Mathematisches Forschungsinstitut Oberwolfach

Report No. 35/2016

DOI: 10.4171/OWR/2016/35

Topologie

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17 July – 23 July 2016

Abstract. The Oberwolfach conference "Topologie" is one of only a few opportunities for researchers from many different areas in algebraic and geometric topology to meet and exchange ideas. This year we emphasized two topics of recent interest: representation stability and motivic homotopy theory, with their respective applications to arithmetic, classical homotopy theory as well as algebraic geometry. Double lectures on each topic where given by Benson Farb and Dan Isaksen. The rest of the program spanned a wide range of topics ranging from topological Hochschild homology to obstruction theory of positive scalar curvature, via, to name a few, K–theory of C^*–algebras, modular characteristic classes, Goodwillie calculus, 2-Segal spaces and deformation quantization.

Mathematics Subject Classification (2010): 55-xx, 57-xx, 18Axx, 18Bxx.

Introduction by the Organisers

This topology conference in Oberwolfach was organized by an organizing committee consisting of Mark Behrens, Peter Teichner, Nathalie Wahl and Michael Weiss, the first organizer being new on the team. About 50 mathematicians participated, working in many different areas of algebraic and geometric topology.

The talks were of three types. There were 12 regular one-hour talks, 2 x 2 one-hour talks by the keynote speakers Benson Farb and Dan Isaksen, and 5 half-hour talks.

Farb gave two lectures on representation stability and its applications to arithmetic, while Isaksen talked about motivic homotopy theory and its applications to classical homotopy theory. Both series of lectures where complemented by follow-up talks by other speakers. The remaining talks of the conference covered a variety of topics including stable homology of automorphisms of free groups with twisted coefficients and unstable homology of general linear groups, cellular E_2–algebras and a motivic version of the E_2–operad, 2–Segal spaces, the harmonic compactification of moduli space, the Milnor number, derived induction theory, K–theory and L–theory for C^*–algebras, deformation quantization, higher topological Hochschild homology and topological cyclic homology, Goodwillie calculus for categories and secondary obstructions for positiv scalar curvature. Speakers were instructed to give talks that could be appreciated by an audience of topologists of many different kinds, and they were generally very successful in doing so, also for the shorter talks.

Keynote speaker Benson Farb gave in his first talk a very nice overview of the state of the art of representation stability and the theory of FI–modules, a very fruitful theory of Church-Farb-Ellenberg that allows to describe stable phenomena in the homology of sequences of objects that earlier were thought of as having no stability. He then devoted his second lecture on "point counting for topologists", explaining how the Grothendieck-Lefshetz formula gives a relationship between homological stability for varieties and asymptotic point counting in arithmetic. This idea was the basis of the breakthrough work of Ellenberg-Venkatesh-Westerland about the Cohen-Lenstra heuristics for function fields. These talks were complemented by the talks of Jesse Wolfson and Craig Westerland. Wolfson introduced the concept of "homological densities", inspired by arithmetic, and described results and conjectures about those from his joint work with Farb and Wood. Westerland gave an account of his joint work with Ellenberg and Tran, where they deduce the asymptotic behaviour of the number of points in certain Hurwitz moduli stack from asymptotic behaviour of the homology of the braid groups with certain twisted coefficients.

Keynote speaker Dan Isaksen started by giving an overview lecture about motivic homotopy theory, explaining how it allows to apply homotopy-theoretic methods to the study of algebraic varieties, and for example better approach their algebraic K–theory. This is the circle of ideas that lead to a proof by Voevodsky of the Milnor and Bloch-Kato conjectures which relate the Milnor K-theory of a field with étale cohomology. In his second lecture, Isaksen talked about his very recent work where he uses motivic stable homotopy groups to get computations of classical stable homotopy groups of spheres in a much larger range than so far obtained. This was supplemented by the talks of Kirsten Wickelgren and Geoffroy Horel. Wickelgren described her joint work with Kass, where they show that the local degree around an isolated zero in motivic homotopy theory identifies with the degree of a certain quadratic form. She then explained how this allowed to prove new results about the behavior of singularities. Horel, in his talk, constructed a lift of the E_2–operad to the category of étale motives over $\mathbb{Q}$.

We now describe the themes of the remaining regular one-hour talks.

Oscar Randal-Williams explained his technique for computing the homology of the automorphisms of free groups with a certain type of twisted coefficients, in the stable range, a technique that can also be used to compute the homology with twisted coefficients of certain mapping class groups. Alexander Kupers described his join work with Galatius and Randal-Williams, in which they show that considering families of groups such as general linear groups, automorphisms of free groups or mapping class groups of surfaces as E_2–algebras, and studying their E_2–homology, gave new information about their ordinary homology. Thomas Nikolaus gave an account of his joint work with Peter Scholze where they give a much simpler construction of topological cyclic homology than what was previously known. This theory was conceived by Bokstedt in the 80's to approximate algebraic K–theory of rings, and was used very successfully for this purpose in particular by Hesselholt-Madsen. On a closely related theme, Birgit Richter gave an overview lecture about topological Hochschild and higher Hochschild homology, explaining its role in studying iterated K–theory, but also in distinguishing A_∞ structures or detecting ramification of extensions of ring spectra. Nick Rozenblyum gave an operadic framework for studying deformation quantization, and explained how factorization homology allows to quantize mapping spaces with source a manifold by quantizing the target. Gijs Heuts described a "Goodwillie tower" of categories approximating more and more the category of pointed spaces, starting from that of spectra. He showed how this set-up yields a new point of view on the classical equivalence between commutative and Lie algebras in rational homotopy theory, and gives a telescope analogue of this equivalence. Finally, Rudolf Zeidler gave obstructions to families of positive scalar curvature metrics using embedded submanifolds of various codimensions.

The half-hour talks were given in an intense, but very enjoyable, morning session on Wednesday by Daniela Egas, Akhil Mathew, Claudia Scheimbauer, Markus Land and David Sprehn. Egas gave an account of her joint work with Boes on the computation of the homology of the harmonic compactification of the moduli space of Riemann surfaces. Mathew gave a version of Dress induction in equivariant stable homotopy theory (from his joint work with Naumann and Noel). Scheimbauer talked about her joint work with Bergner, Osorno, Ozornova and Rovelli, where they give in particular an equivalence between double categories and 2–Segal sets, with nice examples coming from partial monoids. Land talked about his joint work with Nikolaus, where they study the relationship (or sometimes lack of relationship!) between K–theory and L–theory. Finally, Sprehn told us about his computation, together with Lahtinen, of non-trivial homology classes in the homology of general linear groups over finite fields at the characteristic. They find non-trivial classes in much lower degrees than previously known.

Once again the Oberwolfach staff, not least the kitchen staff, helped to make this meeting pleasant and memorable. Our thanks go to the institute for creating this atmosphere and making the conference possible.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons

Foundation for supporting Agnès Beaudry and Emily Riehl in the “Simons Visiting Professors” program at the MFO.

Workshop: Topologie

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Abstracts

Cohomology of $\mathrm{Aut}(F_n)$ with twisted coefficients

Oscar Randal-Williams

The stable (co)homology of $\mathrm{Aut}(F_n)$ (with constant coefficients) has been computed by Galatius [1]. His argument exploits a model $\mathcal{G}_n^1 \simeq B\mathrm{Aut}(F_n)$ given by the space of rank n graphs in $\mathbb{R}^\infty$ equipped with a marked point. These form morphism spaces of a graph cobordism category $\mathcal{G}$, and on one hand the group completion theorem and "parametrised surgery on graphs" relate $\mathcal{G}_\infty^1$ with the K-theory $\Omega B|\mathcal{G}|$ of the symmetric monoidal category $\mathcal{G}$. On the other hand, using geometric arguments concerning spaces of non-compact graphs, Galatius identifies this K-theory space with $Q(S^0)$, the free infinite loop space on a point; this is a well-studied object in homotopy theory, and its homology is completely known.

Fixing a based space X, one may consider analogous spaces $\mathcal{G}_n^1(X)$ of graphs with a marked point which are additionally equipped with a based map to X. Using homological stability for $\mathrm{Aut}(F_n)$ with finite-degree twisted coefficients (which we have recently established in joint work with Wahl [6]) and borrowing an argument of Cohen–Madsen [2] from the case of mapping class groups, one can establish homological stability and the analogue of Galatius' theorem in this case.

Theorem A. There is a map

$$\mathcal{G}_n^1(X) \longrightarrow Q_0(X_+)$$

which, if X is simply-connected, is an isomorphism on homology in degrees $* \leq \frac{n-3}{2}$. (The map is induced by the Becker–Gottlieb transfer for the universal family of graphs over $\mathcal{G}_n^1(X)$.)

The relevance of Theorem A to the question of twisted coefficients for $\mathrm{Aut}(F_n)$ is the fibration

$$\mathrm{map}_*(\vee^n S^1, X) \longrightarrow \mathcal{G}_n^1(X) \longrightarrow \mathcal{G}_n^1 \simeq B\mathrm{Aut}(F_n)$$

and its associated Serre spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Aut}(F_n); H^q(\mathrm{map}_*(\vee^n S^1, X))) \Rightarrow H^{p+q}(\mathcal{G}_n^1(X)),$$

whose target may be identified with $H^{p+q}(Q_0(X_+))$ in a range of degrees by Theorem A. Of course, the behaviour of this spectral sequence for any particular X might be very complicated. However, by exploiting the functoriality of the spectral sequence in the variable X, one can severely restrict its behaviour in many situations.

In particular, choosing a $\mathbb{Q}$-vector space V and setting $X = K(V^*, 2)$, one obtains a spectral sequence of $GL(V)$-representations. Analysing the weight decomposition on this spectral sequence induced by the action of $\mathbb{Q}^\times \leq GL(V)$ one may show that it collapses rationally, and analysing the weight decomposition on $H^*(Q_0(K(V^*, 2)_+); \mathbb{Q}) = \mathrm{Sym}^*(\mathrm{Sym}^{*>0}(V[2]))$ allows one to compute the E_2-page completely. It them becomes a problem in representation theory to combine the information so obtained in a useful way, which eventually leads to the following, where we write $H := H_1(F_n;\mathbb{Z})$ and $H_\mathbb{Q} = H\otimes\mathbb{Q}$, considered as $\mathrm{Aut}(F_n)$-modules.

Theorem B. As a $\mathbb{Q}[\Sigma_q]$-module we have

$$H^q(\mathrm{Aut}(F_\infty); H_\mathbb{Q}^{\otimes q}) = \mathbb{Q}\{\text{partitions of } \{1,2,\ldots,q\}\}\otimes\mathbb{Q}^-$$

and the cohomology in all other degrees vanishes.

The Schur–Weyl decomposition $H_\mathbb{Q}^{\otimes q} = \bigoplus_\lambda S^\lambda \otimes S_\lambda(H_\mathbb{Q})$ in terms of Schur functors $S_\lambda(-)$ shows that for a partition λ of q the dimension of the twisted cohomology $H^q(\mathrm{Aut}(F_\infty); S_\lambda(H_\mathbb{Q}))$ is the multiplicity of the Specht module S^λ in the Σ_q-representation given by $\mathbb{Q}\{\text{partitions of } \{1,2,\ldots,q\}\}\otimes\mathbb{Q}^-$. This may be calculated algorithmically by character theory.

The result of Theorem B can also be obtained by combining work of Djament [3] and Vespa [7], who use techniques of functor homology. However the technique we have described is quite general and can be used to obtained related results in several directions. In one direction, one may study $\mathrm{Out}(F_n)$ by the same methods, giving

$$H^q(\mathrm{Out}(F_\infty); H_\mathbb{Q}^{\otimes q}) = \mathbb{Q}\{\text{partitions of } \{1,2,\ldots,q\} \text{ with no parts of size } 1\}\otimes\mathbb{Q}^-.$$

In another direction, one may obtain results about torsion in the twisted cohomology too. For example, if λ is a partition of q, and $p > q$ is a prime number, then one may still make sense of the Schur functor $S_\lambda(-)$ on $\mathbb{Z}_{(p)}$-modules, and similar techniques show that $H^*(\mathrm{Aut}(F_\infty); S_\lambda(H_{(p)}))$ is a free $H^*(\mathrm{Aut}(F_\infty);\mathbb{Z}_{(p)})$-module (with module generators in degrees which may be deduced from Theorem B). In particular, as all prime numbers are greater than 1 we find that $H^*(\mathrm{Aut}(F_\infty); H)$ is a free $H^*(\mathrm{Aut}(F_\infty);\mathbb{Z})$-module on a single generator in degree 1.

In a third direction, the general strategy we have employed may be attempted whenever one has a "Madsen–Weiss theorem with maps to a background space". This is available in many situations, including mapping class groups of surfaces and diffeomorphism groups of high-dimensional manifolds. In particular, one may use this strategy to recover Looijenga's calculation [4] of the stable cohomology with twisted coefficients for mapping class groups of closed surfaces, and to obtain new results for mapping class groups of surfaces with boundary.

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Cellular E_2-algebras

Alexander Kupers
(joint work with Søren Galatius, Oscar Randal-Williams)

We report on work in progress on a new method for proving homological stability results, based on the theory of cellular E_2-algebras. Furthermore, we discuss applications to two examples of E_2-algebras: $\bigsqcup_{g\geq 0} B\Gamma_{g,1}$, where $\Gamma_{g,1}$ denotes the mapping class group of a genus g surface with one boundary component, and $\bigsqcup_{n\geq 0} BGL_n(R)$, where $GL_n(R)$ denotes the rank n general linear group over a ring R.

These E_2-algebra structures endow $\bigoplus_{g\geq 0} H_*(B\Gamma_{g,1})$ and $\bigoplus_{n\geq 0} H_*(BGL_n(R))$ with the structure of an algebra. Ordinary homological stability results can be phrased in terms of this algebra structure. There is a canonical generator σ in $H_0(B\Gamma_{1,1})$ or $H_0(BGL_1(R))$, and multiplying with this class induces stabilization maps $H_*(B\Gamma_{g,1}) \to H_*(B\Gamma_{g+1,1})$ or $H_*(BGL_n(R)) \to H_*(BGL_{n+1}(R))$. Homological stability is then equivalent to $\sigma \cdot -$ being an isomorphism in a range. Both $\bigoplus_{g\geq 0} H_*(B\Gamma_{g,1})$ and $\bigoplus_{n\geq 0} H_*(BGL_n(R))$, for R of finite stable rank, are known to have homological stability (see e.g. [1] and [2]). A novel feature of our approach is to not restrict attention to multiplication by just σ, but to consider the entire E_2-algebra structure. This allows us to improve on these homological stability results.

To understand E_2-algebras we approximate them by cellular E_2-algebras. A cellular E_2-algebra is one obtained by iterated cell attachments in the category of E_2-algebras; just as any space is weakly homotopy equivalent to a cell-complex, any E_2-algebra is weakly equivalent to a cellular one. There exists a homology theory which detects minimal E_2-cell decompositions in favorable circumstances, e.g. for E_2-algebras in connective spectra. It is obtained by deriving the functor that sends an augmented E_2-algebra to its E_2-indecomposables, and is hence closely related to E_2 versions of Quillen homology and cotangent complexes that have been studied before.

Using a version of this homology theory for E_2-algebras in $\mathbb{N}_0$-graded spectra, where the grading keeps track of respectively g and n, we can formulate and prove a vanishing line for E_2-cells of $\Sigma^\infty_+(\bigsqcup_g B\Gamma_{g,1})$ and $\Sigma^\infty_+(\bigsqcup_n BGL_n(R))$. Together with computational input in low degrees, such a vanishing line can be used to improve known homological stability results and to prove new “metastability” results.

Here is an example of how these ideas allow one to improve known results: the rational computations of the homology of $BGL_n(\mathbb{Z})$ as in [3] and our vanishing line imply the following result.

Theorem 1. The stabilization map $H_*(BGL_n(\mathbb{Z});\mathbb{Q}) \to H_*(BGL_{n+1}(\mathbb{Z});\mathbb{Q})$ is an isomorphism for $* < \frac{5}{6}n$.

The stability range of $* < \frac{5}{6}n$ in this theorem is an improvement over the range $* < \frac{1}{2}n$ that comes out of ordinary homological stability arguments as in [2]. We remark it may be possible to assemble this theorem from results on the cohomology of arithmetic groups (see [4] for an introduction to the literature on this subject).

As an example of a metastability result, we use computations of the homology of $B\Gamma_{g,1}$ as in [5] to prove the following theorem:

Theorem 2. There is a class $k \in H_2(B\Gamma_{3,1};\mathbb{Q})$ dual to the MMM-class κ_1, such that multiplication by k induces a map

$$H_*(B\Gamma_{g-2,1}, B\Gamma_{g-3,1};\mathbb{Q}) \to H_{*+2}(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Q})$$

which is an isomorphism for $* \leq \frac{3g-2}{4}$.

This result gives new information about the homology of mapping class groups outside the stable range, and can be used to confirm a claim from a preprint of J. Harer [6].

