (2.1.2) [**DR**, II, 1.15]. Whenever we speak of standard polygons over S as generalized elliptic curves, it is always understood that we use the structure (2.1.2).

Example 2.1.5. By [**DR**, II, 1.10], the automorphism functor of the standard n-gon C_n as a generalized elliptic curve is $\langle \mathrm{inv} \rangle \ltimes \mu_n$, where inv is the unique involution extending inversion on C_n^{sm} and, for $i \in \mathbf{Z}/n\mathbf{Z}$, μ_n acts on the ith fibral component through $[\zeta](t) = \zeta^i t$.

Example 2.1.6. If E is a non-smooth generalized elliptic curve over a field and $G \subseteq E^{\mathrm{sm}}$ is a finite subgroup then the action of G on E has non-trivial isotropy groups at the non-smooth points except when G is étale and has trivial intersection with the identity component of E^{sm}. Thus, the quotient E/G as a scheme is usually not a DR semistable genus-1 curve and the map $E \to E/G$ is usually not flat. When forming a quotient E/G in the setting of non-smooth generalized elliptic curves over a base scheme, we shall therefore always require G to have trivial intersection with the identity component on non-smooth geometric fibers (so G acts freely on E and is étale on non-smooth fibers, and hence E/G has a natural structure of generalized elliptic curve).

Let $f : C \to S$ be a curve. Since f is flat with fibers of pure dimension 1, the relative smooth locus is exactly where $\Omega^1_{C/S}$ admits a single generator. The jth Fitting ideal sheaf encodes the obstruction to admitting j generators [**Eis**, Proposition 20.6], so the first Fitting ideal sheaf of $\Omega^1_{C/S}$ defines a canonical closed subscheme structure on $C^{\mathrm{sing}} = C - C^{\mathrm{sm}}$. The formation of C^{sing} as an S-scheme is compatible with base change on S.

Example 2.1.7. If $S = \operatorname{Spec} A$ and $C = \operatorname{Spec}(A[x,y]/(xy - a))$, then the ideal (x, y) cuts out C^{sing}.

Definition 2.1.8. If $f : C \to S$ is a proper S-curve, then the *locus of non-smoothness* of f is the scheme-theoretic image $S^{\infty,f}$ of C^{sing} in S.

The scheme $S^{\infty,f}$ is a canonical closed subscheme structure on the closed set of points of S over which the proper S-curve $f : C \to S$ has a non-smooth fiber.

Example 2.1.9. By Example 2.1.7, $S^{\infty,f} = S$ when $f : C \to S$ is the standard n-gon over a scheme S.

Lemma 2.1.10 ([DR, II, 1.5]). *Let $f : E \to S$ be a generalized elliptic curve. The closed subscheme $S^{\infty,f}$ is a locally finite (in S) disjoint union of open subschemes $S^{\infty,f}_n$ such that the generalized elliptic curve E is isomorphic to the standard n-gon fppf-locally over $S^{\infty,f}_n$.*

It is generally *not* true that the formation of $S^{\infty,f}$ commutes with (non-flat) base change on S for arbitrary DR semistable genus-1 curves $f : C \to S$. To be precise, if T is an S-scheme and $f_T : C_T \to T$ is the base change of f over T, then there is an inclusion of closed subschemes

$$T^{\infty,f_T} \subseteq S^{\infty,f} \times_S T \tag{2.1.3}$$

inside of T but this can fail to be an equality of subschemes (though it is an equality on underlying topological spaces).

Example 2.1.11. Here is an interesting example in which (2.1.3) is not an isomorphism. Let $S = \operatorname{Spec}(A)$ be a local artin scheme and let $a, a' \in A$ be two elements in the maximal ideal. Consider a twisted version $C_{a,a'}$ of the standard 2-gon such that the singularities look like $tt' = a$ and $uu' = a'$. More specifically, we glue $\operatorname{Spec}(A[t,t']/(tt' - a))$ to

$\mathrm{Spec}(A[u,u']/(uu'-a'))$ along the complements of the origins via

$$(t,t') \mapsto \begin{cases} (1/t, a't) & \text{if } t \neq 0, \\ (a't', 1/t') & \text{if } t' \neq 0, \end{cases}$$

$$(u,u') \mapsto \begin{cases} (1/u, au) & \text{if } u \neq 0, \\ (au', 1/u') & \text{if } u' \neq 0. \end{cases}$$

This gluing is A-flat and respects base change on A, so it is trivially a DR semistable genus-1 curve. (Its fiber over the reduced point of $\mathrm{Spec}(A)$ is the standard 2-gon since a and a' are nilpotent.) Using a Fitting-ideal calculation, one sees that the locus of non-smoothness in the base is defined by the intersection of the ideals (a) and (a'). The formation of such an intersection does not generally commute with base change.

For example, if $A = k[\varepsilon,\varepsilon']/(\varepsilon,\varepsilon')^2$ for a field k, and we choose $a = \varepsilon$ and $a' = \varepsilon'$, then $(a) \cap (a') = 0$. In this case the locus of non-smoothness of $C_{a,a'}$ is the entire base, but over the closed subscheme defined by $\varepsilon = \varepsilon'$ the locus of non-smoothness is $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ rather than the entire (new) base $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$.

The following important fact is implicit (but not stated) in [**DR**] and it underlies the definition of closed substacks 'at infinity' in moduli stacks of generalized elliptic curves. In view of Example 2.1.11, it gives a special property of generalized elliptic curves among all DR semistable genus-1 curves.

Theorem 2.1.12. *Let $f : E \to S$ be a generalized elliptic curve. The formation of the closed subscheme $S^{\infty,f} \hookrightarrow S$ is compatible with base change on S.*

Proof. By Lemma 2.1.10, the base change $E_{/S^{\infty,f}}$ is *fppf*-locally isomorphic to the standard n-gon for Zariski-locally constant n on $S^{\infty,f}$. Let T be an S-scheme. If (2.1.3) is not an isomorphism, then the same is true after any base change to a T-scheme that is an *fppf*-cover of $S^{\infty,f} \times_S T$. Since E becomes a standard polygon *fppf*-locally over $S^{\infty,f}$, by Example 2.1.9 we conclude that (2.1.3) has to be an isomorphism. □

Remark 2.1.13. By [**DR**, II, 2.7], any DR semistable genus-1 curve $C \to S$ with irreducible geometric fibers and a section $e \in C^{\mathrm{sm}}(S)$ admits a (unique) structure of generalized elliptic curve with e as the identity. In contrast, by Theorem 2.1.12 it follows that Example 2.1.11 gives DR semistable genus-1 curves such that the non-smooth geometric fibers are 2-gons and such that there does not exist a generalized elliptic curve structure *fpqc*-locally on the base. This is why it is hard to determine, via purely geometric methods, which infinitesimal flat deformations of such data as bare schemes admit the required group structures. In [**DR**, III, 1.4.2], such problems are avoided in the case of étale level structures by means of an étale quotient argument that passes to the case of geometrically irreducible fibers.

We conclude our review by recalling the definition of *relative contraction* away from a Cartier divisor with support in the smooth locus. Let C be a DR semistable genus-1 curve and let D be a relative effective Cartier divisor on C^{sm} that is finite over S. By [**DR**, IV, 1.2] there is a DR semistable genus-1 curve $\overline{C}_{/S}$ and a proper S-morphism

$$u : C \to \overline{C} \tag{2.1.4}$$

that contracts all geometric fibral irreducible components that are disjoint from D. This 'contraction away from D' is unique up to unique isomorphism [**DR**, IV, 1.2(b)] (but C can have automorphisms over $\overline{C}$). In particular, its formation is compatible with base change. This compatibility implies that the restriction $u : u^{-1}(\overline{C}^{\mathrm{sm}}) \to \overline{C}^{\mathrm{sm}}$ is an isomorphism. Since the open subscheme $\overline{C}^{\mathrm{sm}} \subseteq \overline{C}$ is universally schematically dense over S, it follows from the uniqueness of contractions that for any S-automorphism α of C taking D into D there is a unique S-automorphism $\overline{\alpha}$ of $\overline{C}$ such that $\overline{\alpha} \circ u = u \circ \alpha$. In more precise terms, the contraction is uniquely functorial with respect to isomorphisms. In particular, if C is a generalized elliptic curve and D is a subgroup of C^{sm} then $\overline{C}$ has a unique structure of generalized elliptic curve such that u induces a group morphism from $u^{-1}(\overline{C}^{\mathrm{sm}})$ to $\overline{C}^{\mathrm{sm}}$, and u is equivariant for the actions of $u^{-1}(\overline{C}^{\mathrm{sm}})$ and $\overline{C}^{\mathrm{sm}}$ on C and $\overline{C}$ respectively. Note also that D is identified with an S-subgroup of $\overline{C}^{\mathrm{sm}}$ and it is S-ample in $\overline{C}$.

2.2. Background results

We now record (for later use) a construction principle of Deligne and Rapoport, a general criterion for an Artin stack to be an algebraic space, and a discussion of excellence for stacks.

Consider a proper semistable curve $f : C \to S$ and a section $e \in C(S)$. Since the geometric fibers are reduced and connected, we have $\mathcal{O}_S \simeq f_*\mathcal{O}_C$ universally. Thus, the section e allows us to view the functor $\mathrm{Pic}_{C/S}$ as classifying line bundles on C rigidified along the section e [**BLR**, pp. 204, 205], and it is a locally separated algebraic space group locally of finite presentation over S [**Ar**, Theorem 7.3]. This S-group is smooth by functorial criteria. By [**SGA6**, Exposé XIII, Theorem 4.7] and [**BLR**, 9.2/14] there is a universal line bundle over this algebraic space and the union $\mathrm{Pic}^0_{C/S} = \mathrm{Pic}^{\tau}_{C/S}$ of the fibral identity components is an open subspace of finite presentation over S classifying line bundles with degree-0 restriction to each irreducible component of a geometric fiber of $C \to S$. Since $C \to S$ is a semistable curve, the valuative criterion ensures that $\mathrm{Pic}^0_{C/S}$ is S-separated and thus is a semi-abelian algebraic space group over S by [**BLR**, 9.2/8]. (Deligne proved that $\mathrm{Pic}^0_{C/S}$ is in fact a scheme, but we do not require this; of course, if S is artin local then the scheme property is automatic since algebraic space groups locally of finite type over an artin ring are schemes.)

Lemma 2.2.1. *Let $A \to S$ be a semi-abelian algebraic space over a scheme S. Let g, h be two endomorphisms of $A_{/S}$. There exists an open and closed subscheme $U \subseteq S$ such that for any S-scheme S', the pullback endomorphisms g', h' of $A' = A_{/S'}$ coincide if and only if $S' \to S$ factors through U.*

Proof. If A is a scheme, this is [**DR**, II, 1.14]; the same proof works with A an algebraic space. □

Let $E \to S$ be a generalized elliptic curve. By [**DR**, II, 1.13], the natural action map of algebraic spaces

$$E^{\mathrm{sm}} \times \mathrm{Pic}^0_{E/S} \to \mathrm{Pic}^0_{E/S} \tag{2.2.1}$$

arising from the E^{sm}-action on E must be trivial. An approximate converse is given by the following extremely useful result that underlies our study of moduli stacks of 'non-étale' level structures on generalized elliptic curves.

Theorem 2.2.2 ([DR, II, 3.2]). *Let $f : C \to S$ be a DR semistable* genus-1 *curve and let $e \in C^{\mathrm{sm}}(S)$ be a section. Let G be a commutative flat S-group scheme locally of finite presentation, and let $\rho : G \times C \to C$ be an action of G on C. Assume that G acts trivially on the algebraic space $\mathrm{Pic}^0_{C/S}$ and that $G(\overline{s})$ acts transitively on the set of irreducible components of $C_{\overline{s}}$ for all geometric points $\overline{s}$ of S.*

There exists a unique generalized elliptic curve structure on C with identity section e such that each $g \in G(T)$ acts on $C_{/T}$ via translation by $g(e) \in C^{\mathrm{sm}}(T)$ for all S-schemes T. Moreover, any automorphism α of C commuting with the G-action is translation by $\alpha(e) \in C^{\mathrm{sm}}(S)$.

The following special case of Theorem 2.2.2 suffices for us.

Corollary 2.2.3. *Let $f : C \to S$ be a DR semistable* genus-1 *curve and let $D \hookrightarrow C$ be an S-ample relative effective Cartier divisor supported in C^{sm}. Assume that D is endowed with a structure of commutative S-group scheme and that there is given an action of D on C that extends the group scheme structure on D.*

This extends to a generalized elliptic curve structure on C if and only if the natural induced action of D on the algebraic space $\mathrm{Pic}^0_{C/S}$ is trivial, in which case such a generalized elliptic curve structure on $C_{/S}$ is unique. The triviality of the D-action on $\mathrm{Pic}^0_{C/S}$ may be checked on geometric fibers over S, and the locus of fibers with trivial action is an open and closed subset of S.

Proof. The S-ampleness of $D \subseteq C^{\mathrm{sm}}$ ensures that, on geometric fibers, the action of $D(\overline{s})$ on $C_{\overline{s}}$ is transitive on the set of irreducible components of $C_{\overline{s}}$. Hence, we may use Theorem 2.2.2 (with $G = D$) once we explain why it suffices to check the triviality condition on fibers, and why the locus of fibers with trivial action is open and closed in S. More generally, if A is a semi-abelian algebraic space over S and $G \to S$ is a finite locally free commutative S-group equipped with an action $\alpha : G \times A \to A$ over S then we claim that the condition on S-schemes S' that $G_{/S'}$ acts trivially on $A_{/S'}$ is represented by a Zariski-open and Zariski-closed subscheme $U \subseteq S$ (so G acts trivially on A if and only if G_s acts trivially on A_s for all $s \in S$).

The universal action

$$\alpha : G \times A \to G \times A \tag{2.2.2}$$

given by $(g, a) \mapsto (g, (\alpha(g))(a))$ is an endomorphism of the semi-abelian algebraic space $A_{/G}$ over $T = G$, so by Lemma 2.2.1 there is an open and closed subscheme $V \subseteq G$ such that for any G-scheme $f : T' \to G$ the pullback of α along f is the identity if and only if f factors through V. Thus, if we let $U \subseteq S$ be the Zariski-open and Zariski-closed complement of the Zariski-open and Zariski-closed image of $G - V$ in S then U represents the condition that $G_{/S'}$ acts trivially on $A_{/S'}$ for variable S-schemes S'. □

Corollary 2.2.4. *Let A be an adic noetherian ring with ideal of definition I, and let $A_n = A/I^{n+1}$ for $n \geqslant 0$. The functor $E \rightsquigarrow (E \bmod I^{n+1})_{n \geqslant 0}$ is an equivalence of categories between the category of generalized elliptic curves over* $\operatorname{Spec} A$ *whose degenerate geometric fibers all have a common number of irreducible components and the category of compatible systems $(E_n)_{n \geqslant 0}$ of such generalized elliptic curves over the schemes* $\operatorname{Spec} A_n$.

Loosely speaking, this corollary says that formal generalized elliptic curves $\mathfrak{E}$ over $\operatorname{Spf} A$ admit unique algebraizations over $\operatorname{Spec} A$, provided that the number of irreducible components on non-smooth fibers is fixed (a condition that is always satisfied when $\operatorname{Spec} A$ is local, by Lemma 2.1.10); Corollary 2.2.4 has content because the locus $\mathfrak{E}^{\mathrm{sm}}$ of formal smoothness for $\mathfrak{E}$ over $\operatorname{Spf} A$ is generally not proper over $\operatorname{Spf} A$.

Proof. Let $(E_n)_{n \geqslant 0}$ be a compatible family of generalized elliptic curves over the schemes $\operatorname{Spec} A_n$ such that each E_n has a fixed number of irreducible components for its degenerate geometric fibers. This common number must be the same for all n; we let d be this number, and we define $d = 1$ if the E_n are smooth. Each $D_n = E_n^{\mathrm{sm}}[d]$ is quasi-finite, separated, and flat over $\operatorname{Spec} A_n$ with constant fibral rank (namely, d^2), so by [**DR**, II, 1.19] each D_n is *finite* over $\operatorname{Spec} A_n$. Hence, $D_n \hookrightarrow E_n$ is a relatively ample relative effective Cartier divisor. Thus, Grothendieck's formal GAGA and existence theorems [**EGA**, III$_1$, 5.4.1, 5.4.5] may be applied to uniquely construct a proper flat A-scheme E, a finite flat commutative d-torsion A-group scheme D equipped with a closed immersion into E, and an action

$$\rho : D \times E \to E$$

extending the group law on D such that reduction modulo I^{n+1} recovers E_n, the A_n-group D_n, and the action of D_n on E_n. By construction, D has order d^2. Any open set in $\operatorname{Spec} A$ that contains $\operatorname{Spec} A/I$ is the entire space, due to the fact that A is an adic noetherian ring with ideal of definition I. Thus, since E^{sm} is open in E, it follows from properness of E that D is supported in E^{sm}. Moreover, by Lemma 2.1.3 the proper flat A-curve E is a DR semistable genus-1 curve since its fibers over $\operatorname{Spec} A/I$ are such curves. In particular, $\operatorname{Pic}^0_{E/A}$ exists as a semi-abelian algebraic space.

The D-action on $\operatorname{Pic}^0_{E/A}$ is trivial over $\operatorname{Spec} A/I$. The only open subscheme in $\operatorname{Spec} A$ containing $\operatorname{Spec} A/I$ is the entire space, so by Corollary 2.2.3 we conclude that there exists a unique structure of generalized elliptic curve on E that is compatible with the A-group structure on D and with the action ρ, and moreover this must induce the given generalized elliptic curve structure on each E_n. Every non-empty closed set in $\operatorname{Spec} A$ meets $\operatorname{Spec} A/I$, so it follows from Lemma 2.1.10 that all degenerate geometric fibers of E over $\operatorname{Spec} A$ are d-gons. In particular, $E^{\mathrm{sm}}[d]$ is a finite flat group scheme with order d^2. Since the finite flat subgroup D in E^{sm} is d-torsion with order d^2, the closed immersion $D \hookrightarrow E^{\mathrm{sm}}[d]$ must be an isomorphism.

Due to the functoriality of d-torsion, it is clear (via formal GAGA, Theorem 2.2.2, and the closedness of the non-smooth loci) that the construction of E with its generalized elliptic curve structure is functorial with respect to morphisms in the inverse system $(E_n)_{n\geqslant 0}$ and moreover gives a quasi-inverse functor to the 'completion' functor $E \rightsquigarrow (E \bmod I^{n+1})_{n\geqslant 0}$. □

In [**LMB**, 8.1.1] a necessary and sufficient criterion is given for an Artin stack to be an algebraic space, but the criterion uses scheme-valued points and in practice it is convenient to require only the use of geometric points. The sufficiency of using geometric points is well known, but due to lack of a reference we provide a proof.

Theorem 2.2.5. *Let $\mathcal{M}$ be an Artin stack over a scheme S.*

(1) *$\mathcal{M}$ is an algebraic space if and only if its geometric points have trivial automorphism functors. In particular, $\mathcal{M}$ is an algebraic space if and only if $\mathcal{M}_{\mathrm{red}}$ is an algebraic space.*

(2) *If $\mathcal{M}$ is locally S-separated, then there exists a unique open substack $\mathcal{U} \subseteq \mathcal{M}$ such that the geometric points of $\mathcal{U}$ are exactly the geometric points of $\mathcal{M}$ whose automorphism functor is trivial. The open substack $\mathcal{U}$ is an algebraic space.*

Proof. By [**LMB**, 8.1.1], an Artin stack is an algebraic space precisely when, for every S-scheme U and morphism $u : U \to \mathcal{M}$ over S, the algebraic space group $G = \underline{\mathrm{Aut}}_{\mathcal{M}_U}(u)$ of finite type over U is the trivial group. To prove (1), we must show that it suffices to take U to be a geometric point.

The map $G \to U$ is a separated algebraic space group of finite type, and the hypothesis on automorphism functors of geometric points implies that G has trivial fibers over U. In particular, G is quasi-finite and separated over U, so G is a scheme by [**LMB**, Theorem A.2]. By Nakayama's lemma, $\Omega^1_{G/U} = 0$ since $\Omega^1_{G_{u_0}/u_0} = 0$ for all $u_0 \in U$. It therefore suffices to prove that if $f : X' \to X$ is a finite type map of schemes and $e : X \to X'$ is a section then f is an isomorphism if $\Omega^1_{X'/X} = 0$ and the (necessarily finite) geometric fibers of f have rank 1. The immersion $\Delta_{X'/X}$ is locally finitely presented, so by Nakayama's lemma and the definition of $\Omega^1_{X'/X}$ we see that $\Delta_{X'/X}$ is an open immersion. Hence, every section to f is an open immersion. The section e is therefore an open immersion, yet is it surjective by the hypothesis on geometric fibers. Thus, e is an isomorphism, and so f is an isomorphism.

We now consider (2). By working locally on $\mathcal{M}$, we can assume that $\mathcal{M}$ is S-separated. Hence, the diagonal of $\mathcal{M}$ is proper, so G is a proper U-group. By Nakayama's lemma, the closed support of $\Omega^1_{G/U}$ on G meets each fiber G_{u_0} in the support of $\Omega^1_{G_{u_0}/u_0}$, so the complement of the image of this support in U is an open locus $U' \subseteq U$ that classifies the étale geometric fibers for G over U. Hence, by replacing $\mathcal{M}$ with an open substack we may suppose that the maps $G \to U$ are proper with étale fibers, and so (by [**LMB**, Corollary A.2.1]) G is a finite U-scheme.

Openness of the locus of geometric points of $\mathcal{M}$ with trivial automorphism functor is now reduced to the general claim that if $Y \to Z$ is a finite map of schemes with $\Omega^1_{Y/Z} = 0$ and if there exists a section $e : Z \to Y$ then the set of geometric points $z \in Z$ such that the étale fiber Y_z has one geometric point is an open set in Z. As before, since $\Omega^1_{Y/Z} = 0$ the diagonal $\Delta_{Y/Z}$ has open image. The complement of this image in $Y \times_Z Y$ is a closed set whose image Z' in Z is closed (as $Y \to Z$ is proper), and the complement of Z' in Z is the desired open locus (since the fibers Y_z are étale for all $z \in Z$). This completes the construction of $\mathcal{U}$ as an open substack, and it follows from (1) that $\mathcal{U}$ is an algebraic space. □

Remark 2.2.6. In the preceding proof, we used the result [**LMB**, Theorem A.2] that an algebraic space that is separated and locally quasi-finite over a scheme is a scheme. There is a minor error in the proof of this result in [**LMB**]: in the notation of that proof, the deduction $x \in X_1$ near the end of the second paragraph is only true under the hypothesis that Z_1 is non-empty (which is to say $x \in f(Y)$). The correct deduction in general is that f is finite over a retrocompact open set in X, and this suffices for the proof of [**LMB**, Theorem A.2].

Corollary 2.2.7. *A map $f : \mathcal{M}' \to \mathcal{M}$ between Artin stacks is representable in algebraic spaces if and only if its geometric fibers are algebraic spaces.*

Proof. Necessity is clear, and for sufficiency we may assume that $\mathcal{M}$ is an algebraic space. By Theorem 2.2.5, we have to prove that every geometric point $s : \operatorname{Spec} k \to \mathcal{M}'$ (with k an algebraically closed field) has trivial automorphism functor. Since $\mathcal{M}$ is an algebraic space, this automorphism functor is unaffected by replacing $\mathcal{M}'$ with its geometric fiber over the composite $\operatorname{Spec} k \to \mathcal{M}' \to \mathcal{M}$. This geometric fiber is an algebraic space by hypothesis, so the automorphism functor is indeed trivial. □

We conclude by considering the notion of excellence for Artin stacks. In [**EGA**, IV_4, 18.7.7], there is given an example of a semi-local noetherian ring that is not universally catenary (and hence not excellent) but that admits a finite étale cover that is excellent. The properties of excellence and being universally catenary are therefore not local for the étale topology on locally noetherian schemes. Thus, these notions do not admit a reasonable definition (in terms of one smooth chart) for algebraic spaces, nor for Artin stacks. However, there is an aspect of excellence that does make sense for Artin stacks.

Theorem 2.2.8. *Let $\mathcal{S}$ be a locally noetherian Artin stack and let $X \to \mathcal{S}$ be a smooth covering by a scheme.*

(1) *If all local rings X are G-rings, then the same holds for any scheme smooth over $\mathcal{S}$.*

(2) *If A is a local noetherian ring, then it is a G-ring if and only if a strict henselization A^{sh} is a G-ring.*

Recall that a noetherian ring A is a *G-ring* if the morphism $\operatorname{Spec}(A^\wedge_{\mathfrak{p}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is regular; that is, (it is flat and) its fiber over any $x \in \operatorname{Spec}(A_{\mathfrak{p}})$ is regular and remains so under arbitrary finite extension on the residue field at x. It suffices to work with maximal $\mathfrak{p}$ in this definition (see [**CRT**, § 32] for more details). Theorem 2.2.8 is also true for the property of being universally Japanese (in the sense of [**EGA**, IV_2, 7.6, 7.7]), as is easily proved by direct limit arguments.

Proof. By [**EGA**, IV$_2$, 7.4.4] (or [**CA**, Theorem 77]), if all local rings on X are G-rings then the same holds for the local rings on any scheme locally of finite type over X. Conversely, since smooth morphisms are regular, by [**CRT**, 32.1] it follows that if X admits a smooth covering whose local rings are G-rings then the same holds for X. This settles (1).

Now consider (2). If A^{sh} is a G-ring, then since the map $A \to A^{\mathrm{sh}}$ is regular it follows (again using [**CRT**, 32.1]) that the composite

$$A \to \widehat{A} \to (A^{\mathrm{sh}})^{\wedge}$$

is regular, and consequently $A \to \widehat{A}$ is regular. Thus, A is a G-ring. Conversely, by [**Gre**, Theorem 5.3(i)], if A is a G-ring then A^{sh} is a G-ring. $\square$

2.3. Drinfeld structures on generalized elliptic curves

In [**KM**, Chapters 1–6], the theory of Drinfeld structures on smooth commutative curve groups is developed. Although results in [**KM**, Chapter 1] are applicable to generalized elliptic curves, many proofs in [**KM**, Chapters 2–6] only work for elliptic curves because the arguments use p-divisible groups, finiteness of torsion, and quotients by possibly non-étale finite locally free subgroups. Due to our intended applications, we shall now extend some of these results to generalized elliptic curves. We refer the reader to [**KM**, §§ 1.5, 1.9, 1.10] for the intrinsic and extrinsic notions of *A-generator* of a finite locally free commutative group scheme $G \to S$, with A a finite abelian group; this is a group homomorphism $\phi : A \to G(S)$ satisfying certain properties, and for $A = \mathbf{Z}/N\mathbf{Z}$ the N-torsion section $\phi(1) \in G(S)$ is called a *$\mathbf{Z}/N\mathbf{Z}$-structure* on G. The following lemma of Katz and Mazur concerning A-generators is extremely useful for our purposes.

Lemma 2.3.1. *Let S be a scheme, and let $0 \to H \to G \to E \to 0$ be a short exact sequence of finite, locally free, commutative S-group schemes with constant rank. Assume that E is étale. Let A be a finite abelian group, let K be a subgroup of A, and consider a commutative diagram of groups*

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, "\phi_K"'] & A \arrow[r] \arrow[d, "\phi"] & A/K \arrow[r] \arrow[d, "\phi_{A/K}"] & 0 \\ 0 \arrow[r] & H(S) \arrow[r] & G(S) \arrow[r] & E(S) & \end{tikzcd} \tag{2.3.1}$$

Assume that the map $A \to E(\overline{s})$ has kernel K for all geometric points $\overline{s}$ of S. The map ϕ is an A-generator of G if and only if

- *K has order equal to that of H,*
- *E has order equal to that of A/K,*
- *ϕ_K is a K-generator of H, $\phi_{A/K}$ is an A/K-generator of E.*

Proof. This is [**KM**, 1.11.2]. (We can drop the connectedness assumption on S in this reference because we assume the group schemes have constant rank.) $\square$

Theorem 2.3.2. *Let $E \to S$ be a generalized elliptic curve.*

(1) *If P is a $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm} then $(N/d)P$ is a $\mathbf{Z}/d\mathbf{Z}$-structure on E^{sm} for any $d \mid N$.*

(2) *If $\{P, Q\}$ is a Drinfeld $\mathbf{Z}/N\mathbf{Z}$-basis of $E^{\mathrm{sm}}[N]$ then P is a $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm} and the ordered pair $\{(N/d)P, (N/d)Q\}$ is a Drinfeld $\mathbf{Z}/d\mathbf{Z}$-basis of $E^{\mathrm{sm}}[d]$ for any $d \mid N$.*

Proof. By [**KM**, 1.3.7], for any S-finite relative effective Cartier divisor D in a smooth and commutative curve group $H_{/S}$, there is a finitely presented closed subscheme of S that is universal for the base change of D to be a subgroup scheme of H. The theorem claims that for certain D this closed subscheme coincides with S. Thus, for the proof of the theorem we may (without loss of generality) assume that the base S is artin local with algebraically closed residue field. The smooth case is an immediate consequence of [**KM**, 5.5.2, 5.5.7], so we shall consider the non-smooth case.

For any finite abelian group A, by [**KM**, 1.7.3] the scheme of A-structures on a finite locally free commutative group scheme G naturally decomposes into a product in a manner that is compatible with the primary decompositions of A and G. Thus, since the hypothesis in (2) forces $E^{\mathrm{sm}}[N]$ to have p-primary part $E^{\mathrm{sm}}[p^{\mathrm{ord}_p(N)}]$ for all primes p, we may assume that N is a prime power. The case when N is not divisible by the residue characteristic is an immediate consequence of [**KM**, 1.4.4]. Hence, we can assume that the residue characteristic p is positive and that $N = p^r$ with $r \geqslant 1$.

Consider the first part of the theorem. By descending induction on r, we may assume $d = p^{r-1}$. Let $G = \sum_{i \in \mathbf{Z}/p^r\mathbf{Z}} [iP]$ denote the order-p^r subgroup scheme 'generated' by P. The classification of degenerate generalized elliptic curves over the algebraically closed residue field implies that G is an extension of a cyclic constant group by a connected multiplicative group over the artin local base.

If G is not connected, then from the connected-étale sequence one sees that there is a unique short exact sequence

$$0 \to K \to G \to \mathbf{Z}/p\mathbf{Z} \to 0$$

sending P to 1. We may use Lemma 2.3.1 to conclude that pP is a $\mathbf{Z}/p^{r-1}\mathbf{Z}$-generator of K, settling (1) in this case. If G is connected then $G \simeq \mu_{p^r}$ and the scheme of $\mathbf{Z}/p^r\mathbf{Z}$-generators of G is the scheme of zeros of the p^rth cyclotomic polynomial Φ_{p^r} [**KM**, 1.12.9]. Thus, to settle (1) in this case we just have to note that if B is a ring and $r \geqslant 1$ then any $b \in B$ satisfying $\Phi_{p^r}(b) = 0$ also satisfies $\Phi_{p^{r-1}}(b^p) = 0$.

Now consider a $\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z}$-structure $\{P, Q\}$ that is a Drinfeld basis for $E^{\mathrm{sm}}[p^r]$. In particular, $E^{\mathrm{sm}}[p^r]$ has order p^{2r}. Since the base is artin local with algebraically closed residue field of characteristic p, the connected-étale sequence of $E^{\mathrm{sm}}[p^r]$ must have the form

$$0 \to \mu_{p^r} \to E^{\mathrm{sm}}[p^r] \to \mathbf{Z}/p^r\mathbf{Z} \to 0. \tag{2.3.2}$$

We want to prove that if $\{P, Q\}$ is a $\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^r\mathbf{Z}$-generator of $G \overset{\mathrm{def}}{=} E^{\mathrm{sm}}[p^r]$ then the Cartier divisors

$$\sum_{j \in \mathbf{Z}/p^r\mathbf{Z}} [jP] \quad \text{and} \quad \sum_{i,j \in \mathbf{Z}/p^{r-1}\mathbf{Z}} [ipP + jpQ]$$

with support in G are subgroup schemes of G, with the second of these two subgroups equal to $E^{\mathrm{sm}}[p^{r-1}] = G[p^{r-1}]$. By [**KM**, 1.9, 1.10], for a finite locally free commutative group scheme H there are *equivalent* intrinsic and extrinsic notions of 'A-structure' on H, with the extrinsic notion depending on an arbitrary choice of isomorphism of H onto a closed subgroup in a smooth commutative curve group (e.g., $G \hookrightarrow E^{\mathrm{sm}}$). Thus, our problem for $G \hookrightarrow E^{\mathrm{sm}}$ is *equivalent* to the same problem posed in terms of a closed immersion of G into any other smooth commutative curve group.

