Algebraic Models in Homotopy Theory

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Homotopy Theory

Fundamental Problem

Are spaces X and Y are homotopy equivalent?

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Methods

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- Define algebraic invariants

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Is it always possible to find an algebraic invariant that distinguishes between non-equivalent spaces?

For simply connected spaces: **Yes!** E_∞ DGA

Outline

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1. Homotopy, Homology, and Cohomology
2. Warm-up Examples
3. Rational Homotopy Theory - CDGAs
4. Cochains and E_∞ DGAs
5. Homotopy Algebras and Homotopy Theory

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Commutative
Differential
Graded
Algebra

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Algebraic Topology

Fundamental Problem

Are spaces X and Y ~~are~~ homotopy equivalent?

It is up to you to produce maps in both directions and homotopies between the composite maps. $X \to Y \quad Y \to X$

Whitehead (1949): This simplifies for “nice” spaces.

Nice spaces: Spaces arising for their geometry
CW complexes. Examples:

- Manifolds
- Polytopes, polyhedra
- Simplicial complexes

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Now “space” means “nice space”
plus maybe a finiteness/compactness hypothesis

The Whitehead Theorem

Theorem (The Whitehead Theorem)

A map $X \to Y$ is a homotopy equivalence if and only if it induces an isomorphism on homotopy groups.

Given a map, you “just” have to check what happens on some algebraic invariants. But can’t usually compute homotopy groups.

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Theorem (The Whitehead Theorem II)

A map $X \to Y$ between simply connected *spaces is a homotopy equivalence if and only if it induces an isomorphism on homology or (equivalently) cohomology.*

How much does (co)homology say about a simply connected space?

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How much does (co)homology say about a simply connected space?

Example: Homology Spheres

Any simply connected space with the homology/cohomology of the sphere S^n ($n > 1$)

dim	0	1	$\cdots$	$n-1$	n	$n+1$	$\cdots$
H^*	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}$	0	$\cdots$

is homotopy equivalent to the sphere.

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Theorem (Hurewicz Theorem)

For a simply connected space, if $H_q X$ is trivial for $1 \leq q < n$, then $\pi_q X = 0$ for $q < n$ and the Hurewicz map $\pi_n X \to H_n X$ is an isomorphism.

$S^n \to X$

Example: $\mathbb{C}P^2$

$H^*(\mathbb{C}P^2)$ looks like: look like this:

dim	0	1	2	3	4	5	6	7	$\cdots$
H^*	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\cdots$

Other spaces also have cohomology like this, e.g., $S^2 \vee S^4$

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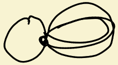

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dim	0	1	2	3	4	5	6	7	$\cdots$
$\mathbb{C}P^2$	1	0	x	0	$y = x^2$	0	0	0	$\cdots$
$S^2 \vee S^4$	1	0	x	0	$y, x^2 = 0$	0	0	0	$\cdots$

Classification

For every n, there is a space X_n with cohomology

dim	0	1	2	3	4	5	6	7	$\cdots$
X_n	1	0	x	0	$y, x^2 = ny$	0	0	0	$\cdots$

Every space with cohomology

$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	$\ldots$

is homotopy equivalent to one of these.

$$\boxed{X_m \simeq X_n \text{ if and only if } m = \pm n}$$

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$S^2 \to X$

$\pi_3 S^2 = \underset{n}{\mathbb{Z}}$

$X_n \to S$

$\shortparallel$

$S^2 \underset{S^3}{\cup} D^4 \to X$

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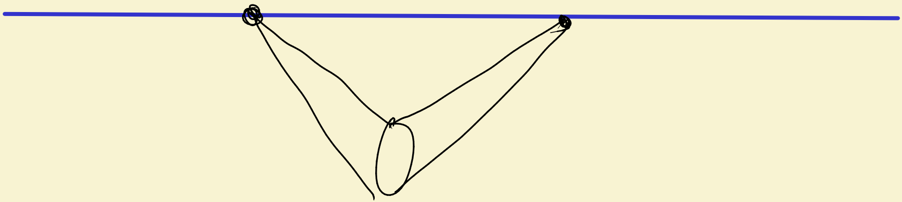
Example: $\Sigma\mathbb{C}P^2$

Suspension – take $\mathbb{C}P^2 \times [0,1]$ and collapse each of $\mathbb{C}P^2 \times \{0\}$ and $\mathbb{C}P^2 \times \{1\}$ to a point.



