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WITT VECTORS AND EQUIVARIANT RING SPECTRA APPLIED TO COBORDISM

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Abstract

Given a finite group G we show that Dress and Siebeneicher's ring of G-typical Witt vectors on the Lazard ring, that is, on the polynomial ring on countably many indeterminates over the integers, embeds as a subring of the unitary cobordism ring of G-manifolds. We also show that the ring of G-typical Witt vectors on the Lazard ring embeds as a subring of the ring of homotopy groups of the G-fixed point spectrum of the spectrum MU representing cobordism. The above results are derived by exploiting the interaction between restriction, additive transfer and multiplicative transfer. This interaction is described by two Mackey functors satisfying a distributivity relation encoded in a formalism developed by Tambara.

1. *Introduction*

This paper has two parts. Firstly, there is a quite formal part studying multiplicative transfers between zeroth homotopy groups of an E_∞ ring G-spectrum for a finite group G. Secondly, it contains an application of multiplicative transfers in equivariant cobordism to show that for every finite group, the ring $\mathbb{W}_G(\pi_*MU)$ of G-typical Witt vectors, in the sense of Dress and Siebeneicher, on the Lazard ring $\pi_*MU = [\mathbb{S}, MU]$, both embeds as a subring of the equivariant cobordism ring $\pi_*^G(MU) = [\mathbb{S}, MU]_G$ and as a subring of the unitary cobordism ring $\mathcal{U}_*^G$ of G-manifolds. The underlying set of the ring $\mathbb{W}_G(\pi_*MU)$ is the same as the underlying set of the ring $\prod_{[H]\in\mathcal{O}} \pi_*MU$ given by the product of one factor of π_*MU for each conjugacy class of subgroups of G, and there is an injective homomorphism of rings from $\mathbb{W}_G(\pi_*MU)$ to the ring $\prod_{[H]\in\mathcal{O}} \pi_*MU$.

Tambara has studied the interaction of trace, norm and restriction maps connecting Burnside rings of different subgroups of a finite group G. He observed that certain relations between the trace, norm and restriction maps satisfied by these Burnside rings are also satisfied by the trace, norm and restriction maps between even cohomology groups and between representation rings of subgroups of G. A collection of abelian groups with trace, norm and restriction maps satisfying such relations was called a TNR-functor in [**30**]. In [**4**] we called such a collection a G-Tambara functor. Here we call it simply a Tambara functor.

The main result of this paper states that every E_∞ ring G-spectrum T has an associated Tambara functor $\widetilde{T}$ with

$$\widetilde{T}(X) = [\Sigma^\infty X_+, T]_G.$$

Here $\Sigma^\infty X_+$ denotes the suspension spectrum on a finite G-set X with a disjoint added basepoint, and $[\Sigma^\infty X_+, T]_G$ denotes the morphisms from $\Sigma^\infty X_+$ to T in the equivariant stable homotopy category associated to a complete universe of G-representations in the sense of Lewis, May and Steinberger [**21**].

The main result of [**4**] is that there is a homomorphism $\tau_S : \mathbb{W}_G(S(G/e)) \to S(G/G)$ for every Tambara functor S. Here $\mathbb{W}_G(S(G/e))$ is Dress' and Siebeneicher's ring of G-typical

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Witt vectors on the commutative ring $S(G/e)$ introduced in [**8**]. Together these two results give a homomorphism

$$\tau_{\widetilde{T}} : \mathbb{W}_G([\mathbb{S}, T]) \longrightarrow [\mathbb{S}, T]_G$$

for every E_∞ ring G-spectrum T. Here $\mathbb{S}$ denotes the sphere spectrum, we have identified $[\Sigma^\infty G_+, T]_G$ with $[\mathbb{S}, T]$, and we have identified $\Sigma^\infty(G/G)_+$ with $\mathbb{S}$. The group $[\mathbb{S}, T]$ is the zeroth homotopy group $\pi_0(T)$ of T and the group $[\mathbb{S}, T]_G$ is the zeroth homotopy group $\pi_0(T^G)$ of the G-fixed point spectrum T^G associated to T. The following theorem of Dress and Siebeneicher [**8**] shows that in some cases $\tau_{\widetilde{T}}$ gives information on $[\mathbb{S}, T]_G$.

Theorem 1.1. *The homomorphism $\tau_{\widetilde{\mathbb{S}}} : \mathbb{W}_G([\mathbb{S}, \mathbb{S}]) \to [\mathbb{S}, \mathbb{S}]_G$ is an isomorphism.*

Hesselholt and Madsen have extended this result to topological Hochschild homology in the special case where G is a finite cyclic group [**15**]. Given an E_∞ ring spectrum R, $\mathrm{THH}(R)$ denotes the topological Hochschild homology of R considered as an E_∞ ring G-spectrum.

Theorem 1.2. *If G is a finite cyclic group and R is an E_∞ ring spectrum, then the homomorphism $\tau_{\widetilde{\mathrm{THH}(R)}} : \mathbb{W}_G([\mathbb{S}, \mathrm{THH}(R)]) \to [\mathbb{S}, \mathrm{THH}(R)]_G$ is an isomorphism.*

Let MP denote the G-equivariant E_∞ ring spectrum representing the periodic version of the unitary Thom spectrum introduced by Strickland [**29**]. (This spectrum is intimately related to cobordism of G-manifolds.) For $H \leqslant G$ the group $[\mathbb{S}, MP]_H$ is the sum of the even-dimensional homotopy groups of MU^H. Using the fact that the spectrum MP has a restriction map similar to the restriction map for topological Hochschild homology we prove the following.

Theorem 1.3. *The homomorphism $\tau_{\widetilde{MP}} : \mathbb{W}_G([\mathbb{S}, MP]) \to [\mathbb{S}, MP]_G$ is injective.*

The homomorphism $\tau_{\widetilde{MP}}$ is not an isomorphism. Already in the case where G is a cyclic group of prime order it is not surjective. The reason for this is that the ring $[\mathbb{S}, MP]_G$ can be considered as a $\mathbb{Z}$-graded ring with non-zero elements, for example the Euler classes, in negative degrees, and that the image of every element in $\mathbb{W}_G([\mathbb{S}, MP])$ is a sum of elements of non-negative degree.

It is a direct consequence of Theorem 1.3 that the composition

$$[\mathbb{S}, \mathbb{S}]_G \cong \mathbb{W}_G(\mathbb{S}, \mathbb{S}) \longrightarrow \mathbb{W}_G([\mathbb{S}, MU]) \xrightarrow{\tau_{\widetilde{MU}}} [\mathbb{S}, MU]_G$$

is injective. This is in a strong contrast to the situation for equivariant unitary K-theory, where the composition

$$[\mathbb{S}, \mathbb{S}]_G \cong \mathbb{W}_G(\mathbb{S}, \mathbb{S}) \longrightarrow \mathbb{W}_G([\mathbb{S}, KU]) \xrightarrow{\tau_{\widetilde{KU}}} [\mathbb{S}, KU]_G$$

is the generally non-injective homomorphism from the Burnside ring of G to the complex representation ring of G taking a finite G-set X to the free $\mathbb{C}$-vector space on X.

Unfortunately, we have not been able to construct multiplicative transfers between the equivariant unitary cobordism ring $\mathcal{U}_*^H$ of H-manifolds for subgroups H of G. However, we shall show that the equivariant unitary cobordism ring $\mathcal{U}_*^G$ of G-manifolds is related to the non-equivariant cobordism ring $\mathcal{U}_* \cong [\mathbb{S}, MP]$ as follows.

Theorem 1.4. *The ring $\mathbb{W}_G(\mathcal{U}_*)$ embeds as a subring of $\mathcal{U}_*^G$.*

In order to construct the Tambara functor $\widetilde{T}$ associated to an E_∞ ring G-spectrum T we need to study ordinary induction and smash induction of equivariant spectra. Since ordinary induction is constructed by wedge sums and smash induction is constructed by smash products, the interaction between smash products and wedge sums will play an important role for us. One way to encode the relations between wedge sums and smash products is in Laplaza's concept of a bimonoidal category [**20**] defined by a number of commutative diagrams. In order to apply Laplaza's coherence result we shall reformulate it in such a way that it becomes clear that commutative ring objects in bimonoidal categories give rise to Tambara functors. This reformulation involves an approach to higher coherences via partial pseudo-functors.

For the passage from categorical constructions to homotopy groups we need a homotopical analysis of the smash induction of spectra. In particular, we must control the operation taking a (non-equivariant) spectrum X to its Gth smash power $X^{\wedge G}$ with G acting by permuting the smash factors. This is gained by examining the interaction between cofibrantions and smash induction. Smash induction is sensitive to the choice of category of equivariant spectra. In this paper we have chosen to work with the orthogonal spectra of [**24**] and [**23**]. We warn the reader that the smash induction of an $\mathbb{S}$-module in the sense of [**12**] might have equivariant homotopy type different from the smash induction of its associated orthogonal spectrum. However, the results of this paper also apply to the category of equivariant symmetric spectra with the model structure considered in [**18**, **17**, **22**]. In a joint paper in preparation with M. Lydakis we show that they also apply to the category of equivariant Gamma spaces.

For our construction of the Tambara functor $\widetilde{T}$ we actually need T to be a strictly commutative orthogonal ring spectrum with action of G. However it is well known that every orthogonal spectrum with an action of an E_∞-operad is stably equivalent to a strictly commutative ring spectrum. (See [**23**, Lemma III.8.4], [**24**, Remark 0.14] and the proof of [**12**, Proposition II.4.3]). We circumvent technical difficulties caused by the fact that in general the commutative replacement of T can not be chosen to be cofibrant in the model structure considered in this paper.

The paper is organized as follows. In Section 2 we give some preliminaries on Tambara functors. Section 3 introduces Tambara categories. These are category-valued coherent Tambara functors playing a key role for the construction of Tambara functors from commutative ring objects in bimonoidal categories. In Section 4 we give a criterion for cofibrations to be preserved under smash induction. This section is the main technical part of the construction of $\widetilde{T}$ from T. Section 5 explains how to construct Tambara functors from commutative ring objects in a bimonoidal category. In Section 6 we show that the category of orthogonal spectra satisfies the criteria of Section 4. As an aside we show that there is a Tambara category of chain complexes. Section 7 ends the construction of $\widetilde{T}$. In Section 8 we study the equivariant version of Strickland's spectrum MP and prove Theorems 1.3 and 1.4. Section 9 is an appendix on filtered objects in symmetric monoidal categories.

2. *Tambara functors*

For the convenience of the reader we recollect parts of the work [**30**] of D. Tambara where he introduced Tambara functors.

A Tambara functor S consists of an abelian group-valued Mackey functor $M = (M_*, M^*)$ and a commutative monoid-valued Mackey functor $M' = (M'_*, M'^*)$ satisfying firstly that M^* and M'^* agree and secondly an elaborate version of the distributive law for commutative rings. The contravariant functor $M^* = M'^*$ is denoted S^* and called the *restriction*. The functor M_* is denoted S_+ and called the *trace*, and the functor M'_* is denoted $S_\bullet$ and called the *norm*.

Let us recall that in the formulation of Dress [**7**] a set-valued *Mackey functor* $M = (M_*, M^*)$ consists of a functor $M_* : \mathcal{F}_G \to \mathcal{S}et$ from the category $\mathcal{F}_G$ of finite left G-sets to the category $\mathcal{S}et$ of sets and a functor $M^* : \mathcal{F}_G^{\mathrm{op}} \to \mathcal{S}et$, where $\mathcal{F}_G^{\mathrm{op}}$ is the opposite category of $\mathcal{F}_G$.

These functors are subject to the following conditions:

(1) the values $M_*(X)$ and $M^*(X)$ of M_* and M^* agree for every object $X \in \mathcal{F}_G$;
(2) if $X = X_1 \cup X_2$ is a disjoint union of finite G-sets, then the homomorphism

$$M^*(X) \longrightarrow M^*(X_1) \times M^*(X_2),$$

induced by the inclusions of X_1 and X_2 in X, is a bijection;
(3) if the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d, "g"'] & X_1 \arrow[d, "g'"] \\ Y_0 \arrow[r, "f'"] & Y_1 \end{tikzcd}$$

is a pull-back diagram in $\mathcal{F}_G$, then $M_*(f) \circ M^*(g) = M^*(g') \circ M_*(f')$.

As a consequence of the conditions (1)–(3) the functors M_* and M^* can be considered as functors into the category of commutative monoids.

The category $\mathcal{F}_G$ of finite left G-sets is small, contains finite sums, finite products and pull-backs. In particular, $\mathcal{F}_G$ contains an initial object $\emptyset$ and a final object $*$. It is well known that for every $f : X \to Y$ in $\mathcal{F}_G$, the pull-back functor

$$\begin{aligned} \mathcal{F}_G/Y &\longrightarrow \mathcal{F}_G/X, \\ (B \to Y) &\longmapsto (X \times_Y B \to X) \end{aligned}$$

has a right adjoint

$$\begin{aligned} \Pi_f : \mathcal{F}_G/X &\longrightarrow \mathcal{F}_G/Y, \\ (A \xrightarrow{p} X) &\longmapsto (\Pi_f A \xrightarrow{\Pi_f p} Y). \end{aligned}$$

The G-map $\Pi_f p$ has fibers $(\Pi_f p)^{-1}(y) = \prod_{x \in f^{-1}(y)} p^{-1}(x)$.

Definition 2.1. A diagram in $\mathcal{F}_G$ isomorphic to a diagram of the form

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, "p"'] & X \times_Y \Pi_f A \arrow[l, "e"'] \arrow[d, "f'"] \\ Y \arrow[r, equal] & Y & \Pi_f A \arrow[l, "\Pi_f p"'] \end{tikzcd}$$

where f' is the projection and e is adjoint to the identity on $\Pi_f A$, is called an *exponential diagram*.

Two diagrams $X \leftarrow A \to B \to Y$ and $X \leftarrow A' \to B' \to Y$ in $\mathcal{F}_G$ are *equivalent* if there exist isomorphisms $A \to A'$ and $B \to B'$ in $\mathcal{F}_G$ making the diagram

$$\begin{tikzcd} X \arrow[d, equal] & A \arrow[l] \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & Y \arrow[d, equal] \\ X & A' \arrow[l] \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

commutative. The category U_+^G has finite G-sets as objects, morphisms from X to Y given by equivalence classes $[X \leftarrow A \to B \to Y]$ of diagrams $X \leftarrow A \to B \to Y$, and composition

$\circ : U_+^G(Y, Z) \times U_+^G(X, Y) \to U_+^G(X, Z)$ defined by the formula

$$[Y \leftarrow C \to D \to Z] \circ [X \leftarrow A \to B \to Y] = [X \leftarrow A'' \to \widetilde{D} \to Z],$$

where the maps on the right are composites of the maps in the diagram

$$\begin{tikzcd}
X & A \arrow[l] \arrow[d] & A' \arrow[l] \arrow[d] & A'' \arrow[l] \arrow[d] \\
& B \arrow[d] & B' \arrow[l] \arrow[d] & \widetilde{C} \arrow[l] \arrow[d, equal] \\
& Y & C \arrow[l] \arrow[d] & \widetilde{C} \arrow[d] \\
& Z & D \arrow[l] & \widetilde{D} \arrow[l]
\end{tikzcd}$$

Here the three squares are pull-back diagrams and the diagram

$$\begin{tikzcd}
C \arrow[d] & B' \arrow[l] & \widetilde{C} \arrow[l] \arrow[d] \\
D & \widetilde{D} \arrow[l] \arrow[r, equal] & \widetilde{D}
\end{tikzcd}$$

is an exponential diagram. Given $f : X \to Y$ in $\mathcal{F}_G$ we denote by R_f, T_f and N_f the morphisms

$$R_f = \left[Y \xleftarrow{f} X \xrightarrow{=} X \xrightarrow{=} X\right],$$

$$T_f = \left[X \xleftarrow{=} X \xrightarrow{=} X \xrightarrow{f} Y\right],$$

$$N_f = \left[X \xleftarrow{=} X \xrightarrow{f} Y \xrightarrow{=} Y\right]$$

in U_+^G. Every morphism in U_+^G can be written as a composition of morphisms on the form R_f, T_f and N_f.

Proposition 2.2. (i) *For every sum diagram*

$$X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$$

in $\mathcal{F}_G$, the diagram

$$X_1 \xleftarrow{R_{i_1}} X \xrightarrow{R_{i_2}} X_2$$

is a product diagram in U_+^G. The object $\emptyset$ is final in U_+^G.

(ii) *Let $\nabla : X \amalg X \to X$ denote the fold morphism of an object X of $\mathcal{F}_G$ and let $i : \emptyset \to X$. Then X, considered as an object of U_+^G, is a semi-ring object with addition T_∇, additive unit T_i, multiplication N_∇ and multiplicative unit N_i.*

(iii) *If $f : X \to Y$ is a morphism in $\mathcal{F}_G$, then the morphisms R_f, T_f and N_f of U_+^G preserve the above structures of commutative semi-ring, additive monoid and multiplicative monoid on X and Y, respectively.*

DEFINITION 2.3. The category of *semi-Tambara functors* is the category of set-valued product-preserving functors S: $U_+^G \to \mathcal{S}et$ on U_+^G with morphisms given by natural transformations.

DEFINITION 2.4. The category of *Tambara functors* is the full subcategory of the category of semi-Tambara functors consisting of those product-preserving functors $S : U_+^G \to \mathcal{S}et$ satisfying the condition that the underlying additive monoid of $S(X)$ is an abelian group for every finite G-set X.

Given a semi-Tambara functor S and $\phi \in U_+^G(X, Y)$ we obtain a function

$$S(\phi) : S(X) \longrightarrow S(Y).$$

Since S is product-preserving, it follows from (ii) of Proposition 2.2 that $S(X)$ is a semi-ring. Given a morphism $f : X \to Y$ in $\mathcal{F}_G$ we let $S^*(f) = S(R_f)$, $S_+(f) = S(T_f)$ and $S_\bullet(f) = S(N_f)$. It follows from Proposition 2.2(iii) that $S^*(f)$ is a homomorphism of semi-rings, that $S_+(f)$ is an additive homomorphism, and that $S_\bullet(f)$ is multiplicative. A semi-Tamara functor S is uniquely determined by the functions $S^*(f)$, $S_+(f)$ and $S_\bullet(f)$ for f in $\mathcal{F}_G$.

EXAMPLE 2.5. For every finite G-set Y, the assignment $X \mapsto U_+^G(Y, X)$ defines a semi-Tambara functor.

