

# Morava $E$ -homology of Bousfield-Kuhn functors on odd-dimensional spheres

YIFEI ZHU

As an application of Behrens and Rezk’s spectral algebra model for unstable  $v_n$ -periodic homotopy theory, we give explicit presentations for the completed  $E$ -homology of the Bousfield-Kuhn functor on odd-dimensional spheres at chromatic level 2, and compare them to the level 1 case. The latter reflects earlier work in the literature on  $K$ -theory localizations.

## 1 Introduction

The rational homotopy theory of Quillen and Sullivan studies unstable homotopy types of topological spaces modulo torsion, or equivalently, after inverting primes. Such homotopy types are computable by means of their *algebraic models*. In particular, Quillen showed that there are equivalences of homotopy categories

$$\mathrm{Ho}_{\mathbb{Q}}(\mathrm{Top})_2 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGL})_1 \simeq \mathrm{Ho}_{\mathbb{Q}}(\mathrm{DGC})_2$$

between simply-connected rational spaces, connected differential graded Lie algebras over  $\mathbb{Q}$ , and simply-connected differential graded (cocommutative) coalgebras over  $\mathbb{Q}$  [Quillen1969, Theorem I].

Working integrally, one has  $p$ -adic analogues where equivalences detected through rational homotopy  $\pi_*(-) \otimes \mathbb{Q}$  are replaced by those through  $\pi_*(-) \otimes \mathbb{Z}_p^\wedge$ . Various algebraic models for  $p$ -adic homotopy types of spaces were developed [Kříž1993, Goerss1995, Mandell2001]. In the modern language of homotopy theory, these models are often formulated in terms of “spectral” algebra. For example, Mandell’s model is given by the functor that takes a pointed nilpotent  $p$ -complete space  $X$  of finite  $p$ -type, to the  $\mathbb{F}_p$ -cochains  $H\mathbb{F}_p^X$  (i.e. the function spectrum  $F(\Sigma^\infty X, H\mathbb{F}_p)$ ). The latter is a commutative  $H\mathbb{F}_p$ -algebra spectrum.

More generally, through the prism of chromatic homotopy theory, Behrens and Rezk have established spectral algebra models for unstable  $v_n$ -periodic homotopy types

[Behrens-Rezk2015, Behrens-Rezk2016] (cf. [Arone-Ching2015, Heuts2016]). Here, instead of inverting primes, they invert classes of maps called “ $v_n$ -self maps” (the case of  $n = 0$  recovers rational homotopy). Correspondingly, they work with the  $n$ ’th unstable monochromatic category  $M_n^f \text{Top}_*$ , in the sense of [Bousfield2001], and study the functor

$$(1.1) \quad \text{Ho}(M_n^f \text{Top}_*) \rightarrow \text{Ho}(\text{Alg}_{\text{Comm}}(\text{Sp}_{T(n)}))$$

that sends a space  $X$  to the  $\mathbf{S}_{T(n)}$ -valued cochains  $\mathbf{S}_{T(n)}^X$ . This last spectrum is an algebra for the (reduced) commutative operad  $\text{Comm}$ , in modules over the localization  $\mathbf{S}_{T(n)}$  of the sphere spectrum with respect to a telescope of  $v_n$ -self maps.

Considering a variant of localization with respect to the Morava  $K$ -theory  $K(n)$ , Behrens and Rezk have obtained an equivalence

$$\Phi_{K(n)}(X) \xrightarrow{\sim} \text{TAQ}_{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^X)$$

of  $K(n)$ -local spectra, on a class of spaces  $X$  including spheres [Behrens-Rezk2015, Theorem 8.1] (cf. [Behrens-Rezk2016]). In more detail, the left-hand side arises from computing homotopy groups in the source category of (1.1), where  $\Phi_{K(n)} = L_{K(n)}\Phi_n$  is a version of the Bousfield-Kuhn functor (cf. [Kuhn2008]). This side is a derived realization of morphisms in the source. The right-hand side is the topological André-Quillen cohomology of  $\mathbf{S}_{K(n)}^X$  as an algebra over the operad  $\text{Comm}$  in  $\mathbf{S}_{K(n)}$ -modules. It is a derived realization of images of morphisms under the functor (1.1) in the target category. Via a suitable Koszul duality between  $\text{Comm}$  and the Lie operad, we may view the spectrum  $\text{TAQ}_{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^X)$  as a Lie algebra model for the unstable  $v_n$ -periodic homotopy type of  $X$ .

## 1.1 Main results

The purpose of this paper is to make available calculations that apply Behrens and Rezk’s theory to obtain quantitative information about unstable  $v_n$ -periodic homotopy types, in the case of  $n = 2$ .

Let  $E$  be a Morava  $E$ -theory spectrum of height 2, with  $E_* \cong \mathbb{W}\overline{\mathbb{F}}_p[[a]][u^{\pm 1}]$  where  $|a| = 0$  and  $|u| = -2$ . Recall that the *completed  $E$ -homology* functor is defined as  $E_*^\wedge(-) := \pi_*(E \wedge -)_{K(2)}$ . It is  $E_0$ -linear dual to  $E^*(-)$  with more convenient properties than  $E_*(-)$  (see [Rezk2009, Section 3]).