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Representation Stability: A survey of recent progress

Benson Farb

This talk was a survey of the theory of representation stability. This theory was initiated in the papers [2] and [1]. A survey up to Spring 2014 is given in [3], but progress in the area has exploded since that time.

Representation stability is a phenomenon whereby the structure of certain sequences X_n of spaces can be seen to stabilize when viewed through the lens of representation theory. One view of this area is as a marriage of representation theory and homological stability.

A key example is that of an FI-module. An *FI-module* V is a functor from the category of finite sets and injections to k-modules, where k is a Noetherian ring. An FI-module gives a sequence V_n of S_n-representations with many linear maps between them. An FI-module is *finitely generated* if there is a finite subset of

$\coprod V_i$ so that no proper sub-FI-module of V contains S. There are many examples of finitely-generated FI-modules, a key example being $H^i(\mathrm{PConf}_n(M);\mathbb{Q})$, the i^{th} cohomology of the space of ordered n-tuples of distinct points in a connected, oriented manifold M which is either compact or the interior of a compact manifold with boundary.

There are many consequences of finite generation of an FI-module V, such as:

1. Representation stability: The sequence V_n is representation stable in the sense of [2].

2. Polynomial characters: There exists a polynomial $Q(X_1,\ldots,X_r)$ in the cycle counting functions X_i on symmetric groups so that for all $n \geq D(i)$:

$$\chi_{H^i(\mathrm{PConf}_n(M);\mathbb{Q})}(\sigma) = Q(X_1(\sigma),\ldots,X_r(\sigma)) \quad \text{for all } \sigma \in S_n.$$

where $\deg(Q) \leq i$ if $\dim Y > 1$ and $\deg(Q) \leq 2i$ if $\dim Y = 1$.

After introducing this concept, the rest of the talk was devoted to the recent developments. These include the following:

(1) Applications to the modular representation theory of symmetric groups S_n. (Harman, Nagpal).
(2) The idea of FI-groups. This gives new finiteness theorems in group theory. (Church-Putman, Day-Putman).
(3) Arithmetic applications, including étale representation stability (Church-Ellenberg-Farb, Farb-Wolfson, Chen, Gadish, Casto, Harman, ...).
(4) Replacing FI with other categories (Sam-Snowden, Wilson, Gadish, and many others).
(5) A huge homological development, for example resolutions of FI-modules, FI-homology, depth, Castelnuevo-Mumford regularity, etc (Sam-Snowden, Ramos, Church-Ellenberg, Nagpal, Gan, Li, and many others).

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Motivic homotopy theory

Dan Isaksen

Motivic homotopy theory is a homotopy theory of algebraic varieties [2]. Voevodsky used this framework to prove the Milnor Conjecture and Bloch-Kato Conjecture, which state that the map

$$K_*^M(F)/n \to H^*_{\text{ét}}(F;\mu_n^{\otimes *})$$

from Milnor K-theory to étale cohomology is an isomorphism. In particular, Voevodsky used "motivic Steenrod operations" to establish the conjectures [1].

In order to construct unstable motivic homotopy theory, start with the category of smooth varieties over a field F. Then adjoin formal colimits to obtain a category of simplicial presheaves. Next, repair some desirable colimits. Namely, if $\{U, V\}$ is a Nisnevich cover of X, then the map $U \coprod_{U \cap V} V \to X$ is declared to be a weak equivalence. Finally, the projection maps $X \times \mathbb{A}^1 \to X$ are declared to be weak equivalences. The result of these steps is unstable motivic homotopy theory.

Motivic homotopy theory has two kinds of circles: $S^{1,0}$ is the usual simplicial circle, and $S^{1,1}$ is the algebraic circle $\mathbb{A}^1 - 0$. Other spheres are formed by taking appropriate smash products of these two circles.

Using the properties of motivic homotopy theory described above, one can show that $\mathbb{A}^n - 0$ is equivalent to the sphere $S^{2n-1,n}$. Moreover, the quotient object $\mathbb{P}^n/\mathbb{P}^{n-1}$ is equivalent to the sphere $S^{2n,n}$.

In addition to the essential properties of unstable homotopy homotopy theory given above, there is one additional essential property of the theory. Let U be open in X. Then there is a cofiber sequence

$$U \to X \to \mathrm{Th}(N),$$

where N is the normal bundle of $X - U$ in X, and the Thom space $\mathrm{Th}(N)$ is defined to be the total space of N modulo the complement of the zero section.

Stable motivic homotopy theory is obtained by stabilizing with respect to the bigraded family of spheres. Then motivic spectra represent generalized cohomology theories for algebraic varieties. For example, algebraic K-theory is represented by a spectrum KGL. The algebraic cobordism spectrum MGL represents a cohomology theory previously unknown to algebraic geometers. Finally, motivic cohomology can be defined to be represented by the infinite symmetric powers of motivic spheres.

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Cyclotomic Structures and Factorization Homology

Thomas Nikolaus
(joint work with Peter Scholze)

Let R be a ring, we want to compute algebraic K-theory groups $\mathrm{K}_*(R)$. This turns out to be very hard, thus one tries to approximate those by more computable invariants. One has the following square

$$\begin{tikzcd}
\mathrm{K}_*(R) \arrow[r] & \mathrm{TC}_*(R) \arrow[r] \arrow[d] & \mathrm{CH}^-_*(R) \arrow[d] \\
& \mathrm{THH}_*(R) \arrow[r] & \mathrm{HH}_*(R)
\end{tikzcd}$$

all of whose corners we explained in the talk. The Hochschild homology groups $\mathrm{HH}_*(R)$ are given by the homology groups of the Hochschild chain complex which is given by the cyclic Bar construction

$$\mathrm{HH}(R) := \mathrm{colim}_{n\in\Delta}\Big(\cdots \Rrightarrow R[0]\otimes R[0]\otimes R[0] \Rrightarrow R[0]\otimes R[0] \rightrightarrows R[0] \Big)$$

Here $R[0] \in \mathrm{Ch}_{\mathbb{Z}}$ is considered as a chain complex concentrated in degree 0. The topological Hochschild homology groups $\mathrm{THH}_*(R)$ are given similarly by the homotopy groups of the topological Hochschild homology spectrum

$$\mathrm{THH}(R) := \mathrm{colim}_{n\in\Delta}\Big(\cdots \Rrightarrow HR\otimes HR\otimes HR \Rrightarrow HR\otimes HR \rightrightarrows HR \Big)$$

where $HR \in \mathrm{Sp}$ is the Eilenberg-MacLane spectrum associated to R.

We explained how these constructions can be seen as instances of factorization homology of S^1:

$$\mathrm{HH}(R) \simeq \int_{S^1} R[0] \qquad\qquad \mathrm{THH}(R) \simeq \int_{S^1} HR$$

The circle group $\mathbb{T}$ acts on S^1 by rotation and by functoriality of factorization homology the Hochschild chain complex (as well as the topological Hochschild homology spectrum) inherits an induced $\mathbb{T}$-action. The negative cyclic homology chain complex is defined as the homotopy fixed point chain complex for this action:

$$\mathrm{CH}^-(R) := \mathrm{HH}(R)^{h\mathbb{T}}$$

For topological cyclic homology (which should really be called negative topological cyclic homology) one has to take an additional structure besides the $\mathbb{T}$-action on $\mathrm{THH}(R)$ into account: the cyclotomic structure. For the formulation of the cyclotomic structure we will use the notion of Tate spectrum X^{tG} for a finite G-action on a spectrum X.

Proposition 1.
- *For every spectrum X and every prime p there is a 'diagonal' map $\Delta : X \to (X\otimes ... \otimes X)^{tC_p}$ which is natural in X.*

- *Let R be an $\mathbb{E}_n$-ring spectrum and $E \xrightarrow{C_p} M$ be a principal C_p-bundle over a framed n-manifolds M. Then there is a map*

$$\int_M R \to \left(\int_E R\right)^{tC_p}$$

which is natural in M and for $M = \mathbb{R}^n$ given by the diagonal Δ as above

It is a remarkable fact that the last proposition is not correct in the category of chain complexes. That is the reason that one has to work in spectra to see the cyclotomic structure. We explained how to prove this non-existence based on the 'Frobenius' of $\mathbb{E}_\infty$-ring spectra. Using the p-fold self-covers of the circle we get the following immediate corollary.

Corollary 2. *For every ring spectrum R the spectrum* $\mathrm{THH}(R)$ *has the following structure:*

- *An action by the circle group $\mathbb{T}$*
- *For every prime p a $\mathbb{T}$-equivariant map*

$$\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$$

where the action on the target uses the identification $\mathbb{T}/C_p \cong \mathbb{T}$.

This structure is what we call a cyclotomic structure. Given this structure extra structure on THH we can give the following formula for topological cyclic homology:

$$\mathrm{TC}(R) := \mathrm{fib}\left(\mathrm{THH(R)}^{h\mathbb{T}} \xrightarrow{\prod_p \mathrm{can}-\varphi_p} \prod_p \left(\mathrm{THH(R)}^{tC_p}\right)^{h\mathbb{T}/C_p}\right)$$

Here can denotes the map induced on homotopy $\mathbb{T}/C_p$-fixed points from the canonical map $\mathrm{THH}(R)^{hC_p} \to \mathrm{THH}(R)^{tC_p}$.

Theorem 3. *This definition of topological cyclic homology is for a connective ring spectrum R equivalent to the old one (as given by Bökstedt-Hsian-Madsen in the p-completed case and Goodwillie in the integral case).*

More generally we prove that the ∞-category of connective cyclotomic spectra as defined this way is equivalent to the ∞-category underlying the classical description of cyclotomic spectra using genuine equivariant homotopy theory (in the incarnation given by Blumberg-Mandell).

Our main result allows to give simpler descriptions and computations for a lot of results in the area, in particular of the cyclotomic trace.

Homotopical aspects of quantization and invariants of manifolds

Nick Rozenblyum

It has been observed that many moduli spaces of interest, such as the character variety of a surface, have natural symplectic structures which play an important role in the theory. A conceptual explanation for the appearance of these symplectic structures is via the theory of shifted symplectic and Poisson structures in derived algebraic geometry, developed in [3] and [2]. Roughly speaking, an n-shifted symplectic structure on a derived stack Y is a closed 2-form of cohomological degree n, which is non-degenerate. One subtlety in this setting is that the notion of closed is defined up to coherent homotopy.

There are a number of important examples of shifted symplectic stacks. Any variety with an algebraic symplectic structure is in particular a 0-shifted symplectic stack. Another fundamental example is the stack BG for an algebraic group G, which has a natural 2-shifted symplectic structure (upon choice of an Ad-invariant symmetric bilinear form on the Lie algebra of G).

A key result in this theory, called the AKSZ construction, states that if X is a compact oriented m-dimensional manifold (or an m-dimensional smooth and proper Calabi-Yau variety) and Y is an n-shifted symplectic stack, then the derived mapping stack $\mathrm{Maps}(X, Y)$ has a natural $(n-m)$-shifted symplectic structure. For instance, if X is an oriented surface and $Y = BG$, then one gets a (0-shifted) symplectic structure on the derived enhancement of the character variety of X. This gives an enhancement of the classical (as studied by Atiyah-Bott, Goldman-Millson, and others) symplectic structure on the character variety of the surface.

Given a symplectic or Poisson manifold, an important question is to classify possible quantizations. Classically, the problem of deformation quantization is formulated as that of deforming the algebra of functions (which is a Poisson algebra) to an associative algebra. For example, the character variety of a surface admits a natural quantization, called the skein quantization, which plays an important role in the study of quantum invariants of 3-manifolds.

1. P_n and BD_n operads and formality

In the context of shifted symplectic stacks, the algebra of functions on an n-shifted symplectic (or Poisson) stack forms a homotopy coherent algebra over the P_{n+1}-operad, which is the operad with a commutative multiplication and a Poisson bracket of homological degree n. Following [1], we can formulate the problem of deformation quantization in this homotopical setting using deformations of operads. In what follows, we will always work over a field k of characteristic zero. Now, for $n \geq 0$, the k-linear little n-disks operad E_n has a filtration such that the associated graded is the operad P_n. For $n \geq 2$, this is given by the Postnikov filtration. We can apply the Rees construction to this filtration to obtain operads BD_n over $k[[h]]$, which interpolate between P_n-algebras and E_n-algebras. The problem of deformation quantization can then be formulated as deforming a P_n-algebra to a BD_n-algebra.

In some sense, the most interesting case is $n = 0$; an E_0-algebra is just a vector space together with a vector, but a P_0-algebra has a lot more structure. The BD_0-operad can be described explicitly as an operad over $k[[h]]$ having a commutative multiplication $\cdot$, a Poisson bracket $\{\ \}$ of homological degree -1 and the additional relation $d(\cdot) = h\{\ \}$.

A fundamental property of the E_n-operads is Dunn's additivity theorem: the Boardman-Vogt tensor product $E_m \otimes_{BV} E_n \simeq E_{n+m}$; i.e. the ($\infty$-)category of E_m-algebras in the category of E_n-algebras is equivalent to that of E_{m+n}-algebras. We prove that this equivalence is compatible with the above filtrations; namely,

Theorem 1. There is a natural (weak) equivalence of operads over $k[[h]]$

$$E_m \otimes_{BV} BD_n \simeq BD_{m+n}.$$

Quotienting by h, we obtain:

Theorem 2. There is a natural (weak) equivalence of operads over k

$$E_m \otimes_{BV} P_n \simeq P_{m+n}.$$

One immediate application of the above is the formality theorem for higher E_n. Kontsevich and Tamarkin proved that the E_n-operads are formal for $n \geq 2$, i.e. that for $n \geq 2$, $BD_n \simeq P_n \otimes k[[h]]$. Starting with this equivalence for $n = 2$, we immediately obtain the corresponding equivalence for $n > 2$ by induction from the above. In particular, upon choosing a formality isomorphism for $n = 2$, we obtain natural formality isomorphisms for all $n > 2$ as well.

2. Manifold topology

To apply the above results to the topology of manifolds, we use factorization homology. Recall that given an E_n-algebra A and a (framed) n-dimensional manifold M, we can form the factorization homology $\int_M A$. Moreover, if A is a commutative algebra, then $\int_M A$ is the algebra of functions on the (derived) mapping stack $\mathrm{Maps}(M, \mathrm{Spec}(A))$.

Applying this fact to P_n and BD_n-algebras, thought of as $E_n \otimes_{BV} P_0$ and $E_n \otimes_{BV} BD_0$-algebras, respectively, we obtain the following enhancement of the PTVV/AKSZ construction:

Theorem 3. Let M be an compact, oriented m-dimensional manifold and Y an n-shifted Poisson stack. Then the (derived) mapping stack $\mathrm{Maps}(M, Y)$ has a natural $(n - m)$-shifted Poisson structure. Moreover, a quantization of Y gives a quantization of the mapping stack $\mathrm{Maps}(M, Y)$.

There are a number of interesting applications of this theorem and its variants. These include:

- If X is a closed, oriented surface and $Y = BG$ for G reductive (e.g. $G = SL_2$) the 2-shifted symplectic structure on BG gives a symplectic structure on the character variety. The stack BG has a natural quantization given by the quantum group, and by the above this gives a quantization of the character variety. This quantization is exactly given by the skein algebra.

- A holomorphic variant of the above results gives the following. Let E be an elliptic curve with a holomorphic volume form, and Y a smooth Calabi-Yau variety. Then the mapping stack $\mathrm{Maps}(E, T^*Y)$ from E to the cotangent bundle of Y is given by the shifted cotangent stack $T^*[-1]\,\mathrm{Maps}(E, T^*Y)$ together with its (-1)-shifted symplectic structure. In this situation a quantization of T^*Y is given by the vertex algebra of chiral differential operators on Y and a quantization of $T^*[-1]\,\mathrm{Maps}(E, T^*Y)$ gives a differential form on Y. This differential form is exactly the Witten class of Y associated to the elliptic curve E.

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Point counting for topologists

Benson Farb

In this talk I explained Weil's dictionary between function fields over $\mathbb{C}$ and function fields over $\mathbb{F}_q$. I explained the statement of the Weil conjectures, proved by Dwork, Grothendieck and Deligne.

Let X be a scheme defined over $\mathbb{Z}$. The Weil conjectures provide a fundamental link between the topology of $X(\mathbb{C})$ and the arithmetic of $X(\mathbb{F}_q)$. As first indicated by work of Ellenberg-Venkatesh-Westerland [2], followed by Vakil-Wood [3], Church-Ellenberg-Farb [1] and others, this correspondence should convert homological stability phenomena in topology to asymptotic point counts on the arithmetic side. We summarize this in the following table, with the rows going from least to most general.

Topology	Arithmetic
$H^*(X(\mathbb{C}))$	$\lvert X(\mathbb{F}_q)\rvert$
homological stability of X_n	asymptotics of $\lvert X_n(\mathbb{F}_q)\rvert$ as $n \to \infty$
representation stability	asymptotics of arithmetic statistics on $X_n(\mathbb{F}_q)$

In this talk I concentrated on the examples $\mathrm{PConf}_n(X)$ (resp. $\mathrm{UConf}_n(X)$) of configurations of ordered (resp. unordered) n-tuples of points on a smooth variety X.

