Derived mapping spaces as models for localizations

by

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Abstract

This work focuses on a generalization of the models for rational homotopy theory developed by D. Sullivan and D. Quillen and p-adic homotopy developed by M. Mandell to $K(1)$-local homotopy theory. The work is divided into two parts.

The first part is a reflection on M. Mandell's model for p-adic homotopy theory. Reformulating M. Mandell's result in terms of an adjunction between p-complete, nilpotent spaces of finite type and a subcategory of commutative $H\overline{\mathbb{F}}_p$-algebras, the main theorem shows that the unit of this adjunction induces an isomorphism between the unstable $H\mathbb{F}_p$ Adams spectral sequence and the $H\overline{\mathbb{F}}_p$ Goerss–Hopkins spectral sequence.

The second part generalizes M. Mandell's model for p-adic homotopy theory to give a model for $K(1)$-localization. The main theorem gives a model for the $K(1)$-localization of an infinite loop space as a certain derived mapping space of $K(1)$-local ring spectra. This result is proven by analyzing a more general functor from finite spectra to a mapping space of $K_p^\wedge$-algebras using homotopy calculus, and then taking the continuous homotopy fixed points with respect to the prime to p Adams operations.

Thesis Supervisor: Mark Behrens
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Contents

Chapter 1

Introduction

Unstable homotopy theory and the pursuit of computational methods to gain insight into the homotopy groups of spheres has long been an area of interest for algebraic topologists. One important reduction suggested by J. P. Serre in the study of homotopy theory is the local study of homotopy groups. We can study rational and p-adic homotopy, in much the same way that number theorists study the integers $\mathbb{Z}$ by studying the p-adic integers $\mathbb{Z}_p^\wedge$ for all primes p and the rational numbers $\mathbb{Q}$.

One of the key ideas in studying the homotopy groups of spaces locally is the notion of Bousfield localization [7]. Localization is an idempotent functor from the homotopy category spaces to the homotopy category of spaces. Given a cohomology theory E and a sufficiently nice space X, the localization X_E of X yields a space whose homotopy groups are recoverable from information about the E cohomology E^*X of X.

Local homotopy groups can used to build global, or integral homotopy groups in some instances. Let $\mathrm{H}\mathbb{Q}$ be the Eilengerg–MacLane spectrum representing rational homology, and let $\mathrm{H}\mathbb{F}_p$ denote the spectrum representing homology with $\mathbb{F}_p$ coefficients. Given a simply connected space X, we can recover the homotopy groups of X from its $\mathrm{H}\mathbb{Q}$ and $\mathrm{H}\mathbb{F}_p$ localizations for all primes p. The local homotopy groups glue to form information about the integral homotopy groups in a manner prescribed by the maps in the homotopy pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_p X_{\mathrm{H}\mathbb{F}_p} \arrow[d] \\ X_{H\mathbb{Q}} \arrow[r] & (\prod_p X_{\mathrm{H}\mathbb{F}_p})_{H\mathbb{Q}}. \end{tikzcd}$$

Rational homotopy is the study of homotopy groups after forgetting all torsion. A rational equivalence is a map of spaces that induces an isomorphism on rational homology. Thus rational homotopy is really the study of spaces up to rational equivalence, or $\mathrm{H}\mathbb{Q}$-local spaces. Specifically, for finite type, nilpotent spaces, rational homotopy is the portion of homotopy that can be recovered from the rational cohomology $\mathrm{H}\mathbb{Q}^*X$ [24]. Similarly, the study of p-adic homotopy is the study of $\mathrm{H}\mathbb{F}_p$-local spaces.

D. Quillen completely characterized rational homotopy types of simply connected spaces by constructing a faithful embedding of the rational homotopy category into the category of commutative differential graded algebras over $\mathbb{Q}$ [24]. It was shown by B. Shipley that there is a Quillen equivalence between the category of commutative differential graded algebras over $\mathbb{Q}$ and commutative $H\mathbb{Q}$-algebras [29]. Thus we like to think of D. Quillen's result as giving a model for the rationalization of a nilpotent, simply connected space X as:

$$X \longrightarrow \mathrm{Alg}_{H\mathbb{Q}}(F(\Sigma^\infty X_+, H\mathbb{Q}), H\mathbb{Q}) .$$

M. Mandell generalized this idea and showed that the $\overline{\mathbb{F}}_p$ cochains of a space give a faithful embedding of the category of p-complete, nilpotent spaces of finite type into the category of E_∞ algebras over $\overline{\mathbb{F}}_p$ [23]. M. Mandell also proved that there is a Quillen equivalence between the category of E_∞-$\overline{\mathbb{F}}_p$-algebras and commutative $\mathrm{H}\overline{\mathbb{F}}_p$-algebras. Thus we can think of this result as giving a model for the $\mathrm{H}\mathbb{F}_p$-localization of a nilpotent, finite type space X as:

$$X \longrightarrow \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(F(\Sigma^\infty X_+, \mathrm{H}\overline{\mathbb{F}}_p), \mathrm{H}\overline{\mathbb{F}}_p) .$$

There is a complication in the fact that the $H\mathbb{F}_p$-localization of a space is difficult to understand. Just as we broke apart integral homotopy groups into smaller local pieces, we can break up p-adic homotopy groups into its various v_n-periodic homotopy groups. We think of the v_n-periodic homotopy groups $v_n^{-1}\pi_*X$ as being layers of the p-adic homotopy group as it only sees a certain part of the p-complete homotopy groups of X. Stably, the relationship of v_n-periodic homotopy groups and $K(n)$- and $E(n)$-localizations and the possible implications was introduced in the paper of D. Ravenel [27]. In this foundational paper, D. Ravenel introduced 7 conjectures, all of which were proved subsequently in the following years except for the conjecture detailing the relationship between v_n-periodic homotopy groups and $K(n)$-local homotopy groups.

A. Bousfield sets up the foundations for a theory of unstable periodic homotopy theory in [6]. Unstably, the unanswered conjecture of Ravenel translates as an open problem called the unstable telescope conjecture [9]. One way to state this conjecture is that virtual $K(n)$-equivalences are v_n-periodic homotopy equivalences. Here a virtual $K(n)$-equivalence is a map $X \to Y$ such that $\Omega X\langle n+2\rangle \to \Omega Y\langle n+2\rangle$ is a $K(n)$-equivalence. As the unstable telescope conjecture is equivalent to the stable telescope conjecture [9], it is true for $n = 1$, but unknown and suspected to be false for $n > 1$. Thus there is some interest in understanding the $K(n)$-local category unstably. In particular, we are interested in developing a model for the $K(1)$-localization of a space analogous to the models developed by D. Quillen for rationalization and M. Mandell for the $H\mathbb{F}_p$-localization.

Chapter 2 is a reflection on the faithful embedding of p-complete, finite p-type, nilpotent spaces and commutative $H\overline{\mathbb{F}}_p$-algebras developed by M. Mandell. This is accomplished by comparing the two spectral sequences: the unstable $H\mathbb{F}_p$-based Adams spectral sequence, which computes the homotopy groups of the p-completion of a nilpotent space, and the $H\overline{\mathbb{F}}_p$-based Goerss–Hopkins spectral, which computes the homotopy groups of the derived mapping space of two $H\overline{\mathbb{F}}_p$-algebras. The goal of Chapter 2 is to prove that the map in the adjunction described by M. Mandell induces an isomorphism between these spectral sequences.

The focus of Chapter 3 is to develop a model for the $K(1)$-localization of simply connected infinite loop spaces. We show that given a *nice* 2-connected spectrum E, the simply connected cover of the derived mapping space of E_∞-$K(1)$-local ring spectra from the function spectrum $F(\Sigma^\infty\Omega^\infty E_+, S_{K(1)})$ to $S_{K(1)}$ is a model for the $K(1)$-localization of $\Omega^\infty E$. That is we obtain a model of the $K(1)$-localization of a *nice* 2-connected infinite loop space as the lift of the map:

$$\Omega^\infty E \longrightarrow \mathrm{E}_\infty(F(\Sigma^\infty\Omega^\infty E_+, S_{K(1)}), S_{K(1)})$$

to the universal covering space of this derived mapping space.

Chapter 2

A comparison of spectral sequences computing unstable homotopy groups of p-complete, nilpotent spaces.

2.1 Introduction

Let Mod_S denote a good model category of spectra, where S denotes the sphere spectrum— either symmetric spectra as in [21] or S-modules with the model structure described in [14]. Write Alg_S for the category of *commutative* S-algebras. Similarly, given an S-algebra R, we may consider the categories of spectra Mod_R and Alg_R. By Alg_R we will always mean *commutative* R-algebras, the category of S-algebras under R. Given an R-module M, write $\mathbb{P}(M)$ for the free commutative R-algebra on M.

Write $\overline{\mathbb{F}}_p$ for the algebraic closure of the field $\mathbb{F}_p$. Let $\mathrm{H}\overline{\mathbb{F}}_p$ denote the Eilenberg–Maclane spectrum whose homotopy groups are $\pi_*\mathrm{H}\overline{\mathbb{F}}_p = \overline{\mathbb{F}}_p$ for $* = 0$ and is 0 otherwise. In particular, we are interested in the category $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ of commutative $\mathrm{H}\overline{\mathbb{F}}_p$-algebras, where p is a prime.

Unstably, we are interested in p-complete spaces. There are two common definitions for the p-completion of a space: one being the $\mathrm{H}\mathbb{F}_p$ localization of the space, and the other being the Bousfield–Kan p-completion. In general, these do not agree. However, for a *nilpotent* space X, both notions of p-completion coincide [12, Proposition V.4.2]. A space X is *nilpotent* if $\pi_1 X$ is nilpotent and acts nilpotently on the higher homotopy groups $\pi_n X$ for $n > 1$.

The *p-completion* of a space X is the homotopy inverse limit of the diagram of spaces:

$$\mathbb{F}_p(X) \rightleftarrows \mathbb{F}_p(\mathbb{F}_p(X)) \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \cdots .$$

The maps in this cosimplicial diagram are discussed in Section 2.4.1. We denote the $\mathrm{H}\mathbb{F}_p$-localization of a space X by X_p.

Let $\mathscr{T}$ denote the category of (unpointed) simplicial sets. Note that all constructions carried out in this chapter have analogs when considering $\mathscr{T}$ to be the category of weak Hausdorff topological spaces. We take the model of simplicial sets in order to make certain constructions more transparent. The author uses the convention of referring to simplicial sets as spaces. Recall that there is a functor $(-)_+$ from unpointed to pointed spaces given by adding a disjoint basepoint, which is left adjoint to the forgetful functor.

Note that $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is enriched over spaces, and the model category structure satisfies Quillen's corner axiom. Thus given any two $\mathrm{H}\overline{\mathbb{F}}_p$-algebras A and B, there is a derived mapping space $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(A, B)$. In this chapter we follow the convention that given a model category $\mathscr{C}$ enriched in spaces, $\mathscr{C}(A, B)$ denotes the derived mapping space.

Consider the functor from $\mathscr{T}$ to $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ discussed in [23, Appendix C]:

$$\begin{aligned} F : \mathscr{T} &\to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}, \\ X &\mapsto \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}. \end{aligned}$$

This is a functor of simplicial categories. In [23], M. Mandell proves that F induces an equivalence onto its image when restricted to the subcategory of p-complete, nilpotent spaces of finite type. In particular, the homotopy inverse of F is given by the right adjoint

$$\begin{aligned} G : \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p} &\to \mathscr{T}, \\ A &\mapsto \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(A, \mathrm{H}\overline{\mathbb{F}}_p). \end{aligned}$$

Thus, given a nilpotent space of finite type, the composite $\psi := G \circ F : \mathscr{T} \to \mathscr{T}$ induces an $\mathrm{H}\mathbb{F}_p$-localization:

$$\psi : X \to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\mathrm{Sp}X_+}, \mathrm{H}\overline{\mathbb{F}}_p).$$

More is true. Applying the functor $\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty(-)_+}$ to an unstable map of simplicial sets gives rise to a map of $\mathrm{H}\overline{\mathbb{F}}_p$-algebras. Thus given a mapping space $\mathscr{T}(Y, X)$, there is an induced functor to the mapping space of $\mathrm{H}\overline{\mathbb{F}}_p$-algebras. The theorem as proved by M. Mandell in [23] can be rewritten in the language of $\mathrm{H}\overline{\mathbb{F}}_p$-algebras as the following:

Theorem 2.1.1 ([23]). *Let X and Y be nilpotent and of finite type. Then the map*

$$\psi : \mathscr{T}(Y, X_p) \to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \tag{2.1}$$

is a weak equivalence.

The unstable homotopy groups of a (p-complete) space can be computed using the unstable Adams spectral sequence. The homotopy groups of the space of $\mathrm{H}\overline{\mathbb{F}}_p$-algebra maps can be computed using a Goerss–Hopkins spectral sequence. The main question we set out to answer is whether or not ψ induces an isomorphism between these two spectral sequences. The answer is yes and is the main application of the main theorem in this chapter.

The goal of this chapter is to prove a general theorem and arrive at the isomorphism of the two unstable spectral sequences as a corollary. There is a canonical simplicial resolution $P(F(\Sigma^\infty X_+, \mathrm{H}\overline{\mathbb{F}}_p))_\bullet$ given by the cofibrant replacement of the constant simplicial object in the E_2 model category structure on simplicial $\mathrm{H}\overline{\mathbb{F}}_p$-algebras. The cosimplicial space $\mathbb{F}_p^\bullet(X)$ is obtained by iteratively applying a the monad $\mathbb{F}_p$ sending X to the underlying simplicial set on the free simplicial $\mathbb{F}_p$-algebra on X. Let Y be a space and consider the diagram of cosimplicial spaces:

$$\begin{tikzcd}
\mathscr{T}(Y, \mathbb{F}_p^\bullet X) \arrow[r] & \mathrm{diag}\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(F(\Sigma^\infty \mathbb{F}_p^\bullet X_+, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}))_\bullet, \mathrm{H}\overline{\mathbb{F}}_p) \\
 & \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(F(\Sigma^\infty X_+, \mathrm{H}\overline{\mathbb{F}}_p))_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}). \arrow[u]
\end{tikzcd}$$

The main theorem is that both maps are equivalences when X and Y are of finite type. In particular, this theorem offers another proof of Theorem 2.1.1 upon totalization of this cosimplicial diagram. It should be pointed out that although this proof is different, the main computation used in proving the main theorem is the same as in M. Mandell's original proof. The corollary exhibits an isomorphism between the E_2 terms of the unstable $\mathrm{H}\mathbb{F}_p$-Adams spectral sequence and the Goerss–Hopkins spectral sequence when conditions of the theorem are satisfied.

In Section 2.2 we define two categories of unstable algebras, over the Steenrod algebra and the E_∞ Steenrod algebra. We define André–Quillen cohomology theories in these categories. The purpose of Section 2.3 is to prove that there is a natural isomorphism between the derived functors of derivations of unstable $\overline{\mathbb{F}}_p$-algebras over the E_∞ Steenrod algebra with the derived functors of derivations of unstable algebras over the Steenrod algebra when the algebras in question have a particular form. This isomorphism is the main tool we use to identify the unstable Adams spectral sequence with the Goerss–Hopkins spectral sequence from the E_2-term onward. In Section 2.4 we recall the unstable Adams spectral sequence and the identification of its E_2 term as an André–Quillen cohomology group of unstable algebras over the mod-p Steenrod algebra. In Section 2.5 we recall the Goerss–Hopkins spectral sequence and identify its E_2 term algebraically as the André–Quillen cohomology of unstable algebras over the E_∞ Steenrod algebra. In Section 2.6 we begin to analyze a map between two spectral sequences induced by applying the map ψ from Theorem 2.1.1 to the standard cosimplicial resolution of a space X constructed in Section 2.4. The work in this section proves that $\pi^s\pi_t$ of the image under application of the map ψ to the Bousfield–Kan cosimplicial $\mathbb{F}_p$ resolution of a space X can in fact be identified as the E_2 term of the Goerss–Hopkins spectral sequence. This identification combined with the descent theorem of Section 2.3 shows that ψ induces an isomorphism of spectral sequences from the E_2 term onward.

