

Localization of André–Quillen–Goodwillie towers, and the periodic homology of infinite loopspaces[☆]

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Received 15 February 2005; accepted 23 February 2005

Communicated by Michael Hopkins

Available online 9 April 2005

Abstract

Let $K(n)$ be the n th Morava K -theory at a prime p , and let $T(n)$ be the telescope of a v_n -self map of a finite complex of type n . In this paper we study the $K(n)_*$ -homology of $\Omega^\infty X$, the 0th space of a spectrum X , and many related matters.

We give a sampling of our results.

Let $\mathbb{P}X$ be the free commutative S -algebra generated by X : it is weakly equivalent to the wedge of all the extended powers of X . We construct a natural map

$$s_n(X) : L_{T(n)}\mathbb{P}(X) \rightarrow L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+$$

of commutative algebras over the localized sphere spectrum $L_{T(n)}S$. The induced map of commutative, cocommutative $K(n)_*$ -Hopf algebras

$$s_n(X)_* : K(n)_*(\mathbb{P}X) \rightarrow K(n)_*(\Omega^\infty X),$$

satisfies the following properties.

It is always monic.

It is an isomorphism if X is n -connected, $\pi_{n+1}(X)$ is torsion, and $T(i)_*(X) = 0$ for $1 \leq i \leq n-1$. It is an isomorphism only if $K(i)_*(X) = 0$ for $1 \leq i \leq n-1$.

[☆]This research was partially supported by a grant from the National Science Foundation.

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It is universal. The domain of $s_n(X)_*$ preserves $K(n)_*$ -isomorphisms, and if F is any functor preserving $K(n)_*$ -isomorphisms, then any natural transformation $F(X) \rightarrow K(n)_*(\Omega^\infty X)$ factors uniquely through $s_n(X)_*$.

The construction of our natural transformation uses the telescopic functors constructed and studied previously by Bousfield and the author, and thus depends heavily on the Nilpotence Theorem of Devinatz, Hopkins, and Smith. Our proof that $s_n(X)_*$ is always monic uses Topological André-Quillen Homology and Goodwillie Calculus in nonconnective settings.

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MSC: primary 55P43; 55P47; 55N20; secondary 18G55

Keywords: Infinite loop space; Morava k-theory; Periodic homotopy; Goodwillie tower; Topological Andre Quillen Homology

1. Introduction and main results

In algebraic topology, homotopical aspects of topological spaces are studied by means of generalized homology and cohomology theories. Such theories are themselves determined by spectra, the objects of the stable category. One can then pass between the worlds of unstable and stable homotopy by means of the adjoint pair of functors $(\Sigma^\infty, \Omega^\infty)$, where $\Sigma^\infty Z$ denotes the suspension spectrum of a based space Z , and $\Omega^\infty X$ denotes the 0th infinite loop space of the spectrum X .

Though Σ^∞ preserves homology (and cohomology), the homological behavior of Ω^∞ is much more subtle, and one has the basic problem: given a spectrum E , to what extent, and in what way, is $E_*(\Omega^\infty X)$ determined by $E_*(X)$?

There is a related, more subtle, problem: to what extent, and in what way, is $L_E \Sigma^\infty \Omega^\infty X$ determined by $L_E X$? Here L_E denotes Bousfield localization with respect to E_* .

In this paper, we develop new techniques allowing for a thorough study of these questions when E_* is a periodic homology theory. The key to our methods is to combine two of the major strands of homotopy theory of the past two decades: the flowering of powerful new techniques in homotopical algebra, many following the conceptual model offered by T. Goodwillie's calculus of functors [G1,G2,G3], and the deepening of our understanding of homotopy as organized from the chromatic point of view, in the wake of the Nilpotence Theorem of Devinatz et al. [DHS].

The tools from modern homotopical algebra that we use are Topological André-Quillen homology of E_∞ -ring spectra, as developed via the Goodwillie calculus framework, together with a good theory of Bousfield localization of structured objects. These concepts require that we work within a nice model category of spectra. Thus, for us, spectra will mean objects in \mathcal{S} , the category of S -modules as in [EKMM], and so, e.g., commutative S -algebras will serve as E_∞ -ring spectra.

The input from chromatic homotopy theory comes from our use of the telescopic functors, constructed by Bousfield and the author in [K3,B1,B4], which factor certain periodic localization functors through Ω^∞ .

The classic stable splitting of the space $\Omega^\infty \Sigma^\infty Z$ [Kah] provides a model for our main results when presented as follows. Let Z_+ denote the union of a space Z with a disjoint basepoint. If X is an S -module, $\Sigma^\infty(\Omega^\infty X)_+$ is naturally a commutative S -algebra. Another example is $\mathbb{P}X$, the free commutative algebra generated by X . This is weakly equivalent to the wedge, running over $r \geq 0$, of $D_r X = E \Sigma_{r+} \wedge_{\Sigma_r} X^{\wedge r}$, the r th extended power of X . Then, for all spaces Z , there is a natural map of commutative S -algebras

$$s(Z) : \mathbb{P}(\Sigma^\infty Z) \rightarrow \Sigma^\infty(\Omega^\infty \Sigma^\infty Z)_+$$

satisfying the following properties.

- (1) $s(Z)_* : E_*(\mathbb{P}(\Sigma^\infty Z)) \rightarrow E_*(\Omega^\infty \Sigma^\infty Z)$ is monic for all theories E_* .
- (2) $s(Z)$ is an equivalence if Z is connected.

Now let $K(n)$ be the n th Morava K -theory spectrum at a fixed prime p , and $T(n)$ its ‘telescopic’ variant: the telescope of a v_n -self map of a finite complex of type n . (See [R] for background on this material.) The Telescope Conjecture is the statement that $\langle K(n) \rangle \geq \langle T(n) \rangle$, i.e. $K(n)_*$ -acyclics are $T(n)_*$ -acyclics; in any case, the converse holds, so that $L_{K(n)} L_{T(n)} = L_{K(n)}$.

We show that, for all spectra X , there is a natural map of commutative $L_{T(n)} S$ -algebras

$$s_n(X) : L_{T(n)} \mathbb{P}(X) \rightarrow L_{T(n)} \Sigma^\infty(\Omega^\infty X)_+$$

satisfying the following properties.

- (1) $s_n(Z)_* : E_*(\mathbb{P}(X)) \rightarrow E_*(\Omega^\infty X)$ is monic for all X , if $\langle T(n) \rangle \geq \langle E \rangle$.
- (2) $s_n(X)$ is an equivalence if X is suitably connected and $T(i)_*(X) = 0$ for $1 \leq i \leq n-1$. It is an equivalence only if $K(i)_*(X) = 0$ for $1 \leq i \leq n-1$.
- (3) s_n is universal in the sense that any natural transformation from a functor invariant under $T(n)_*$ -equivalences to $L_{T(n)} \Sigma^\infty(\Omega^\infty X)_+$ will canonically factor through s_n .

Homological consequences are most precise when $E_* = K(n)_*$. The first property then says that, for all X ,

$$s_n(Z)_* : K(n)_*(\mathbb{P}(X)) \rightarrow K(n)_*(\Omega^\infty X)$$

is an inclusion of commutative, cocommutative, $K(n)_*$ -Hopf algebras. The second property and work of Hopkins et al. [HRW] then combine to say that, if X is an S -module with $T(i)_*(X) = 0$ for $1 \leq i \leq n-1$, then there is an isomorphism of $K(n)_*$ -Hopf algebras

$$K(n)_*(\mathbb{P}X) \simeq K(n)_*(\Omega^\infty X \langle n+1 \rangle)$$

and a short exact sequence of $K(n)_*$ -Hopf algebras

$$K(n)_*(\Omega^\infty X\langle n+1 \rangle) \rightarrow K(n)_*(\Omega^\infty X) \rightarrow \bigotimes_{j=0}^{n+1} K(n)_*(K(\pi_j(X), j)).$$

(Here $\Omega^\infty X\langle n+1 \rangle$ denotes the $(n+1)$ th connected cover of $\Omega^\infty X$.)

Our main theorems also have consequences for E_n^* , where E_n is fundamental p -complete integral height n complex oriented commutative S -algebra appearing in the work of Hopkins and his collaborators, since it is known [H1] that $K(n)_*(X) = 0$ if and only if $E_n^*(X) = 0$. Indeed our work here, combined with work by many, beginning with [HKR,Hu], on $E_n^*(D_r X)$, gets us most of the way towards calculations of $E_n^*(\Omega^\infty X)$ generalizing Bousfield's extensive functorial calculations [B4] of $E_1^*(\Omega^\infty X) = K^*(\Omega^\infty X; \mathbb{Z}_p)$.

Our theory of localized André–Quillen towers also yields the following theorem, a significant generalization of the main result of [K1]: if Z is a connected space, and $f: \Sigma^\infty Z \rightarrow X$ is an E_* -isomorphism, then $(\Omega^\infty f)_*: E_*(\Omega^\infty \Sigma^\infty Z) \rightarrow E_*(\Omega^\infty X)$ is monic.

We describe our results in detail in the next section.

Versions of our main theorem, Theorem 2.5, date from 2000, have been reported on in various seminar and conference talks since then in both the United States and Europe.

During the last 20 months, while the first (June, 2003) version of this paper was *not* being refereed by the well-known journal to which it was submitted, the author has had published a number of related results: in [K6] I show that the Goodwillie towers of endofunctors of spectra (like $\Sigma^\infty \Omega^\infty X$) always split after $T(n)$ -localization, and in [K7] there is a variant of Theorem 2.4 for mapping spaces. Finally, survey paper [K8] includes a discussion of how one might try to disprove the Telescope Conjecture by using results proved here.

2. Main results

In this section, we describe our results. It ends with a discussion of the organization of the remainder of the paper, where proofs and more detail are given.

Throughout we will use the following convention: if A and B are objects in a model category \mathcal{C} , a *weak* map $A \xrightarrow{f} B$ will mean either a pair $A \xleftarrow{\sim} C \xrightarrow{h} B$, or a pair $A \xrightarrow{h} C \xleftarrow{\sim} B$. A weak map in \mathcal{C} induces a well-defined morphism in the homotopy category $ho(\mathcal{C})$, and we say that a diagram of weak maps in \mathcal{C} commutes if the induced diagram in $ho(\mathcal{C})$ does.

2.1. Commutative S -algebras and the stable splitting of QZ , revisited

Let Z_+ denote the union of a space Z with a disjoint basepoint. If X is an S -module, $\Sigma^\infty(\Omega^\infty X)_+$ is naturally a commutative S -algebra augmented over the sphere spectrum S . We denote by \mathcal{Alg} the category of such objects. Another example is $\mathbb{P}X$, the free commutative algebra generated by X . There is a natural weak equivalence:

$$\mathbb{P}X \simeq \bigvee_{r=0}^{\infty} D_r X.$$

Given $A \in \mathcal{Alg}$, let $I(A)$ be the homotopy fiber of the augmentation $A \rightarrow S$. We view $I(A)$ as the augmentation ideal, and we are interested in two associated objects. The first is $\widehat{A} \in \mathcal{Alg}$, which arises as the inverse limit of an *André–Quillen tower* in \mathcal{Alg} , and can be viewed as the $I(A)$ -adic completion of A . The second is an S -module $\mathrm{taq}(A)$, a form of Topological André–Quillen homology, and can be viewed as $I(A)/I(A)^2$. There is a convergence result: if A is 0-connected then the canonical map $A \rightarrow \widehat{A}$ is an equivalence.

Applied to the examples above, if A is either $\mathbb{P}X$ or $\Sigma^\infty(\Omega^\infty X)_+$, with X -1 -connected in the latter case, then $\mathrm{taq}(A) \simeq X$. The natural map $I(A) \rightarrow \mathrm{taq}(A)$ identifies in the first case with projection onto the first factor, and in the second with $\varepsilon(X) : \Sigma^\infty \Omega^\infty X \rightarrow X$, the counit of the adjunction. There is also a natural weak equivalence:

$$\widehat{\mathbb{P}}X \simeq \prod_{r=0}^{\infty} D_r X.$$

We observe:

Proposition 2.1. *Let $f : A \rightarrow B$ be a map in \mathcal{Alg} . If $\mathrm{taq}(f) : \mathrm{taq}(A) \rightarrow \mathrm{taq}(B)$ is an equivalence, so is $\widehat{f} : \widehat{A} \rightarrow \widehat{B}$. Thus, in this case, there is a factorization by weak algebra maps*

$$\begin{array}{ccc} A & \xrightarrow{\text{canonical}} & \widehat{A} \\ & \searrow f & \nearrow \\ & B & \end{array}$$

where the unlabelled weak map is $B \rightarrow \widehat{B} \xleftarrow[\sim]{\widehat{f}} \widehat{A}$.

Observations like this are the basis for various of our splitting theorems. We first illustrate this idea by giving a new formulation and proof of a very highly structured version of the classical splitting of $\Sigma^\infty QZ$, where, as usual, QZ denotes $\Omega^\infty \Sigma^\infty Z$. (See [K4, Appendix B] for a discussion of some of the different proofs this theorem of D.S. Kahn.)

Let $\eta(Z) : Z \rightarrow QZ$ be the unit of the adjunction. In a straightforward way, this then induces a weak map in $\mathcal{A}lg$

$$s(Z) : \mathbb{P}(\Sigma^\infty Z) \rightarrow \Sigma^\infty(QZ)_+.$$

Theorem 2.2. *For all spaces Z , the map $s(Z)$ induces an isomorphism on completions, and thus there is a natural factorization of weak algebra maps*

$$\begin{array}{ccc} \mathbb{P}(\Sigma^\infty Z) & \xrightarrow{\text{canonical}} & \widehat{\mathbb{P}}(\Sigma^\infty Z). \\ & \searrow s(Z) \quad \nearrow t(Z) & \\ & \Sigma^\infty(QZ)_+ & \end{array}$$

If Z is 0-connected then all of these maps are weak equivalences.

In more down-to-earth terms, our theorem gives us a factorization by weak maps

$$\begin{array}{ccc} \bigvee_{r=0}^\infty \Sigma^\infty D_r Z & \xrightarrow{\text{canonical}} & \prod_{r=0}^\infty \Sigma^\infty D_r Z \\ & \searrow s(Z) \quad \nearrow t(Z) & \\ & \Sigma^\infty(QZ)_+ & \end{array}$$

in which both the infinite wedge and product are equivalent to E_∞ -ring spectra, such that all maps in the diagram are E_∞ .

Our general theory leads to a short proof of the theorem as follows. The diagram

$$\begin{array}{ccc} & \Sigma^\infty \Omega^\infty \Sigma^\infty Z & \\ \Sigma^\infty \eta(Z) \nearrow & & \searrow \varepsilon(\Sigma^\infty Z) \\ \Sigma^\infty Z & \xlongequal{\quad} & \Sigma^\infty Z \end{array} \tag{2.1}$$

commutes for all Z . By construction, this shows that $\text{taq}(s(Z))$ can be identified with the identity map on $\Sigma^\infty Z$ and thus is an equivalence. By the proposition, so also is $\widehat{s(Z)}$, and the theorem follows.

In Appendix A, we check that our stable splitting agrees with others in the literature.

In the case when Z is not connected, our theorem improves upon various weaker versions in the literature, and our proof shows that most of the technical issues confronted in these papers need no longer be part of the story. See Remarks 4.3.

2.2. Bousfield localization and André–Quillen towers

Now we mix Bousfield localization with the general theory. If E is any S -module, $L_E S$ will be a commutative S -algebra, and we define $L_E(\mathcal{A}lg)$ to be the category of commutative $L_E S$ -algebras, which are also augmented over $L_E S$, and are E -local. Up to weak equivalence, objects have the form $L_E A$, with $A \in \mathcal{A}lg$, but *not* all morphisms are homotopic to one of the form $L_E f$, with $f \in \mathcal{A}lg$.

Analogous to the nonlocalized theory, given $A \in L_E(\mathcal{A}lg)$, one gets a completion $\widehat{L}_E A \in L_E(\mathcal{A}lg)$, and an $L_E S$ -module $\text{taq}^E(A)$. For $A \in \mathcal{A}lg$, $L_E(\text{taq}(A)) \simeq \text{taq}^E(L_E A)$, but it is *not* in general true that the natural map $L_E \widehat{A} \rightarrow \widehat{L}_E A$ is an equivalence. This is illustrated by the example

$$\widehat{L}_E \mathbb{P}X \simeq \prod_{r=0}^{\infty} L_E D_r X,$$

which is often different than

$$L_E \widehat{\mathbb{P}}X \simeq L_E \left(\prod_{r=0}^{\infty} D_r X \right).$$

Analogous to Proposition 2.1, we observe:

Proposition 2.3. *Let $g : L_E A \rightarrow L_E B$ be a map in $L_E(\mathcal{A}lg)$. If $\text{taq}^E(g) : L_E \text{taq}(A) \rightarrow L_E \text{taq}(B)$ is an equivalence, so is $\widehat{g} : \widehat{L}_E A \rightarrow \widehat{L}_E B$. Thus, in this case, one gets a factorization by weak algebra maps*

$$\begin{array}{ccc} L_E A & \xrightarrow{\text{canonical}} & \widehat{L}_E A \\ & \searrow g & \nearrow \\ & L_E B & \end{array}$$

In the early 1980's, the author proved that if $f : \Sigma^\infty Z \rightarrow \Sigma^\infty W$ is an E_* -isomorphism, with Z and W connected, then $\Omega^\infty f$ is an E_* -monomorphism [K1]. As

a first application of our general theory of localized towers, we deduce the following stronger version.

Theorem 2.4. *Let Z be a connected space. If a map of spectra $f : \Sigma^\infty Z \rightarrow X$ is an E_* -isomorphism, then*

$$(\Omega^\infty f)_* : E_*(QZ) \rightarrow E_*(\Omega^\infty X)$$

is a monomorphism.

We leave to the reader the proof that the hypotheses can be weakened slightly: the domain of f need just be ‘spacelike’, i.e. a wedge summand of a suspension spectrum. See also Appendix A for a version of the theorem for non-connected Z .

Examples illustrating this theorem were given in [K1]. Besides these, see also its use in Appendix B.

2.3. The main theorem

Fixing a prime p and $n \geq 1$, we now apply the proposition of the last subsection to the case when $E = T(n)$. Our theorem has a statement and proof analogous to Theorem 2.2.

