Unstable Chromatic Homotopy Theory

by

Guozhen Wang

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

PhD of Mathematics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2015

Author .... Signature redacted ....
Department of Mathematics
May 18, 2015

Certified by .... Signature redacted ....
Mark Behrens
Professor
Thesis Supervisor

Certified by .... Signature redacted ....
Haynes Miller
Professor
Thesis Supervisor

Accepted by .... Signature redacted ....
Alexei Borodin
Chairman, Department Committee on Graduate Thesis

Unstable Chromatic Homotopy Theory

by

Guozhen Wang

Submitted to the Department of Mathematics
on May 18, 2015, in partial fulfillment of the
requirements for the degree of
PhD of Mathematics

Abstract

In this thesis, I study unstable homotopy theory with chromatic methods. Using the v_n self maps provided by the Hopkins-Smith periodicity theorem, we can decompose the unstable homotopy groups of a space into its periodic parts, except some lower stems. For fixed n, using the Bousfield-Kuhn functor Φ_n we can associate to any space a spectrum, which captures the v_n-periodic part of its homotopy groups. I study the homotopy type of the spectra $L_{K(n)}\Phi_n S^k$, which would tell us much about the v_n-periodic part of the homotopy groups of spheres provided we have a good understanding of the telescope conjecture. I make use the Goodwillie tower of the identity functor, which resolves the unstable spheres into spectra which are the Steinberg summands of classifying spaces of the additive groups of vector spaces over $\mathbb{F}_p$. By understanding the attaching maps of the Goodwillie tower after applying the Bousfield-Kuhn functor, we would be able to determine the homotopy type of $L_{K(n)}\Phi_n S^k$. As an example of how this works in concrete computations, I will compute the homotopy groups of $L_{K(2)}\Phi_2 S^3$ at primes $p \geq 5$. The computations show that the unstable homotopy groups not only have finite p-torsion, their $K(2)$-local parts also have finite v_1-torsion, which indicates there might be a more general finite v_n-torsion phenomena in the unstable world.

Thesis Supervisor: Mark Behrens
Title: Professor

Thesis Supervisor: Haynes Miller
Title: Professor

Acknowledgments

Foremost, I would like to express my sincere gratitude to my advisor Prof. Mark Behrens for the continuous support of my Ph.D study and research. His guidance helped me in all the time of research and writing of this thesis.

I would also like to thank Prof. Haynes Miller, who has given me continuous support for my studies and research and serves as my local advisor in my last year of Ph.D study.

Many thanks to Prof. Clark Barwick, Prof. Michael Hopkins, Prof. Nicholas Kuhn and Prof. Douglas Ravenel who have given me valuable advises on my project.

I also thank the staffs in our department, who has given me many helps.

Contents

List of Figures

Chapter 1

Introduction

This thesis is concerned about the unstable homotopy groups of spheres. The classical approach is to use the EHP sequence to study it. By using the Goodwillie calculus and the Bousfield-Kuhn functor, we know certian part of the EHP sequence can be actually be determined by stable computations. We will study how these works in understanding unstable homotopy theory.

1.1 The EHP sequence

The unstable homotopy groups of spheres can be approached by the EHP spectral sequence. There are computations of the low dimensional portion of the EHP sequence by Toda [27, 28] for the 2,3-primary part, and Behrens [3], Harper [12] for the 5-primary part.

There are certain stable phenomenon in the EHP sequence. In fact, there is one portion in the E_1-term which are in the stable range, which means the input of the EHP spectral sequence is the stable homotopy groups. This portion is called the metastable range. There is the James-Hopf map mapping the (certain iterated loop space of) spheres to the infinite loop spaces of the suspension spectra of the real projective spaces (for the prime 2, and classifying space of the symmetric group Σ_p for odd prime p), respecting the unstable filtration on the sphere side and skeletal filtration on the projective space side. So we have a comparison map of the EHP spectral sequence and the AHSS of the projective space. Moreover, in the metastable range, this induce an isomorphism of the E_1-term. So the differentials of the EHP spectral sequence in the metastable range are in fact stable, and we can apply stable techniques to compute them. For example, the behavior of the image of J is determined by Mahowald [22] and Gray [10].

Above the metastable range in the EHP spectral sequence, there is the portion which has input the metastable homotopy groups, the meta-metastable range, and we also have the meta-meta-metastable range whose input are the meta-metastable groups, and so on. In fact these pieces can be spitted into separate spectral sequences, each of which is a stable AHSS. So the differentials inside each piece are all stable ones, and only the differentials between different pieces are actually unstable. This splitting is achieved by the technique of the Goodwillie calculus developed in [9]. Using this, any space is decomposed into the homogeneous parts, which are infinite loop spaces. The interrelation of the EHP sequence and the Goodwillie tower of the identity is studied by Behrens [6]. In particular, the EHP spectral sequence in each homogeneous part of the Goodwillie tower is a stable AHSS for certain spectra $L(n)_k$.

1.2 The chromatic approach

We can take the chromatic point of view to study the EHP sequence and the Goodwillie tower.

For example, the rational part of unstable hoomotopy groups is the classical computations by Serre. The v_1-periodic part is essentially the image of J. The unstable behavior of these elements are determined by Mahowald [22] and Gray [10], as mentioned before.

To study higher chromatic things, first observe that the notion of v_n periodicity can be extended to the unstable setting, by desuspending the stable v_n self-maps on finte complexes, provided we only look at the higher stems which lies in the range of desuspension.

We have the useful Bousfield-Kuhn functor defined by Bousfield [8] and Kuhn [17].

This functor associates a spectrum to any space which recovers its v_n-periodic unstable homotopy groups. And if we apply it to an infinite loop space, it gives the delooping after inverting v_n. In particular, when applying this functor to the Goodwillie tower, we find that the Goodwillie differentials, which are unstable, becomes stable maps inside each monochromatic layer.

So most of the differentials in the EHP sequence come from stable maps, and we can apply stable techniques to study them. In this thesis, we will study the $K(2)$-local part of the EHP sequence, and give a computation of the $K(2)$-local part of the three sphere at primes $p \geq 5$.

1.3 Summary of main results

First we will give the BP-homology of the Goodwillie derivatives of the spheres. The Adam spectral sequence for this was computed by [15]. The extension problem can be computed by looking at the Steenrod operations on the ordinary cohomology. The BP-cohomology of $L(n)$, the Goodwillie derivatives of S^1, can be represented by a series of generators indexed by n positive integers. Each generator gives a summand free with $v_n, v_{n+1}, \ldots$.

Since we are mainly interested in the $K(2)$-local part, we will only describe the $p, \ldots, v_{n-1}$ extension problem for these generators in detail. The result can be represented by the following pictures. In these pictures, each circle indicates a generator. The arrows indicated where the generator goes after applying the multiplication, up to a factor which is a unit if we invert v_2. The first one says after multiplication by p, the generator goes to v_2 times the one on its left, plus terms with higher filtration. The second is for the case $p = 3$. In general, after multiplication by v_1, a generator goes to v_2 times the generator up-left p steps, plus the generator below, up to terms with higher filtration. If an arrow goes outside of the picture, then we just ignore that term.

After we understand the homology of the derivatives, we study the Goodwillie differentials. For the case of $K(2)$-local computations, there is only one map to

Figure 1-1: multiplication by p in $L(2)$

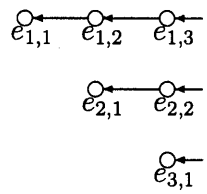
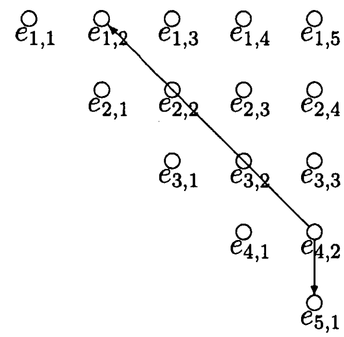


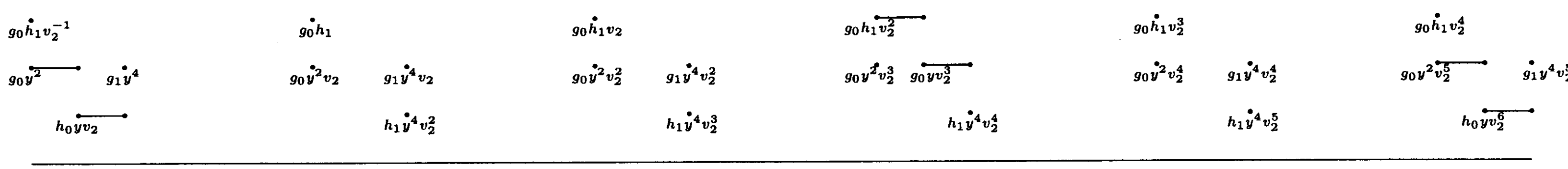
Figure 1-2: multiplication by v_1 in $L(2)$



be studied, which is the James-Hopf map. There are lots of papers studying this map from different perspective. We will describe its effect on Morava E-theory after applying the Bousfield-Kuhn functor. We will show that this is the trace map for the extension $E_{2*} \to E_2^* B\Sigma_p$ divided by p. This description is equivariant under the action of the Morava stabilizer group.

Using this we would be able to compute the E_2-homology of the effect of the Bousfield-Kuhn functor on S^3. It turns out that this can be obtained from E_{2*}/p by adding a p-th root of $\frac{v_1}{v_2}$, and mod out those terms with integral powers. Then we compute the Adams-Novikov spectral sequence using this description. The resulting E_2-term is the following picture (for $p = 5$). It is periodic, the pattern of point reappears each p groups. Each point repents a term $\mathbb{F}_p[\zeta]/\zeta^2$, and each horizontal line indicates a v_1 extension. There is no room for ANSS differentials, so the ANSS collapses, and the E_2-terms is the same as the picture for the homotopy groups.

Figure 1-3: ANSS E_2-term for $\Phi_{K(2)}S^3$ at $p = 5$



Chapter 2

Background

2.1 Stable chromatic homotopy theory

In this section, we will review the stable chromatic homotopy theory.

Fix a prime p. Let BP be the Brown-Peterson theory. Recall that $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. BP is complex oriented, and the associated formal group law is the universal p-typical one. We denote the multiplication of formal group laws by $+_F$. We will use the Araki generators, so that the p-series satisfy the equation

$$[p](x) = px +_F v_1 x^p +_F v_2 x^{p^2} +_F \cdots$$

We have $BP_*BP = BP_*[t_1, t_2, \ldots]$, and (BP_*, BP_*BP) forms a Hopf algebroid. See Appendix 1 of [25] for detail of Hopf algebroid. We will follow the notation there. The right unit η_R satisfies $\eta_R(v_i) \equiv 0 \mod (p, \ldots, v_{i-1})$. The Morava E-theory E_n can be defined using the Landweber exact theorem:

$$E_{n*}X = BP_*(X) \otimes_{BP_*} E_{n*}$$

where $E_{n*} = \mathbb{Z}_p[[v_1, \ldots, v_{n-1}]][v_n^{-1}]$, and the unit map $BP_* \to E_{n*}$ to make E_n into a BP-algebra maps $v_{n+1}, v_{n+2}, \ldots$ to zero. We have $E_{n*}E_n = E_{n*}[t_1, \ldots]/(\eta_R(v_{n+1}), \ldots)$. The Goerss-Hopkins-Miller theorem says there is a unique $\mathcal{E}_\infty$ structure on E_n.

