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THE TOPOLOGICAL q-EXPANSION PRINCIPLE†

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Imitating the classical q-expansion principle we use the elliptic character map to develop the relation between elements in elliptic cohomology and their q-series in K-theory. We show that, under certain exactness conditions, the integrality of elliptic objects is completely controlled by their characters.

As an application, we obtain an interpretation of the cooperations in elliptic cohomology as was conjectured by Clarke and Johnson [11]. It enables us to give a description of the elliptic-based Adams–Novikov spectral sequence in terms of cyclic cohomology of modular forms in several variables, and to set up a higher e-invariant with values in Katz's [23] ring of divided congruences.

We show how the topological q-expansion principle can be used to equip elliptic cohomology with orientations which obey various Riemann–Roch formulas.

INTRODUCTION

The most fundamental result about the relation between modular forms and their q-expansions is known as the q-expansion principle. It captures the fact that the ring over which a modular form is defined is determined by its q-expansion coefficients. Modular forms have entered algebraic topology during the past decade by the construction of elliptic cohomology [15, 32, 41], a complex oriented generalized cohomology theory attached to the universal elliptic curve defined over the ring of modular forms. The point of this work is to carry over the q-expansion principle into the topological framework and to develop the relation between elliptic objects and their q-series in K-theory.

An algebraic technique due to Landweber [29] ensures the existence of a cohomology theory corresponding to any elliptic curve which satisfies certain exactness criteria. For the universal elliptic curves over the ring of modular forms these conditions are most easily verified (Theorem 1.4) by applying the classical q-expansion principle. We set up a character map for these theories using the method of Miller [35]. The topological q-expansion principle (Theorem 1.6) then gives necessary and sufficient conditions on the elliptic homology and K-theory of a space X for the integrality of a homology class to be controlled by its character. Its cohomological version (Theorem 1.12) provides a description of elements in elliptic cohomology in terms of power series in virtual bundles over X which rationally behave in a modular fashion. The assumptions are satisfied for all Landweber exact theories, many classifying spaces and Thom spectra.

As a first application we identify the K-theory of elliptic cohomology (Theorem 2.7) with Katz's universal ring of divided congruences [23, 24, 26]. It allows us to transform topological results into algebraic geometrical ones and vice versa (Corollary 2.8) as was already indicated by Clarke and Johnson [11]. We then prove their conjecture [11] on the structure of the cooperations in elliptic cohomology (Theorem 2.10). It can be viewed as the elliptic

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equivalent to a well known result of Adams [5] on the structure of the ring K_*K. More generally, we show that the homotopy ring of an n-fold smash product of the elliptic spectrum can be interpreted as the ring of integral modular forms (Corollary 2.12) in n variables and that it admits a multivariate q-expansion principle.

Our results enable us to compute the one-line of the elliptic based Adams–Novikov spectral sequence (Theorem 3.6). Moreover, we establish a symmetry relation (Proposition 3.10) to interpret the rich mathematical structure of the two-line as first cyclic cohomology of modular forms in several variables with coefficients in $\mathbb{Q}/\mathbb{Z}$ (Theorem 3.11). An embedding (Theorem 3.9) in a version of Katz's ring of divided congruences leads to the definition of the f-invariant, the higher relative of the classical e-invariant. It associates to each even-dimensional homotopy class s between spheres an inhomogeneous sum of rational modular forms. In the presence of a certain congruence the f-invariant vanishes and s lies in third Adams–Novikov filtration. Its relation to index theorems and to the η-invariant still represents work in progress and will be given somewhere else.

Finally, we show how the topological q-expansion principle can be used to equip elliptic cohomology with orientations which obey various Riemann–Roch formulas (Proposition 4.3). For instance, the original Landweber–Ravenel–Stong elliptic cohomology admits an orientation which leads to the Witten genus [48] of *Spin* manifolds with vanishing $p_1/2$.

1. THE TOPOLOGICAL q-EXPANSION PRINCIPLE

1.1. Rational faithfulness

This section is meant to provide the algebraic context in which the q-expansion principle is going to be developed. We start by looking at maps of abelian groups whose rational behaviour characterizes their sources. We then ask under which algebraic operations this property remains stable.

Definition 1.1. A morphism of abelian groups $f\colon X \to Y$ is rationally faithful if it satisfies one and hence all of the following equivalent conditions:

(i) the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ X \otimes \mathbb{Q} \arrow[r, "f\otimes\mathbb{Q}"] & Y \otimes \mathbb{Q} \end{tikzcd}$$

is a cartesian square.

(ii) ker f is a rational vector space and coker f is torsionfree.

(iii) tors $X \to$ tors Y is iso and cotors $X \to$cotors Y is mono, where the 'cotorsion' cotors X of an abelian group X is $X \otimes \mathbb{Q}/\mathbb{Z}$.

(iv) the sequence

$$0 \longrightarrow X \xrightarrow{(1,f)} (X \otimes \mathbb{Q}) \oplus Y \xrightarrow{(f\otimes\mathbb{Q}\,-\,1)} Y \otimes \mathbb{Q}$$

is exact.

There are two faces to the source of a rationally faithful map f. If X is torsionfree, the elements of X are precisely the elements of its rationalization which lift under $f\otimes\mathbb{Q}$ to Y. On the other hand if $f\otimes\mathbb{Q}$ is injective, we can describe the elements of X as the elements of Y which rationally lift to $X\otimes\mathbb{Q}$.

Example 1.2. Let N be a positive integer and R be a torsionfree ring which contains $1/N$ and a primitive Nth root of unity ζ_N. We denote by $M_*^{\Gamma_1(N)}(R)$ the graded ring of modular forms over R for the congruence subgroup

$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \operatorname{mod} N \right\} \subset Sl_2\mathbb{Z}.$$

Let

$$\lambda_{2k}^{\Gamma_1(N)} : M_{2k}^{\Gamma_1(N)}(R) \to \mathbb{Z}((q)) \otimes R$$

$$f \mapsto f(q) = f(Tate(q^N)_{/Z((q))\otimes R}, \omega_{can}, \zeta_N^i q^j)$$

be the q-expansion ring homomorphism for some choice of N-division point $\zeta_N^i q^j$ $0 \leqslant i, j \leqslant n-1$ and some weight $k \in \mathbb{N}$ (cf. Definition 7). Then the classical q-expansion principle (cf. Theorems 3 and 8) can be stated by saying $\lambda_*^{\Gamma_1(N)}$ is rationally faithful on its homogeneous components.

Lemma 1.3 (i) *Every isomorphism of abelian groups is rationally faithful.*

(ii) *The composite of rationally faithful homomorphisms is rationally faithful.*

(iii) *If the composite* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is rationally faithful and* g *is mono or rationally faithful then* f *is rationally faithful.*

(iv) *Let* $\{f_\alpha : X_\alpha \to Y_\alpha\}$ *be an inverse system of rationally faithful morphisms such that the sources* X_α *are torsionfree. Then* $\lim X_\alpha \to \lim Y_\alpha$ *is rationally faithful.*

(v) *Let* $\{f_\alpha : X_\alpha \to Y_\alpha\}$ *be a direct system of rationally faithful morphisms. Then* $\operatorname{colim} X_\alpha \to \operatorname{colim} Y_\alpha$ *is rationally faithful. In particular, let* $M \to N$ *be a rationally faithful* R*-module homomorphism between* R*-modules* M, N *and* S *be a multiplicative set in* R*. Then* $M[S^{-1}] \to N[S^{-1}]$ *is rationally faithful.*

(vi) *Let* X *be a flat module over some ring* R *and* $f : M \to N$ *a rationally faithful* R*-module map. Then* $f \otimes_R X : M \otimes_R X \to N \otimes_R X$ *is rationally faithful.*

(vii) *Let* M *be a module over a Noetherian ring* R *such that* M *is torsionfree as an abelian group. Then the completion map*

$$c : M \otimes_R R[\![q]\!] \to M[\![q]\!]$$

is mono and rationally faithful.

Proof. The first three claims are immediate from well known properties of cartesian squares. To prove (iv) we use the second characterization of rational faithfulness in Definition 1.1. Sources X_α and targets Y_α are torsionfree. Hence, it is enough to show that the map induced in cortorsion is mono. In the diagram

$$\begin{array}{ccc} \operatorname{cotors}(\lim X_\alpha) & \longrightarrow & \operatorname{cotors}(\lim Y_\alpha) \\ \downarrow & & \downarrow \\ \lim(\operatorname{cotors} X_\alpha) & \longrightarrow & \lim(\operatorname{cotors} Y_\alpha) \end{array}$$

the bottom map is a monomorphism. If we know that the left vertical arrow is another monomorphism, we are done. Let $(x_\alpha) \otimes 1/N \in (\lim X_\alpha) \otimes \mathbb{Q}$ represent an arbitrary element in its kernel. Then $x_\alpha \otimes 1/N = x'_\alpha \otimes 1$ for unique elements $x'_\alpha \in X_\alpha$. The sequence $(x_\alpha{}')$ defines an element in $(\lim X_\alpha)$ which agrees with $(x_\alpha) \otimes 1/N$ in $(\lim X_\alpha) \otimes \mathbb{Q}$.

(v) is immediate from Definition 1.1 (iv) because exactness is preserved by direct limits. The statement (vi) follows trivially from Definition 1.1. (iv) since

$$(M \otimes \mathbb{Q}) \otimes_R X \cong (M \otimes_R X) \otimes \mathbb{Q}.$$

(vii) We first show the injectivity of the completion map for any (not necessarily torsionfree) module N. Without loss of generality we may assume that N is finitely generated since every module is the direct limit of its finitely generated submodules. Now the Artin–Rees lemma applies [7] and states that the completion map for N is iso. It follows that the completion map for M is monic. Also, for every prime p the completion map for M/p is monic. It can be factorized in the form

$$M/p \otimes_R R[\![q]\!] = (M \otimes_R R[\![q]\!])/p \xrightarrow{(c/p)} (M[\![q]\!])/p \to (M/p)[\![q]\!]$$

and so c/p is monic. Thus, $\mathrm{Tor}(\mathit{coker}\ c, \mathbb{Z}/p)$ vanishes in the long exact sequence induced by $Tor(\cdot, \mathbb{Z}/p)$. That is, coker c is torsionfree and the assertion follows from Definition 1.1. (ii). $\square$

1.2. The elliptic character in homology

We are now able to give more examples of rationally faithful homomorphisms. The universal elliptic curves give rise to formal groups over the ring of modular forms

$$M_*^{\Gamma_1(1)} \underset{def}{=} M_*^{\Gamma_1(1)}(\mathbb{Z}[1/6])$$

$$M_*^{\Gamma_1(N)} \underset{def}{=} M_*^{\Gamma_1(N)}(\mathbb{Z}[1/N, \zeta_N]) \quad \text{for } N \geqslant 2.$$

In Appendix A we have chosen parameters such that the exponentials f in these charts take the explicit form

$$f(z) = \begin{cases} -2\wp(z,\tau)/\wp'(z,\tau) & \text{for } N = 1 \\ \dfrac{\Phi(z,\tau)\,\Phi(-2\pi i/N,\tau)}{\Phi(z-(2\pi i/N),\tau)} & \text{for } N \geqslant 2. \end{cases}$$

Formal group laws are classified by Lazard's universal ring. In algebraic topology this ring is presented by the complex cobordism ring Ω_*^U due to the work of Quillen [6, 39]. Hence, each formal group law generates a genus, so there arises a ring homomorphism

$$MU_* = \Omega_*^U \to M_*^{\Gamma_1(N)}.$$

The genera induced by the exponentials f are known as Hirzebruch genera for levels $N \geqslant 2$ and coincide with the Landweber–Ravenel–Stong genus for $N = 2$ (cf. [18]).

Theorem 1.4. (Franke [15], Landweber *et al.* [32]). *There are complex oriented ring spectra $E^{\Gamma_1(N)}$ s.t.*

$$E_*^{\Gamma_1(N)}X \cong M_*^{\Gamma_1(N)} \otimes_{MU_*} MU_*X$$

are natural isomorphisms. $E^{\Gamma_1(N)}$ is unique up to unique isomorphisms in the stable homotopy category.

Level 2 elliptic cohomology was originally defined in [30, 32]. The level 1 case is treated in [9] and an alternative proof was given in [20]. See also [15] for the existence and uniqueness statement of the corresponding ring spectra.

The existence of elliptic cohomology for higher levels can be shown as in [15] with the help of results from algebraic geometry. We are able to give an elementary proof below which only uses the q-expansion principle. In fact, it also shows that there is an elliptic cohomology theory associated to any congruence subgroup $\Gamma \subset Sl_2\mathbb{Z}$ for which a q-expansion principle is available.

Recall that the coefficient ring of complex K-theory is the ring $K_* = \mathbb{Z}[v^{\pm 1}]$ of finite Laurent series in the Bott class $v \in KS^2$. It is convenient to define

$$K_*^{\Gamma_1(N)} \underset{def}{=} \begin{cases} K_*[1/6] & N = 1 \\ K_*[\zeta_N, 1/N] & N \geqslant 2. \end{cases}$$

Then for any choice of cusp and $N \geqslant 2$, the q-expansion induces a map

$$E_{2k}^{\Gamma_1(N)} \cong M_{2k}^{\Gamma_1(N)} \to \mathbb{Z}((q)) \otimes \mathbb{Z}[1/N, \zeta_{\mathrm{N}}] \subset \mathbb{Z}[1/N, \zeta_{\mathrm{N}}]((q)) \cong K_{2k}^{\Gamma_1(N)}((q))$$

which we simply call $\lambda_{2k}^{\Gamma_1(N)}$ again. The case $N = 1$ is treated similarly. That is, we use the coefficient ring of K-theory to keep track of the grading. In the sequel it will not be necessary to refer to the level again and we let Γ be any of the congruence subgroups $\Gamma_1(N)$.

Lemma 1.5. *$\lambda_*^\Gamma : E_*^\Gamma \to K_*^\Gamma((q))$ is rationally faithful.*

Proof. λ_*^Γ is the direct sum of rationally faithful maps. □

Proof of Theorem 1.4. We have to verify the Landweber exactness criterion [29] for the MU_*-modules M_*^Γ. Let p be a prime and u_k the coefficient of x^{p^k} in the $[p]$-series of the formal group law. Then the regularity of sequence $(p, u_1, u_2, \dots)$ is to be shown. First $u_0 = p$ is not a zero divisor since M_*^Γ is torsion free. Next, we claim that multiplication by u_1 is injective modulo p. For that we use the injectivity of the q-expansion map modulo p

$$E_*^\Gamma/p \hookrightarrow K_*^\Gamma((q))/p \hookrightarrow (K_*^\Gamma/p)((q))$$

which is an immediate consequence of the lemma as the cokernel of λ_*^Γ is torsionfree. Hence, it is enough to verify that u_1 does not vanish identically modulo p. In Corollary A.11 we provide an explicit strict isomorphism θ between the formal group law $\lambda_*^\Gamma \hat{F}_\Gamma$ of the universal elliptic curve viewed over the ring of power series and the multiplicative formal group law $\hat{G}_m$. Let u'_k denote the corresponding classes for $\hat{G}_m$. Then we compute modulo p

$$\theta\left(\sum_k \lambda_*^\Gamma(u_k)x^{p^k}\right) \equiv \theta([p]_{\lambda_*^\Gamma \hat{F}_\Gamma}(x)) \equiv [p]_{\hat{G}_m}(\theta(x)) \equiv \sum_k u'_k(\theta(x))^{p^k}$$

and a comparison between coefficients gives

$$\lambda_*^\Gamma(u_1) \equiv u'_1 = v^{p-1}.$$

In particular, u_1 is even invertible over the ring of power series. Finally, it is sufficient to verify that u_2 is invertible modulo (p, u_1). For that we may use the well known proof by contradiction due to J. Franke: Assume u_2 is not invertible modulo (p, u_1) and let m be a maximal ideal containing p, u_1 and u_2. Then the elliptic curve over the residue field $k = M_*^\Gamma/m$ has height greater than two which is impossible (cf. [45]). □

Miller [35] has shown how to make the q-expansion into a map of ring spectra and we recall his work here. Let us be given graded formal group laws F over a ring R and F' over S, a ring homomorphism $\lambda : R \to S$ and a strict isomorphism $\theta : F' \to \lambda F$. Assume further that $(R, F), (S, F')$ satisfy the Landweber exactness conditions. Then there is a natural transformation λ_* s.t. on $X = *$ we have $\lambda_* = \lambda$ and, if L is a line bundle over X and e_R, e_S are the corresponding Euler classes of L, then $\lambda_*(e_R) = \theta(e_S)$. λ is not a map of MU_*-modules. However, we may split λ_* into two parts

$$R \otimes_F MU_*X \xrightarrow{\lambda\otimes 1} S \otimes_{\lambda F} MU_*X \xrightarrow{(id,\theta_*)} S \otimes_{F'} MU_*X$$

in which the first one is. The last two objects are entirely the same theory but with different orientations. The isomorphism θ guarantees that λF is a Landweber formal group. In the case of our ring homomorphims λ^Γ we give $K^\Gamma((q))$ the multiplicative (Todd) orientation, which is isomorphic to the Tate orientation by Corollary A.11. Note that $K^\Gamma_*(X)((q))$ is not represented by a spectrum. However, there is a Landweber theory which agrees with $K^\Gamma_*(X)((q))$ for finite spectra X and serves as intermediary in

$$E^\Gamma_*(X) \xrightarrow{\lambda^\Gamma_*\otimes 1} K^\Gamma_*((q)) \otimes_{K^\Gamma_*} K^\Gamma_*(X) \xrightarrow{c} K^\Gamma_*(X)((q))$$

for arbitrary X. (It is a close relative of the function spectrum $F(BS^1, K^\Gamma) = K^\Gamma_{S^1}$.) The completed λ^Γ_* is the elliptic character. At the low risk of confusion we keep the old notation λ^Γ_*.