Building on and strengthening Rezk’s results in [Rezk2013, §2.13], we obtain the following.

**Theorem 1.2** Given any non-negative integer  $m$ , denote by  $E_*^\wedge(\Phi_2 S^{2m+1})$  the completed  $E$ -homology groups of the Bousfield-Kuhn functor applied to the  $(2m+1)$ -dimensional sphere.

- (i) The group  $E_1^\wedge(\Phi_2 S^{2m+1}) \cong 0$  if  $m = 0$ . As an  $E_0$ -module, it equals  $(E_0/p)^{\oplus p-1}$  if  $m = 1$ . It is a quotient of  $(E_0/p^m)^{\oplus p-1} \oplus E_0/p^{m-1}$  if  $m > 1$ .
- (ii) More explicitly,

$$E_1^\wedge(\Phi_2 S^{2m+1}) \cong \begin{cases} \frac{\bigoplus_{i=1}^{p-1} (E_0/p^m) \cdot x_i \oplus (E_0/p^{m-1}) \cdot x_p}{(r_1, \dots, r_{m-1})} & \text{if } 2 \leq m \leq p+2 \\ \frac{\bigoplus_{i=1}^{p-1} (E_0/p^m) \cdot x_i \oplus (E_0/p^{m-1}) \cdot x_p}{(r_{m-p-1}, \dots, r_{m-1})} & \text{if } m > p+2 \end{cases}$$

where  $r_j = r_j(x_1, \dots, x_p) = w_0^{m-1-j} \sum_{i=1}^p d_{i,j+1} x_i$ . Here, as in [Zhu2015b, Theorem 1.6],

$$d_{i,\tau} = \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1 + \dots + m_{\tau-n} = \tau+i \\ 1 \leq m_s \leq p+1 \\ m_{\tau-n} \geq i+1}} w_{m_1} \cdots w_{m_{\tau-n}}$$

where the coefficients  $w_i \in E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[[a]]$  are defined by the identity

$$\sum_{i=0}^{p+1} w_i b^i = (b-p)(b+(-1)^p)^p - (a-p^2+(-1)^p)b$$

in the variable  $b$ , so that  $w_{p+1} = 1$ ,  $w_1 = -a$ ,  $w_0 = (-1)^{p+1}p$ , and the remaining coefficients

$$w_i = (-1)^{p(p-i+1)} \left[ \binom{p}{i-1} + (-1)^{p+1} p \binom{p}{i} \right]$$

In particular, each relation  $r_j$  contains a term  $(-1)^{j+1} w_0^{m-1-j} w_1^j x_p$ , which equals  $p^{m-1-j} a^j x_p$  up to a sign.

- (iii) The group  $E_0^\wedge(\Phi_2 S^{2m+1}) \cong 0$  for any  $m \geq 0$ .

Since  $E$  is 2-periodic, these determine the completed  $E$ -homology in all degrees.

**Remark 1.3** There is a ring structure on  $E_1^\wedge(\Phi_2 S^{2m+1})$ . Indeed, each generator  $x_i$  is a power  $b^i$  of a certain element  $b$ . See Section 3.1 for details.

**Remark 1.4** In [Wang2015, Sections 5.3–5.4], Wang obtained an equivalent presentation of  $E_*^\wedge(\Phi_2 S^{2m+1})$  for  $m = 1$  and any prime  $p$ . He then used it as the input of a spectral sequence and computed  $\pi_*(\Phi_{K(2)} S^3)$  at  $p \geq 5$ .

**Example 1.5** Write  $B_m := E_1^\wedge(\Phi_2 S^{2m+1})$ . We apply Theorem 1.2 and compute  $B_m$  at  $p = 2$  for small values of  $m$ . We have  $w(a, b) = b^3 - ab - 2$  so that  $w_3 = 1$ ,  $w_2 = 0$ ,  $w_1 = -a$ , and  $w_0 = -2$ .

- When  $m = 2$ , since  $r_1 = 2x_1 - ax_2$ ,  $B_2$  is the quotient of  $(E_0/4) \cdot x_1 \oplus (E_0/2) \cdot x_2$  subject to the relation  $ax_2 = 2x_1$ .
- When  $m = 3$ , we have  $r_1 = -4x_1 + 2ax_2$  and  $r_2 = 2ax_1 - a^2x_2$  so that the relations are

$$a^2x_2 = 2ax_1$$

$$2ax_2 = 4x_1$$

- When  $m = 4$ , we have  $r_1 = 8x_1 - 4ax_2$ ,  $r_2 = -4ax_1 + 2a^2x_2$ , and  $r_3 = 2a^2x_1 + (-a^3 + 4)x_2$ . Thus the relations are