The Morava K-theory is defined by $K(n) = E_n/(p, \dots, v_{n-1})$ as a spectrum, since $p, \dots, v_{n-1}$ is a regular sequence. By convetion $K(0) = H\mathbb{Q}$, the rational homology theory. We have the Bousfield localization functors $L_{K(n)}$ and $L_n = L_{K(0)\vee\dots\vee K(n)}$. We have the chromatic tower $X \to \dots L_2X \to L_1X \to L_0X$ for any spectrum X. By the chromatic convergence theorem, the inverse limit of the chromatic tower is homotopy equivalent to X after localization at p when X is a finite spectrum. We also have the chromatic squares:

$$\begin{array}{ccc} L_nX & \to & L_{n-1}X \\ \downarrow & & \downarrow \\ L_{K(n)}X & \to & L_{n-1}L_{K(n)}X \end{array}$$

By Morava's theorem, the homotopy groups of $L_{K(n)}X$ can be computed using the Adams-Novikov spectral sequence $Ext_{E_{n*}E_n}(E_{n*}, E_{n*}X) \Rightarrow \pi_*L_{K(n)}X$. For more details, see chapter 4 of [25].

Related to the chromatic tower is the periodic tower. We call a finite spectrum W has type n if $K(h)_*W = 0$ for $h < n$ and $K(n)_*W \neq 0$. By Hopkins-Smith periodicity theorem, there exist v_n-self maps on type n complexes which induce a nontrivial multiplication by some power of v_n in $K(n)$-homology. Using this theorem, we can invert v_n on a spectrum X which is nilpotent with respect to $p, \dots, v_{n-1}$. In this way, we can define the periodic tower with layers $v_n^{-1}X/(p^\infty, \dots, v_{n-1}^\infty)$. If we define $T(n)$ to be $v_n^{-1}W$ for any type n spectrum W, then there are analogous periodic squares

$$\begin{array}{ccc} L_{T(0)\vee\dots\vee T(n)}X & \to & L_{T(0)\vee\dots\vee T(n-1)}X \\ \downarrow & & \downarrow \\ L_{T(n)}X & \to & L_{T(0)\vee\dots\vee T(n-1)}L_{T(n)}X \end{array}$$

and the periodic tower becomes $X \to \dots L_{T(0)\vee T(1)\vee T(2)}X \to L_{T(0)\vee T(1)}X \to L_{T(0)}X$. For details see [14].

The relationship between the chromatic tower and the periodic tower is a hard question. In general we have $L_{K(n)} = L_{K(n)}L_{T(n)}$. The telescope conjecture expected them to be the same, which is proved in [23] and [22] for $n = 1$, but people tend to believe the contrary when $n \geq 2$. We can use Adams-Novikov spectral sequence to compute the chromatic things, but currently there is no method to do computations of the periodic things. The range of current stemwise computations are not large enough to show the v_2-periodic picture, except the β-family, in which part the chromatic and periodic points of view seem to agree.

2.2 Periodic unstable homotopy groups

The v_n-self-map on type n spectra can be desuspended, so we can talk about the notion of v_n-multiplication on elements in unstable homotopy groups which are nilpotent with respect to $p, \ldots, v_{n-1}$, provided it is in a high enough stem so that the v_n-self-map desuspends that far. To avoid the complication caused by these desuspension issue, we will only look at unstable homotopy groups with coefficients in type n spectra, in which case we can choose a universal choice of desuspension of a type n spectrum as coefficients.

Let W be a pointed finite CW space whose suspension spectrum is a type n spectrum. Without loss of generality, we assume there is a v_n-self-map $v_n^t : \Sigma^{t|v_n|}W \to W$ on W. For any space X, we define homotopy groups of X with coefficients in W to be $\pi_i(X; W) = [\Sigma^i W, X]$. Using the v_n-self-map on W we can define the notion of multiplication by v_n^t on $\pi_*(X; W)$ by composition with v_n^t on W.

With this multiplication by v_n map, we can define the v_n-periodic homotopy groups of X by inverting v_n^t. We denote this group by $v_n^{-1}\pi_*(X; W)$.

Remark 2.2.1. *By letting W to tend to the sphere stably, (e.g. take W to be some desuspension of $\mathbb{S}/(p^{i_0} \ldots, v_{n-1}^{i_{n-1}})$ for $i_0, \ldots i_{n-1}$ tending to infinity), we can define the notion of v_n-periodic homotopy groups, which capture the information of those parts of the unstable homotopy groups which admit the notion of multiplication by v_n and are not nilpotent with respect to it.*

2.3 The Bousfield-Kuhn functor

The behavior of stable and unstable homotopy groups are different in general. However, the periodic unstable homotopy groups behave like stable ones, because we can shift their dimensions to a higher stem via the periodicity operator, to land in the stable range. To make it precise, we have the Bousfield-Kuhn functor.

The Bousfield-Kuhn functor is a functor from the category of pointed space to the category of spectra. The definition of the functor can be found in [17]. The key property of the functor is as follows:

Theorem 2.3.1. *(Bousfield,Kuhn) There exists a functor $\Phi_n : Spaces_* \to Spectra$, called the Bousfield-Kuhn functor, which has the property:*

1. *If X is a space, and W is a type n space, then $v_n^{-1}\pi(X;W) = \pi_*(X;\Sigma^\infty W)$.*

2. *If Y is a spectrum, then $\Phi_n\Omega^\infty Y = L_{T(n)}Y$.*

So on one hand, this functor gives a canonical delooping in the $T(n)$-local category, without using the infinite loop structure. On the other hand, it transforms the periodic unstable homotopy groups into stable homotopy groups. This enables us to use stable techniques to understand periodic unstable homotopy groups.

As we know little about how to do computations of periodic homotopy groups even in the stable case, we will compute the unstable chromatic homotopy groups instead. By definition, the n^{th} chromatic homotopy groups of a space X is $\pi_*(L_{K(n)}\Phi_n X)$. We will abbreviate $L_{K(n)}\Phi_n$ by $L_{K(n)}$.

2.4 Goodwillie calculus on the identity functor

We will use the Goodwillie tower of the identity functor to understand the homotopy type of $L_{K(n)}S^k$.

Recall that Goodwillie calculus is a functorial way to decompose a functor into homogeneous pieces. The details can be found in [9]. We will apply it to the identity functor in the category of pointed spaces. In particular, we have:

Theorem 2.4.1. *(Goodwillie) There is a functorial way to associate a tower to any pointed space X:*

$$X \to \ldots \to P_3(X) \to P_2(X) \to P_1(X)$$

which has the following property:

1. *The fibers $D_i(X)$ of $P_i(X) \to P_{i-1}(X)$, called the i^{th} derivative, is an infinite loop space.*

2. *$P_1(X) = \Omega^\infty \Sigma^\infty X$.*

3. *The tower converges to X when X is a connected and Z-complete.*

We will apply the Bousfield-Kuhn functor to the Goodwillie towers of spheres. Though the attaching maps in the tower are not infinite loop maps, they become stable maps after applying the Bousfield-Kuhn functor. We will see that these maps can be understand by looking at the Morava E-theory.

The Goodwillie tower for odd spheres are well-studied. We know how to describe its derivatives. In particular, the following is proved in [2]:

Theorem 2.4.2. *Localize at a prime p. The spaces $D_i(S^{2k+1})$ are trivial unless i is a power of p. Moreover, if we apply the Bousfield-Kuhn functor Φ_n, we have $\Phi_n(S^{2k+1}) = \Phi_n(P_{p^n}(S^{2k+1}))$.*

Chapter 3

Homology of Goodwillie derivatives of spheres

In this chapter we will describe the BP-homology of the Goodwillie derivatives of spheres, in such a way that we could compute the coaction of BP_*BP on them, at least in principle. This will enable us to do computations with the Adams-Novikov spectral sequences.

3.1 The Steinberg summand of $B\mathbb{F}_p^n$

There are several descriptions of the Goodwillie derivatives of spheres. We will use the description as the Steinberg summand of $B\mathbb{F}_p^n$, following [1].

Recall that the Steinberg idempotent is the element $e = |GL_n(\mathbb{F}_p)/U_n| \sum_{w \in W} \epsilon(w) n_w \mathcal{B}$ in the group ring $\mathbb{Z}_{(p)}[GL_n(\mathbb{F}_p)]$. Here W is the Weyl group, which is the permutation group in n elements in the case of GL_n, $\epsilon(w)$ is the parity of the permutation, n_w its representative in the normalizer of the split maximum torus, and $\mathcal{B}$ is the sum of elements in the Borel subgroup, i.e. the upper triangular matrices, and U is the unipotent subgroup in the Borel subgroup, i.e. those matrices with diagonals 1. This element generates a projective irreducible representation of $GL_n(\mathbb{F}_p)$. Since e is an idempotent, for any $GL_n(\mathbb{F}_p)$-module we can define its Steinberg summand to be the image of e. This also extends to any spectrum acted by $GL_n(\mathbb{F}_p)$. Concretely, the coefficients in e are in $\mathbb{Z}_{(p)}$, so when localized at p, any such spectrum has a seft map defined by e, and its Steinberg summand is just the fiber of $1-e$.

Let $B\mathbb{F}_p^n$ be the classifying space of the additive group of the vector space over $\mathbb{F}_p$ of dimension n. Then $GL_n(\mathbb{F}_p)$ acts on it. Let $\bar{\rho}$ be the reduced regular representation of $B\mathbb{F}_p^n$, i.e. the sum of all the nontrivial irreducible representations. This representation is $GL_n(\mathbb{F}_p)$-equivariant, and also its multiples $k\bar{\rho}$. So the vector space defined by them are also acted by $GL_n(\mathbb{F}_p)$. By taking the Thom spectra, we construct spectra $(B\mathbb{F}_p^n)^{k\bar{\rho}}$ with $GL_n(\mathbb{F}_p)$ action.

Denote by $L(n)_k$ the Steinberg summand of $(B\mathbb{F}_p^n)^{k\bar{\rho}}$. Then it is shown in [1] that the Goodwillie derivatives of spheres $D_{p^n}S^k$ are homotopy equivalent to $\Omega^\infty\Sigma^{k-n}L(n)_k$. We will abbreviate $L(n)_1$ by $L(n)$.

There is yet another description of the $L(n)$'s. Let $SP^n(S)$ be the n^{th} symmetric power of the sphere spectrum. Then by Dold-Thom theorem, $SP^\infty(S)$ is a model for $H\mathbb{Z}$. There is the filtration $SP^1(S) \to SP^p(S) \to SP^{p^2}(S) \to \dots$. One finds $L(n) = \Sigma^{-n}SP^{p^n}(S)/SP^{p^{n-1}}(S)$.

3.2 The ordinary homology of $L(n)$

The ordinary homology of $L(n)$ was calculated in [24] and [2]:

Theorem 3.2.1. *If we identify the mod p cohomology of $SP^\infty(S) = H\mathbb{Z}$ as the vector space generated by the Steenrod operators $\beta^{\epsilon_1}P^{i_1}\beta^{\epsilon_2}P^{i_2}\beta^{\epsilon_3}\dots P^{i_k}$ with admissible sequences $(\epsilon_1, i_1, \dots, i_k)$, then the filtration defined by the $SP^{p^n}(S)$ induces the filtration defined by the length k of the operator. Consequently, we find the cohomology of $L(n)$ has a set of generators $\beta^{\epsilon_1}P^{i_1}\beta^{\epsilon_2}P^{i_2}\dots P^{i_n}$ with admissible sequences $(\epsilon_1, i_1, \dots, i_n)$.*

For example, the cohomology of $L(1) \cong \Sigma^\infty B\Sigma_p$, has a set of basis of the form $P^1, \beta P^1, P_2, \beta P^2, \dots$. This corresponds to the usual way of constructing the spaces by attaching cells. Note that we have a desuspension in relating $L(n)$ with the symmetric products, so the lowest cell lies in dimension $2(p-1)-1$.

