THE BASIC GEOMETRY OF WITT VECTORS, II
SPACES

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Abstract. This is an account of the algebraic geometry of Witt vectors and arithmetic jet spaces. The usual, "p-typical" Witt vectors of p-adic schemes of finite type are already reasonably well understood. The main point here is to generalize this theory in two ways. We allow not just p-typical Witt vectors but those taken with respect to any set of primes in any ring of integers in any global field, for example. This includes the "big" Witt vectors. We also allow not just p-adic schemes of finite type but arbitrary algebraic spaces over the ring of integers in the global field. We give similar generalizations of Buium's formal arithmetic jet functor, which is dual to the Witt functor. We also give concrete geometric descriptions of Witt spaces and arithmetic jet spaces and investigate whether a number of standard geometric properties are preserved by these functors.

Introduction

Let p be a prime number. For any integer $n \geqslant 0$ and any (commutative) ring A, let $W_n(A)$ denote the ring of p-typical Witt vectors of length n with entries in A. This construction gives a functor W_n from the category of rings to itself. It is an important tool in number theory, especially in the cohomology of varieties over p-adic fields. For example, it is used in the definition of Fontaine's period rings [18] and in the definition of the de Rham–Witt complex, which is an explicit complex that computes crystalline cohomology [31].

The functor W_n has a left adjoint, which we denote by $A \mapsto \Lambda_n \odot A$:

$$\mathrm{Hom}(\Lambda_n \odot A, B) \cong \mathrm{Hom}(A, W_n(B)). \tag{0.0.1}$$

This adjunction was first considered by Greenberg [20][21], but he restricted himself to the case where B is an $\mathbf{F}_p$-algebra, and so he only constructed the special fiber $\mathbf{F}_p \otimes_{\mathbf{Z}} (\Lambda_n \odot A)$. The construction of the full functor had to wait until Joyal [32] and, independently, Buium [9]. It also has applications in number theory, most notably in the study of p-adic points on varieties. For example, see Buium [9] and Buium–Poonen [11] (as well as Buium's earlier work [8] for applications of analogous constructions in complex algebraic geometry). These two adjoint constructions see different sides of the arithmetic of A—the ring $W_n(A)$ sees certain maps into A, and the ring $\Lambda_n \odot A$ sees certain maps out of it.

This paper is part of a general program to analyze varieties over global fields using global analogues of these functors, such as the "big" Witt functors. The first issue one faces is that, even in the p-typical case above, the schemes $\mathrm{Spec}\, W_n(A)$ and $\mathrm{Spec}\, \Lambda_n \odot A$ are not familiar geometric constructions, and it is important that we be able to handle them with ease. The purpose of this paper is to demonstrate that this is possible. The first part [5] developed the affine theory, and this part extends it to arbitrary schemes and algebraic spaces.

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Let us go over the contents in more detail. We will work throughout with certain generalizations of the classical p-typical and big Witt functors. These are the E-typical Witt functors $W_{R,E,n}$ defined in [5]. These functors depend on a ring R, a set E of finitely presented maximal ideals $\mathfrak{m}$ of R with the property that each localization $R_{\mathfrak{m}}$ is discrete valuation ring with finite residue field, and an element $n \in \mathbf{N}^{(E)} = \bigoplus_E \mathbf{N}$. Then $W_{R,E,n}$ is a functor from the category of R-algebras to itself. We recover the p-typical Witt functor when E consists of the single maximal ideal $p\mathbf{Z}$ of $\mathbf{Z}$, and we recover the big Witt functor when E consists of all the maximal ideals of $\mathbf{Z}$. When E consists of the maximal ideal of the valuation ring of a local field, we recover a variant of the p-typical Witt functor due to Drinfeld [14] and to Hazewinkel [29], (18.6.13). While the general E-typical functors are necessary for future applications, all phenomena in this paper occur already in the p-typical case. This is because for foundational questions, the general methods of [5] usually allow one to reduce matters to the case where E consists of a single principal ideal, and in this case, the classical p-typical functor is a representative example.

As in the p-typical case above, the functor $W_{R,E,n}$ has a left adjoint, which in general we will denote by $A \mapsto \Lambda_{R,E,n} \odot A$. But let us write $\Lambda_n \odot A = \Lambda_{R,E,n} \odot A$ and $W_n = W_{R,E,n}$, for short. The first concern of this paper is to extend both of these functors to the category of algebraic spaces over R, including for example all R-schemes. In fact, as explained in Grothendieck–Verdier (SGA 4, exp. III [1]), there is a general way of doing this—we only need to verify that W_n satisfies certain properties. The method is as follows. Let Aff_S denote the category of affine schemes over $S = \operatorname{Spec} R$. Then W_n induces a functor $\mathsf{Aff}_S \to \mathsf{Aff}_S$, which we also denote by W_n. So we have $W_n(\operatorname{Spec} A) = \operatorname{Spec} W_n(A)$. This functor has two important properties. First, if $U \to X$ and $V \to X$ are étale maps in Aff_S, then the induced map

$$W_n(U \times_X V) \longrightarrow W_n(U) \times_{W_n(X)} W_n(V)$$

is an isomorphism. Second, if $(U_i \to X)_{i \in I}$ is a covering family of étale maps, then so is the induced family $\big(W_n(U_i) \to W_n(X)\big)_{i \in I}$. Both of these are consequences of van der Kallen's theorem for E-typical Witt functors, which says that W_n preserves étale maps of R-algebras ([5], theorem B).

It then follows from general sheaf theory that if X is a sheaf of sets on Aff_S in the étale topology, then so is the functor $W_{n*}(X) = X \circ W_n$, thus giving a functor W_{n*} from the category Sp_S of sheaves of sets on Aff_S to itself. By another general theorem, this functor $W_{n*} \colon \mathsf{Sp}_S \to \mathsf{Sp}_S$ has a left adjoint W_n^* satisfying

$$W_n^*(X) = \operatorname*{colim}_U W_n(U),$$

where U runs over the category of affine schemes equipped with a map to X and where we identify the affine scheme $W_n(U)$ with the object of Sp_S it represents. These functors extend the affine functors W_n and $\Lambda_n \odot -$ to Sp_S:

$$W_n^*(\operatorname{Spec} A) = \operatorname{Spec} W_n(A), \qquad W_{n*}(\operatorname{Spec} A) = \operatorname{Spec} \Lambda_n \odot A. \tag{0.0.2}$$

They are the extensions we will consider. In fact, by the discussion above, they are the unique extensions satisfying certain natural properties.

Theorem A. *If $X \in \mathsf{Sp}_S$ is an algebraic space, then so are $W_n^*(X)$ and $W_{n*}(X)$. If X is a scheme, then so are $W_n^*(X)$ and $W_{n*}(X)$.*

We call $W_n^*(X)$ the E-typical Witt space of X of length n, and we call $W_{n*}(X)$ the E-typical arithmetic jet space of X of length n. In certain cases, they have been constructed before. In their appendix, Langer and Zink [35] constructed the p-typical Witt space of a general $\mathbf{Z}_p$-scheme X. For earlier work see Bloch [4], Lubkin [36], and Illusie [30]. Buium [9] has constructed the p-typical arithmetic jet space of a formal $\mathbf{Z}_p$-scheme, extending Greenberg's construction of the special fiber [20][21]. When

R is $\mathbf{Z}$ and E is arbitrary, Buium and Simanca have constructed the arithmetic jet spaces for affine schemes and have constructed certain approximations to it for general schemes [12] (Defintion 2.16).

For the reader who does not have a mind for abstract sheaf theory, let us reinterpret theorem A in the language of covers. The most obvious way of defining the Witt space of a separated scheme X is to choose an affine open cover $(U_i)_{i\in I}$ of X and to define $W_n^*(X)$ to be the result of gluing the affine schemes $W_n(U_i)$ along the affine schemes $W_n(U_i \times_X U_j)$. It is not hard to check that this gives a scheme which is independent of the cover. (If X is arbitrary, then $U_i \times_X U_j$ is separated, and so we can define $W_n^*(X)$ in general by doing this procedure twice.) This is in Langer–Zink [35] in the p-typical case, and the general E-typical case is no harder. When X is an algebraic space and $(U_i)_{i\in I}$ is an étale cover, we need to know that $\coprod_{i,j} W_n(U_i \times_X U_j)$ is an étale equivalence relation on $\coprod_i W_n(U_i)$. This requires van der Kallen's theorem and a more sophisticated gluing argument, but the principle is the same. Instead the approach of this paper is to define $W_n^*(X)$ as an object of Sp_S and to prove later that it is a scheme or an algebraic space. If one cares about W_n^* only for schemes and algebraic spaces, then the difference is mostly a matter of organization.

This method does not work as well with W_{n*}, because it is rarely the case that the $W_{n*}(U_i)$ cover $W_{n*}(X)$. Indeed, generically over $\operatorname{Spec} R$, the space $W_{n*}(X)$ agrees with a certain cartesian power X^N, and of course one cannot usually construct X^N by gluing the U_i^N together. For p-adic formal schemes in the p-typical case, the generic fiber is empty and this method does actually work, but in general it does not. Instead we must use the total space $U = \coprod_i U_i$ of the cover. We will prove below that $W_{n*}(U \times_X U)$ is an étale equivalence relation on $W_{n*}(U)$, and the quotient is $W_{n*}(X)$. If X is quasi-compact and separated, we can assume U and $U \times_X U$ are affine, and then $W_{n*}(X)$ becomes the quotient of a known affine scheme by a known affine étale equivalence relation. And so we could avoid abstract sheaf theory for such X by taking this to be the definition of $W_{n*}(X)$, although it would still take a small argument to prove that W_{n*} is the right adjoint of W_n^* and that it sends schemes to schemes, rather than just algebraic spaces. It would also take some work to remove the assumption that X is quasi-compact and separated, but of course it could be done. Instead we will define $W_{n*}(X)$ in one stroke as an object of Sp_S and then prove the representability properties later.

Another benefit to working with the whole category Sp_S is that is allows us to make the infinite-length constructions

$$W^*(X) = \operatorname*{colim}_n W_n^*(X), \qquad W_*(X) = \lim_n W_{n*}(X). \tag{0.0.3}$$

These constructions are ind-algebraic spaces (resp. pro-algebraic spaces) but are generally not algebraic spaces. While it would be possible to remain in the category of schemes or algebraic spaces by treating them as inductive systems (resp. projective systems), it is convenient to be able to pass to the limit in Sp_S. We will only consider the finite-length constructions in this paper, but it is in fact the infinite-length ones that are of ultimate interest. Further, we will eventually want to consider iterated constructions, such as $W^*W^*(X)$, and so it is convenient to have $W^*(X)$ defined when X ind-algebraic, and to have $W_*(X)$ defined when X is pro-algebraic. At this point, it becomes easier just to let X be any object of Sp_S.

Preservation of properties by W_n^*

We will spend some time looking at whether common properties of algebraic spaces and maps are preserved by W_n^*. Rather than state the results formally, I have arranged them into tables 1 and 2. (Note that we use normalized indexing

Property of algebraic spaces	Preserved by W_n^*?	Reference or counterexample
quasi-compact	yes	16.1
quasi-separated	yes	16.8
affine	yes	10.7
a scheme	yes	15.6
of Krull dimension d	yes	16.5
separated	yes	16.8
reduced and flat over S	yes	16.5
reduced	no	$W_1(\mathbf{F}_p)$
regular, normal	no	$W_1(\mathbf{Z})$
(locally) noetherian	yes^b	16.6 + 16.5
S_k (Serre's property)	yes^b	16.19
Cohen–Macaulay	yes^b	16.19
Gorenstein	no	$W_2(\mathbf{Z})$
local complete intersection	no	$W_2(\mathbf{Z})$

Table 1. This table indicates whether the given property of algebraic spaces X over S is preserved by W_n^* in general. The superscript b means that X is assumed to be locally of finite type over S and that S is assumed to be noetherian. In the counterexamples, W_n denotes the p-typical Witt vectors over $\mathbf{Z}$ of length n (traditionally denoted W_{n+1}).

throughout. So our p-typical Witt functor W_n is what is traditionally denoted W_{n+1}. The reasons for this are explained in [5], 2.5.)

Several results in the p-typical case are folklore or have appeared elsewhere. See, for example, Bloch [4], Illusie [30], or Langer–Zink [35]. Perhaps the most interesting of them is that while smoothness over S and regularity are essentially never preserved by W_n^*, being Cohen–Macaulay always is. As with the work of Ekedahl and Illusie on p-typical Witt vectors of $\mathbf{F}_p$-schemes [16][17][31], this has implications for Grothendieck duality and de Rham–Witt theory, but we will not consider them here.

Preservation of properties by W_{n*}

Preservation results for W_{n*} are typically easier to establish. This is because many common properties of morphisms are naturally expressed in terms of the functor of points, and the functor of points of $W_{n*}(X)$ is described simply in terms of that of X. For the same reason, many of these results extend readily beyond algebraic spaces to the category Sp_S; this is unlike with W_n^*, where we usually need to make representability assumptions.

A number of the results are displayed in table 3. Because we have $W_{n*}(S) = S$, the preservation of properties relative to S is a special case of the preservation of properties of morphisms. This is unlike the case with W_n^*, where we have the right-hand pair of columns in table 2. I have mostly ignored whether absolute properties, such as regularity, are preserved by W_{n*}. This is because such properties are usually not preserved by products over S, and in that case they would fail to be preserved by W_{n*} for the trivial reason that W_{n*} is a product functor away from the ideals of E. This is like the case with W_n^*: properties that are not preserved by disjoint unions, such as connectedness, are not listed in table 1.

Geometric descriptions

Property P of maps $f\colon X \to Y$ of algebraic spaces	Must $W_n^*(f)$ have property P?		When $Y = S$, must $W_n^*(X) \to S$ have property P?	
étale	yes	15.2	no	$\mathbf{Z}$
an open immersion	yes	15.6	no	$\mathbf{Z}$
quasi-compact	yes	16.11	yes	16.7
quasi-separated	yes	16.11	yes	16.8
affine	yes	16.4	yes	+ 16.10
integral	yes	16.4	yes	+ 16.10
a closed immersion	yes	16.4	no	$\mathbf{Z}$
finite étale	yes	16.4	no	$\mathbf{Z}$
separated	yes	16.11	yes	16.8
surjective	yes	16.11	yes	+ 16.10
universally closed	yes	16.11	yes	+ 16.10
locally of finite type	yes[a]	16.13 + 16.6	yes	16.5
of finite type	yes[a]	16.13 + 16.6	yes	16.7
finite	yes[a]	16.13 + 16.6	yes	16.9
proper	yes[a]	16.13 + 16.14	yes	+ 16.10
flat	no	$\mathbf{Z}[x]$	yes	16.5
faithfully flat	no	$\mathbf{Z}[x]$	yes	16.9
Cohen-Macaulay	no	$\mathbf{Z}[x]$	yes[b]	16.19
S_k (Serre's property)	no	$\mathbf{Z}[x]$	yes[b]	16.19
smooth	no	$\mathbf{Z}[x]$	no	$\mathbf{Z}$
finite flat	no	$\mathbf{Z}[\sqrt{p}]$	yes	16.9

Table 2. This table indicates whether the given property P of morphisms of algebraic spaces over S is preserved by W_n^* in general. The central two columns indicate whether P is preserved by W_n^* and give either a reference to the main text or a counterexample. The right columns indicate whether the structure map $W_n^*(X) \to S$ must satisfy P when the structure map $X \to S$ does. The superscript a means that X and Y are assumed to be locally of finite type over S; and b means that also S is assumed to be noetherian. The counterexamples are for W_1^*, the p-typical Witt functor of length 1, with X the spectrum of the given ring and $Y = \operatorname{Spec} \mathbf{Z}$.

As explained above, both $W_n^*(X)$ and $W_{n*}(X)$ can be described in terms of the case where X is affine by using charts. But under some flatness restrictions on X, it is possible to construct $W_n^*(X)$ and $W_{n*}(X)$ in purely geometric terms without mentioning Witt vectors or arithmetic jet spaces at all. I will give the descriptions here in the p-typical case when $n = 1$; the general case is in the body of the paper.

Let us first consider the Witt space $W_1^*(X)$. Assume that X is flat over $\mathbf{Z}$ locally at p. Let X_0 denote the special fiber $X \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{F}_p$. Then the theorem is that $W_1^*(X)$ is the coequalizer in the category of algebraic spaces of the two maps

$$X_0 \underset{i_2}{\overset{i_1 \circ F}{\rightrightarrows}} X \amalg X,$$

where $i_j\colon X_0 \to X \amalg X$ denotes the canonical closed immersion into the j-th component of $X \amalg X$ and where F is the absolute Frobenius endomorphism of X_0. For general n, the space $W_n^*(X)$ can be constructed by gluing $n+1$ copies of X together in a similar but more complicated way along their fibers modulo $p, \dots, p^n$. See 17.3.

Property of maps of algebraic spaces	Preserved by W_{n*}?	Reference or counterexample
(formally) étale, smooth, unram.	yes	11.1
a monomorphism	yes	11.4
an open immersion	yes	11.1 + 11.4
quasi-compact	yes	11.10
quasi-separated	yes	11.10
epimorphism in Sp_S	yes	11.4
affine	yes	13.3
a closed immersion	yes	13.3
integral, finite	no	$\mathbf{Z} \times \mathbf{Z}$
finite étale, finite flat	no	$\mathbf{Z} \times \mathbf{Z}$
(locally) of finite type/pres.	yes	13.3
separated	yes	13.3
smooth and surjective	yes	13.3
surjective	no	$\mathbf{Z}[\sqrt{p}]$
proper, universally closed	no	$\mathbf{Z} \times \mathbf{Z}$
smooth and proper	no	$\mathbf{Z} \times \mathbf{Z}$
flat	no	$\mathbf{Z}[x]/(x^2 - px)$
faithfully flat	no	$\mathbf{Z}[x]/(x^2 - px)$
Cohen-Macaulay	no	$\mathbf{Z}[x]/(x^2 - px)$
S_k (Serre's property)	no	$\mathbf{Z}[x]/(x^2 - px)$

Table 3. This table indicates whether the given property of maps of algebraic spaces over S is preserved by W_{n*} in general. The counterexamples are for the p-typical jet functor W_{1*} applied to the map $\operatorname{Spec} A \to \operatorname{Spec} \mathbf{Z}$, where A is the given ring. See 13.4.

For the arithmetic jet space $W_{1*}(X)$, we need to assume that X is smooth over $\mathbf{Z}$ locally at p. Let I denote the ideal sheaf on $X \times X$ defining the graph of the Frobenius map on the special fiber X_0, and let $\mathcal{B}$ denote the sub-$\mathcal{O}_{X\times X}$-algebra of $\mathcal{O}_{X\times X}[1/p]$ generated by the subsheaf $p^{-1}I$. Then the theorem is that $W_{1*}(X)$ is naturally isomorphic to the relative spectrum $\operatorname{Spec}(\mathcal{B})$ over $X \times X$. (One might hope that it is also worth studying the full blow up of $X \times X$ along I.) In particular, the map $W_{1*}(X) \to X \times X$ is affine and is an isomorphism outside the fiber over p. For general n, the space $W_{n*}(X)$ can be constructed by taking a similar but more complicated affine modification of X^{n+1}. See 18.3.

Absolute algebraic geometry

Let us end with a few words on how the Witt and jet functors relate to the philosophy of absolute algebraic geometry. The first hope of this philosophy is that there exists a category whose relationship to the category of schemes over $\mathbf{Z}$ is analogous to the relationship of $\mathbf{F}_p$ to $\mathbf{F}_p[t]$. It is sometimes called the category of absolute schemes, or schemes over $\mathbf{F}_1$. The second hope is that this category would suggest ways of transporting results in algebraic geometry over $\mathbf{F}_p(t)$ to $\mathbf{Q}$.

There are a number of proposed definitions of this category. One of the general themes is that an absolute scheme could be defined to be a scheme together with some additional structure, which should be interpreted as descent data from $\mathbf{Z}$ to $\mathbf{F}_1$. One precise proposal for this structure is a so-called Λ-structure [6]. If X is a flat scheme over $\mathbf{Z}$, then a Λ-structure is equivalent to a commuting family of maps $\psi_p\colon X \to X$, where p runs over the prime numbers, such that each ψ_p agrees with the Frobenius map on the fiber of X over p. And if X is affine, then a Λ-structure is equivalent to a (special) λ-ring structure on the corresponding ring, in the sense of Grothendieck's Riemann–Roch theory [22].