By [**KM**, 8.10.7(2)], the extension structure (2.3.2) on the p^r-torsion group $G = E^{\mathrm{sm}}[p^r]$ ensures that after some faithfully flat base change we can identify G with the p^r-torsion on an elliptic curve. Since we are trying to prove that certain finite locally free closed subschemes in G are subgroup schemes (and that one of these is equal to $G[p^{r-1}]$), it is harmless to apply a faithfully flat base change. Hence, we reduce to the settled case of elliptic curves. □

Corollary 2.3.3. *Let $G \hookrightarrow E^{\mathrm{sm}}$ be a finite locally free subgroup scheme in a generalized elliptic curve E over a scheme S. Let $0 \to \mu_N \to G \xrightarrow{v} \mathbf{Z}/d\mathbf{Z} \to 0$ be a short exact sequence of group schemes such that $d \mid N$ and G is killed by N. A point $P \in G(S)$ that lies in $v^{-1}(1)$ defines a $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm} if and only if the point $dP \in \mu_{N/d}(S)$ is a $\mathbf{Z}/(N/d)\mathbf{Z}$-generator of $\mu_{N/d}$.*

Proof. The 'only if' direction follows from Theorem 2.3.2, since $\mu_{N/d} \hookrightarrow \mu_N$ is the unique order-N/d finite locally free subgroup scheme. Conversely, suppose dP is a $\mathbf{Z}/(N/d)\mathbf{Z}$-generator of $\mu_{N/d}$. This makes P define a d-torsion section of $G/\mu_{N/d}$, and hence P splits the epimorphism $G/\mu_{N/d} \twoheadrightarrow \mathbf{Z}/d\mathbf{Z}$. Let $H \hookrightarrow G$ be the preimage of the split subgroup scheme $\mathbf{Z}/d\mathbf{Z} \hookrightarrow G/\mu_{N/d}$, so $P \in H(S)$ and there is a short exact sequence $0 \to \mu_{N/d} \to H \to \mathbf{Z}/d\mathbf{Z} \to 0$. By Lemma 2.3.1, P is a $\mathbf{Z}/N\mathbf{Z}$-generator of H. □

Definition 2.3.4. Let E be a generalized elliptic curve over a scheme S. A finite locally free closed subgroup scheme $G \subseteq E^{\mathrm{sm}}$ with constant order N is *cyclic* if it admits a $\mathbf{Z}/N\mathbf{Z}$-generator *fppf*-locally on S.

In the case of elliptic curves, this definition coincides with the notion of cyclicity in [**KM**].

Theorem 2.3.5. *Let G be a cyclic subgroup of order N in the smooth locus of a generalized elliptic curve E over a scheme S. For any $d \mid N$, if two points P and P' in $G(S)$ are each $\mathbf{Z}/N\mathbf{Z}$-generators of G then $(N/d)P$ and $(N/d)P'$ are each $\mathbf{Z}/d\mathbf{Z}$-generators of a common subgroup of G.*

Proof. The case of elliptic curves is [**KM**, 6.7.2]. As usual, in the remaining non-smooth case we may reduce to the case when the base is artin local with algebraically closed residue field, and we can use primary decomposition to reduce to the case when $N = p^r$

with $r > 0$ and p equal to the residue characteristic. By contraction away from G we may assume that G is ample. Let p^s be the number of sides of the closed-fiber polygon (with $0 \leqslant s \leqslant r$), and for $0 \leqslant e \leqslant r$ define the cyclic subgroups $G_{r-e} = \langle p^e P \rangle$ and $G'_{r-e} = \langle p^e P' \rangle$ with order p^{r-e}; Theorem 2.3.2 applied to the connected-étale sequence of G ensures that G_{r-e} and G'_{r-e} are subgroups of G.

By Corollary 2.3.3, both $p^s P$ and $p^s P'$ are $\mathbf{Z}/p^{r-s}\mathbf{Z}$-generators of the same subgroup $\mu_{p^{r-s}}$ that is the p^{r-s}-torsion on the identity component of E^{sm}. Thus, for $0 \leqslant e \leqslant s$ the subgroups $G_{r-e}, G'_{r-e} \subseteq G$ contain the same subgroup $\mu_{p^{r-s}}$ and have quotients in $G/\mu_{p^{r-s}} \simeq \mathbf{Z}/p^s\mathbf{Z}$ with the same order, so these quotients agree. Hence, $G_{r-e} = G'_{r-e}$ for $0 \leqslant e \leqslant s$. We may therefore replace P, P', and G with $p^s P$, $p^s P'$, and $\mu_{p^{r-s}}$ respectively, thereby reducing to the case $G = \mu_{p^{r-s}}$. If $r = s$ then there is nothing to be done, and otherwise we invoke the trivial fact (already used in the proof of Theorem 2.3.2) that the pth-power map carries $\mu_{p^n}^{\times}$ into $\mu_{p^{n-1}}^{\times}$ for $n \geqslant 1$. $\square$

Theorem 2.3.5 permits us to make the following definition.

Definition 2.3.6. Let G be a cyclic subgroup of order N in the smooth locus of a generalized elliptic curve E over a scheme S. For any $d \mid N$, the *standard cyclic subgroup* $G_d \subseteq G$ with order d is *fppf*-locally generated by $(N/d)P$ where P is an *fppf*-local $\mathbf{Z}/N\mathbf{Z}$-generator of G.

For later purposes, we need to check that a cyclicity criterion in [**KM**] for finite locally free subgroups of elliptic curves carries over to finite locally free subgroups in the smooth locus in a generalized elliptic curve.

Theorem 2.3.7. *Let E be a generalized elliptic curve over a scheme S, and let $G \subseteq E^{\mathrm{sm}}$ be a closed S-subgroup that is finite locally free over S with constant rank N. The subgroup G is cyclic if and only if its scheme $G^{\times}$ of $\mathbf{Z}/N\mathbf{Z}$-generators is finite locally free over S with rank $\phi(N)$.*

Proof. If $G^{\times} \to S$ is a finite locally free covering then G is cyclic because it acquires a $\mathbf{Z}/N\mathbf{Z}$-generator after the *fppf* base change $G^{\times} \to S$. For the converse, recall that $G^{\times}$ is *a priori* finite and finitely presented over the base [**KM**, 1.6.5]. We therefore must prove that it is flat and has the expected rank.

Let us first check that on geometric fibers the rank of $G^{\times}$ is $\phi(N)$. To this end, we may suppose $S = \operatorname{Spec} k$ for an algebraically closed field k. The rank of $G^{\times}$ in the case of elliptic curves is $\phi(N)$, by [**KM**, 6.1.1(1)]. Thus, we may suppose that E is non-smooth. If E has d irreducible components then for any multiple d' of d we may embed the k-group E^{sm} into the smooth locus of a standard d'-gon. In this way we can reduce to the case when $E^{\mathrm{sm}}[N]$ is an extension of $\mathbf{Z}/N\mathbf{Z}$ by μ_N. We are over an algebraically closed field k, so $E^{\mathrm{sm}}[N] = \mathbf{Z}/N\mathbf{Z} \times \mu_N$. Hence, there obviously exists an elliptic curve over k with $E^{\mathrm{sm}}[N]$ as its N-torsion subgroup. This identifies G with a cyclic subgroup in an elliptic curve, so the rank of $G^{\times}$ over $\operatorname{Spec} k$ is $\phi(N)$.

The fibral rank of the finite and finitely presented S-scheme $G^{\times}$ has been proved to be $\phi(N)$ for arbitrary S, and we need to prove flatness. In the special case when the base is $\operatorname{Spec} R$ for a discrete valuation ring R, the R-scheme $G^{\times}$ is finite with generic and closed fibers having equal rank. Thus, $G^\times$ is flat in this special case. It follows from the *valuative criterion for flatness* [**EGA**, IV_3, 11.8.1] that $G^\times$ is flat whenever the base is reduced and locally noetherian. In the general case we may assume that the base is artin local with an algebraically closed residue field, and it suffices to prove that such cases may be realized as a base change from a situation over a reduced noetherian base (which we shall achieve by means of deformation theory). We may also make a preliminary finite flat local base change. Thus, we may assume that G admits a $\mathbf{Z}/N\mathbf{Z}$-generator P over the artin local base.

Let $G = \prod G_i$ be the primary decomposition of G. By [**KM**, 1.7.3] and Lemma 2.3.1, each G_i is cyclic (generated by a suitable multiple of P) and $G^\times = \prod G_i^\times$, so we may assume that G has order p^r for some prime p and some $r \geqslant 1$. Let k be the residue field. The case $\mathrm{char}(k) \neq p$ is trivial, so we can assume $\mathrm{char}(k) = p$. We may replace E with its contraction away from G, so we can assume that G is ample. Hence, the closed fiber is a standard p^s-gon with $0 \leqslant s \leqslant r$, and P maps to a generator of the fibral component group.

Since the base is local artinian, we have a short exact sequence of finite *flat* group schemes

$$0 \to \mu_{p^r} \to E^{\mathrm{sm}}[p^r] \xrightarrow{v} \mathbf{Z}/p^s\mathbf{Z} \to 0$$

for some $0 \leqslant s \leqslant r$. By replacing P with a $(\mathbf{Z}/p^r\mathbf{Z})^\times$-multiple we can suppose that P maps to 1 in the étale quotient. By Corollary 2.3.3, an arbitrary point $\widetilde{P} \in (v^{-1}(1))(S)$ is a $\mathbf{Z}/p^r\mathbf{Z}$-structure on E^{sm} if and only if the point $p^s\widetilde{P} \in \mu_{p^{r-s}}(S)$ is a point of $\mu_{p^{r-s}}^\times$. This characterization of when a point in $v^{-1}(1)$ is a $\mathbf{Z}/p^r\mathbf{Z}$-structure will now be used to lift our situation to the case of a reduced noetherian base.

By making a finite flat local base change we may assume that the connected-étale sequence of $E^{\mathrm{sm}}[p^s]$ is split; let $Q \in E^{\mathrm{sm}}[p^s](S)$ be a $\mathbf{Z}/p^s\mathbf{Z}$-structure 'generating' a finite étale subgroup

$$H = \sum_{i \in \mathbf{Z}/p^s\mathbf{Z}} [iQ] \subseteq E^{\mathrm{sm}}$$

of order p^s that gives such a splitting. The quotient $E_0 = E/H$ makes sense as a generalized elliptic curve (see Example 2.1.6), and E_0 has 1-gon closed fiber. By [**DR**, II, 1.17], the infinitesimal deformation theory of the closed fiber of (E, Q) coincides with the infinitesimal deformation theory of the 1-gon E_0 as a generalized elliptic curve. The corresponding universal deformation ring A is therefore formally smooth on one parameter [**DR**, III, 1.2(iii)]; that is, $A \simeq W[\![t]\!]$ with $W = W(k)$ denoting the ring of Witt vectors for k.

We conclude that the closed fiber $(E_0; P_0, Q_0)$ of our initial structure $(E; P, Q)$ has a universal formal deformation ring whose ordinary spectrum is identified with an *fppf* μ_{p^s}-torsor T over the scheme $\mu_{p^{r-s}}^\times$ living on $\mathrm{Spec}\, A$. The universal formal deformation over $\mathrm{Spf}\, A$ uniquely algebraizes over $\mathrm{Spec}\, A$, by Corollary 2.2.4, so the *universal* case deforming $(E_0; P_0, C_0)$ has T as its base. The triple $(E; P, Q)$ over the artin local base S arises via base change on the universal algebraized triple over T. It therefore suffices to prove that the *fppf* μ_{p^s}-torsor T over $\mu_{p^{r-s}/A}^\times$ is reduced, but such reducedness is obvious because $A = W[\![t]\!]$ is a regular local ring with generic characteristic 0. □

The same method of proof shows that if $G \subseteq E^{\mathrm{sm}}$ is cyclic of order N with $\mathbf{Z}/N\mathbf{Z}$-generator P then $G^\times = \sum_{j\in(\mathbf{Z}/N\mathbf{Z})^\times}[jP]$ as closed subschemes of G (see [**KM**, 6.1.1(2)] for the smooth case).

Corollary 2.3.8. *Let E be a generalized elliptic curve over S and let G be a finite locally free subgroup of E^{sm}. If the order of G is squarefree then G is cyclic.*

Proof. By Theorem 2.3.7, it is necessary and sufficient to prove that the finite and finitely presented S-scheme $G^\times$ is locally free with the 'correct' rank. We can assume that the base is artin local with algebraically closed residue field. We may use the primary decomposition of G and the associated decomposition of $G^\times$ to reduce to the case when G has prime-power order, and hence prime order. The étale case is trivial, so we may assume that the residue characteristic p is positive and that G is non-étale with order p. If G is multiplicative then $G \simeq \mu_p$, so $G^\times \simeq \mu_p^\times$ is flat with rank $\deg \Phi_p = \phi(p)$ by inspection. If G is a deformation of α_p then E is smooth, so cyclicity is a special case of [**KM**, 6.8.7]. □

2.4. The moduli problems

Fix $N \geqslant 1$ and a generalized elliptic curve $E \to S$.

Definition 2.4.1. A *$\Gamma_1(N)$-structure* on $E_{/S}$ is an ample Drinfeld $\mathbf{Z}/N\mathbf{Z}$-structure on the smooth separated group scheme E^{sm}; that is, it is a section $P \in E^{\mathrm{sm}}(S)$ such that

- $NP = 0$;
- the relative effective Cartier divisor
$$D = \sum_{j\in\mathbf{Z}/N\mathbf{Z}} [jP] \tag{2.4.1}$$
in E^{sm} is a subgroup scheme;
- D meets all irreducible components of all geometric fibers E_s.

The final condition in Definition 2.4.1 says that the inverse ideal sheaf $\mathcal{O}(D)$ is S-ample on E. Note that this forces all singular geometric fibers to be d-gons for various $d \mid N$. We often write $\langle P\rangle$ for (2.4.1), and call it the *subgroup scheme generated by P*.

Definition 2.4.2. A *$\Gamma(N)$-structure* on $E_{/S}$ is an ample Drinfeld $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$-structure on an N-torsion subgroup in the smooth separated group scheme E^{sm}; that is, it is an ordered pair (P, Q) with $P, Q \in E^{\mathrm{sm}}[N](S)$ such that

- the rank-N^2 Cartier divisor
$$D = \sum_{i,j\in\mathbf{Z}/N\mathbf{Z}} [iP + jQ] \tag{2.4.2}$$
is a subgroup scheme killed by N (so it coincides with $E^{\mathrm{sm}}[N]$);
- D meets all irreducible components of all geometric fibers E_s.

This definition forces all singular geometric fibers to be N-gons. The order-N^2 group scheme D arising in (2.4.2) is denoted $\langle P, Q\rangle$.

Definition 2.4.3. Let N and n be positive integers. For all primes $p \mid \gcd(N,n)$, assume that $\mathrm{ord}_p(n) \leqslant \mathrm{ord}_p(N)$. A $\varGamma_1(N;n)$*-structure* on $E_{/S}$ is a pair (P,C) where

- P is a Drinfeld $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm};
- C is a finite locally free S-subgroup scheme in E^{sm} that is cyclic with order n;
- the degree-Nn relative effective Cartier divisor
$$\sum_{j\in\mathbf{Z}/N\mathbf{Z}} (jP + C) \tag{2.4.3}$$
meets all irreducible components of all geometric fibers;
- for all $p \mid \gcd(N,n)$, with $e_p = \mathrm{ord}_p(n)$, there is an equality of closed subschemes
$$\sum_{j\in\mathbf{Z}/p^e\mathbf{Z}} (j(N/p^{e_p})P + C[p^{e_p}]) = E^{\mathrm{sm}}[p^{e_p}] \tag{2.4.4}$$
in E (in particular, $E^{\mathrm{sm}}[p^{e_p}]$ is a degree-p^{2e_p} relative effective Cartier divisor in E).

Note that $C[p^{e_p}]$ in (2.4.4) is the p-primary part of C, so it is finite locally free and cyclic. For $N = 1$, $\varGamma_1(1;n)$-structures are called $\varGamma_0(n)$-structures; these are cyclic subgroups of order n in E^{sm} that are S-ample in E. For $n = 1$, $\varGamma_1(N;1)$-structures are $\varGamma_1(N)$-structures by another name.

Lemma 2.4.4. *Let $(E;P,C)$ over a scheme S satisfy the first three conditions in Definition 2.4.3, and let $d = \gcd(N,n)$. For $m \mid n$, let $C_m \subseteq C$ be the standard cyclic subgroup of order m. The equality (2.4.4) holds for all $p \mid \gcd(N,n)$ if and only if*
$$\sum_{j\in\mathbf{Z}/d\mathbf{Z}} (j(N/d)P + C_d) = E^{\mathrm{sm}}[d] \tag{2.4.5}$$
as closed subschemes of E. When this holds, then for every $d' \mid d$, we have
$$\sum_{j\in\mathbf{Z}/d'\mathbf{Z}} (j(N/d')P + C_{d'}) = E^{\mathrm{sm}}[d'].$$

Regardless of whether or not (2.4.5) holds, there exists a finite locally free morphism $S' \to S$ of rank $\phi(n)$ over which C universally acquires a $\mathbf{Z}/n\mathbf{Z}$-generator Q. If such a Q exists over the given base and $(E;P,C)$ is a $\varGamma_1(N;n)$-structure then $\{(N/d')P, (n/d')Q\}$ is a $\mathbf{Z}/d'\mathbf{Z} \times \mathbf{Z}/d'\mathbf{Z}$-generator of $E^{\mathrm{sm}}[d']$ for all $d' \mid d$.

Proof. The existence of the covering $S' \to S$ is immediate from Theorem 2.3.7: take S' to be the scheme $C^{\times}$ of $\mathbf{Z}/n\mathbf{Z}$-generators of C.

To relate the conditions (2.4.4) and (2.4.5) we may work *fppf*-locally on S, so we can assume that a $\mathbf{Z}/n\mathbf{Z}$-generator Q exists for the cyclic subgroup C. In this case, for $p \mid d$ we have (with $e = \mathrm{ord}_p(n)$) that $(N/p^e)Q$ is a $\mathbf{Z}/p^e\mathbf{Z}$-structure on the p-primary part $C[p^e] = C_{p^e}$ by [**KM**, 1.10.14], so the equivalence of (2.4.4) and (2.4.5) comes down to the statement that a homomorphism $A \to E^{\mathrm{sm}}(S)$ from a finite commutative group A is an A-generator of some finite locally free subgroup $G \hookrightarrow E^{\mathrm{sm}}$ if and only if each ℓ-primary part A_ℓ of A has such a property relative to some finite locally free subgroup $G_\ell \hookrightarrow E^{\mathrm{sm}}$, in which case G_ℓ is the ℓ-primary part of G. This assertion on primary decomposition of generators follows from [**KM**, 1.7.3].

Now we assume that (2.4.5) holds, so upon choosing Q (when it exists over the given base) we see that (2.4.5) is the statement that the pair $\{(N/d')P, (n/d')Q\}$ is a $\mathbf{Z}/d'\mathbf{Z} \times \mathbf{Z}/d'\mathbf{Z}$-generator of $E^{\mathrm{sm}}[d']$ for all $d' \mid d$. Thus, renaming d as N and the sections $(N/d)P$ and $(N/d)Q$ as P' and Q', it remains to show that if E is a generalized elliptic curve such that $E^{\mathrm{sm}}[N]$ is finite locally free over S and a pair of points $P', Q' \in E^{\mathrm{sm}}[N](S)$ forms a Drinfeld $\mathbf{Z}/N\mathbf{Z}$-basis, then for every $d \mid N$ the points $(N/d)P'$ and $(N/d)Q'$ form a Drinfeld $\mathbf{Z}/d\mathbf{Z}$-basis of $E^{\mathrm{sm}}[d]$. This is Theorem 2.3.2 (2). □

Let $(E; P, C)$ be a $\Gamma_1(N;n)$-structure, and let C_d denote the standard cyclic subgroup in C of order d for each $d \mid n$. By Lemma 2.4.4, if $m \mid n$ then $(E; P, C_{n/m})$ is a $\Gamma_1(N;m)$-structure provided that

$$\sum_{j \in \mathbf{Z}/n\mathbf{Z}} (jP + C_{n/m})$$

is relatively ample.

An important consequence of the definition of $\Gamma_1(N;n)$-structures is the following theorem.

Theorem 2.4.5. *If $(E; P, C)$ is a $\Gamma_1(N;n)$-structure over a scheme S, then the relative effective Cartier divisor $D = \sum_{j \in \mathbf{Z}/N\mathbf{Z}} (jP + C)$ is a subgroup scheme in E^{sm}.*

Proof. As usual, we may assume that S is artin local with algebraically closed residue field. Making a finite flat base change as in Lemma 2.4.4, we may choose a $\mathbf{Z}/n\mathbf{Z}$-generator Q of C. The assertion to be shown is equivalent to the claim that $\{P, Q\}$ is a $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/n\mathbf{Z}$-structure on E^{sm}.

The ampleness condition in the definition of $\Gamma_1(N;n)$-structures will not be used (or we may apply auxiliary contractions), so we may decompose our problem into primary pieces. More precisely, by [**KM**, 1.7.3] the case $\gcd(N, n) = 1$ is trivial and we reduce to the case where $N, n > 1$ are powers of the same prime p. Thus, $N = p^r$ and $n = p^e$ with $r \geqslant e \geqslant 1$. In particular, in the non-smooth case the number of irreducible components is divisible by p. When $E^{\mathrm{sm}}[p]$ has a non-trivial étale quotient and the standard order-p subgroup C_p is non-étale, then P is trivially an 'étale point of order p^r' and everything is clear. Thus, we may assume that C_p is étale when the closed fiber of E is not a supersingular elliptic curve. We treat separately the cases when the closed fiber is an elliptic curve and when it is not.

Suppose that E is an elliptic curve. The case of residue characteristic distinct from p is trivial, so suppose the artin local base has residue characteristic p. The data we are given consist of a $\mathbf{Z}/p^r\mathbf{Z}$-structure P on E and a $\mathbf{Z}/p^e\mathbf{Z}$-structure Q on E such that $\{p^{r-e}P, Q\}$ is a Drinfeld $\mathbf{Z}/p^e\mathbf{Z}$-basis of $E[p^e]$. We must prove that $\{P, Q\}$ is a $\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^e\mathbf{Z}$-structure on E.

After a finite flat surjective base change we can find $Q' \in E(S)$ with $p^{r-e}Q' = Q$. By [**KM**, 5.5.7(4)], Q is a $\mathbf{Z}/p^r\mathbf{Z}$-structure on E. Since $\{p^{r-e}P, p^{r-e}Q'\}$ is a $\mathbf{Z}/p^e\mathbf{Z}\times\mathbf{Z}/p^e\mathbf{Z}$-structure on E, it follows from [**KM**, 5.5.7(3)] that $\{P, Q'\}$ is a Drinfeld $\mathbf{Z}/p^r\mathbf{Z}$-basis of $E[p^r]$. Thus, by [**KM**, 5.5.2], Q' defines a $\mathbf{Z}/p^r\mathbf{Z}$-structure on the elliptic curve $E/\langle P\rangle$ and hence by [**KM**, 5.5.7(2)] the point $Q = p^{r-e}Q'$ defines a $\mathbf{Z}/p^e\mathbf{Z}$-structure on $E/\langle P\rangle$. Let $\overline{G} \hookrightarrow E/\langle P\rangle$ be the order-p^e group scheme generated by Q, and consider the preimage G of $\overline{G}$ in E, so G has order p^{r+e}. We claim that $(P, Q) : \mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^e\mathbf{Z} \to G$ is a $\mathbf{Z}/p^r\mathbf{Z} \times \mathbf{Z}/p^e\mathbf{Z}$-generator. Since P is a $\mathbf{Z}/p^r\mathbf{Z}$-generator of $\langle P\rangle \hookrightarrow G$ and Q induces a $\mathbf{Z}/p^e\mathbf{Z}$-generator of the cokernel $\overline{G}$, we can use the generalization [**KM**, 1.11.3] of the 'if' direction of Lemma 2.3.1 (dropping the étale condition on E and the hypothesis on the kernel of $A \to E(\overline{s})$).

We now suppose that the closed fiber of E is not smooth, so in particular (by the above reductions) we may also suppose that the standard subgroup C_p of order p is étale. Working on the geometric closed fiber, we may infer that C is étale. In particular, the finite étale group scheme C of order p^e has trivial intersection with the identity component of E since this identity component is a torus (as the base is artin local) and the residue characteristic is p. It follows that C gives a splitting of the connected-étale sequence of $E^{\mathrm{sm}}[p^e]$. We may therefore use the universal deformation technique (as in the proof of Theorem 2.3.7) to get to the case when the base is a reduced noetherian ring A with generic characteristics equal to 0. By flatness and separatedness over the base, a relative effective Cartier divisor in E^{sm} is an A-subgroup if it pulls back to a subgroup on the generic fibers over the reduced base $\mathrm{Spec}\, A$. This brings us to the trivial case when the base is a field of characteristic 0. $\square$

Definition 2.4.6. For $\Gamma \in \{\Gamma(N), \Gamma_1(N; n)\}$, the *moduli stack of Γ-structures* on generalized elliptic curves is denoted $\mathcal{M}_\Gamma$. If $\Gamma = \Gamma(1)$ then $\mathcal{M}_1$ denotes $\mathcal{M}_\Gamma$.

The closed substack $\mathcal{M}_\Gamma^\infty \hookrightarrow \mathcal{M}_\Gamma$ is the locus of non-smoothness for the universal generalized elliptic curve with Γ-structure. The open substack $\mathcal{M}_\Gamma - \mathcal{M}_\Gamma^\infty$ is denoted $\mathcal{M}_\Gamma^0$.

The usefulness of $\mathcal{M}_\Gamma^\infty$ as a closed substack rests on Theorem 2.1.12. Each $\mathcal{M}_\Gamma$ is an *fpqc* stack in groupoids over $\mathrm{Spec}\,\mathbf{Z}$ because the Γ-structure provides an ample line bundle. For later purposes we require the following basic lemma whose proof goes as in the higher-genus case in [**DM**, 1.2].

Lemma 2.4.7. *Let $f : C \to S$ be a DR semistable genus-1 curve, and let D be a relative effective Cartier divisor in C with degree $d \geqslant 1$. Assume that D is supported in C^{sm}, and that D meets all irreducible components of all geometric fibers of C over S.*

For all $r \geqslant 1$, $\mathcal{O}(rD)$ is S-ample and $\mathrm{R}^1 f_*(\mathcal{O}(rD)) = 0$. Moreover, $f_*\mathcal{O}(rD)$ is locally free of rank rd and its formation is compatible with base change. If $r \geqslant 3$ then the natural *map $f^*f_*\mathcal{O}(rD) \to \mathcal{O}(rD)$ is surjective, so for such r there is a natural map*

$$C \simeq \operatorname{Proj}\Bigl(\bigoplus_{n\geqslant 0} f_*\mathcal{O}(nD)\Bigr) \to \mathbf{P}(f_*\mathcal{O}(rD))$$

and it is a closed immersion.

2.5. Formal and algebraic Tate curves

We now review Raynaud's construction of Tate curves via formal schemes and algebraization (cf. [**DR**, VII]), and we prove a uniqueness theorem for these curves.

Let us first recall some standard notation from the theory of formal schemes. Let R be a noetherian ring that is separated and complete for the topology defined by an ideal J. The ring $R\{\{T_1,\dots,T_n\}\}$ of restricted powers series over the topological ring R is the J-adic completion of $R[T_1,\dots,T_n]$, and it is the topological ring of formal power series $\sum a_I T^I$ over R such that $a_I \to 0$ in R as the total degree $\|I\| = i_1 + \cdots + i_n$ of the multi-index $I = (i_1,\dots,i_n)$ tends to infinity (this ring is given the J-adic topology). If $r \in R$ is an element then $R\{\{1/r\}\}$ denotes the quotient $R\{\{T\}\}/(1-rT)$ that is the J-adic completion of $R[T]/(1-rT) = R_r$, and if $\mathfrak{X}$ denotes the formal spectrum $\operatorname{Spf}(R)$ then we also write $\mathfrak{X}\{1/r\}$ to denote $\operatorname{Spf}(R\{\{1/r\}\})$.

Choose an integer $n > 1$. For $i \in \mathbf{Z}/n\mathbf{Z}$, define the formal annulus Δ_i to be

$$\operatorname{Spf}\mathbf{Z}[\![q^{1/n}]\!]\{\{X_i, Y_i\}\}/(X_iY_i - q^{1/n})$$

over $\operatorname{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$. For each such i, we define the open formal subschemes $\Delta_i^+ = \Delta_i\{1/X_i\}$ and $\Delta_i^- = \Delta_i\{1/Y_i\}$ in Δ_i, and we identify Δ_i^- with Δ_{i+1}^+ via $Y_i = X_{i+1}$, $X_i = Y_{i+1}$. The resulting glued formal scheme is called the *formal n-gon Tate curve* $\widehat{\underline{\mathrm{Tate}}}_n$ over $\operatorname{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$. Its reduction modulo $q^{1/n} = 0$ is the standard n-gon over $\mathbf{Z}$, so it is proper. By construction it is flat over $\operatorname{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$. A closed immersion $\mathbf{Z}/n\mathbf{Z} \hookrightarrow \widehat{\underline{\mathrm{Tate}}}_n$ is defined by $i \mapsto (1, q^{1/n}) \in \Delta_i(\mathbf{Z}[\![q^{1/n}]\!])$, and this lifts the standard copy of $\mathbf{Z}/n\mathbf{Z}$ in the standard n-gon over $\operatorname{Spec}\mathbf{Z}$. There is an evident 'formal rotation' action

$$\mathbf{Z}/n\mathbf{Z} \times \widehat{\underline{\mathrm{Tate}}}_n \to \widehat{\underline{\mathrm{Tate}}}_n \tag{2.5.1}$$

by the formal finite constant group $\mathbf{Z}/n\mathbf{Z}$ over $\operatorname{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$. This extends the formal group law on the constant group $\mathbf{Z}/n\mathbf{Z}$ and lifts the standard rotation action given by the generalized elliptic curve structure on the standard n-gon over $\operatorname{Spec}\mathbf{Z}$. We let $\widehat{\underline{\mathrm{Tate}}}_n^{\mathrm{sm}}$ denote the open formally smooth locus in $\widehat{\underline{\mathrm{Tate}}}_n$ over $\operatorname{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$.

For any adic noetherian ring R and topologically nilpotent element $r \in R$, $\widehat{\underline{\mathrm{Tate}}}_{n,R}(r)$ denotes the formal scheme over $\operatorname{Spf} R$ obtained via base change of $\widehat{\underline{\mathrm{Tate}}}_n$ by the map $\mathbf{Z}[\![q^{1/n}]\!] \to R$ carrying $q^{1/n}$ to r (in other words, we carry out the gluing of formal annuli over $\operatorname{Spf} R$, using r in the role of $q^{1/n}$). If R is understood from context, we write $\widehat{\underline{\mathrm{Tate}}}_n(r)$; in particular, $\widehat{\underline{\mathrm{Tate}}}_n(q^{1/m})$ arises from the isomorphism $\mathbf{Z}[\![q^{1/n}]\!] \simeq \mathbf{Z}[\![q^{1/m}]\!]$ carrying $q^{1/n}$ to $q^{1/m}$.

