The standard resolution of A as an (A,Γ)-comodule

Let's progress from general to specific. Fix a Hopf algebroid (A,Γ) with units η_L and η_R and comultiplication $\Delta\colon\ \Gamma \to \Gamma^R{\otimes^L}_A\Gamma$, and a comodule N over this Hopf algebroid with structure map (of left A-modules) $\phi\colon\ N \to \Gamma^R{\otimes}_A N$. (From now on all the tensor products will be implicitly over A; if Γ appears on the left side of the tensor it is implicitly using η_R, the right unit, as the right module structure, and similarly using η_L if it appears on the opposite side.) Cf def 5.12 on p18 of [coctalos]. In particular, $\phi = \eta_L$ if $N = A$ (see below), and $\phi = \Delta$ if $N = \Gamma$.

We have an augmented cosimplicial object $C(\Gamma,\Gamma,N) = X$ with

$$X^p = \Gamma \otimes \underbrace{\Gamma \otimes \cdots \otimes \Gamma}_{p \text{ times}} \otimes N$$

The coface maps $d^i\colon\ X^p \to X^{p+1}$ for $0 \le i \le p+1$ are given by

$$d^i = \begin{cases} 1 \otimes \cdots \otimes \Delta \otimes \cdots \otimes 1 & \text{if } i \le p \\ 1 \otimes \cdots \otimes \phi & \text{if } i = p+1 \end{cases}$$

For example, when $p = 0$ the two maps $\Gamma \otimes N \to \Gamma \otimes \Gamma \otimes N$ are $d^0 = \Delta \otimes 1$ and $d^1 = 1 \otimes \phi$. The augmentation of this cosimplicial object X is the structure map $\phi\colon\ N \to \Gamma \otimes N$ (cf pp274-5 of [ha]; in particular, $d^0\phi = d^1\phi$ by coassociativity). Codegeneracies are given by applying the augmentation of the Hopf algebroid $\epsilon\colon\ \Gamma \to A$ to one of the middle factors of Γ.

(If you prefer, a cosimplicial object is supposed to be a functor from nonempty finite ordered sets. In this case, if $S = \{0, 1, \ldots, p\}$, then the tensor symbols in X^p are in bijection with the elements of S; this helps me to remember the effect of a general map of ordered sets.)

We thus get a cobar **resolution**

$$0 \to N \to \Gamma \otimes N \to \Gamma \otimes \Gamma \otimes N \to \cdots$$

where the coboundary maps are ϕ, $\Delta\otimes 1 - 1\otimes\phi$, $\Delta\otimes 1\otimes 1 - 1\otimes\Delta\otimes 1 + 1\otimes 1\otimes\phi$, et cetera. (Cf pp274-5 of [ha]. Generally we just get an augmented cochain complex. The exactness comes from the extra degeneracy, ie applying the augmentation $\epsilon\colon\ \Gamma \to A$ to the leftmost factor of Γ.) If N is A (whose comodule structure $A \to \Gamma$ is the *left* unit η_L because it's supposed to be a map of left A-modules), then you simply replace ϕ with η_L.

Before talking about the specific quadratic Hopf algebroid, let me just say what happens when we're going to compute Ext.

Applying $\mathrm{Hom}_{(A,\Gamma)}(A,-)$ takes an induced comodule $\Gamma \otimes N$ to $A \otimes N \cong N$, and has a straightforward effect on maps - specifically, $f\colon \Gamma \otimes N \to \Gamma \otimes N'$ becomes $(\epsilon \otimes 1) \circ f \circ (\eta_R \otimes 1)$. (Lemma 12.4 on p38 of [coctalos] says that we have an adjoint pair $forget\colon$ (A,Γ)-comodules $\leftrightarrow$ A-modules : $\Gamma \otimes_A -$. In particular, $\mathrm{Hom}_{(A,\Gamma)}(A, \Gamma \otimes N) \cong \mathrm{Hom}_A(A,N) \cong N \cong A \otimes N$ (cf Wed-12/23-1). To see $\mathrm{Hom}_{(A,\Gamma)}(A,-)$ takes $\Gamma \otimes N \xrightarrow{f} \Gamma \otimes N'$ to $A \otimes N \xrightarrow{\eta_R \otimes 1} \Gamma \otimes N \xrightarrow{f} \Gamma \otimes N' \xrightarrow{\epsilon \otimes 1} A \otimes N'$, we can interpret $\eta_R \otimes 1$ as the functor $\Gamma \otimes_A -$ because Γ becomes a right A-module via η_R, and $\epsilon \otimes 1$ as the forgetful functor because of the counital property: $N \xrightarrow{\phi} \Gamma \otimes N \xrightarrow{\epsilon \otimes 1} N$ is the identity.) Applying this to the cobar resolution we get

$$0 \to \mathrm{Hom}_{(A,\Gamma)}(A,N) \to A \otimes N \to A \otimes \Gamma \otimes N \to \cdots$$

and we drop the left-hand term (cf p50 of [ha]) and have the cobar complex

$$0 \to A \otimes N \to A \otimes \Gamma \otimes N \to A \otimes \Gamma \otimes \Gamma \otimes N \to \cdots,$$

which computes Ext.

