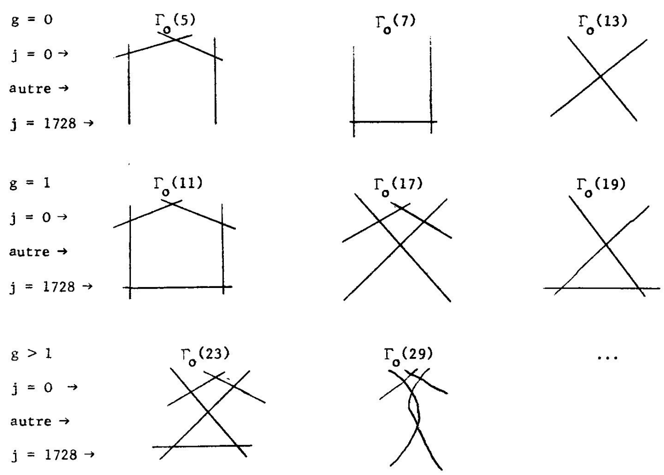
1 Canonical modular polynomials



[Deligne-Rapoport1973, Section VI.6]

Using Magma's calculator, we reproduce the first few canonical modular polynomials from its Modular Polynomial Databases.[1] We note the following, to return to.

(i) Difference from $j = 744$ equals $\#\mathrm{Aut}(E_{\mathrm{s.s.}}/\overline{\mathbb{F}}_p)$.

(ii) Constant term equals p^s where $s = 12/\gcd(p-1, 12) \stackrel{?}{=} \prod \left(\#\mathrm{Aut}(E_{\mathrm{s.s.}}/\overline{\mathbb{F}}_p)/2\right)$.

(iii) Linear coefficient reduces to supersingular j-invariants in the diagram above. (What is the pattern for their exponents?)

- $g = 0$

$$
\begin{aligned}
\Gamma_0(2) \qquad & x^3 + 48x^2 + (768 - j)x + 2^{12} \\
& \equiv x(x^2 - j) \mod 2 \\
\Gamma_0(3) \qquad & x^4 + 36x^3 + 270x^2 + (756 - j)x + 3^6 \\
& \equiv x(x^3 - j) \mod 3 \\
\Gamma_0(5) \qquad & x^6 + 30x^5 + 315x^4 + 1300x^3 + 1575x^2 + (750 - j)x + 5^3 \\
& \equiv x(x^5 - j) \mod 5 \\
\Gamma_0(7) \qquad & x^8 + 28x^7 + 322x^6 + 1904x^5 + 5915x^4 + 8624x^3 + 4018x^2 + (748 - j)x + 7^2 \\
& \equiv x\big(x^7 - (j - 1728)\big) \mod 7 \\
\Gamma_0(13) \qquad & x^{14} + 26x^{13} + 325x^{12} + 2548x^{11} + 13832x^{10} + 54340x^9 + 157118x^8 \\
& + 333580x^7 + 509366x^6 + 534820x^5 + 354536x^4 + 124852x^3 + 15145x^2 \\
& + (746 - j)x + 13^1 \\
& \equiv x\big(x^{13} - (j - 5)\big) \mod 13
\end{aligned}
$$

[1] available up to $\Gamma_0(127)$

- $g = 1$

$$
\begin{aligned}
\Gamma_0(11) \qquad & x^{12} - 5940x^{11} + 14701434x^{10} + (-139755j - 19264518900)x^9 \\
& + (723797800j + 13849401061815)x^8 + (67496j^2 - 1327909897380j \\
& - 4875351166521000)x^7 + (2291468355j^2 + 1036871615940600j \\
& + 400050977713074380)x^6 + (-5346j^3 + 4231762569540j^2 \\
& - 310557763459301490j + 122471154456433615800)x^5 + (161201040j^3 \\
& + 755793774757450j^2 + 17309546645642506200j \\
& + 6513391734069824031615)x^4 + (132j^4 - 49836805205j^3 \\
& + 6941543075967060j^2 - 64815179429761398660j \\
& + 104264884483130180036700)x^3 + (468754j^4 + 51801406800j^3 \\
& + 214437541826475j^2 + 77380735840203400j + 804140494949359194)x^2 \\
& + (-j^5 + 3732j^4 - 4586706j^3 + 2059075976j^2 - 253478654715j \\
& + 2067305393340)x + 11^6 \\
\equiv\ & x\big(x^{11} - j^2(j-1728)^3\big) \mod 11 \\
\Gamma_0(17) \qquad & x^{18} + 510x^{17} + 125001x^{16} + 19248080x^{15} + 2058738420x^{14} \\
& + (10846j + 160172066760)x^{13} + (6027384j + 9242645403716)x^{12} \\
& + \cdots \\
& + (-j^4 + 2982j^3 - 2547081j^2 + 567877726j - 8730057090)x + 17^3 \\
\equiv\ & x\big(x^{17} - j(j-8)^3\big) \mod 17 \\
\Gamma_0(19) \qquad & x^{20} - 152x^{19} + 11020x^{18} - 509732x^{17} + 16884502x^{16} - 423717176x^{15} \\
& + 8284685786x^{14} + (-950j - 127757600560)x^{13} \\
& + \cdots \\
& + (-j^3 + 2236j^2 - 1075910j + 37507528)x + 19^2 \\
\equiv\ & x\big(x^{19} - (j-7)(j-1728)^2\big) \mod 19
\end{aligned}
$$

