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Special Values of Modular Functions on Hecke Groups

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1 Introduction

For $m \geq 3$ a positive integer put $\lambda_m := 2\cos\frac{\pi}{m}$ and denote by $G(\lambda_m)$ the so-called Hecke subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by

$$\begin{pmatrix} 1 & \lambda_m \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These groups operate properly discontinuously on the complex upper half-plane $\mathcal{H}$ and the associated compact Riemann surfaces have genus zero. The group $G(\lambda_m)$ is arithmetic if and only if $m = 3, 4$ or 6 (i.e. $\lambda = 1, \sqrt{2}$ or $\sqrt{3}$, respectively), for $m = 3$ one has the usual modular group $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$.

Let j be the classical modular invariant on $\Gamma(1)$ and let j_n ($n \in \mathbb{N}$) be the unique monic polynomial in j such that $j_n(z) = q^{-n} + \mathcal{O}(q)$, where $q = e^{2\pi iz}$ ($z \in \mathcal{H}$). In [2] it was shown that a certain average sum of the values of j_n over the points of the divisor of a non-zero meromorphic modular function f of weight $k \in \mathbb{Z}$ on $\Gamma(1)$ – up to an additive correction term – is equal to the sum $\sum_{d|n} de(d)$ where $e(n)$ ($n \in \mathbb{N}$) are the exponents in the q-product expansion of f. The correction term was given explicitly as $2k\sigma_1(n)$ where $\sigma_1(n)$ is the sum of the positive divisors of n. Several arithmetical consequences of this result in various directions were given in [2].

The main purpose of this paper is to generalize the above quoted statement for $\Gamma(1)$ to the case of the groups $G(\lambda_m)$, with j replaced by an appropriate uniformizer for $G(\lambda_m)$ and with f a non-zero meromorphic modular function of a given signature (in the terminology of Hecke) on $G(\lambda_m)$ (Sect. 3, Theorem). The corresponding "correction term" is rational in the arithmetic cases and can be given explicitly in terms of divisor functions; in the non-arithmetic case it is transcendental as follows from work of WOLFART [11]. In the arithmetic cases, one can derive several similar consequences as in [2], however, details are left to the reader.

The proof of our result follows the same pattern as in [2], in other words follows from an appropriate modification of the proof of the classical valence formula for $G(\lambda_m)$.

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It is almost certain that our result in fact can be generalized to certain triangular subgroups with cusps of which the Hecke groups form an important subclass. We briefly discuss this in Sect. 4 of the paper.

2 Modular functions on Hecke groups

In this section we will recall several well-known facts about modular functions on $G(\lambda_m)$. For details we refer to [1, 4, 6, 8, 11].

Throughout we fix a positive integer $m \geq 3$ and write $\lambda = \lambda_m$. We put $q_\lambda := e^{2\pi iz/\lambda}$ $(z \in \mathcal{H})$ and $\omega := -e^{-\pi i/m}$. We often suppress the dependence on λ in the notation when there is no danger of confusion.

Let $k \in \mathbb{R}$ and $C = \pm 1$. Recall that a meromorphic function $f : \mathcal{H} \to \mathbb{C}$ is called a meromorphic modular function of signature (λ, k, C) if

i) $f(z+\lambda) = f(z)$,
ii) $f\left(-\frac{1}{z}\right) = C\left(\frac{z}{i}\right)^k f(z)$

for all $z \in \mathcal{H}$ and f has an expansion

$$f(z) = \sum_{n\in\mathbb{Z}} c(n) q_\lambda^n \qquad (0 < |q_\lambda| < \varepsilon,\ \varepsilon > 0)$$

such that almost all $c(n)$ with $n < 0$ are zero. (As usual we understand that complex powers are defined by the principal branch of the complex logarithm.)

Such a function is called a modular form if it is holomorphic on $\mathcal{H}$ and at ∞. If in addition $c(0) = 0$ it is called a cusp form.

