Finite subgroups and isogenies of one-parameter formal Lie groups*

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The purpose of this paper is to tie up certain loose ends from [2]; to prove the conjectures of the last section there, and to complete the proofs promised in the correction-note [3]. The objects under study here are the same as in [2], namely one-parameter formal Lie groups F defined over the ring of integers $\mathfrak{o}$ in a finite extension of the field $\mathbf{Q}_p$ of p-adic numbers. However the point of view is somewhat different here for we now make essential use of the analytic group structure which the power series F furnishes to the maximal ideal in any complete local $\mathfrak{o}$-algebra.

The relationship between lattices in the Tate vector space $V(F)$ and formal groups isogenous to F is an example of a phenomenon which seems to be well known, at least in the parallel case of abelian varieties, but I was first made aware of this relationship by a remark in a letter of J-P. Serre to J. Tate. The same letter contains an outline of a proof of Lemma 1.3 which is somewhat better than my own proof, and which I have made use of here.

For definitions the reader is referred to [2]; as there, I use here the expression *group law* as abbreviation for *one-parameter commutative formal Lie group*.

1. Generalities: finite subgroups of a formal group

1.0. Let p be a prime number. The completion of the field of rational numbers with respect to the p-adic valuation will be denoted $\mathbf{Q}_p$, and its ring of integers $\mathbf{Z}_p$. If K is a field which is algebraic over $\mathbf{Q}_p$, the integral closure of $\mathbf{Z}_p$ in K will be denoted $I(K)$. The algebraic closure of a field K will be called $\bar{K}$ and, if L is normal over K, the Galois group will be called Aut (L/K). Local rings will not necessarily be noetherian; and, if A is a local ring, $M(A)$ will denote its maximal ideal.

Throughout this paper, k will be a finite (algebraic) extension of $\mathbf{Q}_p$. We will be concerned with group laws F defined over $\mathfrak{o} = I(k)$, and the *height* of F will be the height of its reduction modulo $M(\mathfrak{o})$. If K is any finite extension of k, with $\mathfrak{O} = I(K)$, then the set $M(\mathfrak{O})$ has a group structure given by F: for $\alpha, \beta \in M(\mathfrak{O})$, $\alpha +_F \beta = F(\alpha, \beta)$; the identity element is 0 and the inverse of α is $[-1]_F(\alpha)$. These series converge because $\mathfrak{O}$ is complete. This group, which

* This work was supported in part by a grant from the Research Corporation, and in part by NSF Grant #GP-5098.

is also an $\mathrm{End}_{\mathfrak{o}}(F)$-module, will be denoted $F(\mathfrak{O})$. Even though $\bar{\mathfrak{o}} = I(\bar{k})$ is not complete, $+_F$ is defined in $M(\bar{\mathfrak{o}})$ because any two elements of $\bar{k}$ are in some finite extension K of k, and consequently we can speak of $F(\bar{\mathfrak{o}})$.

We see immediately that if F is a group law defined over $\mathfrak{o}$, then every finite subgroup of $F(\bar{\mathfrak{o}})$ is a p-group, because if $[mp^n]_F(\alpha) = 0$ for $\alpha \in M(\bar{\mathfrak{o}})$ and m prime to p, then $[m^{-1}]_F \in \mathrm{End}_{\mathfrak{o}}(F)$ and $[p^n]_F(\alpha) = 0$.

The following form of the abstract Weierstrass preparation theorem, essentially as found in [1, p. 72], is quoted here without proof.

LEMMA 1.1. *Let A be a complete noetherian local domain. Let $f(x) \in A[[x]]$ have first unit coefficient in degree d. Then there is a monic polynomial $P(x) \in A[x]$ of degree d with all lower degree coefficients in $M(A)$ and a power series $u(x) \in A[[x]]$ with $u(0) \notin M(A)$ such that $f = Pu$.*

The number d will be called the *Weierstrass degree* of f, wideg (f), as in [1].

