Algebraic Models in Homotopy Theory

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Homotopy Theory

Fundamental Question

Are spaces X and Y are homotopy equivalent?

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Methods

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- Define algebraic invariants

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- Compute those invariants

Is it always possible to find an algebraic invariant that distinguishes between non-equivalent spaces?

For simply connected spaces: **Yes!**

Cochains

E_∞ DGA

Outline

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1. Homotopy, Homology, and Cohomology
2. Warm-up Examples
3. Rational Homotopy Theory - CDGAs
4. Cochains and E_∞ DGAs
5. Homotopy Algebras and Homotopy Theory

commutative differential graded algebra

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Algebraic Topology

Fundamental Question

Are spaces X and Y are homotopy equivalent?

It is up to you to produce maps in both directions and homotopies between the composite maps.

Whitehead (1949): This simplifies for "nice" spaces.

Nice spaces: Spaces arising for their geometry
CW complexes. Examples:

- Manifolds
- Polytopes, polyhedra
- Simplicial complexes

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- **Simplicial complexes**

Now "space" means "nice space"
plus maybe a finiteness/compactness hypothesis

The Whitehead Theorem

Theorem (The Whitehead Theorem)

A map $X \to Y$ is a homotopy equivalence if and only if it induces an isomorphism on homotopy groups.

Given a map, you “just” have to check what happens on some algebraic invariants. But can’t usually compute homotopy groups.

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Theorem (The Whitehead Theorem II)

A map $X \to Y$ between simply connected *spaces is a homotopy equivalence if and only if it induces an isomorphism on or (equivalently) cohomology.*

homology

How much does (co)homology say about a simply connected space?

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How much does (co)homology say about a simply connected space?

Example: Homology Spheres

Any simply connected space with the homology/cohomology of the sphere S^n ($n > 1$)

dim	0	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$
H^*	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}$	0	$\cdots$

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Theorem (Hurewicz Theorem) For simply conn spaces

If $\cancel{\pi} H_q X$ is trivial for $1 \leq q < n$, then the Hurewicz map $\pi_n X \to H_n X$ is an isomorphism. then $\pi_q = 0$

$S^n \to X$

Example: $\mathbb{C}P^2$

$H^*(\mathbb{C}P^2)$ looks like: look like this:

dim	0	1	2	3	4	5	6	7	$\cdots$
H^*	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\cdots$

Other spaces also have cohomology like this, e.g., $S^2 \vee S^4$

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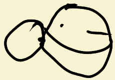

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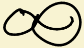

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dim	0	1	2	3	4	5	6	7	$\cdots$
$\mathbb{C}P^2$	1	0	x	0	$y = x^2$	0	0	0	$\cdots$
$S^2 \vee S^4$	1	0	x	0	$y, x^2 = 0$	0	0	0	$\cdots$

Classification

For every n, there is a space X_n with cohomology

dim	0	1	2	3	4	5	6	7	$\cdots$
X_n	1	0	x	0	$y, x^2 = ny$	0	0	0	$\cdots$

Every space with cohomology

$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\ldots$

is homotopy equivalent to one of these.

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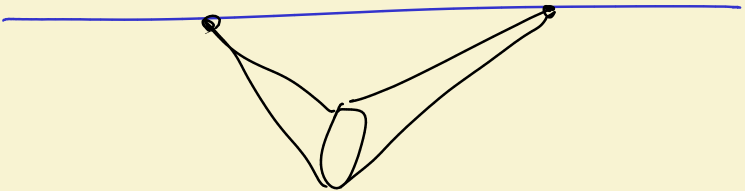
S^2 $\quad$ $\pi_3 S^2 = \mathbb{Z}$ $\quad$ $S^2 \cup D^4$

$S^3 \to S^2$ $\quad$ S^3

[This page intentionally left blank]

Example: $\Sigma\mathbb{C}P^2$

Suspension – take $\mathbb{C}P^2 \times [0,1]$ and collapse each of $\mathbb{C}P^2 \times \{0\}$ and $\mathbb{C}P^2 \times \{1\}$ to a point.



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This shifts cohomology groups up.

dim	0	1	2	3	4	5	6	7	8	$\cdots$
H^*	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\cdots$
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It also kills the cup product.

But not the Steenrod operations on $H^*(-;\mathbb{Z}/2)$.

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$$Sq^{2} : H^n(X;\mathbb{Z}/2) \to H^{n+2}(X;\mathbb{Z}/2)$$

In this case remembers cup product from $\mathbb{C}P^2$

Classification

Every space with cohomology

$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	...

is homotopy equivalent to exactly one of $\Sigma\mathbb{C}P^2$ or $S^3 \vee S^5$.

$\mathbb{Z}/2 \quad 0 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad 0 \quad 0$

Sq^2

0 or $\cong$

$S^3 \vee S^5 \qquad \Sigma\mathbb{C}P^2$

$\pi_4 S^3 = \mathbb{Z}/2$

The Steenrod / Grothendieck Problem

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Problem

Find structure on cohomology or cochains that classifies simply connected spaces up to homotopy equivalence.