Also note that the cohomology of $L(2)$ has a set of basis of the form:

$$P^pP^1, \beta P^pP^1, P^{p+1}\beta P^1, \beta P^{p+1}\beta P^1, P^{p+1}P^1, \beta P^{p+1}P^1, P^{p+2}\beta P^1, \beta P^{p+2}\beta P^1, \ldots,$$
$$P^{2p}P^2, \beta P^{2p}P^2, P^{2p+1}\beta P^2, \beta P^{2p+1}\beta P^2, \ldots$$

So in particular, the dimensions of the cells of $L(2)$ lie in

$$|v_2|-2, |v_2|-1, |v_2|+|v_1|-1, |v_2|+|v_1|, |v_2|+|v_1|-2, |v_2|+|v_1|-1, |v_2|+2|v_1|-1, |v_2|+2|v_1|, \ldots,$$
$$2|v_2|-2, 2|v_2|-2, 2|v_2|-1, 2|v_2|+|v_1|-1, 2|v_2|+|v_1|, \ldots$$

We have the unstable filtration on $L(n)$, and by dualizing the result in [2], we find that it induces the filtration by the last component of the admissible sequences on cohomology.

3.3 The BP-cohomology of $L(n)$

The Adams spectral sequence to determine the BP-homology of $L(n)$ was computed in [15]. We will also give the v_n-extensions in this section.

Recall that the Milnor elements of the Steenrod algebra are defined inductive by the formula: $Q_0 = \beta$ and $Q_{k+1} = [P^{p^k}, Q_k]$. For example, $Q_1 = P^1\beta + \beta P^1$.

By [11], the Steenrod algebra is Kozul with generators the Steenrod operators P^i and the Milnor elements Q_i, and relations: the usual Adem relations for the generators P_i, and $Q_kQ_l = -Q_lQ_k$, $P^nQ_k = Q_kP^n + Q_{k+1}P^{n-p^k}\ if\, n > p^k$, $P^{p^k}Q_k = Q_kP^{p^k} + Q_{k+1}$, $P^nQ_k = Q_kP^n\ if\, n < p^k$.

With the map $BP \to H\mathbb{Z}$, we can identify the ordinary mod p cohomology of BP with the quotient of the Steenrod algebra by the ideal generated by the Q_i's. The Adams spectral sequence computing $BP^*L(n)$ has E_2-term $Ext_A(H\mathbb{F}_p^*BP, H\mathbb{F}_p^*L(n))$. The Q_i's generates a sub-exterior-algebra $E[Q_i]$ in the Steendrod algebra. By the change of rings theorem, we can identify the E_2-term with $Ext_{E[Q_i]}(\mathbb{F}_p, H\mathbb{F}_p^*L(n))$.

In the following we will do the computation for $L(2)$. The general case is similar except for more complicated notations.

In $H\mathbb{F}_p^*L(2)$, we have the elements with names P^iP^j. These elements generate a $\mathbb{F}_p$-sub-vector space U. We find $Q_1P^iP^j = -\beta P^{i+1}P^j + P^{i+1}\beta P^j$. Note the first term has excess one more. Thus we find that the map $E[Q_0.Q_1] \otimes U \to H\mathbb{F}_p^* L(2)$ is an isomorphism.

We also note that $Q_0 Q_1 P^j P^j$ are annihilated by any Q_i, since the composition has three Q_i's, hence the expansion using the Adem relations has at least three β's in each term, in particular with length at least three.

So we conclude that the E_2-term has the form $\mathbb{F}_p[v_2, v_3, \dots] \otimes U$. Here v_i corresponds to $[Q_i^*]$ in the bar (or Kozul) complex, and the generators should be the form $v_2^{i_2} v_3^{i_3} \dots v_k^{i_k} [Q_0 Q_1 P^i P^j]$ in the Kozul complex. Moreover, there is no room for higher differentials. What remains is to determine the v_i-extensions.

Note $v_3, v_4, \dots$ appear explicitly, so their extensions are clear. The reason why v_0 and v_1 do not appear explicitly comes from the fact that anything with v_1, such as $v_1[P^i P^j]$, is contained as one term of the Kozul differential $d([Q_0 P^i P^j]) = \sum \pm v_i [Q_i Q_0 P^i P^j]$. So we conclude that there is a filtration such that the graded pieces are the $\mathbb{F}_p[v_2, v_3, \dots]$, with generators $x_{2;i,j}$ corresponding to $[Q_0 Q_1 P^i P^j]$. Here we will abbreviate $x_{2;i,j}$ as $x_{i,j}$ if no confusion arises.

Using the commutation relations, one finds the following relation:

$$Q_2 P^i P^j + Q_1 (P^{i+p-1} P^{j+1} + P^{i+p} P^j) + Q_0 P^{i+p} P^{j+1} = P^{i+p} P^{j+1} \beta$$

Note that the right hand side has length 3, hence zero in $H\mathbb{F}_p^* L(2)$. Also note that every term is admissible except $P^{i+p-1} P^{j+1}$ when $i = pj$. But in the latter case we find $P^{jp+p-1} P^{j+1} = 0$ from the Adem relations.

We have the following differential in the Kozul complex:

$$d([Q_1 P^i P^j]) = v_0 [Q_0 Q_1 P^i P^j] + v_2 [Q_2 Q_1 P^i P^j] + \dots$$

Using the previous relation, we get $Q_2 Q_1 P^i P^j = Q_1 Q_0 P^{i+p} P^{j+1}$. This shows that we have the following relation in the ext group:

$$p x_{i,j} = \pm v_2 x_{i+p,j+1} + \textit{higher terms}$$

Similarly we have the relation

$$v_1 x_{i,j} = \pm v_2(x_{i+p-1,j+1} + x_{i+p,j}) + \textit{higher terms}$$

Here we understand that the inadmissible term is zero if occurs.

For general $L(n)$, the same computation gives the existence of a filtration on cohomology, with the associated grade pieces direct sum of $BP^*/(p, \ldots, v_{n-1})$'s, and generators $x_{n;i_1,\ldots,i_n}$, with $(i_1, \ldots, i_n)$ admissible, of the form $[Q_0 Q_1 \ldots Q_{n-1} P^{i_1} P^{i_2} \ldots P^{i_n}]$ in the Adams spectral sequence, in terms of the Kozul complex.

Using the commutation relations, we have the equation

$$Q_n P^{i_1} P^{i_2} \ldots P^{i_n} = -Q_{n-1} P^{i_1+p^{n-1}} P^{i_2} \ldots P^{i_n} + P^{i_1+p^{n-1}} Q_{n-1} P^{i_2} \ldots P^{i_n}$$

With induction, we can transform the term $Q_{n-1} P^{i_2} \ldots P^{i_n}$ into the sum of terms with one Q_i on the left with $i < n-1$, and a term of length n. We also assert that the first superscript on P of each term do not exceed $i_2 + p^{n-2}$. Then we can transform $P^{i_1+p^{n-1}} Q_i$ into $Q_{i-1} P^{i_1+p^{n-1}} + Q_{i+1} P^{i_1+p^{n-1}-p^i}$. In this way, we transform the original one into terms with one Q_i on the left with $i < n$, and a term of length $n+1$. Note the inadmissible term might arise from $P^{i_1+p^{n-1}-p^i} P^{i_2+j}$ with $j \leq p^{n-2}$. But the Adem relations show that this term vanishes.

So we arrive at the conclusion that we can play the same trick as before to obtain the formula for multiplication by p, v_1, $\ldots$, v_{n-1}. In fact the formula can be obtained inductively using the above procedure. In particular, we have

$$p x_{n;i_1,\ldots,i_n} = v_n x_{n;i_1+p^{n-1}, i_2+p^{n-2}, \ldots, i_n+1} + \textit{higher terms}$$

3.4 The Dickson-Mui generators

The computations in the previous section only give the generators up to leading term, so we can only have the leading terms of the formulas. In this section, we will construct generators in the BP-cohomology of $L(n)$ which would enable us to get the complete formula for the extensions as well as the action of BP_*BP, at least in principle.

We will use the description of the Steinberg summand in [16], as a quotient of the invariants of the Borel subgroup. As before we will do the computation for $L(2)$, and the general case is analogues.

Recall that $BP^*B\mathbb{F}_p = BP^*[u]/[p](u)$. One finds that this ring is Landweber exact, since it is free when any v_i is inverted, using the Weierstrass preparation theorem.

So $BP^*B\mathbb{F}_p^2 = BP^*[u,v]/([p](u),[p](v))$ by the Künneth formula. Note that if we use the filtration by the powers u, v, and ignore low dimensional irregularities (i.e. if we concentrate on the irreducible part of the relations) , the graded pieces are $BP^*/(p, v_1)[u, v]$.

From [16], the Steinberg summand is isomorphic to the invariants of the Borel subgroup modulo the sum of the invariants of the minimal parabolic subgroups.

Now we can define the Dickson-Mui invariants for GL_2. Define $D_1 = \prod_{i\neq 0}[i](v)$, and $D_2 = \prod_{(i,j)\neq(0,0)}([i](x) +_F [j](y))$. (In general, we have the generators $D_m = \prod_{(i_1,\ldots,i_m)\neq(0,\ldots,0)}([i_1](x_1)+_F\cdots+_F[i_m](x_m))$ for $L(n)$ when $m \leq n$.) They are invariant under the Borel subgroup. We note that $D_1 = v^{p-1}$, and D_2 in invariant under $GL_2(\mathbb{F}_p)$. Also note that D_2 is the top Chern class of the complex reduced regular representation.

Recall we can relate the different Thom spectra by inclusion of the zero section.

Theorem 3.4.1. *The inclusion of $BP^*L(2)_1$ in $BP^*L(2)_0$ is injective, and the image can be described as generated by terms of the form $D_1^{l_1}D_2^{l_2}$, with $l_1, l_2 \geq 1$. Moreover, $D_1^{l_1}D_2^{l_2}$ corresponds to the generator $x_{pl_2+l_1-1,l_2}$ in the previous subsection. The filtration $L(2)_1 \to L(2)_3 \to \cdots$ induced by the unstable filtration, is defined by the powers of D_2.*

Proof. The first assertion comes from the previous section. The others follow from the fact that the generators P^iP^j in ordinary cohomology has the same expression in terms of ordinary Dickson-Mui invariants. □

To understand the multiplication by p and v_1. We know the equations for u and v: $pu+v_1u^p+v_2u^{p^2}+\dots=0$, $pv+v_1v^p+v_2v^{p^2}+\dots=0$. Let $t=uv^p-vu^p$, then the leading term for D_2 is t^{p-1}. Combining the equations of u and v to cancel out terms involving p gives $v_1t+v_2(v^{p^2}u-u^{p^2}v)+\dots=0$. Next we have the Dickson invariant has leading term $(v^{p^2}u-u^{p^2}v)/(v^pu-u^pv)=(u^{p-1})^p+(u^{p-1})^{p-1}v^{p-1}+\dots+(v^{p-1})^p$. Modulo p, one finds that the right hand side reduces to $(v^{p-1})^p+(\frac{t}{v})^{p-1}$. Plugging in this formula we get the equation $v_1t+v_2t(D_1^p+\frac{D_2}{D_1})+\dots=0$. Hence we have the equation for multiplication of v_1 on the generators: $v_1D_1^{l_1}D_2^{l_2}=-v_2(D_1^{l_1+p}D_2^{l_2}+D_1^{l_1-1}D_2^{l_2+1})+\dots$, which is the same relation for the generators $x_{2;i,j}$ obtained previously, but now we can also work out the higher terms with more computations.

Using the equation for v we have the following: $pD_1^{l_1}D_2^{l_2}=(pv)v^{l_1(p-1)-1}D_2^{l_2}=-(v_1v^p+v_2v^{p^2}+\dots)v^{l_1(p-1)-1}D_2^{l_2}=-(v_1D_1^{l_1+1}+v_2D_1^{l_1+(p+1)})D_2^{l_2}+\dots$. With the formula for multiplication by v_1, we get the following equation: $v_1D_1^{l_1+1}D_2^{l_2}=-v_2(D_1^{l_1+1+p}D_2^{l_2}+D_1^{l_1}D_2^{l_2+1})+\dots$. The first term cancels the other term in the previous formula, and we arrive at the formula $pD_1^{l_1}D_2^{l_2}=v_2D_1^{l_1}D_2^{l_2+1}+\dots$, as expected.