1.2. Suppose now that F and G are group laws over $\mathfrak{o}$ with F of finite height, and that $0 \neq f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$. We can see now that $f\colon F(\bar{\mathfrak{o}}) \to G(\bar{\mathfrak{o}})$ is onto and has finite kernel. Let $\beta \in M(\bar{\mathfrak{o}})$, so $\beta \in M(\mathfrak{O})$ for $\mathfrak{O} = I(K)$, K being some finite extension of k. Then $f(x) - \beta \in \mathfrak{O}[[x]]$ is not zero modulo $M(\mathfrak{O})$, in view of [2, Lem. 2.3.1], so $1 \leqq \mathrm{wideg}(f-\beta) < \infty$, and so $f-\beta$ has roots in $M(\bar{\mathfrak{o}})$. Let us call F_1 the derivative of F with respect to the lefthand argument. It has constant term 1, so that for $\gamma \in M(\bar{\mathfrak{o}})$, $F_1(0, \gamma)$ is a unit in $\bar{\mathfrak{o}}$. Then if $f(\alpha) = \beta$, α is a simple root of $f(x) - \beta$ because on differentiating the identity $F(f(x), \beta) = f(F(x, \alpha))$ and setting $x = 0$ we get $F_1(0, \beta)c(f) = f'(\alpha)F_1(0, \alpha)$ so that $f'(\alpha) \neq 0$ because [2, Lem. 2.1.1] $c(f) \neq 0$. Thus $f(x) - \beta$ has wideg (f) roots, and in particular the kernel of $[p^n]_F$ has p^{hn} elements if h is the height of F.

The rest of this section is devoted to showing the converse, that every finite subgroup of $F(\bar{\mathfrak{o}})$ arises as the kernel of some $f\colon F(\bar{\mathfrak{o}}) \to G(\bar{\mathfrak{o}})$, and that the kernel has certain functorial properties.

LEMMA 1.3. *Let A be a power series ring over $\mathfrak{o}$, in finitely many variables. Let F be a group law defined over $\mathfrak{o}$, and Γ a finite subgroup of $F(\mathfrak{o})$. Let the group Γ operate on $A[[x]]$ by: for $\gamma \in \Gamma$ and $f(x) \in A[[x]]$, $f^{\gamma}(x) = f(F(x, \gamma))$. Then the subring of $A[[x]]$ of fixed elements is $A[[\xi]]$, where $\xi = \prod_{\gamma \in \Gamma} F(x, \gamma)$.*

PROOF. Let $B = A[[\xi]]$ and $C = A[[x]]$ have the fraction fields K and L respectively. One checks that Γ acts as a group of automorphisms of C. This action extends to L, and we can see that K is the fixed field. The power series $-\xi + \prod_{\gamma \in \Gamma} F(X, \gamma) \in B[[X]]$ has Weierstrass degree $p^s =$ order of Γ, since wideg $(F(X, \gamma)) = 1$. Lemma 1.1 shows that this power series can be written

$P(X)u(X)$ where P is a monic polynomial over B of degree p^s. Since x generates L over K, and x is a root of P, $[L: K] \leqq p^s$. But K is contained in the fixed field of Γ, over which L is of degree p^s, so that K is the fixed field; and $B = K \cap C$ because C is a free B-module, a basis being $\{1, x, \cdots, x^{p^s-1}\}$.

THEOREM 1.4. *Let K be a finite extension of k, and $\mathfrak{O} = I(K)$. Let F be a group law defined over $\mathfrak{o}$, and Γ a finite subgroup of $F(\mathfrak{O})$. Then there is a group law G defined over $\mathfrak{O}$ and an $f \in \mathrm{Hom}_{\mathfrak{O}}(F, G)$ with $\ker(f) = \Gamma$. Furthermore, if Γ is stable under the action of $\mathrm{Aut}(\bar{k}/k)$, then G and f can be defined over $\mathfrak{o}$.*

PROOF. We set $f(x) = \prod_{\gamma \in \Gamma} F(x, \gamma) \in \mathfrak{O}[[x]]$ so that $f(F(x, y)) = \prod_{\gamma \in \Gamma} F(F(x, y), \gamma) \in \mathfrak{O}[[x, y]]$. We can apply Lemma 1.3 to $f(F(x, y))$ by taking $A = \mathfrak{O}[[x]]$ and $A = \mathfrak{O}[[f(y)]]$ in turn to see that $f(F(x, y)) \in \mathfrak{O}[[f(x), f(y)]]$; i.e., there is $G(x, y) \in \mathfrak{O}[[x, y]]$ such that $f(F(x, y)) = G(fx, fy)$. Since $G(x, y) = f(F(f^{-1}x, f^{-1}y))$, G is a group law and $f \in \mathrm{Hom}_{\mathfrak{O}}(F, G)$. By our construction, $\ker(f) = \Gamma$. If Γ is stable under the action of $\mathrm{Aut}(\bar{k}/k)$, then f is clearly fixed under the action of $\mathrm{Aut}(\bar{k}/k)$ on $\mathfrak{O}[[x]]$, so $f \in \mathfrak{o}[[x]]$, and consequently G is defined over $\mathfrak{o}$ as well.

