# HOPF ALGEBROIDS

GABRIELLA BÖHM

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## 1. INTRODUCTION

If it should be formulated in one sentence what a Hopf algebroid is, it should be described as a generalisation of a Hopf algebra to a non-commutative base algebra. More precisely, the best known examples of Hopf algebroids are Hopf algebras. Clearly, the notion of a Hopf algebroid turns out to be a successful generalisation only if a convincing amount of results about Hopf algebras extend to Hopf algebroids. But one expects more: working with Hopf algebroids should be considered to be useful if in this way one could solve problems that could not be solved in terms of Hopf algebras. Hopf algebroids provide us with results of both types, ones which extend known results about Hopf algebras and also ones which are conceptually new.

Hopf algebras have been intensively studied, and successfully applied in various fields of mathematics and even physics, for more than fifty years. Without aiming at a complete list, let us mention a few applications. Hopf algebras were used to construct invariants in topology and knot theory. In connection with solutions of the quantum Yang-Baxter equation, quantum groups i.e. certain Hopf algebras play a central role. In (low dimensional) quantum field theory Hopf algebras are capable of describing internal symmetry of some models. In non-commutative differential geometry (faithfully flat) Galois extensions by a Hopf algebra are interpreted as non-commutative

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*Date*: May 2008.

principal bundles. Although the theory of Hopf algebras was (is!) extremely successful, in the 1990's there arose more and more motivations for a generalisation.

Originally the term 'Hopf algebroid' was used for cogroupoid objects in the category of commutative algebras. These are examples of Hopf algebroids in our current note, with commutative underlying algebra structure. They found an application e.g. in algebraic topology [50]. As a tool of a study of the geometry of principal fiber bundles with groupoid symmetry, recently more general, non-commutative Hopf algebroids have been used, but still over commutative base algebras [51]. For some applications it is still not the necessary level of generality. In Poisson geometry, solutions of the dynamical Yang-Baxter equation correspond to dynamical quantum groups, which are not Hopf algebras [31], [53], [46], [75], [30], [42]. In topology, invariants obtained in [52] do not fit the Hopf algebraic framework. In transverse geometry extensions of Hopf algebras by non-commutative base algebras occurred [26]. In low dimensional quantum field theories non-integral values of the statistical (also called quantum-) dimensions in some models exclude a Hopf algebra symmetry [12]. Another field where important questions could not be answered in the framework of Hopf algebras is non-commutative geometry, i.e. Hopf Galois theory. Thinking about classical Galois extensions of fields by a finite group, such an extension can be characterised without explicitly mentioning the Galois group. A (unique, upto isomorphism) Galois group is determined by a Galois field extension. In the case of Hopf Galois extensions no such intrinsic characterisation, without explicit use of a Hopf algebra, is known. Also, although the Hopf algebra describing the symmetry of a given Hopf Galois extension is known to be non-unique, the relation between the possible choices is not known. These questions have been handled by allowing for non-commutative base algebras [4], [40]. On the other hand, as the study of Hopf algebroids has a quite short past, there are many aspects of Hopf algebras that have not yet been investigated how to extend to Hopf algebroids. It has to be admitted that almost nothing has been done yet e.g. towards a classification and structure theory of Hopf algebroids.

What does it mean that the base algebra $\mathbf{R}$ of a Hopf algebroid is non-commutative? Recall that a bialgebra over a commutative base ring $k$ is a $k$-module, with compatible algebra and coalgebra structures. By analogy, in a bialgebroid the coalgebra structure is replaced by a coring over any not necessarily commutative $k$-algebra $\mathbf{R}$. Also the algebra structure is replaced by a ring over a non-commutative base algebra. However, in order to formulate the compatibility between the ring and coring structures, the base algebra of the ring has to be not $\mathbf{R}$ but $\mathbf{R} \otimes_k \mathbf{R}^{op}$. A Hopf algebra is a bialgebra with an additional antipode map. In the Hopf algebroid case, the antipode relates two different bialgebroid structures, over the base algebras $\mathbf{R}$ and $\mathbf{R}^{op}$, respectively.

In these notes we arrive at the notion of a Hopf algebroid after considering all constituent structures. In Section 2 $\mathbf{R}$-rings and $\mathbf{R}$-corings are introduced. They are seen to generalise algebras and coalgebras, respectively. Emphasis is put on their duality. Section 3 is devoted to a study of bialgebroids, generalising bialgebras. Several equivalent descriptions are given and examples are collected. In particular, constructions of new bialgebroids from known ones are presented. Some of them change the base algebra of a bialgebroid, so they have no counterparts for bialgebras. Although bialgebroid axioms are not manifestly self-dual, duals of finitely generated and projective bialgebroids are shown to be bialgebroids. Key properties of a bialgebroid are monoidality of the categories of modules and comodules. This is explained in some detail. Section 3 is closed by a most important and most successful application, Galois theory of bialgebroids. Hopf algebroids are the subject of Section 4. After presenting the definition, listing some examples and deriving some immediate consequences of the axioms, we discuss the theory of comodules. Since in a Hopf algebroid there are two bialgebroids (hence corings) present, comodules of the Hopf algebroid comprise comodule structures of both. The relation between the categories of comodules of a Hopf algebroid, and comodules of the constituent bialgebroids, is investigated. The category of comodules of a Hopf algebroid is proven to be monoidal, what is essential from the point of view of Galois theory. Next we turn to a study of the theory of integrals. It is a good example of results that are obtained by using some new ideas, but that extend analogous results for Hopf algebras in a reassuring way. The structure of Galois extensions by Hopf algebroids is investigated. Useful theorems are presented about situations when surjectivity of a canonical map implies Galois

property. They extend known results about Hopf Galois extensions. While there seems to be an accord in the literature that the right generalisation of a bialgebra to a non-commutative base is a bialgebroid, there is some discussion about the right generalisation of a Hopf algebra. We close these notes by collecting and comparing notions suggested by various authors.

In order to keep the list of references perspicuous, we do not refer in these notes to papers containing classical results about Hopf algebras, which are generalised hereby. We believe it is more useful to give here a detailed bibliography of those papers which deal with structures over non-commutative base. A very good and detailed bibliography of the literature of Hopf algebras can be found e.g. in Chapter "Hopf Algebras" of Handbook of Algebra [24].

**Notations and conventions.** Throughout $k$ is an associative and commutative unital ring. All algebras are associative and unital $k$-algebras. A $k$-algebra is denoted by $\mathbf{A}$ and the underlying $k$-module is denoted by $A$. On elements of $A$, multiplication is denoted by juxtaposition. Unit element is denoted by $1_\mathbf{A}$. For an algebra $\mathbf{A}$, with multiplication $(a, a') \mapsto aa'$, $\mathbf{A}^{op}$ denotes the opposite of $\mathbf{A}$. As a $k$-module it is equal to $A$ and multiplication is $(a, a') \mapsto a'a$. The category of right (resp.left) modules of an algebra $\mathbf{A}$ is denoted by $\mathcal{M}_\mathbf{A}$ (resp. $_\mathbf{A}\mathcal{M}$). Hom sets are denoted by $\mathrm{Hom}_\mathbf{A}(-, -)$ (resp. $_\mathbf{A}\mathrm{Hom}(-, -)$). The category of $\mathbf{A}$-bimodules is denoted by $_\mathbf{A}\mathcal{M}_\mathbf{A}$ and its hom sets by $_\mathbf{A}\mathrm{Hom}_\mathbf{A}(-, -)$. Often we identify left $\mathbf{A}$-modules with right $\mathbf{A}^{op}$-modules, but in every such case it is explicitly said. Action on a (say right) module $M$ of a $k$-algebra $\mathbf{A}$, if evaluated on elements $m \in M$ and $a \in A$, is denoted by $\varrho_M : m \otimes a \mapsto m \cdot a$.

For coproducts in a coalgebra and, more generally, in a coring $C$, Sweedler's index notation is used. That is, for an element $c \in C$, we write $c \mapsto c_{(1)} \otimes c_{(2)}$ (or sometimes $c \mapsto c^{(1)} \otimes c^{(2)}$) for the coproduct, where implicit summation is understood. Similarly, for a (say right) coaction on a comodule $M$ of a coring, evaluated on an element $m \in M$, the notation $\varrho^M : m \mapsto m_{[0]} \otimes m_{[1]}$ (or $m \mapsto m^{[0]} \otimes m^{[1]}$) is used, where implicit summation is understood. The category of right (resp. left) comodules of a coring $C$ is denoted by $\mathcal{M}^C$ (resp. $^C\mathcal{M}$). Hom sets are denoted by $\mathrm{Hom}^C(-, -)$ (resp. $^C\mathrm{Hom}(-, -)$).

In any category $\mathcal{A}$, the identity morphism at an object $A$ is denoted by the same symbol $A$. Hom sets in $\mathcal{A}$ are denoted by $\mathrm{Hom}_\mathcal{A}(-, -)$.

In a monoidal category $(\mathcal{M}, \otimes, U)$ we allow for non-trivial coherence isomorphisms $(- \otimes -) \otimes - \cong - \otimes (- \otimes -)$ and $- \otimes U \cong - \cong U \otimes -$, but do not denote them explicitly. (Such monoidal categories are called in the literature sometimes *lax* monoidal.) The opposite of a monoidal category $(\mathcal{M}, \otimes, U)$, denoted by $(\mathcal{M}, \otimes, U)^{op}$, means the same category $\mathcal{M}$ with opposite monoidal product. A functor $\mathbb{F}$ between monoidal categories $(\mathcal{M}, \otimes, U)$ and $(\mathcal{M}', \otimes', U')$ is said to be monoidal if there exist natural transformations $\mathbb{F}^2 : \mathbb{F}(-) \otimes' \mathbb{F}(-) \to \mathbb{F}(- \otimes -)$ and $\mathbb{F}^0 : U' \to \mathbb{F}(U)$, satisfying usual compatibility conditions. $\mathbb{F}$ is said to be op-monoidal if there exist compatible natural transformations $\mathbb{F}_2 : \mathbb{F}(- \otimes -) \to \mathbb{F}(-) \otimes' \mathbb{F}(-)$ and $\mathbb{F}_0 : \mathbb{F}(U) \to U'$. A monoidal functor $(\mathbb{F}, \mathbb{F}^2, \mathbb{F}^0)$ is strong monoidal if $\mathbb{F}^2$ and $\mathbb{F}^0$ are isomorphisms, and it is strict monoidal if $\mathbb{F}^2$ and $\mathbb{F}^0$ are identity morphisms.

## 2. $\mathbf{R}$-RINGS AND $\mathbf{R}$-CORINGS

A *monoid* in a monoidal category $(\mathcal{M}, \otimes, U)$ is a triple $(A, \mu, \eta)$. Here $A$ is an object and $\mu : A \otimes A \to A$ and $\eta : U \to A$ are morphisms in $\mathcal{M}$, satisfying associativity and unitality conditions

$$(2.1) \qquad \mu \circ (\mu \otimes A) = \mu \circ (A \otimes \mu) \qquad \text{and} \qquad \mu \circ (\eta \otimes A) = A = \mu \circ (A \otimes \eta).$$

The morphism $\mu$ is called a *multiplication* (or *product*) and $\eta$ is called a *unit*. An algebra over a commutative ring $k$ can be described as a monoid in the monoidal category $(\mathcal{M}_k, \otimes_k, k)$ of $k$-modules.

A *right module* of a monoid $(A, \mu, \eta)$ is a pair $(V, \nu)$, where $V$ is an object and $\nu : V \otimes A \to V$ is a morphism in $\mathcal{M}$, such that

$$\nu \circ (V \otimes \mu) = \nu \circ (\nu \otimes A) \qquad \text{and} \qquad \nu \circ (V \otimes \eta) = V.$$

Left modules are defined symmetrically.

Reversing all arrows in the definition of a monoid, we arrive at the dual notion of a comonoid. A *comonoid* in a monoidal category $(\mathcal{M}, \otimes, U)$ is a triple $(C, \Delta, \epsilon)$. Here $C$ is an object and $\Delta : C \to C \otimes C$ and $\epsilon : C \to U$ are morphisms in $\mathcal{M}$, satisfying coassociativity and counitality conditions

$$(2.2) \qquad (\Delta \otimes C) \circ \Delta = (C \otimes \Delta) \circ \Delta \qquad \text{and} \qquad (\epsilon \otimes C) \circ \Delta = C = (C \otimes \epsilon) \circ \Delta.$$

The morphism $\Delta$ is called a *comultiplication* (or *coproduct*) and $\epsilon$ is called a *counit*. A coalgebra over a commutative ring $k$ can be described as a comonoid in the monoidal category $(\mathcal{M}_k, \otimes_k, k)$ of $k$-modules. Dualising the definition of a module of a monoid, one arrives at the notion of a comodule of a comonoid.

Many aspects of the theory of algebras and their modules, or coalgebras and their comodules, can be extended to monoids or comonoids in general monoidal categories. Here we are interested in monoids and comonoids in a monoidal category $(_\mathbf{R}\mathcal{M}_\mathbf{R}, \otimes_\mathbf{R}, \mathbf{R})$ of bimodules over a $k$-algebra $\mathbf{R}$. These monoids and comonoids are called $\mathbf{R}$-*rings* and $\mathbf{R}$-*corings*, respectively.

2.1. **R-rings.** Generalising algebras over commutative rings, we study monoids in bimodule categories.

DEFINITION 2.1. For an algebra $\mathbf{R}$ over a commutative ring $k$, an $\mathbf{R}$-*ring* is a triple $(A, \mu, \eta)$. Here $A$ is an $\mathbf{R}$-bimodule and $\mu : A \otimes_\mathbf{R} A \to A$ and $\eta : R \to A$ are $\mathbf{R}$-bimodule maps, satisfying the associativity and unit conditions (2.1). A *morphism of* $\mathbf{R}$-*rings* $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is an $\mathbf{R}$-bimodule map $f : A \to A'$, such that $f \circ \mu = \mu' \circ (f \otimes_\mathbf{R} f)$ and $f \circ \eta = \eta'$.

For an $\mathbf{R}$-ring $(A, \mu, \eta)$, the *opposite* means the $\mathbf{R}^{op}$-ring $(A^{op}, \mu^{op}, \eta)$. Here $A^{op}$ is the same $k$-module $A$. It is understood to be a left (resp. right) $\mathbf{R}^{op}$-module via the right (resp. left) $\mathbf{R}$-action. Multiplication is $\mu^{op}(a \otimes_{\mathbf{R}^{op}} a') := \mu(a' \otimes_\mathbf{R} a)$ and unit is $\eta$.

A most handy characterisation of $\mathbf{R}$-rings comes from the following observation.

LEMMA 2.2. *There is a bijective correspondence between* $\mathbf{R}$-*rings* $(A, \mu, \eta)$ *and* $k$-*algebra homomorphisms* $\eta : \mathbf{R} \to \mathbf{A}$.

Indeed, starting with an $\mathbf{R}$-ring $(A, \mu, \eta)$, a multiplication map $A \otimes_k A \to A$ is obtained by composing the canonical epimorphism $A \otimes_k A \to A \otimes_\mathbf{R} A$ with $\mu$. Conversely, starting with an algebra map $\eta : \mathbf{R} \to \mathbf{A}$, an $\mathbf{R}$-bilinear associative multiplication $A \otimes_\mathbf{R} A \to A$ is obtained by using the universality of the coequaliser $A \otimes_k A \to A \otimes_\mathbf{R} A$.

An $\mathbf{R}$-ring $(A, \mu, \eta)$ determines monads on the categories of right and left $\mathbf{R}$-modules (i.e. monoids in the monoidal categories of endofunctors on $\mathcal{M}_\mathbf{R}$ and $_\mathbf{R}\mathcal{M}$, respectively). They are given by $- \otimes_\mathbf{R} A : \mathcal{M}_\mathbf{R} \to \mathcal{M}_\mathbf{R}$ and $A \otimes_\mathbf{R} - : _\mathbf{R}\mathcal{M} \to _\mathbf{R}\mathcal{M}$, respectively.

DEFINITION 2.3. A *right module* for an $\mathbf{R}$-ring $(A, \mu, \eta)$ is an algebra for the monad $- \otimes_\mathbf{R} A$ on the category $\mathcal{M}_\mathbf{R}$. A *right module morphism* is a morphism of algebras for the monad $- \otimes_\mathbf{R} A$.

A *left module* for an $\mathbf{R}$-ring $(A, \mu, \eta)$ is an algebra for the monad $A \otimes_\mathbf{R} -$ on the category $_\mathbf{R}\mathcal{M}$. A *left module morphism* is a morphism of algebras for the monad $A \otimes_\mathbf{R} -$.

Left modules of an $\mathbf{R}$-ring are canonically identified with right modules for the opposite $\mathbf{R}^{op}$-ring. Analogously to Lemma 2.2, modules for $\mathbf{R}$-rings can be characterised as follows.

LEMMA 2.4. *A* $k$-*module* $M$ *is a (left or right) module of an* $\mathbf{R}$-*ring* $(A, \mu, \eta)$ *if and only if it is a (left or right) module of the corresponding* $k$-*algebra* $\mathbf{A}$ *in Lemma 2.2. Furthermore, a* $k$-*module map* $f : M \to M'$ *is a morphism of (left or right) modules of an* $\mathbf{R}$-*ring* $(A, \mu, \eta)$ *if and only if it is a morphism of (left or right) modules of the corresponding* $k$-*algebra* $\mathbf{A}$ *in Lemma 2.2.*

The situation when the (left or right) regular $\mathbf{R}$-module extends to a (left or right) module of an $\mathbf{R}$-ring $(A, \mu, \eta)$ is of particular interest.

LEMMA 2.5. *The right regular module of a* $k$-*algebra* $\mathbf{R}$ *extends to a right module of an* $\mathbf{R}$-*ring* $(A, \mu, \eta)$ *if and only if there exists a* $k$-*module map* $\chi : A \to R$, *obeying the following properties.*

(i) $\chi(a\eta(r)) = \chi(a)r$, *for* $a \in A$ *and* $r \in R$ *(right* $\mathbf{R}$-*linearity)*,
(ii) $\chi(aa') = \chi((\eta \circ \chi)(a)a')$, *for* $a, a' \in A$ *(associativity)*,

(iii) $\chi(1_{\mathbf{A}}) = 1_{\mathbf{R}}$ *(unitality).*

*The map $\chi$ obeying these properties is called a* right character *on the $\mathbf{R}$-ring $(A, \mu, \eta)$.*

In terms of a right character $\chi$, a right $\mathbf{A}$-action on $R$ is given by $r \cdot a := \chi(\eta(r)a)$. Conversely, in terms of a right $\mathbf{A}$-action on $R$, a right character is constructed as $\chi(a) := 1_{\mathbf{R}} \cdot a$. Symmetrically, one can define a *left character* on an $\mathbf{R}$-ring $(A, \mu, \eta)$ via the requirement that the left regular $\mathbf{R}$-module extends to a left module for $(A, \mu, \eta)$.

DEFINITION 2.6. Let $(A, \mu, \eta)$ be an $\mathbf{R}$-ring possessing a right character $\chi : A \to R$. The *invariants* of a right module $(M, \varrho_M)$ with respect to $\chi$ are the elements of the $k$-submodule

$$M_\chi := \{\, m \in M \mid \varrho_M(m \underset{\mathbf{R}}{\otimes} a) = \varrho_M(m \underset{\mathbf{R}}{\otimes} (\eta \circ \chi)(a)), \quad \forall a \in A \,\} \cong \operatorname{Hom}_{\mathbf{A}}(R, M),$$

where the isomorphism $M_\chi \to \operatorname{Hom}_{\mathbf{A}}(R, M)$ is given by $m \mapsto (r \mapsto m \cdot \eta(r))$. In particular, the invariants of $R$ are the elements of the subalgebra

$$\mathbf{B} := R_\chi = \{\, b \in R \mid \chi(\eta(b)a) = b\chi(a), \quad \forall a \in A \,\}.$$

Associated to a character $\chi$, there is a canonical map

(2.3)
$$A \to {}_{\mathbf{B}}\operatorname{End}(R), \qquad a \mapsto \big(\, r \mapsto \chi(\eta(r)a) \,\big).$$

The $\mathbf{R}$-ring $(A, \mu, \eta)$ is said to be a *Galois $\mathbf{R}$-ring* (with respect to the character $\chi$) provided that the canonical map (2.3) is bijective.

2.2. **$\mathbf{R}$-corings.** The theory of $\mathbf{R}$-corings is dual to that of $\mathbf{R}$-rings. A detailed study can be found in the monograph [20].

DEFINITION 2.7. For an algebra $\mathbf{R}$ over a commutative ring $k$, an *$\mathbf{R}$-coring* is a triple $(C, \Delta, \epsilon)$. Here $C$ is an $\mathbf{R}$-bimodule and $\Delta : C \to C \otimes_{\mathbf{R}} C$ and $\epsilon : C \to R$ are $\mathbf{R}$-bimodule maps, satisfying the coassociativity and counit conditions (2.2). A *morphism of $\mathbf{R}$-corings* $f : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ is an $\mathbf{R}$-bimodule map $f : C \to C'$, such that $\Delta' \circ f = (f \otimes_{\mathbf{R}} f) \circ \Delta$ and $\epsilon' \circ f = \epsilon$.

For an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$, the *co-opposite* means the $\mathbf{R}^{op}$-coring $(C_{cop}, \Delta_{cop}, \epsilon)$. Here $C_{cop}$ is the same $k$-module $C$. It is understood to be a left (resp. right) $\mathbf{R}^{op}$-module via the right (resp. left) $\mathbf{R}$-action. Comultiplication is $\Delta_{cop}(c) := c_{(2)} \otimes_{\mathbf{R}^{op}} c_{(1)}$ and counit is $\epsilon$.

An $\mathbf{R}$-coring $(C, \Delta, \epsilon)$ determines comonads on the categories of right and left $\mathbf{R}$-modules (i.e. comonoids in the monoidal categories of endofunctors on $\mathcal{M}_{\mathbf{R}}$ and ${}_{\mathbf{R}}\mathcal{M}$, respectively). They are given by $- \otimes_{\mathbf{R}} C : \mathcal{M}_{\mathbf{R}} \to \mathcal{M}_{\mathbf{R}}$ and $C \otimes_{\mathbf{R}} - : {}_{\mathbf{R}}\mathcal{M} \to {}_{\mathbf{R}}\mathcal{M}$, respectively.

DEFINITION 2.8. A *right comodule* for an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$ is a coalgebra for the comonad $- \otimes_{\mathbf{R}} C$ on the category $\mathcal{M}_{\mathbf{R}}$. That is, a pair $(M, \varrho^M)$, where $M$ is a right $\mathbf{R}$-module and $\varrho^M : M \to M \otimes_{\mathbf{R}} C$ is a right $\mathbf{R}$-module map satisfying the coassociativity and counit conditions

(2.4)
$$(\varrho^M \underset{\mathbf{R}}{\otimes} C) \circ \varrho^M = (M \underset{\mathbf{R}}{\otimes} \Delta) \circ \varrho^M \qquad \text{and} \qquad (M \underset{\mathbf{R}}{\otimes} \epsilon) \circ \varrho^M = M.$$

A *right comodule morphism* $f : (M, \varrho^M) \to (M', \varrho^{M'})$ is a morphism of coalgebras for the comonad $- \otimes_{\mathbf{R}} C$. That is, a right $\mathbf{R}$-module map $f : M \to M'$, satisfying $\varrho^{M'} \circ f = (f \otimes_{\mathbf{R}} C) \circ \varrho^M$.

Symmetrically, a *left comodule* is a coalgebra for the comonad $C \otimes_{\mathbf{R}} -$ on the category ${}_{\mathbf{R}}\mathcal{M}$. A *left comodule morphism* is a morphism of coalgebras for the comonad $C \otimes_{\mathbf{R}} -$.

Left comodules of an $\mathbf{R}$-coring are canonically identified with right comodules for the co-opposite $\mathbf{R}^{op}$-coring.

The situation when the (left or right) regular $\mathbf{R}$-module extends to a (left or right) comodule of an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$ is of particular interest, see [16, Lemma 5.1].

LEMMA 2.9. *The (left or right) regular $\mathbf{R}$-module extends to a (left or right) comodule of an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$ if and only if there exists an element $g \in C$, obeying the following properties.*

(i) $\Delta(g) = g \otimes_{\mathbf{R}} g$.
(ii) $\epsilon(g) = 1_{\mathbf{R}}$.

*The element $g$ obeying these properties is called a* grouplike element *in the $\mathbf{R}$-coring $(C, \Delta, \epsilon)$.*

Having a grouplike element $g$ in an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$, a right coaction on $R$ is constructed as a map $R \to C$, $r \mapsto g \cdot r$. Conversely, a right coaction $\varrho^R : R \to C$ determines a grouplike element $\varrho^R(1_{\mathbf{R}})$.

DEFINITION 2.10. Let $(C, \Delta, \epsilon)$ be an $\mathbf{R}$-coring possessing a grouplike element $g \in C$. The *coinvariants* of a right comodule $(M, \varrho^M)$ with respect to $g$ are the elements of the $k$-submodule

$$M^g := \{\, m \in M \mid \varrho^M(m) = m \underset{\mathbf{R}}{\otimes} g \,\} \cong \mathrm{Hom}^C(R, M),$$

where the isomorphism $M^g \to \mathrm{Hom}^C(R, M)$ is given by $m \mapsto (r \mapsto m \cdot r)$. In particular, the coinvariants of $R$ are the elements of the subalgebra

$$\mathbf{B} := R^g = \{\, b \in R \mid b \cdot g = g \cdot b \,\}.$$

Associated to a grouplike element $g$, there is a canonical map

$$(2.5) \qquad\qquad R \underset{\mathbf{B}}{\otimes} R \to C, \qquad r \underset{\mathbf{B}}{\otimes} r' \mapsto r \cdot g \cdot r'.$$

The $\mathbf{R}$-coring $(C, \Delta, \epsilon)$ is said to be a *Galois* $\mathbf{R}$-coring (with respect to the grouplike element $g$) provided that the canonical map (2.5) is bijective.

Let $(C, \Delta, \epsilon)$ be an $\mathbf{R}$-coring possessing a grouplike element $g$. Put $\mathbf{B} := R^g$. For any right $C$-comodule $M$, $M^g$ is a right $\mathbf{B}$-module. Furthermore, any right $C$-comodule map $M \to M'$ restricts to a right $\mathbf{B}$-module map $M^g \to M'^g$. There is an adjoint pair of functors

$$(2.6) \qquad\qquad - \underset{\mathbf{B}}{\otimes} R : \mathcal{M}_{\mathbf{B}} \to \mathcal{M}^C \qquad \text{and} \qquad (-)^g : \mathcal{M}^C \to \mathcal{M}_{\mathbf{B}}.$$

If $(C, \Delta, \epsilon)$ is a Galois coring (with respect to $g$), then $\mathcal{M}^C$ is equivalent to the category of descent data for the extension $\mathbf{B} \subseteq \mathbf{R}$. Hence the situation, when the functors (2.6) establish an equivalence, is interesting from the descent theory point of view.

2.3. **Duality.** Beyond the formal duality between algebras and coalgebras, it is well known that the $k$-dual of a coalgebra over a commutative ring $k$ possesses a canonical algebra structure. The converse is true whenever a $k$-algebra is finitely generated and projective as a $k$-module. In what follows we recall analogues of these facts for rings and corings over an arbitrary algebra $\mathbf{R}$.

PROPOSITION 2.11. *Let $\mathbf{R}$ be an algebra over a commutative ring $k$.*
  (1) *For an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$, the left dual $^*C := {}_{\mathbf{R}}\mathrm{Hom}(C, R)$ possesses a canonical $\mathbf{R}$-ring structure. Multiplication is given by $(\phi\psi)(c) := \psi\big(c_{(1)} \cdot \phi(c_{(2)})\big)$, for $\phi, \psi \in {}^*C$ and $c \in C$. Unit map is $\mathbf{R} \to {}^*C$, $r \mapsto \epsilon(-)r$.*
  (2) *For an $\mathbf{R}$-ring $(A, \mu, \eta)$, which is a finitely generated and projective right $\mathbf{R}$-module, the right dual $A^* := \mathrm{Hom}_{\mathbf{R}}(A, R)$ possesses a canonical $\mathbf{R}$-coring structure. In terms of a dual basis $(\{a_i \in A\}, \{\alpha_i \in A^*\})$, comultiplication is given by $\xi \mapsto \sum_i \xi(a_i -) \otimes_{\mathbf{R}} \alpha_i$, which is independent of the choice of a dual basis. Counit is $A^* \to R$, $\xi \mapsto \xi(1_{\mathbf{A}})$.*
  (3) *For an $\mathbf{R}$-coring $(C, \Delta, \epsilon)$, which is a finitely generated and projective left $\mathbf{R}$-module, the second dual $(^*C)^*$ is isomorphic to $C$ as an $\mathbf{R}$-coring.*
  (4) *For an $\mathbf{R}$-ring $(A, \mu, \eta)$, which is a finitely generated and projective right $\mathbf{R}$-module, the second dual $^*(A^*)$ is isomorphic to $A$ as an $\mathbf{R}$-ring.*

Applying Proposition 2.11 to the co-opposite coring and the opposite ring, analogous correspondences are found between right duals of corings and left duals of rings.

PROPOSITION 2.12. *Let $C$ be a coring over an algebra $\mathbf{R}$.*
  (1) *Any right $C$-comodule $(M, \varrho^M)$ possesses a right module structure for the $\mathbf{R}$-ring $^*C$,*

$$(2.7) \qquad\qquad m \cdot \phi := m_{[0]} \cdot \phi(m_{[1]}), \qquad \text{for } m \in M, \ \phi \in {}^*C.$$

   *Any right $C$-comodule map becomes a $^*C$-module map. That is, there is a faithful functor $\mathcal{M}^C \to \mathcal{M}_{^*C}$.*
  (2) *The functor $\mathcal{M}^C \to \mathcal{M}_{^*C}$ is an equivalence if and only if $C$ is a finitely generated and projective left $\mathbf{R}$-module.*

There is a duality between Galois rings and Galois corings too.