2.2 Unstable algebras over $\mathscr{A}$ and $\mathscr{B}$

The two spectral sequences we wish to compare have E_2 terms that can be identified as certain André–Quillen cohomology groups. When restricted to spaces of finite type, the E_2-term of the unstable Adams spectral sequence can be identified as derived functors of derivations of unstable algebras over the Steenrod algebra $\mathscr{A}$. The E_2-term of the Goerss–Hopkins spectral sequence is naturally identified as derived functors of derivations of unstable algebras over the E_∞ Steenrod algebra $\overline{\mathscr{B}}$ over $\overline{\mathbb{F}}_p$.

The goal of this section is to describe these categories of unstable algebras and André–Quillen cohomology in these categories. This sets up the necessary background for Section 2.3 where we give a descent condition relating the André–Quillen cohomology theories over these categories.

2.2.1 Grading convention

Throughout this chapter, we are dealing with objects graded over the integers. We opt to use a chomological grading convention. This choice is two-fold. The first is that a number of function spectra are the main objects of study, and there is an identification

$$\pi_{-*}\mathrm{H}\mathbb{F}_p^{\Sigma^\infty X_+} \cong \mathrm{H}\mathbb{F}_p^* X.$$

The second is that we describe in detail graded unstable modules and algebras over the E_∞ Steenrod algebra. In order for the power operations P^i to correspond to the usual power operations is the Steenrod algebra for $i \geq 0$, and to correspond to the Dyer–Lashoff operations for $i < 0$ and act accordingly to the grading, it is more convenient to choose this scheme.

In terms of notation, to denote a (positive) shift in the cohomological direction, we use the symbol Σ^n. In particular, we have an isomorphism

$$\pi_* \mathrm{H}\mathbb{F}_p^{\Sigma^n X} \cong \Sigma^n \pi_* \mathrm{H}\mathbb{F}_p^X.$$

Similarly, throughout this section, there are a number of graded objects. In keeping with the cohomological grading convention, we highlight this choice by denoting graded a object M by M^* when it seems useful to bring attention to the grading.

2.2.2 The Steenrod algebra

Let $\mathscr{A}$ denote the Steenrod algebra over $\mathbb{F}_p$. If p is odd, there is a basis for $\mathscr{A}$ given by *admissible* sequences. Let the symbols $\epsilon_k \in \{0,1\}$. A sequence $I = (\epsilon_1, i_1, \ldots, \epsilon_n, i_n)$ of nonnegative integers is admissible if $i_{k-1} \leq p i_k + \epsilon_k$. The $\mathbb{F}_p$-module basis for $\mathscr{A}$ is given by $\{P^I : I \text{ is admissible}\}$, where $P^{\epsilon_j} P^{i_j}$ denotes $\beta^{\epsilon_j} P^{i_j}$. If $p = 2$, we consider sequences of nonnegative integers $I = (i_1, \ldots, i_n)$, which are admissible when $i_k \leq 2 i_{k+1}$. Then a $\mathbb{F}_2$ module basis for $\mathscr{A}$ is given by $\{Sq^I : I \text{ is admissible}\}$.

An $\mathscr{A}$-module is a graded $\mathbb{F}_p$ vector space M^* with an action of $\mathscr{A}$. Define the excess of a monomial I to be

$$\begin{aligned} e(I) =& i_n - \sum_{k=1}^{n-1} i_k && p = 2 \\ e(I) =& 2i_k + \epsilon_k - \sum_{k=1}^{n-1} (2(p-1)i_k + \epsilon_k) && p \neq 2. \end{aligned}$$

Then M^* is an *unstable* $\mathscr{A}$-module if for all x in M^*, $P^I(x) = 0$ whenever $e(I) > |x|$. There is a pair of adjoint functors from graded $\mathbb{F}_p$ vector spaces $gr\mathrm{Mod}_{\mathbb{F}_p}$ to unstable $\mathscr{A}$ modules $\mathrm{Mod}_{U\mathscr{A}}$

$$\mathscr{F}_0 : gr\mathrm{Mod}_{\mathbb{F}_p} \rightleftarrows \mathrm{Mod}_{U\mathscr{A}} : U$$

where U simply forgets the unstable $\mathscr{A}$-module structure, and the left adjoint $\mathscr{F}_0$ is the free functor. This adjoint pair induces a Quillen adjunction between simplicial $\mathbb{F}_p$ vector spaces and simplicial unstable $\mathscr{A}$-modules. We can define $\mathscr{F}_0$ explicitly by choosing an $\mathbb{F}_p$ basis $\{x_j\}_{j\in J}$ of our vector space M^*. Then $\mathscr{F}_0(M^*)$ is the unstable $\mathscr{A}$ module with $\mathbb{F}_p$-module basis given by

$$\{P^I x_j : j \in J, I \text{ is admissible, and } e(I) \leq |x|\}.$$

An unstable $\mathscr{A}$-module B is an unstable $\mathscr{A}$-algebra if is a ring, and satisfies the instability condition $P^i x = x^p$ whenever $2i = |x|$ (or $Sq^i x = x^2$ whenever $i = |x|$ if $p = 2$). That is, P^i acts as the Frobenius on the $2i^{\text{th}}$ graded piece of B. The natural example of an unstable $\mathscr{A}$-algebras is $\mathrm{H}\mathbb{F}_p$ cohomology ring of a space X.

There is an adjoint pair of functors from unstable $\mathscr{A}$-modules to unstable $\mathscr{A}$-algebras:

$$E_0 : \mathrm{Mod}_{U\mathscr{A}} \rightleftarrows \mathrm{Alg}_{U\mathscr{A}} : U,$$

where the right adjoint U is the forgetful functor, and the left adjoint E_0 takes the free commutative algebra on a module M^* and identifies $P^i m$ with m^p whenever the degree of m is $2i$ for odd primes, and identifies $Sq^i m$ with m^2 when the degree of m is i at the prime 2.

There are a model category structure on simplicial unstable $\mathscr{A}$-algebras and simplicial unstable $\mathscr{A}$-modules so that E_0 and U form a Quillen adjunction.

Lemma 2.2.1. *Suppose we have unstable $\mathscr{A}$-algebras A^* and B^*, a map $\varphi : A^* \to B^*$, and an unstable $\mathscr{A}$-module, M^*, that is also B^*-module. Suppose M^* satisfies the stricter instability condition that for $m \in M^*$, the power operation $P^I(m) = 0$ whenever $e(I) \geq |m|$ (rather than just greater than). Then $B^* \ltimes M^*$ is naturally an unstable $\mathscr{A}$-algebra over B^*. There is an isomorphism*

$$\mathrm{Alg}_{U\mathscr{A}\downarrow B_*}(A^*, B^* \ltimes M^*) \cong \mathrm{Der}_{U\mathscr{A}/B^*}(A^*, M^*),$$

where $\mathrm{Der}_{U\mathscr{A}/B^*}$ is defined to be $\mathbb{F}_p$- linear derivations relative to B^* that commute *with $\mathscr{A}$.*

Thus the derived functors of derivations are computed by taking a cofibrant replacement for A^* in the category of simplicial unstable $\mathscr{A}$-algebras. Such a cofibrant replacement is a free unstable $\mathscr{A}$-algebra resolution $F(A^*)_\bullet \to A^*$. Let M^* be a B^*-module in unstable $\mathscr{A}$-modules such that whenever $|m| = 2i$, the operation $P^i m = 0$. Define the André–Quillen cohomology as:

$$D^*_{U\mathscr{A}/B^*}(A^*, M^*) := \pi^* \mathrm{Der}_{U\mathscr{A}/B^*}(F(A^*)_\bullet, M^*).$$

2.2.3 The E_∞ Steenrod algebra

Let $\mathscr{B}$ denote the E_∞ Steenrod algebra over $\mathbb{F}_p$. Recall that at an odd prime p, the ring $\mathscr{B}$ over $\mathbb{F}_p$ is the quotient of the free $\mathbb{F}_p$ algebra generated by P^i and βP^i for i any integer. This is a highly noncommutative ring as the products of these generators satisfy the Adem relations. At the prime 2 we write $\mathscr{B}$ as the quotient of $\mathbb{F}_2\langle \ldots, Sq^{-1}, Sq^0, Sq^1, \ldots \rangle$ modulo the ideal generated by the Adem relations [23].

A sequence of *integers* $I = (\epsilon_1, s_1, \epsilon_2, s_2, \ldots, \epsilon_k, s_k)$ is called admissible if $s_{j-1} \leq ps_j + \epsilon_j$. There is an additive basis for $\mathscr{B}$ given by $\{P^I : I \text{ is admissible}\}$. Recall that $\mathscr{B}$ is a graded ring with $|P^i| = 2i(p-1)$ and Böckstein $|\beta| = 1$. The excess of a sequence $e(I)$ is defined exactly as in Section 2.2.2.

We say that a module M^* over $\mathscr{B}$ is unstable if for all x in M^*, the formula $P^I(x) = 0$ whenever $e(I) > |x|$. There is a pair of adjoint functors from graded $\mathbb{F}_p$ vector spaces $gr\mathrm{Mod}_{\mathbb{F}_p}$ to unstable $\mathscr{B}$ modules $\mathrm{Mod}_{U\mathscr{B}}$

$$\mathscr{F} : gr\mathrm{Mod}_{\mathbb{F}_p} \rightleftarrows \mathrm{Mod}_{U\mathscr{B}} : U$$

where U forgets the unstable $\mathscr{B}$-module structure, and the left adjoint $\mathscr{F}_0$ is the free functor. We can define $\mathscr{F}$ explicitly by choosing an $\mathbb{F}_p$ basis $\{x_j\}_{j \in J}$ of our vector space M^*. Then $\mathscr{F}(M^*)$ is the unstable $\mathscr{B}$ module with $\mathbb{F}_p$-module basis given by $\{P^I x_j : j \in J, I \text{ is admissible, and } e(I) \leq |x|\}$. The main example of an unstable $\mathscr{B}$-module appearing in nature is the homotopy groups of a (non-connective) $\mathrm{H}\mathbb{F}_p$-

module.

Every unstable $\mathscr{B}$-module is naturally a graded restricted Lie algebra — a graded $\mathbb{F}_p$-module with a restriction map $\Phi : M^* \to M^{p*}$ that multiplies degree by p. In the case that the restricted module comes from an unstable $\mathscr{B}$-module, at odd primes p, the restriction map is given by $P^m : M^{2m} \to M^{2mp}$ in even degrees, and is identically zero on odd degrees. At the prime 2, the restriction map is given by $Sq^i : M^i \to M^{2i}$. There is an adjoint pair of functors from unstable $\mathscr{B}$-modules to unstable $\mathscr{B}$-algebras

$$E : \mathrm{Mod}_{U\mathscr{B}} \rightleftarrows \mathrm{Alg}_{U\mathscr{B}} : U. \tag{2.2}$$

The left adjoint E is given by the enveloping algebra functor on the underlying abelian graded restricted Lie algebra. This has the effect of equating the restriction map with the Frobenius. The right adjoint is the forgetful functor. These functors induce a Quillen pair on the associated simplicial categories.

2.2.4 Relations between $\mathscr{A}$ and $\mathscr{B}$

Lemma 2.2.2 ([23, Proposition 12.5]). *For any graded $\mathbb{F}_p$ vector space V, there is a natural short exact sequence of $\mathscr{B}$-modules:*

$$0 \longrightarrow \mathscr{F}(V) \xrightarrow{1-P^0} \mathscr{F}(V) \xrightarrow{q} \mathscr{F}_0(V) \longrightarrow 0,$$

which is split exact on the underlying graded restricted $\mathbb{F}_p$-modules.

Note that in particular, this Lemma says that we can recover the Steenrod algebra $\mathscr{A}$ from $\mathscr{B}$ by taking the quotient of $\mathscr{B}$ by the left ideal (which is incidentally a two-sided ideal) generated by $(1 - P^0)$ [23, Proposition 11.4]. Thus we can view an unstable $\mathscr{A}$-module as an unstable $\mathscr{B}$-module via the quotient map.

Recall that the homotopy groups of any $H\mathbb{F}_p$-algebra naturally form an unstable $\mathscr{B}$-algebra. But in the case that the $H\mathbb{F}_p$-algebra is free commutative algebra on a free $H\mathbb{F}_p$-module, the homotopy groups naturally form a free unstable $\mathscr{B}$-algebra.

Lemma 2.2.3. *If Z is an $\mathrm{H}\mathbb{F}_p$-module, then*

$$\pi_*\mathbb{P}(Z) \cong E\mathscr{F}(\pi_*Z).$$

The proof in the case of the Steenrod algebra can be found in [28] or in the case of the Dyer–Lashoff algebra in [13, Theorem IX.2.1]. The proof for the E_∞ Steenrod algebra follows from these proofs.

Applying the previous two lemmas, we find that $\pi_*\mathbb{P}\mathrm{H}\mathbb{F}_p^{S^n_+} \cong E\mathscr{F}(\mathbb{F}_p[n] \oplus \mathbb{F}_p[0])$. Since we know that $\mathrm{H}\mathbb{F}_p^* K(\mathbb{F}_p, n)$ is the free unstable $\mathscr{A}$ algebra on a generator of cohomological degree n, the we find the following natural extension of Lemma 2.2.2.

Lemma 2.2.4. *The map*

$$1 - P^0 : E\mathscr{F}(\mathbb{F}_p[n]) \to E\mathscr{F}(\mathbb{F}_p[n])$$

is injective and the target is a projective module over the source via this map. Moreover, there is a pushout diagram in the category of unstable $\mathscr{B}$-algebras:

$$\begin{tikzcd}
E\mathscr{F}(\mathbb{F}_p[n]) \arrow[r, "1-P^0"] \arrow[d] & E\mathscr{F}(\mathbb{F}_p[n]) \arrow[d] \\
\mathbb{F}_p \arrow[r] & E\mathscr{F}_0(\mathbb{F}_p[n]).
\end{tikzcd}$$

Proof of this Lemma can be found in [23]. It is important to note that $E\mathscr{F}_0(V) \cong E_0\mathscr{F}_0(V)$ since by definition the negative power operations in $\mathscr{B}$ act trivially on $\mathscr{F}_0(V)$. This is a consequence of P^0 acting trivially on $F_0(V)$ and the Adem relations.

In fact, even more is true. Let $\mathbb{F}_p[n]$ denote the graded $\mathbb{F}_p$−module with one $\mathbb{F}_p$ in cohomological degree n. Then we use notation from [23] and denote $\mathscr{F}(\mathbb{F}_p[n])$, the free $\mathscr{B}$−module on a generator in degree n, as $\mathscr{B}^n$. As an example, the enveloping algebra on a free unstable $\mathscr{B}$−module $\mathscr{B}^n$ is given by the free commutative algebra on the symbols $\{P^I i_n : e(I) < n\}$ and $P^n i_n = i_n^p$, where i_n denotes the generator of $\mathscr{B}^n$ as a free $\mathscr{B}$−module.