From this point of view, the functor that forgets the Λ-structure should be thought of as base change from $\mathbf{F}_1$ to $\mathbf{Z}$. Therefore its left adjoint should be thought of as the base-forgetting functor, and its right adjoint the Weil restriction of scalars. In fact, it is possible to say explicitly what these adjoints are. Let E be the set of all maximal ideals of $\mathbf{Z}$. Then for any space $X \in \mathsf{Sp}_{\mathbf{Z}}$, the infinite-length Witt and jet spaces $W^*(X)$ and $W_*(X)$ of (0.0.3) carry natural Λ-structures, and hence give functors from spaces over $\mathbf{Z}$ to those over $\mathbf{F}_1$. The first is the left adjoint of base change and the second is the right adjoint. Thus it is natural to interpret the Witt space $W^*(X) \in \mathsf{Sp}_S$ as $X \times_{\mathbf{F}_1} \operatorname{Spec} \mathbf{Z}$ and the arithmetic jet space $W_*(X) \in \mathsf{Sp}_S$ as the base change to $\mathbf{Z}$ of the Weil restriction of scalars of X to $\mathbf{F}_1$. One would interpret the truncated versions $W_n^*(X)$ and $W_{n*}(X)$ as approximations.

This theme is discussed in more detail in the preprint [6] and will developed in forthcoming work.

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$$\begin{tikzcd}
 & 11 \arrow[r] \arrow[dr] & 12 \arrow[r] \arrow[dr] & 13 \\
10 \arrow[ur] \arrow[dr] & & 17 \arrow[r] & 18 \\
 & 14 \arrow[r] & 15 \arrow[r] \arrow[u] & 16
\end{tikzcd}$$

Figure 1. Dependence between sections

Conventions

This paper is a continuation of [5]. When we need to refer to results in [5], we will generally not mention the paper itself and instead simply refer to the subsection or equation number. There is no risk of confusion because the numbering of this paper is a continuation of the numbering of [5] (see Fig. 1). We will also keep the general conventions of [5].

10. Sheaf-theoretic foundations

The purpose of this section is to set up the basic global definitions. The approach is purely sheaf theoretic in the style of SGA 4 [1].

10.1. *Spaces.* Let Aff denote the category of affine schemes equipped with the étale topology: a family $(X_i \to X)$ is a covering family if each $X_i \to X$ is étale and their images cover X. Let Sp denote the category of sheaves of sets on Aff. We will call its objects *spaces*.[1] Any scheme represents a contravariant set-valued functor on Aff, and this functor is a sheaf. In this way, the category of schemes can be identified with a full subcategory of Sp.

For any object $S \in \mathsf{Sp}$, let Sp_S denote the subcategory of objects over S and let Aff_S denote the full subcategory of Sp_S consisting of objects X over S, where X is affine. When S is a scheme, define AffRel_S to be the full subcategory of Aff_S consisting of objects X whose structure map $X \to S$ factors through an affine open subscheme of S. Observe that the inclusion $\mathsf{AffRel}_S \to \mathsf{Sp}_S$ induces an equivalence between Sp_S and the category of sheaves of sets on AffRel_S, and for convenience, we will typically identify the two. (The reason for using the site AffRel_S, rather than the more common one Aff_S, is that the E-typical Witt functors $W_{R,E,n}$ are defined in terms of the base R; so it is more convenient to use a generating site in which the objects have an affine base available.) In the important special case where S itself is affine, $S = \operatorname{Spec} R$, we will often write AffRel_R and Sp_R, and of course we have $\mathsf{AffRel}_S = \mathsf{Aff}_S$.

10.2. *Supramaximal ideals.* For the rest of this paper, S will denote a separated scheme. (In all applications, S will be an arithmetic curve. The extra generality we work in will not create any more work.) Define a supramaximal ideal on S to be a finitely presented ([23], 0 (5.3.1)) ideal sheaf $\mathfrak{m}$ in $\mathcal{O}_S$ corresponding to either

(a) a closed point whose local ring $\mathcal{O}_{S,\mathfrak{m}}$ is a discrete valuation ring with finite residue field, or
(b) the empty subscheme.

When S is affine, this agrees with the earlier definition in 1.2.

We will generally fix the following notation: Let $(\mathfrak{m}_\alpha)_{\alpha\in E}$ denote a family of supramaximal ideals of S which are pairwise coprime, that is, for all $\alpha, \beta \in E$ with $\alpha \neq \beta$, we have $\mathfrak{m}_\alpha + \mathfrak{m}_\beta = \mathcal{O}_S$. For each α, let q_α be the cardinality of the ring $\mathcal{O}_{S,\mathfrak{m}_\alpha}/\mathfrak{m}_\alpha$. Finally, n will be an element of $\mathbf{N}^{(E)} = \bigoplus_E \mathbf{N}$.

10.3. *Definition of $W^*_{S,E,n}(X)$ and $W_{S,E,n*}(X)$.* Let $X = \operatorname{Spec} A$ be an object of AffRel_S, and let $\operatorname{Spec} R$ be an affine open subscheme of S which contains the image of the structure map $X \to S$. Then $W_{R,E,n}(A)$ is independent of R, up to a coherent family of canonical isomorphisms. (Here we abusively conflate E and n with their restrictions to R.) Indeed, let $\operatorname{Spec} R'$ be another such subscheme of S. Since S is

[1] As always with large sites, there are set-theoretic subtleties. So, precisely, let Υ be a universe containing the universe of discourse. The term *presheaf* will mean a functor from AffRel_S to the category of Υ-small sets, and the term *sheaf* will mean a presheaf satisfying the sheaf condition. Because AffRel_S is an Υ-small category, we can sheafify presheaves. On the other hand, the categories of sheaves and presheaves are not true categories because their hom-sets are not necessarily true sets, but only Υ-small sets. A possible way of avoiding set-theoretic issues would be to consider only sheaves subject to certain set-theoretic smallness conditions, but to my knowledge, no one has pursued this.

separated, we can assume $\operatorname{Spec} R' \subseteq \operatorname{Spec} R$ and can then apply (2.6.2). Thus we can safely define

$$W_{S,E,n}(X) = \operatorname{Spec} W_{R,E,n}(A).$$

Now we will pass from $X \in \mathsf{AffRel}_S$ to $X \in \mathsf{Sp}_S$. The functor $W_{S,E,n}$ preserves étale fiber products. Indeed, let $f\colon \operatorname{Spec} A' \to \operatorname{Spec} A$ and $g\colon \operatorname{Spec} A'' \to \operatorname{Spec} A$ be étale maps in AffRel_S, and let $\operatorname{Spec} R$ be an affine open subscheme of S containing the image of $\operatorname{Spec} A$, and hence those of $\operatorname{Spec} A'$ and $\operatorname{Spec} A''$; then by 9.4, we have

$$W_{R,E,n}(A' \otimes_A A'') = W_{R,E,n}(A') \otimes_{W_{R,E,n}(A)} W_{R,E,n}(A'').$$

Similarly, $W_{S,E,n}$ preserves covering families, by 9.2 and 6.9. It follows from general sheaf theory (see the footnote to SGA 4 III 1.6 [1], say) that for any sheaf X, the presheaf $X \circ W_{S,E,n}$ is a sheaf. Let us write

$$W_{S,E,n*}\colon \mathsf{Sp}_S \longrightarrow \mathsf{Sp}_S$$

for the functor $X \mapsto X \circ W_{S,E,n}$. Again by general sheaf theory (SGA 4 III 1.2 [1]), the functor $W_{S,E,n*}$ has a left adjoint

$$W^*_{S,E,n}\colon \mathsf{Sp}_S \longrightarrow \mathsf{Sp}_S$$

constructed in the usual way. For any affine open subscheme $\operatorname{Spec} R$ of S and any R-algebra A, it satisfies

$$W^*_{S,E,n}(\operatorname{Spec} A) = W_{S,E,n}(\operatorname{Spec} A) = \operatorname{Spec} W_{R,E,n}(A). \tag{10.3.1}$$

By the adjunction between W_n and $\Lambda_n \odot -$, we further have

$$W_{S,E,n*}(\operatorname{Spec} A) = \operatorname{Spec}(\Lambda_{R,E,n} \odot A), \tag{10.3.2}$$

for R and A as above.

We call $W^*_{S,E,n}(X)$ the *E-typical Witt space of X of length n* and $W_{S,E,n*}(X)$ the *E-typical (arithmetic) jet space of X of length n*. We will often use shortened forms such as $W^*_{S,n}$, W^*_n, and so on.

(Note that $W^*_{S,E,n}$ does not generally commute with finite products. For example, see 9.5. So, despite the notation, $W^*_{S,E,n}$ is essentially never the inverse-image functor in a map of toposes.)

10.4. *Restriction of S.* Let $j\colon S' \to S$ be a flat monomorphism of schemes (especially an open immersion or a localization at a point). There are certain isomorphisms of functors

$$W^*_{S',n} \circ j^* \xrightarrow{\sim} j^* \circ W^*_{S,n} \tag{10.4.1}$$

$$W_{S,n*} \circ j_* \xrightarrow{\sim} j_* \circ W_{S',n*} \tag{10.4.2}$$

$$W^*_{S,n} \circ j_! \xrightarrow{\sim} j_! \circ W^*_{S',n} \tag{10.4.3}$$

$$W_{S',n*} \circ j^* \xrightarrow{\sim} j^* \circ W_{S,n*} \tag{10.4.4}$$

$$j_! \circ W_{S',n*} \xrightarrow{\sim} W_{S,n*} \circ j_! \tag{10.4.5}$$

which we will find useful. The map (10.4.1) restricted to the site AffRel_S was constructed in (2.6.3); and it was shown to be an isomorphism in 6.1. It therefore induces an isomorphism (10.4.2) on the whole sheaf category $\mathsf{Sp}_{S'}$ and, by adjunction, (10.4.1) on Sp_S. Similarly, (10.4.3) was constructed on $\mathsf{AffRel}_{S'}$ in (2.6.2); and this induces (10.4.4) on the sheaf category Sp_S and, by adjunction, (10.4.3) on $\mathsf{Sp}_{S'}$.

Finally (10.4.5) is defined to be the composition

$$j_! \circ W_{S',n*} \xrightarrow{\sim} j_! \circ W_{S',n*} \circ j^* \circ j_! \xrightarrow{(10.4.1)} j_! \circ j^* \circ W_{S,n*} \circ j_! \longrightarrow W_{S,n*} \circ j_!$$

where the first map is induced by the unit of the adjunction $j_! \dashv j^*$ and the last by the counit. Let us show the last map isomorphism. It is enough to show that, for any $X' \in \mathsf{Sp}_{S'}$, the structure map $W_{S,n*}(j_!(X')) \to S$ factors through S'. To do this, it is enough to assume $X' = S'$, and in this case, we will show $W_{S,n*}(S') = j_!(W_{S',n*}(S'))$. It suffices to show this locally on S, by (10.4.4), and so we can assume S is affine. Since S is separated, S' is also affine, in which case we can apply (2.6.4).

It will be convenient to refer to the following simplified expressions of the isomorphisms above:

$$W^*_{S',n}(S' \times_S X) = S' \times_S W^*_{S,n}(X) \tag{10.4.6}$$

$$W_{S,n*}\big(j_*(X')\big) = j_*\big(W_{S',n*}(X')\big) \tag{10.4.7}$$

$$W^*_{S,n}(X') = W^*_{S',n}(X') \tag{10.4.8}$$

$$W_{S',n*}(S' \times_S X) = S' \times_S W_{S,n*}(X) \tag{10.4.9}$$

$$W_{S',n*}(X') = W_{S,n*}(X') \tag{10.4.10}$$

for $X \in \mathsf{Sp}_S$, $X' \in \mathsf{Sp}_{S'}$.

10.5. *Restriction of E.* Observe that if we let E' denote the support in E of $n \in \mathbf{N}^{(E)}$, then we have $W_{S,E',n} = W_{S,E,n}$, and hence $W^*_{S,E',n} = W^*_{S,E,n}$ and $W_{S,E',n*} = W_{S,E,n*}$. So without loss of generality, we can assume that E equals the support of n and hence that E is finite. (This is no longer true in the infinite-length case, but that does not appear in this paper.)

10.6. *Natural maps.* Natural transformations between Witt vector functors for rings extend naturally to natural transformations of their sheaf-theoretic variants and, by adjunction, of the arithmetic jet spaces.

For example, for any partition $E = E' \amalg E''$, the natural isomorphism (5.4.2) induces natural isomorphisms

$$W^*_{S,E'',n''}(W^*_{S,E',n'}(X)) \xrightarrow{\sim} W^*_{S,E,n}(X), \tag{10.6.1}$$

$$W_{S,E,n*}(X) \xrightarrow{\sim} W_{S,E',n'*}(W_{S,E'',n''*}(X)), \tag{10.6.2}$$

for any $X \in \mathsf{Sp}_S$.

Similarly, the natural projections $W_{n+i}(A) \to W_n(A)$, induced by the inclusion $\Lambda_n \subseteq \Lambda_{n+i}$, induce natural maps

$$W^*_n(X) \xrightarrow{r_{n,i}} W^*_{n+i}(X), \tag{10.6.3}$$

$$W_{n+i*}(X) \xrightarrow{s_{n,i}} W_{n*}(X), \tag{10.6.4}$$

which we usually just call the natural inclusion and projection; and the natural transformations ψ_i of (2.4.8) induce natural maps

$$W_{n+i*}(X) \xrightarrow{\psi_i} W_{n*}(X), \tag{10.6.5}$$

$$W^*_n(X) \xrightarrow{\psi_i} W^*_{n+i}(X). \tag{10.6.6}$$

The affine ghost maps $w_i \colon W_n(A) \to A$ (for $i = 0, \dots, n$) and $w_{\leqslant n} \colon W_n(A) \to A^{[0,n]}$ of (2.4.3) and (2.4.4) induce general ghost maps

$$X \xrightarrow{w_i} W^*_n(X), \tag{10.6.7}$$

$$\coprod_{[0,n]} X \xrightarrow{w_{\leqslant n}} W^*_n(X), \tag{10.6.8}$$

and, by adjunction, the co-ghost maps

$$W_{n*}(X) \xrightarrow{\kappa_i} X, \tag{10.6.9}$$

$$W_{n*}(X) \xrightarrow{\kappa_{\leqslant n}} X^{[0,n]}. \tag{10.6.10}$$

Observe that if every ideal in E is the unit ideal, then $w_{\leqslant n}$ and $\kappa_{\leqslant n}$ are isomorphisms, simply because they are induced by isomorphisms between the site maps, by for example 2.7.

When E consists of a single ideal $\mathfrak{m}$, the reduced affine ghost maps $\bar{w}_n \colon W_n(A) \to A/\mathfrak{m}^{n+1}A$ of (4.6.1) extend similarly to natural maps

$$S_n \times_S X \xrightarrow{\bar{w}_n} W_n^*(X), \tag{10.6.11}$$

where $S_n = \operatorname{Spec} \mathcal{O}_S/\mathfrak{m}^{n+1}$. Indeed, both sides commute with colimits in X, and so, since every $X \in \mathsf{Sp}_S$ is the colimit of the objects of AffRel_S mapping to it, the maps in the affine case naturally induce maps in general.

Let $\bar{\kappa}_n$ denote the *reduced co-ghost map*

$$\bar{\kappa}_n \colon S_n \times_S W_{n*}(X) \xrightarrow{\bar{w}_n} W_n^* W_{n*}(X) \xrightarrow{\varepsilon} X, \tag{10.6.12}$$

where ε is the counit of the evident adjunction.

Finally, we have natural plethysm and co-plethysm maps

$$W_m^* W_n^*(X) \xrightarrow{\mu_X} W_{m+n}^*(X), \tag{10.6.13}$$

$$W_{m+n*}(X) \longrightarrow W_{n*} W_{m*}(X), \tag{10.6.14}$$

which are induced by (2.4.5).

10.7. Proposition. *If X is affine, then so is $W_n^*(X)$.*

Proof. First observe that when S is affine, this was established already in (10.3.1).

For general S, we will apply Chevalley's theorem (in the final form, due to Rydh [38], Theorem (8.1)), to the ghost map $w_{\leqslant n}$ of (10.6.8). To apply it, it is enough to verify that $\coprod_{[0,n]} X$ is affine, that $W_n^*(X)$ is a scheme, and that $w_{\leqslant n}$ is integral and surjective.

The first statement is clear. Let us check the second. Let $(S_i)_{i \in I}$ be an open affine cover of S. Since $W_n^*(X)$ is the quotient of $\coprod_i S_i \times_S W_n^*(X)$ by the equivalence relation $\coprod_{j,k} S_j \times_S S_k \times_S W_n^*(X)$, it is enough to show that each of the summands in each expression is affine. And since S is separated, it is enough to show the single statement that, for any affine open subscheme S' of S, the space $S' \times_S W_n^*(X)$ is affine. By (10.4.6), we have $S' \times_S W_n^*(X) = W_{S',n}^*(S' \times_S X)$. Further, because X and S' are affine and S is separated, $S' \times_S X$ is affine. Therefore $W_{S',n}^*(S' \times_S X)$ is affine, by the case mentioned in the beginning, and hence so is $S' \times_S W_n^*(X)$.

Now let us check that $w_{\leqslant n}$ is integral and surjective. It is enough to show this for each base-change map $S_i \times_S w_{\leqslant n}$. By (10.4.6) again, this map can be identified with the ghost map $\coprod_{[0,n]} (S_i \times_S X) \to W_{S_i,n}^*(S_i \times_S X)$. In other words, we may assume S is affine, in which case we can conclude by applying 10.8 below. □

10.8. Lemma. *Suppose S is affine, $S = \operatorname{Spec} R$. For any R-algebra A, the map $w_{\leqslant n} \colon W_{R,n}(A) \to A^{[0,n]}$ is integral, and its kernel I satisfies $I^{2^N} = 0$, where $N = \sum_{\mathfrak{m}} n_{\mathfrak{m}}$.*

Proof. We may assume that E equals the support of n, which is finite, and then reason by induction on the cardinality of E. When E is empty, $w_{\leqslant n}$ is an isomorphism. Now suppose E contains an element $\mathfrak{m}$. Write $E' = E - \{\mathfrak{m}\}$ and let n' denote the restriction of n to E'. Then the map $w_{\leqslant n}$ factors as follows:

$$\begin{tikzcd}
W_{E,n}(A) \arrow[rr, "w_{\leqslant n}"] \arrow[d, "(5.4.2)"', "\sim"] & & \left(A^{[0,n_{\mathfrak{m}}]}\right)^{[0,n']} \\
W_{\mathfrak{m},n_{\mathfrak{m}}}(W_{E',n'}(A)) \arrow[r, "w_{\leqslant n_{\mathfrak{m}}}"] & W_{E',n'}(A)^{[0,n_{\mathfrak{m}}]} \arrow[r, "\sim"] & W_{E',n'}(A^{[0,n_{\mathfrak{m}}]}). \arrow[u, "w_{\leqslant n'}"']
\end{tikzcd}$$

So it is enough to show $w_{\leqslant n_{\mathfrak{m}}}$ and $w_{\leqslant n'}$ are integral and their kernels are nilpotent of the appropriate degree. For $w_{\leqslant n'}$, it follows by induction on E. For a, it follows by induction on the integer $n_{\mathfrak{m}}$, using 8.1(a)–(b). □

10.9. *Algebraic spaces.* We will define the category of algebraic spaces to be the smallest full subcategory of $\mathsf{Sp}_{\mathbf{Z}}$ which contains $\mathsf{Aff}_{\mathbf{Z}}$ and which is closed under arbitrary (set indexed) disjoint unions and quotients by étale equivalence relations. This obviously exists. It also agrees with the category of algebraic spaces as defined in Toën–Vaquié [39], section 2. This follows from Toën–Vezzosi [40], Corollary 1.3.3.5, as does the fact that this category is closed under finite limits. (Note that [40] is written in the homotopical language of higher stacks, but it is possible to translate the arguments by substituting the word *space* for *stack* and so on.) Indeed, as mentioned in their Remark 2.6, their category has the defining closure property of ours.

