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Enumeration of uni-singular algebraic hypersurfaces

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Abstract

We enumerate complex algebraic hypersurfaces in $\mathbb{P}^n$, of a given (high) degree with one singular point of a given singularity type. Our approach is to compute the (co)homology classes of the corresponding equisingular strata in the parameter space of hypersurfaces. We suggest an inductive procedure, based on an intersection theory combined with liftings and degenerations. The procedure computes the (co)homology class in question, whenever a given singularity type is properly defined and the stratum possesses good geometric properties. We consider in detail the generalized Newton-non-degenerate singularities. We also give examples of enumeration in some other cases.

1. *Introduction*

1.1. *Preface*

This paper is a generalization of the previous one [**15**], where plane singular curves (with one singular point) were enumerated. The results for curves were as follows:

- For a large class of singularity types (the so-called linear singularities) we gave a method of writing immediately explicit formulae to solve the enumeration problem.
- For all other singularity types we gave an algorithm (which is quite efficient and for every particular singularity type gives the final answer in a bounded number of steps).

The aim of this work is to generalize the method to the case of (uni-singular, complex, algebraic) hypersurfaces in $\mathbb{P}^n$.

The theory of singular hypersurfaces is much richer and complicated than that of curves. Correspondingly, the enumeration is much more difficult both technically and conceptually. It seems that there does not exist one (relatively) easy method applicable to all types of singularities. We propose a method of calculation applicable to a class of the *generalized Newton-non-degenerate* singularities (this includes, in particular, the A, D, E types and all the singularities with $\mu \leqslant 14$). The method can also be applied to some other singularity types; we consider the examples in Appendix A.

1.2. *General settings*

We work with (complex) algebraic hypersurfaces in the ambient space $\mathbb{P}^n$. A hypersurface is defined by a polynomial equation $f(x) = 0$ of degree d in the homogeneous coordinates of $\mathbb{P}^n$. The parameter space of such hypersurfaces (the space of homogeneous polynomials of total degree d in $(n+1)$ variables) is a projective space. We denote it by $\mathbb{P}_f^{N_d}$ (here $N_d = \binom{d+n}{n} - 1$ and d is assumed to be sufficiently high).

The *discriminant*, $\Sigma \subset \mathbb{P}_f^{N_d}$, is the (projective) subvariety of the parameter space, with points that correspond to the singular hypersurfaces (generic points of the discriminant correspond to hypersurfaces with one node). Everywhere in this paper (except for Appendix A) we consider only *isolated* singularities. Even more, we consider only hypersurfaces with just *one* singular point.

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When working with a singular point (specifying its type, parameters etc.) we usually pass from the category of projective hypersurfaces to that of hypersurface germs and consider everything in a small neighbourhood in classical topology.

Consider the classification by (local embedded) topological type: two hypersurface germs $(V_i, 0) \subset (\mathbb{C}^n, 0)$ are of the same type if there exists a homeomorphism $\mathbb{C}^n \stackrel{\phi}{\to} \mathbb{C}^n$ such that $\phi(V_1) = V_2$. Two singular projective hypersurfaces are said to be of the same (local embedded topological) singularity type $\mathbb{S}$, if the corresponding germs are. For a given topological type $\mathbb{S}$, consider the stratum of (projective) hypersurfaces $\Sigma_{\mathbb{S}}$ with a singular point of this type.

Unlike the case of curves the notions of the topological equivalence and the corresponding equisingular stratification are quite complicated for hypersurfaces (as is discussed shortly in § 2.3.1). For the purposes of enumeration we use a more restricted equivalence: by the Newton diagram. We work always with *commode* (convenient) diagrams.

A singular hypersurface germ is called *generalized Newton-non-degenerate* if it can be brought to a Newton-non-degenerate form by locally analytic transformations. A hypersurface with one singular point is called generalized Newton-non-degenerate if the corresponding germ is generalized Newton-non-degenerate. (For the precise definitions and discussion of the relevant notions from singularities cf. § 2.3). Everywhere in this paper (except for Appendix A) we consider generalized Newton-non-degenerate hypersurfaces.

Definition 1.1. Two (generalized Newton-non-degenerate) hypersurface-germs are called *ND-equivalent* if they can be brought by locally analytic transformations to Newton-non-degenerateforms with the same Newton diagram.

For a given Newton diagram $\mathbb{D}$, the *equisingular family* $\Sigma^{d,n}_{\mathbb{D}}$ is defined as the set of all the points in the parameter space $\mathbb{P}^{N_d}_f$ corresponding to generalized Newton-non-degenerate hypersurfaces of degree d that can be brought by locally analytic transformations to $\mathbb{D}$.

Note that for generalized Newton-non-degenerate hypersurfaces this equivalence implies equivalence by the embedded topological type. In general this equivalence is weaker than the (contact) analytic equivalence. Therefore we call this equivalence (and the corresponding families) *ND-topological*. From now on (except for § 2.3.1), by singularity type we mean the ND-topological type $\mathbb{D}$.

As the degree d of the hypersurfaces and the dimension n are always fixed, we omit them. For enumeration purposes we always work with the *topological closure* of the strata $(\bar{\Sigma}_{\mathbb{D}} \subset \mathbb{P}^{N_d}_f)$; to simplify the formulae we usually omit the closure sign (for example, Σ_{A_k}, Σ_{D_k}, Σ_{E_k} etc.).

The so-defined closures sometimes coincide with the closures of the topological equisingular strata (§ 2.3.1). For example, this is the case for A, D, E singularities [**2**, **24**].

To specify the ND-topological singularity type (that is, to construct the corresponding diagram) we usually give a representative. In the simplest cases this representative (the normal form) is classically fixed (cf. tables in [**2**, Chapter 1]).

Example 1.2. The normal forms of some the simplest singularities (since we do not consider analytical equivalence, the moduli are omitted):

$$\begin{aligned}
&A_k : z_1^{k+1},\ \ D_k :\ z_1^2 z_2 + z_2^{k-1},\ \ E_{6k} : z_1^3 + z_2^{3k+1},\ \ E_{6k+1} : z_1^3 + z_1 z_2^{2k+1},\\
&\ E_{6k+2} : z_1^3 + z_2^{3k+2}\\
&P_8 : z_1^3 + z_2^3 + z_3^3,\ \ X_9 : z_1^4 + z_2^4,\ \ J_{10} : z_1^3 + z_2^6,\ \ T_{p,q,r} : z_1^p + z_2^q + z_3^r + z_1 z_2 z_3,\\
&\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1\\
&Q_{10} : z_1^3 + z_2^4 + z_2 z_3^2,\ \ S_{11} : z_1^4 + z_2^2 z_3 + z_1 z_3^2,\ \ U_{12} : z_1^3 + z_2^3 + z_3^4.
\end{aligned} \tag{1}$$

Here we consider the normal form up to stable equivalence, that is, up to the (non-degenerate) quadratic forms $f(z_1,\dots,z_k)+\sum_{i=k+1}^{n} z_i^2 \sim f(z_1,\dots,z_k)$.

Everywhere in the paper we assume that the degree of the hypersurfaces, d, is high. A sufficient condition is: d is bigger than the degree of determinacy for a given singularity type (the latter is often the maximal coordinate of the Newton diagram). This condition is not necessary; for example, in the case of curves the method works also in the irregular region (of small d), although the algorithm should be slightly modified [**15**, Section 5].

The so-defined equisingular strata are (almost by construction) non-empty, algebraic, pure dimensional and irreducible (cf. Proposition 2.11). Therefore the enumerative problem is well defined.

Every equisingular stratum is (by construction) embedded into $\mathbb{P}_f^{N_d}$ and we identify the stratum with its embedding. Its closure has the homology class in the corresponding integer homology group:

$$[\overline{\Sigma}_{\mathbb{D}}] \in H_{2\dim(\Sigma_{\mathbb{D}})}(\mathbb{P}_f^{N_d},\mathbb{Z}) \approx \mathbb{Z}. \tag{2}$$

The degree of this class is the degree of the stratum. The *cohomology class of the stratum*, $[\overline{\Sigma}_{\mathbb{D}}] \in H^{2N_d-2\dim(\Sigma_{\mathbb{D}})}(\mathbb{P}_f^{N_d},\mathbb{Z})$, is the class dual (by Poincare duality) to the above homology class.

As we denote the homology and cohomology class by the same letter, no confusion should arise.

1.3. *The aim of the paper, motivation and main results*

The aim of this paper is to provide a method to enumerate hypersurfaces, that is, to calculate the cohomology classes for the strata corresponding to hypersurfaces with one singular point of a given ND-topological type (for generalized Newton-non-degenerate hypersurface-germs).

The discriminant and, more generally, the varieties of equisingular hypersurfaces (Severi-type varieties) have been the subject of study for a long time. Already in 19th century it was known that the (closure of the) variety of nodal hypersurfaces of degree d in $\mathbb{P}^n$ (that is, the discriminant) is an irreducible subvariety of $\mathbb{P}_f^{N_d}$ of codimension 1 and degree

$$(n+1)(d-1)^n. \tag{3}$$

Any further progress was difficult. Work was mainly concentrated on the enumeration of curves on surfaces [**18**] with many simple singularities.

The present situation in the enumeration of singular hypersurfaces seems to be as follows. (This is not a complete/historical review; for a much better description cf. [**14**, **16**, **17**].)

- In 1998 Aluffi [**1**] calculated the degrees of the strata of cuspidal and bi-nodal hypersurfaces (Σ_{A_2}, Σ_{2A_1})
- In 2001 Hernández and Vázquez-Gallo [**10**] enumerated most of the singularities of cubic surfaces in $\mathbb{P}^3$.
- In 2003 Vainsencher [**23**] calculated the degrees of some strata of multi-nodal hypersurfaces (Σ_{rA_1} for $r \leqslant 6$).
- In 2000–2003 Kazarian, in a series of papers [**11**–**14**], used a topological approach to prove that there exist *universal* formulae for the degrees of equisingular strata. In the spirit of Thom [**22**], they are (unknown) polynomials in some combinations of the Chern classes of the ambient space and the linear family. (In our case these are the Chern classes of $\mathbb{P}^n$ and $\mathbb{P}_f^{N_d}$). The coefficients of those polynomials depend only on the singularity type (and not on the degree or dimension of hypersurfaces). The enumerative answer for a particular question is obtained just by substituting the corresponding Chern classes into the universal polynomial.

Kazarian has developed a method to calculate these Thom polynomials, from which one gets the degrees for the strata of singular hypersurfaces. In particular, he gives the answers for all the possible combinations of types up to codimension 7.

His method enumerates all the singularity types of the given codimension simultaneously. Therefore it needs a preliminary classification of the singularities of a given codimension. However, the computations are not-effective when one needs the answer for just one type (for example, A_k)

Our motivation was to show that the approach suggested in [**15**] (and used there to completely solve the problem for plane uni-singular curves) generalizes to the case of hypersurfaces. In particular, from Theorems 1.6 and 1.7 we have the following corollary.

COROLLARY 1.3. *The proposed method of degenerations* (*the algorithm*) *allows enumeration of any* (*generalized Newton-non-degenerate*) *singularity* (*in a bounded number of steps*).

The result of the enumeration procedure is the cohomology class of a stratum $Si_{\mathbb{D}}$: a polynomial of degree n (the dimension of the ambient space, $\mathbb{P}^n$) in d (the degree of hypersurfaces). The coefficients are functions of n and of the singularity type. As an example of calculations we have the following proposition.

PROPOSITION 1.4. *The cohomology classes of the lifted strata in the following cases are given in Appendix* C*: ordinary multiple points, reducible multiple points* (*see the definition in Appendix* A)*,* $A_{k\leqslant 4}$, $D_{k\leqslant 6}$, E_6, P_8, X_9, Q_{10}, S_{11}, U_{12}.

From our results one can obtain some restrictions on universal Thom polynomials. We should note, however, that from our results it is impossible to recover Thom polynomials completely when the $\mu \geqslant 7$ (cf. Appendix C).

1.4. *Description of the method*

Here we briefly describe the method; it is considered in greater detail in § 3.1. We start from the components and then formulate the enumeration theorems.

We begin in a naive way, trying to work with (locally) complete intersections of hypersurfaces defined by explicit equations. The resulting cohomology class is obtained as the product of the classes of hypersurfaces, with various corrections subtracted. Recall that we always work with closed strata.

1.4.1. *Liftings.* To write explicit equations, lift a given equisingular stratum (which initially lies in $\mathbb{P}^{N_d}_f$) to a bigger space ($\mathrm{Aux} \times \mathbb{P}^{N_d}_f$). Here Aux is an auxiliary space that traces the parameters of the singularity (singular point, tangent cone etc.).

EXAMPLE 1.5. The *minimal lifting* (*partial desingularization*) is just the universal hypersurface

$$\widetilde{\Sigma}_{\mathbb{D}}(x) := \overline{\left\{(x,f)\middle|\text{The hypersurface defined by } f(x)=0 \text{ has singularity type } \mathbb{D} \text{ at the point } x\right\}} \subset \mathbb{P}^n_x \times \mathbb{P}^{N_d}_f \qquad (4)$$

Here $\mathbb{P}^n_x$ is the ambient space of singular hypersurfaces (the subscript x emphasizes that the point of the space is denoted by x).

The cohomology class of the lifted version ($\widetilde{\Sigma}_{\mathbb{D}}$) is easier to calculate (for example, for an ordinary multiple point the lifted stratum is just a complete intersection, cf. § 1.5.1). The (co)homology class is now not just a number, but a polynomial (in the generators of the cohomology ring of the bigger ambient space). Hence we have the multidegree of $\widetilde{\Sigma}$. This provides, of course, much more information about a particular stratum.

Once the class $[\widetilde{\Sigma}_{\mathbb{D}}]$ has been calculated, the cohomology class of the original stratum ($[\Sigma_{\mathbb{D}}]$) is obtained using the Gysin homomorphism. Namely, the projection $\mathrm{Aux} \times \mathbb{P}_f^{N_d} \xrightarrow{\pi} \mathbb{P}_f^{N_d}$ induces the projection on homology $H_i(\mathrm{Aux} \times \mathbb{P}_f^{N_d}) \xrightarrow{\pi_*} H_i(\mathbb{P}_f^{N_d})$ and by Poincare duality in cohomology $H^{i+2\dim(\mathrm{Aux})}(\mathrm{Aux} \times \mathbb{P}_f^{N_d}) \xrightarrow{\pi_*} H^i(\mathbb{P}_f^{N_d})$. The component $H^k(\mathbb{P}_f^{N_d}) \otimes H^{2\dim(\mathrm{Aux})}(\mathrm{Aux})$ is sent isomorphically to $H^k(\mathbb{P}_f^{N_d})$ and all other cohomology classes are sent to zero.

From the calculational point of view, we should just extract from the cohomology class $[\widetilde{\Sigma}_{\mathbb{D}}]$ (which is a polynomial in the cohomology ring of $\mathrm{Aux} \times \mathbb{P}_f^{N_d}$) the coefficient of the maximal powers of the generators of $H^*(\mathrm{Aux})$.

In the above example of minimal lifting, this homomorphism is

$$H^{k+2n}\bigl(\mathbb{P}_x^n \times \mathbb{P}_f^{N_d}\bigr) \longmapsto H^k\bigl(\mathbb{P}_f^{N_d}\bigr),$$

and from $[\widetilde{\Sigma}_{\mathbb{D}}]$ one should extract the coefficient of the nth power of the generator of $H^*(\mathbb{P}_x^n)$.

Summarizing: the cohomology class of a stratum $\Sigma_{\mathbb{D}}$ is completely fixed by that of its lifted version $\widetilde{\Sigma}_{\mathbb{D}}$.

1.4.2. *Degenerations from the simple to the complicated.* The lifted stratum can often be globally defined by some explicit equations (the case of *linear singularity*; cf. Definition 2.23 in § 2.3.3). Unfortunately it is usually only a locally (but not globally) complete intersection. Therefore, if one chooses a locally defining set of hypersurfaces, the intersections contain some residual (unnecessary) pieces, and the contribution of the pieces to the cohomology class should be subtracted. The serious difficulty is that these residual pieces can be of a dimension *bigger* than the true stratum.

In this case we proceed as follows. Let $\mathbb{D}$ be the singularity under consideration, let $\mathbb{D}_0$ be some singularity type to which $\mathbb{D}$ is adjacent and for which the enumeration is already done (the trivial choice for $\mathbb{D}_0$ is just an ordinary multiple point of the same multiplicity as $\mathbb{D}$). Represent $\mathbb{D}$ as a chain of successive degenerations (that is, adjacencies), starting from $\mathbb{D}_0$. At each step the codimension of the variety grows by 1, the stratum being intersected by a hypersurface. Each intersection can be non-transversal somewhere, and the resulting variety of the intersection is usually reducible. In addition to the required (true) variety, the intersection contains some residual varieties. We emphasize that at each step the intersection is with a hypersurface, and therefore the dimensions of the residual pieces are not bigger than that of the true variety. Thus, at each step the contribution of the residual pieces can be removed from the cohomology class of the intersection. In this process one should check the following.

- Where the non-transversality occurs. (This question is considered in § 3.1.1.2 using Proposition 2.31.)
- What the residual pieces produced in the intersection are. (This question is considered in §§ 3.1.1 and 3.1.2.)
- How 'to remove' their contributions from the answer. (This amounts to calculation of the cohomology classes of residual varieties and is considered in §§ B.2 and B.3).

All the above can be formulated as a proposition (proved in § 3.1.2).

THEOREM 1.6. • *For a given linear type $\mathbb{D}$ and an auxiliary type $\mathbb{D}_0$ the algorithm forms the chain of degenerations $\mathbb{D}_0 \to \cdots \to \mathbb{D}$. All the vertices correspond to linear types* (*fixed by the choice of* $\mathbb{D}$, $\mathbb{D}_0$)*. The number of vertices equals* $\operatorname{codim}(\Sigma_{\mathbb{D}}) - \operatorname{codim}(\Sigma_{\mathbb{D}_0})$.

• *Each edge* $\mathbb{D}_i \to \mathbb{D}_{i+1}$ *(a codimension* 1 *degeneration) provides a linear expression for the class* $[\mathbb{D}_{i+1}]$ *in terms of* $[\mathbb{D}_i]$ *and the classes of residual cycles. The residual cycles are fixed by the geometry of the degeneration.*

• *The number of steps required to achieve the result is not bigger than the number of points under the Newton diagram.*

1.4.3. *Degenerations from the complicated to the simple (simplifying degenerations).* In most cases, even the lifted stratum is difficult to define explicitly. This is the case of *non-linear* singularities (defined in § 2.3.3). Then, instead of trying to arrive at the required singularity $\mathbb{D}$ by degenerations of some simpler singularity, we degenerate the $\mathbb{D}$ itself. The aim is to arrive at some singularity ($\mathbb{D}'$) of higher codimension (or higher Milnor number), which is however simple to work with (for example, linear singularity).

Or, geometrically, we intersect the lifted stratum ($\widetilde{\Sigma}_{\mathbb{D}}$) with a cycle in the ambient space so that the cohomology class of the resulting stratum ($\widetilde{\Sigma}_{\mathbb{D}'}$) is easier to calculate. Then (if the intersection is transversal) we have the equation for the cohomology classes:

$$[\widetilde{\Sigma}_{\mathbb{D}}] \times [\text{degenerating cycle}] = [\widetilde{\Sigma}_{\mathbb{D}'}] \in H^*(\mathrm{Aux} \times \mathbb{P}_f^{N_d}). \tag{5}$$

We choose the degenerating cycle in the appropriate manner, so that equation (5) of cohomology classes fixes the class of $\widetilde{\Sigma}_{\mathbb{D}}$ uniquely (cf. § 2.4.5).

In fact the situation is more complicated.

• The resulting stratum in general is reducible and not reduced. Its reduced components enter with different multiplicities (since the initial stratum $\widetilde{\Sigma}_{\mathbb{D}}$ is singular at these loci). The resulting strata and their multiplicities are obtained from the defining ideal by an explicit check of the equations.

• The required stratum $\widetilde{\Sigma}_{\mathbb{D}'}$ is usually not a complete intersection. Thus, on the right-hand side in equation (**5**) there can appear some residual pieces. In this case also one should remove their contributions.

The result for non-linear (generalized Newton-non-degenerate) singularities is (proved in § 3.1.2).

THEOREM 1.7. • *For each (non-linear) singularity type* $\mathbb{D}$ *a tree of degenerations is constructed. The root of the tree is the original type* $\mathbb{D}$*, the leaves are some linear singularities, adjacent to the original stratum. This tree is constructed from the Newton diagram of the given singularity type, without any preliminary classification or preliminary knowledge of adjacent strata.*

• *Every edge of the tree corresponds to a degeneration, resulting in a pure dimensional variety. The corresponding equation for cohomology classes is of the form*

$$[\widetilde{\Sigma}_{\mathbb{D}_i}] \times [\textit{degenerating divisor}] = \sum a_j [\widetilde{\Sigma}_{\mathbb{D}_{i+1,j}}] + [\textit{residual piece}].$$

The cohomology class of the residual piece is calculated by a standard procedure. The cohomology class of the stratum $[\widetilde{\Sigma}_{\mathbb{D}_i}]$ *is restored uniquely from this equation.*

• *If the initial non-linear singularity has order of determinacy* k *and multiplicity* p*, then the number of vertices in this tree is not greater than* $\binom{k+n}{n} - \binom{p-1+n}{n}$.