We have a left adjoint from graded $\mathbb{F}_p$-modules to unstable $\mathscr{B}$-modules. By [25, Theorem II.4.4], there is a model category structure on simplicial unstable $\mathscr{B}$ modules such that the fibrations and weak equivalences are the maps that are fibrations and weak equivalences on the underlying simplicial graded $\mathbb{F}_p$ modules. Consider the homotopy pushout diagram in chain complexes of unstable $\mathscr{B}$-modules:

$$\begin{tikzcd} \mathscr{B}^n[0] \arrow[r, "1-P^0"] \arrow[d] & \mathscr{B}^n[0] \arrow[d] \\ 0 \arrow[r] & C_*, \end{tikzcd}$$

where C_* is the chain complex of unstable $\mathscr{B}$-modules given by

$$C_* = \cdots 0 \to \mathscr{B}^n[1] \to \mathscr{B}^n[0].$$

Considering the image under the Dold–Kan correspondence, we obtain the simplicial unstable $\mathscr{B}$-module $T_\bullet$ as the homotopy pushout of simplicial unstable $\mathscr{B}$-modules:

$$\begin{tikzcd} \mathscr{B}^n_\bullet \arrow[r] \arrow[d] & \mathscr{B}^n_\bullet \arrow[d] \\ 0 \arrow[r] & T_\bullet. \end{tikzcd} \tag{2.3}$$

Proposition 2.2.5. *There is a cofibrant simplicial resolution of $\pi_*\mathrm{H}\mathbb{F}_p^{K(\mathbb{F}_p,n)_+}$ by free unstable $\mathscr{B}$-algebras obtained by applying E to the homotopy pushout $T_\bullet$, which defined by the diagram (2.3):*

$$ET_\bullet \to \pi_*\mathrm{H}\mathbb{F}_p^{K(\mathbb{F}_p,n)_+}.$$

Proof. There is an augmentation map $T_\bullet \to \mathscr{F}(\mathbb{F}_p[n])$, which is a weak equivalence of simplicial $\mathscr{B}$-algebras by viewing $\mathscr{F}(\mathbb{F}_p[n])$ as a constant simplicial object. Since E is a left Quillen functor, taking homotopy pushouts of unstable $\mathscr{B}$-modules to homotopy pushouts of unstable $\mathscr{B}$-algebras, $E(T_\bullet)$ is the bar construction on the map:

$$E(1-P^0) : E\mathscr{F}(\mathbb{F}_p[n]) \to E\mathscr{F}(\mathbb{F}_p[n]).$$

Thus the homotopy groups of $ET_\bullet$ are given by

$$\pi_* ET_\bullet = \mathrm{Tor}_*^{E\mathscr{F}(\mathbb{F}_p[n])}(\mathbb{F}_p, E\mathscr{F}(\mathbb{F}_p[n])).$$

Since $E\mathscr{F}(\mathbb{F}_p[n])$ is projective as a $E\mathscr{F}(\mathbb{F}_p[n])$ module via the module map $E(1-P^0)$ by Proposition 2.2.4, the higher homotopy groups vanish. Thus

$$\pi_* ET_\bullet \cong E\mathscr{B}^n \otimes_{E\mathscr{B}^n} \mathbb{F}_p.$$

But we already know from [23] that this right hand side is $\mathrm{H}\mathbb{F}_p^* K(\mathbb{F}_p, n)$, which is what we wanted to show. □

2.2.5 Unstable $\overline{\mathscr{B}}$-algebras.

Now we extend the discussion above to include $\mathrm{H}\overline{\mathbb{F}}_p$-algebras and unstable $\mathscr{B}$-algebras over $\overline{\mathbb{F}}_p$, which we will denote as $\overline{\mathscr{B}}$-algebras. This is the context which is important for identifying the E_2 term of the Goerss–Hopkins spectral sequence.

Let $f : \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p$ denote the Frobenius map on $\overline{\mathbb{F}}_p$. Note that this map is injective and$f$ is the generator of the Galois group of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$. An unstable $\mathscr{B}$-algebra over $\overline{\mathbb{F}}_p$ is an unstable $\mathscr{B}$ algebra A, whose underlying graded vector space is an $\overline{\mathbb{F}}_p$-module. Given any $a \in A$ and any $\lambda \in \overline{\mathbb{F}}_p$, the action of an element P^i is defined as:

$$P^i(\lambda a) := f(\lambda) P^i a.$$

In particular, P^0 acts as the Frobenius on $\overline{\mathbb{F}}_p$. An example of unstable $\mathscr{B}$-algebras over $\overline{\mathbb{F}}_p$ is the homotopy groups of an $\mathrm{H}\overline{\mathbb{F}}_p$-algebra. We denote the category of unstable $\overline{\mathscr{B}}$-algebras as $\mathrm{Alg}_{U\overline{\mathscr{B}}}$.

Lemma 2.2.6. *If Z is an $\mathrm{H}\overline{\mathbb{F}}_p$-module, then*

$$\pi_* \mathbb{P}(Z) \cong E(\mathscr{F}(\pi_* Z)),$$

is a free unstable $\mathscr{B}$-algebra over $\overline{\mathbb{F}}_p$.

We have the following diagram of adjunctions, where only the left adjoints are labeled:

$$\begin{tikzcd}
\mathrm{Alg}_{U\mathscr{B}} \arrow[r, shift left, "-\hat{\otimes}\overline{\mathbb{F}}_p"] & \mathrm{Alg}_{U\overline{\mathscr{B}}} \arrow[l, shift left] \arrow[d, shift left] \\
\mathrm{Mod}_{U\mathscr{B}} \arrow[u, shift left, "E"] \arrow[r, shift left, "-\hat{\otimes}\overline{\mathbb{F}}_p"] & \mathrm{Mod}_{U\overline{\mathscr{B}}} \arrow[l, shift left] \arrow[u, shift left, "E"] \arrow[d, shift left] \\
gr\mathrm{Mod}_{\mathbb{F}_p} \arrow[u, shift left, "\mathscr{F}"] \arrow[r, shift left, "-\hat{\otimes}\overline{\mathbb{F}}_p"] & gr\mathrm{Mod}_{\overline{\mathbb{F}}_p} \arrow[l, shift left] \arrow[u, shift left, "\mathscr{F}"]
\end{tikzcd}$$

Write

$$\overline{E} := E(-)\hat{\otimes}_{\mathbb{F}_p}\overline{\mathbb{F}}_p : \mathrm{Mod}_{U\mathscr{B}} \to \mathrm{Alg}_{U\overline{\mathscr{B}}}$$

for the free functor from unstable $\mathscr{B}$-modules to unstable $\overline{\mathscr{B}}$-algebras. This is a left adjoint to the forgetful functor.

Lemma 2.2.6 above shows that the commutative operad in $\mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is *adapted* to π_* as in [17, Definition 1.4.13]. By [17, Example 1.4.14] this means that the commutative operad is simplicially adapted to π_*. Thus, given $Z_\bullet$, a cofibrant simplicial $\mathrm{H}\overline{\mathbb{F}}_p$-module, there is an isomorphism of simplicial unstable $\overline{\mathscr{B}}$-algebras

$$\pi_*\mathbb{P}(Z_\bullet) \cong E\mathscr{F}(\pi_* Z_\bullet). \tag{2.4}$$

We record here an extension of Lemma 2.2.4 to finite type complexes extended over $\overline{\mathbb{F}}_p$. This Proposition will be crucial in constructing the map of spectral sequences.

Proposition 2.2.7. *For every graded $\mathbb{F}_p$ vector space of finite type,*

$$\overline{E}(1 - P^0) : \overline{E}(\mathscr{F}(V)) \to \overline{E}(\mathscr{F}(V))$$

is injective, and $\overline{E}(\mathscr{F}(V))$ is projective as a module over itself via $\overline{E}(1 - P^0)$. More-

over, there is a pushout diagram in the category of unstable $\mathscr{B}$-algebras over $\overline{\mathbb{F}}_p$.

$$\begin{tikzcd} \overline{E}(\mathscr{F}(V)) \arrow[r, "1-P^0"] \arrow[d] & \overline{E}(\mathscr{F}(V)) \arrow[d] \\ \overline{\mathbb{F}}_p \arrow[r] & \overline{E}\mathscr{F}_0(V) \end{tikzcd}$$

Proof. The proof follows from Lemma 2.2.4 and Lemma 2.2.6. $\square$

We also extend Proposition 2.2.5 to spaces of finite type. This resolution will be key in order to compute the E_2 term of the Goerss–Hopkins spectral sequence.

Proposition 2.2.8. *Let V be an $\mathbb{F}_p$ vector space. Then viewing $\mathscr{F}(V)_\bullet$ as a constant simplicial object, let $T_\bullet$ be defined by the homotopy cofiber sequence*

$$\mathscr{F}(V)_\bullet \xrightarrow{1-P^0} \mathscr{F}(V)_\bullet \longrightarrow T_\bullet.$$

Then

$$\overline{E}(T_\bullet) \longrightarrow \overline{E}\mathscr{F}_0(V)$$

is a resolution of $\mathrm{H}\overline{\mathbb{F}}_p^*\mathscr{F}_0(W)$ by a simplicial free $\overline{\mathscr{B}}$-algebra.

Proof. The proof is immediate as $\overline{E}$ is a left Quillen functor, and is completely analogous to the proof of Proposition 2.2.5. $\square$

2.2.6 Derivations of unstable $\overline{\mathscr{B}}$-algebras.

Lemma 2.2.9. *Given unstable $\overline{\mathscr{B}}$-algebras A^* and B^*, a map $\varphi : A^* \to B^*$, and an unstable $\overline{\mathscr{B}}$-module, M^*, that is also a B^*-module. Suppose further that M^* satisfies the strict instability condition that for m in M^*, the power operation $P^I(m) = 0$ whenever $e(I) \geq |m|$. Then $B^* \ltimes M^*$ is naturally an unstable $\overline{\mathscr{B}}$-algebra over B^*. There is an isomorphism*

$$\mathrm{Alg}_{U\overline{\mathscr{B}}\downarrow B_*}(A^*, B^* \ltimes M^*) \cong \mathrm{Der}_{U\overline{\mathscr{B}}/B^*}(A^*, M^*),$$

where $\mathrm{Der}_{U\overline{\mathscr{B}}/B^*}$ *is defined to be* $\overline{\mathbb{F}}_p$*- linear derivations relative to* B^* *that commute with* $\mathscr{B}$.

The derived functors of derivations are computed by taking a cofibrant replacement for A^* in the category of simplicial unstable $\mathscr{B}$-algebras over $\overline{\mathbb{F}}_p$. Such a cofibrant replacement is a free unstable $\mathscr{B}$-algebra resolution $F(A^*)_\bullet \to A^*$. Define the André–Quillen cohomology as:

$$D^*_{U\overline{\mathscr{B}}/B^*}(A^*, M^*) := \pi^* \mathrm{Der}_{U\overline{\mathscr{B}}/B^*}(F(A^*)_\bullet, M^*). \tag{2.5}$$

2.3 A descent theorem.

Proposition 2.3.1. *Let* V *be a graded* $\mathbb{F}_p$*-vector space. Let* M *be an* $\overline{E}\mathscr{F}_0(V)$*-module in the category of unstable* $\mathscr{B}$*-algebras over* $\overline{\mathbb{F}}_p$. *Let* $\overline{E}\mathscr{F}(V)$ *be augmented over the* $U\overline{\mathscr{B}}$*-algebra* A. *Then there is a long exact sequence in André–Quillen cohomology:*

$$\cdots \to D^{s-1}_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}(V), M) \to D^{s}_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}_0(V), M) \to$$
$$D^{s-1}_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}(V), M) \to D^{s-1}_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}(V), M) \to \cdots .$$

Proof. Proposition 2.2.8 gives a homotopy cofiber sequence of simplicial unstable $\overline{\mathscr{B}}$-algebras:

$$\overline{E}\mathscr{F}(V) \to \overline{E}\mathscr{F}(V) \to \overline{E}T_\bullet \tag{2.6}$$

where the first two simplicial objects are constant. Combining this with Proposition 2.2.7, which says that $\overline{E}T_\bullet$ is a cofibrant simplicial resolution of $\overline{E}\mathscr{F}_0(V)$ and the fact that the two free constant simplicial objects are already cofibrant, the result is immediate. □

Of course, since $\overline{E}\mathscr{F}(V)$ is already cofibrant, this implies that the higher cohomology vanishes. Thus this long exact sequences reduces to the exact sequence

$$0 \to D^0_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}_0(V), M) \to D^0_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}_0(V), M) \to \tag{2.7}$$

$$D^0_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}(V), M) \to D^1_{U\overline{\mathscr{B}}/A}(\overline{E}\mathscr{F}_0(V), M) \to 0. \tag{2.8}$$

Theorem 2.3.2. *Let V_0 be a graded $\mathbb{F}_p$-module, and suppose $E\mathscr{F}_0(V_0)$ is augmented over the unstable $\mathscr{A}$-algebra A_0. Write $A := \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} A_0$. Let M_0 be an $\overline{E}\mathscr{F}_0(V_0)$-module in the category of unstable $\mathscr{A}$-algebras. Define $V = \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} V_0$ and $M = \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M_0$.*

1. $D^1_{U\overline{\mathscr{B}}/E\mathscr{F}_0(V)}(E\mathscr{F}_0(V), M) = 0$

2. *There is a natural isomorphism*

$$D^0_{U\overline{\mathscr{B}}/A}(E\mathscr{F}_0(V), M) \cong D^0_{U\mathscr{A}/A_0}(E\mathscr{F}_0(V_0), M_0).$$

Proof. Note that $\overline{E}\mathscr{F}(V_0) \cong E\mathscr{F}(V)$, which is augmented over the unstable $\overline{\mathscr{B}}$-algebra A. We compute $D^0_{U\overline{\mathscr{B}}/A}(E\mathscr{F}(V), M)$ directly, and determine the effect of the map $E(1 - P^0)$. The kernel and cokernel of the map $E(1 - P^0)$ are precisely the groups we are trying to compute. The computation shows:

$$\begin{aligned}
D^0_{U\overline{\mathscr{B}}/A}(E\mathscr{F}(V), M) &= \mathrm{Der}_{U\overline{\mathscr{B}}/A}(E\mathscr{F}(V), M) \\
&= \mathrm{Alg}_{U\overline{\mathscr{B}}\downarrow A}(E\mathscr{F}(V), A \ltimes M) \\
&\cong gr\mathrm{Mod}_{\overline{\mathbb{F}}_p\downarrow A}(V, A \ltimes M) \\
&\cong gr\mathrm{Mod}_{\overline{\mathbb{F}}_p\downarrow A}(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} V_0, A \ltimes (\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} M_0)) \\
&\cong gr\mathrm{Mod}_{\mathbb{F}_p\downarrow A_0}(V_0, \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} (A_0 \ltimes M_0)).
\end{aligned}$$

The endomorphism $E(1 - P^0)$ in the short exact sequence (2.7) is reduced by the adjunction in line 3 to the endomorphism $1 - P^0$ acting on the source, where now P^0 is simply the Frobenius map on $\overline{\mathbb{F}}_p$ in the category of graded $\overline{\mathbb{F}}_p$-modules. This map acting on the source is equivalent to the map acting on the target. This map is zero except on $\overline{\mathbb{F}}_p$, where P^0 acts as the Frobenius. Note that $1 - P^0$ is surjective on $\overline{\mathbb{F}}_p$

as P^0 is the generator of the Galois group of $\overline{\mathbb{F}}_p$ over $\mathbb{F}_p$. Thus the kernel of the map on $\overline{\mathbb{F}}_p$ is $\mathbb{F}_p$, and the cokernel is zero.

The map being surjective shows that

$$D^1_{U\overline{\mathscr{B}}/\overline{E}\mathscr{F}_0(V)}(\overline{E}\mathscr{F}_0(V), M) = \mathrm{Coker}(E(1 - P^0)) = 0.$$

The kernel being $\mathbb{F}_p$ gives rise to the following identifications:

$$\begin{aligned} D^0_{U\overline{\mathscr{B}}/\overline{E}\mathscr{F}_0(V)}(E\mathscr{F}_0(V), M) &\cong \mathrm{Ker}(E(1 - P^0)) \\ &\cong gr\mathrm{Mod}_{\mathbb{F}_p\downarrow A_0}(V_0, A_0 \ltimes M_0) \\ &\cong \mathrm{Alg}_{U\mathscr{A}\downarrow A_0}(E\mathscr{F}_0 V_0, A \ltimes M_0) \\ &\cong \mathrm{Der}_{U\mathscr{A}/A_0}(E\mathscr{F}_0 V_0, M_0), \end{aligned}$$

which is the second point. □

2.4 The unstable Adams spectral sequence

The unstable $H\mathbb{F}_p$ based Adams spectral sequence abuts to the unstable homotopy groups of the $\mathbb{F}_p$-completion of a space. When a space is p-good, the p-completion of the space is equivalent to the localization of the space with respect to the ring spectrum $H\mathbb{F}_p$. In particular, a space is p-good when it is nilpotent. The following description comes out [11], which is described in terms of simplicial sets and is sufficient for our purposes here. For analogous constructions defined for more general cohomology theories and on the category of spaces rather than simplicial sets, see [2]. These should be considered the references for details on the construction and convergence of the unstable Adams spectral sequence.