EXAMPLE 2.6. A commutative ring with an action of G gives rise to a Tambara functor with value $\mathrm{Map}_G(X, R)$ on a finite G-set X, as explained below in Example 5.3 and in [**30**, Example 3.1].

EXAMPLE 2.7. Given a Tambara functor S, the commutative ring $S(G/e)$ has an obvious action of G. The forgetful functor from the category of Tambara functors to the category of commutative rings with action of G has a left adjoint functor L. The commutative ring $L(R)(G/G)$ is closely related to the generalized Witt vectors of Dress and Siebeneicher [**8**]. In fact it was shown in [**4**] that if G acts trivially on R, then $L(R)(G/G)$ is isomorphic to the ring $\mathbb{W}_G(R)$ of G-Witt vectors on R.

EXAMPLE 2.8. Let A be an E_∞ ring G-spectrum. In Section 7 we show that there is a Tambara functor $\widetilde{A}$ with $\widetilde{A}(X) = [\Sigma^\infty X_+, A]_G$. The special case where A is the sphere spectrum is [**30**, Example 3.2]. In Section 8 we examine the case $A = MP$.

3. *Tambara categories*

Given a commutative monoid M and a cocommutative co-monoid C in a bimonoidal category $\mathcal{C}$ we can form the commutative monoid $\mathcal{C}(C, M)$. In a similar way we shall explain in Section 5 how Tambara functors are obtained from commutative semi-ring objects and cocommutative co-semi-ring objects in lax category-valued Tambara functors. In order to explain what a lax category-valued Tambara functor is, it will be helpful to consider partial categories as defined below and to consider a particular partial category sU_+ codifying Laplaza's coherence result for bimonoidal categories [**20**]. Some authors use the name *distributive categories* for bimonoidal categories.

DEFINITION 3.1. A *partial category* $\mathcal{G}$ consists of a class $\mathrm{ob}\,\mathcal{G}$ of objects, a class $\mathrm{mor}\,\mathcal{G}$ of arrows, domain and codomain functions $d, c : \mathrm{mor}\,\mathcal{G} \to \mathrm{ob}\,\mathcal{G}$, an identity function $\mathrm{id} : \mathrm{ob}\,\mathcal{G} \to \mathrm{mor}\,\mathcal{G}$, a subclass $\mathrm{com}(\mathcal{G}) \subseteq \mathrm{mor}\,\mathcal{G} \times_{\mathrm{ob}\,\mathcal{G}} \mathrm{mor}\,\mathcal{G}$ of *composable arrows* and a composition $\circ : \mathrm{com}(\mathcal{G}) \to \mathrm{mor}\,\mathcal{G}$ subject to the following associativity and unit axioms.

Associativity. For a diagram

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{k} d$$

of composable arrows in $\mathcal{G}$ the relation $k \circ (g \circ f) = (k \circ g) \circ f$ holds.

Unit. For arrows $f : a \to b$ and $g : b \to c$ the pairs (f, id_b) and (id_b, g) of arrows are composable with $\mathrm{id}_b \circ f = f$ and $g \circ \mathrm{id}_b = g$.

Definition 3.2. A *partial functor* $F : \mathcal{C} \to \mathcal{D}$ of partial categories consists of functions $F : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ and $F : \mathrm{mor}\,\mathcal{C} \to \mathrm{mor}\,\mathcal{D}$ compatible with the domain and codomain functions for $\mathcal{C}$ and $\mathcal{D}$ taking every pair (f, g) of composable arrows of $\mathcal{C}$ to a pair $(F(f), F(g))$ of composable arrows of $\mathcal{D}$ with $F(g) \circ F(f) = F(g \circ f)$.

Definition 3.3. A natural transformation $t : F \to G$ between partial functors

$$F, G : \mathcal{C} \longrightarrow \mathcal{D}$$

consists of arrows $t_c : F(c) \to G(c)$ in $\mathcal{D}$ for every object c of $\mathcal{C}$ subject to the condition that for every arrow $f : a \to b$ of $\mathcal{C}$ the pairs $(G(f), t_a)$ and $(t_b, F(f))$ of arrows in $\mathcal{D}$ are composable with $G(f) \circ t_c = t_b \circ F(f)$.

Definition 3.4. An object c of a partial category $\mathcal{C}$ is the product of the objects a and b if there exist arrows $p_a : c \to a$ and $p_b : c \to b$ with the property that given arrows $f : x \to a$ and $g : x \to b$ there exists a unique arrow $h : x \to c$ such that the pairs of arrows (p_a, h) and (p_b, h) are composable with $p_a \circ h = f$ and $p_b \circ h = g$.

Remark 3.5. Products in partial categories are unique up to a unique isomorphism, and the notions of general limits and colimits make perfect sense in partial categories.

Example 3.6. Given a category $\mathcal{D}$ there are many partial categories contained in it. For example if $F : \mathcal{C} \to \mathcal{D}$ is a functor then $F(\mathcal{C}) \subseteq \mathcal{D}$ is a partial category, but it is not in general a category.

In order to use partial categories to codify coherence we need to consider partial pseudo-functors. The definition below is a variation of Grothendieck's concept of pseudo-functors as presented for example in [**2**, Section 7.5].

Definition 3.7. A partial pseudo-functor F defined on a partial category $\mathcal{D}$ consists of a function $F : \mathrm{ob}\,\mathcal{D} \to \mathrm{ob}\,\mathrm{Cat}$, a function $F : \mathrm{mor}\,\mathcal{D} \to \mathrm{mor}\,\mathrm{Cat}$, for composable arrows $a \xrightarrow{f} b \xrightarrow{g} c$ of $\mathcal{D}$ a natural isomorphism $\gamma_{g,f} : F(f) \circ F(g) \to F(f \circ g)$ and for every object a of $\mathcal{D}$ a natural isomorphism $\delta_a : 1_{Fa} \to F(1_a)$. These natural isomorphisms are required to satisfy the following coherence axioms for associativity and unit.

Associativity. For every triple

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$$

of composable arrows in $\mathcal{D}$ the diagram

$$\begin{tikzcd}
Fh \circ Fg \circ Ff \arrow[r, "\mathrm{id}_{Fh} * \gamma_{g,f}"] \arrow[d, "\gamma_{h,g} * \mathrm{id}_{Ff}"'] & Fh \circ F(gf) \arrow[d, "\gamma_{h,gf}"] \\
F(hg) \circ Ff \arrow[r, "\gamma_{hg,f}"] & F(hgf)
\end{tikzcd}$$

commutes.

Unit. For every $f : a \to b$ in $\mathcal{D}$ the diagrams

$$\begin{tikzcd}
Ff \circ 1_{Fa} \arrow[r, "\mathrm{id}_{Ff} * \delta_a"] \arrow[d, "\mathrm{id}_{Ff}"'] & Ff \circ F1_a \arrow[d, "\gamma_{f,1_a}"] \\
Ff \arrow[r, "\mathrm{id}_{Ff}"] & F(f \circ 1_a)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{Fb} \circ Ff \arrow[r, "\delta_b * \mathrm{id}_{Ff}"] \arrow[d, "\mathrm{id}_{Ff}"'] & F1_b \circ Ff \arrow[d, "\gamma_{1_b,f}"] \\
Ff \arrow[r, "\mathrm{id}_{Ff}"] & F(1_b \circ f)
\end{tikzcd}$$

commute.

The following concept of simple morphisms was suggested by Steiner in [**28**].

DEFINITION 3.8. A morphism

$$\phi = [X \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B \xrightarrow{\phi_3} Y]$$

in U_+^G is called *simple* if firstly, for every $b \in B$, the restriction of ϕ_1 to $\phi_2^{-1}(b)$ is injective, and secondly $\phi_3(b) = \phi_3(b')$ and $\phi_1(\phi_2^{-1}(b)) = \phi_1(\phi_2^{-1}(b'))$ implies $b = b'$. We denote by sU_+^G the partial category with morphisms given by the class of simple morphisms in U_+^G.

LEMMA 3.9. *Given a product-preserving set-valued partial functor F on sU_+^G there exists a unique extension of F to a semi-Tambara functor.*

Proof. By [**30**, Proposition 7.3] the category U_+^G has a presentation involving only simple morphisms. □

DEFINITION 3.10. A *Tambara category* is a product-preserving partial pseudo-functor on sU_+^G.

We shall mostly be interested in Tambara categories arising as homotopy categories of Quillen model categories. Given a Quillen model category $\mathcal{D}$, there exists a localization $\gamma : \mathcal{D} \to \mathrm{ho}\,\mathcal{D}$ of $\mathcal{D}$ with respect to the class of weak equivalences. The category $\mathrm{ho}\,\mathcal{D}$ is the *homotopy category* of $\mathcal{D}$. Given a functor $F : \mathcal{D} \to \mathcal{E}$ from a Quillen model category to a category $\mathcal{E}$, a total left-derived functor of F consists of a functor $LF : \mathrm{ho}\,\mathcal{D} \to \mathcal{E}$ and a natural transformation $t : LF \circ \gamma \to F$ with the following universal property: for every functor $G : \mathrm{ho}\,\mathcal{D} \to \mathcal{E}$ and natural transformation $s : G \circ \gamma \to F$, there exists a unique natural transformation $s' : G \to LF$ such that $s = t \circ (s' * \gamma)$.

The following results will be used to construct homotopy Tambara categories.

PROPOSITION 3.11. *Let $\mathcal{C}$ be a Tambara category with a Quillen model structure on $\mathcal{C}(X)$ for every object X in U_+^G. Suppose that*

(i) *for every $\phi \in sU_+^G$ the functor $\mathcal{C}(\phi)$ has a total left-derived functor and*
(ii) *for composable morphisms ϕ and ψ in sU_+^G, the composite of the total left-derived functors of $\mathcal{C}(\phi)$ and $\mathcal{C}(\psi)$ is a total left-derived functor of $\mathcal{C}(\psi) \circ \mathcal{C}(\phi)$.*

Then there is a Tambara category $\mathrm{ho}\,\mathcal{C}$ *with* $\mathrm{ho}\,\mathcal{C}(\phi)$ *defined to be a (chosen) total left-derived functor of the functor $\mathcal{C}(\phi)$.*

Proof. Condition (ii) implies that for composable arrows $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ of sU_+^G there is an isomorphism

$$L\mathcal{C}(\psi) \circ L\mathcal{C}(\phi) \cong L(\mathcal{C}(\psi) \circ \mathcal{C}(\phi)) \cong L(\mathcal{C}(\psi \circ \phi)).$$

Since any two total left-derived functors of $\mathcal{C}(\psi \circ \phi)$ are isomorphic by a unique isomorphism, the axioms for a Tambara category are readily verified. □

COROLLARY 3.12. *Let $\mathcal{C}$ be a Tambara category with a Quillen model category structure on $\mathcal{C}(X)$ for every object X in U^G_+. Suppose that for every $\phi \in sU^G_+(X,Y)$ the functor $\mathcal{C}(\phi)$ preserves cofibrant objects and that it takes acyclic cofibrations between cofibrant objects to weak equivalences. Then there is a Tambara category* $\mathrm{ho}\,\mathcal{C}$ *with* $\mathrm{ho}\,\mathcal{C}(\phi)$ *defined to be a (chosen) total left-derived functor of the functor $\mathcal{C}(\phi)$.*

Proof. K. Brown's lemma (see for example [**11**, Lemma 9.9]) implies that $\mathcal{C}(\phi)$ preserves weak equivalences between cofibrant objects. Therefore (see for example [**11**, Proposition 9.3]) $\mathcal{C}(\phi)$ has a total left-derived functor $L\mathcal{C}(\phi)$ and for every cofibrant object c of $\mathcal{C}(X)$ the map $L\mathcal{C}(\phi)(c) \to \mathcal{C}(\phi)(c)$ is an isomorphism in $\mathrm{ho}\,\mathcal{C}(Y)$. Since $\mathcal{C}(\phi)$ preserves cofibrant objects, it follows that, given $\psi \in sU^G_+$ such that ϕ and ψ are composable, the composite of the total left-derived functors of $\mathcal{C}(\phi)$ and $\mathcal{C}(\psi)$ is a total left-derived functor of $\mathcal{C}(\psi) \circ \mathcal{C}(\phi)$. □

Given a G-set X we consider X as the object set of the *translation category* X with $X(x,y) = \{g \in G : gx = y\}$. Composition in X is given by multiplication in G.

The following construction is of fundamental importance for us. It is related to, and inspired by, Even's construction of multiplicative induction in [**13**]. Greenlees and May made a similar construction in [**14**]. Our construction mainly differs from the previous ones by the fact that we do not work with wreath products. Given a set Y, the *free* $\{+,\cdot\}$*-algebra* on Y is the set $\coprod_{k\geqslant 1} \underline{A}(Y)_k$ where $\underline{A}(Y)_1 = Y$ and

$$\begin{aligned} \underline{A}(Y)_k = \{&w_1 + w_2 : w_i \in \underline{A}(Y)_{k_i} \text{ and } k_1 + k_2 = k\} \\ &\amalg \{w_1 \cdot w_2 : w_i \in \underline{A}(Y)_{k_i} \text{ and } k_1 + k_2 = k\} \end{aligned}$$

for $k > 1$.

CONSTRUCTION 3.13. Given a bimonoidal category $(\mathcal{C}_0, \square, \diamond, n_\square, n_\diamond)$ in the sense of Laplaza [**20**] we construct a Tambara category $\mathcal{C} = \mathcal{C}(\mathcal{C}_0)$. Here $\square$ is the additive operation and $\diamond$ is the multiplicative operation so one of the isomorphisms for distributivity takes the form

$$c_1 \diamond (c_2 \square c_3) \cong (c_1 \diamond c_2) \square (c_1 \diamond c_3).$$

Recall that $\square$ and $\diamond$ are functors from $\mathcal{C}_0 \times \mathcal{C}_0$ to $\mathcal{C}_0$ and that $n_\square$ and $n_\diamond$ are functors from the trivial category $*$ to C_0.

Given a finite G-set X we let $\mathcal{C}(X)$ denote the category of functors from the translation category of X to $\mathcal{C}_0$. The function $X \mapsto \mathcal{C}(X)$ from the set of objects of U^G_+ to the class of categories clearly preserves products. Below we define $\mathcal{C}$ on morphisms.

We follow Laplaza and let $\underline{A}\{X\}$ denote the free $\{+,\cdot\}$-algebra on $X \amalg \{n_+, n_.\}$. Given $g \in G$ we denote by g_* the endomorphism of $\underline{A}\{X\}$ induced by the action $g\cdot : X \to X$.

The category $\mathrm{Fun}(\mathcal{C}(X), \mathcal{C}_0)$ of functors from $\mathcal{C}(X)$ to $\mathcal{C}_0$ has the structure of a bimonoidal category with operations defined pointwise. There is a function ev from $\underline{A}\{X\}$ to the set of objects of $\mathrm{Fun}(\mathcal{C}(X), \mathcal{C}_0)$ defined by letting $\mathrm{ev}_{n_+} = n_\square$, $\mathrm{ev}_{n_.} = n_\diamond$ and $\mathrm{ev}_x(\alpha) = \alpha(x)$ for $x \in X$ and $\alpha \in \mathcal{C}(X)$ and by requiring that $\mathrm{ev}_{w_1 \cdot w_2} = \mathrm{ev}_{w_1} \diamond \mathrm{ev}_{w_2}$ and $\mathrm{ev}_{w_1 + w_2} = \mathrm{ev}_{w_1} \square \mathrm{ev}_{w_2}$ for $w_1, w_2 \in \underline{A}\{X\}$. We also consider the natural epimorphism $\mathrm{supp} : \underline{A}\{X\} \to \mathbb{N}[X]$, from $A\{X\}$ to the free commutative semi-ring $\mathbb{N}[X]$, defined by letting $\mathrm{supp}(n_+) = 0$, $\mathrm{supp}(n_.) = 1$ and $\mathrm{supp}(x) = x$ for $x \in X$ and by requiring that $\mathrm{supp}(w_1 + w_2) = \mathrm{supp}(w_1) + \mathrm{supp}(w_2)$ and $\mathrm{supp}(w_1 \cdot w_2) = \mathrm{supp}(w_1) \cdot \mathrm{supp}(w_2)$ for $w_1, w_2 \in \underline{A}\{X\}$.

The coherence theorem of Laplaza [**20**, Section 7] implies that if $\mathrm{supp}(w) = \mathrm{supp}(w') \in \mathbb{Z}[X]$ is simple, then there exists a preferred natural transformation $\beta_{w,w'} : \mathrm{ev}_w \to \mathrm{ev}_{w'}$, and if further $\mathrm{supp}(w'') = \mathrm{supp}(w')$ then $\beta_{w',w''} \circ \beta_{w,w'} = \beta_{w,w''}$.

We choose for every morphism

$$\phi = [X \xleftarrow{\phi_1} A \xrightarrow{\phi_2} B \xrightarrow{\phi_3} Y]$$

in $sU_+^G(X,Y)$ and every $y \in Y$ an element $w_{\phi,y} \in \underline{A}\{X\}$ with $\mathrm{supp}(w_{\phi,y}) = \phi_y$, where

$$\phi_y = \sum_{\phi_3(b)=y} \prod_{\phi_2(a)=b} \phi_1(a) \in \mathbb{N}[X].$$

If ϕ is an identity morphism then we insist on choosing $w_{\phi,y} = y$ for $y \in Y$. Note that $\mathrm{supp}(g_*(w_{\phi,y})) = \mathrm{supp}(w_{\phi,gy})$. Given $\alpha \in \mathcal{C}(X)$, the maps $\alpha(g) : \alpha(x) \to \alpha(gx)$ for $x \in X$ induce a map $g_*(\alpha) : \mathrm{ev}_{w_{\phi,y}}(\alpha) \to \mathrm{ev}_{g_*(w_{\phi,y})}(\alpha)$.

We define $\mathcal{C}(\phi) : \mathcal{C}(X) \to \mathcal{C}(Y)$ on objects by constructing $\mathcal{C}(\phi)(\alpha) : Y \to \mathcal{C}_0$ for $\alpha \in \mathcal{C}(X)$ as follows: let $\mathcal{C}(\phi)(\alpha)(y) = \mathrm{ev}_{w_{\phi,y}}(\alpha)$ for $y \in Y$ and define $\mathcal{C}(\phi)(\alpha)(g) : \mathcal{C}(\phi)(\alpha)(y) \to \mathcal{C}(\phi)(\alpha)(gy)$ as the composition

$$\mathrm{ev}_{w_{\phi,y}}(\alpha) \xrightarrow{g_*(\alpha)} \mathrm{ev}_{g_*(w_{\phi,y})}(\alpha) \xrightarrow{\beta_{g_*(w_{\phi,y}),w_{\phi,gy}}} \mathrm{ev}_{w_{\phi,gy}}(\alpha)$$

for $y \in Y$ and $g \in G$. We define $\mathcal{C}(\phi)$ on a morphism $t : \alpha \to \alpha'$ in $\mathcal{C}(X)$ as follows: the maps $t_x : \alpha(x) \to \alpha'(x)$ for $x \in X$ induce maps $\mathrm{ev}_{w_{\phi,y}}(t) : \mathrm{ev}_{w_{\phi,y}}(\alpha) \to \mathrm{ev}_{w_{\phi,y}}(\alpha')$ for $y \in Y$, and these maps define the morphism $\mathcal{C}(\phi)(t) : \mathcal{C}(\phi)(\alpha) \to \mathcal{C}(\phi)(\alpha')$.