We come now to the main result of this section, which we call the topological q-expansion principle. It gives an equivalence between the Landweber exactness criteria and the q-expansion principle.

THEOREM 1.6. *Let X be a spectrum. The elliptic character*

$$\lambda^\Gamma_* : E^\Gamma_* X \to K^\Gamma_* X((q))$$

is rationally faithful if the following two conditions are satisfied:

(i) *$E^\Gamma_* X$ and $K^\Gamma_* X$ are torsionfree,*
(ii) *for each prime p the multiplication by the Hasse invariant u_1 on $E^\Gamma_* X/p$ is monic.*

Moreover, if X has the weak homotopy type of a countable CW-spectrum then the converse statement holds.

Proof. Let the two conditions be satisfied. Then the map

$$\lambda^\Gamma_* \otimes 1 : E^\Gamma_* X \cong E^\Gamma_* \otimes_{E^\Gamma_*} E^\Gamma_* X \to K^\Gamma_*((q)) \otimes_{E^\Gamma_*} E^\Gamma_* X$$

is an inclusion by assumption (i) since it is rationally monic. Moreover, it remains injective modulo p. To see this, note that by assumption (ii) we may as well replace $E^\Gamma_* X$ by

$$u_1^{-1} E^\Gamma_* X \underset{def}{=} \operatorname{colim}(E^\Gamma_* X \xrightarrow{u_1} E^\Gamma_* X \xrightarrow{u_1} \cdots)$$

and show that for any finite subspectrum $Y \subset X$ the map

$$u_1^{-1}(\lambda^\Gamma_* \otimes 1)/p : u_1^{-1} E^\Gamma_* Y/p \to K^\Gamma_*((q)) \otimes_{E^\Gamma_*} E^\Gamma_* Y/p$$

is an inclusion. Furthermore, if we write $M(p)$ for the mod p Moore space and put $Z = Y \wedge M(p)$, then by the universal coefficient theorem it suffices to verify the injectivity of

$$u_1^{-1}(\lambda^\Gamma_* \otimes 1) : u_1^{-1} E^\Gamma_* Z \to K^\Gamma_*((q)) \otimes_{E^\Gamma_*} E^\Gamma_* Z.$$

A theorem of Landweber [31] says that MU_*Z is a finitely presented MU_*MU-comodule and admits a filtration of MU_*MU-subcomodules

$$MU_*Z = F_0 \supset F_1 \cdots \supset F_s = 0$$

so that for $0 \leqslant i < s$, F_i/F_{i+1} is isomorphic to the comodule

$$MU_*/(u_0, u_1, \cdots, u_n)$$

for some n which will in general depend on i. Tensoring the filtration with the Landweber exact MU_*-modules $u_1^{-1}E_*^\Gamma$ and $K_*^\Gamma((q)) \otimes_{E_*^\Gamma} E_*^\Gamma$ gives a filtration of source and target of $u_1^{-1}(\lambda_*^\Gamma \otimes 1)$. Moreover, the dummy version of the elliptic character $u_1^{-1}(\lambda_*^\Gamma \otimes 1)$ respects this filtration as it is a map of left MU_*-modules. Hence, without loss of generality we may assume $E_*^\Gamma Z$ to be of the form $E_*^\Gamma/(u_0, u_1, \ldots, u_n)$ for some n. Then in the only non-trivial cases $n = -1$ and $n = 0$ the map in question is an inclusion as a consequence of the q-expansion principle Lemma 1.5.

We now conclude that

$$\begin{aligned}\lambda_*^\Gamma : E_*^\Gamma X \to K_*^\Gamma((q)) \otimes_{E_*^\Gamma} E_*^\Gamma X &\cong K_*^\Gamma((q)) \otimes_{E_*^\Gamma} E_*^\Gamma \otimes_{MU_*} MU_* X \\ &\cong K_*^\Gamma((q)) \otimes_{Tate} MU_* X \overset{(id,\theta)}{\cong} K_*^\Gamma((q)) \otimes_{Todd} MU_* X \\ &\cong K_*^\Gamma((q)) \otimes_{K_*^\Gamma} K_*^\Gamma(X).\end{aligned}$$

is rationally faithful as follows: In the long exact sequence induced by $\mathrm{Tor}_*(\lambda_*^\Gamma, \mathbb{Z}/p)$ we have

$$\mathrm{Tor}_1(\mathrm{coker}\ \lambda_*^\Gamma, \mathbb{Z}/p) = 0$$

since λ_*^Γ/p is monic and $K_*^\Gamma((q)) \otimes_{K_*^\Gamma} K_*^\Gamma(X)$ is torsionfree. Thus, coker λ_*^Γ is torsionfree and Definition 1.1 (iii) applies.

Finally, the composite with the completion

$$K_*^\Gamma((q)) \otimes_{K_*^\Gamma} K_*^\Gamma(X) \xrightarrow{c} K_*^\Gamma(X)((q))$$

gives the elliptic character. The assertion now follows from Lemma 1.3 (vii) after a localization at q since K_*^Γ is Noetherian and $K_*^\Gamma X$ is torsionfree.

To show the converse, let X be of the weak homotopy type of a countable CW-spectrum and λ_*^Γ be rationally faithful. Then the induced map in torsion is an isomorphism. Hence, condition (i) is a consequence of the fact that *tors* $E_*^\Gamma X$ is countable (cf. [13, section VII 3]) whereas each torsion element in $K_*^\Gamma X$ generates uncountably many torsion series in $K_*^\Gamma X((q))$. Furthermore, λ_*^Γ is monic modulo p as its cokernel is torsionfree. Hence, mod p multiplication by u_1 is monic as it is invertible in the ring of power series $(K_*^\Gamma X/p)((q))$.

□

Corollary 1.7. *The elliptic character*

$$\lambda_*^\Gamma : E_*^\Gamma X \to K_*^\Gamma X((q))$$

is rationally faithful if $E_*^\Gamma X$ *is a flat* E_*^Γ*-module.*

Proof. Multiplication by p on E_*^Γ and $K_*^\Gamma((q))$ and by $u_1 \bmod p$ on E_*^Γ are monic and remain so after tensoring with the flat module $E_*^\Gamma X$ over E_*^Γ. Since K^Γ is a direct summand in $K^\Gamma((q))$ we also verified the absence of torsion in $K_*^\Gamma X$ so that 1.6 applies. □

Corollary 1.8. *The elliptic character* $\lambda_*^\Gamma : E_*^\Gamma F \to K_*^\Gamma F((q))$ *is rationally faithful for any Landweber theory* F.

This is a consequence of

Lemma 1.9. *Let* E and F *be Landweber exact theories. Then* E_*F *is a flat* E_**-module.*

Proof. The lemma is well known but for the reader's convenience we repeat the argument. Let $M \to N \to O$ be an exact sequence of E_*-modules and hence of MU_*-modules. Then tensoring with the flat MU_*-module MU_*MU yields an exact sequence

$$M \otimes_{MU_*} MU_*MU \to N \otimes_{MU_*} MU_*MU \to O \otimes_{MU_*} MU_*MU$$

of MU_*MU-comodules. The comodule structure of each object comes from MU_*MU. If we tensor the exact sequence with F_* from the right it still remains exact since

$$\mathrm{Tor}_1^{MU_*MU}(F_*, O \otimes_{MU_*} MU_*MU) = 0.$$

To see the vanishing of Tor, one writes $O \otimes_{MU_*} MU_*MU$ as direct limit of finitely presented comodules as in [36, 2.1.1] and uses the Landweber exactness of F_*. Finally, $M \otimes_{MU_*} MU_*F \cong M \otimes_{E_*} E_*F$ for any E_*-module M. □

There are interesting universal examples which are not Landweber theories. Let $M\,String = MO\,\langle 8 \rangle$ be the Thom spectrum of the 7-connected cover of BO. (The word "String" in this context is due to H. Miller and will be explained later.)

Corollary 1.10. *The elliptic character*

$$\lambda_*^\Gamma : E_*^\Gamma\, M\,String[1/2] \to K_*^\Gamma\, M\,String[1/2]((q))$$

is rationally faithful.

Proof. Let $p \geqslant 5$ be a prime. Ravenel and Hovey [21] have shown that the reduced powers in the Steenrod algebra act freely on the Thom class in the mod p cohomology of $MString$. This means $MString$ is a wedge of BP's when localized at p. In particular $E_*^\Gamma M\,String_{(p)}$ is a flat E_*^Γ module and Corollary 1.7 applies.

3-locally $M\,String \wedge X$ splits into a wedge of BP's where X is a finite spectrum with cells in dimension 0,4 and 8 [21]. It suffices to prove that $E_*^\Gamma X$ is a free E_*^Γ-module, for then $E_*^\Gamma(MString \wedge X)$ is just $E_*^\Gamma MString \otimes_{E_*^\Gamma} \mathrm{E}_*^\Gamma X$, and hence $E_*^\Gamma MString$ is flat. Let Y be the bottom two cells. Then we have a long exact sequence

$$\cdots \to E_*^\Gamma S^3 \to E_*^\Gamma S^0 \to E_*^\Gamma Y \to E_*^\Gamma S^4 \to E_*^\Gamma S^1 \to \cdots$$

and $E_*^\Gamma S^0$ is evenly graded. It follows that the sequence is actually short exact, and then it has to split since $E_*^\Gamma S^3$ is free. Similarly, we have a long exact sequence

$$\cdots \to E_*^\Gamma S^7 \to E_*^\Gamma Y \to E_*^\Gamma X \to E_*^\Gamma S^8 \to E_*^\Gamma \Sigma Y \to \cdots$$

and the same argument applies. So $E_*^\Gamma X$ is free, and we are done. □

1.3. The elliptic character in cohomology

We now turn to the picture in cohomology. For finite spectra the topological q-expansion principle gives criteria for the rational faithfulness of the elliptic character by applying Spanier–Whitehead duality. However, we are mainly interested in infinite spectra.

Definition 1.11. Let F, G be contravariant functors from a category D to abelian groups. A natural transformation $\lambda: F \to G$ is pro-rationally faithful if

$$\begin{tikzcd} \lim_{X \in D} F(X) \arrow[r] \arrow[d] & \lim_{X \in D} G(X) \arrow[d] \\ \lim_{X \in D}(F(X) \otimes \mathbb{Q}) \arrow[r] & \lim_{X \in D}(G(X) \otimes \mathbb{Q}) \end{tikzcd}$$

is a cartesian square.

Let X, F, G be spectra and $\lambda: F \to G$ be a map. We say $\lambda^*: F^*X \to G^*X$ is pro-rationally faithful if it is so as natural transformation on the category of finite subspectra of X.

THEOREM 1.12. *The elliptic character* $\lambda_\Gamma^*: E_\Gamma^* X \to K_\Gamma^* X((q))$ *is pro-rationally faithful if* $E_*^\Gamma X_{(p)}$ *is a projective* E_*^Γ*-module for each rime* p.

COROLLARY 1.13. *The elliptic character*

$$\lambda_\Gamma^*: E_\Gamma^*\, M\,String[1/2] \to \mathrm{K}_\Gamma^*\, M\,String[1/2]((q))$$

is pro-rationally faithful.

We prepare the proof of Theorem 1.12 with three lemmas.

LEMMA 1.14. *The p-local elliptic character*

$$\lambda_{\Gamma(p)}^*: E_{\Gamma(p)}^* X \to K_{\Gamma(p)}^* X((q))$$

is pro-rationally faithful if $E_{(p)*}^\Gamma X$ *is a projective* $E_{(p)*}^\Gamma$*-module.*

Proof. Let E be an evenly graded Landweber exact theory such that E_*X is a projective E_*-module. Then the universal coefficient spectral sequence [4, 15]

$$E_2^{p,q} = \mathrm{Ext}_{MU_*}^{p,q}(MU_*X, E_*) \Rightarrow E^{p+q}X$$

collapses and the edge homomorphism $E^*X \to Hom_{MU_*}(MU_*X, E_*)$ is an isomorphism. In particular, $\lim^1$ vanishes since it is the kernel of

$$E^*X \to \lim_{Y \subset X} E^*Y \to \lim_{Y \subset X} \mathrm{Hom}_{MU_*}(MU_*Y, E_*) \cong \mathrm{Hom}_{MU_*}(MU_*X, E_*).$$

We may take E to be $E_{(p)}^\Gamma$, $K_{(p)}^\Gamma((q))$ or their rationalizations as with $E_*^\Gamma X_{(p)}$ also

$$K_*^\Gamma((q)) \otimes_{E_*^\Gamma} E_*^\Gamma X_{(p)} \cong K^\Gamma((q))_* X_{(p)}$$

are projective. Hence, when evaluating the left exact functor $\mathrm{Hom}_{MU_*}(MU_*X, \_)$ on the rationally faithful λ^Γ of 1.5, we obtain the cartesian square

$$\begin{tikzcd} \mathrm{Hom}_{MU_*}(MU_*X, E_{*(p)}^\Gamma) \arrow[r] \arrow[d] & \mathrm{Hom}_{MU_*}(MU_*X, K_{*(p)}^\Gamma((q))) \arrow[d] \\ \mathrm{Hom}_{MU_*}(MU_*X, E_*^\Gamma \otimes \mathbb{Q}) \arrow[r] & \mathrm{Hom}_{MU_*}(MU_*X, K_{*(p)}^\Gamma((q)) \otimes \mathbb{Q}) \end{tikzcd}$$

in which each corner may be replaced by $\lim_{Y \subset X} E^*Y$. That is, there is a pro-rationally faithful map. However, it is not yet the elliptic character. To finish the proof we have to compose it with the natural automorphism on $\lim_{Y \subset X} K^*_\Gamma Y_{(p)}((q))$ induced by (id, θ) for each $K^*_\Gamma Y((q)) \cong K^\Gamma_* DY((q))$ as we did in Theorem 1.6. (Here DY denotes the Spanier-Whitehead dual of Y.) □

LEMMA 1.15. *Let A be an abelian group and $P(A)$ the set of all primes p s.t. there is an element of order p in A. Assume $P(A)$ is finite. Then the diagonal map*

$$A \xrightarrow{\Delta} \prod_p A_{(p)}$$

is rationally faithful. Moreover, the completion

$$\left(\prod_p A_{(p)}\right) \otimes \mathbb{Q} \to \prod_p (A \otimes \mathbb{Q})$$

is mono.

Proof. We use the third characterization of rational faithfulness in Definition 1.1. There is a splitting

$$\prod_p A_{(p)} = \bigoplus_{p \in P(A)} A_{(p)} \oplus \prod_{p \notin P(A)} A_{(p)} \qquad (*)$$

in which the second summand is a torsion free group. Now it is clear that the torsion parts map isomorphically. It remains to show the monomorphy in cotorsion. For that compute

$$cotors\, A = A \otimes \mathbb{Q}/\mathbb{Z} = \bigoplus_p A \otimes \mathbb{Z}/p^\infty = \bigoplus_p A_{(p)} \otimes \mathbb{Q}/\mathbb{Z} = \bigoplus_p cotors\, A_{(p)}$$

Thus, the composition of $cotors\, \Delta$ with $cotors \prod A_{(p)} \to \prod cotors\, A_{(p)}$ is mono as it is just the inclusion of the sum in the product of cotorsions. The last statement is easily verified by using $(*)$. □

LEMMA 1.16. *Let M be a finitely generated module over a Noetherian ring R. Then $P(M)$ is finite.*

Proof. Every $p \in P(M)$ lies in some associated prime ideal a_p of M. That is, there exists an element $x_p \in M$ whose annihilator is the prime ideal a_p. Any other prime $q \in P(M)$ is not contained in a_p as x_p has precise order p. We conclude that there are at least as many associated prime ideals in M as primes in $P(M)$. It is a well known fact that the set of associated primes of a finitely generated module over a Noetherian ring is finite (cf. [28, VI 1.4, 4.9, 5.5]). □

Proof Theorem 1.12. From the first lemma we know that

$$\lambda_{\Gamma(p)} : (E^*_\Gamma Y)_{(p)} \to (K^*_\Gamma Y)_{(p)}((q))$$

is pro-rationally faithful on the category of finite subspectra Y of X. Let us now be given a sequence $(z^Y) \in \lim_Y K^*_\Gamma Y((q))$ which rationally lifts to a sequence $(y^Y) \in \lim_Y E^*_\Gamma Y \otimes \mathbb{Q}$. Then for every prime p there is a unique sequence $(x^Y_{(p)}) \in \lim_Y (E^*_\Gamma Y)_{(p)}$ which agrees with (y^Y) rationally and whose character is $(z^Y_{(p)}) \in \lim_Y (K^*_\Gamma Y)_{(p)}((q))$. Thus, if we apply Lemma 1.15 to $(x^Y_{(p)}) \in \Pi_p (E^*_\Gamma Y)_{(p)}$ for each finite complex Y, we find a unique sequence $(x^Y) \in \lim_Y E^*_\Gamma Y$ with the desired properties. This is possible since $E^*_\Gamma Y$ is finitely generated over the Noetherian ring E^*_Γ. □

2. DIVIDED CONGRUENCES AND MULTIPLE EXPANSIONS

2.1. Formal Characters

In the previous sections we understood the elliptic cohomology of various spectra by exploring the properties of the elliptic character map. It is useful to generalize this concept slightly.

