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On the 2-primary v_1-periodic homotopy groups of spaces

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Abstract

We develop foundations of a general approach for calculating p-primary v_1-periodic homotopy groups of spaces using their p-adic KO-cohomologies and K-cohomologies with particular attention to the case $p = 2$. As a main application, we derive a method for calculating v_1-periodic homotopy groups of simply connected compact Lie groups using their complex, real, and quaternionic representation theories. This method has been applied very effectively by Davis in recent work. We rely heavily on the v_1-stabilization functor Φ_1 from spaces to spectra. Roughly speaking, we obtain the p-primary v_1-periodic homotopy of a space X from the p-adic KO-cohomology of $\Phi_1 X$, which we obtain from the p-adic KO-cohomology and K-cohomology of X by a v_1-stabilization process under suitable conditions.

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1. Introduction

The p-primary v_1-periodic homotopy groups $v_1^{-1}\pi_* X$ of a pointed space X, as defined by Davis and Mahowald [26], are a localization of the portion of the homotopy groups of X detected by p-adic K-theory. In [19], we showed that the groups $v_1^{-1}\pi_* X$ are naturally isomorphic to stable homotopy groups $\pi_*\tau_p\Phi_1 X$, where $\tau_p\Phi_1 X$ is the p-torsion part of the spectrum $\Phi_1 X$ obtained using the v_1-stabilization functor Φ_1 constructed in [15,20,27,31]. Moreover, in [19], we developed an approach for calculating

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$v_1^{-1}\pi_*X \cong \pi_*\tau_p\Phi_1X$ from $K^*(X;\hat{\mathbb{Z}}_p)$ via $K^*(\Phi_1X;\hat{\mathbb{Z}}_p)$ when p is an odd prime and X is a suitable space such as a simply connected finite H-space. This approach has been applied very successfully by Davis to simply connected compact Lie groups in [24]. After considerable effort, we recently found that most of the results of [19] can be extended to the case $p=2$ with some modifications and restrictions, and these new results have already been applied by Davis to complete his 13-year project of computing the v_1-periodic homotopy groups of all simply connected compact Lie groups [25]. In this paper and its sequel, we shall develop the promised 2-primary extensions of results of [19], giving a general approach for calculating v_1-periodic homotopy groups of suitable spaces. When possible, we work at an arbitrary prime p, although our main concern is with $p=2$.

We begin in Section 2 by discussing the general theory of v_1-periodic homotopy groups and of the functor Φ_1. This functor carries a pointed space X to a K/p_*-local spectrum Φ_1X such that $v_1^{-1}\pi_*X \cong \pi_*\tau_p\Phi_1X$ and, in fact, such that $v_1^{-1}\pi_*(X;W) \cong [W,\Phi_1X]_*$ for each finite p-torsion spectrum W. Thus, the study of v_1-periodic homotopy groups may be centered around the spectra Φ_1X. These spectra are especially well-behaved when X is a simply connected finite H-space or is spherically resolved or, more generally, when Φ_1X has an exponent. In such cases, we show that Φ_1X is periodic (see Theorem 2.6); we show that X becomes v_1-periodically equivalent to the infinite loop space $\Omega^\infty\Phi_1X$ after finite looping (see Theorem 2.10); and we show that the ordinary homotopy groups of X eventually map splittably onto the v_1-periodic homotopy groups of X (see Theorem 2.11).

Since the spectra Φ_1X are K/p_*-local, they may be studied by the methods of united K-theory [16]. After establishing a Pontrjagin duality between the united p-adic K-cohomology and K-homology theories of spectra (see Theorem 3.1), we show that the groups $v_1^{-1}\pi_*X \cong \pi_*\tau_p\Phi_1X$ are determined up to extension by $KO^*(\Phi_1X;\hat{\mathbb{Z}}_p)$ together with its Adams operations (see Theorem 3.2).

For many interesting spaces X, including simply connected compact Lie groups, the spectra Φ_1X have cohomologies $K^*(\Phi_1X;\hat{\mathbb{Z}}_p)$ concentrated in even or odd degrees. We show that such K/p_*-local spectra Φ_1X are completely classified by their united p-adic K-cohomologies

$$K^*_{\mathrm{CR}}(\Phi_1X;\hat{\mathbb{Z}}_p) \cong \{K^*(\Phi_1X;\hat{\mathbb{Z}}_p), KO^*(\Phi_1X;\hat{\mathbb{Z}}_p)\}$$

equipped with the complexification, realification, conjugation, and Adams operations (see Theorem 5.3). Moreover, when $K^{n-1}(\Phi_1X;\hat{\mathbb{Z}}_p)=0$, we find that $K^*_{CR}(\Phi_1X;\hat{\mathbb{Z}}_p)$ is largely determined by a small part

$$K^n_\Delta(\Phi_1X;\hat{\mathbb{Z}}_p) = \{K^n(\Phi_1X;\hat{\mathbb{Z}}_p), KO^n(\Phi_1X;\hat{\mathbb{Z}}_p), KO^{n-4}(\Phi_1X;\hat{\mathbb{Z}}_p)\}$$

which we call a Δ-module (see Theorems 4.3 and 4.11). We also show that $K^n_\Delta(\Phi_1X;\hat{\mathbb{Z}}_p)$ has a crucial exactness property which facilitates comparisons with other Δ-modules (see Theorem 4.4). This work allows us to approach the v_1-periodic homotopy groups $v_1^{-1}\pi_*X$ of suitable spaces X by seeking to calculate the associated Δ-modules $K^n_\Delta(\Phi_1X;\hat{\mathbb{Z}}_p)$.

For a pointed space X and integer n, we approach the stable K-theoretic Δ-module $K^n_\Delta(\Phi_1X;\hat{\mathbb{Z}}_p)$ by starting with the corresponding unstable K-theoretic Δ-module

$$\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p) = \{\widetilde{K}^n(X;\hat{\mathbb{Z}}_p), \widetilde{KO}^n(X;\hat{\mathbb{Z}}_p), \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)\}$$

and dividing out by its "Φ_1-trivial" part, where the tilde indicates reduced cohomology. In preparation, we develop an array of unstable operations in the p-adic KO-cohomologies and K-cohomologies of spaces using Atiyah's real K-theory (see Theorems 6.4 and 6.5). We then show that the v_1-stabilization homomorphism $\Phi_1\colon \widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p) \to K^n_\Delta(\Phi_1 X;\hat{\mathbb{Z}}_p)$ must annihilate or linearize these unstable operations, allowing us to calculate $K^n_\Delta(\Phi_1 X;\hat{\mathbb{Z}}_p)$ from $\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p)$ under suitable conditions (see Theorem 7.2). When combined with the preceding work, this gives a v_1-stabilization method for calculating $v_1^{-1}\pi_* X$ from $\widetilde{K}^*_{CR}(X;\hat{\mathbb{Z}}_p)$ for suitable spaces X (see 7.6).

To illustrate the v_1-stabilization method, we calculate the groups $v_1^{-1}\pi_* S^{2n+1}$ at $p=2$ and recover results of Mahowald and Davis [23] (see Theorem 8.9). As a main application of the method, we obtain an approach for calculating 2-primary v_1-periodic homotopy groups of simply connected compact Lie groups using their complex, real, and quaternionic representation theories. This approach grew, in part, from our extensive correspondence with Bendersky and Davis, who were developing another 2-primary approach using complex representation theory together with the Bendersky–Thompson spectral sequence (see [11]). Our approach eliminates major difficulties with differentials and has now been applied very effectively by Davis [25] as previously noted. For a simply connected compact Lie group G, our main result (Theorem 9.3) expresses $KO^*(\Phi_1 G;\hat{\mathbb{Z}}_2)$ in terms of the representation theory of G. This result confirms a general conjecture cited in [25, Conjecture 2.2].

This paper provides foundations for a sequel in which we shall continue to develop 2-primary extensions of results of [19]. In particular, we shall obtain explicit 2-primary constructions of the v_1-stabilization $\Phi_1 X$ and of the localization $X_{K/2}$ for various spaces X, including many simply connected compact Lie groups.

Throughout this paper, we generally follow the terminology of [21], so that "space" means "simplicial set," and we let Ho_* (resp. Ho^s) denote the homotopy category of pointed spaces (resp. spectra). The paper is divided into the following sections:

1. Introduction
2. The general theory of v_1-periodic homotopy groups and the functor Φ_1
3. Pontrjagin duality for united p-adic K-theory
4. Even and odd CR-modules
5. Even and odd K/p_*-local spectra
6. Unstable operations in p-adic K-theory
7. The v_1-stabilization homomorphism
8. The v_1-stabilizations of odd spheres
9. The v_1-stabilizations of simply connected compact Lie groups

2. The general theory of v_1-periodic homotopy groups and the functor Φ_1

Working at an arbitrary prime p, we first recall the v_1-periodic homotopy groups $v_1^{-1}\pi_* X$ of a pointed space X and explain how they are captured by the spectrum $\Phi_1 X$. We then obtain stronger results on $v_1^{-1}\pi_* X$ and $\Phi_1 X$ when X is a simply connected finite H-space or is spherically resolved or, more generally, when $\Phi_1 X$ has an exponent. In such cases, we show that the spectrum $\Phi_1 X$ is periodic; we show that X becomes v_1-periodically equivalent to the infinite loop space $\Omega^\infty \Phi_1 X$ after finite looping; and we show that the ordinary homotopy groups of X eventually map splittably onto the v_1-periodic homotopy groups of X. These results build on insights of Davis and Mahowald in [23,26,27].

2.1. The v_1-periodic homotopy groups. For a finite p-torsion spectrum $W \in \mathrm{Ho}^{\mathrm{s}}$, a v_1-*map* is a $K(1)_*$-equivalence ($=K/p_*$-equivalence) ω: $\Sigma^d W \to W$ with $d > 0$ such that $K(n)_*\omega = 0$ for $n > 1$, where $K(n)_*$ is the nth Morava K-theory at p. The Hopkins–Smith Periodicity Theorem (see [30] or [36]) ensures that each finite p-torsion spectrum W has a v_1-map, which becomes unique after sufficient iteration and in fact becomes natural. Since the sequence $W \xleftarrow{\omega} \Sigma^d W \xleftarrow{\omega} \Sigma^{2d} W \xleftarrow{\omega} \cdots$ in Ho^{s} eventually desuspends uniquely in Ho_*, we may define the v_1-*periodic homotopy groups* of a space $X \in \mathrm{Ho}_*$ relative to W by

$$v_1^{-1}\pi_*(X; W) \cong \operatorname*{colim}_m [\Sigma^{md} W, X]_*$$

with naturality in both $X \in \mathrm{Ho}_*$ and $W \in \mathrm{Ho}^{\mathrm{s}}$. Following Davis and Mahowald [26], we may also define the (absolute) v_1-*periodic homotopy groups* of $X \in \mathrm{Ho}_*$ by

$$v_1^{-1}\pi_* X = \operatorname*{colim}_k \, v_1^{-1}\pi_{*+1}(X; \mathbb{Z}/p^k) = \operatorname*{colim}_k \, v_1^{-1}\pi_*(X; S\mathbb{Z}/p^k)$$

using the Moore spectra $S\mathbb{Z}/p^k = S^0 \cup_{p^k} e^1$ with the canonical maps $S\mathbb{Z}/p^{k+1} \to S\mathbb{Z}/p^k$.

2.2. The functor Φ_1. By [20] or by earlier work in [15,27,31], there is a v_1-*stabilization functor* or v_1-*periodic spectrum functor* Φ_1: $\mathrm{Ho}_* \to \mathrm{Ho}^{\mathrm{s}}$ such that:

(i) for a space X and finite p-torsion spectrum W, there is a natural isomorphism $v_1^{-1}\pi_*(X; W) \cong [W, \Phi_1 X]_*$;
(ii) $\Phi_1 X$ is K/p_*-local for each space X;
(iii) for a spectrum E, there is a natural equivalence $\Phi_1(\Omega^\infty E) \cong E_{K/p}$;
(iv) Φ_1 preserves homotopy fiber squares;
(v) for a space X and finite complex A in Ho_*, there is a natural equivalence $\Phi_1(X^A) \cong (\Phi_1 X)^A$.

2.3. p-Torsion parts and p-completions of spectra. A spectrum E has a natural p-*torsion part* $\tau_p E \to E$ given by the homotopy fiber of the localization $E \to E[1/p]$ away from p with $\tau_p E \simeq E \wedge \tau_p S \simeq E \wedge S^{-1}\mathbb{Z}_{p^\infty}$. It also has a natural p-*completion* $E \to \hat{E}_p$ given by the $S\mathbb{Z}/p_*$-localization [13] with $\hat{E}_p \simeq F(S^{-1}\mathbb{Z}_{p^\infty}, E)$. The functors τ_p: $\mathrm{Ho}^{\mathrm{s}} \leftrightarrows \mathrm{Ho}^{\mathrm{s}}$: $(\hat{-})_p$ are adjoint and restrict to equivalences between the homotopy categories of p-complete spectra and of p-torsion spectra, with the K/p_*-local spectra corresponding to the p-torsion K_*-local spectra. Thus $\Phi_1 X$ corresponds to the p-torsion K_*-local spectrum $\tau_p \Phi_1 X$, and by Bousfield [19] we have:

2.4. Theorem. *For a space X, there is a natural isomorphism $v_1^{-1}\pi_* X \cong \pi_* \tau_p \Phi_1 X$.*

In some important cases, the spectrum $\Phi_1 X$ has an exponent and a periodicity, which are automatically inherited by $v_1^{-1}\pi_* X$.

2.5. Spaces with Φ_1-exponents. A space X is said to have a Φ_1-*exponent* p^r if $p^r \simeq 0$: $\Phi_1 X \to \Phi_1 X$. Note that whenever two of the spaces in a fiber sequence have Φ_1-exponents (at p), then so does the third by 2.2 (iv). A space is also said to have an *eventual H-space exponent* p^r if $p^r \simeq 0$: $(\Omega_0^N X)_{(p)} \to (\Omega_0^N X)_{(p)}$ for sufficiently large N. This easily implies that X has a Φ_1-exponent p^r and holds (for suitable values of r) whenever X is a sphere or a simply connected finite H-space (see [35,37]). Thus the following theorem will apply to spherically resolved spaces and to simply-connected finite H-spaces.

2.6. Theorem. *If a space X has a Φ_1-exponent p^r, then the spectrum $\Phi_1 X$ is periodic with $\Sigma^{p^r q_r}\Phi_1 X \simeq \Phi_1 X$ where $q_r = \max\{8, 2^{r-1}\}$ for $p = 2$ and $q_r = 2(p-1)p^{r-1}$ for p odd.*

Proof. We rely on the K_*^{CRT}-Adams spectral sequence of [16] and use the associated notation. Since the united K-homology $K_*^{CRT}\Phi_1 X \in ACRT$ has exponent p^r, it has a canonical periodicity ι: $\Sigma^{q_r} K_*^{CRT}\Phi_1 X \cong K_*^{CRT}\Phi_1 X$, and the element

$$d_2\iota \in \mathrm{Ext}^{2,q_r+1}_{ACRT}(K_*^{CRT}X, K_*^{CRT}X)$$

is the obstruction to realizing ι by a map $\Sigma^{q_r}\Phi_1 X \to \Phi_1 X$. Since d_2 acts as a derivation on compositions and since $p^r(d_2\iota) = 0$, we find that $d_2(\iota^{p^r}) = 0$, and hence ι^{p^r} may be realized by a map $\Sigma^{p^r q_r}\Phi_1 X \to \Phi_1 X$. This gives the required periodicity. $\square$

Before developing further properties of v_1-periodic homotopy groups, we need:

2.7. v_1-Periodic equivalences. As in [19], we say that a map $f\colon X \to Y$ in Ho_* is a *v_1-periodic equivalence* when it satisfies the following equivalent conditions:

(i) $\Phi_1 f\colon \Phi_1 X \simeq \Phi_1 Y$;
(ii) $f_*\colon v_1^{-1}\pi_* X \cong v_1^{-1}\pi_* Y$;
(iii) $f_*\colon v_1^{-1}\pi_*(X; \mathbb{Z}/p) \cong v_1^{-1}\pi_*(Y; \mathbb{Z}/p)$;
(iv) $f_*\colon v_1^{-1}\pi_*(X; W) \cong v_1^{-1}\pi_*(Y; W)$ for each finite p-torsion spectrum W.

A v_1-periodic equivalence may also be characterized homologically. We let $X\langle m\rangle \to X$ denote the m-connected cover of a space X, and we call $K/p_* X\langle m\rangle$ the *generic K/p-homology* of X when $m \geqslant 3$, since it does not depend on the choice of such m. By Bousfield [18, Sections 11.5 and 11.11], we have:

2.8 Theorem. *A map f in Ho_* is a v_1-periodic equivalence if and only if Ωf is a generic K/p_*-equivalence. In this case, f is also a generic K/p_*-equivalence.*

2.9. Examples of v_1-periodic equivalences. By [18, Corollary 11.2], each K/p_*-equivalence of H-spaces is a v_1-periodic equivalence. Thus, each K/p_*-equivalence f of spaces suspends to a v_1-periodic equivalence Σf because $\Omega\Sigma f$ is a K/p_*-equivalence of H-spaces. For example, whenever a map $\alpha\colon \Sigma^d A \to A$ of spaces represents a v_1-map $\Sigma^\infty\alpha$ of spectra, then α suspends to a v_1-periodic equivalence $\Sigma\alpha$, although α itself need not be a v_1-periodic equivalence by Langsetmo and Stanley [32]. Perhaps the most striking example of a v_1-periodic equivalence is the Snaith map

$$s\colon \Omega_0^{2n+1}S^{2n+1} \longrightarrow \Omega^\infty\Sigma^\infty RP^{2n}$$

for $n \geqslant 0$ at $p = 2$, discovered by Mahowald [33] and extended to odd primes by Thompson [39]. This gives

$$\Phi_1 S^{2n+1} \simeq \Sigma^{2n+1}(\Sigma^\infty RP^{2n})_{K/2}$$

by 2.2 and leads to a computation of the groups $v_1^{-1}\pi_* S^{2n+1} = \pi_*\Phi_1 S^{2n+1}$ at $p = 2$ (see [23] and Theorem 8.9). Proceeding more generally, we now show that each space with a Φ_1-exponent (including each spherically resolved space and each simply-connected finite H-space) becomes v_1-periodically equivalent to an infinite loop space after finite looping.