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Motivic stable homotopy groups

Dan Isaksen

The computation of stable homotopy groups is one of the most fundamental problems in homotopy theory. At odd primes, the Adams-Novikov spectral sequence is the most effective computational tool. At the prime 2, the relationship between the Adams spectral sequence and the Adams-Novikov spectral sequence can be leveraged to great effect, especially in the presence of $\mathbb{C}$-motivic computations [1].

The first step is to identify the $\mathbb{C}$-motivic cohomology of a point and the $\mathbb{C}$-motivic Steenrod algebra [2] [3]. The answer is that

$$\mathbb{M}_2 = H^{*,*}(\mathrm{pt}; \mathbb{F}_2) = \mathbb{F}_2[\tau],$$

and

$$A_*^{\mathbb{C}} = \mathbb{M}_2[\tau_i, \xi_i]/\tau_i^2 = \tau\xi_i.$$

The next step is to compute the Adams E_2-page $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$. This can be done by hand to the 70-stem, or by computer to hundreds of stems.

The third step is to compute Adams differentials. Traditionally, this has been done through a combination of techniques, including prior knowledge of the image of J, prior knowledge of the spectrum *tmf*, and analysis of Toda brackets. This has been carried out in great detail to the 61-stem, where this approach becomes too cumbersome.

The element τ of $\mathbb{M}_2$ detects a stable homotopy element $\tau : S^{0,-1} \to S^{0,0}$. The cofiber $C\tau$ of τ is a motivic spectrum with some surprisingly good properties. It is an E_∞-ring spectrum whose Adams-Novikov spectral sequence collapses. In other words, the motivic homotopy groups $\pi_{*,*}C\tau$ are isomorphic to the E_2-page of the classical Adams-Novikov spectral sequence! Moreover, the Adams spectral sequence converging to $\pi_{*,*}C\tau$ is equal to the algebraic Novikov spectral sequence converging to the classical Adams-Novikov E_2-page.

Guozhen Wang used a computer to determine the algebraic Novikov spectral sequence in a large range. Thus, computer output tells us precisely the Adams spectral sequence for $C\tau$. Naturality of the Adams spectral sequence applied to the maps in the cofiber sequence

$$S^{0,-1} \to S^{0,0} \to C\tau \to S^{1,-1}$$

then determine a large number of Adams differentials for the motivic sphere spectrum.

From this perspective, computation of the first 60 stems becomes essentially easy, in the sense that the computer does almost all of the work. After that point, things start to get more difficult. So far, these difficulties are manageable. The computation has been extended to the 70 stem so far, with more to come. It remains to be seen how much further the approach will work.

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Sullivan diagrams and π_*-stability

Daniela Egas Santander
(joint work with Felix Boes)

Let $S^m_{g,n}$ denote the genus g oriented surface with n boundary components and m punctures. We denote by $\mathcal{M}^m_{g,n}$ the moduli space of $S^m_{g,n}$ and by $\mathrm{Mod}(S^m_{g,n})$ its corresponding mapping class group. When the surface has at least one boundary component (i.e., $n > 0$) its moduli space is a model for the classifying space of the mapping class group i.e.,

$$\mathrm{BMod}(S^m_{g,n}) \simeq \mathcal{M}^m_{g,n}.$$

We study the homotopy type of the space of Sullivan diagrams, which is a space of fat graphs that has the homotopy type of the Harmonic compactification of Moduli space. Before stating our results, let us briefly describe the relation between Sullivan diagrams and Moduli space.

In [1], Bödigheimer constructs a space $\mathfrak{Rad}_p(S^m_{g,n})$, for $1 \le p \le n$, which is a model for $\mathcal{M}^m_{g,n}$. On the other hand, Godin uses the ideas of Penner, and Igusa to construct a space of fat graphs which is also a model of Moduli Space [5, 6]. For any $1 \le p < n$, the space of fat graphs has a homotopy equivalent subspace, the space of p-admissible fat graphs which we denote $\mathfrak{Fat}^{ad}_p(S^m_{g,n})$. This subspace has a natural quotient which is the space of p-Sullivan diagrams $\mathscr{SD}_p(S^m_{g,n})$. This space has a canonical CW-structure and its cellular complex is the chain complex of p-Sullivan diagrams. There is a cellular homotopy equivalence between the spaces $\mathscr{SD}_p(S^m_{g,n})$ and $\overline{\mathfrak{Rad}}_p(S^m_{g,n})$ [4]. We summarize this statements in diagram below.

$$\begin{tikzcd}
\mathfrak{Fat}^{ad}_p(S^m_{g,n}) \arrow[d, two heads] & \simeq \mathcal{M}^m_{g,n} \simeq & \mathfrak{Rad}_p(S^m_{g,n}) \arrow[d, hook] \\
\mathscr{SD}_p(S^m_{g,n}) \arrow[rr, "\simeq"] & & \overline{\mathfrak{Rad}}_p(S^m_{g,n})
\end{tikzcd}$$

We give concrete computations on the homotopy type of $\mathscr{SD}_1(S^m_{g,n})$ and show that these spaces exhibit π_*-stability with respect to the number of punctures, boundaries and genus. More precisely, we denote by $\mathscr{SD}^m_g$ the space of 1-Sullivan diagrams of the surface $S^m_{g,1}$ where all punctures are unlabeled and $\widetilde{\mathscr{SD}}^m_g$ to be the space of 1 Sullivan diagrams of the surface $S^m_{g,1}$ where all punctures are labeled by the set $\{1, 2, \ldots, m\}$. Similarly, we denote by $\mathscr{S}\mathscr{D}_{g,m}$ and $\widetilde{\mathscr{S}\mathscr{D}}_{g,m}$ the space of 1-Sullivan diagrams of the surface $S_{g,1+m}$ where all boundary components are unlabeled respectively labeled. Let $\mathscr{S}\mathscr{D}(g,m)$ denote any of these spaces. We show the following results:

Theorem (A). *Let $g \geq 0$ and $m \geq 1$.*

(i) The spaces $\mathscr{S}\mathscr{D}(g,m) = \mathscr{S}\mathscr{D}_g^m$, $\widetilde{\mathscr{S}\mathscr{D}}_g^m$, $\mathscr{S}\mathscr{D}_{g,m}$ and $\widetilde{\mathscr{S}\mathscr{D}}_{g,m}$ are highly connected. More precisely,

$$\pi_*(\mathscr{S}\mathscr{D}(g,m)) = 0 \text{ for } * \leq m-2\,.$$

(ii) This result is slightly improved in the unlabeled cases

$$\pi_*(\mathscr{S}\mathscr{D}_g^m) = 0 \text{ and } \pi_*(\mathscr{S}\mathscr{D}_{g,m}) = 0 \text{ for } * \leq m'\,.$$

where m' is the largest even number smaller than m. Furthermore, in case of genus zero the connectivity bound is sharp. More precisely, we have that

$$H_{m'+1}(\mathscr{S}\mathscr{D}_0^m; \mathbb{Z}) = \mathbb{Z} \text{ and } H_{m'+1}(\mathscr{S}\mathscr{D}_{0,m}; \mathbb{Z}) = \mathbb{Z}\,.$$

Recall that by glueing a genus one surface with two boundary components to the unique boundary of $S_{g,1}$ we obtain maps

$$B\mathrm{Mod}(S_{1,1}) \to \mathrm{BMod}(S_{2,1}) \to \ldots \mathrm{BMod}(S_{g,1}) \to \mathrm{BMod}(S_{g+1,1}) \to \ldots$$

In [7], Harer showed that these maps induce an isomorphisms in a range of dimension increasing with genus. We construct a map

$$\Phi : \mathscr{S}\mathscr{D}(g,m) \longrightarrow \mathscr{S}\mathscr{D}(g+1,m)$$

which extends the stabilization map on mapping class groups and show the following:

Theorem (B). *Let $g \geq 0$ and $m > 2$. The stabilization map Φ is $(g+m-2)$-connected. In the unlabeled cases this can be slightly improved in which case the stabilization map is $(g+m'-2)$-connected, where m' is the largest even number smaller than m.*

In the unparametrized, unlabeled case we can give further results for the case $m = 2$.

Proposition (C). *The fundamental group of $\mathscr{S}\mathscr{D}_1(S_{g,1}^2)$ is*

$$\pi_1(\mathscr{S}\mathscr{D}_1(S_{g,1}^2) \cong \begin{cases} \mathbb{Z} & g = 0 \\ \mathbb{Z}/2\mathbb{Z} & g > 0 \end{cases}.$$

The stabilization map $\Phi\colon \pi_1(\mathscr{S}\mathscr{D}_1(S_{g,1}^2)) \to \pi_1(\mathscr{S}\mathscr{D}_1(S_{g+1,1}^2))$ is the quotient map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ for $g = 0$ and an isomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ for $g > 0$. Furthermore, the generators of these groups are in the image induced by the quotient map of spaces

$$\mathrm{BMod}(S_{g,1}^2) \to \mathscr{S}\mathscr{D}_1(S_{g,1}^2)$$

and they correspond to the diffeomorphism that exchanges the two punctures inside a small disk which is contractible inside $S_{g,1}^2$.

The results above imply that, studying the homotopy type and in particular the homology of Sullivan diagrams could give further insight into the unstable homology of the Moduli space of surfaces. On the other hand, the study of the homotopy type of Sullivan diagrams is also of interest in the field of string topology, which studies algebraic structures on the homology of free loop spaces. Let LM be the free loop space of a manifold M. In string topology one constructs operations

$$H_*(LM)^{\otimes n_1} \longrightarrow H_*(LM)^{\otimes n_2}$$

parametrized over a certain *space of operations* and subject to certain compatibility conditions such that they assemble into some sort of field theory. Our chain complex of Sullivan diagrams corresponds to the one defined by Tradler and Zeinalian in [10] and by Wahl and Westerland in [11] to study operations on the Hochschild homology of algebras with a given structure. When restricted to the case with no punctures i.e. $m = 0$ the space of Sullivan diagrams is homeomorphic the the underlying space of the Sullivan PROP described by Kaufmann in [8]. Furthemore, Poirier and Rounds construct string operations using a space of chord diagrams $\overline{SD}$ and they describe a quotient of this space $\overline{SD}/\sim$ through which their operations factor. This quotient space is homeomorphic to $\mathscr{SD}$ [9]. In a sequel [3], Drummond-Cole, Porier, and Rounds use yet another chain complex of chord diagrams. Currently it is unclear the relation between their chain complex and the one we study here. Finally in [2], Cohen and Godin construct string operations using yet another space of chords diagrams. Although the concepts are very closely related, these spaces are not homotopy equivalent nor are their corresponding chain complexes quasi-isomorphic.

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Derived induction and restriction theory

Akhil Mathew

(joint work with Niko Naumann and Justin Noel)

Let M be a *Mackey functor* for the group G (always assumed finite). That is, one has the following structure and properties:

(1) For every finite G-set T, an abelian group $M(T)$.
(2) For every map of finite G-sets $f : T \to T'$, homomorphisms of abelian groups $f^* : M(T') \to M(T)$ and $f_* : M(T) \to M(T')$.
(3) If $T \simeq T_1 \sqcup T_2$, $M(T) \simeq M(T_1) \times M(T_2)$.
(4) f_*, f^* are functorial for the morphism f and satisfy a natural base-change relation.

Suppose that each $M(T)$ has the structure of a commutative ring such that f^* is always a map of commutative rings and f_* a map of $M(T')$-modules. Then M is called a *Green functor.*

Green functors arise frequently in "nature." For example, there is a Green functor that sends the G-set $G/H \mapsto H^*(H; \mathbb{F}_p)$ and there is one that sends $G/H \mapsto R(H)$ where $R(H)$ is the representation ring of H.

Let $\mathcal{F}$ be a *family of subgroups* of G, i.e., a class of subgroups closed under subconjugation. Let $\mathcal{O}_\mathcal{F}(G)$ be the category of all G-sets of the form $G/H, H \in \mathcal{F}$.

Given a Mackey functor M, we have $\mathcal{F}$-restriction and $\mathcal{F}$-induction maps

$$(1) \quad \phi_\mathcal{F}^{\mathrm{Ind}} : \varinjlim_{G/H \in \mathcal{O}_\mathcal{F}(G)} M(G/H) \to M(G/G), \quad \phi_\mathcal{F}^{\mathrm{Res}} : M(G/G) \to \varprojlim_{G/H \in \mathcal{O}_\mathcal{F}(G)^{op}} M(G/H).$$

Theorem 1 (Dress)**.** Suppose M is a Green functor and $\phi_\mathcal{F}^{\mathrm{Ind}}$ has image containing the unit. Then $\phi_\mathcal{F}^{\mathrm{Ind}}$ and $\phi_\mathcal{F}^{\mathrm{Res}}$ are isomorphisms.

In this project, we develop a "spectral" version of the Dress induction theorem and describe various applications. Our main results are in [6, 5].

Definition 2 (Compare Barwick [2])**.** Let Burn_G denote the *effective Burnside 2-category* of G. This is a $(2,1)$-category such that:

(1) The objects are finite G-sets T.
(2) $\mathrm{Hom}_{\mathrm{Burn}_G}(T, T')$ is given by the groupoid of spans of finite G-sets from T to T'. Composition is given by composition of spans.

A *spectral Mackey functor* is a functor $M : \mathrm{Burn}_G^{op} \to \mathrm{Sp}$, where Sp denotes the ∞-category of spectra.

Theorem 3 (Guillou-May [4])**.** There is an equivalence of ∞-categories between G-spectra and spectral Mackey functors.

Given a G-spectrum X, the associated spectral Mackey functor sends the G-set G/H to the H-fixed point spectrum X^H. We thus obtain derived induction and restriction maps

$$(2) \qquad \psi_\mathcal{F}^{\mathrm{Ind}} : \varinjlim_{G/H \in \mathcal{O}_\mathcal{F}(G)} X^H \to X^G, \quad \psi_\mathcal{F}^{\mathrm{Res}} : X^G \to \varprojlim_{G/H \in \mathcal{O}_\mathcal{F}(G)^{op}} X^H.$$

Theorem 4 (Derived Dress induction)**.** Let X be a G-ring spectrum. If $\psi_{\mathcal{F}}^{\mathrm{Ind}}$ has image containing the unit then $\psi_{\mathcal{F}}^{\mathrm{Ind}}, \psi_{\mathcal{F}}^{\mathrm{Res}}$ are homotopy equivalences.

Definition 5. A G-spectrum is *$\mathcal{F}$-nilpotent* if it belongs to the smallest thick $\otimes$-ideal of the homotopy category of G-spectra generated by $\{G/H_+\}_{H\in\mathcal{F}}$.

One shows easily that $\psi_{\mathcal{F}}^{\mathrm{Ind}}, \psi_{\mathcal{F}}^{\mathrm{Res}}$ are equivalences for an $\mathcal{F}$-nilpotent G-spectrum. Our results show that a G-ring spectrum satisfies the condition of Theorem 4 if and only if it is $\mathcal{F}$-nilpotent.

Definition 6. The *derived defect base* of a G-spectrum is the (unique) smallest family $\mathcal{F}$ for which it is $\mathcal{F}$-nilpotent.

We have the following table (cf. [5]) of derived defect bases of some common G-spectra. Given a spectrum E, we write $\underline{E}$ for the associated Borel-equivariant G-spectrum.

G-spectrum	Derived defect base
$\underline{Hk}$, $\mathrm{char}(k) = p$	elementary abelian p-subgroups (Quillen, Carlson, Balmer)
KU_G, KO_G	cyclic subgroups
$\underline{MU}$	abelian l-subgroups, all l
$\underline{E_n}$	abelian p-subgroups of rank $\leq n$
$\underline{S^0}$	l-subgroups, all l
$K\mathbb{R}$	$\{(1)\}$
$\underline{BP\langle n\rangle}$	abelian p-subgroups either rank $\leq n$ or elementary abelian

The conclusion of the above result for $R = \underline{Hk}$ for k a field of characteristic p is a theorem of Quillen [8]. The statement in language equivalent to that of $\mathcal{F}$-nilpotence appears in work of Carlson [3] (also for $\underline{H\mathbb{Z}}$) and more recently of Balmer [1].

Let X be a G-ring spectrum. If the Green functor π_*^X is induced from $\mathcal{F}$, then X is $\mathcal{F}$-nilpotent. The converse fails, but is in some sense not that far from holding.

Theorem 7 (Cf. [5])**.** Let X be a G-ring spectrum which is $\mathcal{F}$-nilpotent.

Then the Green functor $\pi_*^- X[1/|G|]$ is induced from $\mathcal{F}$. The natural map $\pi_* X^G \to \varprojlim_{G/H\in\mathcal{O}_{\mathcal{F}}(G)^{op}} \pi_* X^H$ has the following two properties:

(1) The kernel is nilpotent.
(2) Given x in the codomain, for appropriate $N > 0$, x^N belongs to the image. If X is p-local we can take N to be a power of p.

