TOPOLOGICAL K-THEORY

JOHN ROGNES

0.1. **Real and complex K-theory.** The set of isomorphism classes of real vector bundles over a finite CW complex X forms a commutative monoid with respect to direct (Whitney) sum of vector bundles. The additive group completion of this commutative monoid is denoted $KO(X)$, and consists of formal differences between pairs of real vector bundles over X. The corresponding construction for complex vector bundles leads to the group $KU(X)$ of formal differences of pairs of complex vector bundles. By Bott periodicity, the external tensor product of vector bundles induces natural isomorphisms $KO(X) \otimes KO(S^8) \cong KO(X \times S^8)$ and $KU(X) \otimes KU(S^2) \cong KU(X \times S^2)$. In terms of the reduced K-groups $\widetilde{KO}(X) = \ker(KO(X) \to KO(*))$ and $\widetilde{KU}(X) = \ker(KU(X) \to KU(*))$, for based finite CW-complexes X, this can be expressed as isomorphisms $\widetilde{KO}(X) \cong \widetilde{KO}(\Sigma^8 X)$ and $\widetilde{KU}(X) \cong \widetilde{KU}(\Sigma^2 X)$. Hence there are generalized (reduced) cohomology theories KO^* and KU^* defined by $KO^n(X) = \widetilde{KO}(\Sigma^m X)$, where $n + m \equiv 0 \mod 8$, and $KU^n(X) = \widetilde{KU}(\Sigma^m X)$, where $n + m \equiv 0 \mod 2$. For definiteness, we may assume $0 \le m < 8$ in the real case, and $0 \le m < 2$ in the complex case. The internal tensor product of vector bundles induces products in these cohomology theories. Complexification, i.e, tensoring a real vector bundle with $\mathbb{C}$ over $\mathbb{R}$ to obtain a complex vector bundle, induces a multiplicative homomorphism $c\colon KO^*(X) \to KU^*(X)$. Realification, i.e., only remembering the underlying real vector bundle of a complex vector bundle, induces a homomorphism $r\colon KU^*(X) \to KO^*(X)$, which is not multiplicative, but is linear as a map of modules over the target.

The reduced K-functors $\widetilde{KO}$ and $\widetilde{KU}$ are represented by the infinite loop spaces $\mathbb{Z}\times BO$ and $\mathbb{Z}\times BU$, respectively, where $\mathbb{Z} \times BO \simeq \Omega^8(\mathbb{Z} \times BO)$ and $\mathbb{Z} \times BU \simeq \Omega^2(\mathbb{Z} \times BU)$ by Bott periodicity. The cohomology theories KO^* and KU^* are thus represented by Ω-spectra KO and KU, respectively, with n-th spaces $\underline{KO}_n = \Omega^m(\mathbb{Z}\times BO)$ and $\underline{KU}_n = \Omega^m(\mathbb{Z}\times BU)$, where m is chosen so that $n+m \equiv 0 \mod 8$ and $0 \le m < 8$ in the real case, and $n + m \equiv 0 \mod 2$ and $0 \le m < 2$ in the complex case. The tensor product pairing is represented by pairings of spectra, that make KO and KU into E_∞ ring spectra. The unit $S \to KO$ is generated by a map $S^0 \to \mathbb{Z} \times BO$ that takes the non-base point to a point in $\{1\} \times BO$, and similarly in the complex case. Complexification is represented by a ring spectrum map $c\colon KO \to KU$, and realification is represented by a KO-module map $r\colon KU \to KO$. The homotopy groups of these ring spectra are known, by Bott periodicity, to be

$$\pi_i(KO) = \begin{cases} \mathbb{Z}\{\beta^k\} & \text{for } i = 8k, \\ \mathbb{Z}/2\{\eta\beta^k\} & \text{for } i = 8k+1, \\ \mathbb{Z}/2\{\eta^2\beta^k\} & \text{for } i = 8k+2, \\ \mathbb{Z}\{\alpha\beta^k\} & \text{for } i = 8k+4, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_i(KU) = \begin{cases} \mathbb{Z}\{u^k\} & \text{for } i = 2k \text{ even}, \\ 0 & \text{for } i \text{ odd}. \end{cases}$$

As graded rings, these are

$$\pi_*(KO) = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

with η, α and β in degree 1, 4 and 8, respectively, and

$$\pi_*(KU) = \mathbb{Z}[u^{\pm 1}]$$

with u in degree 2. Complexification is given by $\eta \mapsto 0$, $\alpha \mapsto 2u^2$ and $\beta \mapsto u^4$. Realification is given by $u^{4k} \mapsto 2\beta^k$, $u^{4k+1} \mapsto \eta^2\beta^k$, $u^{4k+2} \mapsto \alpha\beta^k$ and $u^{4k+3} \mapsto 0$.

There are connective, i.e. (-1)-connected, covers of these ring spectra, denotes ko and ku, respectively, with ring spectrum maps $ko \to KO$ and $ku \to KU$ that induce isomorphisms of homotopy groups in

Date: May 11th 2015.

non-negative degrees. Hence $\pi_i(ko) \cong \pi_i(KO)$ for $i \geq 0$ and $\pi_i(ko) = 0$ for $i < 0$, and similarly in the complex case. As graded rings,

$$\pi_*(ko) = \mathbb{Z}[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 - 4\beta)$$

and

$$\pi_*(ku) = \mathbb{Z}[u]\,.$$

The n-th space $\underline{ko}_n$ of the spectrum ko is an $(n-1)$-connected cover of the n-the space $\underline{KO}_n$, and similarly in the complex case. For example, $\underline{ku}_0 \simeq \mathbb{Z} \times BU$, $\underline{ku}_1 \simeq U$, $\underline{ku}_2 \simeq BU$, $\underline{ku}_3 \simeq SU$ and $\underline{ku}_4 \simeq BSU$.

0.2. **Cohomology and homotopy of K-theory spectra.** Recall that $H^*(H) \cong \mathscr{A}$ and $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}Sq^1 = \mathscr{A} \otimes_{A(0)} \mathbb{F}_2 = \mathscr{A}//A(0)$, where $A(0) = E(Sq^1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1.

Let bu denote the 1-connected cover of ku, so that there is a cofiber sequence

$$bu \to ku \xrightarrow{p_0} H\mathbb{Z} \to \Sigma bu$$

and a Bott equivalence $u\colon \Sigma^2 ku \simeq bu$.

Proposition 0.1. *$H^*(ku) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} = \mathscr{A} \otimes_{E(1)} \mathbb{F}_2 = \mathscr{A}//E(1)$, where $Q_1 = [Sq^1, Sq^2] = Sq^3 + Sq^2Sq^1$ and $E(1) = E(Sq^1, Q_1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1 and Q_1. Hence there is a short exact sequence*

$$0 \to \Sigma^3\mathscr{A}//E(1) \longrightarrow \mathscr{A}//A(0) \xrightarrow{p_0^*} \mathscr{A}//E(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^3\mathbb{F}_2 \longrightarrow E(1)//A(0) \longrightarrow \mathbb{F}_2$ of $E(1)$-modules.