Given morphisms $\phi \in U_+^G(X,Y)$ and $\psi \in U_+^G(Y,Z)$ we have $(\psi \circ \phi)_z = \psi_z(\phi_y \,|\, y \in Y)$, that is, $(\psi\phi)_z$ is obtained by substituting the variables $y \in Y$ by the polynomials ϕ_y in the polynomial ψ_z. Thus $\mathrm{supp}(w_{\psi\circ\phi,z}) = \mathrm{supp}(w_{\psi,z}(w_{\phi,y} \,|\, y \in Y))$ provided that ϕ, ψ and $\psi \circ \phi$ are simple. We define $\gamma_{\psi,\phi} : \mathcal{C}(\psi) \circ \mathcal{C}(\phi) \to \mathcal{C}(\psi \circ \phi)$ by $(\gamma_{\psi,\phi})_z = \beta_{w_{\psi,z}(w_{\phi,y}\,|\,y\in Y), w_{\psi\circ\phi,z}}$ for $z \in Z$. Since $\mathcal{C}(1_Y) = 1_{\mathcal{C}(Y)}$ we define $\delta_Y : \mathrm{id}_{\mathcal{C}(Y)} \to \mathcal{C}(\mathrm{id}_Y)$ to be the identity. The coherence axioms for $\mathcal{C}$ follow directly from Laplaza's coherence result.

It will be convenient to use the symbol $\mathcal{C}$ both for the bimonoidal category $(\mathcal{C}_0, \Box, \diamond, n_\Box, n_\diamond)$ and the Tambara category $\mathcal{C}$ of Construction 3.13. For example we shall consider the Tambara category $\mathcal{T}$ of pointed topological spaces with $\mathcal{T}(X) = \mathrm{Fun}(X, \mathcal{T})$.

Definition 3.14. A partial fibration $\mathbb{G} : \mathcal{G} \to \mathcal{D}$ consists of a partial functor $\mathbb{G}$ subject to the following axioms.

Composability. If the domain of $f \in \mathcal{G}$ agrees with the codomain of $g \in \mathcal{G}$ then f and g are composable in $\mathcal{G}$ if and only if $\mathbb{G}(f)$ and $\mathbb{G}(g)$ are composable in $\mathcal{D}$.

Fibration criterion. For every arrow $\alpha : b \to c$ in $\mathcal{D}$ and object z in $\mathcal{G}$ with $\mathbb{G}(z) = c$ there exists an arrow $f : y \to z$ in $\mathcal{G}$ such that $\mathbb{G}(f) = \alpha$ and with the property that given an arrow $g : x \to z$ in $\mathcal{G}$ with $\mathcal{G}(g) = \alpha \circ \beta$ for composable arrows α and β in $\mathcal{D}$, there exists a unique arrow $h : x \to y$ in $\mathcal{G}$ such that $\mathbb{G}(h) = \beta$ and $g = f \circ h$.

The Composability axiom implies that given a partial fibration $\mathbb{G} : \mathcal{G} \to \mathcal{D}$ and an object d of $\mathcal{D}$, the fiber $\mathbb{G}^{-1}(d)$, that is, the partial category with object set $\{x \in \mathcal{G} : \mathbb{G}(x) = d\}$ and morphism set $\{\alpha \in \mathcal{G} : \mathbb{G}(\alpha) = \mathrm{id}_d\}$, are categories, not only partial categories. The fibration criterion is the usual criterion for categorical fibrations in the sense of Gothendieck; see for example [**3**, Chapter 8]. The following construction is well known; see for example loc. cit.

Construction 3.15. Given a partial pseudo-functor F on $\mathcal{D}$ we construct a partial fibration $\mathbb{G}(F) : \mathcal{G}(F) \to \mathcal{D}^{\mathrm{op}}$ whose fiber at $d \in \mathcal{D}$ is precisely the category $F(d)^{\mathrm{op}}$:

– an object of $\mathcal{G}(F)$ is a pair (d, x) where $d \in \mathcal{D}$ and $x \in F(d)$ are, respectively, objects of $\mathcal{D}$ and $F(d)$;
– an arrow $(d, x) \to (d', x')$ in $\mathcal{G}(F)$ is a pair (α, f) where $\alpha : d' \to d$ and $f : F(\alpha)(x') \to x$ are, respectively, arrows of $\mathcal{D}$ and $F(d)$;
– the functor $\mathbb{G}(F) : \mathcal{G}(F) \to \mathcal{D}^{\mathrm{op}}$ is the first component projection; thus $\mathbb{G}(F)(d, x) = d$ and $\mathbb{G}(F)(\alpha, f) = \alpha$.

We must explain how to provide $\mathcal{G}(F)$ with a partial category structure. Consider arrows $(\alpha, f) : (d'', x'') \to (d', x')$ and $(\beta, g) : (d', x') \to (d, x)$ in $\mathcal{G}(F)$. Provided α and β are composable, this yields the composite

$$F(\alpha\beta)(x'') \cong F(\beta)F(\alpha)(x'') \xrightarrow{F(\beta)(f)} F(\beta)(x') \xrightarrow{\quad g \quad} x$$

in $F(d)$, where the first isomorphism is the associativity isomorphism for the partial pseudo-functor F. Writing $g \star f$ for this composite, we define the composition law by the relation $(\alpha, f) \circ (\beta, g) = (\beta \circ \alpha, g \star f)$. The associativity of this composition follows immediately from the associativity axiom for pseudo-functors. On the other hand, the unit axiom for pseudo-functors implies that the pair $(1_d, \delta_{d,x}^{-1})$ is an identity morphism on the object (d, x) of $\mathcal{G}(F)$. Here $\delta_d : 1_{Fd} \to F(1_d)$ is the unit isomorphism for F. This proves that $\mathcal{G}(F)$ is a partial category. The functoriality of $\mathbb{G}(F)$ is obvious.

We leave the proof of the following lemma to the reader.

LEMMA 3.16. *Given a product-preserving partial pseudo-functor $F : \mathcal{D} \to \mathrm{Cat}$, objects d, d' of $\mathcal{D}$ and an object z of $F(d \times d')$, the object $(d \times d', z \in F(d \times d'))$ is the product of $(d, F(\mathrm{pr}_1)z)$ and $(d', F(\mathrm{pr}_2)z)$ in $\mathcal{G}(F)^{\mathrm{op}}$.*

4. *Induction of cofibrations*

In this section we consider a complete and cocomplete symmetric monoidal category $(\mathcal{C}_0, \diamond, u_\diamond)$, with the property that the functor $c \diamond -$ commutes with colimits for every object c of C_0. Denoting the coproduct of $\mathcal{C}_0$ by $\amalg$ and the initial object by $\emptyset$, we have a bimonoidal category $(\mathcal{C}_0, \amalg, \diamond, \emptyset, u_\diamond)$ and we can consider the Tambara category $\mathcal{C} = \mathcal{C}(\mathcal{C}_0)$ of Construction 3.13. Since $\diamond$ preserves colimits we deduce, for every finite G-set X and every $c \in \mathrm{ob}\, \mathcal{C}(X)$, that the functor $c \diamond -$ preserves push-outs and finite coproducts and that $c \diamond \emptyset \cong \emptyset$.

Given a map $f : X \to Y$ of finite G-sets we obtain functors $f_\diamond := C(N_f)$ and $f_\amalg := C(T_f)$ from $C(X)$ to $C(Y)$ and $f^* := C(R_f)$ from $C(Y)$ to $C(X)$.

Further, we fix for every finite G-set X subsets $J(X)$ and $I(X)$ of the class of morphisms in $\mathcal{C}(X)$, and given objects a and b of $\mathcal{C}(X)$ we write $J(X)(a, b)$ and $I(X)(a, b)$ for $J(X) \cap \mathcal{C}(X)(a, b)$ and $I(X) \cap \mathcal{C}(X)(a, b)$ respectively. These subsets are required to satisfy firstly that given a sum diagram

$$X_1 \xrightarrow{i_1} X \xleftarrow{i_2} X_2$$

of finite G-sets, the isomorphism $(i_1^*, i_2^*) : \mathcal{C}(X) \xrightarrow{\cong} \mathcal{C}(X_1) \times \mathcal{C}(X_2)$ induces bijections

$$J(X)(a, b) \cong J(X_1)(i_1^* a, i_1^* b) \times J(X_2)(i_2^* a, i_2^* b),$$
$$I(X)(a, b) \cong I(X_1)(i_1^* a, i_1^* b) \times I(X_2)(i_2^* a, i_2^* b).$$

Secondly, we require that for every finite G-set X the data $(\mathcal{C}(X), I(X), J(X))$ specifies a cofibrantly generated symmetric monoidal Quillen model category with generating cofibrations $I(X)$ and generating acyclic cofibrations $J(X)$, and that for every G-map $f : X \to Y$, the pair $f_\amalg : \mathcal{C}(X) \rightleftarrows \mathcal{C}(Y) : f^*$ is a Quillen adjoint pair.

By abuse of notation we shall let $\mathbb{Z}$ denote the category associated to the underlying linearly ordered set of the integers. The functor $\mathrm{Fun}(\mathbb{Z}, -)$ taking a category $\mathcal{D}$ to the category $\mathrm{Fun}(\mathbb{Z}, \mathcal{D})$ of filtered objects in $\mathcal{D}$ preserves products, so we obtain a Tambara category $\mathbb{Z}\mathcal{C} := \mathrm{Fun}(\mathbb{Z}, \mathcal{C})$. In Section 9 we have collected some basic facts about filtered objects.

DEFINITION 4.1. Given a G-set X and a morphism $c : C_{-1} \to C_0$ in $\mathcal{C}(X)$, we denote by $\overline{C}$ the object in $(\mathbb{Z}\mathcal{C})(X)$ with $\overline{C}(i) = C_{-1}$ for $i \leqslant -1$ and with $\overline{C}(i) = C_0$ for $i \geqslant 0$. For $i = -1$ the morphism $\overline{C}(i, i+1) : \overline{C}(i) \to \overline{C}(i+1)$ is the morphism c and for $i \neq -1$ it is an identity morphism.

THEOREM 4.2. *Suppose for every G-map $g : W \to Z$ that*

(i) *the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is a cofibration for every c in $I(W)$, and*
(ii) *the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is an acyclic cofibration for every c in $J(W)$.*

Then for every map $f : X \to Y$ of finite G-sets the functor $f_\diamond : \mathcal{C}(X) \to \mathcal{C}(Y)$ preserves both cofibrations and acyclic cofibrations between cofibrant objects.

COROLLARY 4.3. *Suppose for every map $g : W \to Z$ of finite G-sets that*

(i) *for every $d : D_{-1} \to D_0$ in $I(Z)$ the map $g^*(D_{-1}) \to g^*(D_0)$ is a cofibration in $\mathcal{C}(W)$,*
(ii) *for every $d : D_{-1} \to D_0$ in $J(Z)$ the map $g^*(D_{-1}) \to g^*(D_0)$ is an acyclic cofibration in $\mathcal{C}(W)$,*
(iii) *for every $c : C_{-1} \to C_0$ in $I(W)$ the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is a cofibration in $\mathcal{C}(Z)$, and*
(iv) *for every $c : C_{-1} \to C_0$ in $J(W)$ the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is an acyclic cofibration in $\mathcal{C}(Z)$.*

Then for every morphism $\phi \in sU_+^G$ the functor $\mathcal{C}(\phi)$ preserves both cofibrations and acyclic cofibrations between cofibrant objects.

Proof. For a map $f : X \to Y$ of finite G-sets, the functor $f^* : \mathcal{C}(Y) \to \mathcal{C}(X)$ has a right adjoint $f_\Pi : \mathcal{C}(X) \to \mathcal{C}(Y)$ with $f_\Pi(C)(y) \cong \prod_{x \in f^{-1}(y)} C(x)$. In particular, f^* preserves relative cell complexes. Since every (acyclic) cofibration is a retract of a relative cell complex, the functor f^* preserves (acyclic) cofibrations. Thus for every map $f : X \to Y$ of finite G-sets, the functors f_Π and f^* preserve (acyclic) cofibrations between cofibrant objects, and so does $f_\diamond$ by Theorem 4.2. By [**30**, Proposition 7.3], for every $\phi \in sU_+^G$ the functor $\mathcal{C}(\phi)$ is isomorphic to a composite of functors of the above sort. □

COROLLARY 4.4. *For every symmetric monoidal category $\mathcal{C}_0$ with symmetric monoidal Quillen model structures on the categories $\mathcal{C}(X) = \mathrm{Fun}(X, \mathcal{C}_0)$ for X in U_+^G satisfying the assumptions of Corollary 4.3, there exists a Tambara category $\mathrm{ho}\,\mathcal{C}$ with $\mathrm{ho}\,\mathcal{C}(\phi)$ defined to be a (chosen) total left-derived functor of the functor $\mathcal{C}(\phi)$.*

Proof. This follows by combining Corollary 4.3 and Corollary 3.12. □

Since we are only dealing with finite G-sets, Theorem 4.2 is a consequence of the following theorem.

THEOREM 4.5. *Suppose for every G-map $g : W \to Z$ with fibres of cardinality less than or equal to n that*

(i) *the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is a cofibration for every c in $I(W)$ and*
(ii) *the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is an acyclic cofibration for every c in $J(W)$.*

Then for every map $f : X \to Y$ of finite G-sets with fibres of cardinality less than or equal to n, the functor $f_\diamond : \mathcal{C}(X) \to \mathcal{C}(Y)$ preserves both cofibrations and acyclic cofibrations between cofibrant objects.

EXAMPLE 4.6. We illustrate Theorem 4.5 by considering a simple example. Let $G = \mathbb{Z}/2\mathbb{Z}$ and consider the map $f : G/e \to G/G$ of G-sets. The functor

$$\mathcal{C}_0 = \mathrm{Fun}(*, \mathcal{C}_0) \longrightarrow \mathrm{Fun}(G/e, \mathcal{C}_0) = \mathcal{C}(G/e)$$

induced by the functor from the translation category of G/e to the final category $*$ is an equivalence of categories. Suppose that

$$\begin{tikzcd} C_{-1} \arrow[r, "c"] \arrow[d] & C_0 \arrow[d] \\ A_{-1} \arrow[r, "a"] & A_0 \end{tikzcd}$$

is a push-out diagram in $\mathcal{C}_0$ and that the map $(f_\diamond\overline{C})(-1) \to (f_\diamond\overline{C})(0)$ is a cofibration in $\mathcal{C}(G/G)$. Then also the lower horizontal map in the push-out diagram

$$\begin{tikzcd} (f_\diamond\overline{C})(-1) \arrow[r, equal] \arrow[d] & C_{-1} \diamond C_0 \coprod_{C_{-1}\diamond C_{-1}} C_0 \diamond C_{-1} \arrow[r] & (f_\diamond\overline{C})(0) \arrow[d] \\ (f_\diamond\overline{A})(-1) \arrow[r, equal] & A_{-1} \diamond A_0 \coprod_{A_{-1}\diamond A_{-1}} A_0 \diamond A_{-1} \arrow[r] & (f_\diamond\overline{A})(0) \end{tikzcd}$$

is a cofibration in $\mathcal{C}(G/G)$. On the other hand, suppose that the map $C_{-1} \diamond C_{-1} \to C_0 \diamond C_{-1}$ is a cofibration in $\mathcal{C}_0$. Then $f_{\amalg}(C_{-1} \diamond C_{-1}) \to f_{\amalg}(C_0 \diamond C_{-1})$ is a cofibration in $\mathcal{C}(G/G)$, and therefore also the lower horizontal map in the push-out diagram

$$\begin{tikzcd} f_{\amalg}(C_{-1} \diamond C_{-1}) \arrow[r] \arrow[d] & f_{\amalg}(C_0 \diamond C_{-1}) \arrow[d] \\ A_{-1} \diamond A_{-1} \arrow[r] & A_{-1} \diamond A_0 \coprod_{A_{-1}\diamond A_{-1}} A_0 \diamond A_{-1} \end{tikzcd}$$

is a cofibration in $\mathcal{C}(G/G)$. Thus the composition

$$f_\diamond(A_{-1}) = A_{-1} \diamond A_{-1} \longrightarrow (f_\diamond\overline{A})(-1) \longrightarrow (f_\diamond\overline{A})(0) = f_\diamond(A_0)$$

is a cofibration.

The rest of this section is devoted to a proof by induction on n of Theorem 4.5.

LEMMA 4.7. *Let $f : X \to Y$ be a map of finite G-sets. Suppose that the map*

$$(f_\diamond\overline{C})(-1) \longrightarrow (f_\diamond\overline{C})(0)$$

is a cofibration in $\mathcal{C}(Y)$ for every $c : C_{-1} \to C_0$ in $I(X)$ and that it is an acyclic cofibration for every $c : C_{-1} \to C_0$ in $J(X)$. Then the map $(f_\diamond\overline{A})(-1) \to (f_\diamond\overline{A})(0)$ is a cofibration in $\mathcal{C}(Y)$ for every cofibration $a : A_{-1} \to A_0$ in $\mathcal{C}(X)$ and it is an acylcic cofibration for every acyclic cofibration $a : A_{-1} \to A_0$ in $\mathcal{C}(X)$.