- $g > 1$

$$
\begin{aligned}
\Gamma_0(23) \qquad & x^{24} + 94392x^{23} + 4240527204x^{22} + (108774498j + 119018915927208)x^{21} \\
& + \cdots \\
& + (-j^{11} + 8196j^{10} - 28368090j^9 + 53962467848j^8 - 61514962720527j^7 \\
& + 43007336651707740j^6 - 18144237478297458590j^5 \\
& + 4374793948754527714200j^4 - 541459535600500383823479j^3 \\
& + 28035152457942175237515676j^2 - 389561380516779182551042062j \\
& + 312190445452533657242901912)x + 23^6 \\
\equiv\ & x\big(x^{23} - j^2(j+4)^6(j-1728)^3\big) \mod 23
\end{aligned}
$$

$$\begin{aligned}
\Gamma_0(29) \qquad & x^{30} - 1218x^{29} + 750375x^{28} - 312177460x^{27} + 97844061669x^{26} \\
& + (-236321j - 24383203360230)x^{25} + (946283688j + 4982726503407419)x^{24} \\
& + \cdots \\
& + (-j^7 + 5214j^6 - 10272861j^5 + 9480438286j^4 - 4108842162480j^3 \\
& + 728011816505784j^2 - 35575638370254161j + 107281337499515022)x + 29^3 \\
\equiv\ & x\big(x^{29} - j(j-2)^3(j+4)^3\big) \mod 29
\end{aligned}$$

We compare these to modular equations for $\big(\Gamma_0(p), \Gamma_1(N)\big)$ computed previously.

- $g = 0$[2]

$$\begin{aligned}
\big(\Gamma_0(2), \Gamma_1(3)\big) \qquad & d^3 - ad - 2 \\
\equiv\ & d(d^2 + a) \mod 2 \\
\big(\Gamma_0(3), \Gamma_1(4)\big) \qquad & \alpha^4 - 6\alpha^2 + (a^2 - 8)\alpha - 3 \\
\equiv\ & \alpha\big(\alpha^3 + (a^2 + 1)\big) \mod 3
\end{aligned}$$

- $g = 1$

$$\begin{aligned}
\big(\Gamma_0(5), \Gamma_1(4)\big) \qquad & \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - (a^4 - 16a^2 + 26)\alpha + 5 \\
\equiv\ & \alpha\big(\alpha^5 - (a^4 - a^2 + 1)\big) \mod 5
\end{aligned} \tag{1.1}$$

$$\begin{aligned}
\big(\Gamma_0(5), \Gamma_1(3)\big) \qquad & \alpha^6 - 5a\alpha^4 + 40\alpha^3 - 5a^2\alpha^2 + (a^4 - 19a)\alpha - 5 \\
\equiv\ & \alpha\big(\alpha^5 + a(a+1)(a^2 - a + 1)\big) \mod 5
\end{aligned}$$

three supersingular points (or two? $A^4 - 19AB$)

Note: higher coefficients (degree > 1) are constants if $(p-1)(N-1)|12$ with a single supersingular point.

For later reference, we rewrite these equations as follows.

$$\begin{aligned}
\big(\Gamma_0(2), \Gamma_1(3)\big) \qquad & d^3 - ad - 2 && \Delta_1(3) = (a-3)(a^2 + 3a + 9) \\
\equiv\ & (d-2)(d+1)^2 \mod (a-3) && \text{unramified cusp} \\
\big(\Gamma_0(3), \Gamma_1(4)\big) \qquad & \alpha^4 - 6\alpha^2 + (a^2 - 8)\alpha - 3 && \Delta_1(4) = a^2(a+4)(a-4) \\
\equiv\ & (\alpha - 3)(\alpha + 1)^3 \mod a && \text{ramified cusp} \\
\equiv\ & (\alpha + 3)(\alpha - 1)^3 \mod (a+4)(a-4) && \text{unramified cusp } (A^2 - 16B) \\
\big(\Gamma_0(5), \Gamma_1(4)\big) \qquad & \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - (a^4 - 16a^2 + 26)\alpha + 5 \\
\equiv\ & (\alpha - 5)(\alpha - 1)^5 \mod a(a+4)(a-4) \\
\big(\Gamma_0(5), \Gamma_1(3)\big) \qquad & \alpha^6 - 5a\alpha^4 + 40\alpha^3 - 5a^2\alpha^2 + (a^4 - 19a)\alpha - 5 \\
\equiv\ & (\alpha + 5)(\alpha - 1)^5 \mod (a-3)
\end{aligned}$$

[2] [Katz-Mazur1985, Theorem 10.13.12], [Diamond-Shurman2005, Section 3.8]: $\frac{p+1}{12}N^2\prod_{\ell|N}\left(1 - \frac{1}{\ell^2}\right) = 2g - 2 + 2c\big(\Gamma_1(N)\big)$

2 Examples

Here are two examples about how to derive a canonical modular polynomial $\mathrm{cmp}(x, j)$ (cf. [Choi2006, Example 2.4]).