Proof. It is known, from calculations in $H^*(SU)$, that the bottom Postnikov k-invariant of ku, i.e., the composite $H\mathbb{Z} \to \Sigma bu \simeq \Sigma^3 ku \to \Sigma^3 H\mathbb{Z}$ viewed as a class in $H^3(H\mathbb{Z};\mathbb{Z})$, is nonzero. This implies that $H\mathbb{Z} \to \Sigma bu$ induces an isomorphism on H^3, so that $bu \to ku$ and $u\colon S^2 \to ku$ induce zero homomorphisms on H^2. It follows that the Bott equivalence $\phi \circ (1 \wedge u)\colon bu \simeq ku \wedge S^2 \to ku \wedge ku \to ku$ induces 0 in cohomology. Hence we have a map of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^3\mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} \arrow[r] \arrow[d, "\Sigma^3 f"'] & \mathscr{A}/\mathscr{A}Sq^1 \arrow[r] \arrow[d, "\cong"'] & \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} \arrow[r] \arrow[d, "f"'] & 0 \\
0 \arrow[r] & H^*(\Sigma bu) \arrow[r] & H^*(H\mathbb{Z}) \arrow[r, "p_0^*"] & H^*(ku) \arrow[r] & 0
\end{tikzcd}$$

It follows by induction that f is an isomorphism in all degrees. □

Let bo, bso, $bspin$ and $bo\langle 8\rangle$ be the 0-, 1-, 3- and 7-connected covers of ko, respectively, so that there are cofiber sequences

$$bo \to ko \xrightarrow{p_0} H\mathbb{Z} \to \Sigma bo$$

$$bso \to bo \xrightarrow{p_1} \Sigma H \to \Sigma bso$$

$$bspin \to bso \xrightarrow{p_2} \Sigma^2 H \to \Sigma bspin$$

$$bo\langle 8\rangle \to bspin \xrightarrow{p_4} \Sigma^4 H\mathbb{Z} \to \Sigma bo\langle 8\rangle$$

and a Bott equivalence $\beta\colon \Sigma^8 ko \simeq bo\langle 8\rangle$.

There is a cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \xrightarrow{c} ku \longrightarrow \Sigma^2 ko\,,$$

where c denotes the complexification map and η denotes multiplication with the Hopf map $\eta\colon S^1 \to S$. The connecting map $ku \to \Sigma^2 ko$ lifts the composite map $\Sigma^2 r \circ u^{-1}\colon KU \to \Sigma^2 KU \to \Sigma^2 KO$. The spectra ko and ku are (E_∞) ring spectra, and c is a ring spectrum map.

Proposition 0.2. *$H^*(ko) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} = \mathscr{A} \otimes_{A(1)} \mathbb{F}_2 = \mathscr{A}//A(1)$, where $A(1)$ is the subalgebra of $\mathscr{A}$ generated by Sq^1 and Sq^2. Hence there is a short exact sequence*

$$0 \to \Sigma^2\mathscr{A}//A(1) \longrightarrow \mathscr{A}//E(1) \xrightarrow{c^*} \mathscr{A}//A(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^2\mathbb{F}_2 \longrightarrow A(1)//E(1) \longrightarrow \mathbb{F}_2$ of $A(1)$-modules.

$H^*(bo) \cong \Sigma\mathscr{A}/\mathscr{A}Sq^2 = \Sigma\mathscr{A} \otimes_{A(1)} A(1)/A(1)Sq^2$, and there is a short exact sequence

$$0 \to \Sigma^2\mathscr{A}/\mathscr{A}Sq^2 \longrightarrow \mathscr{A}//A(0) \xrightarrow{p_0^*} \mathscr{A}//A(1) \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^2 A(1)/A(1)Sq^2 \longrightarrow A(1)//A(0) \longrightarrow \mathbb{F}_2$ of $A(1)$-modules.

$H^*(bso) \cong \Sigma^2\mathscr{A}/\mathscr{A}Sq^3 = \Sigma^2\mathscr{A} \otimes_{A(1)} A(1)/A(1)Sq^3$, and there is a short exact sequence

$$0 \to \Sigma^3\mathscr{A}/\mathscr{A}Sq^3 \longrightarrow \Sigma\mathscr{A} \xrightarrow{p_1^*} \Sigma\mathscr{A}/\mathscr{A}Sq^2 \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^3 A(1)/A(1)Sq^3 \longrightarrow \Sigma A(1) \longrightarrow \Sigma A(1)/A(1)Sq^2$ of $A(1)$-modules.

$H^*(bspin) \cong \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} = \Sigma^4\mathscr{A} \otimes_{A(1)} A(1)/A(1)\{Sq^1, Sq^2Sq^3\}$, and there is a short exact sequence

$$0 \to \Sigma^5\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \longrightarrow \Sigma^2\mathscr{A} \xrightarrow{p_2^*} \Sigma^2\mathscr{A}/\mathscr{A}Sq^3 \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^5 A(1)/A(1)\{Sq^1, Sq^2Sq^3\} \longrightarrow \Sigma^2 A(1) \longrightarrow \Sigma^2 A(1)/A(1)Sq^3$ of $A(1)$-modules.

$H^*(bo\langle 8\rangle) \cong \Sigma^8\mathscr{A}//A(1)$, and there is a short exact sequence

$$0 \to \Sigma^9\mathscr{A}//A(1) \longrightarrow \Sigma^4\mathscr{A}//A(0) \xrightarrow{p_4^*} \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \to 0$$

of $\mathscr{A}$-modules, induced up from the extension $\Sigma^9\mathbb{F}_2 \longrightarrow \Sigma^4 A(1)//A(0) \longrightarrow \Sigma^4 A(1)/A(1)\{Sq^1, Sq^2Sq^3\}$ of $A(1)$-modules.

Proof. The map $\eta\colon S^1 \to S$ induces the zero homomorphism in cohomology, hence so does $\eta\colon \Sigma ko \to ko$, and there is a vertical map of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \Sigma^2\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} \arrow[r] \arrow[d, "\Sigma^2 f"] & \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} \arrow[r] \arrow[d, "\cong"] & \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & H^*(\Sigma^2 ko) \arrow[r] & H^*(ku) \arrow[r, "c^*"] & H^*(ko) \arrow[r] & 0
\end{tikzcd}$$

It follows by induction that f is an isomorphism in all degrees.

The map $p_0\colon ko \to H\mathbb{Z}$ is 0-connected, hence $p_0^*\colon \mathscr{A}/\mathscr{A}Sq^1 \to \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$ is an isomorphism in degree 0 and surjective in all degrees. Hence p_0 is induced up from the surjection $\epsilon\colon A(1)//A(0) \to \mathbb{F}_2$ of $A(1)$-modules, with kernel $\ker(\epsilon) = \mathbb{F}_2\{Sq^2, Sq^3, Sq^2Sq^3\} \cong \Sigma^2 A(1)/A(1)Sq^2$. Hence $\Sigma H^*(bo) \cong \ker(p_0^*) \cong \mathscr{A} \otimes_{A(1)} \Sigma^2 A(1)/A(1)Sq^2 \cong \Sigma^2\mathscr{A}/\mathscr{A}Sq^2$.