THEOREM 1.5. *Let F, G_1, G_2 be group laws defined over $\mathfrak{o}$ with F of finite height, $f_i \in \mathrm{Hom}_{\mathfrak{o}}(F, G_i)$, $(i = 1, 2)$, $f_1 \neq 0$, and suppose that f_2 vanishes on $\ker(f_1) \subset F(\bar{\mathfrak{o}})$. Then there is a unique $g \in \mathrm{Hom}_{\mathfrak{o}}(G_1, G_2)$ such that $g \circ f_1 = f_2$.*

PROOF. Let $\mathfrak{O} = I(K)$ for K a finite extension of k large enough for $\ker(f_1)$ to be contained in $F(\mathfrak{O})$. We apply Lemma 1.3 to f_1 and f_2 with $\Gamma = \ker(f_1)$ and $A = \mathfrak{O}$ to find $\varphi_i \in \mathfrak{O}[[x]]$ such that $f_i = \varphi_i \circ \psi$ where $\psi(x) = \prod_{\gamma \in \Gamma} F(x, \gamma) \in \mathfrak{O}[[x]]$. Now observe first that φ_i has no constant term because neither f_i nor ψ does, and second that $\mathrm{wideg}(\varphi_1) = 1$ because (by virtue of 1.3) $\mathrm{wideg}(f_1) = \mathrm{wideg}(\psi)$. Thus φ_1 has an inverse in $\mathfrak{O}[[x]]$ and, if we set $g = \varphi_2 \circ \varphi_1^{-1} \in \mathfrak{O}[[x]]$, we see that the relation $g = f_2 \circ f_1^{-1} \in k[[x]]$ implies that g is unique and that $g \in \mathrm{Hom}_{\mathfrak{o}}(G_1, G_2)$.

1.6. In particular, if $0 \neq f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$ and $\ker(f)$ has p^s elements, then $[p^s]_F$ vanishes on $\ker(f)$, so if F is of finite height there is $g \in \mathrm{Hom}_{\mathfrak{o}}(G, F)$ such that $g \circ f = [p^s]_F$ (and $f \circ g = [p^s]_G$). Thus a non-zero homomorphism defined on a group law of finite height is an isogeny in the sense of [2, 5.3.1].

2. The Tate groups

2.0. In this section we define the Tate groups and use them for classifying isogenies. If F is a group law defined over $\mathfrak{o}$, let us call $\Lambda(F)$ the set of all elements of $F(\bar{\mathfrak{o}})$ of finite order. Then $\Lambda(F)$ is the union of all $\ker[p^n]_F$. We define the *Tate group* of F, $T(F)$, to be the set of all sequences $a = (a_0, a_1, \cdots)$

for which $a_0 = 0$ and, for each $i \geqq 1$, $a_i \in \Lambda(F)$ and $[p]_F(a_i) = a_{i-1}$. Addition is defined coordinatewise, and if G is another group law defined over $\mathfrak{o}$, coordinatewise application of $f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$ gives $f\colon T(F) \to T(G)$. If F is of finite height, one sees immediately, because of surjectivity of $[p]_F\colon \Lambda(F) \to \Lambda(F)$ that $a \mapsto a_n$ is a surjection of $T(F)$ on $\ker [p^n]_F$, and that $a \in p^n T(F)$ if and only if $a_n = 0$. If $0 \neq f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$ then $f\colon T(F) \to T(G)$ is an injection because: $\ker(f) \subset \ker [p^s]_F$ for some s, and so, if $f(a_{n+s}) = 0$ for all n, we have $[p^s]_F(a_{n+s}) = 0$ for all n, and $a = 0$. We thus see that $T(F)$ is a torsion-free module over $\mathrm{End}_{\mathfrak{o}}(F)$ and over $\mathbf{Z}_p$. If F is of height $h < \infty$, then $T(F)/pT(F)$ is a vector space of dimension h over $\mathbf{Z}/p\mathbf{Z}$ and, since only 0 is in every $p^n T(F)$, $T(F)$ is a free $\mathbf{Z}_p$-module of rank h. Let us define $V(F)$ as the set of all sequences $a = (a_0, a_1, \cdots)$ such that for each $i \geqq 1$, $a_i \in \Lambda(F)$ and $[p]_F(a_i) = a_{i-1}$. Clearly $V(F) \cong T(F) \otimes_{Z_p} \mathbf{Q}_p$ so that $V(F)$ is an h-dimensional $\mathbf{Q}_p$-vector space. We can associate to each $a \in V(F)$ its initial coordinate $a_0 \in \Lambda(F)$ to get a homomorphism which is surjective because each element of $\Lambda(F)$ is, for some n, the n^{th} coordinate of some element of $T(F)$. Thus $\Lambda(F) \cong V(F)/T(F)$.