We can also understand, at least in principle, the action of the Morava stabilizer group on the cohomology, from the knowledge of the action on protective space, since the generators are expressed in terms of the cohomology of complex projective spaces.

As a final remark, note that there are no negative powers of D_i in the set of generators. This does not mean they are set zero. We have mod out the invariants under the minimal parabolic subgroups. So there are relations to transform any term with negative powers into one with only positive powers of D_i's.

Chapter 4

The James-Hopf Map

The attaching maps in the Goodwillie tower can be described using the James-Hopf map. In this chapter, we will study the effect of the Bousfield-Kuhn functor on the James-Hopf map, which would give us information on the Goodwillie differentials.

4.1 The Goodwillie attaching maps and the James-Hopf map

Let X be a connected space. Recall that by Snaith splitting, $\Sigma^\infty\Omega^\infty\Sigma^\infty X \cong \coprod_{i=1}^{\infty}(\Sigma^\infty X)^{\wedge i}_{h\Sigma_i}$ for any connected pointed space X, and the equivalence is induced from the map $\Sigma^\infty X \to \Sigma^\infty(\Omega^\infty\Sigma^\infty X)_+$ by extending it into an $\mathcal{E}_\infty$ map. See [19] for details.

Fix a prime p. The Jame-Hopf map

$$jh : \Omega^\infty\Sigma^\infty X \to \Omega^\infty(\Sigma^\infty X)^{\wedge p}_{h\Sigma_p}$$

is the adjoint of the projection map

$$\Sigma^\infty\Omega^\infty\Sigma^\infty X \to (\Sigma^\infty X)^{\wedge p}_{h\Sigma_p}$$

Using the James-Hopf map, a sequence $\Omega^\infty L(0) \to \Omega^\infty L(1) \to \Omega^\infty L(2) \to \cdots$ was constructed in [21]. The maps in the sequence are essentially the Steinberg summand of the Jame-Hopf map. In particular, the first map is the James-Hopf map for S^1.

Conjecture 4.1.1. *The above sequence is the same as the seuqence of the attaching maps in the Goodwillie tower of S^1.*

It is well-known that the first attaching map in the Goodwillie tower is the James-Hopf map, and in general it is prove in [5] and [20] that the conjecture is true in ordinary homology.

We will describe the James-Hopf map after applying the Bousfield-Kuhn functor. This will determine the monochromatic part of the Goodwillie attaching maps in cases when the above conjecture is true.

4.2 Weakly graded spectra

We will introduce the (homotopy) category of weakly graded spectra. The objects are sequences of spectra $(X_i)_{i\in\mathbb{Z}}$. For morphisms, let $X = (X_i)$ and $Y = (Y_i)$ be two objects in the category. Then a map can be identified with a matrix (f_{ij}) for $f_{ij} : X_i \to Y_j$. Then this map is a morphism of weakly graded spectra, if for fixed i there are only finitely many j with f_{ij} nontrivial, and for fixed j, there are only finitely many i with f_{ij} nontrivial.

We will use D for Spanier-Whitehead dual. For a graded spectrum $(X_i)_{i\in\mathbb{Z}}$, we introduce the restricted Spanier-Whitehead dual functor, denoted by D', to be the direct sum of the Spanier-Whitehead dual of its homogeneous pieces, i.e. the sequence $(Y_i)_{i\in\mathbb{Z}}$ with $Y_i = DX_{-i}$. One sees that the functor D' can be defined on the category of weakly graded spectra.

We also introduce the (homotopy) category of weak infinite loop spaces. The objects are labeled by pointed connected spaces. For any two objects labeled by X and Y, a morphism is a (homotopy class of) pointed map from $\Omega^\infty\Sigma^\infty X$ to $\Omega^\infty\Sigma^\infty Y$ such that the adjoint $\Sigma^\infty\Omega^\infty X \to \Sigma^\infty Y$ has only finitely many (homotopically) nontrivial components when the domain is decomposed using the Snaith splitting. There is a functor to the category of spaces by taking X into $\Omega^\infty\Sigma^\infty X$, and the composition of this functor with infinite suspension lands in the category of weakly graded spectra.

We see that the infinite suspension of the James-Hopf map lands in the category of weakly graded spectra.

4.3 The behavior of the James-Hopf map on cohomology

We will describe a method to determine the map on E_n-cohomology induced by the James-Hopf map after applying the Bousfield-Kuhn functor.

By results of [7], we can identify $\Phi_{K(n)}X$ with

$$TAQ_{L_{K(n)}\mathbb{S}}(L_{K(n)}D\Sigma^\infty\Omega^\infty\Sigma^\infty X_+)$$

the topological André-Quillen cohomology of the $K(n)$-local $\mathcal{E}_\infty$ spectrum $L_{K(n)}D\Sigma^\infty\Omega^\infty\Sigma^\infty X_+$ where the multiplication is induced from the diagonal on $\Omega^\infty\Sigma^\infty X$.

In order to apply the theorem, we need to understand the $\mathcal{E}_\infty$ structure on $D\Sigma^\infty\Omega^\infty\Sigma^\infty X_+$ induced from the diagonal map on $\Omega^\infty\Sigma^\infty X$. The diagonal map is determined by [18]. Essentially, the diagonal map on the component $\Sigma^\infty X$ in the Snaith splitting is the sum of two maps: the primitive part $X \to (\mathbb{S}\wedge X)\coprod(X\wedge\mathbb{S})$, and the other part $X \to X\wedge X$ coming from the diagonal map on X.

We will work at a slightly more general setting. To simplify the argument, we will work in the category of S-modules in the rest of this section, so that we have a symmetric monoidal smash product, and for cofibrant X, the action of Σ_k on $X^{\wedge k}$ is free. We will also work in the $K(n)$-local category, and $K(n)$-localization will be applied implicitly when necessary.

Let X be a cocommutative S-coalgebra with structure map $\psi_X : X \to X\wedge X$. Suppose that X is cofibrant and $E_n^*(X)$ is free of finite rank over E_n^*. Hence in the $K(n)$-local category, X is dualizible by [14]. We can construct the free commutative S-algebra $S(X) = \oplus_k X^{\wedge k}/\Sigma_k$ generated by X. We have a cocommutative S-coalgebra structure on $S(X)$ defined by extending the composition of the following map

$$X \xrightarrow{1\wedge id + id\wedge 1 + \psi_X} (\mathbb{S}\wedge X)\oplus(X\wedge\mathbb{S})\oplus(X\wedge X)\to S(X)\wedge S(X)$$

into a map of commutative S-algebras. (The last map is the inclusion of summand $(\mathbb{S}\wedge X)\oplus(X\wedge\mathbb{S})\oplus(X\wedge X)\to\oplus_{i,j}(X^{\wedge i})\wedge(X^{\wedge j})/\Sigma_i\times\Sigma_j$.)

Note that in this case, we have a natural decreasing filtration on $S(X)$ defined by the powers of X, which is preserved by the coproduct defined above. Taking the restricted Spanier-Whitehead dual, we have the spectrum $D'S(X)=\oplus_k D(X^{\wedge k}/\Sigma_k)$, which is $K(n)$-locally equivalent to $\oplus_k((DX)^{\wedge k})^{\Sigma_k}$ because X is $K(n)$-locally dualizibile. The coproduct above induces a commutative S-algebra structure on $D'S(X)$, since the formula for the product is a finite sum on each summand. Moreover, the product preserves the increasing filtration defined by the powers of DX.

If we take the graded pieces of the S-algebra $D'S(X)$, we get a graded commutative S-algebra, and the following lemma is straightforward:

Lemma 4.3.1. *Suppose we have the graded commutative S-algebra structure over $D'S(X)$ defined by the coproduct induced from*

$$X \xrightarrow{1\wedge id + id\wedge 1} (\mathbb{S}\wedge X)\oplus(X\wedge\mathbb{S})\to S(X)\wedge S(X)$$

by extending it into an S-algebra map. Then the restriction of the iterated product map $(D'S(X))^{\wedge k}/\Sigma_k\to D'S(X)$ to the summand $DX^{\wedge k}/\Sigma_k$ is the norm map to the summand $(DX^{\wedge k})^{\Sigma_k}$:

$$DX^{\wedge k}/\Sigma_k\to(DX^{\wedge k})^{\Sigma_k}\to D'S(X)$$

Corollary 4.3.2. *In the $K(n)$-local category, the inclusion of DX in $D'S(X)$ induces an equivalence of the free commutative S-algebra generated by DX, into $D'S(X)$.*

Proof. It is enough to check the graded pieces, since the induced map $S(DX)\to D'S(X)$ preserves the natural filtrations.

If we look at the corresponding graded commutative S-algebra for $D'S(X)$, we find that the product is as the one in the lemma. Then the lemma shows that the induced map $S(DX) \to D'S(X)$ is the norm map on each homogeneous component. Then the corollary follows. $\square$

Using this corollary, we can identify the topological André-Quillen cohomolgoy of $D'S(X)$ with DX.

Lemma 4.3.3. *Suppose further that X is the suspension spectrum of a space Y with coproduct induced form the diagonal map. Then the effect of the Bousfield-Kuhn functor on the James-Hopf map for Y is the dual of the composite*

$$(DX^{\wedge p})^{\Sigma_p} \to D'S(X) \xleftarrow{\phi} S(DX) \to DX$$

where the map ϕ is the $K(n)$-equivalence in corollary 4.3.2, and the last map is the projection.

Proof. Behrens and Rezk constructed in Section 6 of [7] a natural transformation

$$TAQ(D\Sigma^\infty Z_+) \to D\Phi_n Z$$

Now when Z takes the form $\Omega^\infty\Sigma^\infty Y$, we have a natural transformation $TAQ(D'\Sigma^\infty Z_+) \to TAQ(D\Sigma^\infty Z_+)$ when we regard Z as an object in the category of weak infinite loop spaces labeled by Y. The composition gives a natural transformation $TAQ(D'\Sigma^\infty Z_+) \to D\Phi_n Z$ from the category of weak infinite loop spaces to the category of spectra. Now it is easy to see that, the following natural transformation

$$TAQ(D'\Sigma^\infty\Omega^\infty\Sigma^\infty Y_+) \to D\Phi_n\Omega^\infty\Sigma^\infty Y$$

is an equivalence since both sides can be identified with $D\Sigma^\infty Y$. So we conclude that the natural transformation $TAQ(D'\Sigma^\infty Z_+) \to D\Phi_n Z$ is an equivalence. The lemma follows from the previous corollary using this TAQ description of the Bousfield-Kuhn functor on weak infinite loop spaces. $\square$

Now we will investigate the maps

$$j_i : (DX^{\wedge i})^{\Sigma_i} \to D'S(X) \xleftarrow{\phi} S(DX) \to DX$$

for $i \leq p$. Of course j_1 is the identity map, and what we want to do is to find how the direct summand in $S(DX)$ transport to $D'S(X)$.

Lemma 4.3.4. *For $i < p$, the map j_i factors through the multiplication map $DX^{\wedge i} \to DX$. In fact, for all i, the composition of j_i with the transfer map $DX^{\wedge i} \to (DX^{\wedge i})^{\Sigma_i}$ is a multiple of the multiplication map.*

Proof. We will show this by induction. The case for $i = 1$ is clear. So we will assume that we have the lemma for all j_t for $t < i$.