If $f \neq 0$, then the valence formula (cf. e.g. [6, I-14]) states that

$$\operatorname{ord}_\infty f + \sum_{\tau \in G(\lambda)\backslash\mathcal{H}} e_\tau \operatorname{ord}_\tau f = \frac{k}{2}\left(\frac{1}{2} - \frac{1}{m}\right) \tag{1}$$

where

$$e_\tau := \begin{cases} 1/2, & \text{if } \tau \sim i \\ 1/m, & \text{if } \tau \sim \omega \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

There is a unique meromorphic modular function $j := j(\lambda; \cdot)$ of signature $(\lambda, 0, 1)$ that is holomorphic on $\mathcal{H}$ and with an expansion

$$j(z) = q_\lambda^{-1} + \mathcal{O}(1).$$

It maps the interior of the hyperbolic triangle with vertices ω, i and ∞ conformally onto $\mathcal{H}$, has a zero of order m at ω and takes the value $a = a_\lambda$ doubly at $z = i$ where

$$a_\lambda^{-1} := 2^{-4+2(-1)^m} m^{-2} \prod_{\nu=1}^{m-1} \left(2 - 2\cos\frac{\pi\nu}{m}\right)^{2(-1)^\nu \cos\frac{2\pi\nu}{m}}. \tag{3}$$

Note that our normalization is different from the one in [8, 11] and in the case of the modular group $\Gamma(1)$ agrees with the usual one.

Explicitly one has – for example –

$$a_{\lambda_3} = 12^3, \quad a_{\lambda_4} = 2^8, \quad a_{\lambda_6} = 2^2 \cdot 3^3, \quad a_{\lambda_5} = 2^6 \cdot 5^{5/2}\Big(\frac{\sqrt{5}-1}{2}\Big)^{3\sqrt{5}}.$$

It is known [11] that a_{λ_m} for $m = 5$ or $m \geq 7$ is transcendental.

One knows more precisely that

$$j(z) = q_\lambda^{-1} + \sum_{n\geq 0} c_\lambda(n) a_\lambda^{n+1} q_\lambda^n \qquad (z \in \mathcal{H}) \tag{4}$$

with $c_\lambda(n) \in \mathbb{Q}$. The numbers $c_\lambda(n)$ are polynomials in $\frac{1}{m}$ with rational coefficients which in principle are explicitly computable (for $0 \leq n \leq 3$ cf. [8]).

The space of holomorphic modular forms of signature (λ, k, C) is non-zero if and only if $k = \frac{4\ell}{m-2} + 1 - C$ with ℓ a non-negative integer. In the latter case its dimension is equal to $1 + \big[\frac{\ell+(C-1)/2}{m}\big]$. The graded ring of all modular forms of signatures (λ, k, C) with λ fixed has two algebraically independent generators

$$f_1 := \Big(\frac{(\frac{2\pi i}{\lambda})^2 j'^2}{j(j-a)}\Big)^{1/(m-2)} \quad \text{and} \quad \Big(\frac{(\frac{2\pi i}{\lambda})^m j'^m}{j^{m-1}(j-a)}\Big)^{1/(m-2)} \tag{5}$$

of signatures $\big(\lambda, \frac{4}{m-2}, 1\big)$ and $\big(\lambda, \frac{2m}{m-2}, -1\big)$, respectively.

3 Special values of j

We keep all notations of the preceding sections.

Recall that any complex-valued, λ-periodic and meromorphic function on $\mathcal{H} \cup \{\infty\}$ has a product expansion

$$f(z) = c q_\lambda^h \prod_{n\geq 1} (1 - q_\lambda^n)^{e(n)}$$

convergent in $|q_\lambda| < \varepsilon$ for some $\varepsilon > 0$. Here c is a constant, $h = \mathrm{ord}_\infty f$ and the $e(n)$ are uniquely determined complex numbers [2,3].