The above result follows from analyzing the spectral sequences associated to the maps of (2), which are equivalences since X is $\mathcal{F}$-nilpotent. The spectral sequences collapse at a finite stage with a horizontal vanishing line, from which the result can be deduced.

Theorem 8. Suppose X is a G-ring spectrum with multiplicative Thom isomorphisms for complex G-representations. Then X is $\mathcal{F}$-induced if and only if for all $H \leq G$ with $H \notin \mathcal{F}$, the kernel of $\pi_* X^H \to \prod_{H'<H} \pi_* X^{H'}$ is nilpotent.

We also have the following result, which uses the nilpotence theorem of [7].

Theorem 9. Suppose X is an $\mathbb{E}_\infty$-algebra in G-spectra. Suppose that the Green functor $\pi_*^- X \otimes \mathbb{Q}$ is $\mathcal{F}$-nilpotent. Then for any prime p or height n, the G-ring spectrum $L_n^f X$ (where L_n^f denotes finitary L_n-localization) is $\mathcal{F}$-nilpotent as well.

We note that this result applies to KU_G, since rationally the representation ring Green functor is induced from the cyclic groups (by Artin's theorem). The result provides a means of passing from the $\mathcal{F}$-nilpotence of $X_\mathbb{Q}$ (which is a purely algebraic question) to the much harder $\mathcal{F}$-nilpotence of $L_n^f X$ (which may involve torsion phenomena).

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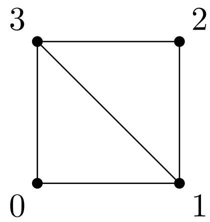
2-Segal spaces and the Waldhausen construction

Claudia Scheimbauer

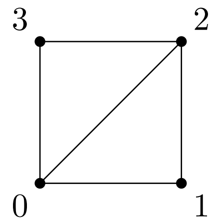
(joint work with Julie Bergner, Angélica Osorno, Viktoriya Ozornova, Martina Rovelli)

The notion of (unital) 2-Segal objects in a model category or $(\infty,1)$-category was introduced by Dyckerhoff and Kapranov in [1] and, independently, for the $(\infty,1)$-category of spaces under the name of decomposition spaces, by Gálvez-Carrillo, Kock, and Tonks [2]. It is a homotopical variant of a category which has a multi-valued composition: a 2-Segal object is a simplicial object $X_\bullet$ satisfying conditions which are 2-dimensional generalizations of the usual Segal condition and are parametrized by triangulations of regular n-gons for $n \geq 3$. For example, for the two triangulations



$\mathcal{T}_1$

and



$\mathcal{T}_2$

of the square the following conditions are required: The triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ determine the two diagrams

$$\begin{tikzcd} X_{\{0,1,3\}} \arrow[dr, "d_0"] & & X_{\{1,2,3\}} \arrow[dl, "d_1"'] \\ & X_{\{1,3\}} & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X_{\{0,1,2\}} \arrow[dr, "d_1"] & & X_{\{0,2,3\}} \arrow[dl, "d_2"'] \\ & X_{\{0,2\}} & \end{tikzcd}$$

which in turn give two maps

$$f_{\mathcal{T}_1} : X_3 \longrightarrow X_{\{0,1,3\}} \underset{X_{\{1,3\}}}{\times} X_{\{1,2,3\}} \quad \text{and} \quad f_{\mathcal{T}_2} : X_3 \longrightarrow X_{\{0,1,2\}} \underset{X_{\{0,2\}}}{\times} X_{\{0,2,3\}}.$$

The 2-Segal condition requires these maps to be weak equivalences. The 0-simplices X_0 should be thought of as the objects of the multivalued category, X_1 as the morphisms, and the span $X_1 \times_{X_0} X_1 \leftarrow X_2 \rightarrow X_1$ as the multivalued composition. An extra condition called unitality ensures that every composition with an identity morphism is (homotopically) unique.

Examples include Segal's nerve of a partial (topological) monoid from [3] and a 2-dimensional cobordism "category" with genus constraints from [6] requiring that the genus of a morphism is $\leq g$ for some fixed $g \in \mathbb{N}$.

Furthermore, both [1] and [2] showed that Waldhausen's $S_\bullet$-construction from [4] provides examples of 2-Segal spaces. In [5] we provide a generalization thereof which proves to be exhaustive in the discrete setting. Let us briefly explain this construction.

The abstract structure needed to define an $S_\bullet$-construction given by diagrams of a certain shape are certain double categories. A double category is a category internal to categories. It has a set of objects, two kinds of morphisms between two objects which we suggestively call "horizontal" and "vertical" morphisms, and 2-morphisms ("squares") which have horizontal source and target morphisms and vertical source and target morphisms:

$$\begin{tikzcd} \bullet \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, tail] & \bullet \end{tikzcd}$$

To define a simplicial object similarly to the one arising from the $S_\bullet$-construction we need the double category to be "pointed", i.e. there is an object 0 which is initial for the horizontal category and terminal for the vertical category. For such a pointed double category $\mathcal{D}$ we let $S_k(\mathcal{D})$ be the set of diagrams of the form

$$\begin{tikzcd}
0 \arrow[r, tail] & a_{00} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & a_{10} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & a_{20} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & \cdots \arrow[r, tail] \arrow[dr, Rightarrow] & a_{n0} \arrow[d, two heads] \\
& 0 \arrow[r, tail] & a_{11} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & a_{21} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, Rightarrow] & \cdots \arrow[r, tail] & a_{n1} \arrow[d, two heads] \\
& & 0 & {} & {} & {} \\
& & & \ddots & & \vdots \\
& & & & {} \arrow[d, two heads] & {} \arrow[d, two heads] \\
& & & & 0 \arrow[r, tail] & a_{nn} \arrow[d, two heads] \\
& & & & & 0
\end{tikzcd}$$

This gives a simplicial set whose face maps are given by deleting a row and column. This simplicial set is 2-Segal if we started with a double category which is "stable", meaning that any 2-morphism is uniquely determined by its horizontal and vertical sources, and also is uniquely determined by its horizontal and vertical targets. Note that $S_0(\mathcal{D}) = \{0\}$. We call a 2-Segal set with this property "reduced".

More generally, we can replace the condition that the double category should be pointed by the data of an "augmentation", which is a certain subset of objects. Then we require the elements on the diagonal to be in the augmentation.

Finally, this generalized $S_\bullet$-construction leads to an equivalence, whose inverse essentially is given by the décalages of the simplicial set.

Theorem (BOORS).
(1) *The generalized $S_\bullet$-construction is an equivalence of categories between pointed, stable double categories and reduced unital 2-Segal sets.*
(2) *The generalized $S_\bullet$-construction is an equivalence of categories between augmented, stable double categories and unital 2-Segal sets.*

A generalization of the above theorem to more homotopical settings such as 2-Segal spaces is work in progress.

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On the relation between K- and L-theory for C^*-algebras

Markus Land
(joint work with Thomas Nikolaus)

Given a C^*-algebra A one can consider the following two invariants. One the one hand there are the classical topological K-theory groups $K_*(A)$ and on the other hand the algebraic L-theory groups $L_*(A)$. To be specific I will mean projective and symmetric L-theory unless stated otherwise. L-theory has its origins in surgery theory where (quadratic) L-groups arise as obstruction groups in the famous surgery exact sequence. By work of Ranicki it is known that the surgery obstruction map is closely related to the assembly map in L-theory which can be written as

$$(L^q\mathbb{Z})^G_*(\underline{\underline{E}}G) \xrightarrow{\text{FJ}} L^q_*(\mathbb{Z}G)$$

where G is a discrete group and $L^q\mathbb{Z}$ denotes quadratic L-theory of the integers. The construction of this assembly map uses on the fact that the group-valued L-theory functor lifts to a functor with values in spectra. The Farrell-Jones conjecture predicts that this map is an isomorphism for all groups G and all involutive rings R (replacing $\mathbb{Z}$).

There is a similar conjecture in topological K-theory, known as the Baum-Connes conjecture. It predicts the assembly map

$$KO^G_*(\underline{E}G) \xrightarrow{\text{BC}} KO_*(C^*_r(G;\mathbb{R}))$$

is an isomorphism for all (countable, discrete) groups G. The spaces $\underline{E}G$ and $\underline{\underline{E}}G$ are the classifying spaces for the family of finite and virtually cyclic subgroups and $C^*_r(G;\mathbb{R})$ denotes the real reduced group C^*-algebra, which is a completion of the group ring $\mathbb{R}G$. For a survey about these conjectures I recommend [3], where the following commutative diagram was envisioned

$$\begin{tikzcd}
KO^G_*(\underline{E}G)[\tfrac{1}{2}] \arrow[rr, "\text{BC}[\frac{1}{2}]"] \arrow[d, dashed, "\cong"'] & & KO(C^*_r(G;\mathbb{R}))[\tfrac{1}{2}] \arrow[d, "\cong"] \\
L\mathbb{R}^G_*(\underline{E}G)[\tfrac{1}{2}] \arrow[r, "\text{FJ}[\frac{1}{2}]"] & L_*(\mathbb{R}G)[\tfrac{1}{2}] \arrow[r, "\text{CC}"] & L_*(C^*_r(G;\mathbb{R}))[\tfrac{1}{2}] \\
L^q\mathbb{Z}^G_*(\underline{E}G)[\tfrac{1}{2}] \arrow[r] \arrow[u, "\cong"] & L^q_*(\mathbb{Z}G)[\tfrac{1}{2}] \arrow[u] &
\end{tikzcd} \tag{1}$$

It relies on the following theorem, see e.g. [5] and [4]

Theorem 1. *Let A be a complex C^*-algebra. Then there are natural isomorphisms*

$$K_n(A) \cong L_n(A) \text{ for all } n \in \mathbb{Z}.$$

If A is a C^*-algebra over $\mathbb{R}$, then the same is true after inverting 2 and not before inverting 2.

To obtain a commutative diagram as above, one needs more than a natural isomorphism of K- and L-groups in order to ensure the dashed arrow to exist rendering the diagram commutative. Precisely one needs the two functors $KO[\frac{1}{2}]$ and $L[\frac{1}{2}]$ to be equivalent as functors from C^*-algebras to spectra.

It was already known that the isomorphism of K- and L-groups in the complex case does not lift to an equivalence of spectra between KA and LA because of their different 2-local behaviour, see [5] and [6].

In [1] we prove the following strengthening of this. We denote by $KU = K(\mathbb{C})$ the topological K-theory spectrum of the C^*-algebra $\mathbb{C}$.

Theorem 2. [1, Theorem 6.1] *Any map between the two spectra KU and $L\mathbb{C}$ is null homotopic. In particular no map will induce an equivalence* after *inverting 2.*

It turns out that if one is willing to work with connective covers or to invert 2, there are natural transformations between K- and L-theory. For this we will work in the setup of ∞-categories as developed by Joyal and Lurie, see e.g. [2]. We thus consider K- and L-theory as functors $\mathrm{NC}^*\mathrm{Alg} \to \mathrm{Sp}$, between the nerve of the category of separable C^*-algebras and the ∞-category Sp of spectra. Let us denote by k and ℓ the composites

$$\mathrm{NC}^*\mathrm{Alg} \xrightarrow[L]{K} \mathrm{Sp} \xrightarrow{\tau_{\geq 0}} \mathrm{Sp}_{\geq 0}$$

given postcomposing K and L with the connective cover functor.

In [1] we prove the following theorems.

Theorem 3. [1, Theorem 3.8]
For every $n \in \mathbb{Z}$ there exists a natural transformation $\tau(n)\colon k \to \ell$, unique up to homotopy, characterized by the property that $\tau(n)_{\mathbb{C}}\colon \pi_0(ku) \to \pi_0(\ell\mathbb{C})$ is given by multiplication by n. More precisely the map

$$\begin{array}{ccc} \pi_0\big(\mathrm{Map}_{\mathrm{Fun}(\mathrm{NC}^*\mathrm{Alg},\mathrm{Sp})}(k,\ell)\big) & \longrightarrow & \mathbb{Z} \\ [\eta] & \longrightarrow & \pi_0(\eta_{\mathbb{C}}) \end{array}$$

is a bijection. Moreover there exists an essentially unique lax symmetric monoidal transformation τ. Its underlying transformation is $\tau(1)$.

Theorem 4. [1, Theorem 4.1] *For all $i \in \{0,1\}$, all $k \geq 0$, and all $A \in \mathrm{C}^*\mathrm{Alg}$ there is an exact sequence*

$$0 \longrightarrow \pi_{2k+i}(kA)_{2^k} \longrightarrow \pi_{2k+i}(kA) \xrightarrow{\tau_A} \pi_{2k+i}(\ell A) \longrightarrow \frac{\pi_{2k+i}(\ell A)}{2^k \cdot \pi_{2k+i}(\ell A)} \longrightarrow 0$$

where the subscript 2^k denotes the subgroup of elements of order 2^k. In particular, τ_A induces an isomorphism on π_0 and π_1.

Theorem 5. [1, Corollary 5.1, Theorem 5.1, and Theorem 5.6] *The functors $K[\frac{1}{2}], L[\frac{1}{2}]\colon \mathrm{NC}^*\mathrm{Alg} \to \mathrm{Sp}$ are equivalent as lax symmetric monoidal functors. Also the two functors $KO[\frac{1}{2}], L[\frac{1}{2}]\colon \mathrm{R}^*\mathrm{Alg} \to \mathrm{Sp}$ are equivalent as lax symmetric monoidal functors. In particular the dashed arrow in diagram (1) exists rendering the diagram commutative.*

I want to end with two open problems. From the commutativity of diagram (1) it follows that the map induced by the completion

$$L_*(\mathbb{R}G)[\tfrac{1}{2}] \xrightarrow{\mathrm{CC}} L_*(C_r^*(G;\mathbb{R}))[\tfrac{1}{2}]$$

is an isomorphism, provided the group G satisfies the Baum-Connes and Farrell-Jones conjecture. We call this the completion conjecture in L-theory:

Conjecture 1. *For every countable discrete group G, the map*

$$L_*(\mathbb{R}G)[\tfrac{1}{2}] \xrightarrow{\mathrm{CC}} L_*(C_r^*(G;\mathbb{R}))[\tfrac{1}{2}]$$

is an isomorphism.

Notice that if the completion conjecture is valid, then the Baum-Connes conjecture and the Farrell-Jones conjecture (for the ring $\mathbb{R}$) are equivalent after inverting 2. It is worthwhile to point out that the corresponding completion conjecture without inverting 2 is not true: a counterexample is given by free abelian groups (of high enough rank).

The reason why we cannot produce an integral transformation $ko \to \ell$ on $\mathrm{R}^*\mathrm{Alg}$, the category of real C^*-algebras, so far is because we do not know yet whether the functor

$$L\colon \mathrm{NR}^*\mathrm{Alg} \to \mathrm{Sp}$$

is KK-invariant, i.e. sends KK-equivalences to equivalences of spectra. If one could prove that L-theory is C^*-stable this would be the case. Again this seems to be a question about continuity of L-theory: Recall that C^*-stability for L-theory asks whether the map

$$LA \to L(\mathcal{K} \otimes A)$$

is an equivalence. Using that $\bigcup M_n(A) \subseteq \mathcal{K} \otimes A$ is dense and Morita invariance for L-theory this reduces to the following general continuity question. Suppose that there is ascending sequence $A_1 \subseteq A_2 \subseteq \dots$ of sub-C^*-algebras of a C^*-algebra A and suppose that the union of all A_i is dense in A.

Question 1. *Under these assumptions, is the map*

$$\operatorname{colim} LA_i \to LA$$

an equivalence? Less general, is L-theory C^*-stable?

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Modular characteristic classes for representations over finite fields

David Sprehn

(joint work with Anssi Lahtinen)

We introduce a new system of modular characteristic classes for representations of groups over finite fields, and use it to construct explicit non-trivial elements in the modular cohomology of the general linear groups over finite fields. The cohomology groups $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F})$ were computed by Quillen [9] in the case where $\mathbb{F}$ is a field of characteristic different from p, but he remarked that determining them in the modular case where the characteristic of $\mathbb{F}$ is p "seems to be a difficult problem once $N \geq 3$" [9, p. 578]. Indeed, the modular cohomology has since resisted computation for four decades. Complete calculations exist only for $N \leq 4$ [1, 12, 11, 2]. Much attention has focused on the case where N is small compared to p, e.g. [3, 4, 5, 10].