Example 2.1 $p = 5$ (cf. [Ahlgren2003, p788])

$$s := \frac{12}{\gcd(p-1, 12)} = 3 \qquad xx' = p^s = 5^3$$

$$u := \frac{p-1}{\gcd(p-1, 12)} = 1$$

$$\begin{aligned}
x' = \phi_p(z) &:= \left(\frac{\eta(z)}{\eta(pz)}\right)^{2s} \\
&= \left(\frac{q^{1/24}\prod_{n=1}^{\infty}(1-q^n)}{q^{5/24}\prod_{n=1}^{\infty}(1-q^{5n})}\right)^6 \\
&= q^{-1}\left(\frac{1-q-q^2+q^5+q^7-q^{12}-q^{15}+\cdots}{1-q^5-q^{10}+q^{25}+q^{35}-q^{60}-q^{75}+\cdots}\right)^6 \qquad c_i = \begin{cases}(-1)^k & \text{if } i = k(3k\pm 1)/2 \\ 0 & \text{otherwise}\end{cases} \\
&= \frac{1}{q} - 6 + 9q + 10q^2 - 30q^3 + 6q^4 - 25q^5 + 96q^6 + 60q^7 - 250q^8 + 45q^9 - 150q^{10} + \cdots \\
&= q^{-u} + \cdots
\end{aligned}$$

a univalent modular function on $\Gamma_0(5)$, with a simple pole at ∞

and a simple zero at 0 (the two cusps of $\Gamma_0(p)$): [Ono2004, Section 1.4, esp. Theorem 1.64], [Apostol1990, Sections 4.7-4.10, esp. Theorems 4.7 and 4.9]

There exists a unique degree-u polynomial $f(j)$ such that $f(j) - x'$ is a cusp form:

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + 333202640600q^5 + \cdots$$

$$f(j) = j - 750$$

Note that

- $j_0 := 750 \equiv 0$ is the unique supersingular j-invariant at 5.
- $$f(j) - x' \equiv 0 \mod 5 \tag{2.2}$$

Questions

- Is $f(j)$ a polynomial of supersingular j-invariants? Cf. [Kaneko-Zagier1998, Theorem 1] and [Milas-Mortenson-Ono2008, Theorems 1.1 and 1.5].

- Is x' a Hasse invariant? Cf. [Zhu2014, Remark 3.4].

Following [Choi2006, (2.4)],[3] we compute that

$$\begin{aligned} j_{up}^{(p)}(z) &= j_5^{(5)}(x',j) \\ &= x'^5 + 30x'^4 + 315x'^3 + 1300x'^2 + 1575x' \\ &= \frac{1}{q^5} - 6 + 5q(\cdots) \qquad\qquad \text{Why??} \end{aligned}$$

and get

$$\begin{aligned} \mathrm{cmp}(x',j') &= x' j_{up}^{(p)} - x'\big(f(j') - x\big) \\ \mathrm{cmp}(x,j) &= x^6 + 30x^5 + 315x^4 + 1300x^3 + 1575x^2 + (750-j)x + 5^3 \end{aligned}$$

More directly, adapting [Choi2006, (2.4)], we have

$$j_{u(p+1)}^{(p)}(x',j') \rightsquigarrow \mathrm{cmp}(x',j')$$

where $j'(z) = j(pz)$. Note that since

$$\begin{aligned} x' &= \frac{1}{q^u} + \cdots \\ j' &= \frac{1}{q^p} + \cdots \end{aligned}$$

and $p \nmid u$, this algorithm for computing $\mathrm{cmp}(x,j)$ <u>always</u> works.

Upshot (cf. [Zhu2015, (3.10)])

$$\begin{aligned} \psi^5 \colon\ E^0 &\to E^0(B\Sigma_5)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![h = j - j_0]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![x,j]\!]/\big(\mathrm{cmp}(x,j)\big) \\ h \mapsto j' - 750 &= x + 1x'^5 + 30x'^4 + 315x'^3 + 1300x'^2 + 1575x' \\ &= x + (h - x^5 - 30x^4 - 315x^3 - 1300x^2 - 1575x)^5 + \cdots \\ &= h^5 + 30h^4 - 787185h^3 - 78654950h^2 + 113706048450h + 9128404218750 \\ &\quad + (-1575h^4 - 209750h^3 + 919941375h^2 + 146313952500h - 53794421543124)x \\ &\quad + (-1300h^4 - 78375h^3 + 765753000h^2 + 83642547500h - 47590693860000)x^2 \\ &\quad + (-315h^4 - 13200h^3 + 185819525h^2 + 17992315500h - 11702653105500)x^3 \\ &\quad + (-30h^4 - 1025h^3 + 17705550h^2 + 1622265375h - 1120917084750)x^4 \\ &\quad + (-1h^4 - 30h^3 + 590310h^2 + 52436200h - 37473142200)x^5 \end{aligned}$$

[3] It generalizes [Ahlgren2003, p788], which in turn generalizes [Bruinier-Kohnen-Ono2004, pp553-554]. In particular, by analogy to the latter, $j_5^{(5)}(z) = j_1^{(5)}(z)|T_0(5)$; in other notation, $h' = T_5 x'$. Compare the Eichler-Shimura congruence from (1.1):

$$T_5\alpha = \sum \alpha_i' = \sum \frac{5}{\alpha_i} = \sum \frac{\alpha_0 \cdots \alpha_5}{\alpha_i} = h = \alpha^5 - 10\alpha^4 + 35\alpha^3 - 60\alpha^2 + 55\alpha + \alpha' \equiv \alpha^5 + \alpha' \mod 5$$

(Hecke operator and involution commute [Atkin-Lehner1970, Lemma 11]).