Solution is E_∞ DGA
Mandell, “Cochains and Homotopy Type”, *Pub. Math. IHÉS*, 2006.

Problem

Given a homotopy invariant (or property or ???), find a structure on cohomology or cochains that determines it. Or vice-versa.

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Rational Homotopy Theory

The Whitehead Theorem: A map $X \to Y$ of simply connected space is a homotopy equivalence if and only if it induces an isomorphism on integral homology.

Definition (Rational Equivalence)

A *rational equivalence* is a map $X \to Y$ that induces an isomorphism on rational homology $H_*(X;\mathbb{Q}) \xrightarrow{\cong} H_*(Y;\mathbb{Q})$ or (equivalently) on rational cohomology $H^*(X;\mathbb{Q}) \xrightarrow{\cong} H^*(Y;\mathbb{Q})$

Rational Homotopy Theory: Make rational equivalences into isomorphisms.

Rational Homotopy Category: Category obtained by formally inverting the rational equivalences.

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Rational Invariants

What information about simply connected spaces is left in the rational homotopy category?

(Anything that takes rational equivalences to isomorphisms)

Lots of rational mapping space data, including *rational homotopy groups*.

$$\boxed{\pi_n X \otimes \mathbb{Q}}$$

More or less anything $\otimes\mathbb{Q}$ that can be computed from spectral sequences.

Serre 1950's: Rational invariants are relatively easy to compute.

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C-theory

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De Rham complex

$H^*(-;\mathbb{Q})$ or $H^*(-;\mathbb{R})$ have a carrier that is a

commutative differential graded algebra (CDGA) A

The De Rham complex of a manifold $\Omega^* M$

$$\omega \wedge \eta = (-1)^{|\omega||\eta|} \eta \wedge \omega$$

$$d\omega \qquad d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$$

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The De Rham complex of a manifold $\Omega^* M$
Piecewise smooth version on a triangulation $\Omega^*_{PS} M$

Piecewise polynomial version using polynomials with coefficients in $\mathbb{Q}$

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$$\underset{\geq}{x_0} + \cdots + \underset{\geq}{x_n} = 1$$

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$\implies \underbrace{\text{Thom–Sullivan De Rham complex } \Omega^*_{TS} M}$ $H^* \Omega_{TS} M = H^*(M;\mathbb{Q})$

Makes sense for any simplicial complex / space.

$$H^*(\Omega^*_{TS} X) \cong H^*(X;\mathbb{Q})$$

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Quillen–Sullivan Theorem

Quasi-isomorphism: A map of CDGAs that induces an isomorphism on cohomology.

equiv rch gps

$A \to B$

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Theorem (Quillen / Sullivan)

Simply connected spaces are rationally equivalent if and only if their Thom–Sullivan De Rham complexes are quasi-isomorphic.

The Thom–Sullivan De Rham complex provides an algebraic model for the rational homotopy type

The rational homotopy groups of X are the André–Quillen cohomology groups of $\Omega^*_{TS}X$.

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Instead of requiring the multiplication to be commutative, require it to be *homotopy* commutative up to “all higher homotopies”

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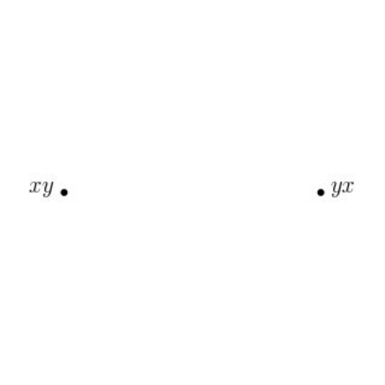

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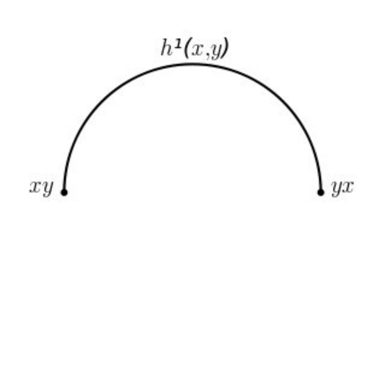

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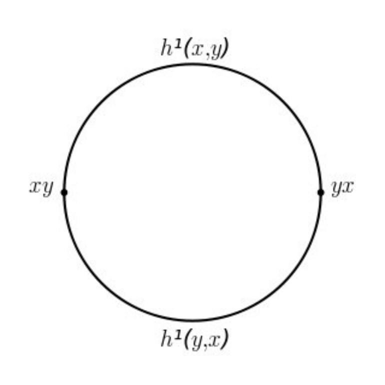

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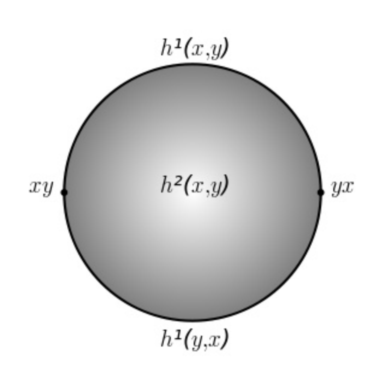


Steenrod Operations

E_∞ DGAs admit Steenrod operations

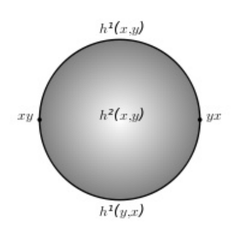
Working over $\mathbb{Z}/2$,

$$dh^n(x,y) = h^{n-1}(x,y) + h^{n-1}(y,x) + h^n(dx,y) + h^n(x,dy)$$

So for $dx \equiv 0 \mod 2$,

$$dh^n(x,x) \equiv h^{n-1}(x,x) + h^{n-1}(x,x) + 0 + 0 \equiv 0 \mod 2$$

$h^n(x,x)$ is a mod 2 cycle, represents $Sq^{2|x|-n}x$.



