MORAVA K-THEORIES AND INFINITE LOOP SPACES

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§1 Introduction and main results

For a fixed prime p, the Morava K-theories $K(0)_{\star}$, $K(1)_{\star}$, $K(2)_{\star}$, ... are a sequence of p-local periodic homology theories generalizing complex K-theory: $K(0)_{\star}$ is ordinary rational homology and $K(1)_{\star}$ is one of the (p-1) isomorphic summands of K-theory with mod p coefficients. These theories have various nice properties - for example, the associated spectra are ring spectra, the coefficients form a graded field, and thus there is a Kunneth formula (see [R2, Chapter 4, §2]). Recently, their central role in stable homotopy has been demonstrated by the work of M. Hopkins and J. Smith on maps between finite complexes [HS], using the remarkable nilpotence theorem of [DHS].

In [B3], A.K. Bousfield proved a beautiful theorem - K(1)localization factors through the 0th space functor - and used this to
reprove and strengthen the various delooping results of Adams-Priddy
[AP] and Madsen-Snaith-Tornehave [MST]. In this paper, I show how the
Hopkins-Smith work allows one to generalize Bousfield's argument to all

To state our main theorem, we need some notation. Let "Spaces" and "Spectra" respectively denote the homotopy categories of p-local spaces and spectra (as in, say, [A2]), and let Ω^{∞} : Spectra + Spaces be the 0th space functor, right adjoint to the suspension Σ^{∞} . Let $L_{K(n)}$: Spectra + Spectra be K(n)-localization [B1]. We will often write $E_{K(n)}$ for $L_{K(n)}$ (E). Recall the characterizing properties: $E + E_{K(n)}$ is a K(n) *-equivalence, and $[X, E_{K(n)}] = 0$ if K(n) *(X) = 0.

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sequences.

We note two pleasant corollaries.

Corollary 1.2 If $\Omega^{\infty}X \simeq \Omega^{\infty}Y$ then $K(n)_{\star}(X) \simeq K(n)_{\star}(Y)$ for all $n \geq 1$.

 $\frac{\text{Proof}}{X_{K(n)}}$ Applying Φ_n to the homotopy equivalence $\Omega^{\infty}X \simeq \Omega^{\infty}Y$ shows that

Corollary 1.3 Let $f: X \to Y$ be a map between spectra. If $\Omega^{\infty}f$ has a section (i.e. a right inverse) then so does $K(n)_{\star}(f)$ for $n \ge 1$. In particular, the K(n)-homology suspension $K(n)_{\star}(\Omega^{\infty}E) \to K(n)_{\star}(E)$ is onto for all E.

<u>Proof</u> Applying Φ_n to the section of $\Omega^{\infty}f$ shows that $f_{K(n)}$ has a section, and the first statement follows. For the second, note that the homology suspension is induced by the evaluation map $\epsilon \colon \Sigma^{\infty}\Omega^{\infty}E \to E$, and $\Omega^{\infty}\epsilon$ has a section.

Note that these corollaries would be false with $K(n)_{\star}$ replaced by ordinary homology, while they are essentially tautologically true (when restricted to (-1)-connected spectra) with $K(n)_{\star}$ replaced by π_{\star}^{S} . The failure of 1.3 for HZ/p is the source of the "unstable" condition for A-modules, thus 1.3 implies that there is no analogous condition for $K(n)_{\star}$.

The key to all of these results is the existence of interesting self maps of finite complexes inducing isomorphisms in K(n). An introduction to the use of such maps in our context occurs in §2. This contains an elementary proof of corollary 1.3 that is independent of both the theory of localizations (and thus Theorem 1.1) and the nilpotence conjecture. The only nontrivial input needed here is

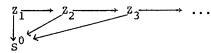
(1.4) There exists a finite complex Z with $K(n)_{\star}(Z) \neq 0$ and a $K(n)_{\star}$ -equivalence $v \colon \Sigma^{d} Z \to Z$ with d > 0.

For small n, this has been known for quite awhile. Adams constructed a $K(1)_{\star}$ -self equivalence of the Moore space in [A1], while, for n = 2 or 3, Toda's spaces V(n) do the job [T]. For arbitrary n, a Z satisfying (1.4) is constructed in [HS], independent of the main theorems of [DHS]. (To paraphrase Mike Hopkins, this Z is the first

kid on the block with an ice cream cone.)

