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The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. (English summary)
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FEATURED REVIEW.

This is an important and beautiful paper. On the one hand, it confirms the senior author's decades-old vision of how unstable homotopy made "periodic" can be calculated from a small amount of stable information. On the other hand, it illustrates how T. Goodwillie's philosophy of how to resolve homotopy functors is leading to a new generation of combinatorial models in homotopy theory, particularly when applied by the junior author.

To set the stage, Goodwillie's previous work (partially available in [K-Theory **4** (1990), no. 1, 1–27; MR1076523; K-Theory **5** (1991/92), no. 4, 295–332; MR1162445; "Calculus. III. The Taylor series of a homotopy functor", in preparation]) implies that, given a pointed space X, there is a unique natural tower of fibrations

$$\cdots \to P_n(X) \to P_{n-1}(X) \to \cdots \to P_1(X) = QX$$

having the following properties. First, there is a map from X to the inverse limit, and this map is a homotopy equivalence if X is connected. Second, there are spectra C_n with an action of the nth symmetric group Σ_n, so that $D_n(X)$, the fiber of $P_n(X) \to P_{n-1}(X)$, is naturally equivalent to $\Omega^\infty((C_n \wedge X^{\wedge n})_{h\Sigma_n})$.

B. Johnson [Trans. Amer. Math. Soc. **347** (1995), no. 4, 1295–1321; MR1297532] had identified C_n as the equivariant S-dual of an explicit finite-dimensional Σ_n-space Δ_n, so that $D_n(X) \simeq \Omega^\infty(\mathrm{Map}_{\mathbb{S}}(\Delta_n, X^{\wedge n})_{h\Sigma_n})$, where $\mathrm{Map}_{\mathbb{S}}$ denotes function spectrum.

The first thing done in the paper under review is to replace Δ_n by a different model better suited for homotopical analysis. Let $\widetilde{K}_n$ denote the category of partitions of a fixed set with n elements, with the initial and terminal partitions discarded. Let K_n denote the unreduced suspension of the classifying space of $\widetilde{K}_n$. Then there is a Σ_n-equivariant map $SK_n \to \Delta_n$ which is a nonequivariant equivalence. Thus $D_n(X) \simeq \Omega^\infty(\mathrm{Map}_{\mathbb{S}}(SK_n, X^{\wedge n})_{h\Sigma_n})$.

The authors now specialize to the case when X is an odd-dimensional sphere. By filtering K_n by skeleta, and computing with homology with rational coefficients and then with mod$p$ coefficients, they are able to deduce that $\mathrm{Map}_{\mathbb{S}}(SK_n, S^{n(2s+1)})_{h\Sigma_n})$, and thus $D_n(S^{2s+1})$, is contractible unless n is a power of a prime p. Furthermore, $D_{p^i}(S^{2a+1})$ is p-local, and, with $L(i,s)$ denoting $\mathrm{Map}_{\mathbb{S}}(SK_{p^i}, S^{n(2s+1)})_{h\Sigma_{p^i}}$, $H^*(L(i,s);\mathbf{Z}/p)$ is free over the subalgebra $A(i-1)$ of the mod p Steenrod algebra A.

An immediate consequence of this last fact is that $D_{p^i}(S^{2a+1})$ has trivial v_k-periodic homotopy if $i > k$. Summarizing, with $R_{i,s} = P_{p^i}(S^{2s+1})$ and with the v_k^{-1}-localization functor applied to everything, there is a finite tower $R_{k,s} \to R_{k-1,s} \to \cdots \to R_{0,s}$ with $S^{2s+1} \to R_k$ a v_k-periodic homotopy equivalence, and with the ith fiber equal to the 0th space of an explicit spectrum $L(i,s)$.

When $k = 0$ we learn that $S^{2s+1} \to QS^{2s+1}$ is a rational equivalence, known since the 1950s. When $k = 1$ and $p = 2$, we learn that S^{2s+1} is equivalent in v_1-periodic homotopy to the fiber of the Hopf invariant $QS^{2s+1} \to Q(\mathbf{R}\mathrm{P}^\infty/\mathbf{R}\mathrm{P}^{2s})$. This was shown previously in major work by Mahowald [Ann. of Math. (2) **116** (1982), no. 1, 65–112; MR0662118] and Mahowald and R. D. Thompson [Topology **31** (1992), no. 1, 133–141; MR1153241].

Making the story even more complete, and giving a second proof of the most technical parts of the paper under review, is subsequent work by Arone and W. Dwyer ["Partition complexes, Tits buildings, and symmetric products", Proc. London Math. Soc., to appear]. They identify the spectra $L(i,s)$ as suitable Thom spectra over the "Steinberg module" part of $B(\mathbf{Z}/p)^i$. Such a result was presaged by work in the early 1980s by the reviewer, S. Priddy, and S. Mitchell [see, e.g., N. J. Kuhn and S. B. Priddy, Math. Proc. Cambridge Philos. Soc. **98** (1985), no. 3, 459–480; MR0803606]. This older works implies, among other things, that the spectral sequence for computing $\pi_*(S^1)$ associated to the tower of $R_{k,0}$'s collapses at E_2. It would be very interesting to know if anything like this carries over to the spectral sequences that calculate $\pi_*(S^{2s+1})$ for $s \geq 1$.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.