PROPOSITION 2.13. *Let $C$ be an $\mathbf{R}$-coring which is a finitely generated and projective left $\mathbf{R}$-module. For an element $g \in C$, introduce the map $\chi_g : {}^*C \to R$, $\phi \mapsto \phi(g)$. The following statements hold.*

(1) *The element $g \in C$ is grouplike if and only if $\chi_g$ is a right character on the $\mathbf{R}$-ring ${}^*C$.*

(2) *An element $b \in R$ is a coinvariant of the right $C$-comodule $R$ (with coaction induced by a grouplike element $g$) if and only if $b$ is an invariant of the right ${}^*C$-module $R$ (with respect to the right character $\chi_g$).*

(3) *The $\mathbf{R}$-coring $C$ is a Galois coring (with respect to a grouplike element $g$) if and only if the $\mathbf{R}$-ring ${}^*C$ is a Galois ring (with respect to the right character $\chi_g$).*

## 3. BIALGEBROIDS

In Section 2 we generalised algebras and coalgebras over commutative rings to monoids and comonoids in bimodule categories. We could easily do so, the category of bimodules over any $k$-algebra $\mathbf{R}$ is a monoidal category, just as the category of $k$-modules. If we try to generalise bialgebras to a non-commutative base algebra $\mathbf{R}$, however, we encounter difficulties. Recall that a $k$-bialgebra consists of an algebra $(B, \mu, \eta)$, and a coalgebra $(B, \Delta, \epsilon)$ defined on the same $k$-module $B$. They are subject to compatibility conditions. Unit and multiplication must be coalgebra maps or, equivalently, counit and comultiplication must be algebra maps. This means in particular that, for any elements $b$ and $b'$ in $B$, multiplication and comultiplication satisfy the condition

$$(3.1) \qquad (bb')_{(1)} \underset{k}{\otimes} (bb')_{(2)} = b_{(1)} b'_{(1)} \underset{k}{\otimes} b_{(2)} b'_{(2)}.$$

Note that (3.1) is formulated in terms of the symmetry $\mathsf{tw}$ in $\mathcal{M}_k$. For any $k$-modules $M$ and $N$, the twist map $\mathsf{tw}_{M,N} : M \otimes_k N \to N \otimes_k M$ maps $m \otimes_k n$ to $n \otimes_k m$. Precisely, (3.1) is equivalent to

$$\Delta \circ \mu = (\mu \underset{k}{\otimes} \mu) \circ (B \underset{k}{\otimes} \mathsf{tw}_{B,B} \underset{k}{\otimes} B) \circ (\Delta \underset{k}{\otimes} \Delta).$$

In the literature one can find generalisations when $\mathsf{tw}$ is replaced by a braiding [74], [47], [61]. (For an approach when $\mathsf{tw}$ is replaced by a *mixed distributive law*, see [48].) However, general bimodule categories are neither symmetric nor braided. There is no natural way to formulate an analogue of (3.1) in a bimodule category. In fact, more sophisticated ideas are needed.

The notion which is known today as a (left) bialgebroid, was introduced (independently) by several authors. The first definition is due to Takeuchi, who used the name $\times_R$-*bialgebra* in [72]. Some twenty years later, with motivation coming from Poisson geometry, in [46] Lu proposed an equivalent definition. The term *bialgebroid* is due to her. A third equivalent set of axioms was invented by Xu in [75]. The equivalence of the listed definitions is far from obvious. It was proven by Brzeziński and Militaru in [18]. Symmetrical notions of left and right bialgebroids were formulated and studied by Kadison and Szlachányi in [41]. The definition presented here is a slightly reformulated version of the one in [46] or [41].

3.1. **Right and left bialgebroids.** In contrast to the definition of a bialgebra, in this section a bialgebroid is not described as a compatible monoid and comonoid in some monoidal category (of bimodules). The ring and coring structures of a bialgebroid are defined over different base algebras: they are a monoid and a comonoid in different monoidal categories. Recall from Lemma 2.2 that an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $A$ (for some algebra $\mathbf{R}$ over a commutative ring $k$) is described by a $k$-algebra map $\eta : \mathbf{R} \otimes_k \mathbf{R}^{op} \to \mathbf{A}$. Equivalently, instead of $\eta$, we can consider its restrictions

$$(3.2) \qquad s := \eta(- \underset{k}{\otimes} 1_{\mathbf{R}}) : \mathbf{R} \to \mathbf{A} \qquad \text{and} \qquad t := \eta(1_{\mathbf{R}} \underset{k}{\otimes} -) : \mathbf{R}^{op} \to \mathbf{A},$$

which are $k$-algebra maps with commuting ranges in $\mathbf{A}$. The maps $s$ and $t$ in (3.2) are called the *source* and *target* maps of an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $A$, respectively. In what follows, an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring will be given by a triple $(\mathbf{A}, s, t)$, where $\mathbf{A}$ is a $k$-algebra (with underlying $k$-module $A$) and $s$ and $t$ are algebra maps with commuting ranges as in (3.2).

DEFINITION 3.1. Let $\mathbf{R}$ be an algebra over a commutative ring $k$. A *right $\mathbf{R}$- bialgebroid* $\mathcal{B}$ consists of an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ and an $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ on the same $k$-module $B$. They are subject to the following compatibility axioms.

(i) The bimodule structure in the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ is related to the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ via

$$(3.3) \qquad\qquad r \cdot b \cdot r' := bs(r')t(r), \quad \text{for } r, r' \in R, \ b \in B.$$

(ii) Considering $B$ as an $\mathbf{R}$-bimodule as in (3.3), the coproduct $\Delta$ corestricts to a $k$-algebra map from $\mathbf{B}$ to

$$(3.4) \qquad B \times_R B := \{ \ \sum_i b_i \underset{\mathbf{R}}{\otimes} b'_i \ | \ \sum_i s(r)b_i \underset{\mathbf{R}}{\otimes} b'_i = \sum_i b_i \underset{\mathbf{R}}{\otimes} t(r)b'_i, \quad \forall r \in R \ \},$$

where $B \times_R B$ is an algebra via factorwise multiplication.

(iii) The counit $\epsilon$ is a right character on the $\mathbf{R}$-ring $(\mathbf{B}, s)$.

REMARKS 3.2. The bialgebroid axioms in Definition 3.1 have some immediate consequences.

(1) Note that the $k$-submodule $B \times_R B$ of $B \otimes_{\mathbf{R}} B$ is defined in such a way that factorwise multiplication is well defined on it. $B \times_R B$ is called the *Takeuchi product*. In fact, it has more structure than that of a $k$-algebra: it is an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring with unit map $\mathbf{R} \otimes_k \mathbf{R}^{op} \to B \times_R B$, $r \otimes_k r' \mapsto t(r') \otimes_{\mathbf{R}} s(r)$. The (corestriction of the) coproduct is easily checked to be an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-bimodule map $B \to B \times_R B$.

(2) Axiom (iii) is equivalent to the requirement that the counit $\epsilon$ is a right character on the $\mathbf{R}^{op}$-ring $(\mathbf{B}, t)$.

(3) Yet another equivalent formulation of axiom (iii) is the following. The map $\theta : \mathbf{B} \to \mathrm{End}_k(R)^{op}$, $b \mapsto \big( \ r \mapsto \epsilon(s(r)b) \ \big)$ is a $k$-algebra map, where $\mathrm{End}_k(R)^{op}$ is an algebra via opposite composition of endomorphisms. The map $\theta$ is called an *anchor map* in [75].

Recall that in a bialgebra over a commutative ring, replacing the algebra with the opposite one, or replacing the coalgebra with the co-opposite one, one arrives at bialgebras again. Analogously, the *co-opposite* of a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ with structure maps denoted as in Definition 3.1, is the following right $\mathbf{R}^{op}$-bialgebroid $\mathcal{B}_{cop}$. $\mathbf{R}^{op} \otimes_k \mathbf{R}$-ring structure is $(\mathbf{B}, t, s)$, $\mathbf{R}^{op}$-coring structure is $(B_{cop}, \Delta_{cop}, \epsilon)$. However, the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}^{op}, t, s)$ and the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ do not satisfy the same axioms in Definition 3.1. Instead, they are subject to a symmetrical version of Definition 3.1.

DEFINITION 3.3. Let $\mathbf{R}$ be an algebra over a commutative ring $k$. A *left $\mathbf{R}$- bialgebroid* $\mathcal{B}$ consists of an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ and an $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ on the same $k$-module $B$. They are subject to the following compatibility axioms.

(i) The bimodule structure in the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ is related to the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ via

$$(3.5) \qquad\qquad r \cdot b \cdot r' := s(r)t(r')b, \quad \text{for } r, r' \in R, \ b \in B.$$

(ii) Considering $B$ as an $\mathbf{R}$-bimodule as in (3.5), the coproduct $\Delta$ corestricts to a $k$-algebra map from $\mathbf{B}$ to

$$(3.6) \qquad B \, {}_R\!\times B := \{ \ \sum_i b_i \underset{\mathbf{R}}{\otimes} b'_i \ | \ \sum_i b_i t(r) \underset{\mathbf{R}}{\otimes} b'_i = \sum_i b_i \underset{\mathbf{R}}{\otimes} b'_i s(r), \quad \forall r \in R \ \},$$

where $B \, {}_R\!\times B$ is an algebra via factorwise multiplication.

(iii) The counit $\epsilon$ is a left character on the $\mathbf{R}$-ring $(\mathbf{B}, s)$.

Since in this note left and right bialgebroids are considered simultaneously, we use two versions of Sweedler's index notation. In a left bialgebroid we use lower indices to denote components of the coproduct, i.e. we write $b \mapsto b_{(1)} \otimes_{\mathbf{R}} b_{(2)}$. In a right bialgebroid we use upper indices to denote components of the coproduct, i.e. we write $b \mapsto b^{(1)} \otimes_{\mathbf{R}} b^{(2)}$. In both cases implicit summation is understood.

Recall that coalgebras over a commutative ring $k$ form a monoidal category with respect to the $k$-module tensor product. Bialgebras over $k$ can be described as monoids in the monoidal category of $k$-coalgebras. In [72] Takeuchi defined bialgebroids ($\times_R$-*bialgebras* in his terminology) as monoids in a monoidal category of certain corings, too. By [73, Definition 3.5], for two $k$-algebras $\mathbf{R}$ and $\mathbf{S}$, an $\mathbf{S}|\mathbf{R}$-*coring* is an $\mathbf{S} \otimes_k \mathbf{R}$-bimodule $C$, together with an $\mathbf{R}$-coring structure $(C, \Delta, \epsilon)$, such that the following identities hold.

$$\Delta((s \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c \cdot (s' \underset{k}{\otimes} 1_{\mathbf{R}})) = c^{(1)} \cdot (s' \underset{k}{\otimes} 1_{\mathbf{R}}) \underset{\mathbf{R}}{\otimes} (s \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c^{(2)} \qquad \text{and}$$

$$(s \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c^{(1)} \underset{\mathbf{R}}{\otimes} c^{(2)} = c^{(1)} \underset{\mathbf{R}}{\otimes} c^{(2)} \cdot (s \underset{k}{\otimes} 1_{\mathbf{R}}), \qquad \qquad \text{for } s, s' \in S, c \in C.$$

Morphisms of $\mathbf{S}|\mathbf{R}$-corings are morphisms of $\mathbf{R}$-corings which are in addition $\mathbf{S}$-bimodule maps. In particular, it can be shown by using the same methods as in [18], that the notion of an $\mathbf{R}|\mathbf{R}$-coring is equivalent to a $\times_R$-*coalgebra* in [72, Definition 4.1]. Part (1) of following Theorem 3.4 is thus a reformulation of [72, Proposition 4.7]. Part (2) states an equivalence of a right $\mathbf{R}$-bialgebroid in Definition 3.1, and a symmetrical version of a $\times_R$-bialgebra in [72, Definition 4.5].

THEOREM 3.4. *For an algebra $\mathbf{R}$ over a commutative ring $k$, the following statements hold.*

(1) $\mathbf{R}|\mathbf{R}$-*corings form a monoidal category. Monoidal product of two objects $(C, \Delta, \epsilon)$ and $(C', \Delta', \epsilon')$ is the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-module tensor product*

$$C \odot C' := C \underset{k}{\otimes} C' / \{ (1_{\mathbf{R}} \underset{k}{\otimes} r) \cdot c \cdot (1_{\mathbf{R}} \underset{k}{\otimes} r') \underset{k}{\otimes} c' - c \underset{k}{\otimes} (r' \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c' \cdot (r \underset{k}{\otimes} 1_{\mathbf{R}}) \mid r, r' \in R \}.$$

$C \odot C'$ *is an $\mathbf{R} \otimes_k \mathbf{R}$-bimodule, via the actions*

$$(r_1 \underset{k}{\otimes} r_2) \cdot (c \odot c') \cdot (r_1' \underset{k}{\otimes} r_2') := (r_1 \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c \cdot (r_1' \underset{k}{\otimes} 1_{\mathbf{R}}) \odot (1_{\mathbf{R}} \underset{k}{\otimes} r_2) \cdot c' \cdot (1_{\mathbf{R}} \underset{k}{\otimes} r_2').$$

*Coproduct and counit in $C \odot C'$ are*

$$c \odot c' \mapsto (c^{(1)} \odot c'^{(1)}) \underset{\mathbf{R}}{\otimes} (c^{(2)} \odot c'^{(2)}) \qquad and \qquad c \odot c' \mapsto \epsilon'\big((\epsilon(c) \underset{k}{\otimes} 1_{\mathbf{R}}) \cdot c'\big).$$

*Monoidal unit is $R \otimes_k R$, with $\mathbf{R}|\mathbf{R}$-coring structure described in Section 3.2.3 below. Monoidal product of morphisms $\alpha_i : (C_i, \Delta_i, \epsilon_i) \to (C_i', \Delta_i', \epsilon_i')$, for $i = 1, 2$, is given by*

$$(\alpha_1 \odot \alpha_2)(c_1 \odot c_2) := \alpha_1(c_1) \odot \alpha_2(c_2).$$

(2) *Monoids in the monoidal category of $\mathbf{R}|\mathbf{R}$-corings are the same as right $\mathbf{R}$-bialgebroids.*

(3) *Monoidal morphisms in the monoidal category of $\mathbf{R}|\mathbf{R}$-corings are the same as maps of $\mathbf{R} \otimes_k \mathbf{R}^{op}$-rings as well as of $\mathbf{R}$-corings.*

Considering a right $\mathbf{R}$-bialgebroid as an $\mathbf{R}|\mathbf{R}$-coring, the $\cdot|\mathbf{R}$ -bimodule structure is given by right multiplications by the source and target maps. The $\mathbf{R}|\cdot$ -bimodule structure is given by left multiplications by the source and target maps.

Theorem 3.4 was extended by Szlachányi in [71]. He has shown that $\mathbf{S}|\mathbf{R}$-corings form a bicategory, monads in which are the same as right bialgebroids. This makes it possible to define (*base changing*) morphisms of bialgebroids as bimodules for the corresponding monads. The constructions described in Section 3.4.1, and also in Definition 3.15, provide examples of bialgebroid morphisms in this sense. For more details we refer to [71].

An $\mathbf{S}|\mathbf{R}$-coring $C$ can be looked at as an $\mathbf{S} \otimes_k \mathbf{S}^{op}$-$\mathbf{R} \otimes_k \mathbf{R}^{op}$ bimodule in a canonical way. Identifying $\mathbf{S}$-bimodules with right $\mathbf{S} \otimes_k \mathbf{S}^{op}$-modules, and $\mathbf{R}$-bimodules with right $\mathbf{R} \otimes_k \mathbf{R}^{op}$-modules, there is an op-monoidal left adjoint functor $- \otimes_{\mathbf{S} \otimes_k \mathbf{S}^{op}} C : {}_{\mathbf{S}}\mathcal{M}_{\mathbf{S}} \to {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$. As a matter of fact, this correspondence establishes a bifunctor from the bicategory of $\mathbf{S}|\mathbf{R}$-corings to the 2-category of op-monoidal left adjoint functors between bimodule categories. For any two 0-cells (i.e. algebras $\mathbf{S}$ and $\mathbf{R}$) it gives an equivalence of the vertical subcategories. So, in addition to a characterisation of bialgebroids as monads in the bicategory of $\mathbf{S}|\mathbf{R}$-corings, they can be described as monads in the bicategory of op-monoidal left adjoint functors between bimodule categories. Monads in the 2-category of op-monoidal functors, (i.e. op-monoidal monads such that multiplication and unit natural transformations of the monad are compatible with the op-monoidal structure) were termed *Hopf monads* in [49] and *bimonads* in [69]. Using the latter terminology, the following characterisation of bialgebroids in [69, Theorem 4.5] is obtained.

THEOREM 3.5. *For an algebra* $\mathbf{R}$*, any right* $\mathbf{R}$*-bialgebroid induces a bimonad on* $_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ *which possesses a right adjoint. Conversely, every bimonad on* $_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ *possessing a right adjoint is naturally equivalent to a bimonad induced by a right* $\mathbf{R}$*-bialgebroid.*

Another aspect of the equivalence in Theorem 3.5 is explained in Section 3.5.

In the paper [27] by Day and Street, op-monoidal monads were studied in the more general framework of pseudo-monoids in monoidal bicategories. Based on Theorem 3.5, a description of bialgebroids as *strong monoidal morphisms between pseudo-monoids* in the monoidal bicategory of bimodules was obtained [27, Proposition 3.3].

3.2. **Examples.** In order to make the reader more familiar with the notion, in this section we list some examples of bialgebroids.

3.2.1. *Bialgebras.* Obviously, a bialgebra $B$ over a commutative ring $k$ determines a (left or right) $k$-bialgebroid in which both the source and target maps are equal to the unit map $k \to \mathbf{B}$. Note, however, that there are $k$-bialgebroids in which the source and target maps are different, or possibly they are equal but their range is not central in the total algebra, hence they are not bialgebras, see e.g. Section 4.1.4.

3.2.2. *Weak bialgebras.* A weak bialgebra over a commutative ring $k$ consists of an algebra and a coalgebra structure on the same $k$-module $B$, subject to compatibility axioms generalising the axioms of a bialgebra [54], [11]. Explicitly, the coproduct $\Delta$ is required to be multiplicative in the sense of (3.1). Unitality of the coproduct $\Delta$ and multiplicativity of the counit $\epsilon$ are weakened to the identities

$$(\Delta(1_{\mathbf{B}}) \underset{k}{\otimes} 1_{\mathbf{B}})(1_{\mathbf{B}} \underset{k}{\otimes} \Delta(1_{\mathbf{B}})) = (\Delta \underset{k}{\otimes} B) \circ \Delta(1_{\mathbf{B}}) = (1_{\mathbf{B}} \underset{k}{\otimes} \Delta(1_{\mathbf{B}}))(\Delta(1_{\mathbf{B}}) \underset{k}{\otimes} 1_{\mathbf{B}}), \qquad \text{and}$$

$$\epsilon(b1_{(1)})\epsilon(1_{(2)}b') = \epsilon(bb') = \epsilon(b1_{(2)})\epsilon(1_{(1)}b'), \qquad\qquad\qquad \text{for } b, b' \in B,$$

respectively. Here $1_{(1)} \otimes_k 1_{(2)}$ denotes $\Delta(1_{\mathbf{B}})$ (which may differ from $1_{\mathbf{B}} \otimes_k 1_{\mathbf{B}}$). The map

$$\sqcap^R : B \to B, \qquad b \mapsto 1_{(1)}\epsilon(b1_{(2)})$$

is checked to be idempotent. Its range is a subalgebra $\mathbf{R}$ of $\mathbf{B}$. $B$ is an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring, with source map given by the inclusion $\mathbf{R} \to \mathbf{B}$ and target map given by the restriction to $R$ of the map $B \to B$, $b \mapsto \epsilon(b1_{(1)})1_{(2)}$. Consider $B$ as an $\mathbf{R}$-bimodule via right multiplication by these source and target maps. Coproduct in an $\mathbf{R}$-coring $B$ is obtained by composing $\Delta : B \to B \otimes_k B$ with the canonical epimorphism $B \otimes_k B \to B \otimes_{\mathbf{R}} B$. It has a counit $\sqcap^R$. The $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring and $\mathbf{R}$-coring structures constructed on $B$ in this way constitute a right $\mathbf{R}$-bialgebroid.

A left $\mathbf{R}^{op}$-bialgebroid structure in a weak bialgebra $B$ is constructed symmetrically. Its source map is the inclusion map into $B$ of the range subalgebra of the idempotent map

$$\sqcap^L : B \to B, \qquad b \mapsto \epsilon(1_{(1)}b)1_{(2)}.$$

Coproduct is obtained by composing the weak coproduct $\Delta : B \to B \otimes_k B$ with an appropriate canonical epimorphism to an $\mathbf{R}^{op}$-module tensor product, too.

As it was observed by Schauenburg in [64] (see also [68, Sections 1.2 and 1.3]), the base algebra $\mathbf{R}$ of a weak bialgebra $B$ is *Frobenius separable*. This means the existence of a $k$-module map $R \to k$ (given by the counit $\epsilon$), possessing a dual basis $\sum_i e_i \otimes_k f_i \in R \otimes_k R$, such that $\sum_i e_i f_i = 1_{\mathbf{R}}$. The dual basis property means $\sum_i e_i \epsilon(f_i r) = r = \sum_i \epsilon(re_i)f_i$, for all $r \in R$. In a weak bialgebra a dual basis is given by $1_{(1)} \otimes_k \sqcap^R(1_{(2)}) \in R \otimes_k R$. In [64], [41] and [68] also the converse is proven: a $k$-module map $\epsilon : R \to k$ on the base algebra $\mathbf{R}$ of a bialgebroid $\mathcal{B}$, with normalised dual basis $\sum_i e_i \otimes_k f_i \in R \otimes_k R$, determines a weak bialgebra structure on the underlying $k$-module $B$. In [41, Proposition 9.3] any separable algebra over a field was proven to be Frobenius separable.

Consider a small category $\mathcal{C}$ with finitely many objects. For a commutative ring $k$, the free $k$-module generated by the morphisms in $\mathcal{C}$ carries a weak bialgebra structure. The product of two morphisms is equal to the composite morphism if they are composable, and zero otherwise. Unit element is a (finite) sum of the identity morphisms for all objects. Extending the product $k$-linearly, we obtain a $k$-algebra. Coproduct is diagonal on all morphisms, i.e. $\Delta(f) = f \otimes_k f$.

Counit maps every morphism to $1_k$. Extending the coproduct and the counit $k$-linearly, we obtain a $k$-coalgebra. The algebra and coalgebra structures constructed in this way constitute a weak bialgebra.

3.2.3. *The bialgebroid* $\mathbf{R} \otimes \mathbf{R}^{op}$. For any algebra $\mathbf{R}$ over a commutative ring $k$, a simplest possible right $\mathbf{R}$-bialgebroid is constructed on the algebra $\mathbf{R} \otimes_k \mathbf{R}^{op}$. Source and target maps are given by the inclusions

$$\mathbf{R} \to \mathbf{R} \underset{k}{\otimes} \mathbf{R}^{op}, \quad r \mapsto r \underset{k}{\otimes} 1_{\mathbf{R}} \qquad \text{and} \qquad \mathbf{R}^{op} \to \mathbf{R} \underset{k}{\otimes} \mathbf{R}^{op}, \quad r \mapsto 1_{\mathbf{R}} \underset{k}{\otimes} r,$$

respectively. Coproduct is

$$\mathbf{R} \underset{k}{\otimes} \mathbf{R}^{op} \to (\mathbf{R} \underset{k}{\otimes} \mathbf{R}^{op}) \underset{\mathbf{R}}{\otimes} (\mathbf{R} \underset{k}{\otimes} \mathbf{R}^{op}), \qquad r \underset{k}{\otimes} r' \mapsto (1_{\mathbf{R}} \underset{k}{\otimes} r') \underset{\mathbf{R}}{\otimes} (r \underset{k}{\otimes} 1_{\mathbf{R}}).$$

Counit is $\mathbf{R} \otimes_k \mathbf{R}^{op} \to R$, $r \otimes_k r' \mapsto r'r$. The corresponding $\mathbf{R}|\mathbf{R}$-coring occurred in Theorem 3.4 (1). The opposite co-opposite of the above construction yields a left $\mathbf{R}^{op}$-bialgebroid structure on $\mathbf{R}^{op} \otimes_k \mathbf{R} \cong \mathbf{R} \otimes_k \mathbf{R}^{op}$.

3.3. **Duality.** In Section 2.3 the duality between $\mathbf{R}$-rings and $\mathbf{R}$-corings has been studied. Now we shall see how it leads to a duality of bialgebroids. Recall that the axioms of a bialgebra over a commutative ring $k$ are self-dual. That is, the diagrams (in the category $\mathcal{M}_k$ of $k$-modules), expressing the bialgebra axioms, remain unchanged if all arrows are reversed. As a consequence, if a bialgebra $B$ is finitely generated and projective as a $k$-module (hence possesses a dual in $\mathcal{M}_k$), then the dual has a bialgebra structure too, which is the transpose of the bialgebra structure of $B$. In contrast to bialgebras, axioms of a bialgebroid are not self dual in the same sense. Although it follows by the considerations in Section 2.3 that the $\mathbf{R}$-dual of a finitely generated and projective bialgebroid possesses an $\mathbf{R}$-ring, and an $\mathbf{R}$-coring structure, it is not obvious that these dual structures constitute a bialgebroid. The fact that they do indeed, was shown first by Schauenburg in [62]. A detailed study can be found also in the paper [41] by Kadison and Szlachányi. Our presentation here is closer to [41, Proposition 2.5].

PROPOSITION 3.6. *Let $\mathbf{R}$ be an algebra. Consider a left bialgebroid $\mathcal{B}$ over $\mathbf{R}$, which is a finitely generated and projective left $\mathbf{R}$-module (via left multiplication by the source map). Then the left $\mathbf{R}$-dual $^*B := {}_{\mathbf{R}}\mathrm{Hom}(B, R)$ possesses a (canonical) right $\mathbf{R}$-bialgebroid structure.*

Applying part (1) of Proposition 2.11 to the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ underlying $\mathcal{B}$, we conclude on the existence of an $\mathbf{R}$-ring structure on $^*B$. Its unit map is $^*s : \mathbf{R} \to {}^*B$, $r \mapsto \epsilon(-)r$. Multiplication is given by

$$(\beta\beta')(b) = \beta'\big( t(\beta(b_{(2)}))b_{(1)} \big), \qquad \text{for } \beta, \beta' \in B,$$

where we used that the right $\mathbf{R}$-module structure in $B$ is given via the target map $t$ in $\mathcal{B}$. Applying (a symmetrical version of) part (2) of Proposition 2.11 to the $\mathbf{R}$-ring $(\mathbf{B}, s)$ underlying $\mathcal{B}$, we conclude on the existence of an $\mathbf{R}$-coring structure in $^*B$. It has a bimodule structure

$$r \cdot \beta \cdot r' = \beta\big( - s(r) \big)r' \qquad \text{for } r, r' \in R, \ \beta \in {}^*B.$$

In particular, $\epsilon \cdot r = {}^*s(r)$, for $r \in R$, as expected. A to-be-target-map is defined as $^*t(r) := r \cdot \epsilon$, for $r \in R$. Counit in the $\mathbf{R}$-coring $^*B$ is $^*\epsilon : {}^*B \to R$, $\beta \mapsto \beta(1_{\mathbf{B}})$. Coproduct is given in terms of a dual basis $(\{ a_i \in B \}, \{ \alpha_i \in {}^*B \})$ as

$$^*\Delta : {}^*B \to {}^*B \underset{\mathbf{R}}{\otimes} {}^*B, \qquad \beta \mapsto \sum_i \alpha_i \underset{\mathbf{R}}{\otimes} \beta(-a_i).$$

Right bialgebroid axioms are verified by direct computations.

One can apply Proposition 3.6 to the co-opposite left bialgebroid, which is a finitely generated and projective right $\mathbf{R}^{op}$-module via left multiplication by the target map. In this way one verifies that the right dual $B^* := \mathrm{Hom}_{\mathbf{R}}(B, R)$ of a left $\mathbf{R}$-bialgebroid $\mathcal{B}$, which is a finitely generated and projective right $\mathbf{R}$-module, possesses a right $\mathbf{R}$-bialgebroid structure $\mathcal{B}^* := (^*(\mathcal{B}_{cop}))_{cop}$. Note that (conventionally), multiplication is chosen in such a way which results in right bialgebroid structures on both duals of a left bialgebroid. Applying the constructions to the opposite bialgebroid, left and right duals of a right bialgebroid, which is a finitely generated and projective $\mathbf{R}$-module on the

appropriate side, are concluded to be left bialgebroids. Our convention is to choose $(\mathcal{B}^{op})^* := (\mathcal{B}^*)^{op}$ and $^*(\mathcal{B}^{op}) := (^*\mathcal{B})^{op}$.

3.4. **Construction of new bialgebroids from known ones.** In addition to the examples in Section 3.2, further (somewhat implicit) examples of bialgebroids are provided by various constructions starting with given bialgebroids.