Fix a simplicial set X. Note that there is an adjunction between simplicial sets and simplicial $\mathbb{F}_p$-algebras:

$$F : \mathscr{T} \rightleftarrows s\mathrm{Mod}_{\mathbb{F}_p} : U.$$

The right adjoint U is the forgetful functor. The left adjoint F is the free functor sending $X \mapsto \mathbb{F}_p(X)$ to the free $\mathbb{F}_p$-algebra on the simplicial set X. The composition $U \circ F$ is a monad on $\mathscr{T}$ and we are interested in the cosimplicial space constructed with this monad. There are two coface maps

$$\mathbb{F}_p(X) \rightrightarrows \mathbb{F}_p(\mathbb{F}_p(X))$$

given by $F(\mathbb{F}_p(-))$ and $\mathbb{F}_p(F(-))$ respectively. There is one codegeneracy map

$$\mathbb{F}_p(X) \longleftarrow \mathbb{F}_p(\mathbb{F}_p(X))$$

given by the addition is simplicial $\mathbb{F}_p$-modules. Thus iterating this process, we obtain a cosimplicial space that we denote $\mathbb{F}_p^{\bullet+1}(X)$ where $\bullet = 0, 1, \ldots,$ and $\mathbb{F}_p^m(X)$ denotes the m fold composition of the monad $U \circ F$. In particular, viewing X as the constant cosimplicial object, we obtain a cosimplicial map

$$r : X \to \mathbb{F}_p^{\bullet+1}(X) \tag{2.9}$$

whose totalization is the Bousfield–Kan p-completion. Thus if we have a mapping space $\mathscr{T}(Y, X)$ with the space X nilpotent, we obtain a cosimplicial space $\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X))$ whose totalization is $\mathscr{T}(Y, X_p)$.

Theorem 2.4.1. *Given spaces X and Y be spaces with X nilpotent, and a map $f : Y \to X$, there is an unstable Adams spectral sequence with E_2 term*

$$E_2^{s,t} := \pi^s \pi_t(\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)), r \circ f) \Longrightarrow \pi_*(\mathscr{T}(Y, X_p), f_p).$$

Convergence of this fringed spectral sequence is a delicate issue. For proof of this theorem and conditions for convergence, see [11]. It is interesting to note that to guarantee convergence of this spectral sequence, one need only suppose a space is p-complete and nilpotent [26].

2.4.1 Identification of E_2 term

We are interested in identifying the E_2 term in the case that X and Y are of finite type. The reason we wish to suppose this is that it allows us to describe the E_2 term of the spectral sequence in terms of the cohomology of X and Y rather than the homology. Throughout this section, we suppose that X and Y are of finite type. These conditions are also necessary for the proof of the main theorem, thus nothing is lost in supposing them here.

We can identify $\pi^*\mathbb{F}_p(X) \cong \mathrm{H}\mathbb{F}_p^*X$, and furthermore

Lemma 2.4.2. *When X is of finite type, there is an isomorphism of unstable $\mathscr{A}$-algebras:*

$$\mathrm{H}\mathbb{F}_p^*\mathbb{F}_p(X) \cong E_0\mathscr{F}_0(\mathrm{H}\mathbb{F}_p^*X).$$

The proof follows from a colimit argument applied to the case that X is a sphere. In this case the lemma says that the cohomology $\mathrm{H}\mathbb{F}_p^*K(\mathbb{F}_p, n)$ is a free unstable $\mathscr{A}$-algebra on a single generator in degree n.

Let $f \in \pi_0\mathscr{T}(Y, X)$ be a basepoint for $\mathscr{T}(Y, X)$. Then consider

$$\pi_t\mathscr{T}(Y, X) \cong \pi_0(Y \downarrow \mathscr{T})(S^t_+ \wedge Y, X),$$

where X is a space under Y via the basepoint map f. There is a Hurewitcz-type homomorphism to the category of unstable algebras over the Steenrod algebra $\mathscr{A}$:

$$\begin{aligned}\pi_0(Y \downarrow \mathscr{T})(S^t_+ \wedge Y, X) \to &\mathrm{Alg}_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(\mathrm{H}\mathbb{F}_p^*X, \mathrm{H}\mathbb{F}_p^*(S^t_+ \wedge Y)) && (2.10)\\ \cong &\mathrm{Der}_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(\mathrm{H}\mathbb{F}_p^*X, \Sigma^t\mathrm{H}\mathbb{F}_p^*Y). && (2.11)\end{aligned}$$

Note that the map (2.10) is an isomorphism if X is of the form $\mathbb{F}_p(X)$. Thus we have an identification of the E_2 term of our spectral sequence:

$$\pi^s\pi_t\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \cong \pi^s\mathrm{Der}_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(\mathrm{H}\mathbb{F}_p^*\mathbb{F}_p^{\bullet+1}(X), \Sigma^t\mathrm{H}\mathbb{F}_p^*Y).$$

Since $\mathrm{H}\mathbb{F}_p^*\mathbb{F}_p^{\bullet+1}(X) \cong \mathscr{F}_0\mathrm{H}\mathbb{F}_p^*\mathbb{F}_p^{\bullet}(X)$ is a simplicial resolution of $\mathrm{H}\mathbb{F}_p^*X$ by free $U\mathscr{A}$-algebras, we can identify $\pi^s\mathrm{Der}_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(\mathrm{H}\mathbb{F}_p^*\mathbb{F}_p^{\bullet+1}(X), \Sigma^t\mathrm{H}\mathbb{F}_p^*Y)$ as

$$\begin{aligned} D^s_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(\mathrm{H}\mathbb{F}_p^*X, \Sigma^t\mathrm{H}\mathbb{F}_p^*Y) &\qquad t > 0, s \geq 1 \\ \mathrm{Alg}_{U\mathscr{A}}(\mathrm{H}\mathbb{F}_p^*X, \mathrm{H}\mathbb{F}_p^*Y) &\qquad (s,t) = (0,0) \end{aligned}$$

where $D^*_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^*Y}(-,-)$ is an André–Quillen type cohomology theory denoting the derived functors of derivations of unstable $\mathscr{A}$-algebras augmented over $\mathrm{H}\mathbb{F}_p^*Y$. Note that we are using the fact that the top Steenrod operations act trivially on the cohomology of suspension spaces. Observe that $\mathrm{H}\mathbb{F}_p^*X$ maps to $\mathrm{H}\mathbb{F}_p^*Y$ via the basepoint map $\mathrm{H}\mathbb{F}_p^*f$.

2.5 The Goerss–Hopkins spectral sequence

The Goerss–Hopkins spectral sequence is a spectral sequence that computes the homotopy groups of a space of maps between E_∞-algebras. In its most general form, we fix an $\mathbb{S}$-algebra A. Let E be an A-algebra such that $\pi_*(E \wedge_A E)$ is flat over π_*E. Given A-algebras M, N and a map of algebras $\phi : M \to N$ giving N the structure of an M-module, the Goerss–Hopkins spectral sequence computes

$$\pi_*(\mathrm{Alg}_A(M, N_E^\wedge), \phi)$$

in terms of E_*M and E_*N. For details on the construction, see [15, 17, 16]. We are interested in the case where both A and E are the Eilenberg–Maclane spectrum $\mathrm{H}\overline{\mathbb{F}}_p$. In this section, we describe the E_2 term of this spectral sequence.

Let $\mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ denote the model category of $\mathrm{H}\overline{\mathbb{F}}_p$-modules. Define $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ to be the category of commutative algebras in $\mathrm{H}\overline{\mathbb{F}}_p$-modules. There is a Quillen adjunction:

$$\mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p} \underset{}{\overset{\mathbb{P}(-)}{\rightleftarrows}} \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$$

with right adjoint the forgetful functor and left adjoint $\mathbb{P}(-)$ the free commutative algebra functor. The model category structure on $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is induced by the model category structure on $\mathrm{Alg}_{\mathbb{S}}$ by identifying $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ with the under category $\mathrm{H}\overline{\mathbb{F}}_p \downarrow \mathrm{Alg}_{\mathbb{S}}$. The model category structure on $\mathrm{Alg}_{\mathbb{S}}$ is induced by the model category structure on a good model for spectra. One may take the model category structure on spectra of $\mathbb{S}$-modules in [14], or one can take the positive projective model category structure on symmetric spectra [21]. The model structure on $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is determined by the fibrations and weak equivalences, which are the maps that are fibrations and weak equivalences on the underlying category of spectra.

In the homotopy category $\mathrm{HoMod}_{\mathrm{H}\overline{\mathbb{F}}_p}$, let $\mathscr{P}$ denote the class of projectives given by the finite cell $\mathrm{H}\overline{\mathbb{F}}_p$-modules. This is the smallest class of spectra that has the following properties:

- $\mathrm{H}\overline{\mathbb{F}}_p \in \mathscr{P}$;
- $\mathscr{P}$ is closed under suspension and desuspension;
- $\mathscr{P}$ is closed under finite wedges;
- For all $P \in \mathscr{P}$, and all $X \in \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}$, the Künneth map

$$[P, X] \to \mathrm{Mod}_{\overline{\mathbb{F}}_p}(\pi_* P, \pi_* X)$$

  is an isomorphism.

Note that this is equivalent to the class of finite coproducts of suspensions and desuspensions of $\mathrm{H}\overline{\mathbb{F}}_p$. This allows us to define the $\mathscr{P}$-projective model category structure on $\mathrm{sMod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ [10, 17]. We say that a map $X_\bullet \to Y_\bullet$ in $\mathrm{sMod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is

- a $\mathscr{P}$-weak equivalence if $[P, X_\bullet] \to [P, Y_\bullet]$ is a weak equivalence of simplicial $\overline{\mathbb{F}}_p$-modules for all $P \in \mathscr{P}$.
- a $\mathscr{P}$-fibration if it is a Reedy fibration and $[P, X_\bullet] \to [P, Y_\bullet]$ is surjective for all $P \in \mathscr{P}$.
- a $\mathscr{P}$-cofibration if it has the left lifting property with respect to all $\mathscr{P}$-fibrations that are also $\mathscr{P}$-weak equivalences.

This model structure is cofibrantly generated; see [17]. Since the $\mathscr{P}$-projective model structure on $s\mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is cofibrantly generated, we obtain a cofibrantly generated model structure on $\mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}$, which we also call the $\mathscr{P}$-projective model category structure, whose weak equivalences and fibrations are those that are $\mathscr{P}$-weak equivalences and $\mathscr{P}$-fibrations on the underlying category $\mathrm{sMod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ [21].

Lemma 2.5.1 ([17, Proposition 1.4.11]). *There exist functorial cofibrant replacements $p : P(X_\bullet) \xrightarrow{\sim} X_\bullet$ in the category of simplicial (commutative) $\mathrm{H}\overline{\mathbb{F}}_p$-algebras such that the degeneracy diagram of $P(X_\bullet)$ has the form $\mathbb{P}(Z_\bullet)$, where $Z_\bullet$ is free as a degeneracy diagram, and each Z_n is a wedge of finite cell $\mathrm{H}\overline{\mathbb{F}}_p$-module spectra.*

We can now describe the Goerss–Hopkins spectral sequence.

Theorem 2.5.2 ([15, Theorem 4.3]). *Given A and B in $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}$, and a map*

$$\phi : A \to B,$$

there is a totalization spectral sequence with E_2 term

$$E_2^{s,t} := \pi^s \pi_t(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(A), B), \phi \circ p) \Longrightarrow \pi_{t-s}(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(A, B), \phi).$$

Proof. Given the standard converges hypotheses, this spectral sequence converges to

$$\pi_{t-s}\mathrm{Tot}(\mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(A), B)),$$

where

$$\mathrm{Tot}(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(A), B)) \simeq \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(A, B).$$

For proof of and the standard technical hypotheses guaranteeing the converges of this spectral sequence, see [15]. $\square$

2.5.1 Identification of the E_2 term

Let A and B be $H\overline{\mathbb{F}}_p$-algebras. Fix an algebra map $\varphi : A \to B$. Note that the E_2 term of the spectral sequence is given by $\pi^s \pi_t \mathrm{Alg}_{H\overline{\mathbb{F}}_p}(P(A), B)$. We begin by analyzing $\pi_t \mathrm{Alg}_{H\overline{\mathbb{F}}_p}(P(A), B)$. There is a Hurewicz-type homomorphism:

$$\begin{aligned}\pi_t(\mathrm{Alg}_{H\overline{\mathbb{F}}_p}(A, B); \varphi) &\cong \pi_0 \mathrm{Alg}_{H\overline{\mathbb{F}}_p \downarrow B}(A, B^{S^t}) \\ &\to \mathrm{Alg}_{U\overline{\mathscr{B}}/\pi_* B}(\pi_* A, \pi_* B[x_t]/(x_t^r)) \\ &\cong \mathrm{Der}_{U\overline{\mathscr{B}}/\pi_* B}(\pi_* A, \Sigma^t \pi_* B).\end{aligned}$$

We show this map is an isomorphism if $A = \mathbb{P}(Z)$, which will give us the desired identification when we replace A by $P(A) \cong \mathbb{P}(Z_\bullet)$.

Let $A = \mathbb{P}(Z)$. Fix an $H\overline{\mathbb{F}}_p$-algebra map $\varphi : \mathbb{P}(Z) \to B$ as the basepoint. There is an induced $H\overline{\mathbb{F}}_p$-module map $\varphi' : Z \to \mathbb{P}(Z) \to B$.

We begin by analyzing

$$\pi_t \mathrm{Alg}_{H\overline{\mathbb{F}}_p}(\mathbb{P}(Z), B) \cong \pi_t \mathrm{Mod}_{H\overline{\mathbb{F}}_p}(Z, B)$$

where the basepoint for the space of $H\overline{\mathbb{F}}_p$−module maps is given by φ'. The homotopy groups of the space of module maps is naturally isomorphic to

$$\pi_0(* \downarrow \mathscr{T})(S^t, \mathrm{Mod}_{H\overline{\mathbb{F}}_p}(Z, B)).$$

Observe that while the category $\mathrm{Alg}_{H\overline{\mathbb{F}}_p}$ is not naturally pointed, the category $\mathrm{Mod}_{H\overline{\mathbb{F}}_p}$ is. Thus the mapping space $\mathrm{Mod}_{H\overline{\mathbb{F}}_p}(Z, B)$ has a natural basepoint 0 given by the zero map. Thus, while we view the pointed space as an object

$$\mathrm{Map}_{H\overline{\mathbb{F}}_p}(Z, B) \in * \downarrow \mathscr{T}$$

in the category of pointed spaces, it is naturally an object in the category $S^0 \downarrow \mathscr{T}$ of spaces under S^0. The map φ' determines a map $S^0 \to \mathrm{Mod}_{H\overline{\mathbb{F}}_p}(Z, B)$. We have an equivalence

$$\pi_0(* \downarrow \mathscr{T})(S^t, \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}(Z, B)) \cong \pi_0(S^0 \downarrow \mathscr{T})(S^t_+, \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}(Z, B)).$$

The category $\mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}$ is tensored and cotensored over $\mathscr{T}$, thus we have the following equivalences:

$$\pi_0(S^0 \downarrow \mathscr{T})(S^t_+, \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p}(Z, B)) \cong \pi_0 \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p \downarrow B^{S^0}}(Z, B^{S^t_+})$$

Here Z is an $\mathrm{H}\overline{\mathbb{F}}_p-$module over B via the map φ', while $B^{S^t_+}$ is via the map of spaces $S^0 \to S^t_+$ sending 0 to $+$ and 1 to the natural basepoint 1 in S^t. Observe that since B is actually an $\mathrm{H}\overline{\mathbb{F}}_p-$algebra, so is $B^{S^t_+}$. Now we can identify this term as

$$\pi_0 \mathrm{Mod}_{\mathrm{H}\overline{\mathbb{F}}_p \downarrow B^{S^0}}(Z, B^{S^t_+}) \cong \mathrm{Mod}_{\overline{\mathbb{F}}_p \downarrow B^*}(\pi_* Z, \pi_* B^{S^t_+}).$$

Since $B^{S^t_+}$ is naturally an $\mathrm{H}\overline{\mathbb{F}}_p-$algebra, its homotopy is naturally the unstable $\overline{\mathscr{B}}-$algebra $B^*(S^t) \cong \pi_* B[x_t]/(x_t^2)$. The algebra structure is given by the product in the cohomology theory B^*. Let (n, m) and (n', m') be elements in $\pi_* B \oplus \Sigma^t \pi_* B$, then

$$(n, m) \cdot (n', m') = (nn', nm' + n'm)$$

since $mm' = 0$ as there are no higher cohomology groups. Thus we might more appropriately write this algebra as the square-zero extension $\pi_* B \ltimes \Sigma^t \pi_* B$. Using the left adjoint to the forgetful functor from $U\overline{\mathscr{B}}$ to $\mathrm{Mod}_{\overline{\mathbb{F}}_p}$ we have an identification

$$\mathrm{Mod}_{\overline{\mathbb{F}}_p \downarrow \pi_* B}(\pi_* Z, \pi_* B \ltimes \Sigma^t \pi_* B) \cong \mathrm{Alg}_{U\overline{\mathscr{B}} \downarrow \pi_* B}(E\mathscr{F}(\pi_* Z), \pi_* B \ltimes \Sigma^t \pi_* B),$$

proving the map in the Hurewicz type homomorphism is in fact an isomorphism.