1.4.4. *Some special simple cases.* In some (very special) cases the enumeration is almost immediate. These are the cases of (mostly) Newton degenerate singularities with reducible jets (§ A.2), that is, $jet_p(f)$ defines a reducible hypersurface. The simplest (non-trivial) example is the degenerate multiple point of order p with hypersurfaces of the form

$f(z_1,\ldots,z_n)=\prod_i \Omega_i^{(p_i)}$ + higher order terms. Here $\Omega_i^{(p_i)}$ are non-degenerate mutually generic homogeneous forms of orders p_i such that $\sum_i p_i = p$.

In this case, the equisingular family is defined by reducibility of the tensor of derivatives (Appendix A). As reducibility is not invariant under topological transformations, the equisingular family in this case does not coincide with the ND-topological stratum. The enumeration goes in the same way as the enumeration of curves [**15**].

1.4.5. *Computer calculations and efficiency.* As we are working with polynomials of a high degree in many variables, the computer is used. The calculations are essentially polynomial algebra: add/subtract polynomials, multiply, open the brackets, eliminate variables, solve a big system linear equations etc. Therefore a restriction arises from the speed and memory of a computer. We discuss some aspects of this step in § C.1.

The restrictions are not severe for linear singularities, but are quite tough for non-linear ones. In particular, for A_8 (a 'very non-linear' case) there was just not enough memory even in the case of curves. We emphasize, however, that this is purely a computer limitation.

1.5. *The simplest examples*

The case of ordinary multiple point is elementary because the defining conditions of the lifted stratum are globally transversal. Usually the best we can hope for is to obtain the locally transversal conditions.

In this case, every time we degenerate, we should check for possible residual varieties and remove their contributions if necessary. This technique is most simply demonstrated by the case of a node, naively defined in affine coordinates.

1.5.1. *Globally complete intersections: ordinary multiple point* $f=\sum z_i^{p+1}$. We work with hypersurfaces $\{f(x)=0\}\subset \mathbb{P}^n_x$ of degree d. The defining condition here is: all the derivatives up to order p should vanish. This can be written as $f|^{(p)}_x = 0$ (tensor of derivatives of order p, in homogeneous coordinates, calculated at the point x). The lifted variety in this case is

$$\widetilde{\Sigma}(x)=\{(f,x)\mid f|^{(p)}_x=0\}\subset \mathbb{P}^{N_d}_f\times\mathbb{P}^n_x. \tag{6}$$

(Recall, we always speak about topological closure of $\Sigma_{\mathbb{D}}$.) This variety is defined by $\binom{p+n}{n}$ transversal conditions. The transversality is proven in general in § 2.4.1. For pedagogical reasons we check it here explicitly.

Note that $\mathbb{P}\mathrm{GL}(n+1)$ acts freely and transitively on $\mathbb{P}^n_x$, therefore it is sufficient to check the transversality at some particular point. For example, fix $x=(1,0,\ldots,0)\in\mathbb{P}^n_x$. Then the conditions of (6) are just linear equations in the space $\mathbb{P}^{N_d}_f$ of all polynomials of the given degree, so the transversality is equivalent to linear independence, and it is checked directly (note that d is sufficiently high).

Thus the variety is a globally complete intersection and its cohomology class is just the product of the classes of defining hypersurfaces. Since all the hypersurfaces have the same class, we obtain

$$\left[\widetilde{\Sigma}(x)\right]=((d-p)X+F)^{\binom{n+p}{p}}. \tag{7}$$

Here F, X are the generators of the cohomology ring of $\mathbb{P}^{N_d}_f\times\mathbb{P}^n_x$.

To obtain the cohomology class of Σ we apply the Gysin homomorphism (as explained in § 1.4.1). From the expression (7) one should extract the maximal non-vanishing power of X, that is, X^n. The coefficient of this term is the cohomology class of the required stratum. This gives the degree

$$\deg(\Sigma)=\binom{\binom{n+p}{p}}{n}(d-p)^n. \tag{8}$$

1.5.2. *Locally complete intersections: nodal hypersurfaces defined in affine coordinates.* Let $x = (z_0 \dots z_n)$ be the homogeneous coordinates in $\mathbb{P}^n_x$. Choose the affine part: $(z_0 \neq 0) \subset \mathbb{P}^n_x$. In local coordinates a hypersurface has a node if the corresponding function vanishes together with its derivatives. Thus we define

$$\widetilde{\Xi}(x) = \{(f,x)|\partial_1 f|_x = \dots = \partial_n f|_x = 0,\ f|_x = 0\} \subset \mathbb{P}^{N_d}_f \times \mathbb{P}^n_x. \tag{9}$$

Over the affine part $\{z_0 \neq 0\} \subset \mathbb{P}^n_x$ this variety coincides with the lifted stratum of nodal curves, $\widetilde{\Sigma}_{A_1}(x)$. However at infinity $(z_0 = 0)$ one can expect some additional pieces. Indeed, the Euler formula (14) shows that equation (9), when translated to the neighbourhood of infinity, is

$$z_0 \partial_0 f|_x = 0 = \partial_1 f|_x = \dots = \partial_n f|_x. \tag{10}$$

That is, the (projective closure of the) variety of (9) is reducible: it is the union of $\widetilde{\Sigma}_{A_1}(x)$ and some residual variety (at $z_0 = 0$), taken with multiplicity one (since z_0 appears in the first degree).

In terms of co-homology classes

$$[\widetilde{\Xi}(x)] = [\widetilde{\Sigma}_{A_1}(x)] + 1[z_0 = 0,\ \partial_1 f = \dots = \partial_n f = 0]. \tag{11}$$

Therefore, to calculate the (co)homological class $[\widetilde{\Sigma}_{A_1}(x)]$ one should subtract from $[\widetilde{\Xi}(x)]$ the (co)homological class of the variety defined by $\{z_0 = 0,\ \partial_1 f = \dots = \partial_n f = 0\}$. Explicit calculation gives the degree of the discriminant (that is, the result of (8) in the case $p = 1$).

1.6. *Organization of material*

The main body of the paper gives the proof of Corollary 1.3.

In § 2 we recall some important definitions, fix the notation and introduce some auxiliary notions used throughout the paper. We discuss the singularity types and the strata (topological and ND-topological, § 2.3.1) and clarify their relation. In § 2.3.3 we introduce *linear singularities*. Then (in § 2.4) we formulate *covariant defining conditions* for linear singularities.

In § 3.1.1 we define the *liftings* of the strata and consider related questions. Then we discuss the problem of non-transversality (§ 3.1.1.2) and obtain the characterization of points of non-transversal intersection.

In § 3.1.2 we prove the main theorems (Theorems 1.6 and 1.7). Essentially just collecting all the results from §§ 2, 3.1 and the Appendix B.

In § 3.2 we demonstrate the algorithm by simple examples: the cusp A_2 and the tacnode A_3. In § 3.3 we consider some higher singularities. We start from the double points of a given co-rank, this corresponds to $A_2, D_4, P_8, \dots$. Then, further degenerations are considered (for example, A_4, D_5, E_6).

In most part of the paper we work with generalized Newton-non-degenerate singularities. In Appendix A we consider a special subclass of Newton-degenerate singularities with reducible jets.

Appendix B is devoted to some results from intersection theory, intensively used throughout the paper (multiplicity of intersections, cohomology classes of some special varieties and restrictions of fibrations).

In Appendix C we give some explicit results (cohomology classes) and discuss the issues of computer calculations and some consistency checks of the formulae.

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2. *Some definitions and auxiliary results*

2.1. *On variables and notation*

2.1.1. *On coordinates.* In this paper we deal with many varieties, embedded into various (products of) projective spaces. Adopt the following convention. If we denote a point in a projective space by the letter x, the corresponding projective space is denoted by $\mathbb{P}^n_x$. The points of projective spaces will be typically denoted by x, y or y_i. For enumerative purposes we use homogeneous coordinates: $x = (z_0 : \ldots : z_n) \in \mathbb{P}^n_x$. When considering a particular singular hypersurface-germ we use local coordinates, centred at the singular point, for example, $(z_1, \ldots, z_n)$ (assuming $z_0 = 1$).

When working with multi-projective space, the point $(x_1, \ldots, x_k) \in \mathbb{P}^n_{x_1} \times \ldots \times \mathbb{P}^n_{x_k}$ is called generic if no subset of the points $x_{i_1}, \ldots, x_{i_l}$ lies in a $(l-2)$-plane. The points $(x_1, \ldots, x_k)$ will always be assumed mutually generic, unless a restriction is explicitly specified. By identifying $\mathbb{P}^{N_d}_f = \mathbb{P}\mathrm{roj}(V)$, we will often consider a point $x \in \mathbb{P}^n_x$ as a vector of $(n+1)$-dimensional vector space V (defined up to a scalar multiplication). For example, the above condition of genericity can be formulated as: the vectors $x_1, \ldots, x_k$ are linearly independent.

A hyperplane in $\mathbb{P}^n$ is defined by a 1-form $l \in (\mathbb{P}^n_l)^*$. Therefore, for example, the incidence variety of hyperplanes and their points is defined as

$$\{(l, x) \,|\, l(x) = 0\} \subset \mathbb{P}^n_x \times (\mathbb{P}^n_l)^*. \tag{12}$$

2.1.2. *On the monomial order.* For the purpose of degeneration we should fix an order on monomials $\mathbf{z}^\mathbf{I}$. Hence, we say that $\mathbf{z}^\mathbf{I} < \mathbf{z}^\mathbf{J}$ if $|I| < |J|$ (the total degrees). For $|I| = |J|$ the order could be defined quite arbitrarily, we chose the lexicographic $z_1 > z_2 > \ldots > z_n$.

2.1.3. *On symmetric forms.* We will often work with symmetric p-forms $\Omega^p \in S^pV^*$ (here S^pV is a symmetric power of $(n+1)$-dimensional vector space). Thinking of a form as being a symmetric tensor with p indices $(\Omega^{(p)}_{i_1,\ldots,i_p})$, we often write $\Omega^{(p)}(\underbrace{x, \ldots, x}_{k})$ as a shorthand for the tensor multiplied k times by a point $x = (z_0, \ldots, z_n) \in \mathbb{P}^n$ (considered here as a vector in V):

$$\Omega^{(p)}(\underbrace{x, \ldots, x}_{k}) := \sum_{0 \leqslant i_1,\ldots,i_k \leqslant n} \Omega^{(p)}_{i_1,\ldots,i_p} z_{i_1} \cdot \ldots \cdot z_{i_k}. \tag{13}$$

Therefore, for example, the expression $\Omega^{(p)}(x)$ is a $(p-1)$-form. Unless stated otherwise, we assume the symmetric form $\Omega^{(p)}$ to be generic (in particular non-degenerate, that is, the corresponding hypersurface $\{\Omega^{(p)}(\underbrace{x,\dots,x}_{p})=0\}\subset\mathbb{P}^n_x$ is smooth).

Symmetric forms will typically occur as tensors of derivatives of order p: for example, $f^{(p)}$ (here f is a homogeneous polynomial defining a hypersurface). Sometimes, to emphasize the point at which the derivative is calculated we assign it. Therefore, for example, $f|^{(p)}_x(\underbrace{y,\dots,y}_{k})$ means: the tensor of derivatives of pth order, calculated at the point x and contracted k times with y.

Throughout the paper we tacitly assume the Euler identity for a homogeneous polynomial of degree d

$$\sum_{i=0}^{n} z_i\partial_i f = df \tag{14}$$

and its consequences (for example, $\sum_i z_i\partial_i\partial_j f=(d-1)\partial_j f$). Therefore, for example, the nodal point, defined by $f|^{(1)}_x=0$, can also be defined by $f|^{(p)}_x(\underbrace{x,\dots,x}_{p-1})=0$.

2.1.4. *On cohomology classes.* The generator of the cohomology ring of $\mathbb{P}^n_x$ is denoted by the corresponding upper case letter X, so that $H^*(\mathbb{P}^n_x)=\mathbb{Z}[X]/(X^{n+1})$. Alternatively, $X=c_1(\mathcal{O}_{\mathbb{P}^n_x}(1))$. By the same letter we also denote the hyperplane class in the homology of $\mathbb{P}^n_x$. Since it is always clear, where we speak about coordinates and where about (co)homology classes, no confusion should arise.

To demonstrate this, consider the hypersurface

$$\Sigma=\{(x,y,f)\mid f(x,y)=0\}\subset\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f. \tag{15}$$

Here f is a polynomial of bi-degree d_x, d_y in homogeneous coordinates

$$x=(z_0:\dots:z_n),\quad y=(w_0:\dots:w_n),$$

the coefficients of f are the homogeneous coordinates of the parameter space $\mathbb{P}^{N_d}_f$. The cohomology class of this variety is

$$[\Sigma]=d_xX+d_yY+F\in H^2(\mathbb{P}^n_x\times\mathbb{P}^n_y\times\mathbb{P}^{N_d}_f). \tag{16}$$

The formulae for the cohomology classes of the lifted strata are polynomials in the generators of the cohomology rings of the products of projective spaces (X for $\mathbb{P}^n_x$, F for $\mathbb{P}^{N_d}_f$ etc.) The polynomials depend actually on some combinations of the generators. For example, F always enters as $F+(d-k)X$ (for some $k\in\mathbb{N}$, which depends on the singularity type only). Correspondingly, we always use the (relative) class

$$Q:=(d-k)X+F \tag{17}$$

(the value of k will be specified or evident from the context).

2.1.5. *On the strata.* We denote an ND-topological stratum by Σ (it will be always clear from the context, which singularity type is meant). The lifted stratum is denoted by $\widetilde{\Sigma}$. Usually there will be many liftings for one stratum; to distinguish between them, we assign the auxiliary parameters. Hence, for example, the stratum defined in (4) is denoted by $\widetilde{\Sigma}(x)$.

We always work with the topological closures of the strata. To simplify the formulae we write just Σ (or $\widetilde{\Sigma}$) for the closure of a (lifted) equisingular stratum. The lifted stratum $\widetilde{\Sigma}$ is often considered as a fibration over the auxiliary space. For a cycle C in the auxiliary space the (scheme-theoretic) restriction of the fibration to the cycle is denoted by $\widetilde{\Sigma}|_C$. For example, if x is a point of the auxiliary space, then $\widetilde{\Sigma}|_x$ is the fibre over x.

2.2. *On the residual varieties and cycles of jump*

We try to represent a stratum as an explicit intersection of hypersurfaces. The intersection will be usually non-transversal. The resulting variety, being reducible, will contain (except for the required stratum) some additional pieces. We call these pieces *residual varieties*.

The intersection process occurs in the space $\mathbb{P}^{N_d}_f \times \mathrm{Aux}$. Here $\mathbb{P}^{N_d}_f$ is the parameter space of hypersurfaces, while Aux is the *auxiliary space*, used to define the lifted stratum explicitly. It will be typically a multi-projective space $\mathrm{Aux} = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ or a projective irreducible subvariety of it. We often consider the lifted stratum as a fibration over the auxiliary space. The fibres correspond to hypersurfaces with some specified parameters of the singularity (for example, chosen singular point, tangent cone etc.) The fibration will be generically locally trivial (in Zariski topology) and this local triviality induces the stratification on the auxiliary space.

Definition 2.1. Let $R_1 \subset \bar{R}_1 = \mathrm{Aux}$ be the maximal (open, dense) subvariety of Aux over which the fibration $\widetilde{\Sigma} \to \mathrm{Aux}$ is locally trivial. By induction, let

$$R_k \subset \bar{R}_{k-1} \backslash R_{k-1} = \mathrm{Aux} \backslash (\bigcup_{i=1}^{k-1} R_i)$$

be the maximal subvariety such that the (scheme) restriction $\widetilde{\Sigma}|_{R_k} \to R_k$ is a locally trivial fibration (over each connected component). The set $\{R_i\}_i$ is called: *the stratification of the auxiliary space by local triviality*.

Let $\bar{R}_k \backslash R_k = \bigcup m_i C_i$ be the decomposition in to a union of closed irreducible subvarieties of Aux, not containing each other. We call these irreducible subvarieties: *the cycles of jump*. The generic fibres of the projection $\widetilde{\Sigma} \to \mathrm{Aux}$ will be linear spaces, therefore the dimension of fibers jumps over the cycles of jump (cf. § 2.3).

Various cycles of jump can contain or intersect other cycles of jump (that appeared from R_k with higher k). It is useful to introduce grading on these cycles.

Definition 2.2. The cycle of jump C_i is assigned grading 1 if it does not contain other (proper) cycles of jump (equivalently if the fibration $\widetilde{\Sigma}|_{C_i} \to C_i$ is locally trivial). A cycle of jump is assigned grading k if it contains a (proper) cycle of grading $(k-1)$ and no cycles of higher grading.

Example 2.3. Quadratic forms of co-rank r (this example is important for the enumeration of singularities as A_2, D_4, $P_8, \ldots$).

Consider the variety of degenerate symmetric matrices (quadratic forms) of co-rank $r \geqslant 2$ acting on $\mathbb{P}^n$. It is a classical determinantal variety [5]. Its lifted version is the incidence variety of degenerate quadratic forms with r vectors of the kernel specified.

$$\widetilde{\Sigma}(x_1, \ldots, x_r) := \overline{\left\{ (\underbrace{x_1, \ldots, x_r}_{\substack{\text{do not lie in an} \\ (r-2)\text{-plane}}} \Omega^{(2)}) \Big| \Omega^{(2)}(x_1) = \ldots = \Omega^{(2)}(x_r) = 0 \right\}} \subset \mathbb{P}^n_{x_1} \times \ldots \times \mathbb{P}^n_{x_r} \times \mathbb{P}_\Omega \tag{18}$$

(here $\mathbb{P}_\Omega$ is the parameter space of quadratic forms, $\dim(\mathbb{P}_\Omega)=\binom{n+2}{2}-1$, $\mathrm{Aux}=\mathbb{P}^n_{x_1}\times\dots\times\mathbb{P}^n_{x_r}$).

The projection: $(x_1,\dots,x_r,\Omega^{(2)})\overset{\pi}{\mapsto}(x_1,\dots,x_r)$ is generically locally trivial fibration. The dimension of the generic fibre is $\binom{n+2}{2}-1-((2n+3-r)/2)r$. The *cycles of jump* here are all the diagonals: $\{(x_{i_1},\dots,x_{i_k})$ are linearly dependent$\}$. The cycles of jump of minimal and maximal grades are

$$\begin{aligned}&C_{\min}=\{x_1=\ldots=x_r\}\subset\mathrm{Aux}\quad \mathrm{codim}_{\mathrm{Aux}}(C_{\min})=n(r-1),\\&C_{\max}=\{(x_1\dots,x_r)\text{ lie in an }(r-2)\text{−plane}\}\subset\mathrm{Aux}\quad \mathrm{codim}_{\mathrm{Aux}}(C_{\max})=n+2-r.\end{aligned}\tag{19}$$

The first important question about cycles of jump is the jump in the dimension of fibres.

Definition 2.4. Let $c\in C$, $x\in\mathrm{Aux}$ be generic points of the cycle of jump and of the auxiliary space. The jump of fibre dimension for the cycle C is $\Delta\dim_C:=\dim(\widetilde{\Sigma}|_c)-\dim(\widetilde{\Sigma}|_x)$.

In the example above the jumps of dimension are

$$\Delta\dim_{C_{\min}}:=\frac{(2n+2-r)(r-1)}{2}-1,\quad \Delta\dim_{C_{\max}}:=n+1-r.\tag{20}$$

The total variety, $\widetilde{\Sigma}$, will be always irreducible, in particular of pure dimension, and therefore we have the following immediate corollary.

Corollary 2.5. *The jump of dimension over a cycle of jump is less than the codimension of the cycle of jump:* $\Delta\dim_C<\mathrm{codim}_{\mathrm{Aux}}(C)$

The restriction of the fibration $\widetilde{\Sigma}\to\mathrm{Aux}$ to the cycles of jump will be the source of residual varieties; therefore we are interested in the cohomology classes of such restrictions $\widetilde{\Sigma}|_C$. This question is considered in § B.3. By now we need the following simple technical result.

Proposition 2.6. *Let C be a cycle of jump and $\{C_i\}_i$ all the cycles of jump that are not contained in C. There exists a hypersurface in the auxiliary space that contains C and does not contain any of C_i (though it can intersect them).*

2.3. *On singularities*

For completeness we recall some notions related to singularities of hypersurfaces [2, 3, 9]. For a given (singular) hypersurface ($f=\sum a_{\mathbf{I}}\mathbf{z}^{\mathbf{I}}$) the Newton polytope is defined as the convex hull of the support of f in $\mathbb{Z}^n$, namely $\mathrm{conv}(\mathbf{I}\in\mathbb{Z}^n\mid a_{\mathbf{I}}\neq 0)$. We always take a generic representative for a given singularity type. Therefore we can assume that the polynomial f contains the monomials of the form $x_i^{d_i}$, for $d_i\gg 0$, and the polytope intersects all the coordinate axes. Such a germ is called *commode* (or convenient). In addition we assume (due to a high degree) that the hypersurface does not contain any line. The upper part of the Newton diagram is defined as $\Gamma_+:=\mathrm{conv}(\bigcup(\mathbf{I}+\mathbb{R}^n_+)\mid a_{\mathbf{I}}\neq 0)$. The Newton diagram (Figure 1) is defined by $\Gamma_f:=\partial\Gamma_+$. By the above assumption it is compact and consists of a finite number of top-dimensional faces.

