

Consider the universal elliptic curve with a choice of 4-torsion point, given by the Weierstrass equation

$$y^2 + axy + acy = x^3 + cx^2$$

over the graded ring  $\mathbb{Z}_{(3)}[a, c]$ , where  $|a| = 1$  and  $|c| = 2$ . To facilitate calculations, we work in the affine coordinate chart  $c = 1$  of the moduli stack  $\mathcal{M}(\Gamma_1(4))$  associated to the level  $\Gamma_1(4)$ -structure on the curve. It is then given by

$$y^2 + axy + ay = x^3 + x^2,$$

with discriminant  $\Delta = a^2(a + 4)(a - 4)$  and Hasse invariant  $h = a^2 + 4$ . In  $uv$ -coordinates, with  $u = x/y$  and  $v = 1/y$ , the equation becomes

$$v + auv + av^2 = u^3 + u^2v.$$

We denote this elliptic curve by  $\mathcal{E}$ .

Let  $F$  be the Morava  $E$ -theory associated to the restriction of  $\mathcal{E}$  to the supersingular locus, so that  $F^0 = \mathbb{Z}_9[[h]]$ .

1. The universal degree 3 isogeny with source  $\mathcal{E}$  is defined over the ring

$$F^0[[t]]/(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2 + 4)t + 3),$$

and has target the elliptic curve

$$y^2 + rxy + ry = x^3 + x^2,$$

where

$$r(a, t) = a^3t^3 + 3a^3t^2 + 3a^3t - 4at + a^3 - 3a.$$

The kernel of this isogeny is generated by the 3-torsion point whose  $x$ -coordinate is  $1/t$ .

2. The power operation  $\psi^3: F^0 \rightarrow F^0[[t]]/(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2 + 4)t + 3)$  is given by

$$\begin{aligned} \psi^3(h) &= (t+1)^3h^3 - (22t^3 + 69t^2 + 75t + 27)h^2 + (128t^3 + 424t^2 + 512t + 201)h \\ &\quad - 16(14t^3 + 49t^2 + 65t + 27), \\ \psi^3(a) &= (t+1)^3a^3 - (4t+3)a. \end{aligned}$$

Let  $A$  be a  $K(2)$ -local commutative  $F$ -algebra. Define  $Q_0, Q_1, Q_2, Q_3: \pi_0 A \rightarrow \pi_0 A$  by

$$\psi^3(x) = Q_0(x) + Q_1(x)\alpha + Q_2(x)\alpha^2 + Q_3(x)\alpha^3,$$

where  $\alpha$  satisfies

$$\alpha^4 - 6\alpha^2 + (a^2 - 8)\alpha - 3 = 0.$$

( $\alpha$  appears as the coefficient of  $u$  in the expression of  $u'$ , comparable to the coefficient  $d$  on page 6 of Rezk's paper "Power operations for Morava  $E$ -theory of height 2 at the prime 2". It is invariant under change of coordinates. The formulas in 1 and 2 above can be written in terms of  $\alpha$  instead of  $t$ . For the formulas below, writing in terms of  $t$  will introduce more denominators, and it will be hard to find the Adem relations.  $\alpha$  and  $t$  are related by  $\alpha = a^2 t^3 + 2a^2 t^2 + a^2 t + 3$ , and  $t = -\alpha/(\alpha + 1)$ , i.e.  $(t + 1)(\alpha + 1) = 1$ .)

Commutation relations:

$$Q_0(hx) = (h^3 - 36h^2 + 390h - 1212)Q_0(x) + (3h^2 - 72h + 360)Q_1(x) + (9h - 108)Q_2(x) + 24Q_3(x),$$

$$Q_1(hx) = (-6h^2 + 144h - 712)Q_0(x) + (-18h + 228)Q_1(x) + (-72)Q_2(x) + (h - 12)Q_3(x),$$

$$Q_2(hx) = (3h - 36)Q_0(x) + 8Q_1(x) + 12Q_2(x) + (-24)Q_3(x),$$

$$Q_3(hx) = (h^2 - 24h + 120)Q_0(x) + (3h - 36)Q_1(x) + 8Q_2(x) + 12Q_3(x);$$

$$Q_0(ax) = (a^3 - 12a + 12a^{-1})Q_0(x) + (3a - 12a^{-1})Q_1(x) + (12a^{-1})Q_2(x) + (-12a^{-1})Q_3(x),$$