((ETC)) □

Theorem 0.3 (Change of rings)**.** *Let A be any algebra, let $B \subset A$ be a subalgebra such that A is flat as a right B-module, let M be a left B-module and let N be a left A-module. Then there is a natural isomorphism*

$$\operatorname{Ext}_A^{*,*}(A \otimes_B M, N) \cong \operatorname{Ext}_B^{*,*}(M, N)\,.$$

Proof. Let $P_* \to M$ be a B-free resolution. Then $A \otimes_B P_* \to A \otimes_B M$ is an A-free resolution. The isomorphism $\operatorname{Hom}_A(A \otimes_B P_*, N) \cong \operatorname{Hom}_B(P_*, N)$ then induces the asserted isomorphism on passage to cohomology. □

Corollary 0.4. *There are Adams spectral sequences*

$$E_2^{s,t} = \operatorname{Ext}_{E(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(ku)_2^\wedge$$

and

$$E_2^{s,t} = \operatorname{Ext}_{A(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(ko)_2^\wedge\,.$$

Proof. The E_2-term of the Adams spectral sequence for ku is

$$\operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(ku), \mathbb{F}_2) \cong \operatorname{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E(1), \mathbb{F}_2) \cong \operatorname{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

and the E_2-term of the Adams spectral sequence for ko is

$$\operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(ko), \mathbb{F}_2) \cong \operatorname{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//A(1), \mathbb{F}_2) \cong \operatorname{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)\,,$$

in both cases by the change-of-rings isomorphism. □

Corollary 0.5. *There is an exact sequence of $A(1)$-modules*

$$0 \to \Sigma^{12}\mathbb{F}_2 \xrightarrow{\eta} \Sigma^7 A(1)//A(0) \xrightarrow{\partial_3} \Sigma^4 A(1) \xrightarrow{\partial_2} \Sigma^2 A(1) \xrightarrow{\partial_1} A(1)//A(0) \xrightarrow{\epsilon} \mathbb{F}_2 \to 0\,.$$

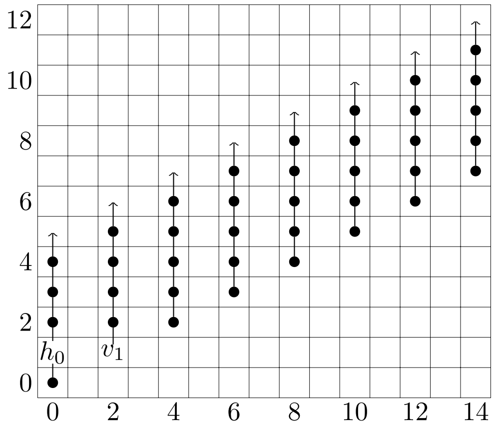

FIGURE 1. The Adams spectral sequence for ku

Proposition 0.6. $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2,\mathbb{F}_2) \cong P(h_0, v_1)$ *where* h_0 *in bidegree* $(s,t)=(1,1)$ *is dual to* Sq^1 *and* v_1 *in bidegree* $(s,t)=(1,3)$ *is dual to* Q_1.

The E_2-term of the Adams spectral sequence for ku is displayed in Figure 1. There is no room for differentials, and the permanent cycles h_0 and v_1 detect 2 and u, respectively, in $\pi_*(ku)^\wedge_2 = \mathbb{Z}_2[u]$.

Proposition 0.7. $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2,\mathbb{F}_2) \cong P(h_0,h_1,v,w_1)/(h_0h_1, h_1^3, h_1v, v^2-h_0^2w_1)$ *where* h_0 *in bidegree* $(s,t)=(1,1)$ *is dual to* Sq^1, *where* h_1 *in bidegree* $(s,t)=(1,2)$ *is dual to* Sq^2, v *is in bidegree* $(s,t)=(3,7)$ *and* w_1 *is in bidegree* $(s,t)=(4,12)$.

Proof. The central extension

$$E(Q_1) \to A(1) \to E(Sq^1, Sq^2)$$

of augmented algebras leads to a Cartan–Eilenberg spectral sequence

$$E_2^{p,q,*} = \mathrm{Ext}^{p,*}_{E(Sq^1,Sq^2)}(\mathbb{F}_2, \mathrm{Ext}^{q,*}_{E(Q_1)}(\mathbb{F}_2,\mathbb{F}_2)) \Longrightarrow \mathrm{Ext}^{p+q,*}_{A(1)}(\mathbb{F}_2,\mathbb{F}_2)$$

where the $E(Sq^1,Sq^2)$-module structure on $\mathrm{Ext}^*_{E(Q_1)}(\mathbb{F}_2,\mathbb{F}_2) = P(h_{01})$ is trivial. Hence the E_2-term can be written as

$$E_2^{*,*,*} = P(h_0,h_1) \otimes P(h_{01})$$

with h_0 in bidegree $(p,q,t)=(1,0,1)$ dual to Sq^1, h_1 in bidegree $(p,q,t)=(1,0,2)$ dual to Sq^2 and h_{01} in bidegree $(p,q,t)=(0,1,3)$ dual to Q_1.

There are differentials $d_2(h_{01}) = h_0h_1$, so that

$$E_3^{*,*,*} = P(h_0,h_1)/(h_0h_1) \otimes P(h_{01}^2)$$

and $d_3(h_{01}^2) = h_1^3$, so that

$$E_4^{*,*,*} = P(h_0,h_1,v,w_1)/(h_0h_1,h_1^3,h_1v,v^2-h_0^2w_1)$$

with $v = h_0h_{01}^2$ and $w_1 = h_{01}^4$. ((Justify the differentials with cobar calculations?)) Then $E_4 = E_\infty$ for degree reasons, and there is no room for multiplicative extensions between the E_∞-term and $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2,\mathbb{F}_2)$. □

The E_2-term of the Adams spectral sequence for ko is displayed in Figure 2. There is no room for differentials, and the permanent cycles h_0, h_1, v and w_1 detect 2, η, α and β, respectively, in $\pi_*(ko)^\wedge_2 = \mathbb{Z}_2[\eta,\alpha,\beta]/(2\eta,\eta^3,\eta\alpha,\alpha^2-4\beta)$.

The unit map $d\colon S \to ko$ induces a ring homomorphism $d_*\colon \pi_*(S)^\wedge_2 \to \pi_*(ko)^\wedge_2$ that takes $\eta \in \pi_1(S)^\wedge_2$ (detected by h_1, dual to the indecomposable Sq^2 in $\mathscr{A}$) to $\eta \in \pi_1(ko)^\wedge_2$ (detected by h_1, dual to the indecomposable Sq^2 in $A(1)$), hence also maps $\eta^2 \in \pi_2(S)^\wedge_2$ to $\eta^2 \in \pi_2(ko)^\wedge_2$. This is the KO-theory d-invariant. The classes α and β are of infinite (additive) order, hence cannot be in the image of the finite groups $\pi_4(S)^\wedge_2$ and $\pi_8(S)^\wedge_2$. However, a calculation of maps of $\mathscr{A}$-module resolutions shows that the homomorphism $d_*\colon \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2,\mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{A(1)}(\mathbb{F}_2,\mathbb{F}_2)$ of Adams E_2-terms for S and ko is an isomorphism in the bidegrees $(t-s,s) = (8k+1, 4k+1)$ and $(t-s,s) = (8k+2,4k+2)$ with $k \ge 0$. Hence the permanent cycles P^kh_1 and $h_1P^kh_1$ in the Adams spectral sequence for S map to the survivors $h_1w_1^k$ and $h_1^2w_1^k$ in the Adams spectral sequence for ko. It follows that there are nonzero classes μ_{8k+1} and

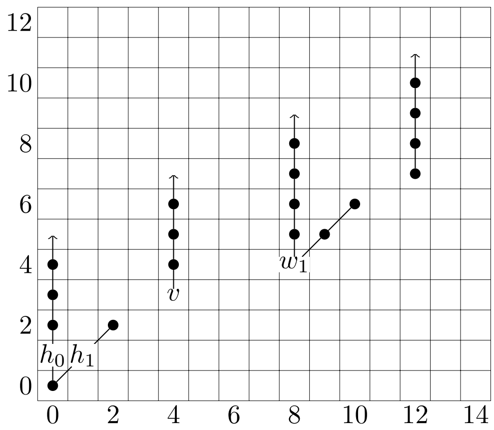

FIGURE 2. The Adams spectral sequence for ko

μ_{8k+2} in $\pi_*(S)^\wedge_2$ that map to $\eta\beta^k$ and $\eta^2\beta^k$, respectively, in $\pi_*(ko)^\wedge_2$. For instance, $\mu_1 = \eta$, $\mu_2 = \eta^2$, $\mu_9 = \mu$ and $\mu_{10} = \eta\mu$, in the notation previously introduced in $\pi_*(S)^\wedge_2$. In general, $\eta\mu_{8k+1} = \mu_{8k+2}$.