For $m \mid n$ with $m > 1$, there is an evident finite étale $\mathrm{Spf}(\mathbf{Z}[\![q^{1/m}]\!])$-map

$$\widehat{\underline{\mathrm{Tate}}}_n(q^{1/m}) \to \widehat{\underline{\mathrm{Tate}}}_m$$

that is the quotient by the formally free action of $m\mathbf{Z}/n\mathbf{Z}$. In this way, the quotient of $\widehat{\underline{\mathrm{Tate}}}_n(q)$ by the action of $\mathbf{Z}/n\mathbf{Z}$ is a proper flat formal curve $\widehat{\underline{\mathrm{Tate}}}_{1,n}$ over $\mathrm{Spf}(\mathbf{Z}[\![q]\!])$ with standard 1-gon reduction modulo $q = 0$ such that for all $n > 1$ the formal curves $\widehat{\underline{\mathrm{Tate}}}_{1,n}$ are canonically identified with each other; we write $\widehat{\underline{\mathrm{Tate}}}_1$ to denote this proper flat formal curve (with a canonical section in its formally smooth locus). The formal curve $\widehat{\underline{\mathrm{Tate}}}_1$ over $\mathrm{Spf}(\mathbf{Z}[\![q]\!])$ is called the *formal* 1*-gon Tate curve*. We let $\widehat{\underline{\mathrm{Tate}}}_1^{\mathrm{sm}}$ denote the open formally smooth locus in $\widehat{\underline{\mathrm{Tate}}}_1$ over $\mathrm{Spf}(\mathbf{Z}[\![q]\!])$.

By Corollary 2.2.3, the action (2.5.1) ensures that for any $n \geqslant 1$ every infinitesimal neighborhood of the reduction modulo $q^{1/n}$ admits a unique structure of generalized elliptic curve compatible with the action of the ample divisor $\mathbf{Z}/n\mathbf{Z}$. By Corollary 2.2.4, $\widehat{\underline{\mathrm{Tate}}}_n$ uniquely algebraizes to a generalized elliptic curve $\underline{\mathrm{Tate}}_n \to \mathrm{Spec}(\mathbf{Z}[\![q^{1/n}]\!])$. A formal Fitting-ideal calculation on the Δ_i shows that its locus of non-smoothness is the zero-scheme of $(q^{1/n})$, so $\underline{\mathrm{Tate}}_n$ is smooth away from $q^{1/n} = 0$. There is a unique isomorphism of $\mathbf{Z}[\![q^{1/n}]\!]$-groups

$$\underline{\mathrm{Tate}}_n^{\mathrm{sm}}[n] \simeq \mu_n \times \mathbf{Z}/n\mathbf{Z} \tag{2.5.2}$$

lifting the canonical isomorphism on the standard n-gon fiber over $q^{1/n} = 0$.

There is a unique isomorphism of formal $\mathrm{Spec}\,\mathbf{Z}[\![q^{1/n}]\!]$-groups

$$(\underline{\mathrm{Tate}}_n)_0^{\wedge} \simeq \widehat{\mathbf{G}}_m \tag{2.5.3}$$

lifting the canonical isomorphism module $q^{1/n}$, where $(\underline{\mathrm{Tate}}_n)_0^{\wedge}$ denotes the formal completion of $\underline{\mathrm{Tate}}_n$ along the identity section. Indeed, for existence we identify $(\underline{\mathrm{Tate}}_n)_0^{\wedge}$ with the formal completion of the $\mathrm{Spf}(\mathbf{Z}[\![q^{1/n}]\!])$-group $\widehat{\underline{\mathrm{Tate}}}_n^{\mathrm{sm}}$ along its identity section and we note that $\widehat{\underline{\mathrm{Tate}}}_n^{\mathrm{sm}}$ contains an open subgroup given by the formal annulus

$$\Delta_0^+\{1/X_0\} = \mathrm{Spf}(\mathbf{Z}[\![q^{1/n}]\!]\{\{X_0, 1/X_0\}\})$$

whose completion along $X_0 = 1$ is $\widehat{\mathbf{G}}_m$. The uniqueness of (2.5.3) is due to the fact that the identity automorphism of $\widehat{\mathbf{G}}_m$ over a noetherian ring R has no non-trivial infinitesimal deformations when all maximal ideals of R have positive residue characteristic.

Example 2.5.1. For any $m, n \geqslant 1$ there is a unique degree-m finite flat map of pointed formal curves $\pi_m : \widehat{\underline{\mathrm{Tate}}}_n(t) \to \widehat{\underline{\mathrm{Tate}}}_n(t^m)$ over $\mathrm{Spf}(\mathbf{Z}[\![t]\!])$ such that π_m induces the mth-power map on the formal groups $\widehat{\mathbf{G}}_m$ along the origin, and $\pi_m^{-1}(1) = \mu_m$. Explicitly, on the formal annuli Δ_i whose gluing defines $\widehat{\underline{\mathrm{Tate}}}_n(t)$ (after perhaps passing to a quotient when $n = 1$), the map π_m is induced by $X_i \mapsto X_i^m$ and $Y_i \mapsto Y_i^m$. Thus, the algebraization of π_m over $\mathrm{Spec}\,\mathbf{Z}[\![t]\!]$ has restriction over $t \neq 0$ that is an m-isogeny of elliptic curves and it has kernel equal to restriction over $t \neq 0$ of the unique subgroup μ_m lifting the m-torsion subgroup μ_m in the identity component of the standard n-gon modulo t.

We conclude this discussion with a uniqueness characterization for $\underline{\mathrm{Tate}}_n$. This will be essential in our study of descent data on the formal completions along cusps over $\mathbf{Z}$.

Theorem 2.5.2. *For all $n \geqslant 1$, $\underline{\mathrm{Tate}}_n \to \mathrm{Spec}(\mathbf{Z}[\![q^{1/n}]\!])$ is (up to unique isomorphism) the unique generalized elliptic curve restricting to $\underline{\mathrm{Tate}}_1 \otimes_{\mathbf{Z}[\![q]\!]} \mathbf{Z}((q^{1/n}))$ over $\mathrm{Spec}\,\mathbf{Z}((q^{1/n}))$ and having n-gon geometric fibers over $q^{1/n} = 0$. This isomorphism over $\mathbf{Z}((q^{1/n}))$ is uniquely characterized by the property that it respects the identifications of $(\underline{\mathrm{Tate}}_n)_0^\wedge$ and $(\underline{\mathrm{Tate}}_1)_0^\wedge$ with $\widehat{\mathbf{G}}_m$ over $\mathrm{Spec}\,\mathbf{Z}((q^{1/n}))$ and $\mathrm{Spec}\,\mathbf{Z}((q))$ respectively.*

Proof. Since $\underline{\mathrm{Tate}}_n$ has n-gon geometric fibers over the zero-locus of $q^{1/n}$, by Corollary 3.2.5 (whose proof does not depend on anything in § 2.5) it follows that $\underline{\mathrm{Tate}}_n$ is uniquely determined by its restriction over $q^{1/n} \neq 0$. This restriction must be an elliptic curve, as the locus of non-smoothness for $\underline{\mathrm{Tate}}_n$ is $q^{1/n} = 0$, so we have to uniquely identify this restriction with $\underline{\mathrm{Tate}}_1(q)_{/\mathbf{Z}((q^{1/n}))}$ as a marked curve such that the identification respects the calculation of formal groups at the origin as $\widehat{\mathbf{G}}_m$ via (2.5.3). The uniqueness of such an identification follows from flatness considerations. The existence of such an identification is a special case of [**DR**, VII, 1.14]. The more precise claim is that the contraction of $\underline{\mathrm{Tate}}_n$ away from its fibral identity components is naturally isomorphic to $\underline{\mathrm{Tate}}_1(q)_{/\mathbf{Z}[\![q^{1/n}]\!]}$ (respecting the identification of $\widehat{\mathbf{G}}_m$ with formal groups at the origin), and by formal GAGA and the compatibility of contraction with base change it suffices to prove the same assertion modulo $(q^{1/n})^m$ compatibly with change in an arbitrary $m \geqslant 1$. This final assertion is physically obvious for compatibly contracting $\underline{\mathrm{Tate}}_{kn}(q^{1/n})$ to $\underline{\mathrm{Tate}}_k(q)$ modulo $(q^{1/n})^m$ for a fixed $k > 1$ and varying $m \geqslant 1$, and so passing to quotients by the free actions of $k\mathbf{Z}/kn\mathbf{Z}$ and $\mathbf{Z}/k\mathbf{Z}$ gives the desired isomorphism. (The compatibility with (2.5.3) may be checked modulo $q^{1/n}$, since all maximal ideals of $\mathbf{Z}$ have positive residue characteristic.) □

3. Global structure of the moduli stacks

3.1. Artin and Deligne–Mumford properties

Choose $\Gamma \in \{\Gamma(N), \Gamma_1(N; n)\}$.

Lemma 3.1.1. *The stack $\mathcal{M}_\Gamma$ has diagonal that is representable by quasi-finite, separated, and finitely presented maps of schemes. In particular, $\mathcal{M}_\Gamma$ is quasi-separated over $\mathrm{Spec}\,\mathbf{Z}$.*

The condition that an isomorphism of generalized elliptic curves carries one Γ-structure into another is represented by a finitely presented closed subscheme of the base [**KM**, 1.3.5]. Hence, to prove Lemma 3.1.1 we may ignore level structures and prove a stronger result.

Theorem 3.1.2. *Let $f : E \to S$ and $f' : E' \to S$ be generalized elliptic curves. The functor $\underline{\mathrm{Isom}}(E, E')$ classifying isomorphisms of generalized elliptic curves over S-schemes is represented by a quasi-finite and separated S-scheme of finite presentation. In particular, it is quasi-affine over S.*

Proof. This is essentially proved in [**DR**, III, 2.5], but since the surrounding discussion there imposes the condition that the number of irreducible components of non-smooth geometric fibers is not divisible by the residue characteristic, let us explain how the proof adapts to the general case.

The restriction of $\underline{\mathrm{Isom}}(E, E')$ over the open subset $S - (S^{\infty,f} \cup S^{\infty,f'})$ is an Isom-functor for elliptic curves, and these are representable and finite (by the theory of Hilbert schemes and the valuative criterion for properness). Hence, it suffices to work near $S^{\infty,f}$ and $S^{\infty,f'}$. By symmetry, we work near $S^{\infty,f}$ and so (Lemma 2.1.10) we can assume that all non-smooth geometric fibers of E are n-gons for some $n \geqslant 1$. Since quasi-finite and separated maps are quasi-affine, by Zariski's main theorem, effectivity for *fpqc* descent for quasi-affine schemes allows us to work *fpqc*-locally. Thus, we may suppose that $E^{\mathrm{sm}}[n]$ is a split extension of $\mathbf{Z}/n\mathbf{Z}$ by μ_n. Let H be the subgroup arising from $\mathbf{Z}/n\mathbf{Z}$ via the splitting.

Any isomorphism from E to E' (after a base change) carries the subgroup H over to a finite locally free subgroup $H' \subseteq E'^{\mathrm{sm}}$ that is étale over the base and relatively ample in E'. Since E' is S-proper and $E'^{\mathrm{sm}}[n]$ is quasi-finite, separated, and finitely presented over the base, the functor that classifies such subgroups H' is represented by a finitely presented and separated S-scheme S' that is quasi-finite over the base (quasi-finiteness is due to the étale condition on H'). Let $\widetilde{H}' \subseteq E'^{\mathrm{sm}}[n]_{S'}$ denote the universal object over S'. We may identify the functor $\underline{\mathrm{Isom}}(E, E')$ on S-schemes with the Isom-functor I' on S'-schemes classifying isomorphisms carrying H to $\widetilde{H}'$. The étale quotients E/H and $E'_{S'}/\widetilde{H}'$ are naturally generalized elliptic curves with geometrically irreducible fibers, so the proof of [**DR**, III, 2.5] shows that I' is represented by a separated and finitely presented S'-scheme. Quasi-finiteness of the representing object is obvious. □

Corollary 3.1.3. *Let $f : E \to S$ be a generalized elliptic curve with n-gon geometric fibers and assume that $S = S^{\infty,f}$. Let $C_n \to S$ be the standard n-gon as a generalized elliptic curve. The map $I = \underline{\mathrm{Isom}}(E, C_n) \to S$ is a finite locally free covering of rank $2n$, étale if $n \in \mathbf{G}_m(S)$, and there is a canonical isomorphism $E_{/I} \simeq C_{n/I}$ as generalized elliptic curves over I.*

In particular, if $E \to S$ has 1*-gon geometric fibers and $S = S^{\infty,f}$ then E becomes canonically isomorphic to the standard* 1*-gon over a degree-*2 *finite étale covering.*

Proof. By Lemma 2.1.10, *fppf*-locally on S we have an isomorphism between I and the Aut-scheme of C_n, and this Aut-scheme is given by Example 2.1.5. □

Theorem 3.1.4. *The stack $\mathcal{M}_\Gamma$ is an Artin stack of finite type over* $\operatorname{Spec} \mathbf{Z}$.

In view of Lemma 3.1.1, it suffices to construct a Γ-structure over a $\mathbf{Z}$-scheme S_Γ of finite type such that the morphism $S_\Gamma \to \mathcal{M}_\Gamma$ is smooth and surjective. To carry out such a construction, we use the technique of universally embedded families.

Fix an integer $d \geqslant 1$ and consider 4-tuples (f, D, ι, ρ) where

- $f : C \to S$ is a DR semistable genus-1 curve,
- D is a degree-d relative effective Cartier divisor in C that is supported in C^{sm} and is ample over S,

- $\iota : f_*\mathcal{O}(3D) \simeq \mathcal{O}_S^{\oplus 3d}$ is an isomorphism (note that $f_*\mathcal{O}(3D)$ is *a priori* locally free of rank $3d$, and by Lemma 2.4.7 its formation is compatible with base change on S),
- $\rho : D \times C \to C$ is an S-morphism that restricts to a commutative group scheme structure on D and is an action of D on C.

We will be interested in cases when C admits a generalized elliptic curve structure compatible with ρ and when D admits a Γ-structure for suitable Γ. We first need to establish the following lemma.

Lemma 3.1.5. *Fix $d \geqslant 1$. There exists a universal 4-tuple (f, D, ι, ρ) as above, over a base scheme that is quasi-projective over* $\mathbf{Z}$.

Proof. Let $f : C \to S$ be a proper flat map of finite presentation and let $\mathcal{L}$ be an invertible sheaf on C. By Lemma 2.1.3, the locus of $s \in S$ such that C_s is a DR semistable genus-1 curve is an open set in S. Likewise, [**EGA**, IV$_3$, 9.6.4] ensures that the locus $U = \{s \in S \mid \mathcal{L}|_{C_s} \text{ is ample}\} \subseteq S$ is open and that $\mathcal{L}|_{f^{-1}(U)}$ is relatively ample over U. Combining these openness properties with Lemma 2.4.7, the theory of Hilbert schemes gives rise to a universal triple (f_0, D_0, ι_0) over a quasi-projective $\mathbf{Z}$-scheme H_0, where we ignore the group scheme data ρ. By base change to a suitable scheme that is of finite type over H_0 with affine structure morphism to H_0, we can endow D_0 with a structure of commutative group scheme in a universal manner.

Before we universally construct the morphism ρ, we need to make some preliminary remarks concerning quasi-compactness of Hom-schemes. For any two *quasi-projective* schemes X and Y over a locally noetherian base S (such as $S = H_0$), with X projective and flat over S, the scheme $\underline{\mathrm{Hom}}(X, Y)$ exists as a countably infinite disjoint union of quasi-projective S-schemes manufactured from the Hilbert scheme of $X \times Y$ over S. If X is finite locally free with constant rank r over S, then for any S-map $h : X \to Y$ the graph $\Gamma_h \hookrightarrow X \times Y$ is finite locally free of rank r over S and hence has constant Hilbert polynomial r for its fibers over S. Thus, only a quasi-compact piece of the Hilbert scheme of $X \times Y$ intervenes in the description of all such graphs Γ_h, so $\underline{\mathrm{Hom}}(X, Y)$ is quasi-compact over S for such X.

For another example of an S-quasi-compact piece of a Hom-scheme, let $X \to S$ be a morphism that is projective and flat as above, and consider the scheme $\underline{\mathrm{Isom}}(X, X)$ that exists as a countably infinite disjoint union of quasi-projective S-schemes. If the fibers of $X \to S$ have dimension less than or equal to 1 then this Isom-scheme is quasi-compact over S and hence is quasi-projective over S. To see this, we may fix a relatively very ample line bundle $\mathcal{L}$ on X with degree $d \geqslant 1$, and with respect to $\mathcal{L}$ we may assume that the fibers of $X_{/S}$ have constant Euler characteristic χ. Since the locus of fibers of a fixed dimension is open in the base (as X is S-flat), and the case of zero-dimensional fibers was treated above, we can assume that the fibers have dimension 1. It suffices to show that if $S = \mathrm{Spec}(k)$ for an algebraically closed field k and $\Gamma_\alpha \hookrightarrow X \times X$ is the graph of an automorphism α of X, then the restriction $\mathcal{L} \otimes \alpha^*\mathcal{L}$ of $p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}$ to $\Gamma_\alpha \simeq X$ has only finitely many possibilities for its Hilbert polynomial. Even better, the Hilbert polynomial is uniquely determined: it must be $2dn + \chi$ since $\mathcal{L} \otimes \alpha^*\mathcal{L}$ has degree $2d$.

Putting these observations together, we conclude that for our universal triple (f_0, D_0, ι_0), the functor

$$\underline{\mathrm{Hom}}(D_0, \underline{\mathrm{Isom}}(C_0, C_0))$$

is represented by a quasi-projective $\mathbf{Z}$-scheme. The action ρ corresponds to a certain kind of group scheme morphism $D_0 \to \underline{\mathrm{Isom}}(C_0, C_0)$. By using suitable Cartesian products of quasi-projective Hom-schemes $\underline{\mathrm{Hom}}(X, Y)$ with $X = D_0, D_0 \times D_0, \dots$, one constructs the desired universal 4-tuple over a quasi-projective $\mathbf{Z}$-scheme. □

For a fixed integer $d \geqslant 1$, consider the universal 4-tuple $(f : C \to H, D, \iota, \rho)$ as in Lemma 3.1.5. If we are to enhance this to a generalized elliptic curve after base change to some H-scheme T, then the 'universal translation' $D \times \mathrm{Pic}^0_{C/H} \to D \times \mathrm{Pic}^0_{C/H}$ defined by

$$(d, \mathcal{L}) \mapsto (d, \rho(d)^*(\mathcal{L})) \tag{3.1.1}$$

must become the identity map after base change to T (cf. (2.2.1)). This universal translation is at least a map of semi-abelian algebraic spaces over $D \times H$, so by Lemma 2.2.1 there exists an open and closed subscheme $U \subseteq D \times H$ that is universal (in the category of $D \times H$-schemes) for (3.1.1) to pull back to the identity map.

For any H-scheme T, it follows that D_T acts trivially on $T \times_H \mathrm{Pic}^0_{C/H} = \mathrm{Pic}^0_{C_T/T}$ if and only if $D \times T \to D \times H$ factors through U. This is equivalent to the finite locally free map $U \to H$ having rank d over the image of $T \to H$, which in turn amounts to $T \to H$ factoring through the open and closed subscheme $V \subseteq H$ over which $U \to H$ has rank d. It follows that the restriction $(f_V, D_V, \iota_V, \rho_V)$ of our 4-tuple over $V \subseteq H$ is the universal 4-tuple (f', D', ι', ρ') satisfying five properties: the four properties listed above Lemma 3.1.5 and the extra condition that ρ' induces the trivial D'-action on $\mathrm{Pic}^0_{C'/S'}$. For any such 4-tuple (not necessarily the universal one), the triviality of the D'-action on $\mathrm{Pic}^0_{C'/S'}$ implies (via Corollary 2.2.3) that C' admits a *unique* structure of generalized elliptic curve such that $D' \hookrightarrow C'^{\mathrm{sm}}$ is a subgroup scheme and ρ' is the D'-action on C' that is induced by the action of C'^{sm} on C'. Applying this in the universal case over V, we have constructed a *universal* triple

$$(f : E \to S, D, \iota) \tag{3.1.2}$$

with $E_{/S}$ a generalized elliptic curve, $\iota : f_*\mathcal{O}(3D) \simeq \mathcal{O}_S^{\oplus 3d}$ a trivialization, and $D \hookrightarrow E^{\mathrm{sm}}$ a subgroup scheme that is finite locally free of rank d over S and is *relatively ample* on $E_{/S}$. The universal base is quasi-projective over $\mathbf{Z}$.

We are now ready to construct universal tri-canonically embedded Γ-structures, and Theorem 3.1.4 will then follow.

Theorem 3.1.6. *There exists a universal generalized elliptic curve $f_\Gamma : E_\Gamma \to S_\Gamma$ equipped with a Γ-structure and a trivialization of ${f_\Gamma}_*\mathcal{O}(3D_\Gamma)$, where $D_\Gamma \hookrightarrow E_\Gamma$ is the ample relative effective Cartier divisor generated by the Γ-structure. The scheme S_Γ is quasi-projective over $\mathbf{Z}$.*

Proof. Fix $\Gamma \in \{\Gamma(N), \Gamma_1(N;n)\}$ and let $d = N^2$ and Nn respectively. Consider the universal triple (f, D, ι) in (3.1.2) for d, with a base that is quasi-projective over $\mathbf{Z}$. Making a suitable finite base change if $\Gamma = \Gamma(N)$ allows us to assume that in this case we instead have the universal triple subject to the extra condition that the group scheme D is killed by N (so $D = E^{\mathrm{sm}}[N]$, by consideration of orders).

First we let $\Gamma = \Gamma(N)$. Adjoining a section (or any specified finite number of sections) to D can be achieved universally after a finite base change. Hence, for a fixed finite constant commutative group A, making such a base change provides a universal group morphism $A \to D$. By [**KM**, 1.3.5], such a group morphism becomes an A-generator of D after a base change if and only if the base change factors through a certain universal closed subscheme in the base. Thus, taking $A = \mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$, we can find a finite base change $(f_\Gamma : E_\Gamma \to S_\Gamma, D_\Gamma, \iota_\Gamma)$ on which there is a universal Γ-structure (generating D_Γ). By construction, the base of this universal family is quasi-projective over $\mathbf{Z}$. This settles the cases $\Gamma = \Gamma(N)$.

Now let $\Gamma = \Gamma_1(N;n)$. Arguing as above and using Theorem 2.4.5, we can find a universal generalized elliptic curve $f : E \to S$ equipped with an ample order-Nn closed subgroup scheme $D \hookrightarrow E^{\mathrm{sm}}$ and a trivialization ι of $f_*\mathcal{O}(3D)$. Making a finite base change on S allows us to universally impose the specification of a section $P \in D[N](S)$. By [**KM**, 1.3.7], passage to a closed subscheme of S gives the further universal condition that the closed subscheme

$$\langle P \rangle \stackrel{\mathrm{def}}{=} \sum_{j \in \mathbf{Z}/N\mathbf{Z}} [jP] \hookrightarrow E^{\mathrm{sm}}$$

is an S-subgroup scheme. Finally, by [**KM**, 1.3.5], replacing S with a suitable closed subscheme universally gives the condition that the relative effective Cartier divisor $\langle P \rangle$ is supported in D. Using Lemma 2.1.10, we can pass to an open subscheme of the base in order to suppose universally that $E^{\mathrm{sm}}[p^e]$ is finite locally free of degree p^{2e} for all primes $p \mid \gcd(N, n)$. The base of this family is quasi-projective over $\mathbf{Z}$.

By working directly with the affine algebra of D over open affines in the base (or by using Hilbert schemes for constant polynomials), the specification of a degree-n relative effective Cartier divisor $C \hookrightarrow D$ is achieved universally by a base change that is quasi-projective. Passing to this new base, we may use [**KM**, 1.3.7] to find a closed subscheme of the base that is universal for C to be a subgroup scheme of E^{sm}. By Theorem 2.3.7, C is cyclic if and only if its finite and finitely presented scheme of $\mathbf{Z}/n\mathbf{Z}$-generators $C^\times$ is locally free with constant rank $\phi(n)$. Since $C^\times$ is finite and finitely presented locally on the base, for every $r \geqslant 0$ (the proof of) Mumford's theorem on flattening stratifications [**Mum1**, Lecture 8] provides a subscheme of the base S that is the universal S-scheme over which the pullback of $C^\times$ is locally free with rank r. Hence, by taking $r = \phi(n)$ we may replace S with this subscheme so as to be in the universal case such that C is cyclic.

The equality of Cartier divisors $\sum_{j \in \mathbf{Z}/N\mathbf{Z}} (jP + C) = D$ is achieved universally upon passage to a closed subscheme of the base [**KM**, 1.3.5], and for the finitely many primes

$p \mid \gcd(N,n)$ we can also require

$$\sum_{j\in \mathbf{Z}/p^e\mathbf{Z}} (j(N/p^e)P + C[p^e]) = E^{\mathrm{sm}}[p^e]$$

universally by passage to a further closed subscheme of the base. By Theorem 2.4.5 this is the desired universal family for $\Gamma = \Gamma_1(N;n)$. The base $S_{\Gamma_1(N;n)}$ is quasi-projective over $\mathbf{Z}$ by construction. □

The morphism $S_\Gamma \to \mathcal{M}_\Gamma$ is a projective-space bundle, so it is smooth and surjective. This proves Theorem 3.1.4. Our next goal is to prove that the Artin stack $\mathcal{M}_\Gamma$ of finite type over $\mathbf{Z}$ is often a Deligne–Mumford stack.

Theorem 3.1.7. *The Artin stack $\mathcal{M}_{\Gamma(N)}$ is Deligne–Mumford. If $\Gamma = \Gamma_1(N;n)$ then $\mathcal{M}_\Gamma$ is Deligne–Mumford along the open substack complementary to the open and closed substack in $\mathcal{M}_\Gamma^\infty$ classifying degenerate triples $(E;P,C)$ in positive characteristics p such that the geometric fibers of the p-part of C are disconnected and non-étale (so $p^2 \mid n$). In particular, $\mathcal{M}_{\Gamma_1(N;n)}$ is Deligne–Mumford if n is squarefree.*

Note in particular that $\mathcal{M}_{\Gamma_1(N)}$ is Deligne–Mumford.

Proof. By [**LMB**, 8.1], an Artin stack is Deligne–Mumford if and only if its diagonal is formally unramified. Since the diagonal is locally of finite type, formal unramifiedness may be checked on geometric fibers. Thus, it is necessary and sufficient to prove that the automorphism scheme of (E,ι) is étale if (E,ι) is a pair consisting of a Γ-structure ι on a generalized elliptic curve E over an algebraically closed field, provided that for $\Gamma = \Gamma_1(N;n)$ we avoid level structures $(E;P,C)$ in positive characteristic p such that the p-part of C has non-trivial connected and étale parts when E is not smooth. This étaleness property is part of the following general lemma. □

Lemma 3.1.8. *Let k be an algebraically closed field and let $\Gamma \in \{\Gamma(N), \Gamma_1(N;n)\}$. Let (E,ι) be a generalized elliptic curve with Γ-structure over k. The k-scheme $\underline{\mathrm{Aut}}(E)$ is finite. Also, $\underline{\mathrm{Aut}}(E,\iota)$ is finite, and it is étale in either of the following cases:*

- *$\Gamma = \Gamma(N)$;*
- *$\Gamma = \Gamma_1(N;n)$, provided that if $\mathrm{char}(k) = p > 0$ and E is non-smooth then the p-part of the cyclic subgroup C of order n is either étale or connected (an automatic property if $p^2 \nmid n$).*

This lemma is well known, but we do not know a reference for the proof (for $\Gamma_1(N;n)$-structures).

Proof. These Aut-schemes are quasi-finite, by Theorem 3.1.2. The étale property of the k-group $\underline{\mathrm{Aut}}(E,\iota)$ is equivalent to the condition that (E,ι) has no non-trivial infinitesimal deformations of the identity. When E is irreducible, $\underline{\mathrm{Aut}}(E)$ is étale over k (use [**Mum2**, Corollary 6.2] in the smooth case and Example 2.1.5 in the non-smooth case). Thus, the subgroup $\underline{\mathrm{Aut}}(E,\iota)$ is étale in these cases.

It remains to consider the case when E is a standard d-gon for some $d \geqslant 2$ and to show that (E, ι) has no non-trivial infinitesimal automorphism lifting the identity. If $\Gamma = \Gamma(N)$ then $d = N$ and an 'ample $\mathbf{Z}/N\mathbf{Z}$-structure' in the $\Gamma(N)$-structure is an 'étale' $\Gamma_1(N)$-structure. We now reduce the case $\Gamma = \Gamma_1(N; n)$ to various $\Gamma_1(M)$-cases, and then we will treat the $\Gamma_1(M)$-cases.

Let (P, C) be a $\Gamma_1(N; n)$-structure on E. If $\langle P \rangle$ is ample then P is a $\Gamma_1(N)$-structure. If $\mathrm{char}(k) \nmid n$ then C is constant, so we can use P and an appropriate multiple of a generator of C to define a $\Gamma_1(M)$-structure for a suitable factor M of Nn (using that $\mathbf{Z}/M\mathbf{Z}$-structures can be analyzed 'one prime at a time'), and the infinitesimal deformations of the identity automorphism of this level structure are the same as that of $(E; P, C)$. Now suppose that the $\mathbf{Z}/N\mathbf{Z}$-structure $\langle P \rangle$ is not ample and that k has positive characteristic p with $p \mid n$. Since the 'prime-to-p' part of our level structures are constant, for the purpose of reducing to the study of infinitesimal deformations of the identity automorphism of $\Gamma_1(M)$-structures we may replace n with its p-part by replacing N with a suitable multiple N' and enhancing the $\mathbf{Z}/N\mathbf{Z}$-structure to a $\mathbf{Z}/N'\mathbf{Z}$-structure. We may also assume $n \neq 1$, so $n = p^e$ with $e \geqslant 1$. Hence, C is either étale or connected. The non-ampleness of $\langle P \rangle$ forces $p \mid d$ since $n = p^e$ with $e \geqslant 1$, so the ampleness of the entire level structure forces C to have non-trivial étale part. Thus, the p-group C is étale and (since we are in characteristic p and E is not smooth) its points must lie on distinct components of E^{sm}. In particular, any infinitesimal automorphism of E that lifts the identity and preserves C must act as the identity on C. We may write $N = N_0 p^r$ with $r \geqslant 0$ and $p \nmid N_0$. Due to the non-ampleness of the $\mathbf{Z}/N\mathbf{Z}$-structure, the N_0-torsion point $p^r P$ and a generator of C define an étale $\Gamma_1(N_0 p^e)$-structure such that it is preserved by any infinitesimal deformation of the identity automorphism of the initial $\Gamma_1(N; n)$-structure.

We are now reduced to showing that if $d \geqslant 2$, $N \in d\mathbf{Z}^+$, and $P \in E^{\mathrm{sm}}(k)$ generates an ample $\mathbf{Z}/N\mathbf{Z}$-structure on the standard d-gon E, then an infinitesimal automorphism of (E, P) lifting the identity must be the identity. Without loss of generality, the generator $P \in E^{\mathrm{sm}}(k)$ of the ample $\mathbf{Z}/N\mathbf{Z}$-structure lies on the component of E^{sm} corresponding to $1 \in \mathbf{Z}/d\mathbf{Z} \simeq \pi_0(E^{\mathrm{sm}})$. By Example 2.1.5 we see that $\underline{\mathrm{Aut}}(E, P) = \{1\}$ except possibly when $d = 2$ (so N is even) and $P = (x, 1)$ with $x^2 \in \mu_2(k)$. Suppose we are in one of these latter cases. If $\mathrm{char}(k) \neq 2$, then $\underline{\mathrm{Aut}}(E)$ is étale and we are done. If $\mathrm{char}(k) = 2$ then $P = (1, 1)$ and an infinitesimal automorphism of E lifting the identity must arise from $\zeta \in \mu_2$. This does not fix P unless $\zeta = 1$. □

Corollary 3.1.9. *Let k be a separably closed field, and choose $\Gamma \in \{\Gamma(N), \Gamma_1(N; n)\}$. Let x be a pair (E, ι) over k, where ι is a Γ-structure on E. If $\mathrm{char}(k) = p > 0$ and E is not smooth, then in the case of a $\Gamma_1(N; n)$-structure $\iota = (P, C)$ assume that the p-part of C is either étale or connected (an automatic condition if $p^2 \nmid n$).*

Let W be a Cohen ring for k. Under the above assumptions, there exists a universal formal deformation ring A_x for the Γ-structure x on the category of complete local noetherian W-algebras with residue field k, and the universal formal deformation is uniquely algebraizable to a Γ-structure on $\mathrm{Spec}\, A_x$.