Now to investigate j_i, we will first find the image of $DX^{\wedge i}/\Sigma_i$ under ϕ, which goes to zero under projection. By [18], the restriction of ϕ to $DX^{\wedge i}/\Sigma_i$ is the sum of the maps

$$\phi_I : DX^{\wedge i}/\Sigma_i \to (DX^{\wedge |I|})^{\Sigma_{|I|}}$$

where I ranges over all partitions of i, into $|I|$ parts. (This corresponds to the special case of a covering by i copies of 1's in the notation of [18].) The maps ϕ_I are described as follows. Let $\Sigma_I \subset \Sigma_i$ be the subgroup of permutations which preserve the (unordered) partition I. For example, $\Sigma_{(p)} = \Sigma_{(1,\ldots,1)} = \Sigma_i$. So if I has i_m components with m elements, then we have $DX^{\wedge i}/\Sigma_I = \wedge_m (DX^{\wedge m}/\Sigma_m)^{\wedge i_m}/\Sigma_{i_m}$. Then ϕ_I is defined to be the composition

$$\begin{aligned} DX^{\wedge i}/\Sigma_i \to DX^{\wedge i}/\Sigma_I = \wedge_m (DX^{\wedge m}/\Sigma_m)^{\wedge i_m}/\Sigma_{i_m} \to \\ \wedge_m DX^{\wedge i_m}/\Sigma_{i_m} \to DX^{\wedge |I|}/\Sigma_{|I|} \to (DX^{\wedge |I|})^{\Sigma_{|I|}} \end{aligned}$$

Here the first map is the transfer, the second induced from the multiplication defined by ψ, the third map is the restriction, and the last map is the norm map.

Now we have the equation $\sum_I j_{|I|} \circ \phi_I = 0$. We compose this equation with the restriction map $DX^{\wedge i} \to DX^{\wedge i}/\Sigma_i$. Then the summand with $|I|= (1, \ldots, 1)$ becomes the composition of j_i with the transfer map, since the composition of the norm map with the restriction is the transfer map. For the other summands, we find that the composition of the restriction map with the map

$$DX^{\wedge i}/\Sigma_i \to DX^{\wedge i}/\Sigma_I = \wedge_m (DX^{\wedge m}/\Sigma_m)^{\wedge i_m}/\Sigma_{i_m} \to$$
$$\wedge_m DX^{\wedge i_m}/\Sigma_{i_m} \to DX^{\wedge |I|}/\Sigma_{|I|}$$

is the sum of maps

$$DX^{\wedge i} \xrightarrow{g} \wedge_m DX^{\wedge i_m} \to DX^{\wedge |I|} \to DX^{\wedge |I|}/\Sigma_{|I|}$$

where the first map rearranges the order of factors according to g for g running through $\Sigma_i/\Sigma_{|I|}$. The second map is the multiplication map, and the last map is the restriction map. So the composition of the map $j_{|I|} \circ \phi_I$ with the transfer map $DX^{\wedge i} \to (DX^{\wedge i})^{\Sigma_i}$ becomes the sum of compositions

$$DX^{\wedge i} \to DX^{\wedge |I|} \to (DX^{\wedge |I|})^{\Sigma_I} \xrightarrow{j_{|I|}} DX$$

where the first map is the multiplication map of type I described above, and the second map is the transfer. The sum is over the set Σ_i/Σ_I of ways to do the multiplication.

So by induction, we proved that the composition of j_i with the restriction $DX^{\wedge i} \to (DX^{\wedge i})^{\Sigma_i}$ is a multiple of the multiplication map. (In fact, we know the coefficient is the alternating sum of ways to do the multiplication, but we do not need this formula.) Now in case $i < p$, the restriction map is projection to a direct summand, so we can find a section, and the lemma is proved.

$\square$

The case $i = p$ is a bit subtle, since then the restriction map is no longer a projection. This is why it is the interesting James-Hopf map. We have the following description:

Theorem 4.3.5. *The map j_p is the sum of two maps. The first map is the negative of* *the map $(DX^{\wedge p})^{\Sigma_p} \leftarrow DX^{\wedge p}/\Sigma_p \to DX$ where the first map is the norm map and the second the multiplication map defined by the map ψ. The other map factors through the multiplication map $DX^{\wedge p} \to DX$.*

Proof. We follow the arguments of the previous lemma. We still have the equation $\sum_I j_{|I|} \circ \phi_I = 0$. Now in this case, the summand corresponding to $I = (1, \ldots, 1)$ is the composition of j_i with the norm map, and the summand corresponding to $I = (p)$ is the multiplication map. So these give the map $(DX^{\wedge p})^{\Sigma_p} \leftarrow DX^{\wedge p}/\Sigma_p \to DX$.

For the other summands, we note that Σ_I is a proper subgroup for $I \neq (1, \ldots, 1), (p)$. Then the restriction map $DX^{\wedge p} \to DX^{\wedge p}/\Sigma_I$ admits a section. As before, we find that the composition

$$DX^{\wedge p} \to DX^{\wedge p}/\Sigma_I \to DX^{\wedge |I|}/\Sigma_{|I|}$$

is the composition

$$DX^{\wedge p} \to DX^{\wedge |I|} \to DX^{\wedge |I|}/\Sigma_{|I|}$$

The only difference is that we no longer need the permutation of factors. The same argument shows that all such summands $j_{|I|} \circ \phi_I$ with $|I| \neq (1, \ldots, 1), (p)$ factors through the multiplication map $DX^{\wedge p} \to DX$. This completes the proof $\square$

Remark 4.3.6. *We do not know how to describe the map factoring the second map through the multiplication map. In ordinary cohomology, it look like the restriction map. However, in the case we are interested in, i.e. for $X = S^{2n+1}$, the multiplication map is homotopically zero nonequivariantly, so in this case the second map has no contributions.*

4.4 Determination of the cohomological James-Hopf map

We will show that Theorem 4.3.5 determines the effect of the James-Hopf map on E_n-cohomology for odd spheres.

We need to understand the map

$$(DX^{\wedge p})^{h\Sigma_p} \leftarrow (DX^{\wedge p})_{h\Sigma_p} \to DX$$

First we look at the case when X is the sphere spectrum. In this case, the multiplication map $\mathbb{S}_{h\Sigma_p} \to \mathbb{S}$ is simply the restriction map along the homomorphism $\Sigma_p \to *$ to the trivial group.

To understand the norm map, we will use the following description. For any Y with an action by a finite group G, the identity map is G-equivariant, so we have a canonical element $id_G \in \pi_0 \mathbb{H}om(Y,Y)^{hG}$, where $\mathbb{H}om(Y,Y)$ is the inner Hom, and G acts on it by conjugation. The following lemma is a consequence of discussions in the appendix of [7]:

Lemma 4.4.1. *The class of the norm map $Y_{hG} \to Y^{hG}$ in π_0 of the spectrum $\mathbb{H}om(Y_{hG}, Y^{hG}) = \mathbb{H}om(Y,Y)^{h(G\times G)}$, is the transfer of id_G along the diagonal homomorphism $G \to G \times G$.*

Now specialize to the case of $\mathbb{S}$. The map id_G is simply the restriction map $\mathbb{S} \to \mathbb{S}^{hG}$ along the map $G \to *$. Now if G is abelien, G is the pull back of the diagram $G \times G \to G \leftarrow *$, where the first map sends (g_1, g_2) to $g_2^{-1}g_1$. In this case, the projection formula says the composition of restriction and transfer equals the composition of transfer and restriction: $\mathbb{S} \xrightarrow{tr} \mathbb{S}^{hG} \xrightarrow{res} \mathbb{S}^{h(G\times G)}$.

If H is a subgroup of G, we consider the composition

$$\mathbb{S} \xrightarrow{res} \mathbb{S}^{hG} \xrightarrow{tr} \mathbb{S}^{h(G\times G)} \xrightarrow{res} \mathbb{S}^{h(H\times H)}$$

of the restriction, transfer and restriction map. Using the double coset formula, the composition of the latter two maps is the sum of maps $\mathbb{S}^{hG} \to \mathbb{S}^{h(H\cap H^g)} \to \mathbb{S}^{h(H\times H)}$, where the sum is over $g \in G/H$, and the first map is restriction, the second map is the transfer along the map $H \cap H^g \to H \times H$ whose first component is the inclusion and the second component is conjugation by g.

Now we take $G = \mathbb{Z}/p$. In this case, G is abelien. So we will look at the composition of $\mathbb{S} \xrightarrow{tr} \mathbb{S}^{hG} \xrightarrow{res} \mathbb{S}^{h(G\times G)}$ of transfer and restriction. After taking E_n-homlogy, we can identify $E_{n*}(\mathbb{S}^{hG}) = E_n^*[[\xi]]/[p](\xi)$ with the ring of functions on the finite flat algebraic group of order p points in the formal group F_n associated with E_n. The transfer map $E_n^*(\mathbb{S}) \to E_n^*(\mathbb{S}_{hG})$ has the following formula:

Lemma 4.4.2. *The transfer of* $1 \in E_n^*(\mathbb{S})$ *is the element*

$$\frac{[p](\xi)}{\xi} \in E_n^*(\mathbb{S}_{hG}) = E_n^*[[\xi]]/[p](\xi)$$

Proof. See page 588 in [13]. $\square$

We know that the addition on $\mathbb{Z}/p$ correspond to the multiplication of the formal group law. So the restriction map along

$$G \times G \to G$$

gives the map on cohomology sending the element

$$f(\xi) \in E_n^*(\mathbb{S}_{hG}) = E_n^*[[\xi]]/[p](\xi)$$

to the element

$$f(\xi -_{F_n} \eta) \in E_n^*(\mathbb{S}_{h(G\times G)}) = E_n^*[[\xi, \eta]]/([p](\xi), [p](\eta))$$

Hence the class of the norm map has Hurewicz image

$$\frac{[p](\xi -_{F_n} \eta)}{\xi -_{F_n} \eta} \in E_n^*(\mathbb{S}_{h(G\times G)})$$

Now we take some embedding $E_n^* \to K$ into some algebraically closed field of characteristic zero. Then over K, the subgroup of order p points becomes the constant group $G \times Spec(K)$. Then the algebra of functions becomes the algebra of functions on the discrete group G with values in K. Then the element $\frac{[p](x)}{x}$ is p times the characteristic function on the unit, since its multiplication with x is zero and have value p when $x = 0$. Hence over K the Hurewicz image of the norm map is p times the characteristic function on the diagonal of $G \times G$.

Now we look at the case when $G = \Sigma_p$, which we are interested in. In this case, we take the subgroup $H = \mathbb{Z}/p \in G$. Then the Weyl group of H in G is $\mathbb{F}_p^\times$. Recall that $E_n^*(\mathbb{S}_{hG})$ can be identified with $\mathbb{F}_p^\times$-invariant elements in $E_n^*(\mathbb{S}_{hH})$ under the restriction map.

We have shown how to compute the map

$$\mathbb{S} \to \mathbb{S}^{hG} \to \mathbb{S}^{h(G\times G)} \to \mathbb{S}^{h(H\times H)}$$

using the double coset formula. There are two case of $H \cap H^g$ according to whether g is contained in the normalizer of H or not. The first case gives the $p-1$ components $S \to S^{h(H\times H)} \to S^{h(H\times H)}$ indexed by elements g in the Weyl group of H, where the first map is the class of the norm map of H, and the second map has *id* on the first component, and conjugation by g on the second component. The other components are $\frac{(p-1)!-(p-1)}{p}$ times the transfer map $S \to S^{h(H\times H)}$. Hence over K, the Hurewicz image of the norm map is the function on $H \times H$ which has value p at any pair $(P, Q) \in H \times H$ if they are both not the unit and Q is a multiple of P, and value zero otherwise unless $(P, Q) = (0, 0)$ in which case we have the value $p!$.