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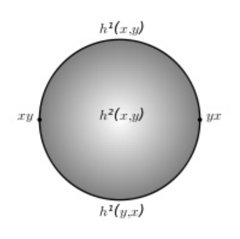

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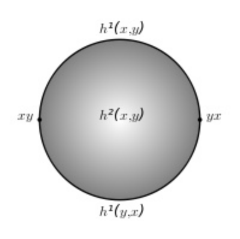
$h^n(x,x)$ is a mod 2 cycle, represents $Sq^{2|x|-n}x$.



Steenrod Operations

E_∞ DGAs admit Steenrod operations

Working over $\mathbb{Z}/2$,

$$dh^n(x,y) = h^{n-1}(x,y) + h^{n-1}(y,x) + h^n(dx,y) + h^n(x,dy)$$

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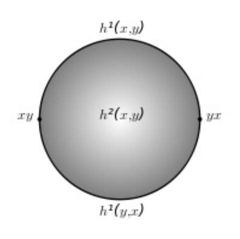
For CDGA $h^n = 0$

so Sq^i is zero



Cochains and E_∞ DGAs

The simplicial (or singular) cochain complex is naturally an E_∞ DGA.

Theorem

Any functor to chain complexes or E_∞ DGAs that satisfies a dimension hypothesis and a weak gluing condition is naturally quasi-isomorphic to the cochain functor with some coefficients.

Example

The Thom–Sullivan De Rham complex $\Omega^*_{TS}X$ is naturally quasi-isomorphic to $C^*(X;\mathbb{Q})$ through maps of E_∞ DGAs.

Consequence

No carrier for integral cohomology can be a CDGA.

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Algebraic Models for Homotopy Theory

Theorem ('2006)

Simply connected spaces are homotopy equivalent if and only if their cochain E_∞ DGAs are quasi-isomorphic.

The cochain complex as an E_∞ DGA provides an algebraic model for homotopy types.

"Can" compute homotopy groups using (e.g.) analogue of the method of Cartan–Serre.

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$C^*(S^2)$ easy to describe as an E_∞ DGA. Beyond a certain range, higher homotopy groups are unknown.

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Homotopy Algebras and Homotopy Theory

Hierarchy of algebraic structures encoding higher homotopies of commutativity.

$$\boxed{E_1 \text{ DGAs}} \subset \boxed{E_2 \text{ DGAs}} \subset \boxed{E_3 \text{ DGAs}} \subset \cdots \subset \boxed{E_\infty \text{ DGAs}}$$

E_1 DGAs are associative DGAs

E_2 DGAs are homotopy commutative plus a little more
Concise definition in terms of brace operations $x\{y_1, \ldots, y_n\}$
($x\{y\}$ is the commutativity homotopy)

E_3 and higher are "even more homotopy commutative"

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Invariants of the E_2 Structure

When we regard C^*X as an E_2 DGA, what information about a simply connected space X remains?

- Homology / cohomology of based loop space as a Hopf algebra
- Homology / cohomology of the free loop space as an H^*X-module.
- Homology / cohomology of mapping space X^M where $M = T^2$ or $\Sigma_g^2 \setminus \{p_1, \ldots, p_n\}$, $n \geq 1$.

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- For X a PD space, the *string topology* BV algebra [?]

E_2 DGAs, DG Hopf Algebras, and Anick's Theorem

Gerstenhaber–Voronov (IMRN 1995): An E_2 DGA is pretty much the same thing as a DG Hopf algebra.

Bar Construction

E_2 DGA structure on $A \Longleftrightarrow$ DG Hopf algebra structure on BA [+/-]

Anick (JAMS 1989) studied BC^*X as a (DG) "Hopf algebra up to homotopy" and proved (for primes p)

Theorem (Anick)

If X is at least c-connected ($c \geq 1$) and at most pd-dimensional, then after inverting $1, \ldots, p-1$ and changing the multiplication up to homotopy, BC^*X is dual to the universal enveloping algebra of a DG Lie algebra.

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Conjecture

If X is at least c-connected ($c \geq 1$) and at most pd-dimensional, then after inverting $1, \ldots, p-1$, the E_2 DGA C^*X is quasi-isomorphic to a commutative DGA.

Consequences

- For highly connected / low dimensional spaces, the E_2 information is relatively accessible.
- On a fixed space, for all but finitely primes, the E_2 information is relatively accessible.

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