((Discuss map $c\colon ko \to ku$ mapping $h_0 \mapsto h_0$, $h_1 \mapsto 0$, $v \mapsto h_0 h_1^2$ and $w_1 \mapsto v_1^4$. Hence $v \mapsto 2u^2$ and $w_1 \mapsto u^4$ in homotopy.))

((After discussing the dual Steenrod algebra, and the calculation of $H_*(ku)$ and $H_*(ku)$, give alternative proof with $A(1)_*$-comodule algebra resolution $\mathbb{F}_2 \to E(\xi_1^2, \bar\xi_2) \otimes P(x_2, x_3)$, with $d(\xi_1^2) = x_2$ and $d(\bar\xi_3) = x_3$.))

0.3. **Adams vanishing.** The subalgebra $A(1)$ inherits the structure of a cocommutative Hopf algebra from $\mathscr{A}$, with the restricted coproduct and conjugation, so that the category of $A(1)$-modules has a symmetric monoidal tensor product given by the diagonal $A(1)$-action.

We start with an easy but not optimal vanishing estimate.

Lemma 0.8. *Let M be connective $A(1)$-module that is free as an $A(0)$-module. Then* $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ *for $t - s < s$.*

Proof. The claim is clear for $s = 0$, since M is concentrated in degrees $* \ge 0$. We prove the claim for $s \ge 1$ by induction.

Note that $A(1)//A(0) = \mathbb{F}_2\{1, Sq^2, Sq^3, Sq^2Sq^3\}$ is concentrated in degrees 0, 2, 3 and 5. The $A(1)$-module action on M induces a short exact sequence

$$0 \to \Sigma^2 K \longrightarrow A(1) \otimes_{A(0)} M \longrightarrow M \to 0$$

of $A(1)$-modules, where also K is connective. Here $A(1) \otimes_{A(0)} M \cong A(1)//A(0) \otimes M$ as $A(1)$-modules, by the untwisting isomorphism [[in the relative case for $A(0) \subset A(1)$]]. Furthermore, $A(1)//A(0) \otimes M$ is a direct sum of suspensions of $A(1)//A(0) \otimes A(0) \cong A(0) \otimes A(1)//A(0)$, as an $A(0)$-module, and the latter $A(0)$-module is free. Hence $A(1) \otimes_{A(0)} M$ is free as an $A(0)$-module, so that $\Sigma^2 K$ is stably free (and projective) as an $A(0)$-module. It follows that K is free as an $A(0)$-module.

Consider the long exact sequence

$$\cdots \to \mathrm{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{A(1)}(A(1) \otimes_{A(0)} M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \to \dots .$$

Here $\mathrm{Ext}^{s,t}_{A(1)}(A(1) \otimes_{A(0)} M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2)$. Since M is free as an $A(0)$-module, $\mathrm{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2) = 0$ for $s \ge 1$, so that the connecting homomorphism δ in the long exact sequence above is surjective. Furthermore, $\mathrm{Ext}^{s-1,t}_{A(1)}(\Sigma^2 K, \mathbb{F}_2) \cong \mathrm{Ext}^{s-1,t-2}_{A(1)}(K, \mathbb{F}_2)$ is 0 for $(t-2) - (s-1) < s - 1$ by the inductive hypothesis, i.e., for $t - s < s$. Hence $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ for $t - s < s$, as asserted. □

((Can we get vanishing also for $t - s = s$ when $s = 3$? If so, we may use $\epsilon'(s) = 2$ for $s \equiv 3 \mod 4$, $\epsilon''(s) = 1$ and 2 for $s \equiv 0$ and $3 \mod 4$, and $\epsilon(s) = 3$ and 2 for $s \equiv 0$ and $1 \mod 4$, in the following results.))

Proposition 0.9. *Let $\epsilon'(s) = 0$, 1, 2 and 3 for $s \equiv 0$, 1, 2 and $3 \mod 4$, respectively, and let M be a connective $A(1)$-module that is free as an $A(0)$-module. Then* $\mathrm{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) = 0$ *for $t - s < 2s - \epsilon'(s)$.*

Proof. As remarked above, we may assume that this has been proved for $0 \le s \le 3$. We prove the claim for $s \ge 4$ by induction.

We tensor the exact sequence from Corollary 0.5 with M, to obtain an exact sequence

$$0 \to \Sigma^{12} M \xrightarrow{1\otimes\eta} \Sigma^7 A(1)//A(0) \otimes M \xrightarrow{1\otimes\partial_3} \Sigma^4 A(1) \otimes M \xrightarrow{1\otimes\partial_2} \Sigma^2 A(1) \otimes M \xrightarrow{1\otimes\partial_1} A(1)//A(0) \otimes M \xrightarrow{1\otimes\epsilon} M \to 0$$

of $A(1)$-modules. It splits into four short exact sequences

$$\begin{gathered} 0 \to \operatorname{im}(1 \otimes \partial_1) \longrightarrow A(1)//A(0) \otimes M \longrightarrow M \to 0 \\ 0 \to \operatorname{im}(1 \otimes \partial_2) \longrightarrow \Sigma^2 A(1) \otimes M \longrightarrow \operatorname{im}(1 \otimes \partial_1) \to 0 \\ 0 \to \operatorname{im}(1 \otimes \partial_3) \longrightarrow \Sigma^4 A(1) \otimes M \longrightarrow \operatorname{im}(1 \otimes \partial_2) \to 0 \\ 0 \to \Sigma^{12} M \longrightarrow \Sigma^7 A(1)//A(0) \otimes M \longrightarrow \operatorname{im}(1 \otimes \partial_3) \to 0 \end{gathered}$$

of $A(1)$-modules, which induce long exact sequences for $\operatorname{Ext}^{*,*}_{A(1)}(-, \mathbb{F}_2)$. By the untwisting isomorphism, $A(1)//A(0) \otimes M \cong A(1) \otimes_{A(0)} M$, and since M is free as an $A(0)$-module, $\operatorname{Ext}^{s,t}_{A(1)}(A(1)//A(0) \otimes M, \mathbb{F}_2) \cong \operatorname{Ext}^{s,t}_{A(0)}(M, \mathbb{F}_2)$ is 0 for all $s \ge 1$. Likewise, $A(1) \otimes M$ is free as an $A(1)$-module, so $\operatorname{Ext}^{s,t}_{A(1)}(A(1) \otimes M, \mathbb{F}_2)$ is 0 for all $s \ge 1$. Hence there is a chain of surjections

$$\begin{aligned} \operatorname{Ext}^{s-4,t-12}_{A(1)}(M, \mathbb{F}_2) = \operatorname{Ext}^{s-4,t}_{A(1)}(\Sigma^{12} M, \mathbb{F}_2) &\xrightarrow{\delta} \operatorname{Ext}^{s-3,t}_{A(1)}(\operatorname{im}(1 \otimes \partial_3), \mathbb{F}_2) \\ &\xrightarrow{\delta} \operatorname{Ext}^{s-2,t}_{A(1)}(\operatorname{im}(1 \otimes \partial_2), \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}^{s-1,t}_{A(1)}(\operatorname{im}(1 \otimes \partial_1), \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2) \end{aligned}$$

for all $s \ge 4$.

