[h2p2, 2.11] and [h2, 10.1]

Tue - 9/3

1. [h2p2, 2.11]
2. [h2, 10.1]
3. connection via a univ coeff ss
4. verification

(1) Let $\gamma \in \Gamma/I$ as a right Γ-module so that

$$\gamma a_0 = 0$$
$$\gamma a_1 = 0$$
$$\gamma a_2 = 0$$

Let $u \in \omega^k$ be a gen so that

	$k=0$	$k=1$	$k=2$	$k=3$
by	2.2	2.4	2.5	
$a_0 u$	1	0	0	$-2u$
$a_1 u$	0	$-u$	0	$-au$
$a_2 u$	0	0	u	0

Then

(truncated) $d_0(\gamma\otimes u)$	0	0	0	0
$d_1(\gamma\otimes q_0\otimes u)$	$\gamma\otimes u$	0	0	$-2\gamma\otimes u$
$d_1(\gamma\otimes q_1\otimes u)$	0	$\gamma\otimes u$	0	$-a\gamma\otimes u$
$d_1(\gamma\otimes q_2\otimes u)$	0	0	$\gamma\otimes u$	0
$d_2(\gamma\otimes r_1\otimes u)$	$\gamma\otimes q_1\otimes u$	$2\gamma\otimes q_2\otimes u$	$2\gamma\otimes q_0\otimes u$	$-2a\gamma\otimes q_0\otimes u+4\gamma\otimes q_1\otimes u$
$d_2(\gamma\otimes r_2\otimes u)$	$\gamma\otimes q_2\otimes u$	$\gamma\otimes q_0\otimes u$	$-a\gamma\otimes q_0\otimes u+2\gamma\otimes q_1\otimes u$	$a^2\gamma\otimes q_0\otimes u-2a\gamma\otimes q_1\otimes u+4\gamma\otimes q_2\otimes$ (note the right R-module strc

Thus

$\mathrm{Tor}_0^\Gamma(\Gamma/I,\omega^k)$	0	0	0	$R/(2,a)$
$\mathrm{Tor}_1^\Gamma(\Gamma/I,\omega^k)$	0	$\dfrac{Rq_0+Rq_2}{R(2q_2)+Rq_0}$	$\dfrac{Rq_0+Rq_1}{R(2q_0)+R(-aq_0+2q_1)}$	$\dfrac{R(aq_0-2q_1)+Rq_2}{R(-2aq_0+4q_1)+R(a^2q_0-2aq_1+4q}$
$\mathrm{Tor}_2^\Gamma(\Gamma/I,\omega^k)$	0	0	0	0

(2) (compare the dual defs of the Koszul cpx in [hzpz, 2.10] and in [hz, 7.3]: $\Gamma[k]$ and $A[k]$ are E_0-linear dual to each other)

Now get to computing

$$\mathrm{Ext}^2_{\Gamma}(\omega^k, \mathrm{nul}) \cong A_1/(s(A_0), b'^k A_1)$$

where, at $p=2$,

$$A_0 = \mathbb{Z}_2[[a]] = E_0$$
$$A_1 = \mathbb{Z}_2[[a]][d]/(d^3 - ad - 2)$$
$$b' = d' = a - d^2$$

We have

$$\mathrm{Ext}^2_{\Gamma}(\omega^0, \mathrm{nul}) = 0$$

For $k=1$,

$$a - d^2$$
$$(a-d^2)d = ad - d^3 \equiv -2$$
$$(a-d^2)d^2 \equiv -2d$$

Be careful about viewing A_1 as a ring vs an E_0-module! (This issue also arose in understandin the Frob cong.)

$$\Rightarrow \mathrm{Ext}^2_{\Gamma}(\omega', \mathrm{nul}) = \frac{E_0 d + E_0 d^2}{E_0(-d^2) + E_0(-2d)} \cong E_0/2$$

For $k=2$,

$$(a-d^2)^2 = a^2 - 2ad^2 + d^4 \equiv a^2 - 2ad^2 + d(ad+2) = a^2 - ad^2 + 2d$$
$$(a-d^2)^2 d \equiv a^2 d - ad^3 + 2d^2 \equiv a^2 d - a(ad+2) + 2d^2 = -2a + 2d^2$$
$$(a-d^2)^2 d^2 \equiv -2ad + 2d^3 \equiv -2ad + 2(ad+2) = 4$$

$$\Rightarrow \mathrm{Ext}^2_{\Gamma}(\omega^2, \mathrm{nul}) = \frac{E_0 d + E_0 d^2}{E_0(-ad^2 + 2d) + E_0(2d^2)}$$