Second attempt: deduce $\psi^5(h)$ by comparing q-expansions

$$
\begin{aligned}
h' &= j' - 750 = \frac{1}{q^5} - 6 + 196884q^5 + \cdots \\
h &= j - 750 = \frac{1}{q} - 6 + 196884q + \cdots \\
x &= \frac{5^3}{x'} = 125q + 750q^2 + 3375q^3 + \cdots
\end{aligned}
$$

$$\Longrightarrow$$

$$h' = h^5 + 30h^4 - 1575h^4x - 787185h^3 + (-1300h^4x^2 + ?h^3x + ?h^2) + \cdots$$

unsuccessful

We do not yet know a nice formula for the higher coefficients in $\mathrm{cmp}(x, j)$ when $g = 0$, though based on the algorithm it is not hard to write down a few terms:

$$
\begin{array}{lll}
w_p = 2sp & = \frac{24p}{p-1} & = 2(s+12) \\
w_{p-1} = sp(2sp - 4s + 3) & = \frac{36p(9p-17)}{(p-1)^2} & = -(s+12)(2s-27) \\
w_{p-2} = \frac{2}{3}sp(2sp - 6s + 1)(sp - 3s + 4) & = \frac{64p(2p-5)(25p-73)}{(p-1)^3} & = \frac{4}{3}(s+12)(s-8)(4s-25) \\
w_{p-3} = \frac{1}{6}sp(2sp - 8s + 1)(2sp - 8s + 3)(sp - 4s + 7) & = \frac{18p(3p-11)(19p-55)(25p-97)}{(p-1)^4} & = -\frac{1}{2}(s+12)(2s-9)(3s-19)(6s-25) \\
w_{p-4} = \frac{1}{15}sp(sp - 5s + 3)(2sp - 10s + 3) & = \frac{576p(5p-21)(9p-41)(34p^2-275p+529)}{5(p-1)^5} & = \frac{4}{15}(s+12)(4s-15)(8s-27) \\
\qquad (2s^2p^2 - 20s^2p + 21sp + 50s^2 - 105s + 4) & & \qquad (8s^2 - 69s + 136)
\end{array}
$$

Example 2.3 $p = 11$

$s = 6 \qquad xx' = 11^6$

$u = 5$

$$
\begin{aligned}
x' = \phi_{11}(z) &= \left(\frac{\eta(z)}{\eta(11z)}\right)^{12} \\
&= \frac{1}{q^5} - \frac{12}{q^4} + \frac{54}{q^3} - \frac{88}{q^2} - \frac{99}{q} + 540 - 418q - 648q^2 + 594q^3 + 836q^4 + 1056q^5 - 4092q^6 - 353q^7 + \cdots
\end{aligned}
$$

a modular function on $\Gamma_0(11)$ with Nebentypus: [Apostol1990, pp86-87]

There exists a unique $f(j)$ such that $f(j) - x'$ is a cusp form $(0 \mod 11)$:

$$
\begin{aligned}
f(j) &= j^5 - 3732j^4 + 4586706j^3 - 2059075976j^2 + 253478654715j - 2067305393340 \\
&\equiv j^2(j-1)^3 \mod 11
\end{aligned}
$$

Adapting [Choi2006, (2.4)], we compute $\mathrm{cmp}(x',j') \rightsquigarrow \mathrm{cmp}(x,j)$ and get

$$\begin{aligned}\mathrm{cmp}(x,j) = x^{12} &- 5940x^{11} + 14701434x^{10} + (-139755j - 19264518900)x^9 \\ &+ \cdots \\ &+ (-j^5 + 3732j^4 - 4586706j^3 + 2059075976j^2 - 253478654715j \\ &+ 2067305393340)x + 11^6 \\ \equiv\ & x\big(x^{11} - f(j)\big) \mod 11\end{aligned}$$

Upshot

$$\begin{aligned}\psi^{11}\colon\ E^0 &\to E^0(B\Sigma_{11})/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_{11}\big)[\![h = j]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_{11}\big)[\![x,j]\!]/\big(\mathrm{cmp}(x,j)\big)\end{aligned}$$

It seems that $\mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![x,j]\!]/\big(\mathrm{cmp}(x,j)\big)$ is not always the correct target of the total power operation for an E-theory, which arises from completion at a single supersingular point; x' needs to split off a factor with q-expansion $\dfrac{1}{q} + \cdots$, to be paired with j as in (2.2).