It will be convenient to use their concept of an algebraic space X being m-geometric, $m \in \mathbf{Z}$. For $m \leqslant -1$, the space X is m-geometric if and only if it is affine, it is 0-geometric if and only if its diagonal map is affine, and every algebraic space is m-geometric for $m \geqslant 1$. (Again, see [39], Remark 2.6. Note that they require $m \geqslant -1$, but it will be convenient for us to allow $m < -1$.) In particular, for $m \geqslant 0$, every m-geometric algebraic space is the quotient of a disjoint union of affine schemes by an étale equivalence relation which is a disjoint union of $(m-1)$-geometric algebraic spaces. (Note however that the converse is not true: over a field of characteristic 0, the quotient group $\mathbf{G}_{\mathrm{a}}/\mathbf{Z}$ is 1-geometric but not 0-geometric.) Let AlgSp_m denote the full subcategory of $\mathsf{Sp}_{\mathbf{Z}}$ consisting of all disjoint unions of m-geometric algebraic spaces. A map $X \to Y$ of spaces is said to be m-representable if for every affine scheme T and every map $T \to Y$, the pull back $X \times_Y T$ is an m-geometric algebraic space.

This definition of algebraic space also agrees with that of Raynaud–Gruson [37]. Indeed, the category of Raynaud–Gruson algebraic spaces contains affine schemes and is closed under disjoint unions and quotients by étale equivalence relations. (See Conrad–Lieblich–Olsson [13], A.1.2.) And conversely, any algebraic space in the sense of Raynaud–Gruson is the quotient of a disjoint union of affine schemes by an étale equivalence relation which is a scheme, necessarily separated; therefore it is 1-geometric.

The difference between these two approaches is that Raynaud–Gruson [37] use schemes as the intermediate class of algebraic spaces, and Toën–Vaquié use algebraic spaces with affine diagonal. The advantage of the second approach is that the two steps—going from affine schemes to the intermediate category, and going from that to general algebraic spaces—are two instances of a single procedure. Thus we can prove results by induction on the geometricity m, and so we do not need to consider the intermediate case separately. Note however that the induction is rather meager in that it terminates after two steps.

Finally, a space is algebraic in the sense of Knutson [33] if and only if it is quasi-separated and algebraic in the sense above.

10.10. *Smoothness properties of maps.* Let us say a map $f\colon X \to Y$ in Sp is formally étale (resp. formally unramified, resp. formally smooth) if the usual nilpotent lifting properties (EGA IV 17.1.2 (iii) [28]) hold: for all nilpotent closed immersions $\bar{T} \to T$ of affine schemes, the induced map

$$X(T) \longrightarrow X(\bar{T}) \times_{Y(\bar{T})} Y(T) \tag{10.10.1}$$

is a bijection (resp. injection, resp. surjection).

Also, let us say that f is locally of finite presentation if for any cofiltered system $(T_i)_{i\in I}$ of Y-schemes, each of which is affine, the induced map

$$\operatorname*{colim}_i \mathrm{Hom}_Y(T_i, X) \longrightarrow \mathrm{Hom}_Y(\lim_i T_i, X) \tag{10.10.2}$$

is a bijection. This definition agrees with the usual one if X and Y are schemes (EGA IV 8.14.2.c [27]).

We call a map étale (resp. unramified, resp. smooth) if it is locally of finite presentation and formally étale (resp. formally unramified, resp. formally smooth). When X and Y are schemes, all these definitions agree with the usual ones.

10.11. *E-flat and E-smooth algebraic spaces.* Let us say that an algebraic space X over S is E-flat (resp. E-smooth) if, for all maximal ideals $\mathfrak{m} \in E$, the algebraic space $X \times_S \operatorname{Spec} \mathcal{O}_{S,\mathfrak{m}}$ is flat (resp. smooth) over $\operatorname{Spec} \mathcal{O}_{S,\mathfrak{m}}$, where $\mathcal{O}_{S,\mathfrak{m}}$ is the local ring of S at $\mathfrak{m}$. If $E = \{\mathfrak{m}\}$, we will often write $\mathfrak{m}$-flat instead of E-flat.

11. Sheaf-theoretic properties of W_{n*}

We continue with the notation of 10.2.

11.1. Proposition. *The functor $W_{n*} \colon \mathsf{Sp}_S \to \mathsf{Sp}_S$ preserves the following properties of maps:*

(a) *locally of finite presentation,*
(b) *formally étale, formally unramified, formally smooth,*
(c) *étale, unramified, smooth.*

Proof. (a): Let $(T_i)_{i\in I}$ be a cofiltered system in Aff_S mapping to $W_{n*}(Y)$, as in (10.10.2). The following chain of equalities, which we will justify below, constitutes the proof:

$$\begin{aligned}\mathrm{Hom}_{W_{n*}(Y)}(\lim_i T_i, W_{n*}(X)) &= \mathrm{Hom}_Y(W_n^*(\lim_i T_i), X)\\ &\stackrel{1}{=} \mathrm{Hom}_Y(\lim_i W_n^*(T_i), X)\\ &\stackrel{2}{=} \operatorname{colim}_i \mathrm{Hom}_Y(W_n^*(T_i), X)\\ &= \operatorname{colim}_i \mathrm{Hom}_{W_{n*}(Y)}(T_i, W_{n*}(X)).\end{aligned}$$

Equality 2 holds because each $W_n^*(T_i)$ is affine (10.7) and because $f \colon X \to Y$ is locally of finite presentation.

To show equality 1, it is enough to show

$$W_n^*(\lim_i T_i) = \lim_i W_n^*(T_i). \tag{11.1.1}$$

Let $(S_j)_{j\in J}$ be an affine open cover of S. Then it is enough to show (11.1.1) after applying each functor $S_j \times_S -$. We can then reduce to the case $S = S_j$, by using (10.4.6), the fact that $S_j \times_S -$ commutes with limits, and the fact that each $S_j \times_S T_i$ is affine (S being separated). In other words, we may assume S is affine, in which case (11.1.1) follows from 6.10 and (10.3.1).

(b): By definition, the map $W_{n*}(X) \to W_{n*}(Y)$ is formally étale (resp. formally unramified, resp. formally smooth) if for any closed immersion $\bar{T} \to T$ of affine schemes defined by a nilpotent ideal sheaf, the induced map

$$W_{n*}(X)(T) \longrightarrow W_{n*}(X)(\bar{T}) \times_{W_{n*}(Y)(\bar{T})} W_{n*}(Y)(T).$$

is bijective (resp. injective, resp. surjective). But this map is the same, by adjunction, as the map

$$X(W_n^*(T)) \longrightarrow X(W_n^*(\bar{T})) \times_{Y(W_n^*(\bar{T}))} Y(W_n^*(T)).$$

Because $f \colon X \to Y$ is formally étale (resp. ...), to show this map is bijective (resp. ...), it is enough to check that the induced map $W_n(\bar{T}) \to W_n(T)$ is also a nilpotent immersion of affine schemes. Affineness follows from 10.7. On the other hand, to show nilpotence, we may work locally. So by (10.4.6), we may assume S and T are affine (since S is separated). We can then apply 6.4.

(c): This follows from (a) and (b) by definition. $\square$

11.2. Proposition. *Suppose E consists of one ideal $\mathfrak{m}$, and set $S_0 = \operatorname{Spec} \mathcal{O}_S/\mathfrak{m}$. Let $f\colon X \to Y$ be a map in Sp_S which is formally étale (resp. formally unramified, resp. formally smooth). Then the map*

$$S_0 \times_S W_{n*}(X) \xrightarrow{\mathrm{id}_{S_0} \times (W_{n*}(f), \kappa_0)} S_0 \times_S W_{n*}(Y) \times_{\kappa_0, Y} X \tag{11.2.1}$$

is an isomorphism (resp. monomorphism, resp. presheaf epimorphism).

Proof. Let Z be an affine S_0-scheme. Then Z is an object of AffRel_S. We have the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_S(Z, W_{n*}(X)) & \xrightarrow{\quad a \quad} & \operatorname{Hom}_S(Z, W_{n*}(Y) \times_Y X) \\ {\scriptstyle c}\downarrow{\scriptstyle \sim} & & {\scriptstyle d}\downarrow{\scriptstyle \sim} \\ \operatorname{Hom}_S(W_n^*(Z), X) & \xrightarrow{\quad b \quad} & \operatorname{Hom}_S(W_n^*(Z), Y) \times_{\operatorname{Hom}_S(Z,Y)} \operatorname{Hom}_S(Z, X), \end{array}$$

where a is induced by (11.2.1), and c and d are given by the evident universal properties. The map $w_0\colon Z \to W_n(Z)$ is a closed immersion defined by a nilpotent ideal, by 6.8 and (10.3.1). Therefore, since f is formally étale (resp. formally unramified, resp. formally smooth), b is bijective (resp. injective, resp. surjective), and hence so is a. In other words, the map in question is an isomorphism (resp. monomorphism, resp. presheaf epimorphism). $\square$

11.3. Corollary. *Let E and S_0 be as in 11.2. Let $(U_i \to X)_{i \in I}$ be an epimorphic family of étale maps in Sp_S. Then the induced family*

$$\Big(S_0 \times_S W_{n*}(U_i) \longrightarrow S_0 \times_S W_{n*}(X)\Big)_{i \in I}$$

is an epimorphic family of étale maps.

Proof. By 11.1, we only need to show the family is epimorphic. By 11.2, each square

$$\begin{array}{ccc} S_0 \times_S W_{n*}(U_i) & \longrightarrow & S_0 \times_S W_{n*}(X) \\ \downarrow{\scriptstyle \kappa_0 \circ \mathrm{pr}_2} & & \downarrow{\scriptstyle \kappa_0 \circ \mathrm{pr}_2} \\ U_i & \longrightarrow & X \end{array}$$

is cartesian. The lower arrows are assumed to form a covering family indexed by $i \in I$, and hence so do the upper arrows. $\square$

11.4. Proposition. *The functor $W_{n*}\colon \mathsf{Sp}_S \to \mathsf{Sp}_S$ preserves epimorphisms and monomorphisms.*

Proof. The statement about monomorphisms follows for general reasons from the fact that W_{n*} has a left adjoint.

Let us now consider the statement about epimorphisms. By (10.6.2), it suffices to assume E consists of one maximal ideal $\mathfrak{m}$. Let $f\colon X \to Y$ be an epimorphism in Sp_S.

First consider the case where X and Y lie in AffRel_S and f is étale. The map $W_{n*}(f)\colon W_{n*}(X) \to W_{n*}(Y)$ is also an étale map between objects of AffRel_S, by 11.1 and (10.3.2). Being epimorphic is then equivalent to being surjective, which can be checked after base change to $s = \operatorname{Spec} \mathcal{O}_S/\mathfrak{m}$ and to $S - s$. For $S - s$, the base change of $W_{n*}(X) \to W_{n*}(Y)$ agrees, by (10.4.9), with that of $X^{[0,n]} \to Y^{[0,n]}$, which is surjective. On the other hand, over s the map $W_{n*}(f)\colon W_{n*}(X)\to W_{n*}(Y)$ is a base change of the map $f\colon X\to Y$, by 11.2, which is also surjective.

Now consider the general case, where $f\colon X\to Y$ is any epimorphism of spaces. It is enough to show that for any object $V\in \mathsf{AffRel}_S$, any given map $a\colon V\to W_{n*}(Y)$ lifts locally on V to $W_{n*}(X)$. Since f is an epimorphism and $W_n^*(V)$ is affine, and hence quasi-compact, there exists a commutative diagram

(11.4.1)
$$\begin{tikzcd} X & Z \arrow[l, "b"'] \arrow[d, two heads, "c"] \\ Y \arrow[from=1-1, "f"] & W_n^*(V), \arrow[l, "a"'] \end{tikzcd}$$

where $Z\in\mathsf{AffRel}_S$, c is an étale cover, and b is some map. Consider the commutative diagram

(11.4.2)
$$\begin{tikzcd} W_{n*}(X) \arrow[d, "W_{n*}(f)"'] & W_{n*}(Z) \arrow[l, "W_{n*}(b)"'] \arrow[d, "W_{n*}(c)"] & W_{n*}(Z)\times_{W_{n*}W_n^*(V)} V \arrow[l, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\ W_{n*}(Y) & W_{n*}W_n^*(V) \arrow[l, "W_{n*}(a)"'] & V, \arrow[l, "\eta"'] \end{tikzcd}$$

where η is the unit of the evident adjunction. By the argument above, $W_{n*}(c)$ is an étale epimorphism between objects of AffRel_S. Therefore pr_2 is the same, and hence the map $V\to W_{n*}(Y)$ lifts locally to $W_{n*}(X)$. □

11.5. Corollary. *Let $(\pi_1,\pi_2)\colon \Gamma\to X\times_S X$ be an equivalence relation on a space $X\in\mathsf{Sp}_S$. Then the map*

$$(W_{n*}(\pi_1),W_{n*}(\pi_2))\colon W_{n*}(\Gamma)\to W_{n*}(X)\times W_{n*}(X)$$

is an equivalence relation on $W_{n*}(X)$, and the induced map

$$W_{n*}(X)/W_{n*}(\Gamma)\longrightarrow W_{n*}(X/\Gamma)$$

is an isomorphism.

Proof. By 11.4, the map $W_{n*}(X)\to W_{n*}(X/\Gamma)$ is an epimorphism of spaces. On the other hand, since W_{n*} has a left adjoint, we have

$$W_{n*}(X)\times_{W_{n*}(X/\Gamma)} W_{n*}(\Gamma)=W_{n*}(X\times_{X/\Gamma}X)=W_{n*}(\Gamma),$$

and so the equivalence relation inducing the quotient $W_{n*}(X/\Gamma)$ is $W_{n*}(\Gamma)$. □

11.6. *Remark.* These results allow us to present $W_{n*}(X)$ using charts, but not in the sense that might first come to mind. For while W_{n*} preserves covering maps (by 11.4), it does not generally preserve covering families. That is, if $(U_i)_{i\in I}$ is an étale covering family of X, then the space $W_{n*}(\coprod_i U_i)$ covers $W_{n*}(X)$, but it is usually not true that $\coprod_i W_{n*}(U_i)$ covers it. For example, consider the p-typical case with $n=1$. On the generic fiber, $W_{1*}(X)$ is just $X\times X$; and of course $\coprod_i U_i\times U_i$ does not generally cover $X\times X$. In particular, $W_{n*}(X)$ cannot be constructed using charts by gluing the spaces $W_{n*}(U_i)$ together along the overlaps $W_{n*}(U_j\times_X U_k)$. This is just a general property of products and not a particular property of Frobenius lifts.

On the other hand, if X is an algebraic space over $\operatorname{Spec}\mathcal{O}_S/\mathfrak{m}^j$, for some integer $j\geqslant 0$, this naive gluing method does work. This is because for any étale cover $(U_i)_{i\in I}$ of X, the family $\big(W_{n*}(U_i)\big)_{i\in I}$ is an étale cover of $W_{n*}(X)$. This is true by 11.1 and 11.3. See 12.8 for the implications this has for Buium's p-jet spaces.

11.7. Proposition. *The functor $W_{n*}\colon \mathsf{Sp}_S\to\mathsf{Sp}_S$ commutes with filtered colimits.*

Proof. By adjunction, this is equivalent to the statement that for any filtered system $(X_i)_{i\in I}$ and any $T \in \mathsf{AffRel}_S$ the map

$$\operatorname{colim}_i \operatorname{Hom}\big(W_n^*(T), X_i\big) \longrightarrow \operatorname{Hom}\big(W_n^*(T), \operatorname{colim}_i X_i\big)$$

is an isomorphism. Because $W_n^*(T)$ is affine (10.3.1), it is quasi-compact and quasi-separated, and so the proposition follows from SGA 4 VI 1.23(ii) [2]. ☐

11.8. Lemma. *Let $(S_t)_{t\in T}$ be an open cover of S, let X be an object of Sp_S, and let $(U_i)_{i\in I}$ be a cover of X by objects of Sp_S. Let J denote the set of finite subsets of I, and for each $j \in J$, write $V_j = \coprod_{i\in j} U_i$. Then $(S_t \times_S V_j)_{(t,j)\in T\times J}$ is a cover of X with the property that $\big(W_{n*}(S_t \times_S V_j)\big)_{(t,j)\in T\times J}$ is a cover of $W_{n*}(X)$.*

Proof. It is clear that $(S_t \times_S V_j)_{t,j}$ is a cover of X. Let us show that $\big(W_{n*}(S_t \times_S V_j)\big)_{t,j}$ is a cover of $W_{n*}(X)$. By (10.4.9) and (10.4.10), it is enough to consider the case where $(S_t)_{t\in T}$ is the trivial cover consisting of S itself. Thus it is enough to show that $\big(W_{n*}(V_j)\big)_j$ is a cover of $W_{n*}(X)$.

Observe that we have a natural isomorphism

$$\coprod_{i\in I} U_i = \operatorname{colim}_{j\in J} V_j; \tag{11.8.1}$$

indeed, both sides have the same universal property. Now consider the commutative diagram

$$\begin{tikzcd}
\coprod_{j\in J} W_{n*}(V_j) \arrow[rr, "a"] \arrow[d, two heads] & & W_{n*}(X) \\
\operatorname{colim}_{j\in J} W_{n*}(V_j) \arrow[r, "\sim", "(11.7)"'] & W_{n*}\big(\operatorname{colim}_{j\in J} V_j\big) \arrow[r, "\sim", "(11.8.1)"'] & W_{n*}\big(\coprod_{i\in I} U_i\big), \arrow[u, "b"']
\end{tikzcd}$$

where a and b are the maps induced by the covering maps $V_j \to X$ and $U_i \to X$. Then a is an epimorphism because b is, which is true by 11.4. ☐

11.9. *W_{n*}-stable covers.* It is useful to have covers $(X_k)_{k\in K}$ of X that are W_{n*}-stable, meaning that $(W_{n*}(X_k))_{k\in K}$ is a cover of $W_{n*}(X)$. While general covers are not W_{n*}-stable (see 11.6), some are. Any singleton cover is, by 11.4, but it is not always enough to have this because there can fail to be singleton covers with desirable properties. For instance, if X is not quasi-compact, it cannot be covered by a single affine scheme. But it often suffices to know only that W_{n*}-stable covers with certain desirable properties exist, and we can sometimes use 11.8 to make them. For instance, we can produce a W_{n*}-stable cover with each X_k affine by taking $K = T \times J$ and $X_{(t,j)} = S_t \times_S V_j$ in 11.8, where the U_i and S_t are affine. If we refine the cover $(S_t)_{t\in T}$ so that each ideal in the support of n is principal on each S_t, then we further have that the image of each X_k in S is contained in an affine open subscheme of S on which each ideal in the support of n is principal. If X is an algebraic space, we can even further arrange for each X_k to be étale over X by taking $(U_i)_{i\in I}$ to be an étale cover of X.

11.10. Proposition. *The functor $W_{n*}\colon \mathsf{Sp}_S \to \mathsf{Sp}_S$ preserves*

(a) *quasi-compactness of objects,*
(b) *quasi-separatedness of objects,*
(c) *quasi-compactness of maps,*
(d) *quasi-separatedness of maps.*

Proof. Let X be an object in Sp_S.

(a): Suppose X is quasi-compact. Let $(U_i)_{i\in I}$ be a finite family in AffRel_S which covers X. (Such a family exists, because the large family of all morphisms from objects of AffRel_S to X covers to X, and therefore has a finite subcover, because X is quasi-compact.) Then the space $U = \coprod_{i\in I} U_i$ is affine. By 11.4, the map $W_{n*}(U) \to W_{n*}(X)$ is an epimorphism. Since $W_{n*}(U)$ is affine, it is quasi-compact. It follows that $W_{n*}(X)$ is quasi-compact (SGA 4 VI 1.3[2]).

(b): Suppose X is quasi-separated. Then for any cover $(U_i)_{i\in I}$ of X, with each $U_i \in \mathsf{AffRel}_S$, each space $U_i \times_X U_j$ is quasi-compact. Therefore, by (a), each space $W_{n*}(U_i) \times_{W_{n*}(X)} W_{n*}(U_j) = W_{n*}(U_i \times_X U_j)$ is quasi-compact. By SGA 4 VI 1.17 [2], this implies that $W_{n*}(X)$ is quasi-separated as long as we can choose the cover $(U_i)_{i\in I}$ such that $(W_{n*}(U_i))_{i\in I}$ is a cover of $W_{n*}(X)$. This is possible by 11.9.

(c): Let $f\colon X \to Y$ be a quasi-compact map of spaces. As above, by 11.9, there exists a cover $(U_i)_{i\in I}$ of Y, with each $U_i \in \mathsf{AffRel}_S$, such that $(W_{n*}(U_i))_{i\in I}$ is an affine cover of $W_{n*}(Y)$. It is then enough to show that each $W_{n*}(U_i)\times_{W_{n*}(Y)} W_{n*}(X)$ is quasi-compact (SGA 4 VI 1.16 [2]), but this agrees with $W_{n*}(U_i \times_Y X)$. Now apply (a).