In place of (2.1), we use the following much deeper theorem: there is a natural factorization by weak S -module maps

$$\begin{array}{ccc} & L_{T(n)} \Sigma^\infty \Omega^\infty X & \\ \eta_n(X) \nearrow & & \searrow L_{T(n)} \varepsilon(X) \\ L_{T(n)} X & \xlongequal{\quad} & L_{T(n)} X. \end{array} \quad (2.2)$$

Such a factorization was constructed in the mid 1980s by Bousfield [B3], when $n = 1$, and by the author, for all $n \geq 1$ [K3]. A recent paper by Bousfield [B7] revisits these constructions. These papers heavily use the classification of stable v_n -self maps of finite complexes, and thus are using (at least for $n > 1$) the work of Devinatz et al. [DHS,HS] on Ravenel’s Nilpotence Conjectures.

The natural transformation $\eta_n(X)$ then induces a weak map

$$s_n(X) : L_{T(n)} \mathbb{P}(X) \rightarrow L_{T(n)} \Sigma^\infty (\Omega^\infty X)_+$$

in $L_{T(n)}(\mathcal{A}lg)$. Proposition 2.3 combines with (2.2) to prove the main theorem of the paper:

Theorem 2.5. *For all spectra X , the map $s_n(X)$ induces an isomorphism on $L_{T(n)}$ -completions, and thus there is a natural factorization of weak algebra maps*

$$\begin{array}{ccc}
 L_{T(n)}\mathbb{P}(X) & \xrightarrow{\text{canonical}} & \widehat{L}_{T(n)}\mathbb{P}(X). \\
 \searrow s_n(X) & & \nearrow t_n(Z) \\
 & L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+ &
 \end{array}$$

In more down-to-earth terms, our theorem gives us a factorization by weak maps

$$\begin{array}{ccc}
 L_{T(n)}\left(\bigvee_{r=0}^\infty D_r X\right) & \xrightarrow{\text{canonical}} & \prod_{r=0}^\infty L_{T(n)} D_r X \\
 \searrow s_n(X) & & \nearrow t_n(X) \\
 & L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+ &
 \end{array}$$

in which both the infinite wedge and product are equivalent to commutative $L_{T(n)}S$ -algebras, such that all maps in the diagram are $L_{T(n)}S$ -algebra maps.

For applications to computing $K(n)_*$ and E_n^* , it is useful to let

$$s_n^K(X) = L_{K(n)}s_n(X) : L_{K(n)}\mathbb{P}(X) \rightarrow L_{K(n)}\Sigma^\infty(\Omega^\infty X)_+.$$

The functor on S -modules sending X to $\mathbb{P}(X)$ preserves E_* -isomorphisms for any generalized homology theory E_* . Our natural transformations $s_n(X)$ and $s_n^K(X)$ yield the *best* possible ‘invariant’ approximations to the functors $L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+$ and $L_{K(n)}\Sigma^\infty(\Omega^\infty X)_+$ in the following sense:

Proposition 2.6. *Let $F : S \rightarrow S$ be any functor preserving $T(n)_*$ -isomorphisms. Then any natural transformation T of the form*

$$T(X) : F(X) \rightarrow L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+$$

factors uniquely through s_n . Similarly, s_n^K is the terminal natural transformation from a functor preserving $K(n)_$ -isomorphisms.*

This proposition will be an easy consequence of results described in §2.5.

2.4. First homological corollaries

It is easily verified that the canonical map from a wedge to the product of a family of spectra induces a monomorphism on any homology theory. Thus Theorem 2.5 has the following theorem as an immediate corollary.

Theorem 2.7. *If E_* is any homology theory such that $T(n)_*$ -acyclics are also E_* -acyclics, then s_n induces a natural monomorphism*

$$s_n(X)_* : E_*(\mathbb{P}X) \rightarrow E_*(\Omega^\infty X).$$

The commutative H -space structure on $\Omega^\infty X$, together with the diagonal, induces a $K(n)_*$ -Hopf algebra structure on $K(n)_*(\Omega^\infty X)$.¹ Meanwhile, the usual product maps,

$$D_i X \wedge D_j X \rightarrow D_{i+j} X,$$

and transfer maps,

$$D_{i+j} X \rightarrow D_i X \wedge D_j X,$$

associated to the inclusion of groups $\Sigma_i \times \Sigma_j \subseteq \Sigma_{i+j}$ make $K(n)_*(\mathbb{P}X)$ into a $K(n)_*$ -Hopf algebra. When specialized to $E_* = K(n)_*$, the last theorem refines as follows.

Theorem 2.8. $s_n(X)_* : K(n)_*(\mathbb{P}X) \rightarrow K(n)_*(\Omega^\infty X)$ is a natural inclusion of commutative, cocommutative $K(n)_*$ -Hopf algebras.

Since the cohomological Bousfield class of E_n is the same as the homological Bousfield class of $K(n)$, Theorem 2.5 also has the following consequence.

Theorem 2.9. *For all spectra X , and all $n \geq 1$, there is a factorization of commutative E_n^* -algebras*

$$\begin{array}{ccc} \bigoplus_{r=0}^{\infty} E_n^*(D_r X) & \xrightarrow{\text{canonical}} & \prod_{r=0}^{\infty} E_n^*(D_r X), \\ & \searrow t_n(X)^* & \nearrow s_n(X)^* \\ & E_n^*(\Omega^\infty X) & \end{array}$$

where the algebra structure on the infinite sum and product is induced by the transfer maps. In particular, $t_n(X)^*$ is a natural inclusion of commutative E_n^* -algebras.

¹ In the twisted sense described in [B4, Appendix], if $p = 2$.

Here the map labelled $t_n(X)^*$ is defined by applying E_n^* to the composite

$$\Sigma^\infty(\Omega^\infty X)_+ \xrightarrow{t_n(X)} \prod_{q=0}^\infty D_q X \rightarrow \prod_{q=0}^r D_q X$$

and then letting r go to ∞ .

We explain the appearance of the transfer maps, e.g. in this last theorem. This is a consequence of the naturality of s_n and t_n , as applied to the diagonal $\Delta : X \rightarrow X \times X$. Since Ω^∞ commutes with products, one sees that the product on $E_n^*(\Omega^\infty X)$ is induced by applying the functor $\Sigma^\infty(\Omega^\infty(-)_+)$ to Δ . Meanwhile, it is well known (see [LMMS, Theorem VII.1.10] or [K2, Proposition A.3]) that applying $D_k(-)$ to the weak map

$$X \xrightarrow{\Delta} X \times X \xleftarrow{\sim} X \vee X$$

yields the product, over $i + j = k$, of the transfer maps

$$D_k X \rightarrow D_i X \wedge D_j X.$$

2.5. When is $s_n(X)$ an equivalence?

One might now wonder how often $s_n(X)$ and $s_n^K(X)$ are equivalences. We have various results which together give a good sense of what is happening.

We define classes of S -modules $\mathcal{S}_n \subseteq \mathcal{S}_n^K$ as follows.

$$\begin{aligned} \text{Let } \mathcal{S}_n &= \{X \in \mathcal{S} \mid s_n(X) \text{ is an equivalence}\} \\ &= \{X \in \mathcal{S} \mid s_n(X)_* : T(n)_*(\mathbb{P}X) \xrightarrow{\sim} T(n)_*(\Omega^\infty X)\}. \end{aligned}$$

$$\begin{aligned} \text{Let } \mathcal{S}_n^K &= \{X \in \mathcal{S} \mid s_n^K(X) \text{ is an equivalence}\} \\ &= \{X \in \mathcal{S} \mid s_n(X)_* : K(n)_*(\mathbb{P}X) \xrightarrow{\sim} K(n)_*(\Omega^\infty X)\}. \end{aligned}$$

We recall that Eilenberg–MacLane spectra are acyclic in $T(n)_*$. From this the following first observations are easily deduced: if S' is either \mathcal{S}_n or \mathcal{S}_n^K , then $X \in S'$ if and only if $X\langle -1 \rangle \in S'$, and furthermore, a necessary condition is that $\pi_0(X) = 0$. Here $X\langle d \rangle$ denotes the d -connected cover of an S -module X .

Our results about \mathcal{S}_n and \mathcal{S}_n^K are most pleasantly described by first introducing two more classes of S -modules.

$$\text{Let } \bar{\mathcal{S}}_n = \{X \in \mathcal{S} \mid X\langle d \rangle \in \mathcal{S}_n \text{ for large } d\},$$

Let $\bar{\mathcal{S}}_n^K = \{X \in \mathcal{S} \mid X\langle d \rangle \in \mathcal{S}_n^K \text{ for large } d\}$.

We recall a concept from [HRW]: say X is *strongly E_* -acyclic* if the spaces $\Omega^\infty \Sigma^c(X\langle -1 \rangle)$ are E_* -acyclic for all large c .² As an important example, it is an observation of Bousfield [B2] that Eilenberg–MacLane spectra are strongly E_* -acyclic if there exists any E_* -acyclic space. Note that this is the case if E_* is either $T(n)_*$ or $K(n)_*$, as finite complexes of type more than n are $T(n)_*$ -acyclic.

Following notation in various papers, e.g. [B4], let L_{n-1}^f denote localization with respect to $T(0) \vee \dots \vee T(n-1)$. We recall that this is smashing: $L_{n-1}^f X \simeq L_{n-1}^f S \wedge X$.

Armed with this terminology, we have the following theorem and proposition.

Theorem 2.10.

- (1) $\bar{\mathcal{S}}_n = \{X \in \mathcal{S} \mid L_{n-1}^f X \text{ is strongly } T(n)_*\text{-acyclic}\}$, and $\mathcal{S}_n \subset \bar{\mathcal{S}}_n$.
- (2) $\bar{\mathcal{S}}_n^K = \{X \in \mathcal{S} \mid L_{n-1}^f X \text{ is strongly } K(n)_*\text{-acyclic}\}$, and $\mathcal{S}_n^K \subset \bar{\mathcal{S}}_n^K$.

Proposition 2.11. *There are implications (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5).*

- (1) $T(i)_*(X) = 0$ for $1 \leq i \leq n-1$.
- (2) $L_{n-1}^f X$ is strongly $T(n)_*$ -acyclic.
- (3) $L_{n-1}^f X$ is strongly $K(n)_*$ -acyclic.
- (4) (with $n \geq 2$) X is strongly $K(n-1)_*$ -acyclic.
- (5) $K(i)_*(X) = 0$ for $1 \leq i \leq n-1$.

Let $c(n)$ denote the smallest integer c such that $\tilde{T}(n)_*(K(\mathbb{Z}/p, c)) = 0$. Then $c(n) \geq n+1$, with equality certainly holding if the Telescope Conjecture is true, and perhaps even if not.

Theorem 2.12.

- (1) Suppose $X \in \bar{\mathcal{S}}_n$. Then $X \in \mathcal{S}_n$ if $\pi_0(X) = 0$, $\pi_j(X)$ is uniquely p -divisible for $0 \leq j \leq c(n)$, and also $\pi_{c(n)+1}(X)/(\text{torsion})$ is uniquely p -divisible.
- (2) Suppose $X \in \bar{\mathcal{S}}_n^K$. Then $X \in \mathcal{S}_n^K$ if and only if $\pi_0(X) = 0$, $\pi_j(X)$ is uniquely p -divisible for $1 \leq j \leq n$, and also $\pi_{n+1}(X)/(\text{torsion})$ is uniquely p -divisible.

Example 2.13. The conditions in Proposition 2.11 trivially hold for all X if $n = 1$. We conclude that $\bar{\mathcal{S}}_1 = \bar{\mathcal{S}}_1^K = \mathcal{S}$, and $\mathcal{S}_1 = \mathcal{S}_1^K$ is determined by the second statement of Theorem 2.12. In particular, there is a natural equivalence of commutative augmented $L_{K(1)}S$ -algebras

$$L_{K(1)}\mathbb{P}(X) \simeq L_{K(1)}\Sigma^\infty(\Omega^\infty X)_+$$

² Note that $\Omega^\infty \Sigma^c(X\langle -1 \rangle)$ is the c th space in the connective cover of the spectrum X .

for all 1-connected X , with torsion π_2 . This fits perfectly with the many results on $K^*(\Omega^\infty X)$ proved by Bousfield beginning with [B1].

Example 2.14. The Telescope Conjecture holds for $n = 1$: $T(1)_*$ -acyclics are $K(1)_*$ -acyclics. Thus the conditions in Proposition 2.11 are all equivalent when $n = 2$. We conclude that $\bar{S}_2 = \bar{S}_2^K = \{X \mid K(1)_*(X) = 0\}$, and S_2^K is the set of S -modules described by the second statement of Theorem 2.12. In particular, there is a natural equivalence of commutative augmented $L_{K(2)}S$ -algebras

$$L_{K(2)}\mathbb{P}(X) \simeq L_{K(2)}\Sigma^\infty(\Omega^\infty X)_+$$

for all $K(1)_*$ -acyclic, 2-connected X , with torsion π_3 .

Example 2.15. These results tell us precisely which p -local finite spectra are in S_n^K and \bar{S}_n^K . All finites are in \bar{S}_1^K , and F is in S_1^K if and only if $\pi_0(F) = \pi_1(F) = 0$ and $\pi_2(F)$ is torsion. If $n \geq 2$, F is in \bar{S}_n^K if and only if F has type at least n , and F is in S_n^K if and only if F has type at least n and also $\pi_j(F) = 0$ for all $0 \leq j \leq n$.

Remarks 2.16. If the Telescope Conjecture is true for a pair (p, n) , then the second statement of Theorem 2.12 improves the first by one dimension. This is due to the fact that in proving this second statement, we use computational methods based on special properties of $K(n)_*$.

Even if the Telescope Conjecture fails, it is still conceivable that some of the conditions in Proposition 2.11 are equivalent. We note that, as observed in [HRW, Theorem 3.14], if X is a BP -module, then condition (5) implies condition (1).

Our results imply that, in general, the two maps

$$s_n^K(\Sigma^\infty Z), L_{K(n)}s(Z) : L_{K(n)}\mathbb{P}(\Sigma^\infty Z) \rightarrow L_{K(n)}\Sigma^\infty QZ_+$$

are *not* homotopic, since, by Theorem 2.2, the second map is an equivalence whenever Z is connected, while the first map need not be. By perturbing the ‘usual’ stable splitting of QZ , we have thus lost homotopy equivalence but gained naturality with respect to *stable* maps between suspension spectra. See Appendix C for more about this.

2.6. More homological corollaries

Hand in hand with the results of the last subsection, are some more homological corollaries.

In the spirit of [B5, §11], if Eilenberg–MacLane spectra are strongly E_* -acyclic, one can define $E_*^{\text{vir}}(Z)$, the *virtual* E_* -homology of a space Z , by the formula

$$E_*^{\text{vir}}(Z) = E_*(Z\langle d \rangle) \text{ for large } d.$$

Methods of [HRW] imply the following illuminating lemma.

Lemma 2.17. *An S -module X is strongly E_* -acyclic if and only if*

$$\tilde{E}_*^{\text{vir}}(\Omega^\infty X) = 0.$$

The fact that $\mathbb{P}(X\langle d \rangle) \rightarrow \mathbb{P}X$ is a $K(n)_*$ -equivalence implies that there is a canonical lifting

$$\begin{array}{ccc} & & K(n)_*^{\text{vir}}(\Omega^\infty X) \\ & \nearrow s_n(X)_* & \downarrow \\ K(n)_*(\mathbb{P}X) & \longrightarrow & K(n)_*(\Omega^\infty X). \end{array}$$

The next theorem is closely related to Theorem 2.10.

Theorem 2.18. *For all X , there is a natural short exact sequence of commutative, cocommutative $K(n)_*$ -Hopf algebras*

$$K(n)_*(\mathbb{P}X) \xrightarrow{s_n(X)_*} K(n)_*^{\text{vir}}(\Omega^\infty X) \rightarrow K(n)_*^{\text{vir}}(\Omega^\infty L_{n-1}^f X).$$

Note that the first term here is a functor of $L_{K(n)}X$; in contrast, the last term is ‘invisible’ to $L_{K(n)}X$, as $K(n)_*(L_{n-1}^f X) = 0$.

Since $K(n)_*(\mathbb{P}X) = 0$ exactly when $K(n)_*(X) = 0$, we have the next corollary, which strengthens [HRW, Corollary 3.13].

Corollary 2.19. *X is strongly $K(n)_*$ -acyclic if and only if X is $K(n)_*$ -acyclic and also $L_{n-1}^f X$ is strongly $K(n)_*$ -acyclic. In particular, if X is $K(n)_*$ -acyclic, and also $T(i)_*$ -acyclic for $1 \leq i \leq n-1$, then X is strongly $K(n)_*$ -acyclic.*

Our next result is a homological variant of Proposition 2.6.

Proposition 2.20. *Let $F : \mathcal{S} \rightarrow K(n)_*$ -modules be any functor preserving $K(n)_*$ -isomorphisms. Then any natural transformation T of the form*

$$T(X) : F(X) \rightarrow K(n)_*(\Omega^\infty X)$$

factors uniquely through s_{n} .*

The $T(n)_*$ variant of this proposition also holds. Similarly, there is also a E_n^* variant that says that s_n^* is the initial functor from $E_n^*(\Omega^\infty X)$ to a functor preserving E_n^* -isomorphisms.

The next theorem has Theorem 2.12(2) as a consequence.

Theorem 2.21. Let $X \in \bar{S}_n^K$, i.e. $L_{n-1}^f X$ is strongly $K(n)_*$ -acyclic. Then each of the maps

$$K(n)_*(\Omega^\infty X\langle j \rangle) \rightarrow K(n)_*(\Omega^\infty X)$$

is an inclusion of a normal sub- $K(n)_*$ -Hopf algebra, and together they induce a decreasing filtration of finite length on $K(n)_*(\Omega^\infty X)$. For each $i \geq -1$, there is a short exact sequence of $K(n)_*$ -Hopf algebras

$$K(n)_*(\mathbb{P}X) \rightarrow K(n)_*(\Omega^\infty X\langle i \rangle) \rightarrow \bigotimes_{j=i+1}^{n+1} K(n)_*(K(\pi_j(X), j)),$$

and these are compatible as i varies.

Example 2.22. When $n = 1$, the theorem says that for all X , there is a short exact sequence of $K(1)_*$ -Hopf algebras

$$K(1)_*(\mathbb{P}X) \rightarrow K(1)_*(\Omega^\infty X) \rightarrow \bigotimes_{j=0}^2 K(1)_*(K(\pi_j(X), j)).$$

Example 2.23. When $n = 2$, the theorem says that, if $K(1)_*(X) = 0$, then there is a short exact sequence of $K(2)_*$ -Hopf algebras

$$K(2)_*(\mathbb{P}X) \rightarrow K(2)_*(\Omega^\infty X) \rightarrow \bigotimes_{j=0}^3 K(2)_*(K(\pi_j(X), j)).$$

Example 2.24. Suppose $L_{n-1}^f X$ is strongly $K(n)_*$ -acyclic, and also X is n -connected with $\pi_{n+1}(X)$ torsion. The theorem implies that

$$K(n)_*(\Omega^\infty X\langle n+1 \rangle) \rightarrow K(n)_*(\Omega^\infty X) \tag{2.3}$$

is an isomorphism.