The restriction of a polynomial f to its Newton diagram $f|_{\Gamma_f}$ is called *the principal part.*

2.3.1. *On the singularity types and strata.* The embedded topological type and ND-topological type were defined in § 1 (Definition 1.1).

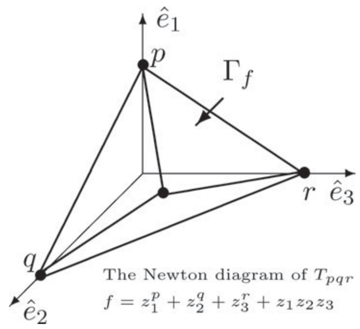

FIGURE 1.

DEFINITION 2.7 [**9**, § 3.4]. • The singular hypersurface $\{f = 0\}$ is called Newton-non-degenerate if the restriction of the polynomial f to every face (of every dimension) of its Newton diagram is non-degenerate (that is, the truncated polynomial has no singular points in the torus $(\mathbb{C}^*)^k$).

• The singular hypersurface $f = 0$ is generalized Newton-non-degenerate if it can be brought to a Newton-non-degenerateform by a locally analytic transformation.

For example, while the hypersurface $(x - y^2)^2 + y^5$ is not Newton-non-degenerate, it is certainly generalized Newton-non-degenerate.

A *topological type* is called Newton-non-degenerateif at least one of its representatives is Newton-non-degenerate(or generalized Newton-non-degenerate). Otherwise it is called Newton-degenerate. The following natural question seems to be open (as always, we assume the topological strata to be algebraic and irreducible).

Let $\{f = 0\}$ be the *generic* representative of a Newton-non-degeneratetopological type. Is f generalized Newton-non-degenerate?

All the simple and the uni-modal singularity types are Newton-non-degenerate [**2**, Chapter 1]. The simplest examples of not-generalized Newton-non-degenerate hypersurfaces are $W^\sharp_{1,p\geqslant 1}$ with Milnor number $\mu = 15 + p$ and $S^\sharp_{1,p\geqslant 1}$ with Milnor number $\mu = 14 + p$. Even if a topological type is represented by a not-generalized Newton-non-degenerate hypersurface (for example, as $W^\sharp_{1,p\geqslant 1}$ before), it is not clear whether *every* representative of this type is not-generalized Newton-non-degenerate (this is the converse of the previous question).

A way to check this in particular cases was pointed to me by G. M. Greuel.

PROPOSITION 2.8. *Let $\{f_\alpha = 0\}$ be a family of hypersurfaces (with one singular point) with the following properties.*

• *They all have the same (local embedded) topological type $\mathbb{S} = \mathbb{S}_{f_\alpha}$.*

• *The family covers all the moduli. Namely, for every value of moduli for the type $\mathbb{S}$, there is a representative $\{f_\alpha = 0\}$ in this family with this value of moduli.*

• *Every member of this family is not generalized Newton-non-degenerate.*

Then every representative of the type $\mathbb{S}$ is not generalized Newton-non-degenerate.

The proof is immediate by observation that the whole stratum is a collection of equimodular orbits under the group of locally analytic transformations and our family intersects every orbit.

For example, from Arnol'd's classification it follows that the topological types $W^\sharp_{1,p\geqslant 1}$ and $S^\sharp_{1,p\geqslant 1}$ are Newton-degenerate (since the modality in both cases is 2 and the families of normal forms indeed cover all the moduli).

Definition 2.9. • The singular hypersurface $\{f=0\}$ is called semi-quasi-homogeneous (SQH) if by a locally analytic transformation it can be brought to a Newton-non-degenerateform the principal part of which is quasi-homogeneous.

- The singularity type is called quasi-homogeneous if it has an SQH representative.

In this case there is a strong result by [**21**].

Proposition 2.10. *Let f be a quasi-homogeneous polynomial of degree d and weights $w_1,\dots,w_n$, defining a singular hypersurface, of topological singularity type $\mathbb{S}_f$. Let another representative of this type (algebraic hypersurface) be defined by an* SQH *polynomial g. Then g is semi-quasi-homogeneous of the same degree d and weights $w_1,\dots,w_n$ as f.*

In the case of curves the embedded topological type and its strata possess all the good properties. For example, the generic representative of the Newton-non-degeneratetype is Newton-non-degenerateand can be brought to the given Newton diagram by locally analytic transformations. Therefore the ND-topological type often coincides with the embedded topological type (the same for the closures of the strata). For high enough degrees of curves the strata are irreducible and smooth in their interior.

For hypersurfaces the situation is much more complicated. By choosing big degree d of hypersurfaces, the non-emptiness of the strata is assured. However, the strata of embedded topological type can behave quite badly. Just to mention, the algebraicity of the strata has not yet been proven in general (though it is known for quite a broad class of types [**24**]). Even assumed to be algebraic, the topological strata can be singular and reducible [**19**] (for large degrees of hypersurfaces). The associated Newton diagram can be non-constant along the equisingular stratum (in the sense that the transformation required to achieve it is not locally analytic but a homeomorphism) [**3**, Example 2.14, Chapter 1]. The constancy of multiplicity along the equisingular stratum has up to now been proved for semi-quasi-homogeneous singularities alone [**6**, **7**] (see also [**4**] for recent results).

The equisingular strata we work with (the ND-topological strata) are chosen especially to possess all the good geometric properties.

Proposition 2.11. *For a given Newton diagram $\mathbb{D}$ the stratum $\Sigma_{\mathbb{D}}$ is a (non-empty) irreducible algebraic variety.*

Indeed, the family of hypersurfaces with a specific diagram (the fibre over the diagram) is defined by linear equations in the parameter space $\mathbb{P}^{N_d}_f$. And then the ND stratum is obtained by the action of algebraic group (locally analytic transformations) on this fibre.

From the irreducibility we get that every invariant defined in an algebraic way (for example, sectional Milnor numbers μ^*) is semi-continuous along the ND-topological strata.

By construction, to every ND-topological type $\mathbb{D}$ an embedded topological type $\mathbb{S}$ is associated with the inclusion of the (closures of the) strata: $\Sigma_{\mathbb{D}}\subset\Sigma_{\mathbb{S}}$. A natural question is: When do the two types of strata coincide?

REMARK 2.12. The simplest example of non-coincidence (or non-uniqueness of ND-topological type for a given embedded topological) is just the case of curves $z_1^p + z_1 x_2^p + z_2^d$, $d \geqslant p+2$.

Therefore, to get the equality of strata $\Sigma_{\mathbb{S}} = \Sigma_{\mathbb{D}}$ one should demand, at least, the minimality of Newton diagram. In more detail, introduce the partial order on the set of Newton diagrams with the same topological type by inclusion. Call a diagram *minimal* if it is not bigger than some other diagram corresponding to the same topological type. In general, it is not clear whether the minimal Newton diagram is unique (up to a permutation of axes). Even if it is unique, it is still unclear whether the two strata coincide.

A constructive way to compare the strata is by codimension.

DEFINITION 2.13. • The codimension of a local embedded topological type $\mathbb{S}$ is the codimension of the topological stratum $\Sigma_{\mathbb{S}}$ in the space of its semi-universal deformation. It equals: $\tau - \sharp(moduli)$ (here τ is the Tjurina number).
• The codimension of an ND-topological type D is the codimension of the stratum $\Sigma_{\mathbb{D}}$ in the parameter space of the hypersurfaces $\mathbb{P}_f^{N_d}$.

Restrict the topological type to be Newton-non-degenerateand assume that the topological stratum is an irreducible algebraic variety. From algebraicity and irreducibility we get the following corollary.

COROLLARY 2.14. *For a Newton-non-degeneratetype $\mathbb{S}$, if the topological stratum $\Sigma_{\mathbb{S}}$ is algebraic, irreducible and the codimension of the topological type $\mathbb{S}$ equals to that of the* ND-*topological type $\mathbb{D}(\mathbb{S})$ then the* (*closure of the*) *strata coincide: $\Sigma_{\mathbb{S}} = \Sigma_{\mathbb{D}}$. Therefore, in this case the generic representative of the topological type $\mathbb{S}$ is generalized Newton-non-degenerate.*

Another observation is the following.

PROPOSITION 2.15. *Let f_α be the family as in Proposition* 2.8. *Assume also that they all have the same Newton diagram. Then the corresponding topological and* ND-*topological strata coincide.*

Using this criterion we get the coincidence of the strata for all singularities with number of moduli at most 2 (in this case the corresponding families are classified in [**2**]).

2.3.2. *On the vector spaces associated to the Newton diagram.* The Newton diagram $\mathbb{D}$ defines a stratification of the tangent space at the origin $T_0\mathbb{C}^n$ as follows. Let $\{f=0\}$ be the generic (generalized Newton-non-degenerate) hypersurface with diagram $\mathbb{D}$, let l be a line through the origin, and let the degree of their intersection be $k_l := \deg(k \cap \{f=0\})$. (We assume the hypersurface to be generic; in particular, it does not contain lines.) The tangent space is stratified: $T_0\mathbb{C}^n = \coprod_k \mathcal{U}_k$ according to the intersection degree $\mathcal{U}_k := \{l \in T_0\mathbb{C}^n | k_l = k\}$. Take the topological closures of $\mathcal{U}_k$ and consider the irreducible components $\bar{\mathcal{U}}_k = \bigcup_j \widetilde{V}_{k,j}$. Call the collection of these components $\widetilde{\mathcal{V}}$.

EXAMPLE 2.16. For an SQH hypersurface, restrict to the principal (quasi-homogeneous) part. Then the so-obtained varieties $\widetilde{V}_{kj}$ are just the vector spaces of a flag of $T_0\mathbb{C}^n$. The flag may be incomplete if some weight of (quasi-homogeneous) variables coincide.

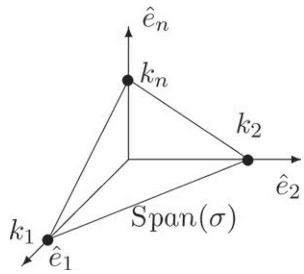

FIGURE 2.

PROPOSITION 2.17. *The varieties $\widetilde{V}_{kj}$ are vector subspaces of $\mathbb{C}^n$ of the form*

$$\{z_{i_1} = \ldots = z_{i_j} = 0\}$$

(*that is, some coordinate planes*). *They satisfy the property that if $\widetilde{V}_{kj} \subsetneqq \widetilde{V}_{k'j'}$ then $k > k'$.*

Proof. Let $z_i = \alpha_i t$ be the parametrization of a line through the origin. Consider its intersection with the hypersurface $\{f = \sum a_{\mathbf{I}} \mathbf{z}^{\mathbf{I}} = 0\}$, with Newton diagram $\mathbb{D}$. Hence, we study the function restricted to the line $\sum a_{\mathbf{I}} (\alpha \mathbf{t})^{\mathbf{I}}$. If the line is contained in $\mathcal{U}_k$ then $\sum_{I_1+\ldots+I_n<k} a_{\mathbf{I}} (\alpha \mathbf{t})^{\mathbf{I}} = 0$. This equation must be satisfied for arbitrary (generic) coefficients of f and for all small values of t. Thus it causes the system of monomial equations of the form $\alpha^{\mathbf{I}} = 0$. Therefore, the irreducible components of $\bar{\mathcal{U}}_k$ are coordinate planes (which do not include one another). $\square$

We need a more-refined stratification. Every top-dimensional face σ of the Newton diagram defines a flag of vector spaces as follows (Figure 2). Let k_i denote the non-zero coordinate of the intersection points of the $(n-1)$ plane Span(σ) with the coordinate axes (so k_i are not necessarily integers). Apply permutation $\sigma \in S^n \subset \mathrm{GL}(n)$ on the coordinate axes to arrange: $k_n \leqslant \ldots \leqslant k_1$. Define vector spaces $\{\tilde{V}_i\}$ inductively:

$$\tilde{V}_n := \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_n), \quad \tilde{V}_{i-1} := \begin{cases} \tilde{V}_i, & \text{if } k_{i-1} = k_i, \\ \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{i-1}) & \text{if } k_{i-1} > k_i. \end{cases} \tag{21}$$

Take now the inverse permutation of axes ($\sigma^{-1} \in S^n \subset \mathrm{GL}(n)$) that restores the initial values of all k_i. Define $V_i := \sigma^{-1}(\tilde{V}_i)$.

DEFINITION 2.18. The sequence of vector spaces $\mathbb{C}^n = V_n \supseteq \ldots \supseteq V_1 \supset \{0\}$ is called the flag of the face σ.

For a general (not SQH) singularity the Newton diagram consists of several top-dimensional faces, each of them defines the corresponding flag. Now combine the flags together into the collection of vector spaces $\{V_\alpha\}_{\alpha \in \mathcal{V}}$. (The coinciding spaces are identified.) We call $\mathcal{V}$ *the collection of vector spaces associated to the Newton diagram.*

EXAMPLE 2.19. • For an SQH hypersurface the collection is just a flag (as in the previous example).

• Consider the hypersurface $z_1^{p_1} + z_2^{p_2} + z_3^{p_3} + z_1 z_2 z_3 + \sum_{i \geqslant 4} z_i^2$ with $1/p_1 + 1/p_2 + 1/p_3 < 1$, $p_1 \lneqq p_2 \lneqq p_3$ and $p_i > 3$.

The top-dimensional faces are $\mathrm{Span}(z_1^{p_1}, z_2^{p_2}, z_1z_2z_3, z_4^2, \dots, z_n^2)$ and similarly for $(2,3)$ and $(1,3)$.
The flag of the top-dimensional face is

$$0 \subset \mathrm{Span}(\hat{e}_2) \subset \mathrm{Span}(\hat{e}_1, \hat{e}_2) \subset \mathrm{Span}(\hat{e}_1, \hat{e}_2, \hat{e}_3) \subset \mathbb{C}^n, \tag{22}$$

similarly for $(2,3)$ and $(1,3)$. The collection of vector spaces is

$$\{\mathrm{Span}(\hat{e}_i)\}_{i=1,2,3}, \quad \{\mathrm{Span}(\hat{e}_i\hat{e}_j)\}_{\substack{i\neq j\\ i,j\leqslant 3}}, \quad \mathrm{Span}(\hat{e}_1\hat{e}_2\hat{e}_3), \quad \mathbb{C}^n. \tag{23}$$

Remark 2.20. The latter stratification is finer than that by the degree of the intersection. As an example, for the hypersurface $z_1^{2p} + z_2^{2p} + z_3^{2p} + z_1^{q-1}z_2^{q-1}z_3^{q-1}(z_1^3 + z_2^3 + z_3^3) + z_1^{p-1}z_2^{p-1}$ with $2p - 2 > 3q$ and $q > 1$ we have $\widetilde{\mathcal{V}} = \bigcup_i \{z_i = 0\}$ while $\mathcal{V} = \bigcup_i \{z_i = 0\} \cup \{z_2 = 0 = z_3\} \cup \{z_3 = 0 = z_1\}$. Therefore in the following we work with the collection $\mathcal{V}$.

As we consider the hypersurfaces of arbitrary dimensions, it is important to check how the collection $\mathcal{V}$ varies with n. More precisely, suppose that f_1 (with $\mathrm{mult}(f_1) > 2$) is stably equivalent to f_2, that is, $f_2(z_1, \dots, z_{n+k}) = f_1(z_1, \dots, z_n) + \sum_{i>n} z_i^2$. The relation between $\mathcal{V}_{f_1}$ and $\mathcal{V}_{f_2}$ is described by the following simple proposition.

Proposition 2.21. • *Let $\sigma_1, \dots, \sigma_r$ be the top-dimensional faces of Γ_{f_1}. Then the top-dimensional faces of Γ_{f_2} are $\{\mathrm{conv}(\sigma_i, \hat{e}_{n+1}., \hat{e}_{n+r})\}_{i=1,\dots,r}$.*

• *Let $\{V_\alpha\}_{\alpha\in\mathcal{V}_{f_1}}$ be the collection of vector spaces associated to Γ_{f_1}. Then the collection $\mathcal{V}_{f_2}$ is obtained as $\{V_\alpha\}_{\alpha\in\mathcal{V}_{f_1}} \cup \{\mathrm{Span}(V_\alpha, \hat{e}_{n+1}., \hat{e}_{n+r})\}_{\alpha\in\mathcal{V}_{f_1}}$.*

Remark 2.22. All the vector spaces above are defined in local coordinates. For enumerative purposes we need their counterparts in homogeneous coordinates. For this, embed every $V \subset \mathbb{C}^n$ into $\mathbb{C}^{n+1}$, by $\mathbb{C}^n \ni (z_1, \dots, z_n) \overset{i}{\mapsto} (0, z_1, \dots, z_n) \in \mathbb{C}^{n+1}$ and define

$$\mathbb{C}^{n+1} \supset \mathbb{V} := \mathrm{Span}(i(V), \hat{e}_0). \tag{24}$$

The corresponding flag $\{\mathbb{V}_i\}$ (or the collection of vector spaces) will be of key importance for writing down the covariant defining equations of the strata.

2.3.3. *Linear singularities.* Fix a Newton diagram and the corresponding ND-topological type $\mathbb{D}$. Consider a (hypersurface) representative of type $\mathbb{D}$. In general, to bring it to the fixed Newton diagram a locally analytic transformation is applied. Split it into the following steps (in local coordinates).

• Move the singular point to the origin. Rotate around the origin to fix the required tangent cone.

• Make the (purely) quadratic transformation $\vec{z} \to \vec{z} + \vec{\Omega}^{(2)}$ (here $\vec{\Omega}^{(2)}$ is an n-tuple of homogeneous quadratic forms) to remove some monomials.

• Make the (purely) cubic transformation ....

For some singularity classes the required Newton diagram is achieved by just the first step (linear transformations). As in the case of curves [15, Section 3], such singularities are much simpler for enumeration purposes. As is shown later, these strata can be lifted to varieties, defined by equations linear in function or its derivatives, and therefore are easy to work with.

Definition 2.23. • For a given ND-topological type $\mathbb{D}$, a hypersurface singularity (with this singularity type) is called *linear*, if it can be brought to the required Newton diagram by linear transformations only.

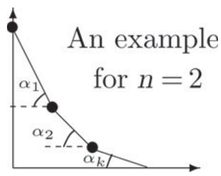

FIGURE 3.

• An ND-topological type/stratum is called *linear* if its generic representative is linear. Otherwise the type/stratum is called *non-linear*.

The simplest linear singularity is the ordinary multiple point (here the required diagram is achieved just by translation of the singular point to the origin).

There is an easy characterization of linear singularities via their Newton diagrams.

LEMMA 2.24. *The Newton-non-degenerate singularity type is linear if and only if all the angles between any face of the Newton diagram and the coordinate hyperplanes have the slope* $\frac{1}{2} \leqslant \mathrm{tg}(\alpha) \leqslant 2$.

Proof. $(\Leftarrow)$: Suppose that all the slopes are bounded as above and a hypersurface germ has been brought to the given Newton diagram by a chain of locally analytic transformations (Figure 3). Start undoing these transformations to achieve the initial germ. Immediate check shows that any non-linear analytic transformation (without linear part) has no effect on the points under the Newton diagram. Correspondingly the monomials of the initial polynomials that lie under the Newton diagram are restored by linear transformation only. However, it means that the germ could be brought to the Newton diagram by linear transformations only.

$(\Rightarrow)$: Suppose at least one of the angles (of the diagram) does not satisfy the condition $\frac{1}{2} \leqslant \mathrm{tg}(\alpha) \leqslant 2$. Then there exists a quadratic shift of coordinates that changes the Newton diagram. Of course, such shift cannot be undone by linear transformations. □

REMARK 2.25. As follows from the lemma, every Newton-non-degenerate singularity of multiplicity p and order of determinacy k, with $k \leqslant 2p$, is linear.

In the case of plane curves there is only one angle for every segment of the diagram, correspondingly the condition on the singularity to be linear is not too restrictive. In the low modality cases the curve singularities brought to a Newton diagram by projective (linear) transformation are (all the notation are from [2]):

• simple singularities (no moduli): $A_{k\leqslant 3}$, $D_{k\leqslant 6}$, $E_{k\leqslant 8}$;
• uni-modal singularities: $X_9(=X_{1,0})$, $J_{10}(=J_{2,0})$, $Z_{k\leqslant 13}$, $W_{k\leqslant 13}$;
• bimodal: $Z_{1,0}$, $W_{1,0}$, $W_{1,1}$, W_{17}, W_{18}.

In the case of hypersurfaces (of dimension at least 3) only a few singularities in each series can be linear. The low modality cases are:

• simple singularities (no moduli): $A_{k\leqslant 3}$, $D_{k\leqslant 5}$, E_6;
• uni-modal singularities: P_8, $X_9(=X_{1,0})$, Q_{10}, S_{11}, U_{12}, $T_{p,q,r}$ $\{p,q,r\} \leqslant 4$.

For surfaces, there are some additional uni-modal linear singularities: $Q_{11}, Q_{12}, S_{12}, Q_{2,0}$, $S_{1,p\leqslant 1}$, $U_{1,q\leqslant 2}$, Q_{17}, Q_{18}, S_{16}, S_{17}.