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This shifts cohomology groups up.

dim	0	1	2	3	4	5	6	7	8	$\cdots$
H^*	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\cdots$

It also kills the cup product.

But not the Steenrod operations on $H^*(-;\mathbb{Z}/2)$.

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$$Sq^2 : H^n(-;\mathbb{Z}/2) \to H^{n+2}(-;\mathbb{Z}/2)$$

$\mathbb{Z}/2 \quad 0 \quad 0 \quad \underset{x}{\mathbb{Z}/2} \quad 0 \quad \underset{y}{\mathbb{Z}/2} \quad 0 \quad -$

$Sq^2 x = y.$

Classification

Every space with cohomology

0	1	2	3	4	5	6	7	
$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	...

is homotopy equivalent to exactly one of $\Sigma\mathbb{C}P^2$ or $S^3 \vee S^5$.

$\Sigma(S^2 \vee S^4)$

$\mathbb{Z}/2 \quad 0 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad 0 \quad 0$

Sq^2

$\pi_2 S^3 = \mathbb{Z}/2$

The Steenrod / Grothendieck Problem

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Problem

Find structure on cohomology or cochains that classifies simply connected spaces up to homotopy equivalence.

Solution is E_∞ DGA
Mandell, "Cochains and Homotopy Type", *Pub. Math. IHÉS*, 2006.

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Given a homotopy invariant (or property or ???), find a structure on cohomology or cochains that determines it. Or vice-versa.

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Rational Homotopy Theory

1950's 1960–1978

The Whitehead Theorem: A map $X \to Y$ of simply connected space is a homotopy equivalence if and only if it induces an isomorphism on integral homology.

Definition (Rational Equivalence)

A *rational equivalence* is a map $X \to Y$ that induces an isomorphism on rational homology $H_*(X;\mathbb{Q}) \xrightarrow{\cong} H_*(Y;\mathbb{Q})$ or (equivalently) on rational cohomology $H^*(X;\mathbb{Q}) \xrightarrow{\cong} H^*(Y;\mathbb{Q})$

Rational Homotopy Theory: Make rational equivalences into isomorphisms.

Rational Homotopy Category: Category obtained by formally inverting the rational equivalences.

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Rational Invariants

What information about simply connected spaces is left in the rational homotopy category?

(Anything that takes rational equivalences to isomorphisms)

Lots of rational mapping space data, including *rational homotopy groups*.

$$\boxed{\pi_n X \otimes \mathbb{Q}}$$

More or less anything $\otimes\mathbb{Q}$ that can be computed from spectral sequences.

Serre 1950's: Rational invariants are relatively easy to compute.

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De Rham complex

$H^*(-;\mathbb{Q})$ or $H^*(-;\mathbb{R})$ have a carrier that is a

commutative differential graded algebra (CDGA)

The De Rham complex of a manifold $\Omega^* M$

$\omega \wedge \eta = (-1)^{|\eta||\omega|} \eta \wedge \omega$

$(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$

$d\omega$

$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|} \omega \wedge d\eta$

$d^2 = 0,$

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$x_0, \dots, x_n \qquad \sum x_i = 1.$

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$\Longrightarrow$ Thom–Sullivan De Rham complex $\Omega^*_{TS} M$

Makes sense for any simplicial complex / space.

$$H^*(\Omega^*_{TS} X) \cong H^*(X;\mathbb{Q})$$

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Theorem (Quillen / Sullivan)

Simply connected spaces are rationally equivalent if and only if their Thom–Sullivan De Rham complexes are quasi-isomorphic.

The Thom–Sullivan De Rham complex provides an algebraic model for the rational homotopy type

The rational homotopy groups of X are the André–Quillen cohomology groups of $\Omega^*_{TS}X$.

