Algebraic topology and arithmetic

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Southern University of Science and Technology

Fudan-Guanghua International Forum for Young Scholars

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

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Cartan formula $\mathrm{Sq}^i(xy) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k}(x)\,\mathrm{Sq}^k(y)$

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Example (complex K-theory)

Adams operations $\psi^i\colon\ K(-) \to K(-)$

Power operation $\psi^p(x) \equiv x^p \mod p$

$\psi^i\psi^j = \psi^{ij}$ $\qquad\qquad$ $\psi^i(xy) = \psi^i(x)\psi^i(y)$

J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962)

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Example (more – a sample)

Voevodsky, *Reduced power operations in motivic cohomology*, 2003.

Lipshitz and Sarkar, *A Steenrod square on Khovanov homology*, 2014.

Feng, *Étale Steenrod operations and the Artin–Tate pairing*, 2018.

Seidel, *Formal groups and quantum cohomology*, 2019.

A connection between Topology and Arithmetic (Quillen '69)

stable homotopy theory $\leftrightsquigarrow$ 1-dim formal group laws

complex-oriented $h^*(-)$ $\qquad$ $F(x,y)$ over $h^*(\mathrm{pt})$

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big)$$

Example

$$H^*(-;\mathbb{Z}) \leftrightsquigarrow \mathbb{G}_a(x,y) = x+y$$

$$K^*(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x+y-xy = 1-(1-x)(1-y)$$

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Background: Chromatic Homotopy Theory 色展同伦论

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Elliptic cohomology and Morava E-theory

Definition (Ando–Hopkins–Strickland '01, Lurie '09, '18)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} E, & C_{E^0(\mathrm{pt})}, \\ \alpha\colon & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \xrightarrow{\sim} \widehat{C} \end{array} \right\}$$

Theorem (Morava '78, Goerss–Hopkins–Miller '90s–'04)

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Power operations for Morava E-theory

$M = E$-module $\qquad \pi_0 M = [S, M]_S \cong [E, M]_E$

$$\mathbb{P}_E(M) = \bigvee_{i \geq 0} \mathbb{P}_E^i(M) = \bigvee_{i \geq 0} (\underbrace{M \wedge_E \cdots \wedge_E M}_{i\text{-fold}})_{h\Sigma_i}$$

$A =$ commutative E-algebra
$\quad =$ algebra for the monad $\mathbb{P}_E$ with $\mu \colon \mathbb{P}_E(A) \to A$

$$\left.\begin{array}{l} \text{total power operation } \psi^i \colon \pi_0 A \to \pi_0\left(A^{B\Sigma_i^+}\right) \\ \forall \eta \in \pi_0 \mathbb{P}_E^i(E), \text{individual po } Q_\eta \colon \pi_0 A \to \pi_0 A \end{array}\right\} \overset{/I}{\rightsquigarrow} \text{additive}$$

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Theorem (Rezk '09, Barthel–Frankland '13)

If $A = K(n)$-local commutative E-algebra, then

$$\pi_* A = \text{graded amplified } L\text{-complete } \Gamma\text{-ring}$$

- $\Gamma =$ twisted bialgebra over E_0 (Dyer–Lashof algebra)
- $\exists\ Q_0 \in \Gamma$ with $Q_0(x) \equiv x^p \mod p$ (Frobenius congruence)

Goal make this structure explicit just as for Dyer–Lashof/Steenrod operations in ordinary homology.
The case of $n = 2$ has been worked out. $\leftsquigarrow$ Arithmetic

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Theorem (Z. '19)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i\colon E^0(-) \to E^0(-)$, $0 \le i \le p$, and quadratic relations

$$Q_i Q_0 = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $1 \le i \le p$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

Remark The first example, for $p = 2$, was calculated by Rezk '08. These have been applied to computations in unstable v_2-periodic homotopy theory (Z. '18 and ongoing joint work with G. Wang).

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Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

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An equivalence of cats (Ando–Hopkins–Strickland '04, Rezk '09)

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Goal Compute one side explicitly to get the other side.

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Frobenius isogenies   power operations

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Frobenius isogenies | power operations

An equivalence of cats (Ando–Hopkins–Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{the Dyer–Lashof algebra} \\ \text{for } E \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Recall E-theory at height n and prime p has an underlying model

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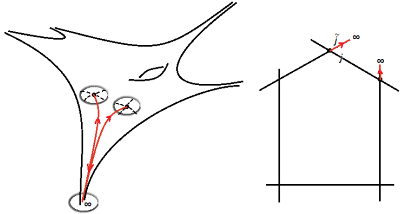

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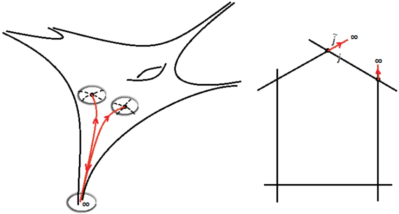

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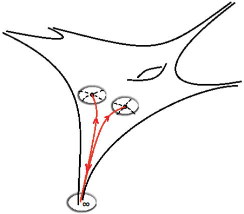

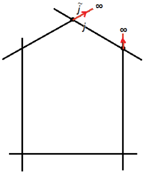

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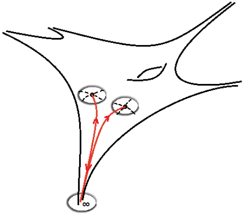