Proof. Assume first that there exists a push-out diagram of the form

$$\begin{tikzcd} C_{-1} \arrow[r, "c"] \arrow[d] & C_0 \arrow[d] \\ A_{-1} \arrow[r, "a"] & A_0 \end{tikzcd}$$

with c in $I(X)$ (respectively $J(X)$). Then by Lemma 9.3 the diagram

$$\begin{tikzcd} (f_\diamond \overline{C})(-1) \arrow[r] \arrow[d] & (f_\diamond \overline{C})(0) \arrow[d] \\ (f_\diamond \overline{A})(-1) \arrow[r] & (f_\diamond \overline{A})(0) \end{tikzcd}$$

is a push-out diagram. Since by assumption the upper horizontal map is a cofibration, so is the lower horizontal map. It follows that the map $(f_\diamond \overline{A})(-1) \to (f_\diamond \overline{A})(0)$ is an (acyclic) cofibration for every relative cell complex $a : A_{-1} \to A_0$. The lemma now follows from the fact that every (acyclic) cofibration is a retract of a relative cell complex. $\square$

Definition 4.8. Given a morphism $c : C_{-1} \to C_0$ in $\mathcal{C}(X)$, we define an object $\widehat{C}$ in $(\mathbb{Z}\mathcal{C})(X)$ as follows: for $i < -1$ we let $\widehat{C}(i) = \emptyset$ be the initial object of $\mathcal{C}(X)$, for $i = -1$ we let $\widehat{C}(i) = C_{-1}$, and for $i \geqslant 0$ we let $\widehat{C}(i) = C_0$. The map $\widehat{C}(i) \to \widehat{C}(i+1)$ is the identity except for $i = -2, -1$. For $i = -2$ it is the unique map $\emptyset = \widehat{C}(-2) \to \widehat{C}(-1)$, and for $i = -1$ it is given by the map c.

Note that for every map $f : X \to Y$ of finite G-sets there are isomorphisms

$$(f_\diamond \overline{C})(-1) \cong (f_\diamond \widehat{C})(-1) \quad \text{and} \quad (f_\diamond \overline{C})(0) \cong (f_\diamond \widehat{C})(0) \cong f_\diamond C_0.$$

Lemma 4.9. *Suppose that for every map $g : W \to Z$ of finite G-sets with fibers of cardinality less than or equal to $n-1$ we have:*

(i) *for every cofibrant object B of $\mathcal{C}(W)$ the object $g_\diamond B$ is cofibrant in $\mathcal{C}(Z)$,*
(ii) *for every $c : C_{-1} \to C_0$ in $I(W)$ the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is a cofibration, and*
(iii) *for every $c : C_{-1} \to C_0$ in $J(W)$ the map $(g_\diamond \overline{C})(-1) \to (g_\diamond \overline{C})(0)$ is an acyclic cofibration.*

Then for every map $f : X \to Y$ of finite G-sets with fibers of cardinality n and every $k \in \mathbb{Z}$ with $-n < k < 0$, the map $(f_\diamond \widehat{A})(k-1) \to (f_\diamond \widehat{A})(k)$ is a cofibration for every cofibration $a : A_{-1} \to A_0$ of cofibrant objects in $\mathcal{C}(X)$, and it is an acyclic cofibration for every acyclic cofibration $a : A_{-1} \to A_0$ of cofibrant objects in $\mathcal{C}(X)$.

Proof of Theorem 4.5. We prove the theorem by induction on n. The theorem holds in the case $n = 0$. Assume that the theorem holds for $n-1$ and note that if $X = X' \amalg X''$, then every map $f : X \to Y$ is of the form $\nabla \circ (f' \amalg f'')$ for G-maps $f' : X' \to Y$ and $f'' : X'' \to Y$, and where $\nabla : Y \amalg Y \to Y$ is the fold map. Using the push-out product axiom in $\mathcal{C}(Y)$ we see that it suffices to show that the theorem holds for every map $f : X \to Y$ of finite G-sets with fibers of cardinality exactly n and every (acyclic) cofibration $a : A_{-1} \to A_0$ of cofibrant objects in $\mathcal{C}(X)$. In that case, $f_\diamond A_{-1} \cong (f_\diamond \widehat{A})(-n)$ and $f_\diamond A_0 \cong (f_\diamond \widehat{A})(0)$. It follows that it suffices to show that for $-n+1 \leqslant k \leqslant 0$ the map $(f_\diamond \widehat{A})(k-1) \to (f_\diamond \widehat{A})(k)$ is an (acyclic) cofibration in $\mathcal{C}(Y)$. The case $k = 0$ is treated in Lemma 4.7, and using our inductive assumption, we see that the case $-n < k < 0$ follows from Lemma 4.9. $\square$

We now introduce some notation needed for the proof of Lemma 4.9. Let $f : X \to Y$ be a map of finite G-sets and let $p : \{-1, 0\} \times X \to X$ denote the projection. We consider an exponential diagram of the form

$$\begin{tikzcd}
X \arrow[d, "f"'] & \{-1,0\} \times X \arrow[l, "p"'] & X' \arrow[l, "e"'] \arrow[d, "f'"] \\
Y \arrow[r, equal] & Y & Y' \arrow[l, "q"']
\end{tikzcd}$$

Given $y' \in Y'$ we let $|y'| := \sum_{x' \in (f')^{-1}(y')} \mathrm{pr}(e(x')) \in \mathbb{Z}$, where $\mathrm{pr} : \{-1, 0\} \times X \to \{-1, 0\}$ is the projection, and given $k \in \mathbb{Z}$ we let $i_k : Y'_k \to Y'$ denote the inclusion of the subset Y'_k consisting of the elements $y' \in Y'$ with $|y'| = k$. We let X'_{-1} and X'_0 be defined as pull-backs in the diagram

$$\begin{tikzcd}
X'_{-1} \arrow[r] \arrow[d] & X' \arrow[d] & X'_0 \arrow[l] \arrow[d] \\
\{-1\} \times X \arrow[r] & \{-1,0\} \times X & \{0\} \times X \arrow[l]
\end{tikzcd}$$

We can now construct a commutative diagram of the form

$$\begin{tikzcd}
X' \arrow[r, equal] \arrow[d, "f'"'] & X'_{-1} \amalg X'_0 \arrow[d, "f'_{-1} \amalg f'_0"] & X'_{-1,k} \amalg X'_{0,k} \arrow[l, "j_{-1,k} \amalg j_{0,k}"'] \arrow[d, "f'_{-1,k} \amalg f'_{0,k}"] \\
Y' & Y' \amalg Y' \arrow[l, "\nabla"'] & Y'_k \amalg Y'_k \arrow[l, "i_k \amalg i_k"']
\end{tikzcd}$$

where ∇ is the fold map and where the right-hand square is a pull-back.

Given a map $a : A_{-1} \to A_0$ in $\mathcal{C}(X)$ we use the isomorphism

$$\mathcal{C}(X) \times \mathcal{C}(X) \cong \mathcal{C}(\{-1\} \times X) \times \mathcal{C}(\{0\} \times X) \cong \mathcal{C}(\{-1, 0\} \times X)$$

to consider $(\emptyset, A_{-1})$ and (A_{-1}, A_0) as objects of $\mathcal{C}(\{-1, 0\} \times X)$. We let

$$A(k)_{-1} = (j_{-1,k} \amalg j_{0,k})^* e^* (\emptyset, A_{-1})$$

and

$$A(k)_0 = (j_{-1,k} \amalg j_{0,k})^* e^* (A_{-1}, A_0) \in \mathcal{C}({X'}_{-1,k} \amalg {X'}_{0,k}).$$

The map a induces a map $a(k) : A(k)_{-1} \to A(k)_0$, and we consider the object $\widehat{A(k)} \in \mathbb{Z}\mathcal{C}({X'}_{-1,k} \amalg {X'}_{0,k})$ of Definition 4.8. Note that considering A_{-1} as a functor from $\mathbb{Z}$ to $\mathcal{C}(X)$ with $A_{-1}(k) = \emptyset$ for $k < 0$ and $A_{-1}(k) = A_{-1}$ for $k \geqslant 0$ we have

$$\widehat{A(k)} = (j^*_{-1,k} e^* p^* A_{-1}, j^*_{0,k} e^* p^* \widehat{A}).$$

We define an object $T_{A,k}$ of $\mathbb{Z}\mathcal{C}(Y)$ by the formula

$$T_{A,k} := q_{\amalg} \circ (\nabla \circ (i_k \amalg i_k) \circ (f'_{-1,k} \amalg f'_{0,k}))_{\diamond}(\widehat{A(k)}).$$

LEMMA 4.10. *For every $k \in \mathbb{Z}$ there is a push-out diagram in $\mathcal{C}(Y)$ of the form*

$$\begin{tikzcd} T_{A,k}(-1) \arrow[r] \arrow[d] & T_{A,k}(0) \arrow[d] \\ (f_\diamond \widehat{A})(k-1) \arrow[r] & (f_\diamond \widehat{A})(k) \end{tikzcd}$$

Proof. Fix $y \in Y$ and let $U = f^{-1}(y)$. There is a bijection $\phi : \{-1,0\}^U \xrightarrow{\cong} q^{-1}(y)$. Let $V_k \subseteq V_{\leqslant k} \subseteq \mathbb{Z}^U$ denote the sets consisting of the maps $a : U \to \mathbb{Z}$ with $\sum_{u \in U} a(u) = k$ and $\sum_{u\in U} a(u) \leqslant k$ respectively. There is a unique functor $T : (\mathbb{Z}^U, \leqslant) \to \mathcal{C}(*)$ with $T(\alpha) = \Diamond_{x \in U} \widehat{A}(\alpha(x))(x)$. For $\alpha : U \to \{-1,0\}$ we have

$$T(\alpha) = \mathop{\Diamond}_{x\in U} A_{\alpha(x)}(x) \cong \mathop{\Diamond}_{x' \in f'^{-1}(\phi(\alpha))} (e^*(A_{-1}, A_0))(x') \cong (f'_\diamond e^*(A_{-1}, A_0))(\phi(\alpha)).$$

Writing out the definitions we get

$$\begin{aligned} (T_{A,k}(0))(y) &\cong \coprod_{y' \in q^{-1}(y)} ((\nabla(i_k \amalg i_k)(f'_{-1,k} \amalg f'_{0,k}))_\diamond \widehat{A(k)})(0)(y') \\ &\cong \coprod_{y' \in q^{-1}(y)} (i_k f'_{-1,k})_\diamond (j^*_{-1,k} e^* p^* A_{-1})(y') \diamond (i_k f'_{0,k})_\diamond (j^*_{0,k} e^* p^* \widehat{A})(0)(y') \\ &\cong \coprod_{\alpha \in V_k \cap \{-1,0\}^U} \left(\mathop{\Diamond}_{x \in \alpha^{-1}(-1)} A_{-1}(x) \right) \diamond \left(\mathop{\Diamond}_{x \in \alpha^{-1}(0)} A_0(x) \right) \\ &\cong \coprod_{\alpha \in V_k \cap \{-1,0\}^U} \left(\mathop{\Diamond}_{x\in U} A_{\alpha(x)}(x) \right) = \coprod_{\alpha \in V_k \cap \{-1,0\}^U} T(\alpha), \end{aligned}$$

and similarly we get

$$\begin{aligned} (T_{A,k}(-1))(y) &\cong \coprod_{\alpha \in V_k \cap \{-1,0\}^U} \left(\mathop{\Diamond}_{x \in \alpha^{-1}(-1)} A_{-1}(x) \right) \diamond \left(\left(\mathop{\Diamond}_{x \in \alpha^{-1}(0)} \widehat{A}(x) \right)(-1) \right) \\ &\cong \coprod_{\alpha \in V_k \cap \{-1,0\}^U} \operatorname*{colim}_{\beta \in V_{<\alpha}} T(\beta), \end{aligned}$$

where $V_{<\alpha} \subseteq \mathbb{Z}^U$ consists of those $\beta \in \mathbb{Z}^U \setminus \alpha$ satisfying $\beta(u) \leqslant \alpha(u)$ for $u \in U$. Note that if $\alpha \notin \{-1,0\}^U$ then the natural map $\operatorname{colim}_{\beta \in V_{<\alpha}} T(\beta) \to T(\alpha)$ is an isomorphism. On the other hand, we have

$$(f_\diamond \widehat{A})(k)(y) \cong \left(\mathop{\Diamond}_{x\in U} \widehat{A}(x) \right)(k) \cong \operatorname*{colim}_{\alpha \in V_{\leqslant k}} T(\alpha),$$

and $(f_\diamond \widehat{A})(k-1)(y) \cong \operatorname{colim}_{\beta \in V_{\leqslant k-1}} T(\beta)$. The lemma now follows from Lemma 9.4. ∎

LEMMA 4.11. *Under the assumptions of Lemma 4.9 the map $T_{A,k}(-1) \to T_{A,k}(0)$ is an (acyclic) cofibration for $-n < k < 0$.*

Proof. Since $-n < k < 0$, the fibers of the maps $f'_{-1,k}$ and $f'_{0,k}$ are of cardinality less than or equal to $n-1$. By assumption the object $(f'_{-1,k})_\diamond j^*_{-1,k} e^* p^* A_{-1}$ is cofibrant in $\mathcal{C}(Y'_k)$.

The map $a(k) = (j_{-1,k} \amalg j_{0,k})^* e^*(\mathrm{id}_{A_{-1}}, a)$ is an (acyclic) cofibration and so is by Lemma 4.7 the map

$$(f'_{-1,k} \amalg f'_{0,k})_\diamond \widehat{A(k)}(-1) \longrightarrow (f'_{-1,k} \amalg f'_{0,k})_\diamond \widehat{A(k)}(0)$$

in $\mathcal{C}(Y'_k)$. Under the isomorphism $\mathcal{C}(Y'_k) \times \mathcal{C}(Y'_k) \cong \mathcal{C}(Y'_k \amalg Y'_k)$ this map can be identified with the map

$$\begin{aligned}((f'_{-1,k})_\diamond j^*_{-1,k} e^* p^* A_{-1}, ((f'_{0,k})_\diamond j^*_{-1,k} e^* p^* \widehat{A})(-1))\\ \longrightarrow ((f'_{-1,k})_\diamond j^*_{-1,k} e^* p^* A_{-1}, ((f'_{0,k})_\diamond j^*_{-1,k} e^* p^* \widehat{A})(0)),\end{aligned}$$

and the map

$$(\nabla(i_k \amalg i_k)(f'_{-1,k} \amalg f'_{0,k}))_\diamond \widehat{A(k)}(-1) \longrightarrow (\nabla(i_k \amalg i_k)(f'_{-1,k} \amalg f'_{0,k}))_\diamond \widehat{A(k)}(0)$$

can be identified with the map

$$\begin{aligned}(i_k \circ f'_{-1,k})_\diamond j^*_{-1,k} e^* p^* A_{-1} \diamond ((i_k \circ f'_{0,k})_\diamond j^*_{0,k} e^* p^* \widehat{A})(-1)\\ \longrightarrow (i_k \circ f'_{-1,k})_\diamond j^*_{-1,k} e^* p^* A_{-1} \diamond ((i_k \circ f'_{0,k})_\diamond j^*_{0,k} e^* p^* \widehat{A})(0).\end{aligned}$$

This map is an (acyclic) cofibration by the push-out product axiom in $\mathcal{C}(Y')$. The result now follows from the fact that as $q_\amalg$ is a left Quillen functor, it preserves (acyclic) cofibrations. □

Proof of Lemma 4.9. By Lemmas 4.10 and 4.11, for $-n < k < 0$ we know that the map $(f_\diamond \widehat{A})(k-1) \to (f_\diamond \widehat{A})(k)$ is a push-out of an (acyclic) cofibration. The statement of Lemma 4.9 now follows since a push-out of an (acyclic) cofibration is an (acyclic) cofibration. □

5. *Constructing Tambara functors*

In this section we study (semi-) Tambara functors arising from Tambara categories. Given a Tambara category $\mathcal{C}$ there is an opposite Tambara category $\mathcal{C}^{\mathrm{op}}$ with $\mathcal{C}^{\mathrm{op}}(X) = \mathcal{C}(X)^{\mathrm{op}}$. In Section 3 we constructed product-preserving partial functors $\mathbb{G}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} : \mathcal{G}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to sU^G_+$ and $\mathbb{G}(\mathcal{C})^{\mathrm{op}} : \mathcal{G}(\mathcal{C})^{\mathrm{op}} \to sU^G_+$. We denote by $\hom_{\mathcal{C}} : (\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{U^{G\,\mathrm{op}}_+} \mathcal{G}(\mathcal{C}))^{\mathrm{op}} \to \mathcal{S}et$ the product-preserving partially defined functor with value $\hom_{\mathcal{C}}((X,x),(X,y)) = \mathcal{C}(X)(x,y)$ on the object $((X,x),(X,y))$ of $(\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{sU^{G\,\mathrm{op}}_+} \mathcal{G}(\mathcal{C}))^{\mathrm{op}}$. Let $s : sU^G_+ \to (\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{sU^{G\,\mathrm{op}}_+} \mathcal{G}(\mathcal{C}))^{\mathrm{op}}$ be a product-preserving section of the projection $(\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{sU^{G\,\mathrm{op}}_+} \mathcal{G}(\mathcal{C}))^{\mathrm{op}} \to sU^G_+$. By Lemma 3.9, there exists a unique semi-Tambara functor $\mathcal{C}(s) : U^G_+ \to \mathcal{S}et$ extending the product-preserving partial functor $\hom_{\mathcal{C}} \circ s : sU^G_+ \to \mathcal{S}et$.

Definition 5.1. A *commutative semi-ring* in a bimonoidal category $\mathcal{C}_0$ is an object A of $(\mathcal{C}_0, \diamond, \square, n_\diamond, n_\square)$ with commutative monoid structures for both $\diamond$ and $\square$ making the following distributivity diagram commutative:

$$\begin{tikzcd}
A \diamond (A \square A) \arrow[r] \arrow[d, "\cong"'] & A \diamond A \arrow[r] & A \arrow[d, equal] \\
(A \diamond A) \square (A \diamond A) \arrow[r] & A \square A \arrow[r] & A
\end{tikzcd}$$

Construction 5.2. Let $(\mathcal{C}_0, \diamond, \square, n_\diamond, n_\square)$ be a bimonoidal category, let $\mathcal{C} = \mathcal{C}(\mathcal{C}_0)$ denote the Tambara category of Construction 3.13, let A be a commutative semi-ring in $\mathcal{C}_0$, and let

B be a commutative semi-ring $\mathcal{C}_0^{\mathrm{op}}$. We shall construct a product-preserving partial functor

$$(s_B, s_A) : sU_+^G \longrightarrow (\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{(sU_+^G)^{\mathrm{op}}} \mathcal{G}(\mathcal{C}))^{\mathrm{op}}.$$

As in the above discussion we get a product-preserving partial functor

$$sU_+^G \xrightarrow{(s_B,s_A)} (\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{(sU_+^G)^{\mathrm{op}}} \mathcal{G}(\mathcal{C}))^{\mathrm{op}} \xrightarrow{\hom_{\mathcal{C}}} \mathcal{S}et$$

and a semi-Tambara functor $\mathcal{C}(B, A) : U_+^G \to \mathcal{S}et$ extending the above partial functor.

