# A modular approach to the K(2)-local sphere at the prime 3

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Outline

Background

Q(2)

The homotopy of Q(2)



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Construction of Q(2)

The homotopy of Q(2)

Some applications and directions

The homotopy of Q(2)

Some applications and directions

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Construction of Q(2)

$$\pi_k S^0 = \pi_{k+n} S^n \otimes \mathbb{Z}_{(p)} \quad \text{(for } n \gg k \geqslant 0\text{)}$$

#### Remark

One strategy is to find other spectra whose homotopy groups approximate  $\pi_*S^0$ .

# Examples

- 1. The spectrum *TMF* of topological modular forms helps account for  $\pi_k S^0$  for  $0 \le k \le 60$ .
- 2. There exist spectra  $\{L_{K(n)}S^0\}_{n=0,1,2,...}$  that play an analogous role.

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# Bousfield localization

#### Theorem

Given a homology theory E, there exists a functor

$$L_E: \mathbf{S} \to \mathbf{S}$$

where  $L_E X$  is "the part of X that E can see."

#### Remark

If  $X \to Y$  induces  $E_*X \cong E_*Y$ , then  $L_EX \simeq L_EY$ .

# Examples

- 1. If E = M is the mod p Moore spectrum,  $L_E X = X_p^{\wedge}$ .
- 2. If  $E = H\mathbb{Q}$ ,  $L_E X = X\mathbb{Q}$ .
- 3. If E = K is complex K-theory,  $\pi_{-2}L_ES^0 = \mathbb{Q}/\mathbb{Z}$ .

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# Homology theories

# Theorem (Landweber exactness)

Given a module M over  $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$ , the functor  $BP_*(-) \otimes M$  is a homology theory iff for each n > 0, multiplication by  $v_n$  in  $BP_*/I_n \otimes M$  is monic, where  $I_n = (p, v_1, v_2, \ldots, v_{n-1})$ .

We will need the following three theories:

- 1. Johnson-Wilson:  $E(n)_* = \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]$
- 2. Lubin-Tate:  $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm 1}]$
- 3. Morava K-theories:  $K(n)_* = \mathbb{Z}/(p)[v_n, v_n^{-1}]$

#### Remark

The Morava K-theories K(n) are not Landweber exact.

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# Chromatic homotopy theory

# Theorem (Chromatic convergence)

Let 
$$L_n = L_{E(n)}$$
. Then

$$S^0 \simeq \mathsf{holim}(L_0 S^0 \leftarrow L_1 S^0 \leftarrow L_2 S^0 \leftarrow \cdots)$$

There is a homotopy pullback square

$$L_{n}S^{0} \longrightarrow L_{K(n)}S^{0}$$

$$\downarrow \qquad \qquad \downarrow$$

$$L_{n-1}S^{0} \longrightarrow L_{n-1}L_{K(n)}S^{0}$$

The building blocks of  $\pi_*S^0$  are the spectra  $L_{K(n)}S^0$ , the K(n)-local spheres.

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▶ **2005:** Goerss, Henn, Mahowald and Rezk showed that  $L_{K(2)}S^0$  at p=3 lies atop a 4-stage tower of fibrations analogous to

$$L_{K(1)}S^0 \rightarrow KO_2^{\wedge} \rightarrow KO_2^{\wedge}$$

at the prime 2.

▶ **2006:** Behrens (following Mahowald and Rezk) built a spectrum Q(2) and proved

$$DQ(2) \rightarrow L_{K(2)}S^0 \rightarrow Q(2)$$

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The homotopy of  $\mathcal{Q}(2)$ 

$$C_{/\mathbb{F}_9}: y^2 = x^3 - x$$

possesses a degree 2 isogeny  $\psi: C \to C$  with kernel H < C and inducing  $\psi^{\wedge}: C^{\wedge} \cong C^{\wedge}$ . We say C has a  $\Gamma_0(2)$  structure.

## Theorem (Goerss-Hopkins-Miller)

There is a functor FGL  $\to$  Spectra sending  $C^{\wedge}$  to  $E_2$ . Since  $\psi^{\wedge}$  is invariant under the action of  $\operatorname{Aut}_{/\mathbb{F}_3}(C,H)\cong D_8$ , it induces a map of spectra

where

$$\psi_d: E_2^{hD_8} \to E_2^{hD_8}$$
 $E_2^{hD_8} \simeq TMF_0(2)$ 

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# A semi-cosimplicial spectrum

We have  $\operatorname{Aut}_{/\mathbb{F}_3}(\mathit{C})\cong \mathit{G}_{24}$ , and

$$E_2^{hG_{24}} \simeq TMF$$

## Proposition (Behrens)

There is a semi-cosimplicial diagram of spectra

$$Q(2)^{\bullet}: TMF \longrightarrow TMF \lor TMF_0(2) \xrightarrow{\psi_d} TMF_0(2)$$

#### Definition

$$Q(2) := \operatorname{holim} Q(2)^{\bullet}$$

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$$\mathcal{M}_{\bullet}: \mathcal{M} \stackrel{\longleftarrow}{\longleftarrow} \mathcal{M} \coprod \mathcal{M}_{0}(2) \stackrel{\psi_{d}}{\longleftarrow} \mathcal{M}_{0}(2)$$

where  $\mathcal{M}$  is the moduli stack of non-singular elliptic curves over  $\mathbb{Z}_{(3)}$ :



and  $\mathcal{M}_0(2)$  is the analogous stack of elliptic curves with a  $\Gamma_0(2)$  structure.