We can take a basis of the E_n cohomology of $\mathbb{S}_{hG}$ over K to be the characteristic functions of the classes of H under the action of $\mathbb{F}_p^\times$. Call them $f_0, \ldots, f_q$, where f_0 is the characteristic function on $0 \in H$. Then the Hurewicz image of the norm map is the element

$$p!\, f_0 \otimes f_0 + pf_1 \otimes f_1 \otimes f_1 + \cdots + pf_q \otimes f_q$$

Hence the nondegenerate quadratic form on $E_n^*(\mathbb{S}_{hG})$ defined by the norm map has the formula

$$< f, g >= \frac{1}{p!} f(0)g(0) + \frac{1}{p(p-1)} \sum_{P \in H\backslash\{0\}} f(P)g(P)$$

over K. Note that this formula also makes sense over E_n^*.

Since the restriction map $\mathbb{S}_{hG} \to \mathbb{S}$ sends 1 to 1 in cohomology, we arrive at the conclusion that the map $\mathbb{S}^{hG} \leftarrow \mathbb{S}_{hG} \rightarrow \mathbb{S}$ has the effect on homology sending any element $f \in E_n^*(S_{hG})$ to the element

$$< f, 1 >= \frac{1}{p!} f(0) + \frac{1}{p(p-1)} \sum_{P \in H \setminus \{0\}} f(P)$$

Now we look at the case of S^k. In that case, we have a commutative diagram

$$\begin{array}{ccccc} S^k \wedge \mathbb{S}_{h\Sigma_p} & \rightarrow & S^k \wedge \mathbb{S}^{h\Sigma_p} & \leftarrow & \Sigma^\infty S^k \\ \downarrow & & \downarrow & & \downarrow \\ (\Sigma^\infty S^{kp})_{h\Sigma_p} & \rightarrow & (\Sigma^\infty S^{kp})^{h\Sigma_p} & \leftarrow & \Sigma^\infty S^k \end{array} \tag{4.1}$$

Here the vertical maps are induced from the diagonal map on S^k. Note that the rows are the dual of the maps we have just studied, since the dual of the norm map is still the norm map of the dual.

The first vertical map is the quotient $P_0 \rightarrow P_k$. We observe that on E_n-cohomology, this map is the inclusion of the ideal generated by $(\xi^{p-1})^m$ if $k = 2m - 1$ is odd. So we get the same formula on cohmology of the second row under this identification.

As we have observed, for odd spheres, the diagonal map is non-equivariantly homotopic to zero, so the other nasty terms disappear. Hence we prove:

Theorem 4.4.3. *The map on E_n-cohomology induced by the James-Hopf map jh is as follows: Let $x_1, x_2, \ldots, x_{p^{n-1}+\cdots+p+1}$ be the roots (over some extension of E_n^*) of the polynomial $q(x)$, defined by the equation $tq(t^{p-1}) = [p](t)$. Then for any polynomial $f(x)$ in $x(E_n^*[x]/q(x))$, we have $jh^*(f(x))$ equals*

$$\frac{1}{p} \sum_i f(x_i)$$

up to a factor which is a unit.

Chapter 5

The Computation of $\Phi_{K(2)}S^3$

In this chapter, we will compute the homotopy groups of $\Phi_{K(2)}S^3$, for $p \geq 5$. As a corollary, we find that the $K(2)$-local unstable homotopy groups of spheres have finite v_1-exponent.

Recall that for any sphere S^k, $K(2)$-locally we have the finite Goodwillie resolution $L(0)_k \to L(1)_k \to L(2)_k$ for $\Phi_{K(2)}S^k$. Since $K(2)$-locally S^1 is trivial, we can quotient out the Goodwillie tower of S^3 by that of S^1, and conclude that $\Phi_{K(2)}S^3$ is the fiber of $L(1)_1^3 \to L(2)_1^3$, for $L(n)_i^j$ the fiber of the suspension map $L(n)_i \to L(n)_{j+1}$.

5.1 Comodules of Hopf algebroids

We will switch between the notion of comodules and group actions. Their relationship is reviewed in this section.

Let X be a scheme. A groupoid over X is a scheme G, together with flat maps $s, t : G \to X$, and a multiplication $G_t \times_X {}_sG \to X$, satisfying the usual axioms of a groupoid. When X and G are affine this is the dual notion of a Hopf algebroid.

A G-sheaf on X is a quasi-coherent sheaf M on X together with a morphism $s^*M \to t^*M$ of quasi-coherent sheaves on G, satisfying transitivity axioms. This is the same as a comodule on Hopf algebroids in the affine case.

Note that the notion of a G-sheaf is encoded by an isomorphism $s^*M \to t^*M$, which corresponds to an isomorphism $M \otimes_A \Delta \to \Delta \otimes_A M$ when $X = Spec(A)$ and

$G = Spec(\Delta)$. So we can encode this data either as a left comodule for Δ, or a right comodule. In this paper, we will use the convention of right comodules, which is different from [25].

Let $f : Y \to X$ be an étale map. Then a groupoid G on X pulls back to a groupoid $G_Y = Y \times_X G \times_X Y$ on Y. In this case, a G-sheaf on X is the same as a G_Y-sheaf on Y. Note that the descent data is automatically contained in G_Y.

Now suppose we have an algebraic group H acting on a scheme Y. Then we can construct a groupoid $H \times Y$ over Y, with the source and target map being the projection and the action respectively. We would say the groupoid splits in this case. In the split case, the notion of an $H \times Y$ sheaf is the same as an H-equivariant sheaf on Y in the usual sense.

Moreover, if K is a finite subgroup of H acting freely on Y so that $Y \to X = Y/K$ is étale, then we can construct a groupoid $K\backslash H \times Y/K$ over X, which pulls back to $H \times Y$ on Y.

Now let $\widehat{E(n)}$ denote the completed Johnson-Wilson theory, with

$$\widehat{E(n)}_* = \mathbb{Z}_p[[v_1, \dots, v_{n-1}]][v_n^{\pm}]$$

Then

$$\widehat{E(n)}_*\widehat{E(n)} = \widehat{E(n)}_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^{p^i} t_i + \dots)$$

Recall E_n is the Morave E-theory, so that $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm}]$. Then the Morava stabilizer group G_n acts on E_n. Moreover, E_{n*} is an étale extension of $\widehat{E(n)}_*$ with Galois group $\mathbb{F}_{p^n}^\times \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \subset G_n$. The split groupoid constructed using this action is dual to the Hopf algebroid $E_{n*}E_n$, which is also the pullback of $\widehat{E(n)}_*\widehat{E(n)}$.

The correspondence goes as follows. Over E_{n*}, the t_i's can be solved out as a power series $t_i = a_i u^{p^i - 1} + \dots$ such that $a_i \in \mathbb{F}_{p^n} \subset W(\mathbb{F}_{p^n})$ and $a_0 \in \mathbb{F}_{p^n}^\times \subset W(\mathbb{F}_{p^n})$. By choosing different choices of the a_i together with an element in the Galois group of $\mathbb{F}_{p^n}$ we get all the points of G_n. Then the formula for the coaction of an $E_{n*}E_n$ comodule corresponds to the action of G_n by substituting the values of t_i in the formula for coaction.

In summary, the data for an E_n-module with compatible G_n-action is the same as that for an $\widehat{E(n)}_*\widehat{E(n)}$ comodule.

5.2 Some $\widehat{E(2)}_*\widehat{E(2)}$ comodules

In this section we will construct several $\widehat{E(2)}_*\widehat{E(2)}$ comodules.

First we have the E_{2*} algebra $E_2^*\mathbb{CP}^\infty$ with compatible action of G_2. As an E_{2*}-module it is $E_{2*}[t]$. The formula for the action can be found in [4].

Next we have the algebra $E_{2*}[t]/[p](t)$ as $E_2^*B\mathbb{Z}/p$. By taking the fixed points of the $\mathbb{F}_p^\times$ action, we have the algebra $E_{2*}[x]/xq(x)$ as $E_2^*B\Sigma_p$, where $q(x)$ satisfies $[p](t) = tq(t^{p-1})$. Now the Morava stabilizer group G_n acts on these. In particular, $(xq(x))$ is an invariant ideal of G_2. We know (x) is also an invariant ideal of G_2. Since $E_{2*}[x]$ is a UFD, we conclude $(q(x))$ is also an invariant ideal, so that we can form the algebra $E_{2*}[x]/q(x)$ with compatible G_2 action.

Now we mod out by p. Then $[p](t) \equiv v_1t^p +_F v_2t^{p^2} \mod p$. So there is some $\bar{q}(x)$ such that $[p](t) \equiv t^p\bar{q}(t^{p-1}) \mod p$. So the same argument shows that we can construct an algebra $E_{2*}/p[x]/\bar{q}(x)$ with G_2 action.

When we assume p to be odd, we have $[-1](t) = -t$, so the ideal $(\bar{q}(x))$ is the same as $(v_1 + v_2x^p)$. So we conclude the algebra $E_{2*}/p[x]/\bar{q}(x)$ is the same as $E_{2*}/p[(\frac{v_2}{v_1})^{\frac{1}{p}}]$.

The last algebra is over $\mathbb{F}_p$, so adding a pth root is purely inseparable, so the action of G_2 extends uniquely. To get the formula for the action, suppose $(t_1, t_2, \dots)$ is a certain set of solutions of the equation for $E_{2*}E_2$ representing an element of $S_2 \subset G_2$. Then it acts trivially on $v_1 \mod p$, and sends v_2 to $\eta_R(v_2) = v_2 + v_1t_1^p - v_1^pt_1$. Hence this element sends $v_2^{\frac{1}{p}}$ to $v_2^{\frac{1}{p}} + v_1^{\frac{1}{p}}t_1 - v_1t_1^{\frac{1}{p}}$. Here $t_1^{\frac{1}{p}}$ is literally the pth root of its value. By the equation $v_2t_1^{p^2} - v_2^pt_1 + \dots$ we can transform $t_1^{\frac{1}{p}}$ into an expression with only integral powers of t_1. This in turn gives the formula for the coaction of $\widehat{E(2)}_*\widehat{E(2)}$.

5.3 Homological computations

Let $R = E_{2*}[y]/q(y)$ with $q(x^{p-1}) = \frac{[p](x)}{x}$ with the action of G_2 defined in the last section. Then $q(y)$ is an irreducible polynomial, and R is a E_{2*}-module, free of rank $p+1$. We have the trace map $tr : R \to E_{2*}$ for the extension $E_{2*} \to R$. We find that $tr(a)$ is divisible by p if $a \in yR$.

Recall that $E_2^*L(1)_{2k-1} = y^k R$. By 4.4.3, the Goodwillie differential on cohomology $E_2^*L(0) \leftarrow E_2^*L(1)$ is the trace map $\frac{tr(-)}{p}$.

From the Whitehead conjecture, proved by [21], and the triviality of $L_{K(2)}L(n)$ for $n \geq 3$, we know $L_{K(2)}\mathbb{S} \to L_{K(2)}L(1)_1 \to L_{K(2)}L(2)_1$ is a split fiber sequence. So we can identify $E_2^*L(2)_1$ with the kernel of $\frac{tr(-)}{p}$.

The following is a consequence of discussions in Chapter 3:

Lemma 5.3.1. *$K(2)$-locally, the unstable filtration on $L(2)_1$ is the same as the filtration induced by powers of p.*

Because every term involved is free on E_{2*}, we can mod out by p, and conclude that $E_2^*L(2)_1^2$ is the kernel of the mod p reduction of $\frac{tr(-)}{p}$.

So $E_2^*\Phi_{K(2)}S^3$ is the intersection of the kernel of the map $\frac{tr(-)}{p} : (yR)/p \to E_{2*}/p$ and the ideal in R/p generated by y^2.

Observe that $\frac{tr(y^s)}{p}$ lies in (p, v_1) for $1 \leq s \leq p$, and $\frac{tr(y^{p+1})}{p}$ is a unit. this shows that $E_2^*\Phi_{K(2)}S^3$ is the module generated by y^s for $s \neq 1 \mod p$.