Let $p_X : X \to *$ denote the G-map from X to a point and let p_X^* denote the functor $\mathcal{C}(R_{p_X}) : \mathcal{C}(*) \to \mathcal{C}(X)$. We define a section

$$s_A : sU_+^G \longrightarrow \mathcal{G}(\mathcal{C})^{\mathrm{op}}$$

of $\mathbb{G}(\mathcal{C})^{\mathrm{op}}$ with $s_A(X) = (X, p_X^* A)$. By Lemma 3.16, s_A preserves products. Let $\phi : X \to Y$ be a morphism of sU_+^G. In order to construct $s_A(\phi) : (Y, p_Y^* A) \to (X, p_X^* A)$ we need to specify a morphism $s_A(\phi)(y) : \mathcal{C}(\phi)(p_X^* A)(y) \to (p_Y^* A)(y) = A$ for every $y \in Y$. In Construction 3.13 we chose $w_{\phi,y} \in \underline{A}\{X\}$ with $\mathrm{supp}(w_{\phi,y}) = \phi_y$ and we defined the element $\mathcal{C}(\phi)(p_X^* A)(y) = \mathrm{ev}_{w_{\phi,y}}(p_X^*)$ as the evaluation in $\mathcal{C}_0$ of the word $w_{\phi,y}$ in the letters A, $\Box$, $\diamond$, $n_\Box$ and $n_\diamond$. Since A is a monoid with respect to both $\Box$ and $\diamond$, there is a morphism $s_A(\phi)(y) : \mathrm{ev}_{w_{\phi,y}}(p_X^*) \to A$, and this morphism is independent of the choice of $w_{\phi,y}$ because A is a commutative semi-ring. This ends the construction of s_A.

Dually, since B is a commutative semi-ring in $\mathcal{C}_0^{\mathrm{op}}$, we obtain a product-preserving section $s_B : sU_+^G \to \mathcal{G}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ of $\mathbb{G}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. Combining s_A and s_B we obtain the desired product-preserving partial functor

$$(s_B, s_A) : sU_+^G \longrightarrow (\mathcal{G}(\mathcal{C}^{\mathrm{op}}) \times_{(sU_+^G)^{\mathrm{op}}} \mathcal{G}(\mathcal{C}))^{\mathrm{op}}.$$

EXAMPLE 5.3. The category $(\mathcal{A}b, \oplus, \otimes, 0, \mathbb{Z})$ of abelian groups is bimonoidal. A semi-ring A in $\mathcal{A}b$ is the same thing as a commutative algebra, and a semi-ring B in $\mathcal{A}b^{\mathrm{op}}$ is the same thing as a commutative coalgebra. For a finite G-set of the form G/H for a subgroup H of G we find that $\hom_{\mathcal{A}b}(B, A)(G/H) = \hom_{\mathbb{Z}}(B, A)^H$ is the ring of H-equivariant $\mathbb{Z}$-linear maps from B to A. In particular, for $B = \mathbb{Z}$ we recover the invariant ring Tambara functor of [**30**, Example 3.1].

EXAMPLE 5.4. A semi-ring in the category $(\mathcal{C}h_{\mathbb{Z}}, \oplus, \otimes, 0, \mathbb{Z})$ of chain complexes of abelian groups is the same thing as a differential graded commutative algebra, and a semi-ring in $\mathcal{C}h_{\mathbb{Z}}^{\mathrm{op}}$ is the same thing as a differential graded commutative coalgebra. For a finite G-set of the form G/H for a subgroup H of G we see that $\hom_{\mathcal{C}h_{\mathbb{Z}}}(B, A)(G/H) = \mathcal{C}h_{\mathbb{Z}}(B, A)^H$ is the ring of H-equivariant $\mathbb{Z}$-linear chain maps from B to A. It is desirable to pass to homology of these chain complexes. Evens does this in the case $B = \mathbb{Z}[T]$ where T in degree 2 has trivial action of G, and where A is concentrated in degree zero [**13**; **30**, Example 3.4]. We shall return to this situation in Example 7.1.

Given a Tambara category $\mathcal{C}$ satisfying the assumptions of Proposition 3.11 there exists a Tambara category $\mathrm{ho}\,\mathcal{C}$ with $\mathrm{ho}\,\mathcal{C}(\phi)$ given by a total left-derived functor of $\mathcal{C}(\phi)$ for $\phi \in sU_+^G$.

PROPOSITION 5.5. *Under the assumptions of Proposition* 3.11 *there is a partial functor* $\gamma : \mathcal{G}(\mathcal{C}) \to \mathcal{G}(\mathrm{ho}\,\mathcal{C})$ *with* $\mathbb{G}(\mathrm{ho}\,\mathcal{C}) \circ \gamma = \mathbb{G}(\mathcal{C})$.

Proof. On objects we define γ to be the identity, and we define γ on morphisms by letting γ take a morphism

$$(X, c) \xleftarrow{(\phi,\alpha)} (Y, d) \quad \text{in } \mathcal{G}(\mathcal{C}),$$

consisting of a morphism $\phi : X \to Y$ in sU_+^G and a morphism $\alpha : \mathcal{C}(\phi)c \to d$ in $\mathcal{C}(Y)$, to the morphism

$$\Big((X, c) \xleftarrow{(\phi,\, \gamma_2(\phi, \alpha))} (Y, d)\Big) \quad \text{in } \mathcal{G}(\mathrm{ho}\,\mathcal{C}),$$

where $\gamma_2(\phi, \alpha)$ is the composite

$$\mathrm{ho}\,\mathcal{C}(\phi)c \xrightarrow{t} \mathcal{C}(\phi)c \xrightarrow{\alpha} d.$$

Here t is part of the structure of the total left-derived functor of $\mathcal{C}(\phi)$. Using the universal property of total left-derived functors, it is easy to see that γ preserves identity morphisms and compositions. □

In the case where we are given a commutative semi-ring in $\mathcal{C}_0$ we can combine Proposition 5.5 and Construction 5.2 to obtain a partial functor $sU_+^G \to \mathcal{G}(\mathcal{C})^{\mathrm{op}} \to \mathcal{G}(\mathrm{ho}\,\mathcal{C})^{\mathrm{op}}$. On the other hand, given a commutative semi-ring in $\mathcal{C}_0^{\mathrm{op}}$, we would like to obtain a partial functor $\sigma : sU_+^G \to \mathcal{G}(\mathrm{ho}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$. This will be possible if the map $t : \mathrm{ho}\,\mathcal{C}(\phi)c \to \mathcal{C}(\phi)c$ is an isomorphism for every object c in the image of σ and every $\phi \in U_+^G$. Note that in this case we can even arrange for $t : \mathrm{ho}\,\mathcal{C}(\phi)c \to \mathcal{C}(\phi)c$ to be an identity morphism by changing our choice of the total left-derived functor $\mathrm{ho}\,\mathcal{C}(\phi)$ of $\mathcal{C}(\phi)$. In § 7.2 we consider a functor $\sigma : sU_+^G \to \mathcal{G}(\mathrm{ho}\,\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ which is not constructed from a commutative semi-ring in $\mathcal{C}^{\mathrm{op}}$.

6. *Homotopical Tambara categories*

In this section we show that the Tambara categories constructed from the categories of pointed topological spaces, orthogonal spectra and chain complexes come with Quillen model structures satisfying the assumptions of Corollary 4.4. In particular, these categories have associated homotopy Tambara categories.

6.1. *Pointed topological spaces*

Let $\mathcal{T}$ denote the category of compactly generated pointed topological spaces. This is a closed symmetric monoidal model category with a set I of generating cofibrations given by standard inclusions of the form $S_+^{n-1} \to D_+^n$ for $n \geqslant 0$. A set J of generating acyclic cofibrations is given by the maps $D_+^n \to (D^n \times [0,1])_+$ induced by the lower inclusion $D^n \cong D^n \times \{0\} \subseteq D^n \times [0,1]$.

Given a G-set X we denote by $\mathcal{L}_X$ the category of set-valued functors F on the translation category of X with the property that for every $x \in X$ the group $X(x, x) \subseteq G$ acts transitively on $F(x)$, and we let L_X denote a set of representatives for the isomorphism classes of $\mathcal{L}_X$. (If $X = \{x\}$ is a one-element G-set, the category $\mathcal{L}_X$ is isomorphic to the category of transitive G-sets, and we could choose L_X to be the family of functors corresponding to the transitive G-sets X/H, with H running over the conjugacy classes of subgroups of G.) We denote by I_X the set of morphisms in $\mathcal{T}(X) = \mathrm{Fun}(X, \mathcal{T})$ of the form $\mathrm{id} \wedge \alpha : F_+ \wedge S_+^{n-1} \to F_+ \wedge D_+^n$ for $F \in L_X$ and $\alpha \in I$. Here $F_+ : X \to \mathcal{T}$ is the functor with $F_+(x)$ given by the pointed topological space obtained by adding a disjoint base point to the set $F(x)$ for $x \in X$. We denote by J_X the set of morphisms in $\mathcal{T}(X)$ of the form

$$\mathrm{id} \wedge \alpha : F_+ \wedge D_+^n \longrightarrow F_+ \wedge (D^n \times [0,1])_+$$

for $F \in L_X$ and $\alpha \in J$. The following theorem is well known. See for example [**10**, Section 1.2] or [**23**, Theorem III.1.8].

THEOREM 6.1. *There is a model structure on $\mathcal{T}(X)$ with I_X and J_X as sets of generating cofibrations and generating acyclic cofibrations respectively. In this model structure a map $f : A \to B$ is a fibration if and only if for every object x of X and every subgroup H of*

$X(x,x)$ the map $A(x)^H \to B(x)^H$ of H-fixpoints induced by f_x is a fibration in $\mathcal{T}$. It is a weak equivalence if and only if for every x and H as above the map $A(x)^H \to B(x)^H$ is a weak equivalence.

PROPOSITION 6.2. *Suppose that $f : X \to Y$ is a map of finite G-sets. Then the map $(f_\wedge\overline{C})(-1) \to (f_\wedge\overline{C})(0)$ is a cofibration for every generating cofibration $c : C_{-1} \to C_0$ in $\mathcal{T}(X)$ and it is an acyclic cofibration for every generating acyclic cofibration $c : C_{-1} \to C_0$ in $\mathcal{T}(X)$.*

Proof. We only prove the statement about acyclic cofibrations. The map c is of the form $F_+ \wedge \alpha$ for an F in L_X and $\alpha : A_{-1} = D^n_+ \to A_0 = (D^n \times [0,1])_+$ in J. We have

$$(f_\wedge\overline{C})(-1) \cong (f_\wedge F_+) \wedge (f_\wedge\overline{A})(-1) \quad \text{and} \quad (f_\wedge\overline{C})(0) \cong (f_\wedge F_+) \wedge (f_\wedge\overline{A})(0).$$

The map $(f_\wedge\overline{A})(-1) \to (f_\wedge\overline{A})(0)$ is an acyclic cofibration in $\mathcal{T}$ by the push-out product axiom, and there is an isomorphism of the form

$$(f_\wedge F_+) \cong (M_1)_+ \vee \ldots \vee (M_n)_+ \quad \text{with } M_1, \ldots, M_n \in L_Y.$$

The result now follows from the fact that the functor $(f_\wedge F_+) \wedge - : \mathcal{T} \to \mathcal{T}(Y)$ commutes with colimits. □

COROLLARY 6.3. *There is a Tambara category* $\operatorname{ho}\mathcal{T}$ *with* $\operatorname{ho}\mathcal{T}(X)$ *given by the homotopy category of $\mathcal{T}(X)$ and with* $\operatorname{ho}\mathcal{T}(\phi)$ *given by a chosen total left-derived functor of the functor $\mathcal{T}(\phi)$ for $\phi \in sU^G_+(X,Y)$.*

Proof. This follows by combining Proposition 6.2 and Corollary 4.4. □

REMARK 6.4. The above results also hold in the context of simplicial sets. The simplicial analogue of Theorem 6.1 is treated in [**9**, Theorem 9.5].

6.2. *Orthogonal spectra*

Let V be a finite-dimensional real inner product space. We denote by S^V the one-point compactification of V with the added point ∞ as base point. The addition in V extends to S^V if one declares $x + y = \infty$ if either x or y is equal to ∞. In other words, $x + y$ is the image of $(x,y) \in S^V \times S^V$ under the composition $S^V \times S^V \to S^V \wedge S^V \cong S^{V\oplus V} \to S^V$, where the last map is induced by addition in V. The group of linear isometries of V is denoted $O(V)$. Let $V \subseteq W$ be an inclusion of finite-dimensional real inner product spaces and let $W - V \subseteq W$ denote the orthogonal complement of V. Given $y \in S^W$ the pointed map $t_y : S^V \to S^W$ is defined by $t_y(x) = x + y$ for $x \in S^V$. Given $k \in O(W)$ we extend k to a pointed map $k : S^W \to S^W$.

Given an inclusion $V \subseteq W$ of finite-dimensional real inner product spaces we let $\mathcal{I}_S(V,W)$ denote the subspace of $\mathcal{T}(S^V, S^W)$ consisting of maps of the form $k \circ t_y$ for $y \in S^{W-V}$ and $k \in O(W)$. Let $y \in S^{W-V}$ and $h \in O(V)$ and pick $h' \in O(W)$ with the property that $h'(x) = h(x)$ for $x \in V$ and such that $h'(y) = y$. Then $t_y \circ h = h' \circ t_y$. In particular,

$$k \circ t_y \circ h \circ t_x = k \circ h' \circ t_{x+y} \in \mathcal{I}_S(U,W)$$

for every $k \in O(W)$, $h \in O(V)$, $y \in S^{W-V}$ and $x \in S^{V-U}$. It follows that the composition $\mathcal{T}(S^V, S^W) \wedge \mathcal{T}(S^U, S^V) \to \mathcal{T}(S^U, S^W)$ induces a composition

$$\mathcal{I}_S(V,W) \wedge \mathcal{I}_S(U,V) \longrightarrow \mathcal{I}_S(U,W)$$

making $\mathcal{I}_S$ a category enriched over $\mathcal{T}$ with the class of finite-dimensional real inner product spaces as object class. Further, a direct sum of vector spaces induces a symmetric monoidal product $\oplus$ on $\mathcal{I}_S$.

Note that $\Phi : O(W)_+ \wedge_{O(W-V)} S^{W-V} \to \mathcal{I}_S(V,W)$ defined by $\Phi([k,y]) = k \circ t_y$ for $k \in O(W)$ and $y \in S^{W-V}$ is a homeomorphism.

The following definition is taken from [**24**, Example 4.4].

Definition 6.5. The category Sp^O of orthogonal spectra is the category of $\mathcal{T}$-functors from $\mathcal{I}_S$ to $\mathcal{T}$.

Defining the smash-product $E \wedge F$ of orthogonal spectra $E, F \in \mathrm{Sp}^O$ as the left Kan extension of the composition

$$\mathcal{I}_S \times \mathcal{I}_S \xrightarrow{E \times F} \mathcal{T} \times \mathcal{T} \xrightarrow{\wedge} \mathcal{T}$$

along $\oplus : \mathcal{I}_S \times \mathcal{I}_S \to \mathcal{I}_S$ as in [**24**, Definition 21.4], and defining internal function objects as in [**24**, Definition 21.6], we see that the category Sp^O of orthogonal spectra becomes a complete and cocomplete closed symmetric monoidal category. Let us abuse notation and denote by Sp^O the Tambara functor with $\mathrm{Sp}^O(X) = \mathrm{Fun}(X, \mathrm{Sp}^O)$ for a finite G-set X.

Given a finite G-set X we let $\mathrm{sk}(\mathcal{I}_S(X^{\mathrm{op}}))$ denote a set containing one representative for each isomorphism class of objects in $\mathcal{I}_S(X^{\mathrm{op}}) = \mathrm{Fun}(X^{\mathrm{op}}, \mathcal{I}_S)$. For A in $\mathcal{I}_S(X^{\mathrm{op}})$ and V in $\mathcal{I}_S$, there is a functor $\mathcal{I}_S(A,V) : X \to \mathcal{T}$ because $\mathcal{I}_S$ is enriched over $\mathcal{T}$. Thus we can consider the object $\mathcal{I}_S(A,-)$ of $\mathrm{Sp}^O(X)$. The set FI_X consists of the morphisms $\mathcal{I}_S(A,-) \wedge \alpha$ in $\mathrm{Sp}^O(X)$ for $A \in \mathrm{sk}(\mathcal{I}_S(X^{\mathrm{op}}))$ and α in I_X of Theorem 6.1. The set FJ_X consists of the morphisms $\mathcal{I}_S(A,-) \wedge \alpha$ in $\mathrm{Sp}^O(X)$ for $A \in \mathrm{sk}(\mathcal{I}_S(X^{\mathrm{op}}))$ and α in J_X of Theorem 6.1.

For the following result we refer to [**9**, Theorem 4.2] and [**24**, Lemma 6.5].

Theorem 6.6. *For every finite G-set X there is a model structure on the category $\mathrm{Sp}^O(X)$ with FI_X as a set of generating cofibrations and with FJ_X as a set of generating acyclic cofibrations. A morphism $\alpha : E \to B$ is a fibration in this model category if and only if for every object A of $\mathcal{I}_S$ the map $\alpha(A) : E(A) \to B(A)$ is a fibration in $\mathcal{T}(X)$, and it is a weak equivalence if and only if the map $\alpha(A)$ is a weak equivalence in $\mathcal{T}(X)$ for every object A of $\mathcal{I}_S$. We call this the projective model structure on $\mathrm{Sp}^O(X)$.*

We refer to fibrations, weak equivalences and acyclic cofibrations in the projective model structure on Sp^O as projective fibrations, projective weak equivalences and projective acyclic cofibrations respectively. The cofibrations in the projective model structure will be referred to simply as cofibrations.

Lemma 6.7. *For every map $f : X \to Y$ of finite G-sets, the map $(f_\wedge \overline{C})(-1) \to (f_\wedge \overline{C})(0)$ is a cofibration in $\mathrm{Sp}^O(Y)$ for every generating cofibration $c : C_{-1} \to C_0$ in $\mathrm{Sp}^O(X)$, and it is a projective acyclic cofibration for every projective acyclic cofibration $c : C_{-1} \to C_0$ in $\mathrm{Sp}^O(X)$.*

Proof. Using the observation $f_\wedge \mathcal{I}_S(A,-) \cong \mathcal{I}_S(f_\oplus A,-)$, we note that the proof is similar to the proof of Proposition 6.2. □

Combining Lemma 6.7 and Corollary 4.3 we obtain the following result.