Proof. Geometric points on Deligne–Mumford stacks admit universal formal deformations, and the (existence and) uniqueness of the algebraization (over the spectrum of the formal deformation ring) follows from Corollary 2.2.4. □

3.2. Properness

We want to prove that $\mathcal{M}_\Gamma$ is proper over $\mathbf{Z}$ (and in § 3.3 we will see that $\mathcal{M}_\Gamma$ is $\mathbf{Z}$-flat with pure relative dimension 1). The first step will be to prove separatedness, which is to say that Isom-schemes for Γ-structures on generalized elliptic curves are proper.

Lemma 3.2.1. *Let $f : E \to S$ be a generalized elliptic curve over an integral henselian local scheme S. Assume that the geometric generic fiber is a d-gon for some $d \geqslant 1$. There exists a finite flat local S-scheme S' such that the base change $E_{/S'}$ is isomorphic to a standard d-gon as a generalized elliptic curve.*

Proof. The locus of non-smoothness $S^{\infty,f}$ is a closed subscheme of the integral S and it contains the generic point, so $S^{\infty,f} = S$. Since S is henselian local, any finite flat S-scheme is a finite disjoint union of finite flat local S-schemes. Thus, Corollary 3.1.3 gives us what we need. □

Theorem 3.2.2. Isom*-schemes of Γ-structures are finite. In particular, $\mathcal{M}_\Gamma$ is separated for all Γ and if (E, ι) and (E', ι') are Γ-structures over a normal scheme S then any isomorphism between them over a dense open uniquely extends to an isomorphism over S.*

This result is essentially due to Deligne and Rapoport, via the valuative criterion for properness, except that they worked in a slightly restrictive setting (with the number of irreducible components on each degenerate geometric fiber required to not be divisible by the residue characteristic). For completeness, we give the argument in detail (adapted to our more general setting).

Proof. The Isom-schemes of Γ-structures are quasi-finite and finitely presented, by Theorem 3.1.2. For the remainder of the proof, we may assume that S is noetherian. We have to check the valuative criterion for properness for the Isom-schemes. Thus, it is enough to work with a base that is a complete discrete valuation ring R with fraction field K and algebraically closed residue field.

Let E_1 and E_2 be generalized elliptic curves over R, equipped with respective Γ-structures ι_1 and ι_2. Let α_K be an isomorphism between $E_{1/K}$ and $E_{2/K}$ that takes $\iota_{1/K}$ to $\iota_{2/K}$. We need to extend α_K to an R-scheme isomorphism α (such an extension automatically respects the group structures and takes ι_1 over to ι_2); uniqueness of such an α is obvious.

We now check that is suffices to construct an R'-scheme morphism α' extending $\alpha_{K'}$ after base change to a discrete valuation ring R' (with fraction field K') that is finite and flat over R. Assuming we can make such a construction over a suitable R' in general, then by applying the same fact with α_K^{-1} in the role of α_K (and replacing R' with a discrete valuation ring that is finite and flat over R' if necessary) we get a scheme morphism over R' extending $\alpha_{K'}^{-1}$. This latter morphism must be inverse to α' because

$\mathrm{Hom}_{R'}(X', Y') \to \mathrm{Hom}_{K'}(X'_{K'}, Y'_{K'})$ is injective for any flat R'-scheme X' and separated R'-scheme Y'. Hence, such an α' is necessarily an isomorphism of R'-schemes and so if α' descends to an R-scheme map $E_1 \to E_2$ then this descent solves our construction problem. To prove that α' descends, more generally we claim that if X is a flat R-scheme and Y is a separated R-scheme then an R'-map $f' : X_{R'} \to Y_{R'}$ whose K'-fiber $f'_{K'}$ descends to a K-map $f_K : X_K \to Y_K$ necessarily descends (uniquely) to an R-map $f : X \to Y$ (with K-fiber f_K). By descent theory it suffices to check that the two pullbacks $p_j^*(f')$ of f' over $R' \otimes_R R'$ coincide. These pullbacks

$$p_1^*(f'), p_2^*(f') : X_{R' \otimes_R R'} \rightrightarrows Y_{R' \otimes_R R'}$$

are a pair of R-maps from a flat R-scheme to a separated R-scheme, so equality of the maps on K-fibers gives the desired equality over R.

The rest of the proof is devoted to finding a finite extension K'/K (with valuation ring R' necessarily finite and flat over R) such that $\alpha_{K'}$ extends to an R'-scheme morphism α' over R'. First consider the case when the common generic fiber is not smooth, and hence is geometrically a d-gon for some $d \geqslant 1$. By Lemma 3.2.1, there exists a finite flat *local* R-algebra A such that $E_{i/A}$ is isomorphic to the standard d-gon. Since R is a complete discrete valuation ring, the normalization of A_{red} is *finite* over R and is trivially R-flat. Thus, any choice of local factor ring of this normalization gives a discrete valuation ring R' that is finite flat over R such that each $E_{i/R'}$ is isomorphic to a standard d-gon. Renaming R' as R, we reduce to the case when $E_1 = E_2 = E$ is a standard d-gon, so $\underline{\mathrm{Aut}}(E)$ is finite (hence proper) over R by Example 2.1.5. Thus, $\underline{\mathrm{Aut}}(E)(K) = \underline{\mathrm{Aut}}(E)(R)$, so the generic fiber automorphism α_K of E_K extends to an automorphism of E. The generically non-smooth case is therefore settled.

Now suppose that the common generic fiber E_K is smooth. When $\Gamma = \Gamma_1(N; n)$, by suitable finite base change on K (and normalizing R) we may assume that all points in the support of the order-n group scheme in the Γ-structure on E_K are K-rational. Hence, in all cases we may assume that the generic fiber Γ-structure Cartier divisor D_K on the K-smooth E_K is a sum of K-rational points. Let D_i be the corresponding ample Cartier divisor on E_i coming from our given Drinfeld structure over R. Note the crucial fact that D_i is supported in E_i^{sm} (or in more concrete terms, D_K does not 'specialize' into the non-smooth locus of the closed fiber). The desired isomorphism over R will be constructed in terms of the minimal regular proper model $\widetilde{E}$ of E_K over R, but first we need to relate this minimal model to the R-curves E_i.

The R-curves $E_{i/R}$ have smooth generic fiber and reduced closed fiber, so they are normal schemes because semistable curves are CM (so Serre's normality criterion applies). Thus, we may use smoothness of the generic fiber and the explicit description of the (geometric!) closed fiber possibilities to adapt the argument in [**DM**, 1.12] to prove that suitable successive blow-up of each E_i at non-smooth points eventually stops at $\widetilde{E}$. Thus, $\widetilde{E}$ *is* a DR semistable genus-1 curve and the composite of blow-ups $\widetilde{E} \to E_i$ is the contraction ('blow-down') of some $\mathbf{P}^1$ in the closed fiber (since the residue field is algebraically closed).

Via scheme-theoretic closure, D_K on E_K uniquely extends to a relative effective Cartier divisor $\widetilde{D}$ on $\widetilde{E}$ that is finite flat over R: since D_K is a sum of K-rational points (with multiplicities), $\widetilde{D}$ is the sum of the corresponding R-rational points. The contraction map $\widetilde{E} \to E_i$ sends non-smooth points to non-smooth points and (by R-flatness reasons) maps $\widetilde{D}$ into D_i. Thus, $\widetilde{D}$ is supported in $\widetilde{E}^{\mathrm{sm}}$ since the R-ample D_i on E_i is supported in the smooth locus (for $i = 1, 2$). We may therefore identify each E_i with the contraction of $\widetilde{E}$ along the closed-fiber irreducible components that are disjoint from $\widetilde{D}$. The resulting unique isomorphism of contractions $E_1 \simeq E_2$ compatible with the contraction maps $\widetilde{E} \twoheadrightarrow E_j$ induces the given generic fiber isomorphism $E_{1/K} \simeq E_{2/K}$, so we have constructed the desired R-isomorphism. $\square$

Our proof of Theorem 1.2.2 will require the construction of *fppf* descent data via the following corollary.

Corollary 3.2.3. *Let (E, ι) and (E', ι') be Γ-structures over a scheme S, and let $D \hookrightarrow S$ be a reduced effective Cartier divisor. An isomorphism between (E, ι) and (E', ι') over $S - D$ uniquely extends over S.*

Proof. We may work locally on S, so we can assume $S = \operatorname{Spec} R$ is an affine scheme and $D = \operatorname{Spec} R/(r)$ for an element $r \in R$ that is not a zero divisor. The finiteness for the Isom-scheme reduces us to proving that R is integrally closed in $R[1/r]$ when $R/(r)$ is reduced. It is enough to show that if $r' \in R$ and r'/r is integral over R then $r' \in (r)$. If the integrality relation has degree $n > 0$ then clearing denominators gives $r'^n \in (r)$. Since $R/(r)$ is reduced, we therefore have $r' \in (r)$ as desired. $\square$

In view of Theorem 3.2.2, it is natural to ask if the Isom-scheme of a pair of generalized elliptic curves $f_1 : E_1 \to S$ and $f_2 : E_2 \to S$ is S-finite. It is easy to construct counterexamples if we allow some intersection $S_{n_1}^{\infty, f_1} \cap S_{n_2}^{\infty, f_2}$ to be non-empty with $n_1 \neq n_2$. (See Lemma 2.1.10 for the definition of the locus $S_n^{\infty, f}$ of n-gon geometric fibers for a generalized elliptic curve $f : E \to S$ and a positive integer n.) Such non-emptiness is the only obstruction to S-finiteness of the Isom-functor, due to the following theorem.

Theorem 3.2.4. *Let $f : E \to S$ and $f' : E' \to S$ be generalized elliptic curves such that $S_n^{\infty, f} \cap S_{n'}^{\infty, f'} = \emptyset$ for all $n \neq n'$. The separated and finitely presented S-scheme $\underline{\mathrm{Isom}}(E, E')$ is S-finite. In particular, if S is normal and $S^* \subseteq S$ is a dense open subset then any isomorphism $E|_{S^*} \simeq E'|_{S^*}$ uniquely extends to an isomorphism over S.*

Proof. As in the proof of Theorem 3.2.2, the proof of Theorem 3.2.4 immediately reduces to checking the valuative criterion for properness over a discrete valuation ring R when the generic fibers are smooth and any non-smooth geometric closed fiber is an n-gon (for some fixed $n \geqslant 1$). The theory of Néron models takes care of the case when at least one (and hence both) of the closed fibers is smooth, so we may assume that both closed fibers are n-gons. We may assume that R is complete with an algebraically closed residue field. Let $S = \operatorname{Spec} R$, and let s denote the closed point and let K denote the fraction field of R. We are given an isomorphism $\alpha_K : E_K \simeq E'_K$ of elliptic curves, and we need to extend it to an isomorphism $\alpha : E \simeq E'$ over R (uniqueness of α is clear).

By contracting away from fibral identity components, Theorem 3.2.2 with $\Gamma = \Gamma(1)$ provides an isomorphism of contractions $c(\alpha) : c(E) \simeq c(E')$ extending α_K. The contraction map $c : E \to c(E)$ induces an isomorphism of group schemes $c^{-1}(c(E)^{\mathrm{sm}}) \simeq c(E)^{\mathrm{sm}}$, so we view $c(E)^{\mathrm{sm}}$ as an open subgroup scheme in E^{sm}. Consider the quasi-finite, flat, and separated group scheme $c(E)^{\mathrm{sm}}[n] \to S$. Since S is henselian local and $c(E)$ has a 1-gon geometric fiber over the closed point $s \in S$, there is a unique open and closed subgroup scheme $\mu \hookrightarrow c(E)^{\mathrm{sm}}[n]$ that is finite flat of order n, and μ_s is the n-torsion on $c(E)^{\mathrm{sm}}_s$. We may therefore use the isomorphism $c(\alpha) : c(E) \simeq c(E')$ to obtain an open and closed immersion of group schemes $\mu \hookrightarrow c(E')^{\mathrm{sm}}[n]$ that induces the canonical isomorphism $\mu_s = c(E)^{\mathrm{sm}}_s[n] \simeq c(E')^{\mathrm{sm}}_s[n]$ on s-fibers.

As in the proof of Corollary 2.2.4, $E^{\mathrm{sm}}[n]$ is a finite flat S-group with rank n^2. Thus, there is a short exact sequence of finite flat S-groups

$$0 \to \mu \to E^{\mathrm{sm}}[n] \to G \to 0, \tag{3.2.1}$$

where $G \to S$ is constant with rank n and μ_s lies in the identity component of E^{sm}_s. By construction, G_s is isomorphic to the component group $\mathbf{Z}/n\mathbf{Z}$ of E^{sm}_s upon identifying E_s with the standard n-gon. There exists a unique isomorphism $G \simeq \mathbf{Z}/n\mathbf{Z}$ lifting the isomorphism on s-fibers. By the same argument, there is an analogous short exact sequence

$$0 \to \mu \to E'^{\mathrm{sm}}[n] \to G' \to 0 \tag{3.2.2}$$

with $G' \simeq \mathbf{Z}/n\mathbf{Z}$. By replacing R with a finite extension, we can assume that these two sequences are split. Due to the existence of $c(\alpha)$, it follows that the K-fibers of the sequences (3.2.1) and (3.2.2) are compatible via α_K. In particular, if $P \in E^{\mathrm{sm}}[n](R)$ maps to a generator of G then $P'_K = \alpha_K(P_K)$ maps to a generator of G'_K. The point $P' \in E'(R)$ extending P'_K lies in $E'^{\mathrm{sm}}[n]$ since $E'^{\mathrm{sm}}[n]$ is finite, so P' maps to a generator of the constant group G' under the quotient map $E'^{\mathrm{sm}}[n] \to G'$.

We conclude that (E, P) and (E', P') are $\Gamma_1(n)$-structures, and α_K is an isomorphism between their generic fibers. Since Isom-functors for $\Gamma_1(n)$-structures are finite (Theorem 3.2.2), by normality of the trait S we obtain the desired extension α of α_K. □

Our study of formal structure along the cusps will require the construction of *fppf* descent data via the following corollary.

Corollary 3.2.5. *Let $f : E \to S$ and $f' : E' \to S$ be generalized elliptic curves over a scheme S and assume that $S^{\infty,f}_n \cap S^{\infty,f'}_{n'} = \emptyset$ for all $n \neq n'$. If $D \hookrightarrow S$ is a reduced effective Cartier divisor then any isomorphism between E and E' over $S - D$ uniquely extends to an isomorphism over S.*

Proof. The proof of Corollary 3.2.3 carries over. □

Since $\mathcal{M}_\Gamma$ is separated and of finite type over $\mathbf{Z}$, to prove it is $\mathbf{Z}$-proper we need to check the valuative criterion for properness. (See [**O**] and [**LMB**, 7.12] for the sufficiency of using discrete valuation rings in this criterion for locally noetherian Artin stacks.)

Lemma 3.2.6. *Let R be a complete discrete valuation ring with fraction field K. Let (E_K, ι_K) be a generalized elliptic curve with Γ-structure over K. After base change to some discrete valuation ring R' that is finite and flat over R, (E_K, ι_K) extends to a Γ-structure over R'.*

Proof. First consider the case when E_K is not smooth. By finite base change we can assume that E_K is the standard d-gon over K for some $d \geqslant 1$. Let E denote the standard d-gon over R. We claim that the scheme-theoretic closure in E of the Γ-structure on E_K is a Γ-structure on E. The M-torsion $E^{\mathrm{sm}}[M]$ is *finite* (flat) over R for any $M \geqslant 1$, so the scheme-theoretic closure D in E of any finite flat *subgroup* scheme $D_K \hookrightarrow E_K^{\mathrm{sm}}$ lies in E^{sm} (and is obviously a subgroup scheme of E^{sm}). Also, by using the explicit description of standard polygons we see that ampleness of D_K on E_K^{sm} forces the subset $D \subseteq E^{\mathrm{sm}}$ to meet all irreducible components of the closed fiber of E. Thus, D is ample. This settles the case when E_K is not smooth.

Now assume E_K is smooth. Let E be the minimal regular proper model of E_K over R. Since an R-section in E must lie in the relative smooth locus, by the R-properness of E we have $E_K(K) = E^{\mathrm{sm}}(R)$. In particular, for any Cartier divisor on E_K that is a sum of K-rational points, its scheme-theoretic closure in E is a relative Cartier divisor over R that is a sum of R-points supported in E^{sm}. By the genus-1 stable reduction theorem [**DR**, IV, 1.6(i),(ii)], we may assume (after suitable finite base change on R and replacing E by the minimal regular proper model over the new base) that E admits a structure of generalized elliptic curve extending that on E_K. Thus, we can use scheme-theoretic closure to extend the Γ-structure on E_K to a Γ-structure on E, up to the problem of ampleness on the closed fiber.

Let $D \subseteq E^{\mathrm{sm}}$ be the underlying Cartier divisor of our 'possibly non-ample Γ-structure' on E (so D is a closed subgroup scheme of E^{sm} that is finite and flat over R). We eliminate lack of ampleness by considering the contraction $c : E \to \overline{E}$ of the closed fiber along the irreducible components disjoint from D. Since $\overline{E}$ is a generalized elliptic curve with generic fiber E_K and $c^{-1}(\overline{E}^{\mathrm{sm}}) \subseteq E^{\mathrm{sm}}$ is an open subgroup scheme *containing* D, composition with the *isomorphism* of group schemes $c : c^{-1}(\overline{E}^{\mathrm{sm}}) \simeq \overline{E}^{\mathrm{sm}}$ defines an ample Γ-structure on $\overline{E}^{\mathrm{sm}}$. Using $\overline{E}$, we get the desired model over R. $\square$

Let us summarize much of what we have proved (accounting for smoothness away from the level [**DR**, III, 2.5(iii)]).

Theorem 3.2.7. *The Artin stack $\mathcal{M}_\Gamma$ is proper over $\mathbf{Z}$. The stack $\mathcal{M}_{\Gamma(N)}$ is Deligne–Mumford, and it is smooth over $\mathbf{Z}[1/N]$. The stack $\mathcal{M}_{\Gamma_1(N;n)}$ is smooth over $\mathbf{Z}[1/Nn]$ and it is Deligne–Mumford away from the open and closed substack in $\mathcal{M}^\infty_{\Gamma_1(N;n)}$ classifying degenerate triples $(E; P, C)$ in positive characteristics p such that the geometric fibers of the p-part of C are both non-étale and disconnected (such structures exist if and only if $p^2 \mid n$). In particular, $\mathcal{M}_{\Gamma_1(N;n)}$ is Deligne–Mumford if and only if n is squarefree.*

3.3. Flatness and fibral properties

Once again, we choose $\Gamma \in \{\Gamma(N), \Gamma_1(N;n)\}$. The key problem for us is to understand the structure of the stack $\mathcal{M}_\Gamma$ along $\mathcal{M}^\infty_\Gamma$ in 'bad' characteristics.

Theorem 3.3.1. *The proper morphism* $\mathcal{M}_\Gamma \to \mathrm{Spec}(\mathbf{Z})$ *is flat and CM with pure relative dimension* 1.

Before we prove Theorem 3.3.1, we introduce a moduli problem to be used in our study of $\mathcal{M}_{\Gamma_1(N;n)}$.

Definition 3.3.2. Let N and n be positive integers such that $\mathrm{ord}_p(n) \leqslant \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N,n)$. A $\widetilde{\Gamma}_1(N;n)$*-structure* on a generalized elliptic curve $E \to S$ is a pair (P,Q) with P a $\mathbf{Z}/N\mathbf{Z}$-structure on E^{sm} and Q a $\mathbf{Z}/n\mathbf{Z}$-structure on E^{sm} such that $(P,\langle Q\rangle)$ is a $\Gamma_1(N;n)$-structure.

By Theorem 2.4.5, a $\widetilde{\Gamma}_1(N;n)$-structure should be viewed as an enhanced version of 'ample $\mathbf{Z}/N\mathbf{Z}\times\mathbf{Z}/n\mathbf{Z}$-structure' on generalized elliptic curves (there is an extra condition on the n-torsion, namely (2.4.4)). Using Lemma 2.4.4 and Theorem 3.1.4, the *fpqc*-stack in groupoids $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ over $\mathrm{Spec}(\mathbf{Z})$ is an Artin stack that is finite and locally free of rank $\phi(n)$ over $\mathcal{M}_{\Gamma_1(N;n)}$, and it is even finite étale of degree $\phi(n)$ over $\mathcal{M}_{\Gamma_1(N;n)}$ after inverting n. In particular, $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ is proper over $\mathbf{Z}$. An important technical advantage of $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ is that it is everywhere Deligne–Mumford, and so in particular its geometric points admit universal deformation rings (as in Corollary 3.1.9), as we now prove.

Lemma 3.3.3. *The Artin stack* $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ *is Deligne–Mumford over* $\mathrm{Spec}\,\mathbf{Z}$.

Proof. By [**LMB**, 8.1], it is necessary and sufficient to prove that $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ has formally unramified diagonal. That is, we must prove that the finite automorphism scheme of a $\widetilde{\Gamma}_1(N;n)$-structure $(E;P,Q)$ over an algebraically closed field is étale. The étale condition says that the identity automorphism has no non-trivial infinitesimal deformations. Since $\widetilde{\Gamma}_1(N;n)$-structures $(E;P,Q)$ 'refine' $\Gamma_1(N;n)$-structures $(E;P,\langle Q\rangle)$, the only cases that we need to investigate are level-structures $(E;P,Q)$ in positive characteristic p with $e=\mathrm{ord}_p(n)\geqslant 1$ such that $\langle P\rangle$ is not ample and E is a d-gon with $p \mid d$. For every prime $\ell \mid \gcd(N,n)$ one of P or Q generates the ℓ-part of the component group, by ampleness of the total level structure. For each such $\ell \neq p$ such that the ℓ-part of $\langle P\rangle$ does not generate the ℓ-part of the component group, we can shift the ℓ-part of $\langle Q\rangle$ into the P-aspect since we are using $\widetilde{\Gamma}_1(N;n)$-structures rather than $\Gamma_1(N;n)$-structures. We may ignore the other ℓ-parts of $\langle Q\rangle$ with $\ell \neq p$, reducing ourselves to the case $n=p^e$ with $e\geqslant 1$; beware that p might not divide N, so the p-part of the level structure may have non-trivial infinitesimal part. If this reduction step causes the new $\langle P\rangle$ to be ample then we are done, so we may still suppose that $\langle P\rangle$ is not ample.

The non-ampleness of $\langle P\rangle$ forces Q to generate the p-part of the component group, so Q and the prime-to-p part of P define a $\Gamma_1(N_0p^e)$-structure, with $N_0 \mid N$ denoting the prime-to-p part of N. Hence, the automorphism functor of interest is a closed subfunctor of the automorphism functor of a $\Gamma_1(N_0p^e)$-structure. Since automorphism functors of $\Gamma_1(M)$-structures over a field are étale, we are done. $\square$

The first step in the proof of Theorem 3.3.1 is the following lemma.

Lemma 3.3.4. *The open substack* $\mathcal{M}^0_\Gamma = \mathcal{M}_\Gamma - \mathcal{M}^\infty_\Gamma$ *classifying smooth elliptic curves with* Γ*-structure is regular and* $\mathbf{Z}$*-flat with fibers of pure dimension* 1 *over* $\mathrm{Spec}(\mathbf{Z})$ *(these fibers are therefore Cohen–Macaulay).*

Proof. The desired results for $\varGamma = \varGamma_1(N)$ and $\varGamma(N)$ are special cases of [**KM**, 5.1.1], so we shall consider $\varGamma = \varGamma_1(N; n)$. The open substack

$$\mathcal{M}^0_{\widetilde{\varGamma}_1(N;n)} \subseteq \mathcal{M}_{\widetilde{\varGamma}_1(N;n)}$$

classifying $\widetilde{\varGamma}_1(N; n)$-structures on elliptic curves is finite locally free of rank $\phi(n)$ over $\mathcal{M}^0_{\varGamma_1(N;n)}$. If $A \to B$ is a flat local map of local noetherian rings and B is regular, then A is regular [**CRT**, 23.7]. Thus, it suffices to check that $\mathcal{M}^0_{\widetilde{\varGamma}_1(N;n)}$ is regular and $\mathbf{Z}$-flat with fibers of pure dimension 1.

If $\gcd(N, n) = 1$ then $\widetilde{\varGamma}_1(N; n)$-structures are the same as $\varGamma_1(Nn)$-structures, by Theorem 2.4.5. Hence, we can assume $\gcd(N, n) > 1$. Choose a prime $p \mid \gcd(N, n)$, so $N = Mp^r$ with $r \geqslant 1$ and $p \nmid M$, and also $n = p^e n'$ with $p \nmid n'$ and $1 \leqslant e \leqslant r$. Since

$$\mathcal{M}^0_{\widetilde{\varGamma}_1(N;n)/\mathbf{Z}[1/p]} \to \mathcal{M}^0_{\varGamma_1(Mn')/\mathbf{Z}[1/p]}$$

and

$$\mathcal{M}^0_{\widetilde{\varGamma}_1(N;n)/\mathbf{Z}_{(p)}} \to \mathcal{M}^0_{\widetilde{\varGamma}_1(p^r;p^e)/\mathbf{Z}_{(p)}}$$

are visibly étale surjective, it remains to prove that $\mathcal{M}^0_{\widetilde{\varGamma}_1(p^r;p^e)}$ is regular and $\mathbf{Z}$-flat with fibers of pure dimension 1.

For such prime-power level, we may use the general regularity criteria of Katz and Mazur [**KM**, 5.2.1, 5.2.2] to reduce to showing that if k is an algebraically closed field of characteristic p and $E_{/k}$ is a supersingular elliptic curve, then the maximal ideal of the universal deformation ring A classifying infinitesimal deformations of the unique $\widetilde{\varGamma}_1(p^r; p^e)$-structure on E is an ideal that is generated by two elements (the existence of the deformation ring A at a 'k-point' follows from the fact that k is separably closed and $\widetilde{\varGamma}_1(N; n)$-structures form a Deligne–Mumford stack). The proof of the analogue for $\varGamma(p^n)$-structures on elliptic curves [**KM**, 5.3.2] readily adapts to the present circumstances: the universal $\widetilde{\varGamma}_1(p^r; p^e)$-structure gives rise to a $\mathbf{Z}/p^r\mathbf{Z}$-structure P and a $\mathbf{Z}/p^e\mathbf{Z}$-structure Q, and the formal group 'coordinates' $X(P)$ and $X(Q)$ generate the maximal ideal of A. $\square$

In view of Lemma 3.3.4, for the proof of Theorem 3.3.1 it remains to look along the closed substack $\mathcal{M}^\infty_\varGamma$. In a finite type $\mathbf{Z}$-scheme, the $\overline{\mathbf{F}}_p$-points for variable primes p are dense and the locus where the scheme is CM over $\mathbf{Z}$ with a fixed pure relative dimension is open (see [**EGA**, IV$_3$, 12.1.1(iv)] for the relative-dimension analysis). The same assertion holds for Artin stacks of finite type over $\mathbf{Z}$, so to prove that $\mathcal{M}_\varGamma$ is CM over $\mathbf{Z}$ with pure relative dimension 1 it suffices to look at $\overline{\mathbf{F}}_p$-points; of course, in some bad characteristics $\mathcal{M}_{\varGamma_1(N;n)}$ is merely an Artin stack along the cusps, so we note that for the remaining study of $\mathcal{M}_{\varGamma_1(N;n)}$ in the proof of Theorem 3.3.1 it suffices to work with the finite locally free covering $\mathcal{M}_{\widetilde{\varGamma}_1(N;n)}$ that is a Deligne–Mumford stack (and so admits universal deformation rings at geometric points).

It remains to prove that universal deformation rings for $\varGamma$-structures on standard polygons over $\overline{\mathbf{F}}_p$ are two dimensional, $W(\overline{\mathbf{F}}_p)$-flat, and Cohen–Macaulay, where

$$\varGamma \in \{\varGamma_1(N), \varGamma(N), \widetilde{\varGamma}_1(N; n)\}$$

(the case $\Gamma_1(N) = \widetilde{\Gamma}_1(N;1)$ is singled out because of its role in reduction steps to follow). To be precise, let k be an arbitrary algebraically closed field and let W be a Cohen ring for k. It suffices to prove that for any Γ-structure (E,ι) on a standard d-gon over k (with $d \geqslant 1$), the universal deformation ring of (E,ι) has a finite faithfully flat cover that is finite flat over $W[\![x]\!]$ as a W-algebra.

The case of most interest to us is $\widetilde{\Gamma}_1(N;n)$ with $\mathrm{char}(k) \mid \gcd(N,n)$. This will be reduced to $\Gamma_1(M)$-cases, which in turn will be reduced to related $\Gamma(M')$-cases, and such reduction steps will be used again later. We shall now treat these cases in reverse order by means of abstract deformation theory.

Case $\boldsymbol{\Gamma = \Gamma(N)}$. Let $\{P_0, Q_0\}$ be a $\Gamma(N)$-structure on the standard N-gon E_0 over k. Since $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ acts functorially on $\Gamma(N)$-structures over any base, without loss of generality we may assume that with respect to the canonical exact sequence

$$0 \to \mu_N \to E_0^{\mathrm{sm}}[N] \xrightarrow{v_0} \mathbf{Z}/N\mathbf{Z} \to 0 \tag{3.3.1}$$

we have $v_0(P_0) = 1$ and $Q_0 \in \mu_N(k)$. By Lemma 2.3.1, the point Q_0 defines a $\mathbf{Z}/N\mathbf{Z}$-structure on μ_N (i.e. $Q_0 \in \mu_N^{\times}(k)$). Let $\zeta(Q_0) \in \mu_N(k)$ be the root of Φ_N corresponding to Q_0, where Φ_N denotes the Nth cyclotomic polynomial. Since the point P_0 in $E_0^{\mathrm{sm}} = \mathbf{G}_m \times (\mathbf{Z}/N\mathbf{Z})$ splits (3.3.1) with coordinate $x_1(P_0)$ on its own $\mathbf{G}_m$-component lying in $\mu_N(k)$, we may apply a unique automorphism to the standard N-gon E_0 to get to the case $x_1(P_0) = 1$. In particular, $\langle P_0 \rangle$ is the kernel of the canonical étale isogeny from the standard N-gon E_0 to the standard 1-gon (i.e. $E_0/\langle P_0\rangle$ is the standard 1-gon).

If E is a deformation of E_0 over an artin local W-algebra R with residue field k, then $E^{\mathrm{sm}}[N]$ is finite flat and hence $E^{\mathrm{sm}}[N] \to \pi_0(E^{\mathrm{sm}}) = \mathbf{Z}/N\mathbf{Z}$ is faithfully flat (as this can be checked over k). The kernel of this faithfully flat map is an artinian deformation of μ_N, so it is uniquely isomorphic to μ_N in a manner that is compatible with the closed-fiber identification in (3.3.1). That is, we can uniquely define a short exact sequence of finite flat R-group schemes

$$0 \to \mu_N \to E^{\mathrm{sm}}[N] \xrightarrow{v} \mathbf{Z}/N\mathbf{Z} \to 0 \tag{3.3.2}$$

lifting (3.3.1). For $P, Q \in E[N](R)$ lifting P_0, Q_0 respectively, we have $P \in v^{-1}(1)$ because $P_0 \in v_0^{-1}(1)$. Thus, by Lemma 2.3.1 the necessary and sufficient condition for this to define a $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$-structure (or equivalently, a $\Gamma(N)$-structure) is that $Q \in \mu_N^{\times}(R) \subseteq \mu_N(R)$. Fix such P and Q.

Since the N-torsion section P splits (3.3.2) and therefore generates an étale closed subgroup scheme $\langle P \rangle = \mathbf{Z}/N\mathbf{Z} \hookrightarrow E^{\mathrm{sm}}[N]$, the quotient $E/\langle P\rangle$ makes sense as a generalized elliptic curve that is an artinian deformation of the standard 1-gon $E_0/\langle P_0\rangle$. The deformation theory of the standard 1-gon as a generalized elliptic curve is formally smooth on one parameter [**DR**, III, 1.2(iii)], so it is pro-represented by $\widehat{\underline{\mathrm{Tate}}_{1/W[\![q]\!]}}$ since this deformation is non-trivial over $k[q]/(q^2)$ (its locus of non-smoothness is the subscheme $\mathrm{Spec}\, k$ defined by $q = 0$, so Example 2.1.9 gives the non-triviality over $k[q]/(q^2)$). By Corollary 2.2.4, we may restate this pro-representability as follows.

