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Localization and periodicity in unstable homotopy theory.
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In this paper, the author studies certain localizations of the (pointed) homotopy category of spaces that he terms the v_n-periodizations, providing a conceptual home for the work of M. E. Mahowald [Ann. of Math. (2) **116** (1982), no. 1, 65–112; MR0662118], Mahowald and R. D. Thompson [Topology **31** (1992), no. 1, 133–141; MR1153241] and others on unstable v_n-periodic homotopy groups. In brief, the author combines his expertise on localization functors with the celebrated theorems of Devinatz, Hopkins, Smith, and Ravenel about stable chromatic phenomena [E. S. Devinatz, M. J. Hopkins and J. H. Smith, Ann. of Math. (2) **128** (1988), no. 2, 207–241; MR0960945; D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Ann. of Math. Stud., 128, Princeton Univ. Press, Princeton, NJ, 1992; MR1192553].

Given a fixed space W, the author calls another space Y W-periodic (or W-local) if $W \to *$ induces an equivalence $Y \cong \mathrm{Map}(W, Y)$, and then calls a map $f\colon A \to B$ a W-periodic equivalence if $f^*\colon\ \mathrm{Map}(B, Y) \cong \mathrm{Map}(A, Y)$ for each W-periodic Y. As in previous work of the author [see, e.g., Topology **14** (1975), 133–150; MR0380779; J. Pure Appl. Algebra **9** (1976/77), no. 2, 207–220; MR0478159] there then exists a natural W-periodization $u\colon X \to P_W X$, a W-periodic equivalence from X to a W-periodic space $P_W X$. The reader is urged to keep in mind the example $W = S^{n+1}$ for which $P_W X$ is the nth Postnikov section of X. The early sections of the paper are devoted to establishing a number of nonobvious properties of the functor P_W. As examples, the author proves the following theorems. Theorem 3.1: The natural map from $P_W(\Omega Y)$ to $\Omega(P_{\Sigma W} Y)$ is a homotopy equivalence. Theorem 4.1: A fiber sequence $F \to X \to B$ will naturally map to a fiber sequence $P_W F \to X' \to P_{\Sigma W} B$, with X' ΣW-periodic and $X \to X'$ a W-periodic equivalence. Theorem 7.1: For appropriate W, the homotopy fiber of the natural map $P_{\Sigma^2 W} Y \to P_{\Sigma W} Y$ is an Eilenberg-Mac Lane space. Theorem 8.1: For these same W, given a fiber sequence $F \to X \to B$, the fiber of the natural map from $P_{\Sigma W} F$ to the fiber of $P_{\Sigma W} X \to P_{\Sigma W} B$ is an Eilenberg-Mac Lane space. Some of these results apparently overlap with new work of E. Dror Farjoun.

Fixing a prime p, the paper then begins to focus on a specific family of periodizations, with P_{v_n} defined to be periodization by a suitable space W_n eventually revealed to have the form, if desired, $V_n \vee K(\mathbf{Z}/p, c(n) - 1)$, where V_n is any finite complex of type $n+1$ (in the complex oriented chromatic sense of Ravenel) and $c(n)$ is an unknown integer that would equal $n+2$ if an appropriate "weak unstable telescope conjecture" were to be true. Then Theorems 13.3 and 13.15, respectively, give homotopical and homological characterizations of when a map between $c(n)$-connected spaces becomes an equivalence after applying P_{v_n}. The specializations of these to the case $n = 1$ are then further refined using the v_1-telescope conjecture (known to be true) yielding (Theorem 14.7) that, given a map $f\colon X \to Y$ between 3-connected spaces, the following conditions are equivalent: (1) $f_*\colon v_1^{-1}\pi_*(X; \mathbf{Z}/p) \to v_1^{-1}\pi_*(Y; \mathbf{Z}/p)$ is an isomorphism; (2) $K_*(\tilde\Omega^k f; \mathbf{Z}/p)$ is an isomorphism for all k; (3) $K_*(\tilde\Omega^k f; \mathbf{Z}/p)$ is an isomorphism for some $k \geq 2$, where $\tilde\Omega X$ denotes the 3-connected cover of ΩX. It is important to note that the implication (1) $\implies$ (2) is independent of the telescope conjecture.

A concrete payoff is the verification of an old conjecture of Mahowald that the James-Hopf-Snaith map $\Omega_0^{2n+1} S^{2n+1} \to Q\mathbf{R}\mathrm{P}^{2n}$ induces an isomorphism in mod 2 K-theory.

Since the K-theory of the target of this map is known, the author has provided us with the key to explicit new calculations of the K-theory of iterated loopspaces.

As is hopefully clear from this description, this reviewer finds this paper full of interesting and important results. Unfortunately, the exposition demands a lot of patience on the part of the reader, as he or she must wade through cascades of lemmas and propositions, alert to those places where key input is injected into the arguments. *N. J. Kuhn*