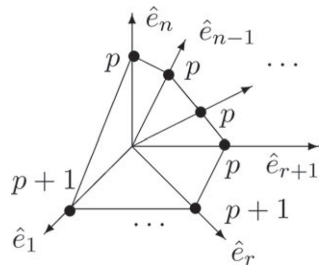

FIGURE 4.

We emphasize, that there are infinite linear singularities. More precisely: for every singularity type $\mathbb{D}_1$, there is a linear singularity $\mathbb{D}_2$, which is adjacent to $\mathbb{D}_1$ (for example, one could take as $\mathbb{D}_2$ an ordinary multiple point of sufficiently high multiplicity).

Even if the singularity is non-linear, one can consider the collection of singular hypersurfaces that can be brought to the given Newton diagram by *linear transformations* alone (or projective transformations in homogeneous coordinates). This defines a subvariety of the non-linear stratum, *the linear substratum*. Our method, of course, enables to calculate the cohomology classes of both the true strata and their linear sub-strata.

2.4. *Defining conditions of the singularities*

For Newton-non-degenerate singularities the defining conditions of a singular point are read from the Newton diagram. Consider the points under the Newton diagram. The corresponding monomials should be absent, that is, the corresponding derivatives should vanish. To define the stratum, one has to write these conditions in a covariant form. For linear singularities this can be done in an especially simple way, since one should achieve the covariance under the group of projective transformations (or linear in local coordinates). Every condition has a form $\{f^{(p)}_{i_1,\dots,i_p}=0\}$ and is transformed by $\mathbb{P}\mathrm{GL}(n+1)$ to $\{f^{(p)}(y_1,\dots,y_p)=0\}$, where $\{y_i\}$ are some points of $\mathbb{P}^n_{y_i}$ (regarded here as $(n+1)$-vectors).

First consider some simple examples.

EXAMPLE 2.26. • An ordinary point of multiplicity p. Here in local coordinates we have

$$f|_x=0,\partial_1 f|_x=\ldots=\partial_n f|_x=0,\ldots,\Big\{\partial_{i_1},\ldots,\partial_{i_{p-1}}f|_x=0\Big\}_{i_1\ldots i_{p-1}}.$$

Passing to the homogeneous coordinates (and using the Euler formula (14) and its consequences) we get the defining conditions in a covariant form $f|^{(p-1)}_x=0$.

• An ordinary point of co-rank r. Consider the singularity with the normal form $\sum_{i=1}^r z_i^{p+1}+\sum_{i=r+1}^n z_i^p$. (For $p=2$ these singularities are $A_2,D_4,P_8\ldots$). The defining conditions are read directly from the Newton diagram (Figure 4):

$$\begin{gathered} f|_x=0,\ \partial_1 f|_x=\ldots=\partial_n f|_x=0,\ \ \ldots,\ \Big\{\partial_{i_1},\ldots,\partial_{i_{p-1}}f|_x=0\Big\}_{i_1,\ldots,i_{p-1}}, \\ \Big\{\partial_{i_1},\ldots,\partial_{i_{p-1}}\partial_j f|_x=0\Big\}_{\substack{1\leqslant i_1,\ldots,i_{p-1}\leqslant n\\ j\leqslant r}}. ; \end{gathered}\tag{25}$$

To transform them to covariant form, introduce the flag of the Newton diagram (Figure 5) (as defined in Definition 2.18):

$$\mathbb{C}^n=V_n=\ldots=V_{r+1}\supset V_r=\mathrm{Span}(\hat{e}_1,\ldots,\hat{e}_r)=\ldots=V_1\supset\{0\}\tag{26}$$

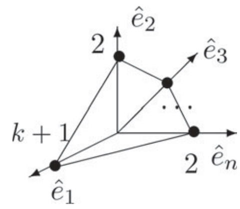

FIGURE 5.

and the corresponding flag $\{\mathbb{V}_i\}_i$ in $\mathbb{C}^{n+1}$. Then the conditions in homogeneous coordinates are

$$f|_x^{(p-1)} = 0, \quad f|_x^{(p)}(y) = 0 \ \forall y \in \mathbb{V}_r \subset \mathbb{C}^{n+1}. \tag{27}$$

- The A_k point: $z_1^{k+1} + \sum_{i=2}^n z_i^2$. Here the flag is $\mathbb{C}^n = V_n = \ldots = V_2 \supset V_1 = \mathrm{Span}(\hat{e}_1) \supset \{0\}$. The conditions are

$$f|_x = 0, \ \partial_1 f|_x = \ldots = \partial_n f|_x = 0, \ \left\{\partial_1^i \partial_* f|_x = 0\right\}_{0<i<(k+1)/2+1}, \ \left\{\partial_1^i f|_x = 0\right\}_{(k+1)/2+1 \leqslant i \leqslant k}. \tag{28}$$

- The D_k point: $z_1^{k-1} + z_2^2 z_1 + \sum_{i=3}^n z_i^2$. We assume for simplicity that k is even. The flag is $\mathbb{C}^n = V_n = \ldots = V_3 \supset V_2 = \mathrm{Span}(\hat{e}_1, \hat{e}_2) \supset V_1 = \mathrm{Span}(\hat{e}_1) \supset \{0\}$. The conditions are

$$f|_x = 0, \ \partial_* f|_x = 0, \ \left\{\partial_2^i \partial_1^{l-i-1} \partial_* f|_x = 0\right\}_{\substack{0 \leqslant i < (k+1-2l)/(k-4) \\ l < (k+1)/2}},$$
$$\left\{\partial_2^i \partial_1^{l-i} f|_x = 0\right\}_{\substack{i < 2(k-1-l)/(k-4). \\ l < k-1}} \tag{29}$$

We describe now the general procedure of formulating the covariant conditions for linear singularities. Recall that in § 2.3.2 the collection of vector spaces associated to a given Newton diagram was defined. Start from the SQH case, where the collection is just a flag in $\mathbb{C}^n$ (Definition 2.18).

2.4.1. *The case of semi-quasi-homogeneous linear singularities.* Let $\hat{e}_1, \ldots, \hat{e}_n$ be the coordinate axes of the lattice (by the same letters we also denote the corresponding unit vectors). Consider the points of intersection of the hyperplane Γ_f with the coordinates axes $\{k_i\}_i$. By renumbering the axes we can assume that $k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n$.

To write the conditions of the Newton diagram explicitly, consider the points of the lattice lying under the Newton diagram (Figure 6). As in the examples above, at each step we consider the points corresponding to partial derivatives of a given order.

Namely, for every $0 \leqslant r \leqslant k_1$ define an $(n-1)$-dimensional simplex

$$\Delta_r := \left\{ x = (m_1, \ldots, m_n) \mid m_i \geqslant 0, \ \sum_{i=1}^n m_i = r, \quad x \text{ lies strictly below } \Gamma_f \right\}. \tag{30}$$

Every integral point of Δ_r corresponds to a vanishing derivative (in local coordinates) $\partial_1^{m_1}, \ldots, \partial_n^{m_n} f \equiv f^{(r)}_{\underbrace{1,\ldots,1}_{m_1},\ldots,\underbrace{n,\ldots,n}_{m_n}} = 0$.

We need a more precise relation between the axes of the diagram and the vector spaces of the flag of the diagram.

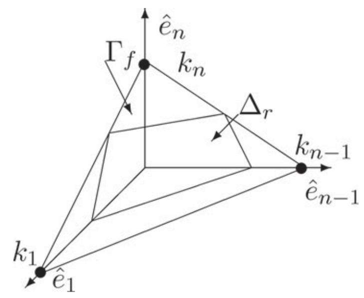

FIGURE 6.

DEFINITION 2.27. Let $\{V_i\}_i$ be the flag of the Newton diagram. For each axis $\hat{e}_i$ define the *associated vector space* as $\mathrm{V}(\hat{e}_i) := V_j$, such that $\hat{e}_i \in V_j$ and $\hat{e}_i \notin V_{j-1}$.

Recall also that each vector space $V_i \subset \mathbb{C}^n$ has its (homogeneous) version $\mathbb{V}_i \subset \mathbb{C}^{n+1}$.

The transition from local conditions (that arise from the given Newton diagram) to the conditions covariant under $\mathrm{PGL}(n+1)$ is done by the following.

LEMMA 2.28. *For an SQH singularity the local set of conditions* $\{\partial_1^{m_1}, \dots, \partial_n^{m_n} f = 0\}$ *for* $\{(m_1, \dots, m_n) \in \Delta_r\}$, *corresponding to the points under the Newton diagram, can be written in a covariant way as*

$$\forall y_i \in \mathbb{V}(\hat{e}_i) \quad \Big\{\forall (m_1, \dots, m_n) \in \Delta_r\Big\}_{r=0}^{k_1} \quad f^{(r)}(\underbrace{y_1, \dots, y_1}_{m_1}, \dots, \underbrace{y_n, \dots, y_n}_{m_n}) = 0. \tag{31}$$

Proof. $(\Rightarrow)$: We should check that the covariant conditions imply the local conditions. This is immediate (just take $y_i = \hat{e}_i$).
$(\Leftarrow)$: Without loss of generality, we can assume that $k_1 \geqslant k_2 \geqslant \dots \geqslant k_n$. Introduce the lexicographic order on the points of Δ_r (for a fixed r):

$$(m_1, \dots, m_n) < (\tilde{m}_1, \dots, \tilde{m}_n) \quad \text{if} \begin{cases} m_i = \tilde{m}_i : & \text{for } i < j, \\ m_j < \tilde{m}_j. \end{cases} \tag{32}$$

Now, if the point $(\tilde{m}_1, \dots, \tilde{m}_n)$ lies under the Newton diagram and $(m_1, \dots, m_n) \leqslant (\tilde{m}_1, \dots, \tilde{m}_n)$, then the point $(m_1, \dots, m_n)$ also lies under the Newton diagram (because $\sum m_i/k_i \leqslant \sum \tilde{m}_i/k_i < 1$).

Now, expanding $y_i = \sum \alpha_i \hat{e}_i$ and substituting into $f^{(r)}(\underbrace{y_1, \dots, y_1}_{m_1}, \dots, \underbrace{y_n, \dots, y_n}_{m_n})$ we get the sum of terms, corresponding to the points $(\tilde{m}_1, \dots, \tilde{m}_n)$, satisfying: $(m_1, \dots, m_n) \geqslant (\tilde{m}_1, \dots, \tilde{m}_n)$. Therefore, if $(m_1, \dots, m_n) \in \Delta_r$ then $f^{(r)}(\underbrace{y_1, \dots, y_1}_{m_1}, \dots, \underbrace{y_n, \dots, y_n}_{m_n}) = 0$. $\square$

EXAMPLE 2.29. Below we write the conditions in several cases considered in Example 2.26.

- Linear singularities of A_k series are $A_{k\leqslant 3}$. In the A_2-case, the covariant conditions are $f|_x^{(1)} = 0$, $\forall y \in \mathbb{V}_1 : f|_x^{(2)}(y) = 0$. In the tacnodal case ($A_3$) the additional condition is $f|_x^{(3)}(y, y, y) = 0$.

- Linear singularities of D_k series are $D_{k\leqslant 5}$. The D_4-case was considered in Example 2.26. Here we consider D_5. The covariant conditions are

$$f|_x^{(1)} = 0, \qquad \forall\, y_1 \in \mathbb{V}_1, \quad y_2 \in \mathbb{V}_2 : f|_X^{(2)}(y_1) = 0 = f|_x^{(2)}(y_2),$$
$$f|_x^{(3)}(y_1, y_1, y_1) = 0 = f|_x^{(3)}(y_1, y_1, y_2).$$

- The singularities of ND-topological type with the representative:

$$\sum_{i=1}^{r_1} z_i^{p+2} + \sum_{i=r_1+1}^{r_2} z_i^{p+1} + \sum_{i=r_2+1}^{n} z_i^p.$$

For $p = 2$ this class contains $A_{k\leqslant 3}$, D_4, E_6 ($r_1 + 1 = r_2 = 2$), P_8 ($r_1 = 0,\ r_2 = 3$), X_9 ($r_1 = r_2 = 2$), U_{12} ($r_1 + 2 = r_2 = 3$) etc. The flag is

$$\begin{aligned}\mathbb{C}^n &= V_n = \ldots = V_{r_2+1} \supset V_{r_2} = \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{r_2})\\ &= \ldots = V_{r_1+1} \supset V_{r_1} = \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{r_1}) = \ldots = V_1 \supset \{0\}.\end{aligned}$$

The point $(m_1, \ldots, m_n) \in \Delta_r$ lies under the Newton diagram provided that

$$\sum_i m_i = r, \quad \sum_{i=1}^{r_1} \frac{m_i}{p+2} + \sum_{i=r_1+1}^{r_2} \frac{m_i}{p+1} + \sum_{i=r_2+1}^{n} \frac{m_i}{p} < 1. \tag{33}$$

The covariant equations are

$$f|_x^{(p-1)} = 0, \quad \forall\, y_1 \in V_{r_1},\ \forall\, y_2 \in V_{r_2} :\ f|_x^{(p)}(y_2) = 0, \quad f^{(p+1)}|_x(\underbrace{y_1, \ldots, y_1}_{k}, \underbrace{y_2, \ldots, y_2}_{l}) = 0,$$
$$\text{for } \frac{2k}{p+2} + \frac{l}{p+1} > 1. \tag{34}$$

- The singularities of ND-topological type with the representative

$$\sum_{i=1}^{r_1} z_i^4 + z_{r_1} z_{r_1+1}^2 + \sum_{i=r_1+2}^{r_2} z_i^3 + \sum_{i=r_2+1}^{n} z_i^2.$$

This class contains D_4, P_8, Q_{10}, $V_{1,0}$ etc. The flag is

$$\begin{aligned}\mathbb{C}^n &= V_n = \ldots = V_{r_2+1} \supset V_{r_2} = \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{r_2}) = \ldots = V_{r_1+2} \supset V_{r_1+1} = \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{r_1+1})\\ &\supset V_{r_1} = \mathrm{Span}(\hat{e}_1, \ldots, \hat{e}_{r_1}) = \ldots = V_1 \supset \{0\}.\end{aligned}$$

The point $(m_1, \ldots, m_n) \in \Delta_r$ lies under the Newton diagram provided that

$$\sum_i m_i = r, \quad \sum_{i=1}^{r_1} \frac{m_i}{4} + \frac{3m_{r_1+1}}{8} + \sum_{i=r_1+2}^{r_2} \frac{m_i}{3} + \sum_{i=r_2+1}^{n} \frac{m_i}{2} < 1. \tag{35}$$

The covariant equations are

$$f|_x^{(1)} = 0, \quad \forall\, y_1 \in V_{r_1},\ \forall\, y_2 \in V_{r_2},\ \forall\, y_3 \in V_{r_1+1} :\ f|_x^{(2)}(y_2) = 0, \quad f|_x^{(3)}(y_1, y_2, y_3) = 0 \tag{36}$$

2.4.2. *The general linear case.* In the linear non-SQH case we have the collection of vector spaces for the top-dimensional faces of the Newton diagram. Therefore, perform the procedure for each top-dimensional face separately (that is, for the hyperplane that contains it). Thus Lemma 2.28 is translated verbatim to a collection of lemmas, for each face. Now combine all the conditions (the coinciding conditions should be identified).

Thus this generalizes the method of obtaining defining conditions to the case of non-quasi-homogeneous singularities.

EXAMPLE 2.30. The singularity of the type T_{pqr} with the normal form

$$z_1^p + z_2^q + z_3^k + z_1z_2z_3 + \sum_{i=4}^{n} z_i^2 \quad (p \geqslant q \geqslant k).$$

It is linear for $p, q, r \leqslant 5$. Here the Newton diagram consists of the three planes

$$L_{pq} := \left\{ \frac{m_1}{p} + \frac{m_2}{q} + m_3\left(1 - \frac{1}{p} - \frac{1}{q}\right) + \sum_{i\geqslant 4} \frac{m_i}{2} = 1 \right\},$$

$$L_{pk} := \left\{ \frac{m_1}{p} + m_2\left(1 - \frac{1}{p} - \frac{1}{k}\right) + \frac{m_3}{k} + \sum_{i\geqslant 4} \frac{m_i}{2} = 1 \right\},$$

$$L_{qk} := \left\{ m_1\left(1 - \frac{1}{k} - \frac{1}{q}\right) + \frac{m_2}{q} + \frac{m_3}{k} + \sum_{i\geqslant 4} \frac{m_i}{2} = 1 \right\}. \tag{37}$$

Therefore, one considers the three flags: $(V_{i,pq}, V_{i,qk}, V_{i,pk})$ and the three sets of polytopes: $(\Delta_{r,pq}, \Delta_{r,qk}, \Delta_{r,pk})$. We get a set of local conditions, which is transformed to the set of covariant conditions, by the prescription of Lemma 2.28. We omit the calculations.

2.4.3. *On the transversality of covariant conditions.* We would like to address here the issue of transversality. All the initial conditions (corresponding to the points under the Newton diagram) are just linear equations on the (derivatives of the) function f; the transversality is equivalent to the linear independence. And this is obvious for conditions that are read from the Newton diagram. The conditions are made covariant by introducing vectors from the collection of vector spaces of the diagram. As the linear independence is preserved under (invertible) linear transformations we obtain that the covariant conditions remain transversal as far as these vectors are mutually generic. More precisely, we have the following proposition.

PROPOSITION 2.31. *The set of conditions* $\{f^{(p_j)}(y_{i_{1,j}}, \dots, y_{i_{p_j},j}) = 0\}_{i,j}$ (*that correspond to the points under the Newton diagram*) *is transversal if by* $\mathrm{PGL}(n+1)$ *the points* $\{y_{i,j}\}_i$ *can be brought to the coordinate axes.*

In particular, one demands that no subset of k variables lies in a $(k-2)$ hyperplane (for any k).

2.4.4. *The non-linear singularities and defining conditions.* Singularity types/strata for which a Newton diagram cannot be achieved by projective (linear) transformations are non-linear. Most types/strata are non-linear.

EXAMPLE 2.32. Consider a hypersurface with the A_4 point. Try to bring it to the Newton diagram of A_4 (that is, that of $z_1^4 + z_2^2 + \ldots + z^2 + n$). The best we can do by projective transformations is to bring it to a form of A_3:

$$f = \sum_{i=2}^{n} \alpha_i z_i^2 + z_1^2 \sum_{i=2}^{n} \beta_i z_i + \gamma z_1^4 + \ldots. \tag{38}$$

To achieve the Newton diagram of A_4 we must do the non-linear shift: $z_i \to z_i + \delta_i z_1^2$ (to kill the monomials $z_2^4, z_1^2 z_i$). Elimination of the parameters of the transformation gives the non-linear equation $\gamma = \sum \beta_i^2/(4\alpha_i)$.

In general, to obtain the locally defining equations of a non-linear singularity one considers the locally analytic transformation of coordinates

$$z_i \to z_i + \sum_{k_i=1}^{\infty} a_{i,k} z_k \quad \text{for } i = 1, \dots, n. \tag{39}$$

Then, demanding that some derivatives (corresponding to the points under the Newton diagram) vanish, one eliminates the parameters of the transformation $\{a_{ik}\}_{i,k}$ to obtain the system of (non-linear) equations. This forms the locally defining ideal of the stratum. The ideal can be quite complicated (since the stratum can be not a locally complete intersection). It is important to trace the whole ideal, that is, all of its generators. Of course the whole procedure is done by computer and is time-consuming.

Note that, as we enumerate the non-linear strata by the *simplifying degenerations* (defined in § 1.4.3), we do not need to convert the defining conditions to a covariant form.

Among the generators of the ideal there are (a finite number of) non-linear expressions in the coefficients of f. The goal of degeneration is to turn these non-linear equations into monomial equations. In this case the ideal would correspond to a collection of linear strata (with some multiplicities).

For enumeration of a non-linear singularity we will be interested in the set of linear singularities to which the given singularity is adjacent.

Definition 2.33. The linear singularity type L is assigned to a given singularity type N if N is adjacent to L (that is, $\Sigma_N \subset \bar{\Sigma_L}$) and the adjacency is minimal. Namely, there is no other linear type L' such that $\Sigma_N \subset \bar{\Sigma'_L} \subsetneqq \bar{\Sigma_L}$.

The assigned type is non-unique in general. As singularities always appear in series, which start from linear singularities, we have a natural way to fix an assigned linear singularity. In the simplest cases the assigned linear singularities are A_3 for $A_{k\geqslant 4}$, D_5 for $D_{k\geqslant 6}$, E_6 for $E_{k>6}$ etc.

2.4.5. *On the invertibility of degenerations.* At each step of the degeneration procedure we get an equation in the cohomology ring of $\mathrm{Aux} \times \mathbb{P}_f^{N_d}$:

$$[\widetilde{\Sigma}_1][\text{degenerating divisor}] = [\widetilde{\Sigma}_2]. \tag{40}$$

An important issue is to check that the degeneration is 'invertible', that is, this equation fixes the cohomology class of the original stratum uniquely. The generators of the cohomology ring are nilpotent (for example, $X^{n+1} = 0 = F^{D+1}$). Thus the solution for $[\widetilde{\Sigma}_1]$ is unique, provided that $\dim(\Sigma_1) + 1 \leqslant D$ and the class of degenerating divisor depends essentially on F. The first condition is always satisfied, while the second means that the degeneration must involve (in an essential way) the function f (or its derivatives). In particular, conditions involving the parameters of the auxiliary space alone (for example, coincidence of points) are non-invertible and will not be used for the degenerations.