In view of the above, for all primes p, we have

$$\begin{aligned}\psi^p\colon\ L_{K(2)}\mathrm{TMF}(?)^0 &\to L_{K(2)}\mathrm{TMF}(?)^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h = f(j)]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![x,j]\!]/\big(\mathrm{cmp}(x,j)\big) \\ h &\mapsto f(j') = x + j^{(p)}_{up}(x',j')\end{aligned}$$

Locally at each supersingular point, the above total power operation splits off a factor

$$\begin{aligned}\psi^p\colon\ E^0 &\to E^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h = j - j_0]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![x_0,j]\!]/\big(\mathrm{cmp}_0(x_0,j)\big) \\ h &\mapsto j' - j_0\end{aligned}$$

In particular,

$$\begin{aligned}x_0' &\equiv j - j_0 \mod p \\ \Longrightarrow\ \mathrm{cmp}_0(x_0',j') &\equiv x_0'\big((x_0')^p - (j' - j_0)\big) \equiv (j - j_0)(j^p - j') \mod p\end{aligned}$$

which symmetrizes to the Kronecker congruence

$$\big(j - (j')^p\big)(j^p - j') \equiv 0 \mod p$$

We check $h = j - j_0$ against the explicit models of $\big(\Gamma_0(p), \Gamma_1(N)\big)$ and see how the q-expansions match up.

- $\Gamma_1(3)\colon\ y^2 + Axy + By = x^3,\ \ |A| = 1,\ \ |B| = 3,\ \ \Delta = B^3(A^3 - 27B),\ \ j = A^3(A^3 - 24B)^3/\Delta$
  At $p = 2$,

$$\begin{cases} H = A \implies \frac{H}{B^{1/3}} = h = j - j_0 = \frac{1}{q} + \cdots \\ j = \frac{\left(\frac{1}{q}B^{1/3}\right)^{12}}{q} + \cdots = \frac{1}{q^{13}}B^4 + \cdots = \frac{1}{q} + \cdots \end{cases} \implies B = q^3 + \cdots$$

At $p=5$, $(p-1)/2=2$,[4]

$$\left\{\begin{array}{l} H = A^4 + 16AB \implies \frac{H}{B^{4/3}} = h = \big((j-j_0)(j-j_1)\big)^2 = \frac{1}{q^4} + \cdots \\ j = \frac{\left(\frac{1}{q^4}B^{4/3}\right)^3}{q} + \cdots = \frac{1}{q^{13}}B^4 + \cdots = \frac{1}{q} + \cdots \end{array}\right. \implies B = q^3 + \cdots$$

- $\Gamma_1(4)$: $y^2+Axy+ABy = x^3+Bx^2$, $|A|=1$, $|B|=2$, $\Delta = A^2B^4(A^2-16B)$, $j=(A^4-16A^2B+16B^2)^3/\Delta$

  At $p=3$, $(p-1)/2=1$,

$$\left\{\begin{array}{l} H = A^2 + 4B \implies \frac{H}{B} = h = j - j_0 = \frac{1}{q} + \cdots \\ j = \frac{\left(\frac{1}{q}B\right)^6}{q} + \cdots = \frac{1}{q^7}B^6 + \cdots = \frac{1}{q} + \cdots \end{array}\right. \implies B = q + \cdots$$

  At $p=5$, $(p-1)/2=2$,

$$\left\{\begin{array}{l} H = A^4 + 24A^2B + 16B^2 \implies \frac{H}{B^2} = h = (j-j_0)^2 = \frac{1}{q^2} + \cdots \\ j = \frac{\left(\frac{1}{q^2}B^2\right)^3}{q} + \cdots = \frac{1}{q^7}B^6 + \cdots = \frac{1}{q} + \cdots \end{array}\right. \implies B = q + \cdots$$

3 Modular equations for Lubin-Tate formal groups

Theorem 3.1 *Let $\mathbb{G}_0$ be a formal group of height 2 over $\overline{\mathbb{F}}_p$, and let $\mathbb{G}$ be its universal deformation. Write A_m for the ring $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G})}$ studied in [Strickland1997], which classifies degree-p^m subgroups of the formal group $\mathbb{G}$. Then $A_0 \cong \mathbb{W}(\overline{\mathbb{F}}_p)[\![h]\!]$ and $A_1 \cong \mathbb{W}(\overline{\mathbb{F}}_p)[\![h,\alpha]\!]/\big(w(h,\alpha)\big)$, where*

$$w(h,\alpha) = (\alpha - p)\big(\alpha + (-1)^p\big)^p - \big(h - p^2 + (-1)^p\big)\alpha$$

Proof Choose a $\mathcal{P}_N$-model for $\mathbb{G}$ as in [Zhu2015, Section 2], and consider a formal neighborhood that contains a single supersingular point in characteristic p with j-invariant j_0 (clearly such a neighborhood is preserved under a deformation of p-power Frobenius, as the Frobenius is an automorphism over $\overline{\mathbb{F}}_p$). Define $h := j - j_0$. By the Serre-Tate theorem and [Zhu2015, Remark 3.2], there exists a unique polynomial

$$w(h,\alpha) = \alpha^{p+1} + \sum_{i=0}^{p} w_i \alpha^i \tag{3.2}$$

with $w_i \in \mathbb{W}(\overline{\mathbb{F}}_p)[\![h]\!]$ such that $A_1 \cong A_0[\alpha]/\big(w(h,\alpha)\big)$. Moreover, by [Ando1995, Theorem 4], we can choose α such that $w_0 = (-1)^{p+1}p$.