$$a^3x_2 = 2a^2x_1 + 4x_2$$

$$2a^2x_2 = 4ax_1$$

$$4ax_2 = 8x_1$$

- When  $m = 5 > p + 2$ , we have  $r_2 = 8ax_1 - 4a^2x_2$ ,  $r_3 = -4a^2x_1 + (2a^3 - 8)x_2$ , and  $r_4 = (2a^3 - 8)x_1 + (-a^4 + 8a)x_2$ . Thus the relations are

$$a^4x_2 = (2a^3 - 8)x_1 + 8ax_2$$

$$2a^3x_2 = 4a^2x_1 + 8x_2$$

$$4a^2x_2 = 8ax_1$$

The relations above show that the bounds for  $p$ -power torsion in Theorem 1.2 are sharp (cf. the notion of *eventual  $H$ -space exponent* for spheres in e.g. [Bousfield2005, §2.5]). Also, as in part (ii) of the theorem, each  $r_j$  contains a term  $2^{m-1-j}a^jx_2$ . Unfortunately, it is impossible to simplify the relations for  $B_m$  into  $2^{m-1-j}a^jx_2 = 0$  by an  $E_0$ -linear change of variables with  $x_i$ . See Remark 2.3.

## 1.2 A comparison to the case of $n = 1$

As an application of Behrens and Rezk's theory, Theorem 1.2 is a step toward the program initiated in [Arone-Mahowald1999] to compute the unstable  $v_n$ -periodic homotopy groups of spheres using stable  $v_n$ -periodic homotopy groups and Goodwillie

calculus. See also [Wang2014, Wang2015]. Given the computations of Davis and Mahowald in the 1980s for the case of  $n = 1$ , we discuss a version of Theorem 1.2 at height 1 according to this program.

Davis and Mahowald showed that,  $K(1)$ -locally at a prime  $p$ , the Moore spectrum  $\mathbf{S}^{-1}/p^m$  with  $i$ 'th space  $S^{i-1} \cup_{p^m} e^i$  is equivalent to the suspension spectrum of a stunted  $B\Sigma_p$ . Via the Goodwillie tower of the identity functor on the category of pointed spaces, the latter can be identified with  $\Phi_1(S^{2m+1})$ , again  $K(1)$ -locally (or  $T(1)$ -locally, due to the validity of the Telescope Conjecture at height 1). We thus obtain a variant of Theorem 1.2.

**Proposition 1.6** *Let  $E$  be a Morava  $E$ -theory spectrum of height 1, with  $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p$ . Assume that  $p \neq 3$  and that, if  $p = 2$ ,  $m \equiv 0, 3 \pmod{4}$ . Then*

$$E_0^\wedge(\Phi_1 S^{2m+1}) \cong E_0/p^m \quad m \geq 0$$

**Proof** For non-negative integers  $k$  and  $b$ , let  $L(k)_b := e_k \Sigma^\infty (B\mathbb{F}_p^k)^{b\bar{\rho}_k}$  be the stable summand of the Thom space  $(B\mathbb{F}_p^k)^{b\bar{\rho}_k}$  associated to the Steinberg idempotent, where  $b\bar{\rho}_k$  denotes the direct sum of  $b$  copies of the  $k$ -dimensional reduced real regular representation of  $\mathbb{F}_p^k$  (cf. [Behrens-Rezk2015, Remark 5.4]). Let  $L(k)_b^t$  be the fiber of the natural map of spectra  $L(k)_b \rightarrow L(k)_{t+1}$  (see [Behrens2012, Chapter 2, esp. Section 2.3]). When  $k = 1$ , it is a stunted  $B\Sigma_p$  (cf. [Mitchell-Priddy1983]). In particular, if  $p = 2$ ,  $L(1)_b^t \simeq P_b^t$ , the suspension spectrum of the stunted real projective space  $\mathbb{R}P^t/\mathbb{R}P^{b-1}$ . In this case, when  $m = 4n$ , we have

$$\begin{aligned} \Phi_1(S^{2m+1}) &\simeq L_{K(1)} \Sigma^{2m+1} L(1)_1^{2m} \quad \text{by [Kuhn2007, Theorem 7.20]} \\ &\simeq L_{K(1)} \Sigma^{2m+1} P_1^{8n} \\ &\simeq L_{K(1)} \Sigma^{2m+1} P_{1-8n}^0 \quad \text{by [Davis-Mahowald1987, Proposition 2.1]} \\ &\simeq L_{K(1)} \Sigma^{2m+1} \mathbf{S}^{-1}/2^m \quad \text{by [Davis-Mahowald1987, proof of Theorem 4.2]} \\ &\simeq L_{K(1)} \mathbf{S}^{2m}/2^m \end{aligned}$$

and thus  $E_0^\wedge(\Phi_1 S^{2m+1}) \cong E_0/2^m$ . It is similar when  $m = 4n + 3$ . For  $p > 3$ , we apply [Davis1986, Corollary 1.7 and Theorem 1.8].  $\square$

**Remark 1.7** There has been extensive work on the case of  $v_1$ -periodic homotopy theory. See [Bousfield1999, §9.11] and [Bousfield2005, §§8.6–8.7] for an alternative approach to a more general result than the above where the assumptions on  $p$  and  $m$  are removed (cf. [Rezk2013, §2.13]). See also [Davis1995, esp. Theorem 3.1] and the references therein for related information in this case.