To our knowledge, when $N > \max\{p, 4\}$, the only previously constructed nonzero elements of $H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$ are those due to Milgram and Priddy [8], in the case $r = 1$. These reside in exponentially high degree: at least p^{N-2}. On the other hand, the cohomology is known to vanish in degrees less than $N/2$, by the stability theorem of Maazen [7] together with Quillen's observation [9] that the stable limit is zero. This leaves a large degree gap where it was not known whether the cohomology groups are nontrivial. We narrow this gap considerably by providing nonzero classes in degrees linear in N. We obtain:

Theorem 1. Let $N \geq 2$, and let n be the natural number satisfying

$$p^{n-1} < N \leq p^n.$$

Then

$$H^*(GL_N\mathbb{F}_{p^r};\mathbb{F}_p)$$

has a nonzero element in degree $r(2p^n - 2p^{n-1} - 1)$. Moreover, it has is a non-nilpotent element in degree $2r(p^n - 1)$ if p is odd and in degree $r(2^n - 1)$ if $p = 2$. □

Notice that the degrees in the theorem grow linearly with N: if d is any of the degrees mentioned in the theorem, then

$$r(N-1) \le d < 2r(pN-1).$$

In the case $r = 1$, we obtain stronger results, for instance:

Theorem 2. For all $N \ge 2$,

$$H^*(GL_N\mathbb{F}_2; \mathbb{F}_2)$$

has a non-nilpotent element of degree d for every d with at least $\lceil \log_2 N \rceil$ ones in its binary expansion. $\square$

Our characteristic classes are defined for representations of dimension $N \ge 2$ over the finite field $\mathbb{F}_{p^r}$, and they are modular in the sense that they take values in group cohomology with coefficients in a field $\mathbb{F}$ of characteristic p. Thus they are interesting even for p-groups. The family of characteristic classes is parametrized by the cohomology of $GL_2\mathbb{F}_{p^r}$. We show that many classes in this family are nonzero by finding representations ρ on which they are nontrivial. This produces a family of nonzero cohomology classes on the general linear groups, namely the "universal classes" obtained by applying the characteristic classes to the defining representation of $GL_N\mathbb{F}_{p^r}$ where N is the dimension of ρ.

The characteristic classes are defined in terms of a push–pull construction featuring a transfer map. This construction was previously studied by the second author in [10], where he proved that it yields an injective map

$$H^*(GL_2\mathbb{F}_{p^r}; \mathbb{F}_p) \to H^*(GL_N\mathbb{F}_{p^r}; \mathbb{F}_p)$$

for $2 \le N \le p$. The present work was inspired by computations of the first author in string topology of classifying spaces [6] featuring similar push–pull constructions.

In addition to the groups $GL_N\mathbb{F}_{p^r}$, our characteristic classes can be used to study other groups with interesting representations over finite fields. For example:

Theorem 3. For all $n \ge 1$,

$$H^*(Aut(F_{p^n}); \mathbb{F}_p) \qquad \text{and} \qquad H^*(GL_{p^n}\mathbb{Z}; \mathbb{F}_p)$$

have a non-nilpotent element of degree $2d$ for every d with the following property: the sum of the p-ary digits of d is equal to $k(p-1)$ for some $k \ge n$. In particular, there is a non-nilpotent element of degree $2p^n - 2$. (For $p = 2$, divide degrees by 2.) $\square$

These classes live in the unstable range where the cohomology groups remain poorly understood.

This work is accessible as a preprint at `arxiv.org/abs/1607.01052`

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Primary and secondary obstructions to positive scalar curvature via submanifolds

Rudolf Zeidler

We study geometric situations, where index invariants on submanifolds are obstructions to existence and concordance of positive scalar curvature (psc) metrics on ambient spin manifolds. The story begins with a recent result due to Hanke, Pape and Schick which extends earlier work of Gromov and Lawson [2, Theorem 7.5]:

Theorem 1 (Hanke–Pape–Schick [3])**.** *Let M be a closed connected spin manifold and $N \subset M$ a closed submanifold of codimension 2 with trivial normal bundle such tha the induced maps $\Lambda := \pi_1(N) \to \pi_1(M) =: \Gamma$ are injective, and $\pi_2(N) \to \pi_2(M)$ surjective, respectively. If $\alpha^\Lambda(N) \neq 0 \in KO_{m-2}(\mathrm{C}^*_\mathrm{r}\Lambda)$, then M does not admit psc.*

This features the Rosenberg index of spin manifolds which is an obstruction to psc. However, the result does not directly show that the Rosenberg index of M is non-vanishing, a point that remains to be clarified.

In the recent past, higher *secondary* index invariants such as the higher Rho-invariant have found many applications distinguishing psc metrics up to bordism or concordance, see for instance [4, 7, 6, 4, 5]. A conceptual way of understanding these invariants is via the following result:

Theorem 2 (Piazza–Schick [4], Xie–Yu [7])**.** *Consider Stolz' sequence of bordism groups of positive scalar curvature metrics. Then there is a commutative diagram* *mapping it to the analytic surgery sequence of Higson and Roe as follows:*

$$\begin{tikzcd}
\Omega^{\mathrm{spin}}_{*+1}(B\Gamma) \arrow[r] \arrow[d, "{[\ ]}"] & R^{\mathrm{spin}}_{*+1}(B\Gamma) \arrow[r] \arrow[d, "\alpha_{\mathrm{APS}}"] & P^{\mathrm{spin}}_{*}(B\Gamma) \arrow[r] \arrow[d, "\rho"] & \Omega^{\mathrm{spin}}_{*}(B\Gamma) \arrow[r] \arrow[d, "{[\ ]}"] & R^{\mathrm{spin}}_{*}(B\Gamma) \arrow[d, "\alpha_{\mathrm{APS}}"] \\
KO_{*+1}(\Gamma) \arrow[r] & KO_{*+1}(\mathrm{C}^*_{\mathrm{r}}\Gamma) \arrow[r, "\partial"] & \mathrm{S}^{\Gamma}_{*}(E\Gamma) \arrow[r] & KO_{*}(\Gamma) \arrow[r, "\alpha"] & KO_{*}(\mathrm{C}^*_{\mathrm{r}}\Gamma)
\end{tikzcd}$$

In particular: Let $\mathcal{R}^+(M)$ denote the set of all metrics of psc on a closed spin manifold M^m. Let $\Gamma = \pi_1(M)$. Then, given $g \in \mathcal{R}^+(M)$, there is the *Rho-invariant* $\rho^\Gamma(g) \in \mathrm{S}^\Gamma_m(E\Gamma)$. Moreover, for $g_0, g_1 \in \mathcal{R}^+(M)$, there is $\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1) \in KO_{m+1}(\mathrm{C}^*_{\mathrm{r}}\Gamma)$ which satisfies $\partial(\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1)) = \rho^\Gamma(g_0) - \rho^\Gamma(g_1)$. The *index difference* $\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1)$ vanishes if g_0 and g_1 are concordant as psc metrics.

We have a secondary companion to Theorem 1 featuring the Rho-invariant:

Theorem 3 ([10, Theorem 4.1.3]). *Let M be a closed spin manifold and $N \subseteq M$ a closed submanifold of codimension 2 with trivial normal bundle. Fix a tubular neighborhood $t\colon N \times D_\varepsilon \hookrightarrow M$. Suppose that the inclusion induces an injection $\Lambda := \pi_1 N \hookrightarrow \pi_1 M =: \Gamma$ and a surjection $\pi_2 N \twoheadrightarrow \pi_2 M$. Let g_0, g_1 be psc metrics on M such that $t^*(g_i) = g_{N,i} \oplus g_{D,i}$, where $g_{D,i}$ is cylindrical near the boundary of the ε-disk D_ε for $i \in \{0, 1\}$.*

Then, if $\rho^\Lambda(g_{N,0}) \neq \rho^\Lambda(g_{N,1}) \in \mathrm{S}^\Lambda_{n-2}(E\Lambda)$, the metrics g_0 and g_1 are not concordant on M.

Similarly as in the primary result, the theorem does not show that the Rho-invariants of the metrics on M are different.

We also have a primary and secondary obstruction theorem featuring submanifolds of codimension 1. Here the picture is more complete and includes a commutative diagram of the respective Higson–Roe sequences:

Theorem 4 ([10, Theorem 4.1.1], see also [9, Theorem 1.7]). *Let M be a closed spin manifold and $\Gamma = \pi_1 M$. Let $N \subset M$ a closed submanifold of codimension* 1 *with trivializable normal bundle. Suppose that the inclusion induces an injection $\pi_1 N =: \Lambda \hookrightarrow \Gamma$. Fix a tubular neighborhood $\iota\colon N \times (-\varepsilon, \varepsilon) \hookrightarrow M$. There exists a commutative diagram,*

$$\begin{tikzcd}
\mathrm{S}^\Gamma_*(\tilde{M}) \arrow[r] \arrow[d, "\tau_{\mathrm{s}}"] & K_*(M) \arrow[r] \arrow[d, "\tau_{\mathrm{t}}"] & K_*(C^*_r\Gamma) \arrow[d, "\tau_{\mathrm{a}}"] \\
\mathrm{S}^\Lambda_{*-1}(\tilde{N}) \arrow[r] & K_{*-1}(N) \arrow[r] & K_{*-1}(C^*_r\Lambda),
\end{tikzcd}$$

with the following properties:

(1) $\tau_{\mathrm{t}}([M]) = [N]$,
(2) *and as a direct consequence of the above* $\tau_{\mathrm{a}}(\alpha^\Gamma(M)) = \alpha^\Lambda(N)$,
(3) $\tau_{\mathrm{s}}(\rho^\Gamma(\tilde{g})) = \rho^\Lambda(\tilde{g}_N)$ *for all $g_M \in \mathcal{R}^+(M)$ with producture structure $\iota^* g = g_N \oplus dt^2$ on the tubular neighborhood of N,*
(4) $\tau_{\mathrm{a}}(\alpha^\Gamma_{\mathrm{diff}}(g_0, g_1) = \alpha^\Lambda_{\mathrm{diff}}(g_{N,0})(g_{N,1})$ *for all $g_0, g_1 \in \mathcal{R}^+(M)$ with product structure $\iota^* g_i = g_{N,i} \oplus dt^2$, $i = 0, 1$, on the tubular neighborhood of N.*

In addition, we have results concerning submanifolds of higher codimensions in special geometric situations like products or fiber bundles over certain aspherical manifolds, see [9, Theorem 1.5].

Under stronger assumptions one can also deal with arbitrary submanifolds of higher codimension due to work of Engel [1, Corollary 4.10]: Let M be a closed spin manifold whose homotopy groups vanish in degrees d with $2 \leq d \leq q$ and let $N \subset M$ be a connected submanifold with trivial normal bundle and of codimension q. Suppose that $\pi_1(N) \to \pi_1(M)$ be injective. In this situation, if we suppose that $\pi_1(M)$ satifies the strong Novikov conjecture, then the higher $\hat{\text{A}}$-genera of N are obstructions to psc on M.

In the talk, we also presented elements of the secondary index theory from [8, 10] involving metrics of partially psc on *non-compact* complete spin manifolds. This included the *secondary partitioned manifold index theorem* for metrics of partially psc, see [8, Theorem 5.15] and [10, Theorem 2.4.6]. These techniques are required in the proofs of Theorems 3 and 4 which were sketched during the talk.

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Galois descent and red shift in algebraic K-theory

Justin Noel

(joint work with Dustin Clausen, Akhil Mathew, Niko Naumann)

In this note we report on recent and upcoming work on some conjectures of Ausoni and Rognes on the algebraic K-theory of commutative ring spectra (i.e., E_∞-ring spectra) [1]. Solutions to these conjectures would generalize certain key results about the algebraic K-theory of commutative rings. The far-reaching goal of these and related conjectures of Rognes is to:

(1) Make the calculation of such algebraic K-theory groups more accessible and
(2) to establish a relationship between algebraic K-theory and the chromatic filtration in stable homotopy theory.

In particular, it would be desirable to have a generalization of the Quillen-Lichtenbaum conjecture (proven by Voevodsky) in this setting. This conjecture asserts that algebraic K-theory satisfies étale hyperdescent in sufficiently large degrees. One can break this up into two intermediate claims: First, establish that algebraic K-theory satisfies étale hyperdescent after a suitable Bousfield localization. Second, show that the localization map has coconnective fiber. As part of the first claim, one expects that $L_T K(-)$ should satisfy Galois descent for $T \in \{T(n), K(n), E(n)\}_{n\geq 0}$. As a consequence of the second claim, one also expects that, for a commutative ring spectrum R with $K(n+i)_* R = 0$ for all $i \geq 1$, $K(n+i)_* K(R) = 0$ for all $i \geq 2$. For discrete commutative rings these two results are due to Thomason [11] and Mitchell [8] respectively.

In [3] we establish the following generalization of Thomason's descent result:

Theorem 1. Let $E \to F$ be a finite G-Galois extension of commutative ring spectra in the sense of Rognes [9]. Then the canonical maps:

$$L_T K(E) \to L_T(K(F)^{hG}) \to (L_T K(F))^{hG}$$

are equivalences for every $T \in \{T(n), K(n), E(n)\}_{n\geq 0}$ (and every implicit prime p) if and only if the transfer map $K_0(F) \to K_0(E)$ is rationally surjective.

The stated condition can be checked in many cases of interest such as the G-Galois extensions $E_n^{hG} \to E_n$, where $G \subsetneq \mathbb{G}_n$ is a finite subgroup of the extended Morava stabilizer group.[1] The condition also holds for any of the finite Galois extensions of various incarnations of topological modular forms. Combining Theorem 1 with known Nisnevich descent results imply Thomason's étale descent theorem. Unfortunately, our methods do not imply an analogous result about hyperdescent.

In upcoming work, we also establish a new method for generalizing Mitchell's theorem to the algebraic K-theory of non-discrete rings. This is based on the following result, which was inspired by the generalized character theory of Hopkins, Kuhn, and Ravenel [4] as well as the results of [7]:

Theorem 2. Fix a prime p and a non-negative integer n. Let E be a commutative ring spectrum and $G = C_p^{\times n}$. Suppose that the sum of the transfer maps :

$$\bigoplus_{H \subsetneq G} E^0(BH) \to E^0(BG)$$

is a rational surjection. Then $L_{T(n+i)}E = L_{K(n+i)}E \simeq *$ for all $i \geq 0$ (and at the implicit prime p).

[1] This depends on joint work of the Lennart Meier, Niko Naumann, and the author.

To obtain analogues of Mitchell's theorem we apply the above result when $E = K(\mathbb{Z})$ ($n = 2$, p an arbitrary prime) and $E = K(KU)$ ($n = 3$, $p \in \{2, 3, 5\}$). To verify the hypothesis of Theorem 2, we prove an analogous statement for the corresponding equivariant algebraic K-theory groups. For the K-theory of the integers, the required hypothesis is a theorem of Swan [10]. We give a new proof of Swan's theorem that also applies to the K-theory of KU. For $K(KU)$ our results depend on a fundamental result of Borel [2], namely that the compact, simply connected, exceptional Lie group E_8 admits non-toral subgroups of the form $C_p^{\times 3}$ precisely when $p \in \{2, 3, 5\}$.

The proofs of the local equivalences in Theorems 1 and 2 crucially depend on the following result of the Mathew, Naumann, and the author used in the proof of May's nilpotence conjecture [5]:

Theorem 3. Let R be a commutative ring spectrum. Then $L_T R \simeq *$ for all $T \in \{T(n), K(n), E(n)\}_{n \geq 0}$ (and every implicit prime p) if and only if $\pi_0 R \otimes \mathbb{Q} = 0$.

The proof of this result makes critical use of the E_∞-ring structure on R and the existence of certain power operations.

Using Theorem 3, the proof of Theorem 1 easily reduces to Thomason's argument for rational Galois descent [11]. The proof of Theorem 2 easily reduces to the case when E is a $K(n+i)$-local E_{n+i}-algebra by using known calculations of the E-cohomology of abelian groups as well as an elementary rank calculation. To apply Theorem 2, we construct finite G-complexes with proper isotropy and whose corresponding equivariant K-classes are equal to the unit up to a non-zero multiple. In the case of $K(KU)$, the calculation of the relevant K-class depends on simplified description of the homotopy theory of KU_{E_8}-modules from [6].

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Coincidences of homological densities, predicted by arithmetic

Jesse Wolfson
(joint work with Benson Farb, Melanie Wood)

Let X be a connected, oriented smooth manifold. Fix $m, n \geq 1$. Let $\vec{\mathbf{d}}$ denote a tuple of natural numbers $(d_1, \ldots, d_m) \in \mathbb{N}^m$ with each $d_i \geq 1$, and let $|\vec{\mathbf{d}}| := \sum_i d_i$. Let $\mathrm{Sym}^d(X) := X^d/S_d$ be the d^{th} symmetric product of X, more generally let $\mathrm{Sym}^{\vec{\mathbf{d}}}(X) := \prod_i \mathrm{Sym}^{d_i}(X)$. Consider the space $\mathcal{Z}_n^{\vec{\mathbf{d}}}(X) \subset \mathrm{Sym}^{\vec{\mathbf{d}}}(X)$ of subsets $D \subset X$ of $|\vec{\mathbf{d}}|$ (not necessarily distinct) points in X such that:

(1) precisely d_i of the points in D are labeled with the "color" i, and
(2) no point of X is labelled with at least n labels of every color.

Such spaces of 0-cycles include several basic examples in topology and geometry. For example:

- $\mathcal{Z}_2^d(X)$ is the configuration space $\mathrm{UConf}_d(X)$ of unordered d-tuples of distinct points in X.
- $\mathcal{Z}_1^{d,d}(\mathbb{C})$ is the space $Rat_d^*(\mathbb{CP}^1)$ of degree d, based rational maps $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ with $f(\infty) = 1$.