3. *Enumeration*

3.1. *Main issues of the method*

3.1.1. *Liftings (desingularizations).* We lift the strata to a bigger ambient space to define them by explicit equations.

The simplest example (minimal lifting), consisting of pairs (the function, the singular point), was considered in § 1.4.1. This lifting is sufficient for ordinary multiple points only. In general one should lift further to the space $\mathrm{Aux} \times \mathbb{P}^D_f$. We will consider the objects: (the function, the singular point, hyperplanes in the tangent cone, some special linear subspaces of the hyperplanes).

In § 2.3.2 we constructed for a given Newton diagram $\mathbb{D}$ the collection of vector spaces (in homogeneous coordinates) $\{\mathbb{V}_\alpha\} \in \mathcal{V}(\mathbb{D})$. Taking their projectivization we arrive at a collection $\{\mathbb{P}^{n_\alpha}_{y_\alpha}\}_{\alpha\in\mathcal{V}}$ (with the conditions that some are subspaces of others). This defines the lifting.

Definition 3.1. For linear ND-topological type the lifting is defined by

$$\widetilde{\Sigma}(x, \{y_\alpha\}_i) := \overline{\left\{\begin{matrix}(x, \{y_\alpha\}, f)\\ \text{generic}\end{matrix}\middle| \begin{matrix}\left\{y_\alpha \in \mathbb{P}^{n_\alpha}_{y_\alpha}\right\}_\alpha \in \mathcal{V}(\mathbb{D}), \quad f \text{ has the prescribed}\\ \text{singularity with } \mathbb{D} \text{ as its Newton diagram}\end{matrix}\right\}} \subset \mathbb{P}^n_x \times \prod_\alpha \mathbb{P}^{n_\alpha}_{y_\alpha} \times \mathbb{P}^{N_d}_f. \tag{41}$$

Example 3.2. In the SQH case the collection of vector spaces is just the flag, therefore,

$$\widetilde{\Sigma}(x, \{y_i\}_i) := \overline{\left\{\begin{matrix}(x, \{y_i\}_i, f)\\ \text{generic}\end{matrix}\middle| \begin{matrix}\left\{\mathbb{V}_i = \mathrm{Span}(y_1, \ldots, y_{\dim(\mathbb{V}_i)})\right\}_i\\ f \text{ has the prescribed singularity with}\\ \mathbb{V}_n \supset \ldots \supset \mathbb{V}_1 \text{ as the flag of the Newton diagram}\end{matrix}\right\}} \subset \mathbb{P}^n_x \times \prod_i \mathbb{P}^n_{y_i} \times \mathbb{P}^{N_d}_f. \tag{42}$$

We emphasize that the tuples $(x, \{y_i\}_i)$ are always taken to be generic (in particular, no k of the points Span a $(k-2)$-plane). For the sake of exposition we will often omit this, but it is always meant.

Remark 3.3. As follows from Lemma 2.28 the so-defined stratum $\widetilde{\Sigma}$ is indeed a lifting. That is, every point of $\widetilde{\Sigma}$ projects to a point of Σ. This definition reduces the enumerative problem for linear singularities to the intersection theory. The intersection theory step has still many complications; related questions are considered in Appendix B.

Example 3.4. For many cases the defining covariant conditions were given in Example 2.29. Therefore, we immediately have the following.

- For multiple point of co-rank r (with normal form $f = \sum_{i=r+1}^n x_i^p$ + higher-order terms),

$$\widetilde{\Sigma}(x, y_1, \ldots, y_r) = \overline{\left\{\begin{matrix}(x, y_1, \ldots, y_r)\\ \text{generic}\end{matrix} \middle| f|^{(p-1)}_x = 0 = \left(f|^{(p)}_x(y_i)\right)^r_{i=1}\right\}} \subset \mathbb{P}^n_x \times \prod_{i=1}^r \mathbb{P}^n_{y_i} \times \mathbb{P}^{N_d}_f. \tag{43}$$

- For singularities with the normal form $\sum_{i=1}^{r_1} z_i^{p+2} + \sum_{i=r_1+1}^{r_2} z_i^{p+1} + \sum_{i=r_2+1}^{n} z_i^p$,

$$\widetilde{\Sigma}(x, y_1, \ldots, y_{r_2}) = \overline{\left\{(x, y_1, \ldots, y_{r_2}) \middle| \begin{matrix} f|^{(p-1)}_x = 0 \ f|^{(p)}_x(y_1) = 0 \text{ for } i \leqslant r_2\\ f^{(p+1)}|_x(y_{i_1}, \ldots, y_{i_k}, y_{j_1}, \ldots, y_{j_l}) = 0\\ \text{for } i_1 \ldots i_k \leqslant r_1, j_1 \ldots j_l \leqslant r_2 \frac{2k}{p+2} + \frac{l}{p+1} > 1\end{matrix}\right\}} \subset \mathbb{P}^n_x \times \prod_{i=1}^{r_2} \mathbb{P}^n_{y_i} \times \mathbb{P}^{N_d}_f. \tag{44}$$

- For singularities with the normal form $\sum_{i=1}^{r_1} z_i^4 + z_{r_1} z_{r_1+1}^2 + \sum_{i=r_1+2}^{r_2} z_i^3 + \sum_{i=r_2+1}^{n} z_i^2$,

$$\widetilde{\Sigma}(x, y_1, \dots, y_{r_2}) = \overline{\left\{ (x, y_1, \dots, y_{r_2}) \left| \begin{array}{l} f|_x^{(1)} = 0 \ f|_x^{(2)}(y_i) = 0 \text{ for } i \leqslant r_2 \\ f|_x^{(3)}(y_i, y_j, y_k) = 0 \text{ for } i \leqslant r_1, j \leqslant r_1 + 1, \ k \leqslant r_2 \end{array} \right. \right\}} \subset \mathbb{P}_x^n \times \prod_{i=1}^{r_2} \mathbb{P}_{y_i}^n \times \mathbb{P}_f^{N_d}. \tag{45}$$

3.1.1.1. *Lifted varieties as fibrations over the auxiliary space.* Note that for a given singular germ $(f = 0)$ only the flag $\mathbb{V}_n \supset \ldots \supset \mathbb{V}_1$ and the assigned polytopes are defined uniquely. The points $\{y_i\}_i$ of the auxiliary space (in Definition 3.1) can vary freely as far as the flag and the polytopes are preserved. Hence we obtain the following important result.

PROPOSITION 3.5. *The projection $\widetilde{\Sigma} \xrightarrow{\pi} \Sigma$ is a fibration with generic fibre the multi-projective space.*

In the example of multiple point of co-rank r the generic fibre is $\mathbb{P}_{y_1}^r \times \ldots \times \mathbb{P}_{y_r}^r$.

Note that the non-zero dimensionality of the fibres already restricts the possible cohomology class of the lifted variety. In fact, let $(X, Y_1, \dots, Y_r, F)$ be the generators of the cohomology rings of $\mathbb{P}_x^n, \mathbb{P}_{y_i}^n, \mathbb{P}_f^{N_d}$ (as defined in § 2.1). The cohomology class of $\widetilde{\Sigma}$ is a polynomial in $(X, Y_1, \dots, Y_r, F)$.

LEMMA 3.6 (The first consistency condition). *Let the fibration with the generic fibre $\mathbb{P}_{y_k}^r \subset \mathbb{P}_{y_k}^n$ be given as in the following diagram; then the variable Y_k that appears in monomials of the polynomial $[\widetilde{\Sigma}(x, y_1, \dots, y_k)]$ has powers not greater than $(n - r)$:*

$$\begin{array}{ccc} \widetilde{\Sigma}(x, y_1, \dots, y_k) \subset & \mathrm{Aux} \times \mathbb{P}_{y_k}^n & \times \mathbb{P}^D \\ \downarrow & \downarrow & \\ \widetilde{\Sigma}(x, y_1, \dots, y_{k-1}) \subset & \mathrm{Aux} & \times \mathbb{P}^D \end{array}$$

Proof. We give the proof in the general case. Consider projective fibration $E \to B$ with the generic fibre $\mathbb{P}_y^r$:

$$\begin{array}{ccc} E & \hookrightarrow & A \times \mathbb{P}_y^n \\ \downarrow & & \downarrow \\ B & \hookrightarrow & A \end{array}$$

Assume that the generic fibre is linearly embedded into $\mathbb{P}_y^n$. Write the cohomology class $[E] \in H^*(A \times \mathbb{P}_y^n)$ as

$$[E] = \sum_i Y^i Q_{\dim(A)+n-\dim(E)-i}. \tag{46}$$

Here Y is a generator of $H^*(\mathbb{P}_y^n)$, while $Q_{\dim(A)+n-\dim(E)-i} \in H^{2(\dim(A)+n-\dim(E)-i)}(A)$. We want to show that terms appearing in the above sum have $i \leqslant n - r$. To see this, multiply $[E]$ by $Y^{n-i}\tilde{Q}_{\dim(E)-n+i}$ (for some arbitrary $\tilde{Q}$). By the duality between homology and cohomology this product corresponds to the intersection of E with (generic) cycle of the form $\mathbb{L}_{\mathbb{P}^n}^{(i)} \times C_A^{\dim(A)-\dim(E)+n-i}$ (here $\mathbb{L}_{\mathbb{P}^n}^{(i)}$ is a linear i-dimensional subspace of $\mathbb{P}^n$, and C a cycle in A). Then (by the dimensional consideration in A) the intersection is empty unless

$$\dim(A) - \dim(E) + n - i + \dim(B) \geqslant \dim(A), \tag{47}$$

which amounts to $n - i \geqslant r$. Therefore we have $Q_{\dim(A)+n-\dim(E)-i}\tilde{Q}_{\dim(E)+i-n} = 0$ for any $\tilde{Q}$, and so $Q_{\dim(A)+n-\dim(E)-i} = 0$ for $i > n - r$. □

Another restriction comes from the symmetry of Definition 3.1 with respect to y_i. Novo $\widetilde{\Sigma}$ is invariant with respect to a subgroup G of group of permutations of y_i that preserves the flag structure. The group has the orbits

$$(y_1, \dots, y_{\dim(V_{n-1})})\ \ (y_{\dim(V_{n-1})+1}, \dots, y_{\dim(V_{n-2})})\ \ \dots\ \ (y_{\dim(V_2)+1}, \dots, y_{\dim(V_1)}). \tag{48}$$

Therefore we have the following corollary.

Corollary 3.7. (The second consistency condition). *The cohomology class of $\widetilde{\Sigma}(x, y_1, \dots, y_r)$ is invariant under the action of the group G (that is, it is a polynomial symmetric with respect to relevant subsets of Y_i).*

Example 3.8. In particular, for a multiple point of co-rank r, equation (43), we have the following statement:

The cohomology class of the lifted stratum of multiple point of co-rank r, expressed in terms of $(Y_1, \dots, Y_r, X)$ and $Q = (d-p)X + F$, is symmetric with respect to $(Y_1, \dots, Y_r, X)$ and the maximal powers of variables $(Y_1, \dots, Y_r, X)$ are not higher than $(n - r)$.

Therefore we have rather restrictive conditions on the possible class of $\widetilde{\Sigma}$. These conditions enable us to avoid lengthy calculations of some parameters (as will be demonstrated in § 3.3.1).

One could also consider the second projection from $\widetilde{\Sigma}$ to the auxiliary space; in Definition 3.1 this space is $\mathbb{P}^n_x \times \prod_i \mathbb{P}^n_{y_i}$.

Example 3.9. For multiple points of co-rank r, the variety defined in (43) is a locally trivial fibration over the auxiliary space $\mathbb{P}^n_x \times \prod_{i=1}^r \mathbb{P}^n_{y_i}$ outside the 'diagonals' (two coinciding points, three points on a line, four points in the plane etc.).

This happens in the general case, where the projection is a locally trivial fibration, outside the cycles of jump.

3.1.1.2. *On the possible cycles of jumps.* The cycles of jump were discussed in general in § 2.2. Here we describe the possible cycles of jump for linear singularities. The lifted stratum is an intersection of hypersurfaces, each being defined by vanishing of a particular derivative. As follows from the discussion in § 2.4.1 the defining equations of the hypersurfaces are of the form

$$f^{(k)}|_x(\underbrace{y_{i_1}, \dots, y_{i_{l_k}}}_{l_k}, \hat{e}_{j_1}, \dots, \hat{e}_{j_{k-l_k}}) = 0 \quad 0 \leqslant j_1 \leqslant \dots \leqslant j_{k-l_k} \leqslant n \tag{49}$$

(where $\hat{e}_{j_1}, \dots, \hat{e}_{j_{k-l_k}}$ are the vectors of the standard basis, defined in § 2.4.1).

The cycles of jumps consist of points of $\mathbb{P}^n_x \times \mathbb{P}^n_{y_1} \times \dots \times \mathbb{P}^n_{y_r}$ over which the intersection is non-transversal. Every equation of the type (49) is linear in the coefficients of f (which are the homogeneous coordinates of the parameter space $\mathbb{P}^{N_d}_f$). Therefore the non-transversality can occur only when the vectors $y_{i_1}, \dots, y_{i_{l_k}}, \hat{e}_{i_1}, \dots, \hat{e}_{i_n - l_k}$ are non-generic with respect to the vectors of other equations. More precisely, the non-transversality can occur only when some of the vectors $y_{i_1},\dots,y_{i_{l_k}},\hat{e}_{i_1},\dots,\hat{e}_{i_n-l_k}$ of one equation belong to the Span of the vectors of other equations.

This condition can be nicely written using projections of vector space. Let $I=\{i_1,\dots,i_k\}$ be an arbitrary (non-empty) subset of $\{0,1,\dots,n\}$. Represent the point of a projective space by its homogeneous coordinates: $y=(z_0,\dots,z_n)\in\mathbb{P}^n_y$. The projection is defined by

$$\pi_I(y):=(z_{i_1},\dots,z_{i_k}). \tag{50}$$

Note that $\pi_I(y)$ is defined up to a scalar multiplication and can have all the entries 0. Immediate check shows that the above condition of non-transversality corresponds to one of the following:

- $\pi_I(y)\in\text{Span }(\pi_I(x),\pi_I(y_{i_1}),\dots,\pi_I(y_{i_k}))$;
- $(\pi_I(x),\pi_I(y_{i_1}),\dots,\pi_I(y_{i_k}))$ are linearly dependent.

(In particular if all the entries of $\pi_I(y)$ are zero, both conditions are trivially satisfied.) Note that the second condition is the closure of the first. Summarizing, we have the following.

Proposition 3.10. *The possible cycles of jump in the auxiliary space*

$$\text{Aux}=\mathbb{P}^n_x\times\mathbb{P}^n_{y_1}\times\ldots\times\mathbb{P}^n_{y_k}$$

are of the form $\big((\pi_I(x),\{\pi_I(y_j\}_{j\in J})$ *are linearly dependent for some* $I\subseteq\{0,\dots,n\}$, $J\subseteq\{1,\dots,k\}\big)$.

Remark 3.11. While the set I is non-empty, J can be empty. In this case the cycle of jump is defined as $\pi_I(x)=(0,\dots,0)$.

3.1.1.3. *On the adjacency.* To each equisingular stratum some strata of higher singularities are adjacent (that is, the strata of higher singularities are included in the closure of the given stratum). For example, $\Sigma_{D_{k+1}}\subset\overline{\Sigma}_{A_k}$.

We constantly use the codimension-1 adjacency, that is, when $\Sigma_{\mathbb{D}'}$ is a divisor in $\Sigma_{\mathbb{D}}$ (or the same for the lifted versions, cf. § 3.1.1). In other words, these are the strata that can be reached by just one degeneration. Many tables of adjacencies are given in [2]. In each particular case the adjacency can be checked by the analysis of Newton diagram of the singularity or of the defining ideal of the singular germ.

The adjacency can depend on moduli [20]. However, if one chooses moduli generically (and we always do that), the adjacency is completely fixed by the topological type.

A more important feature is that the set of relevant adjacent strata can depend on the dimensionality of the ambient space. As an example consider the enumeration of A_4 (cf. § 3.3). It is degenerated by increasing the co-rank of the quadratic form. Then:

- $n=2$, the case of curves $A_4\to D_5$;
- $n\geqslant 3$, the case of hypersurfaces, $A_4\to D_5\cup P_8$.

As will be shown in § 3.3.2, in the second case both D_5 and P_8 are relevant.

In our approach we enumerate separately for each fixed dimension n. By universality, the final answer is given by a unique polynomial (in Cherna classes that depend on n) of a known degree. Therefore it suffices to calculate only for a few required dimensions, and by universality we get the complete answer.

3.1.2. *The ideology of degenerating process.* Here we prove the main theorems stated in § 1 (Theorems 1.6 and 1.7). We must prove the following three statements.

- The degenerating step is always possible and the degeneration is invertible.
- For linear singularities we achieve the cohomology class of the lifted stratum in a finite number of steps and all the intermediate types are linear.

• For non-linear singularities we reduce the problem to enumeration of linear ones (in a finite number of steps).

3.1.2.1. *The degenerating step.* This consists of intersection of a lifted stratum with a hypersurface and subtraction of the residual pieces. In more detail, let $\widetilde{\Sigma}$ be the initial lifted stratum and $\widetilde{\Sigma}_{\mathrm{degen}}$ the stratum we want to reach. The degenerating hypersurface is defined by the equation $f^{(p)}(y_{i_1},\dots,y_{i_k})_{j_1,\dots,j_{p-k}}=0$. For linear singularities this is just the derivative corresponding to a point under the Newton diagram. For non-linear singularities (as is explained in § 2.4.4) the derivative corresponds to a point on the Newton diagram with the minimal distance to the origin. Such a point can be non-unique, in this case we use monomial order from § 2.1.2.

The intersection is in general non-transversal, and the resulting variety is reducible (containing residual pieces in addition to the required degenerated stratum):

$$\widetilde{\Sigma}\cap\{f^{(p)}(y_{i_1},\dots,y_{i_k})_{j_1,\dots,j_{p-k}}=0\}=\widetilde{\Sigma}_{\mathrm{degen}}\cup\{\text{residual pieces}\}. \tag{51}$$

The non-transversality occurs over cycles of jump (described in § 3.1.1.2). The procedure to calculate the cohomology classes of the residual pieces is explained in § B.3. Note that in the above equation the degenerated stratum $\widetilde{\Sigma}_{\mathrm{degen}}$ can be reducible and not-reduced (but is always pure dimensional); this happens for non-linear singularities. The multiplicity of the intersection of $\widetilde{\Sigma}$ with the degenerating hypersurface along $\widetilde{\Sigma}_{\mathrm{degen}}$ is calculated in the classical way (§ B.1).

Summarizing, the degenerating step produces the equation in cohomology:

$$[\widetilde{\Sigma}][f^{(p)}(y_{i_1},\dots,y_{i_k})_{j_1,\dots,j_{p-k}}=0]=[\widetilde{\Sigma}_{\mathrm{degen}}]+[\text{residual pieces}]. \tag{52}$$

By § 2.4.5 the degeneration is invertible, so the equation enables the calculation of either $[\widetilde{\Sigma}]$ or $[\widetilde{\Sigma}_{\mathrm{degen}}]$.

3.1.2.2. *Degenerations for linear singularities* (*from simple to complicated*). The defining set of conditions for linear singularities was described in § 2.4. Starting from the stratum of ordinary multiple point (of the relevant multiplicity), lifted to the space $\mathrm{Aux}\times\mathbb{P}^{N_d}_f$, we apply the defining conditions one by one. Each condition means the absence of a particular monomial, arrange the conditions by the monomial order (defined in § 2.1.2). This guarantees that at each step we get a Newton-non-degenerate type.

The degenerating step was described above. After a restricted number of degenerations (not more than the number of points under the Newton diagram) we arrive at the lifted stratum of the required singularity.

At each intermediate step of the process the singularity type is linear. Indeed by the criterion of Lemma 2.24 all the initial and final slopes are bounded in the interval $[\frac{1}{2},2]$. And in the process of degeneration the slopes change monotonically.

Remark 3.12. Usually it is simpler to start not from the stratum of ordinary multiple point, but from a stratum of a higher (linear) singularity to which the given singularity is adjacent and for which the enumeration problem is already solved. For example, to enumerate the tacnode (A_3) we start from the cusp (A_2).

3.1.2.3. *Degenerations for non-linear singularities* (*from complicated to simple*). As was explained in § 1.4.3 the original non-linear type is degenerated into a combination of linear ones.

The aim of degenerating process is to convert the defining non-linear equations to monomial ones (that is, of the form $\vec{z}^{\vec{m}} = 0$). For this, at each step of the process, we consider a (non-vanishing) monomial of the lowest monomial order (§ 2.1.2).

Suppose that the multiplicity of the singularity is p while the order of determinacy is k. The process goes by first demanding that the derivatives $f^{(p)}_{i_1 \dots i_p}$ vanish (the conditions are applied one by one, at each step we have just the degeneration by a hypersurface). Then one arrives at the singularity of multiplicity $(p+1)$ and order of determinacy k. If the so-obtained singularity (or a collection of singularities) is still non-linear, the process is continued. In the simplest case (Newton-non-degenerate singularity), once we have $k \leqslant 2p$, we necessarily have a collection of linear singularities (by Corollary 2.25). In the worst case (Newton-degenerate singularity) one continues up to the ordinary point of multiplicity k.