3.4.1. *Drinfel'd twist.* A Drinfel'd twist of a bialgebra $B$ over a commutative ring $k$ is a bialgebra with the same algebra structure in $B$, and coproduct deformed (or *twisted*) by an invertible normalised 2-cocycle in $B$ (the so called *Drinfel'd element*). In this section we recall analogous Drinfel'd twists of bialgebroids from [71, Section 6.3]. More general twists, which do not correspond to invertible Drinfel'd elements, are studied in [75]. Such generalised twists will not be considered here.

DEFINITION 3.7. For an algebra $\mathbf{R}$, consider a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.1. An (invertible) element $J$ of the Takeuchi product $B \times_R B$ is called an *(invertible) normalised 2-cocycle in $\mathcal{B}$* provided it satisfies the following conditions.

   (i) $(t(r) \otimes_{\mathbf{R}} s(r'))J = J(t(r) \otimes_{\mathbf{R}} s(r'))$, for $r, r' \in R$, (*bilinearity*),
   (ii) $(J \otimes_{\mathbf{R}} 1_{\mathbf{B}})(\Delta \otimes_{\mathbf{R}} B)(J) = (1_B \otimes_{\mathbf{R}} J)(B \otimes_{\mathbf{R}} \Delta)(J)$ (*cocycle condition*),
   (iii) $(\epsilon \otimes_{\mathbf{R}} B)(J) = 1_{\mathbf{B}} = (B \otimes_{\mathbf{R}} \epsilon)(J)$ (*normalisation*).

PROPOSITION 3.8. *Let $J$ be an invertible normalised 2-cocycle in a right $\mathbf{R}$-bialgebroid $\mathcal{B}$. The $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ in $\mathcal{B}$, the counit $\epsilon$ of $\mathcal{B}$ and the twisted form $\Delta_J := J\Delta(-)J^{-1}$ of the coproduct $\Delta$ in $\mathcal{B}$, constitute a right $\mathbf{R}$-bialgebroid $\mathcal{B}_J$.*

3.4.2. *Cocycle double twist.* Dually to the construction in Section 3.4.1, one can leave the coproduct in a bialgebroid $\mathcal{B}$ unchanged and *twist* multiplication by an invertible normalised 2-cocycle *on* $\mathcal{B}$. For an algebra $\mathbf{R}$ over a commutative ring $k$, consider a left $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.3. Recall from Theorem 3.4 that the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-module tensor product $B \otimes_{\mathbf{R} \otimes_k \mathbf{R}^{op}} B$ (with respect to the right (resp. left) actions given by right (resp. left) multiplications by $s$ and $t$) is an $R$-coring. It has a bimodule structure $r \cdot (b \otimes b') \cdot r' := s(r)t(r')b \otimes b'$, coproduct $b \otimes b' \mapsto (b_{(1)} \otimes b'_{(1)}) \otimes_{\mathbf{R}} (b_{(2)} \otimes b'_{(2)})$ and counit $b \otimes b' \mapsto \epsilon(bb')$. Hence there is a corresponding convolution algebra $_{\mathbf{R}}\mathrm{Hom}_{\mathbf{R}}(B \otimes_{\mathbf{R} \otimes_k \mathbf{R}^{op}} B, R)$ with multiplication $(f \diamond g)(b \otimes b') := f(b_{(1)} \otimes b'_{(1)})g(b_{(2)} \otimes b'_{(2)})$.

DEFINITION 3.9. Let $\mathbf{R}$ be an algebra over a commutative ring $k$ and let $\mathcal{B}$ be a left $\mathbf{R}$-bialgebroid, with structure maps denoted as in Definition 3.3. An (invertible) element of the convolution algebra $_{\mathbf{R}}\mathrm{Hom}_{\mathbf{R}}(B \otimes_{\mathbf{R} \otimes_k \mathbf{R}^{op}} B, R)$ is called an *(invertible) normalised 2-cocycle on $\mathcal{B}$* provided it satisfies the following conditions, for $b, b', b'' \in B$ and $r, r' \in R$.

   (i) $\sigma(s(r)t(r')b, b') = r\sigma(b, b')r'$ (*bilinearity*),
   (ii) $\sigma(b, s(\sigma(b'_{(1)}, b''_{(1)}))b'_{(2)}b''_{(2)}) = \sigma(s(\sigma(b_{(1)}, b'_{(1)}))b_{(2)}b'_{(2)}, b'')$ (*cocycle condition*),
   (iii) $\sigma(1_{\mathbf{B}}, b) = \epsilon(b) = \sigma(b, 1_{\mathbf{B}})$ (*normalisation*).

PROPOSITION 3.10. *Let $\sigma$ be an invertible normalised 2-cocycle on a left $\mathbf{R}$-bialgebroid $\mathcal{B}$, with inverse $\widetilde{\sigma}$. The source and target maps $s$ and $t$ in $\mathcal{B}$, the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ in $\mathcal{B}$, and the twisted product $b \cdot_\sigma b' := s(\sigma(b_{(1)}, b'_{(1)}))t(\widetilde{\sigma}(b_{(3)}, b'_{(3)}))b_{(2)}b'_{(2)}$, for $b, b' \in B$, constitute a left $\mathbf{R}$-bialgebroid $\mathcal{B}^\sigma$.*

3.4.3. *Duality.* The constructions in Section 3.4.1 and Section 3.4.2 are dual of each other, in the following sense.

PROPOSITION 3.11. *For an algebra $\mathbf{R}$, let $\mathcal{B}$ be a left $\mathbf{R}$-bialgebroid which is a finitely generated and projective right $\mathbf{R}$-module via left multiplication by the target map $t$, and consider the right dual right $\mathbf{R}$-bialgebroid $\mathcal{B}^*$. The following statements hold.*

*(1) An element $J = \sum_k \xi_k \otimes_{\mathbf{R}} \zeta_k \in B^* \times_{\mathbf{R}} B^*$ is an invertible normalised 2-cocycle in $\mathcal{B}^*$ if and only if*

$$(3.7) \qquad \sigma_J : B \underset{\mathbf{R} \otimes \mathbf{R}^{op}}{\otimes} B \to R, \qquad \sigma_J(b, b') := \sum_k \xi_k\big(bt(\zeta_k(b'))\big)$$

*is an invertible normalised 2-cocycle on $\mathcal{B}$.*

*(2) Assume the equivalent properties in part (1). The right bialgebroid $(\mathcal{B}^*)_J$, obtained by twisting the coproduct of $\mathcal{B}^*$ by the cocycle $J$, is right dual of the left bialgebroid $\mathcal{B}^{\sigma_J}$, obtained by twisting the product in $\mathcal{B}$ by the cocycle $\sigma_J$ in (3.7).*

The inverse of the construction in (3.7) is given by associating to an invertible normalised 2-cocycle $\sigma$ on $\mathcal{B}$ an invertible normalised 2-cocycle in $\mathcal{B}^*$: $J_\sigma := \sum_i \sigma(-, a_i) \otimes_{\mathbf{R}} \alpha_i$, where $(\{\, a_i \in B \,\}, \{\, \alpha_i \in B^* \,\})$ is a dual basis.

3.4.4. *Drinfel'd double.* For a Hopf algebra $B$ over a commutative ring $k$, such that $B$ is a finitely generated and projective $k$-module, the $k$-module $\mathcal{D}(B) := B \otimes_k B^*$ has a bialgebra (in fact Hopf algebra) structure. It is known as the *Drinfel'd double* of $B$. The category of $\mathcal{D}(B)$-modules is isomorphic (as a monoidal category) to the category of Yetter-Drinfel'd modules for $B$ and also to the monoidal center of the category of $B$-modules. These results were extended to certain bialgebroids by Schauenburg in [62].

In this section let $\mathcal{B}$ be a left bialgebroid over a $k$-algebra $\mathbf{R}$, finitely generated and projective as a right $\mathbf{R}$-module. Assume in addition that the following map is bijective.

$$(3.8) \qquad \vartheta : B \underset{\mathbf{R}^{op}}{\otimes} B \to B \underset{\mathbf{R}}{\otimes} B, \qquad b \underset{\mathbf{R}^{op}}{\otimes} b' \mapsto b_{(1)} \underset{\mathbf{R}}{\otimes} b_{(2)} b',$$

where in the domain of $\vartheta$ module structures are given by right and left multiplications by the target map, and in the codomain module structures are given by left multiplications by the target and source maps. Left bialgebroids, for which the map (3.8) is bijective, were named *(left) $\times_R$-Hopf algebras* in [62] and they are discussed in more detail in Section 4.6.2. In following Proposition 3.12, Sweedler's index notation is used for the coproducts, and also the index notation $\vartheta^{-1}(b \otimes_{\mathbf{R}} 1_{\mathbf{B}}) = b^{\langle 1 \rangle} \otimes_{\mathbf{R}^{op}} b^{\langle 2 \rangle}$, where implicit summation is understood.

PROPOSITION 3.12. *Let $\mathbf{R}$ be an algebra over a commutative ring $k$. Let $\mathcal{B}$ be a left $\times_R$-Hopf algebra which is a finitely generated and projective right $\mathbf{R}$-module via left multiplication by the target map. Denote the structure maps in $\mathcal{B}$ as in Definition 3.3. Consider $B$ as a right $\mathbf{R} \otimes_k \mathbf{R}^{op}$-module via right multiplications by the source and target maps $s$ and $t$ in the left bialgebroid $\mathcal{B}$. Consider the right dual $B^*$ as a left $\mathbf{R} \otimes_k \mathbf{R}^{op}$-module via right multiplications by the target and source maps $t^*$ and $s^*$ in the right bialgebroid $\mathcal{B}^*$. The tensor product $\mathcal{D}(\mathcal{B}) := B \otimes_{\mathbf{R} \otimes \mathbf{R}^{op}} B^*$ has a left $\mathbf{R}$-bialgebroid structure, as follows. Multiplication is given by*

$$(b \bowtie \beta)(b' \bowtie \beta') := bs\big(\beta^{(1)}(b'_{(1)})\big){b'_{(2)}}^{\langle 1 \rangle} \bowtie \beta'\beta^{(2)} s^*\big(\beta^{(3)}({b'_{(2)}}^{\langle 2 \rangle})\big), \qquad for \ b \bowtie \beta, \ b' \bowtie \beta' \in \mathcal{D}(\mathcal{B}).$$

*Source and target maps are*

$$\mathbf{R} \to \mathcal{D}(\mathcal{B}), \qquad r \mapsto 1_{\mathbf{B}} \bowtie t^*(r) \qquad and \qquad \mathbf{R}^{op} \to \mathcal{D}(\mathcal{B}), \qquad r \mapsto 1_{\mathbf{B}} \bowtie s^*(r),$$

*respectively. Coproduct is*

$$\mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{B}) \underset{\mathbf{R}}{\otimes} \mathcal{D}(\mathcal{B}), \qquad b \bowtie \beta \mapsto (b_{(1)} \bowtie \beta^{(1)}) \underset{\mathbf{R}}{\otimes} (b_{(2)} \bowtie \beta^{(2)}),$$

*and counit is $\mathcal{D}(\mathcal{B}) \to R$, $b \bowtie \beta \mapsto \epsilon\big(bs(\beta(1_{\mathbf{B}}))\big)$.*

It is not known if the Drinfel'd double (or the dual) of a $\times_R$-Hopf algebra is a $\times_R$-Hopf algebra too.

3.4.5. *Morita base change.* In contrast to the previous sections, constructions in this section and in the forthcoming ones *change the base algebra of a bialgebroid.*

Let $\mathbf{R}$ be an algebra over a commutative ring $k$ and let $\mathcal{B}$ be a left $\mathbf{R}$-bialgebroid. Denote the structure maps of $\mathcal{B}$ as in Definition 3.3. Let $\widetilde{\mathbf{R}}$ be a $k$-algebra which is Morita equivalent to $\mathbf{R}$. Fix a strict Morita context $(\mathbf{R}, \widetilde{\mathbf{R}}, P, Q, \bullet, \circ)$. Denote the inverse image of $1_{\mathbf{R}}$ under the map $\bullet$ by $\sum_i q_i \otimes_{\widetilde{\mathbf{R}}} p^i \in Q \otimes_{\widetilde{\mathbf{R}}} P$ and denote the inverse image of $1_{\widetilde{\mathbf{R}}}$ under the map $\circ$ by $\sum_j p_j \otimes_{\mathbf{R}} q^j \in P \otimes_{\mathbf{R}} Q$. The $\widetilde{\mathbf{R}}$-$\mathbf{R}$ bimodule $P$ determines a canonical $\mathbf{R}^{op}$-$\widetilde{\mathbf{R}}^{op}$ bimodule $\overline{P}$. Similarly, there is a canonical $\widetilde{\mathbf{R}}^{op}$-$\mathbf{R}^{op}$ bimodule $\overline{Q}$. In [63, Section 5], a left $\widetilde{\mathbf{R}}$-bialgebroid

structure was constructed on the $k$-module $\widetilde{B} := (P \otimes_k \overline{Q}) \otimes B \otimes (Q \otimes_k \overline{P})$, where unadorned tensor product is meant over $\mathbf{R} \otimes_k \mathbf{R}^{op}$. Multiplication is given by

$$[(p_1 \otimes q_1) \otimes b \otimes (q_2 \otimes p_2)][(p_1' \otimes q_1') \otimes b' \otimes (q_2' \otimes p_2')] := (p_1 \otimes q_1) \otimes bs(q_2 \bullet p_1')t(q_1' \bullet p_2)b' \otimes (q_2' \otimes p_2').$$

Source and target maps in $\widetilde{\mathcal{B}}$ are, for $\widetilde{r} \in \widetilde{R}$,

$$\widetilde{r} \mapsto \sum_{j,j'} (\widetilde{r} \cdot p_{j'} \otimes q^j) \otimes 1_\mathbf{B} \otimes (q^{j'} \otimes p_j) \qquad \text{and} \qquad \widetilde{r} \mapsto \sum_{j,j'} (p_{j'} \otimes q^j \cdot \widetilde{r}) \otimes 1_\mathbf{B} \otimes (q^{j'} \otimes p_j),$$

respectively. Coproduct and counit are given on an element $(p_1 \otimes q_1) \otimes b \otimes (q_2 \otimes p_2) \in \widetilde{B}$ as

$$(p_1 \otimes q_1) \otimes b \otimes (q_2 \otimes p_2) \mapsto \sum_{i,j} [(p_1 \otimes q_i) \otimes b_{(1)} \otimes (q_2 \otimes p_j)] \underset{\widetilde{\mathbf{R}}}{\otimes} [(p^i \otimes q_1) \otimes b_{(2)} \otimes (q^j \otimes p_2)],$$

$$(p_1 \otimes q_1) \otimes b \otimes (q_2 \otimes p_2) \mapsto p_1 \cdot \epsilon\big(bs(q_2 \bullet p_2)\big) \circ q_1,$$

respectively. Generalisation, and a more conceptual background of Morita base change in bialgebroids, are presented in Section 3.5.

3.4.6. *Connes-Moscovici's bialgebroids.* The following bialgebroid was constructed in [26] in the framework of transverse geometry. Let $H$ be a Hopf algebra over a commutative ring $k$, with coproduct $\delta : h \mapsto h_{(1)} \otimes_k h_{(2)}$ and counit $\varepsilon$. Let $\mathbf{R}$ be a left $H$-module algebra. Consider the $k$-module $B := R \otimes_k H \otimes_k R$. It can be equipped with an associative multiplication

$$(r_1 \underset{k}{\otimes} h \underset{k}{\otimes} r_2)(r_1' \underset{k}{\otimes} h' \underset{k}{\otimes} r_2') := r_1(h_{(1)} \cdot r_1') \underset{k}{\otimes} h_{(2)}h' \underset{k}{\otimes} (h_{(3)} \cdot r_2')r_2,$$

with unit $1_\mathbf{R} \otimes_k 1_\mathbf{H} \otimes_k 1_\mathbf{R}$. The algebra $\mathbf{B}$ can be made a left $\mathbf{R}$-bialgebroid with source and target maps

$$\mathbf{R} \to \mathbf{B}, \quad r \mapsto r \underset{k}{\otimes} 1_\mathbf{H} \underset{k}{\otimes} 1_\mathbf{R} \qquad \text{and} \qquad \mathbf{R}^{op} \to \mathbf{B}, \quad r \mapsto 1_\mathbf{R} \underset{k}{\otimes} 1_\mathbf{H} \underset{k}{\otimes} r,$$

respectively. Coproduct and counit are

$$r \underset{k}{\otimes} h \underset{k}{\otimes} r' \mapsto (r \underset{k}{\otimes} h_{(1)} \underset{k}{\otimes} 1_\mathbf{R}) \underset{\mathbf{R}}{\otimes} (1_\mathbf{R} \underset{k}{\otimes} h_{(2)} \underset{k}{\otimes} r') \qquad \text{and} \qquad r \underset{k}{\otimes} h \underset{k}{\otimes} r' \mapsto r\varepsilon(h)r'.$$

3.4.7. *Scalar extension.* Let $B$ be a bialgebra over a commutative ring $k$ and let $A$ be an algebra in the category of right-right Yetter-Drinfel'd modules of $B$. Recall that this means that $A$ is a right $B$-module algebra and a right $B$-comodule algebra such that the following compatibility condition holds, for $a \in A$ and $b \in B$.

$$(a \cdot b_{(2)})_{[0]} \underset{k}{\otimes} b_{(1)}(a \cdot b_{(2)})_{[1]} = a_{[0]} \cdot b_{(1)} \underset{k}{\otimes} a_{[1]}b_{(2)}.$$

The category of right-right Yetter-Drinfel'd modules is pre-braided. Assume that $A$ is braided commutative, i.e., for $a, a' \in A$, the identity $a_{[0]}'(a \cdot a_{[1]}') = aa'$ holds. Under these assumptions, it follows by a symmetrical version of [18, Theorem 4.1] that the smash product algebra has a right $\mathbf{A}$-bialgebroid structure. Recall that the smash product algebra is the $k$-module $A\#B := A \otimes_k B$, with multiplication $(a\#b)(a'\#b') := a'(a \cdot b_{(1)}') \underset{k}{\otimes} bb_{(2)}'$. Source and target maps are

$$s : \mathbf{A} \to A\#B, \qquad a \mapsto a_{[0]}\#a_{[1]} \qquad \text{and} \qquad t : \mathbf{A}^{op} \to A\#B, \qquad a \mapsto a\#1_\mathbf{B},$$

respectively. Coproduct is

$$\Delta : A\#B \to (A\#B) \underset{\mathbf{A}}{\otimes} (A\#B), \qquad a\#b \mapsto (a\#b_{(1)}) \underset{\mathbf{A}}{\otimes} (1_\mathbf{A}\#b_{(2)})$$

and counit is given in terms of the counit $\varepsilon$ in $B$ as $A\#B \to A$, $a\#b \mapsto a\varepsilon(b)$. The name *'scalar extension'* comes from the feature that the base algebra $k$ of $B$ (the subalgebra of 'scalars') becomes replaced by the base algebra $\mathbf{A}$ of $A\#B$.

A solution $\mathcal{R}$ of the quantum Yang-Baxter equation on a finite dimensional vector space determines a bialgebra $B(\mathcal{R})$ via the so called FRT construction. In [18, Proposition 4.3] a braided commutative algebra in the category of Yetter-Drinfel'd modules for $B(\mathcal{R})$ was constructed, thus a bialgebroid was associated to a finite dimensional solution $\mathcal{R}$ of the quantum Yang-Baxter equation.

Above construction of a scalar extension was extended in [4, Theorem 4.6]. Following it, a smash product of a right bialgebroid $\mathcal{B}$ over an algebra $\mathbf{R}$, with a braided commutative algebra

$A$ in the category of right-right Yetter-Drinfel'd modules for $\mathcal{B}$, is shown to possess a right $\mathbf{A}$-bialgebroid structure. The fundamental importance of scalar extensions from the point of view of Galois extensions by bialgebroids is discussed in Section 3.7.

3.5. **The monoidal category of modules.** An algebra $\mathbf{B}$ over a commutative ring $k$ is known to have a $k$-bialgebra structure if and only if the category of (left or right) $\mathbf{B}$-modules is monoidal such that the forgetful functor to $\mathcal{M}_k$ is strict monoidal [56]. Generalisation [60, Theorem 5.1] of this fact to bialgebroids is due to Schauenburg. Recall that any (right) module of an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$ is an $\mathbf{R}$-bimodule via the actions by $t(r)$ and $s(r)$, for $r \in R$.

THEOREM 3.13. *For an algebra $\mathbf{R}$ over a commutative ring $k$, the following data are equivalent on an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$.*
    *(1) A right bialgebroid structure on $(\mathbf{B}, s, t)$;*
    *(2) A monoidal structure on the category $\mathcal{M}_{\mathbf{B}}$ of right $\mathbf{B}$-modules, such that the forgetful functor $\mathcal{M}_{\mathbf{B}} \to {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ is strict monoidal.*

Applying Theorem 3.13 to the opposite $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring, an analogous equivalence is obtained between left bialgebroid structures and monoidal structures on the category of left modules. At the heart of Theorem 3.13 lies the fact that the right regular $\mathbf{B}$-module is a generator in the category $\mathcal{M}_{\mathbf{B}}$. Hence the $\mathbf{B}$-module structure on the $\mathbf{R}$-module tensor product of any two $\mathbf{B}$-modules can be expressed in terms of the action on $1_{\mathbf{B}} \otimes_{\mathbf{R}} 1_{\mathbf{B}} \in B \otimes_{\mathbf{R}} B$, which defines a coproduct. In terms of a coproduct $b \mapsto b^{(1)} \otimes_{\mathbf{R}} b^{(2)} := (1_{\mathbf{B}} \otimes_{\mathbf{R}} 1_{\mathbf{B}}) \cdot b$, the right $\mathbf{B}$-action in the $\mathbf{R}$-module tensor product of two right $\mathbf{B}$-modules $M$ and $N$ is written as

$$(3.9) \qquad (m \underset{\mathbf{R}}{\otimes} n) \cdot b = m \cdot b^{(1)} \underset{\mathbf{R}}{\otimes} n \cdot b^{(2)}, \qquad \text{for } m \underset{\mathbf{R}}{\otimes} n \in M \underset{\mathbf{R}}{\otimes} N, \ b \in B.$$

A $\mathbf{B}$-module structure on the monoidal unit $R$ is equivalent to a right character $\epsilon : B \to R$ by Lemma 2.5.

Recall that, for an $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $(\mathbf{B}, s, t)$, the category $\mathcal{M}_{\mathbf{B}}$ is isomorphic to the category of algebras for the monad $- \otimes_{\mathbf{R} \otimes \mathbf{R}^{op}} B$ on ${}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ (where $\mathbf{R}$-bimodules are considered as right $\mathbf{R} \otimes_k \mathbf{R}^{op}$-modules). In light of this fact, Theorem 3.13 is a particular case of a question discussed by Moerdijk in [49]: Having a monad $\mathbb{B}$ on a monoidal category $\mathcal{M}$, the monoidal structure of $\mathcal{M}$ lifts to a monoidal structure on the category $\mathcal{M}_{\mathbb{B}}$ of $\mathbb{B}$-algebras (in the sense that the forgetful functor $\mathcal{M}_{\mathbb{B}} \to \mathcal{M}$ is strict monoidal, as in part (2) in Theorem 3.13) if and only if $\mathbb{B}$ is a monoid in the category of op-monoidal endofunctors on $\mathcal{M}$, i.e. a *Hopf monad* in the terminology of [49] (called a *bimonad* in [69]). Comparing this result with Theorem 3.13, we obtain another evidence for a characterisation of bialgebroids as bimonads in Theorem 3.5.

In the paper [71] by Szlachányi it was investigated what bialgebroids possess monoidally equivalent module categories. That is, a *monoidal Morita theory* of bialgebroids was developed. Based on Theorem 3.5, one main result in [71] can be reformulated as in Theorem 3.14 below. Recall that a bimodule, for two monads $\mathbb{B} : \mathcal{M} \to \mathcal{M}$ and $\mathbb{B}' : \mathcal{M}' \to \mathcal{M}'$, is a functor $\mathbb{M} : \mathcal{M} \to \mathcal{M}'$ together with natural transformations $\varrho : \mathbb{M}\mathbb{B} \to \mathbb{M}$ and $\lambda : \mathbb{B}'\mathbb{M} \to \mathbb{M}$, satisfying the usual compatibility conditions for the right and left actions in a bimodule. If any pair of parallel morphisms in $\mathcal{M}'$ possesses a coequaliser, and $\mathbb{B}'$ preserves coequalisers, then the bimodule $\mathbb{M}$ induces a functor $\overline{\mathbb{M}} : \mathcal{M}_{\mathbb{B}} \to \mathcal{M}'_{\mathbb{B}'}$, between the categories of algebras for $\mathbb{B}$ and $\mathbb{B}'$, respectively, with object map $(V, v) \mapsto \text{Coeq}(\mathbb{M}(v), \varrho_V)$. If both categories $\mathcal{M}$ and $\mathcal{M}'$ possess coequalisers and both $\mathbb{B}$ and $\mathbb{B}'$ preserve them, then one can define the inverse of a bimodule as in Morita's theory. A $\mathbb{B}'$-$\mathbb{B}$ bimodule $\mathbb{M}'$ is said to be the inverse of $\mathbb{M}$ provided that $\overline{\mathbb{M}'}\overline{\mathbb{M}}$ is naturally equivalent to the identity functor on $\mathcal{M}_{\mathbb{B}}$ and $\overline{\mathbb{M}}\overline{\mathbb{M}'}$ is naturally equivalent to the identity functor on $\mathcal{M}_{\mathbb{B}'}$.

THEOREM 3.14. *For two right bialgebroids $\mathcal{B}$ and $\mathcal{B}'$, over respective base algebras $\mathbf{R}$ and $\mathbf{R}'$, the right module categories $\mathcal{M}_{\mathbf{B}}$ and $\mathcal{M}_{\mathbf{B}'}$ are monoidally equivalent if and only if there exists an invertible bimodule in the 2-category of op-monoidal left adjoint functors, for the monads $- \otimes_{\mathbf{R} \otimes \mathbf{R}^{op}} B : {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}} \to {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ and $- \otimes_{\mathbf{R}' \otimes \mathbf{R}'^{op}} B' : {}_{\mathbf{R}'}\mathcal{M}_{\mathbf{R}'} \to {}_{\mathbf{R}'}\mathcal{M}_{\mathbf{R}'}$.*

By standard Morita theory, an equivalence $\overline{\mathbb{M}} : \mathcal{M}_\mathbf{B} \to \mathcal{M}_{\mathbf{B}'}$ is of the form $\overline{\mathbb{M}} = - \otimes_\mathbf{B} M$, for some invertible $\mathbf{B}$-$\mathbf{B}'$ bimodule $M$. In [71] monoidality of the equivalence is translated to properties of the Morita equivalence bimodule $M$.

In [73, Definition 2.1] two algebras $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ over a commutative ring $k$ were said to be $\sqrt{Morita}$-*equivalent* whenever the bimodule categories $_\mathbf{R}\mathcal{M}_\mathbf{R}$ and $_{\widetilde{\mathbf{R}}}\mathcal{M}_{\widetilde{\mathbf{R}}}$ are strictly equivalent as $k$-linear monoidal categories. This property implies that the algebras $\mathbf{R} \otimes_k \mathbf{R}^{op}$ and $\widetilde{\mathbf{R}} \otimes_k \widetilde{\mathbf{R}}^{op}$ are $\sqrt{}$Morita-equivalent (but not conversely). In this situation, any $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring $B$ (i.e. monoid in the category of $\mathbf{R} \otimes_k \mathbf{R}^{op}$-bimodules) determines an $\widetilde{\mathbf{R}} \otimes_k \widetilde{\mathbf{R}}^{op}$-ring $\widetilde{B}$, with underlying $k$-algebra $\widetilde{\mathbf{B}}$ Morita equivalent to $\mathbf{B}$. If $\mathcal{B}$ is a right $\mathbf{R}$-bialgebroid then the forgetful functor $\mathcal{M}_\mathbf{B} \to {}_\mathbf{R}\mathcal{M}_\mathbf{R}$ is strict monoidal by Theorem 3.13. Hence the equivalence $\mathcal{M}_\mathbf{B} \cong \mathcal{M}_{\widetilde{\mathbf{B}}}$ can be used to induce a monoidal structure on $\mathcal{M}_{\widetilde{\mathbf{B}}}$ such that the forgetful functor $\mathcal{M}_{\widetilde{\mathbf{B}}} \to {}_{\widetilde{\mathbf{R}}}\mathcal{M}_{\widetilde{\mathbf{R}}}$ is strict monoidal. By Theorem 3.13 we conclude that there is a right $\widetilde{\mathbf{R}}$-bialgebroid structure on $\widetilde{B}$. In [63], the bialgebroid $\widetilde{\mathcal{B}}$ was said to be obtained from $\mathcal{B}$ via $\sqrt{Morita}$-*base change*. Since Morita equivalent algebras are also $\sqrt{}$Morita-equivalent (but not conversely), the construction in Section 3.4.5 is a special instance of a $\sqrt{}$Morita-base change.