For $k=3$,

$$(a-d^2)^3 = a^3 - 3a^2d^2 + 3ad^4 - d^6 \equiv a^3 - 3a^2d^2 + 3ad(ad+2) - (ad+2)^2 = a^3 + 2ad - a^2d^2 -$$

$$(a-d^2)^3 d \equiv a^3d + 2ad^2 - a^2d^3 - 4d \equiv a^3d + 2ad^2 - a^2(ad+2) - 4d = 2ad^2 - 2a^2 - 4d$$

$$(a-d^2)^3 d^2 \equiv 2ad^3 - 2a^2d - 4d^2 \equiv 2a(ad+2) - 2a^2d - 4d^2 = 4a - 4d^2$$

$$\Rightarrow \operatorname{Ext}^2_\Gamma(\omega^3, \mathrm{nul}) = \frac{E_0 d + E_0 d^2}{E_0(2ad - a^2d^2) + E_0(2ad^2 - 4d) + E_0(-4d^2)}$$

(3) We have

$$\operatorname{Ext}^s_\Gamma(\omega^k, \mathrm{nul}) = 0 \quad \text{for all } k \geq 0,\ s \neq 2$$

We want

$$\operatorname{Tor}^\Gamma_q(\mathrm{nul}', \omega^k) \qquad \text{is a right } \Gamma\text{-module}$$

where $\mathrm{nul}' := \operatorname{Hom}_\Gamma(\mathrm{nul}, E^0)$ w/ $(f\cdot\gamma)(x) := f(\gamma\cdot x)$

Recall [Alaska, I.6.2]: $\exists$ a univ coeff ss

$$E_2^{p,q} = \operatorname{Ext}^{p,q}_{\mathcal{G}}(\underline{J}_*(X), M) \Rightarrow H^n_G(X; M)$$

where

$$\underline{J}_*(X) \cong \underline{H}_*(\underline{C}_\bullet(X))$$

$$H^*_G(X;M) := H_*(C^\bullet_G(X;M))$$

$$= H_*(\operatorname{Hom}_{\mathcal{G}}(\underline{C}_\bullet(X), M))$$

Here we also have a hmlg-to-cohmlg-type univ coeff ss (cf [AT, thm 3.2

$$E_2^{p,q} = \operatorname{Ext}^p_{E_0}(H_q C^\bullet, E_0) \Rightarrow H_{-p-q}(\operatorname{Hom}_{E_0}(C^\bullet, E_0))$$

where

$C^\bullet := \operatorname{Hom}_\Gamma(P_\bullet(\omega^k), \mathrm{nul})$ w/ $P_\bullet(\omega^k)$ the Koszul cpx
($P_\bullet \to \omega^k$ a resn by proj Γ-modules)

so that

$$H_q C^\bullet = \operatorname{Ext}^{-q}_\Gamma(\omega^k, \mathrm{nul})$$

We want to identify $\operatorname{Hom}_{E_0}(\operatorname{Hom}_\Gamma(P_\bullet, \mathrm{nul}), E_0) \cong \mathrm{nul}' \otimes_\Gamma P_\bullet$ $\quad (*)$

so that $H_{-p-q}(\operatorname{Hom}_{E_0}(C^\bullet, E_0)) \cong \operatorname{Tor}^\Gamma_{-p-q}(\mathrm{nul}', \omega^k)$

and the ss then becomes

$$\mathrm{Ext}^*_{E_0}(\mathrm{Ext}^s_\Gamma(\omega^k, \mathrm{nul}), E_0) \Rightarrow \mathrm{Tor}^\Gamma_{s-*}(\mathrm{nul}', \omega^k)$$

which collapses at E_2 and gives

$$\mathrm{Tor}^\Gamma_q(\mathrm{nul}', \omega^k) \cong \mathrm{Ext}^{2-q}_{E_0}(\mathrm{Ext}^2_\Gamma(\omega^k, \mathrm{nul}), E_0).$$

For (*), consider, for left Γ-modules M and N, the E_0-module map

$$\mathrm{Hom}_\Gamma(N, E_0) \otimes_\Gamma M \longrightarrow \mathrm{Hom}_{E_0}(\mathrm{Hom}_\Gamma(M,N), E_0)$$

$$f \otimes m \longmapsto \big(g \mapsto f(g(m))\big)$$

which is an iso if M is fg free as a Γ-module w/ basis $\{m_i\}$
and N is fg free as an E_0-module w/ basis $\{n^j\}$
$(\underset{\Gamma \otimes_{E_0} -}{\Rightarrow}$ fg free as a Γ-module)

so that

$$\begin{pmatrix} n^l \mapsto 1 \\ n^j \mapsto 0,\ j \neq l \end{pmatrix} \otimes m_k \longleftrightarrow \left(\begin{pmatrix} m_k \mapsto n_k \overset{\text{WLOG}}{=} n^l \\ m_i \mapsto 0,\ i \neq k \end{pmatrix} \mapsto 1 \right)$$