By induction, we know that $\operatorname{Ext}^{s-4,t-12}_{A(1)}(M, \mathbb{F}_2) = 0$ for $(t-12) - (s-4) < 2(s-4) - \epsilon'(s-4)$, or equivalently, for $t - s < 2s - \epsilon'(s)$. This completes the inductive step. □

Theorem 0.10. *Let $\epsilon''(s) = 2$, 1, 2 and 3 for $s \equiv 0$, 1, 2 and $3 \mod 4$, respectively, and let M be a connective $\mathscr{A}$-module that is free as an $A(0)$-module. Then $\operatorname{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2) = 0$ for $t - s < 2s - \epsilon''(s)$.*

Proof. Since M is connective, it is clear that $\operatorname{Ext}^{0,t}_{\mathscr{A}}(M, \mathbb{F}_2) = 0$ for $t < 0$, which is stronger than the claim for $s = 0$. We prove the claim for $s \ge 1$ by induction on s. The function ϵ'' is chosen so that $\epsilon'(s) \le \epsilon''(s)$ and $\epsilon''(s-1) - 1 \le \epsilon''(s)$ for all $s \ge 1$.

Note that $\mathscr{A}//A(1) = \mathbb{F}_2\{1, Sq^4, \dots\}$ with the remaining generators in degrees $* \ge 4$. The $\mathscr{A}$-module action on M induces a short exact sequence

$$0 \to \Sigma^4 L \longrightarrow \mathscr{A} \otimes_{A(1)} M \longrightarrow M \to 0$$

of $\mathscr{A}$-modules, where L is connective. Hence there is a long exact sequence

$$\cdots \to \operatorname{Ext}^{s-1,t}_{\mathscr{A}}(\Sigma^4 L, \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathscr{A} \otimes_{A(1)} M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(\Sigma^4 L, \mathbb{F}_2) \to \dots .$$

Here $\operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathscr{A} \otimes_{A(1)} M, \mathbb{F}_2) \cong \operatorname{Ext}^{s,t}_{A(1)}(M, \mathbb{F}_2)$ is 0 for $t - s < 2s - \epsilon'(s)$, by the previous proposition. By induction, $\operatorname{Ext}^{s-1,t}_{\mathscr{A}}(\Sigma^4 L, \mathbb{F}_2) = \operatorname{Ext}^{s-1,t-4}_{\mathscr{A}}(L, \mathbb{F}_2)$ is 0 for $(t-4) - (s-1) < 2(s-1) - \epsilon''(s-1)$, or equivalently, for $t - s < 2s + 1 - \epsilon''(s-1)$. If $t - s < 2s - \epsilon''(s)$ then both inequalities are satisfied, which implies that $\operatorname{Ext}^{s,t}_{\mathscr{A}}(M, \mathbb{F}_2) = 0$. This completes the inductive step. □

Theorem 0.11 (Adams vanishing (weak form))**.** *Let $\epsilon(s) = 4$, 3, 2 and 3 for $s \equiv 0$, 1, 2 and $3 \mod 4$, respectively. Then $\operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t - s < 2s - \epsilon(s)$.*

Proof. Define an $\mathscr{A}$-module M by the short exact sequence

$$0 \to \Sigma^2 M \to \mathscr{A}//A(0) \to \mathbb{F}_2 \to 0 .$$

Recall the basis for $\mathscr{A} = \mathbb{F}_2\{Sq^I\}$ given by the admissible monomials Sq^I, where $I = (i_1, \dots, i_\ell)$ with $i_u \ge 2i_{u+1}$ for each $1 \le u < \ell$, and $i_\ell \ge 1$. The admissible monomials with $i_\ell \ge 2$, including the empty monomial $I = ()$, give a basis for $\mathscr{A}$ as a free right $A(0)$-module, hence also for $\mathscr{A}//A(0)$ as $\mathbb{F}_2$-vector space. The nonempty admissible monomials with $i_\ell \ge 2$ then give a basis for $\Sigma^2 M$. In particular, M is connective. Note now that M is free as a left $A(0)$-module. A basis is given by the Sq^I with I admissible, $i_1 = 2k$ even and $i_\ell \ge 2$, in view of the Adem relation $Sq^1 Sq^{2k} = Sq^{2k+1}$.

Consider the long exact sequence

$$\cdots \to \operatorname{Ext}^{s-1,t}_{\mathscr{A}}(\Sigma^2 M, \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathscr{A} \otimes_{A(0)} \mathbb{F}_2, \mathbb{F}_2) \to \dots .$$

Here $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathscr{A} \otimes_{A(0)} \mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(0)}(\mathbb{F}_2, \mathbb{F}_2)$ is 0 for $t - s \neq 0$. Furthermore, $\mathrm{Ext}^{s-1,t}_{\mathscr{A}}(\Sigma^2 M, \mathbb{F}_2) = \mathrm{Ext}^{s-1,t-2}_{\mathscr{A}}(M, \mathbb{F}_2)$ is 0 for $(t-2)-(s-1) < 2(s-1)-\epsilon''(s-1)$, or equivalently, for $t-s < 2s-1-\epsilon''(s-1)$. We have defined $\epsilon(s) = \epsilon''(s-1)+1$, hence $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $0 < t-s < 2s - \epsilon(s)$, as asserted. $\square$

Remark 0.12. With more work, Adams (1966?) proved that one may deduce the same conclusion with $\epsilon(s) = 1$, 1, 2 and 3 for $s \equiv 0$, 1, 2 and 3 $\mod 4$, respectively, which is the optimal result for $s \geq 1$.

((Can the optimal result be deduced from periodicity and the low-dimensional calculations?))

0.4. **Adams operations.** For each natural number r, Adams (1962) defined natural operations $\psi^r \colon KO(X) \to KO(X)$ and $\psi^r \colon KU(X) \to KU(X)$. For a sum of line bundles, $E = L_1 \oplus \dots \oplus L_k$, the Adams operation is given by the sum of tensor powers $\psi^r(E) = L_1^{\otimes r} \oplus \dots \oplus L_k^{\otimes r}$. This determines its behavior on general vector bundles by naturality and the splitting principle. A recursive construction can be given in terms of exterior powers $\Lambda^i(E)$ of vector bundles, using Newton's identities, by the formula

$$-\psi^r(E) = \sum_{i=1}^{r-1} (-1)^i \Lambda^i(E) \otimes \psi^{r-i}(E) + (-1)^r r \Lambda^r(E) \,.$$

The resulting operation is additive and multiplicative, hence extends over the group completion, to ring operations as indicated above. The real and complex Adams operations are compatible under complexification.

The Adams operations do not commute with the Bott periodicity isomorphisms. In the complex case, the Bott isomorphism $\widetilde{KU}(X) \cong \widetilde{KU}(\Sigma^2 X)$ is induced by the product with the generator $u = 1 - H$ of $\widetilde{KU}(S^2)$, where $KU(S^2) = \mathbb{Z}\{1, H\}$ is generated by the isomorphism classes 1 and H of the trivial and the canonical (Hopf) complex line bundles over $S^2 = \mathbb{C}P^1$, respectively. Here $H + H = 1 + H^2$, so $u^2 = (1-H)^2 = 0$. The complex Adams operation ψ^r maps the generator u to

$$\psi^r(u) = \psi^r(1-H) = 1 - H^r = 1 - (1-u)^r = 1 - (1 - ru) = ru \,,$$

i.e., acts by multiplication by r on $\widetilde{KU}(S^2)$. To extend the Adams operation to the graded groups $KU^n(X) = \widetilde{KU}(\Sigma^m X)$, where $n+m = 2k$, we must localize by inverting r, and define ψ^r on $KU^n(X)[1/r]$ as $(1/r^k)\psi^r$ on $\widetilde{KU}(\Sigma^m X)[1/r]$. The result is a map of ring spectra $\psi^r \colon KU[1/r] \to KU[1/r]$, which restricts to a map of connective ring spectra $\psi^r \colon ku[1/r] \to ku[1/r]$. At the level of homotopy groups, $\psi^r(u^k) = r^k u^k$ in degree $2k$, for all integers k. Similarly, the real Adams operation induces ring spectrum maps $\psi^r \colon KO[1/r] \to KO[1/r]$ and $\psi^r \colon ko[1/r] \to ko[1/r]$. If we complete at a fixed prime p, then $\psi^r \colon ko^\wedge_p \to ko^\wedge_p$ and $\psi^r \colon ku^\wedge_p \to ku^\wedge_p$ are defined for all r that are prime to p. For instance, when $p = 2$, ψ^r is defined for all odd r.