In ongoing work with Benson Farb and Melanie Wood, we study these spaces as sites for the interplay between topology, algebraic geometry and number theory. For the present talk, I will focus on topological analogues of classical density theorems in analytic number theory, which form the main results of [1].

Theorem (in progress) 1 (Homological densities for spaces of 0-cycles). *Let X be a connected, oriented, smooth manifold. Assume that $\dim H^*(X;\mathbb{Q}) < \infty$, for example X compact or is the interior of a compact manifold with boundary. Fix $m, n \geq 1$ and let $\vec{\mathbf{d}} = (d_1, \ldots, d_m)$. In what follows, let $\lim_{\vec{\mathbf{d}} \to \infty}$ mean "as all $d_i \to \infty$", at any rates.*

(1) *If $\dim X$ is even, then*

$$\frac{\sum_{\vec{\mathbf{d}} \in \mathbb{Z}_{\geq 0}^m} \chi(\mathcal{Z}_n^{\vec{\mathbf{d}}}(X)) x^{|\vec{\mathbf{d}}|}}{\sum_{\vec{\mathbf{d}} \in \mathbb{Z}_{\geq 0}^m} \chi(\mathrm{Sym}^{\vec{\mathbf{d}}}(X)) x^{|\vec{\mathbf{d}}|}} = (1 - x^{mn})^{\chi(X)},$$

which in particular only depends on $\chi(X)$ and mn.

(2) *Suppose that the Leray spectral sequence for the inclusion map $\mathcal{Z}_n^{\vec{\mathbf{d}}}(X) \subset \mathrm{Sym}^{\vec{\mathbf{d}}}(X)$ degenerates on the E_2-page. This happens for example if* $\dim(X)$

is odd or X is an open submanifold of $\mathbb{C}^n$ for some n. Then

$$\lim_{\vec{\mathbf{d}}\to\infty} \frac{P_{\mathcal{Z}^{\vec{\mathbf{d}}}_n(X)}(t)}{P_{\mathrm{Sym}^{\vec{\mathbf{d}}}(X)}(t)} \in \mathbb{Z}[[t]]$$

exists in the t-adic topology on the ring $\mathbb{Z}[[t]]$ of formal power series, and this limit depends only on the Betti numbers of X, on $\dim(X)$, and on mn.

(3) *If X is a smooth, complex-algebraic variety then*

$$\lim_{\vec{\mathbf{d}}\to\infty} \frac{\mathrm{HD}_{\mathcal{Z}^{\vec{\mathbf{d}}}_n(X)}(u,v)}{\mathrm{HD}_{\mathrm{Sym}^{\vec{\mathbf{d}}}(X)}(u,v)}$$

exists in the adic topology on $\mathbb{Z}[[u,v]]$, and depends only on the product mn, the mixed Hodge structure on $H^*(X;\mathbb{Q})$, and $\dim X$.

The appearance of a ratios of homological invariants is a surprising aspect of the theorem, and one for which we do not know a natural topological explanation. One can check in a simple example that the ratio is indeed necessary. For $X = \mathbb{C} - 0$, $mn = 2$, we have

$$\lim_{d\to\infty} P_{\mathcal{Z}^{d}_2(X)}(t) = 1 + 2t + 2t^2 + \dots$$
$$\lim_{\vec{\mathbf{d}}=(d_1,d_2)\to\infty} P_{\mathcal{Z}^{\vec{\mathbf{d}}}_1(X)}(t) = 1 + 3t + 4t^2 + 4t^3 + \dots$$

so the numerators themselves are not equal. But, using that $P_{\mathrm{Sym}^\infty(X)}(t) = 1 + t$, the ratios are

$$\frac{1 + 2t + 2t^2 + \dots}{1+t} = \frac{1}{1-t} = \frac{1 + 3t + 4t^2 + \dots}{(1+t)^2}.$$

Roughly, we think of this ratio as measuring the limiting "density" of the space $\mathcal{Z}^{\vec{\mathbf{d}}}_n(X)$ inside the space $\mathrm{Sym}^{\vec{\mathbf{d}}}(X)$. We also still have no explanation *why* these ratios should coincide for different spaces of 0-cycles.

We view the theorem as a topological analogue of classical density results in number theory, suggested by the "number field/function field" dictionary popularized by Weil. Under this dictionary, a manifold X is the analogue of the ring of integers $\mathcal{O}$ in a number field K, and collections of points in X correspond to ideals in $\mathcal{O}$. Since the late 1800s, it has been understood that, e.g. the limiting density of the number of square free integers in the set of all integers equals the limiting density of the set of pairs of coprime integers in the set of all pairs of integers, i.e.

$$\lim_{N\to\infty} \frac{\#\{n \le N \mid \nexists\ a \text{ s.t } a^2|n\}}{N} = \lim_{N\to\infty} \frac{\#\{m,n \le N \mid \gcd(m,n)=1\}}{N^2}$$

Replacing the integers with an arbitrary manifold, and the cardinality of a set with the homology of a space, the dictionary predicts the case $mn = 2$ of Theorem 1, and similar arguments predict the other cases.

On the number field side, the limiting densities we consider are all special values of Dedekind zeta functions of the respective number fields. From this perspective, it is natural to view the limiting homological densities as special values of a "topological zeta function" of the manifold X, i.e. one might define

$$\zeta_X(mn)^{-1} := \lim_{\vec{\mathbf{d}}\to\infty} \frac{P_{\mathcal{Z}_n^{\vec{\mathbf{d}}}(X)}(t)}{P_{\mathrm{Sym}^{\vec{\mathbf{d}}}(X)}(t)} \in \mathbb{Z}[[t]]$$

It is natural to ask whether this analogy can be pushed further.

Question 2. *For a manifold X and $\zeta_X(mn)$ defined as above, does there exist an analytic continuation of $\zeta_X(s)$ for non-integer values of s?*

We deduce Theorem 1 by analyzing the Leray spectral sequence for the inclusion $\mathcal{Z}_n^{\vec{\mathbf{d}}}(X) \to \mathrm{Sym}^{\vec{\mathbf{d}}}(X)$. Methods from algebraic combinatorics, specifically the Björner–Wachs theory of EL-shellability, provide crucial ingredients of this analysis. We find that the E_2-page of this spectral sequence decomposes as a product of two terms, one of which depends only on the cohomology of X and on mn, and the other which is given by the cohomology of the symmetric product $\mathrm{Sym}^{\vec{\mathbf{d}}}(X)$. The homological densities of Theorem 1 amount to three different lenses through which coincidences about the spaces themselves can be deduced from coincidences of the E_2-pages. Put another way, at our present understanding we are only able to identify coincidences of homological densities that neglect the contribution of the differentials in the spectral sequence.

Problem 3. *Extract the "correction terms" from the differentials, and use these to unify and refine the coincidences of Theorem 1.*

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$\mathbb{A}^1$-Milnor number

Kirsten Wickelgren
(joint work with Jesse Leo Kass)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞-function with an isolated zero at the origin. Recall that the local degree $\deg_0 f$ of f at zero is defined as

$$\deg_0 f = \deg(\ \partial B(0,\epsilon) \xrightarrow{f/|f|} \partial B(0,1)\ ) \in \mathbb{Z},$$

where $\epsilon > 0$ is chosen sufficiently small. The *Signature Formula* of Eisenbud-Levine/Khimshiashvili [1] [3] gives a formula for $\deg_0 f$ as the signature of the following real symmetric bilinear form. Define $Q_0(f) = \mathbb{R}[[x_1, \ldots, x_n]]/\langle f_1, \ldots, f_n\rangle$ where f_i denotes the ith coordinate projection of f. Let $J = \det(\frac{\partial f_i}{\partial x_j})$. Choose a $\mathbb{R}$-linear function $\varphi : Q_0(f) \to \mathbb{R}$ such that $\varphi(J) > 0$. Define

$$\langle , \rangle_\varphi : Q_0(f) \times Q_0(f) \to \mathbb{R}$$
$$\langle , \rangle_\varphi(g,h) = \varphi(gh).$$

Theorem. (Eisenbud-Levine/Khimshiashvili Signature Formula)

$$\deg_0 f = \text{signature } \langle , \rangle_\varphi$$

The complex analogue of their theorem was proven earlier by Palamodov. When f is analytic, and hence has a complexification $f \otimes \mathbb{C}$, Palamodov proved

Theorem. (Palamodov)

$$\deg_0 f \otimes \mathbb{C} = \text{rank } \langle , \rangle_\varphi.$$

For an arbitrary field k and a polynomial function f, let

$$Q_0(f) = k[x_1, \ldots, x_n]_{\mathfrak{m}_0} / \langle f_1, \ldots, f_n \rangle,$$

where $\mathfrak{m}_0 = \langle x_1, \ldots, x_n \rangle$, and choose φ to be k-linear such that $\varphi(J) = \dim_k Q_0(f)$. In positive characteristic, assume that $\dim_k Q_0(f)$ is finite and if this dimension is divisible by the characteristic, J is replaced by a distinguished socle element E with $\varphi(E) = 1$. The isomorphism class of $\langle , \rangle_\varphi$ does not depend on the choice of φ.

Eisenbud wrote an AMS Bulletin article about this work [2], and the article ends with some questions. Question 3 [2, p. 763-764] is

> I would propose that the degree of a finite polynomial map $f : k^n \to k^n$, where k is an arbitrary field of characteristic 0 be *defined* to be the equivalence class of the quadratic form $\langle , \rangle_\varphi$ on the local ring of f at 0. ... There is really no reason to stick to characteristic 0 for all this, ... The question is, does this idea of degree have some other interpretation (or usefulness), for example in cohomology theory, as in the case of $\mathbb{R}$ or $\mathbb{C}$

We answer this question "yes:" $\langle , \rangle_\varphi$ is the local degree from Morel-Voevodsky's $\mathbb{A}^1$-homotopy theory [5], appearing before $\mathbb{A}^1$-homotopy theory itself.

Theorem 1. (Kass, W.)

$$\deg_0^{\mathbb{A}^1} f = \langle , \rangle_\varphi$$

About the left hand side: Morel's degree homomorphism in $\mathbb{A}^1$-homotopy theory over a field k takes an endomorphism of a sphere to an element of the Grothendieck Witt group $\mathrm{GW}(k)$ of k. This group is the group completion of the semi-ring of isomorphism classes of non-degenerate symmetric bilinear forms over k. The above equality is in $\mathrm{GW}(k)$. Morel's construction is compatible with the $\mathbb{Z}$-valued topological degree: when we have an embedding $k \hookrightarrow \mathbb{C}$, the topological degree of the $\mathbb{C}$-points of a map is the rank of the bilinear form $\deg^{\mathbb{A}^1}$; the topological degree of the $\mathbb{R}$-points of a map is the signature. (Note the compatibility with the Signature Formula, Palamodov's Theorem and Theorem 1.)

Theorem 1 is proven by reducing to the étale case, where both sides are computed to be equal. To do the reduction, both sides are shown to be unchanged when f is modified by an n-tuple of polynomials in a sufficiently high power of the maximal ideal. We modify f in this way to be able to extend it to an endomorphism G (satisfying certain conditions) of the sphere $\mathbb{P}^n/\mathbb{P}^{n-1}$ in $\mathbb{A}^1$-homotopy theory. We show that $\langle , \rangle_\varphi$ has certain properties of a local degree, namely that there is a global degree which is a sum of local degrees over points of $G^{-1}(x)$ with $x \in \mathbb{A}^n = \mathbb{P}^n - \mathbb{P}^{n-1}$, making this sum independent of x. We can now check the equality of global degrees using an x so that G is étale at every point of $G^{-1}(x)$. When there is no such rational x, we take an odd-degree field extension, which induces an injection on GW.

As an application, we enrich Milnor's equality between the local degree of the gradient of a complex hypersurface singularity and the number of nodes into which the singularity bifurcates [4]. Classically, this common integer is the Milnor number μ. We enrich this to an equality in $\mathrm{GW}(k)$. Specifically, let k be a field of characteristic not 2, and let $g \in k[x_1, \ldots, x_n]$ define a hypersurface with an isolated singularity at 0.

A *node* is a hypersurface singularity isomorphic to $x_1^2 + \ldots + x_n^2$ over k^s where k^s denotes the separable closure of k. Over non-separably closed fields, nodes contain arithmetic information. For example, the isomorphism type of the node of $x_1^2 + ax_2^2 = 0$ depends on the value of a in $k^*/(k^*)^2$. We encode some of this information in a bilinear form. Let $\langle a \rangle$ denote the element of $\mathrm{GW}(k)$ represented by the rank 1 bilinear form $(x, y) \mapsto axy$ for x,y in k. Define the *arithmetic type* of $x_1^2 + ax_2^2 = 0$ to be $\langle a \rangle$ in $\mathrm{GW}(k)$. More generally, for $g = 0$ defining a node at a rational point p, define the arithmetic type to be $\langle H \rangle$, where H is the Hessian $H = \det(\frac{\partial f_i}{\partial x_j}(p))$ evaluated at p. Using descent data, one also defines the arithmetic type when x is not assumed to be rational. (When k is a finite field, we explain the definition later.)

For general $(a_1, ..., a_n) \in \mathbb{A}^n_k(k)$, the family

$$g(x_1, \ldots, x_n) + a_1 x_1 + \cdots + a_n x_n = t$$

over line with coordinate t contains only nodal fibers as singular fibers, and writing these nodes as $p_i \in X_i$, we have:

Theorem 2. (Kass, W.) Suppose $\operatorname{grad} g$ is finite and has only the origin as an isolated zero. Then

$$\mu^{\mathbb{A}^1}(g) = \sum \text{arithmetic type}(p_i) \in X_i \tag{1}$$

in $\mathrm{GW}(k)$, where

$$\mu^{\mathbb{A}^1} g = \deg_0^{\mathbb{A}^1} \operatorname{grad} g$$

and is called the $\mathbb{A}^1$-*Milnor number*.

Let us now analyze Theorem 2 in the special case where $k = \mathbf{F}_q$ is a finite field of characteristic $p \neq 2$. Describing nodal fibers over a finite field is especially tractable because the structure of a finite field is so simple. The stable isomorphism class of a nondegenerate symmetric bilinear form is determined by its rank and discriminant. Furthermore, the discriminant is an element of $k^*/(k^*)^2$, which is a 2 element group that we write as

$$\mathbf{F}_q^*/(\mathbf{F}_q^*)^2 = \{1, u_q\} \text{ for some } u_q \in \mathbf{F}_q^*.$$

In particular, there are two possibilities for the arithmetic type of a node at the origin:

$$\text{the arithmetic type } \langle 1 \rangle \text{ of } x^2 + y^2 \text{ and} \tag{2}$$

$$\text{the arithmetic type } \langle u_q \rangle \text{ of } x^2 + u_q y^2. \tag{3}$$

However, not every collection of nodes $\{x_i \in X_i\}$ satisfying Equation (1) can be realized as the singular fibers of a family. For the example, the equation $f(x, y) = y^3 + x^4$ of the E_6 singularity over $k = \mathbf{F}_5$ satisfies $\mu^{\mathbf{A}^1}(f) = 3 \cdot \mathbf{H}$. We have $3 \cdot \mathbf{H} = 6 \cdot \langle 1 \rangle$ in $\mathrm{GW}(\mathbf{F}_5)$, but there does not exist an (a, b) such that the associated family has 6 fibers with arithmetic type $\langle 1 \rangle$ because $\mathbf{A}^1_{\mathbf{F}_5}(\mathbf{F}_5)$ only has $5 < 6$ elements.

We describe the configurations of nodes occurring in families associated to the singularities in Table 1 for some small finite fields. Table 1 should be read as follows. The `equation` in the second column is the equation of an isolated plane curve singularity, and over the algebraic closure, that singularity is isomorphic to an ADE singularity, specifically the singularity with the `name` in the first column. The $\mathbb{A}^1$`-Milnor number` of the equation is given in the third column. The `discriminant`, considered as an element of $k^*/(k^*)^2$, is listed in the fourth column. The rank of $\mathbf{A}^1$-Milnor number is the integer appearing in the first column (so e.g. for the D_4 singularity, the rank is 4). In the table, $\mathbf{H} = \langle 1, -1 \rangle$ is the class of the standard hyperbolic space.

Table 1. Some singularities over $k = \mathbf{F}_q$, $q = p^n$ for $p > 5$ with A_4 and otherwise $p > 3$

Name	Equation	$\mathbf{A}^1$-Milnor number	Discriminant
A_2	$y^2 + x^3$	$\mathbf{H}$	-1
A_3	$y^2 + vx^4$, $v \in k^*$	$\langle 2 \cdot v \rangle + \mathbf{H}$	$-2 \cdot v$
A_4	$y^2 + x^5$	$2 \cdot \mathbf{H}$	1
D_4	$x^2y + xy^2$	$\langle -2, 2 \cdot 3 \rangle + \mathbf{H}$	3
E_6	$x^4 + y^3$	$3 \cdot \mathbf{H}$	-1

Consider the possible nodal fibers of the family $\mathbf{A}^2_k \to \mathbf{A}^1_k$ defined by $f(x, y) + ax + by = t$. Thus suppose that $x_0 \in X_{t_0}$ is a node of the fiber over the closed point $t_0 \in \mathbf{A}^1_k$. As was mentioned earlier, if $x_0 \in X_0$ has residue field equal to k, then the arithmetic type is the value of the Hessian of f at x_0.