For a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.1, consider $B$ as an $\mathbf{R}$-bimodule (or $\mathbf{R}^{op}$-bimodule) via right multiplications by the source map $s$ and the target map $t$. Both $B \otimes_\mathbf{R} B$ and $B \otimes_{\mathbf{R}^{op}} B$ are left modules for $\mathbf{B} \otimes_k \mathbf{B}$ via the regular actions on the two factors. Associated to $\mathcal{B}$, we construct a category $\mathcal{C}(\mathcal{B})$, with two objects $\circ$ and $\bullet$. Morphisms with source $\circ$ are elements of $B \otimes_\mathbf{R} B$ and morphisms with source $\bullet$ are elements of $B \otimes_{\mathbf{R}^{op}} B$. Morphisms $F$, with target $\circ$ and $\bullet$, are required to satisfy the following ($\mathbf{R}$-centralising) conditions $(T\circ)$ and $(T\bullet)$, respectively, for all $r \in R$.

$$(T\circ) \qquad\qquad (s(r) \underset{k}{\otimes} 1_\mathbf{B}) \cdot F = (1_\mathbf{B} \underset{k}{\otimes} t(r)) \cdot F,$$
$$(T\bullet) \qquad\qquad (t(r) \underset{k}{\otimes} 1_\mathbf{B}) \cdot F = (1_\mathbf{B} \underset{k}{\otimes} s(r)) \cdot F.$$

Via composition given by factorwise multiplication, $\mathcal{C}(\mathcal{B})$ is a category. Unit morphisms at the objects $\circ$ and $\bullet$ are $1_\mathbf{B} \otimes_\mathbf{R} 1_\mathbf{B}$ and $1_\mathbf{B} \otimes_{\mathbf{R}^{op}} 1_\mathbf{B}$, respectively. The range of the coproduct $\Delta$ lies in $\mathrm{Hom}(\circ, \circ) = B \times_R B$ and the range of the co-opposite coproduct $\Delta_{cop}$ lies in $\mathrm{Hom}(\bullet, \bullet) = B_{cop} \times_{R^{op}} B_{cop}$. In terms of the category $\mathcal{C}(\mathcal{B})$, definition of a quasi-triangular bialgebroid as formulated in [30, Proposition 3.13] can be described as follows.

DEFINITION 3.15. For a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.1, let $\mathcal{C}(\mathcal{B})$ be the category constructed above. An invertible morphism $\mathcal{R} = \mathcal{R}^1 \otimes_{\mathbf{R}^{op}} \mathcal{R}^2 \in \mathrm{Hom}(\bullet, \circ)$ (where implicit summation is understood) is a *universal R-matrix* provided that for any $b \in B$ the following identity holds in $\mathrm{Hom}(\bullet, \circ)$

$$\Delta(b)\mathcal{R} = \mathcal{R}\Delta_{cop}(b)$$

and

$$(\Delta_{cop} \underset{\mathbf{R}^{op}}{\otimes} B)(\mathcal{R}) = \mathcal{R} \triangleleft \mathcal{R} \qquad \text{and} \qquad (B \underset{\mathbf{R}^{op}}{\otimes} \Delta_{cop})(\mathcal{R}) = \mathcal{R} \triangleright \mathcal{R},$$

where the (well defined) maps

$$- \triangleleft \mathcal{R} : B \underset{\mathbf{R}^{op}}{\otimes} B \to B \underset{\mathbf{R}^{op}}{\otimes} B \underset{\mathbf{R}^{op}}{\otimes} B, \qquad b \underset{\mathbf{R}^{op}}{\otimes} b' \mapsto b \underset{\mathbf{R}^{op}}{\otimes} \mathcal{R}^1 \underset{\mathbf{R}^{op}}{\otimes} b'\mathcal{R}^2 \qquad \text{and}$$
$$\mathcal{R} \triangleright - : B \underset{\mathbf{R}^{op}}{\otimes} B \to B \underset{\mathbf{R}^{op}}{\otimes} B \underset{\mathbf{R}^{op}}{\otimes} B, \qquad b \underset{\mathbf{R}^{op}}{\otimes} b' \mapsto b\mathcal{R}^1 \underset{\mathbf{R}^{op}}{\otimes} \mathcal{R}^2 \underset{\mathbf{R}^{op}}{\otimes} b'$$

are used. A right bialgebroid $\mathcal{B}$ with a given universal R-matrix $\mathcal{R}$ is called a *quasi-triangular* bialgebroid.

Following Theorem 3.16 was obtained in [30, Theorem 3.15], as a generalisation of an analogous result for quasi-triangular bialgebras.

THEOREM 3.16. *Consider a quasi-triangular right bialgebroid* $(\mathcal{B}, \mathcal{R})$ *over a base algebra* $\mathbf{R}$. *The monoidal category of right B-modules is braided, with braiding natural isomorphism*

$$M \underset{\mathbf{R}}{\otimes} M' \to M' \underset{\mathbf{R}}{\otimes} M, \qquad m \underset{\mathbf{R}}{\otimes} m' \mapsto m' \cdot \mathcal{R}^2 \underset{\mathbf{R}}{\otimes} m \cdot \mathcal{R}^1.$$

3.6. **The monoidal category of comodules.** For a bialgebra over a commutative ring $k$, not only the category of modules, but also the category of (left or right) comodules has a monoidal structure, such that the forgetful functor to $\mathcal{M}_k$ is strict monoidal. In trying to prove an analogue of this result for bialgebroids, the first question is to find a forgetful functor. A right, say, comodule of (the constituent coring in) an **R**-bialgebroid is by definition only a right **R**-module. In order to obtain a forgetful functor to the *monoidal* category of **R**-*bimodules*, following [57, Lemma 1.4.1] is needed.

LEMMA 3.17. *Let* **R** *and* **S** *be two algebras over a commutative ring $k$ and let $C$ be an* **S**|**R***-coring. Any right comodule $(M, \varrho^M)$ of the* **R***-coring $C$ can be equipped with a unique left* **S***-module structure such that $\varrho^M(m)$ belongs to the center of the* **S***-bimodule $M \otimes_\mathbf{R} C$, for every $m \in M$. This unique left* **S***-action makes $M$ an* **S**-**R** *bimodule. Every $C$-comodule map becomes an* **S**-**R** *bimodule map. That is, there is a forgetful functor $\mathcal{M}^C \to {}_\mathbf{S}\mathcal{M}_\mathbf{R}$.*

The left **S**-action on a right comodule $(M, \varrho^M)$ of the **R**-coring $C$ is constructed as

$$(3.10) \qquad s \cdot m := m^{[0]} \cdot \epsilon(m^{[1]} \cdot (s \underset{k}{\otimes} 1_\mathbf{R})), \qquad \text{for } s \in \mathbf{S}, \ m \in M.$$

In particular, (3.10) can be used to equip a right comodule of a right **R**-bialgebroid with an **R**-**R** bimodule structure. Applying the construction to co-opposite and opposite bialgebroids, forgetful functors are obtained from categories of left and right comodules of left and right **R**-bialgebroids to ${}_\mathbf{R}\mathcal{M}_\mathbf{R}$.

THEOREM 3.18. *Let* **R** *be an algebra and let $\mathcal{B}$ be a right* **R**-*bialgebroid. The category $\mathcal{M}^B$ of right $\mathcal{B}$-comodules is monoidal, such that the forgetful functor $\mathcal{M}^B \to {}_\mathbf{R}\mathcal{M}_\mathbf{R}$ is strict monoidal.*

The monoidal unit $R$ in ${}_\mathbf{R}\mathcal{M}_\mathbf{R}$ is a right $\mathcal{B}$-comodule, via a coaction provided by the source map. One has to verify that, for any two right $\mathcal{B}$-comodules $M$ and $N$, the diagonal coaction

$$(3.11) \qquad M \underset{\mathbf{R}}{\otimes} N \to M \underset{\mathbf{R}}{\otimes} N \underset{\mathbf{R}}{\otimes} B, \qquad m \underset{\mathbf{R}}{\otimes} n \mapsto m^{[0]} \underset{\mathbf{R}}{\otimes} n^{[0]} \underset{\mathbf{R}}{\otimes} m^{[1]} n^{[1]}$$

is well defined and that the coherence natural isomorphisms in ${}_\mathbf{R}\mathcal{M}_\mathbf{R}$ are $\mathcal{B}$-comodule maps.

We do not know about a converse of Theorem 3.18, i.e. an analogue of the correspondence (2)⇒(1) in Theorem 3.13 for the category of comodules. A reason for this is that (in contrast to modules of an **R**-bialgebroid $\mathcal{B}$, which are algebras for the monad $- \otimes_{\mathbf{R} \otimes \mathbf{R}^{op}} B$ on the monoidal category of **R**-bimodules), it is not known if comodules can be described as coalgebras of a comonad on ${}_\mathbf{R}\mathcal{M}_\mathbf{R}$.

However, the definition of a bialgebroid can be dualised in the sense of reversing all arrows in the diagrams in $\mathcal{M}_k$, expressing the axioms of a bialgebroid over a $k$-algebra. For a flat $k$-coalgebra $C$, (e.g. when $k$ is a field), $C$-bicomodules constitute a monoidal category ${}^C\mathcal{M}^C$. Monoidal structure is given by *cotensor products* – a notion dual to a module tensor product. That is, for $C$-bicomodules $M$ and $N$, the cotensor product $M\square_C N$ is the equaliser of the maps $\varrho^M \otimes_k N$ and $M \otimes_k {}^N\varrho$, where $\varrho^M$ is the right coaction on $M$ and ${}^N\varrho$ is the left coaction on $N$. Flatness of the $k$-module $C$ implies that $M\square_C N$ is a $C$-bicomodule via the left $C$-coaction on $M$ and right $C$-coaction on $N$. In this case, dualisation of the bialgebroid axioms leads to the notion of a *bicoalgebroid* over the $k$-coalgebra $C$, see [18]. The relation between $C$-bicoalgebroid structures on a comonoid in ${}^C\mathcal{M}^C$, and strict monoidal structures on the forgetful functor from its comodule category to ${}^C\mathcal{M}^C$, is studied in [2] and [3].

Applying Theorem 3.18 to the co-opposite bialgebroid, we conclude on the strict monoidality of the forgetful functor ${}^B\mathcal{M} \to {}_{\mathbf{R}^{op}}\mathcal{M}_{\mathbf{R}^{op}}$, for a right **R**-bialgebroid $\mathcal{B}$. Applying Theorem 3.18 to opposite bialgebroids, it follows that the forgetful functors ${}^B\mathcal{M} \to {}_\mathbf{R}\mathcal{M}_\mathbf{R}$ and $\mathcal{M}^B \to {}_{\mathbf{R}^{op}}\mathcal{M}_{\mathbf{R}^{op}}$ are strict monoidal, for a left **R**-bialgebroid $\mathcal{B}$. Note that, for an **R**-bialgebroid $\mathcal{B}$ which is a finitely generated and projective **R**-module on the appropriate side, the equivalence in Proposition 2.12 between the categories of comodules for $\mathcal{B}$ and modules for its dual is strict (anti-)monoidal.

The reader should be warned that in the paper [15] a different notion of a comodule is used. For an **S**|**R**-coring $C$, the coproduct and the counit of the **R**-coring $C$ project to a coproduct and a counit on the quotient **R**-bimodule $C/\{ (s \otimes_k 1_\mathbf{R}) \cdot c - c \cdot (s \otimes_k 1_\mathbf{R}) \mid c \in C, \ s \in S \}$.

Applying the definition of a comodule of a bimonad in [15, Section 4.1] to a bimonad induced by a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, the resulting notion is a comodule for the corresponding quotient coring $B/\{\, s(r)b - t(r)b \mid b \in B,\ r \in R \,\}$ (where $s$ and $t$ are the source and target map of $\mathcal{B}$, respectively). The category of such comodules is not known to be monoidal.

**3.7. Algebra extensions by bialgebroids. Galois extensions.** In analogy with bialgebra extensions, there are two symmetrical notions of an algebra extension by a bialgebroid $\mathcal{B}$. In the *action picture* one deals with a $\mathcal{B}$-module algebra $M$ and its invariant subalgebra (with respect to a character defined by the counit). In this picture, *Galois property* means Galois property of an associated $\mathbf{M}$-ring. Dually, in the *coaction picture* one deals with a $\mathcal{B}$-comodule algebra $M$ and its coinvariant subalgebra (with respect to a grouplike element defined by the unit). In this picture, *Galois property* means Galois property of an associated $\mathbf{M}$-coring. Although the two approaches are symmetric (and equivalent for finitely generated and projective bialgebroids), the coaction picture is more popular and more developed. We present it in more detail but, for the sake of completeness, we shortly describe the action picture as well.

3.7.1. *The action and coaction pictures.* By Theorem 3.13, the category $\mathcal{M}_{\mathbf{B}}$ of right modules of a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ is monoidal. By definition, a *right $\mathcal{B}$-module algebra* is a monoid $M$ in $\mathcal{M}_{\mathbf{B}}$. Denote the structure maps of $\mathcal{B}$ as in Definition 3.1. In view of Lemma 2.2, a right $\mathcal{B}$-module algebra is the same as an algebra and right $\mathbf{B}$-module $\mathbf{M}$, such that the multiplication in $\mathbf{M}$ is $\mathbf{R}$-balanced and

$$(mm') \cdot b = (m \cdot b^{(1)})(m' \cdot b^{(2)}) \qquad \text{and} \qquad 1_{\mathbf{M}} \cdot b = 1_{\mathbf{M}} \cdot s\big(\epsilon(b)\big), \qquad \text{for } m, m' \in M,\ b \in B,$$

cf. (3.9). Note in passing that, by strict monoidality of the forgetful functor $\mathcal{M}_{\mathbf{B}} \to {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$, a right $\mathcal{B}$-module algebra $M$ has a canonical $\mathbf{R}$-ring structure. Its unit is the map $\mathbf{R} \to \mathbf{M}$, $r \mapsto 1_{\mathbf{M}} \cdot s(r) = 1_{\mathbf{M}} \cdot t(r)$.

For a right $\mathcal{B}$-module algebra $M$, $B \otimes_{\mathbf{R}} M$ has an $\mathbf{M}$-ring structure. It is called a *smash product*, with multiplication $(b \otimes_{\mathbf{R}} m)(b' \otimes_{\mathbf{R}} m') = bb'^{(1)} \otimes_{\mathbf{R}} (m \cdot b'^{(2)})m'$, and unit $m \mapsto 1_{\mathbf{B}} \otimes_{\mathbf{R}} m$. The right character $\epsilon$ on the $\mathbf{R}$-ring $(\mathbf{B}, s)$ determines a right character $\epsilon \otimes_{\mathbf{R}} M$ on the $\mathbf{M}$-ring $B \otimes_{\mathbf{R}} M$. Hence we can consider the invariant subalgebra of the base algebra $\mathbf{M}$, with respect to the right character $\epsilon \otimes_{\mathbf{R}} M$. It coincides with the $\epsilon$-invariants of the $(\mathbf{B}, s)$-module $M$,

$$N := M_\epsilon = \{\, n \in M \mid n \cdot b = n \cdot s\left(\epsilon(b)\right), \quad \forall b \in B \,\}.$$

In the *action picture* the algebra $\mathbf{M}$ is said to be a *right $\mathcal{B}$-Galois extension* of the invariant subalgebra $\mathbf{N}$ provided that $B \otimes_{\mathbf{R}} M$ is a Galois $\mathbf{M}$-ring with respect to the right character $\epsilon \otimes_{\mathbf{R}} M$. That is, the *canonical map*

$$B \underset{\mathbf{R}}{\otimes} M \to {}_{\mathbf{N}}\mathrm{End}(M), \qquad b \underset{\mathbf{R}}{\otimes} m \mapsto \big(\, m' \mapsto (m' \cdot b)m \,\big)$$

is bijective. Left Galois extensions by a left bialgabroid $\mathcal{B}$ are defined symmetrically, referring to a left $\mathcal{B}$-module algebra and its $\epsilon$-invariant subalgebra.

By Theorem 3.18, also the category $\mathcal{M}^B$ of right comodules of a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ is monoidal. By definition, a *right $\mathcal{B}$-comodule algebra* is a monoid in $\mathcal{M}^B$. In view of Lemma 2.2, a right $\mathcal{B}$-comodule algebra is the same as an algebra and right $\mathcal{B}$-comodule $\mathbf{M}$, with coaction $m \mapsto m^{[0]} \otimes_{\mathbf{R}} m^{[1]}$, such that the multiplication in $\mathbf{M}$ is $\mathbf{R}$-balanced and, for $m, m' \in M$,

$$(3.12) \qquad (mm')^{[0]} \underset{\mathbf{R}}{\otimes} (mm')^{[1]} = m^{[0]} m'^{[0]} \underset{\mathbf{R}}{\otimes} m^{[1]} m'^{[1]} \qquad \text{and} \qquad 1_{\mathbf{M}}^{[0]} \underset{\mathbf{R}}{\otimes} 1_{\mathbf{M}}^{[1]} = 1_{\mathbf{M}} \underset{\mathbf{R}}{\otimes} 1_{\mathbf{B}}.$$

Note in passing that, by strict monoidality of the forgetful functor $\mathcal{M}^B \to {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$, a right $\mathcal{B}$-comodule algebra $M$ has a canonical $\mathbf{R}$-ring structure. Its unit is the map $\mathbf{R} \to \mathbf{M}$, $r \mapsto 1_{\mathbf{M}} \cdot r = r \cdot 1_{\mathbf{M}}$.

For a right $\mathcal{B}$-comodule algebra $M$, $M \otimes_{\mathbf{R}} B$ has an $\mathbf{M}$-coring structure with left and right $\mathbf{M}$-actions

$$(3.13) \qquad m_1 \cdot (m \underset{\mathbf{R}}{\otimes} b) \cdot m_2 = m_1 m m_2^{[0]} \underset{\mathbf{R}}{\otimes} b m_2^{[1]}, \qquad \text{for } m_1, m_2 \in M,\ m \underset{\mathbf{R}}{\otimes} b \in M \underset{\mathbf{R}}{\otimes} B,$$

comultiplication $m \otimes_{\mathbf{R}} b \mapsto (m \otimes_{\mathbf{R}} b^{(1)}) \otimes_{\mathbf{M}} (1_{\mathbf{M}} \otimes_{\mathbf{R}} b^{(2)})$ and counit $m \otimes_{\mathbf{R}} b \mapsto m \cdot \epsilon(b)$. The grouplike element $1_{\mathbf{B}}$ in the $\mathbf{R}$-coring $(B, \Delta, \epsilon)$ determines a grouplike element $1_{\mathbf{M}} \otimes_{\mathbf{R}} 1_{\mathbf{B}}$ in the $\mathbf{M}$-coring

$M \otimes_{\mathbf{R}} B$. Hence we can consider the coinvariant subalgebra of the base algebra $\mathbf{M}$, with respect to the grouplike element $1_{\mathbf{M}} \otimes_{\mathbf{R}} 1_{\mathbf{B}}$. It coincides with the $1_{\mathbf{B}}$-coinvariants of the $\mathcal{B}$-comodule $M$,

$$N := M^{1_{\mathbf{B}}} = \{\ n \in M \mid n^{[0]} \underset{\mathbf{R}}{\otimes} n^{[1]} = n \underset{\mathbf{R}}{\otimes} 1_{\mathbf{B}}\ \}.$$

Note that, by right $\mathbf{R}$-linearity of the $\mathcal{B}$-coaction on $M$ and (3.12), for $n \in N$ and $r \in R$,

$$(n \cdot r)^{[0]} \underset{\mathbf{R}}{\otimes} (n \cdot r)^{[1]} = n \underset{\mathbf{R}}{\otimes} s(r) = (r \cdot n)^{[0]} \underset{\mathbf{R}}{\otimes} (r \cdot n)^{[1]}.$$

Hence, for $n \in N$ and $r \in R$,

$$(3.14) \qquad\qquad\qquad n(1_{\mathbf{M}} \cdot r) = n \cdot r = r \cdot n = (1_{\mathbf{M}} \cdot r)n.$$

In the *coaction picture* the algebra $\mathbf{M}$ is said to be a *right $\mathcal{B}$-Galois extension* of the coinvariant subalgebra $\mathbf{N}$ provided that $M \otimes_{\mathbf{R}} B$ is a Galois $\mathbf{M}$-coring with respect to the grouplike element $1_{\mathbf{M}} \otimes_{\mathbf{R}} 1_{\mathbf{B}}$. That is, the *canonical map*

$$(3.15) \qquad\qquad \mathrm{can} : M \underset{\mathbf{N}}{\otimes} M \to M \underset{\mathbf{R}}{\otimes} B, \qquad m \underset{\mathbf{N}}{\otimes} m' \mapsto mm'^{[0]} \underset{\mathbf{R}}{\otimes} m'^{[1]}$$

is bijective. Since for a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ also the category of left comodules is monoidal, there is a symmetrical notion of a left $\mathcal{B}$-Galois extension $\mathbf{N} \subseteq \mathbf{M}$. It is a left $\mathcal{B}$-comodule algebra $M$, with coinvariant subalgebra $\mathbf{N}$, such that an associated $\mathbf{M}$-coring $B \otimes_{\mathbf{R}} M$ is a Galois coring, with respect to the grouplike element $1_{\mathbf{B}} \otimes_{\mathbf{R}} 1_{\mathbf{M}}$. Left and right Galois extensions by left bialgebroids are treated symmetrically.

For a right comodule algebra $M$ of a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, a *right-right relative Hopf module* is a right $M$-module *in $\mathcal{M}^B$*. The category of right-right relative Hopf modules is denoted by $\mathcal{M}_M^B$ and it turns out to be isomorphic to the category of right comodules for the $\mathbf{M}$-coring (3.13). Hence the grouplike element $1_{\mathbf{M}} \otimes_{\mathbf{R}} 1_{\mathbf{B}} \in M \otimes_{\mathbf{R}} B$ determines an adjunction as in (2.6) between $\mathcal{M}_M^B$ and the category $\mathcal{M}_{\mathbf{N}}$ of right modules for the coinvariant subalgebra $\mathbf{N}$ of $M$. It will be denoted as

$$(3.16) \qquad\qquad - \underset{\mathbf{N}}{\otimes} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^B \qquad \text{and} \qquad (-)^{coB} : \mathcal{M}_M^B \to \mathcal{M}_{\mathbf{N}}.$$

Recall from Section 2.2 that, for a $\mathcal{B}$-Galois extension $\mathbf{N} \subseteq \mathbf{M}$, this adjunction is interesting from the descent theory point of view.

For a finitely generated and projective bialgebroid, the action and coaction pictures are equivalent in the sense of Proposition 3.19. This equivalence was observed (in a slightly more restricted context) in [4, Theorem & Definition 3.3].

**PROPOSITION 3.19.** *Let $\mathcal{B}$ be a right $\mathbf{R}$-bialgebroid which is a finitely generated and projective right $\mathbf{R}$-module via right multiplication by the source map.*

*(1) There is a bijective correspondence between right $\mathcal{B}$-module algebra structures and right $(\mathcal{B}^*)^{op}$-comodule algebra structures on a given algebra $\mathbf{M}$.*

*(2) The invariant subalgebra $\mathbf{N}$ of a right $\mathcal{B}$-module algebra $M$ (with respect to the right character given by the counit) is the same as the coinvariant subalgebra of the corresponding right $(\mathcal{B}^*)^{op}$-comodule algebra $M$ (with respect to the grouplike element given by the unit).*

*(3) A right $\mathcal{B}$-module algebra $\mathbf{M}$ is a $\mathcal{B}$-Galois extension of its invariant subalgebra $\mathbf{N}$ in the action picture if and only if $\mathbf{M}$ is a $(\mathcal{B}^*)^{op}$-Galois extension of $\mathbf{N}$ in the coaction picture.*

Part (1) of Proposition 3.19 follows by the strict monoidal equivalence $\mathcal{M}_{\mathbf{B}} \cong \mathcal{M}^{(B^*)^{op}}$. Parts (2) and (3) follow by Proposition 2.13, since the $\mathbf{M}$-ring $B \otimes_{\mathbf{R}} M$, associated to a right $\mathcal{B}$-module algebra $M$, is the left $\mathbf{M}$-dual of the $\mathbf{M}$-coring $M \otimes_{\mathbf{R}} (B^*)^{op}$, associated to the right $(\mathcal{B}^*)^{op}$-comodule algebra $M$.

Consider a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, which is a finitely generated and projective right $\mathbf{R}$-module via the source map. Then, for a right $\mathcal{B}$-module algebra $M$, the category of right-right $(M, (\mathcal{B}^*)^{op})$ relative Hopf modules is equivalent also to the category of right modules for the smash product algebra $B \otimes_{\mathbf{R}} M$.

In the rest of these notes *only coaction picture* of Galois extensions will be used.

3.7.2. *Quantum torsors and bi-Galois extensions.* Following the work of Grunspan and Schauenburg [33], [65], [58], [66], for a bialgebra $B$ over a commutative ring $k$, a faithfully flat right $B$-Galois extension $\mathbf{T}$ of $k$ can be described without explicit mention of the bialgebra $B$. Instead, a *quantum torsor* structure is introduced on $\mathbf{T}$, from which $B$ can be reconstructed uniquely. What is more, a quantum torsor determines a second $k$-bialgebra $B'$, for which $\mathbf{T}$ is a left $B'$-Galois extension of $k$. It is said that any faithfully flat Galois extension of $k$ by a $k$-bialgebra is in fact a *bi-Galois extension*. The categories of (left) comodules for the bialgebras $B$ and $B'$ are monoidally equivalent. Such a description of faithfully flat Galois extensions by bialgebroids was developed in the PhD thesis of Hobst [38] and in the paper [10].

DEFINITION 3.20. For two algebras $\mathbf{R}$ and $\mathbf{S}$ over a commutative ring $k$, an $\mathbf{R}$-$\mathbf{S}$ torsor is a pair $(T, \tau)$. Here $T$ is an $\mathbf{R} \otimes_k \mathbf{S}$-ring with underlying $k$-algebra $\mathbf{T}$ and unit maps $\alpha : \mathbf{R} \to \mathbf{T}$ and $\beta : \mathbf{S} \to \mathbf{T}$ (with commuting ranges in $\mathbf{T}$). Considering $T$ as an $\mathbf{R}$-$\mathbf{S}$ bimodule and as an $\mathbf{S}$-$\mathbf{R}$ bimodule via the maps $\alpha$ and $\beta$, $\tau$ is an $\mathbf{S}$-$\mathbf{R}$ bimodule map $T \to T \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} T$, $t \mapsto t^{\langle 1 \rangle} \otimes_{\mathbf{R}} t^{\langle 2 \rangle} \otimes_{\mathbf{S}} t^{\langle 3 \rangle}$ (where implicit summation is understood), satisfying the following axioms, for $t, t' \in T$, $r \in R$ and $s \in S$.

(i) $(\tau \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} T) \circ \tau = (T \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} \tau) \circ \tau$ (*coassociativity*),
(ii) $(\mu_{\mathbf{R}} \otimes_{\mathbf{S}} T) \circ \tau = \beta \otimes_{\mathbf{S}} T$ and $(T \otimes_{\mathbf{R}} \mu_{\mathbf{S}}) \circ \tau = T \otimes_{\mathbf{R}} \alpha$ (*left and right counitality*),
(iii) $\tau(1_{\mathbf{T}}) = 1_{\mathbf{T}} \otimes_{\mathbf{R}} 1_{\mathbf{T}} \otimes_{\mathbf{S}} 1_{\mathbf{T}}$ (*unitality*),
(iv) $\alpha(r)t^{\langle 1 \rangle} \otimes_{\mathbf{R}} t^{\langle 2 \rangle} \otimes_{\mathbf{S}} t^{\langle 3 \rangle} = t^{\langle 1 \rangle} \otimes_{\mathbf{R}} t^{\langle 2 \rangle} \alpha(r) \otimes_{\mathbf{S}} t^{\langle 3 \rangle}$ and
$t^{\langle 1 \rangle} \otimes_{\mathbf{R}} \beta(s)t^{\langle 2 \rangle} \otimes_{\mathbf{S}} t^{\langle 3 \rangle} = t^{\langle 1 \rangle} \otimes_{\mathbf{R}} t^{\langle 2 \rangle} \otimes_{\mathbf{S}} t^{\langle 3 \rangle}\beta(s)$, (*centrality conditions*)
(v) $\tau(tt') = t^{\langle 1 \rangle}t'^{\langle 1 \rangle} \otimes_{\mathbf{R}} t'^{\langle 2 \rangle}t^{\langle 2 \rangle} \otimes_{\mathbf{S}} t^{\langle 3 \rangle}t'^{\langle 3 \rangle}$ (*multiplicativity*),

where $\mu_{\mathbf{R}}$ and $\mu_{\mathbf{S}}$ denote multiplication in the $\mathbf{R}$-ring $(\mathbf{T}, \alpha)$ and the $\mathbf{S}$-ring $(\mathbf{T}, \beta)$, respectively.

An $\mathbf{R}$-$\mathbf{S}$ torsor $(T, \tau)$ is said to be *faithfully flat* if $T$ is a faithfully flat right $\mathbf{R}$-module and a faithfully flat left $\mathbf{S}$-module.

Note that axiom (iv) in Definition 3.20 is needed in order for the multiplication in axiom (v) to be well defined.

THEOREM 3.21. *For two $k$-algebras $\mathbf{R}$ and $\mathbf{S}$, there is a bijective correspondence between the following sets of data.*

(i) *Faithfully flat $\mathbf{R}$-$\mathbf{S}$ torsors $(T, \tau)$.*
(ii) *Right $\mathbf{R}$-bialgebroids $\mathcal{B}$ and left faithfully flat right $\mathcal{B}$-Galois extensions $\mathbf{S} \subseteq \mathbf{T}$, such that $T$ is a right faithfully flat $\mathbf{R}$-ring.*
(iii) *Left $\mathbf{S}$-bialgebroids $\mathcal{B}'$ and right faithfully flat left $\mathcal{B}'$-Galois extensions $\mathbf{R} \subseteq \mathbf{T}$, such that $T$ is a left faithfully flat $\mathbf{S}$-ring.*

*Furthermore, a faithfully flat $\mathbf{R}$-$\mathbf{S}$ torsor $T$ is a $\mathcal{B}'$-$\mathcal{B}$ bicomodule, i.e. the left $\mathcal{B}'$, and right $\mathcal{B}$-coactions on $T$ do commute.*

Starting with the data in part (ii) of Theorem 3.21, a torsor map on $T$ is constructed in terms of the $\mathcal{B}$-coaction $\varrho^T : T \to T \otimes_{\mathbf{R}} B$, and the inverse of the canonical map (3.15) (with the role of the comodule algebra $M$ in (3.15) played by $T$), as $\tau := (T \otimes_{\mathbf{R}} \mathrm{can}^{-1}(1_{\mathbf{T}} \otimes_{\mathbf{R}} -)) \circ \varrho^T$. Conversely, to a faithfully flat $\mathbf{R}$-$\mathbf{S}$ torsor $(T, \tau)$ (with multiplication $\mu_{\mathbf{S}}$ in the $\mathbf{S}$-ring $(\mathbf{T}, \beta)$) one associates a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, defined on the $\mathbf{R}$-$\mathbf{R}$ bimodule given by the equaliser of the maps $(\mu_{\mathbf{S}} \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} T) \circ (T \otimes_{\mathbf{S}} \tau)$ and $\alpha \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} T : T \otimes_{\mathbf{S}} T \to T \otimes_{\mathbf{R}} T \otimes_{\mathbf{S}} T$.

