SOME RANK TWO BUNDLES ON $P_n\mathbb{C}$, WHOSE CHERN CLASSES VANISH

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H. Grauert and M. Schneider [4] have shown that if $n \geq 4$, every unstable holomorphic rank two vector bundle on $P_n (= P_n\mathbb{C})$ splits as a sum of two line bundles. Moreover, W. Barth [3] has shown that a holomorphic rank two bundle on P_n whose Chern classes vanish is unstable. It follows from these two results that any non-trivial topological rank two bundle on $P_n (n \geq 4)$ whose Chern classes vanish cannot admit a holomorphic structure. Rank two bundles on P_4 were classified in [2] and any such bundle is trivial if its Chern classes vanish. M. Schneider asked me whether there are such non-trivial bundles on P_n for each $n > 4$. The object of this note is to construct such bundles. This is done by using the following result:

<u>Theorem 1</u> For each odd prime p there is a rank two bundle ξ_p on P_{2p-1} such that

1) ξ_p restricted to P_{2p-2} is trivial
2) ξ_p extends (at least) to P_{4p-4}.

The bundle ξ_p is the mod p analogue of the bundle ξ_2 on P_3 considered in [2], this has trivial Chern classes and $\alpha(\xi_2) \neq 0$, however ξ_2 does <u>not</u> extend to P_4.

As a corollary of Theorem 1, one has the required result:

<u>Theorem 2</u> For each $n > 4$, there is a non-trivial rank two bundle on P_n whose Chern classes vanish.

<u>Proof</u> It is enough to show that for each $n > 4$ there is an odd prime p such that $2p-1 \leq n \leq 4p-4$. This follows from Bertrand's postulate [5,p.343] which states that for each prime p there is another prime which is not more than $2p-1$.

The idea for constructing the bundle ξ_p is that there is a non-trivial bundle ζ_p on S^{4p-2}. (From now on all bundles that will be considered have rank two and their Chern classes vanish). The bundle ξ_p is the pull back $q^*\zeta_p$ where $q: P_{2p-1} \to S^{4p-2}$ is the map that collapses P_{2p-2} to a point. By its construction, ξ_p restricted to P_{2p-2} is trivial so its Chern classes vanish. What has to be shown is that it is non-trivial and that it extends.

Bundles on S^n are classified by the group $\pi_{n-1} U(2) \cong$

$\pi_{n-1}SU(2)$, [6]. But $SU(2)$ is isomorphic to S^3 so they are classified by $\pi_{n-1}S^3$. Much information on these groups is given in Toda [8]. In the various homotopy groups considered, we will only be interested in elements whose orders are powers of a fixed prime p. However we will not use any special notation to indicate that we have p-localised although all the quoted results will be about the p-primary components. In particular, recall the following results from Toda [8 Chap.XIII].

Proposition 1. The only non-zero groups $\pi_n S^3$ for $3<n<8p-8$ are when $n=2p,4p-3,4p-2,6p-5$ and $6p-4$. In each of these cases the group is cyclic of order p.

2. The generator α of $\pi_{2p}S^3$ (and its suspensions) is detected by Steenrod's reduced power operation P^1, in the sense that

$$P^1 : H^3(S^3\cup_\alpha e^{2p+1};\, \mathbb{Z}/_p)\to H^{2p+1}(S^3\cup_\alpha e^{2p+1};\, \mathbb{Z}/_p)$$

is non-trivial.

3. The group $\pi_n S^k$ vanishes for $3<k<n<k+4p-6$ except for $n = k+2p-3$ in which case it is cyclic generated by the suspension of $\alpha\in\pi_{2p}S^3$.

Corollary The group $[P_n,S^3]$ has no elements of order p for $n\leq 2p-2$.

Proof The Puppe sequence of the inclusions $P_{n-1}\subset P_n$ gives rise to exact sequences of groups:

$$[P_{n-1},S^3] \leftarrow [P_n,S^3] \leftarrow [S^{2n},S^3] \xleftarrow{r^*} [\Sigma P_{n-1},S^3]$$

By Proposition 1, $\pi_{2n}S^3$ vanishes for $n<p$. Hence for $n\leq p-1$ the corollary follows by induction.