The natural numbers prime to p are dense in the topological group $\mathbb{Z}_p^\times$ of p-adic units, and it is possible to define p-complete Adams operations $\psi^r \colon KU^\wedge_p \to KU^\wedge_p$ for all p-adic units $r \in \mathbb{Z}_p^\times$. This defines actions through E_∞ ring spectrum maps of $\mathbb{Z}_p^\times$ on $KU^\wedge_p$ and $ku^\wedge_p$, with $r \in \mathbb{Z}_p^\times$ acting by $\psi^r(u) = ru$ in homotopy. In particular, ψ^{-1} acts as complex conjugation on KU and ku, taking a complex vector bundle to the same real vector bundle but with the opposite complex structure. There are compatible actions on $KO^\wedge_p$ and $ko^\wedge_p$, with $\psi^r(\alpha) = r^2\alpha$ and $\psi^r(\beta) = r^4\beta$. In this case ψ^{-1} acts as the identity.

0.5. **The image-of-J spectrum.** Let all spectra be implicitly completed at 2. The Adams operation $\psi^3 \colon ko \to ko$ is compatible with the unit map $d \colon S \to ko$, hence the latter lifts to a unit map

$$S \longrightarrow ko^{h\psi^3} = \mathrm{hoeq}(\psi^3, 1 \colon ko \to ko)$$

to the homotopy fixed points of ψ^3 acting on ko. Here $ko^{h\psi^3}$ is an E_∞ ring spectrum, and additively there is a homotopy (co-)fiber sequence

$$\Sigma^{-1} ko \longrightarrow ko^{h\psi^3} \longrightarrow ko \xrightarrow{\psi^3 - 1} ko \,.$$

The unit map $d \colon S \to ko$ is 3-connected, in the sense that $\pi_i(S) \to \pi_i(ko)$ is an isomorphism for $i \geq 2$, and is surjective for $i = 3$. Hence $\psi^3 - 1$ induces the zero homomorphism in degrees $i \leq 3$, so the unit map $S \to ko^{h\psi^3}$ is not an equivalence in low degrees. We correct for this in the following definition. Let j be the E_∞ ring spectrum defined by the right hand pullback square in the following commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "e"] \arrow[d] & j \arrow[r] \arrow[d] & ko^{h\psi^3} \arrow[d] \\
P^2S \arrow[r, equal] & P^2S \arrow[r] & P^2(ko^{h\psi^3})
\end{tikzcd}$$

Here P^2X denotes the second Postnikov section of X, obtained by attaching cells (in the category of E_∞ ring spectra) to kill $\pi_i(X)$ for $i \geq 3$. There is then a homotopy (co-)fiber sequence

$$\Sigma^{-1}bspin \xrightarrow{\partial} j \to ko \xrightarrow{\psi^3-1} bspin\,.$$

Here $\psi^3 - 1$ maps $\alpha\beta^k$ to $3^{2+4k} - 1$ times $\alpha\beta^k$, which is 8 times an odd number, for all $k \geq 0$. Likewise it maps β^k to $3^{4k} - 1$ times β^k, which has 2-valuation $4 + v_2(k)$ for all $k \geq 1$. In other words, $\psi^3 - 1$ multiplies by $16k$ in degree $8k$, up to multiplication and division by odd factors.

We can use this to calculate the homotopy groups of the connective E_∞ ring spectrum $j = j_2^\wedge$:

$$\pi_i(j) = \begin{cases} \mathbb{Z}_2\{\iota\} & \text{for } i = 0, \\ \mathbb{Z}/2\{\eta\} & \text{for } i = 1, \\ \mathbb{Z}/2\{\eta^2\} & \text{for } i = 2, \\ \mathbb{Z}/8\{\nu\} & \text{for } i = 3, \\ 0 & \text{for } i \equiv 4, 5, 6 \mod 8, \\ \mathbb{Z}_2/16k\{\rho_{8k-1}\} & \text{for } i = 8k-1, \\ \mathbb{Z}/2\{\eta\rho_{8k-1}\} & \text{for } i = 8k, \\ \mathbb{Z}/2\{\mu_{8k+1}, \eta^2\rho_{8k-1}\} & \text{for } i = 8k+1, \\ \mathbb{Z}/2\{\eta\mu_{8k+1}\} & \text{for } i = 8k+2, \\ \mathbb{Z}/8\{\zeta_{8k+3}\} & \text{for } i = 8k+3. \end{cases}$$

for $k \geq 1$, where $\rho_{8k-1} = \partial(\beta^k)$ and $\zeta_{8k+3} = \partial(\alpha\beta^k)$. (The case $i = 3$ coincides with the case $i = 8k + 3$ for $k = 0$.)

The map $e\colon S \to j$ induces a homomorphism $e_*\colon \pi_*(S) \to \pi_*(j)$, called the KO-theory e-invariant. As a consequence of the Adams conjecture (proved by Quillen, by Sullivan, and by Becker–Gottlieb), this homomorphism is split surjective in each degree.

Recall that $H^*(ko) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$ and $H^*(bspin) \cong \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\}$.

Proposition 0.13 (Davis, Angeltveit–Rognes, Bruner)**.** *The lift $\psi^3 - 1\colon ko \to bspin$ induces the homomorphism $Sq^4\colon \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} \to \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}$, mapping $\Sigma^4\theta$ to θSq^4. It has kernel Σ^8K where*

$$K = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}\,,$$

and cokernel $C = \mathscr{A}//A(2) = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2, Sq^4\}$. Hence there is an $\mathscr{A}$-module extension

$$0 \to \mathscr{A}//A(2) \longrightarrow H^*(j) \longrightarrow \Sigma^7K \to 0\,.$$

There are precisely two such extensions, and $H^*(j)$ is the nonsplit one. A presentation is

$$H^*(j) = \mathscr{A}\{\iota_0, \iota_7\}/\mathscr{A}\{Sq^1\iota_0, Sq^2\iota_0, Sq^4\iota_0, Sq^8\iota_0 + Sq^1\iota_7, Sq^7\iota_7, (Sq^4Sq^6 + Sq^6Sq^4)\iota_7\}\,.$$

The E_2-term of the Adams spectral sequence for j is shown in Figure 3. In this range, only one pattern of differentials is compatible with the known abutment $\pi_*(j)$, leaving the E_∞-term in Figure 4.

The map $e\colon S \to j$ induces a map

$$e_*\colon \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(H^*(j), \mathbb{F}_2)$$

of Adams spectral sequences, mapping the unit $1 \in E_2^{0,0}$ for S to the generator $1 \in E_2^{0,0}$ for j. Hence the map of E_2-terms is determined by the S-module structure of j and the induced $\operatorname{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2)$-module structure on the Adams E_2-term for j. In this range, this can be directly calculated, and shows that e_* the map of E_∞-terms in surjective for $0 \leq t - s \leq 24$, except for $t - s = 15$, when the map of E_∞-terms is trivial.