This process transforms a non-linear singularity to a collection of linear singularities with enumeration that was described above. As all the degenerations are invertible, this solves the enumerative problem for non-linear singularities.

3.2. *The simplest examples*

We consider here some simplest typical examples to illustrate the method. First we consider the case of cusp. Having enumerated the cusp, one enumerates the tacnode by just one additional degeneration.

3.2.1. *Quadratic forms of co-rank 1 (cuspidal hypersurfaces).* Here we consider singularity with the normal form $\sum_{i=2}^{n} z_i^2 + z_1^3$. The lifted stratum was defined in Example 3.4:

$$\widetilde{\Sigma}_{A_2}(x,y) = \overline{\{(x,y,f) |\ (x \neq y),\ f|_x^{(2)}(x) = 0 = f|_x^{(2)}(y)\}} \subset \mathbb{P}^n_x \times \mathbb{P}^n_y \times \mathbb{P}^{N_d}_f \qquad (53)$$

(here the tensor of second derivatives is calculated at the point x, therefore, $f|_x^{(2)}(x) \sim f|_x^{(1)}$).

We want to represent $\widetilde{\Sigma}_{A_2}(x,y)$ as a (possibly) transversal intersection of hypersurfaces. The $(n+1)$ conditions $f^{(2)}(x) = 0$ are transversal. Suppose that we add one more condition $(f^{(2)}(y))_0 = 0$. Then, the non-transversality occurs over two cycles of jump in $\mathrm{Aux} = \mathbb{P}^n_x \times \mathbb{P}^n_y$.

- $x = y$ (co-dimension n). The jump in the dimension of the fibre is 1.
- $x = (1, 0, \dots, 0)$ (co-dimension n). In this case, since $f^{(2)}(x) = 0$, we already know that the form $f^{(2)}(y)$ annihilates x (that is, $f^{(2)}(y)_0 = 0$). The jump in the dimension of the fibre is 1.

In both cases the dimension of the jump of fibre is less than the codimension of the cycle of jump, so the resulting variety is irreducible. Hence, for the cohomology classes we have

$$[\overline{f^{(2)}(x) = 0, f^{(2)}(y)_0 = 0,\ x \neq y}] = [f^{(2)}(x) = 0] \times [f^{(2)}(y)_0 = 0]. \qquad (54)$$

We continue to degenerate by the conditions in such a way until $(f^{(2)}(y)_{n-1} = 0)$ is reached. At this point for generic (non-coinciding) x, y we have $f^{(2)}(x) = 0 = f^{(2)}(y)$.

The following are the additional pieces that arose.

- Over $x = y$, the conditions of codimension n: over this diagonal the jump in the dimension of fibres is n.
- Over $x = (*, \dots, *, 0)$, the condition of codimension 1: over this subvariety the jump in the dimension of fibres is 1.

On the picture the result of the intersection is shown in a 'log-scale', that is, the dimensionality is transformed to the relative height (Figure 7).

Therefore, the resulting variety is reducible, containing residual pieces over the cycles of jump.

$$\Big\{f|_x^{(2)}(x) = 0 = f|_x^{(2)}(y)\Big\} = \widetilde{\Sigma}_{A_2}(x,y) \cup \Big\{\mathrm{Res}_{x=y}\Big\} \cup \Big\{\mathrm{Res}_{z_n=0}\Big\}. \qquad (55)$$

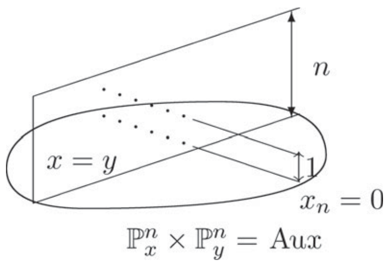

FIGURE 7.

(By direct check we obtain that the multiplicity in both cases is 1.) To remove the contributions from additional pieces we should calculate the classes of the restrictions to the cycles of jump. The method of calculation of the cohomology classes of residual pieces is described in § B.3.

• The residual piece over the diagonal can be described explicitly in a simple manner. As all the intersections over the points of the diagonal are non-transversal, a point over the diagonal corresponds to a nodal hypersurface. Hence, the residual piece is

$$\mathrm{Res}_{x=y} = \{x = y\} \cap \left\{\widetilde{\Sigma}_{A_1}(x) \times \mathbb{P}^n_y\right\},$$

and its cohomology class $[\mathrm{Res}_{x=y}] = [x = y] \times [\widetilde{\Sigma}_{A_1}(x)]$.

• Over the generic point of the cycle $z_n = 0$ (that is, the point for which $z_0 \dots z_{n-1} \neq 0$) all the intersections, except for the last were transversal. Therefore we have

$$\left\{f|^{(2)}_x(x) = 0, \quad z_n = 0\right\} \bigcap_{i=0}^{n-2} \left\{f|^{(2)}_x(y)_i = 0\right\} = \mathrm{Res}_{z_n=0} \cup \mathrm{Res}_{z_n=0=z_{n-1}}, \tag{56}$$

where $\mathrm{Res}_{z_n=0=z_{n-1}}$ is a 'secondary' residual piece. Its description is the same as that of $\mathrm{Res}_{z_n=0}$ and so one has a recursion

$$\left\{f|^{(2)}_x(x) = 0, \quad z_n = 0\right\} \bigcap_{i=0}^{j-2} \left\{f|^{(2)}_x(y)_i = 0\right\} = \mathrm{Res}_{z_n=0=\ldots=z_j} \cup \mathrm{Res}_{z_n=0=\ldots=z_{j-1}}. \tag{57}$$

After completion of the recursion we get the formula

$$[\widetilde{\Sigma}_{A_2}(x, y)] = [f^{(2)}(x) = 0] \left(\sum_{i=0}^{n-1} (-1)^i \left[\prod_{j=n+1-i}^{n} \{x_j = 0\}\right] \left[\prod_{j=0}^{n-1-i} \{f^{(2)}(y)_j = 0\}\right] - [x = y]\right). \tag{58}$$

Or, in terms of cohomology classes,

$$[\widetilde{\Sigma}_{A_2}(x, y)] = (Q + X)^{n+1} \left(\sum_{i=0}^{n} (-1)^i (Q + Y)^{n-i} X^i - \sum_{i=0}^{n} X^i Y^{n-i}\right), \quad Q = (d - 2)X + F. \tag{59}$$

Here (X, Y, F) are the generators of the corresponding cohomology rings. Note that this polynomial (if written in variables (Q, X, Y)) is symmetric in X, Y (even though it was obtained in a very non-symmetric way) as it should be. Also, the terms X^n, Y^n cancel in the polynomial.

To obtain the degree of the variety we should extract the coefficient of $X^n Y^{n-1}$ (after the substitution, $Q = (d - 2)X + F$). We get

$$\deg(\Sigma_{A_2}) = (d - 1)^{n-1}(d - 2)\frac{n(n + 1)(n + 2)}{2} \tag{60}$$

(which of course coincides with the result obtained by Aluffi in [1]).

3.2.2. *The use of adjacency: tacnodal hypersurfaces.* The tacnodal singularity (A_3) has the normal form $f = z_1^4 + \sum_{i=2}^n x_i^2$. The corresponding lifted variety was defined in Example 3.4. We represent the tacnode as a degeneration of the cusp. Correspondingly, we think of the (lifted) stratum of tacnodal hypersurfaces as a subvariety of the cuspidal stratum

$$\widetilde{\Sigma}_{A_3}(x,y) = \overline{\{(x,y,f) \in \widetilde{\Sigma}_{A_2}(x,y),\ x \neq y \ \ f|_x^{(3)}(y,y,y) = 0\}} \subset \widetilde{\Sigma}_{A_2}(x,y) \subset \mathbb{P}_x^n \times \mathbb{P}_y^n \times \mathbb{P}_f^{N_d}. \tag{61}$$

For generic x, y the intersection is $\widetilde{\Sigma}_{A_2}(x,y) \cap \{f^{(3)}(y,y,y) = 0\}$ transversal. The possible non-transversality can occur only when $x = y$. Hence for the cohomology classes we have

$$[\widetilde{\Sigma}_{A_3}(x,y)] = [\widetilde{\Sigma}_{A_2}(x,y)][f^{(3)}(y,y,y) = 0] - [\text{residual piece over } x = y] \tag{62}$$

The method to calculate the cohomology class of the residual piece over $(x = y)$ is given in Appendix B. By Corollary B.3 we have

$$[\text{Residual piece over } x = y] = [x = y]\frac{1}{(n-1)!}\frac{\partial^{n-1}[\widetilde{\Sigma}_{A_2}(x,y)]}{\partial^{n-1}Y}. \tag{63}$$

In this simple case the residual piece can be also easily described explicitly: every point of it corresponds to a hypersurface with a cusp at a given point, but with arbitrary tangent line. Therefore the variety over the diagonal $(x = y)$ is just $\widetilde{\Sigma}_{A_2}(x)$ taken with some multiplicity. The multiplicity can be computed in two ways:

- directly, as the degree of tangency of the non-transversal intersection;
- via consistency condition (as was explained in § 3.1.1). Since the lifted stratum $\widetilde{\Sigma}_{A_3}$ is a fibration over Σ_{A_3} with fibre $\mathbb{P}_y^1$, the corresponding cohomology class should not include terms with Y^n).

Both methods give the multiplicity 3. Thus the cohomology class of the cuspidal stratum is

$$\begin{aligned}
[\widetilde{\Sigma}_{A_3}(x,y)] &= [\widetilde{\Sigma}_{A_2}(x,y)][f^{(3)}(y,y,y) = 0] - 3[x = y][\widetilde{\Sigma}_{A_2}(x)] \\
&= (Q+X)^{n+1}\Bigg(\Big(\sum_{i=0}^n (Q+Y)^{n-i}(-X)^i - \sum_{i=0}^n X^i Y^{n-i}\Big) \\
&\quad \times (Q + 3Y - X) - 3(nQ - 2X)\sum_{i=0}^n X^i Y^{n-i}\Bigg).
\end{aligned} \tag{64}$$

Here, as always $Q = (d-2)X + F$. Note again that (in full accordance with the fibration conditions) the answer (if written in terms of X, Y, Q) is symmetric in X, Y and no terms with X^n or Y^n appear. Finally, the degree is

$$[\Sigma_{A_3}] = \binom{n+2}{3}(d-1)^{n-2}\left(\frac{(3n-1)(n+3)}{2}(d-2)^2 + 2(n-1)(d-2) - 4\right), \tag{65}$$

which for $n = 2$, curves, coincides with the result of Aluffi [1].

3.3. *Further calculations*

We start from homogeneous forms of some (co-)rank (§ 3.3.1). For quadratic forms ($p = 2$) the rank fixes the degeneracy class completely. For higher forms one can impose various additional degeneracy conditions (for example, for $p = 3$ the form $z_1^2 z_2 + \sum_{i=3}^n z_i^3$ is of full rank, but the corresponding singularity is not an ordinary triple point). We consider examples of such singularities in § 3.3.2.

Having calculated the classes of the lifted strata for 2-forms of some rank, we can start enumeration of other singularities. This is done by further degenerations. For example, the tacnode A_3 is the cusp with some degeneracy of the tensor of order-3 derivatives (it was enumerated in § 3.2). We consider here the simplest examples A_4, D_5, E_6.

3.3.1. *The forms of co-rank at least 1.* We first recall the following definition.

DEFINITION 3.13. The homogeneous symmetric form of order p, in n variables, $\Omega^p(z_1,\dots,z_n)$, is called of rank $(n-r)$ (of co-rank r) if by linear transformation of $\mathrm{GL}(n)$ in the space of variables $(\mathbb{C}^n=\{z_1,\dots,z_n\})$ it can be brought to a homogeneous form in $n-r$ variables $\tilde{\Omega}^{(p)}(z_1,\dots,z_{n-r})$.

The collections of such forms are natural generalizations of classical determinantal varieties (symmetric matrices of a given co-rank) [**5**, Chapter 14.3]. To emphasize the co-rank of the form we often assign it as a subscript Ω^p_r.

There are (at least) two approaches to calculate the cohomology classes of the stratum of a given co-rank forms.

- Start from the ordinary multiple point (it corresponds to the non-degenerate form) treated before. Apply the degenerating conditions (one by one) to get the form of the required co-rank. At each step it is necessary to remove the residual pieces. This approach works well for the forms of low co-rank.
- Degenerate the given form to a form of rank 2 or 1. (The forms of rank 2,1 are particular cases of reducible forms; their enumeration is immediate and is treated in Appendix A) Then we will have equation in cohomology of the form

$$\big[\Omega^{(p)}_r\big][\text{degenerating cycle}]=\big[\Omega^{(p)}_2\big]+[\text{residual variety}]. \tag{66}$$

And from this equation the class $[\Omega^{(p)}_r]$ is restored uniquely.

We describe here the first approach. As the computations are extremely involved we solve explicitly only the case of the quadratic forms of arbitrary co-rank.

The lifted stratum was defined in Example 3.4:

$$\begin{aligned}\widetilde{\Sigma}^n_r(x,(y_i)^r_{i=1})=&\overline{\left\{(x,\{y_i\}^r_{i=1},f)\middle|\begin{pmatrix}x,y_1,\dots,y_r\text{ are}\\ \text{linearly independent}\end{pmatrix},\begin{pmatrix}f^{(2)}(x)=0\\(f^{(2)}(y_i)=0)^r_{i=1}\end{pmatrix}\right\}}\subset\mathbb{P}^n_x\\&\times\prod_{i=1}^r\mathbb{P}^n_{y_i}\times\mathbb{P}^{N_d}_f.\end{aligned} \tag{67}$$

As was explained in § 3.1.1, the cohomology class of $\widetilde{\Sigma}^n_r$ is a polynomial in $(X,Y_1,\dots,Y_r,Q=(d-2)X+F)$, symmetric in $(X,Y_1,\dots,Y_r)$ and does not contain powers of Y_i greater than $(n-r)$. The cohomology class of Σ^n_r is just the coefficient of the monomial $Y_1^{n-r}\cdot\ldots\cdot Y_k^{n-r}X^n$ in the cohomology class of $\widetilde{\Sigma}^n_r$.

The enumeration of singularities with quadratic form of co-rank r is completed by the following lemma.

LEMMA 3.14. *The cohomology class of the lifted variety $\widetilde{\Sigma}^n_r(x,(y_i)^r_{i=1})$ can be calculated by successive degenerations starting from $\widetilde{\Sigma}^n_0(x)$ (the nodal hypersurfaces). In particular, the cohomology class of the minimal lifting is*

$$[\widetilde{\Sigma}^n_r(x)]=C_{n,r}Q^{\binom{r}{2}}\sum_{i=0}^r\frac{\binom{n-i}{r-i}\binom{r}{i}}{\binom{2r}{r+i}}Q^{r-i}(-X)^i\quad Q=F+(d-2)X$$

$$C_{n,1}=2,\ \ C_{n,2}=2\binom{n+1}{1},\ \ C_{n,3}=2\binom{n+2}{3},\ \ C_{n,4}=2\binom{n+3}{5}\frac{n+1}{3} \tag{68}$$
$$C_{n,n}=2\binom{2n}{n},\ \ C_{n,n-1}=\frac{2^r\binom{2r}{r}}{n},\ \ C_{n,n-2}=\frac{\binom{2(r+1)}{r+1}\binom{2r}{r}}{\binom{r+2}{2}}.$$

Note that here we give the constant only for some specific values of (n,r). This is due to computer limitations: each time we calculate for a specific value of r and n. Therefore, for every specific n,r one can get the answer (provided the computer has sufficient capacity), but it is not clear how to combine these values into one nice expression.

Proof. The case $r=1$ corresponds to cuspidal hypersurfaces and was considered in § 3.2. The general case is done recursively. Suppose that we have obtained the cohomology class of $\widetilde{\Sigma}^n_r(x,y_1,\dots,y_r)$, as in equation (59). Intersect the variety $\widetilde{\Sigma}^n_r(x,y_1,\dots,y_r)$ with $(n-r)$ hypersurfaces $f^{(2)}(y_{r+1})_0=0,\dots,f^{(2)}(y_{r+1})_{n-r-1}=0$. The possible (significant) non-transversality can occur in two cases, either $y_{r+1}\in\mathrm{Span}(x,y_1,\dots,y_r)$ or $(*,\dots,*,\underbrace{0,\dots,0}_{k})\in\mathrm{Span}(x,y_1,\dots,y_r)$. In both cases one continues as in the case of $r=1$. Thus we get the cohomology class

$$\begin{aligned}[\widetilde{\Sigma}^n_{r+1}]=[\widetilde{\Sigma}^n_r]\Big(&\sum_{i=0}^{n-r}(Q+Y_{r+1})^{n-r-i}(-1)^i\sum_{j_1+\ldots+j_r\leqslant i}Y_1^{j_1}\cdot\ldots\cdot Y_r^{j_r}X^{i-(j_1+\ldots+j_r)}\\&-\sum_{i_1+\ldots+i_{r+1}=n-r-j}X^jY_1^{i_1}\cdot\ldots\cdot Y_{r+1}^{i_{r+1}}\Big)+\text{residual terms}.\end{aligned} \tag{69}$$

Here the residual terms correspond to varieties which occur over the diagonals: $x=y_1$, $y_2\in\mathrm{Span}(x,y_1)$, $\dots$, $y_r\in\mathrm{Span}(x,y_1,\dots,y_{r-1})$. These residual pieces can be calculated by classical intersection theory (as explained in § B.3). However (as happens in all other cases) the classes are actually completely fixed by the consistency conditions (Lemma 3.6 and Corollary 3.7). In this case they read:

The final expression (polynomial in $Q,X,Y_1,\dots,Y_{k+1}$) should be symmetric in $(X,Y_1,\dots,Y_{k+1})$ and should not contain the powers of $X,Y_1,\dots,Y_{k+1}$ that are greater than $n-k-1$.

The explicit calculations are extremely complicated and can be done only by computer. The cohomology classes of the lifted strata are awkward polynomials in many variables.

For example, the cohomology class in the co-rank 2 case (D_4 singularity) is

$$\begin{aligned}[\widetilde{\Sigma}^2_2(x,y_1,y_2)]=&(Q+X)^{n+1}\sum_{i=0}^{n}(Q+Y_1)^{n-i}(-X)^i\\&\times\left(\sum_{j=0}^{n-1}(Q+Y_2)^{n-1-j}(-1)^j\sum_k X^{j-k}Y_1^k-\sum_{j,k}X^jY_1^kY_2^{n-1-j-k}\right)\\&-(Q+X)^{n+1}\sum_{i=0}^{n}X^iY_1^{n-i}\sum_{i=1}^{n}\times\left(\sum_{j=0}^{n-i}\binom{n-j}{i}Q^{n-i-j}(-X)^j-X^{n-i}\right)\\&\times\left(\sum_{j=0}^{i-1}(-1)^{n-j-1}(Q+Y_2)^jX^{i-j-1}-\sum_{j=0}^{i-1}X^{i-1-j}Y_2^j\right).\end{aligned} \tag{70}$$

Extracting the coefficients of $Y_1^{n-r-1}\cdot\ldots\cdot Y_{k+1}^{n-r-1}$ we get the classes of the strata $\widetilde{\Sigma}^n_r(x)$. □

3.3.2. *Some linear singularities.*

(1) Hypersurfaces with a D_5 point: the normal form $f = z_1^4 + z_2^2 z_1 + \sum_{i=3}^n z_i^2$. The lifted stratum was defined in Example 3.4. We represent the stratum $\widetilde{\Sigma}_{D_5}$ as a subvariety of $\widetilde{\Sigma}_{D_4}$:

$$\widetilde{\Sigma}_{D_5}(x,y_1,y_2) = \overline{\{(x,y_1,y_2,f) \in \widetilde{\Sigma}_{D_4}(x,y_1,y_2),\quad f|_x^{(3)}(y_1,y_1,y_1) = 0 = f|_x^{(3)}(y_1,y_1,y_2)\}} \\ \times \subset \widetilde{\Sigma}_{D_4}(x,y_1,y_2). \tag{71}$$

For generic x, y_1, y_2 the intersection is transversal. The non-transversality occurs over diagonals $y_2 \in \mathrm{Span}(x, y_1)$ or $x = y_1$. Note that the first variety is non-closed. We approximate it by the variety (x, y_1, y_2 are linearly dependent); the two varieties coincide for $x \neq y_1$. Correspondingly, over $x = y_1$ we have an additional (secondary) residual piece. Thus the cohomology class is

$$[\widetilde{\Sigma}_{D_5}(x,y_1,y_2)] = [\widetilde{\Sigma}_{D_4}(x,y_1,y_2)][f^{(3)}(y_1,y_1,y_1) = 0][f^{(3)}(y_1,y_1,y_2) = 0] \\ - [x = y_1]A(X,Y_2,Q) - \left[\mathrm{rk}\begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} < 3\right] B(X,Y_1,Y_2,Q). \tag{72}$$

Here A, B are some (homogeneous) polynomials in the generators of the cohomology ring. By the identity in the cohomology ring, $(X - Y)\sum_{i=0}^n X^i Y^{n-i} = X^{n+1} - Y^{n+1} \equiv 0$, we can assume that A does not depend on Y. The only additional condition on A, B is the consistency condition from § 3.1.1:

The cohomology class $[\widetilde{\Sigma}_{D_5}(x,y_1,y_2)]$ should not contain monomials with Y_1^n, Y_2^{n-1}, Y_2^n.

