Research statement

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Algebraic topologists attach algebraic structures, such as groups, rings, and categories, to geometric objects, such as manifolds, simplicial complexes, and even big data sets. In homotopy theory, the main goal is to study *invariants* of geometric objects under "homotopy" transformations. This type of transformations usually turns out to provide the "right" criterion, neither too rigid nor too loose, for gaining useful insights into how these objects look and behave.

Effective *calculational* tools with the associated algebraic structures give homotopy theory its unique flavor. The main players in realizing this process are the various "cohomology theories," each a systematic way of attaching specific algebraic structures to geometric objects, and each successfully capturing some aspects of the objects in question, while being blind to some others [Greenlees1988]. A local-to-global property, manifested by Mayer-Vietoris sequences, makes cohomology theories particularly amenable to computations.

The study of Morava E-theories is like a raindrop in which all of modern homotopy theory is reflected. These cohomology theories are prominent players promoted by the "chromatic viewpoint"—a deep and fruitful relationship between homotopy theory and the theory of one-dimensional formal groups that has been steadily developing and pervading the field since Quillen's work on complex cobordism [Quillen1969]. This approach brings homotopy theorists a "chromatic" view of the stable homotopy category, by filtering cohomology theories through *heights* and *primes* according to their corresponding formal groups. The family of Morava E-theories determines this chromatic filtration via Bousfield localizations [Ravenel1984].

Hendrik Lenstra and Peter Stevenhagen wrote that "nothing can match the clarity of a formula when it comes to conveying a mathematical truth."[1] To understand E-theories in the specific case of height 2, our research has been centering on calculations with their "power operations." These can be viewed as algebraic structures of the algebraic structures attached to geometric objects. They impose restrictions and refinements to the algebraic structures carried by cohomology theories, so as to enhance their ability to recognize and distinguish the subtleties between geometric objects, making those players clear-sighted.

For E-theories at height 2, contacts with algebraic geometry and number theory, particularly through the arithmetic moduli of elliptic curves, avail homotopy theorists effective tools to carry out explicit calculations. In terms of formulas for power operations, this approach imposes vastly intricate data from arithmetic. The goal is to make the cohomology theories more sensitive and powerful in studying geometric questions, with a potential to witness the deep interplay between numbers and spaces.

Below we give an outline of our current research and future plans, with an emphasis on some specific aspects where algebraic topology, algebraic geometry, and number theory interact.

[1] From their book review of *Solving the Pell equation*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 345–351.

1 Elliptic curves: power operation structures at small primes

Cohomology operations have been a calculational tool central to algebraic topology. A classical example that has been extensively studied and widely applied is the Steenrod operations in ordinary cohomology with $\mathbb{Z}/p$-coefficients. When $p = 2$, for all integers $i \geq 0$ and $n \geq 0$, each Steenrod square takes the form $\mathrm{Sq}^i \colon H^n(X; \mathbb{Z}/2) \to H^{n+i}(X; \mathbb{Z}/2)$, natural in spaces X. Together they generate the mod-2 Steenrod algebra subject to a set of axioms, among which, notably, the Adem relations

$$\mathrm{Sq}^i \mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k} \mathrm{Sq}^k \qquad 0 < i < 2j$$

In-depth study of the structure of the Steenrod algebra, and of analogous structures for other cohomology theories such as K-theory and motivic cohomology, has led to spectacular applications: Adams' solution to the problem of counting vector fields on spheres [Adams1962], and Voevodsky's proof of the Milnor conjecture [Voevodsky2003a, Voevodsky2003b], just to name two.

For a Morava E-theory, Rezk computed the first example of an explicit presentation for its algebra of power operations, in the case of height 2 at the prime 2 [Rezk2008]. This algebra is generated over $E^0(\text{point}) \cong \mathbb{W}\big(\overline{\mathbb{F}}_2\big)[\![h]\!]$, a ring of formal power series with coefficients in a Witt ring, by operations $Q_i \colon E^0(X) \to E^0(X)$, $0 \leq i \leq 2$. In particular, they satisfy "Adem relations"

$$Q_1 Q_0 = 2Q_2Q_1 - 2Q_0Q_2 \qquad \text{and} \qquad Q_2Q_0 = Q_0Q_1 + hQ_0Q_2 - 2Q_1Q_2$$

Unlike their classical analogue, these formulas are computed from a specific moduli space of the universal formal deformation of a supersingular elliptic curve over $\overline{\mathbb{F}}_2$. Here, bridging algebraic topology and algebraic geometry are the work of Ando, Hopkins, Strickland, and Rezk [Ando-Hopkins-Strickland2004, Rezk2009] and the theorem of Serre and Tate [Lubin-Serre-Tate1964]. Roughly, power operations correspond to cyclic isogenies of elliptic curves.