**Remark 1.8** For any  $n$ , since  $\Phi_n$  preserves fiber sequences, there is a natural map

$$\Sigma^2 \Phi_n(X) \rightarrow \Phi_n(\Sigma^2 X)$$

In view of the 2-periodicity of  $E$ , this induces a map on completed  $E$ -homology in the same degree. Thus the groups  $\{E_*^\wedge(\Phi_n S^{2m+1})\}_{m \geq 0}$  form a direct system. Homotopy (co)limits of generalized Moore spectra are closely related to various kinds of localizations of the sphere spectrum (see, e.g., [Arone-Mahowald1999, Proposition A.3] and [Hovey-Strickland1999, Proposition 7.10]). On the other hand, note that completed  $E$ -homology does not preserve homotopy colimits [Hovey2008]. Nevertheless, based on computational evidence from Theorem 1.2, Proposition 1.6, and further, we hope to study the relationship between the  $K(n)$ -local sphere and the Bousfield-Kuhn functor on odd-dimensional spheres hinted in [Rezk2016, §§ 3.20–3.21].

### 1.3 Acknowledgements

I thank Mark Behrens, Paul Goerss, Charles Rezk, Guozhen Wang, and Zhouli Xu for helpful discussions. I am especially grateful to Mark for introducing me into the program of computing unstable periodic homotopy groups of spheres and to Charles for sharing his knowledge in power operations.

I thank Lennart Meier for his helpful remarks on this paper, particularly one that saved me from an error in the main results.

## 2 Koszul complexes for modules over the $E$ -theory Dyer-Lashof algebra

Let  $E$  be a Morava  $E$ -theory spectrum of height  $n$  at the prime  $p$ . Its formal group  $\mathrm{Spf} E^0 \mathbb{C}P^\infty$  over  $E_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[[u_1, \dots, u_{n-1}]]$  is the Lubin-Tate universal deformation of a formal group  $\mathbb{G}$  over  $\overline{\mathbb{F}}_p$  of height  $n$ .

Generalizing the Lubin-Tate deformation theory, Strickland shows that for each  $k \geq 0$  there is a ring  $A_k \cong E^0 B\Sigma_{p^k}/I_k$  classifying subgroups of degree  $p^k$  in the universal deformation, where  $I_k$  is a transfer ideal [Strickland1998, Theorem 1.1]. In particular,  $A_0 \cong E_0$ , and there are ring homomorphisms

$$s = s_k, t = t_k: A_0 \rightarrow A_k \quad \text{and} \quad \mu = \mu_{k,m}: A_{k+m} \rightarrow A_k \overset{s}{\otimes}_{A_0} \overset{t}{\otimes} A_m$$

classifying the source and target of an isogeny of degree  $p^k$  on the universal deformation and the composition of two isogenies.

As  $E$  is an  $E_\infty$ -ring spectrum, there are (additive) power operations acting on the homotopy of  $K(n)$ -local commutative  $E$ -algebra spectra. A  $\Gamma$ -module is an  $A_0$ -module  $M$  equipped with structure maps (the power operations)

$$P_k: M \rightarrow {}^t A_k {}^s \otimes_{A_0} M \quad k \geq 0$$

which are a compatible family of  $A_0$ -module homomorphisms. These power operations form the *Dyer-Lashof algebra*  $\Gamma$  for the  $E$ -theory, with graded pieces  $\Gamma[k] := \text{Hom}_{A_0}({}^s A_k, A_0)$ ,  $k \geq 0$ . There is a tensor product  $\otimes$  for  $\Gamma$ -modules [Rezk2013, §4.1].

The structure of a  $\Gamma$ -module is determined by  $P_1$ , subject to a condition involving  $A_2$ , i.e. the existence of the dashed arrow in the diagram

$$(2.1) \quad \begin{array}{ccc} M & \xrightarrow{P_1} & {}^t A_1 {}^s \otimes_{A_0} M \\ \downarrow \text{dashed} & & \downarrow \text{id} \otimes P_1 \\ {}^t A_2 {}^s \otimes_{A_0} M & \xrightarrow{\mu \otimes \text{id}} & {}^t A_1 {}^s \otimes_{A_0} {}^t A_1 {}^s \otimes_{A_0} M \end{array}$$

[Rezk2013, Proposition 7.2]. This manifests the fact that the ring  $\Gamma$  is *Koszul* and, in particular, *quadratic* [Rezk2012].

Let  $D_0 := A_0$ ,  $D_1 := A_1$ , and

$$D_k := \text{coker} \left( \bigoplus_{i=0}^{k-2} A_1^{\otimes i} \otimes A_2 \otimes A_1^{k-i-2} \xrightarrow{\text{id} \otimes \mu \otimes \text{id}} A_1^{\otimes k} \right) \quad k \geq 2$$

Given  $\Gamma$ -modules  $M$  and  $N$ , Rezk defines the *Koszul complex*  $\mathcal{C}^\bullet(M, N)$  by

$$\mathcal{C}^k(M, N) := \text{Hom}_{A_0}(M, D_k \otimes_{A_0} N)$$

with appropriate coboundary maps [Rezk2013, §7.3].