For $n \in \mathbb{N}$, we let j_n be the unique monic polynomial in j such that $j_n = q_\lambda^{-n} + \mathcal{O}(q_\lambda)$. (The uniqueness is clear from (1) and the existence follows in the same way by consecutively substracting appropriate powers of j from j^n.)

We shall prove

Theorem. *Let f be a non-zero meromorphic modular function of signature (λ, k, C). Denote by $e(n)$ $(n \in \mathbb{N})$ the exponents in the q_λ-product expansion of f. Then for each $n \geq 1$ one has*

$$\sum_{d|n} d e(d) = k \cdot t(n) a^n + \sum_{\tau \in G(\lambda)\backslash\mathcal{H}} e_\tau \,\mathrm{ord}_\tau f \cdot j_n(\tau)$$

where $t(n) = t_\lambda(n)$ is a rational number depending only on λ and n and $a = a_\lambda$ is given by (3). Moreover e_τ is given by (2).

Proof. We imitate the classical proof of the valence formula (loc. cit.).

Let C_λ be the following path enclosing all zeros and poles of f in

$$\mathcal{F}(\lambda) := \left\{z \in \mathcal{H} \;\middle|\; |\mathrm{Re}(z)| < \frac{\lambda}{2},\ |z| > 1\right\} :$$

the bottom of C_λ follows the unit circle from ω to $\omega + \lambda$, the right side follows $\mathrm{Re}(z) = \frac{\lambda}{2}$ to $\frac{\lambda}{2} + iT$, the top follows $\mathrm{Im}(z) = T$ to $-\frac{\lambda}{2} + iT$, and the left side follows $\mathrm{Re}(z) = -\frac{\lambda}{2}$ back down to ω, except that one detours around small circular arcs to avoid any zeros or poles of f on the boundary of $\mathcal{F}(\lambda)$.

We then compute the integral

$$\frac{1}{2\pi i}\int_{C_\lambda} \frac{f'(z)}{f(z)} j_n(z)dz$$

in two different ways, first by employing the usual argument principle and secondly by using the transformation formula of f under $G(\lambda)$.

We then find

$$\sum_{\tau \in \mathcal{F}_\lambda \setminus \{\omega, i\}} \mathrm{ord}_\tau\, f \cdot j_n(\tau) = -\frac{1}{2}\,\mathrm{ord}_i\, f \cdot j_n(i) - \frac{1}{m}\,\mathrm{ord}_\omega\, f \cdot j_n(\omega)$$
$$+ \frac{1}{2\pi i}\int_\rho \frac{F'(q_\lambda)}{F(q_\lambda)} J_n(q_\lambda)dq_\lambda - \frac{k}{2\pi i}\int_\sigma \frac{j_n(z)}{z}dz. \tag{6}$$

Here $F(q_\lambda) = f(z)$ and $J_n(q_\lambda) = j_n(z)$. Furthermore, ρ is a small circle around $q_\lambda = 0$ with negative orientation and not containing any zero or pole of $F(q_\lambda)$ except possibly zero, and σ is the part of the unit circle from ω and i.

By [2,3] we have

$$\frac{q_\lambda F'(q_\lambda)}{F(q_\lambda)} = h - \sum_{n \geq 1}\Big(\sum_{d|n} de(d)\Big)q_\lambda^n \tag{7}$$

with $h = \mathrm{ord}_0\, F$. Since by definition $J_n(q_\lambda) = q_\lambda^{-n} + \mathcal{O}(q_\lambda)$ we therefore find that

$$\frac{1}{2\pi i}\int_\rho \frac{F'(q_\lambda)}{F(q_\lambda)} J_n(q_\lambda)dq_\lambda = \sum_{d|n} de(d).$$

Let

$$\Delta_\lambda := f_1^m - f_2^2$$

where f_1 resp. f_2 are defined by (5). One immediately checks that Δ_λ is a non-zero cusp form of signature $(\lambda, \frac{4m}{m-2}, 1)$. By (1), therefore Δ_λ has no zeros on $\mathcal{H}$. From (4) and the definitions of f_1 and f_2 we find that Δ_λ has an expansion

$$\sum_{n \geq 1} \tau_\lambda(n) a_\lambda^n q_\lambda^n$$

with $\tau_\lambda(n) \in \mathbb{Q}$. Hence from (7) we see that $\sum_{d|n} dt_\lambda(d) \in \mathbb{Q}a_\lambda^n$ for all $n \geq 1$ where the $t_\lambda(n)$ are the exponents in the product expansion of Δ_λ.