Note that the ring A_1, with parameters h and α, is precisely the ring A parametrized by T and $\mathbf{N}\big(X(P)\big)$, respectively, in [Katz-Mazur1985, Section 7.7]. We now imitate the derivation from Section 2 of a canonical modular polynomial, with parameters j' and x', and derive a polynomial relation between their counterparts h' and α', which are the images of h and α under the Atkin-Lehner involution.

[4] [Silverman2009, V.4.1a]

By [Katz-Mazur1985, 12.4.1], [Zhu2015, Remark 3.2] and in view of the dehomogenization procedure in [Zhu2015, Example 2.6, Proposition 2.8, and Example 3.4], since $h = j - j_0$, the modular function α' on $\Gamma_0(p)$ has a q-expansion

$$\alpha' = \mu q^{-1} + O(1)$$

for some $\mu \in \mathbb{W}(\overline{\mathbb{F}}_p)^\times \cap \mathbb{Z}$. Thus there exist $w_i' \in p\mathbb{Z}$, $2 \leq i \leq p$ such that

$$(\alpha')^p + w_p'(\alpha')^{p-1} + \cdots + w_2'\alpha' = \mu^p q^{-p} + O(1)$$

On the other hand, since $j'(z) = j(pz)$, we have

$$h' = j' - j_0 = q^{-p} + O(1)$$

Comparing the two displays above, we then have

$$(\alpha')^p + w_p'(\alpha')^{p-1} + \cdots + w_2'\alpha' = \mu^p h' + c + O(q)$$

for some $c \in \mathbb{Z}$. Passing to the mod-p reduction of this identity, we see that $c \in p\mathbb{Z}$. Therefore we can redefine h (and w_i, $2 \leq i \leq p$ in (3.2) accordingly) such that

$$(\alpha')^p + w_p'(\alpha')^{p-1} + \cdots + w_2'\alpha' = h' + O(q)$$

without changing the expressions for A_0 and A_1 (note that α' is independent of the choice of h). From this we obtain

$$(\alpha')^{p+1} + w_p'(\alpha')^p + \cdots + w_2'(\alpha')^2 = h'\alpha' + O(1)$$

In view of the expression for A_1 (under the Atkin-Lehner involution) and the q-expansions for α' and h', we see that the last term $O(1)$ above must be constant. Applying the Atkin-Lehner involution to this polynomial relation between α' and h', we then conclude that the coefficients w_i, $2 \leq i \leq p$ in (3.2) are all constants. It remains to determine their values, which follows from the next proposition. □

Proposition 3.3 *Let M_n and $M_{n,p}$ be the modular schemes in [Katz1973, Section 1.13], the latter being finite and flat over the former of degree $p+1$. In a punctured formal neighborhood of the cusps $\overline{M}_n - M_n$, the scheme $M_{n,p}$, viewed as a relative curve over M_n, has an equation*

$$(\alpha - p)\big(\alpha + (-1)^p\big)^p = 0$$

Proof Choose the particular local coordinate in [Ando1995, Theorem 4] on the universal elliptic curve over $M_{n,p}$, and define the parameter α as in [Zhu2015, Section 3.1, esp. Construction 3.1 (ii) and Remark 3.2]. In view of [Zhu2015, Remark 3.15], the stated equation then follows from the discussion in the first new paragraph on page Ka-23 of [Katz1973] (note that when $p = 2$, the isogeny π in [Katz1973, Section 1.11] differs by a sign from the restriction of the isogeny $\Psi_N^{(p)}$, $N = n$, in [Zhu2015, Section 3] around the ramified cusp 0 of $\Gamma_0(p)$). □

By [Strickland1998, Theorem 1.1] and [Rezk2009, Theorem B], we have the following.

Corollary 3.4 *Let E be a Morava E-theory of height 2 at the prime p. There is a total power operation*

$$\begin{aligned} \psi^p\colon\ E^0 &\to E^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big) \end{aligned}$$

where

$$w(h,\alpha)=(\alpha-p)\big(\alpha+(-1)^p\big)^p-\big(h-p^2+(-1)^p\big)\alpha$$

4 Computations

From Corollary 3.4 we can compute $\psi^p(h)$ via involution as usual ~~algorithmically, if not explicitly~~, which leads to a uniform presentation of the Dyer-Lashof algebra for Morava E-theory at height 2 (cf. [Zhu2015, Remark 6.6]).

Here are two attempts for computing $\psi^p(h)$ explicitly. Assume $p>2$.