Definition 2.1. A multiplicative transformation $\chi: R \to S$ between Landweber exact theories is a formal character if χ is monic in homotopy.

Obviously the elliptic character and the Chern character are formal characters. More examples are given by the following simple

LEMMA 2.2. *Assume the formal groups F over a ring R and F' over S satisfy the Landweber exactness conditions. Then so does F over R_*S and the natural transformation $R \to R \wedge S$ is a formal character.*

Proof. We have to verify the sequence $(u_0, u_1, u_2, \dots)$ to be regular in R_*S. As (R_*, F) suffices the Landweber exactness conditions, u_n is not a zero divisor in $R_*/(u_0, u_1 \dots, u_{n-1})$. Hence, when tensoring with the flat R_*-module R_*S (cf. Lemma 1.9), u_k does not divide zero in $R_*S/(u_0, u_1, \dots, u_{n-1})$ and (R_*S, F) is Landweber exact. In particular, we have that $R \to R_*S \to R_*S \otimes \mathbb{Q} \cong R \otimes S \otimes \mathbb{Q}$ is monic which shows the second claim. □

In case S is rational stable homotopy $S\mathbb{Q}$ and F' is the additive formal group one obtains the Dold character

$$d: R_*X \to (R \wedge S\mathbb{Q})_*X \cong R_* \otimes \pi_*X \otimes \mathbb{Q}.$$

which is an inclusion if X is Landweber exact.

Consequently, if one wishes to determine the structure of the Hopf algebroid R_*R of cooperations, one can do so by embedding R_*R in the trivial Hopf algebroid $R_* \otimes R_* \otimes \mathbb{Q}$ via the Dold character. Before carrying out this program in the case of elliptic cohomology, we compute the ring K_*E^Γ.

2.2 The ring of divided congruences

$K_* \otimes E_*^\Gamma \otimes \mathbb{Q}$ is concentrated in even dimensions and can be identified with the ring of inhomogeneous rational (meromorphic) modular forms Σf_i where f_i has weight i. We omit the redundant powers of the Bott class v to keep the notation easy.

PROPOSITION 2.3. *K_*E^Γ is the ring of sums Σf_i as above which satisfy the following conditions: For each nonzero integers k and for each choice of cusp the sum of q-series $\Sigma k^{-i} f_i(q)$ takes coefficients in $\mathbb{Z}^\Gamma[1/k]$ where $\mathbb{Z}^\Gamma = \pi_0 K^\Gamma$.*

Proof. We first show that the integrality condition is necessary. For that recall the Adams operations ψ^k which were originally constructed as unstable operations in [2]. To obtain a map of spectra one has to introduce coefficients

$$\psi^k: K \to K[1/k].$$

Alternatively, ψ^k may be defined as in Section 1 by the ring homomorphism which maps v to kv and the strict isomorphism

$$\frac{1}{k}[k]: \hat{G}_m \to \psi^k \hat{G}_m.$$

Here $[k]$ denotes the k-series of the multiplicative formal group law $\hat{G}_m$. The Adams operations $\psi^k \wedge 1$ act on $K_0 E^\Gamma[1/k]$ by $\psi^k(\Sigma f_i) = \Sigma k^{-i} f_i$. Hence, if $\mu: K \wedge K \to K$ denotes the multiplication map in K-theory, then $\mu_*((q))\lambda_*^\Gamma(\Sigma k^{-i} f_i)$ is the q-expansion of the whole sum and it takes coefficients in $\mathbb{Z}^\Gamma[1/k]$.

The converse statement is a question about rational faithfulness. By Corollary 1.8 we know that

$$\lambda_0^\Gamma: K_0 E^\Gamma \cong E_0^\Gamma K \to K_0^\Gamma K((q))$$

is rationally faithful. Furthermore, we see from Lemma 2.4 below that

$$\pi_0(\mu(1 \wedge \psi^k))((q)): K_0^\Gamma K((q)) \to \left(\prod_{k \in \mathbb{Z} - 0} \mathbb{Z}^\Gamma\left[\frac{1}{k}\right]\right)((q)) \cong \prod_{k \in \mathbb{Z} - 0} \left(\mathbb{Z}^\Gamma\left[\frac{1}{k}\right]((q))\right)$$

is rationally faithful as well. Now the composite of the two homomorphisms reveals that $K_0 E^\Gamma$ consists of sums Σf_i which satisfy the stated condition. $\square$

LEMMA 2.4. *The map*

$$\pi_0(\mu(\psi^k \wedge 1)): K_0 K^\Gamma \to \prod_{k \in \mathbb{Z} - 0} \left(\mathbb{Z}^\Gamma\left[\frac{1}{k}\right]\right)$$

is rationally faithful.

Proof. Recall the classical result of Adams *et al.* [1, 6] which identifies $K_* K$ with the set of finite Laurent series in $K_* K \otimes \mathbb{Q}$ satisfying

$$f(t, kt) \in \mathbb{Z}[t, t^{-1}, 1/k] \quad \text{for all } k \in \mathbb{Z} - 0.$$

The conditions can be reformulated to the equivalent statement that

$$K_* K \xrightarrow{1 \wedge \psi^k} \prod_{k \in \mathbb{Z} - 0} K_* K\left[\frac{1}{k}\right] \xrightarrow{(\mu_*)} \prod_{k \in \mathbb{Z} - 0} K_*\left[\frac{1}{k}\right]$$

is rationally faithful. Now let Γ be the congruence subgroup $\Gamma_1(N)$ for some $N > 1$. Then

$$K_* K \otimes \mathbb{Z}\left[\zeta_N, \frac{1}{N}\right] \to \left(\prod_{k \in \mathbb{Z} - 0} \pi_0 K\left[\frac{1}{k}\right]\right) \otimes \mathbb{Z}\left[\zeta_N, \frac{1}{N}\right] \cong \left(\prod_{k \in \mathbb{Z} - 0} \pi_0 K\left[\zeta_N, \frac{1}{k}\right]\right) \otimes \mathbb{Z}\left[\frac{1}{N}\right]$$

is rationally faithful by Lemma 1.3 (vi). It remains to show the rational fidelity of the completion

$$\left(\prod_{k \in \mathbb{Z} - 0} \pi_0 K\left[\zeta_N, \frac{1}{k}\right]\right) \otimes \mathbb{Z}\left[\frac{1}{N}\right] \to \prod_{k \in \mathbb{Z} - 0} \left(\mathbb{Z}^\Gamma\left[\frac{1}{k}\right]\right)$$

which is easily verified. The case $N = 1$ is similar. $\square$

Example 2.5. For $n > 1$ and any congruence subgroup Γ define the elements

$$j_{2n} = (B_{2n}/4n)(1 - E_{2n}) \in K_0 E^\Gamma \otimes \mathbb{Q}.$$

The q-expansion of j_{2n} is integral (cf. Eq. (A.1)). In order to check if j_{2n} actually represents an element in K_0E^Γ, it suffices to show the $\mathbb{Z}[1/k]$-integrality of $(B_{2n}/4n)(1-k^{-2n}E_{2n})$ for all non zero integers k. The only term in question is the coefficient of q^0

$$(1-k^{-2n})B_{2n}/4n.$$

Let

$$\alpha^{4n} \in \mathrm{Ext}_K^{1,4n} = \{\alpha \in \pi_{4n}K \otimes \mathbb{Q} : \eta_{\mathrm{L}}(\alpha) - \eta_{\mathrm{R}}(\alpha) \in K_{4n}K\}/\pi_{4n}K$$

be the class with e-invariant $B_{2n}/4n$. Then

$$\begin{aligned}\mu(1 \wedge \psi^k)(\eta_{\mathrm{L}}(\alpha) - \eta_{\mathrm{R}}(\alpha)) &= \mu(\eta_{\mathrm{L}}(v^{-2n}B_{2n}/4n) - \eta_{\mathrm{R}}((kv)^{-2n}B_{2n}/4n)) \\ &= v^{-2n}B_{2n}/4n - (kv)^{-2n}B_{2n}/4n \\ &= v^{-2n}(1-k^{-2n})\, B_{2n}/4n\end{aligned}$$

lies in $\pi_{4n}K[1/k]$ and coincides with the 0-coefficient under the periodicity map

$$\pi_{4n}K \cong \pi_0K \cong \mathbb{Z}.$$

Hence, we have shown $j_{2n} \in K_0E^\Gamma$.

The condition in Proposition 2.3 has a surprising refinement.

LEMMA 2.6. *Let $\sum f_i$ be an inhomogeneous sum of rational modular forms. Then the following statements are equivalent*:

(i) $\sum f_i$ *lies in* K_0E^Γ.
(ii) $\sum f_i(q)$ *takes coefficients in* K_0K^Γ.
(iii) $\sum k^{-i}f_i(q)$ *takes coefficients in* $\mathbb{Z}^\Gamma[1/k]$ *for all nonzero integers* k.
(iv) $\sum h^if_i(q)$ *takes coefficients in* $\mathbb{Z}^\Gamma[1/h]$ *for all nonzero integers* h.
(v) $\sum (h/k)^i f_i(q)$ *takes coefficients in* $\mathbb{Z}^\Gamma[1/hk]$ *for all nonzero integers* h, k.
(vi) $\sum a^if_i(q)$ *takes coefficients in* $\mathbb{Z}^\Gamma_{(p)}$ *for all prime* $p \nmid N$ *and* $a \in \mathbb{Z}^\times_{(p)}$.
(vii) $\sum a^if_i$ *takes coefficients in* $\mathbb{Z}_p^\Gamma$ *for all prime* $p \nmid N$ *and all* $a \in \mathbb{Z}_p^\times$.
(viii) $\sum f_i(q)$ *takes coefficients in* $\mathbb{Z}^\Gamma$.

Proof. The equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) have already been established in Corollary 1.8 and Proposition 2.3. (iv) and (v) are equivalent to the previous by the symmetry $K_0K^\Gamma \cong K_0^\Gamma K$ and Lemma 2.4. (vi) is equivalent to (v) since

$$\bigcap_{p \nmid hkN} \mathbb{Z}^\Gamma{}_{(p)} = \mathbb{Z}^\Gamma\left[\frac{1}{hk}\right].$$

Next, the implication (vii)$\Rightarrow$(vi) follows from $\mathbb{Z}^\times_{(p)} \subset \mathbb{Z}_p^\times$ and $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$. As (viii) is a trivial consequence of (iii), we are left with the statement (viii) $\Rightarrow$ (vii) which is the hard part. Let $F_p[\zeta_N] = F_{p^d}$ be the splitting field of (x^N-1) of degree d (the order of p in $(\mathbb{Z}/N)^\times$). Then $\mathbb{Z}_p^\Gamma$ is the ring of Witt vectors of F_{p^d}. Now the assertion is a consequence of 1.7 in [23]. An alternative proof is provided in the Appendix B.4. □

THEOREM 2.7. *The ring K_0E^Γ coincides with the ring of divided congruences D^Γ of Katz* [23]. *In other words, for any choice of cusp*

$$\lambda_*^\Gamma : K_*E^\Gamma \cong E_*^\Gamma K \xrightarrow{\lambda_*^\Gamma} K_*^\Gamma K((q)) \xrightarrow{\mu_*((q))} K_*^\Gamma((q))$$

is rationally faithful.

Proof. K_0E^Γ is the ring of sums $\sum f_i$ of rational modular forms such that $\sum f_i(q)$ takes coefficients in $\mathbb{Z}^\Gamma$ by Proposition 2.3 and Lemma 2.6. □

The theorem is a q-expansion principle for trivialized modular forms in which form it is due to Katz [23]. Katz identified the ring of divided congruences with the coordinate ring of the moduli space of elliptic curves together with isomorphisms of their formal groups with the multiplicative group. It also can be found in [11]. As a consequence we obtain a proof of a well known result of Serre [11, 42]:

COROLLARY 2.8. *The denominator of $B_{2k}/2k$ is the largest one (away from N) occurring as constant term of any modular form for $\Gamma_1(N)$ of weight $2k$ with q-series in $\mathbb{Q} + \mathbb{Z}^\Gamma((q)) \subset \mathbb{C}((q))$.*

Proof. Let f be a modular form such that

$$f(q) - q^0(f) \in \mathbb{Z}^\Gamma((q))$$

where $q^0(f)$ denotes the 0-coefficient of its q-expansion. Then $f - q^0(f)$ lies in K_0E^Γ. Thus $q^0(f)\,(v^{-2k} - 1)$ represents an element in $\mathrm{Ext}^{1,4n}_{K[1/N]}$ which is well known to be a cyclic group of order equal to the denominator of $B_{2k}/2k$. □

We also get an elliptic form of the Hattori–Stong theorem for free:

COROLLARY 2.9. *The K-Hurewicz map $E^\Gamma_* \to K_*E^\Gamma$ is rationally faithful.*

Proof. Immediate from Lemma 1.3 (vii), Lemma 1.5 and Theorem 2.7. □

2.3. Multivariate modular forms and expansions

We are now prepared to give a description of $E^\Gamma_*E^\Gamma$ in terms of integral modular forms in two variables and a q-expansion principle for such forms.

The theory K^Γ splits into a direct sum of $K[1/N]$-theories. Using Theorem 2.7 we conclude that the elliptic character

$$\tilde{\lambda}^\Gamma_* : K^\Gamma_*E^\Gamma \cong K^\Gamma_* \otimes_{K_*} K_*E^\Gamma \xrightarrow{K^\Gamma_* \otimes \tilde{\lambda}^\Gamma_*} K^\Gamma_* \otimes_{K_*} (K^\Gamma_*((q))) \cong (K^\Gamma_* \otimes_{K_*} K^\Gamma_*)((q))$$

is a rationally faithful map of left K^Γ_*-modules.

Let $\Lambda^n E$ denote a $\wedge$-product of n copies of a theory E. Likewise we use the notation $\otimes^n_R M$ for a product of modules M over a ring R. Then we may consider the $(q_0, q_1, \ldots, q_n)$-expansion

$$\begin{aligned}\pi_*(\Lambda^{n+1}E^\Gamma) \cong E^\Gamma_*(\Lambda^n E^\Gamma) &\xrightarrow{\lambda^\Gamma_*} K^\Gamma_*(\Lambda^n E^\Gamma)((q_0)) \cong \otimes^n_{K^\Gamma_*} K^\Gamma_* E^\Gamma((q_0)) \\ &\otimes\xrightarrow{\tilde{\lambda}^\Gamma_*((q_0))} \Big(\bigotimes\nolimits^{k=1,\ldots,n}_{K^\Gamma_*} (K^\Gamma_* \otimes_{K_*} K^\Gamma_*)((q_k))\Big)((q_0)) \\ &\xrightarrow{c} \Big(\bigotimes\nolimits^{n+1}_{K_*} K^\Gamma_*\Big)((q_0, \ldots, q_n)).\end{aligned}$$

Hence, in even dimension the $(q_0, \ldots, q_n)$-expansion takes coefficients in the tensor product

$$\mathbb{Z}^{(n+1)\Gamma} \underset{def}{=} \bigotimes\nolimits^{n+1} \mathbb{Z}^\Gamma$$

which is simply $\mathbb{Z}[1/2]$ for $\Gamma = \Gamma_1(2)$. For this congruence subgroup and the case $n = 1$ the following multivariate q-expansion principle was conjectured by Clarke and Johnson in [11].

THEOREM 2.10. *The* $(q_0, q_1, \ldots, q_n)$*-expansion is rationally faithful. That is,* $\pi_*(\Lambda^{n+1}E^\Gamma)$ *is given by sums* $\sum f_0 \otimes f_1 \otimes \ldots \otimes f_n$ *of products of rational (meromorphic) modular forms with integral* $(q_0, \ldots, q_n)$*-expansion.*

Proof. The first map λ_*^Γ in the above is rationally faithful by Corollary 1.8. Next, we observe that the tensor product of elliptic characters is the composite of n maps of the form $1 \otimes \cdots \otimes 1 \otimes \tilde{\lambda}_*^\Gamma \otimes 1 \otimes \cdots \otimes 1$ each of which is rationally faithful by Lemma 1.3 (vi) and the discussion above. Hence, the product $\otimes \tilde{\lambda}_*^\Gamma((q_0))$ is so by Lemma 1.3 (iv). Finally, we use Lemma 1.3 (iv), (vii) and an obvious induction to show that the completion map is rationally faithful. □

There is a modular interpretation of the ring $\pi_*\Lambda^n E^\Gamma$.