By Lemma 2.2.6, $\pi_* P(A) \cong \pi_* \mathbb{P}(Z) \cong \overline{E}\mathscr{F}(\pi_* Z)$. By the previous remark, we see that:

$$\pi_t \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(A), B) \cong \pi_t \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathbb{P}(Z_\bullet), B) \cong \mathrm{Der}_{U\overline{\mathscr{B}}/\pi_* B}(\pi_* \mathbb{P}(Z_\bullet), \Sigma^t \pi_* B). \qquad (2.12)$$

Observe that $\pi_* \mathbb{P}(Z_\bullet) \to \pi_* A$ is a cofibrant simplicial resolution of $\pi_* A$ by levelwise free unstable $\overline{\mathscr{B}}$-algebras. Taking π^s of the cosimplicial complex (2.12) we obtain the desired identification of the E_2 term as the right derived functors of derivations of unstable $\overline{\mathscr{B}}$ algebras.

Proposition 2.5.3. *The E_2 term of the Goerss–Hopkins spectral sequence can be identified as*

$$\begin{aligned} \pi^s \pi_t \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(A, B) &\cong D^s_{U\overline{\mathscr{B}}/\pi_* B}(\pi_* A, \Sigma^t \pi_* B) && t > 0 \\ &\cong \mathrm{Alg}_{U\overline{\mathscr{B}}}(\pi_* A, \pi_* B) && s = t = 0. \end{aligned}$$

Given a pointed, p-complete, nilpotent space X, and a nilpotent space Y, we are interested in computing the space $\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$. The of the description of the Goerss–Hopkins spectral sequence computing the homotopy groups of this spaces simplifies.

Proposition 2.5.4. *Given spaces X and Y, the Goerss–Hopkins spectral sequence computing the homotopy groups of $\psi(X)$ has E_2 term given by:*

$$\begin{aligned} E_2^{s,t} =& D^s_{U\overline{\mathscr{B}}/\mathrm{H}\overline{\mathbb{F}}_p^* Y}(\mathrm{H}\overline{\mathbb{F}}_p^* X, \Sigma^t \mathrm{H}\overline{\mathbb{F}}_p^* Y) && t > 0 \\ E_2^{0,0} =& \mathrm{Alg}_{U\overline{\mathscr{B}}}(\mathrm{H}\overline{\mathbb{F}}_p^* X, \mathrm{H}\overline{\mathbb{F}}_p^* Y) \end{aligned}$$

abutting to

$$\pi_{t-s}(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})).$$

2.6 Comparing cosimplicial spaces

Recall that we have the map ψ from (2.1). In particular, ψ induces a map of cosimplicial spaces:

$$\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})). \tag{2.13}$$

The goal is to prove that these two cosimplicial spaces are weakly equivalent. Thus after applying $\pi^s\pi_t$ to this map, we will obtain an isomorphism between the E_2 term of the unstable Adams spectral sequence and the Goerss–Hopkins spectral sequence.

Lemma 2.6.1. *The map*

$$\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+} \to \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+} \tag{2.14}$$

is a $\mathscr{P}$-weak equivalence in $\mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}$.

Proof. Recall that

$$\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+} \to \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}$$

is a $\mathscr{P}$-weak equivalence if

$$[P, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}] \to [P, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}]$$

is a weak equivalence for all $P = \mathbb{P}(Z)$, where Z is in $\mathscr{P}$. This is equivalent to saying

$$\mathrm{Mod}_{\overline{\mathbb{F}}_p}(\pi_* Z, \pi_* \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}) \to \mathrm{Mod}_{\overline{\mathbb{F}}_p}(\pi_* Z, \pi_* \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+})$$

is a weak equivalence of simplicial abelian groups for all Z in $\mathscr{P}$. Moreover, $\pi_* Z$ is projective as an $\overline{\mathbb{F}}_p$-module, thus it suffices to check that

$$\mathrm{H}\overline{\mathbb{F}}_p^* \mathbb{F}_p^{\bullet+1}(X) \longrightarrow \mathrm{H}\overline{\mathbb{F}}_p^* X$$

is a weak equivalence of simplicial abelian groups. But asking whether $X \to \mathbb{F}_p^{\bullet+1}(X)$ is an $\mathrm{H}\mathbb{F}_p$ equivalence is asking whether $\mathbb{F}_p(X) \to \mathbb{F}_p^{\bullet+2}(X)$ is a weak equivalence. This follows since there is an extra degeneracy giving rise to a contracting homotopy; see for example [11, 18]. $\square$

Write $W_\bullet \to W$ for the simplicial resolution (2.14). For each n we obtain a functorial simplicial $\mathscr{P}$-resolution $P(W_n) \to W_n$. Thus we obtain a bisimplicial object $P(W_\bullet)$.

There is an injective model category structure on bisimplicial $\mathrm{H}\overline{\mathbb{F}}_p$-algebras [22, Proposition A.2.8.2] so that a map $A_{\bullet\bullet} \to B_{\bullet\bullet}$ is:

- a cofibration if $A_{n\bullet} \to B_{n\bullet}$ is a $\mathscr{P}$-cofibration for all n,
- a weak equivalence if $A_{n\bullet} \to B_{n\bullet}$ is a $\mathscr{P}$-equivalence for all n,
- and fibrations are determined.

Call this model category structure the injective$-\mathscr{P}$ model structure. The idea is that in the vertical simplicial direction, the cofibrations and weak equivalences are determined levelwise.

Lemma 2.6.2. *In the injective$-\mathscr{P}$ model structure, $P(W_\bullet)$ is cofibrant, and $P(W_\bullet) \to W_\bullet$ is a weak equivalence, where $W_\bullet$ is viewed as a vertically constant bisimplicial $\mathrm{H}\overline{\mathbb{F}}_p$-algebra.*

Proof. This follows since we need only check that levelwise $P(W_n)$ is $\mathscr{P}$-cofibrant and $P(W_n) \to W_n$ is a $\mathscr{P}$-weak equivalence, which is true by construction. $\square$

Define $\widetilde{W_\bullet} := \mathrm{diag}(P(W_\bullet))$.

Lemma 2.6.3. *The diagonal object $\widetilde{W_\bullet}$ is $\mathscr{P}$-cofibrant, and $\widetilde{W_\bullet} \to W_\bullet$ is a $\mathscr{P}$-weak equivalence.*

Proof. The proof is obtained by dualizing the proof found in [10, Lemmas 6.9 and 6.10]. $\square$

We can compose the map (2.13) with the edge map

$$\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(W_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})) \to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}))$$

and we obtain a diagram of cosimplicial spaces:

$$\begin{tikzcd} \mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \arrow[r] & \mathrm{diag}\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})) \\ & \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})). \arrow[u] \end{tikzcd} \tag{2.15}$$

The goal is to prove that when X and Y are of finite type, these maps are all equivalences.

Before we get to the main result of this section, we take a moment to resolve homotopy groups of the spectra W_n as a colimit of free unstable $\mathscr{B}$-algebras over $\overline{\mathbb{F}}_p$. Define

$$W_{-1} := \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}.$$

The proof of Proposition 2.2.7 applied to the spectra in our simplicial resolution gives rise to cofiber sequences:

$$E\mathscr{F}(\pi_* W_{n-1}) \xrightarrow{1-P^0} E\mathscr{F}(\pi_* W_{n-1}) \longrightarrow E\mathscr{F}_0(\pi_* W_{n-1}) \cong \pi_* W_n$$

for each $n \geq 0$.

Proposition 2.6.4. *The totalization spectral sequence*

$$\pi^s \pi^u \pi_t \mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \Longrightarrow \pi^{u+s} \pi_t \mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}),$$

collapses, and induces an isomorphism

$$\pi^s \pi^0 \pi_t \mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \to \pi^s \pi_t \mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$$

Proof. Since $\widetilde{W}_\bullet$ is cofibrant, and its realization is weakly equivalent to W, the right hand side is the E_2 term of the Goerss–Hopkins spectral sequence in Proposition 2.5.4. Observe that for each n and $t > 0$, the left hand side can be identified as

$$\pi^s\pi^u\pi_t\mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_n), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) = \pi^s D^u_{U\mathscr{B}/\mathrm{H}\overline{\mathbb{F}}_p^* Y}(\pi_* W_n, \Sigma^t \mathrm{H}\overline{\mathbb{F}}_p^* Y).$$

Since $\pi_* W_n$ is isomorphic to $E\mathscr{F}_0(\pi_* W_{n-1})$, we can apply Theorem 2.3.2. The consequence of this theorem is that:

$$D^u_{U\overline{\mathscr{B}}/\mathrm{H}\overline{\mathbb{F}}_p^* Y}(\pi_* W_n, \Sigma^t \mathrm{H}\overline{\mathbb{F}}_p^* Y) = 0$$

if $u \neq 0$, and

$$D^0_{U\overline{\mathscr{B}}/\mathrm{H}\overline{\mathbb{F}}_p^* Y}(\pi_* W_n, \Sigma^t \mathrm{H}\overline{\mathbb{F}}_p^* Y) = D^0_{U\mathscr{A}/\mathrm{H}\mathbb{F}_p^* Y}(\pi_* W_{n-1}, \Sigma^t \mathrm{H}\mathbb{F}_p^* Y).$$

□

Proposition 2.6.5. *The map $k : \widetilde{W_\bullet} \to W_\bullet$ induces isomorphisms*

$$\pi^s\pi_t\big(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(W_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}); \varphi\big) \longrightarrow \pi^s\pi_t\big(\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W_\bullet}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}); k \circ \varphi\big).$$

Proof. The proof of Proposition 2.6.4 exhibits an isomorphism

$$\pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(W_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \xrightarrow{\simeq} \pi^s\pi^0\pi_t\mathrm{sAlg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}).$$

The collapse of the spectral sequence in Proposition 2.6.4 gives an isomorphism

$$\pi^s\pi^0\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(P(W_\bullet), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \xrightarrow{\simeq} \pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W_\bullet}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}).$$

The composition yields the desired isomorphism. The fact that it is coming from the map $\widetilde{W_\bullet} \to W$ follows from the fact that the two isomorphisms are induced by maps in the diagram of bisimplicial $\mathrm{H}\overline{\mathbb{F}}_p$-algebras:

$$\begin{tikzcd} P(W_\bullet) \arrow[dr, "\simeq"'] & & \widetilde{W}_\bullet \arrow[ll, "\simeq"'] \arrow[dl] \\ & W_\bullet. & \end{tikzcd}$$

In particular, by the two out of three property, the map $\widetilde{W}_\bullet \to W_\bullet$ is an injective $\mathscr{P}$-weak equivalence of constance bisimplicial spectra, and thus is a $\mathscr{P}$-weak equivalence of simplicial spectra. Thus the map is a $\mathscr{P}$-weak equivalence out of a $\mathscr{P}$-cofibrant object. □

Theorem 2.6.6. *When X and Y are finite type spaces such that X is also nilpotent, the diagram (2.15) is a diagram of weak equivalence of cosimplicial spaces. Moreover, these maps induce weak equivalences*

$$\begin{tikzcd} \mathscr{T}(Y, X_p) \arrow[r] & \mathrm{Tot}\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \\ & \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}). \arrow[u] \end{tikzcd}$$

Proof. Consider the map

$$\pi^s\pi_t\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \to \pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}).$$

From Proposition 2.6.5, we know that the map

$$\pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(W_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \to \pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$$

is an isomorphism. Furthermore, we know that the map

$$\pi^s\pi_t\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \to \pi^s\pi_t\mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(W_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$$

is an isomorphism when X and Y are of finite type since both sides can be identified as derived functors of derivations of unstable $\mathscr{A}$-algebras over $\mathrm{H}\mathbb{F}_p^*Y$.

Similarly, Proposition 2.6.4 exhibits an isomorphism:

$$\pi^s\pi_t\mathrm{Alg}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \leftarrow \pi^s\pi_t\mathrm{Alg}(P(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty X_+}), \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}).$$

Thus by [4, Section 6.3], since there is an isomorphism of the E_2 terms of these spectral sequences, there is an induced weak equivalence on the total spaces in the diagram (2.15).

In particular, by Lemma 2.6.3, the map $\widetilde{W}_\bullet \to W$ is a $\mathscr{P}$-weak equivalence, and $\widetilde{W}_\bullet$ is $\mathscr{P}$-cofibrant. Given the functorial cofibrant replacement $P(W)$ of W guaranteed by Proposition 2.5.1, $P(W) \to W$ is a trivial $\mathscr{P}$-fibration. Thus model category theory guarantees a $\mathscr{P}$-weak equivalence $w : \widetilde{W}_\bullet \to P(W)$ between cofibrant objects. Thus the associated cosimplicial mapping spaces can be naturally identified. □

Remark: This give an alternate proof of Mandell's theorem (Theorem 2.1.1). However, the key computations are the same as in his original proof.

Corollary 2.6.7. *When X and Y are finite type, nilpotent spaces, the map*

$$\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X)) \to \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$$

induces and isomorphism between the unstable Adams spectral sequence and the Goerss–Hopkins spectral sequence.

Proof. Theorem 2.6.6 exhibits an isomorphism when applying $\pi^s\pi_t$ to both of these cosimplicial spaces. Since $\pi^s\pi_t\mathscr{T}(Y, \mathbb{F}_p^{\bullet+1}(X))$ can be identified as the E_2 term of the unstable Adams spectral sequence, all that is left is to identify

$$\pi^s\pi_t\ \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+})$$

as the E_2-term of the Goerss–Hopkins spectral sequence.

Proposition 2.6.5 exhibits an isomorphism

$$\pi^s\pi_t\ \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty \mathbb{F}_p^{\bullet+1}(X)_+}, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}) \cong \pi^s\pi_t\ \mathrm{Alg}_{\mathrm{H}\overline{\mathbb{F}}_p}(\widetilde{W}_\bullet, \mathrm{H}\overline{\mathbb{F}}_p^{\Sigma^\infty Y_+}).$$

By Lemma 2.6.3, the map $\widetilde{W_\bullet} \to W$ is a $\mathscr{P}$-weak equivalence, and $\widetilde{W_\bullet}$ is $\mathscr{P}$-cofibrant. Given the functorial cofibrant replacement $P(W)$ of W guaranteed by Proposition 2.5.1, $P(W) \to W$ is a trivial $\mathscr{P}$-fibration. Thus model category theory guarantees a $\mathscr{P}$-weak equivalence $w : \widetilde{W_\bullet} \to P(W)$ between cofibrant objects. Thus the associated cosimplicial mapping spaces can be naturally identified. In particular, applying $\pi^s\pi_t$ of either space can be naturally identified as the E_2 term of the Goerss–Hopkins spectral sequence.

Thus Proposition 2.6.5 truly exhibits an isomorphism between the unstable Adams spectral sequence and the Goerss–Hopkins spectral sequence. $\square$

Chapter 3

Mapping spaces of $K(1)$-local ring spectra

3.1 The problem

Let Top be denote the model category of topological spaces. Let Sp denote a good category of spectra, such as the category of symmetric spectra [21] or S-modules defined in [14]. Let S denote the $K(1)$-local sphere spectrum unless otherwise specified. Let E_∞ denote a good model category of $K(1)$-local E_∞-algebras. Given any model category $\mathscr{C}$ enriched in spaces, and any two objects A and B in $\mathscr{C}$, we write $\mathscr{C}(A, B)$ for the derived mapping space.