(i) Can a computer manipulate polynomials of indefinite degree? Cf. [Kerner2008, Appendix C].

$$\begin{aligned}
(\alpha-p)(\alpha-1)^p &= (\alpha-p)\sum_{i=0}^{p}\binom{p}{i}\alpha^i(-1)^{p-i}\\
&= \sum_{i=0}^{p}\binom{p}{i}\alpha^{i+1}(-1)^{p-i}-p\sum_{i=0}^{p}\binom{p}{i}\alpha^i(-1)^{p-i}\\
&= \sum_{j=1}^{p+1}\binom{p}{j-1}\alpha^j(-1)^{p-j+1}-p\sum_{j=0}^{p}\binom{p}{j}\alpha^j(-1)^{p-j}\\
&= \sum_{j=2}^{p+1}\left[\binom{p}{j-1}(-1)^{p-j+1}-p\binom{p}{j}(-1)^{p-j}\right]\alpha^j+\cdots \qquad \binom{p}{p+1}=0\\
&= \sum_{j=2}^{p+1}(-1)^{p-j+1}\left[\binom{p}{j-1}+p\binom{p}{j}\right]\alpha^j+\cdots\\
\implies\quad w(h,\alpha) &= (-p)(-1)^p-h\alpha+\sum_{j=2}^{p+1}(-1)^{p-j+1}\left[\binom{p}{j-1}+p\binom{p}{j}\right]\alpha^j\\
&= p-h\alpha+\sum_{j=2}^{p+1}(-1)^{j}\left[\binom{p}{j-1}+p\binom{p}{j}\right]\alpha^j\\
&= p-h\alpha+\sum_{j=2}^{p+1}\left[\binom{p}{j-1}+p\binom{p}{j}\right](-\alpha)^j
\end{aligned}$$

$$\implies \qquad 0 = p - h\alpha + \sum_{j=2}^{p+1}\left[\binom{p}{j-1} + p\binom{p}{j}\right]\alpha^j \qquad\qquad \alpha \mapsto -\alpha,\ h \mapsto -h$$
$$= \alpha^{p+1} + 2p\alpha^p + \cdots - h\alpha + p$$
$$\implies \alpha' = \frac{p}{\alpha} = h - \sum_{j=2}^{p+1}\left[\binom{p}{j-1} + p\binom{p}{j}\right]\alpha^{j-1}$$
$$= h - \sum_{i=1}^{p}\left[\binom{p}{i} + p\binom{p}{i+1}\right]\alpha^{i}$$
$$\implies \qquad h' = \alpha + \sum_{k=1}^{p}\left[\binom{p}{k} + p\binom{p}{k+1}\right]\left(h - \sum_{i=1}^{p}\left[\binom{p}{i} + p\binom{p}{i+1}\right]\alpha^{i}\right)^k$$

Adapting the proof of [Zhu2015, Proposition 6.4] does the trick here!

$$= \alpha + \sum_{k=1}^{p}\left[\binom{p}{k} + p\binom{p}{k+1}\right]\sum_{m_k=0}^{k}\binom{k}{m_k}h^{k-m_k}\left(-\sum_{i=1}^{p}\left[\binom{p}{i} + p\binom{p}{i+1}\right]\alpha^{i}\right)^{m_k}$$
$$= -\alpha^{p^2} - 2p^2\alpha^{p^2-1} + \cdots + \sum_{k=1}^{p}\left[\binom{p}{k} + p\binom{p}{k+1}\right]h^k$$
$$\equiv ?\ \bmod w(h,\alpha)$$

(ii) Change of variables

$$w(h,\alpha) = (\alpha - p)(\alpha-1)^p + (1 + p^2 - h)\alpha$$
$$\begin{cases} \beta = \alpha - 1 & \text{a unit} \\ y = p^2 - h \end{cases} \qquad \Big\Downarrow$$
$$(1 + \beta - p)\beta^p + (1+y)(1+\beta) = 0$$
$$\big(1 - (1+\beta')\big)\beta^p + 1 + y = 0$$
$$1 + y = \beta^p\beta' \qquad\qquad \text{multiplicativity of units}$$
$$y' = (\beta')^p\beta - 1$$
$$= \left(\frac{y+1}{\beta^p}\right)^p \beta - 1$$
$$= \frac{(y+1)^p}{\beta^{p^2-1}} - 1$$

Reducing β^{p^2-1} and then inverting it is complicated. Also,

$$\begin{aligned}
&\begin{cases} \beta^{p+1}+(1-p)\beta^p+(1+y)\beta+(1+y)=\beta^p(\beta+1-p)+(1+y)(\beta+1) \\ \beta^p=(1+y)(\beta+1)\cdot(1+y)^{-1}(\beta^{p-1}-\beta^{p-2}+\beta^{p-3}-\cdots+1)-1 \end{cases} \\
\implies\ & 1=-\beta^p(\beta+1-p)\cdot(1+y)^{-1}(\beta^{p-1}-\beta^{p-2}+\beta^{p-3}-\cdots+1)-\beta^p \\
\implies\ & \beta'=\frac{1+y}{\beta^p} \\
&\quad=-(\beta+1-p)(\beta^{p-1}-\beta^{p-2}+\beta^{p-3}-\cdots+1)-(1+y) \\
&\quad=-\beta^p+p\beta^{p-1}-p\beta^{p-2}+p\beta^{p-3}-\cdots+p-1-(1+y)
\end{aligned}$$

Reducing the p'th power of this last expression seems hard.