We systematically generalized Rezk's methods and obtained results for E-theories at the primes 3 and 5.

Theorem 1.1 ([Zhu2014, Corollary 2.6 and Definition 3.8], [Zhu2015a, Examples 3.4 and 6.1]) *Let E be a Morava E-theory spectrum of height 2 at the prime p. There is a ring homomorphism*

$$\begin{aligned} \psi^p \colon E^0(\text{point}) &\to E^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big) \end{aligned}$$

called a total power operation, *where I is an ideal of transfers.*

(i) *When $p = 3$, we have $w(h,\alpha) = \alpha^4 - 6\alpha^2 + (h-9)\alpha - 3$ and*

$$\psi^3(h) = h^3 - 27h^2 + 201h - 342 + (-6h^2 + 108h - 334)\alpha + (3h-27)\alpha^2 + (h^2 - 18h + 57)\alpha^3$$

(ii) *When $p = 5$, we obtain analogous (but admittedly less readable) formulas.*

(iii) *These lead to presentations for the respective algebra of power operations on $K(2)$-local commutative E-algebra spectra, in terms of explicit generators and quadratic relations.*

Question 1.2 Is there a presentation that applies to *all* primes p for the algebra of power operations in a Morava E-theory at height 2?

Rezk gave such a uniform presentation, *modulo* p, in [Rezk2012b], which relies on explicit formulas for the mod-p reduction of a certain moduli space of elliptic curves from [Katz-Mazur1985]. *Integrally*, we provide an answer in Theorem 3.3 below.

We should emphasize that the arithmetic data extracted from the particular moduli of elliptic curves cannot be homotopy-theoretically meaningful without the aforementioned deep theorems of Ando-Hopkins-Strickland, of Rezk, and of Serre-Tate. In the program of understanding higher chromatic levels, these *computational* experiments supply tangible raw materials to studying *structural* properties of the stable homotopy category. Below are some directions that we have investigated and plan to explore further.

(i) Rezk shows that the algebra of power operations for a Morava E-theory is "Koszul" at all heights and primes [Rezk2012a, Rezk2012b]. For height 2 and small primes, the power operation formulas give rise to explicit Koszul complexes. More generally, the homological algebra of these Koszul complexes has been applied, e.g., to studying Bousfield-Kuhn functors and unstable periodic homotopy groups by Behrens and Rezk [Behrens-Rezk2015, Rezk2013]. In this direction, we have obtained the following.

Pre-Theorem 1.3 ([Zhu2016b]) *Let E be a Morava E-theory spectrum of height 2, with $E_0 \cong \mathbb{W}(\overline{\mathbb{F}}_p)[\![h]\!]$. Then the first completed E-homology of the Bousfield-Kuhn functor applied to an odd-dimensional sphere has a presentation*

$$E_1^\wedge(\Phi_2 S^{2d+1}) \cong \begin{cases} 0 & d = 0 \\ E_0^{\oplus d-1} \oplus (E_0/p^d)^{\oplus p-d} \oplus E_0/(p^{d-1}, p^{d-2}h, \ldots, h^{d-1}) & 1 \leq d \leq p \\ E_0^{\oplus p-1} \oplus E_0/(p^p h^{d-p-1}, p^{p-1}h^{d-p}, \ldots, h^{d-1}) & d > p \end{cases}$$

In particular, $\operatorname{colim}_d E_1^\wedge(\Phi_2 S^{2d+1}) \cong E_0^{\oplus p-1} \oplus E_0/(p^\infty, h^\infty)$.