Lemma 3.3.5. *The Tate curve* $\underline{\mathrm{Tate}}_1 \to \mathrm{Spec}\, W[\![q]\!]$ *is the unique algebraization of the universal formal deformation of the standard* 1*-gon as a generalized elliptic curve over* $\mathrm{Spec}\, k$.

We conclude that there is a unique local W-algebra map $W[\![q]\!] \to R$ and a unique isomorphism of generalized elliptic curves

$$E/\langle P\rangle \simeq \underline{\mathrm{Tate}}_1 \otimes_{W[\![q]\!]} R \tag{3.3.3}$$

lifting the identification of $E_0/\langle P_0\rangle$ with the standard 1-gon over k. Moreover, since $\underline{\mathrm{Aut}}(\mu_N)$ is étale, the isomorphism (3.3.3) over R is compatible with the canonical identifications of N-torsion on each side with μ_N (since such a compatibility holds over k).

Motivated by (3.3.3) and the étaleness of $E \to E/\langle P\rangle$, in order to describe deformations of $(E_0, (P_0, Q_0))$ we will first describe deformations of $E_0/\langle P_0\rangle$ using Tate curves. Then we will lift our analysis through étale isogenies in order to return to the original setting of interest (with N-gon fibers rather than 1-gon fibers).

The specification of Q amounts to giving a $\mathbf{Z}/N\mathbf{Z}$-generator of $\mu_N = (E/\langle P\rangle)[N]$ lifting the $\mathbf{Z}/N\mathbf{Z}$-generator of $\mu_{N/k}$ coming from Q_0. By [**DR**, II, 1.17] and the topological invariance of the étale site, the unique étale isogeny to $E/\langle P\rangle$ that lifts the canonical étale isogeny $C_N \to C_1$ between standard polygons over k is the canonical étale isogeny $E \to E/\langle P\rangle$. Put another way, the deformation E of E_0 is uniquely determined by the deformation $E/\langle P\rangle$ of $E_0/\langle P_0\rangle$.

By Corollary 2.2.4 and [**DR**, II, 1.17], there is a unique étale 'isogeny'

$$\mathcal{E}_N \to \underline{\mathrm{Tate}}_1 \tag{3.3.4}$$

between generalized elliptic curves over $W[\![q]\!]$ such that the reduction over k is the $\mathbf{Z}/N\mathbf{Z}$-quotient map $C_N \to C_1$. The N-torsion on $\mathcal{E}_N^{\mathrm{sm}}$ is finite flat of order N^2 and there is a unique closed immersion $\mu_N \hookrightarrow \mathcal{E}_N^{\mathrm{sm}}[N]$ lifting $\mu_N \hookrightarrow \underline{\mathrm{Tate}}_1[N]$ such that μ_N is supported in the identity component of infinitesimal closed fibers of $\mathcal{E}_N^{\mathrm{sm}}$. Using this closed subgroup, we obtain a unique short exact sequence

$$0 \to \mu_N \to \mathcal{E}_N^{\mathrm{sm}}[N] \xrightarrow{v^{\mathrm{univ}}} \mathbf{Z}/N\mathbf{Z} \to 0 \tag{3.3.5}$$

lifting the analogous filtration on $C_N^{\mathrm{sm}}[N]$. By Lemma 2.3.1, to give a $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$-generator of $\mathcal{E}_N^{\mathrm{sm}}[N]$ extending a given $\mathbf{Z}/N\mathbf{Z}$-generator of μ_N is 'the same' as to give a section of $(v^{\mathrm{univ}})^{-1}(1)$, and this equivalence persists after any base change.

In what is to follow, we define

$$W[\zeta_N] = W[X]/(\Phi_N(X)),$$

where Φ_N is the Nth cyclotomic polynomial and ζ_N is the residue class of X. This is a semi-local ring. We have proved the following lemma.

Lemma 3.3.6. *The datum of a deformation of $(E_0, (P_0, Q_0))$ to R is equivalent to the data consisting of a W-algebra map $W[\zeta_N][\![q]\!] \to R$ with ζ_N mapping to a lift of $\zeta(Q_0) \in \mu_N(k)$ together with a $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$-structure on*

$$\mathcal{E}_N \otimes_{W[\![q]\!]} R$$

that enhances the $\mathbf{Z}/N\mathbf{Z}$-generator of $\mu_N \hookrightarrow \mathcal{E}_N^{\mathrm{sm}}[N]$ arising from the W-algebra map $W[\zeta_N] \to R$.

Put another way, the universal deformation ring of $(E_0, (P_0, Q_0))$ is a local factor ring of the *fppf* μ_N-torsor

$$(v^{\mathrm{univ}})^{-1}(1) \otimes_{W[\![q]\!]} W[\zeta_N][\![q]\!] \tag{3.3.6}$$

over $W[\zeta_N][\![q]\!]$. The distinct local factor rings of (3.3.6) can be seen by setting $q = 0$ and $W = k$: these correspond to choices of $P_0 \in (v_0^{-1}(1))(k)$ and $Q_0 \in \mu_N^\times(k)$. Any such local factor ring is finite flat over $W[\![q]\!]$, thereby settling the analysis of deformation rings for the case $\Gamma = \Gamma(N)$.

Case $\boldsymbol{\Gamma = \Gamma_1(N)}$. This case will be reduced to the 'full level structure' case just treated. We study the deformation theory of a $\mathbf{Z}/N\mathbf{Z}$-structure (E_0, P_0), with E_0 a d-gon over k, $d \mid N$, and

$$P_0 \mapsto 1 \in \mathbf{Z}/d\mathbf{Z} = \pi_0(E_0^{\mathrm{sm}}).$$

Consider the canonical short exact sequence

$$0 \to \mu_d \to E_0^{\mathrm{sm}}[d] \xrightarrow{v_0} \mathbf{Z}/d\mathbf{Z} \to 0.$$

For any deformation E of E_0 over an artin local W-algebra R with residue field k, there exists a unique compatible short exact sequence

$$0 \to \mu_d \to E^{\mathrm{sm}}[d] \xrightarrow{v} \mathbf{Z}/d\mathbf{Z} \to 0 \tag{3.3.7}$$

over $\operatorname{Spec} R$. We are trying to prove that the universal deformation ring of a $\mathbf{Z}/N\mathbf{Z}$-structure over k has (as a W-algebra) a finite flat cover that is finite and flat over a formal power series algebra $W[\![t]\!]$. For the unique algebraization of the universal formal deformation of (E_0, P_0), let v^{univ} be defined via the analogue of (3.3.7) on this universal object. The W-scheme $(v^{\mathrm{univ}})^{-1}(1)$ is finite and faithfully flat over the universal deformation ring, and the affine algebra of this W-scheme is the universal (semi-local) deformation ring for the extra condition of the specification of a splitting of (3.3.7).

For any artinian deformation E of E_0 equipped with a splitting of (3.3.7) via $x \in v^{-1}(1)(R)$, an R-point P lifting P_0 has the form $P = x + \zeta$ in $E^{\mathrm{sm}}(R)$ for a unique $\zeta \in \mu_N(R) \subseteq E^{\mathrm{sm}}[N](R)$. By Corollary 2.3.3, the necessary and sufficient condition for P to determine a $\mathbf{Z}/N\mathbf{Z}$-structure is that $dP \in \mu_{N/d}^\times(R)$. Since $\zeta^d = dP$, we conclude that $(v^{\mathrm{univ}})^{-1}(1)$ is an *fppf* μ_d-torsor over the scheme $\mu_{N/d}^\times$ living on the base given by the disjoint union of spectra of universal deformation rings for all possible $\Gamma_1(d)$-structures on the standard d-gon. Hence, it suffices to study these $\Gamma_1(d)$-deformation rings; that is, we are reduced to the case $d = N$. We now claim that via the map $(P, Q) \mapsto P$, the scheme of Drinfeld $\mathbf{Z}/N\mathbf{Z}$-bases on the N-torsion of the smooth locus of an artinian deformation E of the standard N-gon E_0 is *finite, locally free, surjective* over the scheme of $\mathbf{Z}/N\mathbf{Z}$-structures on E^{sm}. Indeed, since $G \overset{\mathrm{def}}{=} E^{\mathrm{sm}}[N]$ is an extension of $\mathbf{Z}/N\mathbf{Z}$ by μ_N and the schemes of $\mathbf{Z}/N\mathbf{Z}$-structures and $\mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z}$-structures in G are *independent* of the generalized elliptic curve in which G is embedded [**KM**, 1.10.6], we may use faithfully flat base change and [**KM**, 8.10.7] to realize G as the N-torsion scheme on a smooth elliptic curve, in which case our 'finite, locally free, surjective' claim follows from [**KM**, 5.5.3]. It follows that upon *universally* making a finite faithfully flat covering of our artin local base, our $\Gamma_1(N)$-structure can be enhanced to a $\Gamma(N)$-structure, and the $\Gamma(N)$-deformation ring is a finite flat cover of the initial $\Gamma_1(N)$-deformation ring.

Case $\Gamma = \widetilde{\Gamma}_1(N;n)$. It is enough to work with deformations of standard polygons over an algebraically closed field. Since $\widetilde{\Gamma}_1(N;n)$-structures coincide with $\Gamma_1(Nn)$-structures when $\gcd(N,n)=1$, we may assume $d=\gcd(N,n)>1$. If the characteristic of k does not divide nN then we easily reduce to a $\Gamma_1(M)$-case. Thus, we may assume k has positive characteristic p and $p \mid nN$. Choose a generator Q of the constant prime-to-p part C' of C, and use the prime-to-N part of C' to enhance the $\mathbf{Z}/N\mathbf{Z}$-structure to a $\mathbf{Z}/N'\mathbf{Z}$-structure for a multiple N' of N. This reduces us to the case where the prime-to-p part of C is trivial, so $n=p^e$ with $e \geqslant 0$. We may again certainly suppose $\gcd(N,n)>1$, which is to say $e>0$ and $N=Mp^r$ with $r \geqslant e$ and $p \nmid M$.

First suppose that the $\mathbf{Z}/N\mathbf{Z}$-structure $\langle P_0\rangle$ is ample, so to give a $\widetilde{\Gamma}_1(N;p^e)$-structure deformation amounts to giving a $\Gamma_1(N)$-structure deformation and a suitable auxiliary $\mathbf{Z}/p^e\mathbf{Z}$-structure deformation. To be precise, let A be the universal deformation ring for the underlying $\Gamma_1(N)$-structure, with $(\mathcal{E},\mathcal{P})$ the unique algebraization of the universal formal deformation. Recall from Theorem 2.3.2 that $(N/p^e)\mathcal{P}$ is a $\mathbf{Z}/p^e\mathbf{Z}$-structure. We want to study the (non-zero) finite A-algebra B over which $\mathcal{E}$ universally acquires a $\mathbf{Z}/p^e\mathbf{Z}$-structure $Q \in \mathcal{E}^{\mathrm{sm}}[p^e]$ such that $\{(N/p^e)\mathcal{P}, Q\}$ is a $\mathbf{Z}/p^e\mathbf{Z}\times\mathbf{Z}/p^e\mathbf{Z}$-structure. It suffices to show that B is finite flat over A. Note that the ampleness forces $(N/p^e)\mathcal{P}$ to be an étale point of order p^e. Since $\mathcal{E}^{\mathrm{sm}}[p^e]$ is uniquely an extension of $\mathbf{Z}/p^e\mathbf{Z}$ by μ_{p^e} in a manner lifting the extension structure on the closed fiber, we may subtract a unique multiple of $(N/p^e)\mathcal{P}$ from Q to arrange that Q lies in $\mu_{p^e}^{\times}$. Thus, $B = A[T]/(\Phi_{p^e}(T))$.

The final case to consider is a $\widetilde{\Gamma}_1(N;p^e)$-structure $(E_0;P_0,Q_0)$ over k such that $\langle P_0\rangle$ is not ample. This forces Q_0 to be an étale point of exact order p^e such that $\langle Q_0\rangle$ maps isomorphically onto the p-part of the component group. (Thus, the deformation theory of the étale p^e-torsion point Q_0 is the same as that of the étale subgroup $\langle Q_0\rangle$ splitting the connected-étale sequence of the p^e-torsion.) By Theorem 2.3.2, p^rP is a $\mathbf{Z}/M\mathbf{Z}$-structure and MP is a $\mathbf{Z}/p^r\mathbf{Z}$-structure. There is a unique $j \in \mathbf{Z}/p^e\mathbf{Z}$ such that the p^r-torsion point $MP_0 - jQ_0$ lies in the identity component of the closed fiber, so in the study of deformations we may replace the $\mathbf{Z}/N\mathbf{Z}$-structure P with the $\mathbf{Z}/N\mathbf{Z}$-structure $P - jM^{-1}Q$ (see Theorem 2.3.2 (2) for the p-part aspect) to reduce to studying the deformation theory in the case that MP_0 is on the identity component of the closed fiber. By Theorem 2.3.2, the section MP must be a $\mathbf{Z}/p^r\mathbf{Z}$-generator of μ_{p^r}. That is, if we canonically decompose our $\mathbf{Z}/N\mathbf{Z}$-structure into a $\mathbf{Z}/M\mathbf{Z}$-structure and a $\mathbf{Z}/p^r\mathbf{Z}$-structure, then the $\mathbf{Z}/p^r\mathbf{Z}$-structure is in fact a $\mathbf{Z}/p^r\mathbf{Z}$-generator of μ_{p^r}.

The *ample* divisor generated by the $\mathbf{Z}/M\mathbf{Z}$-structure and the étale $\mathbf{Z}/p^e\mathbf{Z}$-structure Q is an étale $\Gamma_1(Mp^e)$-structure. By Theorem 2.3.2, our $\widetilde{\Gamma}_1(N;p^e)$ deformation problem therefore breaks up into two steps: first deform an étale $\Gamma_1(Mp^e)$-structure, and then deform a section of $\mu_{p^r}^{\times}$ such that the two $\mathbf{Z}/p^e\mathbf{Z}$-structures coming from these two steps together define a $\mathbf{Z}/p^e\mathbf{Z}\times\mathbf{Z}/p^e\mathbf{Z}$-structure. This final condition concerning the $\mathbf{Z}/p^e\mathbf{Z}\times\mathbf{Z}/p^e\mathbf{Z}$-structure is forced by the rest, due to Lemma 2.3.1, so we can ignore it. Since $p \nmid M$ and the p-part of the level structure maps isomorphically onto the component group, we may use [**DR**, II, 1.17] to identify the deformation ring of our residual *étale* $\Gamma_1(Mp^e)$-structure with the formally smooth deformation ring $W[\![t]\!]$ of an étale quotient 1-gon. The deformation ring for our $\widetilde{\Gamma}_1(N;p^e)$-deformation problem is therefore isomorphic to the coordinate ring of $\mu_{p^r}^{\times}$ over $W[\![t]\!]$. Since $W[\![t]\!] \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^r}]$ is a regular, W-flat, and two-dimensional local ring, the proof of Theorem 3.3.1 is now complete.

4. Fine structure of the proper stacks $\mathcal{M}_\Gamma$

4.1. Contractions and structure along the boundary

Theorem 4.1.1. *Choose $\Gamma \in \{\Gamma(N), \Gamma_1(N;n)\}$.*

(1) *If $\Gamma = \Gamma(N)$ then the contraction $c_\Gamma : \mathcal{M}_\Gamma \to \mathcal{M}_1$ is relatively representable, finite, and faithfully flat. If $\Gamma = \Gamma_1(N;n)$ then the same holds over $\mathcal{M}_1^0$, and also over $\mathcal{M}_1$ when n is squarefree. Also, the closed substack $\mathcal{M}_\Gamma^\infty$ is a relative effective Cartier divisor in $\mathcal{M}_\Gamma$ over* $\mathrm{Spec}(\mathbf{Z})$ *for all Γ.*

(2) *The stack $\mathcal{M}_\Gamma$ is regular.*

(3) *There is a unique (necessarily proper CM) morphism $\mathcal{M}_{\Gamma(N)} \to$* $\mathrm{Spec}(\mathbf{Z}[\zeta_N])$ *extending the morphism $\mathcal{M}^0_{\Gamma(N)/\mathbf{Z}[1/N]} \to$* $\mathrm{Spec}(\mathbf{Z}[1/N, \zeta_N])$ *defined by the scheme-theoretic Weil pairing.*

Proof. We first show that $\mathcal{M}_\Gamma^\infty$ is a relative Cartier divisor in $\mathcal{M}_\Gamma$ over $\mathrm{Spec}(\mathbf{Z})$. The canonical morphism $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)} \to \mathcal{M}_{\Gamma_1(N;n)}$ is representable and finite locally free, and it pulls the universal generalized elliptic curve back to the universal generalized elliptic curve (the only distinction being the enhancement of the level structure). Thus, for the purpose of the claim concerning relative effective Cartier divisors, the case of $\Gamma_1(N;n)$ is reduced to that of $\widetilde{\Gamma}_1(N;n)$ (for which universal deformation rings exist at geometric points). Our 'Cartier divisor' problem therefore concerns universal deformation rings for level structures on Néron polygons over an algebraically closed field k.

Let W denote a Cohen ring for k. Consider $\Gamma = \Gamma(N)$. Recall the unique degree-n étale isogeny (3.3.4) over $W[\![q]\!]$. We saw in the proof of Lemma 3.3.6 that the deformation rings for $\Gamma(N)$-structures (on Néron polygons over k) are local factor rings of

$$(v^{\mathrm{univ}})^{-1}(1) \otimes_{W[\![q]\!]} W[\zeta_N][\![q]\!], \tag{4.1.1}$$

where $(v^{\mathrm{univ}})^{-1}(1)$ is the *fppf* μ_N-torsor defined by the canonical short exact sequence (3.3.5). We need to check that for $\mathcal{E}_N$ as in (3.3.4) the pullback of $\mathcal{E}_N$ to (4.1.1) has locus of non-smoothness in the base ring that is a relative Cartier divisor over W. To compute this locus of non-smoothness, we shall use Tate curves.

By Grothendieck's algebraization theorem and the construction of $\underline{\mathrm{Tate}}_1$, there is a unique étale isogeny

$$\underline{\mathrm{Tate}}_N(q) \to \underline{\mathrm{Tate}}_1$$

over $W[\![q]\!]$ that lifts the natural étale isogeny of polygons $C_N \to C_1$ on the closed fiber. By [**DR**, II, 1.17] and the uniqueness property characterizing $\mathcal{E}_N$ as a finite étale cover of $\underline{\mathrm{Tate}}_1$, there is a unique isomorphism of generalized elliptic curves

$$\mathcal{E}_N \simeq \underline{\mathrm{Tate}}_N(q) \tag{4.1.2}$$

over $W[\![q]\!]$ lifting the identity on the closed fibers and respecting the étale isogenies to $\underline{\mathrm{Tate}}_1$. In $\mathrm{Spec}(W[\![q]\!])$ the locus of non-smoothness of the map $\underline{\mathrm{Tate}}_N(q) \to \mathrm{Spec}(W[\![q]\!])$ is a relative Cartier divisor over W: it is cut out by (q). This relative Cartier divisor condition over W is preserved by the faithfully flat base change

$$W[\![q]\!] \to (v^{\mathrm{univ}})^{-1}(1) \otimes_{W[\![q]\!]} W[\zeta_N][\![q]\!].$$

Thus, by (4.1.2), this settles the Cartier divisor claim for $\Gamma = \Gamma(N)$.

Using the same arguments as in the proof of Theorem 3.3.1, the cases $\Gamma = \widetilde{\Gamma}_1(N;n)$ are reduced (via consideration of $\Gamma = \Gamma_1(M)$) to the case just handled. Hence, $\mathcal{M}_\Gamma^\infty \hookrightarrow \mathcal{M}_\Gamma$ is always an effective relative Cartier divisor over $\mathrm{Spec}\,\mathbf{Z}$. The rest of Theorem 4.1.1 (1) is given by the following lemma.

Lemma 4.1.2. *The morphism c_Γ is a finite flat covering, provided that for $\Gamma_1(N;n)$ we require n to be squarefree or we work over the open substack $\mathcal{M}_1^0$ in $\mathcal{M}_1$.*

Proof. By Theorem 3.2.7, c_Γ is proper and (trivially) quasi-finite of finite presentation. We now prove that it is representable in algebraic spaces, hence finite. By Corollary 2.2.7 (and Theorem 2.2.5 (2)), it suffices to prove that if (E,ι) is a Γ-structure over an algebraically closed field k then there does not exist a non-trivial automorphism α of (E,ι) that induces the identity map on the contraction $c(E)$ of the non-identity components. (The reason it suffices to work with automorphism groups rather than automorphism group schemes is that the automorphism group schemes at the geometric points under consideration are étale; this follows from Theorem 3.1.7 and the restrictions that we are imposing in the case $\Gamma = \Gamma_1(N;n)$.)

The case of $\Gamma(N)$ immediately reduces to the case of $\Gamma_1(N)$, and the case of irreducible E is obvious. Thus, the case $\Gamma = \Gamma(N)$ reduces to considering a $\Gamma_1(N)$-structure (E,P) on a d-gon $E_{/k}$ with $d \mid N$. These cases follow trivially from Example 2.1.5.

Now consider a $\Gamma_1(N;n)$-structure $(E;P,C)_{/k}$ and an automorphism α inducing the identity on $c(E)$. Without loss of generality E is a d-gon, so n is squarefree by our hypotheses. Since α acts as the identity on $c(E)$, it follows that α must fix E^{sing} pointwise and must act trivially on the component group. On each irreducible component $L \simeq \mathbf{P}^1$ of E, the automorphism α fixes the two points in $L \cap E^{\mathrm{sing}}$ and a point on $L \cap E^{\mathrm{sm}}$ in the ample subgroup $\sum_{j \in \mathbf{Z}/N\mathbf{Z}}(jP + C)$ if $\langle P \rangle$ meets L. This forces α to be the identity on such components. By using contractions, it remains to treat the case $N = 1$ and n squarefree. The connected-étale sequence of C is uniquely split, so we may also replace C with its étale part. Since n is squarefree, the part of C meeting the identity component splits off as a product of primary parts, so we may drop it. Hence, C maps isomorphically onto the component group, and so α fixed C pointwise. Thus, α is the identity. This concludes the discussion of finiteness properties for $\mathcal{M}_\Gamma \to \mathcal{M}_1$.

Since $\mathcal{M}_\Gamma$ is CM over $\mathbf{Z}$ with pure relative dimension 1 (by Theorem 3.3.1) and $\mathcal{M}_1$ is *smooth* over $\mathbf{Z}$ with pure relative dimension 1, the finite map $\mathcal{M}_\Gamma \to \mathcal{M}_1$ over $\mathbf{Z}$ (or over $\mathcal{M}_1^0$ for $\Gamma_1(N;n)$ when n is not squarefree) must be flat, by the standard flatness result in Lemma 4.1.3 below. $\square$

Lemma 4.1.3. *Let $A \to B$ be a local map between local noetherian rings, with A regular and B Cohen–Macaulay. If $\dim B = \dim A + \dim B/\mathfrak{m}_A B$, then B is A-flat.*

Proof. This is [**CRT**, 23.1]. □

We now record a preliminary result in the direction of Theorem 4.1.1 (2). For $\Gamma = \Gamma(N)$ (respectively, $\Gamma = \Gamma_1(N;n)$) we call N (respectively, Nn) the *level* of Γ; our only interest in this terminology will be through its prime factors, so issues of multiplicity are irrelevant for our purposes.

Lemma 4.1.4. *The stack $\mathcal{M}_\Gamma$ is normal. It is also regular away from the part of $\mathcal{M}_\Gamma^\infty$ that is supported in characteristics dividing the level of Γ.*

Proof. For $\Gamma = \Gamma_1(N;n)$ it suffices to work with the finite flat covering $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$. With this modification understood, the $\mathcal{M}_\Gamma$ that we consider are Deligne–Mumford stacks such that $\mathcal{M}_\Gamma \to \mathrm{Spec}(\mathbf{Z})$ is smooth away from the 'level' and is a CM morphism with fibers of pure dimension 1. By Serre's criterion for normality (applied to universal deformation rings at geometric points on the Deligne–Mumford stack $\mathcal{M}_\Gamma$) and by the smoothness of $\mathcal{M}_{\Gamma/\mathbf{Q}}$, it suffices to check regularity away from a relative effective Cartier divisor (as such a divisor cannot contain any 'codimension-1 points' of $\mathcal{M}_\Gamma$ whose residue characteristic is positive). We use the divisor $\mathcal{M}_\Gamma^\infty$: by Lemma 3.3.4, its open complement $\mathcal{M}_\Gamma^0$ is regular. □

We have proved Theorem 4.1.1 (1), as well as Theorem 4.1.1 (2) away from the cusps in bad characteristics.

Remark 4.1.5. With n understood to be squarefree for $\Gamma_1(N;n)$, the contraction $\mathcal{M}_\Gamma \to \mathcal{M}_1$ over $\mathbf{Z}$ is finite and flat. Thus, by regularity of $\mathcal{M}_1$ and normality of $\mathcal{M}_\Gamma$, it follows that the stack $\mathcal{M}_\Gamma$ is *a posteriori* canonically identified with the *normalization* of $\mathcal{M}_1$ in the normal Deligne–Mumford stack $\mathcal{M}_\Gamma|_{\mathbf{Z}[1/\mathrm{level}(\Gamma)]}$ that is finite flat surjective over $\mathcal{M}_1|_{\mathbf{Z}[1/\mathrm{level}(\Gamma)]}$. This proves that the ad hoc compactification technique over $\mathbf{Z}$ that is used in [**DR**] yields moduli stacks for Drinfeld level-structures on generalized elliptic curves (though for $\Gamma_1(N;n)$ when n is not squarefree it cannot recover the 'correct' moduli stack $\mathcal{M}_{\Gamma_1(N;n)}$ because this Artin stack is not Deligne–Mumford along $\mathcal{M}_{\Gamma_1(N;n)}^\infty$ in characteristic p when $p^2 \mid n$).

To prove Theorem 4.1.1 (2) for all Γ, it remains to prove that $\mathcal{M}_\Gamma$ is regular along $\mathcal{M}_\Gamma^\infty$ in 'bad' characteristics. We shall first reduce the regularity problem for $\mathcal{M}_{\Gamma_1(N;n)}$ to regularity in $\Gamma_1(M)$-cases. By Lemma 4.1.4, the only case we need to consider is along $\mathcal{M}_{\Gamma_1(N;n)}^\infty$ in positive characteristic p when $p \mid Nn$. Let $x = (E_0; P_0, C_0)$ be a geometric point on this closed substack in such a characteristic. Since the Deligne–Mumford stack $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ is a flat covering of the Artin stack $\mathcal{M}_{\Gamma_1(N;n)}$, it would suffice to establish regularity of the flat cover, and this is a problem that we can study via universal deformation rings. As we shall now see, this sufficient criterion for regularity will not always work in our favor.

In the proof of Theorem 3.3.1, the analysis of deformation rings on $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ shows that if $p \nmid \gcd(N,n)$ then the deformation rings on $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ at geometric points over x

are identified with deformation rings on Deligne–Mumford stacks of the form $\mathcal{M}_{\Gamma_1(M)}$. Granting that stacks of the form $\mathcal{M}_{\Gamma_1(M)}$ are regular, the case $p \nmid \gcd(N,n)$ is settled.

We now assume $p \mid \gcd(N,n)$, so

$$1 \leqslant e \stackrel{\text{def}}{=} \operatorname{ord}_p(n) \leqslant r \stackrel{\text{def}}{=} \operatorname{ord}_p(N).$$

In the cases where $\langle P_0 \rangle$ and the prime-to-p part of C_0 do *not* generate an ample locus in E_0, the end of the proof of Theorem 3.3.1 shows that the deformation rings on $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ at points over x have the form $W[\![t]\!] \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^r}]$, and hence are regular. Finally, if $\langle P_0 \rangle$ and the prime-to-p part of C_0 do generate an ample locus in E_0 then the deformation rings B on $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ at points over x have the form $A[\zeta_{p^e}] \stackrel{\text{def}}{=} A \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^e}]$ where A is a deformation ring on a stack of the form $\mathcal{M}_{\Gamma_1(M)}$ and where the canonical action of $(\mathbf{Z}/p^e\mathbf{Z})^\times$ on B (through the n-part of the $\widetilde{\Gamma}_1(N;n)$-structure) goes over to the evident action on ζ_{p^e} in $A[\zeta_{p^e}]$. Even granting that the rings A are regular, the rings $A[\zeta_{p^e}]$ often are not regular. However, keep in mind that regularity on $\mathcal{M}_{\widetilde{\Gamma}_1(N;n)}$ is merely sufficient and not necessary for regularity on $\mathcal{M}_{\Gamma_1(N;n)}$.

For a smooth chart $U \to \mathcal{M}_{\Gamma_1(N;n)}$ around x and a geometric point $u \in U$ over x, there is a finite flat covering $U' \to U$ that is universal for imposing a $\mathbf{Z}/n\mathbf{Z}$-generator on the cyclic subgroup of order n. The preceding calculation '$B = A[\zeta_{p^e}]$' implies that for $u' \in U'$ over u there is a natural isomorphism

$$\widehat{\mathcal{O}}_{U',u'} \simeq \widehat{\mathcal{O}}_{U,u} \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_{p^e}]$$

that is equivariant for the action of $(\mathbf{Z}/p^e\mathbf{Z})^\times$, so the subring of invariants is the $\mathbf{Z}$-flat $\widehat{\mathcal{O}}_{U,u}$. However, as we have already noted, the preceding description of B as $A[\zeta_{p^e}]$ is a $(\mathbf{Z}/p^e\mathbf{Z})^\times$-equivariant description and it is also compatible with passage to smooth covers over $\operatorname{Spec} A$. Thus, passing to subrings of invariants, $\widehat{\mathcal{O}}_{U,u}$ is isomorphic to a formally smooth algebra over the complete local ring A, and therefore $\widehat{\mathcal{O}}_{U,u}$ is regular if A is regular. This completes the reduction of regularity of $\mathcal{M}_{\Gamma_1(N;n)}$ to regularity of the $\mathcal{M}_{\Gamma_1(M)}$.

Let us now consider regularity for $\mathcal{M}_\Gamma$ in the cases $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$, so $\mathcal{M}_\Gamma \to \mathcal{M}_1$ is a finite flat covering. In these cases, Lemma 4.1.4 reduces the proof of regularity to calculations in [**KM**], as we shall now explain. Let $\operatorname{Spec}(k)$ be an *algebraic* geometric point of $\operatorname{Spec}(\mathbf{Z})$, and let W be a Cohen ring for k. We need to check regularity of a finite étale cover of the complete local ring at a k-point on $\mathcal{M}_\Gamma^\infty$. Thus, we may use Lemma 3.3.5 to reduce to verifying the regularity of the *scheme* $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} W[\![q]\!]$, where $\operatorname{Spec} W[\![q]\!] \to \mathcal{M}_1$ corresponds to the Tate curve $\underline{\mathrm{Tate}}_1$ over $W[\![q]\!]$. Since $W[\![q]\!]$ is the completion of a strict henselization of $\mathbf{Z}[\![q]\!]$ at the prime ideal (p,q), where $p = \operatorname{char}(k) \geqslant 0$, the natural map $\mathbf{Z}[\![q]\!] \to W[\![q]\!]$ is flat. We may work with $\mathbf{Z}[\![q]\!]$ instead of $W[\![q]\!]$ once we check the following lemma.

Lemma 4.1.6. *The flat map* $\operatorname{Spec}(W[\![q]\!]) \to \operatorname{Spec}(\mathbf{Z}[\![q]\!])$ *is a regular morphism (i.e. has geometrically regular fibers).*

Proof. The ring $\mathbf{Z}[\![q]\!]$ is excellent [**V**, Theorem 9], and in particular it is a G-ring (see the end of § 2.2). Thus, by Theorem 2.2.8 (2), the strict henselization of $\mathbf{Z}[\![q]\!]$ at the prime

(p, q) is a G-ring, and this latter ring has completion $W[\![q]\!]$. The map $\mathbf{Z}[\![q]\!] \to W[\![q]\!]$ is therefore a composite of regular morphisms and so it is regular [**CRT**, 32.1(i)]. □

For $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$, the regularity of the morphism from $\mathbf{Z}[\![q]\!]$ to the universal deformation ring $W[\![q]\!]$ reduces us to proving that the finite flat $\mathbf{Z}[\![q]\!]$-scheme $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ is regular. The verification of this regularity will rest on a direct calculation, and this in turn requires that we first check a weaker property.