Section 3 has the proof of the main theorem - a streamlined version of Bousfield's argument in [B3]. The proof is basically formal, except that (1.4) must be strengthened to

(1.5) There exists a commutative diagram of finite complexes



such that

- (i) $\lim K(n)_{\star}(Z_i) \simeq K(n)_{\star}(S^0)$.
- (ii) each Z_i has a $K(n)_*$ -equivalence $v_i \colon \sum_{i=1}^{d_i} Z_i \to Z_i$ with $d_i > 0$, and these self maps are compatible, for different i, after suitable finite iteration.

This is proved in §4 using the whole strength of Hopkins and Smith's work. It should perhaps be pointed out that the length of [B3] is partly due to the fact that Bousfield had to prove a version of (1.5) pre Devanitz-Hopkins-Smith.

Section 5 contains some questions, conjectures, and examples, e.g., a computation of BP*(g/pl) at the prime 2.

Finally, I wish to thank Pete Bousfield and Mike Hopkins for their help in this project. I came across [B2] after having already discovered Corollary 1.3 and the argument of §2. An exchange of letters (and preprint [B3]) with Pete, and subsequent conversations with Mike, led to Theorem 1.1. It is only excessive modesty that caused each of them to decline joint authorship.

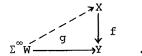
§2 A proof that (1.4) => (1.3)

We begin with the following elementary observation.

Lemma 2.1 Let $f:X \to Y$ be a map between spectra. The following are equivalent:

(1) Ω^{∞} f has a section.

(2) Any map $g: \Sigma^{\infty}W \rightarrow Y$ lifts

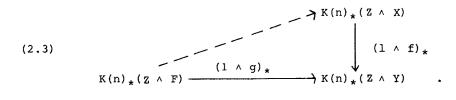


Now we assume that $f: X \to Y$ is a map satisfying the conditions of this last lemma. We wish to show that $K(n)_*(f)$ is onto.

Since Y is a direct limit of its finite subspectra, it suffices to show that given a map $g\colon F\to Y$, where F is a finite complex, there exists an algebraic lifting

(2.2)
$$K(n)_{\star}(F) \xrightarrow{g_{\star}} K(n)_{\star}(Y) .$$

Now let Z be as in (1.4). Because $K(n)_{\star}$ is a field, $K(n)_{\star}(Z) \neq 0$, and $K(n)_{\star}$ satisfies a Kunneth formula, to show that the lifting exists in (2.2), it suffices to show that it exists in the diagram



Now choose N so large that $\Sigma^{\mbox{dN}} \mbox{Z} \wedge \mbox{F} \wedge \mbox{DZ}$ is a suspension spectrum, where DZ denotes the Spanier-Whitehead dual of Z. By our assumption on f, there is a lifting

(2.4)
$$\Sigma^{dN} Z \wedge F \wedge DZ \xrightarrow{g_N} Y ,$$

where g_N is dual to $v^N \wedge g$: $\Sigma^{dN} Z \wedge F \rightarrow Z \wedge Y$. (v^N) is the Nth iterate of v.)

Adjointing yields a diagram

Since $K(n)_{\star}(v)$ is an isomorphism, applying $K(n)_{\star}$ () to (2.5) yields the lifting in (2.3).

§3 Proof of Theorem 1.1

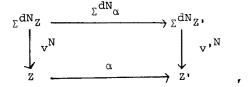
We start with our basic construction.

Construction 3.1 Suppose that Z is a space, and v: $\Sigma^{\vec{d}}Z \to Z$ is a self map with d > 0. We construct a functor

with structure maps

$$\Phi_{Z}^{\prime}(X)_{md} = \text{Map}(Z,X) \xrightarrow{V^{*}} \Omega^{d} \text{Map}(Z,X) = \Omega^{d}\Phi_{Z}^{\prime}(X)_{(m+1)d}$$

Note that a commutative diagram



with N ϵ N, induces a natural transformation.

$$\alpha^*: \Phi_Z^*, \longrightarrow \Phi_Z^*$$
.

We list some basic properties of Φ '.

Proposition 3.2

- (1) $v^*: \Phi_Z^!(X) \simeq \Phi_{\Sigma}^!(X)$, naturally in both Z and X.
- (2) $\Phi_{\mathbf{Z}}^{\bullet}(\mathbf{X})$ preserves fibrations in the variable X, and cofibrations in the variable Z.
- (3) $\Phi_{Z}^{\bullet}(X)$ is periodic with period d.
- (4) $\Phi_Z^{(1)}(\Omega^{\infty}E) \simeq v^{-1}F(Z,E)$, naturally in both Z and X.