The analogue at height 1 is well known from work of Davis and Mahowald [Davis-Mahowald1987, Davis1986]. It would be interesting to understand the occurrence of the free summand in the above colimit, which is a new piece of structure at height 2.

(ii) The power operations at height 2 "descend" to height 1 via a $K(1)$-localization [Zhu2014, Section 4]. At small primes, we observe that the resulting formulas match up numerically to the rings studied by Lubin which parametrize "canonical" subgroups of formal groups [Lubin1979]. These rings can be explicitly determined, one for each height and prime. The patterns re-occur in our study of Rezk's logarithms (see Section 2 below). These suggest a more precise relationship between the first and second chromatic layers from the perspective of power operations and subgroups of formal groups.

(iii) Again, at height 2, the explicit formulas have led to a partial understanding of the "center" for the algebra of power operations in an E-theory [Zhu2015a, Theorem 6.8]. This is related to the Hecke operators that we discuss next.

2 Modular forms: Hecke operators and Rezk's logarithms

In [Rezk2006], using Bousfield-Kuhn functors, Rezk constructed "logarithmic" cohomology operations that naturally act on the units of any strictly commutative ring spectrum. In particular, given a Morava E-theory spectrum E of height n at the prime p, he wrote down a formula for its "logarithm" $\ell_{n,p}\colon\ E^0(X)^\times \to E^0(X)$ in terms of its power operations ψ_A [Rezk2006, Theorem 1.11] and he interpreted this formula in terms of certain "Hecke operators" $T_{j,p}$ as follows:

$$\begin{aligned}\ell_{n,p}(x) &= \frac{1}{p}\log\left(\prod_{j=0}^{n}\prod_{\substack{A\subset(\mathbb{Q}_p/\mathbb{Z}_p)^n[p]\\ |A|=p^j}}\psi_A(x)^{(-1)^j p^{(j-1)(j-2)/2}}\right)\\ &= \sum_{j=0}^{n}(-1)^j p^{j(j-1)/2}\,T_{j,p}(\log x)\end{aligned}\tag{2.1}$$

These Hecke operators are cohomology operations constructed from power operations that were known to act on the E-cohomology of a space [Ando1995]. Based on explicit calculations and a particular choice of parameters in the case of height 2, we revisited these operators to make a precise connection with Hecke operators acting on modular forms. In particular, we obtained the following.

Theorem 2.2 ([Zhu2015a, Proposition 2.8 and Theorem 4.13]) *Let E be a Morava E-theory spectrum of height 2 at the prime p, and let $N>3$ be any integer prime to p.*

(i) *There is a ring homomorphism $\beta_N^{(p)}\colon\ \mathrm{MF}\big(\Gamma_1(N)\big)\to E^0(\mathrm{point})$, where $\mathrm{MF}\big(\Gamma_1(N)\big)$ is the graded ring of weakly holomorphic modular forms on $\Gamma_1(N)$.*

(ii) *Given $f\in\mathrm{MF}\big(\Gamma_1(N)\big)^\times$ with trivial Nebentypus, if its Serre derivative $\vartheta f=0$, then $\beta_N^{(p)}(f)$ is contained in the kernel of the logarithmic operation $\ell_{2,p}\colon\ E^0(\mathrm{point})^\times\to E^0(\mathrm{point})$.*

The logarithms in E-theories at height 2 are critical in the work of Ando, Hopkins, and Rezk on rigidification of the string-bordism elliptic genus [Ando-Hopkins-Rezk2010, Theorem 12.3]. Roughly, in their setting, the kernel of a logarithm contains the desired genera, which they identified with certain Eisenstein series.

Question 2.3 With our result in Theorem 2.2 about the kernel of a logarithmic operation, can we develop an analysis of E_∞-orientations analogous to the work of Ando, Hopkins, and Rezk?

The logarithm of a meromorphic modular form (on which Hecke operators act) appears in Rezk's formula (2.1). Serre's differential operator ϑ appears in Theorem 2.2. In view of these, we ask the following.

Question 2.4 Do these specific pieces of number theory enter homotopy theory in a *structural* way? For example, do Rezk's logarithmic operations bring in a wider class of automorphic functions to homotopy theory? What is present at chromatic level higher than 2?