When $n = 1$, the first hypothesis is always satisfied, and we recover [B1, Theorem 2.4]. This is the key technical theorem of Bousfield's paper. In recent email to the author, Bousfield has observed that (2.3) allows for an addendum to [B7, Theorem 8.1], analogous to the use of [B1, Theorem 2.4] in the proof of [B1, Theorem 3.2].

Some hypotheses are necessary here. In [B7, §8.7], Bousfield notes that if X is the suspension spectrum of the Moore space $M(\mathbb{Z}/p, 3)$, then

$$K(2)_*(\Omega^\infty X\langle 3 \rangle) \rightarrow K(2)_*(\Omega^\infty X)$$

has nonzero kernel.

Example 2.25. Let $k(n)_c = \Omega^\infty \Sigma^c k(n)$, where $k(n)$ is the connective cover of $K(n)$. $k(n) \in \tilde{\mathcal{S}}_n^K$, as it is a BP -module that is $K(i)_*$ -acyclic for $1 \leq i \leq n-1$. Thus the theorem applies to say that the cofibration sequence of spectra

$$\Sigma^{2p^n-2} k(n) \xrightarrow{v} k(n) \rightarrow H\mathbb{Z}/p$$

induces a short exact sequence of commutative, cocommutative $K(n)_*$ -Hopf algebras

$$K(n)_*(k(n)_{2p^n-2+c}) \rightarrow K(n)_*(k(n)_c) \rightarrow K(n)_*(K(\mathbb{Z}/p, c))$$

for all $c > 0$.

This is [BKW, Theorem 1.1].

These last two examples also illustrate our next result.

Theorem 2.26. Suppose $f : X \rightarrow Y$ is map of 0-connected S -modules with cofiber C , such that $\mathbb{P}(f)_* : K(n)_*(\mathbb{P}X) \rightarrow K(n)_*(\mathbb{P}Y)$ is monic. (For example, f might be a $K(n)_*$ -isomorphism.) If $X \in \mathcal{S}_n^K$ then there is a short exact sequence of commutative, cocommutative $K(n)_*$ -Hopf algebras

$$K(n)_*(\Omega^\infty X) \rightarrow K(n)_*(\Omega^\infty Y) \rightarrow K(n)_*(\Omega^\infty C).$$

Remarks 2.27. It seems appropriate to comment on other results in the literature that concern homological calculations of the sort we look at here.

In unpublished work, but in the spirit of [Str], N. Strickland has observed that $E_n^*(\mathbb{P}X)$ is a functor of $E_n^*(X)$, when restricted to X such that $E_n^*(X)$ is appropriately ‘pro-free’ as an E_n^* -module, and also $E_n^*(X)$ is concentrated in even degrees. It strikes the author as likely that the second of these hypotheses is unnecessary, since the work of either McClure [BMMS, Chapter IX] or Bousfield [B4] shows this to be the case when $n = 1$.

Related to this, it appears that when $K(n)_*(X)$ and $K(n)_*(Y)$ are concentrated in even degrees, if $f : X \rightarrow Y$ is a $K(n)_*$ -monomorphism, then so is $\mathbb{P}(f)$.

In various papers culminating in [Ka], Kashiwabara computes $E^*(\Omega^\infty X)$ if $E = BP, E_n$, or $K(n)$, under suitable side hypotheses on X , e.g. if X is -1 -connected with cells only in even dimensions. His methods are very different than ours; in particular, he heavily uses BP -Adams resolutions of his spectra. Most of his results appear to have only limited naturality, and many of his results are specialized to the case when X is a suspension spectrum.

There is most obvious overlap between our results and those of Bousfield, particularly in [B7]. It seems that any proof of our main theorem, Theorem 2.5, will depend crucially on the existence of a Goodwillie tower for $\Sigma^\infty \Omega^\infty X$. But once this theorem has been established, many of our other results allow for alternate proofs using his work. See Appendix B.

2.7. The main theorem for rational homology

For completeness, we note that a version of our main theorem holds when $n = 0$, i.e., with $L_{H\mathbb{Q}}$ replacing $L_{T(n)}$.

Analogous to (2.2), we have the following lemma: for all 0-connected spectra X , there is a natural factorization by weak S -module maps

$$\begin{array}{ccc}
 & L_{H\mathbb{Q}}\Sigma^\infty\Omega^\infty X & \\
 \eta_0(X) \nearrow & & \searrow L_{H\mathbb{Q}}\varepsilon(X) \\
 L_{H\mathbb{Q}}X & \xlongequal{\quad\quad\quad} & L_{H\mathbb{Q}}X.
 \end{array} \tag{2.4}$$

For X just -1 -connected, such a factorization also exists, but cannot be made to be natural.

$\eta_0(X)$ then induces a weak map $s_0(X): L_{H\mathbb{Q}}\mathbb{P}(X) \rightarrow L_{H\mathbb{Q}}\Sigma^\infty(\Omega^\infty X)_+$ in $L_{H\mathbb{Q}}(\mathcal{A}lg)$, natural for 0-connected X . Proposition 2.3 combines with the lemma to prove:

Theorem 2.28. *For all -1 -connected spectra X , the map $s_0(X)$ induces an isomorphism on $L_{H\mathbb{Q}}$ -completions, and thus there is a factorization of weak algebra maps*

$$\begin{array}{ccc}
 L_{H\mathbb{Q}}\mathbb{P}(X) & \xrightarrow{\text{canonical}} & \widehat{L}_{H\mathbb{Q}}\mathbb{P}(X). \\
 s_0(X) \searrow & & \nearrow t_0(Z) \\
 & L_{H\mathbb{Q}}\Sigma^\infty(\Omega^\infty X)_+ &
 \end{array}$$

This is natural for 0-connected X , and, in this case, all three maps are equivalences.

As a corollary, one recovers the known theorem: for 0-connected spectra X , the Hopf algebra $\pi_*^S(\Omega^\infty X) \otimes \mathbb{Q}$ is naturally isomorphic to the graded symmetric algebra primitively generated by $\pi_*(X) \otimes \mathbb{Q}$.

2.8. Organization of the rest of the paper

In §3, we develop the general theory of André–Quillen towers associated to commutative augmented S -algebras, leading to proofs of Propositions 2.1 and 2.3. Included is a subsection summarizing the basic properties of the functor $\Sigma^\infty(\Omega^\infty X)_+$.

The splitting theorems concerning $\Sigma^\infty(\Omega^\infty X)_+$, Theorems 2.2, 2.5, and 2.28, are then easily proved in §4, which includes discussion of (2.2) and (2.4), and Theorem 2.4.

In §5, we explore the extent to which $s_n(X)$ is an equivalence, proving the results in §2.5 and §2.6. Some of the proofs are a bit long and delicate: we hope we have made them comprehensible.

In Appendix A, we check that our stable splitting of QZ agrees with others in the literature. In Appendix B, we compare our constructions and theorems to those of [B7]. In Appendix C, we compare s to s_n , and make some remarks about James–Hopf invariants.

3. The André–Quillen tower of commutative algebras

3.1. Categories of commutative S -algebras

We work always within the topological model category of S -modules as in [EKMM]. This is a symmetric monoidal category with unit the sphere spectrum S , and we recall that associative, commutative, unital S -algebras are modern day versions of E_∞ -algebras. Given such an algebra R , we let $R\text{-Alg}$ denote the category of associative, commutative, unital, augmented R -algebras. When $R = S$ we simplify the notation to Alg .

Given $A \in R\text{-Alg}$, let $I'(A)$ denote the fiber of the augmentation $A \rightarrow R$. As discussed in [Ba], the functor I' takes values in the category $R\text{-Alg}'$ of associative, commutative, nonunital R -algebras, and is the right adjoint of a Quillen equivalence between the model categories $R\text{-Alg}$ and $R\text{-Alg}'$.³ We let $I(A)$ denote a cofibrant replacement in $R\text{-Alg}'$ of $I'(A')$, where A' is a fibrant replacement of $A \in \text{Alg}$.

The categories $R\text{-Alg}$ and $R\text{-Alg}'$ are tensored and cotensored over based topological spaces (see [EKMM] or [K5]). In particular, given $I \in R\text{-Alg}'$, one can form the iterated suspension $S^n \otimes I$ and the iterated looping $\Omega^n I$.

We note that both Ω^n (as a right adjoint) and homotopy colimits over directed systems (see, e.g. [EKMM, Lemma VII.3.10]) commute with the forgetful functor from $R\text{-Alg}'$ to R -modules.

The processes of taking coproducts and suspending (tensoring with S^1) certainly *don't* commute with this forgetful functor. Indeed, the coproduct in $R\text{-Alg}$ is the smash product, and thus in $R\text{-Alg}'$ one has

$$I \amalg J = I \wedge J \vee I \vee J.$$

Regarding suspension, one has the following lemma, which serves as the basis for many convergence results.

Lemma 3.1. *If a cofibrant $I \in \text{Alg}'$ is n -connected, then the natural map*

$$I \rightarrow \Omega(S^1 \otimes I)$$

is $2n + 1$ -connected.

³The left adjoint sends a nonunital R -algebra I to the augmented algebra $I' \vee R$.

We feature two families of examples.

Example 3.2. Given an S -module X , we let $\mathbb{P}X$ denote the free commutative S -algebra generated by X [EKMM, p. 40]. If X is cofibrant then there is a natural weak equivalence [EKMM, p. 64]:

$$\mathbb{P}X \simeq \bigvee_{r=0}^{\infty} D_r X.$$

$\mathbb{P}X$ is naturally augmented, and we let $i : X \rightarrow I(\mathbb{P}X)$ denote the natural weak map. Using the freeness of $\mathbb{P}X$, given any $A \in \mathcal{A}lg$, a weak map of S -modules $f : X \rightarrow I(A)$ induces a weak map in $\mathcal{A}lg$, $\tilde{f} : \mathbb{P}X \rightarrow A$ such that the diagram of weak maps

$$\begin{array}{ccc} X & & \\ \downarrow i & \searrow f & \\ I(\mathbb{P}X) & \xrightarrow{I(\tilde{f})} & I(A) \end{array}$$

commutes.

Given a commutative S -algebra R , and an R -module Y , there is an analogous free object $\mathbb{P}_R Y \in R - \mathcal{A}lg$, satisfying the evident ‘change of rings’ formula

$$\mathbb{P}_R(R \wedge X) = R \wedge \mathbb{P}X.$$

Example 3.3. Given an S -module X , the E_{∞} -structure on the infinite loop space $\Omega^{\infty}X$ implies that $\Sigma^{\infty}(\Omega^{\infty}X)_+$ takes values in $\mathcal{A}lg$. See [M2, Ex.IV.1.10] and [EKMM, §II.4]. The composite

$$I(\Sigma^{\infty}(\Omega^{\infty}X)_+) \rightarrow \Sigma^{\infty}(\Omega^{\infty}X)_+ \rightarrow \Sigma^{\infty}(\Omega^{\infty}X)$$

is a weak equivalence for all X . The natural map $X\langle -1 \rangle \rightarrow X$ of S -modules induces an equivalence in $\mathcal{A}lg$:

$$\Sigma^{\infty}(\Omega^{\infty}X\langle -1 \rangle)_+ \xrightarrow{\sim} \Sigma^{\infty}(\Omega^{\infty}X)_+.$$

3.2. Topological André–Quillen homology

One version of the Topological André–Quillen Homology of $A \in R - \mathcal{A}lg$ is as the set of homotopy groups of the following construction.

Definition 3.4. Given $A \in R - \mathcal{A}lg$, let $taq_R(A) \in R - \mathcal{A}lg'$ be defined by

$$taq_R(A) = \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n(S^n \otimes I(A)).$$

Remark 3.5. An alternative construction, more reminiscent of the work of André and Quillen is to let $taq_R(A) = ZQ(A)$, where $Q : \mathcal{A}lg \rightarrow R - \text{modules}$ is defined by

$$Q(A) = I(A)/I(A)^2,$$

and $Z : R - \text{modules} \rightarrow \mathcal{A}lg'$ is defined by letting $Z(X)$ be a fibrant replacement of X given trivial multiplication. This is the construction explored by Basterra in [Ba], and she and Mandell have an unpublished proof that these two constructions are equivalent.⁴ For our purposes, particularly as in Example 3.9 below, the construction we use is most convenient.

We denote $taq_S(A)$ by $taq(A)$. The following ‘change of rings’ formula follows easily from the definition.

Lemma 3.6. *For all $A \in \mathcal{A}lg$, there is a natural weak equivalence in $R - \mathcal{A}lg'$*

$$R \wedge taq(A) \simeq taq_R(R \wedge A).$$

Another basic property that we will use is the following.

Lemma 3.7. *taq_R takes homotopy cofibration sequences in $R - \mathcal{A}lg$ to homotopy cofibration sequences in R -modules.*

Proof. As will be elaborated on in §3.6, the functor taq_R , defined via stabilization as above, will be 1-excisive in the sense of Goodwillie: it will take homotopy pushout squares to homotopy pullback squares. But in R -modules, homotopy pullbacks are homotopy pushouts. \square

We now calculate $taq(A)$ for our two key examples.

Example 3.8. Corresponding to Example 3.2, we claim that there is a natural equivalence

$$taq(\mathbb{P}X) \simeq X,$$

⁴The proof of the main theorem of [BM] indicates some of the ideas, as does S. Schwede’s earlier paper [Sch].

where X has trivial multiplication, such the natural map $I(\mathbb{P}X) \rightarrow \text{taq}(\mathbb{P}X)$ corresponds to the projection

$$\bigvee_{r=1}^{\infty} D_r X \rightarrow D_1 X = X.$$

To see this, we make two observations.

Firstly, for all based spaces K and S -modules X , there is a natural isomorphism

$$K \otimes \mathbb{P}(X) = \mathbb{P}(K \wedge X).$$

Secondly, for all S -modules X there are natural equivalences

$$\text{hocolim}_{n \rightarrow \infty} \Omega^n D_r(\Sigma^n X) \simeq \begin{cases} X & \text{if } r = 1, \\ * & \text{if } r > 1. \end{cases}$$

(This is clear by a connectivity argument if X is connective, and then note that an arbitrary S -module is equivalent to a hocolimit of connective S -modules.)

Combining these observations, we compute:

$$\begin{aligned} \text{taq}(\mathbb{P}X) &= \text{hocolim}_{n \rightarrow \infty} \Omega^n (S^n \otimes I(\mathbb{P}X)) \\ &= \text{hocolim}_{n \rightarrow \infty} \Omega^n I(\mathbb{P}\Sigma^n X) \\ &\simeq \bigvee_{r=1}^{\infty} \text{hocolim}_{n \rightarrow \infty} \Omega^n D_r(\Sigma^n X) \\ &\simeq X. \end{aligned}$$

Example 3.9. If X is a -1 -connected S -module, corresponding to Example 3.3, we claim that there is a natural weak equivalence

$$\text{taq}(\Sigma^\infty(\Omega^\infty X)_+) \simeq X,$$

such that the natural map $I(\Sigma^\infty(\Omega^\infty X)_+) \rightarrow \text{taq}(\Sigma^\infty(\Omega^\infty X)_+)$ corresponds to the counit

$$\varepsilon : \Sigma^\infty \Omega^\infty X \rightarrow X.$$

To see this, we again make a couple of observations.

Firstly, if Z is an E_∞ -space, let BZ be the associated classifying space [M1]. As surveyed in [K5], there are natural weak equivalences

$$S^1 \otimes I(\Sigma^\infty Z_+) \simeq I(\Sigma^\infty BZ_+)$$

such that the natural map $I(\Sigma^\infty Z_+) \rightarrow \Omega(S^1 \otimes I(\Sigma^\infty Z_+))$ corresponds to $\Sigma^\infty Z \rightarrow \Omega\Sigma^\infty BZ$.

Secondly, given a -1 -connected S -module X , let $X_n = \Omega^\infty \Sigma^n X$. Then there is a natural equivalence $BX_n \xrightarrow{\sim} X_{n+1}$, and a natural weak equivalence

$$\operatorname{hocolim}_{n \rightarrow \infty} \Omega^n \Sigma^\infty X_n \xrightarrow{\sim} X$$

such that the inclusion of $\Sigma^\infty X_0$ into the hocolimit corresponds to ε .

Combining these observations, we compute:

$$\begin{aligned} \operatorname{taq}(\Sigma^\infty(\Omega^\infty X)_+) &= \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n(S^n \otimes I(\Sigma^\infty(X_0)_+)) \\ &\simeq \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n I(\Sigma^\infty(B^n X_0)_+) \\ &\simeq \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n I(\Sigma^\infty(X_n)_+) \\ &\simeq \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n \Sigma^\infty X_n \\ &\simeq X. \end{aligned}$$

3.3. The André–Quillen tower of an augmented R -algebra

It seems that various people have noted the existence of an ‘André–Quillen tower’ associated to $A \in R\text{-Alg}$. Intuitively, this tower is supposed to be the augmentation ideal tower

$$\dots \rightarrow A/I^r \rightarrow \dots \rightarrow A/I^2 \rightarrow A/I$$

constructed in a homotopically meaningful way. The next theorem lists the properties we care about.

In this theorem, $D_r^R Y$ denotes the r th extended power construction in the category of R -modules, i.e. one uses the smash product \wedge_R . Note that there is an isomorphism $R \wedge D_r(X) = D_r^R(R \wedge X)$.

Theorem 3.10. *Given $A \in R - \text{Alg}$, there is a unique natural tower of fibrations in $R - \text{Alg}$ under A*

$$\begin{array}{ccc}
 & & \vdots \\
 & & \downarrow \\
 & & P_{R,2}A \\
 & \nearrow e_2 & \downarrow p_2 \\
 & \nearrow e_1 & P_{R,1}A \\
 & \nearrow e_0 & \downarrow p_1 \\
 A & \longrightarrow & P_{R,0}A,
 \end{array} \tag{3.1}$$

with the following properties.