In (4), F(Z,E) denotes the function spectrum defined, by Brown Representability, so that $[Y \land Z,E] \simeq [Y, F(Z,E)]$, and $v^{-1}F(Z,E)$ is the direct limit lim $\{F(Z,E) \xrightarrow{V} F(\Sigma^d Z,E) \xrightarrow{V} F(\Sigma^{2d} Z,E) \xrightarrow{} \ldots \}$.

<u>Proof of Proposition 3.2</u> Properties (1), (2) and (3) are clear by inspection. For (4), note that $Map(Z, \Omega^{\infty}E) \simeq \Omega^{\infty}F(Z,E)$. Then (4) follows from the next lemma by letting $E(m) = F(\Sigma^{md} Z,E)$ and $d_m = md$.

Lemma 3.3 Suppose given a sequence of spectra

 $\begin{array}{l} {\rm E}(0) \xrightarrow{} {\rm E}(1) \xrightarrow{} {\rm E}(1) \xrightarrow{} {\rm E}(2) \xrightarrow{} \ldots \text{, and an increasing sequence of natural numbers, d}_0 < {\rm d}_1 < {\rm d}_2 < \ldots \text{.} \\ {\rm Define a new spectrum E by} \\ {\rm letting E}_{\rm d_m} = \Omega^{\infty} \ \Sigma^{\rm d_m} \ {\rm E}(m) \text{, with structure maps} \end{array}$

$$\Omega^{\infty} \Sigma^{d_m} f_m : E_{d_m} \rightarrow \Omega^{d_{m+1} - d_m} E_{d_{m+1}}$$
. Then $E \simeq \lim_{n \to \infty} E(m)$.

<u>Proof</u> We can assume $E(m)_i = \Omega^\infty \Sigma^i E(m)$. Thus, for $n \ge m$, the f_m induce maps $E(m)_{d_n} \to E_{d_n}$. These fit together to give a map $\lim_{n \to \infty} E(m) \to E$, which is easily checked to be an isomorphism on homotopy groups.

Note that property (1) of Proposition 3.2 allows us to extend the construction Φ' to any *finite spectrum* Z with self map v: one simply replaces the pair (Z,v) by $(\Sigma^{dN} Z, \Sigma^{dN} v)$ with N large.

Now let $\Phi_{Z} = L_{K(n)} \circ \Phi_{Z}'$: Spaces \rightarrow Spectra.

<u>Proposition 3.4</u> If Z is finite and $K(n)_{\star}(v)$ is an isomorphism, there is a natural equivalence $\Phi_{Z}(\Omega^{\infty} E) \simeq F(Z, E_{K(n)})$.

 $\underline{\operatorname{Proof}}$ By Proposition 3.2 (4), we need to show that there is an equivalence

$$F(Z,E_{K(n)}) \simeq (v^{-1}F(Z,E))_{K(n)}$$
.

First note that $F(Z, E_{K(n)})$ is K(n)-local, since $E_{K(n)}$ is. Thus, to finish the proof, it suffices to show that the natural maps

$$F(Z,E_{K(n)}) \leftarrow F(Z,E) \longrightarrow v^{-1} F(Z,E)$$

induce isomorphisms in $K(n)_{\star}$. But, since Z is finite, these maps are equivalent to the maps

$$DZ \wedge E_{K(n)} \leftarrow DZ \wedge E \longrightarrow v^{-1}(DZ \wedge E)$$
.

Both of these maps are clearly $K(n)_*$ -equivalences, the second because $K(n)_*$ (v) is an isomorphism.

The definition of Φ_n , and the proof of theorem 1.1 are now remarkably easy, assuming (1.5).

Definition 3.5 With
$$Z_1 \rightarrow Z_2 \rightarrow Z_3 \rightarrow \dots$$
 as in (1.5), let $\Phi_n = \lim \Phi_{Z_n}$: Spaces \rightarrow Spectra.