(d): Let $f\colon X \to Y$ be a quasi-separated map of spaces. By definition, its diagonal map Δ_f is quasi-compact. By (c), so is the map $W_{n*}(\Delta_f)$, and this agrees with the diagonal map of $W_{n*}(f)$. □

12. W_{n*} and algebraic spaces

We continue with the notation of 10.2.

12.1. Theorem. *Let X be an algebraic space over S. Then $W_{n*}(X)$ is an algebraic space. If X is a scheme, then so is $W_{n*}(X)$.*

For the proof, see 12.5 below. Observe that when X is quasi-compact and has affine diagonal (e.g. is separated), as is often the case in applications, the algebraicity of $W_{n*}(X)$ follows immediately from 11.5 and 11.1. Thus, for the part of the theorem asserting that $W_{n*}(X)$ is an algebraic space, all the work below is in removing these assumptions.

12.2. Proposition. *For any spaces $X, Y \in \mathsf{Sp}_S$, the diagram*

$$\begin{tikzcd}
W_{n*}(X) \arrow[r, "W_{n*}(j)"] \arrow[d, "\kappa_{\leqslant n}"] & W_{n*}(X \amalg Y) \arrow[d, "\kappa_{\leqslant n}"] \\
X^{[0,n]} \arrow[r, "j^{[0,n]}"] & (X \amalg Y)^{[0,n]},
\end{tikzcd}$$

where $j\colon X \to X \amalg Y$ denotes the canonical summand inclusion, is cartesian.

Proof. It is enough to show that for any object $T \in \mathsf{AffRel}_S$, the functor $\mathrm{Hom}(T, -)$ takes the diagram above to a cartesian diagram. By adjunction, this is equivalent to the existence of a unique dashed arrow making the diagram

$$\begin{tikzcd}
\coprod_{[0,n]} T \arrow[r, "w_{\leqslant n}"] \arrow[d] & W_n^*(T) \arrow[d] \arrow[dl, dashed, "\exists!?"'] \\
X \arrow[r, "j"] & X \amalg Y
\end{tikzcd}$$

commute, for any given vertical arrows making the square commute. It is therefore enough to show $W_n^*(T) \times_{X \amalg Y} Y = \emptyset$.

To do this, we will show that if there exists a map $U \to W_n^*(T) \times_{X \amalg Y} Y$, where U is an affine scheme, then U is empty. Pulling back such a map by $w_{\leqslant n}$, we get a map

$$\Big(\coprod_{[0,n]} T\Big) \times_{W_n^*(T)} U \longrightarrow \Big(\coprod_{[0,n]} T\Big) \times_{W_n^*(T)} W_n^*(T) \times_{(X \amalg Y)} Y.$$

By the commutativity of the square above, the right-hand side is empty. Therefore the left-hand side is empty. But since $w_{\leqslant n}$ is a surjective map of (affine) schemes (by 10.8 and (10.3.2)), U must be empty. □

12.3. *Remark.* It follows from 12.2 that $\coprod_i W_{n*}(X_i)$ is a summand of $W_{n*}(\coprod_i X_i)$. In all but the most trivial cases, the two will not be equal. See 11.6, for example.

12.4. Lemma. *If X is a disjoint union in Sp_S of objects in AffRel_S, then so is $W_{n*}(X)$.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in AffRel_S such that $X \cong \coprod_{i\in I} X_i$. For any function $h\colon [0,n] \to I$, write

$$X^h = \prod_{m\in[0,n]} X_{h(m)}.$$

Then we have

$$X^{[0,n]} = \coprod_h X^h,$$

where h runs over all maps $[0,n] \to I$. Therefore it is enough to show that the pre-image of each X^h under the map $\kappa_{\leqslant n}\colon W_{n*}(X) \to X^{[0,n]}$ is a disjoint union in Sp_S of objects in AffRel_S.

Since this preimage lies over X^h, and hence over $X_{h(0)}$, it lies over an affine open subscheme S' of S. Therefore it is enough to show that this preimage is affine.

Let us first do this when I is finite. Because X^h lies over S', we have

$$X^h \times_{X^{[0,n]}} W_{n*}(X) = X^h \times_{X^{[0,n]}} \big(W_{n*}(X) \times_S S'\big). \tag{12.4.1}$$

Since I is finite, X is affine, and hence so is X^h. On the other hand, $W_{n*}(X) \times_S S'$ is affine, by (10.4.9) and (10.3.2). Therefore the left-hand side of (12.4.1) is affine.

Now suppose I is arbitrary. Let J denote the image of h, and write $Y = \coprod_{i\in J} X_i$. Then the map $X^h \to X^{[0,n]}$ factors through the map $j^{[0,n]}\colon Y^{[0,n]} \to X^{[0,n]}$ induced by the summand inclusion $j\colon Y \to X$. Therefore by 12.2, the right-hand square in the digram

$$\begin{tikzcd}
X^h \times_{Y^{[0,n]}} W_{n*}(Y) \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"] & W_{n*}(Y) \arrow[r, "W_{n*}(j)"] \arrow[d, "\kappa_{\leqslant n}"] & W_{n*}(X) \arrow[d, "\kappa_{\leqslant n}"] \\
X^h \arrow[r] & Y^{[0,n]} \arrow[r, "j^{[0,n]}"] & X^{[0,n]}
\end{tikzcd}$$

is cartesian. Thus $X^h \times_{X^{[0,n]}} W_{n*}(X)$ agrees with $X^h \times_{Y^{[0,n]}} W_{n*}(Y)$, which is affine by the case proved above and the fact that Y is affine. □

12.5. *Proof of 12.1.* It is enough to show that $S' \times_S W_{n*}(X)$ is an algebraic space, or a scheme when X is, for all sufficiently small affine open subschemes S' of S. Therefore by (10.4.9), we may assume that $S = \operatorname{Spec} R$ for some ring R, and that the ideal $\mathfrak{m}$ of R is generated by a single element π.

Let us first show that $W_{n*}(X)$ is an algebraic space. We will show by induction on m that if X is m-geometric, then $W_{n*}(X)$ is an algebraic space. If $m = -1$, then X is affine and so we can apply (10.3.1). Now assume $m \geqslant 0$.

Let $(U_i \to X)_{i\in I}$ be an affine étale cover for which each map $U_i \to X$ is $(m-1)$-representable. Write $U = \coprod_{i\in I} U_i$. Consider the diagrams

$$U \times_X U \rightrightarrows U \longrightarrow X.$$

and

$$W_{n*}(U \times_X U) \rightrightarrows W_{n*}(U) \longrightarrow W_{n*}(X).$$

By 11.5 and 11.1, the space $W_{n*}(U \times_X U)$ is an étale equivalence relation on $W_{n*}(U)$ with quotient $W_{n*}(X)$. Since the category of algebraic spaces is closed under quotients by étale equivalence relations, it is sufficient to show that $W_{n*}(U)$ and $W_{n*}(U \times_X U)$ are algebraic spaces. This holds because of two facts. First, U (resp. $U \times_X U$) is a disjoint union of -1-geometric (resp. $(m-1)$-geometric) algebraic spaces. Second, for $k \leqslant m-1$, the functor W_{n*} applied to a disjoint union of k-geometric algebraic spaces is an algebraic space. Indeed, if $k = -1$, this follows from 12.4; if $k \geqslant 0$, then a disjoint union of k-geometric algebraic spaces is itself a k-geometric algebraic space, and so it follows by induction.

Now suppose X is a scheme. To show that $W_{n*}(X)$ is a scheme, we may assume, by (10.6.2), that E consists of a single ideal $\mathfrak{m}$. Since $W_{n*}(X)$ is an algebraic space, it is enough to show it has an affine open cover.

Let $(V_i)_{i\in I}$ be an affine open cover of X. For any S-space Y, write $Y' = Y \times_S \operatorname{Spec} R[1/\pi]$. Then each V_i' is an affine open subscheme of X. Further, the schemes $V_{i_0}' \times_S \cdots \times_S V_{i_n}'$ cover the product $(X')^{[0,n]} = X' \times_S \ldots \times_S X'$, which agrees with $W_{n*}(X')$ and hence $W_{n*}(X)'$ (by (10.4.9)–(10.4.10)). So all that remains is to show the fiber of $W_{n*}(X)$ over $\mathfrak{m}$ can be covered by open subspaces that are affine schemes.

Since each V_i is affine, each $W_{n*}(V_i)$ is affine, by 10.3.2. Since each map $V_i \to X$ is an étale monomorphism, each map $W_{n*}(V_i) \to W_{n*}(X)$ is an étale monomorphism, by 11.1 and 11.4. Therefore these maps are open immersions, and by 11.3 they cover the fiber of $W_{n*}(X)$ over $\mathfrak{m}$. □

12.6. Corollary. *Let X be an E-smooth (10.11) algebraic space over S. Then $W_{n*}(X)$ is an E-smooth algebraic space over S. In particular, it is E-flat.*

Proof. By 12.1, we know $W_{n*}(X)$ is an algebraic space. Now let us show it is smooth locally at all maximal ideals of E. By (10.4.9), we may assume that X is smooth over S. Then apply 11.1 and the fact that smoothness for a map of algebraic spaces implies flatness. □

12.7. *W_{n*} does not generally preserve flatness.* For example, consider the p-typical jets of length 1. Let $A = \mathbf{Z}[x]/(x^2 - px)$, which is flat over $\mathbf{Z}$ (and happens to be isomorphic to $W_1(\mathbf{Z})$). Then

$$\mathbf{Z}[1/p] \otimes_{\mathbf{Z}} (\Lambda_1 \odot A) = \Lambda_1 \odot (\mathbf{Z}[1/p] \otimes A) = (\mathbf{Z}[1/p] \otimes A)^{\otimes 2} = \mathbf{Z}[1/p]^{2\times 2}.$$

But a short computation using 3.4 shows

$$\Lambda_1 \odot A = \mathbf{Z}[x,\delta]/(x^2 - px, 2x^p\delta + p\delta^2 - x^p - p\delta + p^{p-1}x^p),$$

and hence

$$\mathbf{F}_p \otimes_{\mathbf{Z}} (\Lambda_1 \odot A) = \mathbf{F}_p[x,\delta]/(x^2).$$

So $\operatorname{Spec}(\Lambda_1 \odot A)$ has one irreducible component lying over $\operatorname{Spec} \mathbf{F}_p$. In particular, it is not flat locally at p.

It would be interesting to find a reasonable condition on X that is weaker than smoothness over S but still implies flatness of $W_{n*}(X)$ over S.

12.8. *Relation to Greenberg's and Buium's spaces.* In the case $S = \operatorname{Spec} \mathbf{Z}_p$ and $E = \{p\mathbf{Z}_p\}$, our arithmetic jet space is closely related to previously defined spaces, the Greenberg transform and Buium's p-jet space.

Let X be a scheme locally of finite type over $\mathbf{Z}/p^{n+1}\mathbf{Z}$. Then the Greenberg transform $\mathrm{Gr}_{n+1}(X)$ is a scheme over $\mathbf{F}_p$. (See [20][21], or for a summary in modern language, [7] p. 276.) It is related to $W_{n*}(X)$ by the formula

$$\mathrm{Gr}_{n+1}(Y) = W_{n*}(Y) \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \mathbf{F}_p. \tag{12.8.1}$$

This is simply because they represent, almost by definition, the same functor.

On the other hand, for smooth schemes Y over the completion $\tilde{R}$ of the maximal unramified extension of $\mathbf{Z}_p$, Buium has defined *p-jet spaces* $J^n(Y)$, which are formal schemes over $\tilde{R}$. (See [9], section 2, or [10], section 3.1.) His jet space is related to ours by the formula

$$J^n(Y) = W_{n*}(\hat{Y}), \tag{12.8.2}$$

where $\hat{Y}$ denotes the colimit of the schemes $Y \times_{\operatorname{Spec} \mathbf{Z}} \operatorname{Spec} \mathbf{Z}/p^m\mathbf{Z}$, taken over m and in the category Sp_S. Indeed it is true when Y is affine, by 3.4, and it holds when Y is any smooth scheme over $\tilde{R}$ by gluing. As mentioned in 11.6, the gluing methods used to define W_{n*} and J^n are not the same in general, but here $\hat{Y}$ is p-adically formal; so they agree by the discussion in 11.6.

The following consequence of (12.8.2) is also worth recording: if X is a smooth scheme over $\mathbf{Z}_p$, then we have

$$J^n(X \times_{\operatorname{Spec} \mathbf{Z}_p} \operatorname{Spec} \tilde{R}) = W_{n*}(\hat{X}) \times_{\operatorname{Spf} \mathbf{Z}_p} \operatorname{Spf} \tilde{R}. \tag{12.8.3}$$

13. Preservation of geometric properties by W_{n*}

We continue with the notation of 10.2.

13.1. *Étale-local properties.* Recall that a property P of algebraic spaces over S is said to be *étale-local* if the following hold: whenever X satisfies P, then so does any algebraic space Y which admits an étale map to X; and if $(U_i)_{i\in I}$ is an étale cover of X such that each U_i satisfies P, then so does X.

A property P of maps of algebraic spaces is said to be *étale-local on the target* if for any map $f\colon X \to Y$ the following hold: whenever f satisfies P, then so does any base change $f_V\colon X \times_Y V \to V$ with $V \to Y$ étale; and if $(V_j)_{j\in J}$ is an étale cover of Y such that each base change f_{V_j} satisfies P, then so does f.

Such a property is said to be *étale-local on the source* if, in addition, the following hold: whenever f satisfies P, then so does any composition $U \to X \to Y$ with $U \to X$ étale; and if $(U_i)_{i\in I}$ is an étale cover of X such that each composition $U_i \to X \to Y$ satisfies P, then so does f.

13.2. Proposition. *Let P be a property of maps $f\colon X \to Y$ of algebraic spaces which is étale-local on the target. For W_{n*} to preserve property P, it is sufficient that it do so when E consists of one principal ideal, S is affine, and Y is affine.*

If property P is also étale-local on the source, then we may further restrict to the case where X is affine.

The argument is the same as the one given below for 16.3, except that one takes affine étale covering families of the kind given by 11.9, and one uses the easy fact that W_{n*} preserves fiber products instead of the more difficult 15.2(c). Since the details are given in 16.3, let us omit them here.

13.3. Proposition. *The following properties of maps (étale-local on the target) are preserved by W_{n*}:*

(a) *affine,*

(b) *a closed immersion,*
(c) *locally of finite type,*
(d) *locally of finite presentation,*
(e) *of finite type,*
(f) *of finite presentation,*
(g) *separated,*
(h) *smooth and surjective.*

Proof. Let $f\colon X \to Y$ be such a map. By 13.2, we may assume $S = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$.

(a)–(b): These are affine properties; so we have $X = \operatorname{Spec} B$, for some A-algebra B. By (10.3.2), we have $W_{n*}(X) = \operatorname{Spec} \Lambda_n \odot B$, which is affine. This proves (a). If the structure map $A \to B$ is surjective, then so is the induced map $\Lambda_n \odot A \to \Lambda_n \odot B$, which proves (b).

(c)–(d): These properties are étale-local on the source, and so we may assume $X = \operatorname{Spec} B$, where B is a finitely generated A-algebra. Take an integer $m \geqslant 0$ such that there exists a surjection $A[x]^{\otimes m} \to B$. Then the induced map

$$(\Lambda_n \odot A) \otimes_R (\Lambda_n^{\otimes m}) = \Lambda_n \odot (A \otimes_R R[x_1, \dots, x_m]) \longrightarrow \Lambda_n \odot B$$

is surjective. Therefore it is enough to show that Λ_n is finitely generated as an R-algebra. This was proved in 6.10.

Now suppose that B is finitely presented. Then there exist finitely generated A-algebras A' and A'' and a coequalizer diagram

$$A'' \rightrightarrows A' \longrightarrow B.$$

This then induces a coequalizer diagram

$$\Lambda_n \odot A'' \rightrightarrows \Lambda_n \odot A' \longrightarrow \Lambda_n \odot B.$$

By (c), the first two terms are finitely generated A-algebras; therefore the last term is finitely presented. This proves (d).

(e)–(f): These follow from (c)–(d) and 11.10(c), by definition.

(g): Since the diagonal map $\Delta_f\colon X \to X \times_Y X$ is a closed immersion, (b) implies $W_{n*}(\Delta_f)$ is a closed immersion. This map can be identified with the diagonal map $W_{n*}(X) \to W_{n*}(X) \times_{W_{n*}(Y)} W_{n*}(X)$, and so the result follows.

(h): By 11.1, the map $W_{n*}(f)$ is smooth. By 11.2, it is surjective over $S_0 = \operatorname{Spec} \mathcal{O}/\mathfrak{m}$, and by (10.4.8), it can be identified away from S_0 with the product map $X^{[0,n]} \to Y^{[0,n]}$, which is surjective. Therefore $W_{n*}(f)$ is, too. □

13.4. *Counterexamples.* Consider the p-typical case: $R = \mathbf{Z}$, $E = \{p\mathbf{Z}\}$, where p is a prime number. Then a short computation using 3.4 shows

$$\Lambda_1 \odot (\mathbf{Z} \times \mathbf{Z}) \cong \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z}[1/p] \times \mathbf{Z}[1/p].$$

So W_{n*} does not generally preserve any property which the map $\mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ has and which is at least as strong as integrality: integral, finite, finite flat, finite étale,....

Also, W_{n*} does not generally preserve surjectivity (of schemes), because we have

$$\begin{aligned}\Lambda_1 \odot \mathbf{Z}_p[x]/(x^2 - p) &= \mathbf{Z}_p[x, \delta]/(x^2 - p, p\delta^2 + 2x^p\delta + p^{p-1} - 1)\\ &\cong \mathbf{Q}_p(\sqrt{p}) \times \mathbf{Q}_p(\sqrt{p})\end{aligned}$$

(by 3.4 again) but also $\Lambda_1 \odot \mathbf{Z}_p = \mathbf{Z}_p$.

Finally, as shown in 12.7, the map $\mathbf{Z} \to \mathbf{Z}[x]/(x^2 - px)$ becomes non-flat after the application of $\Lambda_1 \odot -$. So W_{n*} does not generally preserve any property which the map $\mathbf{Z} \to \mathbf{Z}[x]/(x^2 - px)$ has and which is at least as strong as flatness: flat, faithfully flat, Cohen–Macaulay, S_k,....

14. The inductive lemma for W_n^*

We continue with the notation of 10.2, but we restrict to the case where E consists of only one ideal $\mathfrak{m}$. The purpose of this section is to establish the following lemma:

14.1. Lemma. *Let X be an object of AlgSp_m, with $m \in \mathbf{Z}$.*

(a) *$W_n^*(X)$ is an algebraic space; and for any $(m-1)$-representable étale surjection $g\colon U \to X$, where U is a disjoint union of affine schemes, the space $W_n^*(U \times_X U)$ is an étale equivalence relation on $W_n^*(U)$ with respect to the map*

$$(W_n^*(\mathrm{pr}_1), W_n^*(\mathrm{pr}_2))\colon W_n^*(U \times_X U) \longrightarrow W_n^*(U) \times_{W_n^*(X)} W_n^*(U), \tag{14.1.1}$$

and the induced map

$$W_n^*(U)/W_n^*(U \times_X U) \longrightarrow W_n^*(X)$$

is an isomorphism. In particular, (14.1.1) is an isomorphism.

(b) *For any map g as in (a), the diagram*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "w_0"'] & X \arrow[d, "w_0"] \\ W_n^*(U) \arrow[r, "W_n^*(g)"] & W_n^*(X) \end{tikzcd}$$

is cartesian.

(c) *The map*

$$X \times_S S_0 \xrightarrow{w_0 \times \mathrm{id}} W_n^*(X) \times_S S_0 \tag{14.1.2}$$

is a closed immersion defined by a square-zero ideal sheaf, where S_0 denotes $\operatorname{Spec} \mathcal{O}_S/\mathfrak{m}$.

(d) *For any object $X' \in \mathsf{AlgSp}_m$ and any étale map $f\colon X' \to X$, the map*

$$W_n^*(f)\colon W_n^*(X') \to W_n^*(X)$$

is étale; and for any algebraic space Y over X, the map

$$(W_n^*(\mathrm{pr}_1), W_n^*(\mathrm{pr}_2))\colon W_n^*(X' \times_X Y) \longrightarrow W_n^*(X') \times_{W_n^*(X)} W_n^*(Y) \tag{14.1.3}$$

is an isomorphism.