2.10 Theorem. *For a space X with a Φ_1-exponent p^r, there is a natural v_1-periodic equivalence $\lambda\colon \Omega^m X \to \Omega^\infty \Omega^m \Phi_1 X$ for sufficiently large m (depending on p^r).*

This will be proved later in 2.14. We now consider the induced natural homomorphism

$$\lambda_*\colon \pi_i X \longrightarrow v_1^{-1}\pi_i X$$

defined for sufficiently large i when X has a Φ_1-exponent. To show that λ_* is splittably epic for large i, we must assume that $v_1^{-1}\pi_*(X;\mathbb{Z}/p)$, or equivalently $v_1^{-1}\pi_* X$, is of finite type. This condition holds whenever X is spherically resolved [23], or X is a simply-connected compact Lie group (see 9.1 and Theorem 9.3), or X is a rationally associative finite H-space for p odd [19]. Slightly generalizing a result of [26] or [23], we have:

2.11. Theorem. *If X is a space with a Φ_1-exponent p^r and with $v_1^{-1}\pi_*(X;\mathbb{Z}/p)$ of finite type, then $\lambda_*\colon \pi_i X \to v_1^{-1}\pi_1 X$ is splittably epic for sufficiently large i.*

Proof. Whenever an abelian sequence $G_0 \to G_1 \to G_2 \ldots$ has a finitely generated colimit G_∞, the colimit map $G_m \to G_\infty$ must be splittably epic for sufficiently large m. Thus, since $v_1^{-1}\pi_*(X;\mathbb{Z}/p^r)$ is of finite type, the colimit map $\pi_i(X;\mathbb{Z}/p^r) \to v_1^{-1}\pi_i(X;\mathbb{Z}/p^r)$ must be splittably epic for sufficiently large i. In the commutative diagram

$$\begin{array}{ccc}
\pi_i(X;\mathbb{Z}/p^r) & \xrightarrow{\partial_*} & \pi_{i-1}X \\
\downarrow{\scriptstyle\lambda_*} & & \downarrow{\scriptstyle\lambda_*} \\
\pi_i(\Phi_1 X;\mathbb{Z}/p^r) & \xrightarrow{\partial_*} & \pi_{i-1}(\Phi X)
\end{array}$$

for such i, the left λ_* is splittably epic since it is equivalent to the colimit map, and the bottom ∂_* is splittably epic since $p^r \simeq 0\colon \Phi_1 X \to \Phi_1 X$. Hence, the right λ_* is also splittably epic, and the lemma follows. $\square$

We now prepare to prove Theorem 2.10, relying on work in [20].

2.12. The category $\mathscr{W}$. Let d_1 be the integer defined in [20] with $d_1 = 3$ for p odd and $d_1 = 3, 4$, or 5 (but not known) for $p = 2$. Let $\mathscr{W}$ be the category of which an *object* is a d_1-connected finite p-torsion complex $W \in \mathrm{Ho}_*$ equipped with a v_1-map $\omega\colon \Sigma^m W \to W$ in Ho_* for some $m > 0$, such that ω is a v_1-periodic equivalence, and of which a *map* $f\colon (W,\omega) \to (W',\omega')$ is a map $f\colon W \to W'$ in Ho_* with $f\omega^i = (\omega')^j(\Sigma^k f)$ for some $i, j, k > 0$. Note that any finite p-torsion complex with a v_1-map in Ho_* may be suspended to give an object of $\mathscr{W}$ by 2.9, and that the category $\mathscr{W}$ is closed under the suspension functor. Also, for objects (W,ω) and (W',ω') of $\mathscr{W}$, any map $W \to W'$ in Ho_* may be finitely suspended to give a map in $\mathscr{W}$.

Lemma 2.13. *For an object $(W,\omega) \in \mathscr{W}$ and space $X \in \mathrm{Ho}_*$, there is a natural v_1-periodic equivalence $h\colon X^W \to \Omega^\infty(\Phi_1 X)^W$ which respects the suspension in $\mathscr{W}$.*

Proof. Since the map $\omega\colon \Sigma^m W \to W$ is an L_1^f-equivalence by [20, Corollary 4.8], there is a natural isomorphism $[W, L_1^f X] \cong v_1^{-1}\pi_0(L_1^f X; W)$. This composes with the natural isomorphisms

$$v_1^{-1}\pi_0(L_1^f X; W) \cong [W, \Phi_1 L_1^f X] \cong [\Theta_1 \Sigma^\infty W, L_1^f X]$$

of [20, Corollary 5.9] to induce a natural equivalence $\Theta_1 \Sigma^\infty W \simeq L_1^f W$, and hence $(L_1^f X)^W \simeq \Omega^\infty(\Phi_1 X)^W$ by [20, Theorem 5.4]. This combines with the v_1-periodic equivalence $X^W \to (L_1^f X)^W$ to give the desired h. $\square$

2.14. Proof of Theorem 2.10. Choose a sequence of objects (W_i, ω_i) and maps

$$\alpha_i\colon \Sigma^{m_{i+1}-m_i}(W_i, \omega_i) \longrightarrow (W_{i+1}, \omega_{i+1})$$

in $\mathscr{W}$ for $i \geqslant 1$ using Moore spaces

$$W_i = M^{m_i}(p^i) = S^{m_i - 1} \cup_{p^i} e^{m_i}$$

with canonical maps $\alpha_i\colon M^{m_{i+1}}(p^i) \to M^{m_{i+1}}(p^{i+1})$ in Ho_*. Let $q\colon W_i \to S^{m_i}$ be the pinching map. Using a nullhomotopy $p^r \simeq 0\colon \Phi_1 X \to \Phi_1 X$, choose a left inverse γ to the map $q^*\colon \Omega^{m_r}\Phi_1 X \to (\Phi_1 X)^{W_r}$, and let $\gamma\colon (\Phi_1 X)^{W_i} \to \Omega^{m_i}\Phi_1 X$ be the induced map for $i \geqslant r$. Note that when $i \geqslant 2r$, the map γ does not depend on the choice of left inverse. We claim that the composition

$$\Omega^{m_i} X \xrightarrow{q^*} X^{W_i} \xrightarrow{h} \Omega^\infty(\Phi_1 X)^{W_i} \xrightarrow{\Omega^\infty \gamma} \Omega^\infty \Omega^{m_i}\Phi_1 X$$

is a v_1-periodic equivalence for $i \geqslant r$. For this it suffices to show that the map of constant towers

$$\{v_1^{-1}\pi_*\Omega^{m_i} X\}_i \longrightarrow \{v_1^{-1}\pi_*\Omega^\infty\Omega^{m_i}\Phi_1 X\}_i$$

is a pro-isomorphism and hence an isomorphism. This follows since $\{v_1^{-1}\pi_*(-)\}_i$ carries h to an isomorphism by Lemma 2.13 and also carries q^* and $\Omega^\infty\gamma$ to pro-isomorphisms because $\{\pi_*(-)\}_i$ carries $q^*\colon \Omega^{m_i}\Phi_1 X \to (\Phi_1 X)^{W_i}$ to a pro-isomorphism. The required map λ of Theorem 2.10 is now given by $\Omega^{m_i} X \to \Omega^\infty\Omega^{m_i}\Phi_1 X$ for $i = 2r$. $\square$

3. Pontrjagin duality for united p-adic K-theory

Working at a prime p, we wish to study the spectra $\Phi_1 X$ using the methods of united K-theory [16]. In preparation, we now establish a Pontrjagin duality (Theorem 3.1) relating the united p-adic K-cohomology

$$K^*_{CRT}(E; \hat{\mathbb{Z}}_p) = \{K^*(E; \hat{\mathbb{Z}}_p), KO^*(E; \hat{\mathbb{Z}}_p), KT^*(E; \hat{\mathbb{Z}}_p)\} \overset{\text{def}}{=} \{\widehat{K}^* E, \widehat{KO}^* E, \widehat{KT}^* E\}$$

of a spectrum E to the united p-torsion K-homology

$$\begin{aligned} K_*^{CRT}(E; \mathbb{Z}_{p^\infty}) &= \{K_*(E; \mathbb{Z}_{p^\infty}), KO_*(E; \mathbb{Z}_{p^\infty}), KT_*(E; \mathbb{Z}_{p^\infty})\} \overset{\text{def}}{=} \{\overline{K}_* E, \overline{KO}_* E, \overline{KT}_* E\} \\ &\cong \{K_{*-1}(\tau_p E), KO_{*-1}(\tau_p E), KT_{*-1}(\tau_p E)\} = K_{*-1}^{CRT}(\tau_p E) \end{aligned}$$

thereby extending previous work of [5,19, Corollary 2.3; 40]. We also give a basic application of this duality showing that the v_1-periodic homotopy groups of a space X can be extracted from the p-adic KO-cohomology of $\Phi_1 X$ (Theorem 3.2). For a locally compact Hausdorff abelian group G, the *Pontrjagin* *dual* $G^{\#}$ is given by $\mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{R}/\mathbb{Z})$. This restricts to a duality between the categories of discrete abelian groups and compact Hausdorff abelian groups, with the p-torsion groups corresponding to the p-profinite groups. We consider the p-torsion homologies $\{\overline{K}_*E, \overline{KO}_*E, \overline{KT}_*E\}$ and the p-profinite cohomologies $\{\widehat{K}^*E, \widehat{KO}^*E, \widehat{KT}^*E\}$ equipped with stable Adams operations ψ^k for p-local units $k \in \mathbb{Z}_{(p)}^{\times}$ (or more generally for p-adic units $k \in \hat{\mathbb{Z}}_p^{\times}$).

3.1. Theorem. *For a spectrum E, there are natural dualities* $e_C\colon \widehat{K}^*E \cong (\overline{K}_*E)^{\#}$, $e_R\colon \widehat{KO}^{*-4}E \cong (\overline{KO}_*E)^{\#}$, *and* $e_T\colon \widehat{KT}^{*-3}E \cong (\overline{KT}_*E)^{\#}$ *such that*:

(i) *the stable Adams operations* $\psi^k\colon \overline{K}_*E \cong \overline{K}_*E$, $\psi^k\colon \overline{KO}_*E \cong \overline{KO}_*E$, *and* $\psi^k\colon \overline{KT}_*E \cong \overline{KT}_*E$, *respectively, dualize to* $\psi^{1/k}\colon \widehat{K}^*E \cong \widehat{K}^*E$, $k^2\psi^{1/k}\colon \widehat{KO}^{*-4}E \cong \widehat{KO}^{*-4}E$, *and* $k\psi^{1/k}\colon \widehat{KT}^{*-3}E \cong \widehat{KT}^{*-3}E$ *for each* $k \in \mathbb{Z}_{(p)}^{\times}$;

(ii) *the periodicities* $B\colon \overline{K}_*E \cong \overline{K}_{*+2}E$, $B_R\colon \overline{KO}_*E \cong \overline{KO}_{*+8}E$, *and* $B_T\colon \overline{KT}_*E \cong \overline{KT}_{*+4}E$, *respectively, dualize to* $B\colon \widehat{K}^{*+2}E \cong \widehat{K}^*E$, $B_R\colon \widehat{KO}^{*+4}E \cong \widehat{KO}^{*-4}E$, *and* $B_T\colon \widehat{KT}^{*+1}E \cong \widehat{KT}^{*-3}E$;

(iii) *the Hopf operations* $\eta\colon \overline{KO}_*E \to \overline{KO}_{*+1}E$ *and* $\eta\colon \overline{KT}_*E \to \overline{KT}_{*+1}E$, *respectively, dualize to* $\eta\colon \widehat{KO}^{*-3}E \to \widehat{KO}^{*-4}E$ *and* $\eta\colon \widehat{KT}^{*-2}E \to \widehat{KT}^{*-3}E$;

(iv) *the complexification* $c\colon \overline{KO}_*E \to \overline{K}_*E$ *and the realification* $r\colon \overline{K}_*E \to \overline{KO}_*E$, *respectively, dualize to* $rB^2\colon \widehat{K}^*E \to \widehat{KO}^{*-4}E$ *and* $B^{-2}c\colon \widehat{KO}^{*-4}E \to \widehat{K}^*E$;

(v) *the operations* $\varepsilon\colon \overline{KO}_*E \to \overline{KT}_*E$, $\tau\colon \overline{KT}_*E \to \overline{KO}_{*+1}E$, $\zeta\colon \overline{KT}_*E \to \overline{K}_*E$, *and* $\gamma\colon \overline{K}_*E \to \overline{KT}_{*-1}E$, *respectively, dualize to* $\tau\colon \widehat{KT}^{*-3}E \to \widehat{KO}^{*-4}E$, $\varepsilon\colon \widehat{KO}^{*-3}E \to \widehat{KT}^{*-3}E$, $\gamma B^2\colon \widehat{K}^*E \to \widehat{KT}^{*-3}E$, *and* $B^{-2}\zeta\colon \widehat{KT}^{*-4}E \to \widehat{K}^*E$.

Before proving this theorem, we apply part (i) to derive the promised result on the v_1-periodic homotopy groups of a space. Let $r \in \mathbb{Z}_{(p)}^{\times}$ be a unit which generates $(\mathbb{Z}/p)^{\times}$ when p is odd and such that $r \equiv \pm 3 \bmod 8$ when $p = 2$.

3.2. Theorem. *For a space* $X \in \mathrm{Ho}_*$ *and for a spectrum* $E \in \mathrm{Ho}^{\mathrm{s}}$, *there are natural long exact sequences*

$$\cdots \longrightarrow \widehat{KO}^n \Phi_1 X \xrightarrow{\psi^r - r^2} \widehat{KO}^n \Phi_1 X \longrightarrow (v_1^{-1}\pi_{n+3}X)^{\#} \longrightarrow \widehat{KO}^{n+1}\Phi_1 X \xrightarrow{\psi^r - r^2} \cdots$$

$$\cdots \longrightarrow \widehat{KO}^n E \xrightarrow{\psi^r - r^2} \widehat{KO}^n E \longrightarrow (\pi_{n+3}\tau_p E_{K/p})^{\#} \longrightarrow \widehat{KO}^{n+1} E \xrightarrow{\psi^r - r^2} \cdots$$

Proof. For a p-torsion K_*-local spectrum F, there is a natural long exact sequence

$$\cdots \longrightarrow KO_n F \xrightarrow{\psi^r - 1} KO_n F \longrightarrow \pi_{n-1}F \longrightarrow KO_{n-1}F \xrightarrow{\psi^r - 1} \cdots$$

obtained by [13, Corollary 4.4] or by using the K_*^{CRT}-Adams spectral sequence of [16]. After r is replaced by $1/r$ and KO_*F is expressed as $\overline{KO}_{*+1}F$, this sequence dualizes to give a natural long exact sequence

$$\cdots \longrightarrow \widehat{KO}^n F \xrightarrow{r^{-2}\psi^r - 1} \widehat{KO}^n F \longrightarrow (\pi_{n+3}F)^{\#} \longrightarrow \widehat{KO}^{n+1} F \xrightarrow{r^{-2}\psi^r - 1} \cdots$$

by Theorem 3.1. The result now follows by taking $F = \tau_p \Phi_1 X$ or $F = \tau_p E$ and using Theorem 2.4. $\square$

Note that the homotopy groups $\pi_* E_{K/p}$ may be determined from $\pi_* \tau_p E_{K/p}$ as in [13, Proposition 2.5]. The proof of Theorem 3.1 will be based on:

3.3. Brown–Comenetz duality. As in [22], for a spectrum $E \in \mathrm{Ho}^s$, the *Brown–Comenetz dual* $\hat{c}E$ is the function spectrum $F(E, \hat{c}S)$ where $\hat{c}S$ is determined by the natural equivalence $[Y, \hat{c}S] = (\pi_0 Y)^\#$ for $Y \in \mathrm{Ho}^s$. The associated cohomology theory has universal coefficient isomorphisms

$$(\hat{c}E)^n Y \cong (E_n Y)^\#$$

for all $Y \in \mathrm{Ho}^s$ and $n \in \mathbb{Z}$, and thus

$$\pi_n(\hat{c}E) \cong (\pi_{-n} E)^\#.$$

The Brown–Comenetz functor $\hat{c}$ restricts to a contravariant equivalence from the homotopy category of spectra whose homotopy groups are finite direct sums of $\mathbb{Z}/p^j$'s and $\mathbb{Z}_{p^\infty}$'s to the homotopy category of spectra whose homotopy groups are finite direct sums of $\mathbb{Z}/p^j$'s and $\hat{\mathbb{Z}}_p$'s. For a commutative ring spectrum R and an R-module spectrum E, note that the Brown–Comenetz dual $\hat{c}E$ inherits an R-module spectrum structure from E.