As always, this condition itself fixes the polynomials completely. The final cohomology class is given in the Appendix C.

(2) Hypersurfaces with an E_6 point: the normal form $f = z_1^4 + z_2^3 + \sum_{i=3}^n z_i^2$. We represent E_6 as a degeneration of D_5:

$$\widetilde{\Sigma}_{E_6}(x,y_1,y_2) = \overline{\{(x,y_1,y_2,f) \in \widetilde{\Sigma}_{D_5}(x,y_1,y_2),\quad f|_x^{(3)}(y_1,y_2,y_2) = 0\}} \subset \widetilde{\Sigma}_{D_5}(x,y_1,y_2). \tag{73}$$

Again, instead of describing the residual varieties explicitly, we use the consistency conditions, which completely fix the class. The final answer is in the Appendix C.

3.3.3. *Some non-linear singularities.*

(1) The A_4 case. Here we consider the simplest non-linear case. By linear transformation, the singularity germ can be brought to the Newton diagram of A_3:

$$f = \sum_{i=2}^n \alpha_i z_i^2 + z_1^2 \sum_{i=2}^n \beta_i z_i + \gamma z_1^4 + \dots .$$

To achieve the Newton diagram of A_4 we must do the non-linear shift: $z_i \to z_i - (\beta_i/2\alpha_i)z_1^2$ (to get rid of the monomials $z_1^4, z_1^2 z_i$). Elimination gives $\gamma = \sum(\beta_i^2/4\alpha_i)$.

Therefore, degenerating $\alpha_2 = 0$ we get $\beta_2 = 0$ or $\prod_{i\neq 2}\alpha_i = 0$, corresponding to adjacency $\bar{\Sigma}_{A_4} \supset \Sigma_{D_5}, \Sigma_{P_8}$.

In this way we obtain the cohomology class of $\widetilde{\Sigma}_{A_4}(x,y)$. Therefore, we get the equation for cohomology classes:

$$[\widetilde{\Sigma}_{A_4}(x,y_1)][\text{degeneration}] = 2[\widetilde{\Sigma}_{D_5}(x,y_1)] + 2[\widetilde{\Sigma}_{P_8}(x,y_1,y_2)] + [\text{residual piece}]. \tag{74}$$

The final result is in the Appendix C.

(2) The D_6 case. By linear transformation, the singularity germ can be brought to the Newton diagram of D_5: $f = \sum_{i=3}^n \alpha_i z_i^2 + z_1^2 \sum_{i=3}^n \beta_i z_i + \gamma z_1^4 + z_1 z_2^2 + \dots$. To achieve the Newton diagram of D_6 we must do the non-linear shift $z_i \to z_i + \delta_i z_1^2$ (to get rid of the monomials $z_1^4, z_1^2 z_i$). Elimination gives $\gamma = \sum \beta_i^2/(4\alpha_i)$. We degenerate in the same way as in the A_4 case and get

$$[\widetilde{\Sigma}_{D_6}(x,y_1,y_2)][\text{degeneration}] = 2[\widetilde{\Sigma}_{P_8}(x,y_1,y_2,y_3)_{\text{degenerated}}] + [\text{co-rank } 4] + [\text{residual piece}]. \tag{75}$$

We omit the calculations.

Appendix A. *Singularities with reducible jets*

Here we consider singular polynomials with low-order jet that is reducible. The simplest such case is that of the reducible form

$$\Omega^{(p)} = \prod_{i=1}^{k} \left(\Omega_i^{(p_i)}\right)^{r_i}, \quad \sum_{i=1}^{k} r_i p_i = p \tag{76}$$

(here the homogeneous forms $\Omega_i^{(p_i)}$ are irreducible, though they can be degenerate and mutually non-generic). These singularity types are of high codimension, and therefore extremely rare; nevertheless they deserve some attention, being sometimes the final aim of the degenerating process.

Since reducibility is in general not invariant under the topological transformations we (in general) cannot define the corresponding stratum as a topological one. We define the stratum as the collection of hypersurfaces that can be brought (by locally analytic transformation) to a given form.

This stratum is included into the topological stratum. On the other hand it usually contains families of analytical strata, since by the Newton diagram we do not specify the moduli.

In the case of curves ($n = 2$) every singularity of multiplicity p has a reducible p-jet (being homogeneous polynomial of two variables). Therefore the corresponding stratum coincides with the topological equisingular stratum. Some singularities with reducible p-form are $A_1, A_2, D_4, E_6, X_9, Z_{11}, W_{12}, \ldots$. For surfaces ($n = 3$) some singularities of this type are A_2, D_4, P_8, S_{11}, $U_{12}, T_{3,4,4}, T_{4,4,4}, V_{1,0}, V_1', \ldots$. Another example of reducible p-form (for any n) is the p-form of rank 2 or 1. For singularities with reducible jets, the lifted stratum can be explicitly defined by conditions of a very standard type, proportionality of tensors.

A.1. *Reducible homogeneous forms*

We consider here the case of mutually generic forms $\Omega_i^{(p_i)}$ in (76). Every such form defines a hypersurface. By the mutual generality of the forms the hypersurfaces intersect in a generic way; however, each hypersurface can be singular. We restrict to the case of ordinary multiple point of maximal multiplicity (so for each hypersurface the condition is $\Omega_i^{(p_i)}(x) = 0$). The stratum of hypersurfaces with this type of reducibility is defined as

$$\widetilde{\Sigma}\Big(x, (\Omega_i^{(p_i)})_{i=1}^k\Big) = \left\{ \Big(x, (\Omega_i^{(p_i)})_{i=1}^k, f\Big) \Big| f^{(p)} \sim \mathrm{SYM}\Big((\Omega_1^{(p_1)})^{r_1}, \ldots, (\Omega_k^{(p_k)})^{r_k}\Big), \right.$$
$$\left. (\Omega_i^{(p_i)}(x) = 0)_{i=1}^k \right\}. \tag{77}$$

Here SYM means symmetrization of indices. Note that the sets of defining conditions are mutually transversal, for example, f appears in the first proportionality condition only. Therefore the cohomology class of the lifted stratum is just the product of classes of conditions

$$\left[\widetilde{\Sigma}\Big(x, (\Omega_i^{(p_i)})_{i=1}^k\Big)\right] = \left[f^{(p)} \sim \mathrm{SYM}\Big((\Omega_1^{(p_1)})^{r_1}, \ldots, (\Omega_k^{(p_k)})^{r_k}\Big)\right] \prod_i \left[\Omega_i^{(p_i)}(x) = 0\right]. \tag{78}$$

The condition of proportionality of two tensors is considered in § B.2 (equation (92)). In terms of the cohomology ring generators of the ambient space $(X, Q = (d - p)X + F, \Omega_i)$ we have

$$\left[\widetilde{\Sigma}\Big(x, (\Omega_i^{(p_i)})_{i=1}^k\Big)\right] = (\Omega_1 + X)^{\binom{p_1 - 1 + n}{n}}, \ldots, (\Omega_k + X)^{\binom{p_k - 1 + n}{n}}$$
$$\times \sum_{i=0}^{\binom{p+n}{n} - 1} Q^i (r_1\Omega_1 + \ldots + r_k\Omega_k)^{\binom{p+n}{n} - 1 - i}. \tag{79}$$

Note that depending on the singularity type, the projection $\widetilde{\Sigma} \to \Sigma$ cannot be 1:1. The permutation group of forms of equal multiplicity and degeneracy acts on fibres. Thus, to obtain the cohomology class of Σ one should divide the corresponding coefficient by the order of this group, $|\mathrm{Aut}|$.

A.2. *Singularities with reducible jets*

Here we consider singularities of the type

$$f = \prod_{i=1}^{k} f_i(z_1, \ldots, z_n) + \text{higher-order terms} \quad \deg(f_i) = p_i \ \sum_{i=1}^{k} p_i = p. \tag{80}$$

The polynomials f_i are (non-homogeneous) of fixed degrees. We assume that the singularities $f_i = 0$ are linear, in particular they satisfy $2 \times \text{multiplicity} \geqslant p_i$.

In particular, we can assume that all the hypersurfaces $\{f_i = 0\}$ pass through the origin. Introducing factors that do not vanish at the origin leads to hypersurfaces with flexes (the property which is not invariant under local diffeomorphism/homeomorphism).

If the hypersurface $(f_i = 0)$ is smooth and generic with respect to other hypersurfaces (for example, all the normals are in general position, intersection is along generic subvarieties etc.) then $\deg(f_i) = p_i = 1$.

The procedure of enumeration is as in Appendix A.1: the problem is reduced to enumeration of particular singularities (f_i); if some of the hypersurfaces are in a mutually special position, then this should also be taken into account.

We consider some typical situations.

- Mutually generic smooth hypersurfaces. As was explained above, in this case all the degrees are necessarily equal to 1. Therefore, all the factors are linear; this case was treated in Appendix A.1.
- Mutually generic singular hypersurfaces. In this case every singular hypersurface is treated separately, then the results are combined. The simplest case is

$$f = g \prod_{i=1}^{k} f_i + \text{higher-order terms.} \tag{81}$$

Here f_i are homogeneous polynomials, while g is not necessarily homogeneous, with the condition: the lowest-order part of g is completely reducible. This kind of singularity occurs, for example, for curves $(n = 2)$ as $Z_{11}\, \mathrm{jet}_5(f) = z_2(z_1^3 + z_2^4)$, for surfaces $(n = 3)$ as $T_{455}\, \mathrm{jet}_4(f) = z_1(z_2 z_3 + z_1^3)$.
The lifted variety is

$$\widetilde{\Sigma}(x, (l_i)_{i=1}^{q-1}, \Omega^*) = \left\{ (x, (l_i)_{i=1}^{q-1}, \Omega^*, f) \middle| \begin{array}{c} f^{(p)} \sim \mathrm{SYM}(\Omega^{(p_1)}, \ldots, \Omega^{(p_k)}, \Omega^{(q)}) \\ \Omega^{(q)}(x) \sim \mathrm{SYM}(l_1, \ldots, l_{q-1}), \\ l_i(x) = 0 = \Omega^{(p_i)}(x) \end{array} \right\}. \tag{82}$$

- The normals to some of the hypersurfaces are in special position. The following are some simple cases.
- Several coinciding normals: $f = \left(\prod_{i=1}^{k} f_i\right) z_1^r \Omega^{(q)},\ \Omega^{(q)}(x) \sim x_1^{q-1}$. For example, this occurs for curves $(n = 2)$ as $A_3, D_6, E_7, E_8, W_{13}, W_{1,0}, W_{17}, W_{18}$, for surfaces $(n = 3)$ as $X_9 \ldots$. The lifted variety is

$$\widetilde{\Sigma}(x, l, \Omega^*) = \left\{ (x, l, \Omega^*, f) \middle| \begin{array}{l} f^{(p)} \sim \mathrm{SYM}(\Omega^{(p_1)}, \ldots, \Omega^{(p_k)}, l^r, \Omega^{(q)}) \\ \Omega^{(q)}(x) \sim \mathrm{SYM}(l, \ldots, l)\ l(x) \\ \quad = 0 = \Omega^{(p_i)}(x) \end{array} \right\}. \tag{83}$$

• Several normals in one plane: $f = l_1 \dots l_k \Omega^{(p)}$. For $n = 2$ it is, for example, an ordinary multiple point. For $n = 3$ $U_{12} : z_1^3 + z_2^3 + z_3^4$.

$$\widetilde{\Sigma}(x, (l_i)_{i=1}^k, \Omega^{(p)}) = \left\{ (x, (l_i)_{i=1}^k, f) \middle| \begin{array}{l} f^{(p+k)} \sim \mathrm{SYM}(l_1, \dots, l_k, \Omega^{(p)}) \\ (l_i(x) = 0)_{i=1}^3 \quad \Omega^{(p)}(x) = 0 \quad \mathrm{rk}\begin{pmatrix} l_1 \\ \dots \\ l_k \end{pmatrix} < 3 \end{array} \right\}. \tag{84}$$

• The intersection of two hypersurfaces lies in the singular locus of either one. For example, $f = z_1(z_2^2 \Omega^{(p)} + z_1 \Omega^{(p+1)})$. The simplest case is $n = 3 : S_{11}$ $\mathrm{jet}_3(f) = z_3(z_2^2 + z_1 z_3)$. We omit the calculations.

Appendix B. *Some results from intersection theory*

B.1. *The multiplicity of intersection*

At each step of the degenerating process we intersect the lifted stratum $\widetilde{\Sigma}$ with a hypersurface. As the intersection is in general non-transversal the resulting variety will be typically reducible: in addition to the required (degenerated) stratum it will contain a residual variety over a cycle of jump. This residual variety will (in general) enter with a non-trivial multiplicity. The multiplicity is calculated in the classical way. Suppose that the hypersurface is defined by the equation $\{g = 0\}$. Restrict the function g to the stratum $\widetilde{\Sigma}$ and find the vanishing order along the residual variety.

We illustrate this procedure in the following typical example.

EXAMPLE B.1. Degeneration of the ordinary multiple point. Start from the lifted stratum

$$\widetilde{\Sigma} = \{(x, f) \mid f^{(m)}|_x = 0\} \subset \mathbb{P}_x^n \times \mathbb{P}_f^{N_d}. \tag{85}$$

Suppose that we want to degenerate by intersection with the hypersurface

$$S = \{f|_x^{(p)}(\underbrace{y, \dots, y}_{k})_{i_1, \dots, i_{p-k}} = 0\} \quad m + 1 \leqslant p \leqslant l + k. \tag{86}$$

This case occurs, for example, in enumeration of multiple point of co-rank r (in particular A_2 point).

The intersection $S \cap \widetilde{\Sigma}$ is non-transversal over the diagonal $x = y$. To calculate the multiplicity, that is, to find the vanishing order we expand $y = x + \Delta y$. Then restricting to $\widetilde{\Sigma}$ we have (neglecting the numerical coefficients since we are interested in the vanishing order only):

$$f|_x^{(p)}(\underbrace{x + \Delta y, \dots, x + \Delta y}_{k})_{i_1, \dots, i_{p-k}} \sim \sum_{i=0}^{k} f|_x^{(p)}(\underbrace{x \dots x}_{k-i}, \underbrace{\Delta y \dots \Delta y}_{i})_{i_1, \dots, i_{p-k}} \sim f|_x^{(p)}(\underbrace{x, \dots, x}_{p-m-1}, \underbrace{\Delta y, \dots, \Delta y}_{k+1-p+m})_{i_1, \dots, i_{p-k}}$$
$$+ \text{higher-order terms}. \tag{87}$$

Therefore, the function $f|_x^{(p)}(\underbrace{y, \dots, y}_{k})_{i_1, \dots, i_{p-k}}$ has over the diagonal $x = y$ zero of the generic order $(k + m + 1 - p)$. Therefore,

$$\widetilde{\Sigma} \cap X = \widetilde{\Sigma}_{\text{degenerated}} \cup (k + m + 1 - p)\widetilde{\Sigma}|_{x=y}. \tag{88}$$

B.2. *On cohomology classes of cycles of jump*

The possible cycles of jump are described in § 3.1.1.2. Here we present their cohomology classes in the cohomology ring of their ambient space (which is the auxiliary space). The corresponding varieties (degeneracy loci) are known classically; in particular, the cohomology classes are given in [5, Section 14.5].

We define the incidence correspondence:

$$\Sigma = \{(y_1,\dots,y_k)|y_k \in \mathrm{Span}(y_1,\dots,y_{k-1})\} \subset \mathbb{P}^n_{y_1}\times\dots\times\mathbb{P}^n_{y_k}. \tag{89}$$

Note that for $k>2$ this variety is not closed. Its closure is

$$\overline{\Sigma} = \{\text{The points } (y_1,\dots,y_k) \text{ lie in a } (k-2)\text{-plane}\} \subset \mathbb{P}^n_{y_1}\times\dots\times\mathbb{P}^n_{y_k}. \tag{90}$$

The cohomology class of such variety is

$$[\overline{\Sigma}] = \sum_{i_1+\ldots+i_k=n+1-k} Y_1^{i_1}Y_2^{i_2}\cdot\ldots\cdot Y_k^{i_k}. \tag{91}$$

The points of $\overline{\Sigma}\setminus\Sigma$ correspond to configuration $\Big\{y_1,\dots,y_{k-1} \text{ are linearly dependent}\Big\}$. This subvariety (of codimension 1) will also be important in the calculations.

In case the cycle is defined by projection π_I (that is, $(\{\pi_I(y^{(j)}\}_{j\in J})$ are linearly dependent), one continues similarly (thinking of $\pi_I(y^{(j)}$ as being a point in $\pi_I(\mathbb{P}^n)$).

We will often face another condition of a special type: proportionality of symmetric tensors. Let a,b be two symmetric tensors of rank p. By writing their independent components in a row we can think of each of them as being a point (in homogeneous coordinates) of some big projective space $\mathbb{P}^N$. Then the proportionality of the tensors means that a and b coincide as the points in $\mathbb{P}^N$. This condition was considered above. Its cohomology class is

$$[a\sim b] = \sum_{i=0}^{\binom{p+n}{n}-1} A^iB^{\binom{p+n}{n}-1-i}. \tag{92}$$

Here, on the right-hand side, A,B are the cohomology classes of the elements of a,b, that is, the classes of the corresponding hypersurfaces. Equivalently, they are the first Chern classes of the corresponding line bundles.

B.3. *The cohomology class of a restriction of fibration*

As was explained in § 2.2 the fibration $\widetilde{\Sigma}\to \mathrm{Aux}$ is generically locally trivial, it is not locally trivial over the cycles of jump ($C_i\subset \mathrm{Aux}$). The key to degenerating procedure is the calculation of the cohomology class of the restriction $\widetilde{\Sigma}|_{C_i}\subset \mathrm{Aux}\times\mathbb{P}^{N_d}_f$.

The first naive idea is to represent it as a product, $[C]R_{\mathrm{codim}}\,\widetilde{\Sigma}-\mathrm{codim}\,C$, where R is a polynomial representing a class in $H^{2(\mathrm{codim}\,\widetilde{\Sigma}-\mathrm{codim}\,C)}(\mathrm{Aux}\times\mathbb{P}^{N_d}_f)$. This happens only in special cases.

Lemma B.2. • *If C is defined by a set of monomial equations in* Aux (*that is, by a set of the form $\vec{x}^{\vec{m}_1}=\ldots=\vec{x}^{\vec{m}_k}=0$, for $\vec{m}_i$ multi-degrees*), *then* $[\widetilde{\Sigma}|_C]=[C]R_{\mathrm{codim}\,\widetilde{\Sigma}-\mathrm{codim}\,C}$.

• *If C is a 'diagonal' in* Aux (*that is,* $C_k=\{(y_1,\dots,y_k)|\mathrm{linearlydependent}\}$) *then, for a 'flag of sub-diagonals'* ($C_i=\{(y_1,\dots,y_i)|\mathrm{linearlydependent}\}$, $2\leqslant i<k$), *we have*

$$[\widetilde{\Sigma}|_C]=[C_k]R_{\mathrm{codim}\,\widetilde{\Sigma}-\mathrm{codim\,C}}+[C_{k-1}]R_{\mathrm{codim}\,\widetilde{\Sigma}-1-\mathrm{codim}\,C}+\ldots+[C_2]R_{\mathrm{codim}\,\widetilde{\Sigma}-n}.$$

Proof. The first claim is immediate since it follows that (up to the multiplicity) the restriction $\widetilde{\Sigma}|_C$ lies in a linear subspace of $\mathrm{Aux}\times\mathbb{P}^{N_d}_f$. To prove the second, note that $\widetilde{\Sigma}$ is defined by a collection of conditions: $\{f^{(p)}(y_{i_1},\dots,y_{i_r})_{***}\}=0$ (§ 2.4.1). Thus over the open subset of $C: y_r\in\mathrm{Span}(y_1,\dots,y_{r-1})$ the variable y_r can be eliminated from the conditions. Therefore, the two sets of conditions (conditions of $C\subset\mathrm{Aux}$ and conditions of $\widetilde{\Sigma}$ over the open subset of C) are explicitly transversal. The non-transversality can occur only over the 'infinity': $(y_1,\dots,y_{r-1})|$linearly dependent. Therefore, we can write

$$[\widetilde{\Sigma}|_C]=[C_k]R_{\operatorname{codim}\widetilde{\Sigma}-\operatorname{codim}C}+[\text{a piece over }C_{k-1}].$$

Then by recursion we get the statement of the lemma. □

An important case of Lemma B.2 is the simplest case $C=\{y_1=y_2\}\subset \mathrm{Aux}=\mathbb{P}^n_{y_1}\times\mathbb{P}^n_{y_2}$. In this case the residual piece can be written explicitly as follows.