Corollary 6.8. *For every morphism $\phi \in sU_+^G$, the functor $\mathrm{Sp}^O(\phi)$ preserves both cofibrations and projective acyclic cofibrations between cofibrant objects.*

Before defining the stable equivalences in Sp^O we discuss a general construction on a cocomplete symmetric monoidal category $(\mathcal{C}, \diamond, n_\diamond)$. Given morphisms $a : A_{-1} \to A_0$ and $b : B_{-1} \to B_0$ in $\mathcal{C}$ we have the morphism $a \Box b : (\overline{A} \diamond \overline{B})(-1) \to (\overline{A} \diamond \overline{B})(0)$. It is well known that this operation $\Box$ is a symmetric monoidal product on the category $\mathrm{Map}\,\mathcal{C}$ of arrows in

$\mathcal{C}$ where, for a and b as above, a morphism $a \to b$ in $\operatorname{Map}\mathcal{C}$ consists of maps $\phi_i : A_i \to B_i$ for $i = -1, 0$ such that $\phi_0 a = b\phi_{-1}$ (see for example [**16**, p. 109]).

Given a morphism $A \to B$ in $\mathcal{I}_S(X^{\mathrm{op}})$ such that for every $x \in X$ the map $S^{A(x)} \to S^{B(x)}$ is induced by an inclusion of real inner-product spaces, we let S^{A-B} denote the object of $\mathcal{I}_S(X^{\mathrm{op}})$ with $S^{A-B}(x) = S^{A(x)-B(x)}$ for $x \in X$. Since X is a groupoid, there is an isomorphism $X \to X^{\mathrm{op}}$ of categories taking a morphism to its inverse. We consider $\mathcal{I}_S(A, B)$ as a functor from X to $\mathcal{T}$ via the composition

$$X \longrightarrow X \times X \longrightarrow X \times X^{\mathrm{op}} \xrightarrow{A \times B} \mathcal{I}_S^{\mathrm{op}} \times \mathcal{I}_S \xrightarrow{\mathrm{Hom}_{\mathcal{I}_S}} \mathcal{T}$$

and we let $\lambda_{A,B}$ denote the composition

$$\mathcal{I}_S(B, -) \wedge S^{B-A} \subseteq \mathcal{I}_S(B, -) \wedge \mathcal{I}_S(A, B) \longrightarrow \mathcal{I}_S(A, -).$$

We choose a factorization

$$\mathcal{I}_S(B, -) \wedge S^{B-A} \xrightarrow{k_{A,B}} M\lambda_{A,B} \xrightarrow{r_{A,B}} \mathcal{I}_S(A, -)$$

of $\lambda_{A,B}$ by a cofibration $k_{A,B}$ and a projective acyclic fibration $r_{A,B}$. Let E_X denote the set of morphisms of the form $k_{A,B}\square i$ where $k_{A,B}$ is of the above form and $i \in I_X$. We define $K_X = FJ_X \cup E_X$. The following is [**23**, Theorem III.5.3].

THEOREM 6.9. *For every finite G-set X there is a model structure on $\mathrm{Sp}^{\mathcal{O}}(X)$ with FI_X as a set of generating cofibrations and with K_X as a set of generating acyclic cofibrations. This is the stable model structure on $\mathrm{Sp}^{\mathcal{O}}(X)$.*

We refer to fibrations, weak equivalences and acyclic cofibrations in the stable model structure on $\mathrm{Sp}^{\mathcal{O}}(X)$ as stable fibrations, stable weak equivalences and stable acyclic cofibrations respectively. We refer to [**23**, Proposition III.7.5] or [**9**, Lemma 6.27] for the following lemma.

LEMMA 6.10. *The push-out product axiom holds in $\mathrm{Sp}^{\mathcal{O}}(X)$ with the stable model structure.*

The next lemma is a reformulation of [**23**, Proposition V.2.3].

LEMMA 6.11. *Let $f : X \to Y$ be a map of finite G-sets. The functor $f_\vee : \mathrm{Sp}^{\mathcal{O}}(X) \to \mathrm{Sp}^{\mathcal{O}}(Y)$ is a left Quillen functor with respect to the stable model structure.*

The following lemma follows from [**23**, Lemma V.2.2].

LEMMA 6.12. *Let $f : X \to Y$ be a map of finite G-sets. The functor $f^* : \mathrm{Sp}^{\mathcal{O}}(Y) \to \mathrm{Sp}^{\mathcal{O}}(X)$ is a left Quillen functor with respect to the stable model structure.*

PROPOSITION 6.13. *For every map $f : X \to Y$ of finite G-sets and every $c : C_{-1} \to C_0$ in K_X the map $(f_\wedge\overline{C})(-1) \to (f_\wedge\overline{C})(0)$ is an acyclic cofibration in the stable model structure on $\mathrm{Sp}^{\mathcal{O}}(Y)$.*

Proof. If $c \in FJ_X$ then, by Lemma 6.7, the map $f_\square c : (f_\wedge\overline{C})(-1) \to (f_\wedge\overline{C})(0)$ is a projective acyclic cofibration, and in particular it is a stable acyclic cofibration. Thus we are left with the case where c is in E_X. Assume by induction that the proposition holds if f has fibers of cardinality less than or equal to $n-1$. Suppose that f has fibers of cardinality less than or equal to n. By the push-out product axiom in $\mathrm{Sp}^{\mathcal{O}}(Y)$ we can, without loss of generality, assume that the fibers for f all have cardinality n.

Let us start by considering the case $c = k_{A,B}$ for $k_{A,B} : \mathcal{I}_S(B,-) \wedge S^{B-A} \to M\lambda_{A,B}$ as above, where $C_{-1} = \mathcal{I}_S(B,-) \wedge S^{B-A}$ and $C_0 = M\lambda_{A,B}$. With the notation of Definition 4.8 we have a commutative diagram of the form

$$\begin{tikzcd}
(f_\wedge \widehat{C})(-n) \arrow[r, "\gamma"] \arrow[d, "\cong"'] & (f_\wedge \widehat{C})(-1) \arrow[r, "f_\square k_{A,B}"] & (f_\wedge \widehat{C})(0) \arrow[d, "\delta"] \\
f_\wedge(C_{-1}) \arrow[r, "\cong"] & \mathcal{I}_S(f_\oplus B, -) \wedge S^{f_\oplus(B-A)} \arrow[r, "\lambda_{f_\oplus A, f_\oplus B}"] & \mathcal{I}_S(f_\oplus A, -)
\end{tikzcd}$$

Using our inductive assumption and Corollary 6.8 we can apply Lemma 4.9 to conclude that γ is a stable acyclic cofibration. Since $M\lambda_{A,B}$ and $\mathcal{I}_S(A,-)$ are cofibrant, it follows by combining Corollary 6.8 and K. Brown's lemma (see for example [**11**, Lemma 9.9]) that δ is a projective equivalence. In particular, δ is a stable equivalence. The lower horizontal map $\lambda_{f_\oplus A, f_\oplus B}$ is a stable equivalence by definition. It follows that $f_\square k_{A,B}$ is a stable equivalence. Since the cofibrations in the projective and in the stable model structure are the same, we can conclude by Corollary 6.8 that $f_\square k_{A,B}$ is a stable acyclic cofibration. For c of the form $c = k_{A,B} \square i$, the associativity and commutativity isomorphisms for $\square$ induce an isomorphism $f_\square c \cong (f_\square k_{A,B}) \square (f_\square i)$. This is a stable acyclic cofibration by the push-out product axiom. We conclude that for every $c \in E_X$ the map $f_\square c$ is a stable acyclic cofibration. $\square$

Applying Proposition 6.13, Corollary 6.8, Corollary 4.3, Corollary 4.4 and K. Brown's lemma (see for example [**11**, Lemma 9.9]) we obtain the following corollary.

COROLLARY 6.14. *For every morphism $\phi \in sU_+^G$, the functor $\mathrm{Sp}^O(\phi)$ preserves stable equivalences between cofibrant objects. In particular, there is a Tambara category $\mathrm{ho}\,\mathrm{Sp}^O$ with $\mathrm{ho}\,\mathrm{Sp}^O(\phi)$ given by a (chosen) total left-derived functor of $\mathrm{Sp}^O(\phi)$.*

6.3. *Chain complexes*

Let us consider the category $\mathcal{C}h_R$ of chain complexes over a commutative ring R with the projective model structure, that is, the model structure where the weak equivalences are the quasi-isomorphisms and where the fibrations are the surjective chain homomorphisms. Given a G-set X we consider the model structure on the category $\mathcal{C}h_R(X) = \mathrm{Fun}(X, \mathcal{C}h_R)$ where a map $\alpha : A \to B$ is a fibration or weak equivalence if and only if for every $x \in X$ the map $\alpha_x : A(x) \to B(x)$ is so. It is well known that such a model structure exists (compare, for example [**24**, Theorem 6.5]). In particular, $\mathcal{C}h_R(G/G)$ is isomorphic to the category $\mathcal{C}h_{R[G]}$ of chain complexes over the group ring $R[G]$ with the projective model structure. Let us consider the forgetful functor $U : \mathcal{F}_G \to \mathcal{F}$ from finite G-sets to finite sets. Given a finite G-set X there is an inclusion $j_X : UX \subseteq X$ of translation categories. Here UX is a category with only identity morphisms. There is an induced functor

$$j_X^* : \mathcal{C}h_R(X) \to \mathrm{Fun}(UX, \mathcal{C}h_R) = \prod_{x \in X} \mathcal{C}h_R,$$

and a morphism α in $\mathcal{C}h_R(X)$ is a weak equivalence or fibration if and only if $j_X^*(\alpha)$ is so.

PROPOSITION 6.15. *For every $\phi \in sU_+^G(X,Y)$ the functor $\mathcal{C}h_R(\phi)$ preserves weak equivalences between objects A in $\mathcal{C}h_R(X)$ with the property that $j_X^*(A)$ is cofibrant in $\mathcal{C}h_R(UX)$. In particular, if $j_X^*(A)$ is cofibrant in $\mathcal{C}h_R(UX)$, then the object $j_Y^*(\mathcal{C}h_R(\phi)A)$ is cofibrant in $\mathcal{C}h_R(UY)$.*

Proof. Before we treat the general case, let us for a moment assume that G is the trivial group. Then U is simply the identity functor on $\mathcal{F}$. Given a map $f : X \to Y$ of finite sets, the functor $f_\otimes : \mathcal{Ch}_R(X) \to \mathcal{Ch}_R(Y)$ is just given by iterated applications of the tensor product $\otimes : \mathcal{Ch}_R \times \mathcal{Ch}_R \to \mathcal{Ch}_R$. It follows from the push-out product axiom for $\mathcal{Ch}_R$ that the functor $f_\otimes$ preserves weak equivalences between cofibrant objects. The functors $f_\oplus : \mathcal{Ch}_R(X) \to \mathcal{Ch}_R(Y)$ and $f^* : \mathcal{Ch}_R(Y) \to \mathcal{Ch}_R(X)$ are easily seen to preserve weak equivalences between cofibrant objects. Hence the theorem holds in the case where G is the trivial group.

Let us return to the general case where G is a finite group. Given a map $f : X \to Y$ of finite G-sets we obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathcal{Ch}_R(X) \arrow[r, "j_X^*"] \arrow[d, "\mathcal{Ch}_R(\phi)"'] & \mathcal{Ch}_R(UX)VV \arrow[d, "\mathcal{Ch}_R(U\phi)"] \\
\mathcal{Ch}_R(Y) \arrow[r, "j_Y^*"] & \mathcal{Ch}_R(UY)
\end{tikzcd}$$

Since the result holds for the trivial group, we deduce that if j_X^*A is cofibrant, then also $j_Y^*\mathcal{Ch}_R(\phi)A = \mathcal{Ch}_R(U\phi)j_X^*A$ is cofibrant. Further, if j_X^*A and j_X^*B are cofibrant and $\alpha : A \to B$ is a weak equivalence, then also $j_X^*\alpha$ and $\mathcal{Ch}_R(U\phi)(j_X^*\alpha) = j_Y^*\mathcal{Ch}_R(\phi)(\alpha)$ are weak equivalences, and we can conclude that $\mathcal{Ch}_R(\phi)(\alpha)$ is a weak equivalence. $\square$

Combining Propositions 6.15 and 3.11 we get the following result.

Corollary 6.16. *There exists a Tambara category* $\mathrm{ho}\,\mathcal{Ch}_R$ *with* $\mathrm{ho}\,\mathcal{Ch}_R(\phi)$ *given by a total left-derived functor of* $\mathcal{Ch}_R(\phi)$ *for* $\phi \in sU_+^G$.

7. *Homotopical Tambara functors*

In this section we shall use the Tambara categories of Section 6 to construct Tambara functors. In order to do so we apply the construction of Proposition 5.5.

7.1. *Transfer for chain complexes*

Let us return to the situation of Example 5.4, where we considered the category $\mathcal{Ch}_R$ of chain complexes over a commutative ring R and the Tambara category $\mathcal{Ch}_R$ with $\mathcal{Ch}_R(X) = \mathrm{Fun}(X, \mathcal{Ch}_R)$. We saw in Proposition 6.15 that the Tambara category $\mathcal{Ch}_R$ satisfies the assumptions of Proposition 3.11, and therefore, combining Proposition 5.5 and Construction 5.2, we see that a differential graded commutative R-algebra A gives rise to a functor σ_A given by the composition $sU_+^G \to \mathcal{G}(\mathcal{Ch}_R)^{\mathrm{op}} \to \mathcal{G}(\mathrm{ho}\,\mathcal{Ch}_R)^{\mathrm{op}}$. On the other hand, if B is a differential graded cocommutative R-coalgebra and if B is cofibrant considered as a chain complex of R-modules, then by Proposition 6.15, for every $\phi \in U_+^G(X,Y)$ the map $t : \mathrm{ho}\,\mathcal{Ch}_R(\phi)p_X^*B \to \mathcal{Ch}_R(\phi)p_X^*B$ is an isomorphism in $\mathrm{ho}\,\mathcal{Ch}_R(Y)$, and by Construction 5.2 we obtain a partial functor

$$\sigma_B : sU_+^G \longrightarrow \mathcal{G}(\mathrm{ho}\,\mathcal{Ch}_R^{\mathrm{op}})^{\mathrm{op}}$$

with $\sigma_B(X) = (X, p_X^*B)$ and with $\sigma_B(\phi : X \to Y) = (\phi, \sigma_{B,2}(\phi))$ where $\sigma_{B,2}$ is given by the composite

$$\mathrm{ho}\,\mathcal{Ch}_R(\phi)p_X^*B \xleftarrow{t^{-1}} \mathcal{Ch}_R(\phi)p_X^*B \xleftarrow{s_{B,2}(\phi)} p_Y^*B$$

for $s_B(\phi) = (\phi, s_{B,2}(\phi))$ of Construction 5.2. Combining the functors σ_A and σ_B we obtain a product-preserving partial functor $sU_+^G \to \mathcal{G}(\mathrm{ho}\,\mathcal{Ch}_R^{\mathrm{op}})^{\mathrm{op}} \times_{sU_+^G} \mathcal{G}(\mathrm{ho}\,\mathcal{Ch}_R)^{\mathrm{op}} \to \mathcal{S}et$, and by Lemma 3.9 this partial functor extends to a Tambara functor $\mathrm{ho}\,\mathcal{Ch}_R(B,A) : U_+^G \to \mathcal{S}et$.

In the particular case where $B = R[T]$ with T of degree 2 and $A = R$, we obtain a Tambara functor $\mathrm{ho}\,\mathcal{C}h_R(\mathbb{Z}[T], R)$ with

$$\mathrm{ho}\,\mathcal{C}h_R(R[T], R)(G/H) \cong H^{2*}(BH, R) = H^{2*}(H, R).$$

This Tambara functor was considered by Tambara in [**30**, 3.4].

7.2. *Transfer for orthogonal spectra*

We associate a Tambara functor $\widetilde{A}$ such that

$$\widetilde{A}(X) = \mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(X)(p_X^*\mathbb{S}, p_X^*A) \cong \mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(*)((p_X)_\vee p_X^*\mathbb{S}, A) = [\Sigma^\infty X_+, A]_G$$

with every commutative orthogonal G-ring spectrum A, that is, with every commutative monoid A in $\mathrm{Sp}^{\mathcal{O}}(*)$. Here $\mathbb{S} \in \mathrm{Sp}^{\mathcal{O}}(*)$ is the G-sphere spectrum defined by $\mathbb{S}(V) = S^V$ for $V \in \mathcal{I}_S(*)$ and the map $p_X : X \to *$ is the unique map to the terminal object in $\mathcal{F}_G$. We use the notation $(p_X)_\vee = \mathrm{Sp}^{\mathcal{O}}(T_{p_X})$ and $p_X^* = \mathrm{Sp}^{\mathcal{O}}(R_{p_X})$. Note that for $X = G/H$ we have

$$\widetilde{A}(G/H) = [\Sigma^\infty G/H_+, A]_G = \pi_0(A^H),$$

where A^H denotes the H-fixed point spectrum of A.

The construction of $\widetilde{A}$ is complicated by the fact that the sphere spectrum $\mathbb{S}$ is a coalgebra in $\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}$ but not in $\mathrm{Sp}^{\mathcal{O}}$. Note that we do not require A to be cofibrant.

In Construction 5.2 we formed a section $s_A : sU_+^G \to \mathcal{G}(\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}}$ of the partial functor $\mathbb{G}(\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}} : \mathcal{G}(\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}} \to sU_+^G$. Composing with the functor $\mathcal{G}(\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}} \to \mathcal{G}(\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}}$ we obtain a product-preserving section $\sigma_A : sU_+^G \to \mathcal{G}(\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}}$ of the partial functor

$$\mathbb{G}(\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}} : \mathcal{G}(\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}} \longrightarrow sU_+^G.$$

In this section we construct a product-preserving section $\sigma_{\mathbb{S}} : sU_+^G \to \mathcal{G}((\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}})^{\mathrm{op}}$ of the functor $\mathbb{G}((\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}})^{\mathrm{op}}$.

The composite

$$sU_+^G \xrightarrow{(\sigma_{\mathbb{S}}, \sigma_A)} (\mathcal{G}((\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}})^{\mathrm{op}}) \times_{(sU_+^G)^{\mathrm{op}}} \mathcal{G}(\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}))^{\mathrm{op}} \xrightarrow{\mathrm{hom}_{\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}}} \mathcal{S}et$$

is a product-preserving partial functor, and by Lemma 3.9 it defines a Tambara functor $\widetilde{A} = \mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(\mathbb{S}, A)$ with

$$\widetilde{A}(X) = \mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(X)(p_X^*\mathbb{S}, p_X^*A) \cong [\Sigma^\infty X_+, A]_G$$

for X in $\mathcal{F}_G$. The rest of this section contains a construction of $\sigma_{\mathbb{S}}$.