Next we consider the case $n=p$; one has that ΣP_{p-1} is homotopy equivalent to a wedge of spheres because the attaching maps belong to trivial groups (this follows from the Proposition). However $P^1 : H^2(P_p;\mathbb{Z}/_p) \to H^{2p}(P_p;\mathbb{Z}/_p)$ is non-trivial by the definition of P^1 [7]. So ΣP_p is homotopy equivalent to $(S^3\cup_\alpha e^{2p+1})\vee S^5\vee\ldots S^{2p-1}$. This shows that r^* is onto for $n=p$ and so $[P_p,S^3]=0$. The next possible non-zero group $\pi_{2n}S^3$ is when $n=2p-1$ hence the corollary follows.

Proof of Theorem 1

The bundle ζ_p on S^{4p-2} is classified by the generator of $\pi_{4p-3}S^3$. If the bundle ξ_p was trivial, by considering the above Puppe sequence, one sees that the bundle ζ_p would be in the image of

$r^* : [\Sigma P_{2p-2}, BS^3] \longrightarrow [S^{4p-2}, BS^3]$. But, by the Corollary the group $[\Sigma P_{2p-2}, BS^3] = [P_{2p-2}, S^3]$ vanishes.

It remains to show that the bundle ξ_p extends over P_{4p-4}. The obstruction to extending a bundle from P_n to P_{n+1} lies in $\pi_{2n+1}BS^3 = \pi_{2n}S^3$. In the range of dimensions that we are considering the only possible non-zero obstructions lie in $\pi_{4p-2}S^3$ and in $\pi_{6p-4}S^3$. Both these obstructions vanish (for $p=2$ the first does not vanish): consider the first obstruction, it is given by the composition

$$S^{4p-1} \longrightarrow P_{2p-1} \longrightarrow S^{4p-2} \xrightarrow{\zeta_p} BS^3$$

The (p-primary part of the) group $\pi_{4p-1}S^{4p-2}$ vanishes and so this first obstruction vanishes. The second obstruction is given by the composition

$$S^{6p-3} \longrightarrow P_{3p-2} \longrightarrow P_{3p-2}/P_{2p-2} \longrightarrow BS^3$$

From the Proposition, the space P_{3p-3}/P_{2p-2} is homotopy equivalent to a wedge of spheres. The top cell of P_{3p-2}/P_{2p-2} is attached non-trivially because of the non-triviality of P^1 in the $\mathbb{Z}/_p$ cohomology of P_{3p-2}/P_{2p-2}. Choose X so that ΣX is homotopy equivalent to P_{3p-2}/P_{2p-2}. The group $\pi_{6p-4}X$ is cyclic generated by $S^{6p-4} \xrightarrow{\alpha} S^{4p+1} \xrightarrow{i} X$. However the composite $S^{4p+1} \xrightarrow{i} X \xrightarrow{\xi_p} S^3$ is trivial, by the proposition. So the second obstruction also vanishes. This proves the theorem.

In this proof the key role is played by the complexes P_n/P_{2p-2}. For $n>4p-3$ these are no longer homotopy equivalent to suspensions so that some of the homotopy sets are not obviously groups. This would cause problems for any attempts to prove that the bundle ξ_p extended over P_{4p-2} or higher. When $p>3$, the group $\pi_{8p-8}S^3$ vanishes so that it is easy to see that ξ_p extends over P_{4p-3} in this case. When $p=3$, the group $\pi_{16}S^3$ does not vanish and a special analysis shows that ξ_3 does not extend over P_9. I suspect that the bundle ξ_p does not extend indefinitely for any p.

The bundle ξ_2 on P_3 can be generalised in another way to give a bundle on P_{4n+3}. One takes the element $\mu \varepsilon \pi_{8n+5}S^3$ considered by J. F. Adams [1], this map μ induces a non - zero map $\mu^* : KO^{-3}(S^3) \longrightarrow KO^{-3}(S^{8n+5}) = \mathbb{Z}_2$ and the projection map $q : P_{4n+3} \longrightarrow S^{8n+6}$ induces a non-zero map $q^* : KO^{-4}(S^{8n+6}) \longrightarrow KO^{-4}P_{4n+3}$. This proves that the pull back bundle on P_{4n+3} is non-trivial.

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