To understand the homology, we will use the self duality of projective spaces. Observe that the pairing $< a, b >= \frac{tr(ab)}{p}$ defines a perfect pairing between $y^{-k}R$ and $y^{k+1}R$. Modulo p, we find that we have a perfect pairing between $y^{-k+1}\bar{R}$ and $y^{k+1}\bar{R}$. Thus the E_2-homology of $\Phi_{K(2)}S^3$ can be identified with the quotient module of $\bar{R}$ by mod out y^s for s divisible by p.

As a final remark, since $\bar{R}$ is an inseparable extension of E_{2*}, there is a unique extension of the action of the Morava stabilizer group, and all the map above are compatible with the action. Thus from the discussion of the previous sections, the comodule structure of $\widehat{E(2)}_*\Phi_{K(2)}S^3$ has the kind of formula described in the end of the last section.

5.4 Computations of the AHSS differentials

In this section, we will implicitly mod out by p everywhere.

There is a natural filtration on the homology of $\Phi_{K(2)}S^3$ defined by powers of y. To simplify the notations, we will alter the sign of y in this section, so we set $y = (\frac{v_1}{v_2})^{\frac{1}{p}}$. So there is an AHSS to compute its homotopy groups. As a $\mathbb{F}_2$-vector space, $E_{2*}\Phi_{K(2)}S^3$ has generators $(y^k)_s = v_2^s y^k$ for $k \geq 1$ and relations $y^{kp} = 0$.

Since everything is killed by p, the element ζ in the cohomology of the Morava stabilizer group is a permanent cycle, and we will ignore this factor. So we are to compute $H^*(\mathbb{G}_2^1, E_{2*}\Phi_{K(2)}S^3)$.

Recall that $H^*(\mathbb{G}_2^1, \mathbb{F}_2)$ has a basis $1, h_0, h_1, g_0, g_1, h_0g_1 = h_1g_0$. We will do the computation with $\widehat{E(2)}_*\widehat{E(2)}$-comodules. The elements in $Ext_{\widehat{E(2)}_*\widehat{E(2)}}(\widehat{E(2)}_*, M)$ for any comodule M have representatives in the cobar complex. For $H^*(\mathbb{G}_2^1, \mathbb{F}_2) = Ext_{\widehat{E(2)}_*\widehat{E(2)}}(\widehat{E(2)}_*, \mathbb{F}_2)/\zeta$, the representatives are

$$h_0 = [t_1]$$

$$h_1 = [t_1^p]$$

$$g_0 =< h_0, h_0, h_1 >= \frac{1}{2}[t_1^2|t_1^p] + [t_1|t_2]$$

$$g_1 =< h_0, h_1, h_1 >= [t_2|t_1^p] + \frac{1}{2}[t_1|t_1^{2p}]$$

So the E_1 term of the AHSS are the $(y^k)_s$ multiples of these generators. We will compute the differentials.

Lemma 5.4.1. $\eta_R(v_2^{\frac{1}{p}}) = v_2^{\frac{1}{p}} + v_1^{\frac{1}{p}}t_1 - v_1 t_1^{\frac{1}{p}}$.

Proof. This follows from the formula $\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1$. □

To make $t_1^{\frac{1}{p}}$ into an integral expression, note that modulo v_1, $t_i = t_i^{p^2}$, hence we can inductively transform the expression into one without fraction exponent on t_i's.

To trace the effects, we have the following formula:

Lemma 5.4.2. $\eta_R(v_3) = v_3 - v_2^p t_1 + v_2 t_1^{p^2} + v_1 t_2^p + v_1 w_1(v_2, -v_1^p t_1, v_1 t_1^p) - v_1^p t_1^{p^2+1} - v_1^{p^2} t_2 + v_1^{p^2} t_1^{1+p}$, *where*

$$w_1(a, b, c) = -\frac{1}{p}((a+b+c)^p - (a^p + b^p + c^p))$$

Proof. See [25] Chapter 4. □

So we will use the relations $\eta_R(v_3) = 0$ and etc. to transform $t_1^{\frac{1}{p}}$ into an expression with integral powers in t_i's.

Lemma 5.4.3. *For* $1 \le k \le p-1$,

$$\eta_R((y^k)_{1+s}) = v_1^{\frac{k}{p}} (v_2^{\frac{1}{p}} + v_1^{\frac{1}{p}} t_1 - v_1 t_1^{\frac{1}{p}})^{p-k} (v_2 + v_1 t_1^p - v_1^p t_1)^s$$

Proof. This follows from $(y^k)_{1+s} = (v_1^{\frac{k}{p}} v_2^{\frac{p-k}{p}}) v_2^s$. □

In the following, we will describe the AHSS defferentials, denoted by d.

Lemma 5.4.4. *For* $1 \le k \le p-2$,

$$d(y^k)_{1+s} = (p-k) h_0 (y^{k+1})_{1+s}$$

$$d g_1 (y^k)_{1+s} = (p-k) g_1 h_0 (y^{k+1})_{1+s}$$

Proof. This follows from the previous lemma by collecting the leading terms. □

Lemma 5.4.5. *For* $1 \le k \le p-3$,

$$d h_1 (y^k)_{1+s} = -(p-k)(p-k-1) g_0 (y^{k+2})_{1+s}$$

Proof. We have the leading terms:

$$\begin{aligned} d(v_2^s v_1^{\frac{k}{p}} v_2^{\frac{p-k}{p}} [t_1^p]) &= -v_2^s v_1^{\frac{k}{p}} ((p-k) v_2^{\frac{p-k-1}{p}} v_1^{\frac{1}{p}} [t_1 | t_1^p] + \binom{p-k}{2} v_2^{\frac{p-k-2}{p}} v_1^{\frac{2}{p}} [t_1^2 | t_1^p]) \\ &= -(p-k) v_2^s v_1^{\frac{k+1}{p}} v_2^{\frac{p-k-1}{p}} [t_1 | t_1^p] - \binom{p-k}{2} v_2^s v_1^{\frac{k+2}{p}} v_2^{\frac{p-k-2}{p}} [t_1^2 | t_1^p] \end{aligned}$$

We also have

$$d(v_2^s v_1^{\frac{k+1}{p}} v_2^{\frac{p-k-1}{p}}[t_2]) = v_2^s v_1^{\frac{k+1}{p}} v_2^{\frac{p-k-1}{p}}[t_1|t_1^p] - (p-k-1)v_2^s v_1^{\frac{k+2}{p}} v_2^{\frac{p-k-2}{p}}[t_1|t_2]$$

After killing the leading terms, we have the following differential:

$$\begin{aligned} d(v_2^s v_1^{\frac{k}{p}} v_2^{\frac{p-k}{p}}[t_1^p] + (p-k)v_2^s v_1^{\frac{k+1}{p}} v_2^{\frac{p-k-1}{p}}[t_2]) \\ = -(p-k)(p-k-1)v_2^s v_1^{\frac{k+2}{p}} v_2^{\frac{p-k-2}{p}}([t_1|t_2] + \frac{1}{2}[t_1^2|t_1^p]) \\ = -(p-k)(p-k-1)v_2^s v_1^{\frac{k+2}{p}} v_2^{\frac{p-k-2}{p}} g_0 \end{aligned}$$

□

Now we will study the long differentials.

Lemma 5.4.6. *If $s-1$ is not divisible by p, then*

$$d(y^{p-1})_{1+s} = (s-1)h_1(y^{2p-1})_{1+s}$$

If $s-1$ is divisible by p, then

$$d(y^{p-1})_{1+s} = h_0(y^{2p+1})_{2+s}$$

Proof. Up to order $v_1^{\frac{2p+2}{p}}$, we have

$$\begin{aligned} d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}}) = -v_2^s v_1^{\frac{2p-1}{p}} t_1^{\frac{1}{p}} + s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p \\ = -v_2^{s-1} v_1^{\frac{2p-1}{p}} (v_2^{\frac{1}{p}} t_1^p - v_1^{\frac{2}{p}} v_2^{\frac{p-1}{p}} t_1 - \frac{p-1}{2} v_1^{\frac{3}{p}} v_2^{\frac{p-2}{p}} t_1^2) + s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p \\ = (s-1)v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p + v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_1 + \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} t_1^2) \end{aligned}$$

□

Lemma 5.4.7. *For s not divisible by p, we have $dh_0(y)_{1+s} = sg_0(y^{p+2})_{1+s}$.*

Proof. From the previous lemma, we know that

$$d(v_1^{\frac{p-1}{p}} v_2^{\frac{p+1}{p}}) = v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_1 - \frac{1}{2} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{2}} t_1^2 + \cdots =: v_1^2 \eta$$

is a boundary, so in particular a cycle. Hence up to order $v_1^{\frac{p+2}{p}}$, we have

$$d(v_2^s \eta) = -s v_2^{s-1} (v_1^{\frac{p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^p | t_1] - \frac{1}{2} v_1^{\frac{p+2}{p}} v_2^{\frac{p-2}{p}} [t_1^p | t_1^2])$$

And

$$d(v_2^s \eta + s v_2^{s-1} v_1^{\frac{p+1}{p}} v_2^{\frac{p-1}{p}} (-t_2 + t_1^{p+1})) = s v_2^{s-1} v_1^{\frac{p+2}{p}} v_2^{\frac{p-2}{p}} g_0$$

$\square$

Lemma 5.4.8. *If s is divisible by p, then $dh_1(y^{p-1})_{1+s} = g_0(y^{2p+2})_{2+s}$.*

Proof. We have

$$d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_1^p) = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p | t_1^p] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1 | t_1^p] - \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} [t_1^2 | t_1^p])$$

So

$$d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_1^p - \frac{1}{2} v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^{2p} + v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_2) = v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} g_0$$

$\square$

Lemma 5.4.9. *If $s+2$ is not divisible by p, we have*

$$dh_1(y^{p-2})_{1+s} = 2(s+2) g_1(y^{2p-1})_{1+s}$$

If $s+2$ is divisible by p, we have

$$dh_1(y^{p-2})_{1+s} = -2 g_0(y^{2p+1})_{2+s}$$

Proof. Up to order $v_1^{\frac{2p+1}{p}}$, we have

$$\begin{aligned} d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p) = & -2v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} [t_1|t_1^p] + 2v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}}|t_1^p] + 2v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}}|t_1^p] \\ & -sv_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p|t_1^p] - 2sv_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1}|t_1^p] \end{aligned}$$

Following Chapter 4 of [25], we have

$$dt_2 = [t_1|t_1^p] + v_1 T$$

and

$$dt_3^p = v_2^{p^2-1}[t_1^p|t_2] + v_2^{p-1}[t_2^p|t_1^p] + v_2^{p^2} T \mod v_1$$

where $T = w_1([1|t_1], [t_1|1])$.