Consider the functor from spaces to $K(1)$ local E_∞-algebras:

$$\begin{aligned} \mathrm{Top}^{\mathrm{op}} &\to \mathrm{E}_\infty \\ X &\mapsto S^{\Sigma^\infty X_+}. \end{aligned}$$

Given any point x in X, we obtain a E_∞ ring map $S^{\Sigma^\infty X_+} \to S^{S^0}$ in the following way. Identify X as the unpointed mapping space $\mathrm{Map}(*, X)$, then the point x corresponds to a map $x \in \mathrm{Map}(*, X)$ in the obvious way. This induces an unstable map

$S^0 \to X_+$, which induces an E_∞ ring map

$$S^{\Sigma^\infty X_+} \to S^{S^0}.$$

Thus we obtain a natural transformation ψ from the identity functor on pointed topological spaces:

$$\begin{aligned} \psi : \mathrm{Id} \to & \mathrm{E}_\infty(S^{\Sigma^\infty(-_+)}, S) \\ X \simeq \mathrm{Map}(*, X) \mapsto & \mathrm{E}_\infty(S^{\Sigma^\infty X_+}, S) \end{aligned}$$

where the basepoint of $\mathrm{E}_\infty(S^{\Sigma^\infty X_+}, S)$ is the map induced by the basepoint of X. The main question we are interested in is to determine conditions on a topological space X such that the map

$$\psi : X \to E_\infty(S^{\Sigma^\infty X_+}, S) \tag{3.1}$$

is a $K(1)$-localization. Because we can say very little about this natural transformation in general, as is often the case when discussion localizations, we must add constraints to the type of spaces we can talk about. The first constraint we add is to restrict our attention to simply connected spaces. Thus we are really interested in whether or not the map

$$\psi : X \to E_\infty(S^{\Sigma^\infty X_+}, S)\langle 1 \rangle \tag{3.2}$$

is a $K(1)$-localization. Here the map (3.2) is the lift of the map (3.1) to the universal cover of $E_\infty(S^{\Sigma^\infty X_+}, S)$.

The approach is to develop this over two steps. The first step is to show that the map (3.2) is a $K(1)$-localization whenever X is the zeroth space of a 2-connected, finite, p-complete spectrum E such that K^*E is torsion free and of finite type. The second step is to extend this result to spaces whose $K(1)$-local Goodwillie tower for the identity is finite (such as odd dimension spheres), as each layer is an infinite loop space [1]. Two main technical difficulties arise. The first is that before K(1) localization, certain connectivity, torsion free and finiteness hypotheses need be met. The second technical difficulty is convergence criterion for the K(1)-local Eilenberg–Moore spectral sequence. It is precisely these convergence criteria on the spaces in the levels of the Goodwillie tower of a space X that we expect to determine necessary conditions on a space X for the map (3.2) to be a $K(1)$ localization. This chapter is concerned only with the first step.

In Section 3.2, we fix some notation and terminology for the connected covers of spectra and spaces, which play a major role throughout this chapter. Section 3.3 defines the cohomology theories present and the algebraic structure of power operations on $K(1)$-local E_∞-ring spectra and commutative K-algebras. The goal of this section is Theorem 3.3.3, which identifies the K cohomology of certain infinite loop spaces and is instrumental in proving the main theorem. In Section 3.4 we discuss stable and unstable $K(1)$-localization and record the two major pieces of data we use from work of A. Bousfield on the $K(1)$-localization of infinite loop spaces, which we write here as Lemmas 3.4.1 and 3.4.2. Section 3.5 defines the Goerss–Hopkins spectral sequence in Theorem 3.5.3, which is the main computational tool we use. Section 3.6 recalls some background information about Goodwillie calculus, and develops a theory of linear functors from spectra to spaces, which is missing from the literature. The main theorem of this paper appears in Section 3.7.

3.2 Notation and terminology

Given a spectrum Z, we say that Z is n-connected if the all of the homotopy groups of Z satisfy $\pi_i Z = 0$ for all $i \leq n$. Similarly, given any spectrum Z, we write $Z\langle n\rangle$ for the n-connected cover of Z. Note that the n-connected cover of a spectrum fits into a homotopy fiber sequence:

$$Z\langle n\rangle \longrightarrow Z \longrightarrow P^n Z.$$

The spectrum $P^n Z$ is the n-th Postnikov section of Z, so that

$$\pi_i P^n Z = \begin{cases} \pi_i Z & \text{if} \quad i \leq n \\ 0 & \text{if} \quad i > n \end{cases}$$

By abuse of notation, we use exactly the same notation and terminology for Postnikov sections and coverings of spaces. Thus the 1-connected cover of a space X is written as $X\langle 1\rangle$, and is simply the universal covering space of X.

3.3 Cohomology Theories

We are interested in the category of $K(1)$-local spaces, and a way to model the $K(1)$-localization of a space. The model is the derived mapping space of maps between $K(1)$-local ring spectra.

The spectrum $K(1)$ is the first of the Morava K-theories. This is a complex oriented ring spectrum depending on a prime p. Throughout, we will assume that p is odd. The coefficients are given by $K(1)_* = \mathbb{F}_p[v_1^{\pm}]$.

The cohomology theory K refers to complex K-theory completed at the prime p. We leave out notation referencing the completion at p to simplify notation later on. This cohomology theory comes equipped with stable operations induced by the prime to p Adams operations continuously extended to an action of $C_{p-1} \times \mathbb{Z}_p^{\wedge} \cong \mathbb{Z}_p^{\times}$.

Theorem 3.3.1 (Adams–Baird, Devinatz–Hopkins). *The $K(1)$-local sphere spectrum can be realized as the homotopy fixed points of K with respect to the action of $\mathbb{Z}_p^{\times}$. Thus $S \simeq K^{h\mathbb{Z}_p^{\times}}$.*

Remark 3.3.2. *The choice to write S as the homotopy fixed points with respect to a profinite group may be slightly misleading because it is really continuous homotopy fixed points. In this case, these continuous homotopy fixed points can be taken to mean: fix a topological generator ψ^k in $\mathbb{Z}_p^{\times}$ and take the fiber of the map $1 - \psi^k$.*

We write $\mathrm{Sp}_{K(1)}$ for the category of $K(1)$-local spectra. We will write E_∞ for the category of E_∞ ring spectra in the category of $K(1)$-local ring spectra. The $K(1)$-local sphere spectrum S is initial in this category.

3.3.1 Power operations in E_∞

The goal of this section is to define the power operation θ which acts naturally on π_0 of any $K(1)$-local E_∞ ring spectrum.

We recall the construction of power operations and defined in [13]. Let R be in E_∞. We are interested in the pth power operation $\mathscr{P}_p$ which is a map:

$$\mathscr{P}_p : \pi_0 F(S, R) \to \pi_0 F(B\Sigma_{p+}, R).$$

Let e denote the map in Sp defined by the inclusion of the trivial subgroup into Σ_p inducing a map

$$e : S \to B\Sigma_{p+}.$$

We get a composition map:

$$\pi_0 F(S, R) \xrightarrow{\mathscr{P}_p} \pi_0 F(B\Sigma_{p+}, R) \xrightarrow{e^*} \pi_0 F(S, R),$$

such that for any element r in $\pi_0 F(S, R)$, we find that $e^* \mathscr{P}_p(r) = r^p$.

The key point is that $K(1)$-locally, S is equivalent to $B\Sigma_p$. The following construction of the power operations θ and ψ on π_0 of any $K(1)$-local ring spectrum comes from [20].

There is a transfer map:

$$\mathrm{Tr} : B\Sigma_{p+} \longrightarrow E\Sigma_{p+} \simeq S,$$

such that the composite

$$S \longrightarrow B\Sigma_{p+} \xrightarrow{\mathrm{Tr}} S,$$

is an equivalence. Thus we can define a map in the homotopy category

$$\tilde{\theta} : S \to B\Sigma_p$$

such that $\mathrm{Tr} \circ \tilde{\theta} = -\frac{1}{(p-1)!}$. Then the power operation θ is defined by

$$\begin{aligned} \theta, \psi : \pi_0 R &\longrightarrow \pi_0 R \\ r &\mapsto \tilde{\theta}^* \circ \mathscr{P}_p(r). \end{aligned}$$

Note that θ is not a ring map. However, we can define a ring map $\psi = e + p\theta$. In the case $R = K$, the power operation ψ is the pth Adams operation. Let x and y be elements of $\pi_0 R$. Then we have the following formulas:

$$\begin{aligned} \psi(xy) &= \psi(x)\psi(y) \\ \psi(x+y) &= \psi(x) + \psi(y) \\ \theta(xy) &= \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y) \\ \theta(x+y) &= \theta(x) + \theta(y) + \sum_{s=1}^{p-1} \frac{1}{p}\binom{p}{s} x^s y^{p-s} \\ \theta\psi &= \psi\theta \end{aligned}$$

We can think of the operation ψ as being determined from θ and the map e. We choose to use the operation θ throughout. Thus we identify $\pi_0 R$ for any R in E_∞ as a θ-algebra over $\mathbb{Z}_p^\wedge$.

3.3.2 Power operations on commutative K-algebras

Two examples of $K(1)$-local E_∞ ring spectra that we are interested in are K and $F(\Sigma^\infty X_+, K)$, where X is a space. We get operations θ and ψ, where we have seen that ψ is the pth Adams operations ψ^p on $\pi_0 K$ and $\pi_0 F(\Sigma^\infty X_+, K)$.

Let us consider the example $F(\Sigma^\infty X_+, K)$ first. Bott periodicity allows us to extend the power operation θ to all even degrees by requiring that θ commute with the Bott map. In particular, Bott periodicity gives isomorphisms

$$\tilde{K}^{2i}X \cong \tilde{K}^0X$$
$$\tilde{K}^{2i+1}X \cong \tilde{K}^0(S^1 \wedge X).$$

Thus we can also extend the operations to odd degrees. In order to distinguish between the even and odd degrees and simplify formulas, we use θ exclusively as an operation on even degrees and ψ as the operation on odd degrees.

Reducing to the case $F(\Sigma^\infty S^0, K) \cong K$, we see that we have an action of θ that extends to all degrees. We also have stable Adams operations ψ^l where l is prime to p. These extend continuously to give an action $\mathbb{Z}_p^\times$ on K.

Given any commutative K algebra R, we have an action of $\mathbb{Z}_p^\times$ given by the composition

$$\psi^l : R \xrightarrow{\eta\wedge 1} K \wedge R \xrightarrow{\psi^l \wedge 1} K \wedge R \xrightarrow{\mu} R,$$

as well as an action of θ that extends to all degrees.

Bousfield in [5] gives a completely algebraic description of the operations on commutative K-algebras as $\mathbb{Z}/2$ graded p-adic θ rings equipped with prime to p Adams operations. Bott periodicity allows us to view these $\mathbb{Z}$ graded rings as being $\mathbb{Z}/2$ graded.

A $\mathbb{Z}/2$ graded θ ring $R = \{R^0, R^1\}$ is a commutative $\mathbb{Z}/2$ graded ring such that:

- There is a function $\theta : R^0 \to R^0$ such that:

    1. $\theta(1) = 0,$
    2. $\theta(ab) = \theta(a)b^p + a^p\theta(b) + p\theta(a)\theta(b),$
    3. $\theta(a + b) = \theta(a) + \theta(b) - \sum_{s=1}^{p-1} \frac{1}{p}\binom{p}{s} x^s y^{p-s}.$

- There is a map $\psi : R^1 \to R^1$ such that given a in R^0, and x and y in R^1:

    1. $\psi(ax) = (a^p + p\theta(a))\psi(x)$

2. $(xy)^p + \theta(xy) = \psi(x)\psi(y)$.

We say that a $\mathbb{Z}/2$ graded θ ring $R = \{R^0, R^1\}$ is finite p-adic if there is an augmentation map to $\mathbb{Z}_p^\wedge$ such that the augmentation ideal is finite p-torsion and nilpotent. Moreover, for each x in the zeroth graded piece of the augmentation ideal, there is an n such that $\theta^n(\theta x - x) = 0$. For each y in R^1, there is an m such that $\psi^m(\psi y - y) = 0$. A p-adic θ-ring is the inverse limit of finite p-adic θ-rings. Since all p-adic θ-rings that we will encounter will be $\mathbb{Z}/2$-graded, we choose to denote the category of $\mathbb{Z}/2$-graded p-adic θ-rings by Alg_θ.

A finite $\mathbb{Z}/2$-graded p-adic θ-ring which is equipped with a continous action of $\mathbb{Z}_p^\times$ that commutes with the action of θ on R^0 and ψ on R^1 is called a p-adic θ-ring equipped with prime to p Adams operations.

There is a forgetful functor from $\mathbb{Z}/2$-graded p-adic θ rings down to $\mathbb{Z}/2$-graded p-profinite abelian groups by simply forgetting the θ structure. The forgetful functor has a left adjoint, which we write as $\mathscr{F}_\theta$. For example, the free p-adic θ-ring on a single generator a in degree 0 is given by

$$\mathscr{F}_\theta(a) = \mathbb{Z}_p^\wedge[a, \theta(a), \theta^2(a), \dots]_p^\wedge.$$

Similarly, the free p-adic θ ring on a single generator x in degree 1 is given by

$$\mathscr{F}_\theta(x) = \Lambda[x, \psi(x), \psi^2(x), \dots]_p^\wedge.$$

The following theorem of Bousfield plays a major role in this paper.

Theorem 3.3.3 ([5, Corollary 8.6]). *Let E be a 2-connected spectrum such that K^*E is torsion free and of finite type over $\mathbb{Z}_p^\wedge$. Then*

$$K^*\Omega^\infty E \cong \mathscr{F}_\theta(K^*E).$$

3.4 Bousfield localization

3.4.1 Stable $K(1)$-localization

The Morava K-theories are related to the Morava E-theories. The ring spectrum $E(1)$ has coefficients $E(1)_* = \mathbb{Z}_{(p)}[v_1^{\pm}]$. Localization with respect to $E(1)$ is smashing. That is, given a spectrum X, the $E(1)$ localization of X is given by $S_{E(1)} \wedge X$, where $S_{E(1)}$ is the $E(1)$-local sphere spectrum. Stably, localizing with respect to $K(1)$ is equivalent to a composition of smashing with the $K(1)$-local sphere spectrum followed by completing at the prime p. It is important to note that the $K(1)$-localization of a connected spectrum is typically not connected.

3.4.2 $K(1)$-localization of infinite loop spaces

We are interested in unstable $K(1)$-localizations, and in particular models for the $K(1)$-localization of certain infinite loop spaces. In general, $(\Omega^\infty E)_{K(1)}$ is not equivalent to $\Omega^\infty(E_{K(1)})$. Fortunately, any homology localization of an infinite loop space is still an infinite loop space [3, Theorem 1.1].

The goal of this section is to record the following two lemmas. These results follow directly from work of A. Bousfield.

Lemma 3.4.1 ([3, Proposition 1.3]). *Let E be a spectrum. Then $\Omega^\infty(E_{K(1)})$ is $K(1)$-local.*

Given a finite spectrum E, let $E \to \overline{P}^2 E$ be the modified Postnikov section with

$$\pi_i \overline{P}^2 E = \begin{cases} \pi_i E & \text{if} \quad i < 2 \\ \pi_2 E / tors \pi_2 E & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i > 2 \end{cases}$$

Define a spectrum $E^c_{K(1)}$ to be the homotopy pullback in the diagram

$$\begin{tikzcd} E^c_{K(1)} \arrow[r] \arrow[d] & E_{K(1)} \arrow[d] \\ \overline{P}^2 E \arrow[r] & \overline{P}^2 E_{K(1)}. \end{tikzcd}$$

In [8, Theorem 3.8], A. Bousfield proves that given a 0-connected p-complete spectrum E, there is a canonical equivalence $(\Omega^\infty E)_{K(1)} \to \Omega^\infty(E^c_{K(1)})$.