Given a symmetric monoidal category $(\mathcal{C}, \diamond, n_\diamond)$ and $f : X \to Y$ in $\mathcal{F}_G$, the functor

$$f_\diamond : \mathrm{Fun}(X, \mathcal{C}) \longrightarrow \mathrm{Fun}(Y, \mathcal{C})$$

with $f_\diamond(c)(y) \cong \Diamond_{x\in f^{-1}(y)} c(x)$ for $c \in \mathrm{Fun}(X, \mathcal{C})$ is constructed similarly to $\mathcal{C}(N_f)$ of Construction 3.13. In particular, we use this notation for $\mathcal{C} = \mathcal{T}$ with the monoidal products $\amalg$, $\times$ and $\wedge$.

For typographical reasons we introduce the notation $\mathcal{D} = (\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(X))^{\mathrm{op}}$. A partial functor $\sigma = (\mathrm{id}, \sigma_2) : sU_+^G \to \mathcal{G}(\mathcal{D})^{\mathrm{op}}$ is uniquely determined by its values $\sigma(R_f)$, $\sigma(T_f)$ and $\sigma(N_f)$ for $f : X \to Y$ in $\mathcal{F}_G$, and given such values, they extend to a functor σ if and only if the generating relations between R_f, T_f and N_f in sU_+^G considered in [**30**, Proposition 7.2] are respected. Below we define such a functor σ by specifying these values and verifying that the relations of [**30**] are satisfied. Since it is more convenient, we work with a strictly unital smash-product in $\mathrm{Sp}^{\mathcal{O}}$. In particular, this implies $\mathbb{S}_X = \mathrm{Sp}^{\mathcal{O}}(N_{i_X})(*)$ where $i_X : \emptyset \to X$ is the unique map from the initial object of $\mathcal{F}_G$ to X. For $\phi \in sU_+^G(X, Y)$ we choose the total left-derived functor $\mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(\phi)$ such that if M is a cofibrant object in $\mathrm{Sp}^{\mathcal{O}}(X)$, then $t : \mathrm{ho}\,\mathrm{Sp}^{\mathcal{O}}(\phi)M \to \mathrm{Sp}^{\mathcal{O}}(\phi)M$ is the identity map.

Given $f : X \to Y$ in $\mathcal{F}_G$ we have

$$\mathcal{D}(R_f)\mathbb{S}_Y = \mathcal{D}(R_f)\mathcal{D}(R_{p_Y})\mathbb{S} = \mathcal{D}(R_f R_{p_Y})\mathbb{S} = \mathcal{D}(R_{p_X})\mathbb{S} = \mathbb{S}_X,$$

and we let $\sigma(R_f) = (R_f, \mathrm{id})$. Similarly, we have

$$\mathcal{D}(N_f)\mathbb{S}_X = \mathcal{D}(N_f)\mathcal{D}(N_{i_X})* = \mathcal{D}(N_f N_{i_X})* = \mathcal{D}(N_{i_Y})* = \mathbb{S}_Y.$$

and we let $\sigma(N_f) = (N_f, \mathrm{id})$. We let $\sigma(T_f) = (T_f, t_f)$, where the map t_f is the transfer map defined below. If f is the projection $G/H \to G/G$, then t_f is the classical transfer map of Kahn and Priddy [**19**] and of Roush [**27**]. This map has been studied further by, for example, Becker and Gottlieb [**1**]. In order to give the definition of the transfer map in our context we note that the fibers $f^{-1}(y)$ for $y \in Y$ assemble to a functor $f^{-1} : Y \to \mathcal{F}$. We can choose a functor $V : Y \to \mathcal{I}_S$ and an embedding $\iota_f : f^{-1} \hookrightarrow V$, that is, embeddings $\iota_{f,y} : f^{-1}(y) \hookrightarrow V(y)$ for $y \in Y$ such that $\iota_{f,gy} \circ f^{-1}(g) = V(g) \circ \iota_{f,y}$ for every $y \in Y$ and $g \in G$. Identifying small disjoint balls around the elements of $f^{-1}(y)$ with $V(y)$, we can extend the embedding ι_f to an embedding $\iota'_f : f^{-1} \times V \hookrightarrow V$. Collapsing the complement of the image of $f^{-1}(y) \times V(y)$ in $V(y)$ for $y \in Y$ we obtain a map

$$\tau_{f,y} : S^{V(y)} \longrightarrow (f_\vee f^* S^V)(y) = \bigvee_{x \in f^{-1}(y)} S^{V(y)}.$$

These maps define a map $\tau_f : S^V \to f_\vee f^* S^V$. We denote the suspension spectrum of $Z \in \mathrm{Fun}(Y, \mathcal{T})$ by $\mathbb{S}_Y \wedge Z$. We define the map $t_f : \mathbb{S}_Y \to f_\vee \mathbb{S}_X = \mathcal{D}(T_f)\mathbb{S}_X$ in $\mathrm{ho}\,\mathrm{Sp}^O(Y)$ as the composition

$$\begin{aligned}
\mathbb{S}_Y &\xrightarrow{\cong} \hom(\mathbb{S}_Y \wedge S^V, \mathbb{S}_Y \wedge S^V) \\
&\xrightarrow{(\tau_f)_*} \hom(\mathbb{S}_Y \wedge S^V, \mathbb{S}_Y \wedge f_\vee f^* S^V) \\
&\xrightarrow{\cong} \hom(\mathbb{S}_Y \wedge S^V, f_\vee f^* \mathbb{S}_Y \wedge S^V) \\
&\xrightarrow{\cong} f_\vee f^* \mathbb{S}_Y = f_\vee \mathbb{S}_X.
\end{aligned}$$

Here hom denotes the internal hom object in $\mathrm{ho}\,\mathrm{Sp}^O(Y)$. In other words, we have defined a map $t_f : \mathcal{D}(T_f)S_X \to S_Y$ in $\mathcal{D}(Y)$. This ends the construction of the morphisms $\sigma(R_f)$, $\sigma(T_f)$ and $\sigma(N_f)$ for f a morphism in $\mathcal{F}_G$. We move on to the verification of the relations between them.

Note first that $t_f = t_f(\iota_f)$ is independent of the embedding $\iota_f : f^{-1} \hookrightarrow V$. One way to see this is to note that if $\iota_1 : f^{-1} \hookrightarrow V_1$ and $\iota_2 : f^{-1} \hookrightarrow V_2$ are embeddings, we obtain a diagonal embedding $\iota_1 \times \iota_2$ by the composition $f^{-1} \hookrightarrow f^{-1} \times f^{-1} \hookrightarrow V_1 \oplus V_2$, and the diagram

$$\begin{tikzcd}
S^{V_1 \oplus V_2} \arrow[r, "\cong"] \arrow[d, "\tau_f(\iota_1 \times \iota_2)"'] & S^{V_1} \wedge S^{V_2} \arrow[d, "\mathrm{id} \wedge \tau_f(\iota_2)"] \\
f_\vee f^* S^{V_1 \oplus V_2} \arrow[r, "\cong"] & S^{V_1} \wedge (f_\vee f^* S^{V_2})
\end{tikzcd}$$

is homotopy-commutative by a homotopy that shrinks the V_1-coordinate.

Since the relations of [**30**, Proposition 7.2] not involving morphisms of the form T_f are readily verified, we concentrate on the ones involving T_f. We first show that given a pull-back diagram of the form

$$\begin{tikzcd} X & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] & Y' \arrow[l, "g"] \end{tikzcd}$$

in $\mathcal{F}_G$, the relation $\sigma(R_g)\sigma(T_f) = \sigma(T_{f'})\sigma(R_{g'})$ holds. This translates into showing that the diagram

$$\begin{tikzcd} \mathcal{D}(R_g)\mathcal{D}(T_f)\mathbb{S}_X \arrow[r, "\mathcal{D}(R_g)(t_f)"] \arrow[d, "\cong"'] & \mathcal{D}(R_g)\mathbb{S}_Y \arrow[r, "\mathrm{id}"] & \mathbb{S}_{Y'} \arrow[d, equal] \\ \mathcal{D}(T_{f'})\mathcal{D}(R_{g'})\mathbb{S}_X \arrow[r, "\mathrm{id}"] & \mathcal{D}(T_{f'})\mathbb{S}_{X'} \arrow[r, "t_{f'}"] & \mathbb{S}_{Y'} \end{tikzcd}$$

commutes. In order to do so we choose an embedding $f^{-1} \hookrightarrow V$ and consider the induced embedding ${f'}^{-1} \cong g^* f^{-1} \hookrightarrow g^* V$. Tracing back the definition of the transfer using these embeddings we see that it suffices to note that the following diagram in $\mathcal{T}$ is commutative:

$$\begin{tikzcd} g^* f_\vee f^* S^V & g^* S^V \arrow[l, "g^*(\tau_f)"'] \arrow[r, equal] & S^{g^*V} \arrow[d, equal] \\ {f'}_\vee {g'}^* f^* S^V \arrow[u, "\cong"] \arrow[r, equal] & {f'}_\vee {f'}^* S^{g^*V} & S^{g^*V} \arrow[l, "\tau_{f'}"'] \end{tikzcd}$$

Next we show that $\sigma(T_h)\sigma(T_f) = \sigma(T_{hf})$ for $f : X \to Y$ and $h : Y \to Z$ in $\mathcal{F}_G$. This amounts to showing that the diagram

$$\begin{tikzcd} \mathcal{D}(T_h)\mathcal{D}(T_f)\mathbb{S}_X \arrow[r, "\mathcal{D}(T_h)(t_f)"] \arrow[d, "\cong"'] & \mathcal{D}(T_h)\mathbb{S}_Y \arrow[d, "t_h"] \\ \mathcal{D}(T_{hf})\mathbb{S}_X \arrow[r, "t_{hf}"] & \mathbb{S}_Z \end{tikzcd}$$

commutes. In order to do so we choose $V : Z \to \mathcal{I}_S$ and embeddings $\iota_{hf} : (hf)^{-1} \hookrightarrow V$ and $\iota_h : h^{-1} \hookrightarrow V$. We let ι_f denote the embedding $f^{-1} \hookrightarrow h^*(hf)^{-1} \hookrightarrow h^* V$. Tracing back the definition of the transfer with respect to these embeddings we see that it suffices to note that the diagram

$$\begin{tikzcd} h_\vee f_\vee f^* h^* S^V h_\vee & h^* S^V \arrow[l, "h_\vee(\tau_f)"'] \\ (hf)_\vee (hf)^* S^V \arrow[u, "\cong"] & S^V \arrow[l, "\tau_{hf}"'] \arrow[u, "\tau_h"'] \end{tikzcd}$$

commutes.

Finally, we verify the relation $\sigma(T_q)\sigma(N_{f'})\sigma(R_e) = \sigma(N_f)\sigma(T_p)$ for the exponential diagram of Definition 2.1. This amounts to showing that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(T_q)\mathcal{D}(N_{f'})\mathcal{D}(R_e)\mathbb{S}_A \arrow[r, "\cong"] \arrow[d, "\mathrm{id}"'] & \mathcal{D}(N_f)\mathcal{D}(T_p)\mathbb{S}_A \arrow[r, "\mathcal{D}(N_f)(t_p)"] & \mathcal{D}(N_f)\mathbb{S}_X \arrow[d, "\mathrm{id}"] \\
\mathcal{D}(T_q)\mathcal{D}(N_{f'})\mathbb{S}_{X'} \arrow[r, "\mathrm{id}"] & \mathcal{D}(T_q)\mathbb{S}_{Y'} \arrow[r, "t_q"] & \mathbb{S}_Y
\end{tikzcd}$$

In order to do this, we choose an embedding $\iota_p : p^{-1} \hookrightarrow V$ and we let $W = f_\oplus V$. Let ι_q denote the induced embedding $q^{-1} \cong f_\times p^{-1} \subseteq f_\times V = f_\oplus V = W$. Tracing back definitions it suffices to note that we have a commutative diagram of the form

$$\begin{tikzcd}
q_\vee {f'}_\wedge e^*(p^* S^V) & f_\wedge(p_\vee p^* S^V) \arrow[l, "\cong"'] & f_\wedge S^V \arrow[l, "f_\wedge(\tau_p)"'] \arrow[d, equal] \\
q_\vee q^* f_\wedge S^V \arrow[u, "\cong"] \arrow[r, equal] & q_\vee q^* S^W & S^W \arrow[l, "\tau_q"']
\end{tikzcd}$$

8. *Equivariant cobordism*

As an application of our theory we consider equivariant cobordism and the related equivariant spectrum MU. We start by recollecting the following theorem which was proved by Strickland in the context of non-equivariant spectra [**29**].

THEOREM 8.1. *There is a commutative orthogonal ring spectrum MP with action of G of the homotopy type $\bigvee_{r\in\mathbb{Z}} \Sigma^{2r} MU$ considered as a monoid in the G-equivariant stable category.*

Actually we will construct a unitary ring spectrum rather than an orthogonal ring spectrum. There is a strong symmetric monoidal functor from the category of unitary spectra to the category of orthogonal spectra defined by left Kan extension [**24**, Proposition 3.3]. Therefore we can pass from commutative unitary ring spectra to commutative orthogonal ring spectra. We restrict our attention to unitary spectra in this section.

Given a complex inner product space V and $n \in \mathbb{N}$ we let $\mathrm{Grass}(n, V)$ denote the Grassmannian manifold of (complex) n-dimensional subspaces of V. This defines a functor $\mathrm{Grass}(n, -)$ from the category of inner product spaces and injective homomorphisms to the category of topological spaces. Given an inclusion $V \subseteq W$, the induced map $\mathrm{Grass}(n, V) \to \mathrm{Grass}(n, W)$ is $(2\dim(V) - c)$-connected for a constant c not depending on V and W. We denote by $E(n, V)$ the tautological n-plane bundle over $\mathrm{Grass}(n, V)$ consisting of pairs (X, x) of an n-dimensional subspace X of V and a point $x \in X$. The associated Thom space is denoted $T(n, V)$. If G acts on V, then there is an induced action of G on $T(n, V)$. For complex inner product spaces V and W there is a pairing $T(n, V) \wedge T(m, W) \to T(n + m, V \oplus W)$ induced by the pairing $E(n, V) \oplus E(m, W) \to E(n + m, V \oplus W)$ taking $((X, x), (Y, y))$ to $(X \oplus Y, x + y)$.

Let $\mathcal{U}$ be a complete complex G-universe. Given an inner product space V with action of G, we follow tom Dieck [**6**] (see also [**5**]) and let $MU(V) = T(\dim(V), V \oplus \mathcal{U})$. The pairing

$$\begin{aligned}
&T(\dim(V), V \oplus \mathcal{U}) \wedge T(\dim(W), W \oplus \mathcal{U}) \\
&\qquad \longrightarrow T(\dim(V \oplus W), V \oplus \mathcal{U} \oplus W \oplus \mathcal{U}) \cong T(\dim(V \oplus W), V \oplus W \oplus \mathcal{U})
\end{aligned}$$

defines a map from $MU(V) \wedge MU(W)$ to $MU(V \oplus W)$ and the inclusion $V \to E(\dim(V), V \oplus \mathcal{U})$ taking v to $(V \oplus 0, v)$ defines a map $S^V \to MU(V)$. This structure defines a commutative ring G-prespectrum MU in the sense that the prespectrum MU represents a commutative monoid in the G-equivariant stable category. For $r \geqslant 0$ we let $MU_r(V) = MU(V \oplus \mathbb{C}^r)$ and $MU_{-r}(V) = \mathrm{map}(S^{\mathbb{C}^r}, MU(V))$. Thus for $r \in \mathbb{Z}$ there is a stable equivalence $MU_r \simeq \Sigma^{2r} MU$ of G-prespectra.

Proof of Theorem 8.1. Given a finite-dimensional inner product space V, we follow Strickland [**29**] and define

$$MP(V) = \bigvee_{r \in \mathbb{Z}} MP_r(V)$$

where $MP_r(V)$ denotes the space $T(r + \dim(V), V \oplus V)$ for $r \in \mathbb{Z}$. The map

$$\begin{aligned} T(r + \dim(V), V \oplus V) \wedge T(s + \dim(W), W \oplus W)& \\ \longrightarrow T(r + s + \dim(V \oplus W), V \oplus V \oplus W \oplus W)& \\ \cong T(r + s + \dim(V \oplus W), V \oplus W \oplus V \oplus W)& \end{aligned}$$

defines a pairing

$$MP_r(V) \wedge MP_s(W) \longrightarrow MP_{r+s}(V \oplus W)$$

and the map from V to $E(\dim(V), V \oplus V)$ taking v to $(V \oplus 0, v)$ defines a map $S^V \to MP_0(V)$. Together these maps make MP into a commutative unitary ring spectrum. (See for example [**23**, Theorem 3.4] for the translation between unitary ring spectra and this kind of structure.) If G acts on the inner product space V then we obtain an induced action on $MP(V)$, and this way MP is considered as a commutative unitary ring spectrum with action of G.

We need to show that MP represents $\bigvee_{r \in \mathbb{Z}} \Sigma^{2r} \wedge MU$ as a monoid in the G-equivariant stable category. If $r \geqslant 0$ and V is contained in a G-universe $\mathcal{U}$, then the composition

$$T(r + \dim(V), V \oplus V) \longrightarrow T(r + \dim(V), V \oplus \mathcal{U}) \longrightarrow T(\dim(V \oplus \mathbb{C}^r), V \oplus \mathbb{C}^r \oplus \mathcal{U})$$

defines a map $MP_r(V) \to MU(V \oplus \mathbb{C}^r)$ and the composition

$$\begin{aligned} S^{\mathbb{C}^r} \wedge T(-r + \dim(V), V \oplus V) &\longrightarrow T(r, \mathbb{C}^r \oplus \mathcal{U}) \wedge T(-r + \dim(V), V \oplus \mathcal{U}) \\ &\longrightarrow T(\dim(V), \mathbb{C}^r \oplus V \oplus \mathcal{U}) \\ &\longrightarrow T(\dim(V), V \oplus \mathcal{U}) \end{aligned}$$

is adjoint to a map $MP_{-r}(V) \to \mathrm{map}(S^{\mathbb{C}^r}, MU(V))$. The map $MP_r(V)^H \to MU_r(V)^H$ is $(2\dim(V^H) - c)$-connected for a constant c independent of V for every $r \in \mathbb{Z}$ and every subgroup H of G. In particular, the map $MP_r \to MU_r$ of prespectra is a π_*-equivalence and it induces an isomorphism in the G-equivariant stable category. We leave it to the reader to check that these maps are compatible with the ring structures. □

In the rest of this section we prove Theorem 1.3 and Theorem 1.4. Let A be a commutative ring and let $\mathcal{O}$ denote the partially ordered set of conjugacy classes of subgroups of G with $[H] \leqslant [K]$ if and only if there exists $g \in G$ such that $H \subseteq gKg^{-1}$. The ring $\mathbb{W}_G(A)$ of [**8**] has $\mathrm{map}(\mathcal{O}, A)$ as underlying set and ring structure defined through the ghost coordinates $\Phi^A_{[H]} : \mathbb{W}_G(A) \to A$ with

$$\Phi^A_{[H]}(\alpha) = \sum_{[K] \in \mathcal{O}} |(G/K)^H| \cdot \alpha([K])^{(K:H)}$$

for $[H] \in \mathcal{O}$ and $\alpha : \mathcal{O} \to A$. Here $|(G/K)^H|$ denotes the cardinality of the set of H-fixed points of G/K and $(K:H)$ denotes the index of H in gKg^{-1}. These ghost coordinates assemble to a ring homomorphism $\Phi^A : \mathbb{W}_G(A) \to \mathrm{map}(\mathcal{O}, A)$ called the ghost map. If the underlying abelian group of A is torsion free then the ghost map is injective.