In general, the definition of the arithmetic type is more subtle. Colloquially, $x_0 \in X_{t_0}$ corresponds to a Galois orbit of nodes (over, say a large field extension), and if the common arithmetic type of these nodes is α, then the arithmetic type of $x_0 \in X_{t_0}$ is the Scharlau trace $\mathrm{Tr}_{L/k}(\alpha)$.

More formally, suppose first that $k(t_0) = k$ but $k(x_0)/k$ is a nontrivial extension, say $k(x_0) = L$. Then $X_{t_0} \otimes_k L$ has finitely many nodes mapping to x_0, say $\widetilde{x}_1, \ldots, \widetilde{x}_n \in X_{t_0} \otimes_k L$. Each of these nodes has residue field L, and a node's arithmetic type (over L) is computed as the class of a Hessian. Moreover, the $\widetilde{x}_i$'s are transitively permuted by the Galois group $\mathrm{Gal}(L/k)$, so any two nodes have the same type, say $\alpha \in \mathrm{GW}(L)$. We then have

$$\text{the arithmetic type of } x_0 \in X_{t_0} = \mathrm{Tr}_{L/k}(\alpha).$$

Here $\mathrm{Tr}_{L/k} \colon \mathrm{GW}(L) \to \mathrm{GW}(k)$ is the Scharlau trace.

The most general case is where $k(t_0)$ is a nontrivial extension, say L. In this case, t_0 corresponds to a $\mathrm{Gal}(L/k)$-orbit of fibers $\widetilde{X}_{\widetilde{t}_1}, \ldots, \widetilde{X}_{\widetilde{t}_m}$ that are transitively permuted by the Galois group. Each of the points $\widetilde{t}_1, \ldots, \widetilde{t}_m$ has residue field L, so the arithmetic type of a node of $\widetilde{X}_{\widetilde{t}_i}$ is defined as in the previous paragraph. Fixing one fiber, say $\widetilde{X}_{\widetilde{t}_1}$, and defining $\alpha \in \mathrm{GW}(L)$ to be the sum of the arithmetic types of the nodes of $\widetilde{X}_{\widetilde{t}_1}$ that map to x_0, we have

$$\text{the arithmetic type of } x_0 \in X_{t_0} = \mathrm{Tr}_{L/k}(\alpha).$$

For given $k = \mathbf{F}_q$, $f(x,y) \in k[x,y]$, $a, b \in k$, the arithmetic types of the nodal fibers of $f(x,y) + ax + by = t$ can be computed using Gröbner basis techniques. For example, consider the family $x^2y - xy^2 + 2x + y = t$ over $k = \mathbf{F}_{17}$. The singular fibers are the fibers over the points of the closed scheme defined by $k[t] \cap (f(x,y) + ax + by - t, \frac{\partial f}{\partial x} + a, \frac{\partial f}{\partial y} + b)$. A Gröbner basis computation shows that this ideal is generated by $d(t) = t^4 + 14$, an irreducible polynomial. In $L := k[r]/t^4 + 14$, a second Gröbner basis computation shows that X_{t_1} has a node at the point $(4r^3 + 5r, 9r^3)$. The value of the Hessian at this point is $4r^2 = 1$ in $L^*/(L^*)^2$. We conclude that the nodal fibers of the family consists of a Galois orbit of 4 fibers, each with a single node of type $\langle 1 \rangle \in \mathrm{GW}(L)$. Table 2 was generated by similar computations.

The table should be read as follows. The first column describes a `singularity` from Table 1. For a given singularity, the possible singular fibers of a family $f(x,y) + ax + by = t$ with only `nodal fibers` are listed in the second column. The last column is the `count` of the (a,b)'s that define a family with singular fibers as described by the corresponding entry in the second column. (E.g. for the A_2 singularity over $k = \mathbf{F}_5$, there are 5 elements $(a,b) \in \mathbf{A}^2_k(k)$ s.t. $f(x,y)+ax+by = t$ has 2 nodal fibers, each with a node of type $\langle 1 \rangle$.)

TABLE 2. Possible singular fibers of a family

Singularity	Nodal fibers	Count
A_2 with $k = \mathbf{F}_5$	1 orbit of 2 fibers of type $\langle u_{p^2} \rangle$	10
	2 fibers of type $\langle u_p \rangle$	5
	2 fibers of type $\langle 1 \rangle$	5
	Total	20
A_2 with $k = \mathbf{F}_7$	1 orbit of 2 fibers of type $\langle 1 \rangle$	21
	1 fiber of type $\langle 1 \rangle$, 1 fiber of type $\langle u_p \rangle$	21
	Total	42
A_3 with $k = \mathbf{F}_5$, $v = 1$	1 fiber of type $\langle 1 \rangle$, 1 orbit of 2 fibers of type $\langle 1 \rangle$	20
	Total	20
A_3 with $k = \mathbf{F}_5$, $v = 2$	1 node of type $\langle u_p \rangle$, 1 orbit of 2 fibers of type $\langle 1 \rangle$	20
	Total	20
A_3 with $k = \mathbf{F}_7$, $v = 1$	1 orbit of 3 fibers of type $\langle u_{p^3} \rangle$	28
	3 nodes of type $\langle u_p \rangle$	14
	Total	42
A_3 with $k = \mathbf{F}_7$, $v = -1$	3 fibers of type $\langle 1 \rangle$	14
	1 orbit of 3 fibers of type $\langle 1 \rangle$	28
	Total	42
A_4 with $k = \mathbf{F}_7$	1 type $\langle 1 \rangle$ fiber, 1 type $\langle u_p \rangle$ fiber, 1 orbit of 2 type $\langle 1 \rangle$ fibers	21
	2 orbits of 2 fibers of type $\langle 1 \rangle$	21
	Total	42
D_4 with $k = \mathbf{F}_5$	1 orbit of 4 fibers of type $\langle 1 \rangle$	12
	1 orbit of 2 fibers of type $\langle 1 \rangle$, 2 fibers of type $\langle 1 \rangle$	2
	2 fibers of type $\langle u_p \rangle$, 1 fiber of type $\mathrm{Tr}_{\mathbf{F}_{p^2}/\mathbf{F}_p}(\langle 1 \rangle)$	6
	1 orbit of 2 fibers of type $\langle 1 \rangle$, 1 fiber of type $\langle 1, 1 \rangle$	4
	Total	24
E_6 with $k = \mathbf{F}_5$	1 fiber of type $\mathrm{Tr}_{\mathbf{F}_{p^2}/\mathbf{F}_p}(\langle u_p \rangle)$, 2 orbits of 2 fibers of type $\langle u_{p^2} \rangle$	4
	2 fibers of type $\langle 1 \rangle$, 2 orbits of 2 fibers of type $\langle 1 \rangle$	4
	2 fibers of type $\langle u_p \rangle$, 2 orbits of 2 fibers of type $\langle 1 \rangle$	4
	3 orbits of 2 fibers of type $\langle u_{p^2} \rangle$	4
	Total	16

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Lie algebras and v_n-periodic spaces

Gijs Heuts

The goal of this talk was to explain some results on v_n-periodic unstable homotopy theory analogous to Quillen's results on rational homotopy.

Goodwillie towers of ∞-categories. Write $\mathcal{S}_*$ for the (∞-)category of pointed spaces. The *Goodwillie tower of the identity functor* [7] on $\mathcal{S}_*$ gives, for each pointed space X, a tower of spaces

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
\vdots & P_3X \arrow[d] & \\
 & P_2X \arrow[d] & \\
X \arrow[r] \arrow[ur] \arrow[uur] & P_1X \arrow[r, equal] & \Omega^\infty\Sigma^\infty X,
\end{tikzcd}$$

which interpolates between the stable and unstable homotopy type of X. The homotopy fiber of the map $P_nX \to P_{n-1}X$ is usually denoted D_nX and may be expressed as follows:

$$D_nX = \Omega^\infty\big((\partial_n\mathrm{id} \wedge X^{\wedge n})_{h\Sigma_n}\big).$$

Here $\partial_n\mathrm{id}$ is a spectrum carrying an action of the symmetric group Σ_n and is called the *n'th derivative* of the identity functor. Ching [5] showed that the symmetric sequence of derivatives $\partial_*\mathrm{id}$ has a natural operad structure; furthermore, this operad is the cobar construction of the commutative cooperad and as such could be considered as the (desuspension of) the Lie operad in the category of spectra.

In particular, taking integral homology reproduces (a degree shift of) the ordinary Lie operad in the category of abelian groups.

In [9] we constructed the *Goodwillie tower of* $\mathcal{S}_*$, which is a tower of ∞-categories interpolating between $\mathcal{S}_*$ and the ∞-category Sp of spectra:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
\vdots & \mathcal{P}_3\mathcal{S}_* \arrow[d] & \\
 & \mathcal{P}_2\mathcal{S}_* \arrow[d] & \\
\mathcal{S}_* \arrow[r, "\Sigma^\infty"] \arrow[ur, "\Sigma_2^\infty"] \arrow[uur, "\Sigma_3^\infty"] & \mathcal{P}_1\mathcal{S}_* \arrow[r, equal] & \mathrm{Sp}.
\end{tikzcd}$$

All functors in this diagram are left adjoints; we write Ω_n^∞ for the right adjoint of Σ_n^∞. Let us list the following properties:

(1) The identity functor $\mathrm{id}_{\mathcal{P}_n\mathcal{S}_*}$ of $\mathcal{P}_n\mathcal{S}_*$ is an n-excisive functor in the sense of Goodwillie.
(2) The counit $\Sigma_n^\infty\Omega_n^\infty \to \mathrm{id}_{\mathcal{P}_n\mathcal{S}_*}$ canonically factors through a natural transformation $P_n(\Sigma_n^\infty\Omega_n^\infty) \to \mathrm{id}_{\mathcal{P}_n\mathcal{S}_*}$, which is an equivalence.
(3) The unit $\mathrm{id}_{\mathcal{S}_*} \to \Omega_n^\infty\Sigma_n^\infty$ canonically factors through a natural transformation $P_n\mathrm{id}_{\mathcal{S}_*} \to \Omega_n^\infty\Sigma_n^\infty$, which is an equivalence.

In fact, the construction of Goodwillie towers makes sense for a wide class of ∞-categories $\mathcal{C}$, namely those that are *pointed* (meaning they have a zero object) and *compactly generated*. The analogous properties hold in this generality.

Before stating the result we need, recall the definition of Tate spectra: any spectrum E with an action by a finite group G gives rise to a *norm map*

$$E_{hG} \xrightarrow{\mathrm{Nm}} E^{hG},$$

whose cofiber is by definition the Tate spectrum E^{tG}. The analogue of this map in ordinary algebra is 'summing over the group'; in detail, for an abelian group M with G-action, one considers the map $[m] \mapsto \sum_{g\in G} gm$. The following 'degeneration' has a number of useful applications (see [9]):

Theorem. *Let $\mathcal{C}$ be a pointed, compactly generated ∞-category such that all Tate spectra are contractible in the ∞-category* $\mathrm{Sp}(\mathcal{C})$ *of spectra in $\mathcal{C}$. Then for each $n \geq 1$ there is a canonical equivalence*

$$\mathcal{P}_n\mathcal{C} \simeq \mathcal{P}_n\mathrm{Alg}_{\partial_*\mathrm{id}_{\mathcal{C}}}(\mathrm{Sp}(\mathcal{C})),$$

compatible with the functors in the respective Goodwillie towers. $\mathrm{Alg}_{\partial_*\mathrm{id}_{\mathcal{C}}}(\mathrm{Sp}(\mathcal{C}))$ *denotes the ∞-category of algebras over the derivatives of the identity of $\mathcal{C}$.*

Remark. The statement of the theorem just given is informal: we are assuming an operad structure on $\partial_* \mathrm{id}_{\mathcal{C}}$, which is not a given (although it is not a problem for the examples we consider here). A more precise formulation in terms of coalgebras is Corollary 2.23 of [9].

The examples of stable ∞-categories with vanishing Tate spectra we have in mind are the following:

(1) The ∞-category $\mathrm{Sp}_{\mathbb{Q}}$ of rational spectra. Vanishing of Tate spectra is a straightforward consequence of the invertibility of the order of the group.
(2) The ∞-category $\mathrm{Sp}_{K(n)}$ of $K(n)$-local spectra, with $K(n)$ the n'th Morava K-theory at some prime p. The vanishing of Tate spectra in this case is a result of Greenlees and Sadofsky [8].
(3) The ∞-category $\mathrm{Sp}_{T(n)}$ of $T(n)$-local spectra, with $T(n)$ the telescope of a v_n self-map on a p-local finite type n spectrum. The vanishing of Tate spectra in this case is a result of Kuhn [12].

Using example (1) to apply the theorem above to the ∞-category of rational pointed spaces one can reprove some of Quillen's results, comparing rational homotopy theory with the homotopy theories of differential graded Lie algebras or commutative coalgebras over $\mathbb{Q}$. Our focus will be on a certain ∞-category $\mathcal{V}_n$ whose associated stable homotopy theory is (3). Example (2) can be treated in a similar way.

The v_n-periodic homotopy theory of spaces. Let V_n be a p-local finite space of type n with a v_n self-map $v : \Sigma^d V_n \to V_n$. Then for any (p-local) pointed space X we may define its *v-periodic homotopy groups* by

$$v^{-1}\pi_*(X; V_n) := \pi_* \varinjlim(\mathrm{Map}_*(V_n, X) \xrightarrow{v^*} \Omega^d \mathrm{Map}_*(V_n, X) \to \cdots).$$

In fact, the *Bousfield-Kuhn functor* [11] captures these groups in a way that is independent of choices; it is a functor

$$\Phi : \mathcal{S}_* \to \mathrm{Sp}_{T(n)}$$

satisfying $\pi_*(\Phi(X) \wedge \mathbf{D}V_n) \simeq v^{-1}\pi_*(X; V_n)$, with $\mathbf{D}$ denoting Spanier-Whitehead dual. We say a map f of pointed spaces is a *v_n-periodic equivalence* if $\Phi(f)$ is a weak equivalence of spectra (or, equivalently, if $v^{-1}\pi_*(f; V_n)$ is an isomorphism).

In [10] we will construct a subcategory $\mathcal{V}_n$ of $\mathcal{S}_*$ together with a functor $M_n : \mathcal{S}_* \to \mathcal{V}_n$, such that a map f of pointed spaces is a v_n-periodic equivalence if and only if $M_n(f)$ is a weak equivalence. The functor M_n is not a localization, but rather a composition of a colocalization with a localization. The details of its construction rely heavily on the work of Bousfield [3, 4] and Dror Farjoun [6].

Proposition. *The stabilization* $\mathrm{Sp}(\mathcal{V}_n)$ *is equivalent to* $\mathrm{Sp}_{T(n)}$ *and under this identification the derivatives of the identity* $\partial_* \mathrm{id}_{\mathcal{V}_n}$ *are equivalent to* $L_{T(n)} \partial_* \mathrm{id}_{\mathcal{S}_*}$.

Our previous theorem then implies the following [10]:

Theorem. *There are canonical equivalences, natural in $k \geq 1$, as follows:*

$$\mathcal{P}_k\mathcal{V}_n \simeq \mathcal{P}_k\mathrm{Alg}_{\partial_*\mathrm{id}_{\mathcal{S}_*}}(\mathrm{Sp}_{T(n)}) \simeq \mathcal{P}_k\mathrm{coAlg}(\mathrm{Sp}_{T(n)}).$$

Here $\mathrm{coAlg}(\mathrm{Sp}_{T(n)})$ *denotes the* ∞*-category of commutative coalgebras in* $\mathrm{Sp}_{T(n)}$.

Moreover, one deduces that under these identifications the composition

$$\mathcal{P}_k\mathcal{V}_n \xrightarrow{\Omega_k^\infty} \mathcal{V}_n \xrightarrow{\Phi} \mathrm{Sp}_{T(n)}$$

corresponds to the forgetful functor

$$\mathcal{P}_k\mathrm{Alg}_{\partial_*\mathrm{id}_{\mathcal{S}_*}}(\mathrm{Sp}_{T(n)}) \xrightarrow{\Omega_k^\infty} \mathrm{Alg}_{\partial_*\mathrm{id}_{\mathcal{S}_*}}(\mathrm{Sp}_{T(n)}) \xrightarrow{\text{forget}} \mathrm{Sp}_{T(n)}$$

and the 'derived primitives' functor

$$\mathcal{P}_k\mathrm{coAlg}(\mathrm{Sp}_{T(n)}) \xrightarrow{\Omega_k^\infty} \mathrm{coAlg}(\mathrm{Sp}_{T(n)}) \xrightarrow{\text{prim}} \mathrm{Sp}_{T(n)}$$

respectively. The latter is a construction formally dual to the topological André-Quillen homology of ring spectra, which is a form of 'derived indecomposables'. A sample application of this theorem uses a convergence result of Arone-Mahowald [1] to identify $\Phi(S^q)$ with the derived primitives of the commutative coalgebra $L_{T(n)}\Sigma^\infty S^q$, for any sphere S^q. After $K(n)$-localization this reproduces a recent result of Behrens and Rezk [2].