- (1) $P_{R,0}A \simeq R$ so that e_0 identifies with the augmentation.
- (2) For $r \geq 1$, the fiber of $p_r : P_{R,r}A \rightarrow P_{R,r-1}A$ is naturally weakly equivalent to $D_r^R(\text{taq}_R(A))$. Furthermore, $I(e_1)$ identifies with the natural map $I(A) \rightarrow \text{taq}_R(A)$.
- (3) Denoting $P_{S,r}(A)$ by $P_r(A)$, there is a change of rings formula: Given $A \in \text{Alg}$ and R a commutative S -algebra, there is a natural weak equivalence of towers under $R \wedge A$:

$$R \wedge P_r(A) \simeq P_{R,r}(R \wedge A).$$

- (4) If $I(A)$ is 0-connected, then e_r is r -connected.

The uniqueness statement means up to natural weak equivalence. The weak equivalence in the second property is as R -modules.⁵

Minasian constructs a tower with these properties in the preprint [Min], following along the lines of [Ba], and using her version of $\text{taq}_R(A)$.⁶ However, the appearance (finally) of a finished version of [G3] allows the author to feel comfortable with an alternative construction, suggested to him by Arone.⁷

⁵ Though we won't need nor prove it, if one gives $D_r^R(\text{taq}_R(A))$ trivial multiplication, this equivalence is even as objects in $R - \text{Alg}'$.

⁶ In email with the author, Mandell has also sketched this result.

⁷ Added February, 2005: These comments were written in early 2003, before the paper fell into a refereeing purgatory.

Definition 3.11. Let the tower $\{P_{R,r}(\)\}$ denote the Goodwillie tower associated to the identity functor on $R - \mathcal{A}lg$.

In the subsection §3.6 we sketch a proof that this tower has the properties stated in the theorem. Assuming the theorem, we now follow up with some consequences and an example.

Definition 3.12. Let $\widehat{A} = \text{holim}_{r \rightarrow \infty} P_{R,r}A$.

Corollary 3.13. If $I(A)$ is 0-connected, then the natural map $A \rightarrow \widehat{A}$ is an equivalence.

Example 3.14. Corresponding to Example 3.2, we have

$$P_r(\mathbb{P}X) \simeq \left(\bigvee_{q=0}^{\infty} D_q X \right) / \left(\bigvee_{q=r+1}^{\infty} D_q X \right) \quad (3.2)$$

so that there is a natural equivalence

$$\widehat{\mathbb{P}}X \simeq \prod_{r=0}^{\infty} D_r X.$$

One way to see this is to note that both sides of (3.2), viewed as towers of functors from S -modules to $\mathcal{A}lg$, have the correct form to be a Goodwillie tower of the functor sending X to $\mathbb{P}X$, and thus agree, up to natural weak equivalence.

The following lemma is well known and much used.

Lemma 3.15. The functor D_r^R preserves weak equivalences of R -modules.

Proof. Both \wedge_R and $(\)_{h\Sigma_r}$ (homotopy Σ_r -orbits) preserve weak equivalences. \square

Proposition 3.16. Let $f : A \rightarrow B$ be a map in $R - \mathcal{A}lg$. If $\text{taq}_R(f) : \text{taq}_R(A) \rightarrow \text{taq}_R(B)$ is a weak equivalence, so is $P_{R,r}(f) : P_{R,r}(A) \rightarrow P_{R,r}(B)$ for all r , and thus also $\widehat{f} : \widehat{A} \rightarrow \widehat{B}$. Thus, in this case, one gets a factorization by weak R -algebra maps

$$\begin{array}{ccc} A & \xrightarrow{\text{canonical}} & \widehat{A}, \\ & \searrow f & \nearrow \\ & B & \end{array}$$

where the unlabelled weak map is $B \rightarrow \widehat{B} \xleftarrow[\sim]{\widehat{f}} \widehat{A}$.

Proof. Using the lemma, one proves this by induction up the André–Quillen tower. \square

This proposition specializes to Proposition 2.1 when $R = S$.

3.4. A summary of the properties of $\mathbb{P}(X)$ and $\Sigma^\infty(\Omega^\infty X)_+$

In this subsection, we use our work thus far to summarize basic properties of $\mathbb{P}(X)$ and $\Sigma^\infty(\Omega^\infty X)_+$, viewed as functors from S -modules to $\mathcal{A}lg$.

Proposition 3.17. *The functor \mathbb{P} satisfies the following properties.*

- (1) \mathbb{P} takes homotopy colimits in S -modules to homotopy colimits in $\mathcal{A}lg$.
- (2) $I(\mathbb{P}(X)) \rightarrow \text{taq}(\mathbb{P}(X))$ identifies with $\bigvee_{r=1}^\infty D_r X \rightarrow D_1 X = X$. Thus the composite $X \xrightarrow{i} I(\mathbb{P}(X)) \rightarrow \text{taq}(\mathbb{P}(X))$ is an equivalence.
- (3) $\mathbb{P}(X) \rightarrow \widehat{\mathbb{P}}(X)$ identifies with $\bigvee_{r=0}^\infty D_r X \rightarrow \prod_{r=0}^\infty D_r X$.

Proof. The first property follows formally from the fact that \mathbb{P} is left adjoint to the forgetful functor from $\mathcal{A}lg$ to S -modules, since the model category on $\mathcal{A}lg$ is defined so that algebra maps are fibrations or weak equivalences exactly when they are fibrations or weak equivalences when considered as maps of S -modules. The other two properties were established above in Examples 3.8 and 3.14. \square

Proposition 3.18. *The functor $\Sigma^\infty(\Omega^\infty)_+$ satisfies the following properties.*

- (1) $\Sigma^\infty(\Omega^\infty)_+$ takes filtered homotopy colimits in S -modules to filtered homotopy colimits in $\mathcal{A}lg$.
- (2) $\Sigma^\infty(\Omega^\infty)_+$ takes coproducts in S -modules to coproducts in $\mathcal{A}lg$.
- (3) If $X \rightarrow Y \rightarrow Z$ is a cofibration sequence S -modules, with X and Y -1 -connected and Z 0 -connected, then

$$\Sigma^\infty(\Omega^\infty X)_+ \rightarrow \Sigma^\infty(\Omega^\infty Y)_+ \rightarrow \Sigma^\infty(\Omega^\infty Z)_+$$

is a cofibration sequence in $\mathcal{A}lg$.

- (4) $I(\Sigma^\infty(\Omega^\infty X)_+) \rightarrow \text{taq}(\Sigma^\infty(\Omega^\infty X)_+)$ identifies with $\varepsilon : \Sigma^\infty \Omega^\infty X \rightarrow X\langle -1 \rangle$.

Proof. The last property was established above in Example 3.9.

To see that the first property holds, we first note that filtered homotopy colimits in $\mathcal{A}lg$ are detected by viewing them as being in S -modules. (Compare with [EKMM, §II.7].) But, as a functor to S -modules, $\Sigma^\infty(\Omega^\infty)_+$ certainly commutes with filtered homotopy colimits.

Thanks to the first property, it suffices to prove the second property for finite coproducts. In $\mathcal{A}lg$, we have equivalences

$$\Sigma^\infty(\Omega^\infty(X \vee Y))_+ \xrightarrow{\sim} \Sigma^\infty(\Omega^\infty(X \times Y))_+ = \Sigma^\infty(\Omega^\infty X)_+ \wedge \Sigma^\infty(\Omega^\infty Y)_+,$$

which is the coproduct in $\mathcal{A}lg$ of $\Sigma^\infty(\Omega^\infty X)_+$ and $\Sigma^\infty(\Omega^\infty Y)_+$.

The proof of the third property is more delicate. A cofiber sequence of S -modules

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

will induce a commutative diagram in $\mathcal{A}lg$:

$$\begin{array}{ccccc} \Sigma^\infty(\Omega^\infty X)_+ & \longrightarrow & \Sigma^\infty(\Omega^\infty Y)_+ & \longrightarrow & A \\ \parallel & & \parallel & & \downarrow h \\ \Sigma^\infty(\Omega^\infty X)_+ & \longrightarrow & \Sigma^\infty(\Omega^\infty Y)_+ & \longrightarrow & \Sigma^\infty(\Omega^\infty Z)_+ \end{array}$$

where A is the cofiber in $\mathcal{A}lg$ of $\Sigma^\infty(\Omega^\infty f)_+$. We wish to show the map h is an equivalence.

If X , Y , and Z , are all -1 -connected, then applying taq to this diagram yields the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & taq(A) \\ \parallel & & \parallel & & \downarrow taq(h) \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \end{array}$$

where we have used property (4) above. The bottom horizontal sequence is given as a cofibration sequence; by Lemma 3.7, so is the top horizontal sequence. We conclude that $taq(h)$ is an equivalence. By Proposition 2.1, we conclude that \hat{h} is an equivalence. Under our connectivity hypothesis that Z is also 0-connected (so that $\pi_0(f)$ is onto), one can deduce that both A and $\Sigma^\infty(\Omega^\infty Z)_+$ have 0-connected augmentation ideals, and thus are equivalent to their completions, by Theorem 3.10(4). We conclude that h is an equivalence. \square

Remark 3.19. If one regards $\Sigma^\infty \Omega^\infty X_+$ just as a functor taking values in S -modules, rather than in $\mathcal{A}lg$, the fact that its Goodwillie tower has r th fiber equivalent to $D_r(X)$ has been known for awhile by Goodwillie and others working with the calculus of functors. This tower appears explicitly in the literature in [AK].

3.5. The localized tower

If E is an S -module, let L_E denote Bousfield localization with respect to E_* . It has long been usefully observed that various constructions in infinite loop space theory behave well with respect to Bousfield localization. For example, [K1] heavily used the following analogue of Lemma 3.15.

Lemma 3.20 ([K1, Corollary 2.3]). *The functor D_r^R preserves E_* -isomorphisms.*

Proof. Both \wedge_R and $(\)_{h\Sigma_r}$ preserve E_* -isomorphisms. \square

This same fact is behind the beautiful and much more recent theorem that if R is a commutative S -algebra, so is L_ER [EKMM, Chapter VIII].

Lemma 3.21. *The functor taq_R preserves E_* -isomorphisms.*

Proof. [K5, Corollary 7.5] says that $taq_R(A)$ is the colimit of an increasing filtration $F_1taq_R(A) \rightarrow F_2taq_R(A) \rightarrow \dots$ by cofibrations, and identifies the cofibers: there is an equivalence

$$F_dtaq_R(A)/F_{d-1}taq_R(A) \simeq (\Sigma K_d \wedge_R I(A)^{\wedge R^d})_{h\Sigma_d},$$

where K_d is a certain partition complex appearing in [AM]. The functor on the right of this equivalence certainly preserves E_* -isomorphisms, and thus so does $taq_R(A)$. \square

The lemmas combine with induction up the André–Quillen tower to prove

Corollary 3.22. *The functors $P_{R,r}$ preserve E_* -isomorphisms.*

Definition 3.23. Let $L_E(\mathcal{A}lg)$ be the full subcategory of $L_ES - \mathcal{A}lg$ consisting of E -local objects.

In the spirit of these last results, we have the following proposition.

Proposition 3.24. $L_E : \mathcal{A}lg \rightarrow L_E(\mathcal{A}lg)$ commutes with homotopy pushout squares and filtered homotopy colimits in the following sense:

(1) *the natural map*

$$L_E(\mathrm{hocolim}\{B \leftarrow A \rightarrow C\}) \rightarrow L_E(\mathrm{hocolim}\{L_E B \leftarrow L_E A \rightarrow L_E C\})$$

is an equivalence, and

(2) *the natural map $L_E(\mathrm{hocolim}_i A_i) \rightarrow L_E(\mathrm{hocolim}_i L_E A_i)$ is an equivalence.*

Proof. As discussed on [EKMM, p. 162], a model for the pushout of a diagram of R -algebras of the form $B \leftarrow A \rightarrow C$, where both maps are cofibrations, is given by an appropriate bar construction $\beta^R(B, A, C)$. This construction preserves E_* -isomorphisms

in all variables; in particular

$$L_E(\beta^S(B, A, C)) \rightarrow L_E(\beta^{L_E S}(L_E B, L_E A, L_E C))$$

is an equivalence, establishing (1). The proof of (2) is similar. \square

Remark 3.25. It is easy to see that $B \in L_E S - \mathcal{A}lg$ is E -local if and only if it is weakly equivalent $L_E A$, for some $A \in \mathcal{A}lg$. More precisely, B is E -local if and only if the natural weak map $L_E(I(B) \vee S) \xrightarrow{\sim} B$ is an equivalence. Thus $L_E(\mathcal{A}lg)$ is equivalent to the category $L'_E(\mathcal{A}lg)$ with objects $A \in \mathcal{A}lg$, and with morphisms from A to B equal to the $L_E S - \mathcal{A}lg$ maps from $L_E A$ to $L_E B$.

Definitions 3.26. We define functors with domain $L'_E(\mathcal{A}lg)$ as follows.

- (1) Let $taq^E(A) = L_E(taq_{L_E S}(L_E A))$.
- (2) Define the natural tower of fibrations in $L_E(\mathcal{A}lg)$ under $L_E A$,

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & P_2^E A \\ & \nearrow e_2^E & \downarrow p_2^E \\ & \nearrow e_1^E & P_1^E A \\ & \nearrow e_0^E & \downarrow p_1^E \\ A & \xrightarrow{\quad} & P_0^E A, \end{array} \quad (3.3)$$

to be the tower obtained by applying L_E to the André–Quillen tower (3.1) $\{P_{L_E S, r}(L_E A)\}$.

- (3) Let $\widehat{L}_E A = \lim_{r \rightarrow \infty} P_r^E(A)$.

We have the following analogue of Proposition 3.16. This proposition is a slight elaboration of Proposition 2.3 of the introduction.

Proposition 3.27. Let $f : L_E A \rightarrow L_E B$ be a map in $L_E(\mathcal{A}lg)$. If $taq^E(f) : taq^E(A) \rightarrow taq^E(B)$ is a weak equivalence, so is $P_r^E(f) : P_r^E(A) \rightarrow P_r^E(B)$ for all r , and thus also $\widehat{f} : \widehat{L}_E A \rightarrow \widehat{L}_E B$. Thus, in this case, one gets a factorization by weak $L_E S$ -

algebra maps

$$\begin{array}{ccc}
 L_E A & \xrightarrow{\text{canonical}} & \widehat{L}_E A, \\
 & \searrow f & \nearrow \\
 & L_E B &
 \end{array}$$

where the unlabelled weak map is $L_E B \rightarrow \widehat{L}_E B \xleftarrow[\sim]{\widehat{f}} \widehat{L}_E A$.

Proof. This is proved by induction on r , using Lemma 3.20 and Theorem 3.10(2). \square

This is given added punch when combined with the next proposition.

Proposition 3.28. Given $A \in \mathcal{A}lg$ there are natural weak equivalences

- (1) $taq^E(A) \simeq L_E(taq(A))$.
- (2) $\{P_r^E(A)\} \simeq \{L_E P_r(A)\}$, as towers.

Proof. First note that if X is an S -module, then each of the maps

$$X \rightarrow L_E S \wedge X \rightarrow L_E X$$

is an E_* -isomorphism.

To prove (1), we have

$$\begin{aligned}
 taq^E(A) &= L_E(taq_{L_E S}(L_E A)) \\
 &\simeq L_E(taq_{L_E S}(L_E S \wedge A)) \text{ by Lemma 3.21} \\
 &\simeq L_E(L_E S \wedge taq(A)) \text{ by Lemma 3.6} \\
 &\simeq L_E(taq(A)).
 \end{aligned}$$

To prove (2), we have

$$\begin{aligned}
 P_r^E(A) &= L_E(P_{L_E S, r}(L_E A)) \\
 &\simeq L_E(P_{L_E S, r}(L_E S \wedge A)) \text{ by Corollary 3.22} \\
 &\simeq L_E(L_E S \wedge P_r(A)) \text{ by Theorem 3.10(3)} \\
 &\simeq L_E(P_r(A)). \quad \square
 \end{aligned}$$

In contrast to the equivalences in this last proposition, we note that it is *not* necessarily true that the natural weak map

$$L_E(\widehat{A}) \rightarrow \widehat{L}_E(A)$$

is an equivalence (particularly when E is not connective), and thus the convergence of the localized tower is very problematic. For example, if $A = \mathbb{P}X$, this map has the form

$$L_E\left(\prod_{r=0}^{\infty} D_r X\right) \rightarrow \prod_{r=0}^{\infty} L_E D_r X,$$

which would usually not be an equivalence.

3.6. Proof of the properties of the André–Quillen tower

In the series of papers [G1,G2,G3], Tom Goodwillie has developed his theory of polynomial resolutions of homotopy functors. Although [G3] only explicitly studies such resolutions of functors

$$F : \mathcal{A} \rightarrow \mathcal{B}$$

with \mathcal{A} and \mathcal{B} either spaces or spectra, essentially everything in his paper makes sense in a much broader setting. In particular, his concepts and constructions certainly make sense if \mathcal{A} and \mathcal{B} are (based) topological model categories (a model category tensored over based topological spaces, satisfying properties as in [EKMM, VII.4]), and \mathcal{B} furthermore is a category in which a directed hocolimit of homotopy cartesian cubical diagrams is again a homotopy cartesian cubical diagram. One such category is $R\text{-Alg}$. Another is $R\text{-Mod}$, the category of R -modules.

Recall that the tower $\{P_{R,r}(\)\}$ is defined to be the Goodwillie tower associated to the identity functor on $R\text{-Alg}$. In this subsection we indicate why this tower has the properties given in Theorem 3.10. We do this by summarizing the main points of Goodwillie's work as they apply to Theorem 3.10. Throughout we are citing the version of [G3] of June, 2002.

As in [G2], a functor is said to be *r-excisive* if it takes strongly homotopy cocartesian $(r+1)$ -cubical diagrams to homotopy cartesian cubical diagrams. In [G3], given a functor F , the tower $\{P_r F(\)\}$ is defined so that $F \rightarrow P_r F$ is the universal arrow to an *r-excisive* functor, up to weak equivalence.

Goodwillie proves the existence of such a tower by an explicit construction which amounts to modifying F so as to visibly *force* certain strongly homotopy cocartesian $(r+1)$ -cubical diagrams to transform to homotopy cartesian diagrams. Readers looking to apply his paper in the setting where the domain and range of F are topological model categories should write ' $U_+ \otimes X$ ' whenever Goodwillie writes ' $X \times U$ ', with U

a finite set, and also recall that the domain category has an initial/terminal object. [G3, Theorem 1.8] says that the tower constructed as he describes has the desired universal properties.