We may now view $\hat{c}(K\mathbb{Z}_{p^\infty})$, $\hat{c}(KO\mathbb{Z}_{p^\infty})$, $\hat{c}(KT\mathbb{Z}_{p^\infty})$ as module spectra over the commutative ring spectra $\widehat{K}_p$, $\widehat{KO}_p$, and $\widehat{KT}_p$, since they are obtained from the module spectra $K\mathbb{Z}_{p^\infty} \simeq \widehat{K}_p\mathbb{Z}_{p^\infty}$, $KO\mathbb{Z}_{p^\infty} \simeq \widehat{KO}_p\mathbb{Z}_{p^\infty}$, and $KT\mathbb{Z}_{p^\infty} \simeq \widehat{KT}_p\mathbb{Z}_{p^\infty}$. Let $\varepsilon_C \in \pi_0\,\hat{c}(K\mathbb{Z}_{p^\infty})$, $\varepsilon_R \in \pi_{-4}\,\hat{c}(KO\mathbb{Z}_{p^\infty})$, and $\varepsilon_T \in \pi_{-3}\,\hat{c}(KT\mathbb{Z}_{p^\infty})$ be the elements corresponding to $1 \in \hat{\mathbb{Z}}_p$ under the isomorphisms

$$\hat{\mathbb{Z}}_p \cong (\pi_0 K\mathbb{Z}_{p^\infty})^\# \cong (\pi_4 KO\mathbb{Z}_{p^\infty})^\# \cong (\pi_3 KT\mathbb{Z}_{p^\infty})^\#$$

induced by $rB^2\colon \pi_0 K\mathbb{Z}_{p^\infty} \cong \pi_4 KO\mathbb{Z}_{p^\infty}$ and $\tau\colon \pi_3 KT\mathbb{Z}_{p^\infty} \cong \pi_4 KO\mathbb{Z}_{p^\infty}$. Then let $e_C\colon \widehat{K}_p \to \hat{c}(K\mathbb{Z}_{p^\infty})$ be the $\widehat{K}_p$-module map with $e_C(1) = \varepsilon_C$; let $e_R\colon \Sigma^{-4}\widehat{KO}_p \to \hat{c}(KO\mathbb{Z}_{p^\infty})$ be the $\widehat{KO}_p$-module map with $e_R(1) = \varepsilon_R$; and let $e_T\colon \Sigma^{-3}\widehat{KT}_p \to \hat{c}(KT\mathbb{Z}_{p^\infty})$be the $\widehat{KT}_p$-module map with $e_T(1) = \varepsilon_T$.

3.4. Lemma. *The maps* $e_C\colon \widehat{K}_p \to \hat{c}(K\mathbb{Z}_{p^\infty})$, $e_R\colon \Sigma^{-4}\widehat{KO}_p \to \hat{c}(KO\mathbb{Z}_{p^\infty})$, *and* $e_T\colon \Sigma^{-3}\widehat{KT}_p \to \hat{c}(KT\mathbb{Z}_{p^\infty})$ *are equivalences.*

Proof. Using [16, Section 2.5], we see that: $\pi_*\hat{c}(K\mathbb{Z}_{p^\infty})$ is a free $\pi_*\widehat{K}_p$-module on ε_C; $\pi_*\hat{c}(KO\mathbb{Z}_{p^\infty})$ is a free $\pi_*\widehat{KO}_p$-module on ε_R; and $\pi_*\hat{c}(KT\mathbb{Z}_{p^\infty})$ is a free $\pi_*\widehat{KT}_p$-module on ε_T. Thus the given maps are π_*-equivalences. □

This lemma combines with 3.3 to give natural duality isomorphisms

$$e_C\colon K^*(E; \hat{\mathbb{Z}}_p) \cong K_*(E; \mathbb{Z}_{p^\infty})^\#$$

$$e_R\colon KO^{*-4}(E; \hat{\mathbb{Z}}_p) \cong KO_*(E; \mathbb{Z}_{p^\infty})^\#$$

$$e_T\colon KT^{*-3}(E; \hat{\mathbb{Z}}_p) \cong KT_*(E; \mathbb{Z}_{p^\infty})^\#$$

for a spectrum E.

3.5. Lemma. *Parts* (ii)–(v) *of Theorem* 3.1 *hold for the above dualities* e_C, e_R, *and* e_T.

Proof. Each of the homology operations in (ii)–(v) is represented by some $\widehat{KO}_p$-module map $\phi \in [E\mathbb{Z}_{p^\infty}, F\mathbb{Z}_{p^\infty}]_*$ for spectra $E, F \in \{K, KO, KT\}$. The map ϕ dualizes via Lemma 3.4 to a $\widehat{KO}_p$-module map $\hat{c}\phi \in [\widehat{F}_p, \widehat{E}_p]_*$ which must be shown equal to some specified $\widehat{KO}_p$-module map $\phi' \in [\widehat{F}_p, \widehat{E}_p]_*$. In each case, we check that $\hat{c}\phi$ and ϕ' induce the same π_*-homomorphisms, and we conclude that $\hat{c}\phi = \phi'$ since all $\widehat{KO}_p$-module maps in $[\widehat{F}_p, \widehat{E}_p]_*$, except for some irrelevant ones in $[\widehat{KT}_p, \widehat{KT}_p]_{4n+2}$, are detected by π_* (see [16, Section 1.9]). □

We must deal separately with part (i) of Theorem 3.1 since it involves p-adic Adams operations which are not represented by $\widehat{KO}_p$-module maps. We first show:

3.6. Lemma. *For a spectrum $E \in \{K, KO, KT\}$, the homology E_*E is a free E_*-module on generators in degree* 0 *for $E = K$, in degree* 0 *for $E = KO$, and in degrees* 0 *and* 3 *for $E = KT$.*

Proof. This was shown for $E = K$ by Adams-Clarke [4] and is presumably known for $E = KO$. In general, we may rely on the formula $K_*^{CRT}E = \bar{U}(\pi_*^{CRT}E)$ of [16, Theorem 8.2], using the exact functor $\bar{U}\colon \mathscr{I}nv \to \mathscr{A}$ of [16, Proposition 6.6]. The result of [4] now shows that $\bar{U}(\mathbb{Z}\oplus\psi^{-1}\mathbb{Z})$ is free abelian, and hence $\bar{U}(M)$ is also free abelian for $M = \mathbb{Z}$ with $\psi^{-1} = \pm 1$ since $M \subset \mathbb{Z}\oplus\psi^{-1}\mathbb{Z}$. Thus $K_*E \cong \bar{U}(\pi_*^C E)$ is free abelian for each $E \in \{K, KO, KT\}$, and $KO_*E \cong \bar{U}(\pi_*^R E)$ is the underlying KO_*-module of a free E_*-module on zero-dimensional generators. Hence, $K_*^{CRT}E$ is a free CRT-module by [16, Theorem 3.2] on generators in K_0E for $E = K$, in KO_0E for $E = KO$, and in KT_3E for $E = KT$ by the structural results of [16, Section 2.4]. The lemma now follows from these same structural results. □

3.7. Lemma. *The maps in $[\widehat{K}_p, \widehat{K}_p]_i$, $[\widehat{KO}_p, \widehat{KO}_p]_i$, and $[\widehat{KT}_p, \widehat{KT}_p]_i$ are all detected by rational homotopy groups $\mathbb{Q}\otimes\pi_*$ with the exception of the maps in $[\widehat{KO}_2, \widehat{KO}_2]_i$ for $i \equiv 1, 2 \bmod 8$ and in $[\widehat{KT}_2, \widehat{KT}_2]_i$ for $i \equiv 1, 2 \bmod 4$.*

Proof. Let E denote K, KO, or KT. Since $[\widehat{E}_p, \widehat{E}_p]_i \cong [E, \Sigma^{-i}\widehat{E}_p]$, it suffices to show that the maps in $[E, N]$ are detected by $\mathbb{Q}\otimes\pi_*$ whenever N is an E-module spectrum with $\pi_0 N$ torsion-free and also with $\pi_3 N$ torsion-free when $E = KT$. There are natural universal coefficient isomorphisms

$$[E, N] \cong \mathrm{Hom}_{CRT}(K_*^{CRT}E, \pi_*^{CRT}N) \cong \mathrm{Hom}_{E_*}(E_*E, \pi_*N)$$

obtained from [16, Section 9.6] (or [3] when $E = K$ or $E = KO$) using the freeness results of Lemma 3.6. Thus the rationalization $N \to NQ$ induces a monomorphism $[E, N] \to [E, NQ]$, and hence the maps in $[E, N]$ are detected by $\pi_* \otimes Q$ as required. □

3.8. Proof of Theorem 3.1. Using the dualities e_C, e_R, and e_T of Lemma 3.4, we have proved parts (ii)–(v) of the theorem in Lemma 3.5, and we easily deduce part (i) from Lemma 3.7. □

4. Even and odd *CR*-modules

For many interesting spaces X, including simply connected compact Lie groups, the complex p-adic K-theory $K^*(\Phi_1 X; \hat{\mathbb{Z}}_p)$ is concentrated in even or odd degrees. In such cases, the united p-adic K-theory of $\Phi_1 X$ is completely captured by

$$K^*_{CR}(\Phi_1 X; \hat{\mathbb{Z}}_p) = \{K^*(\Phi_1 X; \hat{\mathbb{Z}}_p), KO^*(\Phi_1 X; \hat{\mathbb{Z}}_p)\}$$

without $KT^*(\Phi_1 X; \hat{\mathbb{Z}}_p)$, and in fact is largely captured by a small part of $K^*_{CR}(\Phi_1 X; \hat{\mathbb{Z}}_p)$ which we call a Δ-module. In this section, we first recall the underlying theory of *CR*-modules from [16, Section 4.7] using cohomological indexing and working over an arbitrary abelian category. We then develop some crucial special properties of even and odd *CR*-modules, and finally introduce the theory of Δ-modules.

4.1. CR**-modules.** A *CR-module* over an abelian category $\mathscr{M}$ consists of a pair $M = \{M^*_C, M^*_R\}$ of $\mathbb{Z}$-graded objects in $\mathscr{M}$ with operations

$$B\colon M^*_C \cong M^{*-2}_C, \quad t\colon M^*_C \cong M^*_C, \quad B_R\colon M^*_R \cong M^{*-8}_R,$$

$$\eta\colon M^*_R \to M^{*-1}_R, \quad c\colon M^*_R \to M^*_C, \quad r\colon M^*_C \to M^*_R$$

satisfying the relations

$$2\eta = 0, \quad \eta^3 = 0, \quad \eta B_R = B_R \eta, \quad \eta r = 0, \quad c\eta = 0,$$

$$t^2 = 1, \quad tB = -Bt, \quad rt = r, \quad tc = c, \quad cB_R = B^4 c,$$

$$rB^4 = B_R r, \quad cr = 1 + t, \quad rc = 2, \quad rBc = \eta^2, \quad rB^{-1}c = 0.$$

We may sometimes write $\psi^{-1}z$ or z^* in place of tz for $z \in M^*_C$. We let $\mathscr{CR}_{\mathscr{M}}$ denote the abelian category of *CR*-modules over $\mathscr{M}$. For a *CR*-module *M*, we call

$$\cdots \longrightarrow M^{*+1}_R \xrightarrow{\eta} M^*_R \xrightarrow{c} M^*_C \xrightarrow{rB^{-1}} M^{*+2}_R \xrightarrow{\eta} \cdots$$

the *Bott sequence* and call *M Bott exact* when this chain complex is exact. We also call *M CR-exact* [16, Section 4.7] when it is Bott exact and the chain complex

$$\cdots \longrightarrow M^{*+1}_R/r \xrightarrow{\eta} M^*_R/r \xrightarrow{\eta} M^{*-1}_R/r \xrightarrow{\eta} \cdots$$

is exact. Some examples of Bott exact *CR*-modules are

$$K^*_{CR}(E; G) = \{K^*(E; G), KO^*(E; G)\} \quad K^{CR}_*(E; G) = \{K_*(E; G), KO_*(E; G)\}$$

for arbitrary spectra *E* and coefficients *G*, where the operations come from standard maps for the spectra *K* and *KO* [16, Section 1.9] with Bott exactness shown by [16, Section 1.11]. Whenever $K^*_{CR}(E; G)$ or $K^{CR}_*(E; G)$ is *CR*-exact, it prolongs canonically to give the groups $KT^*(E; G)$ or $KT_*(E; G)$, which become superfluous [16, Theorem 4.15].

4.2. Even and odd Bott exact *CR*-modules. Let *M* be a Bott exact *CR*-module over an abelian category $\mathscr{M}$. We call *M even* or *odd* when M^*_C vanishes in the opposite degrees, allowing us to determine M^*_C periodically from a single term M^n_C where $M^{n-1}_C = 0$. This implies that *M* is *CR*-exact since $\eta\colon M^i_R/r \cong M^{i-1}_R/r$ for $i \equiv n \bmod 2$. Moreover, by the following theorem, we can largely determine M^*_R from the single triad of terms $\{M^n_C, M^n_R, M^{n-4}_R\}$ with operations

$$t\colon M^n_C \to M^n_C \quad c\colon M^n_R \to M^n_C \quad r\colon M^n_C \to M^n_R \quad c'\colon M^{n-4}_R \to M^n_C \quad q\colon M^n_C \to M^{n-4}_R$$

where $c' = B^{-2}c$ and $q = rB^2$. We use the notation $A\backslash f$ and B/f for the kernel and cokernel of a homomorphism $f\colon A \to B$.

4.3. Theorem. *Suppose M is a Bott exact CR-module over $\mathscr{M}$ with $M^{n-1}_C = 0$ for some n. Then there are natural isomorphisms*

$$M^{n-i}_R \cong \begin{cases} M^n_R & \text{for } i \equiv 0 \bmod 8, \\ M^n_R/r & \text{for } i \equiv 1 \bmod 8, \\ M^{n-4}_R\backslash c' & \text{for } i \equiv 3 \bmod 8, \\ M^{n-4}_R & \text{for } i \equiv 4 \bmod 8, \\ M^{n-4}_R/q & \text{for } i \equiv 5 \bmod 8, \\ M^n_R\backslash c & \text{for } i \equiv 7 \bmod 8. \end{cases}$$

Moreover, the terms $M^{n+8k-2}_R \cong M^{n-2}_R$ and $M^{n+8k-6}_R \cong M^{n-6}_R$ belong to natural extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^n_R/r & \xrightarrow{\eta^2} & M^{n-2}_R & \xrightarrow{B^{-1}c} & M^n_C\backslash r & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle q} & & \\ 0 & \longrightarrow & M^n_C/c' & \xrightarrow{rB} & M^{n-2}_R & \xrightarrow{\eta^2} & M^{n-4}_R\backslash c' & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{n-4}_R/q & \xrightarrow{\eta^2} & M^{n-6}_R & \xrightarrow{B^{-3}c} & M^n_C\backslash q & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c'} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & M^n_C/c & \xrightarrow{rB^3} & M^{n-6}_R & \xrightarrow{B_R^{-1}\eta^2} & M^n_R\backslash c & \longrightarrow & 0 \end{array}$$

where the horizontal maps satisfy $(rB)(B^{-1}c) = 2$, $(B^{-1}c)(rB) = 1 - t$, $(rB^3)(B^{-3}c) = 2$, and $(B^{-3}c)(rB^3) = 1 - t$.

Proof. This follows easily from Bott exactness and the *CR*-module relations. □

We shall see later (Theorem 4.11) that the "difficult" terms M^{n-2}_R and M^{n-6}_R may actually be recovered up to isomorphism (though not functorially) from the given triad $\{M^n_C, M^n_R, M^{n-4}_R\}$, at least when $\mathscr{M}$ is the category of abelian groups or *p*-profinite abelian groups. We now obtain a crucial exactness result for $\{M^n_C, M^n_R, M^{n-4}_R\}$.

4.4. Theorem. *Suppose M is a Bott exact CR-module over $\mathscr{M}$ with $M^{n-1}_C = 0$ for some n. Then there is an exact sequence*

$$\cdots \longrightarrow M^n_C \xrightarrow{(r,q)} M^n_R \oplus M^{n-4}_R \xrightarrow{c-c'} M^n_C \xrightarrow{1-t} M^n_C \xrightarrow{(r,q)} M^n_R \oplus M^{n-4}_R \longrightarrow \cdots$$

Proof. This follows from the exact sequences

$$0 \longrightarrow M^n_R/r \xrightarrow{c} M^n_C/c' \xrightarrow{1-t} M^n_C\backslash r \xrightarrow{q} M^{n-4}_R\backslash c' \longrightarrow 0$$
$$0 \longrightarrow M^{n-4}_R/q \xrightarrow{c'} M^n_C/c \xrightarrow{1-t} M^n_C\backslash q \xrightarrow{r} M^n_R\backslash c \longrightarrow 0$$

associated with the ladders of extensions in Theorem 4.3. □

In view of the preceding theorems, we introduce:

4.5. Δ-Modules. A Δ-*module* over an abelian category $\mathscr{M}$ consists of a triad $A=\{A_C, A_R, A_H\}$ of objects in $\mathscr{M}$ with operations

$$t\colon A_C \to A_C, \quad c\colon A_R \to A_C, \quad r\colon A_C \to A_R, \quad c'\colon A_H \to A_C, \quad q\colon A_C \to A_H$$

satisfying the relations

$$t^2=1, \quad cr=1+t, \quad rc=2, \quad tc=c, \quad rt=r,$$
$$c'q=1+t, \quad qc'=2, \quad tc'=c', \quad qt=q.$$

We may sometimes write $\psi^{-1}z$ or z^* in place of tz for $z \in A_C$. We let $\Delta_{\mathscr{M}}$ denote the abelian category of Δ-modules over $\mathscr{M}$, and we say that a Δ-module A is *exact* when the chain complex

$$\cdots \longrightarrow A_C \xrightarrow{(r,q)} A_R \oplus A_H \xrightarrow{c-c'} A_C \xrightarrow{1-t} A_C \xrightarrow{(r,q)} A_R \oplus A_H \longrightarrow \cdots$$

is exact. Hence, if two Δ-modules in a short exact sequence are exact, then so is the third. For a CR-module $M \in CR_{\mathscr{M}}$ and integer n, we obtain a Δ-module

$$\Delta^n M = \{M^n_C, M^n_R, M^{n-4}_R\} \in \Delta_{\mathscr{M}}$$

which is exact whenever M is Bott exact with $M^{n-1}_C = 0$.