4 Based on $\mathrm{Ext}^2_\Gamma(\omega^k, \mathrm{nul})$ in 2', we compute

$$\mathrm{Ext}^*_{E_0}(\mathrm{Ext}^2_\Gamma(\omega^k, \mathrm{nul}), E_0)$$

and check that it agrees w/ $\mathrm{Tor}^\Gamma_{2-*}(\mathrm{nul}', \omega^k)$ in 1'.

<u>$k=0$</u> ✓

<u>$k=1$</u> $\mathrm{Ext}^*_{E_0}(E_0/2, E_0)$

$$0 \to E_0 \xrightarrow{\cdot 2} E_0 \to E_0/2 \to 0$$

$$\Rightarrow \quad 0 \to \mathrm{Hom}_{E_0}(E_0, E_0) \to \mathrm{Hom}_{E_0}(E_0, E_0) \to 0$$

$$f \mapsto 2f$$

$$\Rightarrow \quad \begin{array}{cccc} * & 0 & 1 & 2 \\ \mathrm{Ext}^* & 0 & E_0/2 & 0 \\ & & \cong & \\ & & \dfrac{Rq_0 + Rq_2}{R(2q_2)+Rq_0} & \end{array}$$

$\underline{k=2}$ $\quad \mathrm{Ext}^*_{E_0}\left(\dfrac{E_0x + E_0y}{E_0(-ay+2x)+E_0(2y)}, E_0 \right)$

$$0 \to E_0\alpha + E_0\beta \to E_0x + E_0y \to \frac{E_0x+E_0y}{E_0(-ay+2x)+E_0(2y)} \to 0$$

$$\alpha \mapsto -ay+2x$$
$$\beta \mapsto 2y$$

$$\Rightarrow \quad 0 \to \mathrm{Hom}_{E_0}(E_0x+E_0y, E_0) \to \mathrm{Hom}_{E_0}(E_0\alpha+E_0\beta, E_0) \to 0$$

$$\Rightarrow \quad \begin{array}{cccc} * & 0 & 1 & 2 \\ \mathrm{Ext}^* & 0 & \dfrac{E_0\alpha^* + E_0\beta^*}{E_0(2\alpha^*)+E_0(-a\alpha^*+2\beta^*)} & 0 \\ & & \cong & \\ & & \dfrac{Rq_0 + Rq_1}{R(2q_0)+R(-aq_0+2q_1)} & \end{array}$$

$\underline{k=3}$ $\quad \mathrm{Ext}^*_{E_0}\left(\dfrac{E_0 x + E_0 y}{E_0(2ax - a^2 y) + E_0(2ay - 4x) + E_0(-4y)} \, , \, E_0 \right)$

$$0 \to E_0 \zeta \to E_0\alpha + E_0\beta + E_0\gamma \to E_0 x + E_0 y \to \frac{E_0 x + E_0 y}{E_0(2ax - a^2 y) + E_0(2ay-4x) + E_0(-4y)} \to 0$$

$$\zeta \mapsto 2\alpha + a\beta \qquad \begin{array}{lcl} \alpha & \mapsto & 2ax - a^2 y \\ \beta & \mapsto & 2ay - 4x \\ \gamma & \mapsto & -4y \end{array}$$

$\Rightarrow$

$$0 \to \mathrm{Hom}_{E_0}(E_0 x + E_0 y, E_0) \to \mathrm{Hom}_{E_0}(E_0\alpha + E_0\beta + E_0\gamma, E_0) \to \mathrm{Hom}_{E_0}(E_0\zeta, E_0) \to 0$$

$\Rightarrow$

$*$	0	1	2
Ext^*	0	$\dfrac{E_0(a\alpha^* - 2\beta^*) + E_0\gamma^*}{E_0(2a\alpha^* - 4\beta^*) + E_0(-a^2\alpha^* + 2a\beta^* - 4\gamma^*)}$	$\dfrac{E_0\zeta^*}{E_0(2\zeta^*) + E_0(a\zeta}$
		$\cong$	$\cong$
		$\dfrac{R(aq_0 - 2q_1) + Rq_2}{R(-2aq_0 + 4q_1) + R(a^2 q_0 - 2aq_1 + 4q_2)}$	$R/(2,a)$