Proposition 0.14. *The permanent cycles h_0^k for $k \geq 0$, h_1, h_1^2, $h_0^kh_2$ for $0 \leq k \leq 2$, $h_0^kh_3$ for $0 \leq k \leq 3$, c_0, h_1c_0, Ph_1, h_1Ph_1, $h_0^kPh_2$ for $0 \leq k \leq 2$, Pc_0, h_1Pc_0, P^2h_1, $h_1P^2h_1$, $h_0^kP^2h_2$ for $0 \leq k \leq 2$, (h_1Pd_0,) $h_0^{k+2}i$ for $0 \leq k \leq 3$ and P^2c_0 in the Adams spectral sequence for S map to (nonzero) survivors in the Adams spectral sequence for j, hence are themselves (nonzero) survivors.*

Corollary 0.15. *h_2h_4 and g are permanent cycles.*

Proof. These classes could only support differentials hitting h_1Pc_0, P^2h_1 or $h_0^kP^2h_2$ for $0 \le k \le 2$, which we have now shown are not the targets of differentials. □

Remark 0.16. In degree $n = 15$ (and more generally, in all degrees $n \equiv 15 \mod 32$) the homomorphism $e_* \colon \pi_n(S) \to \pi_n(j)$ induces a zero homomorphism of E_∞-terms. Nonetheless e_* is split surjective. This is a case of a shift in Adams filtration. There is a class $\rho \in \pi_{15}(S)$ that is represented by $h_0^3h_4$ in Adams filtration $s = 4$, and which maps to a generator of $\pi_{15}(j)$, which is represented in Adams filtration $s = 5$. Once we prove that $\eta\rho$ is represented by Pc_0, so that there is a hidden η-multiplication in the Adams spectral sequence for S, then since $e_*(\eta\rho)$ generates $\pi_{16}(j)$, it is clear that $e_*(\rho)$ must generate $\pi_{15}(j)$.

0.6. **The next fifteen stems.**

Theorem 0.17. (14) *$\pi_{14}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa, \sigma^2\}$, with κ represented by d_0 and σ^2 represented by h_3.*

(15) *$\pi_{15}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\kappa\} \oplus \mathbb{Z}/32\{\rho\}$, with $\eta\kappa$ represented by h_1d_0 and $\rho = \rho_{15}$ represented by $h_0^3h_4$.*

(16) *$\pi_{16}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\rho, \eta^*\}$, with $\eta\rho$ represented by Pc_0 and $\eta^* = \eta_4$ represented by h_1h_4. ((Check that $\eta\rho \ne 0$.)) ((Is $\sigma\mu = \eta\rho$?))*

(17) *$\pi_{17}(S)^\wedge_2 = \mathbb{Z}/2\{\bar\mu, \eta^2\rho, \nu\kappa, \eta\eta^*\}$, with $\bar\mu = \mu_{17}$ represented by P^2h_1, $\eta^2\rho$ represented by h_1Pc_0, $\nu\kappa$ represented by h_2d_0 and $\eta\eta^*$ represented by $h_1^2h_4$. ((Check that $2\nu\kappa = 0$.))*

(18) *$\pi_{18}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\bar\mu\} \oplus \mathbb{Z}/8\{\nu^*\}$, with $\eta\bar\mu$ represented by $h_1P^2h_1$ and ν^* represented by h_2h_4.*

(19) *$\pi_{19}(S)^\wedge_2 = \mathbb{Z}/8\{\bar\zeta\} \oplus \mathbb{Z}/2\{\bar\sigma\}$, with $\bar\zeta = \zeta_{19}$ represented by P^2h_2 and $\bar\sigma$ represented by c_1.*

(20) *$\pi_{20}(S)^\wedge_2 = \mathbb{Z}/8\{\bar\kappa\}$, with $\bar\kappa$ represented by $g = g_1$.*

(21) *$\pi_{21}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\bar\kappa, \nu\nu^*\}$, with $\eta\bar\kappa$ represented by h_1g and $\nu\nu^*$ represented by $h_2^2h_4$. ((Check that $2\nu\nu^* = 0$, which follows from $\eta^2\bar\kappa \ne 0$.))*

(22) *$\pi_{22}(S)^\wedge_2 = \mathbb{Z}/2\{\eta^2\bar\kappa, \nu\bar\sigma\}$, with $\eta^2\bar\kappa$ represented by Pd_0 and $\nu\nu^*$ represented by h_2c_1. ((Check that $\eta^2\bar\kappa \ne 0$ and that $2\nu\bar\sigma = 0$. The latter follows from $\eta^2\bar\kappa \ne 0$, since then $\eta^3\bar\kappa \ne 0$.))*

(23) *$\pi_{23}(S)^\wedge_2 = \mathbb{Z}/16\{\bar\rho\} \oplus \mathbb{Z}/8\{\nu\bar\kappa\} \oplus \mathbb{Z}/2\{\sigma\eta^*\}$, with $\bar\rho = \rho_{23}$ represented by h_0^2i, $\nu\bar\kappa$ represented by h_2g, $2\nu\bar\kappa$ represented by h_0h_2g, $4\nu\bar\kappa = \eta^3\bar\kappa$ represented by h_1Pd_0, and $\sigma\eta^*$ represented by h_4c_0. ((Check that $\sigma\eta^*$ is represented by h_4c_0.))*

(24) *$\pi_{24}(S)^\wedge_2 = \mathbb{Z}/2\{\sigma\bar\mu\} \oplus \mathbb{Z}/2\{\eta\sigma\eta^*\}$, with $\sigma\bar\mu$ represented by P^2c_0 and $\eta\sigma\eta^*$ represented by $h_1h_4c_0$. ((Check that $\eta\bar\rho \ne 0$.)) ((Is $\sigma\bar\mu = \mu\rho = \eta\bar\rho$?))*

(25) *$\pi_{25}(S)^\wedge_2 = \mathbb{Z}/2\{\mu_{25}, \eta^2\bar\rho\}$, with μ_{25} represented by P^3h_1 and $\eta^2\bar\rho$ represented by $h_1P^2c_0$.*

(26) *$\pi_{26}(S)^\wedge_2 = \mathbb{Z}/2\{\eta\mu_{25}, \nu^2\bar\kappa\}$, with $\eta\mu_{25}$ represented by $h_1P^3h_1$ and $\nu^2\bar\kappa$ represented by h_2^2g.*

(27) *$\pi_{27}(S)^\wedge_2 = \mathbb{Z}/8\{\zeta_{27}\}$, with ζ_{27} represented by P^3h_2, $2\zeta_{27}$ represented by $h_0P^3h_2$ and $4\zeta_{27} = \eta^2\mu_{25}$ represented by $h_0^2P^3h_2$.*

(28) *$\pi_{28}(S)^\wedge_2 = \mathbb{Z}/2\{\kappa^2\}$, with κ^2 represented by d_0^2.*

(29) *$\pi_{29}(S)^\wedge_2 = 0$. ((This assumes that the differential $d_3(r) = h_1d_0^2$ is known.))*

(30) *$\pi_{30}(S)^\wedge_2 = \mathbb{Z}/2\{\theta_4\}$, with θ_4 represented by h_4^2. ((This assumes that the differentials from $t-s = 31$ are known.))*

Alternatively, we might just list $\ker(e_*) \subset \pi_*(S)^\wedge_2$, also known as the cokernel of J. These are the homotopy groups of the homotopy fiber $c = \mathrm{hofib}(e)$. Note that e_* maps both ϵ and $\eta\sigma$ to the generator of $\pi_8(j)$, so $\bar\nu = \epsilon + \eta\sigma$ generates $\pi_8(c)$. Here $\eta\bar\nu = \nu^3$. ((Is $\nu\nu^* = \sigma^3$?))