Proof. We will prove all parts at once by induction on m. For clarity, write $(\mathrm{a})_m$ for the statement (a) above, and so on.

First consider the case where $m \leqslant -1$. Here we use the fact that W_n^* preserves coproducts together with the analogous affine results: $(\mathrm{a})_m$ follows from (10.3.1), 9.2, 6.9, and 9.4; $(\mathrm{c})_m$ follows from 6.8; and $(\mathrm{d})_m$ follows from 9.2 and 9.4.

It remains to prove $(\mathrm{b})_m$. It is enough to assume U and X are affine. Consider the map

$$a = (g, w_0)\colon U \longrightarrow X \times_{W_n^*(X)} W_n^*(U).$$

By assumption, the source is étale over X, and by 9.2 so is the target. Therefore a itself is étale, and so to show it is an isomorphism, it is enough by 14.3 below to show that the maps $(S' \times_S a)_{\mathrm{red}}$ and $(S_0 \times_S a)_{\mathrm{red}}$ are isomorphisms, where $S' = S - S_0$. On the one hand, $S' \times_S a$ agrees, by (10.4.6), with $S' \times_S -$ applied to the evident map

$$U \longrightarrow X \times_{(\coprod_{[0,n]} X)} \coprod_{[0,n]} U,$$

which is an isomorphism. On the other hand, by 6.8, the map $(S_0 \times a)_{\mathrm{red}}$ agrees with b_{red}, where b is the evident map $S_0 \times_S U \to S_0 \times_S (X \times_X U)$. Since b is an isomorphism, so is $(S_0 \times a)_{\mathrm{red}}$. This proves $(\mathrm{b})_m$ and hence the lemma for $m \leqslant -1$.

From now on, assume $m \geqslant 0$.

(a)$_m$: First, observe that it follows from the rest of (a) that $W_n^*(X)$ is an algebraic space. Indeed, because X is in AlgSp_m, there exists a map g as in (a). Assuming the rest of (a), we have

$$W_n^*(X) \cong W_n^*(U)/W_n^*(U \times_X U),$$

and so it is enough to show that $W_n^*(U)$ and $W_n^*(U \times_X U)$ are algebraic spaces. This follows from (a)$_{m-1}$ because $U, U \times_X U \in \mathsf{AlgSp}_{m-1}$.

The diagram

$$U \times_X U \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} U \xrightarrow{f} X$$

is a coequalizer diagram and, since W_n^* commutes with colimits, so is

$$W_n^*(U \times_X U) \underset{W_n^*(\mathrm{pr}_2)}{\overset{W_n^*(\mathrm{pr}_1)}{\rightrightarrows}} W_n^*(U) \xrightarrow{W_n^*(f)} W_n^*(X). \tag{14.1.4}$$

Thus, all that remains is to show that $W_n^*(U \times_X U)$ is an étale equivalence relation on $W_n^*(U)$ under the structure map (14.1.1). By (d)$_{m-1}$, the projections $W_n^*(\mathrm{pr}_i)$ in (14.1.4) are étale. Let us now show that $W_n^*(U \times_X U)$ is an equivalence relation.

Let t denote the map (14.1.1). Let us first show that t is a monomorphism. We can view this as a map of algebraic spaces over $W_n^*(U)$ by projecting onto the first factor, say. Then since $W_n^*(U \times_X U)$ is étale over $W_n^*(U)$, it is enough, by 14.2 below, to show that $t \times_S S'$ and $(t \times_S S_0)_{\mathrm{red}}$, are monomorphisms. For $t \times_S S'$, we may assume $\mathfrak{m}$ is the unit ideal, by (10.4.6); then t agrees with the evident map

$$\coprod_{[0,n]} U \times_X U \longrightarrow \Big(\coprod_{[0,n]} U\Big) \times_S \Big(\coprod_{[0,n]} U\Big),$$

which is a monomorphism. On the other hand, by (c)$_{m-1}$, the map $(t \times_S S_0)_{\mathrm{red}}$ can be identified with u_{red}, where u is the evident map

$$U \times_X U \times_S S_0 \longrightarrow U \times_S U \times_S S_0,$$

which is a monomorphism. This proves t is a monomorphism.

Now let us show that $W_n^*(U \times_X U)$ is an equivalence relation on $W_n^*(U)$. It is reflexive and symmetric, simply because W_n^* is a functor and $U \times_X U$ is reflexive and symmetric relation on U. Let us show transitivity.

Write

$$\Gamma_1 = U \times_X U \quad \text{and} \quad \Gamma_2 = W_n^*(U) \times_S W_n^*(U).$$

Then by definition, $W_n^*(\Gamma_1)$ is transitive if and only if there exists a map c' making the right-hand square in the diagram

$$\begin{tikzcd} W_n^*(\Gamma_1 \times_U \Gamma_1) \arrow[r, "t'"] \arrow[dr, "W_n^*(c_1)"'] & W_n^*(\Gamma_1) \times_{W_n^*(U)} W_n^*(\Gamma_1) \arrow[r, "t \times t"] \arrow[d, dashed, "c'"] & \Gamma_2 \times_{W_n^*(U)} \Gamma_2 \arrow[d, "c_2"] \\ & W_n^*(\Gamma_1) \arrow[r, tail, "t"] & \Gamma_2 \end{tikzcd}$$

commute, where each c_i is the transitivity map for the equivalence relation Γ_i. If we define $t' = (W_n^*(\mathrm{pr}_1), W_n^*(\mathrm{pr}_2))$, then the perimeter commutes. Therefore it is enough to show that t' is an isomorphism. This follows from (d)$_{(m-1)}$, which we can apply because we have $\Gamma_1, U \in \mathsf{AlgSp}_{m-1}$, as discussed above.

(b)$_m$: To show that the map

$$(g, w_0)\colon U \longrightarrow X \times_{W_n^*(X)} W_n^*(U)$$

is an isomorphism, it suffices to do so after applying $U \times_X -$. We can do that as follows:

$$\begin{aligned} U \times_X U &\overset{1}{=} U \times_{W_n^*(U)} W_n^*(U \times_X U) \\ &\overset{2}{=} U \times_{W_n^*(U)} W_n^*(U) \times_{W_n^*(X)} W_n^*(U) \\ &= U \times_{W_n^*(X)} W_n^*(U). \end{aligned}$$

Equality 2 follows from $(\mathrm{a})_m$. Thus it suffices to show equality 1.

Let $h\colon V \to U \times_X U$ be an $(m-2)$-representable étale cover, where $V \in \mathsf{AlgSp}_{-1}$; this exists because $U \times_X U \in \mathsf{AlgSp}_{m-1}$. Consider the following diagram:

$$\begin{tikzcd} V \arrow[r, "h"] \arrow[d, "w_0"] & U \times_X U \arrow[r, "\mathrm{pr}_1"] \arrow[d, "w_0"] & U \arrow[d, "w_0"] \\ W_n^*(V) \arrow[r, "W_n^*(h)"] & W_n^*(U \times_X U) \arrow[r, "\mathrm{pr}_1"] & W_n^*(U). \end{tikzcd}$$

By $(\mathrm{b})_{m-1}$, the left-hand square is cartesian, since $U \times_X U \in \mathsf{AlgSp}_{m-1}$. Further, the perimeter is cartesian; this is because U and V are disjoint unions of affine schemes, and so we can apply $(\mathrm{b})_{-1}$ on each component. Therefore the induced map

$$U \times_X U \longrightarrow U \times_{W_n^*(U)} W_n^*(U \times_X U) \tag{14.1.5}$$

becomes an isomorphism when we apply the functor $- \times_{W_n^*(U \times_X U)} W_n^*(V)$. But the map $W_n^*(V) \to W_n^*(U \times_X U)$ is an étale cover, by $(\mathrm{a})_{m-1}$. So this implies (14.1.5) is an isomorphism.

$(\mathrm{c})_m$: Let $g\colon U \to X$ be an $(m-1)$-representable étale cover, where $U \in \mathsf{AlgSp}_{-1}$. Then $W_n^*(g)$ is an étale cover, by $(\mathrm{a})_m$. Therefore it is enough to show that (14.1.2) becomes a closed immersion defined by a square-zero ideal after base change from $W_n^*(X)$ to $W_n^*(U)$—indeed, this is an étale-local property. But by $(\mathrm{b})_m$, this map can be identified with

$$w_0 \times \mathrm{id}_{S_0}\colon U \times_S S_0 \longrightarrow W_n^*(U) \times_S S_0,$$

which has the required property by $(\mathrm{c})_{-1}$.

$(\mathrm{d})_m$: Let $u'\colon U' \to X'$ and $u\colon U \to X$ be $(m-1)$-representable étale covers, where $U', U \in \mathsf{AlgSp}_{-1}$, such that the map $f\colon X' \to X$ lifts to a map $h\colon U' \to U$, necessarily étale. Then we have a commutative diagram

$$\begin{tikzcd} W_n^*(U') \arrow[r, "W_n^*(u')"] \arrow[d, "W_n^*(h)"'] & W_n^*(X') \arrow[d, "W_n^*(f)"] \\ W_n^*(U') \arrow[r, "W_n^*(u)"] & W_n^*(X). \end{tikzcd}$$

By $(\mathrm{a})_m$, all the spaces in this diagram are algebraic, and the horizontal maps are étale covers. By $(\mathrm{d})_{-1}$, the map $W_n^*(h)$ is étale. Therefore $W_n^*(f)$ is étale.

Let us now show that (14.1.3) is an isomorphism. The map

$$\mathrm{pr}_2\colon W_n^*(X') \times_{W_n^*(X)} W_n^*(Y) \longrightarrow W_n^*(Y)$$

is étale, because it is a base change of $W_n^*(f)$. The map

$$W_n^*(\mathrm{pr}_2)\colon W_n^*(X' \times_X Y) \longrightarrow W_n^*(Y)$$

is also étale. Indeed, by the above, we only need to verify $X' \times_X Y \in \mathsf{AlgSp}_m$. This holds because AlgSp_m is stable under pull back, by [40], Corollary 1.3.3.5.

Therefore, we can view (14.1.3), which we will denote t, as a map of étale algebraic spaces over $W_n^*(Y)$. So, to show t is an isomorphism, it is enough by 14.3 below to show $(t \times_S S')_{\mathrm{red}}$ and $(t \times_S S_0)_{\mathrm{red}}$ are isomorphisms. This can be done as in the proof of (b): for $t \times_S S'$, use (10.4.6) to reduce the question to one about ghost components; for $(t \times_S S_0)_{\mathrm{red}}$, use $(\mathrm{c})_m$. □

14.2. Lemma. *Consider a commutative diagram of algebraic spaces*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & & Y \arrow[dl] \\ & Z, & \end{tikzcd}$$

where g is étale. Then the following hold:

(a) *f is a monomorphism if and only if f_{red} is.*
(b) *Let Z_0 be a closed algebraic subspace of Z, and let Z' be its complement. Then f is a monomorphism if and only if $f \times_Z Z_0$ and $f \times_Z Z'$ are.*

Proof. The only-if parts of both statements follow immediately from the fact that both closed and open immersions are monomorphisms.

(a): It is enough to show that for any affine scheme T and any maps $a, b\colon T \to X$ such that $f \circ a = f \circ b$, we have $a = b$. Then we have $f_{\mathrm{red}} \circ a_{\mathrm{red}} = f_{\mathrm{red}} \circ b_{\mathrm{red}}$. Since f_{red} is assumed to be a monomorphism, we have $a_{\mathrm{red}} = b_{\mathrm{red}}$. Since X is étale over Z, we have $a = b$ (EGA IV 18.1.3 [28]).

(b): Again, let T be an affine scheme with maps $a, b\colon T \to X$ such that $f \circ a = f \circ b$. Let $\bar{T}$ denote the equalizer of a and b. It is an algebraic subspace of T. By the assumptions on f, we have $\bar{T} \times_Z Z_0 = T \times_Z Z_0$ and $\bar{T} \times_Z Z' = T \times_Z Z'$. Therefore $\bar{T}$ is a closed subscheme of T defined by a nil ideal. As above, since X is étale over Z, there is at most one extension of the Z-morphism $\bar{T} \to X$ to T. Therefore $a = b$. □

14.3. Lemma. *Let $f\colon X \to Y$ be an étale map of algebraic spaces, and let Y_0 be a closed algebraic subspace of Y with complement Y'. Then f is an isomorphism if and only if $(f \times_Y Y_0)_{\mathrm{red}}$ and $(f \times_Y Y')_{\mathrm{red}}$ are.*

Proof. The only-if statement is clear. Now consider the converse. It follows from 14.2 that f is a monomorphism, and so it is enough to show that f is an epimorphism. To do this, it is enough to show that any étale map $V \to Y$, with V affine, lifts to a map $V \to X$. Thus, by changing base to V and relabeling $Y = V$, we may assume Y is affine. (The property of being an isomorphism after applying $(-)_{\mathrm{red}}$ is stable under base change.) Now let $(U_i)_{i \in I}$ be an étale cover of X, where each U_i is an affine scheme. Then each composition $U_i \to X \to Y$ is an étale morphism of affine schemes, and the union of images of these maps covers Y. Therefore the induced map $\coprod_i U_i \to Y$ is an epimorphism, and hence so is f. □

15. W_n^* and algebraic spaces

The purpose of this section is to give a number of useful consequences of the inductive lemma in the previous section. We continue with the notation of 10.2.

15.1. Theorem. *Let X be an algebraic space over S. Then $W_n^*(X)$ is an algebraic space.*

Proof. By (10.6.1), we may assume E consists of one ideal, in which case we can apply 14.1(a). □

15.2. Theorem. *Let $f\colon X' \to X$ be an étale map of algebraic spaces over S. Then the following hold.*

(a) *The induced map $W_n^*(f)\colon W_n^*(X') \to W_n^*(X)$ is étale.*
(b) *If f is surjective, then so is $W_n^*(f)$.*

(c) *For any algebraic space Y over X, the map $(W_n^*(\mathrm{pr}_1), W_n^*(\mathrm{pr}_2))$*

$$W_n^*(X' \times_X Y) \longrightarrow W_n^*(X') \times_{W_n^*(X)} W_n^*(Y)$$

is an isomorphism.

Proof. By (10.6.1), we may assume E consists of one ideal. Then parts (a) and (c) follow from 14.1(d). For part (b), it is enough, by passing to an étale cover of X', to assume $X' \in \mathsf{AlgSp}_{-1}$. Then we can apply 14.1(a), because f is 1-representable. □

15.3. Corollary. *Let $(U_i)_{i \in I}$ be an étale cover of an algebraic space X over S. Then $\big(W_n^*(U_i)\big)_{i \in I}$ is an étale cover of $W_n^*(X)$, and for each pair $(i,j) \in I^2$, the map*

$$W_n^*(U_i \times_X U_j) \longrightarrow W_n^*(U_i) \times_{W_n^*(X)} W_n^*(U_j)$$

given by $\big(W_n^*(\mathrm{pr}_1), W_n^*(\mathrm{pr}_2)\big)$ is an isomorphism.

In other words, $W_n^*(X)$ can be constructed by charts in the étale topology.

Proof. Because W_n^* is a left adjoint, it preserves disjoint unions. Then apply 15.1(b) to the induced map $\coprod_i U_i \to X$ and 15.1(c) to $U_i \times_X U_j$. □

15.4. Corollary. *Let $f\colon X \to Y$ be an étale map of algebraic spaces. Then the following diagrams are cartesian, where the horizontal maps are the ones defined in 10.6:*

(a) *for $i \in \mathbf{N}^{(E)}$,*

$$\begin{tikzcd} W_n^*(X) \arrow[r, "\psi_i"] \arrow[d] & W_{n+i}^*(X) \arrow[d] \\ W_n^*(Y) \arrow[r, "\psi_i"] & W_{n+i}^*(Y); \end{tikzcd}$$

(b) *for $i = \mathbf{N}^{(E)}$,*

$$\begin{tikzcd} W_n^*(X) \arrow[r, "r_{n,i}"] \arrow[d] & W_{n+i}^*(X) \arrow[d] \\ W_n^*(Y) \arrow[r, "r_{n,i}"] & W_{n+i}^*(Y); \end{tikzcd}$$

(c) *for $i \in [0,n]$,*

$$\begin{tikzcd} X \arrow[r, "w_i"] \arrow[d] & W_n^*(X) \arrow[d] \\ Y \arrow[r, "w_i"] & W_n^*(Y); \end{tikzcd}$$

(d) *when $E = \{\mathfrak{m}\}$ and $i \in \mathbf{N}$,*

$$\begin{tikzcd} X \times_S S_n \arrow[r, "\bar{w}_i"] \arrow[d] & W_n^*(X) \arrow[d] \\ Y \times_S S_n \arrow[r, "\bar{w}_i"] & W_n^*(Y), \end{tikzcd}$$

where $S_n = \operatorname{Spec} \mathcal{O}_S/\mathfrak{m}^{n+1}$.

Proof. By (10.6.1), we may assume E consists of one ideal $\mathfrak{m}$. All four parts are proved by the same method. Let us give the details for (a) and leave the rest to the reader.

We want to show that the induced map

$$g\colon W_n^*(X) \longrightarrow W_n^*(Y) \times_{W_{n+i}^*(Y)} W_{n+i}^*(X)$$

is an isomorphism. By 15.2, this is a map of étale algebraic spaces over $W_n^*(Y)$. Therefore, to show it is an isomorphism, it is enough by 14.3 to show that $(g \times_S S_0)_{\mathrm{red}}$ and $(g \times_S S')_{\mathrm{red}}$ are isomorphisms, where $S_0 = \operatorname{Spec} \mathcal{O}_S/\mathfrak{m}$ and $S' = S - S_0$.

It is easy check that $g \times_S S'$ is an isomorphism. Write $X_0 = X \times_S S_0$, $Y_0 = Y \times_S S_0$, and let F denote the q-th power Frobenius map. Then the map $(g \times_S S_0)_{\mathrm{red}}$, can be identified with h_{red}, where $h\colon X_0 \to Y_0 \times_{F^i, Y_0} X_0$ is the map induced by the diagram

$$\begin{tikzcd} X_0 \arrow[r, "F^i"] \arrow[d] & X_0 \arrow[d] \\ Y_0 \arrow[r, "F^i"] & Y_0. \end{tikzcd}$$

This diagram is cartesian (SGA 5 XV §1, Proposition 2(c)[3]), and so h and h_{red} are isomorphisms. □

15.5. Corollary. *Let X be an algebraic space over S.*

(a) *Let*

$$U \longrightarrow Y \rightrightarrows Z$$

be an equalizer diagram of algebraic spaces over X. If Z is étale over X, then the induced diagram

$$W_n^*(U) \longrightarrow W_n^*(Y) \rightrightarrows W_n^*(Z)$$

is also an equalizer diagram.

(b) *Let $(Y_i)_{i \in I}$ be a finite diagram of étale algebraic X-spaces. Then the following natural map is an isomorphism:*

$$W_n^*(\lim_{i \in I} Y_i) \xrightarrow{\sim} \lim_{i \in I} W_n^*(Y_i).$$

Here the limits are taken in the category of X-spaces.

Proof. (a): Since the structure map $Z \to X$ is étale, so is the diagonal map $Z \to Z \times_X Z$. And since U is $Y \times_{Z \times_X Z} Z$, we have by 15.2(c)

$$\begin{aligned} W_n^*(U) &= W_n^*(Y) \times_{W_n^*(Z \times_X Z)} W_n^*(Z) \\ &= W_n^*(Y) \times_{(W_n^*(Z) \times_{W_n^*(X)} W_n^*(Z))} W_n^*(Z) \end{aligned}$$

Thus $W_n^*(U)$ is the equalizer of the two induced maps $W_n^*(Y) \rightrightarrows W_n^*(Z)$.

(b): To show a functor preserves finite limits, it is sufficient to show it preserves finite products and equalizers of pairs of arrows. The first follows from 15.2(c), and the second from part (a) above. □

15.6. Corollary. *Let $j\colon U \to X$ be an open immersion of algebraic spaces. Then the map $W_n^*(j)\colon W_n^*(U) \to W_n^*(X)$ is an open immersion. If X is a scheme, then so is $W_n^*(X)$.*

Proof. An open immersion is the same as an étale monomorphism. By 15.2, $W_n^*(j)$ is étale, and so we only need to show it is a monomorphism or, equivalently, that its diagonal map is an isomorphism. By 15.2(c), the diagonal map of $W_n^*(j)$ agrees with $W_n^*(\Delta_j)$, where Δ_j is the diagonal map $U \to U \times_X U$ of j. Because j is a monomorphism, Δ_j is an isomorphism, and hence so is $W_n^*(\Delta_j)$. Therefore the diagonal map of $W_n^*(j)$ is an isomorphism, and so $W_n^*(j)$ is a monomorphism.