THEOREM 3.22. *For two $k$-algebras $\mathbf{R}$ and $\mathbf{S}$, consider a faithfully flat $\mathbf{R}$-$\mathbf{S}$ torsor $(T, \tau)$. Let $\mathcal{B}$ and $\mathcal{B}'$ be the associated bialgebroids in Theorem 3.21. Assume that $T$ is a faithfully flat right $\mathbf{S}$-module and $B'$ is a flat right $\mathbf{S}$-module. Then the categories of left $\mathcal{B}$-, and $\mathcal{B}'$-comodules are monoidally equivalent.*

Note that the assumptions made about the right $\mathbf{S}$-modules $T$ and $B'$ in Theorem 3.22 become redundant if working with one commutative base ring $\mathbf{R} = \mathbf{S}$ and equal unit maps $\alpha = \beta$. The equivalence in Theorem 3.22 is given by $T \square_{\mathcal{B}} - : {}^B\mathcal{M} \to {}^{B'}\mathcal{M}$, a cotensor product with the $\mathcal{B}'$-$\mathcal{B}$ bicomodule $T$. (Recall that the notion of a *cotensor product* is dual to the one of module tensor

product. That is, for a right $\mathcal{B}$-comodule $(M, \varrho^M)$ and a left $\mathcal{B}$-comodule $(N, {}^N\varrho)$, $M\square_{\mathcal{B}}N$ is the equaliser of the maps $\varrho^M \otimes_{\mathbf{R}} N$ and $M \otimes_{\mathbf{R}} {}^N\varrho$.)

3.7.3. *Galois extensions by finitely generated and projective bialgebroids.* The bialgebra $B$, for which a given algebra extension is $B$-Galois, is non-unique. Obviously, there is even more possibility for a choice of $\mathcal{B}$ if it is allowed to be a bialgebroid. Still, as a main advantage of studying Galois extensions by bialgebroids, in an appropriately finite case all possible bialgebroids $\mathcal{B}$ can be related to a canonical one. Following Theorem 3.24 is a mild generalisation of [4, Proposition 4.12].

DEFINITION 3.23. Let $\mathcal{B}$ be a right bialgebroid over an algebra $\mathbf{R}$. A *right-right Yetter-Drinfel'd module* for $\mathcal{B}$ is a right $\mathcal{B}$-module and right $\mathcal{B}$-comodule $M$ (with one and the same underlying $\mathbf{R}$-bimodule structure), such that the following compatibility condition holds.

$$(m \cdot b^{(2)})^{[0]} \underset{\mathbf{R}}{\otimes} b^{(1)}(m \cdot b^{(2)})^{[1]} = m^{[0]} \cdot b^{(1)} \underset{\mathbf{R}}{\otimes} m^{[1]}b^{(2)}, \qquad \text{for } m \in M, \ b \in B.$$

It follows by a symmetrical version of [62, Proposition 4.4] that the category of right-right Yetter-Drinfel'd modules of a right bialgebroid $\mathcal{B}$ is isomorphic to the weak center of the monoidal category of right $\mathbf{B}$-modules. Hence it is monoidal and pre-braided. Following the paper [4], the construction in Section 3.4.7 can be extended to a braided commutative algebra $A$ in the category of right-right Yetter-Drinfel'd modules of a right $\mathbf{R}$-bialgebroid $\mathcal{B}$. That is, the smash product algebra $A\#B$ can be proven to carry the structure of a right $\mathbf{A}$-bialgebroid, called a *scalar extension* of $\mathcal{B}$ by $A$.

In following Theorem 3.24, the center of a bimodule $M$ of an algebra $\mathbf{R}$ is denoted by $M^{\mathbf{R}}$.

THEOREM 3.24. *For an algebra $\mathbf{R}$ consider a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ which is a finitely generated and projective left $\mathbf{R}$-module via right multiplication by the target map. Let $\mathbf{N} \subseteq \mathbf{M}$ be a right $\mathcal{B}$-Galois extension. Then $\mathbf{N} \subseteq \mathbf{M}$ is a right Galois extension by a right bialgebroid $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$ over the base algebra $M^{\mathbf{N}}$. What is more, the bialgebroid $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$ is isomorphic to a scalar extension of $\mathcal{B}$.*

In proving Theorem 3.24 the following key ideas are used. First a braided commutative algebra structure, in the category of right-right Yetter-Drinfel'd modules for $\mathcal{B}$, is constructed on $M^{\mathbf{N}}$. The $\mathcal{B}$-coaction on $M^{\mathbf{N}}$ is given by restriction of the $\mathcal{B}$-coaction on $M$. The $\mathbf{B}$-action on $M^{\mathbf{N}}$ is of the Miyashita-Ulbrich type, i.e. it is given in terms of the inverse of the canonical map (3.15). Introducing an index notation $\mathrm{can}^{-1}(1_{\mathbf{M}} \otimes_{\mathbf{R}} b) = b^{\{1\}} \otimes_{\mathbf{N}} b^{\{2\}}$, for $b \in B$ (implicit summation is understood), the right $\mathbf{B}$-action on $M^{\mathbf{N}}$ is $a \cdot b := b^{\{1\}}ab^{\{2\}}$, for $b \in B$ and $a \in M^{\mathbf{N}}$. Since in this way $M^{\mathbf{N}}$ is a braided commutative algebra in the category of right-right Yetter-Drinfel'd modules for $\mathcal{B}$, there exists a right $M^{\mathbf{N}}$-bialgebroid $M^{\mathbf{N}} \otimes_{\mathbf{R}} B$ (cf. Section 3.4.7). Restriction of the $\mathcal{B}$-canonical map (3.15) establishes a bijection $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}} \to M^{\mathbf{N}} \otimes_{\mathbf{R}} B$. Hence it induces an $M^{\mathbf{N}}$-bialgebroid structure on $(M\otimes_{\mathbf{N}} M)^{\mathbf{N}}$, and also an $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$-comodule algebra structure on $M$. After checking that coinvariants of the $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$-comodule $M$ are precisely the elements of $\mathbf{N}$, the $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$-Galois property of the extension $\mathbf{N} \subseteq \mathbf{M}$ becomes obvious: the $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$-canonical map differs from the $\mathcal{B}$-canonical map (3.15) by an isomorphism.

3.7.4. *Depth two algebra extensions.* Classical finitary Galois extensions of fields can be characterised inherently, by normality and separability properties, without referring to the Galois group $G$. That is, a (unique upto isomorphism) finite Galois group $G := \mathrm{Aut}_K(F)$ is *determined* by any normal and separable field extension $F$ of $K$. While no such inherent characterisation of Galois extensions by (finitely generated and projective) bialgebras is known, a most important achievement in the Galois theory of bialgebroids is a characterisation of Galois extension by finitely generated and projective bialgebroids. A first result in this direction was [4, Theorem 3.7]. At the level of generality presented here, it was proven in [40, Theorem 2.1].

Following definition in [41, Definition 3.1] was abstracted from depth 2 extensions of $C^*$-algebras.

DEFINITION 3.25. Consider an extension $\mathbf{N} \subseteq \mathbf{M}$ of algebras. It is said to satisfy the *right (resp. left) depth 2* condition if the $\mathbf{M}$-$\mathbf{N}$ bimodule (resp. $\mathbf{N}$-$\mathbf{M}$ bimodule) $M \otimes_{\mathbf{N}} M$ is a direct summand in a finite direct sum of copies of $M$.

Note that the right depth 2 property of an algebra extension $\mathbf{N} \subseteq \mathbf{M}$ is equivalent to the existence of finitely many elements $\gamma_k \in {}_{\mathbf{N}}\mathrm{End}_{\mathbf{N}}(M) \cong {}_{\mathbf{M}}\mathrm{Hom}_{\mathbf{N}}(M \otimes_{\mathbf{N}} M, M)$ and $c_k \in (M \otimes_{\mathbf{N}} M)^{\mathbf{N}} \cong {}_{\mathbf{M}}\mathrm{Hom}_{\mathbf{N}}(M, M \otimes_{\mathbf{N}} M)$, the so called *right depth 2 quasi-basis*, satisfying the identity

$$(3.17) \qquad\qquad \sum_k m\gamma_k(m')c_k = m \underset{\mathbf{N}}{\otimes} m' \qquad \text{for } m, m' \in M.$$

DEFINITION 3.26. An extension $\mathbf{N} \subseteq \mathbf{M}$ of algebras is *balanced* if all endomorphisms of $M$, as a left module for the algebra $\mathcal{E} := \mathrm{End}_{\mathbf{N}}(M)$, are given by right multiplication by some element of $\mathbf{N}$.

THEOREM 3.27. *For an algebra extension* $\mathbf{N} \subseteq \mathbf{M}$, *the following properties are equivalent.*
(i) $\mathbf{N} \subseteq \mathbf{M}$ *is a right Galois extension by some right* $\mathbf{R}$*-bialgebroid* $\mathcal{B}$, *which is a finitely generated and projective left* $\mathbf{R}$*-module via right multiplication by the target map.*
(ii) *The algebra extension* $\mathbf{N} \subseteq \mathbf{M}$ *is balanced and satisfies the right depth 2 condition.*

If $\mathbf{N} \subseteq \mathbf{M}$ is a right Galois extension by a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, then $M \otimes_{\mathbf{N}} M \cong M \otimes_{\mathbf{R}} B$ as $\mathbf{M}$-$\mathbf{N}$ bimodules. Hence the right depth two condition follows by finitely generated projectivity of the left $\mathbf{R}$-module $B$. A left $\mathcal{E}$-module endomorphism of $M$ is given by right multiplication by an element $x \in M^{\mathbf{R}}$, by the right $\mathbf{N}$-linearity of the maps, given by left multiplication by an element $m \in M$, and right multiplication by $r \in R$. Since by (3.14) also the right action (2.7) on $M$ by $\phi \in {}^*B$ is a right $\mathbf{N}$-module map, it follows that $x^{[0]}\phi(x^{[1]}) = x\phi(1_{\mathbf{B}})$, for all $\phi \in {}^*B$. Together with the finitely generated projectivity of the left $\mathbf{R}$-module $B$, this implies that $x$ belongs to the coinvariant subalgebra $\mathbf{N}$, hence the extension $\mathbf{N} \subseteq \mathbf{M}$ is balanced.

By Theorem 3.24, if $\mathbf{N} \subseteq \mathbf{M}$ is a right Galois extension by some finitely generated projective right $\mathbf{R}$-bialgebroid $\mathcal{B}$, then it is a Galois extension by the (finitely generated projective) right $M^{\mathbf{N}}$-bialgebroid $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$. Hence in the converse direction a balanced algebra extension $\mathbf{N} \subseteq \mathbf{M}$, satisfying the right depth 2 condition, is shown to be a right Galois extension by a right $M^{\mathbf{N}}$-bialgebroid $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$, constructed in [41, Section 5]. (In fact, in [41] both the left and right depth two properties are assumed. It is proven in [40] that the construction works for one sided depth two extensions as well.) The coproduct in $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$ and its coaction on $M$ are constructed in terms of the right depth 2 quasi-basis (3.17). Let us mention that the only point in the proof, where the balanced property is used, is to show that the $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$-coinvariants in $M$ are precisely the elements of $N$.

## 4. Hopf algebroids

A Hopf algebra is a bialgebra $H$ equipped with an additional *antipode* map $H \to H$. The antipode is known to be a bialgebra map from $H$ to the opposite co-opposite of $H$. It does not seem to be possible to define a Hopf algebroid based on this analogy. Starting with a, say left, bialgebroid $\mathcal{H}$, its opposite co-opposite $\mathcal{H}_{cop}^{op}$ is a right bialgebroid. There is no sensible notion of a bialgebroid map $\mathcal{H} \to \mathcal{H}_{cop}^{op}$. If we choose as a guiding principle the antipode of a Hopf algebroid $\mathcal{H}$ to be a bialgebroid map $\mathcal{H} \to \mathcal{H}_{cop}^{op}$, then $\mathcal{H}$ and $\mathcal{H}_{cop}^{op}$ need to carry the same, say left, bialgebroid structure. This means that the underlying algebra $\mathbf{H}$ must be equipped both with a left, and a right bialgebroid structure. The first definition fulfilling this requirement was proposed in [13, Definition 4.1], where however the antipode was defined to be bijective. Bijectivity of the antipode was relaxed in [7, Definition 2.2]. Here we present a set of axioms which is equivalent to [7, Definition 2.2], as it was formulated in [9, Remark 2.1].

DEFINITION 4.1. For two algebras $\mathbf{R}$ and $\mathbf{L}$ over a commutative ring $k$, a *Hopf algebroid* over the base algebras $\mathbf{R}$ and $\mathbf{L}$ is a triple $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$. Here $\mathcal{H}_L$ is a left $\mathbf{L}$-bialgebroid and $\mathcal{H}_R$ is a right $\mathbf{R}$-bialgebroid, such that their underlying $k$-algebra $\mathbf{H}$ is the *same*. The *antipode* $S$ is a $k$-module map $H \to H$. Denote the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring structure of $\mathcal{H}_R$ by $(\mathbf{H}, s_R, t_R)$ and its $\mathbf{R}$-coring structure by $(H, \Delta_R, \epsilon_R)$. Similarly, denote the $\mathbf{L} \otimes_k \mathbf{L}^{op}$-ring structure of $\mathcal{H}_L$ by $(\mathbf{H}, s_L, t_L)$ and its $\mathbf{L}$-coring structure by $(H, \Delta_L, \epsilon_L)$. Denote the multiplication in the $\mathbf{R}$-ring $(\mathbf{H}, s_R)$ by $\mu_R$ and denote the multiplication in the $\mathbf{L}$-ring $(\mathbf{H}, s_L)$ by $\mu_L$. These structures are subject to the following compatibility axioms.

(i) $s_L \circ \epsilon_L \circ t_R = t_R$, $\quad t_L \circ \epsilon_L \circ s_R = s_R$, $\quad s_R \circ \epsilon_R \circ t_L = t_L$ and $t_R \circ \epsilon_R \circ s_L = s_L$.

(ii) $(\Delta_L \otimes_{\mathbf{R}} H) \circ \Delta_R = (H \otimes_{\mathbf{L}} \Delta_R) \circ \Delta_L$ and $(\Delta_R \otimes_{\mathbf{L}} H) \circ \Delta_L = (H \otimes_{\mathbf{R}} \Delta_L) \circ \Delta_R$.

(iii) For $l \in L$, $r \in R$ and $h \in H$, $S(t_L(l)ht_R(r)) = s_R(r)S(h)s_L(l)$.

(iv) $\mu_L \circ (S \otimes_{\mathbf{L}} H) \circ \Delta_L = s_R \circ \epsilon_R$ and $\mu_R \circ (H \otimes_{\mathbf{R}} S) \circ \Delta_R = s_L \circ \epsilon_L$.

REMARKS 4.2. Hopf algebroid axioms in Definition 4.1 require some interpretation.

(1) By the bialgebroid axioms, all maps $s_L \circ \epsilon_L$, $t_L \circ \epsilon_L$, $s_R \circ \epsilon_R$ and $t_R \circ \epsilon_R$ are idempotent maps $H \to H$. Hence the message of axiom (i) is that the ranges of $s_L$ and $t_R$, and also the ranges of $s_R$ and $t_L$, are coinciding subalgebras of $\mathbf{H}$. These axioms imply that the coproduct $\Delta_L$ in $\mathcal{H}_L$ is not only an $\mathbf{L}$-bimodule map, but also an $\mathbf{R}$-bimodule map. Symmetrically, $\Delta_R$ is an $\mathbf{L}$-bimodule map, so that axiom (ii) makes sense.

(2) The $k$-module $H$ underlying a (left or right) bialgebroid is a left and right comodule via the coproduct. Hence a $k$-module $H$ underlying a Hopf algebroid is a left and right comodule for the constituent left and right bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$, via the two coproducts $\Delta_L$ and $\Delta_R$. Axiom (ii) expresses a property that these regular coactions do commute, i.e. $H$ is an $\mathcal{H}_L$-$\mathcal{H}_R$ bicomodule and also an $\mathcal{H}_R$-$\mathcal{H}_L$ bicomodule.

Alternatively, considering $H$ and $H \otimes_{\mathbf{L}} H$ as right $\mathcal{H}_R$-comodules via the respective coactions $\Delta_R$ and $H \otimes_{\mathbf{L}} \Delta_R$, the first axiom in (ii) expresses that $\Delta_L$ is a right $\mathcal{H}_R$-comodule map. Symmetrically, this condition can be interpreted as a left $\mathcal{H}_L$-comodule map property of $\Delta_R$. Similarly, the second axiom in (ii) can be read as right $\mathcal{H}_L$-colinearity of $\Delta_R$ or left $\mathcal{H}_R$-colinearity of $\Delta_L$.

(3) Axiom (iii) formulates the $\mathbf{R}$-$\mathbf{L}$ bimodule map property of the antipode, needed in order for axiom (iv) to make sense.

(4) Analogously to the Hopf algebra axioms, axiom (iv) tells us that the antipode is convolution inverse of the identity map $H$, in some generalised sense. The notion of convolution products in the case of two different base algebras $\mathbf{L}$ and $\mathbf{R}$ is discussed in Section 4.5.2.

Since in a Hopf algebroid $\mathcal{H}$ there are two constituent bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$ present, in these notes we use two versions of Sweedler's index notation in parallel, to denote components of the coproducts $\Delta_L$ and $\Delta_R$. We will use lower indices in the case of a left bialgebroid $\mathcal{H}_L$, i.e. we write $\Delta_L(h) = h_{(1)} \otimes_{\mathbf{L}} h_{(2)}$, and upper indices in the case of a right bialgebroid $\mathcal{H}_R$, i.e. we write $\Delta_R(h) = h^{(1)} \otimes_{\mathbf{R}} h^{(2)}$, for $h \in H$, where implicit summation is understood in both cases. Analogously, we use upper indices to denote components of a coaction by $\mathcal{H}_R$, and lower indices to denote components of a coaction by $\mathcal{H}_L$.

4.1. **Examples and constructions.** Before turning to a study of the structure of Hopf algebroids, let us see some examples.

4.1.1. *Hopf algebras.* A Hopf algebra $H$ over a commutative ring $k$ is an example of Hopf algebroids over base algebras $\mathbf{R} = k = \mathbf{L}$. Both bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$ are equal to the $k$-bialgebra $H$ and the Hopf algebra antipode of $H$ satisfies the Hopf algebroid axioms.

Certainly, not every Hopf algebroid over base algebras $\mathbf{R} = k = \mathbf{L}$ is a Hopf algebra (see e.g. Section 4.1.4). Examples of this kind have been constructed in [25], as follows. Let $H$ be a Hopf algebra over $k$, with coproduct $\Delta_L : h \mapsto h_{(1)} \otimes_k h_{(2)}$, counit $\epsilon_L$ and antipode $S$. Let $\chi$ be a character on $H$, i.e. a $k$-algebra map $H \to k$. The coproduct $\Delta_R : h \mapsto h_{(1)} \otimes_k \chi\big(S(h_{(2)})\big) h_{(3)}$ and the counit $\epsilon_R := \chi$ define a second bialgebra structure on the $k$-algebra $\mathbf{H}$. Looking at these two bialgebras as left and right $k$-bialgebroids respectively, we obtain a Hopf algebroid with a *twisted* antipode $h \mapsto \chi(h_{(1)})S(h_{(2)})$. This construction was extended in [6, Theorem 4.2], where new examples of Hopf algebroids were constructed by twisting a (bijective) antipode of a given Hopf algebroid.

4.1.2. *Weak Hopf algebras.* A *weak Hopf algebra* over a commutative ring $k$ is a weak bialgebra $H$ equipped with a $k$-linear antipode map $S : H \to H$, subject to the following axioms [11]. For $h \in H$,

$$h_{(1)}S(h_{(2)}) = \sqcap^L(h), \qquad S(h_{(1)})h_{(2)} = \sqcap^R(h), \qquad S(h_{(1)})h_{(2)}S(h_{(3)}) = S(h),$$

where the maps $\sqcap^L$ and $\sqcap^R$ were introduced in Section 3.2.2.

The right $\mathbf{R}$-bialgebroid and the left $\mathbf{R}^{op}$-bialgebroid, constructed for a weak Hopf algebra $H$ in Section 3.2.2, together with the antipode $S$, satisfy the Hopf algebroid axioms.

In particular, consider a small groupoid with finitely many objects. By Section 3.2.2, the free $k$-module spanned by its morphisms is a weak $k$-bialgebra. It can be equipped with an antipode by putting $S(f) := f^{-1}$ for every morphism $f$, and extending it $k$-linearly. Motivated by this example, weak Hopf algebras, and sometimes also Hopf algebroids, are called *quantum groupoids* in the literature.

Weak Hopf algebras have a nice and well understood representation theory. The category of finite dimensional modules of a finite dimensional semisimple weak Hopf algebra $H$ over a field $k$ is a $k$-linear semisimple category with finitely many inequivalent irreducible objects, with all finite dimensional hom spaces. It is a monoidal category with left and right duals. A category with the listed properties is termed a *fusion category*. Conversely, based on Tannaka-Krein type reconstruction theorems in [68] and [36], it was proven in [32] that any fusion category is equivalent to the category of finite dimensional modules of a (non-unique) finite dimensional semisimple weak Hopf algebra.

4.1.3. $\mathbf{R} \otimes \mathbf{R}^{op}$. The left and right bialgebroids on an algebra of the form $\mathbf{R} \otimes_k \mathbf{R}^{op}$, constructed for any $k$-algebra $\mathbf{R}$ in Section 3.2.3, form a Hopf algebroid together with the antipode $r \otimes_k r' \mapsto r' \otimes_k r$.

4.1.4. *The algebraic quantum torus.* Consider an algebra $\mathbf{T}_q$ over a commutative ring $k$, generated by two invertible elements $U$ and $V$, subject to the relation $UV = qVU$, where $q$ is an invertible element in $k$. $\mathbf{T}_q$ possesses a right bialgebroid structure over the commutative subalgebra $\mathbf{R}$ generated by $U$. Both the source and target maps are given by the inclusion $\mathbf{R} \to \mathbf{T}_q$. Coproduct and counit are defined by $\Delta_R : V^m U^n \mapsto V^m U^n \otimes_{\mathbf{R}} V^m$ and $\epsilon_R : V^m U^n \mapsto U^n$, respectively. Symmetrically, there is a left $\mathbf{R}$-bialgebroid structure given by the coproduct $\Delta_L : U^n V^m \mapsto U^n V^m \otimes_{\mathbf{R}} V^m$ and counit $\epsilon_L : U^n V^m \mapsto U^n$. Together with the antipode $S : U^n V^m \mapsto V^{-m} U^n$ they constitute a Hopf algebroid.

4.1.5. *Scalar extension.* Consider a Hopf algebra $H$ and a braided commutative algebra $A$ in the category of right-right Yetter-Drinfel'd modules of $H$. As it was seen in Section 3.4.7, the smash product algebra $A \# H$ carries a right $\mathbf{A}$-bialgebroid structure. If the antipode $S$ of $H$ is bijective, then the $\mathbf{A}$-bialgebroid structure of $A \# H$ extends to a Hopf algebroid. Indeed, $A \# H$ is a left $\mathbf{A}^{op}$-bialgebroid via the source map $a \mapsto a_{[0]} \cdot S(a_{[1]}) \# 1_{\mathbf{B}}$, target map $a \mapsto a_{[0]} \# a_{[1]}$, coproduct $a \# h \mapsto (a \# h_{(1)}) \otimes_{\mathbf{A}^{op}} (1_{\mathbf{A}} \# h_{(2)})$ and counit $a \# h \mapsto a_{[0]} \cdot S^{-1}\big(hS^{-1}(a_{[1]})\big)$. The (bijective) antipode is given by $a \# h \mapsto a_{[0]} \cdot S(h_{(2)}) \# a_{[1]} S(h_{(1)})$.

4.2. **Basic properties of Hopf algebroids.** In this section some consequences of Definition 4.1 of a Hopf algebroid will be recalled from [7, Section 2].

The opposite co-opposite of a Hopf algebra $H$ is a Hopf algebra, with the same antipode $S$ of $H$. If $S$ is bijective, then also the opposite and the co-opposite of $H$ are Hopf algebras, with antipode $S^{-1}$. A generalisation of these facts to Hopf algebroids is given below.

PROPOSITION 4.3. *Consider a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ over base algebras $\mathbf{L}$ and $\mathbf{R}$. The following hold true.*

    (1) *The triple $((\mathcal{H}_R)^{op}_{cop}, (\mathcal{H}_L)^{op}_{cop}, S)$ is a Hopf algebroid over the base algebras $\mathbf{R}^{op}$ and $\mathbf{L}^{op}$.*

    (2) *If the antipode $S$ is bijective then $((\mathcal{H}_R)^{op}, (\mathcal{H}_L)^{op}, S^{-1})$ is a Hopf algebroid over the base algebras $\mathbf{R}$ and $\mathbf{L}$, and $((\mathcal{H}_L)_{cop}, (\mathcal{H}_R)_{cop}, S^{-1})$ is a Hopf algebroid over the base algebras $\mathbf{L}^{op}$ and $\mathbf{R}^{op}$.*

Proposition 4.4 states the expected behaviour of the antipode of a Hopf algebroid with respect to the underlying ring and coring structures. Consider a Hopf algebroid $\mathcal{H}$ over base algebras $\mathbf{L}$ and $\mathbf{R}$, with structure maps denoted as in Definition 4.1. It follows immediately by axiom (i) in Definition 4.1 that the base algebras $\mathbf{L}$ and $\mathbf{R}$ are anti-isomorphic. Indeed, there are inverse algebra isomorphisms

(4.1) $$\epsilon_L \circ s_R : \mathbf{R}^{op} \to \mathbf{L} \qquad \text{and} \qquad \epsilon_R \circ t_L : \mathbf{L} \to \mathbf{R}^{op}.$$

Symmetrically, there are inverse algebra isomorphisms

$$(4.2) \qquad \epsilon_R \circ s_L : \mathbf{L}^{op} \to \mathbf{R} \qquad \text{and} \qquad \epsilon_L \circ t_R : \mathbf{R} \to \mathbf{L}^{op}.$$

PROPOSITION 4.4. *Let $\mathcal{H}$ be a Hopf algebroid over base algebras $\mathbf{L}$ and $\mathbf{R}$, with structure maps denoted as in Definition 4.1. The following assertions hold.*

(1) *The antipode $S$ is a homomorphism of $\mathbf{R} \otimes_k \mathbf{R}^{op}$-rings*

$$( \ \mathbf{H} \ , \ s_R \ , \ t_R \ ) \to ( \ \mathbf{H}^{op} \ , \ s_L \circ (\epsilon_L \circ s_R) \ , \ t_L \circ (\epsilon_L \circ s_R) \ )$$

*and also a homomorphism of $\mathbf{L} \otimes_k \mathbf{L}^{op}$-rings*

$$( \ \mathbf{H} \ , \ s_L \ , \ t_L \ ) \to ( \ \mathbf{H}^{op} \ , \ s_R \circ (\epsilon_R \circ s_L) \ , \ t_R \circ (\epsilon_R \circ s_L) \ ).$$

*In particular, $S$ is a $k$-algebra anti-homomorphism $\mathbf{H} \to \mathbf{H}$.*
(2) *The antipode $S$ is a homomorphism of $\mathbf{R}$-corings*

$$( \ H \ , \ \Delta_R \ , \ \epsilon_R \ ) \to ( \ H \ , \ \Delta_L^{cop} \ , \ (\epsilon_R \circ s_L) \circ \epsilon_L \ ),$$

*where $\Delta_L^{cop}$ is considered as a map $H \to H \otimes_{\mathbf{L}^{op}} H \cong H \otimes_{\mathbf{R}} H$, via the isomorphism induced by the algebra isomorphism (4.2). Symmetrically, $S$ is a homomorphism of $\mathbf{L}$-corings*

$$( \ H \ , \ \Delta_L \ , \ \epsilon_L \ ) \to ( \ H \ , \ \Delta_R^{cop} \ , \ (\epsilon_L \circ s_R) \circ \epsilon_R \ ),$$

*where $\Delta_R^{cop}$ is considered as a map $H \to H \otimes_{\mathbf{R}^{op}} H \cong H \otimes_{\mathbf{L}} H$, via the isomorphism induced by the algebra isomorphism (4.1).*

A Hopf algebroid has a number of module structures over its base algebras $\mathbf{L}$ and $\mathbf{R}$. They turn out to be strongly related.