Lemma 3.4.2. *Given a 2-connected, finite, p-complete spectrum E, there is a canonical isomorphism*

$$(\Omega^\infty E)_{K(1)} \xrightarrow{\simeq} (\Omega^\infty E_{K(1)})\langle 1\rangle.$$

Proof. First we observe that since E is 2-connected, $\overline{P}^2 E \simeq 0$. Secondly, since E is a finite spectrum, E is p-complete and $\pi_2 E_{K(1)}$ is all torsion. Thus $\overline{P}^2 E_{K(1)} \simeq P^1 E_{K(1)}$. Consequently, we identify $E^c_{K(1)}$ as the homotopy fiber

$$E^c_{K(1)} \longrightarrow E_{K(1)} \longrightarrow P^1 E_{K(1)}\ .$$

Hence there is a natural identification: $E^c_{K(1)} \simeq E_{K(1)}\langle 1\rangle$. The natural isomorphism $\Omega^\infty(E_{K(1)}\langle 1\rangle) \simeq (\Omega^\infty E_{K(1)})\langle 1\rangle$ provides the desired result. □

3.5 A Goerss–Hopkins spectral sequence

The main tool for analyzing mapping spaces of $K(1)$-local E_∞-ring spectra is a spectral sequence developed by P. Goerss and M. Hopkins in [17]. We are interested in computing the homotopy groups of a mapping space $\mathrm{E}_\infty(A, B)$ where B is a commutative K-algebra. In particular, we are only going to be interested in the case that $B \simeq K$ and $A \simeq F(\Sigma^\infty\Omega^\infty E_+, K)$.

There is an adjunction between $K(1)$-local E_∞ ring spectra and commutative

K-algebras:

$$(K \wedge -) : \mathrm{E}_\infty \rightleftarrows \mathrm{Alg}_K : U$$

where U is the forgetful functor, and smashing with K is the left adjoint. Let $\mathbb{P} : \mathrm{Mod}_K \to \mathrm{Alg}_K$ denote the left adjoint to the forgetful functor from commutative K-algebras to K-modules. Then, given a K-module M such that $\pi_* M$ is torsion free over $\mathbb{Z}_p^\wedge$, there is an isomorphism:

$$\mathscr{F}_\theta(\pi_* M) \cong \pi_* \mathbb{P}(M).$$

Note that since θ commutes with the prime to p Adams operations, this free functor does not affect this action.

Let $\mathscr{P}$ be a class of K-modules such that for P in $\mathscr{P}$ and any K-module M, the Künneth map

$$[P, M] \longrightarrow \mathrm{Mod}_{K_*}(\pi_* P, \pi_* M)$$

is an isomorphism. That is, $\mathscr{P}$ is the smallest class of spectra that contains K, is closed under suspension, desuspension and finite wedges whose homotopy groups are torsion free and of finite type over $\mathbb{Z}_p^\wedge$.

There is a $\mathscr{P}$-projective model structure on the category of simplicial K-modules such that a map $M_\bullet \to N_\bullet$ is

- a $\mathscr{P}$-weak equivalence if $[P, M_\bullet] \to [P, N_\bullet]$ is a weak equivalence of simpical K_*-modules for all P in $\mathscr{P}$.
- a $\mathscr{P}$-fibration if $[P, M_\bullet] \to [P, N_\bullet]$ is surjective for all P in $\mathscr{P}$.
- a $\mathscr{P}$-cofibration if $[P, M_\bullet] \to [P, N_\bullet]$ has the left lifting property with respect to all trivial $\mathscr{P}$-fibrations.

This model structure is cofibrantly generated. If $\{Z\}$ is a class of cofibrant generators, we obtain a model structure on the category of simplicial K-algebras so that it is cofibrantly generated with generators $\{\mathbb{P}(Z)\}$ and the forgetful functor to K-modules and the functor $\mathbb{P}$ induce a Quillen adjoint pair on the simplicial categories.

3.5.1 The spectral sequence

There is a Hurewicz map:

$$\pi_t \mathrm{Alg}_K(K \wedge A, B) \longrightarrow \mathrm{Alg}_\theta(\pi_* K \wedge A, \pi_* F(S^t_+, B)). \tag{3.3}$$

Lemma 3.5.1. *If $K \wedge A \simeq \mathbb{P}(Z)$ where Z is a K-module whose homotopy groups are torsion free, then the Hurewicz map (3.3) is an isomorphism.*

Proof. Fix a map $\mathbb{P}(Z) \to B$ to be used as the basepoint. Then we have the following isomorphisms:

$$\begin{aligned}
\pi_t \mathrm{Alg}_K(\mathbb{P}(Z), B) &\cong \pi_0 \mathrm{Top}_+(S^t, \mathrm{Alg}_K(\mathbb{P}(Z), B)) \\
&\cong \pi_0 \mathrm{Top}_+(S^t, \mathrm{Mod}_K(Z, B)) \\
&\cong \pi_0 S^0 \downarrow \mathrm{Top}(S^t_+, \mathrm{Mod}_K(Z, B)) \\
&\cong \pi_0 \mathrm{Mod}_{K \downarrow B}(Z, F(S^t_+, B)) \\
&\cong \mathrm{Mod}_{K_*/\pi_* B}(\pi_* Z, \pi_* F(S^t_+, B)) \\
&\cong \mathrm{Alg}_{\theta/\pi_* B}(\mathscr{F}_\theta(\pi_* Z), \pi_* F(S^t_+, B)) \\
&\cong \mathrm{Alg}_{\theta/\pi_* B}(\pi_* K \wedge A, \pi_* F(S^t_+, B)),
\end{aligned}$$

where the isomorphism third isomorphism corresponds to the face that the category of K-modules is already pointed, and all other isomorphisms follow from adjunctions or assumptions. $\square$

Lemma 3.5.2. *There is an identification of $\mathbb{Z}/2$-graded θ-algebra maps over B_* as derivations of θ-algebras relative to B_*:*

$$\mathrm{Alg}_{\theta/\pi_* B}(\pi_* K \wedge A, \pi_* F(S^t_+, B)) \cong \mathrm{Der}_{\theta/B_*}(\pi_* K \wedge A, \Sigma^t B_*).$$

Note that a derivation of θ-algebras is a derviation of $\mathbb{Z}_p^\wedge$-modules that commute with the action of θ.

The proof is routine and can be found in [17, Section 2.4.3].

Consider the mapping space $\mathrm{E}_\infty(A, B) \simeq \mathrm{Alg}_K(K \wedge A, B)$. There exists a functorial cofibrant replacement of $K \wedge A$ in the category of simplicial K-algebras such that $K \wedge A^{cof} \simeq \mathbb{P}(Z_\bullet)$ [17, Proposition 1.4.11]. We obtain a cosimplicial mapping space by replacing the source by its simplicial cofibrant replacement. There is a totalization spectral sequence

$$\pi^s \pi_t \mathrm{Alg}_K(\mathbb{P}(Z_\bullet), B)) \longrightarrow \pi_{t-s}\mathrm{Tot}\mathrm{Alg}_K(\mathbb{P}(Z_\bullet), B).$$

Furthermore, this computes the homotopy groups of our original mapping space as there is a weak equivalence

$$\mathrm{Tot}\mathrm{Alg}_K(\mathbb{P}(Z_\bullet), B) \simeq \mathrm{Alg}_K(K \wedge A, B).$$

In light of Lemma 3.5.1, we can identify the E_2 term as

$$\pi^s \mathrm{Der}_{\theta/B_*}(\pi_*\mathbb{P}(Z_\bullet), \Sigma^t B_*),$$

which should be thought of as the right derived functors of derivations of θ-algebras from K_*A to $\Sigma^t B_*$ in light of the fact that $\pi_*\mathbb{P}(Z_\bullet)$ is a cofibrant simplicial resolution of K_*A by free θ-algebras. In order to simplify notation, we write the derived functors of these θ-algebra derivations as

$$R^s \mathrm{Der}_{\theta/\pi_* B} =: D^s_{\theta/\pi * B}.$$

Theorem 3.5.3. *There is an identification of the E_2 term spectral sequence abutting* *to the homotopy groups $\pi_{t-s}\mathrm{Alg}_K(K \wedge A, B)$ with E_2 term:*

$$\begin{aligned} E_2^{s,t} = & D^s_{\theta/\pi_* B}(K_* A, \Sigma^t \pi_* B) && \text{if } s, t > 0 \\ & \mathrm{Alg}_\theta(K_* A, \pi_* B) && \text{if } s = t = 0. \end{aligned}$$

Proof. The proof follows immediately by applying Lemma 3.5.2 to the E_2 term defined above. □

Remark 3.5.4. *Note that the spectral sequence used in [17] is different from the spectral sequence we use here. In [17], the goal is to compute the $\pi_* \mathrm{E}_\infty(A, B)$, where A and B are $K(1)$-local E_∞-ring spectra. Thus the spectral sequence they use begins with the K-homology of A and B and computes the homotopy groups of the derived mapping space. The spectral sequence developed uses the fact that the K-homology of a $K(1)$-local E_∞-ring spectrum has the structure of a $\mathbb{Z}/2$-graded θ ring with an action by $\mathbb{Z}_p^\times$. Since we are computing the homotopy groups of the space of K-algebra maps, the action of $\mathbb{Z}_p^\times$ does not come into play in our spectral sequence.*

3.6 Homotopy Calculus

We are interested in understanding the functor

$$X \to \mathrm{E}_\infty(S^{\Sigma^\infty X_+}, S)$$

from spaces to spaces. The main result focuses on understanding this functor when restricted to infinite loop spaces that are the zeroth space of a 2-connected spectrum satisfying certain torsion free and finiteness hypotheses. Thus we are interested in an associated functor from spectra to spaces. The approach is to use the theory of homotopy calculus developed by T. Goodwillie in [19]. We recall the general theory and terminology before recording the results needed.

A functor F is a *homotopy functor* if it takes weak equivalences to weak equivalences. A homotopy functor is *reduced* if $F(*) \simeq *$.

Homotopy calculus is a method that given a functor F from a pointed model category to a pointed model category, naturally produces a tower of fibrations of functors

$$\begin{tikzcd}
 & P_nF \arrow[d, dotted] & D_nF \arrow[l] \\
 & P_3F \arrow[d] & D_3F \arrow[l] \\
 & P_2F \arrow[d] & D_2F \arrow[l] \\
F \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & P_1F & D_1F, \arrow[l, "\simeq"']
\end{tikzcd}$$

where each functor P_nF is n-excisive. The homotopy fibers of each vertical map $P_nF \to P_{n-1}F$ is an n-homogeneous functor D_nF. A functor is n-homogeneous is an n-excisive functor whose $n-1$ excisive component is trivial.

We are interested in functors from finite spectra to spaces, or from finite spectra to spectra.

3.6.1 Linear functors from spectra to spaces or spectra

We are only interested in P_1F, so we specialize the discussion to the linearization. A 1-excisive functor, or *linear* functor, is a functor which takes homotopy pushout squares to a homotopy pullback squares. The construction of P_1 is as follows. Suppose that F is a reduced homotopy functor from finite spectra to spaces. Define a functor $T_1F(X)$ as the homotopy limit of the diagram

$$\begin{tikzcd}
 & F(CX) \arrow[d] \\
F(CX) \arrow[r] & F(\Sigma X).
\end{tikzcd}$$

Since we have assumed that F is reduced, and $F(CX) \simeq F(*) \simeq *$, we see that $T_1F(X) \simeq \Omega F(\Sigma X)$. Then the functor P_1F is defined as the homotopy colimit

$$P_1F(X) := \operatorname{hocolim}\{F(X) \longrightarrow T_1F(X) \longrightarrow T_1T_1F(X) \longrightarrow \cdots\}. \tag{3.4}$$

This construction is due to T. Goodwillie in [19].

In general, n-homogeneous functors are are functors such that $P_nF \simeq F$ and $P_{n-1}F \simeq *$. When $n = 1$, following the convention that $P_0F \simeq *$ for all reduced functors F, 1-homogenous functors correspond precisely with linear functors. The goal here is to classify 1-homogeneous functors from finite spectra to spaces.

Write $L_1(\mathrm{Sp}^f, \mathrm{Top})$ for 1-homogeneous, or linear reduced homotopy functors from finite spectra to spaces. Write $L_1(\mathrm{Sp}^f, \mathrm{Sp})$ for linear reduced homotopy functors from finite spectra to spectra. Given G in $L_1(\mathrm{Sp}^f, \mathrm{Sp})$, we get a linear functor from finite spectra to spaces by composing with Ω^∞. This follows from the fact that Ω^∞ commutes with holim.

Lemma 3.6.1. *There is a functor*

$$B : L_1(\mathrm{Sp}^f, \mathrm{Top}) \to L_1(\mathrm{Sp}^f, \mathrm{Top})$$

such that $\Omega BG \simeq G$ for all linear functors G.

Proof. Let G be a linear reduced functor from finite spectra to spaces. Define $B(G(X))$ to be $G(\Sigma X)$. Then BG is linear by definition. We need only show that $\Omega BG \simeq G$.

The category of finite spectra is triangulated. In particular, there is a distinguished triangle

$$X \xrightarrow{1} X \longrightarrow * \longrightarrow \Sigma X,$$

where

$$X \longrightarrow * \longrightarrow \Sigma X$$

is a homotopy cofiber sequence. Since we assume that G is a reduced linear functor, we get a homotopy fiber sequence

$$G(X) \longrightarrow * \longrightarrow G(\Sigma X),$$

where by definition $G(\Sigma X) = BG(X)$.

For the identification of BG as a de-looping, we write ΣX as the homotopy pushout of the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX . \\ CX & \end{tikzcd}$$

Since G is linear, it takes homotopy pushout diagrams to homotopy pullback diagrams. Thus we get a homotopy pullback diagram

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d] & G(*) \arrow[d] \\ G(*) \arrow[r] & B(G(X)). \end{tikzcd}$$

However, the homotopy pullback of this diagram is more commonly known as $\Omega BG(X)$, thus we obtain a weak equivalence $G(X) \xrightarrow{\simeq} \Omega B(G(X))$. □

Theorem 3.6.2. *The map*

$$L_1(\mathrm{Sp}^f, \mathrm{Sp}) \xrightarrow{\Omega^\infty} L_1(\mathrm{Sp}^f, \mathrm{Top})$$

has an inverse.

Proof. The proof is identical to the proof of the analogous theorem about 1-homogeneous functors from spaces to spaces found in [19]. We write B^∞ for the inverse of Ω^∞. Given a linear functor G from finite spectra to spaces, we obtain the functor $B^\infty G(X)$ as the spectrification of the pre-spectrum $\{B^n G(X)\} \simeq \{G(\Sigma^n X)\}$. □

Theorem 3.6.3 (Classification of homogeneous functors). *If F is a reduced homotopy* *functor from finite spectra to spaces, then for all finite spectra E,*

$$P_1F(E) \simeq \Omega^\infty(\delta_1 \wedge E).$$

This is a fundamental theorem of T. Goodwillie, and we refer the reader to [19] for more details.

Theorem 3.6.4. *Given a functor F from finite spectra to spaces, the spectrum δ_1 in the classification of homogeneous functors is the spectrum given by the spectrification of the pre-spectrum $\{F(S^n)\}$.*

Proof. In [19, Theorem 3.5], the spectrum δ_1 is identified as the spectrum $(B^\infty P_1F)(S)$, where $(B^\infty P_1F)$ is a functor from finite spectra to spectra such that $P_1F(X) \simeq \Omega^\infty B^\infty P_1F(X)$. Thus the proof follows by identifying $B^\infty P_1F(S)$ as the spectrification of the pre-spectrum $\{F(S^n)\}$.

We know that δ_1 is the spectrification of the pre-spectrum $\{P_1F(S^n)\}$. We have natural maps

$$F(S^n) \to P_1F(S^n) \simeq \text{hocolim}\Omega^m F(S^{m+n})$$

for all n. The goal is to show that the spectra defined by the spectrification of these pre-spectra are weakly equivalent. This follows immediately from a constructive definition of spectrification. $\square$

3.7 Main theorem

We state the main theorem we wish to prove.

Theorem 3.7.1. *Let E be a finite, p-complete, 2-connected spectrum such that K^*E is torsion free and of finite type. Then the map*

$$\psi : \Omega^\infty E \to \mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty E_+}, S)\langle 1\rangle$$

is a $K(1)$-localization.