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Instead of requiring the multiplication to be commutative, require it to be *homotopy* commutative up to "all higher homotopies"

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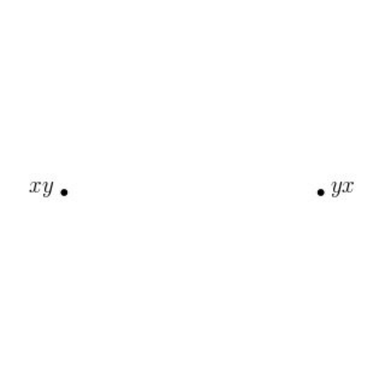

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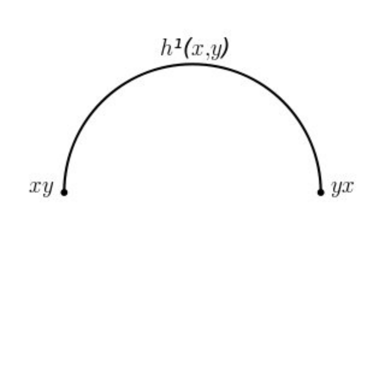

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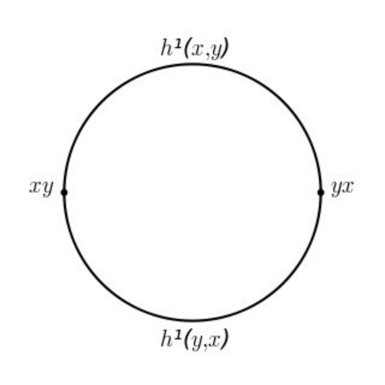

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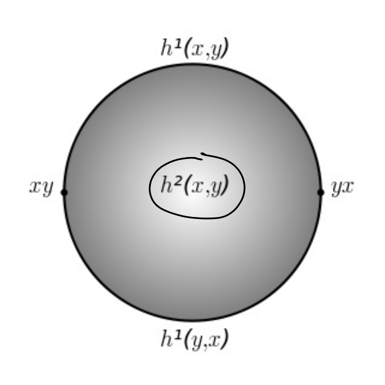


$h^2(y,x)$

Steenrod Operations

E_∞ DGAs admit Steenrod operations

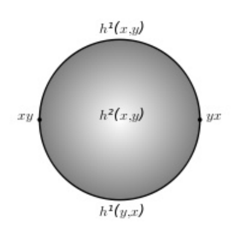
Working over $\mathbb{Z}/2$,

$$dh^n(x,y) = h^{n-1}(x,y) + h^{n-1}(y,x) + h^n(dx,y) + h^n(x,dy)$$

So for $dx \equiv 0 \mod 2$,

$$dh^n(x,x) \equiv h^{n-1}(x,x) + h^{n-1}(x,x) + 0 + 0 \equiv 0 \mod 2$$

$h^n(x,x)$ is a mod 2 cycle, represents $Sq^{2|x|-n}x$.



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$h^{n-1} : C \otimes C \to C$

$dh^n = h^{n-1} + h^{n-1} \circ \tau$

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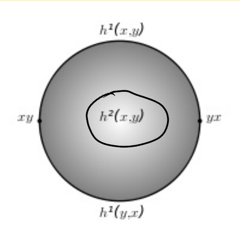

Steenrod Operations

E_∞ DGAs admit Steenrod operations

Working over $\mathbb{Z}/2$,

$$dh^n(x,y) = h^{n-1}(x,y) + h^{n-1}(y,x) + h^n(dx,y) + h^n(x,dy)$$

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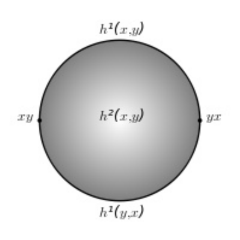

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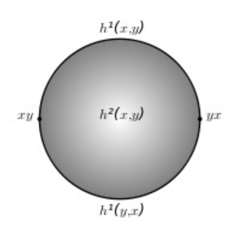
For a CDGA $h^n = 0$

Steenrod opns are zero



Cochains and E_∞ DGAs

The simplicial (or singular) cochain complex is naturally an E_∞ DGA.

