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2-PRIMARY EXPONENTS FOR THE HOMOTOPY GROUPS OF SPHERES

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§1. INTRODUCTION

WE SAY that p^r is a p-primary exponent for an abelian group G if $p^r \cdot (p\text{-torsion of } G) = 0$. The best possible p-primary exponent for G is least such p^r and will be denoted $\exp_p(G)$. In [7], James proved that

$$\exp_2(\pi_*(S^{2n+1})) \leq 4 \exp_2(\pi_*(S^{2n-1}))$$

and thus by induction on n obtained the exponent 2^{2n} for $\pi_*(S^{2n+1})$. This exponent has been improved slightly to 2^{2n-2} by Cohen[1] for $n \geq 4$. Barratt and Mahowald conjecture that the 2-primary exponent of $\pi_*(S^{2n+1})$ increases with n according to the scheme

$$\exp_2(\pi_*(S^{2n+1})) = \lambda \exp_2(\pi_*(S^{2n-1}))$$

where

$$\lambda = \left\{ \begin{array}{ll} 4 & n \equiv 1 \pmod 4 \\ 2 & n \equiv 0 \text{ or } 2 \pmod 4 \\ 1 & n \equiv 3 \pmod 4 \end{array} \right\}.$$

(See [4]). Mahowald[8] has shown that the exponent of $\pi_*(S^{2n+1})$ is at least as large as this conjecture indicates. In this paper we prove that

$$\exp_2(\pi_*(S^{2n+1})) \leq 2 \exp_2(\pi_*(S^{2n-1}))$$

for $n \equiv 0 \pmod 4$ thus reducing the proof of the Barratt–Mahowald conjecture to the proof of the case $n \equiv 3 \pmod 4$. Combined with James' result we obtain the exponent $2^{(3/2)n+\epsilon}$ for $\pi_*(S^{2n+1})$, where

$$\epsilon = \left\{ \begin{array}{ll} 0 & n \equiv 0 \pmod 2 \\ \frac{1}{2} & n \equiv 1 \pmod 2 \end{array} \right\}.$$

The problem of computing the best possible exponents for the homotopy groups of spheres at odd primes has been completely solved in the papers[2, 3, 5, 9–12].

§2. PROOF OF THE MAIN THEOREM

Throughout this section all spaces and maps will be assumed to have been localized at 2.

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Given a homotopy associative H-space X and a positive integer k, let $X\{k\}$ denote the homotopy–theoretic fibre of the kth power map $k: X \to X$ which takes x to x^k. We denote the James Hopf–invariant map by $H: \Omega S^m \to \Omega S^{2m-1}$. This map was constructed by James in [7], where he showed that

$$S^{m-1} \xrightarrow{E} \Omega S^m \xrightarrow{H} \Omega S^{2m-1}$$

is a homotopy–theoretic fibration.

Let Q denote the homotopy–theoretic pullback of H: $\Omega S^{2n+1} \to \Omega S^{4n-1}$ and the canonical map i: $\Omega S^{4n+1}\{2\} \to \Omega S^{4n+1}$. Looping gives the following commutative diagram in which all rows and columns are homotopy–theoretic fibrations

$$\begin{tikzcd}
 & \Omega S^{2n} \arrow[r, equal] \arrow[d, "a"] & \Omega S^{2n} \arrow[d] \\
\Omega^3 S^{4n+1} \arrow[r, "j"] \arrow[d, equal] & \Omega Q \arrow[r] \arrow[d, "b"] & \Omega^2 S^{2n+1} \arrow[d, "\Omega H"] \\
\Omega^3 S^{4n+1} \arrow[r, "k"] & \Omega^2 S^{4n+1}\{2\} \arrow[r, "i"] & \Omega^2 S^{4n+1}.
\end{tikzcd}$$

Since $2\Omega H$: $\Omega^2 S^m \to \Omega^2 S^{2m-1}$ is null homotopic when m is odd (see Cohen[1], Lemma 3.1) the principal fibration

$$\Omega^3 S^{4n+1} \xrightarrow{j} \Omega Q \to \Omega^2 S^{4n+1}$$

splits to give

$$\Omega Q \approx \Omega^3 S^{4n+1} \times \Omega^2 S^{4n+1}.$$

Let s: $\Omega Q \to \Omega^3 S^{4n+1}$ be such that $sj \simeq 1_{\Omega^3 S^{4n+1}}$.

Remark. The fact $2\Omega H \simeq 0$ for m odd was proved independently by M. Barratt and F. Cohen. The only proof which is in print is Cohen[1]. Notes taken by Neisendorfer at a lecture by Barratt in 1978 show that Barratt had previously proved this fact but by 1980 Barratt had forgotten both the proof and the fact that he had once proved this statement. The proof was then rediscovered by Cohen.

THEOREM 1. *If* $2^r \pi_q(S^{4n-1}) = 0$ *for* $q > 4n - 1$ *then* $2^{r+1} \pi_q(S^{4n+1}) = 0$ *for* $q > 4n + 1$.

Remark. This theorem was proved in the cases $n = 1$ and $n = 2$ by Cohen ([1], Corollary 3.5).

Proof. Suppose that $2^r \pi_t(\Omega S^{4n-1}) = 0$ where $t > 4n - 2$. Let $x \in \pi_t(\Omega^3 S^{4n+1})$. We have $b_\# j_\#(2x) = k_\#(2x) = 0$ since $k \circ 2$ is the composition of consecutive maps in a fibration sequence. Thus $j_\#(2x) = a_\#(y)$ for some $y \in \pi_t(\Omega S^{2n})$. Since 2^r is an exponent for $\pi_t(\Omega S^{4n-1})$, from James' fibration

$$S^{2n-1} \xrightarrow{E} \Omega S^{2n} \xrightarrow{H} \Omega S^{4n-1}$$

we conclude that $H_{\#}(2^r y) = 2^r H_{\#}(y) = 0$ and so $2^r y = E_{\#}(z)$ for some $z \in \pi_t(\Omega S^{2n-1})$. Thus

$$s_{\#} a_{\#} E_{\#}(z) = s_{\#} a_{\#}(2^r y) = 2^r s_{\#} j_{\#}(2x) = 2^{r+1} x.$$

On the other hand $saE: S^{2n-1} \to \Omega^3 S^{4n+1}$ is null homotopic for degree reasons, so $2^{r+1}x = 0$ as desired. □

COROLLARY: $2^{(3/2)n+\epsilon}\pi_q(S^{2n+1}) = 0$ for $q > 2n + 1$ where

$$\epsilon = \begin{Bmatrix} 0 & n \equiv 0 \pmod 2 \\ 1 & n \equiv 1 \pmod 2 \end{Bmatrix}.$$ □

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