**Proposition 2.2** *If  $M$  is projective as an  $A_0$ -module, then*

$$H^k \mathcal{C}^\bullet(M, N) \cong \text{Ext}_\Gamma^k(M, N)$$

*In particular, if  $k > n$ ,  $D_k \cong 0$  and so  $\text{Ext}_\Gamma^k(M, N) \cong 0$ .*

**Proof** This is [Rezk2013, Proposition 7.4]. □

## 2.1 The case of $n = 2$

Choose a *preferred*  $\mathcal{P}_N$ -model for  $E$  in the sense of [Zhu2015a, Definition 3.29] so that the formal group of  $E$  is isomorphic to the formal group of a universal deformation of a supersingular elliptic curve satisfying a list of properties.

Using the theory of dual isogenies of elliptic curves, Rezk identifies that  $D_2 \cong A_1/s(A_0)$  [Rezk2013, Proposition 9.3]. He also classifies  $\Gamma$ -modules of rank 1 in this case [Rezk2013, Proposition 9.7]. In particular, each of them takes the form  $L_\beta$  with structure map

$$\begin{aligned} P: L_\beta &\rightarrow {}^t A_1 {}^s \otimes_{A_0} L_\beta \\ x &\mapsto \beta \otimes x \end{aligned}$$

where  $x$  is a generator for the underlying  $A_0$ -module, and  $\beta \in A_1$  is such that  $\iota(\beta) \cdot \beta \in s(A_0)$  with  $\iota(-)$  the Atkin-Lehner involution (this condition on  $\beta$  corresponds to the condition in (2.1)). Moreover,  $L_1$  is the unit object in the symmetric monoidal category of  $\Gamma$ -modules with respect to  $\otimes$ , and  $L_{\beta_1} \otimes L_{\beta_2} \cong L_{\beta_1 \beta_2}$ . Thus  $L_\beta$  is  $\otimes$ -invertible as a  $\Gamma$ -module if and only if  $\beta \in A_1^\times$ .

Now let  $M = L_\alpha$  and  $N = L_\beta$ . We have identifications

$$\begin{aligned} A_0 &\xrightarrow{\sim} \mathcal{C}^0(M, N) = \text{Hom}_{A_0}(M, N) & f &\mapsto (x \mapsto f y) \\ A_1 &\xrightarrow{\sim} \mathcal{C}^1(M, N) = \text{Hom}_{A_0}(M, {}^t A_1 {}^s \otimes_{A_0} N) & g &\mapsto (x \mapsto g \otimes y) \\ A_1/s(A_0) &\xrightarrow{\sim} \mathcal{C}^2(M, N) = \text{Hom}_{A_0}\left(M, {}^{\iota^2 s} (A_1/s(A_0)) {}^s \otimes_{A_0} N\right) & h &\mapsto (x \mapsto h \otimes y) \end{aligned}$$

Thus the Koszul complex in this case is

$$A_0 \xrightarrow{d_0} A_1 \xrightarrow{d_1} A_1/s(A_0)$$

with  $d_0 f = \iota(f)\beta - f\alpha$  and  $d_1 g = \iota(g)\beta + g\iota(\alpha)$  [Rezk2013, §9.18].

More explicitly, we have identifications

$$A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[[a]] \quad \text{and} \quad A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[[a, b]]/(w(a, b))$$

where

$$w(a, b) = \sum_{i=0}^{p+1} w_i b^i = (b-p)(b+(-1)^p)^p - (a-p^2+(-1)^p)b$$

[Zhu2015b, Theorem 1.2]. Note that the parameters  $a$  and  $b$  are chosen as in [Rezk2013, §9.15], and they correspond precisely to  $h$  and  $\alpha$  in [Zhu2015a, Zhu2015b]. In particular, the  $\Gamma$ -module of invariant 1-forms is  $\omega = L_b$ .



**Remark 2.3** As we will see in Section 4, the generators  $x_i$  in Theorem 1.2 (ii) depend on the choice of the parameter  $b$  for  $A_1$ . We do not know if a different choice would make the presentations simpler.

The ring homomorphism  $s: A_0 \rightarrow A_1$  is simply the inclusion of scalars, viewing  $A_1$  as a free left module over  $A_0$  of rank  $p + 1$ . We will thus abbreviate  $s(A_0)$  as  $A_0$ . Following [Rezk2013], we will also abbreviate  $\iota(x)$  as  $x'$ , which is written as  $\tilde{x}$  in [Zhu2015b]. Note that  $w_{p+1} = 1$ ,  $p|w_i$  for  $2 \leq i \leq p$ ,  $w_1 = -a$ , and

$$(2.4) \quad w_0 = (-1)^{p+1}p = bb'$$

[Zhu2015a, (3.30)].