PROPOSITION 4.5. *Let $\mathcal{H}$ be a Hopf algebroid over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, with structure maps denoted as in Definition 4.1. If the antipde is bijective then the following statements hold.*

(1) *The right $\mathbf{L}$-module $H$, given by left multiplication by $t_L$, is finitely generated and projective if and only if left $\mathbf{R}$-module $H$, given by right multiplication by $t_R$, is finitely generated and projective.*
(2) *The left $\mathbf{L}$-module $H$, given by left multiplication by $s_L$, is finitely generated and projective if and only if the right $\mathbf{R}$-module $H$, given by right multiplication by $s_R$, is finitely generated and projective.*

The $k$-dual $H^*$ of a finitely generated and projective Hopf algebra $H$ over a commutative ring $k$ is a Hopf algebra. The antipode in $H^*$ is the transpose of the antipode of $H$. No generalisation of this fact for Hopf algebroids is known. Although the dual $\mathcal{H}^*$ of a finitely generated and projective Hopf algebroid $\mathcal{H}$ has a bialgebroid structure (cf. Section 3.3), the transpose of the antipode in $\mathcal{H}$ is not an endomorphism of $\mathcal{H}^*$. Duals only of *Frobenius* Hopf algebroids are known to be (Frobenius) Hopf algebroids, see Section 4.4.2.

4.3. **Comodules of Hopf algebroids.** In a Hopf algebroid, the constituent left and right bialgebroids are defined on the *same* underlying algebra. Therefore, modules for the two bialgebroids coincide. This is not the case with comodules: the two bialgebroids have different underlying corings (over anti-isomorphic base algebras, cf. (4.1)-(4.2)), that have a priori different categories of (say, right) comodules. We take the opportunity to call here the reader's attention to a regrettable error in the literature. Based on [17, Theorem 2.6], whose proof turned out to be *incorrect*, the categories of (right) comodules of two constituent bialgebroids in a Hopf algebroid were claimed to be strict monoidally isomorphic in [9, Theorem 2.2]. Since it turned out recently that in [17, Theorem 2.6] there are some assumptions missing, the derived result [9, Theorem 2.2] needs not hold either at the stated level of generality. (There is a similar error in [8, Proposition 3.1]). Regrettably, this error influences some results also in [1], [9] and [14]. In the current section and in 4.5 we present the corrected statements.

4.3.1. *Comodules of a Hopf algebroid and of its constituent bialgebroids.* Since, as it is explained above, comodules of the two constituent bialgebroids in a Hopf algebroid are in general different notions, none of them can be expected to be a well working definition of a comodule of a Hopf algebroid. The following definition of a comodule for a Hopf algebroid, as a compatible comodule of both constituent bialgebroids, was suggested in [8, Definition 3.2] and [4, Section 2.2].

DEFINITION 4.6. For a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, denote the structure maps as in Definition 4.1. A *right $\mathcal{H}$-comodule* is a right $\mathbf{L}$-module as well as a right $\mathbf{R}$-module $M$, together with a right $\mathcal{H}_R$-coaction $\varrho_R : M \to M \otimes_{\mathbf{R}} H$ and a right $\mathcal{H}_L$-coaction $\varrho_L : M \to M \otimes_{\mathbf{L}} H$, such that $\varrho_R$ is an $\mathcal{H}_L$-comodule map and $\varrho_L$ is an $\mathcal{H}_R$-comodule map. Explicitly, $\varrho_R : m \mapsto m^{[0]} \otimes_{\mathbf{R}} m^{[1]}$ is a right $\mathbf{L}$-module map in the sense that

$$(m \cdot l)^{[0]} \underset{\mathbf{R}}{\otimes} (m \cdot l)^{[1]} = m^{[0]} \underset{\mathbf{R}}{\otimes} t_L(l) m^{[1]}, \qquad \text{for } m \in M, \ l \in L,$$

$\varrho_L : m \mapsto m_{[0]} \otimes_{\mathbf{L}} m_{[1]}$ is a right $\mathbf{R}$-module map in the sense that

$$(m \cdot r)_{[0]} \underset{\mathbf{L}}{\otimes} (m \cdot r)_{[1]} = m_{[0]} \underset{\mathbf{L}}{\otimes} m_{[1]} s_R(r), \qquad \text{for } m \in M, \ r \in R,$$

and the following compatibility conditions hold.

$$(\varrho_R \underset{\mathbf{L}}{\otimes} H) \circ \varrho_L = (M \underset{\mathbf{R}}{\otimes} \Delta_L) \circ \varrho_R \qquad \text{and} \qquad (\varrho_L \underset{\mathbf{R}}{\otimes} H) \circ \varrho_R = (M \underset{\mathbf{L}}{\otimes} \Delta_R) \circ \varrho_L.$$

Morphisms of $\mathcal{H}$-comodules are $\mathcal{H}_L$- as well as $\mathcal{H}_R$-comodule maps.

The category of right comodules of a Hopf algebroid $\mathcal{H}$ is denoted by $\mathcal{M}^{\mathcal{H}}$.

It is not difficult to see that the right $\mathbf{R}$- and $\mathbf{L}$-actions on a right $\mathcal{H}$-comodule $M$ necessarily commute. That is, $M$ carries the structure of a right $\mathbf{L} \otimes_k \mathbf{R}$-module.

Note that by Definition 4.1 (i) and (ii), the right $\mathbf{R} \otimes_k \mathbf{L}$-module $H$, with $\mathbf{R}$-action via the right source map $s_R$ and $\mathbf{L}$-action via the left target map $t_L$, is a right comodule of the Hopf algebroid $\mathcal{H}$, via the two coactions given by the two coproducts $\Delta_R$ and $\Delta_L$.

Left comodules of a Hopf algebroid $\mathcal{H}$ are defined symmetrically and their category is denoted by $^{\mathcal{H}}\mathcal{M}$.

REMARK 4.7. The antipode $S$ in a Hopf algebroid $\mathcal{H}$ defines a functor $\mathcal{M}^{\mathcal{H}} \to {}^{\mathcal{H}}\mathcal{M}$. Indeed, if $M$ is a right $\mathcal{H}$-comodule, with $\mathcal{H}_R$-coaction $m \mapsto m^{[0]} \otimes_{\mathbf{R}} m^{[1]}$ and $\mathcal{H}_L$-coaction $m \mapsto m_{[0]} \otimes_{\mathbf{L}} m_{[1]}$, then it is a left $\mathcal{H}$-comodule with left $\mathbf{R}$-action $r \cdot m = m \cdot \epsilon_L(t_R(r))$, left $\mathbf{L}$-action $l \cdot m = m \cdot \epsilon_R(t_L(l))$ (where the notations in Definition 4.1 are used) and respective coactions

$$m \mapsto S(m_{[1]}) \underset{\mathbf{R}}{\otimes} m_{[0]} \qquad \text{and} \qquad m \mapsto S(m^{[1]}) \underset{\mathbf{L}}{\otimes} m^{[0]}.$$

Right $\mathcal{H}$-comodule maps are also left $\mathcal{H}$-comodule maps for these coactions.

A functor $^{\mathcal{H}}\mathcal{M} \to \mathcal{M}^{\mathcal{H}}$ is constructed symmetrically.

Although comodules of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ can not be described as comodules of a coring, the free-forgetful adjunction (cf. [20, 18.13(2)]), corresponding to the $L$-coring underlying $\mathcal{H}_L$, lifts to an adjunction between the categories $\mathcal{M}^{\mathcal{H}}$ and $\mathcal{M}_{\mathbf{L}}$. Indeed, the forgetful functor $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}_{\mathbf{L}}$ has a right adjoint $- \otimes_{\mathbf{L}} H : \mathcal{M}_{\mathbf{L}} \to \mathcal{M}^{\mathcal{H}}$. Unit and counit of the adjunction are given, for a right $\mathcal{H}$-comodule $(M, \varrho_L, \varrho_R)$ and a right $\mathbf{L}$-module $\mathbf{N}$, by the maps

$$(4.3) \qquad\qquad \varrho_L : M \to M \underset{\mathbf{L}}{\otimes} H \qquad \text{and} \qquad N \underset{\mathbf{L}}{\otimes} \epsilon_L : N \underset{\mathbf{L}}{\otimes} H \to N,$$

respectively, where $\epsilon_L$ is the counit of $\mathcal{H}_L$. There is a similar adjunction between the cateories $\mathcal{M}^{\mathcal{H}}$ and $\mathcal{M}_{\mathbf{R}}$.

Our next task is to look for situations when the category of comodules of a Hopf algebroid coincides with the comodule category of any of the constituent bialgebroids. Recall from Remark 4.2 (2) that for a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, the underlying $k$-module $H$ is an $\mathcal{H}_L$-$\mathcal{H}_R$ bicomodule and an $\mathcal{H}_R$-$\mathcal{H}_L$ bicomodule, via the coactions given by the coproducts. In appropriate situations, taking *cotensor products* with a bicomodule induces a functor between the comodule categories of the two corings, see [20, 22.3] and its Erratum. In Theorem 4.8 functors of this type are considered.

Recall from Section 3.6 that any right comodule of a right bialgebroid over a $k$-algebra $\mathbf{R}$ possesses a unique $\mathbf{R}$-bimodule structure such that any comodule map is $\mathbf{R}$-bilinear. Thus if $\mathcal{H}$ is a Hopf algebroid over the anti-isomorphic base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, then any right comodule of the constituent right (or left) bialgebroid can be regarded as a right $\mathbf{L} \otimes_k \mathbf{R}$-module and the coaction is a right $\mathbf{R} \otimes_k \mathbf{L}$-module map.

THEOREM 4.8. *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid, over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, with structure maps denoted as in Definition 4.1. Consider the $\mathbf{R}$- and $\mathbf{L}$-actions on $H$ that define its coring structures, cf. Definitions 3.1 and 3.3. If the equaliser*

$$(4.4) \qquad M \xrightarrow{\ \varrho\ } M \underset{\mathbf{R}}{\otimes} H \ \underset{M \otimes_{\mathbf{R}} \Delta_R}{\overset{\varrho \otimes_{\mathbf{R}} H}{\rightrightarrows}} \ M \underset{\mathbf{R}}{\otimes} H \underset{\mathbf{R}}{\otimes} H$$

*in the category $\mathcal{M}_{\mathbf{L}}$ of right $\mathbf{L}$-modules is preserved by both functors $-\otimes_{\mathbf{L}} H \otimes_{\mathbf{L}} H$ and $-\otimes_{\mathbf{L}} H \otimes_{\mathbf{R}} H :$ $\mathcal{M}_{\mathbf{L}} \to \mathcal{M}_k$, for any right $\mathcal{H}_R$-comodule $(M, \varrho)$, then the forgetful functor $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}^{\mathcal{H}_R}$ is an isomorphism.*

By standard terminology, the conditions in Theorem 4.8 are phrased as the equaliser (4.4) in $\mathcal{M}_{\mathbf{L}}$ is $H \otimes_{\mathbf{L}} H$-*pure* and $H \otimes_{\mathbf{R}} H$-*pure*. Symmetrical conditions imply that the forgetful functor to the category of right $\mathcal{H}_L$-comodules is an isomorphism.

Theorem 4.8 is proven by constructing the inverse of the forgetful functor, i.e. by equipping any right $\mathcal{H}_R$-comodule with an $\mathcal{H}$-comodule structure. Any right $\mathcal{H}_R$-comodule $M$ is isomorphic, as a right $\mathbf{R}$-module, to the cotensor product $M \square H$ of $M$ with the $\mathcal{H}_R$-$\mathcal{H}_L$ bicomodule $H$. Since under the assumptions in Theorem 4.8, $M \square H$ is a right $\mathcal{H}_L$-comodule via $M \square \Delta_L : M \square H \to M \square (H \otimes_{\mathbf{L}} H) \cong (M \square H) \otimes_{\mathbf{L}} H$, the isomorphism $M \cong M \square H$ induces a right $\mathcal{H}_L$-coaction on $M$. Moreover, by the assumptions in Theorem 4.8, we have isomorphisms $M \square (H \otimes_{\mathbf{L}} H) \cong M \otimes_{\mathbf{L}} H$, $M \square (H \otimes_{\mathbf{R}} H) \cong M \otimes_{\mathbf{R}} H$, $M \square (H \otimes_{\mathbf{R}} H \otimes_{\mathbf{L}} H) \cong M \otimes_{\mathbf{R}} H \otimes_{\mathbf{L}} H$ and $M \square (H \otimes_{\mathbf{L}} H \otimes_{\mathbf{R}} H) \cong M \otimes_{\mathbf{L}} H \otimes_{\mathbf{R}} H$. The compatibility conditions between the $\mathcal{H}_R$- and $\mathcal{H}_L$-coactions follow by these isomorphisms, the Hopf algebroid axioms Definition 4.1 (ii) and functoriality of the cotensor product. With similar methods any $\mathcal{H}_R$-comodule map is checked to be also $\mathcal{H}_L$-colinear.

The purity conditions in Theorem 4.8 are checked to hold in all of the examples in Section 4.1. Moreover, if a Hopf algebroid with a bijective antipode is finitely generated and projective in all of the four senses in Proposition 4.5 then it satisfies all purity conditions in Theorem 4.8 since taking tensor products with flat modules preserves any equaliser.

4.3.2. *Coinvariants in a comodule of a Hopf algebroid.* By Definition 4.6, a comodule $M$ of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is a comodule for both constituent bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$. Since the unit element $1_H$ is grouplike for both corings underlying $\mathcal{H}_L$ and $\mathcal{H}_R$, one can speak about coinvariants

$$M^{co\mathcal{H}_R} = \{\ m \in M \mid \varrho_R(m) = m \underset{\mathbf{R}}{\otimes} 1_{\mathbf{H}}\ \}$$

of $M$ with respect to the $\mathcal{H}_R$-coaction $\varrho_R$, or coinvariants

$$M^{co\mathcal{H}_L} = \{\ m \in M \mid \varrho_L(m) = m \underset{\mathbf{L}}{\otimes} 1_{\mathbf{H}}\ \}$$

with respect to the $\mathcal{H}_L$-coaction $\varrho_L$. Proposition 4.9 relates these two notions. It is of crucial importance from the point of view of Galois theory, see Section 4.5.

PROPOSITION 4.9. *Let $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ be a Hopf algebroid and $(M, \varrho_L, \varrho_R)$ be a right $\mathcal{H}$-comodule. Then any coinvariant of the $\mathcal{H}_R$-comodule $(M, \varrho_R)$ is coinvariant also for the $\mathcal{H}_L$-comodule $(M, \varrho_L)$.*

*If moreover the antipode $S$ is bijective then coinvariants of the $\mathcal{H}_R$-comodule $(M, \varrho_R)$ and the $\mathcal{H}_L$-comodule $(M, \varrho_L)$ coincide.*

For a right $\mathcal{H}$-comodule $(M, \varrho_L, \varrho_R)$, consider the map

$$(4.5) \qquad \Phi_M : M \underset{\mathbf{R}}{\otimes} H \to M \underset{\mathbf{L}}{\otimes} H, \qquad m \otimes_{\mathbf{R}} h \mapsto \varrho_L(m) \cdot S(h),$$

where (using the notations in Definition 4.1) $H$ is a left $\mathbf{L}$-module via the source map $s_L$ and a left $\mathbf{R}$-module via the target map $t_R$, and $M \otimes_{\mathbf{L}} H$ is understood to be a right $H$-module via the

second factor. Since $\Phi_M(\varrho_R(m)) = m \otimes_{\mathbf{L}} 1_H$ and $\Phi_M(m \otimes_{\mathbf{R}} 1_H) = \varrho_L(m)$, we have the first claim in Proposition 4.9 proven. In order to prove the second assertion, note that if $S$ is an isomorphism then so is $\Phi_M$, with inverse $\Phi_M^{-1}(m \otimes_{\mathbf{L}} h) = S^{-1}(h) \cdot \varrho_R(m)$, where $M \otimes_{\mathbf{R}} H$ is understood to be a left $H$-module via the second factor.

4.3.3. *Comodule algebras of a Hopf algebroid.* As it was explained in Section 3.7, from the point of view of Galois theory (in the coaction picture) monoidality of the category of comodules is of central importance. Theorem 4.10 replaces *unjustified* [9, Theorem 2.2] (cf. first paragraph of Section 4.3).

By Definition 4.6 a right comodule of a Hopf algebroid $\mathcal{H}$, over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, is a right $\mathbf{L} \otimes_k \mathbf{R}$-module. Since $\mathbf{L}$ and $\mathbf{R}$ are anti-isomorphic algebras, we may regard, alternatively, any $\mathcal{H}$-comodule as an $\mathbf{R}$-bimodule by translating the right $\mathbf{L}$-action to a left $\mathbf{R}$-action via the algebra anti-isomorphism (4.1).

THEOREM 4.10. *For a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, the category $\mathcal{M}^{\mathcal{H}}$ of right $\mathcal{H}$-comodules is monoidal. Moreover, there are strict monoidal forgetful functors, rendering commutative the following diagram.*

$$
\begin{array}{ccc}
\mathcal{M}^{\mathcal{H}} & \longrightarrow & \mathcal{M}^{\mathcal{H}_R} \\
\downarrow & & \downarrow \\
\mathcal{M}^{\mathcal{H}_L} & \longrightarrow & {}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}.
\end{array}
$$

Commutativity of the diagram in Theorem 4.10 follows by comparing the unique $\mathbf{R}$-actions that make $\mathbf{R}$-bilinear the $\mathcal{H}_R$-coaction and the $\mathcal{H}_L$-coaction in an $\mathcal{H}$-comodule, respectively. Strict monoidality of the functor on the right hand side was proven in Theorem 3.18. Strict monoidality of the functor in the bottom row follows by applying Theorem 3.18 to the opposite of the bialgebroid $\mathcal{H}_L$ and identifying $\mathbf{L}^{op}$-bimodules and $\mathbf{R}$-bimodules via the algebra isomorphism (4.1). In order to see strict monoidality of the remaining two functors, recall that by Theorem 3.18 – applied to $\mathcal{H}_R$ and the opposite of $\mathcal{H}_L$ –, the $\mathbf{R}$-module tensor product of any two $\mathcal{H}$-comodules is an $\mathcal{H}_R$-comodule and an $\mathcal{H}_L$-comodule, via the diagonal coactions, cf. (3.11). It is straightforward to check compatibility of these coactions in the sense of Definition 4.6. Similarly, $\mathbf{R}(\cong \mathbf{L}^{op})$ is known to be an $\mathcal{H}_R$-comodule and an $\mathcal{H}_L$-comodule, and compatibility of the coactions is obvious. Finally, the $\mathbf{R}$-module tensor product of $\mathcal{H}$-comodule maps is an $\mathcal{H}_R$-comodule map and an $\mathcal{H}_L$-comodule map by Theorem 3.18. Thus it is an $\mathcal{H}$-comodule map. By Theorem 3.18 also the coherence natural transformations in ${}_{\mathbf{R}}\mathcal{M}_{\mathbf{R}}$ are $\mathcal{H}_R$- and $\mathcal{H}_L$-comodule maps, so $\mathcal{H}$-comodule maps, what proves Theorem 4.10.

Theorem 4.10 enables us to introduce comodule algebras of Hopf algebroids.

DEFINITION 4.11. A *right comodule algebra* of a Hopf algebroid $\mathcal{H}$ is a monoid in the monoidal category $\mathcal{M}^{\mathcal{H}}$ of right $\mathcal{H}$-comodules. Explicitly, an $\mathbf{R}$-ring $(M, \mu, \eta)$, such that $M$ is a right $\mathcal{H}$-comodule and $\eta : R \to M$ and $\mu : M \otimes_{\mathbf{R}} M \to M$ are right $\mathcal{H}$-comodule maps. Using the notations $m \mapsto m^{[0]} \otimes_{\mathbf{R}} m^{[1]}$ and $m \mapsto m_{[0]} \otimes_{\mathbf{L}} m_{[1]}$ for the $\mathcal{H}_L$- and $\mathcal{H}_R$-coactions, respectively, $\mathcal{H}$-colinearity of $\eta$ and $\mu$ means the identities, for all $m, m' \in M$,

$$
1_{\mathbf{M}}{}^{[0]} \underset{\mathbf{R}}{\otimes} 1_{\mathbf{M}}{}^{[1]} = 1_{\mathbf{M}} \underset{\mathbf{R}}{\otimes} 1_{\mathbf{H}}, \qquad (mm')^{[0]} \underset{\mathbf{R}}{\otimes} (mm')^{[1]} = m^{[0]} m'^{[0]} \underset{\mathbf{R}}{\otimes} m^{[1]} m'^{[1]}
$$

$$
1_{\mathbf{M}[0]} \underset{\mathbf{L}}{\otimes} 1_{\mathbf{M}[1]} = 1_{\mathbf{M}} \underset{\mathbf{L}}{\otimes} 1_{\mathbf{H}}, \qquad (mm')_{[0]} \underset{\mathbf{L}}{\otimes} (mm')_{[1]} = m_{[0]} m'_{[0]} \underset{\mathbf{L}}{\otimes} m_{[1]} m'_{[1]}.
$$

Symmetrically, a *left $\mathcal{H}$-comodule algebra* is a monoid in ${}^{\mathcal{H}}\mathcal{M}$.

The functors in Remark 4.7 induced by the antipode are checked to be strictly anti-monoidal. Therefore, the opposite of a right $\mathcal{H}$-comodule algebra, with coactions in Remark 4.7, is a left $\mathcal{H}$-comodule algebra and conversely. Thus there are four different categories of modules of a comodule algebra of a Hopf algebroid.

DEFINITION 4.12. Let $\mathcal{H}$ be a Hopf algebroid and $M$ be a right $\mathcal{H}$-comodule algebra. Left and right $M$-modules in $\mathcal{M}^{\mathcal{H}}$ are called left-right and right-right *relative Hopf modules*, respectively.

Their categories are denoted by $_M\mathcal{M}^{\mathcal{H}}$ and $\mathcal{M}_M^{\mathcal{H}}$, respectively. Left and right $M^{op}$-modules in $^{\mathcal{H}}\mathcal{M}$ are called right-left and left-left relative Hopf modules, respectively, and their categories are denoted by $^{\mathcal{H}}\mathcal{M}_M$ and $_M^{\mathcal{H}}\mathcal{M}$, respectively.

Explicitly, e.g. a right-right $(M, \mathcal{H})$-relative Hopf module is a right module $W$ for the **R**-ring $M$, such that the action is a right $\mathcal{H}$-comodule map $W \otimes_{\mathbf{R}} M \to W$. Using index notations, with superscripts for the $\mathcal{H}_R$-coactions and subscripts for the $\mathcal{H}_L$-coactions, both on $W$ and $M$, this means the identities, for $w \in W$ and $m \in M$,

$$(w \cdot m)^{[0]} \underset{\mathbf{R}}{\otimes} (w \cdot m)^{[1]} = w^{[0]} \cdot m^{[0]} \underset{\mathbf{R}}{\otimes} w^{[1]} m^{[1]} \qquad \text{and} \qquad (w \cdot m)_{[0]} \underset{\mathbf{L}}{\otimes} (w \cdot m)_{[1]} = w_{[0]} \cdot m_{[0]} \underset{\mathbf{L}}{\otimes} w_{[1]} m_{[1]}.$$

In contrast to relative Hopf modules of bialgebroids in Section 3.7.1, relative Hopf modules of Hopf algebroids can not be identified with comodules of a coring. Still, they determine an adjunction, very similar to (3.16). Consider a right comodule algebra $M$ of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, over base algebras **L** and **R**. Denote the $\mathcal{H}_R$-coinvariant subalgebra of $M$ by **N**. It follows from Proposition 4.9 that for any right **N**-module $V$, $V \otimes_{\mathbf{N}} M$ is a right-right relative Hopf module via the second factor. The resulting functor $- \otimes_{\mathbf{N}} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}}$ turns out to have a right adjoint: Any object $W$ in $\mathcal{M}_M^{\mathcal{H}}$ can be regarded as an object in $\mathcal{M}_M^{\mathcal{H}_R}$, so we can take its $\mathcal{H}_R$-coinvariants (cf. Section 4.3.2). These considerations lead to an adjoint pair of functors

$$(4.6) \qquad - \underset{\mathbf{N}}{\otimes} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}} \qquad \text{and} \qquad (-)^{co\mathcal{H}_R} : \mathcal{M}_M^{\mathcal{H}} \to \mathcal{M}_{\mathbf{N}}.$$

The unit of the adjunction is given, for any right **N**-module $V$, by the map

$$(4.7) \qquad V \to (V \underset{\mathbf{N}}{\otimes} M)^{co\mathcal{H}_R}, \qquad v \mapsto v \underset{\mathbf{N}}{\otimes} 1_{\mathbf{M}}$$

and the counit is given, for an $(M, \mathcal{H})$-relative Hopf module $W$, by

$$(4.8) \qquad W^{co\mathcal{H}_R} \underset{\mathbf{N}}{\otimes} M \to M, \qquad w \underset{\mathbf{N}}{\otimes} m \mapsto w \cdot m.$$

The message of this observation is that studying *descent theory* of Galois extensions of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, one can examine both the adjunction in (4.6), corresponding to an $\mathcal{H}$-comodule algebra $M$, and also the adjunction (3.16), determined by $M$, regarded as an $\mathcal{H}_R$-comodule algebra. Proposition 4.13 is obtained by observing that the units of the two adjunctions coincide, and the counit of the adjunction in (4.8) is obtained by restricting to the objects in $\mathcal{M}_M^{\mathcal{H}}$ the counit of the adjunction (3.16).

PROPOSITION 4.13. *Consider a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ and a right $\mathcal{H}$-comodule algebra $M$. Denote the $\mathcal{H}_R$-coinvariant subalgebra of $M$ by **N**.*

*(1) The functor $- \otimes_{\mathbf{N}} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}_R}$ is fully faithful if and only if the functor $- \otimes_{\mathbf{N}} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}}$ is fully faithful.*

*(2) If the functor $- \otimes_{\mathbf{N}} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}_R}$ is an equivalence then also the functor $- \otimes_{\mathbf{N}} M : \mathcal{M}_{\mathbf{N}} \to \mathcal{M}_M^{\mathcal{H}}$ is an equivalence.*

4.3.4. *The Fundamental Theorem of Hopf modules.* In this section we investigate the adjunction (4.6) in a special case.

The coproducts $\Delta_L$ and $\Delta_R$ in a Hopf algebroid $\mathcal{H}$ make the underlying algebra **H** a right $\mathcal{H}$-comodule algebra. Corresponding right-right relative Hopf modules are called simply Hopf modules and their category is denoted by $\mathcal{M}_H^H$. Coinvariants of the right $\mathcal{H}_R$-comodule algebra $H$ are the elements $t_R(r)$, for $r \in R$, where $t_R$ is the target map. If $\mathcal{H}_R$ is the underlying bialgebroid in a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, then $t_R : \mathbf{R}^{op} \to \mathbf{H}$, equivalently, $s_L : \mathbf{L} \to \mathbf{H}$, is a right $\mathcal{H}_R$-Galois extension, cf. Section 4.6.2. Hence Theorem 4.14, known as the *Fundamental Theorem of Hopf modules*, can be interpreted as a Descent Theorem for this Galois extension $\mathbf{L} \cong \mathbf{R}^{op} \subseteq \mathbf{H}$.

THEOREM 4.14. *For a Hopf algebroid $\mathcal{H}$, over base algebras **L** and **R**, the functor $- \otimes_{\mathbf{L}} H : \mathcal{M}_{\mathbf{L}} \to \mathcal{M}_H^H$ is an equivalence.*

Theorem 4.14 is proven by constructing the inverses of the unit (4.7) and the counit (4.8) of the relevant adjunction. Use the notations for the structure maps of a Hopf algebroid in Definition 4.1. For a right **L**-module $V$, the inverse of (4.7) is the map $(V \otimes_{\mathbf{L}} H)^{co\mathcal{H}_R} \to V$, $\sum_i v_i \otimes_{\mathbf{L}} h_i \mapsto$

$\sum_i v_i \cdot s_L(\epsilon_L(h_i))$. For a Hopf module $W$, denote the $\mathcal{H}_R$-coaction by $w \mapsto w^{[0]} \otimes_{\mathbf{R}} w^{[1]}$ and for the $\mathcal{H}_L$-coaction write $w \mapsto w_{[0]} \otimes_{\mathbf{L}} w_{[1]}$. Then an epimorphism $W \to W^{co\mathcal{H}_R}$ is given by $w \mapsto w^{[0]} \cdot S(w^{[1]})$. The inverse of (4.8) is the map $W \to W^{co\mathcal{H}_R} \otimes_{\mathbf{L}} H$, $w \mapsto w_{[0]}{}^{[0]} \cdot S(w_{[0]}{}^{[1]}) \otimes_{\mathbf{L}} w_{[1]}$.

4.4. **Integral theory.** In a Hopf algebra $H$, over a commutative ring $k$, integrals are invariants of the regular module of the underlying $k$-algebra $\mathbf{H}$, with respect to a character given by the counit. The study of integrals provides a lot of information about the structure of the $k$-algebra $\mathbf{H}$. Most significantly, ($k$-relative) semisimplicity of $\mathbf{H}$ is equivalent to its separability over $k$, and also to the existence of a normalised integral in $H$ [22], [45]. Since this extends Maschke's theorem about group algebras, it is known as a Maschke type theorem. Its dual version relates cosemisimplicity and coseparability of the $k$-coalgebra underlying $H$ to the existence of normalised cointegrals [22]. Another group of results concerns Frobenius property of $\mathbf{H}$, equivalent to the existence of a non-degenerate integral in $H$ [44], [55]. Integral theory of Hopf algebras was generalised to Hopf algebroids in [7].