2.1. We saw in § 1 that, if F is a group law of height $h < \infty$ defined over $\mathfrak{o}$, finite subgroups of $\Lambda(F)$ classify isogenies (i.e. non-zero homomorphisms) defined on F; if Γ is such a subgroup, Γ is the kernel of some isogeny; and if $\Gamma_1 \subset \Gamma_2 \subset \Lambda(F)$ with $\Gamma_i = \ker(f_i)$, then there is a unique g such that $g \circ f_1 = f_2$. In particular, if $\Gamma_1 = \Gamma_2$ then this g is an isomorphism. We now examine the relation between group laws isogenous with F and *lattices* in $V(F)$; i.e., free $\mathbf{Z}_p$-submodules of $V(F)$ which are of rank $h = \dim(V(F))$.

THEOREM 2.2. *Let F be a group law of finite height defined over $\mathfrak{o}$, and L a lattice in $V(F)$ such that $p^r T(F) \subset L \subset p^{-s} T(F)$, for non-negative r and s. Let Γ be the group of all elements of $\Lambda(F)$ which appear as r-coordinates of elements of L, and let K be a finite extension of k such that $\Gamma \subset \mathfrak{D} = I(K)$. Then there is a group law G defined over $\mathfrak{D}$, $f \in \mathrm{Hom}_{\mathfrak{D}}(F, G)$ with $\ker(f) = \Gamma$, and $g \in \mathrm{Hom}_{\mathfrak{D}}(G, F)$ with $g \circ f = [p^{r+s}]_F$. If L is stable under the action of $\mathrm{Aut}(\bar{k}/k)$ on $V(F)$, then G, f, and g may be defined over $\mathfrak{o}$. Finally, if L_1 and L_2 are two such lattices with $L_1 \subset L_2$ where L_i gives rise to G_i, f_i, and g_i, then there is a unique $\varphi\colon G_1 \to G_2$ such that $\varphi \circ f_1 = f_2$ and $g_2 \circ \varphi = g_1$.*

$$\begin{tikzcd}
F \arrow[d, "{[p^r]}"'] \arrow[dr, "f_1"] \arrow[drr, "f_2"] & & \\
F \arrow[d, "{[p^s]}"'] & G_1 \arrow[r, "\varphi"] \arrow[dl, "g_1"'] & G_2 \arrow[dll, "g_2"] \\
F & &
\end{tikzcd}$$

PROOF. The case $r > 0$ follows from the case $r = 0$.

The isomorphism of $\Lambda(F)$ with $V(F)/T(F)$ induces a one-to-one inclusion-preserving correspondence between finite subgroups of $\Lambda(F)$ and lattices in $V(F)$ which contain $T(F)$, and such a lattice is stable under the action of $\mathrm{Aut}\,(\bar{k}/k)$ if and only if the corresponding finite subgroup is stable also. Notice that if $s \geqq 0$, then $p^{-s}T(F)$ is a lattice in $V(F)$ containing $T(F)$ which corresponds to $\ker\,[p^s]_F$. Thus the case $r = 0$ follows from the parallel facts about finite subgroups of $\Lambda(F)$, namely Theorems 1.4 and 1.5.

