Nathaniel Stapleton's REGS Report

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Stable homotopy theory is the study of generalized cohomology theories. A generalized cohomology theory is a rule that associates to a topological space some kind of algebraic structure so that if two spaces are homotopy equivalent the attached algebraic structures are isomorphic. For an arbitrary cohomology theory it is very hard to compute the cohomology of a space. However, a particularly tractable class of cohomology theories are the complex oriented ones. They are tractable because each of them has an associated group scheme (a formal group) that controls much of the computation. The group schemes have an associated height, a natural number analagous to the order of a finite group, that can be used to organize the complex oriented cohomology theories. Those of higher height are considered more complicated than those of lower height. For decades these cohomology theories have been studied with great success from the perspective of algebraic geometry.

For every finite group G, associated to every cohomology theory E there is an equivariant cohomology theory E_G defined by $E_G(X) = E(EG \times_G X)$ for G-spaces X. Work of Hopkins, Kuhn, and Ravenel in the 80's and 90's described a way of constructing, for every finite group G, a map from particularly nice equivariant complex oriented theories of this sort to a height 0 equivariant theory of their creation. This eases some computations dramatically and is interesting in its own right because it gives a way of understanding things of one height in terms of a lower height. My research has focused around generalizing their results in order to obtain maps with the same domain but where the equivariant cohomology theory in the codomain can be constructed to have any height h between 0 and n.

Let K be complex K-theory and let $R(G)$ be the complex representation ring of a finite group G. Considering a representation as a G-vector bundle over a point there is a natural map $R(G) \to K^0(BG)$. This takes a virtual representation to a virtual vector bundle over BG by applying the Borel construction $EG \times_G -$. Work of Atiyah[1] in the 50's and 60's showed that this map becomes an isomorphism after completing $R(G)$ with respect to the ideal of virtual bundles of dimension 0.

Let L be the smallest characteristic 0 field containing all roots of unity and let $Cl(G;L)$ be the ring of class functions on G taking values in L. A classical result in representation theory states that the character map

$$\chi : R(G) \to Cl(G, L)$$

taking a virtual representation to the sum of its characters induces an isomorphism $L \otimes R(G) \to Cl(G;L)$.

In the 80's and 90's Hopkins, Kuhn, and Ravenel[2] described how the two previous results can be combined. They work completed at a prime p. Let

$G_p \subseteq G$ be the subgroup of elements that are p^k-torsion for some k . In terms of K-theory they found a ring L and a map

$$\Phi : K^*(BG) \to Cl(G_p, L)$$

that induces an isomophism when the left-hand side is tensored with L.

In fact they discovered much more: The ring L is flat over K^0 and thus can be used to create a new cohomology theory. They proved that there is a map of equivariant cohomology theories (theories that take G-spaces as input) that outputs Φ when evaluated at a point. But even more, their theorem applies to all of the Morava E-theories E_n, a sequence of cohomology theories that are intimately related to the stable sphere and organize a large portion of stable homotopy theory. E_1 is p-complete K-theory. They build a map

$$\Phi : E_n^*(BG) \to Cl(G_p, L).$$

A different L is constructed for each n.

Associated to each theory E_n is a formal group F_{E_n} of height n (and dimension 1) in the sense of formal algebraic geometry. Cohomology theories that have a small height succumb to geometric interpretation more easily than those with large height. For example, de Rahm cohomology has height 0 and can be described in terms of differential forms, and topological K-theory has height 1 and can be described in terms of vector bundles. Understanding maps from a cohomology theory to one of lower height offers the possibility of using geometric methods in calculation and might even suggest a way to give a geometric description of the first theory. One famous map of this sort is the Chern character $ch : K \to H\mathbb{Q}$.

The codomain of the map Φ constructed by Hopkins, Kuhn, and Ravenel is a rational cohomology theory (p was inverted in the process of creating L). It has height 0. In my research during the REGS this summer I have completed my main goal: I have discovered, for all finite groups G, analogues of Φ for each height h between 0 and n,

$$\Phi_h : E_n^*(BG) \to S_h(G_p, D).$$

Φ_h and Φ have the same domain, but the codomain of Φ_h, S_h, is an equivariant cohomology theory of intermediate height. D is a ring analagous to L and, in order to give a flavor of the work, it is described below. As in the case of Φ, I have shown that each Φ_h comes from a map of equivariant cohomology theories evaluated at the point.

For $G \cong \mathbb{Z}/p^k$, the map Φ admits an algebro-geometric interpretation. The domain of Φ, $E_n^*(BG)$, is exactly the p^k-torsion of F_{E_n}. Formal groups do not behave particularly well under pullback; the pullback of a formal group need not be a formal group but is always a p-divisible group $\mathbb{G}$ of the same height. Associated to each p-divisible group that I work with is an exact sequence

$$0 \longrightarrow \mathbb{G}_0 \hookrightarrow \mathbb{G} \longrightarrow \mathbb{G}_{et} \longrightarrow 0$$

where $\mathbb{G}_0$ is the connected component of the identity (the formal part) of $\mathbb{G}$, and $\mathbb{G}_{et}$ is the etale quotient. The ring L constructed by Hopkins, Kuhn, and Ravenel is the universal place where F_{E_n} is constant and thus isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ where $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are the p-adic integers and rationals respectively. In other words, F_{E_n} is isomorphic to a constant group scheme after base extension to L and L is the initial ring extension of E_n^* that does this. If $G \cong \mathbb{Z}/p^k$, the map Φ can be interpreted as the global sections of the map on p^k-torsion

$$\Phi^* : (\mathbb{Q}_p/\mathbb{Z}_p)^n[p^k] \to F_{E_n}[p^k].$$

Thus the ring L is the universal place where the formal part of the group vanishes and the etale part becomes discrete.

The main idea in the construction of D is, instead of finding the universal ring that discretizes the group, to find the universal ring that causes the formal group to split as a product of a formal part and a discrete part and to work there. To be more precise, for each $0 < h < n$, I have found the universal example of a ring D such that

$$D \otimes F_{E_n} \cong \mathbb{G}_0 \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^{n-h}.$$

D is the universal place where F_{E_n} splits as a sum of a height h formal group and a constant p-divisible group of height $n-h$.

For $G \cong \mathbb{Z}/p^k$, the map Φ_h above is precisely the global sections of the following map:

$$\Phi^* : \mathbb{G}_0 \oplus \mathbb{Q}_p^{n-h}/\mathbb{Z}_p^{n-h}[p^k] \to F_{E_n}[p^k].$$

When E_n is tensored with D this map becomes an isomorphism of cohomology theories. Another interesting aspect of the work is the construction of the map Φ_h. The construction uses aspects of the theory of G-spaces as well as the algebraic properties of the ring D. When I began the summer 2010 REGS I was only able to construct Φ_h for finite abelian G. The main goal of the summer REGS, which was achieved, was giving this map for aribtrary finite G. This has led to several related questions that I am now exploring.

References

[1] M. F. Atiyah. Characters and cohomology of finite groups. *Publ. Math., Inst. Hautes Etud. Sci.*, 9:247–288, 1961.

[2] M. J. Hopkins, N. J. Kuhn, and D. C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Am. Math. Soc.*, 13(3):553–594, 2000.