### 3 Computations

Recall that  $\omega = L_b$  is the  $\Gamma$ -module of invariant 1-forms defined in Section 2. Write  $\text{nul} := L_0$ , the  $\Gamma$ -module annihilated by  $\Gamma$ . By Proposition 2.2,

$$\text{Ext}_{\Gamma}^*(\omega^m, \text{nul}) \cong H^* \mathcal{C}^*(L_{b^m}, L_0)$$

where

$$\mathcal{C}^*(L_{b^m}, L_0): A_0 \xrightarrow{-b^m} A_1 \xrightarrow{b'^m} A_1/A_0$$

**Proposition 3.1** For all  $m \geq 0$ ,  $H^0 \mathcal{C}^*(L_{b^m}, L_0) \cong 0$ .

**Proof** We need to show that  $A_0 \xrightarrow{-b^m} A_1$  is injective. Given  $f(a) \in A_0 \cong \mathbb{W}\overline{\mathbb{F}}_p[[a]]$ , suppose  $-b^m \cdot f(a) = 0 \in A_1 \cong \mathbb{W}\overline{\mathbb{F}}_p[[a, b]]/(w(a, b))$ .

If  $0 \leq m \leq p$ , since  $w(a, b)$  is a polynomial in  $b$  of degree  $p + 1$  with coefficients in  $A_0$ , clearly  $f(a)$  must be 0.

If  $m > p$ , we need only show that  $b^m \not\equiv 0 \pmod{w(a, b)}$ . Since  $w(a, b) \equiv b(b^p - a) \pmod{p}$ , we have  $b^{p+1} \equiv ab \pmod{(p, w)}$ , and thus  $b^m \not\equiv 0 \pmod{(p, w)}$ .  $\square$

**Proposition 3.2** For all  $m \geq 0$ ,  $H^1 \mathcal{C}^*(L_{b^m}, L_0) \cong 0$ .

**Proof** Let  $g(a, b)$  be a polynomial in  $b$  of degree at most  $p$  with coefficients in  $A_0$  that represents an element in  $A_1$ . Suppose  $b'^m \cdot g(a, b) = 0 \in A_1/A_0$ . We need to show that  $g(a, b) \equiv -b^m \cdot f(a) \pmod{w(a, b)}$  for some  $f(a) \in A_0$ .

We do this by induction on  $m$ . The case of  $m = 0$  is clear. Let  $m \geq 1$ . By the induction hypothesis, since  $b'^{m-1} \cdot b'g(a, b) = 0 \in A_1/A_0$ , we have  $b'g(a, b) \equiv -b^{m-1} \cdot f(a) \pmod{w(a, b)}$ . Multiplying both sides by  $b$ , in view of (2.4), we get

$$(3.3) \quad w_0g(a, b) \equiv -b^mf(a) \pmod{w}$$

and thus

$$(3.4) \quad 0 \equiv -b^mf(a) \pmod{(p, w)}$$

Since  $b^{p+1} \equiv ab \pmod{(p, w)}$ , (3.4) implies that  $p \mid f(a)$  in  $A_1$ . As  $p$  is not a zero-divisor in  $A_1$ , (3.3) implies that  $g(a, b) \equiv -b^m\tilde{f}(a) \pmod{w}$  for some  $\tilde{f}(a) \in A_0$ .  $\square$

### 3.1 Computing the second cohomology

Finally, we compute  $H^2\mathcal{C}^*(L_{b^m}, L_0)$ . Write  $B_m := H^2\mathcal{C}^*(L_{b^m}, L_0) \cong A_1/(A_0 + b'^m A_1)$ .<sup>1</sup> Clearly,  $B_0 \cong 0$ . Let  $m > 0$  for the rest of this section.

As a free module over  $A_0$ , the ring  $A_1$  has a basis consisting of

$$(3.5) \quad 1, b, b^2, \dots, b^p$$

**Proposition 3.6** *In the  $A_0$ -module  $B_m$ ,  $p^m b^i = 0$  for  $1 \leq i \leq p-1$  and  $p^{m-1} b^p = 0$ .*

**Proof** In view of (2.4), we have  $w_0^m b^i = b'^m b^m b^i = b'^m b^{m+i} = 0$  and

$$(3.7) \quad \begin{aligned} w_0^{m-1} b^p &= w_0^{m-1} (-b' - w_p b^{p-1} - \dots - w_2 b) \\ &= -b'^{m-1} b'^m - w_0^{m-1} w_p b^{p-1} - \dots - w_0^{m-1} w_2 b \\ &= -w_0^{m-1} w_p b^{p-1} - \dots - w_0^{m-1} w_2 b \end{aligned}$$

Since  $p \mid w_i$  for  $2 \leq i \leq p$ , the last expression has a factor of  $w_0^m$  and so vanishes as we have just shown.  $\square$

Let  $1 \leq m \leq p$ . Under the map of multiplication by  $b'^m$ , the elements in (3.5) become

$$(3.8) \quad b'^m, w_0 b'^{m-1}, w_0^2 b'^{m-2}, \dots, w_0^{m-1} b', w_0^m, w_0^m b, \dots, w_0^m b^{p-m}$$

Note that  $w_0^{m-1} b' = 0$  in  $B_m$  is equivalent to (3.7). Thus, as a quotient of  $(A_0/p^m)^{\oplus p-1} \oplus A_0/p^{m-1}$  from the above proposition,  $B_m$  has relations given precisely by the vanishing of the first  $(m-1)$  terms in (3.8).