DEFINITION 4.15. For an algebra $\mathbf{R}$, consider a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.1. *Right integrals* in $\mathcal{B}$ are the invariants of the right regular module of the underlying $\mathbf{R}$-ring $(\mathbf{B}, s)$, with respect to the right character $\epsilon$. Equivalently, invariants of the right regular module of the $\mathbf{R}^{op}$-ring $(\mathbf{B}, t)$, with respect to $\epsilon$. That is, the elements of

$$B_\epsilon = \{\ i \in B \mid ib = is(\epsilon(b)), \quad \forall b \in B\ \} = \{\ i \in B \mid ib = it(\epsilon(b)), \quad \forall b \in B\ \} \cong \operatorname{Hom}_{\mathbf{B}}(R, B).$$

A right integral $i$ is *normalised* if $\epsilon(i) = 1_{\mathbf{R}}$.

Symmetrically, *left integrals* in a left $\mathbf{R}$-bialgebroid are defined as invariants of the left regular module of the underlying $\mathbf{R}$-ring or $\mathbf{R}^{op}$-ring, with respect to a left character defined by the counit. Note that a left integral $i$ in a left bialgebroid $\mathcal{B}$ is a left integral also in $\mathcal{B}_{cop}$ and a right integral in the right bialgebroids $\mathcal{B}^{op}$ and $\mathcal{B}^{op}_{cop}$.

Since the counit in a right (resp. left) bialgebroid is a right (resp. left) character, there is no way to consider left (resp. right) integrals in a right (resp. left) bialgebroid. On the contrary, the base algebra in a right bialgebroid $\mathcal{B}$ is both a right and a left $\mathcal{B}$-comodule, via coactions given by the source and target maps, respectively. Hence there are corresponding notions of left and right cointegrals.

DEFINITION 4.16. For an algebra $\mathbf{R}$, consider a right $\mathbf{R}$-bialgebroid $\mathcal{B}$, with structure maps denoted as in Definition 3.1. A *right cointegral* on $\mathcal{B}$ is an element of

$$\operatorname{Hom}^B(B, R) = \{\ \iota \in \operatorname{Hom}_{\mathbf{R}}(B, R) \mid (\iota \underset{\mathbf{R}}{\otimes} B) \circ \Delta = s \circ \iota\}.$$

A right cointegral $\iota$ is *normalised* if $\iota(1_{\mathbf{B}}) = 1_{\mathbf{R}}$.

Symmetrically, a *left cointegral* on $\mathcal{B}$ is an element of ${}^B\operatorname{Hom}(B, R)$.

Left and right cointegrals on a left $\mathbf{R}$-bialgebroid $\mathcal{B}$ are defined analogously, as left and right comodule maps $B \to R$.

If a right $\mathbf{R}$-bialgebroid $\mathcal{B}$ is finitely generated and projective as a, say right, $\mathbf{R}$-module (via right multiplication by the source map), then the isomorphism $\operatorname{Hom}_{\mathbf{R}}(R, B^*) \cong \operatorname{Hom}_{\mathbf{R}}(B, R)$ induces an isomorphism ${}_{B^*}\operatorname{Hom}(R, B^*) \cong \operatorname{Hom}^B(B, R)$. Hence in this case left integrals in the left $\mathbf{R}$-bialgebroid $\mathcal{B}^*$ are the same as right cointegrals on $\mathcal{B}$. Similar statements hold for all other duals of left and right bialgebroids.

For a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, left (resp. right) cointegrals on $\mathcal{H}_L$ and $\mathcal{H}_R$ can be shown to be left (resp. right) $\mathcal{H}$-comodule maps.

4.4.1. *Maschke type theorems.* Recall that an $\mathbf{R}$-ring $B$ is said to be *separable* provided that the multiplication map $B \otimes_{\mathbf{R}} B \to B$ is a split epimorphism of $B$-bimodules. The $\mathbf{R}$-ring $B$ is said to be left (resp. right) *semisimple* (or sometimes $\mathbf{R}$-relatively semisimple) if every left (resp. right) $B$-module is $\mathbf{R}$-relative projective. That is, every $B$-module epimorphism, which has an $\mathbf{R}$-module section, is a split epimorphism of $B$-modules. By a classical result due to Hirata and Sugano [37], a separable $\mathbf{R}$-ring is left, and right semisimple. For Hopf algebroids also the converse can be proven.

THEOREM 4.17. *For a Hopf algebroid* $(\mathcal{H}_L, \mathcal{H}_R, S)$ *over base algebras* **L** *and* **R***, denote the structure maps as in Definition 4.1. The following properties are equivalent.*

   (i) *The* **R***-ring* $(\mathbf{H}, s_R)$ *underlying* $\mathcal{H}_R$ *is separable.*
   (ii) *The* $\mathbf{R}^{op}$*-ring* $(\mathbf{H}, t_R)$ *underlying* $\mathcal{H}_R$ *is separable.*
   (iii) *The* **L***-ring* $(\mathbf{H}, s_L)$ *underlying* $\mathcal{H}_L$ *is separable.*
   (iv) *The* $\mathbf{L}^{op}$*-ring* $(\mathbf{H}, t_L)$ *underlying* $\mathcal{H}_L$ *is separable.*
   (v) *The* **R***-ring* $(\mathbf{H}, s_R)$ *underlying* $\mathcal{H}_R$ *is right semisimple.*
   (vi) *The* $\mathbf{R}^{op}$*-ring* $(\mathbf{H}, t_R)$ *underlying* $\mathcal{H}_R$ *is right semisimple.*
   (vii) *The* **L***-ring* $(\mathbf{H}, s_L)$ *underlying* $\mathcal{H}_L$ *is left semisimple.*
   (viii) *The* $\mathbf{L}^{op}$*-ring* $(\mathbf{H}, t_L)$ *underlying* $\mathcal{H}_L$ *is left semisimple.*
   (ix) *There exists a normalised right integral in* $\mathcal{H}_R$*.*
   (x) *There exists a normalised left integral in* $\mathcal{H}_L$*.*
   (xi) *The counit* $\epsilon_R$ *in* $\mathcal{H}_R$ *is a split epimorphism of right* **H***-modules.*
   (xii) *The counit* $\epsilon_L$ *in* $\mathcal{H}_L$ *is a split epimorphism of left* **H***-modules.*

Since the source map in a right **R**-bialgebroid is a right **R**-module section of the counit, implication (v)⇒(xi) is obvious. For a right **H**-module section $\nu$ of the counit, $\nu(1_\mathbf{R})$ is a normalised right integral. Thus (xi)⇒(ix). The antipode in a Hopf algebroid maps a normalised right integral in $\mathcal{H}_R$ to a normalised left integral in $\mathcal{H}_L$, and vice versa. So (ix)⇔(x). If $i$ is a normalised left integral in $\mathcal{H}_L$, then the map $H \to H \otimes_\mathbf{R} H$, $h \mapsto hi^{(1)} \otimes_\mathbf{R} S(i^{(2)}) = i^{(1)} \otimes_\mathbf{R} S(i^{(2)})h$ is an **H**-bimodule section of the multiplication in the **R**-ring underlying $\mathcal{H}_R$ (where the index notation $\Delta_R(h) = h^{(1)} \otimes_\mathbf{R} h^{(2)}$ is used, for $h \in H$.). This proves (x)⇒(i). The remaining equivalences follow by symmetry. Note that equivalences (iv)⇔(viii)⇔(x) ⇔(xii) hold also for a $\times_L$-Hopf algebra $\mathcal{H}_L$ (discussed Section 4.6.2).

As an alternative of Theorem 4.17, one can ask about properties of the $\mathbf{R} \otimes_k \mathbf{R}^{op}$-ring, underlying a right bialgebroid $\mathcal{H}_R$, and the $\mathbf{L} \otimes_k \mathbf{L}^{op}$-ring, underlying a left bialgebroid $\mathcal{H}_L$, in a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$. Theorem 4.18 is obtained by application of [15, Theorem 6.5]. For a $k$-algebra **L**, consider a left **L**-bialgebroid $\mathcal{H}_L$. Denote its $\mathbf{L} \otimes_k \mathbf{L}^{op}$-ring structure by $(\mathbf{H}, s_L, t_L)$ and its **L**-coring structure by $(H, \Delta_L, \epsilon_L)$. Look at $L$ as a left $\mathbf{L} \otimes_k \mathbf{L}^{op}$-module, with action given by left and right multiplications. Look at $H$ as a right $\mathbf{L} \otimes_k \mathbf{L}^{op}$-module, with action given by right multiplications by $s_L$ and $t_L$. Note that

$$(4.9) \qquad H \otimes_{\mathbf{L} \otimes_k \mathbf{L}^{op}} L \cong H/\{\ hs_L(l) - ht_L(l) \mid h \in H,\ l \in L\ \}$$

is an **L**-coring (via quotient maps of $\Delta_L$ and $\epsilon_L$) and a left **H**-module. Hence we can speak about the invariants of $H \otimes_{\mathbf{L} \otimes_k \mathbf{L}^{op}} L$ with respect to $\epsilon_L$. An invariant of $H \otimes_{\mathbf{L} \otimes_k \mathbf{L}^{op}} L$ is said to be *normalised* if the quotient of $\epsilon_L$ maps it to $1_\mathbf{L}$.

THEOREM 4.18. *Consider a Hopf algebroid* $(\mathcal{H}_L, \mathcal{H}_R, S)$*, over base $k$-algebras* **L** *and* **R***. The following assertions are equivalent.*

   (i) *The* $\mathbf{R} \otimes_k \mathbf{R}^{op}$*-ring underlying* $\mathcal{H}_R$ *is separable.*
   (ii) *The* $\mathbf{L} \otimes_k \mathbf{L}^{op}$*-ring underlying* $\mathcal{H}_L$ *is separable.*
   (iii) *The* $\mathbf{R} \otimes_k \mathbf{R}^{op}$*-ring underlying* $\mathcal{H}_R$ *is right semisimple.*
   (iv) *The* $\mathbf{L} \otimes_k \mathbf{L}^{op}$*-ring underlying* $\mathcal{H}_L$ *is left semisimple.*
   (v) *There is a normalised invariant in the right* **H***-module* $R \otimes_{\mathbf{R} \otimes_k \mathbf{R}^{op}} H$*.*
   (vi) *There is a normalised invariant in the left* **H***-module* $H \otimes_{\mathbf{L} \otimes_k \mathbf{L}^{op}} L$*.*

For a $\times_L$-Hopf algebra $\mathcal{H}_L$ (discussed in Section 4.6.2), equivalences (ii)⇔(iv)⇔(vi) in Theorem 4.18 hold true.

Recall that an **R**-coring $B$ is said to be *coseparable* provided that the comultiplication map $B \to B \otimes_\mathbf{R} B$ is a split monomorphism of $B$-bicomodules. The **R**-coring $B$ is said to be left (resp. right) *cosemisimple* (or sometimes **R**-relatively cosemisimple) if every left (resp. right) $B$-comodule is **R**-relative injective. That is, every $B$-comodule monomorphism, which has an **R**-module retraction, is a split monomorphism of $B$-comodules. A coseparable **R**-coring is left, and right cosemisimple. For Hopf algebroids also the converse can be proven.

THEOREM 4.19. *For a Hopf algebroid* $(\mathcal{H}_L, \mathcal{H}_R, S)$ *over base algebras* **L** *and* **R**, *the following properties are equivalent.*

   (i)   *The* **R**-*coring underlying* $\mathcal{H}_R$ *is coseparable.*
   (ii)  *The* **L**-*coring underlying* $\mathcal{H}_L$ *is coseparable.*
   (iii) *The* **R**-*coring underlying* $\mathcal{H}_R$ *is right cosemisimple.*
   (iv)  *The* **R**-*coring underlying* $\mathcal{H}_R$ *is left cosemisimple.*
   (v)   *The* **L**-*coring underlying* $\mathcal{H}_L$ *is right cosemisimple.*
   (vi)  *The* **L**-*coring underlying* $\mathcal{H}_L$ *is left cosemisimple.*
   (vii) *There exists a normalised right cointegral on* $\mathcal{H}_R$.
  (viii) *There exists a normalised left cointegral on* $\mathcal{H}_R$.
   (ix)  *There exists a normalised right cointegral on* $\mathcal{H}_L$.
   (x)   *There exists a normalised left cointegral on* $\mathcal{H}_L$.
   (xi)  *The source map in* $\mathcal{H}_R$ *is a split right* $\mathcal{H}_R$-*comodule monomorphism.*
   (xii) *The target map in* $\mathcal{H}_R$ *is a split left* $\mathcal{H}_R$-*comodule monomorphism.*
  (xiii) *The source map in* $\mathcal{H}_L$ *is a split left* $\mathcal{H}_L$-*comodule monomorphism.*
   (xiv) *The target map in* $\mathcal{H}_L$ *is a split right* $\mathcal{H}_L$-*comodule monomorphism.*

4.4.2. *Frobenius Hopf algebroids.* It was proven by Larson and Sweedler in [44] that every finite dimensional Hopf algebra over a field is a Frobenius algebra. Although this is not believed to be true for any finitely generated projective Hopf algebra over a commutative ring, Frobenius Hopf algebras form a distinguished class. A Hopf algebra is known to be a Frobenius algebra if and only if it possesses a non-degenerate integral [55]. It is a self-dual property: a non-degenerate integral determines a non-degenerate cointegral, i.e. a non-degenerate integral in the dual Hopf algebra.

In a Hopf algebroid there are four algebra extensions present: the ones given by the source and target maps of the two constituent bialgebroids. Among Hopf algebroids, those in which these are Frobenius extensions, play an even more distinguished role. Although the dual of any finitely generated projective Hopf algebroid is not known to be a Hopf algebroid, duals of Frobenius Hopf algebroids are Frobenius Hopf algebroids.

While every finitely generated and projective Hopf algebra over a commutative ring $k$ was proven by Pareigis to be a quasi-Frobenius $k$-algebra in [55], an analogous statement fails to hold for Hopf algebroids. In [7, Section 5] Hopf algebroids were constructed, which are finitely generated and projective over their base algebras (in all the four senses in Proposition 4.5), but are not quasi-Frobenius extensions of the base algebra.

Recall (e.g. from [39]) that an **R**-ring $(\mathbf{H}, s)$ is said to be *Frobenius* provided that $H$ is a finitely generated and projective left **R**-module and $^*H := {}_{\mathbf{R}}\mathrm{Hom}(H, R)$ is isomorphic to $H$ as an **H**-**R** bimodule. Equivalently, if $H$ is a finitely generated and projective right **R**-module and $H^* := \mathrm{Hom}_{\mathbf{R}}(H, R)$ is isomorphic to $H$ as an **R**-**H** bimodule. These properties are equivalent also to the existence of an **R**-bimodule map $\psi : \mathbf{H} \to \mathbf{R}$, the so called Frobenius functional, possessing a dual basis $\sum_i e_i \otimes_{\mathbf{R}} f_i \in H \otimes_{\mathbf{R}} H$, satisfying, for all $h \in H$, $\sum_i e_i \cdot \psi(f_i h) = h = \sum_i \psi(h e_i) \cdot f_i$. The following characterisation of Frobenius Hopf algebroids was obtained in [7, Theorem 4.7], see the *Corrigendum.*

THEOREM 4.20. *Consider a Hopf algebroid* $\mathcal{H}$, *over base algebras* **L** *and* **R**, *with structure maps denoted as in Definition 4.1. Assume that* $H$ *is finitely generated and projective as a right* $R$-*module via right multiplication by* $s_R$, *as a left* $R$-*module via right multiplication by* $t_R$, *as a left* $L$-*module via left multiplication by* $s_L$ *and as a right* $L$-*module via left multiplication by* $t_L$. *The following statements are equivalent.*

   (i)   *The* **R**-*ring* $(\mathbf{H}, s_R)$ *is Frobenius.*
   (ii)  *The* $\mathbf{R}^{op}$-*ring* $(\mathbf{H}^{op}, t_R)$ *is Frobenius.*
   (ii)  *The* **L**-*ring* $(\mathbf{H}, s_L)$ *is Frobenius.*
   (iv)  *The* $\mathbf{L}^{op}$-*ring* $(\mathbf{H}^{op}, t_L)$ *is Frobenius.*
   (v)   *There exists a right cointegral* $\iota$ *on* $\mathcal{H}_R$, *such that the map* $\widetilde{\iota} : H \to \mathrm{Hom}_R(H, R)$, $h \mapsto \iota(h-)$ *is bijective.*

(vi) *There exists a left cointegral $v$ on $\mathcal{H}_L$, such that the map $\widetilde{v} : H \to {}_{\mathbf{L}}\mathrm{Hom}(H, L)$, $h \mapsto v(-h)$ is bijective.*

(vii) *There exists a right integral $i$ in $\mathcal{H}_R$, such that the maps $\widetilde{i} : {}_{\mathbf{L}}\mathrm{Hom}(H, L) \to H$, $\psi \mapsto t_L(\psi(i_{(2)}))i_{(1)}$ and is bijective.*

(viii) *There exists a left integral $j$ in $\mathcal{H}_L$, such that the map $\widetilde{j} : \mathrm{Hom}_{\mathbf{R}}(H, R) \to H$, $\phi \mapsto j^{(2)}t_R(\phi(j^{(1)}))$ is bijective.*

*A Hopf algebroid for which these equivalent conditions hold is said to be* Frobenius, *and a right (resp. left) integral obeying property (vii) (resp. (viii)) is said to be* non-degenerate.

*In a Frobenius Hopf algebroid the antipode $S$ is bijective.*

If $j$ is a non-degenerate left integral in $\mathcal{H}_L$ then $\iota := (\widetilde{j})^{-1}(1_{\mathbf{H}})$ is a right cointegral on $\mathcal{H}_R$ and a Frobenius functional for the $\mathbf{R}$-ring $(\mathbf{H}, s_R)$ with dual basis $j^{(1)} \otimes_{\mathbf{R}} S(j^{(2)})$. Thus (viii)$\Rightarrow$(i). If property (i) holds, then the right cointegrals on $\mathcal{H}_R$ are shown to form a free rank one left $\mathbf{R}$-module $\mathcal{I}$, via the action $r \cdot \iota = \iota(t_R(r)-)$. Using finitely generated projectivity of the right $\mathbf{R}$-module $H$, the dual $H^* := \mathrm{Hom}_{\mathbf{R}}(H, R)$ can be equipped with a Hopf module structure, with coinvariants $\mathcal{I}$. Hence Theorem 4.14 implies an isomorphism $H^* \cong H \otimes_{\mathbf{R}} \mathcal{I}$. This isomorphism is used to show that the cyclic generator $\iota$ of the $\mathbf{R}$-module $\mathcal{I}$ satisfies condition (v). If there is a right cointegral $\iota$ as in part (v), then a non-degenerate left integral $j$ as in part (viii) is constructed in terms of $(\widetilde{\iota})^{-1}$ and a dual basis for the finitely generated projective right $\mathbf{R}$-module $H$. It is shown to satisfy $(\widetilde{j})^{-1} = \widetilde{\iota} \circ S$, which implies bijectivity of $S$. The remaining equivalences follow by relations between the source and target maps in $\mathcal{H}_L$ and $\mathcal{H}_R$, and symmetrical versions of the arguments above.

For a Frobenius Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, over base algebras $\mathbf{L}$ and $\mathbf{R}$, all the four duals $\mathrm{Hom}_{\mathbf{R}}(H, R)$, ${}_{\mathbf{R}}\mathrm{Hom}(H, R)$, $\mathrm{Hom}_{\mathbf{L}}(H, L)$ and ${}_{\mathbf{L}}\mathrm{Hom}(H, L)$ possess (left or right) bialgebroid structures. A left integral $j$ in $\mathcal{H}_L$, such that the map $\widetilde{j}$ in part (viii) of Theorem 4.20 is bijective, determines further similar isomorphisms ${}_{\mathbf{R}}\mathrm{Hom}(H, R) \to H$, $\mathrm{Hom}_{\mathbf{L}}(H, L) \to H$ and ${}_{\mathbf{L}}\mathrm{Hom}(H, L) \to H$. What is more, putting $\iota := (\widetilde{j})^{-1}(1_{\mathbf{H}})$, there is an algebra automorphism of $H$,

$$(4.10) \qquad\qquad \zeta : H \to H, \qquad h \mapsto h^{(2)}t_R(\iota(j\, h^{(1)})).$$

These isomorphisms combine to bialgebroid (anti-) isomorphisms between the four duals of $H$, cf. [13, Theorem 5.16].

Frobenius property of a Hopf algebroid was shown to be self-dual in [13, Theorem 5.17 and Proposition 5.19], in the following sense.

THEOREM 4.21. *Consider a Frobenius Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, over base algebras $\mathbf{L}$ and $\mathbf{R}$. Let $j$ be a left integral in $\mathcal{H}_L$, such that the map $\widetilde{j}$ in part (viii) of Theorem 4.20 is bijective. Then the left $\mathbf{R}$-bialgebroid $H^* := \mathrm{Hom}_{\mathbf{R}}(H, R)$ extends to a Hopf algebroid. A bijective antipode is given in terms of the map (4.10), by $S^* := (\widetilde{j})^{-1} \circ S \circ \zeta \circ \widetilde{j}$. The right bialgebroid structure is determined by the requirement that $S^*$ is a bialgebroid anti-isomorphism in the sense of Proposition 4.4. This dual Hopf algebroid is Frobenius, with non-degenerate left integral $(\widetilde{j})^{-1}(1_{\mathbf{H}}) \in \mathcal{H}^*$.*

In a paper [70] by Szlachányi, an equivalent description of a Frobenius Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ was proposed, via so called *double algebras*. In this picture the isomorphism $\widetilde{j}$ in part (viii) of Theorem 4.20 is used to transfer the multiplication in $H^* := \mathrm{Hom}_{\mathbf{R}}(H, R)$ to a second algebra structure in $H$, with unit $j$. In this way four Frobenius ring structures are obtained on $H$. Note that the coproducts in $\mathcal{H}_L$ and $\mathcal{H}_R$ correspond canonically to the Frobenius ring structures transferred from $H^*$. In this approach the Hopf algebroid axioms are formulated as compatibility conditions between the two algebra structures on $H$.

4.5. **Galois theory of Hopf algebroids.** Galois extensions by Hopf algebras are the same as Galois extensions by the constituent bialgebra. Still, since the structure of a Hopf algebra is more complex than that of a bialgebra, it allows to derive stronger results. On the contrary, Galois theory of a Hopf algebroid is more conceptually different from Galois theory of the constituent bialgebroids: As it is discussed in Section 4.3, comodules of a Hopf algebroid carry more structure

than comodules of any constituent bialgebroid. Consequently, Galois theory of Hopf algebroids, discussed in the current section, is significantly richer than the theory of bialgebroids. In particular, for a comodule algebra of a Hopf algebroid (cf. Section 4.3.3), several theorems concerning an equivalence between the category of relative Hopf modules and the category of modules of the coinvariant subalgebra – i.e. descent theorems – can be proven.

By Definition 4.11 and Theorem 4.10, a right comodule algebra $M$ of a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ is both an $\mathcal{H}_L$-comodule algebra and an $\mathcal{H}_R$-comodule algebra. Denote the $\mathcal{H}_R$-coinvariant subalgebra of $M$ by $\mathbf{N}$. In light of Proposition 4.9, there are two corresponding canonical maps

$$(4.11) \qquad M \underset{\mathbf{N}}{\otimes} M \to M \underset{\mathbf{R}}{\otimes} H, \qquad m \underset{\mathbf{N}}{\otimes} m' \mapsto mm'^{[0]} \underset{\mathbf{R}}{\otimes} m'^{[1]} \qquad \text{and}$$

$$(4.12) \qquad M \underset{\mathbf{N}}{\otimes} M \to M \underset{\mathbf{L}}{\otimes} H, \qquad m \underset{\mathbf{N}}{\otimes} m' \mapsto m_{[0]} m' \underset{\mathbf{L}}{\otimes} m_{[1]},$$

where $m \mapsto m^{[0]} \otimes_R m^{[1]}$ and $m \mapsto m_{[0]} \otimes_L m_{[1]}$ denote the $\mathcal{H}_R$-coaction and the $\mathcal{H}_L$-coaction on $M$, respectively. In general, bijectivity of the two canonical maps (4.11) and (4.12), are not known to be equivalent. Only a partial result [8, Lemma 3.3] is known.

PROPOSITION 4.22. *If the antipode $S$ in a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ is bijective, then the $\mathcal{H}_R$-canonical map (4.11) is bijective if and only if the $\mathcal{H}_L$-canonical map (4.12) is bijective.*

This follows by noting that the two canonical maps differ by the isomorphism $\Phi_A$ in (4.5).

By Proposition 4.9 and Proposition 4.22, a for right comodule algebra $M$ of a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ with a bijective antipode, an algebra extension $\mathbf{N} \subseteq \mathbf{M}$ is $\mathcal{H}_L$-Galois if and only if it is $\mathcal{H}_R$-Galois.

REMARK 4.23. Consider a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ over base algebras $\mathbf{L}$ and $\mathbf{R}$, which is finitely generated and projective in the equivalent senses in Proposition 4.5 (2). Then $H$ is in particular flat as a left $\mathbf{L}$-module and as a left $\mathbf{R}$-module. Therefore, by Theorem 4.8 and Theorem 4.10, the category of right $\mathcal{H}$-comodules is strict monoidally isomorphic to the category of comodules for any of the constituent left and right bialgebroids $\mathcal{H}_L$ and $\mathcal{H}_R$. Thus comodule algebras for $\mathcal{H}$ coincide with comodule algebras for $\mathcal{H}_L$ or $\mathcal{H}_R$.

If furthermore the antipode is bijective, we conclude by Proposition 4.9 and Proposition 4.22 that an algebra extension $\mathbf{N} \subseteq \mathbf{M}$ is a right $\mathcal{H}_R$-Galois extension if and only if it is a right $\mathcal{H}_L$-Galois extension.

4.5.1. *Depth two Frobenius extensions.* An analogue of Theorem 3.27 for Frobenius Hopf algebroids is [4, Theorem 3.6].

THEOREM 4.24. *An algebra extension $\mathbf{N} \subseteq \mathbf{M}$ is a right Galois extension by some Frobenius Hopf algebroid (i.e. by any of its constituent bialgebroids) if and only if it is a Frobenius extension, it is balanced, and satisfies the (left and right) depth 2 conditions.*

In a case of a Frobenius extension $\mathbf{N} \subseteq \mathbf{M}$, left and right depth 2 properties are equivalent. By Remark 4.23, for a Frobenius Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, $\mathcal{H}_L$- and $\mathcal{H}_R$-Galois properties of an extension are equivalent. By Theorem 3.24, a right depth 2 and balanced algebra extension $\mathbf{N} \subseteq \mathbf{M}$ is a Galois extension by a right bialgebroid $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$. If in addition $\mathbf{N} \subseteq \mathbf{M}$ is a Frobenius extension then $(M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$ is shown to be a constituent right bialgebroid in a Frobenius Hopf algebroid. A non-degenerate (left and right) integral $\sum_i m_i \otimes_{\mathbf{N}} m_i' \in (M \otimes_{\mathbf{N}} M)^{\mathbf{N}}$ is provided by the dual basis of a Frobenius functional $\psi : M \to N$. A non-degenerate (left and right) integral in the dual Hopf algebra $_{\mathbf{N}}\text{End}_{\mathbf{N}}(M)$ is $\psi$. In the converse direction, note that a right comodule $M$ of a Hopf algebroid $\mathcal{H}$ is an $\mathcal{H}^*$-module. If $\mathcal{H}$ is a Frobenius Hopf algebroid and $\mathbf{N} \subseteq \mathbf{M}$ is an $\mathcal{H}$-Galois extension, a Frobenius functional $M \to N$ is given by the action by a non-degenerate integral in $\mathcal{H}^*$.

4.5.2. *Cleft extensions by Hopf algebroids.* For an algebra $\mathbf{M}$ and a coalgebra $C$ over a commutative ring $k$, $\text{Hom}_k(C, M)$ is a $k$-algebra via the convolution product

$$(4.13) \qquad (f \diamond g)(c) := f(c_{(1)})g(c_{(2)}), \qquad \text{for } f, g \in \text{Hom}_k(C, M), \ c \in C.$$

A comodule algebra $M$ of a $k$-Hopf algebra $H$ is said to be a cleft extension of its coinvariant subalgebra $\mathbf{N}$ provided that there exists a convolution invertible map $j \in \mathrm{Hom}_k(H, M)$ which is an $H$-comodule map. The relevance of cleft extensions by Hopf algebras stems from Doi and Takeuchi's observation in [29] that $\mathbf{N} \subseteq \mathbf{M}$ is a cleft extension if and only if it is a Galois extension and an additional normal basis property holds, i.e. $M \cong N \otimes_k H$ as a left $\mathbf{N}$-module right $H$-comodule. What is more, (establishing an even stronger similarity with Galois extensions of fields), $\mathbf{N} \subseteq \mathbf{M}$ is a cleft extension if and only if $\mathbf{M}$ is isomorphic to a crossed product of $\mathbf{N}$ with $H$ with respect to an invertible 2-cocycle [29], [5].