Proof of Theorem 1.1 There are natural equivalence

$$\begin{array}{lll} \Phi_{\mathbf{n}}(\Omega^{\infty} \ E) & \cong & \lim\limits_{\longleftarrow} \Phi_{\mathbf{Z}_{\mathbf{i}}} & (\Omega^{\infty} \ E) \\ & \cong & \lim\limits_{\longleftarrow} F(\mathbf{Z}_{\mathbf{i}}, E_{K(\mathbf{n})}) \\ & \cong & F(\lim\limits_{\longrightarrow} \mathbf{Z}_{\mathbf{i}}, E_{K(\mathbf{n})}) \\ & \cong & F(\mathbf{S}^{\mathbf{0}}, E_{K(\mathbf{n})}) \\ & \cong & E_{K(\mathbf{n})} & \end{array}$$

Here the second equivalence follows from Proposition 3.4. The fourth equivalence holds because $\lim_{i \to S} S^{0}$ is a $K(n)_{\star}$ -equivalence ((1.5) (i)), and $E_{K(n)}$ is K(n)-local.

Finally, that $\boldsymbol{\Phi}_n$ preserves fibrations is a direct consequence of Proposition 3.2 (2).

Exercise Prove Corollary 1.2 (without using Theorem 1.1!), by just using the Φ_{Z}^{\bullet} construction applied to the pair (Z,v) of (1.4).

§4 $\frac{c_{n}\text{-resolutions}}{c_{n}}$

In this section we develop the theory of what we dub " \mathcal{C}_n -resolutions": approximations to a fixed finite complex by complexes admitting K(n)*-equivalences. A special case will be (1.5).

We need some notation and definitions from [HS]. Let $\mathcal C$ be the p-local, stable homotopy category of finite complexes, and let $\mathcal C_n$ be the full subcategory consisting of the K(n-l)*-acyclic complexes.

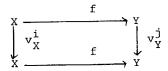
<u>Definition 4.1</u> For $X \in \mathcal{C}$, a map $v: \Sigma^d X \to X$ is a v_n -self map if $K(n)_*(v)$ is an isomorphism, and $K(m)_*(v)$ is nilpotent for $m \neq n$.

For the rest of this section we will repress suspensions " Σ^d " (i.e. view morphisms as having possibly nonzero degrees).

The main theorem of [HS] is

Theorem 4.2

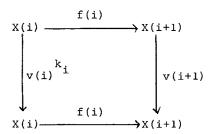
- (1) $X \in C_n$ if and only if X has a v_n -self map.
- (2) Given X,Y \in C_n, with respective v_n-self maps v_X, v_Y, and f: X \rightarrow Y, there exist integers i, j such that



commutes.

Note that this has the following consequence.

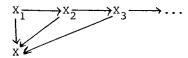
such that



commutes for all i.

We now define our approximations.

<u>Definition 4.4</u> For X ϵ C, a C_n-resolution of X (written X_{*} \rightarrow X) is a commutative diagram



such that

- (1) $X_i \in C_n$
- (2) $\lim_{\longrightarrow} K(m)_{*}(X_{1}) \rightarrow K(m)_{*}(X)$ is an isomorphism for $m \ge n$.

In light of Corollary 4.3, (1.5) is essentially the case $X = S^0$ of the following theorem.

Theorem 4.5 Every X ϵ C has a c_n -resolution.

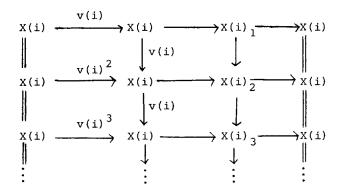
It is handy to have the following notion: A map $f_{\star}\colon X_{\star} \to Y_{\star}$ of \mathcal{C}_n -resolutions, over $f\colon X \to Y$, is a collection of maps $f_n\colon X_n \to Y_n$ making all the obvious diagrams commute. Similarly, given a commutative diagram D of spectra in \mathcal{C} , there is an obvious notion of a commutative diagram of \mathcal{C}_n -resolutions over D.

As a first step towards proving Theorem 4.5, we prove

Proposition 4.6 Given
$$X(1) \xrightarrow{f(1)} X(2) \xrightarrow{f(2)} X(3) \xrightarrow{f(3)} \dots$$
 with

X(i) ϵ $c_{\rm n}$, there exists a commutative diagram of $c_{\rm n+1}$ -resolutions

 \underline{Proof} Let v(i) and $k_{\underline{i}}$ be as in Corollary 4.3. For each i, we have a diagram of cofibration sequences, defining $X(i)_{\underline{i}}$:



Furthermore, we then get a commutative diagram

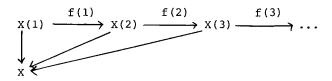
by letting $f(i)_{j}$ be the composite $X(i)_{j} \rightarrow X(i)_{k_{i}j} \rightarrow X(i+1)_{j}$.