Definition 2.11. A *test object* in n variables over a ring R is a sequence

$$(E_{0/S}, \omega_0, P_0) \xrightarrow{\varphi_0} (E_{1/S}, \omega_1, P_1) \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \omega_{n-1}, P_{n-1})$$

consisting of

(i) elliptic curves E_i over an $(\otimes^n R)$-algebra S for each $0 \leqslant i < n$
(ii) nowhere vanishing sections ω_i on E_i
(iii) points P_i of exact order N on E_i
(iv) isomorphisms $\varphi_i : \hat{E}_i \to \hat{E}_{i+1}$ of formally completed formal groups s.t. $\varphi_i^* \omega_{i+1} = \omega_i$ when viewed over $\hat{E}_i$.

A modular form for $\Gamma_1(N)$ over R in n variables of weight k is a rule f which assigns to each test object an element

$$f((E_{0/S}, \omega_0, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \omega_{n-1}, P_{n-1})) \in S$$

satisfying the following conditions
(i) f only depends on the S-isomorphism class of the sequence,
(ii) the formation $f((E_{0/S}, \omega_0, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \omega_{n-1}, P_{n-1}))$ commutes with arbitrary base change,
(iii)

$$f((E_{0/S}, \lambda\omega_0, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \lambda\omega_{n-1}, P_{n-1}))$$
$$= \lambda^{-k} f((E_{0/S}, \omega_0, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \omega_{n-1}, P_{n-1}))$$

We denote by $M_k^{n\Gamma_1(N)}(R)$ the $(\otimes^n R)$-algebra of such forms.

The fundamental test object is the sequence of Tate curves

$$(Tate^n(q^N), \omega_{can}, \{P_i\}) \underset{def}{=} (Tate(q_0^N), \omega_{can}, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-1}} (Tate(q_{n-1}^N), \omega_{can}, P_{n-1}) \quad (2.1)$$

over $\mathbb{Z}^{n\Gamma_1(N)}((q_0, \ldots, q_{n-1}))$ with the canonical isomorphisms $\varphi_i : q_i \mapsto q_{i+1}$ between them and any choice of N-division points

$$P_i = \zeta_{N,i}^{k_i} q_i^{l_i} \quad \text{for } 0 \leqslant k_i, l_i < N.$$

Let R be a ring in which N is invertible and which contains a primitive Nth root of unity ζ_N. Then the $(q_0, q_1, \ldots, q_n)$-expansion at the sequence of cusps $\{P_i\}$ is the ring homomorphism

$$M_*^{(n+1)\Gamma}(R) \to Z((q_0, \ldots, q_n)) \otimes \bigotimes^{n+1} R \xrightarrow{c} \left(\bigotimes^{n+1} R\right)((q_0, \ldots, q_n))$$

defined by

$$f \mapsto f(q) = f\left(Tate^n(q^N)_{\mathbb{Z}((q_0, \ldots, q_n)) \otimes (\bigotimes^{n+1} R)}, \{P_i\}\right).$$

We say f is holomorphic if $f(q)$ already q-expands in $(\bigotimes^{n+1} R)[\![q_0, \ldots, q_n]\!]$ and write $\bar{M}_*^{(n+1)\Gamma}(R)$ for the graded ring of such forms.

Corollary 2.12. *There is a canonical isomorphism*

$$\pi_* \Lambda^n E^\Gamma \cong M_*^{n\Gamma}(\mathbb{Z}^\Gamma) \underset{def}{=} M_*^{n\Gamma}$$

and the following q-expansion principle holds: If for some cusp $f \in M_k^{n\Gamma}(\mathbb{Q}[\zeta_N])$ has all its $(q_0, \ldots, q_{n-1})$-expansion coefficients in the ring $\mathbb{Z}^{n\Gamma}$ then it does so at all cusps and there is a unique $\tilde{f} \in M_k^{n\Gamma}$ which gives rise to f by extension of scalars.

Proof. $\pi_*(\Lambda^n E^\Gamma)$ corepresents the functor from graded rings to sets given by the following data

$$R_* \mapsto \left\{ \begin{array}{l} \text{(i) isomorphism classes} (E_{i/R}, \omega_i, P_i) \text{ of } n \text{ elliptic curves } E_i, \\ \text{nowhere vanishing sections } \omega_i \text{ and points } P_i \text{ of order } N. \\ \text{(ii) a sequence of graded isomorphisms } \hat{F}_0 \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-2}} \hat{F}_{n-1} \\ \text{where } \hat{F}_i \text{ is the formal group law of } \hat{E}_i \text{ in the formal parameter} \\ z = z(E_i, \omega) \text{ under which } \omega = dz. \end{array} \right\}$$

This is a consequence of the universal property of $\bigotimes_{MU_*}^{n-1} MU_*MU$ and the following commutative diagram

$$\begin{tikzcd}
\bigotimes^n MU_* \arrow[r] \arrow[d] & \bigotimes^n E_*^\Gamma \arrow[d] \arrow[dr, "\eta"] & \\
\bigotimes_{MU_*}^{n-1} MU_*MU \arrow[r] & \bigotimes_{E_*^\Gamma}^{n-1} E_*^\Gamma E^\Gamma \arrow[r, "\phi"] & R_*
\end{tikzcd}$$

Any ring homomorphism ϕ gives rise to n elliptic curves via η, n graded formal group laws together with a sequence of isomorphisms of graded formal group laws classified by $\bigotimes_{MU_*}^{n-1} MU_*MU$ and vice versa.

Likewise, there is a canonical map

$$\bigotimes^n M_*^\Gamma \to M_*^{n\Gamma}$$

given by

$$\left(\sum f^0 \otimes \cdots \otimes f^n\right)\left((E_{0/S}, \omega_0, P_0) \xrightarrow{\varphi_0} \cdots \xrightarrow{\varphi_{n-2}} (E_{n-1/S}, \omega_{n-1}, P_{n-1})\right)$$
$$= \sum f^0(E_{0/S}, \omega_0, P_0) \cdots f^{n-1}(E_{n-1/S}, \omega_{n-1}, P_{n-1})$$

which induces n elliptic curves and graded formal group laws $\hat{F}_i$ over $M_*^{n\Gamma}$. Choosing the identity isomorphism beween them, we get a graded ring homomorphism

$$\pi_* \Lambda^n E^\Gamma \to M_*^{n\Gamma}.$$

On the other hand, any modular form f in n variables over $\mathbb{Z}^{n\Gamma}$ can be evaluated on the obvious universal test object over $\pi_* \Lambda^n E^\Gamma$. One readily verifies that this gives a well defined inverse.

The q-expansion principle for modular forms in n variables is now an immediate consequence of Theorem 2.1. □

3. MODULAR HOMOTOPY INVARIANTS

3.1. The elliptic based Adams–Novikov spectral sequence

Recall the construction of the Adams–Novikov spectral sequence (ANSS) [4]. Let E be a ring spectrum with unit $\eta_E : S \to E$. Let $\bar{E}$ denote the fibre of η_E and

$$\begin{tikzcd} \bar{E} \arrow[d] & \\ S \arrow[r] & E \arrow[ul, dotted] \end{tikzcd}$$

be the exact triangle. Define spectra $\bar{E}^s = \Lambda^s \bar{E}$ and $E^s = E \wedge \Lambda^s \bar{E}$. By smashing the exact triangle with E^s and applying the functor π_* one obtains an exact couple. The associated filtration of π_*^S is given by

$$\mathscr{F}^s = \operatorname{im}(\pi_* \bar{E}^s \to \pi_* S).$$

Our goal is an intrinsic description of the E_2-term in terms of the coefficients E_* by means of a formal character. Such a description is only known for the character $\eta_{\mathrm{E}} \wedge E$: $E \cong S \wedge E \to E \wedge E$.

LEMMA 3.1. *Let E, F be flat ring theories with torsion free coefficient groups. Let $r : E \to F$ be a map such that $r_*(1) = 1$. Under these assumptions the group of s-cycles Z_*^s satisfies*

$$Z_*^s = \textit{equalizer}\left(\begin{tikzcd}[column sep=small] \pi_* E^s \arrow[rr, "\pi_*(\eta_F \wedge E^s)"] \arrow[dr, "\pi_*(r \wedge \bar{E}^s)"'] & & \pi_*(F \wedge E^s) \\ & \pi_*(F \wedge \bar{E}^s) \arrow[ur, "\pi_*(F \wedge \eta_E \wedge \bar{E}^s)"'] & \end{tikzcd} \right).$$

Proof. r fits into a commutative solid rhombus

$$\begin{tikzcd} & E \arrow[dr, "\eta_F \wedge E"] \arrow[dd, dotted, "r"] & \\ S \arrow[ur, "\eta_E"] \arrow[dr, "\eta_F"'] & & F \wedge E \\ & F \arrow[ur, "F \wedge \eta_E"'] & \end{tikzcd}$$

and makes the left triangle commute. After applying the functor $\pi_*(\_ \wedge \bar{E}^s)$ we have to show that the cycles are precisely the classes in the upper corner $\pi_* E^s$ which make the right triangle commute.

Let us refer to the elements in the image of $\pi_* \bar{E}^s \to \pi_* E^s$ as permanent cycles even though they may be boundaries in the E_2-term. A permanent cycle certainly lies in the equalizer by what we have said so far.

Next, let z be a arbitrary cycle in $\pi_* E^s$. *Then* z lies in the kernel of the differential $\pi_*(d \wedge \bar{E}^s)$ where

$$d: E \to \Sigma\bar{E} \to \Sigma E \wedge \bar{E}.$$

When smashing with F the last map in d becomes an inclusion of a direct summand since the suspension of

$$F \wedge E \wedge \bar{E} \xrightarrow{f\wedge r\wedge\bar{E}} F \wedge F \wedge \bar{E} \xrightarrow{\mu\wedge\bar{E}} F \wedge \bar{E}$$

provides a retraction map. Thus, z generates an element in the kernel of the F-Hurewicz map $\pi_*\Sigma\bar{E}^{s+1} \to F_*\Sigma\bar{E}^{s+1}$. As rationally this map is an inclusion, we conclude that z gives a torsion element in $\pi_*\Sigma\bar{E}^{s+1}$. Hence, by exactness Nz is a permanent cycle for some integer N. As we have seen earlier this implies that $N(\eta_F \wedge E^s - r \wedge \eta_E \wedge \bar{E}^s)_* z$ vanishes in the torsion free group $F_* \wedge E^s$. Consequently, any cycle lies in the equalizer.

Finally, let z lie in the equalizer. Then by exactness, z is annihilated by $\pi_*(\eta_F \wedge d \wedge \bar{E}^s)$. Since the F-Hurewicz map $\pi_*\Sigma E^{s+1} \to F_*\Sigma E^{s+1}$ is injective z has to be a cycle. □

Note that if F is rational stable homotopy $S\mathbb{Q}$ then a natural transformation $r: E \to S\mathbb{Q}$ with $r_*1 = 1$ always exists. It is convenient to think of $\pi_*\Sigma\bar{E} \otimes \mathbb{Q}$ as the quotient $(E_* \otimes \mathbb{Q})/\mathbb{Q}$. More generally, define the tensor algebra

$$T_* = (\textstyle\bigoplus_{k=0}^{\infty} \bigotimes^k E_* \otimes \mathbb{Q})/I$$

of the algebra $E_* \otimes \mathbb{Q}$. Here I is the two sided ideal generated by the relation $[1] = 1$ and $\oplus$ is the direct sum of rational vector spaces. Then $(\pi_*\bar{E}^s) \otimes \mathbb{Q}$ naturally agrees with the degree s part of the graded algebra G_* associated to the filtration $\mathrm{im}(\bigoplus_{n\leqslant k}^{n} \bigotimes^n E_* \otimes \mathbb{Q} \to T_*)$. In detail, we have

$$(\pi_*\Sigma^s\bar{E}^s) \otimes \mathbb{Q} \cong \textstyle\bigotimes^s E_* \otimes \mathbb{Q} / \sum_{i=1}^{s} \bigotimes^{i-1} E_* \otimes \mathbb{Q} \bigotimes^{s-i} E_* \cong G_*^s.$$

Proposition 3.2. *Let E be flat with torsionfree coefficients. Let Z, B be the groups of cycles and boundaries of the AN-E_1 term, respectively. Then there are canonical isomorphisms*

$$Z_*^s \cong \{z \in G_*^s : 1 \otimes z \in \mathrm{im}(E_*\textstyle\bigwedge^s E \to E_* \otimes \bigotimes^s E_* \otimes \mathbb{Q} \to E_* \otimes G_*^s)\}$$

$$B_*^s \cong \mathrm{im}(E_*\textstyle\bigwedge^{s-1} E \to \bigotimes^s E_* \otimes \mathbb{Q} \to G_*^s)$$

induced by

$$\rho \underset{def}{=} \pi_*(r \wedge \Sigma^s\bar{E}^s).$$

Proof. In the commutative diagram

$$\begin{tikzcd} Z_*^s \arrow[r] \arrow[d, "\rho"'] & E_*\Sigma^s\bar{E}^s \arrow[d] & E_*\bigwedge^s E \arrow[l, two heads] \arrow[d] \\ G_*^s \arrow[r] & E_* \otimes G^s & E_* \otimes \bigotimes^s E_* \otimes \mathbb{Q} \arrow[l, two heads] \end{tikzcd}$$

the left square is a pullback of monomorphisms by Lemma 3.1. Hence, a short diagram chase gives the first isomorphism. Similarly, one sees from

$$\begin{tikzcd} E_*\bigwedge^{s-1} E \arrow[r, two heads] \arrow[d] & E_*\Sigma^{s-1}\bar{E}^{s-1} \arrow[r, "\pi_* d"] \arrow[d] & E_*\Sigma^s\bar{E}^s \arrow[d, "\rho"] \\ \bigotimes^s E_* \otimes \mathbb{Q} \arrow[r, two heads] & E_*\Sigma^{s-1}\bar{E}^{s-1} \otimes \mathbb{Q} \arrow[r] & G^s \end{tikzcd}$$

that the subgroups of boundaries correspond. □

In the case of elliptic cohomology we can take $r: E^{\Gamma} \to S\mathbb{Q} \cong H\mathbb{Q}$ to be the 0-coefficient in the elliptic character followed by the 0-dimensional term of the complexified Chern character. Then r is not a ring map, but $r_*(1) = 1$. Each class of degree n of the graded algebra G_* contains an essentially unique representative $f = \sum f^1 \otimes \ldots \otimes f^n$ where none of the rational modular forms f^i is constant.

COROLLARY 3.3. *The n-line of the E^{Γ}-based AN-E_2 term consists of sums*

$$f = \sum f^1 \otimes \cdots \otimes f^n \in G_*^n$$

of products of rational modular forms f^k such that $1 \otimes f \in E_*^{\Gamma} \otimes G_*^n$ admits a representative with integral $(q_0, q_1, \ldots, q_n)$-expansion.

The group of boundaries is given by sums which admit a representative with integral $(q_0, \ldots, q_n)$-expansion.

COROLLARY 3.4. *Let $I = (i_1, i_2 \ldots, i_n)$ be a multiindex with only nonzero entries. Let q^I: $G_*^n \to \mathbb{Q}$ be the map which sends a sum of products of modular forms to the $q^I = q_0^{i_1} \cdots q_n^{\mathrm{in}}$-coefficient of its expansion. Then $q^I(f)$ is integral for every cycle f.*

Proof. $1 \otimes f$ has a representative of the form

$$v = 1 \otimes \sum f^1 \otimes \ldots \otimes f^n + \sum_{i=1}^n \sum g \otimes h^1 \otimes \cdots \otimes h^{i-1} \otimes 1 \otimes h^{i+1} \otimes \ldots \otimes h^n$$

with integral $(q_0. \ldots, q_n)$-expansion. Hence,

$$q^I(f) = q^{(0,I)}(1 \otimes f) = q^{(0,I)}(v)$$

is integral. □

Example 3.5. We can ask whether the divided Eisenstein series

$$\bar{E}_{2k} \underset{def}{=} E_{2k} B_{2k}/4k$$

are nonzero elements in the 1-line. $1 \otimes \bar{E}_{2k} \in E_*^{\Gamma} \otimes G_*^1$ admits the representative $1 \otimes \bar{E}_{2k} - \bar{E}_{2k} \otimes 1$ with integral $(q_{\mathrm{L}}, q_{\mathrm{R}})$-expansion. Hence it lies in the E_2-term. $\bar{E}_{2k}$ is non trivial since the 0-coefficient is not integral. Indeed, for level N modular forms we can define the homomorphism

$$\iota^1 : E_2^{1,4k} \cong Z_{4k}^1/B_{4k}^1 \to \mathbb{Z}^{\Gamma} \otimes \mathbb{Q}/\mathbb{Z}; \quad k > 0$$

sending a form to the 0-coefficient of its q-expansion. ι^1 is well-defined for positive dimensions. Any boundary has integral q-coefficients as it is its only representative in G_{4k}^1. We claim that ι^1 is a monomorphism. Let f be an element in its kernel. Then by the corollary all coefficients of f are integral. That is, f is an integral modular form and thus bounds.