Theorem

Any functor to chain complexes or E_∞ DGAs that satisfies a dimension axiom, a homotopy condition, and a weak gluing condition is naturally quasi-isomorphic to the cochain functor with some coefficients.

Example

The Thom–Sullivan De Rham complex $\Omega^*_{TS}X$ is naturally quasi-isomorphic to $C^*(X;\mathbb{Q})$ through maps of E_∞ DGAs.

Consequence

No carrier for integral cohomology can be a CDGA.

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Algebraic Models for Homotopy Theory

Theorem (M. 2006)

Simply connected spaces are homotopy equivalent if and only if their cochain E_∞ DGAs are quasi-isomorphic.

The cochain complex as an E_∞ DGA provides an algebraic model for homotopy types.

"Can" compute homotopy groups using (e.g.) analogue of the method of Cartan–Serre.

Example

$C^*(S^2)$ easy to describe as an E_∞ DGA. Beyond a certain range, higher homotopy groups are unknown.

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Homotopy Algebras and Homotopy Theory

Hierarchy of algebraic structures encoding higher homotopies of commutativity.

$$\boxed{E_1 \text{ DGAs}} \subset \boxed{E_2 \text{ DGAs}} \subset \boxed{E_3 \text{ DGAs}} \subset \cdots \subset \boxed{E_\infty \text{ DGAs}}$$

E_1 DGAs are associative DGAs

E_2 DGAs are homotopy commutative plus a little more

Concise definition in terms of brace operations $x\{y_1, \ldots, y_n\}$ ($x\{y\}$ is the commutativity homotopy)

E_3 and higher are "even more homotopy commutative"

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Invariants of the E_2 Structure

When we regard C^*X as an E_2 DGA, what information about a simply connected space X remains?

- Homology / cohomology of based loop space as a Hopf algebra
- Homology / cohomology of the free loop space as an H^*X-module.
- Homology / cohomology of mapping space X^M where $M = T^2$ or $\Sigma_g^2 \setminus \{p_1, \ldots, p_n\}$, $n \geq 1$.
- For X a PD space, the *string topology* BV algebra

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Working with the E_2 Structure

Let p be a prime number.

Conjecture

If X is at least c-connected ($c \geq 1$) and at most pc-dimensional, then after inverting $1, \ldots, p-1$, the E_2 DGA C^*X is quasi-isomorphic to a commutative DGA.

Consequences

- For highly connected / low dimensional spaces, the cochain E_2 DGA is equivalent to a commutative DGA.
- On a fixed space, for all but finitely primes, the cochain E_2 DGA is equivalent to a commutative DGA.

Homotopy theory becomes relatively accessible.

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Quillen Sullivan

Rational Htpy Cat

$\simeq$

CDGAs (cohomology looks right)

- finite dim in each degree
- and in right dimensions

Fix p.

p-adic htpy

$\simeq$ $\quad$ $P^0 = \mathrm{Id}$

E_∞ DGAS (cohomology looks right)

But over $\mathbb{Z}$.

htpy cat $\not\simeq$ E_∞ DGAS

maps $C^*Y \to C^*X$ surject onto maps $X \to Y$ (but not always inject)

$E_\infty DGA(-, \mathbb{Z})$

$\mathcal{C}[W^{-1}]$ $\qquad$ $L_W\mathcal{C}(-,-)$

$\mathcal{C}[W^{-1}] = \pi_0 L_W\mathcal{C}(-,-)$

$L_W\mathcal{C}(X,Y)$

$L_W\mathcal{C}(-,\mathbb{Z})$ $\qquad$ $L_W\mathcal{C}(Y,X)$

$L_W\mathcal{C}(X,X)$

$$\underline{\underline{A}} \to \underline{\underline{C^* X}}$$

$$E_\infty\left(A,\ C^*(\underline{\underline{\Delta[\cdot]}})\right)$$