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(Higher) Topological Hochschild homology – an overview

Birgit Richter

When topological Hochschild homology, THH, of rings and ring spectra was first defined by Bökstedt in the mid 80's [5], there was no symmetric monoidal category of spectra developed, yet. Bökstedt used the diagram category of finite sets and injections in order to give a model for THH. Since the mid 90's there are other models, for instance one that mimics the definition of the Hochschild complex, one using a Tor-like definition and one using a suitable bar construction (see [13, chapter IX]). It was shown that THH of a ring is isomorphic to MacLane homology [19] and to stable K-theory [12]. The Dennis trace map $tr\colon K_*(R) \to HH_*(R)$ factors over $\mathsf{THH}_*(R)$ and the latter is a better approximation to algebraic K-theory than $HH_*(R)$; it also serves as the input for the construction for topological cyclic homology, $TC(R)$, and this approximates $K_*(R)$ very well in many cases.

Bökstedt calculated THH of the integers and of $\mathbb{F}_p$ [6]. His famous spectral sequence was used for instance by McClure and Staffeldt to determine the mod p homotopy groups of THH of the connective Adams summand [17]. We know THH in many more examples, for instance for local fields [14], number rings [15], $\mathbb{Z}/p^n$ [7] and connective complex topological K-theory [3].

For a discrete R-algebra A (R commutative), the center of A over R can be identified with the endomorphisms of A in the category of A-bimodules over R. Topological Hochschild cohomology of an R-algebra spectrum A can be defined as the derived spectrum of self-maps of A over the enveloping algebra $A \wedge_R^L A^o$ and can hence be viewed as a derived center of A over R. Angeltveit showed that this derived center depends on the chosen A_∞-structure, for instance different A_∞-structure of Morava K-theory, K_n, over Morava E-theory, E_n, give different $THH_{E_n}(K_n)$ [2].

Let A be a commutative R-algebra spectrum. Rognes defined in [20] when A is unramified over R and showed that in this case the canonical map $A \to THH^R(A)$ is a weak equivalence. We use this to show that the complexification map $ko \to ku$ is wildly ramified [11, Theorem 5.2]: $THH_*^{ko}(ku)$ is not equivalent to ku_* and it behaves like Hochschild homology of the Gaussian integers.

In the discrete case Weibel and Geller showed [22] that for an étale extension of commutative rings $R \to A$ Hochschild homology satisfies étale descent, $\mathsf{HH}_*(A) \cong A \otimes_R \mathsf{HH}_*(R)$, and if $R \to A$ is G-Galois for a finite group G this implies $\mathsf{HH}_*(A)^G \cong \mathsf{HH}_*(R)$. Both properties do not carry over to ring spectra: Akhil Mathew shows [16] that there is a C_p-Galois extension of commutative ring spectra for which étale descent fails for THH. In joint work with Ausoni we show that for the $H\mathbb{Q}$-dual of the Hopf map $\eta^*\colon F(S^2_+, H\mathbb{Q}) \to F(S^3_+, H\mathbb{Q})$ the S^1 homotopy fixed points of $THH(F(S^3_+, H\mathbb{Q}))$ are *not* homotopy equivalent to $THH(F(S^2_+, H\mathbb{Q}))$ although η^* is an S^1-Galois extension.

The category of commutative ring spectra is tensored over (pointed) simplicial sets. For a commutative ring spectrum A the standard simplicial model of

$\mathsf{THH}(A)$ can be directly identified with $A \otimes S^1$ where $S^1 = \Delta^1/\partial\Delta^1$ is the standard simplicial model of the 1-sphere.

For any pointed simplicial set X we call $\pi_*(A \otimes X)$ the *X-homology of A*. In the discrete case this was defined by Pirashvili [18], but mentioned earlier for spheres for instance by Anderson [1] in the context of iterated Eilenberg-Moore spectral sequences. Basterra-McCarthy showed that topological André-Quillen homology can be viewed as the stabilization of the $A \otimes S^n$'s [4].

Higher topological Hochschild homology of order n of A is S^n-homology of A and denoted by $\mathsf{THH}^{[n]}(A)$. another important special case is torus homology [8]: if one considers n-fold iterated algebraic K-theory of A, $K^n(A)$, then the iteration of the trace map has $A \otimes (S^1)^n$ as the target.

We know $THH^{[n]}$ in some cases for all $n \geq 1$. For instance we show in [10, 3.6] that

$$\mathsf{THH}_*^{[n]}(H\mathbb{F}_p) \cong \mathsf{Tor}_{*,*}^{\mathsf{THH}_*^{[n-1]}(H\mathbb{F}_p))}(\mathbb{F}_p, \mathbb{F}_p), \quad n \geq 2.$$

These Tor-algebras were determined by Cartan [9] and can be explicitly written down as graded commutative $\mathbb{F}_p$-algebras. This result was also known to Basterra and Mandell. We also show in [11] that for all primes

$$\mathsf{THH}_*^{[2]}(H\mathbb{Z}_{(p)}) \cong \mathbb{Z}_{(p)}[x_1, x_2, \ldots]/p^n x_n = 0, x_n^p = p x_{n+1}, \quad |x_1| = 2p.$$

Schlichtkrull gives a general identification for X-homology of commutative Thom spectra [21].

Ongoing work by Ausoni and Dundas makes progress on Rognes' red-shift conjecture using torus homology. They show that the generator v_{n-1} of connective Morava K-theory is not in the kernel of the unit map

$$k(n-1)_* \to k(n-1)_* K^n(H\mathbb{F}_p).$$

They prove this by showing that v_{n-1} is detected in $k(n-1)_*(H\mathbb{F}_p \otimes (S^1)^n)^{h(S^1)^n}$. It turns out that $\pi_*(H\mathbb{F}_p \otimes (S^1)^n)$ can be described by higher THH of $H\mathbb{F}_p$ because in this case torus homology does not see the attaching maps in the CW structure of the torus. In order to prove the red-shift conjecture for $\mathbb{F}_p$ they have to show that all powers of v_{n-1} also survive.

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Fox-Neuwirth cells, quantum shuffle algebras, and Malle's conjecture for function fields

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(joint work with Jordan S. Ellenberg, TriThang Tran)

Malle conjectured in [1] an asymptotic formula for the growth of number fields with specified Galois group. Specificially, for a fixed integer n and transitive subgroup $G \leq S_n$, one may define a function

$$N_G(X) := \#\{K/\mathbb{Q} \text{ of degree } n \text{ with } |\Delta_K| \leq X \text{ and } Gal(K/\mathbb{Q}) \cong G\}$$

where Δ_K is the discriminant of the number field K. Conjecturally, $N_G(X)$ grows asymptotically as a polynomial in X and $\log(X)$. Specifically, Malle asserts the existence of constants a, b, and C with

$$N_G(X) \sim CX^a(\log X)^{b-1}.$$

The constants a and b are given in terms of properties of conjugacy classes in the group G, their action via S_n, and the action of the absolute Galois group on G via the cyclotomic character.

We can reformulate this conjecture in the function field setting by replacing $\mathbb{Q}$ with $\mathbb{F}_q(t)$, the function field of $\mathbb{A}^1(\mathbb{F}_q)$. An extension K of $\mathbb{F}_q(t)$ corresponds to a ramified covering $\Sigma \to \mathbb{A}^1(\mathbb{F}_q)$. We may regard the (absolute value of the) discriminant $|\Delta_K|$ as q^r, where r is the ramification index of the covering (i.e., the number of branch points), since in the number field setting, a prime ramifies in an extension if and only if it divides the discriminant. This suggests that we define

$$N_{G,q}(X) := \#\{\pi : \Sigma \to \mathbb{A}^1(\mathbb{F}_q) \text{ of degree } n \text{ with } q^r \leq X \text{ and } Aut(\Sigma/\mathbb{A}^1) \cong G\}.$$

We should insist that the Σ being enumerated are geometrically connected, in order to more directly connect to Malle's original conjecture. If $c \subseteq G$ is a conjugation-invariant subset, we may define a more refined function $N^c_{G,q}(X)$ to be the cardinality of the subset where all ramification has monodromy lying in c. Then Malle's conjecture in this setting amounts to the claim that

$$N^c_{G,q}(X) \sim CX^a(\log X)^{b-1}.$$

where a, b, and C now depend upon both G and c.

In the function field setting, unlike the original arithmetic conjecture, there is a scheme whose $\mathbb{F}_q$ points are enumerated by $N^c_{G,q}(X)$. Specifically, for a given r, there is a Hurwitz moduli scheme $Hn^c_{G,r}$ which parameterizes geometrically connected G-branched covers of $\mathbb{A}^1$ with precisely r branch points, with monodromy in c. Then

$$N^c_{G,q}(X) = \sum_{r=0}^{\log_q X} \#Hn^c_{G,r}(\mathbb{F}_q),$$

and Malle's conjecture amounts to the assertion that $\#Hn^c_{G,r}(\mathbb{F}_q) \sim C' q^{ar} r^{b-1}$. In the case, for instance, that $G = S_n$, and c is the conjugacy class of transpositions, a and b are both predicted to be 1. We give an asymptotic upper bound on the growth of this quantity that is slightly larger than Malle's:

Theorem 1 (Ellenberg-Tran-W.)**.** *If $G = S_n$, and c is the conjugacy class of transpositions, there is a positive integer d (depending upon n) with the property that*

$$\lim_{r\to\infty} \frac{\#Hn^c_{G,r}(\mathbb{F}_q)}{q^r r^d} = 0.$$

This result holds for a larger class of (G, c), but for brevity, we leave out the somewhat complicated condition which is required. Our main tool in proving this result is the Grothendieck-Lefschetz fixed point theorem, allowing us to compute the number of $\mathbb{F}_q$ points of a scheme in terms of the trace of Frobenius on its étale cohomology. The main input to this machine is a computation of the rational singular homology of the complex points of the scheme.

In the case at hand,

$$H_*(Hn^c_{G,r}(\mathbb{C}), \mathbb{Q}) = H_*(B_r, W_r)$$

can be described as the homology of the r^{th} braid group B_r with coefficients in a rational representation W_r. Specifically, W_r is a summand of $V^{\otimes r}$, where $V = \mathbb{Q}c$ is the vector space generated by c, and the braid action on $V^{\otimes r}$ is built from V's structure as a *braided vector space*. That is, there is an automorphism σ of $V \otimes V$ which obeys the braid equation on $V^{\otimes 3}$; this yields an action to B_r on $V^{\otimes r}$. For any such V, we prove

Theorem 2 (Ellenberg-Tran-W.)**.** *There is an isomorphism*

$$H_j(B_r; V^{\otimes r}) \cong Ext^{r-j,r}_{A(V_\epsilon)}(k,k).$$

Here V_ϵ is V with its braiding altered by a sign, and $A(V_\epsilon)$ is the *quantum shuffle algebra* that it generates. This is a braided Hopf algebra whose coalgebra structure is that of the tensor algebra on V_ϵ, and whose multiplication mimics the classical shuffle product, but is deformed by a lift of the set of shuffles to representatives in the braid group. The proof relies heavily on Fox-Neuwirth's cellular stratification of the configuration space of points in the plane [2].

This reformulates the braid group homology as a computation in the homological algebra of this braided Hopf algebra. For a class of examples, we develop tools to ensure that the indicated Ext algebra grows at worst exponentially in j and polynomially in r. These give the bounds needed for Theorem 1.

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A motivic little two-disks operad

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The little 2-disks operad $\mathcal{D}$ is an operad in topological spaces whose n-th space has the homotopy type of the space $\mathrm{Conf}_n(\mathbb{C})$ of ordered configurations of n points in the complex plane. This space is the also the complex analytic space underlying the scheme Conf_n:

$$\mathrm{Conf}_n := \mathrm{Spec}(\mathbb{Q}[x_1, \ldots, x_n][(x_i - x_j)^{-1}, i < j])$$

By Artin's comparison theorem between étale and singular cohomology, we obtain for any integer m an equivalence of E_∞-algebras:

$$C^*_{et}(\mathrm{Conf}_n \times_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Z}/m) \simeq C^*(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Z}/m)$$

The E_∞-algebra on the left hand side has an action of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which can thus be transferred on the right hand side via the above equivalence.

It seems impossible to give an operad structure on the collection of algebraic varieties $\{\mathrm{Conf}_n, n \in \mathbb{Z}_{\geq 0}\}$ that realizes the little 2-disks operad when one takes the underlying analytic space. Nevertheless, we have the following theorem (*cf.* [2]).

Theorem 1. The action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the E_∞-algebras $C^*(\mathrm{Conf}_n(\mathbb{C}), \mathbb{Z}/m)$ extends to an action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the Hopf cooperad $C^*(\mathcal{D}, \mathbb{Z}/m)$.

Let us recall the definition of the ∞-category of Hopf cooperads over a commutative ring R. The ∞-category of E_∞-algebras in chain complexes over R has coproducts given by the tensor product $\otimes_R$. It then follows that the opposite of this ∞-category has products. We may therefore define the ∞-category of Hopf cooperads over R as the category of operads in the ∞-category opposite to that of E_∞-algebras.

This theorem strongly suggests that the operad of little 2-disks comes from an algebro-geometric object which is defined over $\mathbb{Q}$. The main goal of this talk is to show that this is the case for a certain explicit choice of interpretation of this question.

We introduce the category $\mathbf{DA}(\mathbb{Q})$. This is the category of étale motives over $\mathrm{Spec}(\mathbb{Q})$ with coefficients in $\mathbb{Z}$. This is a symmetric monoidal stable ∞-category. It can roughly be described as the category obtained from the derived category of étale sheaves of abelian group over the category of smooth schemes over $\mathbb{Q}$ by imposing $\mathbb{A}^1$-invariance and invertibility of the Tate motive. Given a smooth scheme X over $\mathbb{Q}$, we denote by $M(X)$ its image in $\mathbf{DA}(\mathbb{Q})$.

There exists a symmetric monoidal left adjoint functor called the Betti realization which takes values in $\mathbf{D}_\mathbb{Z}$, the derived category of $\mathbb{Z}$. For X a smooth scheme over $\mathbb{Q}$, the motive $M(X)$ is sent to an object equivalent to $C_*(X(\mathbb{C}), \mathbb{Z})$ by the Betti realization functor. The main result of this talk is the following theorem.

Theorem 2. There exists a Hopf cooperad $\underline{\mathcal{D}}(n)$ in the category $\mathbf{DA}(\mathbb{Q})$ whose Betti realization is weakly equivalent to $C^*(\mathcal{D}, \mathbb{Z})$ and such that for each n, we have an equivalence between $\underline{\mathcal{D}}(n)$ and $M(\mathrm{Conf}_n)^\vee$, the linear dual of $M(\mathrm{Conf}_n)$.

This theorem is proved by mean of a fracture square. It can be shown that any object M in $\mathbf{DA}(\mathbb{Q})$ is obtained by gluing together the rationalization $M \otimes \mathbb{Q}$ and the profinite completion $\widehat{M}$. Hence we are reduced to constructing $\underline{\mathcal{D}} \otimes \mathbb{Q}$ and $\widehat{\underline{\mathcal{D}}}$.

By the Suslin rigidity theorem (*cf.* [3]), the category of étale motives with finite coefficients can be identified with the category of chain complexes with an action of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Therefore, the construction of $\widehat{\underline{\mathcal{D}}}$ essentially follows from Theorem 1

The category of étale motives with rational coefficients is more mysterious. However, if we restrict to the full subcategory of mixed Tate motives, this can be understood as the category of representations of a certain affine group scheme called the motivic Galois group. Following ideas of Drinfel'd, Fresse proves in [1] that there exists a model for $C^*(\mathcal{D}, \mathbb{Q})$ as a Hopf cooperad with an action of that group. This gives us a Hopf cooperad in mixed Tate motives which we can in turn see as a Hopf cooperad in the category of motives with rational coefficients.

Using the theory of weights and this motivic little two-disks operad, one can prove the following theorem.

Theorem 3. *Let p be a prime number. The non-symmetric coooperad $C^*(\mathcal{D}, \mathbb{F}_p)$ is formal.*

I conjecture that the same statement is true for the little n-disks operads when $n > 2$. One strategy for proving this for $n = 2m$ an even integer could be to construct a motivic little $2m$-disks operad by taking the m-fold Boardman-Vogt tensor product of $\underline{\mathcal{D}}$ with itself.

Such a formality result would have important consequences in the calculation of the homology of the space of long knots in $\mathbb{R}^n$ with coefficients in a finite field following the approach used by Sinha, Lambrechts, Turchin, Volić in the rational case(*cf.* [4, 5]).

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