For example, there is a strongly cocartesian diagram

$$\begin{array}{ccc} S^0 & \longrightarrow & D^+ \\ \downarrow & & \downarrow \\ D^- & \longrightarrow & S^1, \end{array}$$

representing the circle as the union of two 1-disks D^+ and D^- . Then $P_1 F(X)$ is defined to be the homotopy colimit of

$$F(X) \rightarrow T_1 F(X) \rightarrow T_1 T_1 F(X) \rightarrow \dots$$

where $T_1 F(X)$ is the homotopy pullback of

$$\begin{array}{ccc} & F(D^+ \otimes X) & \\ & \downarrow & \\ F(D^- \otimes X) & \longrightarrow & F(S^1 \otimes X). \end{array}$$

Since D^+ and D^- are contractible, $F(D^+ \otimes X)$ and $F(D^- \otimes X)$ are equivalent to the initial/terminal object in the domain category, so that

$$T_1 F(X) \simeq \Omega F(S^1 \otimes X)$$

and

$$P_1 F(X) \simeq \operatorname{hocolim}_{n \rightarrow \infty} \Omega^n F(S^n \otimes X).$$

Already, just using this part of the theory, various parts of Theorem 3.10 are evident. Statement (1), saying that $P_{R,0} A \simeq R$, is clear. Statement (3), the change of rings formula, is also clear, noting that $R \wedge ()$ takes strongly homotopy cocartesian cubes of S -algebras to strongly homotopy cocartesian cubes of R -algebras, and homotopy cartesian cubes of S -algebras to homotopy cartesian cubes of R -algebras (see below). Finally, part of statement (2), that the fiber of $p_1 : P_{R,1} A \rightarrow P_{R,0} A$ is naturally equivalent to $\operatorname{taq}_R(A)$, follows from the above description of $P_1 F$.

In [G2,G3], Goodwillie develops general theory and examples allowing for connectivity estimates to be made for the maps $F(X) \rightarrow P_r F(X)$ in terms of the connectivity of X . In particular, statement (4), stating that $e_r : A \rightarrow P_{R,r} A$ is r -connected if $I(A)$ is 0-connected, can be deduced from Lemma 3.1.

We are left needing to show the rest of statement (2): that $D_{R,r}(A)$, the fiber of p_r , is weakly equivalent to the r th extended power of $taq(A)$, the fiber of p_1 .

To show this, we begin by noting that homotopy pullback diagrams in $R - Alg$ are just diagrams in $R - Alg$ that are homotopy pullbacks in R -modules. Thus the tower $\{P_{R,r}(\)\}$, with algebra structures forgotten, is the Goodwillie tower of the inclusion functor

$$\mathcal{I} : R - Alg \hookrightarrow R - Mod.$$

The category R -modules is a stable model category, in the sense of [H2]; in particular, homotopy cocartesian cubical diagrams are equivalent to homotopy cartesian cubical diagrams. Thus one can apply Goodwillie's analysis in [G3] of how $D_r F(A)$, the fiber of $P_r F(A) \rightarrow P_{r-1} F(A)$, can be computed by means of cross effects.

The bits of the general theory we need are the following. Let

$$F : \mathcal{A} \rightarrow \mathcal{B}$$

be a functor between topological model categories as above, with \mathcal{B} stable.

Let $\mathbf{r} = \{1, 2, \dots, r\}$. In [G3, §3], $\chi_r F$, the r th cross effect of F , is defined to be the functor of r variables given as the total homotopy fiber

$$\chi_r F(A_1, \dots, A_r) = \text{TotFib } F \left(\coprod_{T \subset \mathbf{r}} A_i \right). \quad (3.4)$$

Then [G3, Theorem 6.1] says that $D^{(r)} F$, the r th multilinearization of F , can be computed by the formula

$$D^{(r)} F(A_1, \dots, A_r) \simeq \text{hocolim}_{n_i \rightarrow \infty} \Omega^{n_1 + \dots + n_r} \chi_r F(S^{n_1} \otimes A_1, \dots, S^{n_r} \otimes A_r). \quad (3.5)$$

Finally, [G3, Theorem 3.5] says that there is a natural weak equivalence

$$D_r F(A) \simeq (D^{(r)} F(A, \dots, A))_{h\Sigma_r}. \quad (3.6)$$

We apply this theory to the case when $F = \mathcal{I}$.

Since the coproduct in $R - Alg$ is just the smash product \wedge_R , we have

$$\chi_r \mathcal{I}(A_1, \dots, A_r) = \text{TotFib}_{T \subset \mathbf{r}} \left(\bigwedge_{i \in \mathbf{r} - T} A_i \right).$$

For example, $\chi_2(A, B)$ is the total homotopy fiber of the square

$$\begin{array}{ccc} A \wedge_R B & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & R \end{array}$$

Recall that the total homotopy fiber is isomorphic to the iterated homotopy fiber. This makes the next lemma easy to check.

Lemma 3.29. *The natural map*

$$I(A_1) \wedge_R \dots \wedge_R I(A_r) \rightarrow \chi_r \mathcal{I}(A_1, \dots, A_r)$$

is an equivalence.

Corollary 3.30. *There are natural weak equivalences of R -modules*

$$D^{(r)} \mathcal{I}(A_1, \dots, A_r) \simeq \mathrm{taq}(A_1) \wedge_R \dots \wedge_R \mathrm{taq}(A_r),$$

and

$$D_r \mathcal{I}(A) = (\mathrm{taq}(A)^{\wedge_{R^r}})_{h\Sigma_r}.$$

We have finished our proof of Theorem 3.10, as this last equivalence is just a restatement of the remaining unproven part of statement (2).

4. Proof of Theorems 2.2, 2.4, 2.5, and 2.28

In this section we use the theory developed in the last section to prove the splitting theorems of the introduction.

4.1. Proof of Theorem 2.2

Definition 4.1. If Z is a space, let $s(Z) : \mathbb{P}(\Sigma^\infty Z) \rightarrow \Sigma^\infty(QZ)_+$ be the natural weak map in $\mathcal{A}lg$ induced by the weak natural map of S -modules

$$\Sigma^\infty Z \xrightarrow{\Sigma^\infty \eta(Z)} \Sigma^\infty QZ \xleftarrow{\sim} I(\Sigma^\infty(QZ)_+).$$

We restate Theorem 2.2.

Theorem 4.2. *For all spaces Z , the map $s(Z)$ induces an isomorphism on completions, and thus there is a natural factorization of weak algebra maps*

$$\begin{array}{ccc} \mathbb{P}(\Sigma^\infty Z) & \xrightarrow{\text{canonical}} & \hat{\mathbb{P}}(\Sigma^\infty Z) \\ & \searrow s(Z) \quad \nearrow t(Z) & \\ & \Sigma^\infty(QZ)_+ & \end{array}$$

If Z is 0-connected then all of these maps are weak equivalences.

Proof. By Proposition 3.16, we just need to show that

$$\text{taq}(s(Z)) : \text{taq}(\mathbb{P}(\Sigma^\infty Z)) \rightarrow \text{taq}(\Sigma^\infty(QZ)_+)$$

is an equivalence. To see this, consider the diagram:

$$\begin{array}{ccccc} \Sigma^\infty Z & \xlongequal{\quad} & & \xlongequal{\quad} & \Sigma^\infty Z \\ & \searrow \eta & & \nearrow \varepsilon & \\ & & \Sigma^\infty QZ & & \\ & \searrow i & \uparrow \wr & & \\ & & I(\Sigma^\infty(QZ)_+) & & \\ & \nearrow I(s(Z)) & & \searrow & \\ I(\mathbb{P}(\Sigma^\infty Z)) & \xrightarrow{\quad} & & & \\ & \searrow & & \nearrow & \\ \text{taq}(\mathbb{P}(\Sigma^\infty Z)) & \xrightarrow{\text{taq}(s(Z))} & & & \text{taq}(\Sigma^\infty(QZ)_+) \end{array}$$

Proposition 3.17(2) says that the left edge is an equivalence, and the left triangle commutes. Similarly, Proposition 3.18(4) shows that the right edge is an equivalence, and the right quadrilateral commutes. Naturality shows that the bottom quadrilateral commutes. The map $s(Z)$ was *defined* so that the middle quadrilateral commutes. Finally, the top triangle is just the categorical factorization (2.1). Thus the diagram commutes, and inspection of the outside square shows that $\text{taq}(s(Z))$ can be identified with the identity map on $\Sigma^\infty Z$. \square

Remarks 4.3. When Z is connected, the realization that a weak equivalence like $s(Z)$ can be taken to be E_∞ dates back to the late 1970s, with the first proof based on a

point set analysis of James–Hopf maps [CMT]. Proofs working on the spectrum level were given in [LMMS, Theorem VII.5.5] and [K1, Proposition 4.3]. Both of these references construct $s(Z)$ using ideas of [C]: see [K4, Appendix B] for an updated account.

If Z is *not* 0-connected, then the fact $t(Z)$ can be constructed to be E_∞ seems to be new, though slightly weaker results were proved in [CMT]. We also manage to show the existence of $t(Z)$ without appealing to properties of group completions: again this is new.

Our proof does *not* use the combinatorial approximation $CZ \rightarrow QZ$, though some of the ideas behind that approximation are obviously lurking in the proofs of needed properties of $\Sigma^\infty(QZ)_+$. We relate our constructions to those using CZ in Appendix A.

4.2. Proof of Theorem 2.4

Suppose that Z is a connected space, and $f : \Sigma^\infty Z \rightarrow X$ a map inducing an isomorphism on E_* -homology. The maps in $\mathcal{A}lg$,

$$\mathbb{P}(\Sigma^\infty Z) \xrightarrow{s(Z)} \Sigma^\infty(QZ)_+ \xrightarrow{\Sigma^\infty \Omega^\infty f} \Sigma^\infty(\Omega^\infty X)_+,$$

induce a commutative diagram

$$\begin{array}{ccccc} L_E \mathbb{P}(\Sigma^\infty Z) & \xrightarrow{\sim} & L_E \Sigma^\infty(QZ)_+ & \xrightarrow{L_E \Sigma^\infty(\Omega^\infty f)_+} & L_E \Sigma^\infty(\Omega^\infty X)_+ \\ \downarrow & & \downarrow & & \downarrow \\ \widehat{L}_E \mathbb{P}(\Sigma^\infty Z) & \xrightarrow{\sim} & \widehat{L}_E \Sigma^\infty(QZ)_+ & \xrightarrow{\sim} & \widehat{L}_E \Sigma^\infty(\Omega^\infty X)_+. \end{array}$$

In this diagram, the top left horizontal maps is an equivalence by Theorem 2.2, while the bottom left horizontal map is similarly an equivalence using Proposition 2.3. The lower right horizontal map is an equivalence by Proposition 2.3, since our hypothesis that f is an E_* -isomorphism implies that $\text{taq}^E(f)$ is an equivalence.

As the left vertical map is an E_* -monomorphism, we conclude that so is the upper right horizontal map. Otherwise said,

$$(\Omega^\infty f)_* : E_*(QZ) \rightarrow E_*(\Omega^\infty X)$$

is monic.

4.3. Telescopic functors and the proof of Theorem 2.5

We fix a prime p and work p -locally. For $n \geq 1$, let $K(n)$ be the n th Morava K -theory, and $T(n)$ be the telescope of a v_n -self map of a finite complex of type n . It

is known that the Bousfield class of $T(n)$ is independent of choice of v_n -self map, and that $\langle K(n) \rangle \leq \langle T(n) \rangle$, with equality holding if and only if the Telescope Conjecture holds (see [B4] for background and more references).

Theorem 4.4. *There exists a functor $\Phi_n : \text{Spaces} \rightarrow S\text{-modules}$ and a natural weak equivalence $\Phi_n \Omega^\infty X \simeq L_{T(n)} X$.*

With the result stated at the level of homotopy categories, and with $K(n)$ replacing $T(n)$, this is the main theorem of [K3]. However the sorts of constructions given there, and in [B1] (for $n = 1$), yield the theorem as stated. In particular, in the recent paper [B4], Bousfield proves the theorem as stated using the model category of spectra of [BF]. But this category is known [SS] to be Quillen equivalent to the S -modules of [EKMM].

As a corollary, we obtain a proof of (2.2), which we restate here.

Corollary 4.5 (Compare with [K3]). *There is a natural factorization by weak S -module maps*

$$\begin{array}{ccc} & L_{T(n)} \Sigma^\infty \Omega^\infty X & \\ \eta_n(X) \nearrow & & \searrow L_{T(n)} \varepsilon(X) \\ L_{T(n)} X & \xlongequal{\quad\quad\quad} & L_{T(n)} X. \end{array}$$

Proof. Apply Φ_n to the commutative diagram

$$\begin{array}{ccc} & \Omega^\infty \Sigma^\infty \Omega^\infty X & \\ \eta(\Omega^\infty X) \nearrow & & \searrow \Omega^\infty \varepsilon(X) \\ \Omega^\infty X & \xlongequal{\quad\quad\quad} & \Omega^\infty X. \end{array} \quad (4.1) \quad \square$$

Definition 4.6. If X is an S -module, let

$$s_n(X) : L_{T(n)} \mathbb{P}(X) \rightarrow L_{T(n)} \Sigma^\infty (\Omega^\infty X)_+$$

be the natural weak map in $L_{T(n)}(\mathcal{A}lg)$ induced by the weak natural map of S -modules

$$L_{T(n)} X \xrightarrow{\eta_n(X)} L_{T(n)} \Sigma^\infty \Omega^\infty X \xleftarrow{\sim} L_{T(n)} I(\Sigma^\infty (\Omega^\infty X)_+).$$

We restate Theorem 2.5.

Theorem 4.7. *For all spectra X , the map $s_n(X)$ induces an isomorphism on $L_{T(n)}$ -completions, and thus there is a natural factorization of weak algebra maps*

$$\begin{array}{ccc}
 L_{T(n)}\mathbb{P}(X) & \xrightarrow{\text{canonical}} & \widehat{L}_{T(n)}\mathbb{P}(X) \\
 \searrow s_n(X) & & \nearrow t_n(Z) \\
 & L_{T(n)}\Sigma^\infty(\Omega^\infty X)_+ &
 \end{array}$$

Proof. Denote $L_{T(n)}$ by L . By Proposition 3.27, we just need to show that

$$\text{taq}(s_n(X)) : L\text{taq}(\mathbb{P}(X)) \rightarrow L\text{taq}(\Sigma^\infty(\Omega^\infty X)_+)$$

is an equivalence. To see this, consider the diagram:

$$\begin{array}{ccccc}
 LX & \xlongequal{\quad} & LX & & LX \\
 \downarrow \wr & \searrow i & \nearrow L\epsilon & & \downarrow \wr \\
 & & L\Sigma^\infty\Omega^\infty X & & \\
 & & \uparrow \wr & & \\
 & & LI(\Sigma^\infty(\Omega^\infty X)_+) & & \\
 & \nearrow I(s_n(X)) & & \searrow & \\
 LI(\mathbb{P}(X)) & \xrightarrow{\quad} & LI(\Sigma^\infty(\Omega^\infty X)_+) & & \\
 \swarrow & & \searrow & & \\
 L\text{taq}(\mathbb{P}(X)) & \xrightarrow{\text{taq}(s_n(X))} & L\text{taq}(\Sigma^\infty(\Omega^\infty X)_+) & &
 \end{array}$$

The top triangle commutes by Corollary 4.5, and the map $s_n(X)$ was defined so that the middle quadrilateral commutes. The rest of the diagram commutes for the same reasons as in the proof of Theorem 2.2. \square

4.4. Rational spectra and the proof of Theorem 2.28

We restate (2.4) as a lemma.

Lemma 4.8. *For all 0-connected spectra X , there is a natural factorization by weak S -module maps*

$$\begin{array}{ccc} & L_{H\mathbb{Q}}\Sigma^\infty\Omega^\infty X & \\ \eta_0(X) \nearrow & & \searrow L_{H\mathbb{Q}}\varepsilon(X) \\ L_{H\mathbb{Q}}X & \xlongequal{\quad\quad\quad} & L_{H\mathbb{Q}}X. \end{array}$$

For X just -1 -connected, such a factorization also exists, but cannot be made to be natural.

Assuming this for the moment, we continue as we did in the last subsection.

Definition 4.9. If X is a -1 -connected S -module, let

$$s_0(X) : L_{H\mathbb{Q}}\mathbb{P}(X) \rightarrow L_{H\mathbb{Q}}\Sigma^\infty(\Omega^\infty X)_+$$

be the weak map in $L_{H\mathbb{Q}}(\mathcal{A}lg)$ induced by the weak natural map of S -modules

$$L_{H\mathbb{Q}}X \xrightarrow{\eta_0(X)} L_{H\mathbb{Q}}\Sigma^\infty\Omega^\infty X \xleftarrow{\sim} L_{H\mathbb{Q}}I(\Sigma^\infty(\Omega^\infty X)_+).$$

We restate Theorem 2.28.

Theorem 4.10. *For all -1 -connected spectra X , the map $s_0(X)$ induces an isomorphism on $L_{H\mathbb{Q}}$ -completions, and thus there is a factorization of weak algebra maps*

$$\begin{array}{ccc} L_{H\mathbb{Q}}\mathbb{P}(X) & \xrightarrow{\text{canonical}} & \widehat{L}_{H\mathbb{Q}}\mathbb{P}(X). \\ & \searrow s_0(X) & \nearrow t_0(Z) \\ & L_{H\mathbb{Q}}\Sigma^\infty(\Omega^\infty X)_+ & \end{array}$$

This is natural for 0-connected X , and, in this case, all three maps are equivalences.

The theorem follows from the lemma in the usual way.

Proof of Lemma 4.8. The idea behind this lemma is that passing to homotopy groups is a full and faithful process on the homotopy category of $H\mathbb{Q}$ -local S -modules.

A version of the lemma then follows easily as follows. Let

$$\eta(X)_* : \pi_*(X) \rightarrow \pi_*(\Sigma^\infty \Omega^\infty X)$$

be the map induced by the canonical inclusion

$$\eta(X) : \Omega^\infty X \rightarrow Q\Omega^\infty X.$$

Note that $\eta(X)_*$ is a map of abelian groups if $* > 0$, but is only a map of sets if $* = 0$. However, for all $* \geq 0$, we have that $\varepsilon(X)_* \circ \eta_*(X)$ is the identity.

Thus if X is 0-connected, there is a canonical natural homotopy class of maps

$$\eta_0(X) : L_{H\mathbb{Q}}X \rightarrow L_{H\mathbb{Q}}\Sigma^\infty \Omega^\infty X$$

realizing $\eta(X)_* \otimes \mathbb{Q}$.