4.6. Other examples of exact Δ-modules. For a compact Lie group G, there is a Δ-module $R_\Delta(G) = \{R(G), R_R(G), R_H(G)\}$ consisting of the complex representation ring $R(G)$ with its real and quaternionic parts $R_R(G), R_H(G) \subset R(G)$ linked by the standard operations. This Δ-module is always exact since it is freely generated by irreducible representations of complex, real, and quaternionic types (see [2,12]). For an abelian group N with involution $t\colon N \cong N$, we let $N^+ = N\backslash(1-t)$ and $N_+ = N/(1-t)$. Then there are exact Δ-modules $\{N, N^+, N_+\}$ and $\{N, N_+, N^+\}$ with obvious operations. When $t=1\colon N \cong N$ and the map $N\backslash 2 \to N/2$ is trivial (e.g. when $t=1\colon \mathbb{Z}/2^k \cong \mathbb{Z}/2^k$ for $k>1$), there are also exact Δ-modules $\{2N, N, 4N\}$ and $\{2N, 4N, N\}$ with obvious operations and there are generally many more.

For an object $N \in \mathscr{M}$ with involution $t\colon N \cong N$, we write $h^+N = \ker(1-t)/\operatorname{im}(1+t)$ and $h^-N = \ker(1+t)/\operatorname{im}(1-t)$ for the associated cohomologies. It is straightforward to show:

4.7. Lemma. *For an exact Δ-module $M \in \Delta_{\mathscr{M}}$, there are isomorphisms*

$$c+c'\colon M_R/r \oplus M_H/q \cong h^+M_C,$$
$$(r,q)\colon h^-M_C \cong M_R\backslash c \oplus M_H\backslash c'.$$

We suspect that exact $\varDelta$-modules are determined up to isomorphism by their complex parts together with their direct sum splittings of h^+ and h^- terms, but we shall not pursue this here. Instead, we use these terms to give comparison lemmas for exact $\varDelta$-modules and *CR*-modules.

4.8. Lemma. *Suppose $f: L \to M$ is a map of exact $\varDelta$-modules over $\mathscr{M}$. Then*

(i) *f is an isomorphism if and only if $f_C: L_C \to M_C$ is an isomorphism*;
(ii) *f is epic if and only if $f_C: L_C \to M_C$ and $f_*: h^+L_C \to h^+M_C$ are both epic*;
(iii) *f is monic if and only if $f_C: L_C \to M_C$ and $f_*: h^-L_C \to h^-M_C$ are both monic*.

Proof. In view of Lemma 4.7, part (ii) follows by comparing the sequences $L_C \to L_R \to L_R/r \to 0$ and $M_C \to M_R \to M_R/q \to 0$, as well as their quaternionic counterparts. The other parts follow similarly. $\square$

4.9. Lemma. *Suppose $f: L \to M$ is a map of Bott exact CR-modules over $\mathscr{M}$ with $L_C^{n-1} = 0$ and $M_C^{n-1} = 0$ for some n. Then*

(i) *f is an isomorphism if and only if $f_C: L_C^n \to M_C^n$ is an isomorphism*;
(ii) *f is epic* (*resp. monic*) *if and only if $f: L_C^n \to M_C^n$, $f_*: h^+L_C^n \to h^+M_C^n$, and $f_*: h^-L_C^n \to h^-M_C^n$ are all epic* (*resp. monic*).

Proof. This follows by combining Lemma 4.8 with Theorems 4.3 and 4.4. $\square$

In the remainder of this section, we shall establish a very close correspondence between exact $\varDelta$-modules and even or odd Bott exact *CR*-modules. We first consider

4.10. Adjoints of $\varDelta^n$. For any *n*, the functor $\varDelta^n: \mathscr{CR}_{\mathscr{M}} \to \varDelta_{\mathscr{M}}$ has a left adjoint $CR(-,n): \varDelta_{\mathscr{M}} \to \mathscr{CR}_{\mathscr{M}}$ where $CR(M,n)_C^{n-i} \cong M_C$ for *i* even, $CR(M,n)_C^{n-i} \cong 0$ for *i* odd, and

$$CR(M,n)_R^{n-i} \cong \begin{cases} M_R & \text{for } i \equiv 0 \bmod 8, \\ M_R/r & \text{for } i \equiv 1 \bmod 8, \\ M_C/c' & \text{for } i \equiv 2 \bmod 8, \\ 0 & \text{for } i \equiv 3, 7 \bmod 8, \\ M_H & \text{for } i \equiv 4 \bmod 8, \\ M_H/q & \text{for } i \equiv 5 \bmod 8, \\ M_C/c & \text{for } i \equiv 6 \bmod 8. \end{cases}$$

Hence $\varDelta^n(CR(M,n)) = M$, and if M is an exact $\varDelta$-module with $h^-M_C = 0$ (i.e. with c and c' monic), then $CR(M,n)$ is a Bott exact *CR*-module. The functor $\varDelta^n: \mathscr{CR}_{\mathscr{M}} \to \varDelta_{\mathscr{M}}$ also has a right adjoint $CR'(-,n): \varDelta_{\mathscr{M}} \to \mathscr{CR}_{\mathscr{M}}$ where $CR'(M,n)_C^{n-i} \cong M_C$ for *i* even, $CR'(M,n)_C^{n-i} \cong 0$ for *i* odd, and

$$CR'(M,n)_R^{n-i} \cong \begin{cases} M_R & \text{for } i \equiv 0 \bmod 8, \\ 0 & \text{for } i \equiv 1, 5 \bmod 8, \\ M_C \backslash r & \text{for } i \equiv 2 \bmod 8, \\ M_H \backslash c' & \text{for } i \equiv 3 \bmod 8, \\ M_H & \text{for } i \equiv 4 \bmod 8, \\ M_C \backslash q & \text{for } i \equiv 6 \bmod 8, \\ M_R \backslash c & \text{for } i \equiv 7 \bmod 8. \end{cases}$$

Hence $\varDelta^n(CR'(M,n)) = M$, and if M is an exact $\varDelta$-module with $h^+M_C = 0$ (i.e. with r and q epic), then $CR'(M,n)$ is a Bott exact CR-module.

We now address the general problem of prolonging an exact $\varDelta$-module $M \in \varDelta_{\mathscr{M}}$ to give a Bott exact CR-module $\overline{M} \in \mathscr{CR}_{\mathscr{M}}$ with $\varDelta^n\overline{M} = M$ and $\overline{M}_C^{n-1} = 0$ for some integer n. When $h^-M_C = 0$ or $h^+M_C = 0$, such an $\overline{M}$ will be given by the above $CR(M,n)$ or $CR'(M,n)$. However, when M is the exact $\varDelta$-module $\{\mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/2\}$ with $c = 1$ and $q = 1$, such a prolongation $\overline{M}$ cannot exist in the category of $\mathbb{Z}/2$-modules. Fortunately, this trouble disappears in our preferred abelian categories.

4.11. Theorem. *Suppose $\mathscr{M}$ is the category of abelian groups or p-profinite abelian groups. For an exact $\varDelta$-module $M \in \varDelta_{\mathscr{M}}$ and integer n, there exists a Bott exact CR-module $\overline{M} \in \mathscr{CR}_{\mathscr{M}}$ with $\varDelta^n\overline{M} = M$ and $\overline{M}_C^{n-1} = 0$. Moreover, $\overline{M}$ is unique up to (noncanonical) isomorphism.*

Thus, the isomorphism classes of even or odd Bott exact CR-modules over $\mathscr{M}$ correspond to the isomorphism classes of exact $\varDelta$-modules over $\mathscr{M}$ via the functor $\varDelta^n$ for an even or odd n. This theorem will be proved below in 4.13 using:

4.12. Special resolutions. In the category of abelian groups, we consider the following elementary exact $\varDelta$-modules: $F(\mathbb{C}) = \{\mathbb{Z} \oplus t\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\}$ with $c = 1 + t$ and $c' = 1 + t$; $F(\mathbb{R}) = \{\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\}$ with $t = 1$, $c = 1$, and $q = 1$; $F(\mathbb{H}) = \{\mathbb{Z}, \mathbb{Z}, \mathbb{Z}\}$ with $t = 1$, $c' = 1$, and $r = 1$; $F'(\mathbb{R}) = \{\mathbb{Z}, \mathbb{Z}/2, 0\}$ with $t = -1$ and r onto; and $F'(\mathbb{H}) = \{\mathbb{Z}, 0, \mathbb{Z}/2\}$ with $t = -1$ and q onto. A $\varDelta$-module is called *free* when it is a direct sum on copies of $F(\mathbb{C})$, $F(\mathbb{R})$, and $F(\mathbb{H})$; it is called *parafree* when it is a direct sum of copies of $F'(\mathbb{R})$ and $F'(\mathbb{H})$. For an abelian exact $\varDelta$-module M, we use Lemma 4.7 to construct a parafree $\varDelta$-module F' and map $F' \to M$ inducing an isomorphism $h^-F'_C \cong h^-M_C$. We then find a free $\varDelta$-module F and map $F \to M$ such that $F \oplus F' \to M$ is onto. This determinesa short exact sequence of $\varDelta$-modules $0 \to \widetilde{F} \to F \oplus F' \to M \to 0$ called a *special resolution* of M. The $\varDelta$-module $\widetilde{F}$ is exact with $\widetilde{F}_C$ free abelian and with $h^-\widetilde{F}_C = 0$. Hence $\widetilde{F}$ is a free $\varDelta$-module by [16, Proposition 4.8] applied to $CR(\widetilde{F}, 0)$.

4.13. Proof of Theorem 4.11. It suffices to prove the theorem for abelian groups, since it then follows for p-profinite abelian groups by Pontrjagin dualization. Let $0 \to \widetilde{F} \to F \oplus F' \to M \to 0$ be a special resolution for the exact $\varDelta$-module M. Then the induced map of CR-modules $CR(\widetilde{F}, n) \to CR(F,n) \oplus CR'(F',n)$ is monic by Lemma 4.9, and one easily checks that its cokernel is the required $\overline{M}$. The desired uniqueness follows from a more general property of this $\overline{M}$. Namely, for any Bott exact CR-module N with $N_C^{n-1} = 0$, we claim that each $\varDelta$-module map $M \to \varDelta^n N$ prolongs (nonuniquely) to a CR-module map $\overline{M} \to N$. For this, it is fairly straightforward to check that the induced map $F \oplus F' \to \varDelta^n N$ prolongs (nonuniquely) to a map $CR(F,n) \oplus CR'(F',n) \to N$, which must be trivial on $CR(\widetilde{F}, n)$. The desired map $\overline{M} \to N$ is now obtained by dividing out $CR(\tilde{F}, n)$. $\square$

We conclude with a technical lemma for later use.

4.14. Lemma. *Suppose $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of exact $\varDelta$-modules over an abelian category $\mathscr{M}$ with $h^-M'_C = 0$ and $h^-M_C = 0$. Then there are natural* *exact sequences*

$$\begin{aligned}
&0 \longrightarrow M''_R\backslash c \longrightarrow M'_C/c \longrightarrow M_C/c \longrightarrow M''_C/c \longrightarrow 0,\\
&0 \longrightarrow M''_R\backslash c \longrightarrow M'_H/q \longrightarrow M_H/q \longrightarrow M''_H/q \longrightarrow 0,\\
&0 \longrightarrow M''_H\backslash c' \longrightarrow M'_C/c' \longrightarrow M_C/c' \longrightarrow M''_C/c' \longrightarrow 0,\\
&0 \longrightarrow M''_H\backslash c' \longrightarrow M'_R/r \longrightarrow M_R/r \longrightarrow M''_R/r \longrightarrow 0.
\end{aligned}$$

Proof. Since $M'_R\backslash c=0$ and $M_R\backslash c=0$, we obtain the first exact sequence by the serpent lemma. We then obtain the second exact sequence from the first by using the isomorphism of the left and middle kernels in the ladder of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & M'_H/q \arrow[r, "c'"] \arrow[d] & M'_C/c \arrow[r] \arrow[d] & M'_C/M'^{+}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_H/q \arrow[r, "c'"] & M_C/c \arrow[r] & M_C/M^{+}_C \arrow[r] & 0
\end{tikzcd}$$

where the right kernel is trivial. We obtain the third and fourth exact sequences similarly. □

Roughly speaking, the exact sequences in this lemma combine to give a long exact sequence for the real parts of the associated Bott exact *CR*-modules.

5. Even and odd K/p_*-local spectra

Using results of [16], we now obtain an algebraic classification of the K/p_*-local spectra E whose complex K-cohomologies $K^*(E;\hat{\mathbb{Z}}_p)$ are concentrated in even or odd degrees. This classification will depend only on the *CR*-modules $K^*_{CR}(E;\hat{\mathbb{Z}}_p)$ together with their stable Adams operations and will apply to most of the spectra $E=\Phi_1 X$ of interest on this paper.

5.1. Stable p-adic Adams modules. By a *finite stable p-adic Adams module* we mean a finite abelian p-group G with automorphisms $\psi^k\colon G\cong G$ for $k\in\mathbb{Z}^\times_{(p)}$ such that:

(i) $\psi^1=1$ and $\psi^j\psi^k=\psi^{jk}$ for all $j,k\in\mathbb{Z}^\times_{(p)}$;
(ii) for a sufficiently large integer n, the condition $j\equiv k \bmod p^n$ implies $\psi^j=\psi^k$ on G.

By a *stable p-adic Adams module* we mean the topological inverse limit of an inverse system of finite stable p-adic Adams modules. Such a module G has an underlying p-profinite abelian group structure with continuous automorphisms $\psi^k\colon G\cong G$ for $k\in\mathbb{Z}^\times_{(p)}$. In fact, a stable p-adic Adams module is just the Pontrjagin dual of a stable p-torsion Adams module in the sense of [14, Section 1] or [16, Section 5.1]. We let $\hat{\mathscr{A}}$ be the abelian category of stable p-adic Adams modules, and we let $\bar{S}^i\colon\hat{\mathscr{A}}\to\hat{\mathscr{A}}$, for $i\in\mathbb{Z}$, be the functor with $\bar{S}^iG$ equal to G as a group but with $\psi^k\colon \bar{S}^iG\cong\bar{S}^iG$ equal to $k^i\psi^k\colon G\cong G$ for $k\in\mathbb{Z}^\times_{(p)}$.

5.2. $\hat{A}CR$-modules. By an $\hat{A}CR$*-module* we mean a *CR*-module *M* of stable *p*-adic Adams modules with $\psi^{-1}=t$ in M^*_C and with $\psi^{-1}=1$ in M^*_R, where the homomorphisms

$$B\colon \bar{S}M^*_C \cong M^{*-2}_C, \quad B_R\colon \bar{S}^4 M^*_R \cong M^{*-8}_R, \quad \eta\colon M^*_R \to M^{*-1}_R,$$
$$c\colon M^*_R \to M^*_C, \quad r\colon M^*_C \to M^*_R$$

are all maps in $\hat{\mathscr{A}}$. Equivalently, an $\hat{A}CR$-module is just the Pontrjagin dual of a *p*-torsion *ACR*-module in the sense of [16, Section 5.5], where the duality is taken with respect to the componentwise rules of Theorem 3.1. We let $\hat{\mathscr{A}}\mathscr{C}\mathscr{R}$ be the abelian category of $\hat{A}CR$-modules. The main examples of $\hat{A}CR$-modules are the cohomologies

$$K^*_{CR}(E;\hat{\mathbb{Z}}_p) = \{K^*(E;\hat{\mathbb{Z}}_p), KO^*(E;\hat{\mathbb{Z}}_p)\}$$

for arbitrary spectra *E*, which are Pontrjagin dual to the *p*-torsion *ACR*-modules

$$K^{CR}_*(E;\mathbb{Z}_{p^\infty}) = \{K_*(E;\mathbb{Z}_{p^\infty}), KO_*(E;\mathbb{Z}_{p^\infty})\} \cong \{K_{*-1}(\tau_p E), KO_{*-1}(\tau_p E)\} = K^{CR}_{*-1}(\tau_p E)$$

by Theorem 3.1. We can now give our main classification theorem for even or odd K/p_*-local spectra.