We claim that $X(i)_{\star} \to X(i)$ is a C_{n+1} -resolution. By construction $X(i)_{j} \in C_{n+1}$ for all j. For $m \ge n+1$, $K(m)_{\star}(v(i))$ is nilpotent. Thus applying $K(m)_{\star}$ to (4.7) yields an isomorphism

$$\lim_{\stackrel{\rightarrow}{j}} K(m)_{\star}(X(i)_{j}) \xrightarrow{\sim} K(m)_{\star}(X(i)).$$

Proof of Theorem 4.5 We prove this by induction on n. Assuming that

there is a C_n -resolution of X,



we will construct a C_{n+1} -resolution.

Find C_{n+1} -resolutions of the X(i) as in the last proposition. Then let $Y_i = X(i)_i$, and let $Y_i \to Y_{i+1}$ be the composite $X(i)_i \to X(i+1)_i \to X(i+1)_{i+1}$. We claim that $Y_\star \to X$ is a C_{n+1} -resolution. This is easy to check: $\{Y_i\}$ are cofinal in $\{X(i)_j\}$, so that, for $m \ge n+1$,

§5 Examples and Conjectures

Our first example uses a consequence of Corollary 1.2. Recall a definition from [R1]: Harmonic localization is localization with respect to V K(n). Ravenel shows that BP is harmonic, as is any $n \ge 0$ finite complex.

<u>Proposition 5.1</u> Suppose that $\Omega^{\infty}X$ is homotopic to a weak product of spaces $\Omega^{\infty}Y_{\mathbf{i}}$, where each $Y_{\mathbf{i}}$ has only finitely many non zero homotopy groups. Then

$$x \longrightarrow x_Q$$

is harmonic localization.

<u>Proof</u> Eilenberg-MacLane spectra are K(n)*-acyclic for n \geq 1[R1], thus so are spectra with only a finite number of homotopy groups. By hypothesis, $\Omega^{\infty} X \simeq \Omega^{\infty} Y$ where Y is a wedge of such spectra. It follows that K(n)*(X) = 0 for n \geq 1, so that X \rightarrow X_Q is a K(n)*-equivalence for all n \geq 0.

Example 5.2 Let g, pl, and top be the usual spectra with 0th spaces G, PL and Top, as in [MM]. (G is the group of stable homotopy equivalences of spheres, etc.) By [MM, Theorem 4.8 and remark 4.36], if X is either g/pl or g/top, then X satisfies the hypothesis of the last proposition at the prime 2. It follows, e.g., that, at the prime 2, BP*(g/pl) \cong BP*((g/pl) $_{\odot}$).

Remark 5.3 We would like to thank Frank Adams for explaining to us how easy it is to calculate [HQ,E]. In particular, $BP^*(\Sigma^{-1} \ HQ) \ \simeq \ Z_{\widehat{D}} \ / \ Z_{(p)} \ \otimes \ BP^*, \ from \ which \ one \ can \ calculate \ BP^*(X_Q).$

Example 5.4 $K(n)_{\star}(g) \simeq K(n)_{\star}(s^{0})$ for $n \ge 1$.

Proof Let Q_0S^0 be the basepoint component of $\Omega^\infty\Sigma^\infty S^0$, so that $Q_0S^0=\Omega^\infty(S<0>)$ where S<0> is the 0-connected cover of $\Sigma^\infty S^0$. Then $\Omega^\infty g=G\simeq Q_0S^0\times Z/2=\Omega^\infty(S<0>\vee HZ/2)$. Thus 1.2 implies that $K(n)_*(g)\simeq K(n)_*(S<0>\vee HZ/2)$. But HZ/2 and HZ are both $K(n)_*$ -acyclic, so $K(n)_*(S<0>\vee HZ/2)\simeq K(n)_*(S^0)$.

Our next examples are applications of Corollary 1.3. As in [K2], call a sequence of spectra ... \rightarrow X₂ \rightarrow X₁ \rightarrow X₀ \rightarrow E₀ exact if it is formed by splicing together cofibration sequences $E_{i+1} \rightarrow X_i \xrightarrow{f_i} E_i$ where each $\Omega^{\infty}f_i$ has a section. Corollary 1.3 implies

<u>Proposition 5.4</u> If ... \rightarrow $X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow E$ is an exact sequence of spectra, then, for all $n \ge 1$, there is a long exact sequence

...
$$\rightarrow$$
 K(n)_{*}(X₁) \rightarrow K(n)_{*}(X₀) \rightarrow K(n)_{*}(E) \rightarrow 0.