THEOREM 3.6. *The* 1-*line $E_2^{1,4k}$ of the E^{Γ}-based ANSS is the cyclic group of order $m(2k)$ generated by the Eisenstein series $\bar{E}_{2k}$. Here, $m(t)$ is the numerical function with*

$$v_p(m(t)) = \begin{cases} 0 & \text{if} \quad t \not\equiv 0 \mod (p-1) \ \text{or} \ 1/p \in \mathbb{Z}^{\Gamma} \\ 1 + v_p(t) & \text{if} \quad t \equiv 0 \mod (p-1) \ \text{and} \ 1/p \notin \mathbb{Z}^{\Gamma} \end{cases}$$

for all odd primes p and

$$v_2(m(t)) = \begin{cases} 0 & \text{if } 1/2 \in \mathbb{Z}^\Gamma \\ 1 & \text{if } t \not\equiv 0 \mod (p-1) \text{ and } 1/2 \notin \mathbb{Z}^\Gamma \\ 2 + v_p(t) & \text{if } t \equiv 0 \mod (p-1) \text{ and } 1/2 \notin \mathbb{Z}^\Gamma. \end{cases}$$

In this notation $v_p(n)$ is the exponent to which the prime p occurs in the decomposition of n into prime powers, so that

$$n = 2^{v_2(n)} 3^{v_3(n)} 5^{v_5(n)} \cdots .$$

Proof. We have just seen that $\bar{E}_{2k}$ generates a cyclic subgroup of $E_2^{1,4k}$ of order equal to the denominator of $B_{2k}/4k$ which is $m(2k)$ up to invertible elements in $\mathbb{Z}^\Gamma$ (cf. [2]). Now let z be any nonconstant rational meromorphic modular form of weight $2k$. If z is a cycle and is represented by some f, then all non zero q-coefficients are integral by Corollary 3.4. Furthermore, $q^0(f)$ lies in $\mathbb{Q} + \mathbb{Z}^\Gamma \subset \mathbb{C}$ as one easily verifies. It follows with Corollary 2.8 that there is a multiple $a \in \mathbb{Z}[1/N]$ such that $q^0(f) = q^0(a\bar{E}_{2k}) \bmod \mathbb{Z}^\Gamma$. We conclude that $f - a\bar{E}_{2k}$ lies in the kernel of ι^1 and thus bounds. That is, z belongs to the cyclic subgroup of $E_2^{1,4k}$ generated by $\bar{E}_{2k}$. □

3.2. The 2-line and cyclic cohomology

In order to identify the second line of the E^Γ-based ANSS we set up the higher analogue of the monomorphism ι^1. For that a better description of the group of boundaries is necessary.

LEMMA 3.7. *Let z be a 2-cycle of degree $2k$. Then there is a representative $\sum f \otimes g$ which satisfies*

$$q^{(0,0)}\left(\sum f \otimes g\right) \in \mathbb{Q} + \mathbb{Z}^{2\Gamma}$$

Proof. Let $\bar{f}$ denote the modular form over $\mathbb{Q}$ obtained from $f \in M_*^{\Gamma_1(N)}(\mathbb{Q}[\zeta_N])$ by setting $\zeta_N = 1$. Let $\sum a \otimes b$ be any representative of z. We claim that

$$\sum f \otimes g \underset{def}{=} \sum a \otimes b - \bar{b}a \otimes 1 - \sum 1 \otimes \bar{a}b$$

will do the job. Let u, w, s and t be such that

$$\sum 1 \otimes a \otimes b + u \otimes 1 \otimes w + s \otimes t \otimes 1 \equiv 0 \bmod \mathbb{Z}^{3\Gamma}.$$

In particular, we have

$$\sum 1 \otimes a \otimes \bar{b} + u \otimes 1 \otimes \bar{w} + s \otimes t \otimes 1 \equiv 0 \bmod \mathbb{Z}^{3\Gamma}$$

$$\sum 1 \otimes \bar{a} \otimes b + u \otimes 1 \otimes w + s \otimes \bar{t} \otimes 1 \equiv 0 \bmod \mathbb{Z}^{3\Gamma}$$

Next, observe that for any f, g

$$\begin{aligned} \sum q^0(fg) &= \sum \sum_{i \in \mathbb{Z}} q^i(f) q^{-i}(g) = \sum \sum_{i \in \mathbb{Z}} q^{(i,-i)}(f \otimes g) \\ &= \sum_{i \in \mathbb{Z}} q^{(i,-i)}\left(\sum f \otimes g\right) \overset{3.1.4}{\equiv} q^{(0,0)}\left(\sum f \otimes g\right) \\ &= \sum q^{(0,0)}(f \otimes g) = \sum q^0(f) q^0(g) \bmod \mathbb{Z}^{2\Gamma} \end{aligned} \tag{3.1}$$

Hence, we have

$$q^{(0,0,0)}(1\otimes\sum f\otimes g)\equiv\sum 1\otimes(q^0(a)\otimes q^0(b)-q^0(\bar{b})\,q^0(a)\otimes 1-q^0(\bar{a})\,1\otimes q^0(b))$$

$$\equiv\sum q^0(u)\otimes 1\otimes q^0(\bar{w})+q^0(s)\otimes q^0(\bar{t})\otimes 1.$$

Setting $\zeta_N\otimes 1\otimes 1=1$, we conclude

$$q^{(0,0)}(\sum f\otimes g)\equiv(q^0(\bar{u})q^0(\bar{w})+q^0(\bar{s})q^0(\bar{t})\in\mathbb{Q}\bmod\mathbb{Z}^{2\Gamma}. \qquad \square$$

Hence, when representing 2-cycles we may always assume them to be of the above form. Note also, for every cycle $\sum q^0(g)\,f$ expands in $(\mathbb{Q}\otimes\mathbb{Z}^\Gamma)\,((q_{\mathrm{L}},q_{\mathrm{R}}))$ ($\sum q^0(f)g$ in $(\mathbb{Z}^\Gamma\otimes\mathbb{Q})\,((q_{\mathrm{L}},q_{\mathrm{R}}))$ respectively) up to series in $\mathbb{Z}^{2\Gamma}((q_{\mathrm{L}},q_{\mathrm{R}}))$.

Lemma 3.8. *Let $\sum f\otimes g$ represent a 2-cycle of degree $2k$. Then the following statements are equivalent*:

(i) *$\sum f\otimes g$ is a boundary.*

(ii) *$\sum f\otimes g+v\otimes 1+1\otimes w\equiv 0\bmod\mathbb{Z}^{2\Gamma}((q_{\mathrm{L}},q_{\mathrm{R}}))$ for some v, w of weight k.*

(iii) *there is a system of equations*

$$\sum f\otimes q^0(g)+v\otimes 1+1\otimes q^0(w)\equiv 0\bmod\mathbb{Z}^{2\Gamma}((q))$$

$$\sum q^0(f)\otimes g+1\otimes w+q^0(v)\otimes 1\equiv 0\bmod\mathbb{Z}^{2\Gamma}((q))$$

for some v, w of weight k.

(iv) *$\sum q^0(g)f\equiv h\bmod\mathbb{Q}+\mathbb{Z}^\Gamma((q))$ for some h of weight k.*

(v) *$\sum q^0(\bar{f})g\equiv h\bmod\mathbb{Q}+\mathbb{Z}^\Gamma((q))$ for some h of weight k.*

Proof. The equivalence of the first two statements has already been established in Corollary 3.3. The third statement follows trivially from the second by looking only at the left and right series. On the other hand (iii) implies (ii) Corollary 3.4. (iv) and (v) are trivial consequences of (iii) by letting h be $-v$ and $-w$ respectively.

Now let (iv) be satisfied. Using Corollary 3.4 and Eq. (3.1) we know that

$$\sum f\otimes g-\sum q^0(g)(f\otimes 1)-\sum q^0(f)(1\otimes g)+\sum q^0(fg)$$

has integral $(q_{\mathrm{L}},q_{\mathrm{R}})$-expansion. Setting $q_{\mathrm{L}}=q_{\mathrm{R}}=q$ and $\zeta_N\otimes 1=1\otimes\zeta_{\mathrm{N}}$ we conclude

$$\sum fg-\sum q^0(g)f-\sum q^0(f)g+\sum q^0(fg)\equiv 0\bmod\mathbb{Z}^\Gamma((q)). \qquad (3.2)$$

Hence, we define $v=-h$ and $w=h-\sum fg$ and compute

$$\sum f\otimes q^0(g)+v\otimes 1+1\otimes q^0(w)$$

$$\equiv\sum f\otimes q^0(g)-h\otimes 1+1\otimes q^0(h-fg)$$

$$\equiv\sum f\otimes q^0(g)+q^0(h)\otimes 1-q^0(g)\,f\otimes 1$$

$$+q^0(f)\,q^0(g)\otimes 1+1\otimes q^0(h)-1\otimes q^0(f)\,q^0(g)\equiv 0$$

$$
\begin{aligned}
&\sum q^0(f) \otimes g + 1 \otimes w + q^0(v) \otimes 1 \\
&\quad \equiv \sum 1 \otimes q^0\,(f)g + 1 \otimes h - 1 \otimes fg - q^0(h) \otimes 1 \\
&\quad \equiv \sum -1 \otimes q^0(g)f + 1 \otimes q^0(fg) + 1 \otimes h - q^0\,(h) \otimes 1 \\
&\quad \equiv \sum 1 \otimes (-q^0(g)f + h + q^0\,(f)\,q^0(g) - q^0(h)) \equiv 0.
\end{aligned}
$$

(v) is similar. □

Filter the ring of divided congruences D^Γ by

$$D_k^\Gamma \underset{def}{=} \operatorname{im}(\textstyle\bigoplus_{l=0}^k M_l^\Gamma \otimes \mathbb{Q} \to D^\Gamma \otimes \mathbb{Q}) \cap D^\Gamma$$

and define the subgroups of $D_k^\Gamma \otimes \mathbb{Q}$

$$D_k^\Gamma \underset{def}{=} \mathbb{Q} + D_k^\Gamma$$

$$D_k^\Gamma \underset{def}{=} \mathbb{Q} + D_k^\Gamma + M_k^\Gamma \otimes \mathbb{Q}$$

Proposition 3.9. *The homomorphism*

$$\iota^2 : E_2^{2,k} \cong Z_k^2 / B_k^2 \to \underline{\underline{D}}_k^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

sending $\sum f \otimes g$ *to* $\sum q^0(g)\, f$ *is injective.*

Proof. ι is well defined and injective by the equivalence (i)$\Leftrightarrow$(iv) of the lemma. □

We may draw another useful consequence of eq. (3.2).

Proposition 3.10. *A cycle represented by* $\sum f \otimes g$ *satisfies the identity*

$$\sum f \otimes g - fg \otimes 1 = -\sum g \otimes f + 1 \otimes fg \bmod \mathbb{Z}^{2\Gamma}.$$

In particular, we have

$$\iota^2(\textstyle\sum f \otimes g) = -\iota^2(\sum g \otimes f).$$

Proof. We have to verify the integrality of

$$q^I(\textstyle\sum f \otimes g - fg \otimes 1 + g \otimes f - 1 \otimes fg)$$

which is immediate from Eq. (3.2) and Corollary 3.4. □

The proposition suggests that we may restrict ourselves to antisymmetric cycles. For that define the cocyclic object $M_*^{(s+1)\Gamma}$ with faces d_i for $i = 0, \ldots, s$, degeneracies s_j for $j = 0, \ldots, s$ and cyclic operators t_s given by

$$d_i : M_*^{s\Gamma} \to M_*^{(s+1)\Gamma};\ \sum f^0 \otimes \ldots \otimes f^{s-1} \mapsto \sum f^0 \otimes \ldots f^{i-1} \otimes 1 \otimes f^i \ldots \otimes f^{s-1}$$

$$s_j : M_*^{(s+2)\Gamma} \to M_*^{(s+1)\Gamma};\ \sum f^0 \otimes \ldots \otimes f^{s+1} \mapsto \sum f^0 \otimes \ldots \otimes f^j f^{j+1} \otimes \ldots \otimes f^{s+1}$$

$$t_s : M_*^{(s+1)\Gamma} \to M_*^{(s+1)\Gamma};\ \sum f^0 \otimes \ldots \otimes f^s \mapsto (-1)^s \sum f^s \otimes \ldots \otimes f^s \otimes f^0.$$

Then one easily verifies the identities

$$d_j d_i = d_i d_{j-1} \quad \text{for } i < j$$

$$s_j s_i = s_i s_{j+1} \quad \text{for } i \leqslant j$$

$$s_j d_i = \begin{cases} d_i s_{j-1} & \text{for } i < j \\ id & \text{for } i = j, i = j+1 \\ d_{i-1} s_j & \text{for } i > j+1 \end{cases}$$

$$t_s d_i = -d_{i-1} t_{s-1} \quad \text{for } 1 \leqslant i \leqslant s, \quad t_s d_0 = (-1)^s d_s$$
$$t_s s_i = -s_{i-1} t_{s+1} \quad \text{for } 1 \leqslant i \leqslant s, \quad t_s s_0 = (-1)^s s_s t_{s+1}^2, \quad t_s^{s+1} = id.$$

The difference between cyclic and cocyclic objects is not serious. Each cocyclic object gives rise to cyclic one and vice versa (cf. [22]). Dualizing Connes' original definition, we define the cocyclic bicomplex $CC\,M_*^{*\Gamma}$

$$\begin{tikzcd}
\uparrow & \uparrow & \uparrow & \uparrow & \\
M^{3\Gamma} \arrow[r, "1-t"] \arrow[u] & M^{3\Gamma} \arrow[r, "N"] \arrow[u] & M^{3\Gamma} \arrow[r, "1-t"] \arrow[u] & M^{3\Gamma} \arrow[r, "N"] \arrow[u] & {} \\
M^{2\Gamma} \arrow[r, "1-t"] \arrow[u, "b"] & M^{2\Gamma} \arrow[r, "N"] \arrow[u, "-b'"'] & M^{2\Gamma} \arrow[r, "1-t"] \arrow[u, "b"] & M^{2\Gamma} \arrow[r, "N"] \arrow[u, "-b'"'] & {} \\
M^{\Gamma} \arrow[r, "1-t"] \arrow[u, "b"] & M^{\Gamma} \arrow[r, "N"] \arrow[u, "-b'"'] & M^{\Gamma} \arrow[r, "1-t"] \arrow[u, "b"] & M^{\Gamma} \arrow[r, "N"] \arrow[u, "-b'"'] & {}
\end{tikzcd}$$

where

$$b \underset{def}{=} \sum_{i=0}^{s} (-1)^i d_i; \quad b' \underset{def}{=} \sum_{i=0}^{s-1} (-1)^i d_i$$

and $N = 1 + t + \cdots + t^s$ is the norm map corresponding to the cyclic operator $t = t_s$. Let M^Γ in the left hand corner have bidegree (0, 0) and write

$$HC^s(M^{*\Gamma}) = H^s(Tot\,CCM^{*\Gamma})$$

for the s-cohomology of the total complex $Tot\,CCM^\Gamma$.

Theorem 3.11. *The canonical map*

$$can\colon H^s(Tot\,CCM_*^{*\Gamma}, \mathbb{Q}/\mathbb{Z}) \to E_2^{s+1,*}$$

induced by the projection onto the first column

$$Tot\,CCM_*^{*\Gamma} \longrightarrow (M^\Gamma \xrightarrow{b} M^{2\Gamma} \xrightarrow{b} M^{3\Gamma} \xrightarrow{b} \cdots)$$

is an isomorphism in dimensions $s = 0$ and $s = 1$.

Proof. Let $z = (f, \alpha) \in M_*^{(s+1)\Gamma} \oplus \bigoplus_{1 \leqslant r \leqslant s} M^{r\Gamma} = Tot^s CCM_*^{*\Gamma}$ be a cycle. Then $b(f)$ is an integral representative of $1 \otimes f \in M_*^{*\Gamma} \otimes G_*^{s+1}$ and thus a cycle in $E_1^{s+1,*}$. Hence, can is well defined as any $b(g) \in M_*^{(s+1)\Gamma}$ bounds in $E_1^{s+1,*}$. In dimension $s = 0$ we have $1 - t = 0$ and the two cycle conditions $1 \otimes f - f \otimes 1 = 0 \bmod \mathbb{Z}^{2\Gamma}$ obviously coincide.

Next, let $s = 1$ and let $z = \sum f \otimes g + h \in Tot^2\, CC\,M_*^{*\Gamma}$ be a cycle of degree $2k$. Then $N(h) = h$ vanishes. We wish to verify the injectivity of can and assume $can[z]$ is a boundary in E_1. That is, mod $\mathbb{Z}^{2\Gamma}$ we have

$$\sum f \otimes g = v \otimes 1 + 1 \otimes w$$

for some v, w of degree $2k$. Then

$$0=(1-t)\left(\sum f\otimes g\right)=\sum f\otimes g+g\otimes f=(v+w)\otimes 1+1\otimes(v+w)$$

shows $v-q^0(v)=q^0(w)-w$. Using

$$0=q^{(0,0,0)}\left(b\left(\sum f\otimes g\right)\right)=\sum q^0(f)\,q^0(g)=q^0(v)+q^0(w)$$

we conclude that $\sum f\otimes g=v\otimes 1-1\otimes v=b(v)$ bounds.