5 More about the eta-quotient

For the genus-zero primes p (each with a single supersingular j-invariant), it is straightforward to show that

$$\text{constant term of } -\phi_p(z)=2s=\#\mathrm{Aut}(E_{\mathrm{s.s.}}/\overline{\mathbb{F}}_p)$$

as noted in Section 1 ($768-j$, etc.).

In view of the construction of ϕ_p from Δ, it is not surprising that ϕ_p has vanishing "p-local Serre derivative" [Ahlgren2003, Section 4] (cf. [Zhu2015, (4.10)]):

$$\begin{aligned}
&\text{"}G_p:=\frac{\theta\phi_p}{\phi_p}+\frac{pE_2|V(p)-E_2}{p-1}=0\text{"} \\
\iff\ & \vartheta_p\phi_p:=D\phi_p+\left(\frac{k/12-h}{p-1}\cdot p\mathcal{E}_2|V(p)+\frac{h-pk/12}{p-1}\cdot\mathcal{E}_2\right)\phi_p=0
\end{aligned}$$

where weight $k=0$, $h=-u=-1$ [Ahlgren2003, Lemma 2.2]. This generalizes to all primes in [Choi2006, Theorem 3.4]. In fact, there is a version of p-local Serre derivative for each (l,p)-type sequence $\{g_m^{(p)}\}$.[5]

It is illegitimate to deduce from this and [Zhu2015, Theorem 4.13] that

$$\begin{aligned}
\ell_{2,p}(x')&=\frac{1}{p}\log\frac{(x')^p\cdot x'}{x_0\cdots x_p} \\
&=\frac{1}{p}\log\frac{(x')^{p+1}}{p^s}
\end{aligned}$$

[5] Sequences of modular functions:

[Bruinier-Kohnen-Ono2004]	$\mathrm{SL}_2(\mathbb{Z})$	$\{j_m(z)\}$	$(-1,1)$-type	$j_1(z)=j(z)-744$	$H_\tau(z)$
[Ahlgren2003]	$\Gamma_0(p),\ g=0$	$\{j_m^{(p)}(z)\}$	$(-1,p)$-type	$j_1^{(p)}(z)=\phi_p(z)\equiv f(j) \mod p$	$H_\tau^{(p)}(z)$
[Choi2006, (2.4)]	$\Gamma_0(p),\ g>0$	$\{j_m^{(p)}(z)\}$	$(-u,p)$-type	$j_1^{(p)}(z)=j(z)$ $j_u^{(p)}(z)=\phi_p(z)\equiv f(j) \mod p$	
[Choi2006, Definition 3.1]	$\Gamma_0(N)$	$\{g_m^{(N)}(z)\}$	(l,p)-type		$[g_m^{(N)}]_\tau(z),\ [g_m^{(N)}]_t(z),\ [g_m^{(N)}]_\infty(z)$

must then be zero, because ϕ_p is not on $\Gamma_1(N)$ and is not a unit either.

However, the above does seem to make sense $K(1)$-locally. Recall from [Lubin1979, Definition above Theorem E] that the universal ring parametrizing canonical (degree-p) subgroups is

$$\mathfrak{A}_2 \cong \mathbb{Z}_p\left[\!\left[t, \frac{p}{t}, \frac{p^p}{t^{p+1}}\right]\!\right] \sim \mathbb{Z}_p\left[\!\left[\alpha, \alpha', \frac{(\alpha')^{p+1}}{p}\right]\!\right] \sim \mathbb{Z}_p\left[\!\left[x, x', \frac{(x')^{p+1}}{p^s}\right]\!\right]$$

We know that $K(1)$-localization inverts $h \sim \alpha' \sim x'$, so that

$$\ell_{2,p}(x') = 0 \implies \frac{(x')^{p+1}}{p^s} \in \mu_p \implies A_1 \cong \mathfrak{A}_2$$

meaning, tautologically, that $K(1)$-locally the unique degree-p subgroup is the canonical subgroup.

Question: What does $\vartheta_p \circ x' = 0$ indicate, if anything, for the diagram below? [Katz1973, Appendices 1 and 2]

$$\begin{tikzcd}
0 \arrow[r] & \omega \arrow[r] \arrow[d, "\cdot \alpha' \sim x'"'] & H^1_{\mathrm{DR}}(C/S) \arrow[r] \arrow[d] & \omega^{-1} \arrow[r] \arrow[d, "\cdot h'"] & 0 \\
0 \arrow[r] & \omega^{(p)} \arrow[r] \arrow[d, "\vartheta_p"'] & H^1_{\mathrm{DR}}(C^{(p)}/S) \arrow[r] \arrow[d, "\nabla(D)"] & \omega^{(-p)} \arrow[r] & 0 \\
 & {} & H^1_{\mathrm{DR}}(C^{(p)}/S) \otimes \omega^{(2p)} & &
\end{tikzcd}$$

♠ Look into more of [Katz1973] for a structural explanation of the appearance of Serre derivatives in homotopy-theoretic context.

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