<sup>1</sup>This duplicates the notation in Section 1. The reader may take it as a convention within this section. We will show in the next section that the symbol  $B_m$  indeed has a consistent meaning.

To write down these relations explicitly, with notation as in [Zhu2015b, Theorem 1.6 (ii)], we have

$$b'^k = d_{p,k}b^p + d_{p-1,k}b^{p-1} + \cdots + d_{0,k} \quad 2 \leq k \leq m \leq p$$

(cf. [Zhu2015b, Section 4.1] for  $k > p$ ). In particular, the formula for the coefficient  $d_{p,k}$  has a leading term  $(-1)^k w_1^{k-1} w_{p+1}$ . Thus setting  $w_0^{m-k} b'^k$  to be zero in  $B_m$  gives an expression for  $w_0^{m-k} w_1^{k-1} b^p$  in terms of an  $A_0$ -linear combination of  $b^{p-1}, \dots, b$ , and, possibly,  $b^p$  itself if there are more than one term in  $d_{p,k}$  not divisible by  $p^{m-1}$ .

The case of  $m > p$  is similar.

## 4 Proof of Theorem 1.2

Recall that given a Morava  $E$ -theory  $E$  of height  $n$ , the completed  $E$ -homology functor is defined as  $E_*^\wedge(-) := \pi_*(E \wedge -)_{K(n)}$ . In particular,

$$\begin{aligned} E_*^\wedge(\Phi_{K(n)}X) &= \pi_*(E \wedge \Phi_{K(n)}X)_{K(n)} \\ &= \pi_*(L_n \Phi_{K(n)}X)_{K(n)} \\ &= \pi_*(L_n L_{K(n)} \Phi_n X)_{K(n)} \\ &= \pi_*(L_n \Phi_n X)_{K(n)} \\ &= E_*^\wedge(\Phi_n X) \end{aligned} \tag{4.1}$$

In [Rezk2013], Rezk sets up a composite functor spectral sequence (CFSS) followed by a mapping space spectral sequence (MSSS) to compute the homotopy groups of derived mapping spaces  $\widehat{\mathcal{R}}_E(A, B)$  between  $K(n)$ -local augmented commutative  $E$ -algebras  $A$  and  $B$ . He identifies the  $E_2$ -term in the CFSS as Ext-groups over the Dyer-Lashof algebra  $\Gamma$ . The CFSS converges to the  $E_2$ -term in the MSSS.

In particular, [Rezk2013, §2.13] shows that this setup specializes to compute the  $E$ -cohomology of the topological André-Quillen homology  $\mathrm{TAQ}^{\mathbf{S}_{K(n)}}(\mathbf{S}_{K(n)}^{S^{2m+1}})$ , and that the two spectral sequences both collapse at the  $E_2$ -term when  $n = 2$ . Here  $A = E_+^{S^{2m+1}} := F(\Sigma_+^\infty S^{2m+1}, E)$  and  $B = E \rtimes E$  is a *square-zero extension* in the sense of [Rezk2013, §5.10].

Now, by [Behrens-Rezk2015, Theorem 8.1] and (4.1), we identify the abutment of the MSSS as

$$\pi_{t-s} \widehat{\mathcal{R}}_E(E_+^{S^{2m+1}}, E \rtimes E) \cong \pi_{t-s} F(\mathrm{TAQ}^{\mathbf{S}_{K(2)}}(\mathbf{S}_{K(2)}^{S^{2m+1}}), E) \cong E_{t-s}^\wedge(\Phi_2 S^{2m+1})$$

For a fixed  $t$ , Rezk identifies the possibly nonzero terms on the  $E_2$ -page of the CFSS as  $\mathrm{Ext}_{\mathrm{Mod}_{\Gamma}^*}^s(\omega^m, \omega^{(t-1)/2} \otimes \mathrm{nul})$ , where  $\mathrm{Mod}_{\Gamma}^*$  is the category of  $\mathbb{Z}/2$ -graded  $\Gamma$ -modules in the sense of [Rezk2013, §5.6]. Thus for degree and periodicity reasons, we may set  $t = 1$  and the calculations in Section 3 then complete the proof, with  $b^i$  written as  $x_i$  in Theorem 1.2.  $\square$