In [Zhu2015a, Section 5], we have started investigating certain aspects of the aforementioned type of elliptic functions, not totally modular, in the framework of "logarithmic q-series" originally studied by Knopp and Mason [Knopp-Mason2011]. It has a curious relationship to mock modular forms [Zhu2015a, Remark 5.2].

3 Formal groups

3.1 Modular equations for Lubin-Tate deformations

Classically, the Kronecker congruence

$$\left(\widetilde{j}-j^p\right)\left(\widetilde{j}^p-j\right)\equiv 0 \mod p$$

gives a (local) equation, reduced modulo p, for the curve that represents $[\Gamma_0(p)]$, the moduli problem of finite flat subgroup schemes of rank p for elliptic curves. Indeed, this is precisely the formula that underlies Rezk's uniform presentation for the mod-p reduction of the power operation algebra (see Section 1).

Strickland studied various moduli problems for formal groups of finite height [Strickland1997] and he applied them to the study of power operations in Morava E-theories [Strickland1998]. At height 2, we have obtained an integral lift of the Kronecker congruence above, in a different pair of parameters.

Theorem 3.1 ([Zhu2015b, Theorem 1.2]) *Let $\mathbb{G}_0$ be a formal group over $\overline{\mathbb{F}}_p$ of height 2, and let $\mathbb{G}$ be its universal deformation. Write A_m for the ring $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G})}$ studied in [Strickland1997], which classifies degree-p^m subgroups of the formal group $\mathbb{G}$. In particular, write $A_0\cong\mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!]$ according to the Lubin-Tate theorem [Lubin-Tate1966].*

Then the ring $A_1\cong\mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big)$ is determined by the polynomial

$$w(h,\alpha)=(\alpha-p)\big(\alpha+(-1)^p\big)^p-\big(h-p^2+(-1)^p\big)\alpha \tag{3.2}$$

which reduces to $\alpha(\alpha^p-h)$ modulo p.

This gives an explicit description of $[\Gamma_0(p)]$ at a supersingular point (cf. [Katz-Mazur1985, Section 7.7]). It is *not* an equation for the modular curve over Spec $\mathbb{Z}$, which, as hinted in [Rezk2014], might connect to power operations for a "globally equivariant" elliptic cohomology. It would be interesting to explore this local-global relationship, intertwined by the actions of the Morava stabilizer groups and of the modular group, which unites the chromatic and equivariant perspectives on homotopy theory. We have started investigating related functorial constructions, such as Witt ring schemes [Hazewinkel1978], plethories [Borger-Wieland2005], and topological modular forms with level structure [Hill-Lawson2016].

Put in the context of homotopy theory, Theorem 3.1 yields the following answer to Question 1.2.

Theorem 3.3 ([Zhu2015b, Theorems 1.6 and 1.7]) *Continue with the notation in Theorem 1.1.*

(i) *In the total power operation $\psi^p\colon\ E^0(\text{point})\to E^0(B\Sigma_p)/I\cong\mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big)$, the polynomial*

$$w(h,\alpha)=w_{p+1}\alpha^{p+1}+\cdots+w_1\alpha+w_0 \qquad\qquad w_i\in\mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!]$$

can be given as (3.2) from Theorem 3.1 above. In particular, $w_{p+1}=1$, $w_1=-h$, $w_0=(-1)^{p+1}p$, and the remaining coefficients

$$w_i=(-1)^{p\,(p-i+1)}\left[\binom{p}{i-1}+(-1)^{p+1}p\binom{p}{i}\right]$$

(ii) *The image $\psi^p(h)=\sum_{i=0}^{p}Q_i(h)\,\alpha^i$ is then given by*

$$\psi^p(h)=\alpha+\sum_{i=0}^{p}\alpha^i\sum_{\tau=1}^{p}w_{\tau+1}\,d_{i,\tau}$$

where

$$d_{i,\tau}=\sum_{n=0}^{\tau-1}(-1)^{\tau-n}\,w_0^n\sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau+i\\ 1\le m_s\le p+1\\ m_{\tau-n}\ge i+1}}w_{m_1}\cdots w_{m_{\tau-n}}$$

In particular, $Q_0(h)\equiv h^p \mod p$.