5.3. Theorem. *Suppose M is an even or odd Bott exact $\hat{A}CR$-module. Then there exists a K/p_*-local spectrum E with $K^*_{CR}(E;\hat{\mathbb{Z}}_p) \cong M$, and E is unique up to* (*noncanonical*) *equivalence.*

Proof. The Pontrjagin dual $M^\#$ is a *p*-torsion Bott exact *ACR*-module which prolongs canonically to a *p*-torsion *CRT*-exact *ACRT*-module by [16, Lemma 4.14]. Hence, there exists a *p*-torsion K_*-local spectrum *X* with $K^{CR}_*(X;\mathbb{Z}_{p^\infty}) \cong K^{CR}_{*-1}X \cong M^\#$ by [16, Theorem 10.1], and the spectrum $E=\hat{X}_p$ has the desired properties. It is unique by Theorem 5.4 below. □

Theorem 5.3 shows that the homotopy types of even or odd K/p_*-local spectra correspond to the isomorphism classes of even or odd Bott exact $\hat{A}CR$-modules. We have used:

5.4. Theorem. *Suppose E and F are K/p_*-local spectra with $K^{n-1}(E;\hat{\mathbb{Z}}_p)=0$ and $K^{n-1}(F;\hat{\mathbb{Z}}_p)=0$ for some n. Then, for each $\hat{A}CR$-module homomorphism $\phi\colon K^*_{CR}(F;\hat{\mathbb{Z}}_p) \to K^*_{CR}(E;\hat{\mathbb{Z}}_p)$, there exists a map $f\colon E\to F$ with $\phi=f^*$.*

Proof. It suffices to prove the corresponding result for the *p*-torsion *K*-local spectra $\tau_p E$ and $\tau_p F$, and that result follows from [16, Section 9.8] by dualization as in the proof of Theorem 5.3. □

We remark that the map *f* in this theorem is generally not unique. For instance, there is a map $f\neq 0\colon S_{K/2}\to S_{K/2}$ of order 2 with $f^*=0$ on $K^*_{CR}(S_{K/2};\hat{\mathbb{Z}}_2)$.

6. Unstable operations in *p*-adic *K*-theory

For a pointed space *X* and integer *n*, we may approach the stable *K*-theoretic Δ-module

$$K^n_\Delta(\Phi_1 X;\hat{\mathbb{Z}}_p) = \{K^n(\Phi_1 X;\hat{\mathbb{Z}}_p), KO^n(\Phi_1 X;\hat{\mathbb{Z}}_p), KO^{n-4}(\Phi_1 X;\hat{\mathbb{Z}}_p)\}$$

in favorable cases by starting with the corresponding unstable K-theoretic Δ-module

$$\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p)=\{\widetilde{K}^n(X;\hat{\mathbb{Z}}_p),\widetilde{KO}^n(X;\hat{\mathbb{Z}}_p),\widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)\}$$

and dividing out by its "Φ_1-trivial" part, where the tilde indicates reduced cohomology. In preparation, we now discuss various unstable operations in p-adic K-theory. Although some of these operations are well-known, others may not be, and we shall explain how they may be constructed using Atiyah's Real K-theory [7]. We start by recalling:

6.1. λ-Rings without identity. A *λ-ring without identity* consists of a commutative ring A without identity together with functions $\lambda^m\colon A\to A$ for $m=1,2,3,\ldots$ where $\lambda^1(a)=a$ and where the usual expressions for $\lambda^m(a+b)$, $\lambda^m(ab)$, and $\lambda^m\lambda^n(a)$ hold when $a,b\in A$ and $m,n\geqslant 1$ (see [10]). We note that such an A may be viewed as the augmentation ideal of a λ-ring $\mathbb{Z}\oplus A$ with identity $1\in\mathbb{Z}$. We call such an A *semigraded* when it has the form $A=A^0\oplus A^1$ with the $\mathbb{Z}/2$-gradation properties that for all elements $x,y\in A^0$, $u,v\in A^1$, and $m,k\geqslant 1$, the following conditions hold: $xy\in A^0$, $xu\in A^1$, $uv\in A^0$, $\lambda^m x\in A^0$, $\lambda^{2k}u\in A^0$, and $\lambda^{2k-1}u\in A^1$. Of course, $uv=vu$ instead of $uv=-vu$ for $u,v\in A^1$.For a compact Lie group G, we note that the augmentation ideals $\widetilde{R}(G)$ and $\widetilde{R}_R(G)\oplus\widetilde{R}_H(G)$ are λ-rings without identity, where the latter is semigraded. We may view $\widetilde{R}_\Delta(G)=\{\widetilde{R}(G),\widetilde{R}_R(G),\widetilde{R}_H(G)\}$ as a prototype for:

6.2. $\Delta\lambda$-Rings. A *$\Delta\lambda$-ring* A consists of a λ-ring A_C without identity and a semigraded λ-ring $A_R\oplus A_H$ without identity together with a Δ-module structure on $\{A_C,A_R,A_H\}$ such that the following conditions hold for all elements $z,w\in A_C$, $x,y\in A_R$, $u,v\in A_H$, and $n,k\geqslant 1$:

(i) $c(xy)=(cx)(cy)$, $c(uv)=(c'u)(c'v)$, $c'(xu)=(cx)(c'u)$, and $(zw)^*=z^*w^*$ where $-^*$ denotes $t-$;
(ii) $(rz)x=r(z(cx))$, $(rz)u=q(z(c'u))$, $(qz)x=q(z(cx))$, and $(qz)u=r(z(c'u))$;
(iii) $c(\lambda^m x)=\lambda^m(cx)$, $c(\lambda^{2k}u)=\lambda^{2k}(c'u)$, $c'(\lambda^{2k-1}u)=\lambda^{2k-1}(c'u)$, and $(\lambda^m z)^*=\lambda^m(z^*)$;
(iv) the operation $\bar{\phi}\colon A_C\to A_R$ given by $\bar{\phi}(z)=\lambda^2(rz)-r(\lambda^2 z)$ has the properties $c\bar{\phi}(z)=z^*z$, $\bar{\phi}(zw)=(\bar{\phi}z)(\bar{\phi}w)$, and $\bar{\phi}(z+w)=\bar{\phi}z+\bar{\phi}w+r(z^*w)$;
(v) using the operation $\bar{\phi}\colon A_C\to A_R$, we have

$$\lambda^{2k}(qz)=\lambda^{2k}(rz)=r(\lambda^{2k}z)+\bar{\phi}(\lambda^k z)+r\sum_{i=1}^{k-1}(\lambda^i z)(\lambda^{2k-i}z^*),$$

$$\lambda^{2k-1}(rz)=r(\lambda^{2k-1}z)+r\sum_{i=1}^{k-1}(\lambda^i z)(\lambda^{2k-1-i}z^*),$$

$$\lambda^{2k-1}(qz)=q(\lambda^{2k-1}z)+q\sum_{i=1}^{k-1}(\lambda^i z)(\lambda^{2k-1-i}z^*).$$

6.3. p-Adic $\Delta\lambda$-rings. We say that a $\Delta\lambda$-ring $A=\{A_C,A_R,A_H\}$ is of *finite type* when A_C, A_R, A_H are finitely generated as abelian groups, and we say that A is *γ-nilpotent* when A_C and $A_R\oplus A_H$ are γ-nilpotent, i.e., when they are nilpotent with vanishing operations γ^m for sufficiently large m (see [17, Section 4]). For a $\Delta\lambda$-ring A of finite type and a fixed prime p, the tensor product with the λ-ring $\hat{\mathbb{Z}}_p$

gives a $\Delta\lambda$-ring $A \otimes \hat{\mathbb{Z}}_p$ whose underlying Δ-module is p-profinite and whose operations are all continuous. By a *weak* (resp. *strong*) *p-adic* $\Delta\lambda$*-ring*, we mean the topological inverse limit $\lim_\alpha(A_\alpha \otimes \hat{\mathbb{Z}}_p)$ of an inverse system of $\Delta\lambda$-rings $A_\alpha \otimes \hat{\mathbb{Z}}_p$ where each A_α is of finite type (resp. of finite type and γ-nilpotent). For a strong p-adic $\Delta\lambda$-ring B, we note that $\hat{\mathbb{Z}}_p \oplus B_C$ and $\hat{\mathbb{Z}}_p \oplus B_R \oplus B_H$ are p-adic λ-rings in the sense of [17, Section 5].

Our main topological examples of p-adic $\Delta\lambda$-rings will be given by

$$\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p) = \{\widetilde{K}^n(X;\hat{\mathbb{Z}}_p), \widetilde{KO}^n(X;\hat{\mathbb{Z}}_p), \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)\}$$

for a space X and integer n. We define an *internal multiplication* $*$ on $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$ by $z*w=zw$ when $n=0$ and $z*w=0$ when $n\neq 0$ for elements $z, w \in \widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$. We also define an *internal multiplication* $*$ on $\widetilde{KO}^n(X;\hat{\mathbb{Z}}_p) \oplus \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)$ by the following formulae for elements $x, y \in \widetilde{KO}^n(X;\hat{\mathbb{Z}}_p)$ and $u, v \in \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)$: (i) $x*y=\eta^n xy$ when $n\geqslant 0$ and $x*y=0$ when $n<0$; (ii) $x*u=\eta^n xu$ when $n\geqslant 0$ and $x*u=0$ when $n<0$; and (iii) $u*v=B_R^{-1}\eta^n uv$ when $n\geqslant 0$ and $u*v=0$ when $n<0$. Note that $\eta^n=0$ in $\widetilde{KO}^*(X;\hat{\mathbb{Z}}_p)$ unless $p=2$ and $n\leqslant 2$.

6.4. Theorem. *For a space $X \in \mathrm{Ho}_*$ and integer n, $\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p)$ has a natural weak p-adic $\Delta\lambda$-ring structure with the above internal multiplication, where the structure is strong whenever X is connected or $n\neq 0$. Moreover, the $\Delta\lambda$-ring $\widetilde{K}^n_\Delta(X;\hat{\mathbb{Z}}_p)$ is isomorphic to $\widetilde{K}^0_\Delta(\Sigma^{|n|}X;\hat{\mathbb{Z}}_p)$ for $n\leqslant 0$, while the λ-ring $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$ is isomorphic to $\widetilde{K}^0(\Sigma^{|n|}X;\hat{\mathbb{Z}}_p)$ for all n. Finally, the operation $\bar{\phi}$: $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^n(X;\hat{\mathbb{Z}}_p)$ is trivial for $n>0$.*

This will be proved in 6.10, and it provides a wide array of exterior power operations in $\widetilde{K}^*(X;\hat{\mathbb{Z}}_p)$ and $\widetilde{KO}^*(X;\hat{\mathbb{Z}}_p)$. The following theorem expresses the operation $\bar{\phi}$: $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^n(X;\hat{\mathbb{Z}}_p)$ for $n\leqslant 0$ in terms of a more basic operation ϕ: $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^0(X;\hat{\mathbb{Z}}_p)$.

6.5. Theorem. *For a space $X \in \mathrm{Ho}_*$ and $n\leqslant 0$, there is a natural operation ϕ: $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^0(X;\hat{\mathbb{Z}}_p)$ with the following properties for elements $z, w \in \widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$:*

(i) $c\phi(z) = B^n(z^*z)$;
(ii) $\phi(zw) = (\phi z)(\phi w)$ allowing elements of different degrees;
(iii) $\phi(z+w) = \phi z + \phi w + rB^n(z^*w)$;
(iv) $\phi(Bz) = -\phi(z)$;
(v) $\bar{\phi}(z) = \eta^{|n|}\phi(z)$.

This extends a result of Seymour [38] and will be proved in 6.11 using:

6.6. Atiyah's Real K-theory. In [7], Atiyah introduced a common generalization $KR(-)$ of real and complex K-theory. It applies to a *compact Real space* Y, which consists of a compact Hausdorff space Y equipped with a map τ: $Y \to Y$ such that $\tau^2=1$. A *Real vector bundle* over Y consists of a complex vector bundle q: $E \to Y$ equipped with a map τ: $E \to E$ such that $\tau^2=1$ and $\tau q = q\tau$ with τ: $E_y \to E_{\tau y}$ antilinear for each $y \in Y$. The ring $KR(Y)$ is then obtained by applying the Grothendieck construction to the semiring of real vector bundles on Y. Atiyah–Segal [9] and Dupont [28] extended the ring $KR(Y)$ to a semigraded ring $KM(Y)=KR(Y)\oplus KH(Y)$ using the same definitions, but with the condition $\tau^4=1$ on vector bundles in place of $\tau^2=1$, where the summand $KH(Y)$ is generated by the vector bundles with $\tau^2=-1$, which are called *Symplectic*.

For a compact Real space Y, we now obtain a Δ-module $K_\Delta(Y)=\{K(Y), KR(Y), KH(Y)\}$ with operations as follows:

(i) $t\colon K(Y)\to K(Y)$ is defined on vector bundles by $t(E)=\tau^*\bar{E}$;
(ii) $c\colon KR(Y)\to K(Y)$ and $c'\colon KH(Y)\to K(Y)$ are defined on vector bundles by forgetting the τ-actions;
(iii) $r\colon K(Y)\to KR(Y)$ and $q\colon K(Y)\to KH(Y)$ are defined on vector bundles by sending E to $E\oplus\tau^*\bar{E}$ with the natural real or symplectic τ-action.

When Y is connected or has a specified base component (closed under τ), we define augmentations $\varepsilon\colon K(Y)\to Z$, $\varepsilon\colon KR(Y)\to Z$, and $\varepsilon\colon KH(Y)\to Z$ sending vector bundles to their complex dimensions over the base component, and we let $\widetilde{K}_\Delta(Y)=\{\widetilde{K}(Y), \widetilde{KR}(Y), \widetilde{KH}(Y)\}$ be the Δ-module of augmentation kernels. We now obtain a precursor to Theorem 6.4.

6.7. Theorem. *For a compact Real space Y with a specified base component, the Δ-modules $K_\Delta(Y)$ and $\widetilde{K}_\Delta(Y)$ have natural $\Delta\lambda$-ring structures. Moreover, if Y is a finite complex, then $K_\Delta(Y)$ and $\widetilde{K}_\Delta(Y)$ are of finite type, and if Y is a connected finite complex, then $\widetilde{K}_\Delta(Y)$ is γ-nilpotent.*

Proof. In [28, Theorem 2], Dupont showed that the classical splitting principle [8, Corollary 2.7.11] generalizes to real and symplectic vector bundles over compact real spaces, using real and symplectic "line bundles" of complex dimension 1. Thus, we obtain λ-ring structures on $K(Y)$ and $KR(Y)\oplus KH(Y)$ by the usual constructions for vector bundles, and these give $\Delta\lambda$-ring structures on $K_\Delta(Y)$ and $\widetilde{K}_\Delta(Y)$ by straightforward arguments. When Y is a finite complex, $K^*(Y)$ is of finite type, and hence $KR(Y)$ is finitely generated abelian by Segal's spectral sequence (see [38, Theorem 3.1]). More generally, $KR(Y)\oplus KH(Y)$ is finitely generated abelian since it is additively isomorphic to $KR(Y\times S^{3,0})$ by Dupont [28, Theorem 1]. When Y is a connected finite complex, the ideal $\widetilde{K}(Y)$ and the kernel of $c\colon \widetilde{KR}(Y)\to\widetilde{K}(Y)$ are nilpotent by Seymour [38, Theorem 3.1]. Hence, $\widetilde{KR}(Y)\oplus\widetilde{KH}(Y)$ is also nilpotent. Moreover, the elements of $\widetilde{KR}(Y)$ and $\widetilde{KH}(Y)$ must have finite γ-dimension by the splitting principle, since a real or symplectic line bundle ω has $\gamma^m(\omega-1)=0$ for $m>1$ and has $\gamma^m(1-\omega)=(1-\omega)^m=0$ for sufficiently large m. Hence, $\widetilde{K}_\Delta(Y)$ is γ-nilpotent. □

We also obtain a precursor to Theorem 6.5. This applies to a compact real space Y with basepoint (fixed under τ), and it involves the compact real spaces $Y\ddot{\times}Y$ and $Y\ddot{\wedge}Y$ given by $Y\times Y$ with $\tau(y_1,y_2)=(\tau y_2,\tau y_1)$ and by $Y\wedge Y$ with $\tau(y_1\wedge y_2)=\tau y_2\wedge\tau y_1$.