Corollary B.3. *Suppose that the projection $\widetilde{\Sigma}(y_1,y_2)|_{y_1=y_2}\mapsto\widetilde{\Sigma}(y_1)$ has the generic fibre $\mathbb{P}^r$, $0\leqslant r<n$. Then*

$$[\widetilde{\Sigma}(y_1,y_2)|_C]=\frac{[y_1=y_2]}{(n-r)!}\frac{\partial^{n-r}[\widetilde{\Sigma}(y_1,y_2)]}{\partial Y_2^{n-r}}.$$

Proof. Over the diagonal $(y_1=y_2)$ the variable y_2 can be completely eliminated from the defining conditions of $\widetilde{\Sigma}$. This corresponds to the projection $\mathrm{Aux}=\mathbb{P}^n_{y_1}\times\mathbb{P}^n_{y_2}\to\mathbb{P}^n_{y_1}$. Then the class of the image is obtained by the Gysin homomorphism (§ 1.4.1) from the initial class. □

In general, the calculation of the cohomology class of the restriction is done as follows. As will be explained later we can assume $\widetilde{\Sigma}|_{C_i}$ to be irreducible (reduced). The calculation is in fact a typical procedure from intersection theory and does not use any property of $\widetilde{\Sigma}$ related to the singularity theory.

Therefore, let $\widetilde{\Sigma}\subset\mathrm{Aux}\times\mathbb{P}^{N_d}_f$ be an irreducible (reduced) projective variety. In general $\widetilde{\Sigma}$ is not a globally complete intersection; however, we assume that we can calculate the cohomology class $[\widetilde{\Sigma}]\in H^*(\mathrm{Aux}\times\mathbb{P}^{N_d}_f)$ by the classical intersection theory (that is, by intersecting various hypersurfaces and subtracting the contributions of the residual pieces). Let $\{C_i\}_i$ be all the cycles of jump.

Lemma B.4. *The classes $[\widetilde{\Sigma}|_{C_i}]\in H^*(\mathrm{Aux}\times\mathbb{P}^{N_d}_f)$ can all be calculated recursively using the following data:*

- *the cohomology class of $\widetilde{\Sigma}$ obtained by the classical intersection theory;*
- *the cohomology classes of $C_i\subset\mathrm{Aux}$ obtained by the classical intersection theory.*

Proof. The proof goes by the induction on the grading of the cycles of jump and by the recursion on the dimensionality of the auxiliary space Aux.

We first calculate $[\widetilde{\Sigma}|_C]$ for C the cycle of jump of grading 1 (see Definition 2.2). As follows from Proposition 2.6 for such a cycle there exists a hypersurface $\{g=0\}$ containing C and not containing any other cycle. Therefore, consider the intersection $\widetilde{\Sigma}\cap\{g=0\}$. There can be two possibilities for the jump of dimension of fibre and the codimension of the cycle in $\mathrm{Aux}=\mathrm{Aux}_0$ (by Corollary 2.5).

- $\Delta\dim_{C_i}<\operatorname{codim}_{\mathrm{Aux}}(C)-1$. In this case the intersection gives just the restriction of the fibration $\widetilde{\Sigma}\cap(g=0)=\widetilde{\Sigma}|_{g=0}$, without any residual terms. Then decomposing the polynomial into irreducible factors $g=\prod_i g_i^{n_i}$ we get the union of restrictions $\widetilde{\Sigma}|_{g=0}=\bigcup_i n_i\widetilde{\Sigma}|_{g_i}$, each restriction again being irreducible.

Consider now the cycle C as a subvariety of the new (smaller) auxiliary space $Aux_1=\{g=0\}\subset Aux_0$. Intersect the cycle with the next hypersurface and so on.

• $\Delta\dim_C = \operatorname{codim}_{\mathrm{Aux}}(C) - 1$. In this case the intersection brings residual piece (of the same dimension):

$$\widetilde{\Sigma} \cap (g = 0) = \widetilde{\Sigma}|_{g=0} \cup \alpha\widetilde{\Sigma}|_C. \tag{93}$$

Here α is the multiplicity with which the residual piece enters. Note that the residual piece consists of the restriction to the cycle C only, because we have chosen C to be of grade 1 (and then all other restrictions are excluded by codimension). In this case we actually obtain the required restriction as a residual piece, so the problem is reduced to the calculation of $[\widetilde{\Sigma}|_{g=0}]$.

Thus in both cases the calculation is reduced to enumeration in a smaller auxiliary space $\mathrm{Aux}_1 = \{g = 0\} \subset \mathrm{Aux}_0$.

By the assumption of the lemma, the class $[\widetilde{\Sigma}]$ is obtained by the classical intersection procedure (consisting of intersections and removing the contribution of residual varieties). It follows that the class $[\widetilde{\Sigma}|_{g=0}]$ can be obtained by the same procedure. In the course of calculation there will appear new cycles of jump; however, the dimension of the auxiliary space has been reduced by 1. In this way, by recursion we obtain the class of restriction $[\widetilde{\Sigma}|_C]$ for any cycle C of grade 1.

The case of general grade is treated by induction. Suppose that we have calculated the classes of restriction $[\widetilde{\Sigma}|_{C_i}]$ for all the cycles of grades up to k. When doing the procedure for a cycle C of grade $(k+1)$ the only difference will be that equation (93) is replaced by a more general equation

$$\widetilde{\Sigma} \cap (g = 0) = \widetilde{\Sigma}|_{g=0} \cup \alpha\widetilde{\Sigma}|_C \cup \alpha_i\widetilde{\Sigma}|_{C_i}, \tag{94}$$

that is, on the right-hand side there appear restrictions to other cycles. However (as was noted above), by Lemma 2.6 other cycles will be of grade at most k, the case already solved. Therefore, from the above equation we get the class of the required restriction. □

B.4. *On use of consistency conditions*

In the preceding sections we have described how to calculate the cohomology classes of residual varieties. The method is recursive and is often quite cumbersome (though it is always possible to perform the calculations using a computer). Often lengthy calculations can be avoided by using (heavily) the consistency conditions.

The consistency conditions were stated in § 3.1.1 (Lemma 3.6 and Corollary 3.7). An 'experimental' observation is that they are very restrictive and in fact often *fix the cohomology classes of residual varieties*. (This happens for all the examples considered in the paper).

In general, the consistency conditions fix the cohomology class in the following equation.

$$\begin{aligned}[\widetilde{\Sigma}][\text{degeneration}] = [\widetilde{\Sigma}_{\text{degenerated}}] &+ [(y_1 \dots y_k)|\text{linearly dependent}][R_k] \\ &+ \dots + [(y_1, y_2)|\text{linearly dependent}][R_2]. \end{aligned} \tag{95}$$

Here the classes of the initial stratum $(\widetilde{\Sigma})$ and degenerating divisor/cycle are known, while the class of degenerated stratum $(\widetilde{\Sigma}_{\text{degenerated}})$ satisfies some consistency conditions (symmetric in some variables, with no terms of $Y^i, i > n - k$). The 'experimental fact' (which occurs in all the examples considered in the paper) is that the above equation, together with consistency conditions, has a unique solution.

The general formal method of calculation has already been explained in detail (§§ 3.1.1.2, 3.1.2, B.3). Thus we do not consider the above equation in detail and do not prove any general statement of uniqueness of solution. We emphasize, however, that all the results of this paper can be (and in fact were) obtained using only the consistency conditions.

Appendix C. *Some explicit formulae for cohomology classes of singular strata*

C.1. *Computer calculations*

As was already mentioned, except for the simplest cases (ordinary multiple points, $A_{k\leqslant 3}$) the computation should be done on computer (systems as Singular or Mathematica can be used). The specific programs can be obtained from the author.

Here we meet the following difficulty of a purely software nature. The calculation consists of addition/subtraction and multiplication of polynomials of indefinite degree and indefinite number of variables (that is, both the degree and the number of variables are parameters). For example, when enumerating the double point of co-rank r (§ 3.3.1) the multidegree of $\widetilde{\Sigma}^n_r$ is a polynomial in $(f, x, y_1, \dots, y_r)$ of degree $n+1+r(2n-1+r/2)$. Here both r and n are *parameters*. Another task is elimination and solution of systems of big linear equations. To the best of our knowledge, neither Singular nor Mathematica can in general process such expressions (that is, open the brackets, simplify, extract the coefficient of, say, $y_1^{n-r} \cdot \dots \cdot y_k^{n-r}$).

However the programs solve perfectly the problem for any fixed values of n, r. Thus to obtain the final answer (which is a polynomial in n, r) one should calculate separately for a sufficient number of pairs (n, r) and then interpolate. For the interpolation to be rigorous, we must know the degree of the polynomial we want to re-create. This degree is known by universality [**12**].

We emphasize that this problem arises only due to the current state of the software, and is not of any mathematical origin.

C.2. *On the possible checks of numerical results*

As in every problem, in which the answer is an explicit numerical formula, it is important to have some ways to check the numerical results. Our results 'successfully pass' the following checks.

- Comparison to the already known degrees. The most extensive 'database' in this case is Kazarian's tables of Thom classes for singularities of codimension up to 7. Our results are obtained by specializations from the general case to the case of complete linear system hypersurfaces of degree d in $\mathbb{P}^n$.

Very few of Kazarian's results were known before. The degree for ordinary multiple point is a classical result (known probably from the 19th century). Another check is for the cusp (A_2), enumerated by Aluffi.

- Comparison to the known results for curves ($n=2$). By universality, the substitution $n=2$ to the formulae must give the degrees of the strata for curves. This enables to check, for example, $A_3, A_4, D_4, D_5, E_6, X_9, Z_{11}, W_{12}$.
- Comparison to the case when the jet is reducible. For example, for singularity with degenerate quadratic form of co-rank k, the substitution $k=n-2$ or $k=n-1$ gives singularities with reducible two-jet. And in these cases the enumeration is immediate.

C.3. *Cohomology classes for hypersurfaces*

We present here the cohomology classes of the (minimally) lifted strata:

$$\widetilde{\Sigma}(x) = \overline{\{(x,f) | \text{the hypersurface defined by } f \text{ has singularity of the given type at the point } x\}} \subset \mathbb{P}^n_x \times \mathbb{P}^{N_d}_f. \tag{96}$$

The classes $[\widetilde{\Sigma}(x)]$ are expressed in terms of the generators of the cohomology ring of the ambient space ($H^*(\mathbb{P}^n_x) = \mathbb{Z}[X]/X^{n+1}$, $H^*(\mathbb{P}^{N_d}_f) = \mathbb{Z}[F]/F^{D+1}$). Therefore, the polynomials represent the multi-degrees of the lifted strata. The degree of the stratum itself $[\Sigma]$ is the coefficient of X^n.

All the notation are from [2]. Here (as anywhere in the paper) d is the degree of singular hypersurfaces, n is the dimensionality of the ambient space (thus hypersurfaces are of dimension $n-1$).

PROPOSITION C.1. *The following are the cohomology classes of the lifted strata and the degrees of the strata in several simplest cases.*

- Ordinary multiple point: $f=\sum_i z_i^{p+1}$. This includes, for $n=2$: $A_1, D_4, X_9, \ldots$; for $n=3$: $A_1, P_8, \ldots$.

$$[\widetilde{\Sigma}(x)] = Q^{\binom{n+p}{p}}, \quad Q=(d-p)X+F, \quad [\Sigma] = \binom{\binom{n+p}{p}}{n}(d-p)^n.$$

- Degenerate multiple point (with reducible defining form) (Defined in Appendix A): $\mathrm{jet}_p(f)=\prod_{i=1}^k \left(\Omega_i^{(p_i)}\right)^{r_i}$ $\sum_{i=1}^k r_i p_i = p$. This includes:
  - curves $n=2$:

$$\begin{gathered} A_1(k=2, r_i=1,\ p_i=1), \quad A_2(k=1, r_1=2,\ p_1=1),\\ D_4(k=3, r_i=1,\ p_i=1), \quad E_6(k=1, r_1=3,\ p_1=1),\\ X_9(k=4, r_i=1,\ p_i=1), \quad Z_{11}(k=2, r_1=1, r_2=3,\ p_i=1),\\ W_{12}(k=1, r_1=4,\ p_1=1), \ldots. \end{gathered}$$

  - surfaces $n=3$:

$$\begin{gathered} A_2(k=2, r_i=1,\ p_i=1), \quad D_4(k=1, r_1=2,\ p_1=1),\\ T_{3,4,4}(k=2, r_i=1,\ p_1=1, p_2=2), \quad T_{4,4,4}(k=3, r_i=1,\ p_i=1),\\ V_{1,0}(k=2, r_1=1, r_2=2,\ p_i=1), \quad V_1'(k=1, r_1=3,\ p_1=1)\ldots. \end{gathered}$$

Here the forms $\Omega_i^{(p_i)}$ are mutually generic (that is, the corresponding hypersurfaces intersect generically near the singular point). The cohomology (multi-)class was given in equation (79). To obtain the answer we should extract from the equation the coefficient of maximal (non-zero) powers of $\Omega_1^{(p_1)}\ldots\Omega_k^{(p_k)}$.

$$\begin{aligned} [\widetilde{\Sigma}(x)] = \frac{1}{|\mathrm{Aut}|} & \sum_{i=0}^{\binom{p+n}{n}-1} Q^{\binom{p+n}{n}-1-i} X^{k+i-\sum_{j=1}^k \binom{p_j-1+n}{p_j}} \\ & \times \sum_{\substack{i_1+\ldots+i_k=i\\ \binom{p_j-1+n}{p_j-1}-1\leqslant i_j\leqslant\binom{p_j+n}{n}-1}} \binom{i}{i_1\ldots i_k} \prod_{j=1}^k r_j^{i_j} \binom{\binom{p_j-1+n}{n}}{\binom{p_j+n}{n}-1-i_j}. \end{aligned} \tag{97}$$

Here Aut is the group of automorphisms of the branches, $\binom{i}{i_1\ \ldots\ i_k}$ is the multinomial coefficient from expansion of $(\ldots)^i$ and $Q=(d-p)X+F$.

- Singularity with degenerate quadratic form Σ_k^n :

$$f=\sum_{i=1}^k z_i^3 + \sum_{i=k+1}^n z_i^2. \quad (k=1: A_2, k=2: D_4, k=3: P_8\ldots).$$

$$[\widetilde{\Sigma}^n_k(x)] = C_{n,k}(Q+X)^{n+1}Q^{\binom{k}{2}}\sum_{i=0}^{k}\frac{\binom{n-i}{k-i}\binom{k}{i}}{\binom{2k}{k+i}}Q^{k-i}(-x)^i \quad Q=F+(d-2)X$$

$$C_{n,1}=2,\ C_{n,2}=2\binom{n+1}{1},\quad C_{n,3}=2\binom{n+2}{3},\quad C_{n,4}=2\binom{n+3}{5}\frac{n+1}{3}$$

$$C_{n,n}=2\binom{2n}{n},\quad C_{n,n-1}=\frac{2^k\binom{2k}{k}}{n},\quad C_{n,n-2}=\frac{\binom{2(k+1)}{k+1}\binom{2k}{k}}{\binom{k+2}{2}}.$$

In particular:

- ∘ $[\Sigma_{A_2}] = 3\binom{n+2}{3}(d-1)^{n-1}(d-2)$
- ∘ $[\Sigma_{D_4}] = (n+1)/8\binom{n+1}{3}(d-1)^{n-3}(d-2)^2\Big((d-2)(n^3+n^2+10n+8)+4(n^2+6)\Big)$
- ∘ $[\Sigma_{P_8}] = \binom{n+2}{3}\binom{n+4}{7}(d-1)^{n-6}(d-2)^3\Big(\binom{n+7}{3}(d-2)^3/10+\binom{n+6}{2}(d-2)^2 + (n+5)9(d-2)/2+12\Big)$

• $A_3: f = z_1^4+\sum_{i=2}^n z_i^2, \quad Q=F+(d-2)X, \quad [\widetilde{\Sigma}_{A_3}(x)] = (Q+X)^{n+1}(nQ-2X)(Q\frac{3n-1}{2}-4X).$

$$[\Sigma_{A_3}] = \binom{n+2}{3}(d-1)^{n-2}\left(\frac{(3n-1)(n+3)}{2}(d-2)^2+2(n-1)(d-2)-4\right).$$

• $A_4: f = z_1^5+\sum_{i=2}^n z_i^2.$

$$\begin{aligned}[\widetilde{\Sigma}_{A_4}(x)] = (Q+X)^{n+1}\Big(&\frac{n(5n^2-5n+2)}{2}Q^3-4(5n^2-3n+1)Q^2x\\ &+4(13n-5)\quad Qx^2-48x^3\Big)\quad Q=F+(d-2)X\end{aligned}$$

$$[\Sigma_{A_4}] = \frac{1}{8}\binom{n+2}{3}(d-1)^{n-3}\begin{pmatrix} d^3(24-46n+27n^2+30n^3+5n^4)\\ -d^2(+96-184n+138n^2+160n^3+30n^4)\\ +d(144-316n+192n^2+280n^3+60n^4)-136\\ +224n-48n^2-160n^3-40n^4\end{pmatrix}.$$

• D_5: $f = z_1^4+z_1z_2^2+\sum_{i=3}^n z_i^2, \quad Q=(d-2)X+F.$

$$[\widetilde{\Sigma}_{D_5}(x)] = (Q+X)^{n+1}\frac{Q(n+1)}{6}\Big((3n-2)Q-10X\Big)\begin{pmatrix}(n^2-n)Q^2-6(n-1)QX\\ +12X^2\end{pmatrix}.$$

• D_6: $f = z_1^5+z_1z_2^2+\sum_{i=3}^n z_i^2, \quad Q=(d-2)X+F$

$$\begin{aligned}[\widetilde{\Sigma}_{D_6}(x)] &= (Q+X)^{n+1}(1+n)Q\\ &\times\begin{pmatrix}\frac{4(n-1)n(3n^2-5n+3)}{15}Q^4-\frac{2(n-1)(16n^2-19n+9)}{3}Q^3x\\ +\frac{2(83n^2-140n+69)}{3}Q^2x^2-136(n-1)Qx^3+136x^4\end{pmatrix}.\end{aligned}$$

• E_6: $f = z_1^4+z_2^3+\sum_{i=3}^n z_i^2, \quad Q=(d-2)X+F.$

$$\begin{aligned}[\widetilde{\Sigma}_{E_6}(x)] &= (Q+X)^{n+1}Q(n+1)\\ &\times\begin{pmatrix}\frac{n(n-1)(12n^2-15n+2)}{40}Q^4-\frac{(n-1)n(15n-14)}{4}Q^3X\\ +\frac{37n^2-52n+12}{2}Q^2X^2-6(7n-5)QX^3+36X^4\end{pmatrix}.\end{aligned}$$

- X_9: $f = z_1^4 + z_2^4 + \sum_{i=3}^n z_i^2, \quad Q = (d-2)X + F.$

$$[\widetilde{\Sigma}_{X_9}(x)] = (Q+X)^{n+1}Q(n+1)$$
$$\times \begin{pmatrix} \frac{n(n-1)(3n+1)(33n^3-102n^2+102n-20)}{1680}Q^6 \\ -\frac{(n-1)n(138n^3-309n^2+153n+86)}{120}Q^5X \\ +\frac{(379n^4-1004n^3+719n^2+166n-160)}{40}Q^4X^2 \\ -\frac{126n^3-247n^2+58n+72}{2}Q^3X^3 \\ +\frac{625n^2-670n-216}{6}Q^2x^4 - 16(8n-1)QX^5 + 48X^6 \end{pmatrix}.$$

- Q_{10}: $z_1^4 + z_2^3 + z_1 z_3^2 + \sum_{i=4}^n z_i^2, \quad Q = (d-2)X + F.$

$$[\widetilde{\Sigma}_{Q_{10}}(x)] = (Q+X)^{n+1}\binom{n+2}{3}Q^3\Big((n-1)Q - 4X\Big)^2$$
$$\times \left(\frac{3}{5}\binom{n}{3}Q^3 - \frac{12}{5}\binom{n-1}{2}Q^2X + 6(n-2)QX^2 - 12X^3\right).$$

- S_{11}: $z_1^4 + z_2^2 z_3 + z_1 z_3^2 + \sum_{i=4}^n z_i^2, \quad Q = (d-2)X + F.$

$$[\widetilde{\Sigma}_{S_{11}}(x)] = (Q+X)^{n+1}6\binom{n+2}{3}Q^3\Big((n-1)Q - 4x\Big)$$
$$\times \begin{pmatrix} \frac{(n-2)(n-1)n(51n^2-98n+31)}{3360}Q^5 + \frac{(56n-79)}{2}Qx^4 \\ -\frac{(n-2)(n-1)(253n^2-392n+75)}{840}Q^4x - 27x^5 \\ +\frac{(n-2)(103n^2-190n+67)}{40}Q^3x^2 \\ -\frac{(47n^2-130n+75)}{4}Q^2x^3 \end{pmatrix}.$$

- U_{12}: $z_1^3 + z_2^3 + z_3^4 + \sum_{i=4}^n z_i^2, \quad Q = (d-2)X + F.$

$$[\widetilde{\Sigma}_{U_{12}}(x)] = (Q+X)^{n+1}\binom{n+2}{3}Q^3$$
$$\times \begin{pmatrix} \binom{n}{3}\frac{(n-1)(117n^3-328n^2+207n+12)}{1120}Q^7 \\ -\binom{n-1}{2}\frac{(323n^4-1057n^3+1021n^2-231n-72)}{336}Q^6x \\ +\binom{n-1}{2}\frac{(991n^3-2545n^2+1525n-27)}{84}Q^5x^2 \\ +\frac{(n-1)(2161n^2-5606n+2553)}{12}Q^3x^4 \\ -\frac{(3491n^4-16290n^3+25396n^2-14838n+2577)}{84}Q^4x^3 \\ +28(25n-31)Qx^6 - (475n^2-1150n+631)Q^2x^5 - 448x^7 \end{pmatrix}.$$

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