2.3. Theorem 2.2 shows that lattices containing $T(F)$ correspond to equivalence classes of isogenies defined on F and sublattices of $T(F)$ correspond to equivalence classes of isogenies into F, two such isogenies being considered equivalent if the homomorphism φ above is an isomorphism. Let us notice that if $L \subset T(F)$ and L gives rise to $g\colon G \to F$, then in fact $L = g(T(G))$. To see this, let $p^r T(F) \subset L \subset T(F)$, so that we get an $f\colon F \to G$ whose kernel consists of all r-coordinates of elements of L, and such that $f \circ g = [p^r]_G$. Let $b \in T(G)$: then the r-coordinate of $g(b)$ is $g(b_r)$ which is annihilated by f, so that $g(b)$ and some element of L have the same r-coordinate, and their difference then is in $p^r T(F) \subset L$. Consequently $g(T(G)) \subset L$. On the other hand, if $a \in L$, the r-coordinate of $f(a)$ is zero, so $f(a) \in p^r T(G)$ and $a \in g(T(G))$.

3. Applications

3.0. We now have in $T(F)$ a tool for investigating the endomorphism rings of group laws. For instance, if F and G are group laws of finite height over $\mathfrak{o}$ which are isogenous over $\mathfrak{o}$, then the rings $c(\mathrm{End}_{\mathfrak{o}}(F))$ and $c(\mathrm{End}_{\mathfrak{o}}(G))$ have the same fraction field. Indeed, if $0 \neq f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$, then there is r such that $p^r T(G) \subset f(T(F))$; and so if $[\zeta]_G \in \mathrm{End}_{\mathfrak{o}}(G)$, then $[p^r\zeta]_G(T(G)) \subset f(T(F))$. Then the lattice inclusion $([p^r\zeta]_G \circ f)(T(F)) \subset f(T(F)) \subset T(G)$ implies existence of $\varphi \in \mathrm{End}_{\mathfrak{o}}(F)$ such that $f \circ \varphi = [p^r\zeta]_G \circ f$, and φ is necessarily $[p^r\zeta]_F$. Thus $p^r c(\mathrm{End}_{\mathfrak{o}}(G)) \subset c(\mathrm{End}_{\mathfrak{o}}(F))$ and symmetrically $p^s c(\mathrm{End}_{\mathfrak{o}}(F)) \subset c(\mathrm{End}_{\mathfrak{o}}(G))$ for some s.

Suppose that F is a group law of finite height defined over $\mathfrak{o}$ and that Σ is the fraction field of $c(\mathrm{End}_{\mathfrak{o}}(F))$. Since $V(F)$ is an $\mathrm{End}_{\mathfrak{o}}(F)$-module, it is also a Σ-module: for $\zeta \in \Sigma$ and $a \in V(F)$, ζa is $p^{-n}[p^n\zeta]_F(a)$ for large enough n. If F and G are both defined over $\mathfrak{o}$ and isogenous over $\mathfrak{o}$, then Σ operates on both $V(F)$ and $V(G)$, and any $f \in \mathrm{Hom}_{\mathfrak{o}}(F, G)$ induces a Σ-linear map $f\colon V(F) \to V(G)$.

THEOREM 3.1. *Let F be a group law of finite height defined over $\mathfrak{o}$, and L an* $\mathrm{Aut}\,(\bar{k}/k)$*-stable lattice in $V(F)$ giving rise to a group law G defined over $\mathfrak{o}$. Let Σ be the fraction field of* $c(\mathrm{End}_{\mathfrak{o}}(F))$. *Then* $\zeta \in c(\mathrm{End}_{\mathfrak{o}}(G))$ *if and only if $\zeta \in \Sigma$ and $\zeta L \subset L$.*

PROOF. First, in case $L = T(F)$, suppose $\zeta \in \Sigma$ and $\zeta T(F) \subset T(F)$. Then for n so large that $p^n\zeta \in c(\mathrm{End}_{\mathfrak{o}}(F))$, we have $[p^n\zeta]_F(T(F)) \subset [p^n]_F(T(F))$ which means that there is $\varphi \in \mathrm{End}_{\mathfrak{o}}(F)$ with $[p^n]_F \circ \varphi = [p^n\zeta]_F$. But φ is necessarily $[\zeta]_F$. The converse is immediate.

The case of general L comes down immediately to the case $L \subset T(F)$, when $L = g(T(G))$, $g: G \to F$. Since g is a Σ-isomorphism of $V(G)$ onto $V(F)$, $\zeta g(T(G)) \subset g(T(G))$ if and only if $\zeta T(G) \subset T(G)$. q.e.d.