6.8. Theorem. *For a compact real space Y with basepoint, there are natural operations $\ddot{\phi}\colon K(Y)\to KR(Y\ddot{\times}Y)$ and $\ddot{\phi}\colon\widetilde{K}(Y)\to\widetilde{KR}(Y\ddot{\wedge}Y)$ with the following properties on elements of $K(Y)$ or $\widetilde{K}(Y)$:*

(i) $c\ddot{\phi}(z)=z^*\times z$;
(ii) $\ddot{\phi}(zw)=(\ddot{\phi}z)(\ddot{\phi}w)$;
(iii) $\ddot{\phi}(z+w)=\ddot{\phi}z+\ddot{\phi}w+r(z^*\times w)$;
(iv) $\Delta^*\ddot{\phi}(z)=\bar{\phi}(z)$ *where Δ is the diagonal $Y\to Y\ddot{\times}Y$ or $Y\to Y\ddot{\wedge}Y$.*

Proof. The operation $\ddot{\phi}$: $K(Y) \to KR(Y\ddot{\times}Y)$ is defined on vector bundles by sending E to $\tau^*\bar{E} \otimes E$ with the twisting τ-action, and this induces an operation $\ddot{\phi}$: $\widetilde{K}(Y) \to \widetilde{KR}(Y\ddot{\wedge}Y)$. Properties (i)–(iv) are easily verified on vector bundles. $\square$

As a final preparation for our main proofs, we use Real K-theory to approach:

6.9. The p-adic K-cohomology of spaces. For $m, n \geqslant 0$, let $\Sigma^{m,n}$ be the pointed Real $(m+n)$-sphere obtained as the 1-point compactification of $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ with $\tau(x, y) = (-x, y)$. If X is a pointed finite complex then the results of Atiyah [7], Atiyah-Segal [9], Dupont [28], or Seymour [38] give a natural isomorphism of Δ-modules

$$\widetilde{K}^n_\Delta(X; \mathbb{Z}) \cong \begin{cases} \widetilde{K}_\Delta(\Sigma^{0,|n|} \wedge X) & \text{for } n \leqslant 0, \\ \widetilde{K}_\Delta(\Sigma^{n,0} \wedge X) & \text{for } n \geqslant 0. \end{cases}$$

Moreover, the internal multiplication in $\widetilde{K}^n_\Delta(X; \mathbb{Z})$ (defined by the formulae preceding Theorem 6.4) agrees with the real K-theoretic multiplication in $\widetilde{K}_\Delta(\Sigma^{0,|n|} \wedge X)$ or $\widetilde{K}_\Delta(\Sigma^{n,0} \wedge X)$. This follows easily for $n \leqslant 0$ and follows for $n > 0$ since the diagonal $\Sigma^{n,0} \to \Sigma^{n,0} \wedge \Sigma^{n,0} = \Sigma^{2n,0}$ is equivariantly homotopic to the standard inclusion and hence induces the operator

$$\eta^n\colon \widetilde{KO}^{2n}(X) \oplus \widetilde{KO}^{2n-4}(X) \longrightarrow \widetilde{KO}^{n}(X) \oplus \widetilde{KO}^{n-4}(X)$$

by Atiyah [7, Proposition 3.2]. More generally, if X is a pointed complex (possibly infinite), then there is a natural isomorphism of p-adic Δ-modules

$$\widetilde{K}^n_\Delta(X; \hat{\mathbb{Z}}_p) \cong \begin{cases} \lim\limits_{\alpha} \widetilde{K}_\Delta(\Sigma^{0,|n|} \wedge X_\alpha) \otimes \hat{\mathbb{Z}}_p & \text{for } n \leqslant 0, \\ \lim\limits_{\alpha} \widetilde{K}_\Delta(\Sigma^{n,0} \wedge X_\alpha) \otimes \hat{\mathbb{Z}}_p & \text{for } n \geqslant 0. \end{cases}$$

where X_α ranges over the finite pointed subcomplexes of X. Moreover, the internal multiplication in $\widetilde{K}^n_\Delta(X; \hat{\mathbb{Z}}_p)$ agrees with the limit of real K-theoretic multiplications.

6.10. Proof of Theorem 6.4. By Theorems 6.7 and 6.9, $\widetilde{K}^n_\Delta(X; \hat{\mathbb{Z}}_p)$ has a natural weak p-adic $\Delta\lambda$-ring structure obtained as an inverse limit of the structures of $\widetilde{K}_\Delta(\Sigma^{0,|n|} \wedge X_\alpha) \otimes \hat{\mathbb{Z}}_p$ or $\widetilde{K}_\Delta(\Sigma^{n,0} \wedge X_\alpha) \otimes \hat{\mathbb{Z}}_p$ for the finite pointed $X_\alpha \subset X$. This structure is strong when X is connected (with a single vertex for simplicity) or $n \neq 0$, since the complexes $\Sigma^{0,|n|} \wedge X_\alpha$ and $\Sigma^{n,0} \wedge X_\alpha$ are then connected. Moreover, the operation $\bar{\phi}$: $\widetilde{K}^n(X; \hat{\mathbb{Z}}_p) \to \widetilde{KO}^n(X; \hat{\mathbb{Z}}_p)$is trivial for $n > 0$ by Theorem 6.8, since the diagonals

$$\Delta\colon \Sigma^{n,0} \longrightarrow \Sigma^{n,0}\ddot{\wedge}\Sigma^{n,0} \cong \Sigma^{n,n},$$

$$\Delta\colon \Sigma^{n,0} \wedge X_\alpha \longrightarrow (\Sigma^{n,0} \wedge X_\alpha)\ddot{\wedge}(\Sigma^{n,0} \wedge X_\alpha)$$

are equivariantly nullhomotopic. $\square$

6.11. Proof of Theorem 6.5. For a pointed finite complex W and $n \leqslant 0$, there is a natural operation ϕ: $\widetilde{K}^n(W; \mathbb{Z}) \to \widetilde{KO}^0(W; \mathbb{Z})$ given by the composition

$$\widetilde{K}(\Sigma^{0,|n|} \wedge W) \xrightarrow{\ddot{\phi}} \widetilde{KR}((\Sigma^{0,|n|}\ddot{\wedge}\Sigma^{0,|n|}) \wedge (W\ddot{\wedge}W)) \xrightarrow{(h\wedge\Delta)^*} \widetilde{KR}(\Sigma^{|n|,|n|} \wedge W) \overset{\mu}{\cong} \widetilde{KR}(W),$$

where $h\colon \Sigma^{|n|,|n|} \cong \Sigma^{0,|n|} \mathbin{\ddot{\wedge}} \Sigma^{0,|n|}$ is the standard equivariant homeomorphism and where μ comes from Atiyah's (1, 1)-Periodicity Theorem [7]. Moreover, the operations $\phi\colon \widetilde{K}^n(W;\mathbb{Z}) \to \widetilde{KO}^0(W;\mathbb{Z})$ satisfy the conditions of Theorem 6.5(i)–(v) by Theorem 6.8 and by Atiyah [7, Proposition 3.2], since the diagonal $\Delta\colon \Sigma^{0,|n|} \to \Sigma^{0,|n|}\ddot{\wedge}\Sigma^{0,|n|}$ is equivalent to the standard inclusion $\Sigma^{0,|n|} \subset \Sigma^{|n|,|n|}$. The desired operations $\phi\colon \widetilde{K}(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^0(X;\hat{\mathbb{Z}}_p)$ are now obtained by tensoring with $\hat{\mathbb{Z}}_p$ and passing to inverse limits. □

7. The v_1-stabilization homomorphism

For a space X and prime p, we now introduce the v_1-stabilization homomorphism $\Phi_1\colon \widetilde{K}^*_{CR}(X;\hat{\mathbb{Z}}_p) \to K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ and explain how it may be used to determine $K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ under suitable conditions (see 7.6). This will be applied to odd spheres and to simply connected compact Lie groups in Sections 8 and 9.

7.1. The v_1-stabilization homomorphism. For a K/p_*-local spectrum E (such as $\widehat{K}_p$ or $\widehat{KO}_p$) and an integer n, the cohomology $\widetilde{E}^n(X)$ of a pointed space X is represented by the space $\underline{E}_n = \Omega^\infty(\Sigma^n E)$, which has $\Phi_1\underline{E}_n \simeq \Sigma^n E$ by 2.2. Thus there is a natural v_1-*stabilization homomorphism*

$$\Phi_1\colon \widetilde{E}^n(X) \longrightarrow E^n(\Phi_1 X)$$

sending each $f\colon X \to \underline{E}_n$ to $\Phi_1 f\colon \Phi_1 X \to \Sigma^n E$. Equivalently, this is obtained by applying E^n to the map $\Phi_1 X \to \Phi_1\Omega^\infty\Sigma^\infty X \simeq (\Sigma^\infty X)_{K/p}$ induced by the adjunction unit $X \to \Omega^\infty\Sigma^\infty X$. The homomorphism Φ_1 respects the cohomology suspension σ, so that the diagram

$$\begin{array}{ccc} \widetilde{E}^n(X) & \xrightarrow{\Phi_1} & E^n(\Phi_1 X) \\ \downarrow{\scriptstyle\sigma} & & {\scriptstyle\cong}\downarrow{\scriptstyle\sigma} \\ \widetilde{E}^{n-1}(\Omega X) & \xrightarrow{\Phi_1} & E^{n-1}(\Phi_1\Omega X) \end{array}$$

commutes, and hence an element $x \in \widetilde{E}^n(X)$ has $\Phi_1 x = 0$ whenever $\sigma^i x = 0$ for some $i > 0$. Any element $x \in \widetilde{E}^n(X)$ with $\Phi_1 x = 0$ is called Φ_1-*trivial*. For K/p_*-local spectra $D, E \in \mathrm{Ho}^{\mathrm{s}}$ and integers $m, n \in \mathbb{Z}$, let $\omega\colon \widetilde{D}^m(X) \to \widetilde{E}^n(X)$ be a natural cohomology operation with representing map $\omega\colon \underline{D}_m \to \underline{E}_n$. Then ω induces a commutative diagram

$$\begin{array}{ccc} \widetilde{D}^m(X) & \xrightarrow{\Phi_1} & D^m(\Phi_1 X) \\ \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\Phi_1\omega} \\ \widetilde{E}^n(X) & \xrightarrow{\Phi_1} & E^n(\Phi_1 X) \end{array}$$

and we note that Φ_1 preserves stable cohomology operations, that is, it gives $\Phi_1\omega = v$ whenever $\omega = \Omega^\infty v$.

For a pointed space X, the homomorphisms

$$\Phi_1\colon \widetilde{K}^*(X;\hat{\mathbb{Z}}_p) \longrightarrow K^*(\Phi_1 X;\hat{\mathbb{Z}}_p),$$

$$\Phi_1\colon \widetilde{KO}^*(X;\hat{\mathbb{Z}}_p) \longrightarrow KO^*(\Phi_1 X;\hat{\mathbb{Z}}_p),$$

now combine to give the v_1-*stabilization homomorphism*

$$\Phi_1\colon \widetilde{K}^*_{CR}(X;\hat{\mathbb{Z}}_p) \longrightarrow K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$$

of p-profinite CR-modules. For each integer n, this restricts to a homomorphism

$$\Phi_1\colon \widetilde{K}^n_{\Delta}(X;\hat{\mathbb{Z}}_p) \longrightarrow K^n_{\Delta}(\Phi_1 X;\hat{\mathbb{Z}}_p)$$

of p-profinite Δ-modules, which we may use to determine $K^n_{\Delta}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ and eventually $K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ under favorable conditions.

7.2. Theorem. *For a pointed space X and integer n, suppose that $K^{n-1}(\Phi_1 X;\hat{\mathbb{Z}}_p) = 0$ and suppose that $M \subset \widetilde{K}^n_{\Delta}(X;\hat{\mathbb{Z}}_p)$ is a Φ_1-trivial p-profinite Δ-submodule such that $\widetilde{K}^n_{\Delta}(X;\hat{\mathbb{Z}}_p)/M$ is an exact Δ-module with $\Phi_1\colon \widetilde{K}^n(X;\hat{\mathbb{Z}}_p)/M_C \cong K^n(\Phi_1 X;\hat{\mathbb{Z}}_p)$. Then $\Phi_1\colon \widetilde{K}^n_{\Delta}(X;\hat{\mathbb{Z}}_p)/M \cong K^n_{\Delta}(\Phi_1 X;\hat{\mathbb{Z}}_p)$.*

Proof. This follows by Lemma 4.8 since the Δ-module $K^n_{\Delta}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ is exact by Theorem 4.4. □

To construct the needed Φ_1-trivial elements in $\widetilde{K}^n_{\Delta}(X;\hat{\mathbb{Z}}_p)$, we use operations obtained from Theorems 6.4 and 6.5.

7.3. Lemma. *For a pointed space X and integers n and k with $k > 0$, the internal λ-ring operations λ^{pk}, ψ^{pk}, and θ^p in $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$ and in $\widetilde{KO}^n(X;\hat{\mathbb{Z}}_p) \oplus \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)$ are all annihilated by Φ_1. Moreover, when $n \leqslant 0$, the operation $\phi\colon \widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^0(X;\hat{\mathbb{Z}}_p)$ is annihilated by Φ_1.*

Here, θ^p is the natural λ-ring operation with $\psi^p x = x^p + p\theta^p x$ as in [17]. This lemma will be proved in 7.8.

7.4. Examples of Φ_1-trivial elements. For a pointed space X, we obtain from 7.1 and Lemma 7.3 the following useful examples of Φ_1-trivial elements in $\widetilde{K}^*(X;\hat{\mathbb{Z}}_p)$ and $\widetilde{KO}^*(X;\hat{\mathbb{Z}}_p)$: all graded decomposables in $\widetilde{K}^*(X;\hat{\mathbb{Z}}_p)$ and their realifications in $\widetilde{KO}^*(X;\hat{\mathbb{Z}}_p)$; all graded decomposables in $\widetilde{KO}^*(X;\hat{\mathbb{Z}}_p)$; all image elements of λ^p or θ^p in $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)$ and in $\widetilde{KO}^n(X;\hat{\mathbb{Z}}_p) \oplus \widetilde{KO}^{n-4}(X;\hat{\mathbb{Z}}_p)$ for $n = -1, 0, 1$; all image elements of $\phi\colon \widetilde{K}^n(X;\hat{\mathbb{Z}}_p) \to \widetilde{KO}^0(X;\hat{\mathbb{Z}}_p)$ for $n = -1, 0$; and all elements generated by the preceding ones using the CR-module operations. We may obtain other examples by relaxing the conditions on n, but these are generally superfluous.

7.5. $\widehat{K}\Phi_1$-good spaces. Dividing $\widetilde{K}^*(X;\hat{\mathbb{Z}}_p)$ by its submodule of known Φ_1-trivial elements, we obtain a v_1-stabilization homomorphism

$$\Phi_1\colon \hat{Q}K^*(X;\hat{\mathbb{Z}}_p)/\theta^p \longrightarrow K^*(\Phi_1 X;\hat{\mathbb{Z}}_p)$$

as in [19, Lemma 7.10], where $\hat{Q}K^*(X;\hat{\mathbb{Z}}_p)$ denotes the p-profinite quotient of $\widetilde{K}^*(X;\hat{\mathbb{Z}}_p)$ by its graded decomposables, and where $\hat{Q}K^*(X;\hat{\mathbb{Z}}_p)/\theta^p$ denotes the Bott periodic quotient of $\hat{Q}K^*(X;\hat{\mathbb{Z}}_p)$ by $\theta^p\hat{Q}K^0(X;\hat{\mathbb{Z}}_p)$ and $\theta^p\hat{Q}K^{-1}(X;\hat{\mathbb{Z}}_p)$. A pointed space X will be called $\widehat{K}\Phi_1$-*good* if

$$\Phi_1\colon \hat{Q}K^*(X;\hat{\mathbb{Z}}_p)/\theta^p \cong K^*(\Phi_1 X;\hat{\mathbb{Z}}_p).$$

Here, we could equivalently replace θ^p by λ^p since each $\theta^p x$ is congruent to $(-1)^{p+1}\lambda^p x$ modulo decomposables. We also note that the operation θ^p in $\widetilde{K}^{-1}(X;\hat{\mathbb{Z}}_p)$ corresponds to the operation ψ^p of [19] in $\widetilde{K}^1(X;\hat{\mathbb{Z}}_p)$. Working at an odd prime p in [19, Theorem 9.2], we proved the $\widehat{K}\Phi_1$-goodness of an arbitrary 1-connected H-space X with $H_*(X;\mathbb{Q})$ associative and with $H_*(X;\mathbb{Z}_{(p)})$ finitely generated over $\mathbb{Z}_{(p)}$. As explained below in 8.1 and 9.1, we can also prove the $\widehat{K}\Phi_1$-goodness of an odd sphere and of a simply connected compact Lie group at the prime $p=2$. In these cases and others, we may apply:

7.6. The general v_1-stabilization method. Suppose that X is a $\widehat{K}\Phi_1$-good pointed space with $K^{n-1}(\Phi_1 X;\hat{\mathbb{Z}}_p)=0$ for a suitable n. Then, under favorable conditions, we may apply a version of Theorem 7.2 to determine $K^n_\Delta(\Phi_1 X;\hat{\mathbb{Z}}_p)$; then apply Theorems 4.3 and 4.11 to determine $K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$; and finally apply Theorem 3.2 or methods of united K-theory to determine the v_1-periodic homotopy groups $v_1^{-1}\pi_* X \cong \pi_*\tau_p\Phi_1 X$. This general method should provide additional information on the spectrum $\Phi_1 X$ since the $\hat{\mathscr{A}}CR$-module $K^*_{CR}(\Phi_1 X;\hat{\mathbb{Z}}_p)$ determines the homotopy type of $\Phi_1 X$ by Theorem 5.3.

We conclude this section by proving Lemma 7.3 using the following Φ_1-triviality criterion, which applies to a natural cohomology operation $\omega\colon \widetilde{D}^m(X)\to\widetilde{E}^n(X)$ with representing map $\omega\colon \underline{D}_m\to\underline{E}_n$ for K/p_*-local spectra $D, E\in \mathrm{Ho}^s$ and integers $m,n\in\mathbb{Z}$. Using the p-torsion subgroup functor t_p, we say that $t_p\pi_*\underline{E}_n$ has *exponent* p^s if $p^s t_p\pi_i\underline{E}_n=0$ for all $i>0$. We also say that $\omega_*\colon \pi_*\underline{D}_m/t_p\to\pi_*\underline{E}_n/t_p$ *becomes boundlessly p-divisible* if, for each power p^j, it becomes p^j divisible in sufficiently high dimensions.