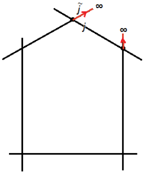

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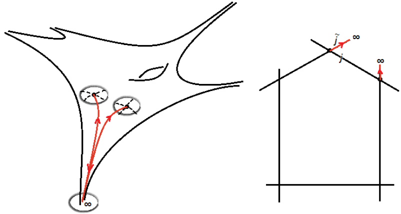

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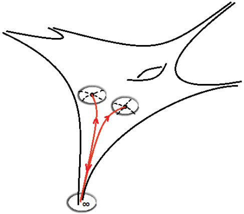

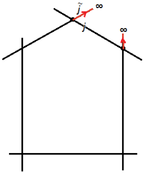

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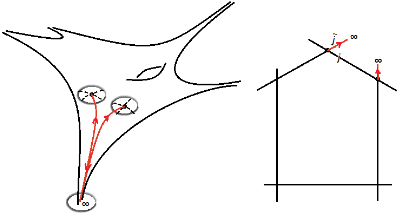

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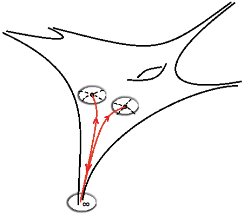

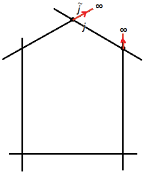

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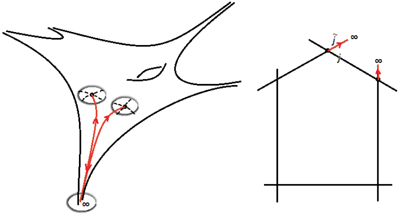

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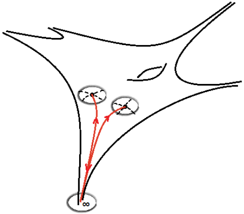

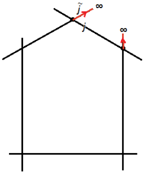

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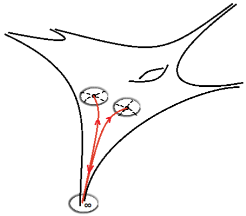

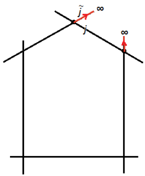

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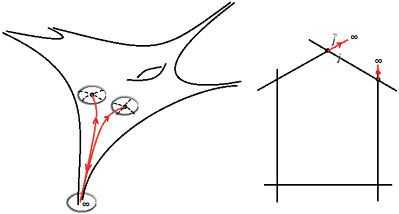

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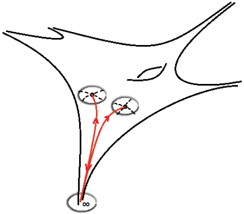

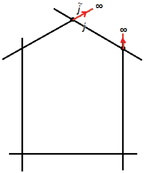

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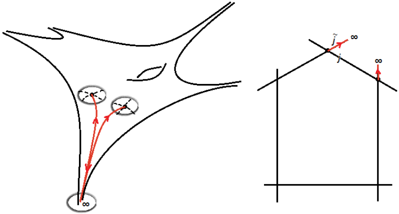

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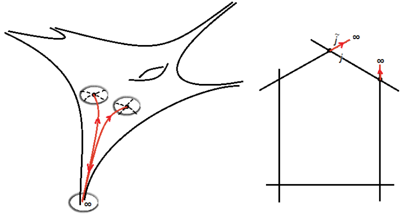

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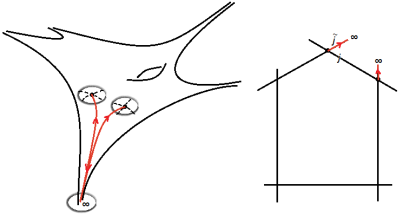

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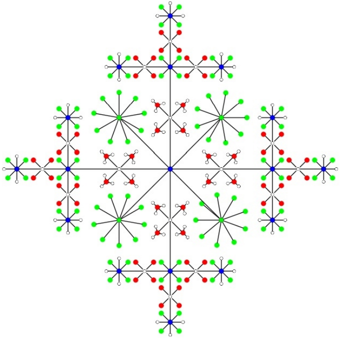

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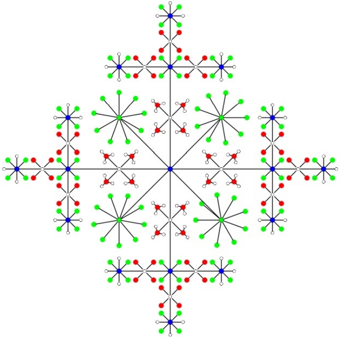

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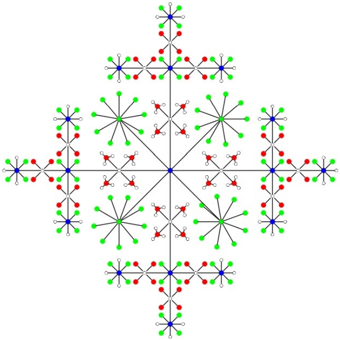

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