Lemma 4.1.7. *Let $\Gamma = \Gamma_1(N)$ or $\Gamma(N)$. The scheme $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ is normal, and it is regular in characteristics not dividing N.*

Proof. Since $\mathbf{Z}[\![q]\!]$ is q-adically separated and complete, its maximal ideals contain q. In particular, a non-empty closed set in the scheme $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ must meet the fiber over (p, q) for some prime $p > 0$. Since the normal and regular loci in $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ are open (by excellence of $\mathbf{Z}[\![q]\!]$), by Lemma 4.1.6 it suffices to check normality (respectively, regularity) after base change to $W[\![q]\!]$, for $W = W(\overline{\mathbf{F}}_p)$ (respectively, with $p \nmid N$). Such base changes of the scheme $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ are spectra of finite products of complete local rings at geometric points on $\mathcal{M}_\Gamma$ (respectively, in characteristics not dividing N). By excellence arguments, the normality (respectively, regularity) of such rings follows from the normality (respectively, regularity) of $\mathcal{M}_\Gamma$ (respectively, of $\mathcal{M}_\Gamma$ in residue characteristics away from the 'level') as in Lemma 4.1.4. □

We conclude that the *finite flat* $\mathbf{Z}[\![q]\!]$-scheme $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!]$ is the normalization of $\mathbf{Z}[\![q]\!]$ in

$$\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}(\!(q)\!)$$

(which is itself normal and finite flat over $\mathbf{Z}(\!(q)\!)$). The $\mathbf{Z}(\!(q)\!)$-scheme

$$\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}(\!(q)\!)$$

is the scheme of Γ-structures on the elliptic curve $\underline{\mathrm{Tate}}_1$ over $\mathbf{Z}(\!(q)\!)$. Such level-structure schemes are analyzed in [**KM**], so our task comes down to making the detailed analysis from [**KM**] explicit in our cases of interest and using this to compute the normalization of $\mathbf{Z}[\![q]\!]$ in $\mathcal{M}_\Gamma \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}(\!(q)\!)$. These normalizations will be observed to be regular by inspection.

When $\Gamma = \Gamma(N)$, [**KM**, 10.8.2] computes the scheme of Γ-structures over $\mathbf{Z}(\!(q)\!)$ to be a finite disjoint union of copies of $\operatorname{Spec}(\mathbf{Z}[\zeta_N](\!(q^{1/N})\!))$, in which the normalization of $\operatorname{Spec}(\mathbf{Z}[\![q]\!])$ is a finite disjoint union of copies of the regular scheme $\operatorname{Spec}(\mathbf{Z}[\zeta_N][\![q^{1/N}]\!])$. This settles the case of $\Gamma(N)$.

Let us briefly digress and use the regularity of $\mathcal{M}_{\Gamma(N)}$ to settle part (3) in Theorem 4.1.1. We must construct a unique CM morphism $\mathcal{M}_{\Gamma(N)} \to \operatorname{Spec}(\mathbf{Z}[\zeta_N])$ extending the Weil pairing morphism

$$\mathcal{M}^0_{\Gamma(N)/\mathbf{Z}[1/N]} \to \operatorname{Spec}(\mathbf{Z}[1/N, \zeta_N]). \tag{4.1.3}$$

That is, we need to uniquely extend ζ_N to a solution of $\Phi_N(X) = 0$ in the ring of global functions on $\mathcal{M}_{\Gamma(N)}$. Since $\mathcal{M}_{\Gamma(N)}$ is $\mathbf{Z}$-flat and $\mathcal{M}^0_{\Gamma(N)}$ is the complement of a relative effective Cartier divisor, normality of $\mathcal{M}_{\Gamma(N)}$ provides the desired unique extension. It is automatic from Theorem 3.3.1 that the resulting morphism $\mathcal{M}_{\Gamma(N)} \to \mathrm{Spec}(\mathbf{Z}[\zeta_N])$ is proper and flat. Since source and target are Cohen–Macaulay, this flat map is automatically CM. This proves part (3).

Regularity in the case $\Gamma = \Gamma_1(N)$ will be more complicated than in the case $\Gamma = \Gamma(N)$ that has already been discussed, but the key computation is again to be found in [**KM**]. By [**KM**, 7.4.3, 10.5.1(2)], the scheme of $\Gamma_1(N)$-structures on $\underline{\mathrm{Tate}}_{1/\mathbf{Z}((q))}$ is

$$\coprod_{\Lambda_i} \mathrm{Spec}\, \mathbf{Z}[\zeta_N]((q^{1/N}))^{G \cap \mathrm{Fix}(\Lambda_i)}, \tag{4.1.4}$$

where

$$G \stackrel{\mathrm{def}}{=} \left\{ \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \right\},$$

$\{\Lambda_i\}$ is a set of representatives for the quotient $\mathrm{HomSurj}((\mathbf{Z}/N\mathbf{Z})^2, \mathbf{Z}/N\mathbf{Z})/G$ of the set of surjective linear functionals on $(\mathbf{Z}/N\mathbf{Z})^2$, and

$$\mathrm{Fix}(\Lambda_i) \stackrel{\mathrm{def}}{=} \{g \in \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \mid \Lambda_i \circ g = \Lambda_i\}.$$

¿From the definition in [**KM**, 10.5] (to which we refer for explicit formulae), the action of $\mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z})$ with respect to which we take invariants in (4.1.4) is defined on the level of 'uncompleted' rings $\mathbf{Z}[\zeta_N][q^{1/N}]$.

The normalization of $\mathbf{Z}[\![q]\!]$ in (4.1.4) is obviously the disjoint union of normal schemes of the form

$$\mathrm{Spec}\, \mathbf{Z}[\zeta_N][\![q^{1/N}]\!]^{G \cap \mathrm{Fix}(\Lambda_i)},$$

so it suffices to check regularity of these schemes at points with residue characteristic $p \mid N$. By [**EGA**, IV_2, 7.8.3(v)], for any excellent ring A and *any* ideal I of A, the map $A \to \widehat{A}$ to the I-adic completion is a regular map. In particular, if A is regular then so is $\widehat{A}$. Since flat base change (e.g., $(\cdot) \otimes_{\mathbf{Z}[q]} \mathbf{Z}[\![q]\!]$) commutes with formation of invariants under the action of a finite group, it is therefore enough to check regularity of the schemes

$$\mathrm{Spec}\, \mathbf{Z}_{(p)}[\zeta_N][q^{1/N}]^{G \cap \mathrm{Fix}(\Lambda_i)}$$

with $p \mid N$.

Using the *method* of proof of [**KM**, 10.10.3], we reduce to replacing N by its p-part, and then [**KM**, 10.10.4] settles the three cases $p \neq 2$, $N = 2$, and $N = 2^r$ with $\det(G \cap \mathrm{Fix}(\Lambda_i)) \equiv 1 \bmod 4$. There remain the cases in which both the conditions $N = 2^r$ with $r \geqslant 2$ and $\det(G \cap \mathrm{Fix}(\Lambda_i)) \not\equiv 1 \bmod 4$ hold. For such cases we may choose the set of representatives

$$\{\Lambda_i\} = \{(a, 0) \mid 2 \nmid a\} \cup \{(a, 1) \mid 2 | a\},$$

so we have

$$G \cap \mathrm{Fix}(a, 0) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \right\} \subseteq \mathrm{GL}_2(\mathbf{Z}/N\mathbf{Z}) \tag{4.1.5}$$

for odd a and

$$G \cap \mathrm{Fix}(a,1) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1-ax \end{pmatrix} \;\middle|\; x \in \mathbf{Z}/2^r\mathbf{Z} \right\} \subseteq \mathrm{GL}_2(\mathbf{Z}/2^r\mathbf{Z}) \tag{4.1.6}$$

for even a. Since $\det(Q \cap \mathrm{Fix}(\Lambda_i)) \not\equiv 1 \bmod 4$ for each i, when $2 \mid a$ we see that $a/2$ is odd.

For the case of odd a, it follows from [**KM**, 10.3.2] that the elements in (4.1.5) act through the determinant character on roots of unity and leave $q^{1/N}$ invariant, so the fixed subring is the regular ring $\mathbf{Z}_{(2)}[q^{1/N}]$.

Now suppose that a is even. We use the 'upper triangularization' construction [**KM**, 10.3.4] to put our groups into a form with respect to which it is easier to compute the subrings of invariants. Applying this construction to (4.1.6) yields the group

$$\left\{ \begin{pmatrix} 1-ax & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbf{Z}/2^r\mathbf{Z} \right\}. \tag{4.1.7}$$

The group (4.1.7) acts in a very simple manner on $\mathbf{Z}_{(2)}[\zeta_N][q^{1/N}]$:

$$\begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix} : (\zeta_N, q^{1/N}) \mapsto (\zeta_N^r, \zeta_N^s q^{1/N}).$$

We have $a = 2a'$ for odd a'. The subgroup of (4.1.7) with *even* x has subring of invariants

$$\mathbf{Z}_{(2)}[i][\zeta(q^{1/2^r})^2]$$

for some $\zeta \in \mu_{2^r}$, with $i = \zeta_{2^r}^{2^{r-2}}$. It is not difficult to check that $\zeta = \zeta_{2^r}^{1/a'} \cdot i^m$ for some m (with $1/a' \in (\mathbf{Z}/2^r\mathbf{Z})^\times$ denoting the inverse of a'). The power of i does not affect this subring of invariants, so we may take $\zeta = \zeta_{2^r}^{1/a'}$. We need to compute the subring of invariants under the action of

$$\begin{pmatrix} 1-a & 1 \\ 0 & 1 \end{pmatrix}$$

on $\mathbf{Z}_{(2)}[i][\zeta(q^{1/2^r})^2]$. This action is determined by

$$i \mapsto i^{1-a} = i^{1-2a'} = -i,$$
$$\zeta(q^{1/2^r})^2 \mapsto \zeta^{1-a}(\zeta_{2^r} q^{1/2^r})^2 = \zeta_{2^r}^2 \zeta^{-a} \zeta(q^{1/2^r})^2 = \zeta(q^{1/2^r})^2.$$

Thus, the invariant subring is $\mathbf{Z}_{(2)}[\zeta(q^{1/2^r})^2]$, and this is regular. $\square$

4.2. Schematic loci

Theorem 4.2.1. *Let $\Gamma = \Gamma_1(N;n)$ (respectively, $\Gamma(N)$) and $S = \mathrm{Spec}(\mathbf{Z})$ (respectively, $\mathrm{Spec}(\mathbf{Z}[\zeta_N])$), and let $\mathcal{M}$ denote $\mathcal{M}_\Gamma$ considered as an Artin stack over S.*

(1) *The proper flat map $\mathcal{M} \to S$ has geometrically connected fibers.*

(2) *Assume n is squarefree if $\Gamma = \Gamma_1(N;n)$. There exists an open subscheme $\mathcal{M}^{\rm sch}$ whose geometric points are exactly those of $\mathcal{M}$ with trivial automorphism group. If $d \mid N$ and $d \geqslant 5$ (respectively, $d \geqslant 3$) then $\mathcal{M}_{/S[1/d]}$ lies inside of $\mathcal{M}^{\rm sch}$ and the relative effective Cartier divisor $\mathcal{M}^{\infty}_{/S[1/d]} \hookrightarrow \mathcal{M}_{/S[1/d]}$ is relatively ample.*

Remark 4.2.2. This theorem implies that for 'large' N, the proper flat $\mathbf{Z}$-schemes $X_1(N)_{/\mathbf{Z}}$ and $X(N)^{\rm can}_{/\mathbf{Z}[\zeta_N]}$ defined in [**KM**] coincide respectively with the stacks $\mathcal{M}_{\Gamma_1(N)}$ and $\mathcal{M}_{\Gamma(N)/\mathbf{Z}[\zeta_N]}$, so these are fine moduli schemes for Drinfeld structures on generalized elliptic curves.

Before we prove Theorem 4.2.1, we must record a lemma concerning contraction maps to 'lower level'. Let M be a positive integer, and let n be a squarefree integer. Assume $M \geqslant 5$ (respectively, $M \geqslant 3$), and let (E, ι) be a generalized elliptic curve with $\Gamma_1(M;n)$-structure (respectively, $\Gamma(M)$-structure) over an algebraically closed field k with characteristic not dividing M. The pair (E, ι) has no non-trivial automorphisms; this is well known for the cases of $\Gamma_1(M)$ and $\Gamma(M)$, and the case of $\Gamma_1(M;n)$ is reduced to that of $\Gamma_1(N)$ by the final part of the following lemma.

Lemma 4.2.3. *Let $N \geqslant 1$ be an integer and let n be a squarefree integer. Each of the following 'contraction maps' is finite, flat, and surjective with constant rank:*

(1) $\mathcal{M}_{\Gamma(N)} \to \mathcal{M}_{\Gamma_1(N)}$ *defined by $(E; P, Q) \rightsquigarrow (c(E), P)$, with $c(E)$ the contraction away from $\langle P \rangle$;*

(2) $\mathcal{M}_{\Gamma(N)} \to \mathcal{M}_{\Gamma(d)}$ *and* $\mathcal{M}_{\Gamma_1(N)} \to \mathcal{M}_{\Gamma_1(d)}$ *for $d \mid N$, via contraction away from 'standard d-torsion subgroups';*

(3) $\mathcal{M}_{\Gamma_1(N;n)} \to \mathcal{M}_{\Gamma_1(N)}$ *defined by $(E; P, C) \rightsquigarrow (c(E), P)$, with $c(E)$ the contraction away from $\langle P \rangle$.**

Proof. Let $c : \mathcal{M} \to \mathcal{M}'$ denote any of the contraction maps under consideration. This morphism is compatible with the contraction map from each side to $\mathcal{M}_1$. These latter maps to $\mathcal{M}_1$ are relatively representable, finite, flat, and surjective by Theorem 4.1.1. Thus, c must be finite in such cases. As both the source and target of c are $\mathbf{Z}$-flat Deligne–Mumford stacks with pure relative dimension 1, with target regular and source CM, flatness of c is automatic by Lemma 4.1.3. To see that the finite flat c has constant (positive) rank, we reduce to the trivial case of working over the Zariski-dense open locus in $\mathcal{M}^0_{\Gamma}$ where the 'level' is invertible. $\square$

In [**O**] the theorem on formal functions and coherence for higher direct images are proved for proper morphisms of Artin stacks. Hence, the Zariski connectedness theorem holds for Artin stacks: Stein factorizations of proper maps between locally noetherian Artin stacks have geometrically connected fibers. The map $f : \mathcal{M} \to S$ as in Theorem 4.2.1 is proper and flat with smooth generic fiber, so it follows from connectedness

* Please see 'Note added in proof', which precedes the reference list.

and normality of S that f is its own Stein factorization, and hence has geometrically connected fibers, if the geometric generic fiber of f is connected. Since $\mathcal{M}^\infty$ is a relative effective Cartier divisor in $\mathcal{M}$, and $\mathcal{M}$ has fibers of pure dimension 1 over $\operatorname{Spec} \mathbf{Z}$, the geometric connectivity of a fiber is a consequence of geometric connectivity for the complement of $\mathcal{M}^\infty$ in that fiber. The connectedness of the complex fiber of $\mathcal{M}^0$ follows (upon adjoining a little level) from comparison with the complex-analytic theory. This completes the proof that $\mathcal{M} \to S$ has geometrically connected fibers. This settles Theorem 4.2.1 (1).

To see the existence of the open subscheme $\mathcal{M}^{\mathrm{sch}}$ of geometric points in $\mathcal{M} = \mathcal{M}_\Gamma$ with trivial automorphism group, we just need to prove its existence as a separated open algebraic subspace and then exhibit $\mathcal{M}$ as quasi-finite over a separated scheme. The existence as an algebraic space follows from Theorem 2.2.5 since the automorphism functors of geometric points are étale and $\mathcal{M}$ is separated. By Lemma 4.2.3, this algebraic space contains $\mathcal{M}_{/S[1/d]}$ for any factor d of N as in the statement of Theorem 4.2.1. To exhibit $\mathcal{M}$ as quasi-finite over a separated scheme, it suffices to treat $\mathcal{M}_1$. Since $\mathcal{M}_1$ is smooth over $\mathbf{Z}$ with geometrically connected fibers of dimension 1, it suffices to construct a map from $\mathcal{M}_1$ to a separated and finite type $\mathbf{Z}$-scheme such that the map is non-constant on geometric fibers over $\mathbf{Z}$; one such map is the standard map $j : \mathcal{M}_1 \to \mathbf{P}^1_{\mathbf{Z}}$ defined by the generating sections Δ and $c_4^{\otimes 3}$ of $\omega_{\mathcal{E}/\mathcal{M}_1}^{\otimes 12}$, where $\mathcal{E} \to \mathcal{M}_1$ is the universal generalized elliptic curve and $\omega_{\mathcal{E}/\mathcal{M}_1}$ on $\mathcal{M}_1$ is the pushforward of its relative dualizing sheaf.

We now suppose that there exists $d \mid N$ such that $d \geqslant 5$ when $\Gamma = \Gamma_1(N; n)$ (with squarefree n), and $d \geqslant 3$ when $\Gamma = \Gamma(N)$. It remains to check that $\mathcal{M}^\infty_{/S[1/d]} \hookrightarrow \mathcal{M}_{/S[1/d]}$ is relatively ample over $S[1/d]$. It is enough to check on geometric fibers over $S[1/d]$ [**EGA**, IV$_3$, 9.6.4], so it suffices to prove that the divisor $\mathcal{M}^\infty$ meets all irreducible components of geometric fibers of $\mathcal{M}$ over $S[1/d]$ and does not lie on any intersection of such components. Since $\mathcal{M} \to \mathcal{M}_{1/S}$ is relatively representable, finite, flat, and surjective, for each geometric point $\overline{s}$ of S we see that every irreducible component of $\mathcal{M}_{\overline{s}}$ maps surjectively onto $(\mathcal{M}_1)_{\overline{s}}$. The fiber of the surjection $\mathcal{M}_{\overline{s}} \to (\mathcal{M}_1)_{\overline{s}}$ over the single geometric point of $(\mathcal{M}_1^\infty)_{\overline{s}}$ is the 'closed subset' underlying $\mathcal{M}^\infty_{\overline{s}}$. It therefore suffices to check that, on geometric fibers over S, the intersection of any two fibral geometric irreducible components of $\mathcal{M}$ lies in $\mathcal{M}^0$.

Consider the finite flat contraction map

$$\mathcal{M}_{\Gamma_1(N;n)} \to \mathcal{M}_{\Gamma_1(N)}$$

for squarefree n. Both sides are proper and flat with pure relative dimension 1 over $\mathbf{Z}$. Since the contraction map is finite, for the cases of $\Gamma_1(M)$-structures and $\Gamma_1(M;n)$-structures we reduce to showing that if $N \geqslant 1$ and k is an algebraically closed field then the divisor $\mathcal{M}^\infty_{\Gamma_1(N)/k} \hookrightarrow \mathcal{M}_{\Gamma_1(N)/k}$ does not meet intersections of distinct irreducible components of $\mathcal{M}_{\Gamma_1(N)/k}$. A similar argument applies to $\Gamma(M)$-structures. Thus, by using the finite flat map $\mathcal{M}_{\Gamma(M)} \to \mathcal{M}_{\Gamma_1(M)}$ and the isomorphism

$$\mathcal{M}_{\Gamma(M)} \otimes_{\mathbf{Z}} k \simeq \mathcal{M}_{\Gamma(M)} \otimes_{\mathbf{Z}[\zeta_M]} k[\zeta_M]$$

(with $k[\zeta_M] \stackrel{\mathrm{def}}{=} k \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_M]$) it remains to prove the following lemma.

Lemma 4.2.4. *The closed substack $\mathcal{M}^{\infty}_{\Gamma(N)} \hookrightarrow \mathcal{M}_{\Gamma(N)}$ lies in the smooth locus of $\mathcal{M}_{\Gamma(N)} \to \operatorname{Spec}(\mathbf{Z}[\zeta_N])$ for arbitrary $N \geqslant 1$.*

Proof. Recall from the proof of Theorem 4.1.1 that (for $N \geqslant 1$) there is a finite disjoint union decomposition

$$\mathcal{M}_{\Gamma(N)} \times_{\mathcal{M}_1} \operatorname{Spec} \mathbf{Z}[\![q]\!] \simeq \coprod \operatorname{Spec}(\mathbf{Z}[\zeta_N][\![q^{1/N}]\!]) \tag{4.2.1}$$

as $\mathbf{Z}[\![q]\!]$-schemes. We claim that this isomorphism is compatible with the $\mathbf{Z}[\zeta_N]$-structure on both sides (using Theorem 4.1.1 for the $\mathbf{Z}[\zeta_N]$-structure on the left side). To see this compatibility, by flatness it suffices to check such compatibility after inverting q. Over $\mathbf{Z}(\!(q)\!)$, the isomorphism (4.2.1) becomes (by construction) exactly the isomorphism constructed in [**KM**, 10.8.2]. This isomorphism is *defined* via the Weil pairing on the smooth Tate curve over $\mathbf{Z}(\!(q)\!)$, thereby yielding the desired $\mathbf{Z}[\zeta_N]$-compatibility.

We will use (4.2.1) to establish formal smoothness of universal deformation rings at the cusps. Let $k = \overline{\mathbf{Q}}$ or $\overline{\mathbf{F}}_p$, and let W be the corresponding Cohen ring. We want to study the universal deformation rings of $\mathcal{M}_{\Gamma(N)}$ at a k-point of $\mathcal{M}^{\infty}_{\Gamma(N)}$ (with arbitrary $N \geqslant 1$). To prove that these deformation rings are formally smooth over $W[\zeta_N]$, by (4.2.1) it suffices to show that for each local factor ring A of the finite flat $W[\![q]\!]$-scheme $\mathcal{M}_{\Gamma(N)} \times_{\mathcal{M}_1} \operatorname{Spec} W[\![q]\!]$, the morphism $\operatorname{Spec}(A) \to \mathcal{M}_{\Gamma(N)}$ corresponds to a uniquely algebraized universal deformation of the induced 'geometric point' $\operatorname{Spec}(k) \to \mathcal{M}_{\Gamma(N)}$. Since the morphism $\operatorname{Spec}(W[\![q]\!]) \to \mathcal{M}_1$ is a uniquely algebraized universal deformation of the induced geometric point (by Lemma 3.3.5), we are done because adic completion is compatible with passage to finite algebras over noetherian rings. □

4.3. Formal structure along cusps

We take the level structure Γ to be $\Gamma(N)$ or $\Gamma_1(N;n)$ as usual. We wish to describe the moduli stacks formally near the cusps, at least for $\Gamma_1(N) = \Gamma_1(N;1)$. The relative effective Cartier divisor $\mathcal{M}^{\infty}_{\Gamma}$ is proper, quasi-finite, and flat over $\mathbf{Z}$. By Lemma 2.1.10 and Theorem 2.1.12, we may make the following definition.

Definition 4.3.1. For any positive integer d, $\mathcal{M}^{\infty}_{\Gamma,d}$ is the open and closed substack in $\mathcal{M}^{\infty}_{\Gamma}$ over which the universal generalized elliptic curve has d-gon geometric fibers (and hence is *fppf*-locally isomorphic to the standard d-gon).

Lemma 4.3.2. *The relative Cartier divisor $\mathcal{M}^{\infty}_{\Gamma,d}$ in $\mathcal{M}_{\Gamma}$ is reduced.*

Proof. Since all stacks being considered are flat over $\mathbf{Z}$, it suffices to work with the fiber over the generic point $\operatorname{Spec}(\mathbf{Q})$, or even over a geometric generic point. For any local artin ring B with algebraically closed residue field of characteristic 0, $B^{\times}$ is a divisible group. Thus, by Lemma 2.1.10, any generalized elliptic curve over B that is *fppf*-locally isomorphic to a standard d-gon is in fact isomorphic to a standard d-gon. The automorphism scheme of the standard d-gon over $\mathbf{Q}$ is étale, so we conclude from considerations with universal deformation rings that the fiber of $\mathcal{M}^{\infty}_{\Gamma,d}$ over $\operatorname{Spec}(\mathbf{Q})$ is étale. Thus, $\mathcal{M}^{\infty}_{\Gamma,d}$ is reduced. □

In what follows we work out the case $\Gamma = \Gamma_1(N)$, and the other cases can be done by similar arguments. (For $\Gamma = \Gamma_1(N;n)$ it is necessary to avoid geometric points $(E;P,C)$ in positive characteristic p when E is non-smooth and C has p-part that is neither étale nor connected, cases that can occur if and only if $p^2 \mid n$.)

For arbitrary $N \geqslant 1$ and $d \mid N$, each geometric point of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ has trivial automorphism functor if $d > 2$ (as we noted near the end of the proof of Lemma 3.1.8), so by Theorem 4.2.1 the closed substack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ lies inside of the maximal open subscheme $\mathcal{M}^{\mathrm{sch}}_{\Gamma_1(N)} \subseteq \mathcal{M}_{\Gamma_1(N)}$ (over $\mathbf{Z}$!) if $d > 2$. Thus, it makes sense to contemplate the formal completion of $\mathcal{M}_{\Gamma_1(N)}$ along $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ if $d > 2$. We will directly compute this formal completion over $\mathbf{Z}$. The case $d \leqslant 2$ is more subtle. For $N \geqslant 1$, it is straightforward to check (via study of automorphism functors of geometric points) that $\mathcal{M}^{\infty}_{\Gamma_1(N),1}$ is not a scheme in characteristic $p > 0$ if and only if $N \in \{p^s, 2p^s\}$ (with $s \geqslant 0$). Likewise, $\mathcal{M}^{\infty}_{\Gamma_1(N),2}$ is not a scheme in characteristic $p > 0$ if and only if $N = 2p^s$ (with $s \geqslant 0$).

Definition 4.3.3. We refer to the cases

$$(N,d) \in \{(1,1),(2,1),(2,2),(4,2)\}$$

(i.e. $d \leqslant 2$ and $N/d \leqslant 2$) as the *nowhere-schematic cases* and

$$(N,d) \in (\{(p^s,1) \mid s > 0\} \cup \{(2p^s,1),(2p^s,2) \mid s > 0\}) - \{(1,1),(2,1),(2,2),(4,2)\}$$

as the *partially schematic cases* (and define $\delta_{N,d} = p$ in these latter cases). All other cases are called *schematic* (with $\delta_{N,d} \overset{\mathrm{def}}{=} 1$). We do not define $\delta_{N,d}$ in the nowhere-schematic cases.

In all schematic cases $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is a scheme, and in the partially schematic cases $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is a scheme when restricted over $\mathbf{Z}[\delta_{N,d}^{-1}]$. In the nowhere-schematic cases, $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is nowhere a scheme.

Theorem 4.3.4. *Choose $d \mid N$. The stack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is regular of pure relative dimension 0 over $\operatorname{Spec}\mathbf{Z}$, and it is proper, quasi-finite, and flat over $\mathbf{Z}$.*

If $d > 2$ then the scheme $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is a finite non-empty disjoint union of copies of $\operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}])$, indexed by pairs (b,r) where $b \in (\mathbf{Z}/d\mathbf{Z})^{\times}$ and r is a positive divisor of d such that r reduces to a unit in $\mathbf{Z}/(N/d)\mathbf{Z}$.

Suppose that $d \leqslant 2$. If we are in one of the schematic cases then

$$\mathcal{M}^{\infty}_{\Gamma_1(N),d} \simeq \operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}]^{+}),$$

where $\mathbf{Z}[\zeta_M]^{+}$ denotes the ring of integers of the maximal totally real subfield of $\mathbf{Q}(\zeta_M)$. In general there is a relatively representable degree-2 finite flat covering

$$\operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}]) \to \mathcal{M}^{\infty}_{\Gamma_1(N),d}. \tag{4.3.1}$$

In the nowhere-schematic cases this map is finite étale if $d = 1$ and it is finite étale after inverting 2 if $d = 2$. In the partially schematic cases, the restriction of (4.3.1) over $\mathbf{Z}[\delta_{N,d}^{-1}]$ is the degree-2 finite étale covering

$$\operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}][\delta_{N,d}^{-1}]) \to \operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}]^{+}[\delta_{N,d}^{-1}]).$$

Proof. The Deligne–Mumford stack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ over $\mathbf{Q}$ is a scheme if and only if $d > 2$ or $N/d > 2$. We shall begin by computing the residue fields at all (possibly stacky) points on this stack. Any $\Gamma_1(N)$-structure on the standard d-gon C_d over $\overline{\mathbf{Q}}$ can clearly be defined over $\mathbf{Q}(\zeta_N)$. Thus, all residue fields of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ over $\mathbf{Q}$ lie inside of $\mathbf{Q}(\zeta_N)$.

First we treat $d \leqslant 2$. By using the embeddings $\mathbf{Q}(\zeta_N) \hookrightarrow \overline{\mathbf{Q}}$, the $\Gamma_1(N)$-structure $(C_1, \zeta_N)_{/\mathbf{Q}(\zeta_N)}$ induces all $\Gamma_1(N)$-structures on C_1 over $\overline{\mathbf{Q}}$. Moreover, two such distinct embeddings yield isomorphic structures if and only if they are related by complex conjugation on $\mathbf{Q}(\zeta_N)$ since $\mathrm{Aut}(C_1) = \langle \mathrm{inv} \rangle$. Thus, $\mathcal{M}^{\infty}_{\Gamma_1(N),1}$ over $\mathbf{Q}$ consists of a single (possibly stacky) point with residue field given by $\mathbf{Q}(\zeta_N)^+$. When $d = 2$, the $\Gamma_1(N)$-structure $(C_2, (\zeta_N, 1))_{/\mathbf{Q}(\zeta_N)}$ induces all $\Gamma_1(N)$-structures on C_2 over $\overline{\mathbf{Q}}$, up to isomorphism (note that $(C_2, (\zeta_N, 1))$ and $(C_2, (-\zeta_N, 1))$ are isomorphic to each other). To compute the residue field, we need to determine which powers ζ_N^a for $a \in (\mathbf{Z}/N\mathbf{Z})^\times$ lie in $\{\pm\zeta_N, \pm\zeta_N^{-1}\}$. If $N/2$ is odd then $-\zeta_N$ is not a primitive Nth root of unity, so the residue field is $\mathbf{Q}(\zeta_N)^+ = \mathbf{Q}(\zeta_{N/2})^+$ for such N. If $N/2$ is even then $-\zeta_N = \zeta_N^{1+N/2}$ is a primitive Nth root of unity, so the residue field is again $\mathbf{Q}(\zeta_{N/2})^+$.

Now suppose $d > 2$. The geometric possibilities are $(C_d, (\zeta_N^r, b))$ where $b \in (\mathbf{Z}/d\mathbf{Z})^\times$, $r \in \mathbf{Z}/N\mathbf{Z}$ maps to a unit in $\mathbf{Z}/(N/d)\mathbf{Z}$, and $\zeta_N \in \overline{\mathbf{Q}}$ is a fixed choice of primitive Nth root of unity. Using automorphisms of $\overline{\mathbf{Q}}$, we may scale r by $(\mathbf{Z}/N\mathbf{Z})^\times$ so as to reduce to considering the list of geometric possibilities with $r \mid N$, $r > 0$. Since r is a unit modulo N/d, the pair (b, r) is as in the statement of the theorem. Since $b \in (\mathbf{Z}/d\mathbf{Z})^\times$ and $d > 2$, clearly b and $-b$ are distinct in $\mathbf{Z}/d\mathbf{Z}$. Thus, the possibilities isomorphic to $(C_d, (\zeta_N^r, b))$ are those of the form $(C_d, (\zeta_N^{ra}, b))$ where $a \in (\mathbf{Z}/N\mathbf{Z})^\times$ and $\zeta_N^{ra-r} \in \mu_d(\overline{\mathbf{Q}})$. That is, we require $(N/d) \mid r(a-1)$. Since r is relatively prime to N/d, we get exactly the condition $a \equiv 1 \bmod N/d$. This subgroup of $(\mathbf{Z}/N\mathbf{Z})^\times$ has fixed field $\mathbf{Q}(\zeta_{N/d})$ inside of $\mathbf{Q}(\zeta_N)$. This determines all residue fields at points of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ over $\mathbf{Q}$, so by Lemma 4.3.2 this settles the situation over $\mathrm{Spec}(\mathbf{Q})$ (except that in the nowhere-schematic cases we have only determined the residue field at the unique stacky point).