(iii) *These lead to a presentation for the algebra of power operations on $K(2)$-local commutative E-algebra spectra. In particular, the generators $Q_i\colon\ E^0(X)\to E^0(X)$ satisfy quadratic relations*

$$Q_kQ_0=-\sum_{j=1}^{p-k}w_0^j\,Q_{k+j}Q_j-\sum_{j=1}^{p}\sum_{i=0}^{j-1}w_0^i\,d_{k,j-i}\,Q_iQ_j\qquad\qquad 1\le k\le p$$

where the first summation is vacuous if $k=p$.

Question 3.4 To study power operations in Morava E-theories at height greater than 2, can we generalize Theorem 3.1 for the ring A_2? How to formulate this for p-divisible groups in general?

A difficulty for such a generalization lies in the current methods for proving Theorem 3.1: we argue with q-expansions of certain Hauptmoduln and their Hecke translates, as in [Choi2006, Example 2.4] and [Katz1973, Section 1.11], which are specific to heights 1 and 2. Moreover, we are in much need of results from concrete "computational experiments" at higher chromatic levels, indeed, at height 3 (see [Lawson2015, Meier2014]). It would be interesting to work on this.

3.2 Norm-coherent coordinates and structured orientations

The various technical choices made in establishing Theorems 2.2 and 3.1 forced us to carefully study a specific coordinate on the Lubin-Tate universal deformation in question, characterized by a "norm coherence" property. Roughly, it is functorial under pushforward along any isogeny, with respect to a norm map coming from a Galois action by the kernel of the isogeny.

In [Ando1995], Ando constructed norm-coherent coordinates for deformations of the Honda formal groups over $\mathbb{F}_p$. Given these, he established the existence and uniqueness of H_∞ complex orientations for the corresponding Morava E-theories. However, as the underlying formal groups in Theorems 2.2 and 3.1 come from supersingular elliptic curves over algebraic extensions of $\mathbb{F}_p$, Ando's results do not apply to our cases. Placing the problem in a more flexible context, we have obtained analogous results for Morava E-theories associated to any formal group (of any height) as follows.

Let $\mathrm{FG}_{\mathrm{isog}}$ be the category with objects

$$\begin{tikzcd} G \arrow[d] \\ \operatorname{Spec} k \end{tikzcd} \qquad \begin{aligned} k &= \text{perfect field of characteristic } p \\ G &= \text{formal group of height } n \end{aligned}$$

and morphisms

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & G' \times_{k'} k \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G' \arrow[d] \\ \operatorname{Spec} k \arrow[r, equal] & \operatorname{Spec} k \arrow[r] & \operatorname{Spec} k' \end{tikzcd}$$

where f is an isogeny of formal groups over k. Let $\text{Ando}\colon \text{FG}_{\text{isog}} \to \text{Set}$ be the bivariant functor

$$G/k \mapsto \{\text{norm-coherent coordinates on } F/E_n\}$$

where F is the universal deformation of G over the Lubin-Tate ring E_n. It respects pullback and base change contravariantly and pushforward covariantly. Let $\text{Coord}\colon \text{FG}_{\text{isog}} \to \text{Set}$ be the bivariant functor

$$G/k \mapsto \{\text{coordinates on } G\} \subset \mathcal{O}_G$$

Pre-Theorem 3.5 ([Zhu2016a]) *The natural transformation*

$$\text{Ando} \to \text{Coord}$$

of restriction to the special fiber is an isomorphism. Moreover, it respects Galois descent: given a Galois extension K/k, the following diagram commutes.

$$\begin{tikzcd} \text{Ando}(G/K) \arrow[r, "\sim"] \arrow[d, "(-)^{\text{Gal}(K/k)}"'] & \text{Coord}(G/K) \arrow[d, "(-)^{\text{Gal}(K/k)}"] \\ \text{Ando}(G/k) \arrow[r, "\sim"] & \text{Coord}(G/k) \end{tikzcd}$$

As a consequence, any Morava E-theory admits a unique H_∞ complex orientation.

We would like to know if this orientation rigidifies to be E_∞, as H_∞ maps are the elements on the zero line of a certain spectral sequence for getting E_∞ maps. The higher terms on the spectral sequence is related to André-Quillen cohomology and might be relatively easy to compute.

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