It remains to show the surjectivity of *can*. Let $\tilde z=\sum f\otimes g$ be a cycle in E_1. Then

$$z\underset{def}{=}\sum f\otimes g-\sum fg\otimes 1$$

satisfies $(1-t)(z)=0$ by the antisymmetry relation of 3.2.4. Moreover,

$$b(z)=\sum 1\otimes f\otimes g-f\otimes 1\otimes g+f\otimes g\otimes 1-1\otimes fg\otimes 1$$

is easily checked to be integral using Eqs. (3.2) and (3.1). □

The complex

$$C_*^\Gamma: M^\Gamma\xrightarrow{b}M^{2\Gamma}\xrightarrow{b}M^{3\Gamma}\xrightarrow{b}\cdots$$

coincides with the well known cobar complex $C_{M_*^{2\Gamma}}(M_*^\Gamma, M_*^\Gamma)$ (cf. [40, A.1.2.11]) under the isomorphism

$$M_*^\Gamma\otimes_{M_*^\Gamma}\bigotimes{}^s_{M_*^\Gamma}M_*^{2\Gamma}\otimes_{M_*^\Gamma}M_*^\Gamma\to M_*^{s\Gamma}$$

$$(e(f^1\otimes g^1)|f^2\otimes g^2|\ldots|(f^s\otimes g^s)h)\mapsto ef^1\otimes g^1f^2\otimes\ldots\otimes g^{s-1}f^s\otimes g^s h.$$

Hence, for positive s we have

$$\begin{aligned}E_2^{s,*}&\cong H^sC_{M_*^{2\Gamma}}(M_*^\Gamma, M_*^\Gamma)\cong Cotor^s_{M_*^{2\Gamma}}(M_*^\Gamma, M_*^\Gamma)\\&\cong Cotor^{s-1}_{M_*^{2\Gamma}}(M_*^\Gamma, M_*^\Gamma\otimes Q/\mathbb{Z})\cong H^sC_{M_*^{2\Gamma}}(M_*^\Gamma, M_*^\Gamma\otimes\mathbb{Q}/\mathbb{Z})\\&\cong H^{s-1}(C_*^\Gamma,\mathbb{Q}/\mathbb{Z})\end{aligned}$$

by the exactness of *Cotor*. Most results of this section can also be deduced from the cobar complex. However, it is important to have explicit geometric isomorphisms at hand.

3.3. *d, e and f-Invariants*

In this section we shall define some basic invariants of stable homotopy groups. Throughout, we assume that E is a flat ring theory with evenly graded coefficients E_*.

Suppose given a stable homotopy class $s\in\pi_nS$ in the filtration group $\mathscr{F}^s$ of the E-based ANSS. Then we can consider its image under the map

$$e_2^{s,n+s}:\mathscr{F}_n^s\to\mathscr{F}_n^s/\mathscr{F}_n^{s+1}\cong E_\infty^{s,n+s}\hookrightarrow E_2^{s,n+s}.$$

If we take $n=s=0$ then the invariant

$$e_2^{0,0}:\pi_0S=\mathscr{F}_0^0\to E_2^{0,0}=\pi_0E$$

gives the degree d of s. Now let the dimension of s be positive. Then $e_2^{0,n}$ vanishes as π_nS is all torsion. Hence, we get the 'Hopf–Steenrod invariant'

$$e_2^{1,n+1}:\pi_nS=\mathscr{F}_n^1\to E_2^{1,n+1}.$$

In case E is elliptic cohomology E^Γ and $n = 4k - 1$ the invariant may be composed with the monomorphism ι^1 of Proposition 3.9 for some choice of cusp to yield

$$e:\pi_n S \xrightarrow{e_2^{1,n+1}} E_2^{1,n+1} \xrightarrow{\iota^1} \mathbb{Z}^\Gamma \otimes \mathbb{Q}/\mathbb{Z}$$

PROPOSITION 3.12. *e coincides with the classical Adams invariant* [3] *in dimensions* $4k-1$ *for* $\Gamma = \Gamma_1(2)$ *at the cusp* 0.

Proof. Let $s \in \pi_{4k-1}S$ be represented by a manifold M with a framing on its stable tangent bundle. Then M is the boundary of some Spin manifold N since the Spin-cobordism group vanishes in dimension $4k-1$. The e-invariant of s is given by the relative $\hat{A}$-genus of N (cf. [8]). The manifold N represents an element in the first line of the $MSpin$-based ANSS (cf. [12]). Now ι^1 takes N to the 0-coefficient in

$$\pi_*\overline{MSpin} \otimes \mathbb{Q} \to \pi_*\overline{E}^{\Gamma_1(2)} \otimes \mathbb{Q} \xrightarrow{\lambda_*^{\Gamma_1(2)}} \pi_*\overline{K}^{\Gamma_1(2)}((q)) \otimes \mathbb{Q}.$$

Its value is the relative $\hat{A}$-genus of N as $\lambda^{\Gamma_1(2)}(0)$ sends δ to $-\frac{1}{8}$ and ε to 0 [18]. □

The e-invariant vanishes in even dimensions. Hence, for even $n > 0$ we have

$$e_2^{2,n+2}: \pi_n S = \mathscr{F}_n^2 \to E_2^{2,n+2}$$

which we may compose with the monomorphism ι^2 of Proposition 3.9 to obtain

$$f:\pi_n S \xrightarrow{e_2^{2,n+2}} E_2^{2,n+2} \overset{\iota^2}{\hookrightarrow} \underline{\underline{D}}_n^\Gamma \otimes \mathbb{Q}/\mathbb{Z}.$$

The f-invariant permits a description in terms of Hirzebruch genera on manifolds with corners, but the details shall be developed somewhere else. The next result shows that the f-invariant already takes values in the smaller group of holomorphic divided congruences.

PROPOSITION 3.13. *The f-invariant admits a factorization*

$$\begin{tikzcd} \pi_n S \arrow[rr, "f"] \arrow[dr, "\bar{f}"'] & & \underline{\underline{D}}_n^\Gamma \otimes \mathbb{Q}/\mathbb{Z} \\ & \underline{\underline{\bar{D}}}_n^\Gamma \otimes \mathbb{Q}/\mathbb{Z} \arrow[ur, "can"'] & \end{tikzcd}$$

where

$$\underline{\underline{\bar{D}}}_n^\Gamma \underset{def}{=} \operatorname{im}\left(\bigoplus_{l=0}^n \bar{M}_l^\Gamma \otimes \mathbb{Q} \to \underline{\underline{D}}^\Gamma \otimes \mathbb{Q}\right) \cap D_n^\Gamma.$$

The kernel consists of elements in the higher filtration group $\mathscr{F}^3$.

Proof. Let $s \in \pi_n S$ be a stable class of even dimension $n > 0$. Then s lies in second MU-filtration and we can find a $\tilde{s} \in \pi_n(\bar{M}U \wedge \bar{M}U)$ which projects to s. Now recall that the elliptic genus $o:\Omega_*^U \to M_*^\Gamma$ already takes values in the subring $\bar{M}_*^\Gamma$ of holomorphic modular forms (cf. [18, I.7 6.4]). Moreover, by looking at the (q^L, q^R)-expansion we see that the map

$$\pi_*(o \wedge o): MU_*MU \to E_*^\Gamma E^\Gamma = M_*^{2\Gamma}$$

already takes values in the subring $\bar{M}^{2\Gamma}$ of holomorphic modular forms in 2 variables. Hence, we obtain the commutative diagram

$$\begin{tikzcd}
\pi_*(\Sigma^2 \bar{MU} \wedge \bar{MU}) \arrow[r] \arrow[d, "\kappa"] & \pi_*(\Sigma^2 \bar{E}^\Gamma \wedge \bar{E}^\Gamma) \arrow[d] \\
\frac{\pi_*(\Sigma^2 \bar{MU} \wedge \bar{MU}) \otimes \mathbb{Q}}{MU_*(\Sigma \bar{MU})} \arrow[r] \arrow[d, "="] & \frac{\pi_*(\Sigma^2 E^\Gamma \wedge E^\Gamma) \otimes \mathbb{Q}}{E^\Gamma_*(\Sigma \bar{E}^\Gamma)} \arrow[d, "="] \\
\frac{MU_* \otimes MU_* \otimes \mathbb{Q}}{MU_* MU + MU_* \otimes \mathbb{Q} + \mathbb{Q} \otimes MU_*} \arrow[r] \arrow[d] & \frac{M^\Gamma_* \otimes M^\Gamma \otimes \mathbb{Q}}{M^{2\Gamma}_* + M^\Gamma_* \otimes \mathbb{Q} + \mathbb{Q} \otimes M^\Gamma_*} \arrow[d, "="] \\
\frac{\bar{M}^\Gamma_* \otimes \bar{M}^\Gamma \otimes \mathbb{Q}}{M^{2\Gamma}_* + M^\Gamma_* \otimes \mathbb{Q} + \mathbb{Q} \otimes M^\Gamma_*} \arrow[r] \arrow[d] & \frac{M^\Gamma_* \otimes M^\Gamma \otimes \mathbb{Q}}{M^{2\Gamma}_* + M^\Gamma_* \otimes \mathbb{Q} + \mathbb{Q} \otimes M^\Gamma_*} \arrow[d, "\iota^2"] \\
\underline{\underline{\bar{D}}}^\Gamma_* \otimes \mathbb{Q}/\mathbb{Z} \arrow[r, "can"] & \underline{\underline{D}}^\Gamma_* \otimes \mathbb{Q}/\mathbb{Z}
\end{tikzcd}$$

in which $\tilde{s}$ is mapped to the f-invariant of s in the lower right corner. Since $\kappa(\tilde{s}) = e_2^{2,n+2}(s)$ in the MU-based ANSS we obtain a well defined factorization of the f invariant. The last statement is clear. $\square$

We are now going to compute the f-invariant of the periodic family β which was first considered by Smith in [44]. Recall that the Hazewinkel generators v_n are congruent to u_n modulo the ideal generated by $(p, u_1, \dots, u_{n-1})$ (cf. [30]).

THEOREM 3.14 (Smith [47])

(i) *Let $V(0)$ be the cofibre of $p: S \to S$ and p be an odd prime. Then there is a selfmap*

$$\Phi_0 : \Sigma^{2(p-1)} V(0) \to V(0)$$

inducing multiplication by v_1 in complex bordism.

(ii) *Let $V(1)$ be the cofibre of Φ_0 and $p \geqslant 5$. Then there is a selfmap*

$$\Phi_1 : \Sigma^{2(p^2-1)} V(1) \to V(1)$$

inducing multiplication by v_2 in complex bordism.

(iii) *Let β be the composite*

$$S^{2(p^2-1)} \xrightarrow{i_0} \Sigma^{2(p^2-1)} V(0) \xrightarrow{i_1} \Sigma^{2(p^2-1)} V(1) \xrightarrow{\Phi_1} V(1) \xrightarrow{p_1} \Sigma^{2p-1} V(0) \xrightarrow{p_0} S^{2p}$$

where i_0, i_1, p_0, p_1 come from the cofibre sequences above and $p \geqslant 5$. Then β represents a nontrivial permanent cycle.

A useful result in this context is the *geometric boundary theorem*:

THEOREM 3.15 (Ravenel [40]). *Let E be a flat ring spectrum and E_* be commutative. Let*

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} \Sigma W$$

be a cofibre sequence of finite spectra with $E_*(h) = 0$. Assume further that $[s] \in E_2^{t, *+t}(Y)$ converges to $s \in \pi_t(Y)$. Then $\delta[s]$ converges to $h_*(s) \in \pi^{st}_{t-1}(W)$ where δ is the connecting *homomorphism to the short exact sequence of chain complexes*

$$0 \to E_1(W) \to E_1(X) \to E_1(Y) \to 0.$$

Consider the case $E = E^{\Gamma_1(1)}$. In [30] it was shown that for $p > 3$

$$v_2 \equiv (-1)^{(p-1)/2} \Delta^{(p^2-1)/12} \quad \bmod (p, v_1)$$

and we saw earlier already

$$v_1 \equiv E_{p-1} \bmod p.$$

$\Phi_1 i_1 i_0 : \Sigma^{2(p^2-1)} S \to V(1)$ is represented by

$$v_2 \in E_2^{0,2(p^2-1)}(V(1)) \subset \pi_{2(p^2-1)} E^{\Gamma_1(1)}/(p, v_1).$$

Since $E_*^{\Gamma_1(1)}(p_1) = 0$ we may apply the geometric boundary theorem to the cofibre sequence

$$\Sigma^{2(p-1)} V(0) \xrightarrow{\Phi_1} V(0) \longrightarrow V(1) \xrightarrow{p_1} \Sigma^{2p-1} V(0).$$

Thus, the boundary $\delta(v_2)$ converges to $p_1 \phi_1 i_1 i_0$. To determine the boundary, we view v_2 as an element in

$$E^{\Gamma_1(1)}_{2(p^2-1)} V(0) = \pi_{2(p^2-1)} E^{\Gamma_1(1)}/p,$$

compute its differential and divide by v_1. It is customary to identify $E_*^{\Gamma_1(1)} \Sigma \bar{E}^{\Gamma_1(1)}$ with the augmentation ideal in the Hopf algebroid $E_*^{\Gamma_1(1)} E^{\Gamma_1(1)}$. Then the first differential becomes the difference $\eta_L - \eta_R$ between left and right unit. Let m_i be the coefficient of x^{p^i} in the logarithm of the $p-$typicalized formal group law $\widetilde{F}_{E_{\Gamma_1(1)}}$. Let t_i be the image of the standard generators in $BP_* BP$ under the classifying map. Then we have the formulas (cf. Appendix of [34, 40])

$$\eta_R(m_k) = \sum_{i+j=k} m_i t_j^{p^i} \quad (m_0 = t_0 = 1)$$

$$p\, m_{n+1} = \sum_{i+j=n} m_j v_{i+1}^{p^j}.$$

Hence, we obtain

$$\begin{aligned} \eta_R(v_1) &= p\, \eta_R(m_1) = p\, t_1 + v_1 \\ \eta_R(v_2) &= p\, \eta_R(m_2) - \eta_R(m_1)\, \eta_R(v_1^p) \\ &= p\, t_2 + v_1 t_1^p + (v_2 + v_1^{p+1}/p) - (t_1 + v_1/p)(p\, t_1 + v_1)^p \\ &\equiv v_1 t_1^p + v_2 - t_1 v_1^p \bmod \mathrm{p} \end{aligned}$$

and thus

$$\delta(v_2) = t_1^p - v_1^p t_1 \in E_2^{1,2(p^2-1)}(V(0)).$$

Finally we apply Theorem 3.15 to the cofibre sequence

$$S^{2p-1} \xrightarrow{p} S^{2p-1} \to \Sigma^{2p-1} V(0) \to S^{2p}$$

and see that β is represented by

$$1/p\, d(t_1^p - v_1^p t_1) \in E_2^{2,2((p^2-1)-p+1)}.$$

In our old notation this element carries the name

$$1/p(1 \otimes (v_1/p)^p - v_1^{p-1} \otimes (v_1/p)) \equiv -1/p^2(v_1^{p-1} \otimes v_1) = -1/p^2(E_{p-1}^{p-1} \otimes E_{p-1}).$$

We conclude from Proposition 3.10:

PROPOSITION 3.16.

$$f(\beta) = -p^{-2}E_{p-1}^{p-1} = p^{-2}E_{p-1}.$$

The Deligne congruence $E_{p-1} \equiv 1 \bmod p$ shows that β has order at most p. The nontriviality of β is equivalent to the nonexistence of a congruence

$$1/p\,(E_{p-1} - 1) \equiv w - q^0(w) \bmod p$$

for any integral $w - q^0(w)$ of weight $(p^2 - p)$.

4. ORIENTATIONS AND RIEMANN–ROCH FORMULAS

We can use the topological q-expansion principle to equip E^Γ with orientations which differ from the original complex ones. Recall that the Landweber–Ravenel–Stong elliptic genus originated from the signature operator on the loop space of a compact *Spin* manifold M after a transformation of variables. More precisely, Witten [48] formally identified the S^1-equivariant signature with the index of

$$\partial^+(TM) \otimes (\bigotimes^{k>0} \bigwedge\nolimits_{q^k} TM \otimes S_{q^k}\, TM \otimes \mathbb{C})$$

using the Lefschetz fixed point formula. Here $\partial^+(TM)$ is the Dirac operator of M and we abbreviated $\Lambda_t E \underset{def}{=} \sum_{k=0}^\infty (\Lambda^k E)t^k$ and $S_t E \underset{def}{=} \sum_{k=0}^\infty (S^k E)t^k$.

As the signature is only a twisted version of the Dirac operator one might expect to obtain a more powerful genus by using the Dirac operator itself. The corresponding expression

$$\partial^+(TM) \otimes \bigotimes^{k>0} S_{q^k}(TM - \dim M) \otimes \mathbb{C}$$

leads to the Witten genus (the factor $q^{-4k/24}\Delta^{4k/24}$ included). The Witten genus gives an integral modular form for every $M\,String$ manifold. We can ask if it arises from a map of ring spectra $w : MString \to E^\Gamma$ when restricting to the coefficients.