((ETC))

References

[Mim65] Mamoru Mimura, *On the generalized Hopf homomorphism and the higher composition. II. $\pi_{n+i}(S^n)$ for $i = 21$ and 22*, J. Math. Kyoto Univ. **4** (1965), 301–326. MR0177413 (31 #1676)

[MMO75] Mamoru Mimura, Masamitsu Mori, and Nobuyuki Oda, *Determination of 2-components of the 23- and 24-stems in homotopy groups of spheres*, Mem. Fac. Sci. Kyushu Univ. Ser. A **29** (1975), no. 1, 1–42. MR0375300 (51 #11496)

[MT63] Mamoru Mimura and Hirosi Toda, *The $(n + 20)$-th homotopy groups of n-spheres*, J. Math. Kyoto Univ. **3** (1963), 37–58. MR0157384 (28 #618)

[Tod62] Hirosi Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962. MR0143217 (26 #777)

Department of Mathematics, University of Oslo, Norway
E-mail address: rognes@math.uio.no
URL: http://folk.uio.no/rognes

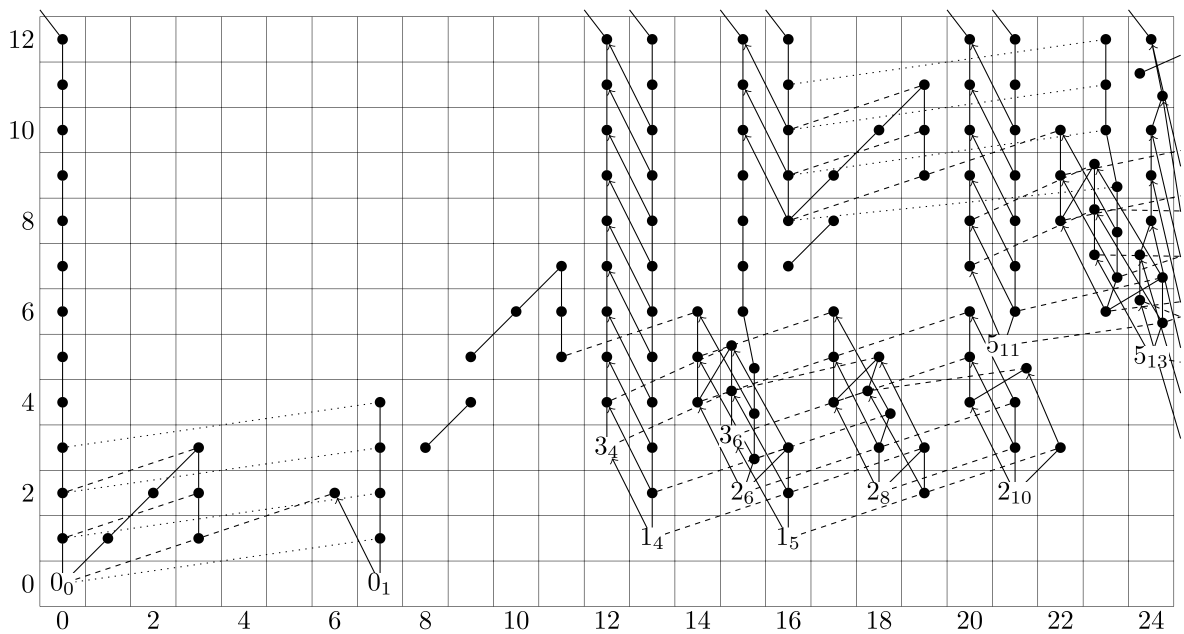

Figure 3. Adams (E_2, d_2)-term for j

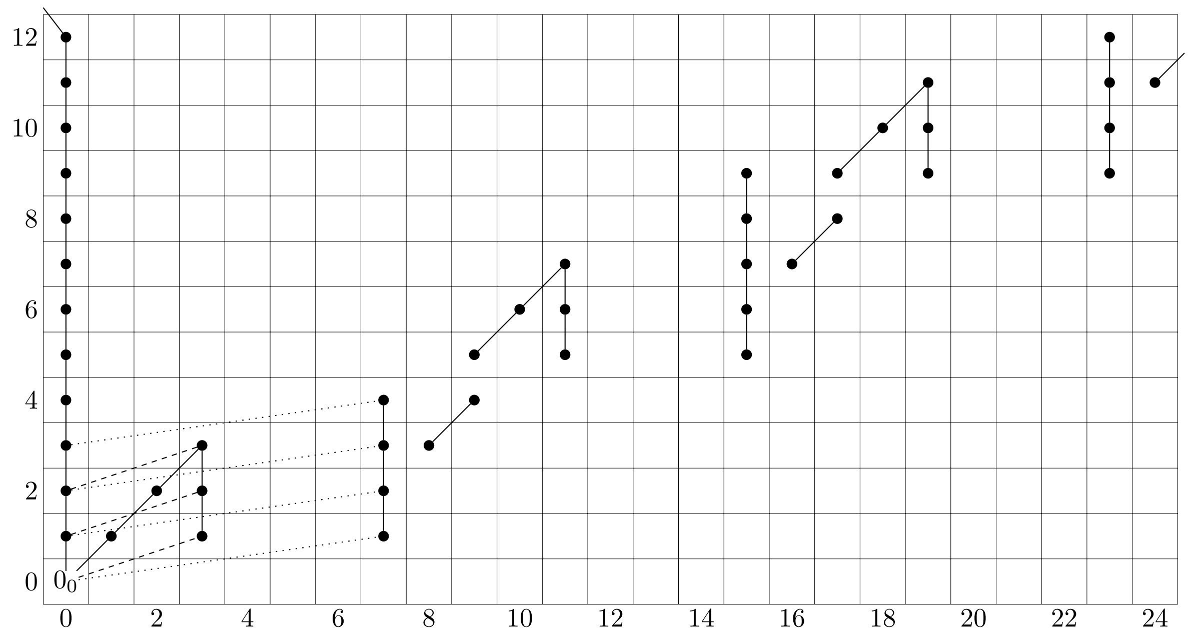

Figure 4. Adams E_∞-term for j

n	$\pi_n(c)$	gen.	rep.
6	$\mathbb{Z}/2$	ν^2	h_2^2
8	$\mathbb{Z}/2$	$\bar{\nu}$	h_1h_3
9	$\mathbb{Z}/2$	$\eta\bar{\nu}$	$h_1^2h_3$
14	$(\mathbb{Z}/2)^2$	κ	d_0
		σ^2	h_3^2
15	$\mathbb{Z}/2$	$\eta\kappa$	h_1d_0
16	$\mathbb{Z}/2$	η^*	h_1h_4
17	$(\mathbb{Z}/2)^2$	$\nu\kappa$	h_2d_0
		$\eta\eta^*$	$h_1^2h_4$
18	$\mathbb{Z}/8$	ν^*	h_2h_4
19	$\mathbb{Z}/2$	$\bar{\sigma}$	c_1
20	$\mathbb{Z}/8$	$\bar{\kappa}$	g
21	$(\mathbb{Z}/2)^2$	$\eta\bar{\kappa}$	h_1g
		$\nu\nu^*$	$h_2^2h_4$
22	$(\mathbb{Z}/2)^2$	$\eta^2\bar{\kappa}$	Pd_0
		$\nu\bar{\sigma}$	h_2c_1
23	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\nu\bar{\kappa}$	h_2g
		$\sigma\eta^*$	h_4c_0
24	$\mathbb{Z}/2$	$\eta\sigma\eta^*$	$h_1h_4c_0$
26	$\mathbb{Z}/2$	$\nu^2\bar{\kappa}$	h_2^2g
28	$\mathbb{Z}/2$	κ^2	d_0^2
30	$\mathbb{Z}/2$	θ_4	h_4^2