If X is just -1 -connected, one can still choose a \mathbb{Q} -linear section to $\varepsilon(X)_* \otimes \mathbb{Q}$, and then realize this, defining $\eta_0(X)$. However, this cannot possibly be taken to be natural by the following argument, which the author learned from Pete Bousfield. If $\eta_0(X)$ were natural, then the maps $\eta_0(HV)$, with V a \mathbb{Q} -vector space, would define a natural section to the augmentation

$$\mathbb{Q}[V] \rightarrow V$$

defined on the category of \mathbb{Q} -vector spaces. (Here $\mathbb{Q}[V]$ denotes the \mathbb{Q} -vector space with basis V .) But it is well known, and easily verified, that there exist no nontrivial natural transformations $V \rightarrow \mathbb{Q}[V]$.

A careful reader may be wondering if, in the 0-connected case, one can construct a natural weak section at the model category level, and not just a natural section in the homotopy category. This is also possible: one can apply [McC, Theorem 4], which implies that if $P^1(X)$ is the codegree 1 approximation to $L_{H\mathbb{Q}}(\Sigma^\infty \Omega^\infty X)$ (in the sense of dual calculus), then the composite

$$P^1(X) \rightarrow L_{H\mathbb{Q}}\Sigma^\infty \Omega^\infty X \xrightarrow{L_{H\mathbb{Q}}\varepsilon(X)} L_{H\mathbb{Q}}X$$

is an equivalence for 0-connected X . \square

5. When $s_n(X)$ is an equivalence and related matters

Recall that $S_n = \{S\text{-modules } X \mid s_n(X) \text{ is an equivalence}\}$ and that $S_n^K = \{S\text{-modules } X \mid s_n^K(X) \text{ is an equivalence}\}$. Recall also that $c(n)$ denotes the smallest

integer c such that $\tilde{T}(n)_*(K(\mathbb{Z}/p, c)) = 0$. The analogous integer associated to $K(n)_*$ is $n+1$: by the calculations in [RW], $\tilde{K}(n)_*(K(\mathbb{Z}/p, c)) = 0$ if and only if $c \geq n+1$.⁸

The starting point for the detailed results about \mathcal{S}_n and \mathcal{S}_n^K given in §2.5 is the following partial result.

Theorem 5.1. *Let X be an S -module such that $L_{n-1}^f X \simeq *$.*

- (1) $X \in \mathcal{S}_n$ if X is $c(n)$ -connected.
- (2) $X \in \mathcal{S}_n^K$ if X is $n+1$ -connected.

The proof of this theorem is slightly long and delicate. We organize it into the following steps:

Step 1: We show that if F is a finite S -module of type n , then $\Sigma^d F \in \mathcal{S}_n$ for $d \gg 0$.

Step 2: Assuming Step 1, we show that if F is a 0-connected finite S -module of type n , then $\Sigma^{c(n)} F \in \mathcal{S}_n$, and $\Sigma^{n+1} F \in \mathcal{S}_n^K$.

Step 3: We show that the classes \mathcal{S}_n and \mathcal{S}_n^K are closed under various constructions.

Step 4: We show that, starting from the finite S -modules shown in Step 2 to be in \mathcal{S}_n and \mathcal{S}_n^K , one can build all X 's as in Theorem 5.1 using the constructions of Step 3.

These will be proved in the next four subsections. Then, armed with Theorem 5.1, we will systematically work our way through the proofs of the various results stated in sections §2.5 and §2.6.

5.1. Proof of Theorem 5.1: step 1

Lemma 5.2. *Let Z be a space whose suspension spectrum is finite of type at least n . Then, for $d \gg 0$, $L_{T(n)} s(\Sigma^d Z) \simeq s_n(\Sigma^\infty \Sigma^d Z)$.*

Postponing the proof momentarily, we note the following corollary.

Corollary 5.3. *Let F be a finite of type at least n . Then for $d \gg 0$, $s_n(\Sigma^d F)$ is an equivalence.*

Proof. Replacing F by a high suspension of F if needed, we can assume that $F \simeq \Sigma^\infty Z$, for some space Z . By the lemma, for large enough d , $s_n(\Sigma^d F)$ will be homotopic to $L_{T(n)} s(\Sigma^d Z)$, which is an equivalence by Theorem 2.2. \square

Proof of Lemma 5.2. Let Z be a space. By construction, the two maps of $L_{T(n)} S$ -algebras

$$s_n(\Sigma^\infty Z), L_{T(n)} s(Z) : L_{T(n)} \mathbb{P}(\Sigma^\infty Z) \rightarrow L_{T(n)} \Sigma^\infty (QZ)_+$$

⁸ When $p = 2$, the reference is [JW, Appendix].

will be homotopic if and only if the two maps of $L_{T(n)}S$ -modules

$$\eta_n(\Sigma^\infty Z), L_{T(n)}\Sigma^\infty \eta(Z) : L_{T(n)}\Sigma^\infty Z \rightarrow L_{T(n)}\Sigma^\infty QZ$$

are homotopic.

Now suppose that $\Sigma^\infty Z$ has type at least n , and is at least 0-connected. Then $\Sigma^\infty Z$ is $T(i)_*$ -acyclic for $0 \leq i \leq n-1$, and thus the same is true for $\Sigma^\infty QZ$, since by Theorem 2.2, $\Sigma^\infty(QZ)_+ \simeq \mathbb{P}(\Sigma^\infty Z)$.

If a spectrum X is $T(i)_*$ -acyclic for $0 \leq i \leq n-1$, then $L_{T(n)}X \simeq L_n^f X \simeq L_n^f S \wedge X$. Applying this to our situation, we see that $\eta_n(\Sigma^\infty \Sigma^d Z)$ and $L_{T(n)}\Sigma^\infty \eta(\Sigma^d Z)$ correspond to homotopy classes of maps of S -modules

$$x_n(d), x(d) \in [\Sigma^\infty Z, L_n^f S \wedge \Sigma^{-d} \Sigma^\infty Q\Sigma^d Z].$$

We now show that, if d is very large, then $x(d) = x_n(d)$. This follows from three observations.

Firstly, the naturality of η and η_n ensures that $x(d)$ maps to $x(d+1)$, and $x_n(d)$ maps to $x_n(d+1)$, under the homomorphism

$$[\Sigma^\infty Z, L_n^f S \wedge \Sigma^{-d} \Sigma^\infty Q\Sigma^d Z] \rightarrow [\Sigma^\infty Z, L_n^f S \wedge \Sigma^{-(d+1)} \Sigma^\infty Q\Sigma^{d+1} Z]$$

induced by the evaluation map $\Sigma Q\Sigma^d Z \rightarrow Q\Sigma^{d+1} Z$.

Secondly, the colimit

$$\operatorname{colim}_{d \rightarrow \infty} [\Sigma^\infty Z, L_n^f S \wedge \Sigma^{-d} \Sigma^\infty Q\Sigma^d Z]$$

can be identified with $[\Sigma^\infty Z, L_n^f S \wedge \Sigma^\infty Z]$.

Thirdly, the key properties of η and η_n , (2.1) and (2.2), imply that under this identification, $\operatorname{colim}_{d \rightarrow \infty} x(d)$ and $\operatorname{colim}_{d \rightarrow \infty} x_n(d)$ each correspond to the canonical element: the unit $S \rightarrow L_n^f S$ smashed with the identity on $\Sigma^\infty Z$.

We conclude that $\operatorname{colim}_{d \rightarrow \infty} (x(d) - x_n(d))$ is zero in the colimit, and thus $x(d) - x_n(d)$ is zero at a finite stage of this colimit. Otherwise said, for d large, we have $x(d) = x_n(d)$. \square

5.2. Proof of Theorem 5.1: step 2

Proposition 5.4. *Let F be a -1 -connected finite S -module of type $n+1$.*

- (1) $\tilde{T}(n)_*(\Omega^\infty \Sigma^{c(n)} F) = 0$.
- (2) $\tilde{K}(n)_*(\Omega^\infty \Sigma^{n+1} F) = 0$.

As a corollary, one deduces the assertion of step 2.

Corollary 5.5. *Let F be a 0-connected finite S -module of type n .*

- (1) $\Sigma^{c(n)} F \in \mathcal{S}_n$.
- (2) $\Sigma^{n+1} F \in \mathcal{S}_n^K$.

Proof. Let F be 0-connected and finite of type n . Let $v : \Sigma^d F \rightarrow F$ be a v_n -map, an isomorphism in both $T(n)_*$ and $K(n)_*$. Let W be the fiber. By statement (1) of the proposition, $\Omega^\infty \Sigma^{c(n)} W$ is $T(n)_*$ -acyclic. By an Atiyah–Hirzebruch–Serre spectral sequence argument, it follows that $\Sigma^c v$ induces an isomorphism

$$v_* : T(n)_*(\Omega^\infty \Sigma^{d+c} F) \xrightarrow{\sim} T(n)_*(\Omega^\infty \Sigma^c F)$$

if $c \geq c(n)$.

Now consider the diagram

$$\begin{array}{ccc} L\mathbb{P}(\Sigma^{dN+c(n)} F) & \xrightarrow{s_n(\Sigma^{dN+c(n)} F)} & L\Sigma^\infty(\Omega^\infty(\Sigma^{dN+c(n)} F))_+ \\ \downarrow & & \downarrow \\ L\mathbb{P}(\Sigma^{c(n)} F) & \xrightarrow{s_n(\Sigma^{c(n)} F)} & L\Sigma^\infty(\Omega^\infty \Sigma^{c(n)} F)_+ \end{array}$$

where $L = L_{T(n)}$, and the vertical maps are induced by v^N . The above discussion implies that the right vertical map is an equivalence. The left vertical map is an equivalence because v is a $T(n)_*$ -isomorphism. Finally, for large enough N the top horizontal map is an equivalence as a consequence of Step 1. Thus the bottom map is an equivalence, proving statement (1) of the corollary. Statement (2) is proved similarly. \square

To prove Proposition 5.4, we use ideas from [HRW].

If $H\mathbb{Z}/p$ is strongly E_* -acyclic, let $c_p(E)$ be the smallest c such that $\tilde{E}_*(K(\mathbb{Z}/p, c)) = 0$. The two statements of Proposition 5.4 are then just special cases of the following proposition.

Proposition 5.6. *Suppose a -1 -connected spectrum X has p -torsion homotopy groups. If X is strongly E_* -acyclic, then $\tilde{E}_*(\Omega^\infty \Sigma^{c_p(E)} X) = 0$.*

Proof. We argue as in [HRW, §3].

Firstly, the argument proving [HRW, Proposition 3.4] shows that if $\Omega^\infty \Sigma^c X$ is E_* -acyclic, then so is $\Omega^\infty \Sigma^c(X \wedge Y)$ for any -1 -connected spectrum Y .

Now let statement (c) be the statement that $\tilde{E}_*(\Omega^\infty \Sigma^c X) = 0$. By assumption, statement (c) is true if c is very large. We complete the proof of the proposition by showing that, if $c \geq c_p(E)$, then statement (c+1) implies statement (c).

Consider the fibration sequence of S -modules

$$\bar{H} \rightarrow S \rightarrow H\mathbb{Z}.$$

This induces a fibration sequence of spaces

$$\Omega^\infty \Sigma^{c+1}(X \wedge \Sigma^{-1} \bar{H}) \rightarrow \Omega^\infty \Sigma^c X \rightarrow \Omega^\infty \Sigma^c (X \wedge H\mathbb{Z}).$$

By our inductive assumption, and the observation above, the first of these spaces is E_* -acyclic. Thus the second map is an E_* -isomorphism. But $\Omega^\infty \Sigma^c (X \wedge H\mathbb{Z})$ will be a weak product of Eilenberg–MacLane spaces of type $K(A, c)$ where A is p -torsion and $c \geq c_p(n)$. Thus this space is also E_* -acyclic, and we conclude the same for $\Omega^\infty \Sigma^c X$. \square

5.3. Proof of Theorem 5.1: step 3

The following proposition says that \mathcal{S}_n and \mathcal{S}_n^K are closed under various constructions.

Proposition 5.7. *Let \mathcal{S}' be either \mathcal{S}_n or \mathcal{S}_n^K .*

- (1) $X \in \mathcal{S}'$ if and only if $X\langle -1 \rangle \in \mathcal{S}'$.
- (2) If $X \in \mathcal{S}'$, then $\pi_0(X) = 0$.
- (3) Let $X \rightarrow Y \rightarrow Z$ be a cofibration sequence of S -modules with X and Y -1 -connected and Z 0 -connected. Then $X, Y \in \mathcal{S}'$ implies that $Z \in \mathcal{S}'$. In particular, if $X \in \mathcal{S}'$ is -1 -connected, then $\Sigma X \in \mathcal{S}'$.
- (4) Let X be the filtered homotopy colimit of S -modules X_i . If $X_i \in \mathcal{S}'$ for all i , then $X \in \mathcal{S}'$.

Proof. We will prove the statements when $\mathcal{S}' = \mathcal{S}_n$; the proofs when $\mathcal{S}' = \mathcal{S}_n^K$ are similar. Let L denote $L_{T(n)}$.

The first two properties follow from the fact that Eilenberg–MacLane spectra are $T(n)_*$ -acyclic.

In more detail, $X\langle -1 \rangle \rightarrow X$ is a $T(n)_*$ -isomorphism, and thus so is $\mathbb{P}(X\langle -1 \rangle) \rightarrow \mathbb{P}(X)$. Also $\Omega^\infty X\langle -1 \rangle = \Omega^\infty X$. Thus in the square

$$\begin{array}{ccc} T(n)_*(\mathbb{P}(X\langle -1 \rangle)) & \xrightarrow{s_{n*}(X\langle -1 \rangle)} & T(n)_*(\Omega^\infty(X\langle -1 \rangle)) \\ \downarrow & & \downarrow \\ T(n)_*(\mathbb{P}(X)) & \xrightarrow{s_{n*}(X)} & T(n)_*(\Omega^\infty X) \end{array}$$

both vertical maps are equivalences and (1) follows.

For (2), consider the commutative square

$$\begin{array}{ccc}
 T(n)_*(\mathbb{P}(X\langle 0 \rangle)) & \xrightarrow{s_{n*}(X\langle 0 \rangle)} & T(n)_*(\Omega^\infty(X\langle 0 \rangle)) \\
 \downarrow & & \downarrow \\
 T(n)_*(\mathbb{P}(X)) & \xrightarrow{s_{n*}(X)} & T(n)_*(\Omega^\infty X).
 \end{array}$$

The left vertical map is an isomorphism, and the horizontal maps are monic by Theorem 2.5. As $\Omega^\infty(X\langle 0 \rangle)$ is just one of the path components of $\Omega^\infty X$, the right vertical map is only epic if $\pi_0(X) = 0$, and (2) follows.

Properties (3) and (4) follow by combining Propositions 3.17, 3.18, and 3.24. \square

5.4. Proof of Theorem 5.1: step 4

It is convenient to make the following definition, a variant on similar notions in the literature.

Definition 5.8. Let \mathcal{C} be any collection of S -modules. Say that an S -module X is *built from \mathcal{C}* if $X \simeq \text{hocolim}_i X_i$, for some sequence $X_0 \rightarrow X_1 \rightarrow \dots$ such that X_0 is a wedge of S -modules in \mathcal{C} , and, for all $i \geq 0$, X_{i+1} is the cofiber of a map $W_i \rightarrow X_i$ with W_i a wedge of S -modules in \mathcal{C} .

Note that properties (3) and (4) of Proposition 5.7 imply that if \mathcal{C} is any subset of 0-connected S -modules in \mathcal{S}_n or \mathcal{S}_n^K , then any S -module built from \mathcal{C} is also in \mathcal{S}_n or \mathcal{S}_n^K .

The following proposition is a variant of a well known consequence (as in [Mil]) of the Nilpotence Theorems. I would like to thank Pete Bousfield for suggesting the simple proof.

Proposition 5.9. $L_{n-1}^f X \simeq *$ and X is c -connected if and only if X can be built from c -connected finite S -modules of type n .

To prove this we first need a lemma.

Lemma 5.10. Suppose $L_{n-1}^f X \simeq *$. Then any $f : Y \rightarrow X$, with Y finite of type at most n and c -connected, can be factored as a composite $Y \rightarrow F \rightarrow X$ such that F is a finite S -module that is both c -connected and of type n .

Proof. We prove this by downwards induction on the type of Y . The induction is begun by noting that there is nothing to prove if Y has type n .

So suppose the lemma has been established for all $g : Z \rightarrow X$, where Z is c -connected of type at least $i+1$. Let $f : Y \rightarrow X$, where Y has type i and is c -connected.

By [HS], there exists a v_i -self map $v : \Sigma^d Y \rightarrow Y$. Since $T(i)_*(X) = 0$, there exists an N such that $\Sigma^{Nd} Y \xrightarrow{v^N} Y \xrightarrow{f} X$ is null. Letting Z be the cofiber of v^N , it follows that f factors as a composite $Y \rightarrow Z \xrightarrow{g} X$. Since Z is of type $i + 1$ and is still c -connected, g , and thus f , factors as needed. \square

Proof of Proposition 5.9. The Nilpotence Theorem implies that $T(i)_*(F) = 0$ whenever F is finite of type greater than i . Thus one implication is clear: if X can be built from c -connected finite S -modules of type at least n , then $L_{n-1}^f X \simeq *$ and X is c -connected.

Conversely, suppose that $L_{n-1}^f X \simeq *$ and X is c -connected. We describe how to construct a diagram

$$\begin{array}{ccccccc}
 X_0 & \xrightarrow{j_0} & X_1 & \xrightarrow{j_1} & X_2 & \xrightarrow{j_2} & \cdots \\
 & & \swarrow g_1 & \searrow g_2 & \nearrow & & \\
 g_0 \downarrow & & & & & & \\
 & & X & & & &
 \end{array}$$

showing that X is built from c -connected finites of type at least n .

First choose a wedge of c -connected spheres T and a map $f : T \rightarrow X$ that is onto in π_* . By the last lemma, this factors as a composite $T \rightarrow X_0 \xrightarrow{g_0} X$ with X_0 a wedge of c -connected finites of type n .

Assume $g_i : X_i \rightarrow X$ has been constructed with $\pi_*(g_i)$ onto, and X_i c -connected. Let Y_i be the fiber of g_i . Choose a wedge of c -connected spheres T and a map $f : T \rightarrow Y_i$ that is onto in π_* . By the last lemma, this factors as a composite $T \rightarrow W_i \xrightarrow{g_0} Y_i$ with W_i a wedge of c -connected finites of type n . If we then let X_{i+1} be the cofiber of the composite $W_i \rightarrow Y_i \rightarrow X_i$, it follows that g_i will factor as a composite $X_i \xrightarrow{j_i} X_{i+1} \xrightarrow{g_{i+1}} X$.