So we have

$$d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_2) = v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} [t_1|t_1^p] + v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}}|t_2] - sv_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p|t_2] + v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} T$$

Then

$$\begin{aligned} d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p + 2v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_2) = & \ 2v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}}|t_1^p] + 2v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}}|t_1^p] \\ & +2v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}}|t_2] + 2v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} T - sv_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p|t_1^p] \\ & -2sv_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1}|t_1^p] - 2sv_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p|t_2] \end{aligned}$$

Also

$$v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}}|t_1^p] = v_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p|t_1^p] + v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_2|t_1^p] - \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^2|t_1^p]$$

$$v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}}|t_1^p] = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1}|t_1^p] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^2|t_1^p]$$

$$v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}}|t_2] = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p|t_2] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1|t_2]$$

We also have

$$d(v_2^{s-1}v_1^{\frac{2p-2}{p}}v_2^{\frac{2}{p}}t_1^{2p}) = -2v_2^{s-1}v_1^{\frac{2p-1}{p}}v_2^{\frac{1}{p}}[t_1|t_1^{2p}] + 2v_2^{s-1}v_1^{\frac{2p-2}{p}}v_2^{\frac{2}{p}}[t_1^p|t_1^p]$$

So

$$\begin{aligned} d(v_2^s v_1^{\frac{p-2}{p}}v_2^{\frac{2}{p}}t_1^p + 2v_2^s v_1^{\frac{p-1}{p}}v_2^{\frac{1}{p}}t_2 + (\frac{s}{2}-1)v_2^{s-1}v_1^{\frac{2p-2}{p}}v_2^{\frac{2}{p}}t_1^{2p}) = \\ v_2^{s-1}v_1^{\frac{2p-1}{p}}v_2^{\frac{1}{p}}(2([t_1|t_1^{2p}] + [t_2|t_1^p] + [t_1^{p+1}|t_1^p] + [t_1^p|t_2] + v_2 T) - s([t_1|t_1^{2p}] + 2[t_1^{p+1}|t_1^p] + 2[t_1^p|t_2])) \\ -v_2^{s-1}v_1^{\frac{2p+1}{p}}v_2^{\frac{p-1}{p}}([t_1^2|t_1^p] + 2[t_1|t_2]) \end{aligned}$$

Observe that in the E_2 term we have, modulo v_1,

$$2[t_2|t_1^p] + [t_1|t_1^{2p}] = 2 < h_0, h_1, h_1 >= 2g_1$$

$$[t_1^p|t_2] - [t_2|t_1^p] + [t_1^{p+1}|t_1^p] =< h_1, h_0, h_1 >= -2g_1$$

and T is homologous to $-v_2^{-1} < h_1, h_0, h_1 >= 2v_2^{-1}g_1$.

Collecting the terms, we find that, when $s+2$ is not divisible by p, we have

$$d(v_2^s v_1^{\frac{p-2}{p}}v_2^{\frac{2}{p}}t_1^p + \dots) = 2(s+2)v_2^{s-1}v_1^{\frac{2p-1}{p}}v_2^{\frac{1}{p}}g_1$$

And when $s+2$ is divisible by p, we have

$$d(v_2^s v_1^{\frac{p-2}{p}}v_2^{\frac{2}{p}}t_1^p + \dots) = -2v_2^{s-1}v_1^{\frac{2p+1}{p}}v_2^{\frac{p-1}{p}}g_0$$

□

Lemma 5.4.10. *If s is not divisible by p, then $dg_0(y)_{-2+s} = sg_0h_1(y^{p+1})_{-2+s}$.*

Proof. From the previous lemma, we know $g_0(y)_{-2}$ is a permanent cycle, so

$$dg_0(y)_{-2+s} = sg_0h_1v_1v_2^{-1}(y)_{-2+s}$$

□

Lemma 5.4.11. *If $s+3$ is divisible by p, then $dg_1(y^{p-1})_{1+s} = h_0 g_1(y^{2p+1})_{2+s}$.*

Proof. We have

$$d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} g_1) = -v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}}|g_1] + s v_2^{s-1} v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} [t_1^p|g_1] - v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [T|t_1^p]$$

We know that up to order v_1, T is homologous to $2v_2^{-1}g_1$ and $h_1 g_1$ is homologous to 0. Also up to order $v_1^{\frac{2p+1}{p}}$, we have

$$v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}}|g_1] = v_2^{s-1} v_1^{\frac{2p-1}{p}} [t_1^p|g_1] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1|g_1]$$

The lemma follows. □

Now we have computed all the differential, so we have:

Theorem 5.4.12. *$H^*(\mathbb{G}_2^1, E_{2*}\Phi_{K(2)}S^3)$ is a vector space over $\mathbb{F}_p$, with a set of basis $h_0(y)_{1+pt}$, $v_1 h_0(y)_{1+pt}$, $h_1(y^{p-1})_{1+s}$, $g_0(y)_{-2+pt}$, $v_1 g_0(y)_{-2+pt}$, $g_0(y^2)_s$, $g_0(y^2)_{pt}$, $v_1 g_0(y^2)_{pt}$, $g_1(y^{p-1})_{-2+s}$, $g_0 h_1(y)_{-3+s}$, $g_0 h_1(y)_{-3+pt}$, $v_1 g_0 h_1(y)_{-3+pt}$. Here s runs over integers not divisible by p, and t runs over all integers.*

One can see that there are no differentials in ANSS, so the homotopy groups $\pi_*\Phi_{K(2)}S^3$ is a free module over $\mathbb{F}_p[\zeta]/\zeta^2$ with the above generators.

Remark 5.4.13. *We have the boundary map from $\Phi_{K(2)}S^3$ to $L_{K(2)}M(p)$. On E_2-term of ANSS, this is the boundary map coming from the exact sequence*

$$E_{2*}/p \xrightarrow{v_1} y\bar{R} \to E_{2*}\Phi_{K(2)}S^3$$

One can show this map, after projecting to the top cell in $M(p)$, is essentially the stabilization map $\Omega^3 S^3 \to \Omega^\infty \Sigma^\infty S^0$.

5.5 v_1-exponent of unstable spheres

We find that, from the computations in the last section, the homotopy groups of $\Phi_{K(2)}S^3$ is killed by v_1^2. This implies that all the $K(2)$-local unstable homotopy groups of spheres have finite v_1-exponent.

Definition 5.5.1. *We say that a spectrum X have finite v_n-exponent, if there exists a fixed type $n+1$ complex V_n, such that any map $S^k \to X$ can be lifted to a map $\Sigma^k V_n \to X$.*

The following lemma is straightforward:

Lemma 5.5.2. *A spectrum X has finite v_n-exponent, if and only if the following holds:*

1. *X has finite v_{n-1}-exponent. Let V_{n-1} be a choice of type n complex admitting liftings. Choose a v_n-self map v_n^k on V_{n-1}.*

2. *There exists a number N, such that for any map $f : V_{n-1} \to X$, the composition $f \circ (v_2^k)^N = 0$.*

Obviously, any complex of type $n+1$ has finite v_n-exponent. We also note that the class of spectra with finite v_n-exponent is closed under taking fibers.

We will show that for all $k \geq 1$, $\Phi_{K(2)} S^k$ has finite v_1-exponent. It is enough to treat the odd sphere case.

Theorem 5.5.3. *$\Phi_{K(2)} S^{2k+1}$ has finite v_1-exponent at prime $p \geq 5$.*

Proof. We will show this with induction. The case for S^1 is trivial, and the case for S^3 is already proved. Now let $W(k)$ be the fiber of $S^{2k+1} \to S^{2k+3}$. Then using the secondary suspension, one finds that the fiber of the secondary suspension $\Phi_{K(2)} W(k) \to \Phi_{K(2)} W(k+1)$ is $K(2)$-locally equivalent to a type 2 complex. So we prove inductively all the $W(k)$ has finite v_1-exponent, and the theorem follows with another induction. □

Remark 5.5.4. *We can actually show that the v_1-exponent in the E_2-term of ANSS is bounded by $\frac{k(k+3)}{2}$ on S^{2k+1}.*

Remark 5.5.5. *Using the tmf resolution, one can also prove the $p=3$ case of the theorem.*

This theorem leads to the following conjecture, generalizing the Cohen-Moore-Neisendorfer theorem for the p-exponent of unstable spheres:

Conjecture 5.5.6. *The v_n-torsion of unstable groups of spheres have finite v_n-exponent for every n.*

Remark 5.5.7. *The $K(2)$-local computations suggest the v_1-exponent of S^3 might be 2 at primes $p \geq 5$. This exponent was also suggested in [26] from different reasons.*

Bibliography

[1] G. Z. Arone and W. G. Dwyer. Partition complexes, Tits buildings and symmetric products. *Proc. London Math. Soc. (3)*, 82(1):229–256, 2001.

[2] Greg Arone and Mark Mahowald. The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres. *Invent. Math.*, 135(3):743–788, 1999.

[3] Mark Behrens. Table of 5-primary EHP sequence and AHSS of the Goodwillie tower.

[4] Mark Behrens. Root invariants in the Adams spectral sequence. *Trans. Amer. Math. Soc.*, 358(10):4279–4341, 2006.

[5] Mark Behrens. The Goodwillie tower for S^1 and Kuhn's theorem. *Algebr. Geom. Topol.*, 11(4):2453–2475, 2011.

[6] Mark Behrens. The Goodwillie tower and the EHP sequence. *Mem. Amer. Math. Soc.*, 218(1026):xii+90, 2012.

[7] Mark Behrens and Charles Rezk. The Bousfield-Kuhn functor and topological André-Quillen cohomology. 2012.

[8] A. K. Bousfield. On the telescopic homotopy theory of spaces. *Trans. Amer. Math. Soc.*, 353(6):2391–2426 (electronic), 2001.

[9] Thomas G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711 (electronic), 2003.

[10] Brayton Gray. Unstable families related to the image of J. *Math. Proc. Cambridge Philos. Soc.*, 96(1):95–113, 1984.

[11] Brayton Gray. The periodic lambda algebra. In *Stable and unstable homotopy (Toronto, ON, 1996)*, volume 19 of *Fields Inst. Commun.*, pages 93–101. Amer. Math. Soc., Providence, RI, 1998.

[12] John Harper. Table of 5-primary EHP sequence.

[13] Michael J. Hopkins, Nicholas J. Kuhn, and Douglas C. Ravenel. Generalized group characters and complex oriented cohomology theories. *J. Amer. Math. Soc.*, 13(3):553–594 (electronic), 2000.

[14] Mark Hovey and Neil P. Strickland. Morava K-theories and localisation. *Mem. Amer. Math. Soc.*, 139(666):viii+100, 1999.

[15] Masateru Inoue. On the Hurewicz image of the Steinberg summand $M(n)$. *Topology Appl.*, 157(15):2362–2372, 2010.

[16] Nicholas J. Kuhn. Chevalley group theory and the transfer in the homology of symmetric groups. *Topology*, 24(3):247–264, 1985.

[17] Nicholas J. Kuhn. Morava K-theories and infinite loop spaces. In *Algebraic topology (Arcata, CA, 1986)*, volume 1370 of *Lecture Notes in Math.*, pages 243–257. Springer, Berlin, 1989.

[18] Nicholas J. Kuhn. Stable splittings and the diagonal. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 169–181. Amer. Math. Soc., Providence, RI, 2001.

[19] Nicholas J. Kuhn. Localization of André-Quillen-Goodwillie towers, and the periodic homology of infinite loopspaces. *Adv. Math.*, 201(2):318–378, 2006.

[20] Nicholas J Kuhn. The Whitehead conjecture, the tower of S^1 conjecture, and Hecke algebras of type A. *Journal of Topology*, page jtu019, 2014.

[21] Nicholas J. Kuhn and Stewart B. Priddy. The transfer and Whitehead's conjecture. *Math. Proc. Cambridge Philos. Soc.*, 98(3):459–480, 1985.

[22] Mark Mahowald. The image of J in the EHP sequence. *Ann. of Math. (2)*, 116(1):65–112, 1982.

[23] Haynes R. Miller. On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space. *J. Pure Appl. Algebra*, 20(3):287–312, 1981.

[24] Minoru Nakaoka. Cohomology mod p of symmetric products of spheres. *J. Inst. Polytech. Osaka City Univ. Ser. A*, 9:1–18, 1958.

[25] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press, Inc., Orlando, FL, 1986.

[26] Dai Tamaki. On a space realizing the v_1-torsion part of the mod p homotopy groups of S^3. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 229–267. Amer. Math. Soc., Providence, RI, 1994.

[27] Hirosi Toda. *Composition methods in homotopy groups of spheres*. Annals of Mathematics Studies, No. 49. Princeton University Press, Princeton, N.J., 1962.

[28] Hirosi Toda. Unstable 3-primary unstable homotopy groups of spheres. *Faculty of Econoinformatics*, 2003.