Koszul resolutions

September 27

History

Theorem (H.Hopf, H.Samelson 1941)

Let G be a compact connected Lie group and

$$P_G = \{\alpha \in H^{\geq 1}(G;\mathbb{R}) | \Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1\}$$

be the space of primitive elements (here, Δ is induced on $H^*(G;\mathbb{R})$ by the multiplication on G). Then $H^*(G;\mathbb{R}) \simeq \bigwedge^* P_G$.

Consider a principle G-bundle $G \overset{j}{\hookrightarrow} E \xrightarrow{\pi} M$. Let $\{x_i\}$ be a basis of P_G. There are G-invariant differential forms $\xi_i \in A^*(E)$ on E such that

- $j^*(\xi_i)$ represent x_i
- $d\xi_i = \pi^*(c_i)$ for some closed $c_i \in A^*(M)$.

One equips the graded algebra $\bigwedge^* P_G \otimes A^*(M)$ with the differential d defined by

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The natural map $\psi : \bigwedge^* P_G \otimes A^*(M) \to A^*(E)$ *defined by*

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In particular, if M is formal (say, it is a compact Kähler manifold [DGMS75]) then the de Rham cohomology of E is computed by the complex $\bigwedge^* P_G \otimes H^*(M)$.

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Let $\mathfrak{g}$ be a Lie algebra over k, A be a left $\mathfrak{g}$-module and C be a right $\mathfrak{g}$-module

Definition

$$H_*(\mathfrak{g}, A) = \mathrm{Tor}_*^{U(\mathfrak{g})}(A, k), \quad H^*(\mathfrak{g}, A) = \mathrm{Ext}^*_{U(\mathfrak{g})}(k, C),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Tor and Ext can be computed using the standard *bar-complex*, but there is a smaller complex that does the job. We equip $V_p(\mathfrak{g}) = U(\mathfrak{g}) \otimes_k \bigwedge^p g$ with the differential $d(u \otimes x_1 \wedge \dots x_p) = \theta_1 + \theta_2$, where

$$\theta_1 = \sum_{i=1}^{p} (-1)^{i+1} u x_i \otimes x_1 \dots \hat{x}_i \dots x_p$$

$$\theta_2 = \sum_{i<j} (-1)^{i+j} u \otimes [x_i, x_j] x_1 \dots \hat{x}_i \dots \hat{x}_j \dots x_p.$$

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Theorem

The complex $V_*(\mathfrak{g})$ (known as the Chevalley-Eilenberg complex) is an acyclic subcomplex of the bar-resolution.

Thus, $H_*(\mathfrak{g}, A)$ (resp. $H^*(\mathfrak{g}, A)$) can be computed as the homology of the complex $A \otimes_{U(\mathfrak{g})} V(\mathfrak{g})$ (resp. $\mathrm{Hom}(V(\mathfrak{g}), C)$).

Minimal free resolutions

Let A be a (non-commutative) non-negatively graded associative augmented k-algebra such that $\dim(A_i) < \infty$ and $A_0 = k$. One often needs to have a free resolution of the ground field k. Moreover, for practical purposes such a resolution should be "small".

Definition

A bounded above complex of free graded A-modules

$$\cdots \to P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \to 0$$

is called *minimal* provided all the induced maps $k \otimes_A P_{i+1} \to k \otimes_A P_i$ vanish. In other words, $d(P_i) \subset A_+ P_{i-1}$.

Lemma

A free resolution is minimal iff for each i, a basis of P_{i-1} maps onto a minimal set of generators of $\mathrm{coker}(d_i)$.

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A resolution

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of a graded A-module by free graded A-modules is called a *linear free resolution* if each P_i is generated in degree i.

Lemma

Any linear free resolution is minimal.

Proof. Since d is assumed to be homogeneous of degree zero, then the image $d(P_i) \subset P_{i-1}$ is sitting in degrees $\geq i$, that is, $d(P_i) \subset A_+ P_{i-1}$.

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Koszul algebras

Definition

The algebra A is called Koszul iff k admits a linear free resolution.

Example

The symmetric algebra $S(V)$ over a f.d. k-vector space is Koszul. The linear free resolution of k as a trivial $S(V)$-module is given by the so-called standard (or tautological) Koszul complex:

$$\cdots \bigwedge^{3}(V^*) \otimes S(V) \to \bigwedge^{2}(V^*) \otimes S(V) \to V^* \otimes S(V) \to S(V) \to k$$

where the differential is $a^* \otimes a \mapsto \sum_{i=1}^{n} (a^* \wedge e_i^*) \otimes (e_i a)$.

Quadratic algebras

Definition

An associative k-algebra A is called *quadratic* if $A \simeq T(V)/(R)$, where $T(V)$ is the tensor algebra over a k-vector space V and $R \subset V \otimes V$ is a subspace.

Example

1. The tensor algebra $T(V)$ is quadratic ($R = 0$).
2. The symmetric algebra $S(V)$ is quadratic ($R = \langle e_i \otimes e_j - e_j \otimes e_i \rangle$).
3. The exterior algebra $\bigwedge(V)$ is quadratic ($R = \langle e_i \otimes e_j + e_j \otimes e_i \rangle$).
4. The quantum plane is the quadratic algebra generated by $V = \langle x, y \rangle$ and $R = \langle x \otimes y - qy \otimes x \rangle$).
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Lemma

A Koszul algebra A is quadratic.

Proof. Invstigate the first three terms of the linear free resolution of k:

$$\dots P_2 = A^{b_2} \to P_1 = A^{b_1} \to P_0 = A \to k \to 0$$