Consider the universal elliptic curve with a choice of 4-torsion point, given by the Weierstrass equation

$$y^2 + axy + acy = x^3 + cx^2$$

over the graded ring $\mathbb{Z}_{(3)}[a, c]$, where $|a| = 1$ and $|c| = 2$. To facilitate calculations, we work in the affine coordinate chart $c = 1$ of the moduli stack $\mathcal{M}\big(\Gamma_1(4)\big)$ associated to the level $\Gamma_1(4)$-structure on the curve. It is then given by

$$y^2 + axy + ay = x^3 + x^2,$$

with discriminant $\Delta = a^2(a + 4)(a - 4)$ and Hasse invariant $h = a^2 + 4$. In uv-coodinates, with $u = x/y$ and $v = 1/y$, the equation becomes

$$v + auv + av^2 = u^3 + u^2v.$$

We denote this elliptic curve by $\mathcal{E}$.

Let F be the Morava E-theory associated to the restriction of $\mathcal{E}$ to the supersingular locus, so that $F^0 = \mathbb{Z}_9[\![h]\!]$.

1. The universal degree 3 isogeny with source $\mathcal{E}$ is defined over the ring

   $$F^0[\![t]\!]/\big(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2 + 4)t + 3\big),$$

   and has target the elliptic curve

   $$y^2 + rxy + ry = x^3 + x^2,$$

   where

   $$r(a, t) = a^3t^3 + 3a^3t^2 + 3a^3t - 4at + a^3 - 3a.$$

   The kernel of this isogeny is generated by the 3-torsion point whose x-coordinate is $1/t$.

2. The power operation $\psi^3\colon F^0 \to F^0[\![t]\!]/\big(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2 + 4)t + 3\big)$ is given by

   $$\psi^3(h) = (t+1)^3h^3-(22t^3+69t^2+75t+27)h^2+(128t^3+424t^2+512t+201)h$$
   $$-16(14t^3 + 49t^2 + 65t + 27),$$
   $$\psi^3(a) = (t + 1)^3a^3 - (4t + 3)a.$$

Let A be a $K(2)$-local commutative F-algebra. Define $Q_0, Q_1, Q_2, Q_3\colon\ \pi_0 A \to \pi_0 A$ by

$$\psi^3(x) = Q_0(x) + Q_1(x)\alpha + Q_2(x)\alpha^2 + Q_3(x)\alpha^3,$$

where α satisfies

$$\alpha^4 - 6\alpha^2 + (a^2 - 8)\alpha - 3 = 0.$$

(α appears as the coefficient of u in the expression of u', comparable to the coefficient $-d$ on p6 of your paper "Power operations for Morava E-theory of height 2 at the prime 2". It is invariant under change of coordinates. The formulas in 1 and 2 above can be written in terms of α instead of t. For the formulas below, writing in terms of t will introduce more denominators, and it will be hard to find the Adem relations.)

Commutation relations:

$Q_0(hx) = (h^3 - 36h^2 + 390h - 1212)Q_0(x) + (3h^2 - 72h + 360)Q_1(x) + (9h - 108)Q_2(x) + 24Q_3(x)$,

$Q_1(hx) = (-6h^2 + 144h - 712)Q_0(x) + (-18h + 228)Q_1(x) + (-72)Q_2(x) + (h - 12)Q_3(x)$,

$Q_2(hx) = (3h - 36)Q_0(x) + 8Q_1(x) + 12Q_2(x) + (-24)Q_3(x)$,

$Q_3(hx) = (h^2 - 24h + 120)Q_0(x) + (3h - 36)Q_1(x) + 8Q_2(x) + 12Q_3(x)$;

$Q_0(ax) = (a^3 - 12a + 12a^{-1})Q_0(x) + (3a - 12a^{-1})Q_1(x) + (12a^{-1})Q_2(x) + (-12a^{-1})Q_3(x)$,

$Q_1(ax) = (-6a + 20a^{-1})Q_0(x) + (-20a^{-1})Q_1(x) + (-a + 20a^{-1})Q_2(x) + (4a - 20a^{-1})Q_3(x)$,

$Q_2(ax) = (4a^{-1})Q_0(x) + (-4a^{-1})Q_1(x) + (4a^{-1})Q_2(x) + (-a - 4a^{-1})Q_3(x)$,

$Q_3(ax) = (a - 4a^{-1})Q_0(x) + (4a^{-1})Q_1(x) + (-4a^{-1})Q_2(x) + (4a^{-1})Q_3(x)$.

Adem relations:

$Q_1Q_0(x) = (-6)Q_0Q_1(x) + (6h - 72)Q_0Q_2(x) + (-6h^2 + 144h - 747)Q_0Q_3(x) + 18Q_1Q_2(x) + 3Q_2Q_1(x) + (-18h + 216)Q_1Q_3(x) + (-54)Q_2Q_3(x) + (-9)Q_3Q_2(x)$,

$Q_2Q_0(x) = (-3)Q_0Q_2(x) + (3h - 36)Q_0Q_3(x) + 9Q_1Q_3(x) + 3Q_3Q_1(x)$,

$Q_3Q_0(x) = Q_0Q_1(x) + (-h + 12)Q_0Q_2(x) + (h^2 - 24h + 126)Q_0Q_3(x) +$

$(-3)Q_1Q_2(x) + (3h-36)Q_1Q_3(x) + 9Q_2Q_3(x)$.

Cartan formula:

$$Q_0(xy) = Q_0(x)Q_0(y) + 3[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + 18Q_3(x)Q_3(y),$$

$$Q_1(xy) = [Q_0(x)Q_1(y) + Q_1(x)Q_0(y)] + (-h+12)[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + 3[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + (-6h+72)Q_3(x)Q_3(y),$$

$$Q_2(xy) = [Q_0(x)Q_2(y) + Q_1(x)Q_1(y) + Q_2(x)Q_0(y)] + 6[Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y)] + (-h+12)[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + 39Q_3(x)Q_3(y),$$

$$Q_3(xy) = [Q_0(x)Q_3(y) + Q_1(x)Q_2(y) + Q_2(x)Q_1(y) + Q_3(x)Q_0(y)] + 6[Q_2(x)Q_3(y) + Q_3(x)Q_2(y)] + (-h+12)Q_3(x)Q_3(y).$$

Additivity:

$$Q_i(x+y) = Q_i(x) + Q_i(y).$$

Action on scalars:

$$Q_0(1) = 1, Q_1(1) = Q_2(1) = Q_3(1) = 0;$$

$$Q_0(h) = h^3 - 36h^2 + 390h - 1212,$$

$$Q_1(h) = -6h^2 + 144h - 712,$$

$$Q_2(h) = 3h - 36,$$

$$Q_3(h) = h^2 - 24h + 120;$$

$$Q_0(a) = a^3 - 12a + 12a^{-1},$$

$$Q_1(a) = -6a + 20a^{-1},$$

$$Q_2(a) = 4a^{-1},$$

$$Q_3(a) = a - 4a^{-1}.$$

Frobenius congruence:

$$Q_0(x) \equiv x^3 \bmod 3,$$

$\theta\colon \pi_0 A \to \pi_0 A$ such that $Q_0(x) = x^3 + 3\theta(x)$.