$$Q_1(ax) = (-6a + 20a^{-1})Q_0(x) + (-20a^{-1})Q_1(x) + (-a + 20a^{-1})Q_2(x) + (4a - 20a^{-1})Q_3(x),$$

$$Q_2(ax) = (4a^{-1})Q_0(x) + (-4a^{-1})Q_1(x) + (4a^{-1})Q_2(x) + (-a - 4a^{-1})Q_3(x),$$

$$Q_3(ax) = (a - 4a^{-1})Q_0(x) + (4a^{-1})Q_1(x) + (-4a^{-1})Q_2(x) + (4a^{-1})Q_3(x).$$

Adem relations:

$$\alpha' = -\alpha^3 + 6\alpha + (-h + 12);$$

$$\begin{aligned} \Psi(x) = & Q_0Q_0(x) + (-h+12)Q_0Q_1(x) + (h^2-24h+126)Q_0Q_2(x) + (-3)Q_1Q_1(x) + \\ & (-h^3 + 36h^2 - 396h + 1296)Q_0Q_3(x) + (3h - 36)Q_1Q_2(x) + (-3h^2 + 72h - \\ & 378)Q_1Q_3(x) + 9Q_2Q_2(x) + (-9h + 108)Q_2Q_3(x) + (-27)Q_3Q_3(x); \end{aligned}$$

$$Q_1Q_0(x) = (-6)Q_0Q_1(x) + (6h-72)Q_0Q_2(x) + (-6h^2+144h-747)Q_0Q_3(x) + 18Q_1Q_2(x) + 3Q_2Q_1(x) + (-18h+216)Q_1Q_3(x) + (-54)Q_2Q_3(x) + (-9)Q_3Q_2(x),$$

$$Q_2Q_0(x) = (-3)Q_0Q_2(x) + (3h-36)Q_0Q_3(x) + 9Q_1Q_3(x) + 3Q_3Q_1(x),$$

$$Q_3Q_0(x) = Q_0Q_1(x) + (-h+12)Q_0Q_2(x) + (h^2-24h+126)Q_0Q_3(x) + (-3)Q_1Q_2(x) + (3h-36)Q_1Q_3(x) + 9Q_2Q_3(x).$$

Cartan formula:

$$Q_0(xy) = Q_0(x)Q_0(y) + 3[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + 18Q_3(x)Q_3(y),$$

$$Q_1(xy) = [Q_0(x)Q_1(y) + Q_1(x)Q_0(y)] + (-h+12)[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + 3[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + (-6h+72)Q_3(x)Q_3(y),$$

$$Q_2(xy) = [Q_0(x)Q_2(y) + Q_1(x)Q_1(y) + Q_2(x)Q_0(y)] + 6[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + (-h+12)[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + 39Q_3(x)Q_3(y),$$

$$Q_3(xy) = [Q_0(x)Q_3(y) + Q_1(x)Q_2(y) + Q_2(x)Q_1(y) + Q_3(x)Q_0(y)] + 6[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + (-h+12)Q_3(x)Q_3(y).$$

Additivity:

$$Q_i(x+y) = Q_i(x) + Q_i(y).$$

Action on scalars:

$$Q_0(1) = 1, Q_1(1) = Q_2(1) = Q_3(1) = 0;$$

$$Q_0(h) = h^3 - 36h^2 + 390h - 1212,$$

$$Q_1(h) = -6h^2 + 144h - 712,$$

$$Q_2(h) = 3h - 36,$$

$$Q_3(h) = h^2 - 24h + 120;$$

$$Q_0(a) = a^3 - 12a + 12a^{-1},$$

$$Q_1(a) = -6a + 20a^{-1},$$

$$Q_2(a) = 4a^{-1},$$

$$Q_3(a) = a - 4a^{-1}.$$

Frobenius congruence:

$$Q_0(x) \equiv x^3 \pmod{3},$$

$$\theta: \pi_0 A \rightarrow \pi_0 A \text{ such that } Q_0(x) = x^3 + 3\theta(x).$$