We now apply (6) with $f = \Delta_\lambda$. Then the assertion of the Theorem follows immediately (note that we have absorbed the factor $\frac{4m}{m-2}$ into the rational constant).

□

Remarks. i) The Fourier coefficients of j in principle are computable for any fixed λ and n [8], hence also the numbers $t_\lambda(n)$.
ii) In the arithmetic cases $m = 3, 4, 6$ one can give the values $t_\lambda(n)a_\lambda^n$ in closed arithmetic form for all n, since the product expansion of Δ_λ can be given explicity. The case $m = 3$ of course is the classical case of Ramanujan's discriminant function Δ. By [7], for $m = 4$ and $m = 6$ the function Δ_λ (up to normalization) is given by

$$q_\lambda \prod_{n\geq 1} (1 - q_\lambda^n)^{4(3+(-1)^n)}$$

and

$$q_\lambda^2 \prod_{\substack{n\geq 1 \\ n\equiv 0 \pmod 3}} (1 - q_\lambda^n)^{24} \cdot \prod_{\substack{n\geq 1 \\ n\not\equiv 0 \pmod 3}} (1 - q_\lambda^n)^{12},$$

respectively. From this using (7) we conclude that

$$t_\lambda(n)a_\lambda^n = \begin{cases} 2\sigma_1(\frac{n}{2}) + \sigma_1(n), & \text{if } m = 4 \\ 3\sigma_1(\frac{n}{3}) + \sigma_1(n), & \text{if } m = 6. \end{cases}$$

4 The more general case of triangular groups with cusps

For some details on triangular groups and their automorphic functions the reader is referred to [5, Chap. I, 1.G] and to [10, 12].

Let F be a hyperbolic triangle in $\mathcal{H}$ with vertices A_1, A_2, A_3 and angles $\frac{\pi}{e_1}$, $\frac{\pi}{e_2}$, $\frac{\pi}{e_3}$, respectively where $e_1, e_2, e_3 \in \mathbb{N} \cup \{\infty\}$ and $\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} < 1$. Denote by F' the triangle with vertices A_1, A_3, A_4 obtained from F by hyperbolic reflection along the side $\overline{A_1A_3}$.

The triangle group of "signature" (e_1, e_2, e_3) by definition is the subgroup of $\mathrm{SL}_2(\mathbb{R})$ generated by the matrices γ_1, γ_3 and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, where γ_1 and γ_3 are uniquely determined up to sign by the properties $\gamma_1 \circ A_1 = A_1$, $\gamma_3 \circ A_3 = A_3$ and $\gamma_1 \circ A_2 = \gamma_3 \circ A_2 = A_4$. (Note that the word "signature" here as a different meaning than in Sect. 2.) This group is uniquely determined up to conjugation by the triple (e_1, e_2, e_3) and we simply denote it by $\Gamma = \Gamma(e_1, e_2, e_3)$.

If at least one of the e_ν ($\nu = 1, 2, 3$) is equal to ∞, then Γ is called a triangular group with cusps.

The group Γ operates properly discontinuously on $\mathcal{H}$ and $F \cup F'$ is a fundamental domain for Γ.

Note that in the above terminology the Hecke group $G(\lambda_m)$ has signature $(\infty, m, 2)$. According to [9], up to conjugation there are exactly nine *arithmetic* triangular groups with cusps, namely those with signatures (∞, e_2, e_3) where

$$(e_2, e_3) \in \{(\infty, \infty), (\infty, 2), (\infty, 3), (3, 2), (4, 2), (6, 2), (4, 4), (3, 3), (6, 6)\}$$

(note that the signature (∞, ∞, ∞) corresponds to the principal congruence subgroup of $\Gamma(1)$ of level 2).