LEMMA 4.1. *There is a unique map of ring spectra $\omega : MString \to K^\Gamma((q))$ s.t. the Riemann–Roch formula*

$$\omega f_!^{MString}(\alpha) = f_!^{\hat{A}}(\bigotimes^{k>0} S_{q^k}(\dim \nu_M - \nu_M) \otimes \omega(\alpha) \otimes \mathbb{C})$$

holds for all String-oriented maps $f : M \to X$ between smooth compact manifolds and all $\alpha \in MString^* M$. Here ν_M is the normal bundle of M.

Proof. We first show uniqueness. Let X be a smooth compact manifold. Then by the work of Quillen [39] every element in $MString^{4d}X$ is of the form $f_!^{MString}(1)$ for a certain $f : M \to X$ as above. The value of ω on $f_!^{MString}(1)$ is given by the Riemann–Roch formula. Furthermore every finite CW-complex has the homotopy type of a smooth manifold. Hence, ω is determined when restricted to the cofinal system of finite subspectra in $MString$. We have seen earlier that all $\lim^1$ vanish locally at each prime. Thus they vanish globally and there can only be one map ω with the required properties.

We could take the Riemann–Roch formula to show existence. However, there is a more natural approach. Let $\tilde{A} : MString \to MSpin \to K^\Gamma((q))$ be the usual orientation which induces the (complexified) $\hat{A}$-genus on *MString* manifolds. Let $u^{\hat{A}}(\xi)$ be the induced Thom class for an arbitrary *MString* bundle ξ. Then $u^{\omega}(\xi) = u^{\hat{A}}(\xi) \otimes \bigotimes^{k>0} S_{q^k}(\dim \xi - \xi)$ is another natural Thom class as the twisting factor is a unit in the ring of power series. The limit of all such Thom classes yields a map of spectra $\omega: MString \to K^{\Gamma}((q))$. Moreover, the new Thom class is multiplicative. To see this, let η be another *M String* bundle. Then certainly $u^{\hat{A}}(\xi+\eta) = u^{\hat{A}}(\xi) u^{\hat{A}}(\eta)$. It remains to check the exponential behaviour of the symmetric powers

$$S_{q^k}(\bar{\xi} + \bar{\eta}) = \sum_{n=0}^{\infty} S^n(\bar{\xi} + \bar{\eta})\, q^{kn} = \sum_{n=0}^{\infty} \sum_{r+s=n} S^r(\bar{\xi}) \otimes S^z(\bar{\eta})\, q^{kn}$$

$$= \sum_{r=0}^{\infty} S^r(\bar{\xi})\, q^{kr} \otimes \sum_{s=0}^{\infty} S^s(\bar{\xi})\, q^{kz} = S_{q^k}(\bar{\xi}) \otimes S_{q^k}(\bar{\eta}).$$

Thus ω is a map of ring spectra. The correct Riemann–Roch transformation follows as in [14]. □

THEOREM 4.2. *There is a unique map of ring spectra* $w: M\,String \to E_{\Gamma}[1/2]$ *s.t.* $\lambda_{\Gamma} w = \omega$. *In particular, the Riemann–Roch formula holds*:

$$ch\, \lambda_{\Gamma} w f_!^{MString}(\alpha) = f_!^{H\mathbb{Q}}(\hat{A}(TM)\, ch(\textstyle\bigotimes^{k>0} S_{q^k}(TM - \dim TM) \otimes w(\alpha) \otimes \mathbb{C})).$$

Proof. Consider the cartesian diagram of Corollary 1.13

$$\begin{tikzcd}
E_{\Gamma}^{*} M\,String[1/2] \arrow[r, "\lambda_{\Gamma}"] \arrow[d, "d"] & K_{\Gamma}^{*} M\,String[1/2]((q)) \arrow[d, "ch"] \\
H^{*}(M\,String, E_{\Gamma}^{*} \otimes \mathbb{Q}) \arrow[r, "\lambda_{\Gamma}"] & H^{*}(M\,String, K_{\Gamma}^{*} \otimes \mathbb{Q})((q))
\end{tikzcd}$$

in which the left vertical arrow is the Dold character. The lemma provides an element $\omega \in K_{\Gamma} M\,String((q))$. In order to lift ω to $E_{\Gamma} MString[1/2]$ it is enough to prove that

$$ch\, \omega f_!^{M\,String}(1) = f_!^{HQ}(\hat{A}(TM)\, ch(\textstyle\bigotimes^{k>0} S_{q^k}(TM - \dim TM) \otimes \mathbb{C}))$$

gives an element in $H^0(X, E_{\Gamma}^{*} \otimes \mathbb{Q})$ for every *M String* oriented map $M \to X$. The argument is well known (cf. [18] Eq. (6.3)) but customly stated in terms of Chern numbers instead of Chern classes. The computation of

$$\hat{A}(TM)\, ch(\textstyle\bigotimes^{k>0} S_{q^k}(TM - \dim TM) \otimes \mathbb{C})$$

in formal Chern roots x_i of TM leads to

$$\prod_{i=1}^{2m} \frac{x_i/2}{\sinh(x_i/2)} \prod_{k=1}^{\infty} \frac{(1-q^k)^2}{(1-q^k e^{x_i})(1-q^k e^{-x_i})}$$

$$= \prod_{i=1}^{2m} \frac{x_i/2}{\sinh(x_i/2)} \left(\prod_{k=1}^{\infty} \frac{(1-q^k)^2}{(1-q^k e^{x_i})} \right) e^{-\bar{E}_2(\tau) x_i^2} = \prod_{i=1}^{2m} \frac{x_i}{\sigma_{L_\tau}(x_i)}.$$

Here, the first identity holds since the first Pontrjagin class of v_M and hence of TM vanish. $\bar{E}_2$ denotes the divided second Eisenstein series. In the last equality σ_{L_τ} is the Weierstraß σ-function for the lattice L_τ (cf. [49]). The coefficients of the rth homogeneous part of the last expression are homogeneous lattice functions of weight r, and so modular forms with respect to any congruence subgroup $\Gamma \subset Sl_2(\mathbb{Z})$. Since $f_!^{H\mathbb{Q}}$ is linear over the coefficients its pushforward lies in $H^*(X, E_\Gamma^* \otimes \mathbb{Q})$. It remains to show that w is a map of ring spectra. Obviously,

$$E_\Gamma^*(M\,String \wedge M\,String)[1/2] \xrightarrow{\lambda_\Gamma} K_\Gamma^*(M\,String \wedge M\,String)[1/2]((q))$$

is injective. Thus the assertion follows from Lemma 4.1. □

The Witten genus w of E_Γ in the above theorem coincides with the canonical orientation recently provided in [19] by the theorem of the cube. Our approach is more direct and elementary but less conceptual than the one of Mike Hopkins.

We do not wish to invert 2 but $MString$ is not well understood at the prime 2. However, the method described above applies for other Thom spectra as well.

PROPOSITION 4.3. *Let MG be a Thom spectrum and $E_*^\Gamma MG_{(p)}$ be projective for all prime p. Let $F: E^\Gamma \to K^\Gamma((q))$ be an exponential class s.t.*

$$ch\,F(V) \in H^0(X, E_*^\Gamma \otimes \mathbb{Q})$$

for all G-oriented vector bundles V over X. Then there is a unique orientation

$$\omega: MG \to E^\Gamma$$

s.t. the Riemann–Roch formula

$$\lambda_\Gamma\,\omega f_!^{MG}(\alpha) = f_!^{Todd}(F(v_M) \otimes \lambda_\Gamma \omega(\alpha))$$

holds for all G-oriented $f: M \to X$ and all $\alpha \in MG^*M$.

Proof. The topological q-expansion principle (Theorem 1.12) applies. □

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APPENDIX A. THE CLASSICAL q-EXPANSION PRINCIPLE

In this appendix we recall the classical q-expansion principles of the basic elliptic curves which lead to Landweber exact theories. The main references are the books of Silverman [45, 46], Serre [44] and the articles of Katz [23, 26].

A.1. Weierstraß cubics

Classically a complex modular form of weight k is a function on the upper half-plane $\mathfrak{h} = \{\tau \in \mathbb{C};\ \mathrm{Im}(\tau) > 0\}$ which obeys certain transformation laws and is holomorphic in a suitable sense. In order to obtain the notion of a modular form over any ring, we will view them as certain kind of "distributions". The test objects are elliptic curves together with nowhere vanishing invariant differentials.

Let

$$GL^+ = \{(\omega_1, \omega_2) \in \mathbb{C}^2;\ \mathrm{Im}(\omega_2/\omega_1) > 0\}$$

be the space of oriented $\mathbb{R}$-bases of $\mathbb{C}$. Its orbit space L under the right action of $Sl_2(\mathbb{Z})$ is the space of lattices $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ in $\mathbb{C}$. Weierstraß theory establishes a correspondence between points L of L and elliptic curves given as cubics in the complex projective plane by the inhomogeneous equations

$$E_{/\mathbb{C}} : y^2 = 4x^3 - g_2(L)x - g_3(L)$$

where

$$g_2(L) = 60 \textstyle\sum_{0 \neq l \in L} 1/l^4; \quad g_3(L) = 140 \sum_{0 \neq l \in L} 1/l^6.$$

Let $\Delta = g_3^2 - 27g_2^3$ be the discriminant of $E_{/\mathbb{C}}$. Then L becomes the open subspace $Spec(\mathbb{C}[g_2, g_3, \Delta^{-1}])$ of $\mathbb{C}^2 = Spec(\mathbb{C}[g_2, g_3])$. Physically, the Weierstraß cubic associated to L is analytically isomorphic to the torus $\mathbb{C}/L$:

$$z \in \mathbb{C}/L \mapsto (x = \wp(z, L),\ y = \wp'(z, L))$$

where $\wp$ is the Weierstraß function

$$\wp(z, L) = \frac{1}{z^2} + \sum_{0 \neq l \in L} \left(\frac{1}{(z - l)^2} - \frac{1}{l^2} \right).$$

Under this map the translation invariant 1-form dz is sent to the nowhere vanishing holomorphic differential

$$\omega = dx/y \in H^1(E_{/\mathbb{C}}, \Omega^1_{E_{/\mathbb{C}}}).$$

Conversely, any Weierstraß cubic with invariant differential ω generates a lattice of periods (cf. [23])

$$L = \left\{ \int_\gamma \omega;\ \gamma \in H^1(E_{/\mathbb{C}}, \mathbb{Z}) \right\}.$$

We will freely think of the $\mathbb{C}^\times$-space L as lattices with stretching action of $\mathbb{C}^\times$ or as the space of pairs $(E_{/C}, \omega)$ with $a(E_{/\mathbb{C}}, \omega) = (E_{/\mathbb{C}}, a\,\omega)$ for $a \in \mathbb{C}^\times$. A function $f\colon \mathsf{L} \to \mathbb{C}$ is said to be homogeneous of weight $k \in \mathbb{Z}$ if it is equivariant under the action on $\mathbb{C}$ defined by $z \mapsto a^{-k}z$.

For instance, the global coordinates g_2 and g_4 have weight 4 and 6 respectively. They are closely related to the Eisenstein series of weight $2k$

$$G_{2k}(L) = \textstyle\sum_{0 \neq l \in L} l^{-2k}/2\zeta(2l); \quad \zeta(s) = \sum_{n \geqslant 1} n^{-s}$$

as $G_4 = 12 g_2$ and $G_6 = 216 g_3$. A homogeneous function f gives rise to a periodic function $f(\tau) = f(2\pi i(\mathbb{Z} + \tau\mathbb{Z}))$ on $\mathfrak{h}$. If f viewed as a function of $q = \exp(2\pi i\tau)$ extends to a holomorphic function of q in $|q| < 1$ it is called a holomorphic modular form. In this case, its Fourier series or q-expansion determines f completely.

g_2 and g_3 are of particular importance as they freely generate the ring of holomorphic modular forms

$$\bar{M}^{\Gamma_1(1)}(\mathbb{C}) = \mathbb{C}[G_2, G_3].$$

We associate to each modular form twice its weight, i.e. $|G_{2k}| = 4k$, so that $\bar{M}_*^{\Gamma_1(1)}(\mathbb{C})$ becomes a graded ring. Furthermore, each Weierstraß cubic has the structure of a one-dimensional group scheme. Its formal completion is a formal group. As it turns out, the Weierstraß parametrization is in fact an isomorphism of groups. Near the identity element $\infty = [0, 1, 0]$ the formal parameter

$$t = -2x/y = -2\mathfrak{p}(z, L)/\mathfrak{p}'(z, L) = z + \sum_{n \geqslant 2} a_n z^n$$

is the exponential of the group law in this chart. The series $t = f(z)$, a priori with coefficients $a_n \in \bar{M}_{2n}^{\Gamma_1(1)}(\mathbb{C}) \subset \mathbb{C}[\![q]\!]$, actually gives an element in $\mathbb{Z}[1/6][\![w, q]\!]$ for $w = 1 - \exp(-z)$. This can be seen from the q-expansion of $\mathfrak{p}$ (cf. [18])

$$\mathfrak{p}(z, \tau) = \sum_{n \in \mathbb{Z}} \frac{1}{(q^{n/2} e^{z/2} - q^{-n/2} e^{-z/2})^2} - \left(-\frac{1}{12} + \sum_{0 \neq n \in \mathbb{Z}} \frac{1}{(q^{n/2} - q^{-n/2})^2} \right)$$

It is more convenient to equip $L_\tau = 2\pi i(\mathbb{Z} + \tau\mathbb{Z})$ with the differential $2\pi i\, dz$ rather than dz. Then $f(2\pi i z)$ is the unique parameter in terms of which $dz = dx/y$ for the universal Weierstraß cubic

$$y^2 = 4x^3 - e_4 x + e_6.$$

In this notation

$$12 e_4 = E_4;\ 216 e_3 = E_6$$

where $E_{2k} = (2\pi i)^{2k} G_{2k}$ are the normalized *Eisenstein series*. We would like to give an interpretation of the ring $\mathbb{Z}[1/6, E_4, E_6]$ in terms of integral modular forms.

Definition A.1. Let S be a R-algebra. A test object is a pair $(E_{/S}, \omega)$ where E is an elliptic curve over S (an abelian scheme of dimension one) and ω is a nowhere vanishing differential on E. A modular form over R is a rule which assigns to a test object $(E_{/S}, \omega)$ an element

$$f(E_{/S}, \omega) \in S$$

satisfying the following conditions:

(i) $f(E_{/S}, \omega)$ only depends on the isomorphism class of the pair $(E_{/S}, \omega)$
(ii) the formation of $f(E_{/S}, \omega)$ commutes with base change
(iii) for any $a \in S^\times$ we have $f(E_{/S}, a\,\omega) = a^{-k} f(E_{/S}, \omega)$

We denote by $M_k^{\Gamma_1(1)}(R)$ the R-algebra of such forms.

Alternatively, they can be thought of as global sections of certain invertible sheaves over the moduli space of elliptic curves, but we will not pursue this point of view any further.

We wish to construct a q-expansion map for arbitrary modular forms, i.e. a ring embedding of $M^{\Gamma_1(1)}(R)$ into some power series ring. The original complex q-expansion map can be recovered by evaluating the modular forms on the test object given by the universal Weierstraß cubic over

$$M^{\Gamma_1(1)}(\mathbb{C}) = \mathbb{C}[E_4, E_6, \Delta^{-1}]$$

and its section pushed over the ring of finite tailed Laurent series $\mathbb{C}((q))$ via the Fourier expansion. We could use this process to define a q-expansion for modular forms over any ring in which 6 is invertible by the following observation. Recall the development of the Eisenstein series [44]

$$E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{2k-1} \right) q^n. \tag{A.1}$$

Here B_{2k} denote the Bernoulli numbers determined by

$$x/(e^x - 1) = \sum_{i=0}^{\infty} (B_i/i!) x^i.$$

An inspection of this formula at $k = 2, 3$ shows that the Weierstraß test object is already defined over $\mathbb{Z}[1/6]((q))$. This leaves us with rings R which do not contain 1/6. Under the change of variables

$$x \mapsto x + 1/12 \qquad y \mapsto x + 2y$$

the Weierstraß equation takes the form

$$y^2 + xy = x^3 + B(q)x + C(q)$$

where

$$B(q) = -1/48\,(E_4(q) - 1) \quad C(q) = 1/496(E_4(q) - 1) - 1/864(E_6(q) - 1)$$

are power series with integral coefficients (cf. [26]). This is the famous *Tate curve* which is defined over $\mathbb{Z}((q))$ and restricts to the Weierstraß curve if 6 is inverted.

Definition A.2. The q-expansion map is the ring homomorphism

$$\lambda_*^{\Gamma_1(1)} : M_*^{\Gamma_1(1)}(R) \to \mathbb{Z}((q)) \otimes R \subset R((q))$$

given by

$$f \mapsto f(q) \underset{def}{=} f(Tate_{/\mathbb{Z}((q))\otimes R}, \omega_{can} = dx/(2y + x)).$$

Needless to say, the ring of complex meromorphic modular forms $M_*^{\Gamma_1(1)}(\mathbb{C})$ is just $\mathbb{C}[E_4, E_6, \Delta^{-1}]$ and the two notions of q-expansions agree. If we define the ring of holomorphic modular forms $\bar{M}_*^{\Gamma_1(1)}(R)$ to be the subring of $M_*^{\Gamma_1(1)}(R)$ with q-expansion in $\mathbb{Z}[[q]] \otimes R$, we again have $\bar{M}_*^{\Gamma_1(1)}(\mathbb{C}) = \mathbb{C}[E_4, E_6]$.