7.7. Lemma. *Suppose that $t_p\pi_*\underline{E}_n$ has exponent p^s for some $s>0$; suppose that $\omega_*\colon \pi_*\underline{D}_m/t_p\to\pi_*\underline{E}_n/t_p$ becomes boundlessly p-divisible; and suppose that the maps in $[D,E]_{m-n}$ are all detected by $Q\otimes\pi_*$. Then $\Phi_1\omega=0$ and the image elements of $\omega\colon\widetilde{D}^m(X)\to\widetilde{E}^n(X)$ are Φ_1-trivial for each space X.*

Proof. For each $k>0$, the image of $\omega_*\colon\pi_*(\underline{D}_m;\mathbb{Z}/p^k)\to\pi_*(\underline{E}_n;\mathbb{Z}/p^k)$ has exponent p^{2s} in sufficiently high dimensions, and thus the image of $\omega_*\colon v_1^{-1}\pi_*\underline{D}_m\to v_1^{-1}\pi_*\underline{E}_n$ has exponent p^{2s}. Hence, by Theorem 2.4, the image of $\tau_p\Phi_1\omega_*\colon\pi_*\tau_p\Phi_1\underline{D}_m\to\pi_*\tau_p\Phi_1\underline{E}_n$ has exponent p^{2s}, and the image of $\Phi_1\omega_*\colon\pi_*\Phi_1\underline{D}_m\to\pi_*\Phi_1\underline{E}_n$ has exponent p^{4s}. Since the maps in $[D,E]_{m-n}$ are detected by $\mathbb{Q}\otimes\pi_*$, we conclude that $\Phi_1\omega=0$, and the lemma follows. □

7.8. Proof of Lemma 7.3. When X is a sphere, the operations λ^{pk}, ψ^{pk}, and θ^p are easily determined modulo torsion, since they are preserved by complexification and are known in the λ-ring $\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)\cong\widetilde{K}^0(\Sigma^{|n|}X;\hat{\mathbb{Z}}_p)$. The results that $\Phi_1\lambda^{pk}=0$, $\Phi_1\psi^{pk}=0$, and $\Phi_1\theta^p=0$ now follow by Lemma 7.7, using Lemma 3.7 to verify the $\mathbb{Q}\otimes\pi_*$ condition. When $n\leqslant 0$, the operation $c\phi\colon\widetilde{K}^n(X;\hat{\mathbb{Z}}_p)\to\widetilde{K}^0(X;\hat{\mathbb{Z}}_p)$ is annihilated by Φ_1 since the elements $c\phi(z)=B^n(z^*z)$ are Φ_1-trivial by 7.1. Thus, $c_*(\Phi_1\phi)=0$ in the Bott exact sequence

$$[K\hat{\mathbb{Z}}_p, KO\hat{\mathbb{Z}}_p]_{n-1}\xrightarrow{\eta_*}[K\hat{\mathbb{Z}}_p, KO\hat{\mathbb{Z}}_p]_n\xrightarrow{c_*}[K\hat{\mathbb{Z}}_p, K\hat{\mathbb{Z}}_p]_n$$

and $\Phi_1\phi=0$ since $\eta_*=0$. □

8. The v_1-stabilizations of odd spheres

We now illustrate the v_1-stabilization method of 7.6 by applying it to an odd sphere S^{2n+1} at the prime $p=2$. In particular, we show that S^{2n+1} is $\widehat{K}\Phi_1$-good; we determine the 2-adic united K-theory and other homotopical properties of $\Phi_1 S^{2n+1}$; and we recover the v_1-periodic homotopy groups $v_1^{-1}\pi_* S^{2n+1} \cong \pi_* \tau_2 \Phi_1 S^{2n+1}$ which were originally determined by Mahowald using other methods (see [23]). For simplicity, we rely on certain results of Mahowald and Thompson to show the $\widehat{K}\Phi_1$-goodness of S^{2n+1} at $p=2$, although we hope to give a more general self-contained account in a subsequent paper.

8.1. Theorem. *For $n \geqslant 1$, the sphere S^{2n+1} is $\widehat{K}\Phi_1$-good at an arbitrary prime p.*

Proof. At p odd, this follows by [19, Theorem 9.2]. At $p=2$, we have $\Phi_1 S^{2n+1} \simeq \Sigma^{2n+1}(\Sigma^\infty RP^{2n})_{K/2}$ by 2.9, and hence

$$K^i(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \begin{cases} 0 & \text{for } i=0 \\ \mathbb{Z}/2^n & \text{for } i=-1 \end{cases}$$

by Adams [1]. Since these groups agree with $\hat{Q}K^i(S^{2n+1}; \hat{\mathbb{Z}}_2)/\theta^2$, it suffices to show that Φ_1: $\widetilde{K}^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2) \to K^{-1}(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is onto, and this follows since the map from $\Phi_1 S^{2n+1}$ to $(\Sigma^\infty S^{2n+1})_{K/2}$ has homotopy cofiber $(\Sigma^\infty D_2 S^{2n+1})_{K/2}$ with $K^0(\Sigma^\infty D_2 S^{2n+1}; \hat{\mathbb{Z}}_2) = 0$ by Mahowald and Thompson [34] and 2.2. □

Focusing on the case $p=2$, we now apply our v_1-stabilization method to determine $KO^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ from the Δ-module $\widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2)$ or $\widetilde{K}_\Delta^{1}(S^{2n+1}; \hat{\mathbb{Z}}_2)$ with its internal operation $\theta^2 = -\lambda^2$. We deal separately with the four possible cases of n modulo 4.

8.2. Determining $KO^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ for $n \equiv 1 \bmod 4$. We consider the Δ-module $\widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\}$ with its operations $t=1$, $c=2$, $r=1$, $c'=1$, $q=2$, $\theta_C^2 = 2^n$, $\theta_R^2 = 2^n$, and $\theta_{HR}^2 = 2^{n-1}$, using the notation θ_C^2, θ_R^2, and θ_{HR}^2 for the complex, real, and quaternionic-to-real components of θ^2. This Δ-module has a Φ_1-trivial submodule

$$M = \{2^n\hat{\mathbb{Z}}_2,\ 2^{n-1}\hat{\mathbb{Z}}_2,\ 2^{n+1}\hat{\mathbb{Z}}_2\} = \{\operatorname{im}\theta_C^2,\ \operatorname{im}\theta_{HR}^2,\ \operatorname{im} q\theta_C^2\}$$

giving an exact quotient Δ-module $\{\mathbb{Z}/2^n, \mathbb{Z}/2^{n-1}, \mathbb{Z}/2^{n+1}\}$ whose complex component goes isomorphically to $K^{-1}(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ by Theorem 8.1. Thus, we obtain v_1-stabilization isomorphisms

$$K_\Delta^{-1}(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2)/M \cong \{\mathbb{Z}/2^n,\ \mathbb{Z}/2^{n-1},\ \mathbb{Z}/2^{n+1}\},$$

and we can now apply Theorems 4.3 and 4.11 to determine $K_{CR}^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$. In fact, using our knowledge of the stable Adams operations ψ^k in $\widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2)$ for $k \in \mathbb{Z}_{(2)}^\times$, we find that $KO^i(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is: $\mathbb{Z}/2^{n-1}$ for $i=-1$ with $\psi^k = k^{n+1}$; $\mathbb{Z}/2^{n+1}$ for $i=-5$ with $\psi^k = k^{n+3}$; $\mathbb{Z}/2$ for $i=-3,-4,-6,-7$ with $\psi^k=1$; and 0 for $i=-2,-8$.

8.3. Determining $KO^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ for $n \equiv 2 \bmod 4$. We consider the Δ-module $\widetilde{K}_\Delta^{1}(S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\}$ with its operations $t=1$, $c=2$, $r=1$, $c'=1$, $q=2$, $\theta_C^2 = 2^n$, $\theta_R^2 = 2^n$, and $\theta_{HR}^2 = 2^{n-1}$. Then, as in 8.2, we determine $K_{CR}^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ and find that $KO^i(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is: $\mathbb{Z}/2^{n-1}$ for $i=1$

with $\psi^k = k^n$; $\mathbb{Z}/2^{n+1}$ for $i = -3$ with $\psi^k = k^{n+2}$; $\mathbb{Z}/2$ for $i = -1, -2, -4, -5$ with $\psi^k = 1$; and 0 for $i = 0, -6$.

8.4. Determining $KO^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ **for** $n \equiv 3 \bmod 4$. We consider the Δ-module $\widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\}$ with its operations $t = 1$, $c = 1$, $r = 2$, $c' = 2$, $q = 1$, $\theta_C^2 = 2^n$, $\theta_R^2 = 2^n$, and $\theta_{HR}^2 = 2^{n+1}$. This Δ-module has a Φ_1-trivial submodule

$$M = \{2^n \hat{\mathbb{Z}}_2,\ 2^n \hat{\mathbb{Z}}_2,\ 2^n \hat{\mathbb{Z}}_2\} = \{\operatorname{im} \theta_C^2,\ \operatorname{im} \theta_R^2,\ \operatorname{im} q\theta_C^2\}$$

giving an exact quotient Δ-module $\{\mathbb{Z}/2^n, \mathbb{Z}/2^n, \mathbb{Z}/2^n\}$ whose complex component goes isomorphically to $K^{-1}(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ by Theorem 8.1. Thus, we obtain v_1-stabilization isomorphisms

$$K_\Delta^{-1}(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2)/M \cong \{\mathbb{Z}/2^n,\ \mathbb{Z}/2^n,\ \mathbb{Z}/2^n\},$$

and we can now apply Theorems 4.3 and 4.11 to determine $K_{CR}^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$. In fact, using our knowledge of ψ^k in $\widetilde{K}_\Delta^{-1}(S^{2n+1}; \hat{\mathbb{Z}}_2)$ for $k \in \mathbb{Z}_{(2)}^\times$, we find that $KO^i(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is: $\mathbb{Z}/2^n$ for $i = -1$ with $\psi^k = k^{n+1}$; $\mathbb{Z}/2^n$ for $i = -5$ with $\psi^k = k^{n+3}$; $\mathbb{Z}/2$ for $i = -2, -4$ with $\psi^k = 1$; $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ for $i = -3$ with $\psi^k = 1$ by Yosimura [41, Lemma 4.5]; and 0 for $i = -6, -7, -8$.

8.5. Determining $KO^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ **for** $n \equiv 0 \bmod 4$. We consider the Δ-module $\widetilde{K}_\Delta^1(S^{2n+1}; \hat{\mathbb{Z}}_2) \cong \{\hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2, \hat{\mathbb{Z}}_2\}$ with its operations $t = 1$, $c = 1$, $r = 2$, $c' = 2$, $q = 1$, $\theta_C^2 = 2^n$, $\theta_R^2 = 2^n$, and $\theta_{HR}^2 = 2^{n+1}$. Then, as in 8.4, we determine $K_{CR}^*(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ and find that $KO^i(\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is: $\mathbb{Z}/2^n$ for $i = 1$ with $\psi^k = k^n$; $\mathbb{Z}/2^n$ for $i = -3$ with $\psi^k = k^{n+2}$; $\mathbb{Z}/2$ for $i = 0$ or -2 with $\psi^k = 1$; $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ for $i = -1$ with $\psi^k = 1$ by Yosimura [41, Lemma 4.5]; and 0 for $i = -4, -5, -6$.

We could now apply Theorem 3.2 to determine the v_1-periodic homotopy groups $v_1^{-1}\pi_* S^{2n+1}$ up to extension. However, to circumvent extension problems, we prefer to treat $\Phi_1 S^{2n+1}$ as a K-theoretic two-cell spectrum built from:

8.6. The K-theoretic sphere and pseudosphere. Suppose that X is a K_*-local spectrum with $K_0 X \cong \mathbb{Z}$, $K_1 X \cong 0$, and $XQ \simeq SQ$. Then, by [16, Proposition 10.6], either $X \simeq S_K$ or $X \simeq T_K$ for the spectrum $T = S^0 \cup_\eta e^2 \cup_2 e^3$. We call T_K the K-theoretic *pseudosphere* and find that $KO_* T_K$ is a free KO_*-module on a generator of degree 4, instead of the usual degree 0. We note that the above classification implies that $T_K \wedge T_K \simeq S_K$, and hence the functor $T_K \wedge -$ acts as an equivalence on the homotopy category of K_*-local spectra. We let $g\colon S_K \to T_K$ be the bottom cell map and find that it gives the doubling homomorphism on K_0 using the standard isomorphisms $K_0 S_K \cong \mathbb{Z}$ and $K_0 T_K \cong \mathbb{Z}$. Moreover, g generates the group $[S_K, T_K] \cong \pi_0 T_K \cong \mathbb{Z}$ by [13, Corollary 4.4]. We can now give the promised K-theoretic two-cell model for the homotopy type of $\Phi_1 S^{2n+1}$.

8.7. Theorem. *The spectrum $\Omega^{2n+1}\Phi_1 S^{2n+1}$ is equivalent to the homotopy fiber of $2^{n-1}g\colon S_{K/2} \to T_{K/2}$ for $n \equiv 1, 2 \bmod 4$ and of $2^n\colon S_{K/2} \to S_{K/2}$ for $n \equiv 0, 3 \bmod 4$.*

Proof. Using the results of 8.2–8.5, it is straightforward to show that the $\hat{A}CR$-module $K^*(\Omega^{2n+1}\Phi_1 S^{2n+1}; \hat{\mathbb{Z}}_2)$ is isomorphic to $K^*(F_n; \hat{\mathbb{Z}}_2)$ for the required homotopy fiber F_n, and hence $\Omega^{2n+1}\Phi_1 S^{2n+1} \simeq F_n$ by Theorem 5.3. □

The above model for $\Phi_1 S^{2n+1}$ might also be obtained from Yosimura's analysis of $(\Sigma^\infty RP^{2n})_K$ in [41] or from Mahowald and Thompson's work in [34]. To describe the resulting homotopy groups, we let $v_2(j)$ denote the greatest power of 2 dividing j (where $v_2(0) = +\infty$).

8.8. Thoerem. *For each i, we have*:

$$\pi_i S_{K/2} \simeq \begin{cases} \hat{\mathbb{Z}}_2 & \text{if } i = -1, \\ \hat{\mathbb{Z}}_2 \oplus \mathbb{Z}/2 & \text{if } i = 0, \\ \mathbb{Z}/2 & \text{if } i \equiv 0, 2 \bmod 8 \text{ with } i \neq 0, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } i \equiv 1 \bmod 8, \\ \mathbb{Z}/8 & \text{if } i \equiv 3 \bmod 8, \\ 0 & \text{if } i \equiv 4, 5, 6 \bmod 8, \\ \mathbb{Z}/2^{v_2(j)+4} & \text{if } i = 8j - 1 \text{ with } j \neq 0, \end{cases}$$

$$\pi_i T_{K/2} \simeq \begin{cases} \hat{\mathbb{Z}}_2 & \text{if } i = 0, -1, \\ 0 & \text{if } i \equiv 0, 1, 2 \bmod 8 \text{ with } i \neq 0, \\ \mathbb{Z}/8 & \text{if } i \equiv 3 \bmod 8, \\ \mathbb{Z}/2 & \text{if } i \equiv 4, 6 \bmod 8, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } i \equiv 5 \bmod 8, \\ \mathbb{Z}/2^{v_2(j)+4} & \text{if } i = 8j - 1 \text{ with } j \neq 0. \end{cases}$$

Proof. The results for $\pi_i S_{K/2}$ follow from [13, Corollary 4.5], while those for $\pi_i T_{K/2}$ follow from Theorem 3.2 except in the case $i \equiv 5 \bmod 8$ where there is an extension problem. To solve this problem, and for later use, we note that $T_{K/2} \wedge S/2 \simeq T_K \wedge (S/2)_K \simeq \Sigma^4 (S/2)_K$ by Theorem 5.3 or [16, Proposition 10.5]. We then deduce the required splitting of $\pi_i T_{K/2}$ for $i \equiv 5 \bmod 8$ from the orders of the groups $\pi_*(T_{K/2} \wedge S/2) \cong \pi_{*-4}(S/2)_K$ calculated using Theorem 3.2. $\square$

We can now recover the result of Mahowald and Davis [23, p. 1041] on the v_1-periodic homotopy groups of S^{2n+1} at $p = 2$.