In Section 7.2 we constructed the Tambara functor $\widetilde{MP}$. The homomorphism

$$\tau_{\widetilde{MP}} : \mathbb{W}_G([\mathbb{S}, MP]) = \prod_{[K]\in\mathcal{O}} (\widetilde{MP}(G/e)) \longrightarrow \widetilde{MP}(*) \cong [\mathbb{S}, MP]_G$$

defined by the formula

$$\tau_{\widetilde{MP}} = \sum_{[K]\in\mathcal{O}} \widetilde{MP}(T_{p^G_K} \circ N_{p^K_e})$$

is called the *Teichmüller homomorphism* in [**4**]. Here p^G_K and p^K_e denote the G-maps

$$G/e \xrightarrow{p^K_e} G/K \xrightarrow{p^G_K} G/G.$$

THEOREM 8.2. *There exists a homomorphism*

$$R : [\mathbb{S}, MP]_G \longrightarrow \mathrm{map}(\mathcal{O}, [\mathbb{S}, MP])$$

of commutative rings with $R \circ \tau_{\widetilde{MP}} = \Phi^{[\mathbb{S},MP]}$.

Proof of Theorem 1.3. By Milnor's and Novikov's calculation [**25**, **26**] the underlying abelian group of $[\mathbb{S}, MP] = \pi_0(MP)$ is torsion free. It follows that the ghost map $\Phi^{[\mathbb{S},MP]}$ is injective and hence, by Theorem 8.2, $\tau_{\widetilde{MP}}$ is injective. □

In order to construct the homomorphism R, it is convenient to note that $MP(V)$ is the Thom space of the canonical bundle over the space $\coprod_{r\in\mathbb{Z}} \mathrm{Grass}(r, V \oplus V)$ of subspaces of $V \oplus V$. Let H be a subgroup of G and $\mathcal{U}$ denote a complete G-universe. An element of $MP(V)^H$ consists of a pair (X, x) of an H-invariant subspace X of $V \oplus V$ and an element $x \in X^H$. To this pair we associate the pair (X^H, x) considered as an element of $MP(V^H)$. This defines a map $r^H : MP(V)^H \to MP(V^H)$. As defined in [**23**, Definition III.3.2] the group $[\mathbb{S}, MP]_H$ is the colimit over all finite-dimensional subspaces $V \subseteq \mathcal{U}$ of the abelian groups $\pi_0(\mathrm{map}(S^V, MP(V))^H)$. The composition

$$\mathrm{map}(S^V, MP(V))^H \longrightarrow \mathrm{map}(S^{V^H}, MP(V)^H) \xrightarrow{\mathrm{map}(S^{V^H},\ r^H)} \mathrm{map}(S^{V^H}, MP(V^H))$$

induces a map $r^H : [\mathbb{S}, MP]_H \to [\mathbb{S}, MP]$.

REMARK 8.3. In other words, we have constructed a map of orthogonal spectra

$$r_G : \Phi^G MP \longrightarrow MP$$

from the geometric fixed point spectrum of MP back to MP. (See [**23**, Section V.4] for a definition of the geometric fixed point spectrum.) Given $W \subseteq \mathcal{U}$, there is a map

$$s_G(W) : MP(W^G) \longrightarrow MP(W)^G$$

taking a pair (x, X) with $X \subseteq W^G \oplus W^G$ to the same pair with X considered as a subspace of $W \oplus W$. These maps define a map $s_G : MP \to \Phi^G MP$ with $r_G \circ s_G = \mathrm{id}$.

LEMMA 8.4. *We have* $r^H \circ T_{p^H_K} = 0$ *for* $K < H$.

Proof. This follows from the fact that $(H/K_+ \wedge S^V)^H = *$. □

LEMMA 8.5. *We have* $r^H \circ N_{p_e^H} = \mathrm{id}$ *for* $H \leqslant G$.

Proof. This follows from the facts that the H-fixed points space of $(S^V)^{\wedge H}$ is S^V and that the H-fixed point sub-vector space of $\mathbb{C}[H] \otimes V = \bigoplus_{h \in H} V$ is isomorphic to V. $\square$

Proof of Theorem 8.2. We define R by $R(a)([H]) = r^H(\widetilde{MP}(R_{p_H^G})(a))$ for $[H] \in \mathcal{O}$ and $a \in [\mathbb{S}, MP]_G$. For $\alpha : \mathcal{O} \to [\mathbb{S}, MP]$ we have

$$r^H(\widetilde{MP}(R_{p_H^G})(\tau_{\widetilde{MP}}(\alpha))) = \sum_{[K] \in \mathcal{O}} r^H(\widetilde{MP}(R_{p_H^G} \circ T_{p_K^G} \circ N_{p_e^K})(\alpha([K]))).$$

Using Lemma 8.4, the additive double coset formula, Lemma 8.5 and the multiplicative double coset formula respectively, we compute for $H \leqslant K$ that

$$\begin{aligned} r^H(\widetilde{MP}(R_{p_H^G} \circ T_{p_K^G} \circ N_{p_e^K})(a)) &= |(G/K)^H| \cdot r^H(\widetilde{MP}(R_{p_H^K} \circ N_{p_e^K})(a)) \\ &= |(G/K)^H| \cdot a^{(K:H)} \end{aligned}$$

for $a \in [\mathbb{S}, MP]$. For general subgroups H and K of G we have

$$r^H(\widetilde{MP}(R_{p_H^G} \circ T_{p_K^G} \circ N_{p_e^K})(a)) = |(G/K)^H| \cdot a^{(K:H)}.$$

In particular $R \circ \tau_{\widetilde{MP}} = \Phi^{[\mathbb{S}, MP]}$. $\square$

Proof of Theorem 1.4. This proof is similar to the proof of Theorem 1.3. However we are not able to prove that there is a Tambara functor with value $\mathcal{U}_*^H$ on G/H. Instead we consider the auxillary semi-Tambara functor $\mathcal{M}_*$ with $\mathcal{M}_*(X)$ given by the set of isomorphism classes of functors from the translation category of X to the category of almost complex manifolds. We let $\widehat{\mathcal{M}}_*$ denote the Tambara functor obtained by group-completing $\mathcal{M}_*$. By [**4**, Theorem 3.6] we have the Teichmüller homomorphism $\tau_{\widehat{\mathcal{M}}_*} : \mathbb{W}_G(\widehat{\mathcal{M}}_*(G/e)) \to \widehat{\mathcal{M}}_*(G/G)$. Choosing a ring homomorphism $\sigma : \mathcal{U}_* \to \widehat{\mathcal{M}}_*(G/e)$ such that σ is a section of the surjective ring homomorphism $q : \widehat{\mathcal{M}}_*(G/e) \to \mathcal{U}_*$, and denoting by q^G the surjection $\widehat{\mathcal{M}}_*(G/G) \to \mathcal{U}_*^G$ we obtain a commutative diagram of the form

$$\begin{tikzcd}
\mathbb{W}_G(\widehat{\mathcal{M}}_*(G/e)) \arrow[r, "\tau_{\widehat{\mathcal{M}}_*}"] & \widehat{\mathcal{M}}_*(G/G) \arrow[r, "\mathrm{Fix}_{\widehat{\mathcal{M}}_*}"] \arrow[d, "q^G"] & \mathrm{map}(\mathcal{O}, \widehat{\mathcal{M}}_*(G/e)) \arrow[d, "\mathrm{map}(\mathcal{O}, q)"] \\
\mathbb{W}_G(\mathcal{U}_*) \arrow[u, "\mathbb{W}_G(\sigma)"] \arrow[r, "q^G \circ \tau_{\widehat{\mathcal{M}}_*} \circ \mathbb{W}_G(\sigma)"] & \mathcal{U}_*^G \arrow[r, "\mathrm{Fix}_{\mathcal{U}_*}"] & \mathrm{map}(\mathcal{O}, \mathcal{U}_*)
\end{tikzcd}$$

where

$$(\mathrm{Fix}_{\mathcal{U}_*}[M])([K]) = [M^K] \quad \text{and} \quad (\mathrm{Fix}_{\widehat{\mathcal{M}}_*}([M]))([K]) = [M^K].$$

Note that we have $\Phi^{\mathcal{M}_*(G/e)} = \mathrm{Fix}_{\widehat{\mathcal{M}}_*} \circ \tau_{\widehat{\mathcal{M}}_*}$. Since $\Phi_{[K]}$ is given by an integral polynomial, we have

$$q \circ \Phi_{[K]}^{\widehat{\mathcal{M}}_*(G/e)} \circ \mathbb{W}_G(\sigma) = q \circ \Phi_{[K]}^{\widehat{\mathcal{M}}_*(G/e)} \circ \mathrm{map}(\mathcal{O}, \sigma) = \Phi_{[K]}^{\mathcal{U}_*} \circ \mathrm{map}(\mathcal{O}, q \circ \sigma) = \Phi_{[K]}^{\mathcal{U}_*},$$

and therefore

$$\begin{aligned} \mathrm{Fix}_{\mathcal{U}_*} \circ q^G \circ \tau_{\widehat{\mathcal{M}}_*} \circ \mathbb{W}_G(\sigma) &= \mathrm{map}(\mathcal{O}, q) \circ \mathrm{Fix}_{\widehat{\mathcal{M}}_*} \circ \tau_{\widehat{\mathcal{M}}_*} \circ \mathbb{W}_G(\sigma) \\ &= \mathrm{map}(\mathcal{O}, q) \circ \Phi^{\widehat{\mathcal{M}}_*(G/e)} \circ \mathbb{W}_G(\sigma) \\ &= \Phi^{\mathcal{U}_*}. \end{aligned}$$

It now follows from the injectivity of $\Phi^{\mathcal{U}_*}$ that $\tau_{\mathcal{U}_*} := q^G \circ \tau_{\widehat{\mathcal{M}}_*} \circ \mathbb{W}_G(\sigma)$ is injective. $\square$

9. *Filtered objects*

We shall let $\mathbb{Z} = (\mathbb{Z}, \leqslant)$ denote the partially ordered set of integers considered as a symmetric monoidal category with monoidal operation given by the sum in $\mathbb{Z}$.

DEFINITION 9.1. A functor $X : \mathbb{Z} \to \mathcal{C}$ is called a *filtered object* in $\mathcal{C}$. The category $\mathbb{Z}\mathcal{C}$ of functors from $\mathbb{Z}$ to $\mathcal{C}$ is the *category of filtered objects in* $\mathcal{C}$. If $(\mathcal{C}, \diamond, u)$ is a cocomplete symmetric monoidal category, there is a symmetric monoidal structure on $\mathbb{Z}\mathcal{C}$ induced from the symmetric monoidal structures on $\mathbb{Z}$ and $\mathcal{C}$ by the usual left Kan extension: given $A, B : \mathbb{Z} \to \mathcal{C}$, the monoidal product $A \diamond B : \mathbb{Z} \to \mathcal{C}$ is the left Kan extension of the composition

$$\mathbb{Z} \times \mathbb{Z} \xrightarrow{A \times B} \mathcal{C} \times \mathcal{C} \xrightarrow{\diamond} \mathcal{C}$$

along the functor $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ with $(A \diamond B)(i) = \operatorname{colim}_{\alpha+\beta \leqslant i} A(\alpha) \diamond B(\beta)$.

LEMMA 9.2. *Let $E \to F$ be a map of filtered objects in a symmetric monoidal category $(\mathcal{C}, \diamond, u)$ satisfying the condition that for every $c \in \mathcal{C}$, the functor $c \diamond -$ preserves push-outs. Suppose that D is a filtered object in $\mathcal{C}$ and that for every $i \in \mathbb{Z}$ the square*

$$\begin{tikzcd}
E(i) \arrow[r] \arrow[d] & E(i+1) \arrow[d] \\
F(i) \arrow[r] & F(i+1)
\end{tikzcd}$$

is a push-out in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd}
(E \diamond D)(i) \arrow[r] \arrow[d] & (E \diamond D)(i+1) \arrow[d] \\
(F \diamond D)(i) \arrow[r] & (F \diamond D)(i+1)
\end{tikzcd}$$

is a push-out in $\mathcal{C}$ for every $i \in \mathbb{Z}$.

Proof. We can factor the second diagram of the lemma as

$$\begin{tikzcd}
(E \diamond D)(i) \arrow[r] \arrow[d, "="'] & (F \diamond D)(i) \arrow[d, "="] \\
\operatorname*{colim}_{\alpha_1+\alpha_2 \leqslant i} E(\alpha_1) \diamond D(\alpha_2) \arrow[r] \arrow[d] & \operatorname*{colim}_{\alpha_1+\alpha_2 \leqslant i} F(\alpha_1) \diamond D(\alpha_2) \arrow[d] \\
\operatorname*{colim}_{\alpha_1+\alpha_2 \leqslant i} E(\alpha_1+1) \diamond D(\alpha_2) \arrow[r] \arrow[d, "\cong"'] & \operatorname*{colim}_{\alpha_1+\alpha_2 \leqslant i} F(\alpha_1+1) \diamond D(\alpha_2) \arrow[d, "\cong"] \\
(E \diamond D)(i+1) \arrow[r] & (F \diamond D)(i+1)
\end{tikzcd}$$

The middle square in the above diagram is a push-out because $- \diamond D(\alpha_2)$ and colimits preserve push-outs. $\square$

LEMMA 9.3. *Let $(\mathcal{C}_0, \diamond, u)$ be a cocomplete symmetric monoidal category and suppose that for every $c \in \mathcal{C}_0$, the functor $c \diamond -$ preserves push-outs. Let $f : X \to Y$ be a map of finite G-sets and let $E \to F$ be a map in the category $(\mathbb{Z}\mathcal{C})(X)$ of functors from the translation category of X to $\mathbb{Z}\mathcal{C}_0$. Consider the functor $f_\diamond : (\mathbb{Z}\mathcal{C})(X) \to (\mathbb{Z}\mathcal{C})(Y)$ with*

$$f_\diamond(C)(y)(i) = \Bigl(\Diamond_{x\in f^{-1}(y)} C(x)\Bigr)(i).$$

If the square

$$\begin{tikzcd} E(x)(i) \arrow[r] \arrow[d] & E(x)(i+1) \arrow[d] \\ F(x)(i) \arrow[r] & F(x)(i+1) \end{tikzcd}$$

is a push-out in $\mathcal{C}$ for every $i \in \mathbb{Z}$ and $x \in X$, then the square

$$\begin{tikzcd} (f_\diamond E)(y)(i) \arrow[r] \arrow[d] & (f_\diamond E)(y)(i+1) \arrow[d] \\ (f_\diamond F)(y)(i) \arrow[r] & (f_\diamond F)(y)(i+1) \end{tikzcd}$$

is a push-out in $\mathcal{C}$ for every $i \in \mathbb{Z}$ and $y \in Y$.

Proof. We have to check that for every $i \in \mathbb{Z}$ and $y \in Y$ the diagram

$$\begin{tikzcd} \Bigl(\Diamond_{x\in f^{-1}(y)} E(x)\Bigr)(i) \arrow[r] \arrow[d] & \Bigl(\Diamond_{x\in f^{-1}(y)} E(x)\Bigr)(i+1) \arrow[d] \\ \Bigl(\Diamond_{x\in f^{-1}(y)} F(x)\Bigr)(i) \arrow[r] & \Bigl(\Diamond_{x\in f^{-1}(y)} F(x)\Bigr)(i+1) \end{tikzcd}$$

is a push-out diagram in $\mathcal{C}$. This follows from Lemma 9.2. $\square$

The following lemma can be used to identify filtration quotients of the form

$$(D_1 \diamond D_2 \diamond \ldots \diamond D_n)(k)/(D_1 \diamond D_2 \diamond \ldots \diamond D_n)(k-1)$$

for filtered objects $D_1, \ldots, D_n$. We leave its proof to the reader.

LEMMA 9.4. *Let U be a finite set and consider, for $k \in \mathbb{Z}$, the sets $V_k \subseteq V_{\leqslant k} \subseteq \mathbb{Z}^U$ consisting of maps $\alpha : U \to \mathbb{Z}$ satisfying the conditions that $\sum_{u\in U} \alpha(u) = k$ and $\sum_{u\in U} \alpha(u) \leqslant k$ respectively. Let us consider these sets as partially ordered sets with $\beta \leqslant \alpha$ if and only if $\beta(u) \leqslant \alpha(u)$ for every $u \in U$. Given $\alpha \in \mathbb{Z}^U$ we let $V_{<\alpha}$ denote the partially ordered set consisting of those $\beta \in \mathbb{Z}^U \setminus \{\alpha\}$ such that $\beta \leqslant \alpha$. For every cocomplete category $\mathcal{C}$ and every* *functor $T : (\mathbb{Z}^U, \leqslant) \to \mathcal{C}$ we have a push-out diagram of the form*

$$\begin{tikzcd}
\coprod_{\alpha\in V_k} \operatorname{colim}_{\beta\in V_{<\alpha}} T(\beta) \arrow[r] \arrow[d] & \coprod_{\alpha\in V_k} T(\alpha) \arrow[d] \\
\operatorname{colim}_{\beta\in V_{\leqslant k-1}} T(\beta) \arrow[r] & \operatorname{colim}_{\alpha\in V_{\leqslant k}} T(\alpha)
\end{tikzcd}$$

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