Example 5.5 The K(n)-homology suspension epimorphism can be extended, using the canonical resolution based on the adjoint pair $(\Sigma^{\infty}, \Omega^{\infty})$:

...
$$\rightarrow K(n) \cdot (Q\Omega^{\infty}E) \rightarrow K(n) \cdot (\Omega^{\infty}E) \xrightarrow{\varepsilon_{\star}} K(n) \cdot E \rightarrow 0$$
.

(Here QX denotes $\Omega^{\infty}\Sigma^{\infty}X$.)

Example 5.6 For a "smaller" bound on the kernel of ϵ_{\star} , we use the main theorem of [K4]: If E is 0-connected, at the prime 2 there is an exact sequence

$$\Sigma^{\infty} D_{2} \Omega^{\infty} E \xrightarrow{f_{2}} \Sigma^{\infty} \Omega^{\infty} E \xrightarrow{\varepsilon} E .$$

Here $D_2(X)$ is the quadratic construction on X, and f_2 is the composite $\Sigma^{\infty}D_2(\Omega^{\infty}E) \longrightarrow \Sigma^{\infty}Q(\Omega^{\infty}E) \xrightarrow{\Sigma^{\infty}\Omega^{\infty}E} \Sigma^{\infty}\Omega^{\infty}E$. (See [K4] for the odd primary

analogue.) It follows that there is an exact sequence:

$$K(n)_{\star}(D_{2}(\Omega^{\infty}E)) \xrightarrow{f_{2}\star} K(n)_{\star}(\Omega^{\infty}E) \xrightarrow{\varepsilon_{\star}} K(n)_{\star}(E) \rightarrow 0$$

Example 5.7 In [K1, KP1], we constructed a "minimal spacelike resolution" of HZ_(D) extending the Kahn-Priddy epimorphism:

...
$$\rightarrow$$
 L(2) \rightarrow L(1) \rightarrow L(0) \rightarrow HZ_(p)

Here $L(0) = \Sigma^{\infty}S^{0}$, $L(1) = \Sigma^{\infty}B\Sigma_{p}$, and, in general, L(m) is an indecomposable stable wedge summand of $B(Z/p)_{+}^{m}$. L(m) is $K(n)_{*}$ -acyclic if m > n [W, K5]. It follows that, for all $n \ge 1$, there is an exact sequence

(5.8)
$$0 \rightarrow K(n)_{\star}(L(n)) \rightarrow ... \rightarrow K(n)_{\star}(L(1)) \rightarrow K(n)_{\star}(L(0)) \rightarrow 0$$
.

This generalizes the well known isomorphism $K(1)_*(B\Sigma_p) \xrightarrow{\sim} K(1)_*(S^0)$ (see e.g. [K3]).

Note that (5.8) implies that

$$\sum_{m=0}^{n} (-1)^{m} \dim_{K(n)_{\star}} K(n)_{\star} (L(m)) = 0.$$

This was observed computationally in [K5], and first caused us to try to prove Corollary 1.3.

With end this section with some questions and conjectures, aimed at making stronger use of Theorem 1.1.

Question 5.8 How faithful is the functor $VL_{R(n)}$: Spectra \rightarrow Spectra?

Clearly, one should begin by restricting to the harmonic subcategory. Bousfield [B4] has pointed out that the cofibers of maps $S_K^{-2} \to S_K^0$ provide infinitely many distinct K-local spectra all having identical K(0) and K(1) localizations.

With Mike Hopkins, we conjecture

Conjecture 5.9 If X and Y are finite spectra, then

$$X_{K(n)} \simeq Y_{K(n)}$$
 for all $n \Rightarrow X \simeq Y$.

A consequence of Theorem 1.1 and the validity of this conjecture would be:

 $\Omega^{\infty} X \simeq \Omega^{\infty} Y \Longrightarrow X \simeq Y \text{ for all finite } X \text{ and } Y.$

We repeat a question from [R1].

Question 5.10 Is every suspension spectrum harmonic?

Conjecture 5.11 QX \approx QY => Σ^{∞} X \approx Σ^{∞} Y.

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