8.9. Theorem (*Mahowald and Davis*). *If* $n \equiv 1, 2 \bmod 4$, *then*

$$v_1^{-1} \pi_{2n+1+i} S^{2n+1} \cong \begin{cases} \mathbb{Z}/2 & \text{if } i \equiv 0, 5 \bmod 8, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } i \equiv 1, 4 \bmod 8, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{\min(3, n+1)} & \text{if } i \equiv 2, 3 \bmod 8, \\ \mathbb{Z}/2^{\min(n-1, v_2(j)+4)} & \text{if } i = 8j - 2 \text{ or } 8j - 1. \end{cases}$$

If $n \equiv 0, 3 \bmod 4$, *then*

$$v_1^{-1} \pi_{2n+1+i} S^{2n+1} \cong \begin{cases} \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{if } i \equiv 0, 1 \bmod 8, \\ \mathbb{Z}/8 \oplus \mathbb{Z}/2 & \text{if } i \equiv 2 \bmod 8, \\ \mathbb{Z}/8 & \text{if } i \equiv 3 \bmod 8, \\ 0 & \text{if } i \equiv 4, 5 \bmod 8, \\ \mathbb{Z}/2^{\min(n, v_2(j)+4)} & \text{if } i = 8j - 2, \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2^{\min(n, v_2(j)+4)} & \text{if } i = 8j - 1. \end{cases}$$

Proof. The map $g^*\colon KO^i(T_{K/2}; \hat{\mathbb{Z}}_2) \to KO^i(S_{K/2}; \hat{\mathbb{Z}}_2)$ is given by 1 for $i \equiv 0 \bmod 8$, given by 4 for $i \equiv 4 \bmod 8$, and given by 0 otherwise. Hence, using Theorems 3.2 and 8.8, we find that $g_*\colon \pi_i S_{K/2} \to$

$\pi_i T_{K/2}$ is given by (1, 0) for $i = 0$, given by 4 for $i \equiv 3 \bmod 8$, given by 1 for $i \equiv 7 \bmod 8$, and given by 0 otherwise. Using the homotopy fiber sequence $(\Sigma S/2)_{K/2} \to S_{K/2} \xrightarrow{g} T_{K/2}$ and the fact that $S/2$ has exponent 4, we can now easily calculate the homotopy groups

$$v_1^{-1}\pi_{3+i} S^3 \cong \pi_i(\Omega^3 \Phi_1 S^3) \cong \pi_i(\Sigma S/2)_{K/2}$$

to confirm the theorem for $n = 1$. We can next calculate the homotopy groups $v_1^{-1}\pi_{2n+i} S^{2n+1} \cong \pi_i(\Omega^{2n+1}\Phi_1 S^{2n+1})$ for $n > 1$ up to extension by using the homotopy exact sequences from Theorem 8.7. Finally we conclude that all of these extensions split since the homomorphisms

$$(1 \wedge 2^{n-1} g)_*\colon \pi_*(S/2 \wedge S_{K/2}) \longrightarrow \pi_*(S/2 \wedge T_{K/2}),$$

$$(1 \wedge 2^n)_*\colon \pi_*(S/2 \wedge S_{K/2}) \longrightarrow \pi_*(S/2 \wedge S_{K/2})$$

are trivial for $n > 1$, as seen from the exponents of the homotopy groups $\pi_*(S/2 \wedge S_{K/2}) \cong \pi_*(S/2)_{K/2}$ and $\pi_*(S/2 \wedge T_{K/2}) \cong \pi_*(\Sigma^4 S/2)_{K/2}$ in each dimension. $\square$

9. The v_1-stabilizations of simply connected compact Lie groups

Finally, we apply the v_1-stabilization method to simply connected compact Lie groups at the prime $p = 2$. For such a group G, our main result (Theorem 9.3) will express $KO^*(\Phi_1 G; \hat{\mathbb{Z}}_2)$ in terms of the representation theory of G, assuming that G is $\widehat{K}\Phi_1$-good. Since we can prove the $\widehat{K}\Phi_1$-goodness of all simply connected compact Lie groups, this will confirm our general conjecture, which Davis presented in [25, Conjecture 2.2] and used so effectively to calculate v_1-periodic homotopy groups. We start by discussing:

9.1. The $\widehat{K}\Phi_1$-goodness of simply-connected compact Lie groups. In [19, Theorem 9.2], we proved that each simply connected compact Lie group G is $\widehat{K}\Phi_1$-good at an odd prime p;in fact, we gave explicit constructions of $G_{K/p}$ and $\Phi_1 G$. After considerable effort, we recently showed that this work extends to the prime $p = 2$ provided that G satisfies a certain Technical Condition involving its representation ring. This result is made useful by recent work of Davis [25, Theorem 1.3] showing that a simply connected compact simple Lie group satisfies our Technical Condition if and only if it is *not* E_6 or $\mathrm{Spin}(4k+2)$ with k not a 2-power. Hence, all simply connected compact simple Lie groups are $\widehat{K}\Phi_1$-good, except possibly for E_6 and $\mathrm{Spin}(4k+2)$ with k not a 2-power. Fortunately, we can prove that these remaining groups are $\widehat{K}\Phi_1$-good by a careful analysis of $(E_6/F_4)_{K/2}$ and by fibration arguments. We plan to include a detailed account of this work in a subsequent paper showing the $\widehat{K}\Phi_1$-goodness of all simply connected compact Lie groups. To state our main theorem, we use:

9.2. Indecomposables of representation rings. For a simply-connected compact Lie group G, we first let $Q(G) = \widetilde{R}(G)/\widetilde{R}(G)^2$ denote the indecomposables of the complex representation ring $R(G)$. We then let $Q_R(G) \subset Q(G)$ and $Q_H(G) \subset Q(G)$ denote the real and symplectic indecomposables given by the images of $\widetilde{R}_R(G)$ and $\widetilde{R}_H(G)$ in $Q(G)$. These indecomposables combine to give a $\Delta\lambda$-ring

$$Q_\Delta(G) = \{Q(G), Q_R(G), Q_H(G)\}$$

whose structure is inherited from the $\Delta\lambda$-ring $\{\widetilde{R}(G), \widetilde{R}_R(G), \widetilde{R}_H(G)\}$ of 6.1 and 6.2. Since $Q_\Delta(G)$ has trivial multiplication, its operations λ^k are additive for $k \geqslant 1$. By standard results presented in [12, Sections II.6 and VI.4] or [25, Theorem 2.3], $Q(G)$ is a finitely generated free abelian group on generators $\tilde{\rho} = \rho - \dim \rho$ where ρ ranges over the basic representations of G; moreover, $Q_\Delta(G)$ is a free Δ-module on the generators $\tilde{\rho}$ where ρ ranges over the complex, real, and quaternionic basic representations in $Q(G)$, $Q_R(G)$, and $Q_H(G)$ respectively. In particular, $Q_\Delta(G)$ is an exact Δ-module such that $c\colon Q_R(G) \subset Q(G), c'\colon Q_H(G) \subset Q(G), Q_R(G) \cap Q_H(G) = \mathrm{im}(1+t)$, and $Q_R(G) + Q_H(G) = \ker(1-t)$ for the conjugation $t\colon Q(G) \to Q(G)$.

We can now state our main theorem in a form derived directly from Davis [25, Conjecture 2.2]. For brevity, we write $Q = Q(G)$, $Q_R = Q_R(G)$, $Q_H = Q_H(G)$, and $\widehat{KO}^*(-) = KO^*(-; \hat{\mathbb{Z}}_2)$.

9.3. Theorem. *If G is a simply connected compact Lie group which is $\widehat{K}\Phi_1$-good at $p = 2$, then there is a long exact sequence of abelian groups*

$$\begin{aligned}
&\cdots \longrightarrow 0 \longrightarrow \widehat{KO}^0(\Phi_1 G) \longrightarrow Q/(Q_R + Q_H) \xrightarrow{\lambda^2} Q/Q_R \longrightarrow \widehat{KO}^1(\Phi_1 G)\\
&\quad\longrightarrow 0 \longrightarrow Q_H/(Q_R \cap Q_H) \longrightarrow \widehat{KO}^2(\Phi_1 G) \longrightarrow Q_R \cap Q_H \xrightarrow{\lambda^2} Q_H\\
&\longrightarrow \widehat{KO}^3(\Phi_1 G) \longrightarrow 0 \longrightarrow 0 \longrightarrow \widehat{KO}^4(\Phi_1 G) \longrightarrow Q/(Q_R \cap Q_H) \xrightarrow{\lambda^2} Q/Q_H\\
&\quad\longrightarrow \widehat{KO}^5(\Phi_1 G) \longrightarrow (Q_R + Q_H)/(Q_R \cap Q_H) \xrightarrow{\lambda^2} Q_R/(Q_R \cap Q_H)\\
&\quad\longrightarrow \widehat{KO}^6(\Phi_1 G) \longrightarrow Q_R + Q_H \xrightarrow{\lambda^2} Q_R \longrightarrow \widehat{KO}^7(\Phi_1 G) \longrightarrow 0\\
&\quad\longrightarrow 0 \longrightarrow \widehat{KO}^8(\Phi_1 G) \longrightarrow Q/(Q_R + Q_H) \xrightarrow{\lambda^2} Q/Q_R \longrightarrow \cdots
\end{aligned}$$

which continues by Bott periodicity. Moreover, for any integer i and odd integer $k \geqslant 1$, the Adams operation ψ^k in $\widehat{KO}^{2i-1}(\Phi_1 G)$ and $\widehat{KO}^{2i-2}(\Phi_1 G)$ corresponds to $k^{-i}\psi^k$ (or equivalently $k^{-i+1}\lambda^k$) in the Q-terms under the morphisms of the exact sequence.

As explained in 7.6 and [25], this leads to calculations of the v_1-periodic homotopy groups $v_1^{-1}\pi_* G$. In cases where G satisfies the Technical Condition (see 9.1), we originally obtained the above exact sequence (tensored with $\hat{\mathbb{Z}}_2$) as the $\widehat{KO}^*$-cohomology exact sequence of a stable (co)fiber sequence coming from our explicit construction of $\Phi_1 G$ at $p = 2$. We now proceed to prove Theorem 9.3 in general using:

9.4. The v_1-stabilization of $Q_\Delta(G)$. For a simply connected compact Lie group G, we first obtain a Δ-module homomorphism

$$\alpha\colon \widetilde{R}_\Delta(G) \longrightarrow K_\Delta^{-1}(\Phi_1 G; \hat{\mathbb{Z}}_2)$$

by composing the canonical $\Delta\lambda$-ring homomorphism $\widetilde{R}_\Delta(G) \to \widetilde{K}^0_\Delta(BG; \hat{\mathbb{Z}}_2)$ with the Δ-module homomorphisms

$$\begin{array}{ccc} \widetilde{D}^m(X) & \xrightarrow{\Phi_1} & D^m(\Phi_1 X) \\ \downarrow \omega & & \downarrow \Phi_1\omega \\ \widetilde{E}^n(X) & \xrightarrow{\Phi_1} & E^n(\Phi_1 X) \end{array}$$

This α factors through the quotient homomorphism $\widetilde{R}_\Delta(G) \twoheadrightarrow Q_\Delta(G)$ since it vanishes on the terms $\widetilde{R}(G)^2$, $\widetilde{R}_R(G)^2$, $\widetilde{R}_H(G)^2$, $\widetilde{R}_R(G)\widetilde{R}_H(G)$, $r\widetilde{R}(G)^2$, $q\widetilde{R}(G)^2$, and $\phi\widetilde{R}(G)$ by 7.4. We let

$$\alpha\colon\ Q_\Delta(G) \longrightarrow K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$$

denote the induced v_1-stabilization homomorphism for the exact Δ-module $Q_\Delta(G)$.

9.5. The exact Δ-modules $Q'_\Delta(G)$ and $Q''_\Delta(G)$. We now modify the exact Δ-module $Q_\Delta(G) = \{Q(G), Q_R(G), Q_H(G)\}$ to give an exact Δ-module

$$Q'_\Delta(G) = \{Q(G),\ \ Q_R(G) + Q_H(G),\ \ Q_R(G) \cap Q_H(G)\}$$

with the obvious operations $c\colon Q_R(G) + Q_H(G) \subset Q(G)$, $c'\colon Q_R(G) \cap Q_H(G) \subset Q(G)$, $t\colon Q(G) \to Q(G)$, $r = 1 + t\colon Q(G) \to Q_R(G) + Q_H(G)$, $q = 1 + t\colon Q(G) \to Q_R(G) \cap Q_H(G)$. We note that $Q'_\Delta(G)$ has epic q since $Q_R(G) \cap Q_H(G)$ equals the image of $1 + t\colon Q(G) \to Q(G)$. The monic operation $\lambda^2\colon Q(G) \to Q(G)$ now induces a Δ-module monomorphism $\lambda^2\colon Q'_\Delta(G) \to Q_\Delta(G)$ since $\lambda^2 t = t\lambda^2$, $\lambda^2 Q_R(G) \subset Q_R(G)$, $\lambda^2 Q_H(G) \subset Q_R(G)$, and $\lambda^2(Q_R(G) \cap Q_H(G)) \subset Q_R(G) \cap Q_H(G)$. This determines an exact Δ-module $Q''_\Delta(G) = \operatorname{coker} \lambda^2$ belonging to a short exact sequence

$$0 \longrightarrow Q'_\Delta(G) \xrightarrow{\lambda^2} Q_\Delta(G) \longrightarrow Q''_\Delta(G) \longrightarrow 0$$

where $Q''_\Delta(G)$ has monic c by Lemma 4.14 since $Q'_\Delta(G)$ has epic q. The v_1-stabilization homomorphism $\alpha\colon Q_\Delta(G) \to K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$ now factors through the quotient homomorphism $Q_\Delta(G) \twoheadrightarrow Q''_\Delta(G)$ since it vanishes on image elements of λ^2 by 7.4, and we let

$$\alpha\colon\ Q''_\Delta(G) \longrightarrow K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$$

denote the induced v_1-stabilization homomorphism.

9.6. Lemma. *If G is a simply connected compact Lie group which is $\widehat{K}\Phi_1$-good at $p=2$, then $\alpha\colon Q''_\Delta(G) \cong K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$ and $K^0(\Phi_1 G; \hat{\mathbb{Z}}_2) \cong 0$. Moreover, for any odd integer $k \geqslant 1$, the Adams operations $\{\psi^k, \psi^k, \psi^k\}$ in $K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$ correspond to $\{\psi^k, \psi^k, k^2\psi^k\}$ in $Q''_\Delta(G)$.*

Proof. Since G is $\widehat{K}\Phi_1$-good, there are isomorphisms

$$\Phi_1\colon\ \widehat{Q}K^*(G; \hat{\mathbb{Z}}_2)/\lambda^2 \cong K^*(\Phi_1 G; \hat{\mathbb{Z}}_2)$$

which reduce to isomorphisms $\alpha\colon Q''(G) \cong K^{-1}(\Phi_1 G; \hat{\mathbb{Z}}_2)$ and $0 \cong K^0(\Phi_1 G; \hat{\mathbb{Z}}_2)$ by Atiyah [6] or Hodgkin [29]. Thus $\alpha\colon Q''_\Delta(G) \cong K^{-1}_\Delta(\Phi_1 G; \hat{\mathbb{Z}}_2)$ as in the proof of Theorem 7.2, and the lemma follows easily. $\square$

We can now essentially determine $K^*_{CR}(\Phi_1 G;\hat{\mathbb{Z}}_2)$ from the exact Δ-module $Q''_\Delta(G)$ using the notation $Q''=Q''(G)$, $Q''_R=Q''_R(G)$, and $Q''_H=Q''_H(G)$. We write $A\#B$ for an abelian group belonging to a short exact sequence $0\to A\to A\#B\to B\to 0$.

9.7. Lemma. *If G is a simply connected compact Lie group which is $\widehat{K}\Phi_1$-good at $p=2$, then there are natural isomorphisms*

$$KO^m(\Phi_1 G;\hat{\mathbb{Z}}_2)\cong\begin{cases} Q''_R & \text{for } m\equiv 7 \bmod 8,\\ Q''_R/r & \text{for } m\equiv 6 \bmod 8,\\ (Q''/c')\#(Q''_H\backslash c') & \text{for } m\equiv 5 \bmod 8,\\ Q''_H\backslash c' & \text{for } m\equiv 4 \bmod 8,\\ Q''_H & \text{for } m\equiv 3 \bmod 8,\\ Q''_H/q & \text{for } m\equiv 2 \bmod 8,\\ Q''/c & \text{for } m\equiv 1 \bmod 8,\\ 0 & \text{for } m\equiv 0 \bmod 8.\end{cases}$$

Moreover, for any integer i and odd integer $k\geqslant 1$, the Adams operation ψ^k in $KO^{2i-1}(\Phi_1 G;\hat{\mathbb{Z}}_2)$ and $KO^{2i-2}(\Phi_1 G;\hat{\mathbb{Z}}_2)$ corresponds to the operation $k^{-i}\psi^k$ in the Q-terms.

Proof. By 4.1 and Lemma 9.6, $K^*_{CR}(\Phi_1 G;\hat{\mathbb{Z}}_2)$ is a Bott exact CR-module with $K^0(\Phi_1 G;\hat{\mathbb{Z}}_2)\cong 0$ and $K^{-1}_\Delta(\Phi_1 G;\hat{\mathbb{Z}}_2)\cong Q''_\Delta(G)$ where c is monic in $Q''_\Delta(G)$. Hence, the result follows immediately from Theorem 4.3. □

9.8. Proof of Theorem 9.3. We first obtain exact sequences

$$0\longrightarrow Q/(Q_R+Q_H)\xrightarrow{\lambda^2} Q/Q_R\longrightarrow Q''/c\longrightarrow 0,$$
$$0\longrightarrow Q_H/(Q_R\cap Q_H)\longrightarrow Q''_H/q\longrightarrow 0,$$
$$0\longrightarrow Q_R\cap Q_H\xrightarrow{\lambda^2} Q_H\longrightarrow Q''_H\longrightarrow 0,$$
$$0\longrightarrow Q''_H\backslash c'\longrightarrow Q/(Q_R\cap Q_H)\xrightarrow{\lambda^2} Q/Q_H\longrightarrow Q''/c'\longrightarrow 0,$$
$$0\longrightarrow Q''_H\backslash c'\longrightarrow (Q_R+Q_H)/(Q_R\cap Q_H)\xrightarrow{\lambda^2} Q_R/(Q_R\cap Q_H)\longrightarrow Q''_R/r\longrightarrow 0,$$
$$0\longrightarrow Q_R+Q_H\xrightarrow{\lambda^2} Q_R\longrightarrow Q''_R\longrightarrow 0.$$

by applying Lemma 4.14 to the short exact sequence $0\to Q'_\Delta(G)\to Q_\Delta(G)\to Q''_\Delta(G)\to 0$ of Δ-modules. We then obtain the desired long exact sequence by patching the above exact sequences together and applying Lemma 9.7. The theorem now follows easily. □

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