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Some applications and directions

- 1. The Bousfield-Kan spectral sequence for totalizations of cosimplicial spectra.
- 2. The chromatic spectral sequence

$$E_1^{s,t} = \pi_t(M_sQ(2)) \Rightarrow \pi_{t-s}Q(2)$$

where 
$$M_sQ(2) \rightarrow L_sQ(2) \rightarrow L_{s-1}Q(2)$$
.

3. Compute the Adams-Novikov  $E_2$ -term for Q(2) and add in the differentials using those from the ANSS for TMF.

$$(B,\Gamma) = (\mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}], B[r]/(r^3 + q_2r^2 + q_4r))$$

represents the groupoid of elliptic curves of the form  $y^2 = 4x(x^2 + q_2x + q_4)$ .

#### Theorem

The ANSS  $E_2$ -term for TMF is

$$\mathsf{Ext}^* := \mathsf{Ext}^*_\Gamma(B,B) = H^*(B \to \Gamma \to \Gamma^{\otimes 2} \to \Gamma^{\otimes 3} \to \cdots)$$

The ANSS for  $TMF_0(2)$  is concentrated on the zero line, and gives

$$\pi_{2k}TMF_0(2) = B_k$$

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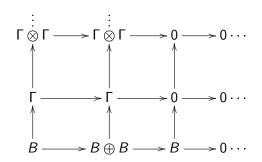


Recall that

$$\textit{Q}(2) = \mathsf{holim}(\textit{TMF} \,\rightarrow\, \textit{TMF} \,\vee\, \textit{TMF}_0(2) \,\rightarrow\, \textit{TMF}_0(2))$$

# Proposition

The ANSS  $E_2$ -term for Q(2) is the cohomology of the totalization of the double cochain complex  $C^{*,*}$  given by



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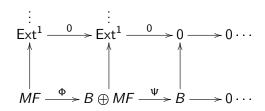
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where  $1728\Delta = c_4^3 - c_6^2$ .

#### Remark

After taking cohomology with respect to the vertical arrows,  $C^{*,*}$  becomes



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# The ANSS for Q(2)

In the cochain complex

$$C^{0,*}:MF \xrightarrow{\Phi} B \oplus MF \xrightarrow{\Psi} B$$

the maps  $\Phi$  and  $\Psi$  are sums of  $\mathbb{Z}_{(3)}$ -module maps corresponding to the maps of spectra in  $Q(2)^{\bullet}$ .

## Example

The map  $\psi_d: TMF_0(2) \to TMF_0(2)$  corresponds to  $\psi_d^*: B \to B$  defined by

$$q_2 \mapsto -2q_2, \quad q_4 \mapsto -4q_4 + q_2^2$$

and  $\Psi = (\psi_d^* + 1) \oplus g$  for  $g : MF \to B$ .

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$$\psi_d: \mathcal{M}_0(2) \to \mathcal{M}_0(2)$$

and on the level of R-points,

$$(C, H) \mapsto (C/H, \widehat{H})$$

where  $\hat{H}$  is the kernel of the dual isogeny  $\hat{\psi}: C/H \to C$ .

#### Remark

The formula for  $\psi_d^*: B \to B$  comes from studying the effect of above map on Weierstrass equations.

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#### Remarks

The only other possibly nontrivial differential is

$$d_2: \operatorname{Ext}^1 \to \operatorname{coker} \Psi$$

The next step is to use the ANSS differentials for TMF to complete the calculation of  $\pi_*Q(2)$ .

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Some applications and directions

For any prime p, there exist spectra Q(N) built from degree N isogenies of elliptic curves, as long as N is a topological generator of  $\mathbb{Z}_p^{\times}$ .

# Theorem (Behrens 2009)

Let  $p \ge 5$ . There is a 1-1 correspondence that associates to each additive generator

$$\beta_{i/j,k} \in \operatorname{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

a modular form  $f_{i/j,k} \in MF_{2i(p^2-1)}$  satisfying certain congruence conditions.

# Conjecture

The above theorem holds at p = 3.

### Greek letter elements

# Theorem (Behrens 2009)

Let  $p \geqslant 5$ . The spectrum Q(N) is E(2)-local, and the images of the homotopy elements  $\alpha_{i,j}$  and  $\beta_{i/j,k}$  under the homomorphism

$$\pi_* L_2 S^0 \to \pi_* Q(N)$$

are non-trivial.

# Conjecture

The homotopy Greek letter elements  $\beta_{i/j,k}^h$  are detected by the spectra Q(N) at the primes 2 and 3.

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# The K(2)-local sphere

## Theorem (Behrens 2006)

At p = 3, there is a cofiber sequence

$$D_{K(2)}Q(2) \to L_{K(2)}S^0 \to Q(2)$$

and the same is true at p = 5.

# Open questions

- 1. Does this theorem for all *p* and *N*? If so, is there a uniform proof?
- 2. Describe the connecting map

$$Q(N) \rightarrow \Sigma D_{K(2)} Q(N)$$

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# The End

Thank you!

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