The coface maps, after applying $\mathrm{Hom}_{(A,\Gamma)}(A,-)$, are mostly the same, with the exception of $\Delta \otimes \cdots \otimes 1$; it becomes $\eta_R \otimes \cdots \otimes 1$. So the coboundary maps in the cobar complex are: $\eta_R \otimes 1 - 1 \otimes \phi$, $\eta_R \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \phi$, et cetera.

There's a lot of redundant notation here when I leave the tensor factors of A on either side; it's more compact to write the cobar complex when $N = A$ as

$$0 \to A \to \Gamma \to \Gamma \otimes \Gamma \to \cdots$$

with coboundary maps

$$\begin{aligned} &\eta_R - \eta_L \\ &\eta_R \otimes 1 - \Delta + 1 \otimes \eta_L \\ &\eta_R \otimes 1 \otimes 1 - \Delta \otimes 1 + 1 \otimes \Delta - 1 \otimes 1 \otimes \eta_L, \end{aligned}$$

et cetera. But it's sometimes handy to remember the implicit factors of A on either side. By convention, we use the shorthand $[\gamma_1|\cdots|\gamma_p]$ for the representative elements $1 \otimes \gamma_1 \otimes \cdots \otimes \gamma_p \otimes 1$ in the cobar complex. Under these conventions, the coboundary maps are:

$$\begin{aligned}
&a \mapsto [\eta_R(a)] - [\eta_L(a)] \\
&[\gamma] \mapsto [1|\gamma] - [\Delta\gamma] + [\gamma|1] \\
&[\gamma_1|\gamma_2] \mapsto [1|\gamma_1|\gamma_2] - [\Delta\gamma_1|\gamma_2] + [\gamma_1|\Delta\gamma_2] - [\gamma_1|\gamma_2|1]
\end{aligned}$$

Note that the left-right tensor product notation means that you can move an element $a \in A$ across the tensor product, but it switches from the left to right units; e.g. $[\gamma_1|\eta_L(a)\gamma_2] = [\eta_R(a)\gamma_1|\gamma_2]$.

OK.

Let's specifically talk about the quadratic Hopf algebroid, which has $A = \mathbb{Z}[b,c]$, $\Gamma = A[r]$, $\eta_L\colon A \to A[r]$ being the standard inclusion, $\eta_R(b) = b + 2r$, $\eta_R(c) = c + br + r^2$, $\Delta(a) = \overset{\text{1st}}{a} \otimes \overset{\text{2nd}}{1}$ for $a \in A$, $\Delta(r) = r \otimes 1 + 1 \otimes r$. Cf Sat-11/14.

Our cobar resolution is isomorphic to

$$0 \to \mathbb{Z}[b,c] \to \mathbb{Z}[b,c,r] \xrightarrow{d^0} \mathbb{Z}[b,c,r_1',r_2'] \xrightarrow{d^1} \mathbb{Z}[b,c,r_1'',r_2'',r_3''] \to \cdots$$

where we've collapsed some identifications. The cobar representatives are given by $r = r$, $r_1' = r \otimes 1$, $r_2' = 1 \otimes r$, et cetera.

The coboundary d^0 is $\Delta \otimes 1 - 1 \otimes \eta_L$. This sends a to $a \otimes 1 - a \otimes 1 = 0$ for $a \in A$, and sends r to $1 \otimes r = r_2'$. You can calculate the image of any element but a general formula is kind of annoying to write down.

The coboundary d^1 is $\Delta \otimes 1 \otimes 1 - 1 \otimes \Delta \otimes 1 + 1 \otimes 1 \otimes \eta_L$. For example, it sends $a \otimes 1$ to $a \otimes 1 \otimes 1$ (which I called "a" in the above identification) for $a \in A$, and $r_1' = r \otimes 1$ to $r \otimes 1 \otimes 1 + 1 \otimes r \otimes 1 = r_1'' + r_2''$.

For a similar discussion of the cobar complex, cf Sat-11/14.