Now suppose X is a scheme. Let $(U_i)_{i \in I}$ be an open cover of X. By 15.3, $W_n^*(X)$ is an algebraic space covered by the $W_n^*(U_i)$, and by the above, each map $W_n^*(U_i) \to W_n^*(X)$ is an open immersion. Therefore X is a scheme. □

15.7. Corollary. *Let X be an algebraic space over S.*

(a) *The map* $w_{\leqslant n}\colon \coprod_{[0,n]} X \to W_n^*(X)$ *is surjective and integral, and the kernel* I *of the induced map*

$$\mathcal{O}_{W_n^*(X)} \to w_{\leqslant n*}(\mathcal{O}_{\coprod_{[0,n]} X})$$

satisfies $I^{2^N} = 0$*, where* $N = \sum_{\mathfrak{m}} n_{\mathfrak{m}}$.

(b) *The map* $w_0\colon X \to W_n^*(X)$ *is a closed immersion.*

Proof. All the properties in question are étale-local on $W_n^*(X)$, and hence S. Therefore, we may assume that S is affine, by (10.4.6), and that X is affine, by 15.4(b). In this case, (a) was proved in 10.8, and (b) follows from the surjectivity of the map $w_0\colon W_n(A) \to A$, which follows from the existence of the Teichmüller section (1.21), say. □

15.8. Corollary. *The functor* $W_n^*\colon \mathsf{AlgSp} \to \mathsf{AlgSp}$ *is faithful.*

Proof. The map w_0 is easily seen to be equal to the composition

$$X \xrightarrow{\varepsilon} W_{n*}W_n^*X \xrightarrow{\kappa_0} W_n^*X,$$

where ε is the unit of the evident adjunction, and κ_0 is as in (10.6.4). Therefore by 15.7(b), the map ε is a monomorphism. Equivalently, W_n^* is faithful. □

15.9. W_n^* *is generally not full.* For example, if we consider the usual p-typical Witt vectors over $\mathbf{Z}$ of length n, and if A and B are $\mathbf{Z}[1/p]$-algebras, then we have

$$\mathrm{Hom}_{W_n(\mathbf{Z})}(W_n(A), W_n(B)) = \mathrm{Hom}_{\mathbf{Z}^{[0,n]}}(A^{[0,n]}, B^{[0,n]}) = \mathrm{Hom}(A,B)^{[0,n]},$$

which is usually not the same as $\mathrm{Hom}(A,B)$. To be sure, the entire point of the theory is in applying W to rings where p is not invertible.

15.10. Corollary. *Let* $f\colon X \to Y$ *be an étale map of algebraic spaces over* S*. Then the diagram*

$$\begin{tikzcd} W_m^*W_n^*(X) \arrow[r, "\mu_X"] \arrow[d, "W_m^*W_n^*(f)"'] & W_{m+n}^*(X) \arrow[d, "W_{m+n}^*(f)"] \\ W_m^*W_n^*(Y) \arrow[r, "\mu_Y"] & W_{m+n}^*(Y) \end{tikzcd} \tag{15.10.1}$$

is cartesian, where μ_X *and* μ_Y *are the plethysm maps of (10.6.13).*

Proof. We use the usual method, as in 15.4. Let us assume that E consists of one ideal $\mathfrak{m}$. This is sufficient by 15.2, (10.6.1), and a short argument we we leave to the reader.

By 15.2, the map

$$W_m^*W_n^*(X) \xrightarrow{g} W_{m+n}^*(X) \times_{W_{m+n}^*(Y)} W_m^*W_n^*(Y) \tag{15.10.2}$$

is étale, and so we only need to show $g \times_S S'$ and $(g \times_S S_0)_{\mathrm{red}}$ are isomorphisms, where $S_0 = \mathrm{Spec}\, \mathcal{O}_S/\mathfrak{m}$ and $S' = S - S_0$.

Consider $g \times_S S'$ first. By (10.4.6), we may assume $\mathfrak{m}$ is the unit ideal. Then diagram (15.10.1) can be identified with the diagram

$$\begin{tikzcd} \coprod_{[0,m]}\coprod_{[0,n]} X \arrow[r, "\mu_X"] \arrow[d] & \coprod_{[0,m+n]} X \arrow[d] \\ \coprod_{[0,m]}\coprod_{[0,n]} Y \arrow[r, "\mu_Y"] & \coprod_{[0,m+n]} Y, \end{tikzcd}$$

where each map μ sends component $(i,j) \in [0,m] \times [0,n]$ identically to component $i+j \in [0,m+n]$. Since this diagram is cartesian, $g \times_S S'$ is an isomorphism.

Now consider $(g \times_S S_0)_{\mathrm{red}}$. Write $(-)'$ for the functor $T \mapsto (S_0 \times_S T)_{\mathrm{red}}$; thus we want to show g' is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd} W_m^* W_n^*(X)' \arrow[r, "g'"] \arrow[dr, "b"'] & \big(W_{m+n}^*(X) \times_{W_{m+n}^*(Y)} W_m^* W_n^*(Y)\big)' \arrow[d, "c"] \\ & W_{m+n}^*(X)' \times_{W_{m+n}^*(Y)'} W_m^* W_n^*(Y)', \end{tikzcd}$$

where c is the evident map induced by the universal property of products. Since the functor $(-)'$ sends étale maps to étale maps, c is étale; also c_{red} is an isomorphism, and so c is an isomorphism. Therefore it is enough to show that b is isomorphism—in other words, that diagram (15.10.1) becomes cartesian after applying $(-)'$.

To do this, it is enough to show that for $T = X, Y$ (or any algebraic space over S), the map $\mu'_T \colon W_m^* W_n^*(T)' \to W_{m+n}^*(T)'$ is an isomorphism. Consider the commutative diagram

$$\begin{tikzcd} & T \arrow[dl, "w_0"'] \arrow[dr, "w_0"] & \\ W_n^*(T) \arrow[r, "w_0"'] & W_m^* W_n^*(T) \arrow[r, "\mu_T"'] & W_{m+n}^*(T). \end{tikzcd}$$

If we apply $S_0 \times -$ to this diagram, all maps labeled w_0 become closed immersions defined by square-zero ideals, by 14.1(c). Thus they all become isomorphisms after applying $(-)'$, and therefore so does μ_T. □

16. Preservation of geometric properties by W_n^*

We continue with the notation of 10.2.

Sheaf-theoretic properties

16.1. Proposition. *Let X be a quasi-compact object of Sp_S. Then $W_n^*(X)$ is quasi-compact.*

Proof. Since X is quasi-compact, it has a finite cover $(U_i)_{i\in I}$ by affine schemes. Therefore $W_n^*(\coprod_i U_i)$ is affine, and since $W_n^*(X)$ is covered by this space, it must be quasi-compact. □

General localization

16.2. Proposition. *Let P be an étale-local property of algebraic spaces X over S. For W_n^* to preserve property P, it is sufficient that it do so when E consists of one principal ideal and both X and S are affine schemes.*

Proof. When E is empty, W_n^* is the identity functor. Therefore by (10.6.1), it is enough to consider the case where E consists of one ideal $\mathfrak{m}$.

Let X be an algebraic space satisfying property P, and let $(U_i)_{i\in I}$ be an étale cover of X such that each U_i is affine and lies over an affine open subscheme S_i of S over which $\mathfrak{m}$ is principal. Because P is étale local, each U_i satisfies P; and since we have $W_{S,n}^*(U_i) = W_{S_i,n}^*(U_i)$, by (10.4.8), so does each $W_n^*(U_i)$. But the spaces $W_n^*(U_i)$ form an affine étale cover of $W_n^*(X)$, by 15.2. Therefore $W_n^*(X)$ satisfies P. □

16.3. Proposition. *Let P be a property of maps $f\colon X \to Y$ of algebraic spaces which is étale-local on the target. For W_n^* to preserve property P, it is sufficient that it do so when E consists of one principal ideal, S is affine, and Y is affine.*

If property P is also étale-local on the source, then we may further restrict to the case where X is affine.

Proof. Let $f\colon X \to Y$ be a map satisfying P. As in the proof of 16.2, it is enough to consider the case where E consists of one ideal $\mathfrak{m}$.

Let us show the first statement. Let $(V_j)_{j\in J}$ be an étale cover of Y such that each V_j is affine and lies over an affine open subscheme subscheme S_j of S on which $\mathfrak{m}$ is principal. Then $\big(W_n^*(V_j)\big)_{j\in J}$ is an étale cover of $W_n^*(Y)$, by 15.2(b). Therefore $W_n^*(f)$ satisfies P if its base change to each $W_n^*(V_j)$ does. By 15.2(c), this base change can be identified with

$$W_n^*(f_{V_j})\colon W_n^*(V_j \times_Y X) \longrightarrow W_n^*(V_j).$$

Since f_{V_j} satisfies P, so does $W_{S',n}^*(f_{V_j})$, by the assumptions of the proposition. By (10.4.8), we have $W_n^*(f_{V_j}) = W_{S',n}^*(f_{V_j})$, and so $W_n^*(f_{V_j})$ also satisfies P.

Now suppose property P is also étale-local on the source. By what we just proved, we may assume Y and S are affine. Let $(U_i)_{i\in I}$ be an étale cover of X, with each U_i affine. Then each composition $U_i \to X \to Y$ satisfies P. Therefore so does each composition $W_n^*(U_i) \to W_n^*(X) \to W_n^*(Y)$. But again by 15.2(b), the spaces $W_n^*(U_i)$ form an étale cover of $W_n^*(X)$. Since P is local on the source, $W_n^*(f)$ satisfies P. □

Affine properties of maps

16.4. Proposition. *The following (affine) properties of maps of algebraic spaces are preserved by W_n^*:*

(a) *affine,*
(b) *a closed immersion,*
(c) *integral,*
(d) *finite étale.*

Proof. Let $f\colon X \to Y$ be a map satisfying one of these properties. In particular, f is affine. Because the properties are local on the target, it is enough, by 16.3, to assume that both Y and S are affine and that E consists of one ideal $\mathfrak{m}$. But because f is affine, X must also be an affine scheme. In other words, it is enough to show W_n^* preserves these properties for maps of affine schemes. For (a), there is nothing to prove, and (b) is true by 6.5.

Let us prove (c). Write $S = \operatorname{Spec} R$. Let A be an R-algebra, and let B be an integral A-algebra. Consider the induced diagram

$$\begin{tikzcd} W_n(B) \arrow[r, "w_{\leqslant n}"] & B^{[0,n]} \\ W_n(A) \arrow[u] \arrow[r, "w_{\leqslant n}"] & A^{[0,n]}. \arrow[u] \end{tikzcd}$$

Since B is integral over A, we know that $B^{[0,n]}$ is integral over $A^{[0,n]}$. By 8.2, $A^{[0,n]}$ is integral over $W_n(A)$, and hence so is $B^{[0,n]}$, and hence so is the image of the ghost map $w_{\leqslant n}$. But the kernel of $w_{\leqslant n}$ is nilpotent; so $W_n(B)$ is integral over $W_n(A)$. □

Absolute properties and properties relative to S

16.5. Proposition. *The following (étale local) properties of algebraic spaces over S are preserved by W_n^*:*

(a) *locally of finite type over S,*
(b) *flat over S,*
(c) *flat over S and reduced,*
(d) *of Krull dimension d.*

Proof. Because these are étale-local properties, by 16.2 we can write $S = \operatorname{Spec} R$, $X = \operatorname{Spec} A$, and $E = \{\pi R\}$ with $\pi \in R$.

(a): Let T be a finite subset of A generating it as an R-algebra, and let B denote the sub-R-algebra of $W_n(A)$ generated by the set $\{[t] : t \in T\}$ of Teichmüller lifts (3.9). It is enough to show that $W_n(A)$ is finitely generated as a B-module. By induction, we may assume $W_{n-1}(A)$ is finitely generated. Therefore it is enough to show that $V^n W_n(A)$ is finitely generated. We will do this by showing that the subset

$$T = \Big\{ V_\pi^n \Big[\prod_{t\in T} t^{a_t} \Big] \ \mid\ 0 \leqslant a_t < q^n \text{ for all } t \in T \Big\} \subseteq V^n W_n(A), \tag{16.5.1}$$

where $[x]$ denotes the Teichmüller lift of x, generates $V^n W_n(A)$ as a B-module. Indeed, for any monomial $\prod_t t^{c_t}$, write $c_t = b_t q^n + a_t$ with $0 \leqslant a_t < q^n$. Then we have by (3.9.1)

$$V_\pi^n \Big[\prod_t t^{c_t} \Big] = \Big(\prod_t [t]^{b_t} \Big) \Big(V_\pi^n \Big[\prod_t t^{a_t} \Big] \Big).$$

(b): Since A is flat over R, the ghost map $w_{\leqslant n} \colon W_n(A) \to A^{[0,n]}$ is injective (2.7). Since $A^{[0,n]}$ is $\mathfrak{m}$-flat (10.11), so is $W_n(A)$. But $R[1/\pi] \otimes_R W_n(A)$ is also flat, because it agrees with $(A[1/\pi])^{[0,n]}$, by (10.4.6). Therefore $W_n(A)$ is flat over R.

(c): By (b), we only need to show if A is flat and reduced over R, then $W_n(A)$ is reduced. Since A is flat over R, the ghost map $w_{\leqslant n} \colon W_n(A) \to A^{[0,n]}$ is injective (2.7). And since A is also reduced, so is $W_n(A)$.

(d): The ghost map $w_{\leqslant n} \colon W_n(A) \to A^{[0,n]}$ is integral and surjective on spectra; so the Krull dimension of $W_n(A)$ agrees with that of $A^{[0,n]}$, which is d. (See EGA 0, 16.1.5 [25].) □

16.6. *Counterexamples with relative finite conditions and noetherianness.* It is not true that W_n^* preserves relative finite generation or presentation in general. For example, consider the usual p-typical Witt vectors. Let $A = \mathbf{Z}[x_1, x_2, \ldots]$, and let $B = A[t]$. It is then a short exercise to show that $W_1(B)$ is not a finitely generated $W_1(A)$-algebra.

For another, perhaps more extreme example, let $C = A[t]/(t^2)$. Then C is finite free as an A-module, but $W_1(C)$ is not finitely generated as a $W_1(A)$-algebra.

Noetherianness is also not preserved. If k is a field of characteristic p, then $W_1(k)$ is a local ring with residue field k and maximal ideal isomorphic to $k^{1/p}$. Therefore $W_1(k)$ is noetherian if and only if k has a finite p-basis.

16.7. Corollary. *The following properties of algebraic spaces over S are preserved by W_n^*:*

(a) *quasi-compact over S,*
(b) *finite type over S.*

Proof. Because these properties are étale local on S, we can assume S is affine, by (10.4.6).

(a): Since the structure map $X \to S$ is quasi-compact and S is affine, X is quasi-compact. Then $W_n^*(X)$ is quasi-compact, by 16.1. Therefore the structure map $W_n^*(X) \to S$ is quasi-compact, since S is affine. (See SGA 4 VI 1.14 [2], say.)

(b): By (a) and 16.5(a). □

16.8. Proposition. *The following properties of algebraic spaces over S are preserved by W_n^*:*

(a) *quasi-separated over S,*
(b) 0*-geometric over S (see 10.9),*
(c) *separated over S,*
(d) *separated.*

Proof. (a): Consider the diagram

$$
\begin{tikzcd}
\coprod_{[0,n]} X \arrow[r, "a"] \arrow[d, "c=w_{\leqslant n}"'] & (\coprod_{[0,n]} X) \times_S (\coprod_{[0,n]} X) \arrow[d, "b=w_{\leqslant n}\times w_{\leqslant n}"] \\
W_n^*(X) \arrow[r, "d"] & W_n^*(X) \times_S W_n^*(X),
\end{tikzcd}
\tag{16.8.1}
$$

where the horizontal maps are the diagonal maps. Because X is quasi-separated, a is quasi-compact, and by 15.7, so is b. Therefore $b \circ a$ is quasi-compact, and hence so is $d \circ c$. Now let U be an affine scheme mapping to $W_n^*(X) \times_S W_n^*(X)$, and let $(V_i)_{i\in I}$ be an étale cover of the pull back $d^*(U)$. Since $d \circ c$ is quasi-compact, there is a finite subset $J \subseteq I$ such that $(c^*V_j)_{j\in J}$ is a cover of $c^*d^*(U)$. In other words, the induced map $v\colon \coprod_{j\in J} V_j \to d^*(U)$ becomes surjective after base change by the map c. Because c is surjective (15.7), v must also be.

(b): Recall that a map is 0-geometric if and only if its diagonal map is affine (10.9). This is equivalent to requiring the existence of an étale cover $(U_i)_{i\in I}$ of X, with each U_i affine, such that $U_i \times_X U_j$ is affine. Fix such a cover of X. By 15.3, the family $\big(W_n^*(U_i)\big)_{i\in I}$ is an étale cover of $W_n^*(X)$. Therefore it is enough to show that $W_n^*(U_i) \times_{W_n^*(X)} W_n^*(U_j)$ is affine, for all $i, j \in I$. By 15.2(c), this agrees with $W_n^*(U_i \times_X U_j)$. Because X has affine diagonal, $U_i \times_X U_j$ is affine. Therefore, by 10.7, so is $W_n^*(U_i \times_X U_j)$.

(c): Let us first assume that X is of finite type over S. Consider diagram (16.8.1) above. To show that d is a closed immersion, it is enough to show that it is a finite monomorphism. (It is a general fact the a finite monomorphism of algebraic spaces is a closed immersion. To prove it, it is enough to work étale locally, which reduces us to the affine case, where it follows from Nakayama's lemma.) Since d has a retraction, it is a monomorphism. Therefore it suffices to show that d is finite. On the other hand, by 16.7(b), the structure map $W_n^*(X) \to S$ is of finite type, and hence so is d. Therefore it is enough to show that d is integral.

Since X is separated, a is a closed immersion and, in particular, is integral. Since b is integral (by 15.7), $b \circ a$ is integral and, hence, so is $d \circ c$. By part (b), the map d is affine. Therefore, by 15.7, the maps c and d can be written étale-locally on $W_n^*(X) \times_S W_n^*(X)$ as

$$\operatorname{Spec} C \xrightarrow{c} \operatorname{Spec} B \xrightarrow{d} \operatorname{Spec} A,$$

where the induced ring map $B \to C$ has nilpotent kernel. We showed above that C is integral over A. Therefore B is integral over A. This proves X is separated over S when it is of finite type.

Now consider the general case. Let us show that we can assume X is quasi-compact. To prove that d is a closed immersion, it is enough to work étale locally. Therefore, it is enough (by 15.2(b)) to show that for any affine schemes U, V with étale maps to X, the base change

$$W_n^*(U) \times_{W_n^*(X)} W_n^*(V) \xrightarrow{d'} W_n^*(U) \times_S W_n^*(V)$$

of d is a closed immersion. By 15.2(c), the source of d' agrees with $W_n^*(U \times_X V)$. Therefore, d' does not change if we replace X with the union of the images of U and V. So in particular, we can assume X is quasi-compact.

Then there exists an affine S-map $h\colon X \to X_0$, where X_0 is some separated algebraic space of finite type over S. Indeed, since X is quasi-compact and separated, by Conrad–Lieblich–Olsson [13], Theorem 1.2.2, there is an affine map $h'\colon X \to X_0'$, where X_0' is a separated algebraic space of finite type over $\mathbf{Z}$. Put $X_0 = S \times_{\mathbf{Z}} X_0'$. Then the induced map $h\colon X \to X_0$ factors as

$$X \longrightarrow S \times_{\mathbf{Z}} X \longrightarrow S \times_{\mathbf{Z}} X_0'.$$

The first map is a base change of the diagonal map $S \to S \times_{\mathbf{Z}} S$, which is a closed immersion, since S is separated. The second map is a base change of h', which is affine. Therefore both maps in the factorization above are affine, and hence so is the composition h.

Since X_0 is of finite type over S, we can apply the argument above to see that $W_n^*(X_0)$ is separated. But $W_n^*(X)$ is affine over $W_n^*(X_0)$, by 16.4(a). Therefore $W_n^*(X)$ is also separated.