In order to work over $\mathbf{Z}$, recall that the stack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is *a priori* proper, quasi-finite, and flat over $\mathbf{Z}$ since it is a relative Cartier divisor inside of $\mathcal{M}_{\Gamma_1(N)}$. Also, the preceding analysis when $d \leqslant 2$ yields canonical (schematic) morphisms $\mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}]) \to \mathcal{M}^{\infty}_{\Gamma_1(N),d}$ that are automatically finite (equivalently proper, quasi-finite, and representable) surjections. In view of our calculations over $\mathbf{Q}$, it therefore remains to prove two things:

(i) the stack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is regular for any $d \mid N$ (so (4.3.1) is finite flat, with degree checked to be 2 by working over $\overline{\mathbf{Q}}$), and

(ii) in the nowhere-schematic cases the map (4.3.1) is étale after inverting d.

The étale property over $\mathbf{Z}[\frac{1}{2}]$ for $d = 2$ is obvious, since any degree-2 covering is étale away from residue characteristic 2. The étale property over $\mathbf{Z}$ for $d = 1$ (and $N \leqslant 2$) follows from the fact that the map (4.3.1) for such (N, d) is an *fppf*-torsor for $\underline{\mathrm{Aut}}(C_1) = \mathbf{Z}/2\mathbf{Z}$ (as C_1 admits a unique $\Gamma_1(N)$-structure over any base scheme when $N \leqslant 2$).

The regularity of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ for all $d \mid N$ will follow in general if we can find a regular scheme that is faithfully flat over the complete local ring of the Deligne–Mumford stack $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ at each $\overline{\mathbf{F}}_p$-point x for any prime p. We shall now construct such regular covers.

Let $W = W(\overline{\mathbf{F}}_p)$ and let $\mathcal{M}_d$ denote the disjoint union of the formal spectra of the complete local rings of $\mathcal{M}_{\Gamma_1(N)}$ at each of the finitely many $\overline{\mathbf{F}}_p$-points of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$. Since $\mathcal{M}_{\Gamma_1(N)}$ is regular, the affine adic algebra A_d of $\mathcal{M}_d$ is regular. By Lemma 2.3.1 and the description of the $\overline{\mathbf{Q}}$-points of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$, we can clearly define a finite set of $\Gamma_1(N)$-structures on $\underline{\mathrm{Tate}}_d$ over $W[\zeta_N][\![q^{1/d}]\!]$ that induce all characteristic-0 geometric points of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$, where

$$W[\zeta_N] \stackrel{\mathrm{def}}{=} W \otimes_{\mathbf{Z}} \mathbf{Z}[\zeta_N].$$

Consider the resulting map of finite $W[\![q]\!]$-algebras

$$\coprod \mathrm{Spec}(W[\zeta_N][\![q^{1/d}]\!]) \to \mathrm{Spec}(A_d). \tag{4.3.2}$$

Both sides admit a finite étale cover that is finite flat over $W[\![q]\!]$ (use Lemma 3.3.5 to see this for A_d). It therefore follows from Lemma 4.1.3 that the finite map (4.3.2) must be *flat*. In fact, (4.3.2) is even faithfully flat. Indeed, since this map is finite flat, hence open and closed, to see the surjectivity it is enough to work modulo q and to pass to the geometric generic fiber of the resulting finite flat W-schemes. Working modulo nilpotents, we verify such surjectivity on the geometric generic fibers over $\mathrm{Spec}\, W$ by noting that the Tate-curve structures in the definition of (4.3.2) were chosen to lift all $\Gamma_1(N)$-structures on the standard d-gon over $\overline{\mathbf{Q}}$. This shows that (4.3.2) is faithfully flat.

By Lemma 4.3.2 and the fact that completion preserves reducedness for excellent schemes, the ideal of topological nilpotents in A_d cuts out the locus of non-smoothness for the universal family over $\mathrm{Spec}(A_d)$. By Theorem 2.1.12, this ideal in A_d has pullback under (4.3.2) that is equal to $(q^{1/d})$ because $\underline{\mathrm{Tate}}_d \to \mathrm{Spec}(\mathbf{Z}[\![q^{1/d}]\!])$ has locus of non-smoothness cut out by $(q^{1/d})$. We conclude via (4.3.2) that each of the complete local rings of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ at an $\overline{\mathbf{F}}_p$-point has a faithfully flat (finite) covering by the regular scheme $\mathrm{Spec}(W[\zeta_N])$. □

Remark 4.3.5. For the reader who is interested in § 4.4 and § 4.5, we remark that the remainder of the present section can be skipped. The only aspect of Theorems 4.3.6 and 4.3.7 that is required in § 4.4 and § 4.5 is the result proved above that universal deformation rings at geometric points of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ admit finite flat covers of the form $W'[\![q^{1/d}]\!]$, with W' a cyclotomic extension of a Cohen ring W, such that the universal $\Gamma_1(N)$-structure over $W'[\![q^{1/d}]\!]$ is $(\underline{\mathrm{Tate}}_d(\zeta q^{1/d}), q^{b/d})$ with $b \in (\mathbf{Z}/d\mathbf{Z})^{\times}$ and $\zeta \in W'^{\times}$ some Nth root of unity.

For a divisor d of N and a unit $r \in (\mathbf{Z}/(N/d)\mathbf{Z})^{\times}$, let

$$\underline{\mathrm{Tate}}_1(\zeta_{N/d}^{-r} q) \to \mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]) \tag{4.3.3}$$

denote the base change of $\underline{\mathrm{Tate}}_1 \to \mathrm{Spec}(\mathbf{Z}[\![q]\!])$ by the continuous map $\mathbf{Z}[\![q]\!] \to \mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]$ determined by the condition $q \mapsto \zeta_{N/d}^{-r} q$. For $d > 1$, the generalized elliptic curve (4.3.3) is not regular and does not have d-gon geometric fibers over $q^{1/d} = 0$. Its 'resolution of singularities' leads to the right generalized elliptic curves to consider for deformations of a d-gon. To be precise, we have the following theorem.

Theorem 4.3.6. *For $N \geqslant 1$, $d \mid N$, and $r \in (\mathbf{Z}/(N/d)\mathbf{Z})^{\times}$, there is a unique generalized elliptic curve*

$$\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/d}^{-r}q) \to \mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]) \tag{4.3.4}$$

that has d-gon geometric fibers over $q^{1/d}=0$ and is isomorphic to

$$\underline{\mathrm{Tate}}_1(\zeta_{N/d}^{-r}q)_{/\mathbf{Z}[\zeta_{N/d}]((q^{1/d}))}$$

over $q^{1/d} \neq 0$. Moreover, its locus of non-smoothness is $(q^{1/d})$ and its base change to $\mathrm{Spec}(\mathbf{Z}[\zeta_N][\![q^{1/d}]\!])$ via $\zeta_{N/d} \mapsto \zeta_N^d$ is isomorphic to

$$\underline{\mathrm{Tate}}_{d/\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]}(\zeta_N^{-r'}q^{1/d}) \tag{4.3.5}$$

for any $r' \in \mathbf{Z}/N\mathbf{Z}$ lifting r.

For $b' \in (\mathbf{Z}/N\mathbf{Z})^{\times}$ lifting $b \in (\mathbf{Z}/d\mathbf{Z})^{\times}$, the $\Gamma_1(N)$-structure '$q^{b/d}$' on

$$\underline{\mathrm{Tate}}_{d/\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]}(\zeta_N^{-r'b'^{-1}}q^{1/d})$$

arising from the canonical splitting (2.5.2) on d-torsion uniquely descends to a $\Gamma_1(N)$-structure (also denoted '$q^{b/d}$') on the generalized elliptic curve $\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/d}^{-rb'^{-1}}q)$.

If $d>2$ and we replace r with a $(\mathbf{Z}/(N/d)\mathbf{Z})^{\times}$-multiple so that r admits a (necessarily unique) lift $r' \mid d$ with $r'>0$, then $\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/d}^{-rb'^{-1}}q)$ equipped with the $\Gamma_1(N)$-structure '$q^{b/d}$' induces a map

$$\mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]) \to \mathcal{M}_{\Gamma_1(N)}$$

that lands inside the maximal open subscheme $\mathcal{M}_{\Gamma_1(N)}^{\mathrm{sch}}$ and identifies $\mathrm{Spf}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!])$ with the (b,r')-component of the formal completion of $\mathcal{M}_{\Gamma_1(N)}$ along $\mathcal{M}_{\Gamma_1(N),d}^{\infty}$. In particular, this computes the algebraization of the formal completion along the (b,r')-component of $\mathcal{M}_{\Gamma_1(N),d}^{\infty}$ for $d>2$.

Corollary 2.3.3 ensures that '$q^{b/d}$' does indeed define a $\Gamma_1(N)$-structure in the setup over $\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]$ prior to descent.

Proof. The uniqueness of (4.3.4) follows from the final part of Theorem 2.5.2. We shall prove existence of (4.3.4) by descent from the faithfully flat cover $\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]$. Over this larger base, the change of variable $q^{1/d} \mapsto \zeta_N^{r'b'^{-1}}q^{1/d}$ transforms the existence question into the setting of Theorem 2.5.2 with the Dedekind domain $A=\mathbf{Z}[\zeta_N]$, which is to say that our problem becomes one of faithfully flat descent of the $\Gamma_1(N)$-structure $(\underline{\mathrm{Tate}}_d(\zeta_N^{-r'b'^{-1}}q^{1/d}), q^{b/d})$ with respect to the finite faithfully flat covering

$$\mathrm{Spec}(\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]) \to \mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]). \tag{4.3.6}$$

By Theorem 2.5.2, this $\Gamma_1(N)$-structure restricts to $\underline{\mathrm{Tate}}_1(\zeta_{N/d}^{-rb'^{-1}}q)$ over $\mathbf{Z}[\zeta_N]((q^{1/d}))$ (endowed with the $\Gamma_1(N)$-structure '$q^{b/d}$'). Thus, there is an evident descent to $\mathbf{Z}[\zeta_{N/d}]((q^{1/d}))$, so we get *fppf* descent data with respect to the map (4.3.6) after inverting $q^{1/d}$. Since the typically non-normal ring $\mathbf{Z}[\zeta_N] \otimes_{\mathbf{Z}[\zeta_{N/d}]} \mathbf{Z}[\zeta_N]$ is *reduced*, Corollary 3.2.5 supplies the *fppf* descent data without inverting $q^{1/d}$. To prove effectivity of the descent of the underlying curve down to $\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]$, we use flatness over $\mathbf{Z}[\![q]\!]$ to check stability of the ample d-torsion under the descent data (by working over $\mathbf{Z}(\!(q)\!)$). An easy flatness argument shows that the generalized elliptic curve structure uniquely descends.

It remains to study the properties of the map

$$\operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]) \to \mathcal{M}_{\Gamma_1(N)} \tag{4.3.7}$$

induced by our descended $\Gamma_1(N)$-structure for $d > 2$. Since (4.3.4) has d-gon geometric fibers along $q^{1/d} = 0$, and the only open subscheme of $\operatorname{Spec}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!])$ that contains $q^{1/d} = 0$ is the entire scheme, it follows from Theorem 4.2.1 that if $d > 2$ then (4.3.7) does land inside of the maximal open subscheme. The locus of non-smoothness of (4.3.4) is the closed subscheme cut out by $(q^{1/d})$ because this is true after the faithfully flat base change to $\mathbf{Z}[\zeta_N][\![q^{1/d}]\!]$. Hence, (4.3.7) induces a map

$$\operatorname{Spf}(\mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]) \to \widehat{\mathcal{M}}_d \tag{4.3.8}$$

to the formal completion of $\mathcal{M}_{\Gamma_1(N)}$ along the closed subset $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ that lies in the open subscheme $\mathcal{M}^{\mathrm{sch}}_{\Gamma_1(N)}$. We need to prove that this map of formal schemes is an open and closed immersion (again, with $d > 2$). By Lemma 4.3.2 and Theorem 4.3.4, the ideal of topological nilpotents on $\widehat{\mathcal{M}}_d$ must pull back to the ideal $(q^{1/d})$, and this latter ideal cuts out the locus of non-smoothness for (4.3.4) over $\operatorname{Spec} \mathbf{Z}[\zeta_{N/d}][\![q^{1/d}]\!]$ because this may be checked after the scalar extension $\mathbf{Z}[\zeta_{N/d}] \to \mathbf{Z}[\zeta_N]$. Upon passing to ordinary closed subschemes cut out by these ideals on both sides of (4.3.8) we get an open and closed immersion (by Theorem 4.3.4), and hence the map (4.3.8) on formal schemes is a closed immersion that is a homeomorphism onto a connected component of the target. Since both source and target have affine adic algebras that are regular and equidimensional of the same dimension, we conclude that (4.3.8) is indeed an open and closed immersion. □

To handle $d \leqslant 2$ with $N/d > 2$ (the partially schematic cases), we need to find the correct analogues of (4.3.4) with the cyclotomic integer ring replaced by its totally real subring. In these cases, the 'universal family' is *not* a Tate curve with a twisted q-parameter as above, but rather is a quadratic twist of such a curve. We shall treat the cases $d = 1$ and $d = 2$ separately.

For $d = 1$ and $N/d > 2$, consider the generalized elliptic curve $\underline{\mathrm{Tate}}_1 \to \operatorname{Spec}(\mathbf{Z}[\zeta_N][\![q]\!])$ equipped with the $\Gamma_1(N)$-structure defined by $\zeta_N \in \mu_N(\mathbf{Z}[\zeta_N])$. Since the map

$$\operatorname{Spec}(\mathbf{Z}[\zeta_N][\delta_{N,1}^{-1}]) \to \operatorname{Spec}(\mathbf{Z}[\zeta_N]^+[\delta_{N,1}^{-1}])$$

is a degree-2 étale Galois covering with covering group determined by $\zeta_N \mapsto \zeta_N^{-1}$, the map

$$\operatorname{Spec}(\mathbf{Z}[\zeta_N][\![q]\!][\delta_{N,1}^{-1}]) \to \operatorname{Spec}(\mathbf{Z}[\zeta_N]^+[\![q]\!][\delta_{N,1}^{-1}]) \tag{4.3.9}$$

is also such a finite étale covering. Base change on $(\underline{\mathrm{Tate}}_1, \zeta_N)$ by $\zeta_N \mapsto \zeta_N^{-1}$ yields the negated $\Gamma_1(N)$-structure on the same curve $\underline{\mathrm{Tate}}_1$, so using 'negation' on generalized elliptic curves [**DR**, II, 2.8] yields étale descent data relative to (4.3.9). The descent is trivially effective, yielding a $\Gamma_1(N)$-structure on a quadratically twisted Tate curve that we denote

$$\underline{\mathrm{Tate}}'_1 \to \mathrm{Spec}(\mathbf{Z}[\zeta_N]^+[\![q]\!][\delta_{N,1}^{-1}]).$$

Now suppose $d = 2$ and $N/d > 2$. Rather than doing a quadratic descent on $\mathbf{Z}[\zeta_{N/2}][\![q^{1/2}]\!][\delta_{N,2}^{-1}]$ relative to the automorphism that fixes $q^{1/2}$ and satisfies $\zeta_{N/2} \mapsto \zeta_{N/2}^{-1}$, we want to do a quadratic descent on $\mathbf{Z}[\zeta_{N/2}, \delta_{N,2}^{-1}][\![q^{1/2}]\!]$ with respect to the automorphism determined by

$$\zeta_{N/2} \mapsto \zeta_{N/2}^{-1}, q^{1/2} \mapsto \zeta_{N/2}^{-1} q^{1/2}. \tag{4.3.10}$$

This automorphism fixes $\zeta_{N/2}^{-1} q$, and the subring of invariants is readily computed to be

$$\mathbf{Z}[\zeta_{N/2}]^+[\delta_{N,2}^{-1}][\![(1 + \zeta_{N/2}^{-1})q^{1/2}]\!]; \tag{4.3.11}$$

this calculation uses the fact that $1 + \zeta_{N/2}^{-1} \in \mathbf{Z}[\zeta_{N/2}, \delta_{N,2}^{-1}]^\times$ (recall $N/d > 2$, so $N \geqslant 5$). If we let

$$t = (1 + \zeta_{N/2}^{-1})q^{1/2},$$

then $(2 + \zeta_{N/2} + \zeta_{N/2}^{-1})^{-1} t^2 = \zeta_{N/2}^{-1} q$, where

$$2 + \zeta_{N/2} + \zeta_{N/2}^{-1} = \zeta_{N/2}(1 + \zeta_{N/2}^{-1})^2 \in \mathbf{Z}[\zeta_{N/2}]^+[\delta_{N,2}^{-1}]$$

is a unit. Hence, (4.3.11) is equal to $\mathbf{Z}[\zeta_{N/2}]^+[\delta_{N,2}^{-1}][\![q^{1/d}]\!]$.

Since the map

$$\begin{aligned}&\mathrm{Spec}(\mathbf{Z}[\zeta_{N/2}, \delta_{N,2}^{-1}][\![q^{1/2}]\!]) \\ &\quad = \mathrm{Spec}(\mathbf{Z}[\zeta_{N/2}, \delta_{N,2}^{-1}][\![(1 + \zeta_{N/2}^{-1})q^{1/2}]\!]) \to \mathrm{Spec}(\mathbf{Z}[\zeta_{N/2}]^+[\delta_{N,2}^{-1}][\![q^{1/2}]\!])\end{aligned} \tag{4.3.12}$$

is a degree-2 étale Galois covering, we can do descent relative to this covering. Consider the $\Gamma_1(N)$-structure on $\mathrm{Spec}(\mathbf{Z}[\zeta_{N/2}, \delta_{N,2}^{-1}][\![q^{1/2}]\!])$ defined by '$q^{1/2}$' on $\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/2}^{-1} q)$. Base change by the non-trivial automorphism of (4.3.12) takes this to the $\Gamma_1(N)$-structure '$\zeta_{N/2}^{-1} q^{1/2}$' on the same curve. This is inversion applied to the original $\Gamma_1(N)$-structure (to see this algebraically, note that making an *fppf* base change to $\mathrm{Spec}(\mathbf{Z}[\zeta_N, \delta_{N,2}^{-1}][\![q^{1/2}]\!])$ and a change of variable $q^{1/2} \mapsto \zeta_N q^{1/2}$ brings us to the visibly inverse $\Gamma_1(N)$-structures $\zeta_N q^{1/2}$ and $\zeta_N^{-1} q^{1/2}$ on $\underline{\mathrm{Tate}}_2$). Thus, we can again carry out a quadratic descent, now getting a $\Gamma_1(N)$-structure on a quadratically twisted descent $\underline{\mathrm{Tate}}'_2 \to \mathrm{Spec}(\mathbf{Z}[\zeta_{N/2}]^+[\delta_{N,2}^{-1}][\![q^{1/2}]\!])$ of $\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/2}^{-1} q)$.

Here is the analogue of Theorem 4.3.6 for $d \leqslant 2$; the first part is a variant on Lemma 3.3.5.

Theorem 4.3.7. *For any case with $d \leqslant 2$, $\underline{\mathrm{Tate}}^{\mathrm{reg}}(\zeta_{N/d}^{-1} q)$ with its $\Gamma_1(N)$-structure '$q^{1/d}$' induces compatible degree-2 finite flat coverings*

$$\mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}][q^{1/d}]/(q^{1/d})^{m+1}) \to \mathcal{M}^{\infty}_{\Gamma_1(N),d,m} \tag{4.3.13}$$

of the mth infinitesimal neighborhood of $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ for all $m \geqslant 0$, and if $N/d > 2$ then these are finite étale of degree 2 after inverting $\delta_{N,d}$.

For $d \leqslant 2$ and $N/d > 2$, $\underline{\mathrm{Tate}}'_d$ with its descended $\Gamma_1(N)$-structure induces a map

$$\mathrm{Spec}(\mathbf{Z}[\zeta_{N/d}]^+[\delta_{N,d}^{-1}][\![q^{1/d}]\!]) \to \mathcal{M}_{\Gamma_1(N)}$$

that lands inside of $\mathcal{M}^{\mathrm{sch}}_{\Gamma_1(N)}$. This map is an algebraization of the formal completion along $(\mathcal{M}^{\infty}_{\Gamma_1(N),d})_{/\mathbf{Z}[\delta_{N,d}^{-1}]}$ and it is compatible with (4.3.13) on infinitesimal neighborhoods.

Proof. The proof is essentially the same as that of Theorem 4.3.6, so we omit the details except to note that the nowhere-schematic cases ($N/d \leqslant 2$) follow from (4.3.1) and the fibral criterion for flatness, and the cases when $N/d > 2$ is neither a prime power nor twice a prime power (that is, $\delta_{N,d} = 1$) are treated separately from the cases when $N/d > 2$ is a prime power p^s or twice such a power for a prime p and an exponent $s > 0$ (that is, $\delta_{N,d} = p$). □

4.4. Integral structure on spaces of modular forms

Let $f : E \to S$ be a generalized elliptic curve and let $e \in E^{\mathrm{sm}}(S)$ denote its identity section. By [**DR**, II, 1.6], the pushforward $\omega_{E/S}$ of the relative dualizing sheaf is an invertible sheaf on S whose formation commutes with base change on S, and there is a canonical isomorphism $\omega_{E/S} \simeq e^*(\Omega^1_{E/S})$ since e is supported in the smooth locus (where $\Omega^1_{E/S}$ is the relative dualizing sheaf). If $D \subseteq E^{\mathrm{sm}}$ is a finite locally free subgroup and $c : E \to \overline{E}$ is the contraction away from D, then c induces an isomorphism near the respective identity sections e and $\overline{e}$. Hence, we get a canonical isomorphism $\omega_{E/S} \simeq \omega_{\overline{E}/S}$. Roughly speaking, the sheaf $\omega_{E/S}$ is insensitive to contraction of E away from a finite subgroup.

For $\Gamma \in \{\Gamma(N), \Gamma_1(N), \Gamma_0(N)\}$, let $\mathcal{E}_\Gamma \to \mathcal{M}_\Gamma$ be the universal generalized elliptic curve and let ω_Γ on $\mathcal{M}_\Gamma$ be the pushforward of the relative dualizing sheaf for $\mathcal{E}_\Gamma$ over $\mathcal{M}_\Gamma$. The $\mathbf{Z}$-module $\mathrm{H}^0(\mathcal{M}_\Gamma, \omega_\Gamma^{\otimes k})$ is finite and free since $\mathcal{M}_\Gamma$ is proper and flat over $\mathbf{Z}$. By Remark 4.1.5, $\mathcal{M}_{\Gamma(N)/\mathbf{Z}[\zeta_N]}$ agrees with the stack $\mathcal{M}_N$ over $\mathrm{Spec}\,\mathbf{Z}[\zeta_N]$ constructed by normalization over $\mathcal{M}_1$ in [**DR**, IV, § 3]. Thus, by [**DR**, VII, § 4], if we use the embedding $\mathbf{Z}[\zeta_N] \hookrightarrow \mathbf{C}$ defined by $\zeta_N \mapsto \mathrm{e}^{2\pi\mathrm{i}/N}$ for a choice of $\mathrm{i} = \sqrt{-1} \in \mathbf{C}$ then we get a canonical isomorphism

$$\mathbf{C} \otimes_{\mathbf{Z}[\zeta_N]} \mathrm{H}^0(\mathcal{M}_{\Gamma(N)}, \omega_{\Gamma(N)}^{\otimes k}) \simeq \mathrm{M}_k(\Gamma(N), \mathbf{C}) \tag{4.4.1}$$

to the space of classical modular forms with full level N and weight $k \geqslant 0$. By the same methods, there are canonical isomorphisms

$$\mathbf{C} \otimes_{\mathbf{Z}} \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega_{\Gamma_1(N)}^{\otimes k}) \simeq \mathrm{M}_k(\Gamma_1(N), \mathbf{C}), \mathbf{C} \otimes_{\mathbf{Z}} \mathrm{H}^0(\mathcal{M}_{\Gamma_0(N)}, \omega_{\Gamma_0(N)}^{\otimes k}) \simeq \mathrm{M}_k(\Gamma_0(N), \mathbf{C}). \tag{4.4.2}$$

These $\mathbf{Z}$-structures are generally *not* the same as those defined via integrality for q-expansions at a single cusp, but they are compatible with the $\mathbf{Z}$-structure (4.4.1) because the canonical maps $\mathcal{M}_{\Gamma(N)} \to \mathcal{M}_{\Gamma_1(N)}$ and $\mathcal{M}_{\Gamma_1(N)} \to \mathcal{M}_{\Gamma_0(N)}$ from Lemma 4.2.3 are finite flat and are pullback-compatible with the line bundles ω on source and target stacks.

Let $\mathrm{M}_{k,\mathbf{Z}} \subseteq \mathrm{M}_k(\Gamma_1(N), \mathbf{C})$ be the image of $\mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega_{\Gamma_1(N)}^{\otimes k})$ under (4.4.2). For the finite flat covering $\pi : \mathcal{M}_{\Gamma_1(N)} \to \mathcal{M}_{\Gamma_0(N)}$ defined by $(E, P) \rightsquigarrow (E, \langle P \rangle)$, the pullback of $\mathcal{E}_{\Gamma_0(N)}$ is naturally identified with $\mathcal{E}_{\Gamma_1(N)}$. Using the canonical isomorphism $\pi^*\omega_{\Gamma_0(N)} \simeq \omega_{\Gamma_1(N)}$, the resulting injection

$$\mathrm{H}^0(\mathcal{M}_{\Gamma_0(N)}, \omega_{\Gamma_0(N)}^{\otimes k}) \to \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega_{\Gamma_1(N)}^{\otimes k})$$

via equation (4.4.2) is compatible with the standard injection of $\mathrm{M}_k(\Gamma_0(N), \mathbf{C})$ into $\mathrm{M}_k(\Gamma_1(N), \mathbf{C})$. There is an evident action of $(\mathbf{Z}/N\mathbf{Z})^\times$ on the universal $\Gamma_1(N)$-structure $\mathcal{E}_{\Gamma_1(N)} \to \mathcal{M}_{\Gamma_1(N)}$, and so there is an induced action of $(\mathbf{Z}/N\mathbf{Z})^\times$ on $\omega_{\Gamma_1(N)}$ covering the action on $\mathcal{M}_{\Gamma_1(N)}$. This is compatible with the diamond-operator action of $(\mathbf{Z}/N\mathbf{Z})^\times$ on $\mathrm{M}_k(\Gamma_1(N), \mathbf{C})$.

Lemma 4.4.1. *The submodule*

$$\mathrm{H}^0(\mathcal{M}_{\Gamma_0(N)}, \omega_{\Gamma_0(N)}^{\otimes k}) \subseteq \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega_{\Gamma_1(N)}^{\otimes k})$$

is the submodule of $(\mathbf{Z}/N\mathbf{Z})^\times$-invariants.

Proof. The isomorphism $\omega_{\Gamma_1(N)} \simeq \pi^*\omega_{\Gamma_0(N)}$ gives rise to an inclusion

$$\omega_{\Gamma_0(N)}^{\otimes k} \subseteq \pi_*(\omega_{\Gamma_1(N)}^{\otimes k}) \simeq \omega_{\Gamma_0(N)}^{\otimes k} \otimes \pi_*\mathcal{O}_{\mathcal{M}_{\Gamma_1(N)}},$$

so it suffices to prove that $\mathcal{O}_{\mathcal{M}_{\Gamma_0(N)}}$ is the subsheaf of $(\mathbf{Z}/N\mathbf{Z})^\times$-invariants in the coherent sheaf $\pi_*\mathcal{O}_{\mathcal{M}_{\Gamma_1(N)}}$. Since the $(\mathbf{Z}/N\mathbf{Z})^\times$-invariant morphism π is a finite flat covering of normal $\mathbf{Z}$-flat Artin stacks, it suffices to work over $\mathbf{Z}[1/N]$. The map $\pi_{\mathbf{Z}[1/N]}$ is a finite étale $(\mathbf{Z}/N\mathbf{Z})^\times$-torsor, so the conclusion is obvious. □

Remark 4.4.2. An element $f \in \mathrm{M}_k(\Gamma_1(N), \mathbf{C})$ lies in $\mathrm{M}_{k,\mathbf{Z}}$ if and only if the q-expansion of f at every cusp has coefficients that are algebraic integers and the q-expansions at the N-gon cusps have coefficients in $\mathbf{Z}$. We leave the proof as an exercise.

We now prove Theorem 1.2.2. Let us recall the statement.

Theorem 4.4.3. *The maps $\pi_1^0, \pi_2^0 : \mathcal{M}_{\Gamma_1(N;p)}^0 \rightrightarrows \mathcal{M}_{\Gamma_1(N)}^0$ defined by*

$$\pi_1^0(E; P, C) = (E, P), \qquad \pi_2^0(E; P, C) = (E/C, P \bmod C)$$

uniquely extend to finite flat morphisms $\pi_1, \pi_2 : \mathcal{M}_{\Gamma_1(N;p)} \rightrightarrows \mathcal{M}_{\Gamma_1(N)}$.

Likewise, the natural map

$$\xi^0 : (\pi_2^0)^*(\omega_{\Gamma_1(N)}|_{\mathcal{M}_{\Gamma_1(N)}^0}) \to \omega_{\Gamma_1(N;p)}|_{\mathcal{M}_{\Gamma_1(N;p)}^0}$$

defined via pullback along the universal p-isogeny uniquely extends to a map $\xi : \pi_2^*\omega_{\Gamma_1(N)} \to \omega_{\Gamma_1(N;p)}$ on $\mathcal{M}_{\Gamma_1(N;p)}$.

Remark 4.4.4. In the definition of π_2^0, note that $P \bmod C$ is a $\mathbf{Z}/N\mathbf{Z}$-structure on E/C by Theorem 2.3.2 (1) and Lemma 2.4.4.

Proof. The uniqueness of π_1 and π_2 follow from the finiteness of Isom-schemes for $\Gamma_1(N;p)$-structures (Theorem 3.2.2) and the normality of $\mathcal{M}_{\Gamma_1(N;p)}$, and the uniqueness of ξ follows from the fact that $\mathcal{M}^{\infty}_{\Gamma_1(N;p)}$ is a Cartier divisor in $\mathcal{M}_{\Gamma_1(N;p)}$. The problem is therefore one of existence. By Lemma 4.2.3, there is a canonical finite flat morphism $\pi_1 : \mathcal{M}_{\Gamma_1(N;p)} \to \mathcal{M}_{\Gamma_1(N)}$ defined by $(E;P,C) \rightsquigarrow (c(E),P)$, where $c(E)$ denotes the contraction away from $\langle P\rangle$. This settles the problem for π_1.