By construction, $\pi_*(g_i)$ is onto and $\ker(\pi_*(j_i)) = \ker(\pi_*(g_i))$ for all i . It follows that $\text{hocolim}_i X_i \simeq X$ as needed. \square

5.5. Virtual homology and the proof of Lemma 2.17

We need a variant of Proposition 5.6.

If Eilenberg–MacLane spectra are strongly E_* -acyclic, let $c(E)$ be the smallest c such that $\tilde{E}_*(K(\mathbb{Z}, c)) = 0$.

Proposition 5.11. *If a -1 -connected spectrum X is strongly E_* -acyclic, then $\tilde{E}_*(\Omega^\infty \Sigma^{c(E)} X) = 0$.*

The proof of Proposition 5.6 goes through without change.

Proof of Lemma 2.17. Suppose X is -1 -connected. We need to show that $\tilde{E}_*(\Omega^\infty \Sigma^c X) = 0$ for large c if and only if $\tilde{E}_*(\Omega^\infty X\langle d \rangle) = 0$ for large d .

Let $P^d X$ denote the d th Postnikov section of X , so there is a cofibration sequence of spectra

$$X\langle d \rangle \rightarrow X \rightarrow P^d X.$$

Then, for all $c \geq 1$, there is a fibration sequence of spaces

$$\Omega^\infty \Sigma^{c-1} P^d X \rightarrow \Omega^\infty \Sigma^c (X\langle d \rangle) \rightarrow \Omega^\infty \Sigma^c X.$$

If $c > c(E)$, this fiber will be E_* -acyclic, and thus there will be an isomorphism

$$E_*(\Omega^\infty \Sigma^c (X\langle d \rangle)) \xrightarrow{\sim} E_*(\Omega^\infty \Sigma^c X).$$

Suppose $\tilde{E}_*(\Omega^\infty X\langle d \rangle) = 0$. Then $\tilde{E}_*(\Omega^\infty \Sigma^c (X\langle d \rangle)) = 0$ for all $c \geq 0$. Thus by our remarks above, $\tilde{E}_*(\Omega^\infty \Sigma^c X) = 0$ for $c > c(E)$.

Conversely, suppose $\tilde{E}_*(\Omega^\infty \Sigma^c X) = 0$ for all large c . Then, for all $d \geq 0$, $\tilde{E}_*(\Omega^\infty \Sigma^c (X\langle d \rangle)) = 0$ for all large c , i.e. $X\langle d \rangle$ is strongly E_* -acyclic. If $d \geq c(E) - 1$, we can apply Proposition 5.11 to the spectrum $\Sigma^{-c(E)}(X\langle d \rangle)$ to conclude that $\tilde{E}_*(\Omega^\infty X\langle d \rangle) = 0$. \square

5.6. Proof of Theorem 2.10 and Proposition 2.11

We start with a general lemma.

Lemma 5.12. Suppose $f : X \rightarrow Y$ is a map between 0 -connected spectra with cofiber C . For any homology theory E_* , there are implications (1) \Rightarrow (2) \Rightarrow (3).

- (1) $\tilde{E}_*(\Omega^\infty \Sigma^{-1} C) = 0$.
- (2) $E_*(\Omega^\infty f)$ is an isomorphism.
- (3) $\tilde{E}_*(\Omega^\infty C) = 0$.

Proof. To see that (1) implies (2), note that $\Omega^\infty \Sigma^{-1} C$ is the fiber of $\Omega^\infty f$.

To see that (2) implies (3), note that

$$\Sigma^\infty(\Omega^\infty X)_+ \xrightarrow{\Omega^\infty f} \Sigma^\infty(\Omega^\infty Y)_+ \rightarrow \Sigma^\infty(\Omega^\infty C)_+$$

is a cofibration sequence in $\mathcal{A}lg$. Thus Proposition 3.24 applies to say that if $\Omega^\infty f$ is an E_* -isomorphism, then $\Sigma^\infty(\Omega^\infty C)_+$ is E_* -equivalent to S . \square

Proof of Theorem 2.10. We prove statement (1); the proof of statement (2) is similar.

We temporarily introduce a new class of spectra: let

$$\tilde{\mathcal{S}}_n = \{X \in \mathcal{S} \mid \Sigma^c(X\langle -1 \rangle) \in \mathcal{S}_n \text{ for large } c\}.$$

Let $C_{n-1,d}(X)$ and f_d be defined so that

$$C_{n-1,d}(X) \xrightarrow{f_d} X\langle d \rangle \rightarrow (L_{n-1}^f X)\langle d \rangle$$

is a fibration sequence of \mathcal{S} -modules.

As L_{n-1}^f is smashing, and $T(i)_*(T(n)) = 0$ for $0 \leq i \leq n-1$, it follows that $L_{n-1}^f X$ is always $T(n)_*$ -acyclic, thus f_d is always a $T(n)_*$ -isomorphism.

Theorem 5.1 then implies that, for all large d and large c ,

$$C_{n-1,d}(X), \Sigma^c C_{n-1,-1}(X) \in \mathcal{S}_n.$$

Now consider the diagram

$$\begin{array}{ccc} T(n)_*(\mathbb{P}(C_{n-1,d}(X))) & \longrightarrow & T(n)_*(\Omega^\infty C_{n-1,d}(X)) \\ \downarrow & & \downarrow T(n)_*(\Omega^\infty f_d) \\ T(n)_*(\mathbb{P}(X\langle d \rangle)) & \longrightarrow & T(n)_*(\Omega^\infty X\langle d \rangle) \end{array}$$

If d is large, both the top map and the left map are isomorphisms, and we conclude that $X\langle d \rangle \in \mathcal{S}_n$ if and only if $T(n)_*(\Omega^\infty f_d)$ is an isomorphism. Thus $X \in \tilde{\mathcal{S}}_n$ if and only if $T(n)_*(\Omega^\infty f_d)$ is an isomorphism for large d .

Similarly $X \in \tilde{\mathcal{S}}_n$ if and only if $T(n)_*(\Omega^\infty \Sigma^c f_{-1})$ is an isomorphism for large c .

The last lemma and Lemma 2.17 now combine to say that

$$X \in \tilde{\mathcal{S}}_n \Leftrightarrow X \in \tilde{\mathcal{S}}_n \Leftrightarrow L_{n-1}^f X \text{ is strongly } T(n)_* \text{ - acyclic.}$$

The inclusion $\mathcal{S}_n \subset \tilde{\mathcal{S}}_n$ is evident, using Proposition 5.7, thus $\mathcal{S}_n \subset \tilde{\mathcal{S}}_n$. \square

Proof of Proposition 2.11. Suppose condition (1) holds: $T(i)_*(X) = 0$ for $1 \leq i \leq n-1$. Then $L_{n-1}^f X \rightarrow L_{\mathbb{Q}} X$ is an equivalence. Since rational spectra are certainly strongly $T(n)_*$ -acyclic, condition (2) holds: $L_{n-1}^f X$ is strongly $T(n)_*$ -acyclic.

Condition (2) obviously implies condition (3).

Suppose condition (3) holds: $L_{n-1}^f X$ is strongly $K(n)_*$ -acyclic. The main theorem of either [B6,W] implies that if a spectrum Y is strongly $K(n)_*$ -acyclic, then it is strongly $K(i)_*$ -acyclic for $1 \leq i \leq n$. Applied to our situation, we deduce that $L_{n-1}^f X$

is strongly $K(n-1)_*$ -acyclic. With notation as in the last proof, Theorem 5.1 implies that $C_{n-1,-1}(X)$ is also strongly $K(n-1)_*$ -acyclic. Thus so is X , i.e. condition (4) holds.

Finally, if X is strongly $K(n-1)_*$ -acyclic, then the Bousfield–Wilson theorem implies that X is $K(i)_*$ -acyclic for $1 \leq i \leq n-1$, and thus condition (5) holds. \square

5.7. Proof of Theorems 2.18 and 2.26

The fact that the Kunneth Theorem holds for $K(n)_*$ allows for special calculational techniques. For example, [EKMM, Theorem 7.7] applies to show that, if $A \rightarrow B \rightarrow C$ is a cofibration sequence in $\mathcal{A}lg$, the bar spectral sequence converging to $K(n)_*(C)$ has

$$E_{p,q}^2 = \mathrm{Tor}_{p,q}^{K(n)_*(A)}(K(n)_*(B), K(n)_*).$$

This has the following consequence of relevance to us.

Lemma 5.13. *Suppose $f : X \rightarrow Y$ is a map between 0-connected spectra with cofiber C . If $K(n)_*(\Omega^\infty f)$ is monic, there is a short exact sequence of $K(n)_*$ -Hopf algebras*

$$K(n)_*(\Omega^\infty X) \xrightarrow{(\Omega^\infty f)_*} K(n)_*(\Omega^\infty Y) \rightarrow K(n)_*(\Omega^\infty C).$$

Proof. $K(n)_*(\Omega^\infty X)$ is in the category of \mathcal{K}/p -Hopf algebras studied by Bousfield in [B4, Appendix]. He shows [B4, Theorem 10.8] that objects in this category are flat over subobjects. It follows that, if $K(n)_*(\Omega^\infty f)$ is monic, the spectral sequence associated to the cofibration sequence in $\mathcal{A}lg$

$$\Sigma^\infty(\Omega^\infty X)_+ \rightarrow \Sigma^\infty(\Omega^\infty Y)_+ \rightarrow \Sigma^\infty(\Omega^\infty C)_+$$

collapses, giving the desired short exact sequence. \square

Proof of Theorem 2.18. Recall that we have cofibration sequences

$$C_{n-1,d}(X) \xrightarrow{f_d} X\langle d \rangle \rightarrow (L_{n-1}^f X)\langle d \rangle,$$

and that $C_{n-1,d}(X) \in \mathcal{S}_n^K$ if d is large.

Now consider the diagram used in the proof of Theorem 2.10:

$$\begin{array}{ccc}
 K(n)_*(\mathbb{P}(C_{n-1,d}(X))) & \longrightarrow & K(n)_*(\Omega^\infty C_{n-1,d}(X)) \\
 \downarrow & & \downarrow (\Omega^\infty f_d)_* \\
 K(n)_*(\mathbb{P}(X\langle d \rangle)) & \xrightarrow{s_n(X\langle d \rangle)_*} & K(n)_*(\Omega^\infty X\langle d \rangle).
 \end{array}$$

The left map is always an isomorphism, as is the top map if d is large. The bottom map is always monic by Theorem 2.8, thus so is the right map, if d is large. The previous lemma applies to say that, for all large d , there is a short exact sequence of $K(n)_*$ -Hopf algebras

$$K(n)_*(\Omega^\infty C_{n-1,d}(X)) \xrightarrow{(\Omega^\infty f_d)_*} K(n)_*(\Omega^\infty X\langle d \rangle) \rightarrow K(n)_*(\Omega^\infty (L_{n-1}^f X)\langle d \rangle).$$

This rewrites as the short exact sequence of the theorem:

$$K(n)_*(\mathbb{P}X) \xrightarrow{s_n(X)_*} K(n)_*^{\text{vir}}(\Omega^\infty X) \rightarrow K(n)_*^{\text{vir}}(\Omega^\infty L_{n-1}^f X). \quad \square$$

Proof of Theorem 2.26. Suppose given $f : X \rightarrow Y$ with $X \in \mathcal{S}_n^K$. In the diagram

$$\begin{array}{ccc}
 K(n)_*(\mathbb{P}X) & \longrightarrow & K(n)_*(\Omega^\infty X) \\
 \downarrow \mathbb{P}(f)_* & & \downarrow (\Omega^\infty f)_* \\
 K(n)_*(\mathbb{P}Y) & \longrightarrow & K(n)_*(\Omega^\infty Y),
 \end{array}$$

we then know that the top map is an isomorphism. Since the bottom map is always monic, if the left map is monic, we deduce that $(\Omega^\infty f)_*$ is also monic. If X and Y are 0-connected and C is the cofiber of f , the lemma applies, yielding the short exact sequence of the theorem. \square

Remark 5.14. This proof makes evident the following $T(n)_*$ variant of Theorem 2.26: if $f : X \rightarrow Y$ is a $T(n)_*$ -isomorphism, and $X \in \mathcal{S}_n$, then $(\Omega^\infty f)_* : T(n)_*(\Omega^\infty X) \rightarrow T(n)_*(\Omega^\infty Y)$ is monic.

5.8. Proof of Theorems 2.12 and 2.21

Since $T(n)_*$ is all p -torsion, [B2, §4] implies

Lemma 5.15. *Let A be an abelian group. $K(A, j)$ is $T(n)_*$ -acyclic if*

(1) $j = 0$ and $A = 0$,

- (2) $1 \leq j \leq c(n) - 1$ and A is uniquely p -divisible,
- (3) $j = c(n)$ and $A/(torsion)$ is uniquely p -divisible, or
- (4) $j > c(n) + 1$.

Proof of Theorem 2.12(1). Given an S -module X , for each $j \geq 0$, there is a fibration sequence of spaces

$$K(\pi_{j+1}(X), j) \rightarrow \Omega^\infty X\langle j+1 \rangle \rightarrow \Omega^\infty X\langle j \rangle.$$

Under the theorem's hypotheses on $\pi_*(X)$, the fiber is $T(n)_*$ -acyclic, so the second map is a $T(n)_*$ -isomorphism. We deduce that $X\langle j+1 \rangle \in \mathcal{S}_n$ if and only if $X\langle j \rangle \in \mathcal{S}_n$.

The hypothesis that $X \in \bar{\mathcal{S}}_n$ means that $X\langle d \rangle \in \mathcal{S}_n$ for all large d . By downward induction on j , we deduce that $X\langle j \rangle \in \mathcal{S}_n$ for all $j \geq 0$. Since $\pi_0(X) = 0$, $X\langle 0 \rangle \in \mathcal{S}_n$ implies $X \in \mathcal{S}_n$. \square

For $K(n)_*$, we have sharper results.

Lemma 5.16. *Let A be an abelian group. $K(A, j)$ is $K(n)_*$ -acyclic if and only if*

- (1) $j = 0$ and $A = 0$,
- (2) $1 \leq j \leq n$ and A is uniquely p -divisible,
- (3) $j = n + 1$ and $A/(torsion)$ is uniquely p -divisible, or
- (4) $j > n + 1$.

Proof of Theorem 2.12(2) and Theorem 2.21. We can assume that $X \in \bar{\mathcal{S}}_n^K$ is 0-connected. As in our proof of Lemma 2.17, let $P^d X$ denote the d th Postnikov section of X , so there is a cofibration sequence of spectra

$$X\langle d \rangle \rightarrow X \rightarrow P^d X.$$

If d is large, then $X\langle d \rangle \in \mathcal{S}_n^K$. Then Theorem 2.26 applies, and we deduce that there is a short exact sequence of $K(n)_*$ -Hopf algebras

$$K(n)_*(\Omega^\infty X\langle d \rangle) \rightarrow K(n)_*(\Omega^\infty X) \rightarrow K(n)_*(\Omega^\infty P^d X).$$

Thus $X \in \mathcal{S}_n^K$ if and only if $\Omega^\infty P^d X$ is $K(n)_*$ -acyclic.

The main theorem of [HRW] says that there is an isomorphism

$$K(n)_*(\Omega^\infty P^d X) \simeq \bigotimes_{j=1}^d K(n)_*(K(\pi_j(X), j)).$$

Theorem 2.21 follows.

By the lemma, this tensor product will be isomorphic to $K(n)_*$ if and only if $\pi_j(X)$ is uniquely p -divisible for $1 \leq j \leq n$, and also $\pi_{n+1}(X)/(\text{torsion})$ is uniquely p -divisible. Theorem 2.12(2) follows. \square

5.9. $s_n(X)$ is universal: proof of Proposition 2.6 and related results

We prove the first part of Proposition 2.6; the proofs of the other variants, including Proposition 2.20, are similar and left to the reader.

Suppose $F : \mathcal{S} \rightarrow \mathcal{S}$ is functor preserving $T(n)_*$ -isomorphisms, and T is a weak natural transformation of the form

$$T(X) : F(X) \rightarrow L_{T(n)} \Sigma^\infty(\Omega^\infty X)_+.$$

We show it uniquely factors through s_n .

Let $C(X) = C_{n-1, c(n)+2}(X)$ defined as in the proof of Theorem 2.10. Then $s_n(C(X))$ is an equivalence, and $C(X) \rightarrow X$ is a $T(n)_*$ -isomorphism.

We simplify notation; let

$$P(X) = L_{T(n)} \mathbb{P}(X) \text{ and } L(X) = L_{T(n)} \Sigma^\infty(\Omega^\infty X)_+.$$

By naturality, we have a commutative diagram

$$\begin{array}{ccccc} F(C(X)) & \xrightarrow{T(C(X))} & L(C(X)) & \xleftarrow{s_n(C(X))} & P(C(X)) \\ \downarrow \wr & & \downarrow & \sim & \downarrow \wr \\ F(X) & \xrightarrow{T(X)} & L(X) & \xleftarrow{s_n(X)} & P(X), \end{array}$$

where the left vertical map is an equivalence since F preserves $T(n)_*$ -isomorphisms. The canonical factorization of T through s_n is evident.

Acknowledgements

Many people deserve thanks for helping me with aspects of this work. I thank Greg Arone and Mike Mandell for many tutorials, taught from complementary perspectives, on André–Quillen–Goodwillie towers of various sorts. What I have learned from them has also been enhanced by the ideas of Randy McCarthy and his students, and by perusing recent versions of [G3] that Tom Goodwillie kindly supplied me. Some of the conversations with Mike Mandell were during a visit to the University of Chicago during the fall of 2000, and I thank the Chicago Mathematics Department for its support. I thank Charles Rezk for helping me through a point of confusion, and Steve

Wilson for helping me reach a point of clarity, both related to proofs of results in §2.5. Takuji Kashiwabara caught an error in the first version. Finally, I need to thank Pete Bousfield. He has been a constant guide to my understanding of the strange behavior of periodic localization. This has been true for 20 years, but email exchanges dating from mid 2000 have particularly helped me keep straight the technical details of this project. I particularly recommend his recent paper [B7] for discussions of problems closely related to those studied here.

Appendix A. Comparison of Theorem 2.2 with other stable splittings

Let CZ be the free E_∞ -space generated by a space Z , as in [M1]. The inclusion $Z \rightarrow QZ$ then induces the standard approximation map $\alpha(Z) : CZ \rightarrow QZ$. The suspension spectrum $\Sigma^\infty(CZ)_+$ has the structure of an object in $\mathcal{A}lg$ such that $\Sigma^\infty(\alpha(Z))_+$ is an algebra map.