## References

- [Arone-Ching2015] Gregory Arone and Michael Ching, *A classification of Taylor towers of functors of spaces and spectra*, Adv. Math. **272** (2015), 471–552. [MR3303239](#)
- [Arone-Mahowald1999] Greg Arone and Mark Mahowald, *The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres*, Invent. Math. **135** (1999), no. 3, 743–788. [MR1669268](#)
- [Behrens2012] Mark Behrens, *The Goodwillie tower and the EHP sequence*, Mem. Amer. Math. Soc. **218** (2012), no. 1026, xii+90. [MR2976788](#)
- [Behrens-Rezk2015] Mark Behrens and Charles Rezk, *The Bousfield-Kuhn functor and topological André-Quillen cohomology*, available at <http://www3.nd.edu/~mbehren1/papers/BKTAQ6.pdf>.
- [Behrens-Rezk2016] Mark Behrens and Charles Rezk, *Spectral algebra models of unstable  $v_n$ -periodic homotopy theory*, available at <http://www3.nd.edu/~mbehren1/papers/BKTAQsurvey2.pdf>.
- [Bousfield1999] A. K. Bousfield, *The  $K$ -theory localizations and  $v_1$ -periodic homotopy groups of  $H$ -spaces*, Topology **38** (1999), no. 6, 1239–1264. [MR1690156](#)
- [Bousfield2001] A. K. Bousfield, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2391–2426 (electronic). [MR1814075](#)
- [Bousfield2005] A. K. Bousfield, *On the 2-primary  $v_1$ -periodic homotopy groups of spaces*, Topology **44** (2005), no. 2, 381–413. [MR2114954](#)
- [Davis1986] Donald M. Davis, *Odd primary bo-resolutions and  $K$ -theory localization*, Illinois J. Math. **30** (1986), no. 1, 79–100. [MR822385](#)
- [Davis1995] Donald M. Davis, *Computing  $v_1$ -periodic homotopy groups of spheres and some compact Lie groups*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 993–1048. [MR1361905](#)
- [Davis-Mahowald1987] Donald M. Davis and Mark Mahowald, *Homotopy groups of some mapping telescopes*, Algebraic topology and algebraic  $K$ -theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 126–151. [MR921475](#)

- [Goerss1995] Paul G. Goerss, *Simplicial chains over a field and  $p$ -local homotopy theory*, Math. Z. **220** (1995), no. 4, 523–544. [MR1363853](#)
- [Heuts2016] Gijs Heuts, *Goodwillie approximations to higher categories*. [arXiv:1510.03304](#)
- [Hovey2008] Mark Hovey, *Morava  $E$ -theory of filtered colimits*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 369–382 (electronic). [MR2342007\(2008g:55007\)](#)
- [Hovey-Strickland1999] Mark Hovey and Neil P. Strickland, *Morava  $K$ -theories and localization*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100. [MR1601906\(99b:55017\)](#)
- [Kříž1993] Igor Kříž,  *$p$ -adic homotopy theory*, Topology Appl. **52** (1993), no. 3, 279–308. [MR1243609](#)
- [Kuhn2007] Nicholas J. Kuhn, *Goodwillie towers and chromatic homotopy: an overview*, Proceedings of the Nishida Fest (Kinosaki 2003), Geom. Topol. Monogr., vol. 10, Geom. Topol. Publ., Coventry, 2007, pp. 245–279. [MR2402789](#)
- [Kuhn2008] Nicholas J. Kuhn, *A guide to telescopic functors*, Homology, Homotopy Appl. **10** (2008), no. 3, 291–319. [MR2475626](#)
- [Mandell2001] Michael A. Mandell,  *$E_\infty$  algebras and  $p$ -adic homotopy theory*, Topology **40** (2001), no. 1, 43–94. [MR1791268](#)
- [Mitchell-Priddy1983] Stephen A. Mitchell and Stewart B. Priddy, *Stable splittings derived from the Steinberg module*, Topology **22** (1983), no. 3, 285–298. [MR710102](#)
- [Quillen1969] Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. [MR0258031](#)
- [Rezk2009] Charles Rezk, *The congruence criterion for power operations in Morava  $E$ -theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379. [MR2591924\(2011e:55021\)](#)
- [Rezk2012] Charles Rezk, *Rings of power operations for Morava  $E$ -theories are Koszul*. [arXiv:1204.4831](#)
- [Rezk2013] Charles Rezk, *Power operations in Morava  $E$ -theory: structure and calculations*, available at <http://www.math.uiuc.edu/~rezk/power-ops-ht-2.pdf>.
- [Rezk2016] Charles Rezk, *Elliptic cohomology and elliptic curves*, Felix Klein Lectures, Bonn 2015, available at <http://www.math.uiuc.edu/~rezk/felix-klein-lectures-notes.pdf>.
- [Strickland1998] N. P. Strickland, *Morava  $E$ -theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779. [MR1607736\(99e:55008\)](#)
- [Wang2014] Guozhen Wang, *The monochromatic Hopf invariant*. [arXiv:1410.7292](#)
- [Wang2015] Guozhen Wang, *Unstable chromatic homotopy theory*, ProQuest LLC, Ann Arbor, MI, 2015, Thesis (Ph.D.)—Massachusetts Institute of Technology. [MR3427198](#)
- [Zhu2015a] Yifei Zhu, *The Hecke algebra action on Morava  $E$ -theory of height 2*, available at <https://yifeizhu.github.io/ho.pdf>.
- [Zhu2015b] Yifei Zhu, *Modular equations for Lubin-Tate formal groups at chromatic level 2*, available at <https://yifeizhu.github.io/me.pdf>.