MR662118 (83i:55019) 55Q40
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The image of J in the EHP sequence.
Ann. of Math. (2) **116** (1982), *no. 1,* 65–112.

A more apt title for this very important paper might be "v_1-periodic unstable homotopy and the J-spectrum". The classical EHP sequence and image of J may be an underlying motivation, but seem somewhat peripheral to the theorems proved.

The v_1-periodicity arose in BP-theory [H. R. Miller, D. C. Ravenel and W. S. Wilson, same journal (2) **106** (1977), no. 3, 469–516; MR0458423]; in this paper it is given an elegant unstable homotopy-theoretic description. Let Y^n denote the 4-cell complex $\Sigma^{n-6}\mathbf{C}P^2\wedge\mathbf{R}P^2$, and define $\pi_n(\ ;Y)=[Y^n,\ ]$, a homotopy functor on topological spaces or spectra. There is a map $v_1\colon Y^{n+2}\to Y^n$, such that v_1 acts on $\pi_*(\ ;Y)$. By localizing with respect to v_1, a theory $V^{-1}\pi_*(\ ;Y)$ is obtained, which is effectively the v_1-periodic homotopy of a space. The reviewer and the author [Amer. J. Math. **103** (1981), no. 4, 615–659; MR0623131] discussed the relationship between v_1-periodic Y-homotopy and ordinary homotopy.

One of the main theorems (1.3) is that the Snaith map $\Omega^{2n}S^{2n+1}\to\Omega^\infty\Sigma^{\infty+1}P^{2n}$ induces an isomorphism in $V^{-1}\pi_*(\ ;Y)$. Note that the domain is unstable homotopy, while the target is effectively stable homotopy. Both are shown to have precisely four v_1-periodic families.

Let J denote the fibre of $\psi^3-1\colon bo\to\Sigma^4 b\mathrm{sp}$, where these spectra are localized at 2. It has been known since Adams' work on $J(X)$ that $\pi_*(J)$ is closely related to the classical image of J. A new discussion of this relationship is given in Section 8. Unfortunately, Theorem 8.4 is marred by serious misprints: The first sentence should say $\mathcal{O}(j)=3,2,5$ if $j\equiv 0,1,3 \pmod 8$. The second sentence need not be included.

The close relationship between J and v_1-periodicity was already noted [the author and the reviewer, op. cit.]. This relationship is expanded in Theorem 6.2: If X is a finite complex, then $\Omega^\infty\Sigma^\infty X\to X\wedge J$ induces an isomorphism in $V^{-1}\pi_*(\ ;Y)$. The special case $X=P^{2n}$ is given as 1.4, the statement of which is somewhat misleading in suggesting that it deals with $\pi_*(P^{2n})$ rather than $\pi_*(\Omega^\infty\Sigma^\infty P^{2n})$. When combined with 1.3, the remarkable result that the v_1-periodic unstable Y-homotopy of odd spheres is isomorphic to that of the easily calculated (stable) $(J\wedge Y)$-homology of projective spaces is obtained.

Another main result (1.0) is in some sense a limiting version of 1.3, but has the added feature of comparing the homotopy theory $V^{-1}\pi_*(\ ;Y)$ with the homology theory $V^{-1}(J\wedge Y)_*(\ )$. It states that there is an isomorphism $V^{-1}\pi_*(\Omega^\infty S^\infty;Y)\to V^{-1}(J\wedge Y)_*(\mathbf{R}P^\infty)$. Several warnings to the reader are in order here. First, 1.1, of which 1.0 is a corollary, is misleading in that its blanks are not to be filled in with the same spaces. Second, the Y which appears in the homology functor $(J\wedge Y)_*$ is really Y^3, i.e. its bottom cell has dimension zero, which is different from what one would expect if thinking of the smash product of the spectra J and Y. Third, underlying the switch of Y from the domain (in $\pi_*(\ ;Y)$) to the target (in $(J\wedge Y)_*(\ )$) is an application of S-duality and the fact that Y is self-dual, ideas which are not mentioned in the paper.

The fourth main result tells the extent to which $s_*\colon\pi_i(\Omega^{2n+1}S^{2n+1})\to\pi_i(P^{2n}\wedge J)$ is surjective. This is true (1.5) outside the stable range except possibly in one Kervaire-invariant dimension. In the stable range ($i<2n+1$) the classes not in the image are given in 7.10. The complete calculation of $\pi_*(P^{2n}\wedge J)$ is given; it is quite elementary. To show that various families of elements are in the image of s_* requires much work, using the preceding results on Y-homotopy and careful consideration of Adams filtration. The Kervaire-invariant question plays a central role; this will be greatly expanded upon in a series of papers by the author, M. Barratt, and J. Jones.

The proofs involve deep methods of unstable homotopy theory such as the Λ-algebra and unstable resolutions. The important ideas involved more than compensate for the occasional lack of clarity. Much of the work is involved in constructing a homomorphism from a spectral sequence of unstable homotopy groups to one of stable groups and in showing that it is, in fact, an isomorphism.

The author's theorem of bo-resolutions [Pacific J. Math. **92** (1981), no. 2, 365–383; MR0618072] is used in calculating the v_1-periodic homotopy groups. An addendum to that paper will appear shortly and will show that minor changes required in that theorem do not affect the applications in this paper.

A section of conjectures and problems appears at the end of the paper. Most deal with torsion and v_1-torsion in homotopy groups. Another discusses the possibility of mimicking some of the results of this paper for v_2-periodicity. *Donald M. Davis*