Let $(\mathcal{E};\mathcal{P},\mathcal{C})$ be the universal $\Gamma_1(N;p)$-structure over $\mathcal{M}_{\Gamma_1(N;p)}$, and let $\mathcal{Z} \subseteq \mathcal{M}^{\infty}_{\Gamma_1(N;p)}$ be the open and closed substack whose geometric points $(E_0;P_0,C_0)$ have C_0 contained in the identity component of E_0^{sm} (that is, C_0 is the subgroup μ_p in the standard polygon E_0); to see that $\mathcal{Z}$ is open and closed we apply Lemma 2.1.10 to the contraction of $\mathcal{E}$ away from $\mathcal{C}$. Concretely, $\mathcal{Z}$ classifies the degenerate triples $(E;P,C)$ such that C has non-trivial intersection with the fibral identity components of E^{sm}. The quotient $\mathcal{E}/\mathcal{C}$ makes sense as a generalized elliptic curve away from $\mathcal{Z}$ (Example 2.1.6), and as in Remark 4.4.4 we have a $\Gamma_1(N)$-structure $(\mathcal{E}/\mathcal{C},\mathcal{P} \bmod \mathcal{C})$ away from $\mathcal{Z}$. This defines a morphism $\pi_2' : \mathcal{M}_{\Gamma_1(N;p)} - \mathcal{Z} \to \mathcal{M}_{\Gamma_1(N)}$ extending π_2^0, and using pullback along the universal degree-p 'isogeny' $\mathcal{E} \to \mathcal{E}/\mathcal{C}$ away from $\mathcal{Z}$ yields a pullback map $\xi' : (\pi_2')^*(\omega_{\Gamma_1(N)}) \to \omega_{\Gamma_1(N;p)}$ over $\mathcal{M}_{\Gamma_1(N;p)} - \mathcal{Z}$ that extends ξ^0. Our problem is to extend π_2' to a morphism $\pi_2 : \mathcal{M}_{\Gamma_1(N;p)} \to \mathcal{M}_{\Gamma_1(N)}$ and to extend ξ' to a map of line bundles $\xi : \pi_2^*\omega_{\Gamma_1(N)} \to \omega_{\Gamma_1(N;p)}$. We also need to prove that π_2 is finite and flat.

Constructing π_2 amounts to extending the $\Gamma_1(N)$-structure $(\mathcal{E}/\mathcal{C},\mathcal{P} \bmod \mathcal{C})$ over $\mathcal{M}_{\Gamma_1(N;p)}$. To construct this (unique) extension, we claim that it suffices to work on complete local rings of $\mathcal{M}_{\Gamma_1(N;p)}$ at geometric points of $\mathcal{Z}$. Such sufficiency follows by taking $B = \operatorname{Spec} \mathbf{Z}$, $\mathcal{S} = \mathcal{M}_{\Gamma_1(N;p)}$, and $\mathcal{M} = \mathcal{M}_{\Gamma_1(N)}$ in the following lemma.

Lemma 4.4.5. *Let B be a scheme, let $\mathcal{S}$ be a normal locally noetherian Deligne–Mumford stack over B whose strictly henselian local rings are G-rings, and let $\mathcal{M}$ be a Deligne–Mumford stack over B that is separated and locally of finite presentation. Let $\mathcal{S}^0 \subseteq \mathcal{S}$ be a dense open substack and let $f^0 : \mathcal{S}^0 \to \mathcal{M}$ be a morphism over B.*

If there exists a morphism $f : \mathcal{S} \to \mathcal{M}$ over B and an isomorphism $\alpha : f|_{\mathcal{S}^0} \simeq f^0$ then (f,α) is unique up to unique isomorphism, and such a pair exists if and only if such a pair exists over the completion of the strictly henselian local ring at each point of $\mathcal{S}$ outside of $\mathcal{S}^0$.

Remark 4.4.6. See the end of § 2.2, especially Theorem 2.2.8, for a discussion of the G-ring condition on Artin stacks; in particular, it is equivalent to require that the local rings on one (or every) smooth scheme covering is a G-ring.

Proof. To prove the uniqueness of (f,α) up to unique isomorphism we may work étale-locally on $\mathcal{S}$, so we can assume that $\mathcal{S} = S$ is a scheme. Let $S^0 = \mathcal{S}^0$, so S^0 is a Zariski-dense open in S. If (f,α) and (f',α') are two solutions to the extension problem then they correspond to a pair of objects x and x' in the fiber category $\mathcal{M}_S$ equipped with an isomorphism $\iota^0 = \alpha' \circ \alpha^{-1} : x|_{S^0} \simeq x'|_{S^0}$ in $\mathcal{M}_{S^0}$. Since $\mathcal{M}$ is a separated Deligne–Mumford stack, the functor $T \rightsquigarrow \mathrm{Isom}_{\mathcal{M}_T}(x_T, x'_T)$ on S-schemes is represented by a finite S-scheme. Hence, since S is normal and S^0 is a dense open subset, an S^0-point ι^0 of this functor uniquely extends to an S-point ι. This settles the uniqueness assertion for (f,α).

We now turn to the necessary and sufficient criterion for the existence of (f, α). Necessity is obvious. Since such a pair over a normal $\mathcal{S}$ is unique up to unique isomorphism when it exists, for the proof of sufficiency we may work locally for the étale topology on $\mathcal{S}$. In particular, we can assume that $\mathcal{S} = S$ is a scheme. Once again, we write S^0 to denote $\mathcal{S}^0$. Since $\mathcal{M}$ is locally of finite presentation over B, so the fiber categories of $\mathcal{M}$ are compatible with limits of affine schemes over B, by uniqueness and 'smearing out' principles at generic points of $S - S^0$ we may use noetherian induction to reduce to the case when $S = \operatorname{Spec} R$ is local and S^0 is the complement of the closed point. We may also replace R with its strict henselization R^{sh} that is a G-ring. Hence, we may assume that we have a solution over $\widehat{R}$ and we need to construct a solution over R when R is (strictly) henselian. Since R is a G-ring, so the flat morphism $R \to \widehat{R}$ is regular, Popescu's theorem [**S**] ensures that $\widehat{R}$ is a direct limit (with local transition maps) of a directed system of essentially smooth and residually trivial local R-algebras. Thus, our extension problem can be solved over a smooth R-algebra A with a rational point z in its closed fiber. There is an R-section through z because R is henselian and A is R-smooth, so pullback along this section gives a solution over R. □

The same technique shows that the problem of extending ξ (once we extend π_2) may also be reduced to a problem on complete local rings at geometric points. Thus, our problem is the following. Let k be an algebraically closed field with associated Cohen ring W, and let $(E_0; P_0, C_0)$ be a $\Gamma_1(N; p)$-structure with universal formal deformation ring A. Assume that E_0 is a standard d-gon and that C_0 is the p-torsion μ_p in the identity component of E_0^{sm} (so $\langle P_0 \rangle$ is ample, and hence $d \mid N$). Let $(E; P, C)$ be the algebraized universal deformation over $\operatorname{Spec} A$, and let I be the ideal corresponding to the locus of non-smoothness for E over A. By Theorem 4.1.1 (1) the ideal I is invertible, say $I = aA$. We want to prove that the $\Gamma_1(N)$-structure $(E/C, P \bmod C)$ over $A[1/a]$ extends to a $\Gamma_1(N)$-structure $(\overline{E}, \overline{P})$ over A and that a generator of $\omega_{\overline{E}/A}$ pulls back to a section of $\omega_{E/A}$ under the isogeny of elliptic curves $E|_{a \neq 0} \to \overline{E}|_{a \neq 0}$.

The condition on $\omega_{E/A}$ and $\omega_{\overline{E}/A}$ says that the induced $A[1/a]$-linear map between cotangent spaces at the origin over the locus $a \neq 0$ extends to an A-linear map of cotangent spaces at the origin over $\operatorname{Spec} A$, so this condition holds if the map on formal groups along the origin over $\operatorname{Spec} A[1/a]$ extends over $\operatorname{Spec} A$. Once we construct a generalized elliptic curve $\overline{E}$ over A such that $\overline{E}$ restricts to E/C over $A[1/a]$ and such that the map $E|_{a \neq 0} \to \overline{E}|_{a \neq 0}$ extends over A on the level of formal groups at the origin, the problem of constructing $\overline{P}$ as a $\Gamma_1(N)$-structure is equivalent to the problem of constructing a section $\overline{P} \in \overline{E}^{\mathrm{sm}}(A)$ that restricts to $P \bmod C$ over $A[1/a]$. Indeed, such an A-section $\overline{P}$ must be a $\mathbf{Z}/N\mathbf{Z}$-structure (as $\overline{E}$ is A-flat and $P \bmod C$ is a $\mathbf{Z}/N\mathbf{Z}$-structure over $A[1/a]$) and we may apply a contraction to force the subgroup $\langle \overline{P} \rangle$ to be relatively ample. (Note also that such a contraction has no impact on formal groups along the origin.)

By Corollary 3.2.3, it suffices to solve our problem after replacing A with a finite flat extension A' if $(A' \otimes_A A')/a(A' \otimes_A A')$ is reduced (reducedness ensures that the descent data over $(A' \otimes_A A')[1/a]$ extends over $A' \otimes_A A'$, thereby permitting us to canonically return to the initial base $\operatorname{Spec} A$). The special form of C_0 implies that (E_0, P_0) is a $\Gamma_1(N)$-structure whose infinitesimal deformation theory coincides with that of $(E_0; P_0, C_0)$.

By Remark 4.3.5, $\operatorname{Spec} A$ admits a finite flat covering $\operatorname{Spec} A'$ with $A' = W'[\![q^{1/d}]\!]$ for a cyclotomic extension W' of W, and over this covering (E, P) is identified with $(\underline{\mathrm{Tate}}_d(\zeta q^{1/d}), q^{b/d})$ for a suitable Nth root of unity $\zeta \in W'^{\times}$ and a suitable $b \in (\mathbf{Z}/d\mathbf{Z})^{\times}$. Let us check that $(A' \otimes_A A')/a(A' \otimes_A A')$ is reduced for this A'. The locus of non-smoothness over $A' = W'[\![q^{1/d}]\!]$ is cut out by the ideal $(q^{1/d})$, so $q^{1/d}$ is a unit multiple of a in A'. Hence, $(A' \otimes_A A')/a(A' \otimes_A A') = W' \otimes_{A/aA} W'$. Since $A \to A' = W'[\![q^{1/d}]\!]$ is finite flat, passing to the quotient modulo aA implies that $A/aA \to W'$ is finite flat. Hence, if k has characteristic 0 then A/aA is a field over which W' is a finite (separable) extension, and if k has positive characteristic then A/aA is a $W(k)$-finite discrete valuation ring over which W' is finite flat. In either case, it is clear that $W' \otimes_{A/aA} W'$ is reduced. It is therefore sufficient to solve our extension problem over $W'[\![q^{1/d}]\!]$.

Upon inverting $q^{1/d}$ we may use Theorem 2.5.2 to identify $\underline{\mathrm{Tate}}_d(\zeta q^{1/d})$ with $\underline{\mathrm{Tate}}_1(\zeta^d q)$ as elliptic curves over $W'(\!(q^{1/d})\!)$. Due to how this identification over $W'(\!(q^{1/d})\!)$ is defined by means of contraction over $W'[\![q^{1/d}]\!]$, it respects the formation of the subgroup μ_p (as in (2.5.2)) and the formation of the identification of $\widehat{\mathbf{G}}_m$ with formal groups along the origin over $W'[\![q^{1/d}]\!]$.

By Example 2.5.1, we may identify the quotient $\underline{\mathrm{Tate}}_1(\zeta^d q)/\mu_p$ over $W'(\!(q^{1/d})\!)$ with the elliptic curve $\underline{\mathrm{Tate}}_1(\zeta^{dp} q^p)$ that extends to the generalized elliptic curve $\overline{E} = \underline{\mathrm{Tate}}_{dp}(\zeta q^{1/d})$ over $W'[\![q^{1/d}]\!]$. (Note that modulo $q^{1/d}$ this is an iterated blow-up of E along components of E^{sing}.) On formal groups along the origin, under this identification the quotient map

$$E = \underline{\mathrm{Tate}}_1(\zeta^d q) \to \underline{\mathrm{Tate}}_1(\zeta^d q)/\mu_p = E/C$$

over $W'(\!(q^{1/d})\!)$ induces the pth power map on $\widehat{\mathbf{G}}_m$ over $W'(\!(q^{1/d})\!)$, and hence the map on formal groups extends over $W'[\![q^{1/d}]\!]$ as desired. By inspection, $P \bmod C$ extends to a section in $\overline{E}(W'[\![q^{1/d}]\!])$ whose reduction modulo $q^{1/d}$ is supported in $\overline{E}^{\mathrm{sm}}$ away from the identity component on fibers. This completes the construction of π_2 and ξ.

Finally, we must check that π_2 is finite and flat. It is clear that π_2 is quasi-finite, so it must be flat (by Lemma 4.1.3). It is likewise clear that π_2 is proper, so π_2 is finite if and only if π_2 is representable in algebraic spaces. By Corollary 2.2.7 (and Theorem 2.2.5 (2)), it suffices to prove that on every geometric π_2-fiber (considered as a stack) the automorphism functor of each geometric point is trivial. Such triviality of the automorphism functors is obvious away from the cusps, and it is also clear at degenerate geometric points $(E_0; P_0, C_0)$ lying outside of $\mathcal{Z}$. Thus, it suffices to study the situation at points $(E_0; P_0, C_0)$ over an algebraically closed field k such that E_0 is a standard polygon and C_0 is the p-torsion μ_p in the identity component of E_0^{sm} (so $\langle P_0 \rangle$ is ample on E_0). The map π_1 presents $\mathcal{M}^{\infty}_{\Gamma_1(N;p)}$ as finite over $\mathcal{M}^{\infty}_{\Gamma_1(N)}$, and for such triples $(E_0; P_0, C_0)$ we have $\pi_1(E_0; P_0, C_0) = (E_0, P_0)$. Thus, $\mathcal{M}^{\infty}_{\Gamma_1(N;p)}$ is an algebraic space near $(E_0; P_0, C_0)$ if $d > 2$ since $\mathcal{M}^{\infty}_{\Gamma_1(N),d}$ is a scheme when $d > 2$. For $d \leqslant 2$, it follows from the construction of π_2 that $\pi_2(E_0; P_0, C_0)$ is a level structure on a standard polygon that is a blow-up of E_0 in its non-smooth locus. In particular, this polygon contains E_0^{sm} as an open subset. Hence, the point $(E_0; P_0, C_0)$ admits no non-trivial automorphisms as a geometric point of its π_2-fiber. Its étale automorphism functor must therefore be trivial. $\square$

The preceding considerations can be adapted to $\Gamma_0(N)$, as follows. For positive integers N and n such that $\mathrm{ord}_p(n) \leqslant \mathrm{ord}_p(N)$ for all primes $p \mid \gcd(N, n)$, a *$\Gamma_0(N; n)$-structure* on a generalized elliptic curve E is a pair (G, C) with $G \subseteq E^{\mathrm{sm}}$ a cyclic subgroup of order N and $C \subseteq E^{\mathrm{sm}}$ a cyclic subgroup of order n such that *fppf*-locally where G admits a $\mathbf{Z}/N\mathbf{Z}$-generator P, the pair (P, C) is a $\Gamma_1(N; n)$-structure. Note that the choice of P does not matter because for any $m \mid N$ the Cartier divisor $\sum_{j \in \mathbf{Z}/m\mathbf{Z}}(j(N/m)P + C)$ is independent of P (proof: use the fact that universal deformation rings for $\Gamma_1(N; n)$-structures are $\mathbf{Z}$-flat and that functors of $\mathbf{Z}/N\mathbf{Z}$-generators of cyclic groups of order N are finite flat over the base (see Theorem 2.3.7)). The map $\mathcal{M}_{\Gamma_1(N;n)} \to \mathcal{M}_{\Gamma_0(N;n)}$ is finite flat with degree $\phi(N)$, and one infers (using our analogous earlier results for $\Gamma_1(N; n)$) that $\mathcal{M}_{\Gamma_0(N;n)}$ is a proper flat Artin stack over $\mathbf{Z}$ that is regular with geometrically connected fibers of pure dimension 1, and that $\mathcal{M}^{\infty}_{\Gamma_0(N;n)}$ is a $\mathbf{Z}$-flat Cartier divisor in $\mathcal{M}_{\Gamma_0(N;n)}$.

The map of Artin stacks

$$\widetilde{\pi}_1 : \mathcal{M}_{\Gamma_0(N;p)} \to \mathcal{M}_{\Gamma_0(N)}$$

defined by the operation $(E; G, C) \rightsquigarrow (c(E), G)$ that ignores C and contracts away from G is easily shown to be representable in algebraic spaces (use Corollary 2.2.7 and Theorem 2.2.5 (2)) and hence it is finite flat (use Lemma 4.1.3 for flatness). The map $\widetilde{\pi}^0_2 : (E; C, G) \rightsquigarrow (E/C, G \bmod C)$ makes sense away from the Cartier divisor $\mathcal{M}^{\infty}_{\Gamma_0(N;p)}$, but to extend it over $\mathcal{M}_{\Gamma_0(N;p)}$ we cannot use Lemma 4.4.5 because $\mathcal{M}_{\Gamma_0(N;p)}$ is often not Deligne–Mumford. We shall circumvent this problem by using the map π_2 that has already been constructed on the finite flat covering $\mathcal{M}_{\Gamma_1(N;p)}$. Since $\mathcal{M}_{\Gamma_0(N;p)}$ is normal and $\mathcal{M}_{\Gamma_1(N;p)} \to \mathcal{M}_{\Gamma_0(N;p)}$ is finite flat, Corollary 3.2.3 reduces the problem of factorizing π_2 (compatibly with $\widetilde{\pi}^0_2$) to checking that

$$\mathcal{M}^{\infty}_{\Gamma_1(N;p)} \times_{\mathcal{M}^{\infty}_{\Gamma_0(N;p)}} \mathcal{M}^{\infty}_{\Gamma_1(N;p)}$$

is reduced. Such reducedness is straightforward because $\mathcal{M}^{\infty}_{\Gamma_1(N;p)} \to \mathcal{M}^{\infty}_{\Gamma_0(N;p)}$ is flat (it is a pullback of $\mathcal{M}_{\Gamma_1(N;p)} \to \mathcal{M}_{\Gamma_0(N;p)}$) and the $\mathbf{Z}$-flat Cartier divisors $\mathcal{M}^{\infty}_{\Gamma_1(N;p)}$ and $\mathcal{M}^{\infty}_{\Gamma_0(N;p)}$ are reduced (Lemma 4.3.2) with generic characteristics equal to 0. This constructs $\widetilde{\pi}_2$. Since $\widetilde{\pi}_2$ is compatible with π_2, considerations with Corollary 2.2.7 and regularity of the Artin stacks show that $\widetilde{\pi}_2$ is representable in algebraic spaces and in fact is finite flat. The map $\widetilde{\xi} : \widetilde{\pi}^*_2\omega_{\Gamma_0(N)} \to \omega_{\Gamma_0(N;p)}$ that is analogous to ξ and extends the evident construction over $\mathcal{M}^0_{\Gamma_0(N;p)}$ is constructed by descent of ξ through the finite flat covering $\mathcal{M}_{\Gamma_1(N;p)} \to \mathcal{M}_{\Gamma_0(N;p)}$.

4.5. Hecke operators over Z and over $\mathbf{F}_p$

We conclude with some applications to Hecke operators on spaces of modular forms on $\Gamma_1(N)$. The formation of ω is insensitive to contraction, so there is a canonical isomorphism $\pi^*_1\omega_{\Gamma_1(N)} \simeq \omega_{\Gamma_1(N;p)}$. Since π_1 is finite and flat, there is a natural trace map

$$\begin{aligned}\mathrm{Tr} : \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N;p)}, \omega^{\otimes k}_{\Gamma_1(N;p)}) = \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \pi_{1*}\pi^*_1\omega^{\otimes k}_{\Gamma_1(N)}) &\to \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega^{\otimes k}_{\Gamma_1(N)}) \\ &= \mathrm{M}_{k,\mathbf{Z}}.\end{aligned}$$

The composite map

$$\mathrm{M}_{k,\mathbf{Z}} = \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)}, \omega_{\Gamma_1(N)}^{\otimes k}) \xrightarrow{\xi^{\otimes k} \circ \pi_2^*} \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N;p)}, \omega_{\Gamma_1(N;p)}^{\otimes k}) \xrightarrow{\mathrm{Tr}} \mathrm{M}_{k,\mathbf{Z}} \tag{4.5.1}$$

recovers the operator pT_p on the complex fiber.

We wish to give a conceptual proof (without the crutch of q-expansions) that (4.5.1) has image in $p\mathrm{M}_{k,\mathbf{Z}}$, so the $\mathbf{Z}$-structure $\mathrm{M}_{k,\mathbf{Z}}$ on $\mathrm{M}_k(\Gamma_1(N), \mathbf{C})$ is preserved under all Hecke operators, and we will use the method of proof to give a direct construction of the T_p-operator on Katz modular forms for $\Gamma_1(N)$ in characteristic $p \nmid N$ with arbitrary weight (especially weight 1). The global assertion that the map (4.5.1) has image in $p\mathrm{M}_{k,\mathbf{Z}}$ follows from the following local assertion:

Theorem 4.5.1. *Let K be a separably closed field and let $x : \mathrm{Spec}\, K \to \mathcal{M}_{\Gamma_1(N)}$ be a map. Let y be a point in the finite K-scheme $\pi_1^{-1}(x)$. Let R_x and R_y denote the corresponding strictly henselian local rings at the corresponding points of the stacks, so R_y is finite flat over R_x. Let $R_{\pi_2(y)}$ denote the strictly henselian local ring at the K-point $\pi_2(y)$ in $\mathcal{M}_{\Gamma_1(N)}$. The composite map*

$$\omega_{\Gamma_1(N),\pi_2(y)} \xrightarrow{\xi_y} \omega_{\Gamma_1(N;p),y} = R_y \otimes_{R_x} \omega_{\Gamma_1(N),x} \xrightarrow{\mathrm{Tr}_{y|x}} \omega_{\Gamma_1(N),x}$$

on stalks has image in $p \cdot \omega_{\Gamma_1(N),x}$.

Proof. We may assume that K has characteristic p, and by normality of the moduli stacks we may ignore the codimension-2 locus of cusps in characteristic p. We may also work with the local rings modulo p (that is, we may work on the stacks over $\mathbf{F}_p$), so our aim is to prove that the composite map on stalks modulo p is 0. The point x corresponds to some $\Gamma_1(N)$-structure (E_0, P_0) on an elliptic curve E_0 over K, and y corresponds to a $\Gamma_1(N;p)$-structure $(E_0; P_0, C_0)$ over a finite extension of K.

If C_0 is multiplicative then the universal p-isogeny over $R_y/(p) = \mathcal{O}^{\mathrm{sh}}_{\mathcal{M}_{\Gamma_1(N;p)},y}$ has kernel μ_p and hence ξ_y induces the zero map under pullback on invariant 1-forms in characteristic p. This gives the desired result in such cases. If C_0 is étale (so E_0 is ordinary) then we claim that the trace map from $R_y/(p)$ to $R_x/(p)$ is 0, so once again we get the desired result. It is equivalent to prove the vanishing the trace map on completed local rings. The $\widehat{R}_x/(p)$-algebra $\widehat{R}_y/(p)$ classifies splittings of the connected-étale sequence of the p-torsion of the universal ordinary elliptic curve over $\widehat{R}_x/(p)$. By *fppf* Kummer theory this cover is given by the extraction of the pth root of a unit, and ring extensions $A \to A[X]/(X^p - u)$ in characteristic p (with $u \in A$) have vanishing trace because such vanishing holds in the universal case $\mathbf{F}_p[u] \to \mathbf{F}_p[u, X]/(X^p - u)$ (whose localization over $\mathbf{F}_p(u)$ is a purely inseparable and non-trivial extension of fields). □

Theorem 4.5.1 has a further interesting application in characteristic p. Consider the problem of defining the Hecke operator T_ℓ on the space $\mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}, \omega_{\Gamma_1(N)/\mathbf{F}_p}^{\otimes k})$ of weight-k Katz modular forms mod p. (The reader may impose the condition $p \nmid N$, but this property is never used in what follows.) Since the finite flat Hecke correspondence and the map ξ are defined over $\mathbf{Z}$, by reducing the Hecke correspondence and ξ modulo p we may use these data to define scaled Hecke operators

$$\ell T_\ell : \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}, \omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p}) \to \mathrm{H}^0(\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}, \omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p})$$

for all primes ℓ. This global procedure is not useful if $\ell = p$. Moreover, if $k = 1$ then such forms do not generally lift into characteristic 0. One way around this problem, used in [**G**, § 4] for $p \nmid N$, is to 'define' T_p by the same q-expansion formula as in characteristic 0. This approach requires an ad hoc procedure (and explicit manipulations with q-expansions) to verify that it is well defined on the space of Katz forms.

We shall now show that the existence of the finite flat Hecke correspondences on moduli stacks over $\mathbf{Z}$ (as in Theorem 1.2.2 and Theorem 4.4.3) provides another approach to the problem that requires no computations, does not use global liftings (perhaps meromorphic along the cusps), tautologically preserves holomorphicity along the cusps, and permits q-expansion formulae to be derived by pure thought from characteristic 0 *a posteriori*. The intervention of division by p upon the modular correspondence does seem to require the use of some lifting in the construction for T_p on Katz forms in characteristic p (lifting to something flat over either $\mathbf{Z}/(p^2)$ or over $\mathbf{Z}$), but we shall only require liftings on henselian local rings at geometric points.

Let ℓ be an arbitrary prime. For any Katz form f of weight k and level N in characteristic p and any morphism $x : \operatorname{Spec} K \to \mathcal{M}_{\Gamma_1(N)}$ with K separably closed of characteristic p, let us first define an element $(f \mid T_\ell)_x$ in the mod-p stalk $\omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p,x}$ over the strictly henselian local ring $\mathcal{O}^{\mathrm{sh}}_{\mathcal{M}_{\Gamma_1(N)},x} = R_x/(p)$. For every point $y \in \pi_1^{-1}(x)$, there is an isomorphism of modules

$$\omega^{\otimes k}_{\Gamma_1(N),\pi_2(y)}/p \cdot \omega^{\otimes k}_{\Gamma_1(N),\pi_2(y)} \simeq \omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p,\pi_2(y)}$$

over $R_{\pi_2(y)}/(p)$, and likewise for $\omega_{\Gamma_1(N;p),y}$ over $R_y/(p)$, so we may lift $f_{\pi_2(y)}$ to an element $F_{\pi_2(y)}$ in the stalk $\omega^{\otimes k}_{\Gamma_1(N),\pi_2(y)}$, and we can form the finite flat trace

$$\mathrm{Tr}_{y|x}(\xi(\pi_2^*(F_{\pi_2(y)}))) \in \omega^{\otimes k}_{\Gamma_1(N),x}.$$

Using Theorem 4.5.1 in case $\ell = p$, this trace lies in $\ell \cdot \omega^{\otimes k}_{\Gamma_1(N),x}$. An alternative procedure is to only lift to the stalk modulo p^2. Either way, modulo ℓp this trace only depends on $f_{\pi_2(y)}$, so if we divide by ℓ and reduce modulo p then the result in characteristic p only depends on $f_{\pi_2(y)}$. We therefore get a well-defined element

$$(f \mid T_\ell)_x = \sum_y (\ell^{-1} \cdot \mathrm{Tr}_{y|x}(\xi(\pi_2^*(F_{\pi_2(y)})))) \bmod p \in \omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p,x}.$$

Since π_2 is finite we have

$$\pi_2^{-1}(\operatorname{Spec} R_x/(p)) = \coprod_y \operatorname{Spec} R_y/(p),$$

so it follows from the construction of $(f \mid T_\ell)_x$ that if x' is a generic point of $\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}$ specializing to x and we choose a map $R_x/(p) \to R_{x'}/(p)$ over $\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}$ then $(f \mid T_\ell)_{x'}$

is the image of $(f \mid T_\ell)_x$ under the corresponding localization map of stalks of $\omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p}$. The stack $\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}$ is Cohen–Macaulay, so the line bundle $\omega^{\otimes k}_{\Gamma_1(N)/\mathbf{F}_p}$ on $\mathcal{M}_{\Gamma_1(N)/\mathbf{F}_p}$ is a Cohen–Macaulay coherent sheaf. Hence, we can 'glue' the stalks $(f \mid T_\ell)_x$ to define a global section $f \mid T_\ell$ via the following lemma.

Lemma 4.5.2. *Let $\mathcal{M}$ be a locally noetherian Deligne–Mumford stack and let $\mathcal{F}$ be a Cohen–Macaulay coherent sheaf on $\mathcal{M}$. If $s_x \in \mathcal{F}_x$ is an element in the stalk module over the strictly henselian local ring at each point x of $\mathcal{M}$, and if the s_x are compatible under localization, then there exists a unique $s \in \mathrm{H}^0(\mathcal{M}, \mathcal{F})$ with stalk s_x at each x.*

Proof. The uniqueness allows us to work locally, so we may assume that $\mathcal{M} = \operatorname{Spec} A$ is affine. Let M be the Cohen–Macaulay finite A-module corresponding to $\mathcal{F}$. We may replace A with $A/\operatorname{ann}(M)$, so M has full support on $\operatorname{Spec} A$. We may assume $M \neq 0$, so $A \neq 0$. We have to prove that a compatible system of elements $s_{\mathfrak{p}} \in M \otimes_A A^{\mathrm{sh}}_{\mathfrak{p}}$ arises from a unique $s \in M$. We may use 'smearing out' and étale descent to construct s, provided that we can test equalities by working in the collection of localizations at minimal primes (for which there is no room to smear out). Hence, we need that every zero-divisor of M in A lies in a minimal prime of A. Let $\mathfrak{q}$ be an associated prime of M, so $M_{\mathfrak{q}}$ is a Cohen–Macaulay finite module with depth 0 and full support over $A_{\mathfrak{q}}$. The Cohen–Macaulay condition says that the depth is equal to the dimension of the support, so $\dim A_{\mathfrak{q}} = 0$ and hence $\mathfrak{q}$ is minimal. □

Example 4.5.3. Let p be a prime not dividing N and let f be a Katz modular form of weight $k \geqslant 1$ for $\Gamma_1(N)$ over $\overline{\mathbf{F}}_p$. Choose a prime ℓ (possibly $\ell = p$) and choose a primitive Nth root of unity $\zeta \in \mu_N(\overline{\mathbf{F}}_p)$, and consider q-expansions via evaluation at $(\underline{\mathrm{Tate}}_1, \zeta)$ over $\overline{\mathbf{F}}_p[\![q]\!]$ by using the basis $(\mathrm{d}t/t)^{\otimes k}$ for $\omega^{\otimes k}_{\underline{\mathrm{Tate}}_1}$. Suppose

$$f(\underline{\mathrm{Tate}}_1, \zeta) = \Big(\sum a_n q^n\Big)(\mathrm{d}t/t)^{\otimes k},$$

and if $\ell \nmid N$ then suppose

$$(f \mid \langle \ell \rangle)(\underline{\mathrm{Tate}}_1, \zeta) = \Big(\sum b_n q^n\Big)(\mathrm{d}t/t)^{\otimes k}.$$

We claim

$$(f \mid T_\ell)(\underline{\mathrm{Tate}}_1, \zeta) = \begin{cases} \Big(\displaystyle\sum_{n\geqslant 1} a_{n\ell} q^n + \ell^{k-1} \sum_{n \geqslant 1} b_n q^{n\ell}\Big)(\mathrm{d}t/t)^{\otimes k} & \text{if } \ell \nmid N, \\ \displaystyle\sum_{n \geqslant 1} a_{n\ell} q^n (\mathrm{d}t/t)^{\otimes k} & \text{if } \ell \mid N. \end{cases}$$

These are the well-known formulae from characteristic 0, and to verify them in characteristic p (especially if $\ell = p$ and $k = 1$) we may work over $\overline{\mathbf{F}}_p(\!(q)\!)$. Since $\underline{\mathrm{Tate}}_1$ over $\overline{\mathbf{F}}_p(\!(q)\!)$ is the reduction of $\underline{\mathrm{Tate}}_1$ over $W(\overline{\mathbf{F}}_p)(\!(q)\!)$, it is immediate from the *method* of construction of $f \mid T_\ell$ via smearing out reductions of étale-local lifts that the formulae in characteristic p are a formal consequence of their validity in characteristic 0. Briefly, the crux of the matter is that since π_1 is finite flat and π_2 is finite, if x' specializes to x then

$$R_{x'} \otimes_{R_x} \left(\prod_{y \in \pi_1^{-1}(x)} R_y \right) \simeq \prod_{y' \in \pi_1^{-1}(x')} R_{y'}$$

(since strict henselization is compatible with passage to finite algebras) and $\pi_2(y')$ specializes to $\pi_2(y)$ if y' specializes to y.

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Note added in proof

By Theorem 2.3.7, the assertion in Lemma 4.2.3 also holds for the morphism $\mathcal{M}_{\Gamma_1(N)} \to \mathcal{M}_{\Gamma_0(N)}$ defined by $(E, P) \rightsquigarrow (E, \langle P \rangle)$.

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