The most fundamental result about modular forms and their q-expansion is known as q-expansion principle.

Theorem A.3 (Deligne and Rapaport [13], Katz [24]). *If the q-expansion of a* (*possibly meromorphic*) *modular form f over S of weight k has all its coefficients in a subring $R \subset S$ then there is a unique modular form $\tilde{f}$ of weight k over R which gives rise to f by extension of scalars. Moreover*, *a modular form of weight k is uniquely detemined by its q-expansion.*

Corollary A.4. *If R is a torsionfree ring*, *then the q-expansion map*

$$\lambda : M_k^{\Gamma_1(1)}(R) \to \mathbb{Z}((q)) \otimes R$$

is rationally faithful.

In fact, the corollary is just a reformulation of the theorem for torsionfree rings R. It is also worth mentioning that the canonical map

$$M_*^{\Gamma_1(1)}(\mathbb{Z}) \otimes R \to M_*^{\Gamma_1(1)}(R)$$

is an isomorphism (cf. [24]) if 2 and 3 are invertible in R.

A study of the q-expansion of the Eisenstein series E_4 and E_6 gives

Corollary A.5. $M_*^{\Gamma_1(1)}(\mathbb{Z}[1/6]) \cong \mathbb{Z}[1/6, E_4, E_6, \Delta^{-1}]$.

A.2. Jacobi quartics and Hirzebruch curves

In the first paragraph we saw that integral modular forms do not behave well at the primes 2 and 3. This is reason enough to enlarge the concept of modular forms by varying the class of test objects.

Consider the family of Jacobi quartics given by the inhomogeneous equation

$$y^2 = 1 - 2\delta x^2 + \varepsilon x^4.$$

Its closure in $\mathbb{C}P^2$ is singular at $\infty = (0, 1, 0)$. However, viewed as curve in $\mathbb{C}P^3$ under the normalisation map $[x, y, 1] \mapsto [1, x, x^2, y]$ it becomes an honest elliptic curve whenever $\Delta = 64\varepsilon(\delta^2 - \varepsilon)$ does not vanish. The universal one of these lives over $\mathbb{Z}[\delta, \varepsilon, \Delta^{-1}]$, i.e. any ring homomorphism into some ring R determines a Jacobi quartic over R.

We wish to investigate their relation with lattices. For that we restrict the action of $Sl_2\mathbb{Z}$ on the space of oriented based lattices GL^+ to the subgroup

$$\Gamma_1(2) \underset{def}{=} \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod 2 \right\}.$$

Then the half basis point $\omega_1/2$ satisfies

$$(a\omega_1 + c\omega_2)/2 \equiv \omega_1/2 \bmod L \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(2).$$

Hence, it is not hard to verify that the orbit space

$$\mathsf{L}_1(2) \underset{def}{=} GL^+/\Gamma_1(2)$$

is the space of lattices together with a distinquished 2-division point. In order to construct the desired correspondence define the function [18]

$$f: \mathbb{C} \times \mathsf{L}_1(2) \to \mathbb{C};\ f(z, L, \omega_1/2) = 1/\sqrt{\wp(z, L) - \wp(\omega_1/2, L)} = z + O(z^2)$$

which is elliptic with respect to the sublattice $\tilde{L} = \mathbb{Z}\omega_1 + \mathbb{Z}2\omega_2$ of order 2 in L. f gives a group homomorphism

$$(z \in \mathbb{C}/\tilde{L}, dz, \omega_1/2) \mapsto ((x = f(z, L, \omega_1/2), y = f'(z, L, \omega_1/2)), dx/y, \infty).$$

Conversely, for arbitrary coefficients δ and ε with $\Delta \neq 0$ we get a differential equation for f which is uniquely solved with power series methods by a function f which is elliptic with respect to a lattice $\tilde{L}$. This implies that $\mathsf{L}_1(2)$ takes the form $Spec\ \mathbb{C}[\delta, \varepsilon, \Delta^{-1}]$.

By picking x as coordinate near the identity we identify f as exponential of the associated formal group law. The distinguished point

$$f(\omega_1/2) = \infty \in {}_2E_{/\mathbb{C}} = ker(E_{/\mathbb{C}} \xrightarrow{2} E_{/C})$$

of order 2 imparts an additional structure to each Jacobi quartic.

There is a natural generalization of this concept to higher levels $N \geqslant 2$. Jacobi quartics belong to the family of Hirzebruch curves

$$\frac{1}{x^N} + a_{2N}x^N = \left(\frac{y}{x}\right)^N + a_1\left(\frac{y}{x}\right)^{N-1} + \dots a_{N-1}\left(\frac{y}{x}\right) + a_N.$$

Let $f(z, L, \omega_1/N)$ be the theta function for which f^N is elliptic with respect to L, which has divisor $\operatorname{div} f = (0) - (2\pi i/N)$ and whose Taylor expansion around 0 is of the form $z + O(z^2)$. Again, f can be taken as exponential of a formal group law and ω_1/N generates a subgroup of order N. Note that given an elliptic curve over an arbitrary ring the existence of a point of order N on a curve over R implies that N is invertible in R (cf. [26]).

Definition A.6. A $\Gamma_1(N)$-test object over R is a triple $(E_{/S}, \omega, P)$ where E is an elliptic curve over an R-algebra S, ω a nowhere vanishing differential on E and P is a point of exact order N. A modular form for the congruence subgroup $\Gamma_1(N)$ of weight k is a rule f which assigns to a test object $(E_{/S}, \omega, P)$ an element

$$f(E_{/S}, \omega, P) \in S$$

satisfying the conditions analogous in Definition A.1. We denote by $M_k^{\Gamma_1(N)}(R)$ the R-algebra of such forms.

Interesting test objects are the *Tate curves* $Tate(q^N)_{/\mathbb{Z}((q))}$ with their canonical differential $\omega_{can} = dx/(2y + x)$ and any point of order N. They are deduced from $(Tate(q), \omega_{can})$ by the extension of scalars $\mathbb{Z}((q)) \to \mathbb{Z}((q)) q \mapsto q^N$. The Tate curves have multiplicative reduction given by

$$(e^{2\pi i z} = u \in \mathbb{C}^*/q^{N\mathbb{Z}}, du/u) \mapsto \Bigg(x = \sum_{k \in \mathbb{Z}} \frac{q^{Nk}u}{(1 - q^{Nk}u)^2} - 2\sum_{k=1}^{\infty} \frac{q^{Nk}}{1 - q^{Nk}},$$

$$y = \sum_{k \in \mathbb{Z}} \frac{(q^{Nk}u)^2}{(1 - q^{Nk}u)^3} + \sum_{k=1}^{\infty} \frac{q^{Nk}}{1 - q^{Nk}}, \mathrm{d}x/2y + x \Bigg).$$

Thus, the points of order N in $Tate(q^N)$ correspond to the points of order N on $\mathbb{C}^*/q^{NZ}$ and have the form

$$\zeta_N^i q^j \quad 0 \leqslant i, j \leqslant n - 1$$

where ζ_N denotes a primitive Nth root of unity. All $\Gamma_1(N)$-structures are defined over $\mathbb{Z}((q)) \otimes \mathbb{Z}[\zeta_N, 1/N]$.

Definition A.7. Let R be a ring in which N is invertible and which contains a primitive N-th root of unity ζ_N. Then the q-expansion map at the cusp $\zeta_N^i q^j$ is the ring homomorphism

$$\lambda_*^{\Gamma_1(N)} : M_*^{\Gamma_1(N)}(R) \to \mathbb{Z}((q)) \otimes R \subset R((q))$$

defined by

$$f \mapsto f(q) \underset{def}{=} f(Tate(q^N)_{/\mathbb{Z}((q)) \otimes R}, \omega_{can}, \zeta_N^i q^j).$$

We say f is holomorphic if $f(q)$ already q-expands in $R[\![q]\!]$ for one and hence for all cusps (cf. [27]) and write $\bar{M}_*^{\Gamma_1(N)}(R)$ for the graded ring of holomorphic $\Gamma_1(2)$-modular forms over R.

THEOREM A.8. (Deligne and Rapaport [13], Katz [26]). *If for some cusp a (possibly meromorphic) $\Gamma_1(N)$-modular form f over S of weight k has all its q-expansion coefficients in a subring $R \subset S$ then it does so at all cusps and there exists a unique $\Gamma_1(N)$-modular form f_0 of* *weight k over R which gives rise to f by extension of scalars. Furthermore, every q-expansion map is injective.*

COROLLARY A.9. *If R is torsionfree, then the q-expansion map $\lambda^{\Gamma_1(N)}$ is rationally faithful on its homogeneous components.*

The expansions of ε and δ at the cusp 0, i.e. $(Tate(q^N), \omega_{can}, q)$, are given by the formulas [18]

$$\delta(\tau) = -\frac{1}{8} - 3\sum_{n\geqslant 1}\left(\sum_{2\nmid d|n} d\right) q^{n/2}; \varepsilon(\tau) = \sum_{n\geqslant 1}\left(\sum_{2\nmid d|n}\left(\frac{n}{d}\right)^3\right) q^{n/2}.$$

From that it is easy to conclude

COROLLARY A.10.

$$M^{\Gamma_1(2)}(\mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2, \delta, \varepsilon, \Delta^{-1}]$$

$$\bar{M}^{\Gamma_1(2)}(\mathbb{Z}[1/2]) \cong \mathbb{Z}[1/2, \delta, \varepsilon]$$

The q-expansion map of $M_*^{\Gamma_1(N)} \underset{def}{=} M_*^{\Gamma_1(N)}(\mathbb{Z}[1/N, \zeta_N])$ can be used to push the formal group $\hat{F}_{\Gamma_1(N)}$ defined by the exponential f to the ring of power series. Explicitly, at the cusp ∞, i.e. at $(Tate(q^N), \omega_{can}, \zeta_N)$, we have the formulas [18, AI.7/6.4]:

$$f(z, \tau) = \frac{\Phi(z, \tau)\,\Phi(-2\pi i/N, \tau)}{\Phi(z - (2\pi i/N), \tau)}$$

where

$$\Phi(z, \tau) = (e^{z/2} - e^{-z/2}) \prod_{n=1}^{\infty} \frac{(1 - q^n e^z)(1 - q^n e^{-z})}{(1 - q^n)^2}.$$

A short calculation shows that f gives a power series in $\mathbb{Z}[1/n, \zeta_N,][\![w, q]\!]$ where $w = 1 - \exp(-z)$. We conclude with the q-expansion principle:

COROLLARY A.11. *There are unique isomorphisms of formal groups over $\mathbb{Z}((q)) \otimes \mathbb{Z}[1/N, \zeta_N]$*

$$\hat{G}_m \cong \lambda_*^{\Gamma_1(N)} \hat{F}_{\Gamma_1(N)} \cong \hat{F}_{Tate(q^N)}.$$

In particular, the ring inclusion

$$M_*^{\Gamma_1(N)} \hookrightarrow M_*^{\Gamma_1(NM)}$$

induces an isomorphism of formal groups. The equivalent statement also holds for

$$M_*^{\Gamma_1(1)} \underset{def}{=} M_*^{\Gamma_1(1)}(\mathbb{Z}[1/6]).$$

APPENDIX B. THE p-ADIC q-EXPANSION PRINCIPLE

Modular forms over the p-adic numbers as defined in Appendix A do not reflect the p-adic topology in a serious way as they are just the tensor product of integral modular forms with $\mathbb{Z}_p$. One wishes to allow limits of such forms in such a manner that forms with highly congruent q-expansion are close. The first approach for such a theory was taken by

Serre [43]. He identified p-adic modular forms with their q-expansion and showed they can be associated a weight which is a character $\chi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$. Serre modular forms form a subring of the ring of trivialized modular forms developed by Katz. Katz's theory is preferable to us as it allows a modular interpretation and his ring is the p-adic counterpart of $K_0 E^\Gamma$.

For a detailed and complete treatment of p-adic modular forms "with growth conditions" the reader is referred to the articles of Katz [23, 24, 26] and the overview given in [16].

B.1. Trivialized modular forms and diamond operators

Trivialized modular forms were first introduced by Katz in [24]. Katz uses the expression "generalized modular forms" as they include honest modular forms, modular forms in the sense of Serre and "modular forms with growth condition 1".

Let R be a p-adic ring. We will always assume that R contains a primitive Nth-root of unity with $p \nmid N$, and that it is a p-adically complete, discrete valuation ring or a quotient of such a ring.

Definition B.1. A trivialized $\Gamma_1(N)$-test object is a triple $(E_{/S}, \varphi, P)$ consisting of an elliptic curve over a p-adically complete and separated R-algebra S, a trivialization of the formal group of $E_{/S}$ by an isomorphism

$$\varphi : \hat{E} \xrightarrow{\sim} \hat{G}_m$$

over S and a point P of exact order N. A modular form for $\Gamma_1(N)$ over R is a rule f which assigns to any trivialized $\Gamma_1(N)$-test object a value

$$f(E_{/S}, \varphi, P) \in S$$

satisfying the following conditions:

(i) $f(E_{/S}, \varphi, P)$ depends only on the S-isomorphism class of $(E_{/S}, \varphi, P)$
(ii) the formation $f(E_{/S}, \varphi, P)$ commutes with arbitrary base change.

We denote by $T^{\Gamma_1(N)}(R)$ the ring of trivialized modular forms over R.

Note that we do not require these modular forms to have a weight. However, there is an action of $\mathbb{Z}_p^\times$ on $T^{\Gamma_1(N)}(R)$: For $a \in \mathbb{Z}_p^\times$ we define the diamond operator

$$([a]f)(E_{/S}, \varphi, P) = f(E_{/S}, a^{-1}\varphi, P).$$

It is clear that an ordinary $\Gamma_1(N)$-modular form f gives rise to a trivialized one by

$$f(E_{/S}, \varphi, P) = f(E_{/S}, \varphi^*(dT/(1+T)), P)$$

where $dT/(1+T)$ is the standard differential on $\hat{G}_m$. The Tate curve admits a canonical trivialization φ_{can} with

$$\varphi_{can}^*(dT/(1+T)) = \omega_{can}$$

in the notation of Appendix A.

Definition B.2. *The* q-expansion map of trivialized modular forms at $\zeta_N^i q^j$ is the ring homomorphism

$$\bar{\chi}^{\Gamma_1(N)} : T^{\Gamma_1(N)}(R) \to \widehat{R((q))}$$

given by

$$f \mapsto f(q) \underset{def}{=} f(Tate(q^N)_{\widehat{R((q))}}, \varphi_{can}, \zeta^i q^j).$$

We say f is holomorphic if it already q-expands in the subring $R[\![q]\!]$ and write $\bar{T}^{\Gamma_1(N)}(R)$ for the resulting ring.

THEOREM B.3. *If a trivialized $\Gamma_1(n)$-modular form over S already q-expands in $\widehat{R((q))}$ for some subring $R \subset S$ then there is a unique modular form $\tilde{f}$ over R which gives rise to f by extension of scalars. Moreover, a trivialized modular form is uniquely determined by its q-expansion.*

In [24] Katz gives two proofs of the theorem. The hard part is to show the injectivity of the q-expansion for a field of characteristic p. It is a consequence of the irreducibility of the moduli space of trivialized test objects for which Katz refers to Igusa or Ribet.

COROLLARY B.4. *Let W be the Witt vectors of a perfect field of characteristic p which contains a primitive Nth root of unity ζ_N. Let $\sum f_i$ be a sum of true $\Gamma_1(N)$-modular forms over $W[1/p]$ which q-expands in $\widehat{W((q))}$ at some cusp. Then so does $\sum a^i f_i$ for all $a \in \mathbb{Z}_p^\times$.*

Proof. Let N be such that $p^N f_i$ is a modular form over W. Then $\sum p^N f_i$ gives rise to an element in $T^{\Gamma_1(N)}(W)$ with q-expansion divisible by p^N. Thus, by the q-expansion principle $\sum p^N f_i$ is uniquely divisible by p^N. We conclude that $\sum f_i$ lies in $T^{\Gamma_1(N)}(W)$ and so does $[a] \sum f_i$. But the effect of the diamond operator is

$$\begin{aligned}([a] \textstyle\sum f_i)(E_{/S}, \varphi, P) &= (\textstyle\sum f_i)(E_{/S}, a^{-1}\varphi, P) \\ &= \textstyle\sum f_i(E_{/S}, a^{-1}\varphi^*(dT/(1+T))) \\ &= \textstyle\sum a^i f_i(E_{/S}, \varphi^*(dT/(1+T))) \\ &= (\textstyle\sum a^i f_i)(E_{/S}, \varphi, P).\end{aligned}$$

Hence, any q-expansion of $\sum a^i f_i$ is integral. $\square$