One can define automorphic functions and forms w.r.t. Γ of real weight k and unitary multiplier system v in the usual way (where in the case of presence of cusps we require that $v(\gamma)$ is a root of unity for all $\gamma \in \Gamma$).

In particular, the "absolute invariant" j of Γ can be defined, being the meromorphic function on $\mathcal{H}$ (uniquely determined up to multiplication with non-zero complex numbers) which maps the open triangle F biholomorphically onto $\mathcal{H}$ and such that $j(A_1) = 0$ and $j(A_3) = \infty$.

A valence formula of the usual type is valid, the space of automorphic forms w.r.t. Γ of a given weight k and multiplier system v is finite-dimensional and in fact the "bi-graded" algebra of automorphic forms w.r.t. Γ can be explicitly described in terms of generators and relations [10, notably Satz 1].

Let us now suppose that $A_1 = \infty$ and that $A_4 - A_2$ is real. The function j when properly normalized then can be shown to have a Fourier expansion similar as in (4) – mutatis mutandis – where the n-th Fourier coefficient is the product of a rational number and the $(n+1)$-th power of a non-zero real number a depending only on the group. If in addition A_2 is purely imaginary, then a is algebraic if Γ is arithmetic and otherwise is transcendental. For all this see [12, notably Satz 3, Satz 4].

From the above facts, together with the explicit description of the generators of the algebra of modular forms for Γ in terms of j [10, p. 182], it is almost certain that with some work one can deduce an analogue of the assertion of the Theorem of Sect. 2 in the present situation (with the obvious definition of the functions j_n, $n \in \mathbb{N}$). We leave the details to the reader.

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References

[1] J. Bogo, W. Kuyk and F. van Oystayen, The graded algebras generated by the modular forms for the Hecke groups $G(\lambda)$, $1 \leq \lambda < 2$. *Bull. Soc. Math. Belgique* **24** (1974), 371–382.

[2] J. H. Bruinier, W. Kohnen and K. Ono, The arithmetic of the values of modular functions and the divisors of modular forms. To appear in *Compos. Math.*

[3] W. Eholzer and N.-P. Skoruppa, Product expansions of conformal characters. *Phys. Lett.* B **388** (1996), 82–89.

[4] E. Hecke, Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung. In: E. Hecke, *Gesammelte Werke* (ed.: Akademie d. Wiss. Göttingen), Vandenhoeck&Ruprecht, Göttingen, 1959, pp. 591–626.

[5] J. Lehner, *Discontinuous groups and automorphic functions*. AMS, Providence, 1964.

[6] A. Ogg, *Modular forms and Dirichlet series*. Benjamin, New York, 1969.

[7] Ch. Pohl, G. Rosenberger and A. Schoofs, Arithmetische Eigenschaften von Eisenstein-Reihen zu den Hecke-Gruppen $G(\sqrt{2})$ und $G(\sqrt{3})$. *Abh. Math. Sem. Univ. Hamburg* **54** (1984), 49–68.

[8] J. Raleigh On the Fourier coefficients of triangle functions. *Acta Arithm.* VII (1962), 107–111.

[9] K. Takeuchi, Arithmetic triangle groups. *J. Math. Soc. Japan* **29** (1977), 91–106.

[10] J. Wolfart, Graduierte Algebren automorpher Formen zu Dreiecksgruppen. *Analysis* I (1981), 177–190.

[11] ———, Transzendente Zahlen als Fourierkoeffizienten von Heckes Modulformen. *Acta Arithm.* XXXIX (1981), 193–205.

[12] ———, Eine arithmetische Eigenschaft automorpher Formen zu gewissen nicht-arithmetischen Gruppen. *Math. Ann.* **262** (1983), 1–21.

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