(d): This follows from (c) because S is assumed to be separated. □

16.9. Proposition. *The following properties are preserved by W_n^*:*

(a) *finite over S,*
(b) *faithfully flat over S.*

Proof. These are local properties on the target S. So, by (10.4.6) and (10.6.1), we can write assume $S = \operatorname{Spec} R$ and $E = \{\mathfrak{m}\}$, for some ring R and ideal $\mathfrak{m}$. Away from $\mathfrak{m}$, the properties are clearly true. Therefore we only need to work locally near $\mathfrak{m}$, and in particular we can assume $\mathfrak{m}$ is not the unit ideal. By (10.4.6) again, we may further assume R agrees with $R_{\mathfrak{m}}$, which is a discrete valuation ring.

Let X be an algebraic space over S having the property in question.

(a): Write $X = \operatorname{Spec} A$. Then $W_n(A)$ is a subring of $A^{[0,n]}$, which is finite over R because A is. Since R is a discrete valuation ring, this implies that $W_n(A)$ is finite over R.

(b): The composition

$$\coprod_{[0,n]} X \xrightarrow{w_{\leqslant n}} W_n^*(X) \longrightarrow S$$

is surjective because $X \to S$ is. Therefore the map $W_n^*(X) \to S$ is surjective. It is flat by 16.5. □

16.10. Corollary. *The structure map $W_n^*(S) \to S$ finite and faithfully flat.*

Relative properties

16.11. Proposition. *The following properties of maps (étale-local on the target) of algebraic spaces are preserved by W_n^*.*

(a) *quasi-compact,*
(b) *universally closed,*
(c) *quasi-separated,*
(d) *separated,*
(e) *surjective.*

Proof. Let $f\colon X \to Y$ be the map in question. By 16.3, we may write $S = \operatorname{Spec} R$ and $E = \{\mathfrak{m}\}$ and we may assume that Y is affine.

(a): Since f is quasi-compact and Y is affine, X is quasi-compact. Then $W_n^*(X)$ is quasi-compact, by 16.1. Since $W_n^*(Y)$ is affine (10.7), $W_n^*(f)$ is quasi-compact. (SGA 4 VI 1.14 [2])

(b): Consider the following square:

$$\begin{array}{ccc} \coprod_{[0,n]} X & \xrightarrow{w_X} & W_n^*(X) \\ \Big\downarrow{\scriptstyle \amalg f} & & \Big\downarrow{\scriptstyle W_n^*(f)} \\ \coprod_{[0,n]} Y & \xrightarrow{w_Y} & W_n^*(Y), \end{array} \tag{16.11.1}$$

where w_X and w_Y denote the ghost maps $w_{\leqslant n}$ for X and Y. To show $W_n^*(f)$ is universally closed, it is enough to show that w_X is surjective and $w_Y \circ \coprod f$ is universally closed. (See EGA II 5.4.3(ii) and 5.4.9 [24].)

But we know w_X is surjective by 15.7; and $w_Y \circ f$ is universally closed because f is universally closed and because w_Y is integral, by 15.7, and hence universally closed. (See EGA II 6.1.10 [24].)

(c)–(d): Because Y is affine, so is $W_n^*(Y)$, by (10.3.1). Therefore being separated or quasi-separated over $W_n^*(Y)$ is equivalent to being so over S. Thus the results then follow from 16.8,

(e): Consider diagram (16.11.1). By 15.7, the map w_Y is surjective. Since f is too, so is $\coprod f$. Therefore $w_Y \circ \coprod f$ and hence $W_n^*(f)$. □

16.12. *Flatness properties.* With the p-typical Witt vectors, say, $W_1(\mathbf{Z}[x])$ is not flat over $W_1(\mathbf{Z})$. So if P is a property of morphisms which is stronger than flatness and which is satisfied by the map $\mathbf{Z} \to \mathbf{Z}[x]$, then it is not generally preserved by W_n. Examples: flat, faithfully flat, smooth, Cohen–Macaulay, and so on.

16.13. Proposition. *Let $f\colon X \to Y$ be a map of algebraic spaces having one of the following properties:*

(a) *locally of finite type,*
(b) *of finite type,*
(c) *finite,*
(d) *proper.*

Then $W_n^*(f)\colon W_n^*(X) \to W_n^*(Y)$ has the same property, as long as Y is locally of finite type over S.

Proof. (a): The composition $X \to Y \to S$ is locally of finite type because the factors are. Therefore, $W_n^*(X)$ is locally of finite type over S, by 16.5. In particular, it is locally of finite type over $W_n^*(Y)$.

(b): Part (a) above plus 16.11(a).

(c): Part (b) above plus 16.4(c).

(d): Part (b) above plus 16.11(b),(e). □

16.14. *W_n^* and properness.* Some hypotheses on Y are needed in 16.13. For example, let f be the canonical projection $\mathbf{P}^1_Y \to Y$, where $Y = \operatorname{Spec} \mathbf{Z}[x_1, x_2, \ldots]$. Then f is proper, but $W_1^*(f)$ (with p-typical Witt vectors, say) is not, because it is not of finite type. Indeed, the map $W_1^*(\mathbf{A}^1_Y) \to W_1^*(\mathbf{P}^1_Y)$ is étale (15.2), and hence of finite type, but the map $W_1^*(\mathbf{A}^1_Y) \to W_1^*(Y)$ is not (16.6).

Depth properties

16.15. Proposition. *Suppose that $S = \operatorname{Spec} R$ for some ring R and that E consists of a single maximal ideal $\mathfrak{m}$ of R. Let A be a local R-algebra whose maximal ideal contains $\mathfrak{m}$. Then $W_n(A)$ is a local ring with maximal ideal $w_0^{-1}(\mathfrak{m})$, where w_0 denotes the usual projection map $W_n(A) \to A$.*

Proof. Let I denote $w_0^{-1}(\mathfrak{m})$; it is a maximal ideal because w_0 is surjective (1.21). Let us show that it is the unique maximal ideal.

Let J be a maximal ideal of $W_n(A)$. By 8.2, the map

$$w_{\leqslant n} : W_n(A) \longrightarrow A^{[0,n]}$$

is integral and its kernel is nilpotent. Therefore, J is the pre-image of a maximal ideal of $A^{[0,n]}$. But every maximal ideal of $A^{[0,n]}$ contains $\mathfrak{m}$, because the maximal ideal of A does. Therefore J contains $\mathfrak{m}$. But I is the only maximal ideal of $W_n(A)$ containing $\mathfrak{m}$, because by 6.8, every element of I is nilpotent modulo $\mathfrak{m}W_n(A)$. Therefore $J = I$. □

16.16. Proposition. *Let $S, R, E, \mathfrak{m}$ be as in 16.15. Let A be an R-algebra, and let $\mathfrak{p}$ be a prime ideal of $W_n(A)$.*

(a) *If $\mathfrak{p}$ does not contain $\mathfrak{m}$, then there is a unique integer $i \in [0, n]$ and a unique prime ideal $\mathfrak{q}$ of A such that $w_i^{-1}(\mathfrak{q}) = \mathfrak{p}$. For this i and $\mathfrak{q}$, the map*

$$A_{\mathfrak{q}} \longrightarrow W_n(A)_{\mathfrak{p}} \tag{16.16.1}$$

induced by w_i is an isomorphism.

(b) *If $\mathfrak{p}$ does contain $\mathfrak{m}$, then there is a unique prime ideal $\mathfrak{q}$ of A such that $w_0^{-1}(\mathfrak{q}) = \mathfrak{p}$. For this $\mathfrak{q}$, there is a unique map of $W_n(A)$-algebras*

$$W_n(A)_{\mathfrak{p}} \longrightarrow W_n(A_{\mathfrak{q}}), \tag{16.16.2}$$

and this map is an isomorphism.

Proof. (a): This holds because the map $w_{\leqslant n}$ is an isomorphism away from $\mathfrak{m}$, by 6.1.

(b): Any such prime ideal $\mathfrak{q}$ contains $\mathfrak{m}$. Therefore to show such a prime ideal $\mathfrak{q}$ exists and is unique, it is enough to show the map

$$\mathrm{id} \otimes w_0 \colon R/\mathfrak{m} \otimes_R W_n(A) \longrightarrow R/\mathfrak{m} \otimes_R A$$

induces a bijection on prime ideals. This holds because $\mathrm{id} \otimes w_0$ is surjective with nilpotent kernel, by 8.2.

Now consider the diagram

$$\begin{tikzcd} A \arrow[r] & A_{\mathfrak{q}} \\ W_n(A) \arrow[u, "w_0"] \arrow[r] & W_n(A_{\mathfrak{q}}). \arrow[u, "w_0"'] \end{tikzcd}$$

By 16.15, the ring $W_n(A_{\mathfrak{q}})$ is a local ring. So, to show there is a unique map of $W_n(A)$-algebras as in (16.16.2), it is enough to that $\mathfrak{p}$ is the pre-image in $W_n(A)$ of the maximal ideal $w_0^{-1}(\mathfrak{m})$ of $W_n(A_{\mathfrak{q}})$. This holds by the commutativity of the diagram above.

Now let us show that (16.16.2) is an isomorphism. By induction, we may assume that the map

$$W_{n-1}(A)_{\mathfrak{p}} \longrightarrow W_{n-1}(A_{\mathfrak{q}})$$

is an isomorphism. By 4.4, it is therefore enough to show that the maps

$$A_{(n)} \otimes_{W_n(A)} W_n(A)_{\mathfrak{p}} \longrightarrow (A_{\mathfrak{q}})_{(n)}$$

are isomorphisms. Write $T_{\mathfrak{p}} = W_n(A) - \mathfrak{p}$ and $T_{\mathfrak{q}} = A - \mathfrak{q}$. Then this map can be identified with the following map of localizations:

$$w_n(T_{\mathfrak{p}})^{-1}A \longrightarrow T_{\mathfrak{q}}^{-1}A. \tag{16.16.3}$$

Thus it is enough to show that $T_{\mathfrak{q}}$ becomes invertible in $w_n(T_{\mathfrak{p}})^{-1}A$. It is therefore enough to show $(T_{\mathfrak{q}})^{q_{\mathfrak{m}}^n} \subseteq w_n(T_{\mathfrak{p}})$, which holds by the following.

We have $[T_{\mathfrak{q}}] \subseteq T_{\mathfrak{p}}$, where $[-]$ denotes the Teichmüller section (of 1.21); indeed, if $[x] \in \mathfrak{p} = w_0^{-1}(\mathfrak{q})$, then $x = w_0([x]) \in \mathfrak{q}$. Therefore, we have

$$w_n(T_{\mathfrak{p}}) \supseteq w_n([T_{\mathfrak{q}}]) = (T_{\mathfrak{q}})^{q_{\mathfrak{m}}^n}.$$

□

16.17. *Remark.* In either case of 16.16, the prime ideal $\mathfrak{q}$ is the pre-image of $\mathfrak{p}$ under the Teichmüller map $t\colon A \to W_n(A)$, $t(a) = [a]$. In fact, the induced function "$\operatorname{Spec} t$": $\operatorname{Spec} W_n(A) \to \operatorname{Spec} A$ is continuous in the Zariski topology. Even further, if we view the structure sheaves on these schemes as sheaves of commutative monoids under multiplication, then the usual Teichmüller map gives "$\operatorname{Spec} t$" the structure of a map of locally monoided topological spaces. If R is an $\mathbf{F}_p$-algebra, for some prime number p, then t is a ring map, and "$\operatorname{Spec} t$" agrees with the scheme map $\operatorname{Spec} t$; but otherwise "$\operatorname{Spec} t$" will not generally be a scheme map. These statements can in fact be promotied to the étale topology, but there it is necessary to work with maps of toposes instead of maps of topological spaces.

16.18. Proposition. *Let $S, R, E, \mathfrak{m}, A$ be as in 16.15. Let $a_1, \dots, a_d \in \mathfrak{m}_A$ be a regular sequence for A. Then $[a_1], \dots, [a_d]$ lie in the maximal ideal of $W_n(A)$ and form a regular sequence for $W_n(A)$.*

Proof. By 16.15, the maximal ideal of $W_n(A)$ is $w_0^{-1}(\mathfrak{m}_A)$, which contains the sequence $[a_1], \dots, [a_d]$. It remains to show that the sequence is regular.

By (10.4.6), we can assume R agrees with $R_{\mathfrak{m}}$, and hence that the ideal $\mathfrak{m}$ is generated by an element π. The argument will now go by induction on n. For $n = 0$, there is nothing to prove; so assume $n \geqslant 1$. For any $W_n(A)$-module M, let $K_{W_n(A)}(M)$ denote the Koszul complex of M with respect the sequence $[a_1], \dots, [a_d] \in W_n(A)$. If this sequence is regular for M, then $H^{d-1}(K_{W_n(A)}(M)) = 0$, and the converse holds if M is finitely generated and nonzero. (See Eisenbud [15], Corollary 17.5 and Theorem 17.6.)

In particular, it is enough to show $H^{d-1}(K_{W_n(A)}(W_n(A))) = 0$. Considering the exact sequence (4.4.1) of $W_n(A)$-modules

$$0 \longrightarrow A_{(n)} \xrightarrow{V_\pi^n} W_n(A) \longrightarrow W_{n-1}(A) \longrightarrow 0,$$

we see it is even sufficient to show

$$H^{d-1}(K_{W_n(A)}(A_{(n)})) = H^{d-1}(K_{W_n(A)}(W_{n-1}(A))) = 0.$$

Observe that we have

$$H^{d-1}\big(K_{W_n(A)}(W_{n-1}(A))\big) = H^{d-1}\big(K_{W_{n-1}(A)}(W_{n-1}(A))\big) = 0,$$

by induction. Therefore, it is enough to prove $H^{d-1}\big((K_{W_n(A)}(A_{(n)}))\big) = 0$, and hence that $[a_1], \dots, [a_d]$ is regular for $A_{(n)}$. This is equivalent to the sequence $a_1^{q^n}, \dots, a_d^{q^n} \in A$ being regular for A—indeed, the product $[a] \cdot x$, where $x \in A_{(n)}$, is by definition $w_n([a])x$, which equals $a^{q^n}x$. We complete the argument with the general fact that any power of a regular sequence for a finitely generated module is again regular ([15], Corollary 17.8). □

16.19. Proposition. *Let k be an integer. Let X be an algebraic space over S such that both X and $W_n^*(X)$ are locally noetherian. Suppose X satisfies one of the following properties:*

(a) *Cohen–Macaulay,*
(b) *Cohen–Macaulay over S,*
(c) *S_k (Serre's condition),*
(d) *S_k over S.*

Then $W_n^*(X)$ satisfies the same property.

16.20. *Remark.* See EGA IV (5.7.1), (6.8.1) [26] for the definition of Cohen–Macaulay and S_k. Typically, these concepts are discussed only for noetherian rings, but because W_n^* does not preserve noetherianness, we must assume $W_n^*(X)$ is noetherian. I do not know if it is possible to remove this assumption by extending the concept of depth beyond the noetherian setting. If so, maybe even the noetherian hypotheses on X could be removed.

Note that, by 16.5(a), the assumptions hold if S is noetherian and X is locally of finite type over S.

16.21. *Proof of 16.19.* The properties are all étale-local (EGA IV (6.4.2) [26]). So by 16.2, we can write $S = \operatorname{Spec} R$, $E = \{\mathfrak{m}\}$, $\mathfrak{m} = \pi R$, and $X = \operatorname{Spec} A$.

(a)–(b): These follow from (c) and (d).

(c): At a prime ideal of $W_n(A)$ not containing π, the local ring agrees with a local ring of A (by 16.16), which satisfies S_k by assumption; so there is nothing to prove. Now let $\mathfrak{p}$ be a prime ideal of $W_n(A)$ containing π. Let $\mathfrak{q}$ be the corresponding prime ideal of A given by 16.16. Then we have $W_n(A)_{\mathfrak{p}} = W_n(A_{\mathfrak{q}})$, so it suffices to assume that A is a local ring with maximal ideal $\mathfrak{q}$. We can therefore also assume that R is a discrete valuation ring with maximal ideal

By 16.18 and 16.5, we have

$$\operatorname{depth} W_n(A) \geqslant \operatorname{depth} A, \quad \dim W_n(A) = \dim A.$$

By the definition of S_k, $\operatorname{depth} A$ is at least k or $\dim A$. Therefore $\operatorname{depth} W_n(A)$ is at least k or $\dim W_n(A)$. In other words, $W_n(A)$ also satisfies S_k.

(d): By 16.5, $W_n^*(A)$ is flat over R. Away from $\mathfrak{m}$, we have $W_n^*(A) = A^{[0,n]}$, the fibers over S of which satisfy S_k. So it suffices to consider the fiber over $\mathfrak{m}$. Therefore, by 16.16, we can assume that A is a local R-algebra whose maximal ideal contains $\mathfrak{m}$. By (10.4.6), we can further assume that R is a discrete valuation ring. We need to show that $W_n(A)/\pi W_n(A)$ satisfies S_k.

Since A and $W_n(A)$ are flat over R, the element π is not a zero divisor in A or $W_n(A)$. Therefore we have

$$\operatorname{depth} W_n(A)/\pi W_n(A) = \operatorname{depth} W_n(A) - 1 \geqslant \operatorname{depth} A - 1 = \operatorname{depth} A/\pi A,$$

by 16.18 (and EGA 0 (16.4.6)(ii) [25], say). Further, by 16.5, we have

$$\dim W_n(A)/\pi W_n(A) = \dim W_n(A) - 1 = \dim A - 1 = \dim A/\pi A.$$

Because $A/\pi A$ satisfies S_k, $\operatorname{depth} A/\pi A$ is at least k or $\dim A/\pi A$. Therefore $\operatorname{depth} W_n(A)/\pi W_n(A)$ is at least k or $\dim W_n(A)/\pi W_n(A)$. In other words, the fiber $W_n(A)/\pi W_n(A)$ satisfies S_k. □

16.22. *Gorenstein, regular, normal.* Consider the p-typical Witt vectors. Then we have

$$W_n(\mathbf{Z}) = \mathbf{Z}[x_1, \dots, x_n]/(x_i x_j - p^i x_j \mid 1 \leqslant i \leqslant j \leqslant n), \tag{16.22.1}$$

with the element x_i corresponding to $V_p^i(1)$, where V_p denotes the usual Verschiebung operator. (See 3.8.)

This presentation gives some easy counterexamples. The ring $W_1(\mathbf{Z})$ agrees with $\mathbf{Z}[x]/(x^2 - px)$, which is not normal. So W_n does not generally preserve regularity or normality.

The property of being Gorenstein is also not preserved by W_n. Indeed, we have

$$\mathbf{F}_p \otimes_{\mathbf{Z}} W_n(\mathbf{Z}) = \mathbf{F}_p[x_1, \dots, x_n]/(x_i x_j \mid 1 \leqslant i \leqslant j \leqslant n).$$

Therefore the socle of $\mathbf{F}_p \otimes_{\mathbf{Z}} W_n(\mathbf{Z})$ (that is, the annihilator of its maximal ideal) is the vector space

$$\mathbf{F}_p x_1 \oplus \cdots \oplus \mathbf{F}_p x_n \quad \text{if } n \geqslant 1,$$

and is $\mathbf{F}_p$ if $n = 0$. Since the sequence $\{p\}$ of length 1 is a system of parameters in $W_n(\mathbf{Z})$ at the prime ideal $\mathfrak{p}$ containing p, the ring $W_n(\mathbf{Z})$ is Gorenstein at $\mathfrak{p}$ if and only if the dimension of the socle is 1. This holds if and only if $n = 0, 1$. When $n = 1$, it is even a complete intersection, but it is not normal. (A basic treatment of these concepts is in Kunz's book [34] (VI 3.18), for example.)

17. Ghost descent and the geometry of Witt spaces

The purpose of this section is to describe the Witt space $W_n^*(X)$ of a flat algebraic space X as a certain quotient, in the category of algebraic spaces, of the ghost space $\coprod_{[0,n]} X$. We continue with the notation of 10.2.

17.1. *Reduced ghost components.* Suppose E consists of one ideal $\mathfrak{m}$, consider the diagram

(17.1.1)
$$S_n \times_S X \overset{i_1 \circ \bar{w}_{n+1}}{\underset{\bar{\imath}_2}{\rightrightarrows}} W_n^*(X) \amalg X \xrightarrow{\alpha_n} W_{n+1}^*(X),$$

where

$\bar{w}_{n+1} \colon S_n \times_S X \to W_n^*(X)$ is as in (10.6.11),
$i_1 \colon W_n^*(X) \to W_n^*(X) \amalg X$ is the inclusion into the first component,
$\bar{\imath}_2$ is the closed immersion of $S_n \times_S X$ into the second component, and
α_n is $r_{n,1}$ on $W_n^*(X)$ and w_{n+1} on X, in the notation of (10.6.3).