The purpose of this appendix is to make the following two observations. Firstly, $s(Z) : \mathbb{P}(\Sigma^\infty Z) \rightarrow \Sigma^\infty(QZ)_+$ refines to a natural map $s^C(Z) : \mathbb{P}(\Sigma^\infty Z) \rightarrow \Sigma^\infty(CZ)_+$. Secondly, $s^C(Z)$ is always an equivalence, and agrees with the standard ‘stable splittings’ in the literature.

This first point is easily checked. Recall that $s(Z)$ is defined to be the natural weak map in $\mathcal{A}lg$ induced by the weak natural map of S -modules

$$\Sigma^\infty Z \xrightarrow{\Sigma^\infty \eta(Z)} \Sigma^\infty QZ \xleftarrow{\sim} I(\Sigma^\infty(QZ)_+).$$

Similarly we define $s^C(Z)$ to be the natural weak map in $\mathcal{A}lg$ induced by the weak natural map of S -modules

$$\Sigma^\infty Z \xrightarrow{\Sigma^\infty \eta(Z)} \Sigma^\infty CZ \xleftarrow{\sim} I(\Sigma^\infty(CZ)_+).$$

Then there is an evident factorization

$$\begin{array}{ccc} \mathbb{P}(\Sigma^\infty Z) & & \\ \downarrow s^C(Z) \searrow s(Z) & & \\ \Sigma^\infty(CZ)_+ & \xrightarrow{\alpha(Z)} & \Sigma^\infty(QZ)_+. \end{array}$$

To check the second point, we begin by observing that s^C admits a slightly different definition. Let $a(Z)$ denote the fiber (in S -modules) of the evident ‘augmentation’ $\Sigma^\infty(Z_+) \rightarrow S$. Note that the composite $a(Z) \rightarrow \Sigma^\infty(Z_+) \rightarrow \Sigma^\infty Z$ is always an equivalence. Then $s^C(Z)$ can alternatively be defined as the natural weak map in $\mathcal{A}lg$

induced by the weak natural map of S -modules

$$\Sigma^\infty Z \xleftarrow{\sim} a(Z) \rightarrow a(CZ) = I(\Sigma^\infty(CZ)_+).$$

Now we need to recall that C can be defined on the spectrum level [LMMS]. Let $S - S$ denote the category of diagrams of S -modules of the form

$$\begin{array}{ccc} & X & \\ \nearrow & & \searrow \\ S & \xlongequal{\quad} & S. \end{array}$$

There is a functor $C : S - S \rightarrow \mathcal{A}lg$ such that

- (1) $C(\Sigma^\infty(Z_+)) = \Sigma^\infty(CZ)_+$, and
- (2) $X \rightarrow X \vee S$ induces $\mathbb{P}(X) = C(X \vee S)$.

Using (2), the commutative diagram

$$\begin{array}{ccccc} \Sigma^\infty Z & \xlongequal{\quad} & \Sigma^\infty Z & \xleftarrow{\sim} & a(Z) \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^\infty Z \vee S & \xrightarrow{\sim} & \Sigma^\infty Z \times S & \xleftarrow{\sim} & \Sigma^\infty(Z_+), \end{array}$$

induces a diagram in $\mathcal{A}lg$

$$\begin{array}{ccccc} \mathbb{P}(\Sigma^\infty Z) & \xlongequal{\quad} & \mathbb{P}(\Sigma^\infty Z) & \xleftarrow{\sim} & \mathbb{P}(a(Z)) \\ \parallel & & \downarrow & & \downarrow \\ C(\Sigma^\infty Z \vee S) & \xrightarrow{\sim} & C(\Sigma^\infty Z \times S) & \xleftarrow{\sim} & C(\Sigma^\infty(Z_+)). \end{array}$$

Now using (1), this shows that $s^C(Z)$ is the natural weak equivalence

$$\mathbb{P}(\Sigma^\infty Z) = C(\Sigma^\infty Z \vee S) \xrightarrow{\sim} C(\Sigma^\infty Z \times S) \xleftarrow{\sim} \Sigma^\infty(CZ)_+.$$

Defined this way, $s^C(Z)$ satisfies the characterization of natural splittings given in [K4, Appendix B].

We end this appendix by noting that the proof of Theorem 2.4 generalizes in a straightforward way to prove the following variant.

Theorem A.1. *If a map of spectra $f : \Sigma^\infty Z \rightarrow X$ is an E_* -isomorphism, then the composite*

$$E_*(CZ) \xrightarrow{\alpha(Z)} E_*(QZ) \xrightarrow{(\Omega^\infty f)_*} E_*(\Omega^\infty X)$$

is a monomorphism.

Appendix B. Comparison with recent work of Bousfield

In this appendix, we show how Theorems 2.2 and 2.5 can be compared by using Bousfield's beautiful natural zig-zag of $L_{T(n)}$ -equivalences relating any S -module X to a suspension spectrum determined by X [B7]. This allows for an alternative proof of Theorem 5.1, and thus of many of the results in §2.5 and §2.6.

Bousfield constructs a functor

$$\Theta_n : S\text{-modules} \rightarrow \text{Spaces}$$

that is a left adjoint of sorts to the telescopic functor

$$\Phi_n : \text{Spaces} \rightarrow S\text{-modules}.$$

Using this adjunction, the equivalence

$$L_{T(n)}X \xrightarrow{\sim} \Phi_n(\Omega^\infty X),$$

corresponds to an equivalence

$$L_n^f \Sigma^\infty \Theta_n(X) \xrightarrow{\sim} M_n^f X,$$

where $M_n^f X$ is the fiber of $L_n^f X \rightarrow L_{n-1}^f X$. Thus $L_{n-1}^f \Sigma^\infty \Theta_n(X) \simeq *$, and there is a natural $T(n)_*$ -equivalence $\Sigma^\infty \Theta_n(X) \rightarrow L_n^f X$. Furthermore, Bousfield observes that $\Theta_n(X)$ is always d_n -connected, where d_n is defined in [B7, §4.3]: one can deduce that $d_n \geq c(n) + 1$ from [B2, Proposition 2.1]. One also has [B7, Theorem 3.3] that $M_n^f L_{T(n)} \simeq M_n^f$ and $L_{T(n)} M_n^f \simeq L_{T(n)}$.

Thus we get a zig-zag of $T(n)_*$ -equivalences:

$$\Sigma^\infty \Theta_n(X) \xrightarrow{\beta(X)} (L_n^f X) \langle d_n \rangle \leftarrow X \langle d_n \rangle \rightarrow X.$$

This allows us to consider the following diagram:

$$\begin{array}{ccc}
 L_{T(n)} \mathbb{P}(X) & \xrightarrow{s_n(X)} & L_{T(n)} \Sigma^\infty \Omega^\infty X_+ \\
 \wr \uparrow & & \uparrow \\
 L_{T(n)} \mathbb{P}(X \langle d_n \rangle) & \xrightarrow{s_n(X \langle d_n \rangle)} & L_{T(n)} \Sigma^\infty \Omega^\infty X \langle d_n \rangle_+ \\
 \wr \downarrow & & \downarrow \wr \\
 L_{T(n)} \mathbb{P}((L_n^f X) \langle d_n \rangle) & \xrightarrow{s_n((L_n^f X) \langle d_n \rangle)} & L_{T(n)} \Sigma^\infty \Omega^\infty (L_n^f X) \langle d_n \rangle_+ \\
 \wr \uparrow & & \uparrow \\
 L_{T(n)} \mathbb{P}(\Sigma^\infty \Theta_n(X)) & \xrightarrow[L_{T(n)} s(\Theta_n(X))]{\sim} & L_{T(n)} \Sigma^\infty Q \Theta_n(X)_+.
 \end{array}$$

Below we will show that the diagram commutes. Thus the classical stable splitting of $Q \Theta_n(X)$ given by Theorem 2.2, corresponds to the splitting of $L_{T(n)} \Sigma^\infty \Omega^\infty X$ given by Theorem 2.5.

A crucial point about this diagram is that, as indicated, the middle vertical map on the right is an equivalence, as Ω^∞ takes L_n^f -equivalences between d_n -connected spectra to $T(n)_*$ -equivalences, thanks to [B7, Corollary 4.8].⁹ Thus, since the diagram commutes, it is clear that $s_n(X)$ is an equivalence on highly connected X if and only if the bottom right vertical map is an equivalence, i.e. $(\Omega^\infty \beta(X))_* : T(n)_*(Q \Theta_n(X)) \rightarrow T(n)_*(\Omega^\infty (L_n^f X) \langle d_n \rangle)$ is an isomorphism. Again appealing to [B7, Corollary 4.8], this will happen if $L_{n-1}^f X \simeq *$, and we have reproved Theorem 5.1 using Bousfield's results.

It is illuminating to note that $\Omega^\infty \beta(X)$ is always a $T(n)_*$ -monomorphism, by virtue of our Theorem 2.4.

⁹ As $d_n \geq c(n) + 1$, this also follows from Theorem 5.1. However, if one wishes to offer an alternative proof of Theorem 5.1, it seems prudent to not argue this way.

The top two squares of the diagram obviously commute. Checking that the bottom square commutes quickly reduces to verifying that the following diagram commutes:

$$\begin{array}{ccc}
 L_{T(n)}(L_n^f X)\langle d_n \rangle & \xrightarrow{\eta_n((L_n^f X)\langle d_n \rangle)} & L_{T(n)}\Sigma^\infty\Omega^\infty((L_n^f X)\langle d_n \rangle) \\
 \wr \uparrow L_{T(n)}\beta(X) & & \uparrow L_{T(n)}\Sigma^\infty\Omega^\infty\beta(X) \\
 L_{T(n)}\Sigma^\infty\Theta_n(X) & \xrightarrow{L_{T(n)}\Sigma^\infty\eta(\Theta_n(X))} & L_{T(n)}\Sigma^\infty Q\Theta_n(X).
 \end{array}$$

We show this using a variant of a proof which was outlined to us in email from Pete Bousfield. It is an exercise in using the various adjunctions constructed in [B7], as summarized in [B7, Theorem 5.14].

By the naturality of η_n , it suffices to verify the following proposition.

Proposition B.1. $\eta_n(\Sigma^\infty\Theta_n(X)) \simeq L_{T(n)}\Sigma^\infty\eta(\Theta_n(X))$.

To prove this, we first observe that $\Sigma^\infty\Theta_n$ preserves $T(n)_*$ -equivalences, and thus so does $\Sigma^\infty Q\Theta_n$. Thus the zig-zag of $T(n)_*$ -equivalences

$$X \rightarrow L_n^f X \leftarrow M_n^f X$$

can be used to reduce the proof of the proposition to the case when $X = M_n^f X$, i.e. $X \in \mathcal{M}_n^f$ in the notation of [B7].

For any space Z , unravelling the definitions reveals that

$$\eta_n(\Sigma^\infty Z) = \Phi_n(\eta(QZ)),$$

while

$$L_{T(n)}\Sigma^\infty\eta(Z) = \Phi_n(Q\eta(Z)).$$

Both of these maps clearly agree after precomposition with

$$\Phi_n(\eta(Z)) : \Phi_n(Z) \rightarrow L_{T(n)}\Sigma^\infty Z.$$

Thus the next lemma will complete the proof of the proposition.

Lemma B.2. *If $X \in \mathcal{M}_n^f$, then*

$$\Phi_n(\eta(\Theta_n(X))) : \Phi_n(\Theta_n(X)) \rightarrow L_{T(n)}\Sigma^\infty\Theta_n(X)$$

is split epic.

To prove this, we will use a very general categorical lemma. Suppose one has two categories \mathcal{A} and \mathcal{B} , and two pairs of adjoint functors

$$\mathcal{A} \begin{array}{c} \xrightarrow{L_1} \\ \xleftarrow{R_1} \end{array} \mathcal{B} \begin{array}{c} \xrightarrow{L_2} \\ \xleftarrow{R_2} \end{array} \mathcal{A}.$$

Let $\eta_1 : 1_{\mathcal{A}} \rightarrow R_1 L_1$ and $\eta_2 : 1_{\mathcal{B}} \rightarrow R_2 L_2$ be the units of the adjunctions.

Now suppose we are also given a natural transformation $\gamma : 1_{\mathcal{A}} \rightarrow R_1 R_2$ with adjoint $\beta : L_2 L_1 \rightarrow 1_{\mathcal{A}}$.

Lemma B.3. *γ is an equivalence if and only if β is an equivalence. In this case, the map*

$$R_1 \eta_2(L_1 A) : R_1 L_1 A \rightarrow R_1 R_2 L_2 L_1 A$$

is split epic for all A .

Proof. The first statement is clear. The second statement then follows from the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_1(A)} & R_1 L_1 A \\ \downarrow \gamma(A) & & \downarrow R_1 \eta_2(L_1 A) \\ R_1 R_2 A & \xleftarrow[\sim]{R_1 R_2 \beta(A)} & R_1 R_2 L_2 L_1 A. \quad \square \end{array}$$

Proof of Lemma B.2. The previous lemma applies to the pair of adjoint functors appearing in [B7, Theorem 5.14] to say that

$$M_n^f \Phi_n(\eta_2(\Theta_n(X))) : M_n^f \Phi_n \Theta_n(X) \rightarrow M_n^f \Phi_n \Omega^\infty(L_n^f \Sigma^\infty \Theta_n(X))\langle d_n \rangle$$

is split epic, where η_2 , defined on a certain category of d_n -connected spaces, has the form

$$\eta_2(Z) : Z \rightarrow \Omega^\infty(L_n^f \Sigma^\infty Z)\langle d_n \rangle.$$

Applying $L_{T(n)}$, one deduces that

$$\Phi_n(\eta_2(\Theta_n(X))) : \Phi_n \Theta_n(X) \rightarrow L_{T(n)}((L_n^f \Sigma^\infty \Theta_n(X))\langle d_n \rangle)$$

is split epic.

Using the zig-zag of $T(n)_*$ -equivalences

$$(L_n^f \Sigma^\infty \Theta_n(X)) \langle d_n \rangle \rightarrow L_n^f \Sigma^\infty \Theta_n(X) \leftarrow \Sigma^\infty \Theta_n(X),$$

it follows that this last map identifies with

$$\Phi_n(\eta(\Theta_n(X))) : \Phi_n \Theta_n(X) \rightarrow L_{T(n)} \Sigma^\infty \Theta_n(X). \quad \square$$

Appendix C. A comparison of s_n and s

One might wonder for what Z the two natural maps

$$s_n(\Sigma^\infty Z), L_{T(n)} s(Z) : L_{T(n)} \mathbb{P}(\Sigma^\infty Z) \rightarrow L_{T(n)} \Sigma^\infty (QZ)_+$$

are homotopic. Here we briefly summarize what we can say about this.

On the positive side, $s_n(\Sigma^\infty Z) \simeq L_{T(n)} s(Z)$ if and only if

$$\eta_n(\Sigma^\infty Z) \simeq L_{T(n)} \eta(Z) : L_{T(n)} \Sigma^\infty Z \rightarrow L_{T(n)} \Sigma^\infty QZ.$$

Thus Proposition B.1 implies

Proposition C.1. *If $Z \simeq \Theta_n(X)$ then $s_n(\Sigma^\infty Z) \simeq L_{T(n)} s(Z)$.*

Lemma 5.2 gave another sufficient condition on Z ; we do not know if the proposition includes this as a special case.

On the negative side, since $s(Z)$ is an equivalence for all connected Z , we have an obvious necessary condition.

Lemma C.2. *If Z is connected and $\Sigma^\infty Z \notin \mathcal{S}_n$ then $s_n(\Sigma^\infty Z) \not\simeq L_{T(n)} s(Z)$.*

Thus, for example, the two maps are distinct for $Z = S^1$, and for all $Z = S^d$ if $n \geq 2$. More examples of suspension spectra not in \mathcal{S}_n can be found using the next simple lemma, which doesn't seem to follow immediately from our other results.

Lemma C.3. *If $X \notin \mathcal{S}_n$ then $\Sigma^\infty \Omega^\infty X \notin \mathcal{S}_n$.*

Proof. The evaluation map $\Sigma^\infty \Omega^\infty X \rightarrow X$ induces a commutative diagram

$$\begin{array}{ccc} T(n)_*(\mathbb{P}\Sigma^\infty \Omega^\infty X) & \xrightarrow{s_n(\Sigma^\infty \Omega^\infty X)} & T(n)_*(Q\Omega^\infty X) \\ \downarrow & & \downarrow \\ T(n)_*(\mathbb{P}X) & \xrightarrow{s_n(X)} & T(n)_*(\Omega^\infty X). \end{array}$$

The horizontal maps are always monic, and the right vertical map is always epic, as it admits an obvious splitting. Thus if the top map is an isomorphism, so is the bottom. \square

When $\eta_n(\Sigma^\infty Z)$ and $L_{T(n)}\eta(Z)$ differ, one can roughly measure the difference by means of James–Hopf invariants. Let

$$t_r(Z) : \Sigma^\infty QZ \rightarrow \Sigma^\infty D_r Z$$

be the r th component of $t(Z)$, as given by Theorem 2.2. In the literature, the adjoint

$$j_r(Z) : QZ \rightarrow QD_r Z$$

is usually called the r th James–Hopf invariant.

Now consider the two maps

$$L_{T(n)}\Sigma^\infty Z \xrightarrow{L_{T(n)}\eta(Z)} L_{T(n)}\Sigma^\infty QZ \xrightarrow{L_{T(n)}t_r(Z)} L_{T(n)}\Sigma^\infty D_r Z,$$

and

$$L_{T(n)}\Sigma^\infty Z \xrightarrow{\eta_n(\Sigma^\infty Z)} L_{T(n)}\Sigma^\infty QZ \xrightarrow{L_{T(n)}t_r(Z)} L_{T(n)}\Sigma^\infty D_r Z.$$

For $r \geq 2$, the former is 0, while the latter is $\Phi_n(j_r(Z))$, as is easily checked.

Comparison with Proposition C.1 implies the next corollary.

Corollary C.4. *If $s_n(\Sigma^\infty Z) \simeq L_{T(n)}s(Z)$, e.g. if $Z \simeq \Theta_n(X)$ for some X , then $\Phi_n(j_r(Z))$ is null for all $r \geq 2$.*

There are some intriguing open questions regarding the natural transformations

$$\Phi_n(j_r(Z)) : L_{T(n)}\Sigma^\infty Z \rightarrow L_{T(n)}\Sigma^\infty D_r Z.$$

For example, they induces natural transformations

$$E_n^*(D_r Z) \rightarrow E_n^*(Z),$$

and one might wonder if these are related to either the constructions in [HKR], or, via duality, to total power operations in E_n^* cohomology.

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