Remark 3.7.2. *The requirement that E be finite is stricter than necessary. Due to the nature of this model, which depends only on the $K(1)$-homotopy type, it is sufficient for E to be $K(1)$-equivalent to a finite spectrum.*

Recall from Theorem 3.3.1 that $S \simeq K^{h\mathbb{Z}_p^\times}$, where $\mathbb{Z}_p^\times$ acts on K via the p-adic Adams operations. Thus using the fact that the profinite group $\mathbb{Z}_p^\times$ has a topological generator, we have the following equivalences:

$$\begin{aligned} \mathrm{E}_\infty(X, S) &\simeq \mathrm{E}_\infty(X, K^{h\mathbb{Z}_p^\times}) \\ &\simeq \mathrm{E}_\infty(X, K)^{h\mathbb{Z}_p^\times} \\ &\simeq \mathrm{Alg}_K(K \wedge X, K)^{h\mathbb{Z}_p^\times}. \end{aligned}$$

In order to understand the mapping space of E_∞-rings, we first understand the space of commutative K-algebra maps, and then take the homotopy fixed points with respect to the action of the prime to p Adams operations.

Let E be a spectrum. We begin by considering the functor

$$\begin{aligned} F : \mathrm{Sp}^f &\to \mathrm{Top} \\ E &\mapsto \mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty E_+}, K). \end{aligned}$$

Lemma 3.7.3. *Given a 2-connected, spectrum E such that K^*E is torsion free and of finite type, there is an isomorphism $\pi_* F(E) \cong K_* E$.*

Proof. We use the Goerss–Hopkins spectral sequence for $K(1)$-local E_∞-ring spectra. Thus we have a spectral sequence abutting to

$$\pi_{t-s}\mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty E_+}, K) \cong \pi_{t-s}\mathrm{Alg}_K(K^{\Sigma^\infty \Omega^\infty E_+}, K)$$

with E_2 term given by the André–Quillen cohomology of p-adic θ-algebras

$$D^s_\theta(K^*(\Omega^\infty E), \Omega^t K^*).$$

Since by [5, Corollary 8.6], given a 2-connected spectrum E with K^*E torsion free and finite as a module over $\mathbb{Z}_p^\wedge$, there is an isomorphism $K^*(\Omega^\infty E) \cong \mathscr{F}_\theta(K^*E)$, it follows that $D_\theta^s(K^*(\Omega^\infty E), \Omega^t K^*) = 0$ for $s > 0$. Using the finite type hypothesis, we find that:

$$\begin{aligned} D_\theta^0(\mathscr{F}_\theta(K^*E), \Omega^t K^*) &\cong \mathrm{Hom}_\theta(\mathscr{F}_\theta(K^*E), K^*S^t) \\ &\cong \mathrm{Hom}_{K^*}(K^*E, K^*S^t) \\ &\cong K_tE. \end{aligned}$$

Thus this spectral sequence collapses and we find that

$$\pi_t \mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty E_+}, K) \cong K_tE.$$

□

We now use homotopy calculus to analyze the functor F. The goal is to first explicitly identify the linearization P_1F, then prove that when E is a 2-connected spectrum such that K^*E is torsion free of finite type, $F(E) \simeq P_1F(E)$. In order to use homotopy calculus, we need the following lemma.

Lemma 3.7.4. *The functor F is a reduced homotopy functor.*

Proof. The functor F is reduced because

$$F(*) \simeq \mathrm{E}_\infty(S^{S^0}, K) \simeq \mathrm{E}_\infty(S, K),$$

and S is the initial object in the K(1) local category of E_∞ ring spectra.

To see that F is a homotopy functor, it suffice to show that F takes weak equivalence to weak equivalences. Let $E \to E'$ be a weak equivalence of finite spectra. Then there is a weak equivalence of $\Omega^\infty E_+ \to \Omega^\infty E'_+$, and a weak equivalence $S^{\Sigma^\infty \Omega^\infty E'_+} \to S^{\Sigma^\infty \Omega^\infty E_+}$ in the category of $K(1)$-local spectra. Then there is a map $\mathrm{E}_\infty(S^{\Omega^\infty E_+}, K) \to \mathrm{E}_\infty(S^{\Omega^\infty E'_+}, K)$ which is a weak equivalence since the derived mapping space depends only on the $K(1)$ homotopy type of the source. □

First we construct a natural transformation $P_1F \to \Omega^\infty(K \wedge -)_+$. Observe that the functor F and the functor $\Omega^\infty(K \wedge -)_+$ are both functors from spectra to spaces. In order to define a natural transformation from $P_1F \to \Omega^\infty(K \wedge -)$, it suffices to define a natural transformation $F(-) \to \Omega^\infty(K \wedge -)$, which naturally factors through P_1F since $\Omega^\infty(K \wedge -)$ is linear. We construct this transformation explicitly.

Given a finite spectrum E and an E_∞ ring map from $F(\Sigma^\infty\Omega^\infty E_+, S)$ to K, we wish to produce a point in $\Omega^\infty(K \wedge E)$. Observe that a point in $\Omega^\infty(K \wedge E)$ is a point in the zeroth space of the spectrum $K \wedge E$. As E is a finite spectrum, this is equivalent to a point in the zeroth space of the spectrum $F(S^E, K)$, which is the same as map of spectra viewed as a point in the derived mapping space of spectra $\mathrm{Map}(S^E, K)$.

There is a spectrum map

$$\Sigma^\infty\Omega^\infty E_+ \longrightarrow \Sigma^\infty\Omega^\infty E \longrightarrow E, \tag{3.5}$$

where the second map is the adjunction of the identity map from $\Omega^\infty E$ to itself. Thus we define a natural transformation

$$\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K) \longrightarrow \mathrm{Map}(S^E, K) \tag{3.6}$$

$$(S^{\Sigma^\infty\Omega^\infty E_+} \to K) \longmapsto (S^E \to S^{\Sigma^\infty\Omega^\infty E_+} \to K) \tag{3.7}$$

by simply pre-composing with the dual of the map (3.5).

Lemma 3.7.5. *The natural transformation $P_1F \to \Omega^\infty(K \wedge -)$ is a weak equivalence.*

Proof. By Theorem 3.6.3, there is an identification

$$P_1F(X) \simeq \Omega^\infty(\delta_1 \wedge X).$$

By Theorem 3.6.4, the spectrum δ_1 is given by the spectrification of the pre-spectrum $\{F(S^n)\}$. In order to check that this natural transformation is a weak equivalence, it suffices evaluate the effect of the map on homotopy groups of these functors applied to spheres.

Thus we need to understand the effect of π_* on the map (3.6) in the case that $E = S^n$. Appealing to Lemma 3.7.3, we find that when we evaluate this functor on S^n for $n \geq 3$:

$$\begin{aligned}\pi_t \mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty S^n_+}, K) &\cong \pi_t \mathrm{Alg}_K(K^{\Sigma^\infty \Omega^\infty S^n_+}, K) \\ &\cong \mathrm{Alg}_\theta(K^*(\Omega^\infty S^n), K^* S^t).\end{aligned}$$

When we consider the left hand side, we find that

$$\begin{aligned}\pi_t \mathrm{Map}(S^{-n}, K) &\cong \pi_t \mathrm{Mod}_K(S^{-n} \wedge K, K) \\ &\cong \pi_0(\mathrm{Mod}_K(S^{-n} \wedge K, K^{S^t}) \\ &\cong \mathrm{Mod}_{\mathbb{Z}_p^\wedge}(\tilde{K}^* S^n, \tilde{K}^* S^t)\end{aligned}$$

However, we know that $K^* \Omega^\infty S^n$ is the free θ-algebra generated by $\tilde{K}^* S^n$. So in particular, we see that the π_t applied to the map (3.6) induces the map:

$$\mathrm{Alg}_\theta(K^*(\Omega^\infty S^n), K^* S^t) \to \mathrm{Mod}_{\mathbb{Z}_p^\wedge}(\tilde{K}^* S^n, \tilde{K}^* S^t),$$

which is precisely the map that sends any θ-algebra map from $K^* \Omega^\infty \Sigma^\infty S^n$ to $K^* S^t$ to the underlying map of modules from $\tilde{K}^* S^n$ to $\tilde{K}^* S^t$. Appealing to the adjunction, we know that isomorphism

$$\mathrm{Alg}_\theta(K^*(\Omega^\infty S^n), K^* S^t) \cong \mathrm{Mod}_{\mathbb{Z}_p^\wedge}(K^* S^n, K^* S^t)$$

is induced by the same map. Thus this natural transformation is in fact a weak equivalence. $\square$

What we have shown is that the spectrum δ_1 that classifies the homogeneous functor $P_1 F$ is homotopy equivalent to K. The proof of Lemma 3.7.5 shows that the map of prespectra

$$\{F(S^n)\} \to \{\Omega^\infty(K \wedge S^n)\}$$

is levelwise a weak equivalence when $n > 2$. Thus the induced map after spectrification $\delta_1 \to K$ is a weak equivalence.

Lemma 3.7.6. *The natural transformation $F \to P_1F \to \Omega^\infty(K \wedge -)$ commutes with the action of $\mathbb{Z}_p^\times$.*

Proof. This follows immediately from the construction of the natural transformation

$$\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K) \to \mathrm{Map}(S^E, K)$$

since the action of any p-adic Adams operation ψ^λ is precisely coming from post-composition with the map $\psi^\lambda : K \to K$ on both sides. □

Proposition 3.7.7. *The natural transformation $F \to P_1F$ is a weak equivalence on finite, 2-connected spectra E satisfying K^*E is torsion free and of finite type over $\mathbb{Z}_p^\wedge$.*

Proof. The fact that $F \to \Omega^\infty(K \wedge -)$ is a weak equivalence follows immediately from the computation in Lemma 3.7.3 along with the explicit construction of this natural transformation (3.6). Lemma 3.7.5 proves that P_1F is weakly equivalent to $\Omega^\infty(K \wedge -)$. Thus by the two out of three property, we find that for suitable 2-connected spectra, F is linear. □

Let E be a spectrum satisfying the hypothesis of Theorem 3.7.1. Consider the composition

$$\Omega^\infty E \xrightarrow{\alpha} \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K) \xrightarrow{\beta} \Omega^\infty(K \wedge E).$$

From Lemma 3.7.6, we know that β commutes with the action of $\mathbb{Z}_p^\times$. The map α also commutes with the action of $\mathbb{Z}_p^\times$, where the action of $\mathbb{Z}_p^\times$ on E is trivial. Thus these maps factor through the homotopy fixed points:

$$\Omega^\infty E \longrightarrow \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K)^{h\mathbb{Z}_p^\times} \longrightarrow \Omega^\infty(K \wedge E)^{h\mathbb{Z}_p^\times}.$$

Proposition 3.7.8. *Let E be a 2-connected, p-complete, finite spectrum such that K^*E is finite and torsion free over $\mathbb{Z}_p^\wedge$. Then the map*

$$\Omega^\infty E \to \Omega^\infty (K \wedge E)^{h\mathbb{Z}_p^\times}\langle 1\rangle$$

is a $K(1)$-localization.

Proof. First note that since E is a finite spectrum, the spectrum $K \wedge E$ is p-complete. Secondly, in order to take the homotopy fixed points, it suffices to choose a topological generator ψ^k, and take the homotopy fiber of the map $1 - \psi^k$. Thus we make the identifications:

$$\Omega^\infty (K \wedge E)^{h\mathbb{Z}_p^\times} \simeq \text{fiber}\{1 - \psi^k\}.$$

Taking the fiber commutes with Ω^∞, thus we are really considering

$$\Omega^\infty(\text{fiber}(1 - \psi^k)).$$

Since E is a finite spectrum with trivial $\mathbb{Z}_p^\times$ action, and by Theorem 3.3.1, we know that the fiber of $1 - \psi^k$ acting on K is the $K(1)$-local sphere, the fiber is given by $\Omega^\infty(S \wedge E)$. Since $S \wedge E$ is the fiber of a p-complete spectrum, it is p-complete. Thus $S \wedge E \simeq E_{K(1)}$, and we have an identification

$$\Omega^\infty (K \wedge E)^{h\mathbb{Z}_p^\times} \simeq \Omega^\infty(E_{K(1)}).$$

Since E is 2-connected, in particular $\Omega^\infty E$ is simply connected, and the map $\Omega^\infty E \to \Omega^\infty E_{K(1)}$ lifts to the universal covering space

$$\begin{tikzcd} & (\Omega^\infty E_{K(1)})\langle 1\rangle \arrow[d] \\ \Omega^\infty E \arrow[ur] \arrow[r] & \Omega^\infty(E_{K(1)}). \end{tikzcd}$$

By Lemma 3.4.2, it follows that we have a weak equivalence

$$\Omega^\infty(E_{K(1)})\langle 1\rangle \simeq (\Omega^\infty E)_{K(1)}.$$

Since $\Omega^\infty(E_{K(1)})$ is $K(1)$-local by Lemma 3.4.1, and the map $\Omega^\infty E \to \Omega^\infty(E_{K(1)})$ factors uniquely through the $K(1)$-localization of $\Omega^\infty E$, by universality of the localization, the map $\Omega^\infty E \to (\Omega^\infty E_{K(1)})\langle 1\rangle$ must in fact be the $K(1)$-localization of $\Omega^\infty E$. $\square$

Proposition 3.7.9. *The map on homotopy fixed points*

$$\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K)^{h\mathbb{Z}_p^\times} \to \Omega^\infty(K\wedge E)^{h\mathbb{Z}_p^\times}$$

is a weak equivalence when applied to 2-connected spectra satisfying the hypotheses of Theorem 3.7.1.

Proof. There is a diagram of homotopy fiber sequences:

$$\begin{tikzcd}
\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S) \arrow[r] \arrow[d] & \Omega^\infty(S\wedge E) \arrow[d] \\
\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K) \arrow[r, "\simeq"] \arrow[d, "1-\psi^k"] & \Omega^\infty(K\wedge E) \arrow[d, "1-\psi^k"] \\
\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, K) \arrow[r, "\simeq"] & \Omega^\infty(K\wedge E).
\end{tikzcd}$$

As in Proposition 3.7.8, we identify the homotopy fixed points as the fibers in the diagram. By the commutativity of this diagram, it follows immediately that the fibers are homotopy equivalent. $\square$

Note that Proposition 3.7.9 gives a homotopy equivalence between the universal covers

$$\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle \xrightarrow{\simeq} \Omega^\infty(S\wedge E)\langle 1\rangle.$$

The proof of Theorem 3.7.1 follows immediately from these two propositions.

Proof of Theorem 3.7.1. The map

$$\psi : \Omega^\infty E \to \mathrm{E}_\infty(S^{\Sigma^\infty \Omega^\infty E_+}, S)$$

lifts to the universal cover, and we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle \arrow[r, "\simeq"] \arrow[d] & \Omega^\infty(S \wedge E)\langle 1\rangle \simeq (\Omega^\infty E_{K(1)})\langle 1\rangle \arrow[d] \\
\Omega^\infty E \arrow[ur] \arrow[urr] \arrow[r] & \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S) \arrow[r, "\simeq"] & \Omega^\infty(S \wedge E),
\end{tikzcd}$$

where the weak equivalences indicated exist by Proposition 3.7.9. By Proposition 3.7.8 it follows that after $K(1)$ localization we obtain a diagram:

$$\begin{tikzcd}
\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle_{K(1)} \arrow[r, "\simeq"] & \Omega^\infty(S \wedge E)\langle 1\rangle \\
(\Omega^\infty E)_{K(1)}. \arrow[u] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

By 2 out of 3, it follows that the map

$$(\Omega^\infty E)_{K(1)} \longrightarrow \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle_{K(1)}$$

is a homotopy equivalence. Thus from the diagram:

$$\begin{tikzcd}
(\Omega^\infty E)_{K(1)} \arrow[r] \arrow[dr, "\simeq"'] & \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle \arrow[r, "\simeq"] \arrow[d] & (\Omega^\infty E_{K(1)})\langle 1\rangle \\
 & \mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle_{K(1)} \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

it follows that $\mathrm{E}_\infty(S^{\Sigma^\infty\Omega^\infty E_+}, S)\langle 1\rangle$ is $K(1)$-local, and is therefore a model for the $K(1)$-localization of $\Omega^\infty E$. $\square$

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