When X is affine, this is the same as the diagram in (8.1.1).

17.2. Proposition. *Let X be an algebraic space over S. Then the map α_n is an effective descent map for the fibered category of algebraic spaces which are both étale and affine over their base. In this case, descent data is equivalent to gluing data with respect to the diagram (17.1.1).*

Proof. Given an étale map $U \to X$ with U affine, consider the following three categories: the category of affine étale algebraic spaces over $W_{n+1}^*(U)$, that of affine étale algebraic spaces over $W_n^*(U) \amalg U$ with descent data with respect to α_n, and that of affine étale algebraic spaces over $W_n^*(U) \amalg U$ with gluing data with respect to the diagram (17.1.1). As U varies, there are obvious transition functors, and these give rise to three fibered categories over the small étale topology of X.

There are also evident morphisms between these fibered categories, and the statement of the corollary is that, for $U = X$, these morphisms are equivalences.

By 15.1, all these fibered categories satisfy effective descent in the étale topology. Thus it is enough (by Giraud [19], II 1.3.6, say) to assume X is affine, in which case the equivalence follows from 8.3. □

17.3. Theorem. *If X is $\mathfrak{m}$-flat (10.11), then (17.1.1) is a coequalizer diagram in the category of algebraic spaces.*

Proof. For any space Z over S, write $Z_n = S_n \times_S Z$.

Let us first reduce to the case where X is affine. Write

(17.3.1)
$$X = \operatorname*{colim}_{i \in I} U_i,$$

where $(U_i)_{i \in I}$ is a diagram of affine schemes mapping by étale maps to X. Then $\big((U_i)_n\big)_{i \in I}$ is a diagram of affine schemes mapping by étale maps to X_n. We also have

(17.3.2)
$$X_n = \operatorname*{colim}_{i} (U_i)_n,$$

because the functor $S_n \times_S -\colon \mathsf{Sp}_S \to \mathsf{Sp}_{S_n}$ has a right adjoint, and hence preserves colimits. In particular, both colimit formulas (17.3.1) and (17.3.2) hold in the category of algebraic spaces, as well as in Sp_S. Therefore, assuming the theorem in the affine case, we can make the following formal computation in the category of algebraic spaces:

$$\begin{aligned}
\operatorname{coeq}\big[X_n \rightrightarrows W_n^*(X) \amalg X\big] &= \operatorname{coeq}\big[\operatorname{colim}_i (U_i)_n \rightrightarrows \operatorname{colim}_i W_n^*(U_i) \amalg \operatorname{colim}_i U_i\big] \\
&= \operatorname{coeq}\big[\operatorname{colim}_i (U_i)_n \rightrightarrows \operatorname{colim}_i \left(W_n^*(U_i) \amalg U_i\right)\big] \\
&= \operatorname{colim}_i \operatorname{coeq}\big[(U_i)_n \rightrightarrows W_n^*(U_i) \amalg U_i\big] \\
&= \operatorname{colim}_i W_{n+1}^*(U_i) \\
&= W_{n+1}^*(\operatorname{colim}_i U_i) \\
&= W_{n+1}^*(X).
\end{aligned}$$

Hence we can assume X is affine.

Let Y be an algebraic space, and let $d\colon W_{n-1}^*(X) \amalg X \to Y$ be a map such that the two compositions in the diagram

$$\begin{tikzcd} S_n \times_S X \arrow[r, shift left, "i_1 \circ \bar{w}_{n+1}"] \arrow[r, shift right, "\bar{\imath}_2"'] & W_n^*(X) \amalg X \arrow[r, "d"] & Y \end{tikzcd}$$

agree. We want to show that d factors through α_n. Because $W_n^*(X) \amalg X$ is affine, by 10.7, there is a quasi-compact open algebraic subspace containing the image of d. Since we can replace Y with it, we may assume Y is quasi-compact.

Take $m \geqslant -1$ such that $Y \in \mathsf{AlgSp}_m$. We will argue by induction on m. When $m = -1$, the space Y is a quasi-compact disjoint union of affine schemes; therefore it is affine. The result then follows because (17.1.1) is a coequalizer diagram in the category of affine schemes, by 8.1.

Now suppose $m \geqslant 0$. Let $e\colon Y' \to Y$ be an étale surjection, where Y' is an affine scheme. Then there is a étale surjection $g\colon X' \to X$, with X' affine, such that d lifts to a map d' as follows:

$$\begin{tikzcd} W_n^*(X') \amalg X \arrow[r, dashed, "d'"] \arrow[d, two heads, "W_n^*(g) \amalg g"'] & Y' \arrow[d, "e"] \\ W_n^*(X) \amalg X \arrow[r, "d"] & Y. \end{tikzcd}$$

Indeed, the existence of such a map d' is equivalent to the existence of a lift f'

$$\begin{tikzcd} X' \arrow[r, dashed, "f'"] \arrow[d, two heads, "g"] & W_{n*}(Y') \times_S Y' \arrow[d, "W_{n*}(e) \times e"] \\ X \arrow[r, "f"] & W_{n*}(Y) \times_S Y, \end{tikzcd}$$

where f is the left adjunct of d. This exists because $W_{n*}(e) \times e$ is an epimorphism of spaces, which is true by 11.4.

Now let us construct the diagram in figure 2. The rows are diagrams of the form (17.1.1); to get d'', take the product of d' with itself over $W_n^*(X) \amalg X$ and then apply 15.2(c). The maps a, a', a'' have not been constructed yet.

Since X' and X are affine, so is $X' \times_X X'$; and since X' is étale over X, which is $\mathfrak{m}$-flat, X' and $X' \times_X X'$ are also $\mathfrak{m}$-flat. Also, since $Y \in \mathsf{AlgSp}_m$, we have

$$Y', Y' \times_Y Y' \in \mathsf{AlgSp}_{m-1}.$$

So, by induction, there are unique maps a' and a'' such that $d' = a' \circ c'$ and $d'' = a'' \circ c''$.

$$
\begin{tikzcd}
X'_n \times_{X_n} X'_n \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, shift right, "\mathrm{pr}_1"'] \arrow[dd, shift left, "\mathrm{pr}_2"] & W^*_n(X' \times_X X') \amalg (X' \times_X X') \arrow[rr, "c''"] \arrow[dr, "d''"] \arrow[dd, shift right, "W^*_n(\mathrm{pr}_1) \amalg \mathrm{pr}_1"'] \arrow[dd, shift left, "W^*_n(\mathrm{pr}_2) \amalg \mathrm{pr}_2"] & & W^*_{n+1}(X' \times_X X') \arrow[dl, dashed, "a''"] \arrow[dd, shift right, "W^*_{n+1}(\mathrm{pr}_1)"'] \arrow[dd, shift left, "W^*_{n+1}(\mathrm{pr}_2)"] \\
& & Y' \times_Y Y' \arrow[dd, shift right, "\mathrm{pr}_1"'] \arrow[dd, shift left, "\mathrm{pr}_2"] & \\
X'_n \arrow[r, shift left] \arrow[r, shift right] \arrow[dd, "g_n"] & W^*_n(X') \amalg X' \arrow[rr, "c'"] \arrow[dr, "d'"] \arrow[dd, "W^*_n(g) \amalg g"'] & & W^*_{n+1}(X') \arrow[dl, dashed, "a'"] \arrow[dd, "W^*_{n+1}(g)"] \\
& & Y' \arrow[dd, "e"] & \\
X_n \arrow[r, shift left] \arrow[r, shift right] & W^*_n(X) \amalg X \arrow[rr, "c"] \arrow[dr, "d"] & & W^*_{n+1}(X) \arrow[dl, dashed, "a"] \\
& & Y &
\end{tikzcd}
$$

FIGURE 2.

Now let us show $a' \circ W^*_{n+1}(\mathrm{pr}_i) = \mathrm{pr}_i \circ a''$, for $i = 1, 2$. It is enough to show

$$a' \circ W^*_{n+1}(\mathrm{pr}_i) \circ c'' = \mathrm{pr}_i \circ a'' \circ c''.$$

Indeed, by induction the coequalizer universal property holds for the top row. Showing this equality is a straightforward diagram chase.

Therefore we have

$$e \circ a' \circ W^*_{n+1}(\mathrm{pr}_1) = e \circ \mathrm{pr}_1 \circ a'' = e \circ \mathrm{pr}_2 \circ a'' = e \circ a' \circ W^*_{n+1}(\mathrm{pr}_2)$$

So, by the universal property of coequalizers applied to the rightmost column, there exists a unique map a such that

$$a \circ W^*_{n+1}(g) = e \circ a'. \tag{17.3.3}$$

Finally, let us verify the equality $d = a \circ c$. Because $W^*_n(g) \amalg g$ is an epimorphism, it is enough to show

$$d \circ \big(W^*_n(g) \amalg g\big) = a \circ c \circ \big(W^*_n(g) \amalg g\big).$$

This follows from (17.3.3) and a diagram chase which is again left to the reader. □

17.4. *Remark.* It is typically not true that (17.1.1) is a coequalizer diagram in the category Sp_S. For example, if we take $X = \operatorname{Spec} \mathbf{Z}_p[\sqrt{p}]$ and consider the usual, p-typical Witt vectors, then α_1 is not an epimorphism in Sp_S.

18. The geometry of arithmetic jet spaces

The main purpose of the section is to prove 18.3. We continue with the notation of 10.2. Let X be an algebraic space over S.

18.1. *Single-prime notation.* Suppose that E consists of one maximal ideal $\mathfrak{m}$. Let

$$W_{n+1*}(X) \xrightarrow{f} W_{n*}(X) \times_S X$$

denote the map $(s_{n,1}, \kappa_{n+1})$ (in the notation of 10.6), and let I denote the ideal sheaf of $\mathcal{O}_{W_{n*}(X) \times_S X}$ defining the closed immersion

$$(\mathrm{pr}_2, \bar{\kappa}_n) \colon S_n \times_S W_{n*}(X) \longrightarrow W_{n*}(X) \times_S X. \tag{18.1.1}$$

Let $\mathcal{B}$ denote the sub-$\mathcal{O}_{W_{n*}(X)\times_S X}$-algebra of $\mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \mathcal{O}_{W_{n*}(X)\times_S X}$ generated by the subsheaf $\mathfrak{m}^{-n-1} \otimes_{\mathcal{O}_S} I$. Observe that $\mathcal{B}$ is $\mathfrak{m}$-flat and satisfies

$$\mathfrak{m}^{n+1}\mathcal{B} \supseteq \mathfrak{m}^{n+1}(\mathfrak{m}^{-n-1} \otimes_{\mathcal{O}_S} I)\mathcal{B} = I\mathcal{B}. \tag{18.1.2}$$

When necessary, we will write $f_X, I_X, \mathcal{B}_X$ to be clear.

18.2. Proposition. *Suppose that E consists of one maximal ideal $\mathfrak{m}$. Let T be an $\mathfrak{m}$-flat algebraic space over $W_{n*}(X) \times_S X$. Then there exists at most one map $\tilde{g}$*

$$\begin{tikzcd} T \arrow[r, dashed, "\tilde{g}"] \arrow[dr, "g"'] & W_{n+1*}(X) \arrow[d, "f"] \\ & W_{n*}(X) \times_S X \end{tikzcd}$$

lifting the structure map g. Such a lift exists if and only if $I\mathcal{O}_T \subseteq \mathfrak{m}^{n+1}\mathcal{O}_T$.

Proof. Giving a map $\tilde{g}\colon T \to W_{n+1*}(X)$ is equivalent to giving a map $W^*_{n+1}(T) \to X$. Such maps can be described using the diagram

$$S_n \times_S T \rightrightarrows W^*_n(T) \amalg T \longrightarrow W^*_{n+1}(T),$$

because it is a coequalizer diagram in the category of algebraic spaces, by 17.3. Therefore giving a map $T \to W_{n+1*}(X)$ is equivalent to giving maps $a\colon T \to W_{n*}(X)$ and $b\colon T \to X$ such that the diagram

$$\begin{tikzcd} S_n \times_S T \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\bar{w}_n"'] & T \arrow[d, "b"] \\ W^*_n(T) \arrow[r, "a'"] & X, \end{tikzcd}$$

where a' is the left adjunct of a, commutes. The commutativity of this diagram is equivalent to that of

$$\begin{tikzcd} S_n \times_S T \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times a"'] & T \arrow[d, "b"] \\ S_n \times_S W_{n*}(X) \arrow[r, "\bar{\kappa}_n"] & X. \end{tikzcd} \tag{18.2.1}$$

This is because the following diagram commutes:

$$\begin{tikzcd} S_n \times_S T \arrow[r, "\mathrm{id}\times a"] \arrow[d, "\bar{w}_n"'] & S_n \times_S W_{n*}(X) \arrow[d, "\bar{w}_n"] \arrow[ddr, "\bar{\kappa}_n"] & \\ W^*_n(T) \arrow[r, "W^*_n(a)"] \arrow[drr, "a'"'] & W^*_n W_{n*}(X) \arrow[dr, "\varepsilon"] & \\ & & X, \end{tikzcd}$$

where ε is the counit of the evident adjunction. (And this diagram commutes by the naturalness of $\bar{w}_n$ and the definitions of a' and $\bar{\kappa}_n$.)

Let us now apply this in the case where we take (a, b) to be g. Then the map $\tilde{g}$ required by the lemma is unique, and it exists if and only if (18.2.1) commutes.

The commutativity of (18.2.1) is equivalent to that of

(18.2.2)
$$\begin{tikzcd} S_n \times_S T \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{id}\times(\mathrm{pr}_1\circ g)"'] & T \arrow[d, "g"] \\ S_n \times_S W_{n*}(X) \arrow[r, "h"] & W_{n*}(X) \times_S X, \end{tikzcd}$$

where h denotes the map $(\mathrm{pr}_2, \bar{\kappa}_n)$ of (18.1.1). Because h is a closed immersion, the commutativity of (18.2.2) is equivalent to requiring that the ideal I defining h pull back to the zero ideal on $S_n \times_S T$, which is equivalent to the containment $I\mathcal{O}_T \subseteq \mathfrak{m}^{n+1}\mathcal{O}_T$. □

18.3. Theorem. *Suppose that E consists of one maximal ideal $\mathfrak{m}$ and that $W_{n+1*}(X)$ is $\mathfrak{m}$-flat (10.11). Let $\mathcal{B}$ be as in 18.1. Then the unique $(W_{n*}(X) \times_S X)$-map*

$$\operatorname{Spec} \mathcal{B} \xrightarrow{\tilde{g}} W_{n+1*}(X),$$

of 18.2 is an isomorphism.

18.4. *Remark.* In other words, we have

(18.4.1)
$$W_{n+1*}(X) = \operatorname{Spec} \mathcal{O}_{W_{n*}(X)\times_S X}[\mathfrak{m}^{-n-1} \otimes_{\mathcal{O}_S} I],$$

which gives a concrete recursive description of $W_{n+1*}(X)$ when it is $\mathfrak{m}$-flat. Note that, by 12.6, this flatness condition is satisfied when X is E-smooth.

18.5. *Proof of 18.3.* Fix, for the moment, an étale algebraic X-space U. Let Y_U denote $W_{n*}(U)\times_S U$, and let Z_U denote $\operatorname{Spec} \mathcal{B}_U$. Let C_U denote the full subcategory of algebraic spaces over Y_U consisting of objects T which are $\mathfrak{m}$-flat and satisfy

(18.5.1)
$$I_U \mathcal{O}_T \subseteq \mathfrak{m}^{n+1}\mathcal{O}_T,$$

where I_U is the ideal sheaf defined in 18.1; let $\mathsf{C}_U^{\mathrm{aff}}$ denote the full subcategory of C_U consisting of objects which are affine over Y_U.

First, observe that $W_{n+1*}(U)$ is the terminal object of C_U. Indeed, by 18.2, it is enough to show that $W_{n+1*}(U)$ is $\mathfrak{m}$-flat; this is true because, by 11.1, it is étale over $W_{n+1*}(X)$, which is $\mathfrak{m}$-flat by assumption.

Second, observe that Z_U is the terminal object of $\mathsf{C}_U^{\mathrm{aff}}$: it is an object of $\mathsf{C}_U^{\mathrm{aff}}$ by 18.1.2, and it is terminal by the definition of *generated*.

Because of these two terminal properties, the theorem is equivalent to the statement that there exists a map $W_{n+1*}(X) \to Z_X$ of Y_X-spaces, which is what we will prove.

Let D be a diagram of étale algebraic spaces U over X (as above) such that each space U in the diagram is an object of AffRel_S and such that the induced map

(18.5.2)
$$\operatorname*{colim}_{U\in\mathsf{D}} W_{n+1*}(U) \longrightarrow W_{n+1*}(X)$$

is an isomorphism. The existence of D follows from 11.9. (One can in fact take D to consist of all such spaces U.) Then, for any map $b\colon V \to U$ of D, the space $W_{n+1*}(V)$ is an object of $\mathsf{C}_U^{\mathrm{aff}}$. Indeed, the induced map $W_{n+1*}(V) \to Y_U$ is affine, because both the source and the target are affine schemes (10.3.2); and (18.5.1) is satisfied because we have $I_U\mathcal{O}_{Y_V} \subseteq I_V$.

Therefore, by the terminal property of Z_U, for any such map $b\colon V \to U$, there is a unique map $F(b)\colon W_{n+1*}(V) \to Z_U$ of Y_U-spaces. In particular, the induced diagram

$$\begin{tikzcd} W_{n+1*}(V) \arrow[r, "F(\mathrm{id}_V)"] \arrow[d, "W_{n+1*}(b)"'] & Z_V \arrow[d] \\ W_{n+1*}(U) \arrow[r, "F(\mathrm{id}_U)"] & Z_U \end{tikzcd}$$

commutes. In particular, the compositions

$$W_{n+1*}(U) \xrightarrow{F(\mathrm{id}_U)} Z_U \longrightarrow Z_X$$

form a compatible family of Y_X-maps, as U runs over D. This induces a Y_X-map

$$\operatorname*{colim}_{U\in\mathsf{D}} W_{n+1*}(U) \to Z_X.$$

On other hand, (18.5.2) is an isomorphism of Y_X-spaces. Thus there exists a map $W_{n+1*}(X) \to Z_X$ of Y_X-spaces, which completes the proof. □

18.6. *Non-smooth counterexample.* We cannot remove the assumption above that $W_{n+1*}(X)$ is $\mathfrak{m}$-flat. Indeed, the example in 12.7 shows that the locus of $W_{n*}(X)$ over the complement of $\operatorname{Spec}\mathcal{O}_S/\mathfrak{m}$ can fail to be dense in $W_{n*}(X)$.

18.7. Corollary. *If X is E-smooth, then the co-ghost map*

$$W_{n*}(X) \xrightarrow{\kappa_{\leqslant n}} X^{[0,n]}$$

is affine. It is an isomorphism away from E.

It would be interesting to know whether this is true for arbitrary algebraic spaces X over S.

Proof. If E is empty, then $\kappa_{\leqslant n}$ is an isomorphism. If not, write $E = E' \amalg E''$, where E'' consists of a single element. Let n' and n'' denote the projections of n onto $\mathbf{N}^{(E')}$ and $\mathbf{N}^{(E'')}$. Then by (10.6.2), the map $\kappa_{\leqslant n}$ can be identified with the composition

$$W_{n''*}\big(W_{n'*}(X)\big) \xrightarrow{\kappa_{\leqslant n''}} \big(W_{n'*}(X)\big)^{[0,n'']} \xrightarrow{(\kappa_{\leqslant n'})^{[0,n'']}} \big(X^{[0,n']}\big)^{[0,n'']}.$$

By 11.1 and (10.4.9), the space $W_{n'*}(X)$ is E-smooth. Therefore, by induction, the second map above is affine and is an isomorphism away from E. Thus to show the first map is affine and is an isomorphism away from E, it is enough to prove the corollary itself in the case where E consists of a single element.

In that case, $\kappa_{\leqslant n}$ factors as follows

$$W_{n*}(X) \xrightarrow{f} W_{n-1*}(X) \times_S X \xrightarrow{f\times\mathrm{id}_X} \big(W_{n-2*}(X) \times_S X\big) \times_S X \longrightarrow \cdots \longrightarrow X^{[0,n]}.$$

Since X is E-smooth, each $W_{i*}(X)$ is E-smooth and hence E-flat. Thus, by 18.3, each of these maps is affine and an isomorphism away from E. Therefore so is their composition $\kappa_{\leqslant n}$. □

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