References

1. A. K. Bousfield, *Types of acyclicity*, J. Pure Appl. Algebra **4** (1974), 293-298. MR0367978
2. A. K. Bousfield, *The localization of spaces with respect to homology*, Topology **14** (1975), 133-150. MR0380779
3. A. K. Bousfield, *Constructions of factorization systems in categories*, J. Pure Appl. Algebra **9** (1977), 207-220. MR0478159
4. A. K. Bousfield, *The Boolean algebra of spectra*, Comment. Math. Helv. **54** (1979), 368-377. MR0543337
5. A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), 257-281. MR0551009
6. A. K. Bousfield, *K-localizations and K-equivalences of infinite loop spaces*, Proc. London Math. Soc. **44** (1982), 291-311. MR0647434
7. A. K. Bousfield, *The simplicial homotopy theory of iterated loop spaces* (in preparation).
8. A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ-spaces, spectra, and bisimplicial sets*, Lecture Notes in Math., vol. 658, Springer-Verlag, Berlin and New York, 1978, pp. 80-130. MR0513569
9. A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math., vol. 304, Springer-Verlag, Berlin and New York, 1972. MR0365573
10. C. Casacuberta, G. Peschke, and M. Pfenninger, *On orthogonal pairs in categories and localizations*, London Math. Soc. Lecture Note Ser., vol. 175, Cambridge Univ. Press, Cambridge and New York, 1992, pp. 211-223.
11. F. Cohen and J. Neisendorfer, *A note on desuspending the Adams map*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 59-64. MR0809498
12. E. Devinatz, M. J. Hopkins, and J. H. Smith, *Nilpotence and stable homotopy*, Ann. of Math. (2) **128** (1988), 207-242. MR0960945
13. E. Dror Farjoun, *The localization with respect to a map and v_1-periodicity*, Lecture Notes in Math., vol. 1509, Springer-Verlag, Berlin and New York, 1992, pp. 104-113. MR1185964
14. E. Dror Farjoun, *Localizations, fibrations, and conic structures* (to appear).
15. E. Dror Farjoun and J. H. Smith, *Homotopy localization nearly preserves fibrations* (to appear). cf. MR1318881
16. M. J. Hopkins, *Global methods in homotopy theory*, London Math. Soc. Lecture Note Ser., vol. 117, Cambridge Univ. Press, Cambridge and New York, 1987, pp. 73-96. MR0932260
17. M. J. Hopkins and J. H. Smith, *Nilpotence and stable homotopy theory* II (to appear). cf. MR1652975
18. D. M. Kan, *On c.s.s. complexes*, Amer. J. Math. **79** (1957), 449-476. MR0090047
19. L. Langsetmo, *The K-theory localization of loops on an odd sphere and application*, Topology **32** (1993), 577-585. MR1231964
20. M. Mahowald, *The image of J in the EHP sequences*, Ann. of Math. (2) **116** (1982),

65-112. MR0662118

21. M. Mahowald and D. Ravenel, *Toward a global understanding of stable homotopy groups of spheres*, Contemp. Math., vol. 58, part II, Amer. Math. Soc., Providence, RI, 1987, pp. 75-88. MR0893848
22. M. Mahowald and W. Richter, *There exists a v_1^4 Adams map $M_{2\iota}^{13} \to M_{2\iota}^5$ of James filtration two* (to appear).
23. M. Mahowald and R. Thompson, *K-theory and unstable homotopy groups*, Contemp. Math., vol. 96, Amer. Math. Soc., Providence, RI, 1989, pp. 273-279. MR1022687
24. M. Mahowald and R. Thompson, *The K-theory localization of an unstable sphere*, Topology **31** (1992), 133-141. MR1153241
25. H. Miller, *The Sullivan conjecture on maps from classifying spaces*, Ann. of Math. (2) **120** (1984), 39-87. MR0750716
26. H. Miller and V. Snaith, *On the K-theory of the Kahn-Priddy map*, J. London Math. Soc. **20** (1979), 339-342. MR0551463
27. H. Miller and V. Snaith, *On $K_*(Q\mathbb{R}P^n;\ Z/2)$*, CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 1982, pp. 233-243. MR0686118
28. J. Milnor, *On the construction FK*, London Math. Soc. Lecture Note Ser., vol. **4**, Cambridge Univ. Press, Cambridge and New York, 1972, pp. 119-136.
29. G. Mislin, *Localizations with respect to K-theory*, J. Pure Appl. Algebra **10** (1977), 201-213. MR0467732
30. S. A. Mitchell, *Finite complexes with $A(n)$-free cohomology*, Topology **24** (1985), 227-246. MR0793186
31. S. Oka, *Existence of the unstable Adams map*, Mem. Fac. Sci. Kyushu Univ. **42** (1988), 95-108. MR0963199
32. D. G. Quillen, *Homotopical algebra*, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin and New York, 1967. MR0223432
33. D. C. Ravenel, *Localization with respect to certain periodic homology theories*, Amer. J. Math. **106** (1984), 351-414. MR0737778
34. D. C. Ravenel, *Progress report on the telescope conjecture*, London Math. Soc. Lecture Note Ser., vol. 176, Cambridge Univ. Press, Cambridge and New York, 1992, pp. 1-21. MR1232195
35. D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Ann. of Math. Stud., no. 128, Princeton Univ. Press, Princeton, NJ, 1992. MR1192553
36. D. C. Ravenel and W. S. Wilson, *The Morava K-theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture*, Amer. J. Math. **102** (1980), 691-748. MR0584466
37. G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293-312. MR0353298
38. E. Spanier, *Function spaces and duality*, Ann. of Math. (2) **69** (1959), 142-198. MR0105106
39. R. Thompson, *A relation between K-theory and unstable homotopy groups with an application to $B\Sigma_p$*, Contemp. Math., vol. 146, Amer. Math. Soc., Providence, RI, 1993, pp. 431-439. MR1224930
40. Z. Yosimura, *Universal coefficient sequences for cohomology theories of CW spectra*, Osaka J. Math. **12** (1975), 305-323. MR0388375

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.