3.2. We can now prove the conjectures 5.3.1 and 5.3.2 of [2]. We use the fact from [2, paragraph 2.3.3] that, if K is an extension of k with $\mathfrak{O} = I(K)$ and F is defined over $\mathfrak{o}$, then $c(\mathrm{End}_{\mathfrak{O}}(F)) = \mathfrak{O} \cap c(\mathrm{End}\,(F))$.

If R is an order in $\mathfrak{o}$, there is a group law G whose absolute endomorphism ring $\mathrm{End}\,(G)$ is isomorphic to R. To construct G, we start with a full group law F defined over $\mathfrak{o}$ such that $c(\mathrm{End}\,(F)) = \mathfrak{o}$. The best construction of such F is found in [4, Th. 1]. Now $T(F)$ is a free $\mathbf{Z}_p$-module of rank $h = [k: \mathbf{Q}_p]$ so that $T(F)$ is a free $\mathrm{End}\,(F)$-module of rank one. Let a basis element be b. Then since R is a free $\mathbf{Z}_p$-module of rank h, Rb is a lattice in $T(F)$ which will be $\mathrm{Aut}\,(\bar{k}/K)$-stable for K a suitable finite extension of k, and Rb gives rise to a group law G defined over $\mathfrak{O} = I(K)$. Then $c(\mathrm{End}_{\mathfrak{O}}(G)) = R$, by Theorem 3.1, and $\mathrm{End}_{\mathfrak{O}}(G) = \mathrm{End}\,(G)$ because $c(\mathrm{End}\,(G)) \subset \mathfrak{o} \subset \mathfrak{O}$. (An argument like the proof of Theorem 3.3 below shows that it is impossible to take $K = k$.)

Another consequence of Theorem 3.1 is that any group law F of finite height defined over $\mathfrak{o}$ is $\mathfrak{o}$-isogenous to a group law G defined over $\mathfrak{o}$ with $\mathrm{End}_{\mathfrak{o}}(G)$ integrally closed. To see this we call Σ the fraction field of $c(\mathrm{End}_{\mathfrak{o}}(F))$ and form $L = I(\Sigma)T(F)$ which is a lattice in $V(F)$, clearly $\mathrm{Aut}\,(\bar{k}/k)$-stable. Certainly if $\zeta \in I(\Sigma)$, we have $\zeta L \subset L$. In particular every almost full group law is isogenous to a full one.

Finally we have the partial replacement of the incorrect Theorem 3.3.1 of [2]. The proof below, more perspicuous than the original proof, as outlined in [3], was suggested by the referee.

THEOREM 3.3. *Let k be unramified over $\mathbf{Q}_p$ and $\mathfrak{o} = I(k)$. If F is a group law of finite height defined over $\mathfrak{o}$, then the absolute endomorphism ring* $\mathrm{End}\,(F)$ *is integrally closed in its fraction field.*

PROOF. The Eisenstein criterion shows that $[p]_F(x)/x$ is irreducible in $\mathfrak{o}[[x]]$, so that (0) is the only proper $\mathrm{Aut}\,(\bar{k}/k)$-submodule of $\ker\,[p]_F$. In view of the exact sequence of Galois modules $0 \to pT(F) \to T(F) \to \ker\,[p]_F \to 0$, there are no proper sublattices L of $T(F)$ containing $pT(F)$ properly, which are stable under $\mathrm{Aut}\,(\bar{k}/k)$. Now consider the absolute endomorphism ring $\mathrm{End}\,(F)$ and its maximal ideal M. They are both $\mathrm{Aut}\,(\bar{k}/k)$-stable, as is $MT(F)$, and

$pT(F) \subset MT(F) \subset T(F)$ where the second inclusion is proper, since $T(F)$ is finitely generated and non-trivial, and End (F) is noetherian. Thus $MT(F) = pT(F)$, so $p^{-1}MT(F) = T(F)$. But by Theorem 3.1, if $c \in p^{-1}M$, then $c \in$ End (F). Thus $M = p$ End (F), so that End (F) is a discrete valuation ring and so integrally closed. q.e.d.

This proof shows again [2, Cor. 3.3.2] that the fraction field of End (F) is unramified over $\mathbf{Q}_p$, if k is unramified over $\mathbf{Q}_p$.

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(Received June 8, 1966)