Above results have been extended to Hopf algebroids in [9]. In order to formulate the definition of a cleft extension, as a first step, a generalised convolution product has to be introduced. Using notations as in Definition 4.1, in a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ there is an $\mathbf{L}$-coring $(H, \Delta_L, \epsilon_L)$ and an $\mathbf{R}$-coring $(H, \Delta_R, \epsilon_R)$ present. Consider an $\mathbf{L} \otimes_k \mathbf{R}$-ring $M$, with multiplications $\mu_{\mathbf{L}} : M \otimes_{\mathbf{L}} M \to M$ and $\mu_{\mathbf{R}} : M \otimes_{\mathbf{R}} M \to M$. For these data, the convolution algebra (4.13) can be generalised to a *convolution category*. It has two objects, conveniently labelled by $\mathbf{L}$ and $\mathbf{R}$. For $P, Q \in \{\mathbf{L}, \mathbf{R}\}$, morphisms from $P$ to $Q$ are $Q$-$P$ bimodule maps $H \to M$, where the bimodule structure of the domain is determined by the ($P$- and $Q$-) coring structures of $H$ and the bimodule structure of the codomain is determined by the ($P$- and $Q$-) ring structures of $M$. For $P, Q, T \in \{\mathbf{L}, \mathbf{R}\}$, and morphisms $f : Q \to P$ and $g : T \to Q$, composition is given by a convolution product

$$(4.14) \qquad\qquad f \diamond g := \mu_Q \circ (f \underset{Q}{\otimes} g) \circ \Delta_Q.$$

Recall from Theorem 4.10 that a right comodule algebra of a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ has a canonical $\mathbf{R}$-ring structure over the base algebra $\mathbf{R}$ of $\mathcal{H}_R$.

DEFINITION 4.25. Consider a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, over base algebras $\mathbf{L}$ and $\mathbf{R}$. A right $\mathcal{H}$-comodule algebra $M$ is said to be a *cleft extension* of the $\mathcal{H}_R$-coinvariant subalgebra $\mathbf{N}$ provided that the following properties hold.

  (i) The canonical $\mathbf{R}$-ring structure of $M$ extends to an $\mathbf{L} \otimes_k \mathbf{R}$-ring structure.
  (ii) There exists an invertible morphism $j : \mathbf{R} \to \mathbf{L}$ in the convolution category (4.14) which is a right $\mathcal{H}$-comodule map.

In an $\mathcal{H}$-cleft extension $\mathbf{N} \subseteq \mathbf{M}$, $\mathbf{N}$ can be proven to be an $\mathbf{L}$-subring of $\mathbf{M}$.

In a Hopf algebroid $\mathcal{H}$, using the notations introduced in Definition 4.1, an $\mathbf{L} \otimes_k \mathbf{R}$-ring is given by $(\mathbf{H}, s_L, s_R)$. The identity map of $H$ is a morphism $\mathbf{R} \to \mathbf{L}$ in the corresponding convolution category (4.14). It is obviously right $\mathcal{H}$-colinear. What is more, the antipode is its inverse by axioms (iii) and (iv) in Definition 4.1. Hence the right regular comodule algebra of a Hopf algebroid is a cleft extension of the coinvariant subalgebra $t_R(\mathbf{R})$. This extends a well known fact that the right regular comodule algebra of a Hopf algebra, over a commutative ring $k$, is a cleft extension of $k$. A further similarity between cleft extensions by Hopf algebras and Hopf algebroids is expressed by the following theorem.

THEOREM 4.26. *Consider a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, over base algebras $\mathbf{L}$ and $\mathbf{R}$. Its right comodule algebra $M$ is a cleft extension of the $\mathcal{H}_R$-coinvariant subalgebra $\mathbf{N}$ if and only if the following properties hold.*

  (i) $\mathbf{N} \subseteq \mathbf{M}$ *is a Galois extension by $\mathcal{H}_R$;*
  (ii) *the* normal basis condition *holds, i.e. $M \cong N \otimes_{\mathbf{L}} H$ as left $\mathbf{N}$-modules right $\mathcal{H}$-comodules.*

Note the appearance of the two base algebras $\mathbf{L}$ and $\mathbf{R}$ in conditions (i) and (ii) in Theorem 4.26.

Another characterisation of a cleft extension by a Hopf algebroid can be given by using the construction of a crossed product.

DEFINITION 4.27. Consider a left bialgebroid $\mathcal{B}$ over a base $k$-algebra $\mathbf{L}$. Denote its structure maps as in Definition 3.3. We say that $\mathcal{B}$ *measures* an $\mathbf{L}$-ring $N$ with unit map $\iota : \mathbf{L} \to N$ if there exists a $k$-module map $\cdot : B \otimes_k N \to N$, the so called *measuring*, such that, for $b \in B$, $l \in L$ and $n, n' \in N$, the following axioms are satisfied.

(i) $b \cdot 1_{\mathbf{N}} = \iota(\epsilon(b))$,

(ii) $(t(l)b) \cdot n = (b \cdot n)\iota(l)$ and $(s(l)b) \cdot n = \iota(l)(b \cdot n)$,

(iii) $b \cdot (nn') = (b_{(1)} \cdot n)(b_{(2)} \cdot n')$.

Note that in Definition 4.27 condition (iii) makes sense in view of (ii).

Consider a left bialgebroid $\mathcal{B}$ over a $k$-algebra $\mathbf{L}$ and denote its structure maps as in Definition 3.3. Let $N$ be an $\mathbf{L}$-ring, with unit $\iota : \mathbf{L} \to \mathbf{N}$, which is measured by $\mathcal{B}$. These data determine a category $\mathcal{C}(\mathcal{B}, N)$ as follows. Consider $B \otimes_k B$ as an $\mathbf{L}$-bimodule, via left multiplication by $s$ and $t$ in the first factor. For an element $f$ in ${}_{\mathbf{L}}\mathrm{Hom}_{\mathbf{L}}(B \otimes_k B, N)$, consider the following ($\mathbf{L}$-balancing) conditions. For $a, b \in B$ and $l \in L$,

$(T\circ)$ $\qquad\qquad\qquad\qquad f(a \underset{k}{\otimes} s(l)b) = f(as(l) \underset{k}{\otimes} b)$

$(S\circ)$ $\qquad\qquad\qquad\qquad f(a \underset{k}{\otimes} t(l)b) = f(at(l) \underset{k}{\otimes} b)$

$(T\bullet)$ $\qquad\qquad\qquad\qquad f(a \underset{k}{\otimes} s(l)b) = (a_{(1)} \cdot \iota(l))f(a_{(2)} \underset{k}{\otimes} b)$

$(S\bullet)$ $\qquad\qquad\qquad\qquad f(a \underset{k}{\otimes} t(l)b) = f(a_{(1)} \underset{k}{\otimes} b)(a_{(2)} \cdot \iota(l))$.

Define a category $\mathcal{C}(\mathcal{B}, N)$ of two objects $\circ$ and $\bullet$. For two objects $X, Y \in \{\circ, \bullet\}$, morphisms $X \to Y$ are elements of ${}_{\mathbf{L}}\mathrm{Hom}_{\mathbf{L}}(B \otimes_k B, N)$, satisfying conditions $(SX)$ and $(TY)$. Composition of morphisms $g : X \to Y$ and $f : Y \to Z$ is given by

$$(f \diamond g)(a \underset{k}{\otimes} b) := f(a_{(1)} \underset{k}{\otimes} b_{(1)})g(a_{(2)} \underset{k}{\otimes} b_{(2)}).$$

Unit morphism at the object $\circ$ is the map $a \otimes_k b \mapsto (ab) \cdot 1_{\mathbf{N}} = \iota(\epsilon(ab))$ and unit morphism at the object $\bullet$ is the map $a \otimes_k b \mapsto a \cdot (b \cdot 1_{\mathbf{N}})$.

DEFINITION 4.28. Consider a left bialgebroid $\mathcal{B}$ over a $k$-algebra $\mathbf{L}$ and denote its structure maps as in Definition 3.3. Let $N$ be a $\mathcal{B}$-measured $\mathbf{L}$-ring, with unit $\iota : \mathbf{L} \to \mathbf{N}$. An $N$-valued *2-cocycle* on $\mathcal{B}$ is a morphism $\circ \to \bullet$ in the category $\mathcal{C}(\mathcal{B}, N)$ above, such that, for $a, b, c \in B$,

(i) $\sigma(1_{\mathbf{B}}, b) = \iota\big(\epsilon(b)\big) = \sigma(b, 1_{\mathbf{B}})$,

(ii) $\big(a_{(1)} \cdot \sigma(b_{(1)}, c_{(1)})\big)\sigma(a_{(2)}, b_{(2)}c_{(2)}) = \sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c)$.

The $\mathcal{B}$-measured $L$-ring $N$ is called a *$\sigma$-twisted $\mathcal{B}$-module* if in addition, for $n \in N$ and $a, b \in B$,

(iii) $1_{\mathbf{B}} \cdot n = n$,

(iv) $\big(a_{(1)} \cdot (b_{(1)} \cdot n)\big)\sigma(a_{(2)}, b_{(2)}) = \sigma(a_{(1)}, b_{(1)})(a_{(2)}b_{(2)} \cdot n)$.

An $N$-valued 2-cocycle $\sigma$ on $\mathcal{B}$ is said to be *invertible* if it is invertible as a morphism in $\mathcal{C}(\mathcal{B}, N)$.

Note that in Definition 4.28 conditions (ii) and (iv) make sense in view of the module map and balanced properties of $\sigma$. If an $\mathbf{L}$-ring $N$ is measured by a left $\mathbf{L}$-bialgebroid $\mathcal{B}$, then the $\mathbf{L}^{op}$-ring $N^{op}$ is measured by the co-opposite left $\mathbf{L}^{op}$-bialgebroid $\mathcal{B}_{cop}$. The inverse of an $N$-valued 2-cocycle $\sigma$ on $\mathcal{B}$ turns out to be an $N^{op}$-valued 2-cocycle on $\mathcal{B}_{cop}$.

For a left bialgebroid $\mathcal{B}$ over an algebra $\mathbf{L}$, with structure maps denoted as in Definition 3.3, the base algebra $\mathbf{L}$ is measured by $\mathcal{B}$, via the left action $b \cdot l := \epsilon(bs(l))$. For this measuring conditions $(S\circ)$ and $(S\bullet)$ are equivalent and also conditions $(T\circ)$ and $(T\bullet)$ are equivalent. Consequently, an $\mathbf{L}$-valued 2-cocycle on $\mathcal{B}$ in the sense of Definition 4.28 is equivalent to a cocycle considered in Section 3.4.2. Extending cocycle double twists in Section 3.4.2, one can consider more general deformations of a Hopf algebroid $\mathcal{H}$ (or a $\times_L$-Hopf algebra $\mathcal{B}$, discussed in Section 4.6.2) by an $N$-valued invertible 2-cocycle $\sigma$ in Definition 4.28, cf. [10, Appendix]. In that construction the base algebra $\mathbf{L}$ of $\mathcal{H}_L$ is replaced by an $\mathcal{H}_L$-measured $\mathbf{L}$-ring $N$. In particular, Connes and Moscovici's bialgebroids in Section 3.4.6 arise in this way.

A *crossed product* $N\#_\sigma B$ of a left $\mathbf{L}$-bialgebroid $\mathcal{B}$ with a $\sigma$-twisted $\mathcal{B}$-module $N$, with respect to an $N$-valued 2-cocycle $\sigma$, is the the $\mathbf{L}$-module tensor product $N \otimes_{\mathbf{L}} B$ (where $B$ is a left $\mathbf{L}$-module via the source map $s$), with associative and unital multiplication

$$(n\#b)(n'\#b') = n(b_{(1)} \cdot n')\sigma(b_{(2)}, b'_{(1)})\#b_{(3)}b'_{(2)}, \qquad \text{for } n\#b, n'\#b' \in N \underset{\mathbf{L}}{\otimes} B.$$

Equivalence classes of crossed products with a bialgebroid were classified in [9, Section 4].

THEOREM 4.29. *A right comodule algebra $M$ of a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$ is a cleft extension of the $\mathcal{H}_R$-coinvariant subalgebra $\mathbf{N}$ if and only if $M$ is isomorphic, as a left $\mathbf{N}$-module and right $\mathcal{H}$-comodule algebra, to a crossed product algebra $N\#_\sigma \mathcal{H}_L$, with respect to some invertible $N$-valued 2-cocycle $\sigma$ on $\mathcal{H}_L$.*

4.5.3. *The structure of Galois extensions by Hopf algebroids.* In the theory of Galois extensions by Hopf algebras (with a bijective antipode) important tools are provided by theorems which state that in appropriate situations *surjectivity* of the canonical map implies its bijectivity, i.e. Galois property of an algebra extension $\mathbf{N} \subseteq \mathbf{M}$. There are two big groups of such theorems. In the first group a Hopf algebra $H$ is assumed to be a flat module over its commutative base ring $k$, and its regular comodule algebra is assumed to be a projective $H$-comodule. These properties hold in particular if $H$ is a finitely generated and projective $k$-module, in which case such a theorem was proven first by Kreimer and Takeuchi [43]. In another group of such results, due to Schneider, $H$ is assumed to be a projective $k$-module and its comodule algebra $M$ is assumed to be a $k$-relative injective $H$-comodule [67].

Analogous results for extensions by a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ were obtained in the papers [8] and [4], and in [1], respectively. A common philosophy behind such theorems originates from a work [66] of Schauenburg (on the Hopf algebra case). The key idea is to investigate a lifting (4.16) of the canonical map (4.11), introduced for an $\mathcal{H}$-comodule algebra $M$ below. By a general result [19, Theorem 2.1] about Galois comodules, split surjectivity of the lifted canonical map (4.16), as a morphism of relative $(M, \mathcal{H}_R)$-Hopf modules, implies $\mathcal{H}_R$-Galois property, i.e. bijectivity of (4.11), whenever $(M \otimes_\mathbf{T} M)^{co\mathcal{H}_R} = M \otimes_\mathbf{T} M^{co\mathcal{H}_R}$, where $(-)^{co\mathcal{H}_R}$ denotes the $\mathcal{H}_R$-coinvariants functor.

For a Hopf algebra $H$ over a commutative ring $k$, and a right $H$-comodule algebra $M$ with coinvariant subalgebra $\mathbf{N}$, the canonical map $M \otimes_\mathbf{N} M \to M \otimes_k H$ can be lifted to a map

$$(4.15) \qquad M \underset{k}{\otimes} M \longrightarrow M \underset{\mathbf{N}}{\otimes} M \xrightarrow{\text{can}} M \underset{k}{\otimes} H \ .$$

More generally, consider a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$ over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$. For a right $\mathcal{H}$-comodule algebra $M$, the canonical map (4.11) can be lifted to

$$(4.16) \qquad M \underset{\mathbf{T}}{\otimes} M \to M \underset{\mathbf{R}}{\otimes} H, \qquad m \underset{\mathbf{T}}{\otimes} m' \mapsto mm'^{[0]} \underset{\mathbf{R}}{\otimes} m'^{[1]},$$

for any $k$-algebra $\mathbf{T}$ such that the $\mathcal{H}_R$-coinvariant subalgebra $\mathbf{N}$ of $M$ is a $\mathbf{T}$-ring. The map (4.16) is a morphism of right-right $(M, \mathcal{H})$-relative Hopf modules.

THEOREM 4.30. *Consider a Hopf algebroid $\mathcal{H}$, over base $k$-algebras $\mathbf{L}$ and $\mathbf{R}$, with a bijective antipode. Denote its structure maps as in Definition 4.1. Assume that $H$ is a flat left $\mathbf{R}$-module (via right multiplication by $t_R$) and a projective right $\mathcal{H}_R$-comodule (via $\Delta_R$). Let $M$ be a right $\mathcal{H}$-comodule algebra with $\mathcal{H}_R$-coinvariant subalgebra $\mathbf{N}$. Under these assumptions the following statements hold.*

   (1) *If the $\mathcal{H}_R$-coinvariants of the right $\mathcal{H}$-comodule $M \otimes_k M$ (with coactions given via the second factor) are precisely the elements of $M \otimes_k N$, then the canonical map (4.11) is bijective if and only if it is surjective.*

   (2) *If the canonical map (4.11) is bijective then $M$ is a projective right $\mathbf{N}$-module.*

Since coinvariants are defined as a kernel, coinvariants of $M \otimes_k M$ are are precisely the elements of $M \otimes_k N$ if e.g. $M$ is a flat $k$-module. In order to have an impression about the proof of part (1) of Theorem 4.30, note that flatness of the left $\mathbf{R}$-module $H$ and projectivity of the right regular $\mathcal{H}_R$-comodule together imply that $M \otimes_\mathbf{R} H$ is projective as a right-right $(M, \mathcal{H}_R)$-relative Hopf module. Hence if the the canonical map (4.11) is surjective then the (surjective) lifted canonical map (4.16) is a split epimorphism of right-right $(M, \mathcal{H}_R)$-relative Hopf modules, for any possible $k$-algebra $\mathbf{T}$. Thus bijectivity of the canonical map (4.11) follows by [19, Theorem 2.1]. Part (2) of Theorem 4.30 follows by exactness of the naturally equivalent functors $\text{Hom}^{\mathcal{H}_R}(H, -) \cong \text{Hom}_\mathbf{N}(M, (-)^{co\mathcal{H}_R})$, which is a consequence of the projectivity of the right regular $\mathcal{H}_R$-comodule.

If in a Hopf algebroid $\mathcal{H}$ with a bijective antipode, $H$ is a finitely generated and projective $\mathbf{L}$-, or $\mathbf{R}$-module in all of the four senses occurring in Proposition 4.5, then it is obviously a flat left

**R**-module. Furthermore, in this case the right regular $\mathcal{H}_R$-comodule can be shown to be projective, by using Theorem 4.14. Following Weak Structure Theorem 4.31 is thus based on Remark 4.23, part (2) of Theorem 4.30 and its application to the right comodule algebra $M^{op}$ of the opposite Hopf algebroid $\mathcal{H}^{op}$, and a theorem [23, Theorem 3.5] about Galois corings.

THEOREM 4.31. *Consider a Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, *over base algebras* **L** *and* **R**, *with a bijective antipode* $S$. *Assume that* $H$ *is a finitely generated and projective* **L**-, *or* **R**-*module in all of the four senses occurring in Proposition 4.5. For a right* $\mathcal{H}$-*comodule algebra* $M$, *with* $\mathcal{H}_R$-*coinvariant subalgebra* **N**, *the following statements are equivalent.*

- (i) *The extension* $\mathbf{N} \subseteq \mathbf{M}$ *is* $\mathcal{H}_R$-*Galois.*
- (ii) $M$ *is a generator in the category* $\mathcal{M}_M^{\mathcal{H}} \cong \mathcal{M}_M^{\mathcal{H}_R} \cong \mathcal{M}_M^{\mathcal{H}_L}$.
- (iii) *The* $\mathcal{H}_R$-*coinvariants functor* $\mathcal{M}_M^{\mathcal{H}} \to \mathcal{M}_{\mathbf{N}}$ *is fully faithful.*
- (iv) *The extension* $\mathbf{N} \subseteq \mathbf{M}$ *is* $\mathcal{H}_L$-*Galois.*
- (v) $M$ *is a generator in the category* $_M\mathcal{M}^{\mathcal{H}} \cong {}_M\mathcal{M}^{\mathcal{H}_R} \cong {}_M\mathcal{M}^{\mathcal{H}_L}$.
- (vi) *The* $\mathcal{H}_L$-*coinvariants functor* $_M\mathcal{M}^{\mathcal{H}} \to {}_{\mathbf{N}}\mathcal{M}$ *is fully faithful.*

*Furthermore, if these equivalent conditions hold then* $M$ *is a projective left and right* **N**-*module.*

It was a key observation by Doi that relative injectivity of a comodule algebra $M$ of a Hopf algebra $H$ is equivalent to the existence of a so called total integral – meaning an $H$-comodule map $j : H \to M$, such that $j(1_{\mathbf{H}}) = 1_{\mathbf{M}}$ [28]. This fact extends to Hopf algebroids as well. Recall (e.g. from [34]) that, for any functor $\mathbb{U} : \mathcal{A} \to \mathcal{B}$, between any categories $\mathcal{A}$ and $\mathcal{B}$, an object $A \in \mathcal{A}$ is said to be $\mathbb{U}$-*injective*, if the map $\mathrm{Hom}_{\mathcal{A}}(g, A) : \mathrm{Hom}_{\mathcal{A}}(Y, A) \to \mathrm{Hom}_{\mathcal{A}}(X, A)$ is surjective, for any objects $X, Y \in \mathcal{A}$, and all such morphisms $g \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ for that $\mathbb{U}(g)$ is a split monomorphism in $\mathcal{B}$. If $\mathbb{U}$ has a right adjoint, then $\mathbb{U}$-injectivity of an object $A$ is equivalent to the unit of the adjunction, evaluated at $A$, being a split monomorphism in $\mathcal{A}$, see [34, Proposition 1]. For example, for a Hopf algebra $H$ over a commutative ring $k$, injective objects with respect to the forgetful functor $\mathcal{M}^H \to \mathcal{M}_k$ are precisely relative injective $H$-comodules.

A version of Theorem 4.32 below was proven in [1, Theorem 4.1], using the notion of *relative separability* of a forgetful functor. Recall from Remark 4.7 that the opposite of a right comodule algebra $M$ of a Hopf algebroid $\mathcal{H}$ has a canonical structure of a left $\mathcal{H}$-comodule algebra.

THEOREM 4.32. *For a right comodule algebra* $M$ *of a Hopf algebroid* $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$, *the following statements are equivalent.*

- (i) *There exists a right* $\mathcal{H}$-*comodule map (resp. right* $\mathcal{H}_L$-*comodule map)* $j : H \to M$, *such that* $j(1_{\mathbf{H}}) = 1_{\mathbf{M}}$.
- (ii) $M$ *is injective with respect to the forgetful functor* $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}_{\mathbf{L}}$ *(resp. with respect to the forgetful functor* $\mathcal{M}^{\mathcal{H}_L} \to \mathcal{M}_{\mathbf{L}}$, *i.e.* $M$ *is a relative injective* $\mathcal{H}_L$-*comodule).*
- (iii) *Any object in the category* $\mathcal{M}_M^{\mathcal{H}}$ *of right-right relative Hopf modules is injective with respect to the forgetful functor* $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}_{\mathbf{L}}$ *(resp. with respect to the forgetful functor* $\mathcal{M}^{\mathcal{H}_L} \to \mathcal{M}_{\mathbf{L}}$).

*If in addition the antipode* $S$ *is bijective then assertions (i)-(iii) are equivalent also to*

- (iv) *There exists a left* $\mathcal{H}$-*comodule map (resp. left* $\mathcal{H}_R$-*comodule map)* $j' : H \to M$, *such that* $j'(1_{\mathbf{H}}) = 1_{\mathbf{M}}$.

*Hence assertions (i)-(iv) are equivalent also to the symmetrical versions of (ii) and (iii).*

The key idea behind Theorem 4.32 is the observation that both forgetful functors $\mathcal{M}^{\mathcal{H}} \to \mathcal{M}_{\mathbf{L}}$ and $\mathcal{M}^{\mathcal{H}_L} \to \mathcal{M}_{\mathbf{L}}$ possess left adjoints $- \otimes_{\mathbf{L}} H$ (cf. (4.3)). A correspondence can be established between comodule maps $j$ as in part (i), and natural retractions of the counit of the adjunction, i.e. of the $\mathcal{H}_L$-coaction.

Based on [21, Theorem 4.7] and Theorem 4.32, also the following Strong Structure Theorem holds.

THEOREM 4.33. *Consider a Hopf algebroid* $(\mathcal{H}_L, \mathcal{H}_R, S)$, *over base algebras* **L** *and* **R**, *with a bijective antipode* $S$. *Assume that* $H$ *is a finitely generated and projective* **L**-, *or* **R**-*module in any*

*(hence all) of the senses listed in Proposition 4.5. For a right $\mathcal{H}_R$-, (equivalently, right $\mathcal{H}_L$-) Galois extension $\mathbf{N} \subseteq \mathbf{M}$, the following statements are equivalent.*

   (i) *$M$ is a faithfully flat right $\mathbf{N}$-module.*

   (ii) *The inclusion $\mathbf{N} \to \mathbf{M}$ splits in $\mathcal{M}_\mathbf{N}$.*

   (ii) *$M$ is a generator of right $\mathbf{N}$-modules.*

   (iii) *$M$ is a faithfully flat left $\mathbf{N}$-module.*

   (iv) *The inclusion $\mathbf{N} \to \mathbf{M}$ splits in $_\mathbf{N}\mathcal{M}$.*

   (v) *$M$ is a generator of left $\mathbf{N}$-modules.*

   (vi) *The functor $- \otimes_\mathbf{N} M : \mathcal{M}_\mathbf{N} \to \mathcal{M}_M^\mathcal{H}$ is an equivalence.*

   (vii) *$M$ is a projective generator in $\mathcal{M}_M^\mathcal{H}$.*

   (viii) *There exists a right $\mathcal{H}$-comodule map $j : M \to H$, such that $j(1_\mathbf{H}) = 1_\mathbf{M}$.*

Note that if a Hopf algebra $H$, over a commutative ring $k$, is a projective $k$-module, then $M \otimes_k H$ is a projective left $\mathbf{M}$-module, for any right $H$-comodule algebra $M$. Thus, denoting the subalgebra of coinvariants in $M$ by $\mathbf{N}$, surjectivity of the canonical map $M \otimes_\mathbf{N} M \to M \otimes_k H$ implies that its lifted version (4.15) is a split epimorphism of left $\mathbf{M}$-modules, so in particular of $k$-modules. By Schneider's result [67, Theorem I], if the antipode of $H$ is bijective and $M$ is a $k$-relative injective right $H$-comodule, then bijectivity of the canonical map follows from the $k$-module splitting of its lifted version (4.15). In order to formulate following generalisation Theorem 4.34 of this result, note that the lifted canonical map (4.16) is an $\mathbf{L}$-bimodule map, with respect to the $\mathbf{L}$-actions $l \cdot (m \otimes_\mathbf{T} m') \cdot l' := m \cdot \epsilon_R(s_L(l)) \otimes_\mathbf{T} \epsilon_R(t_L(l')) \cdot m'$ (recall that the $\mathbf{R}$-, and $\mathbf{T}$-actions on $M$ commute by virtue of (3.14)) and $l \cdot (m \otimes_\mathbf{R} h) \cdot l' := m \otimes_\mathbf{R} s_L(l)t_L(l')h$, on its domain and codomain, respectively, where notations in Definition 4.1 are used.

THEOREM 4.34. *Consider a Hopf algebroid $\mathcal{H}$ with a bijective antipode, over base algebras $\mathbf{L}$ and $\mathbf{R}$. Denote its structure maps as in Definition 4.1. Let $M$ be a right $\mathcal{H}$-comodule algebra with $\mathcal{H}_R$-coinvariants $\mathbf{N}$. Let $\mathbf{T}$ be a $k$-algebra, such that $\mathbf{N}$ is a $\mathbf{T}$-ring. In this setting, if the lifted canonical map (4.16) is a split epimorphism of right $\mathbf{L}$-modules then the following statements are equivalent.*

   (i) *$\mathbf{N} \subseteq \mathbf{M}$ is an $\mathcal{H}_R$-Galois extension and the inclusion $\mathbf{N} \to \mathbf{M}$ splits in $\mathcal{M}_\mathbf{N}$.*

   (ii) *$\mathbf{N} \subseteq \mathbf{M}$ is an $\mathcal{H}_R$-Galois extension and the inclusion $\mathbf{N} \to \mathbf{M}$ splits in $_\mathbf{N}\mathcal{M}$.*

   (iii) *There exists a right $\mathcal{H}$-comodule map $j : H \to M$, such that $j(1_\mathbf{H}) = 1_\mathbf{M}$.*

   (iv) *$M \otimes_\mathbf{N} - : {}_\mathbf{N}\mathcal{M} \to {}_M^\mathcal{H}\mathcal{M}$ is an equivalence and the inclusion $\mathbf{N} \to \mathbf{M}$ splits in $\mathcal{M}_\mathbf{N}$.*

*Furthermore, if the equivalent properties (i)-(iii) hold then $M$ is a $\mathbf{T}$-relative projective right $\mathbf{N}$-module.*

Note that by Proposition 4.9 and Proposition 4.22, in parts (i) and (ii) $\mathcal{H}_R$-Galois property can be replaced equivalently by $\mathcal{H}_L$-Galois property. Also, by Theorem 4.32, the existence of a unit preserving right $\mathcal{H}$-comodule map in part (iii) can be replaced equivalently by the existence of a unit preserving left $\mathcal{H}$-comodule map.

The most interesting part of Theorem 4.34 is perhaps the claim that if property (iii) holds then right $\mathbf{L}$-module splitting of the lifted canonical map (4.16) implies $\mathcal{H}_R$-Galois property. The proof of this fact is based on an observation, originated from [1], that assertion (iii) is equivalent to relative separability of the forgetful functor $\mathcal{M}^\mathcal{H} \to \mathcal{M}_\mathbf{L}$, with respect to the forgetful functor $\mathcal{M}_M^\mathcal{H} \to \mathcal{M}^\mathcal{H}$. A relative separable functor reflects split epimorphisms in the sense that if $f$ is a morphism in $\mathcal{M}_M^\mathcal{H}$, which is a split epimorphism of right $\mathbf{L}$-modules, then it is a split epimorphism of right $\mathcal{H}$-comodules. This proves that, under the assumptions made, the lifted canonical map (4.16) is a split epimorphism of right $\mathcal{H}$-comodules. Furthermore, the forgetful functor $\mathcal{M}_M^\mathcal{H} \to \mathcal{M}^\mathcal{H}$ possesses a left adjoint $- \otimes_\mathbf{R} M$. Hence the right-right $(M, \mathcal{H})$-relative Hopf module $M \otimes_\mathbf{R} H$, which is isomorphic to $H \otimes_\mathbf{R} M$ by bijectivity of the antipode, is relative projective in the sense that a split epimorphism $g$ in $\mathcal{M}^\mathcal{H}$, of codomain $H \otimes_\mathbf{R} M \cong M \otimes_\mathbf{R} H$, is a split epimorphism in $\mathcal{M}_M^\mathcal{H}$. This proves that in the situation considered the lifted canonical map (4.16) is a split epimorphism of right-right $(M, \mathcal{H})$-relative Hopf modules. Then it is a split epimorphism of right-right $(M, \mathcal{H}_R)$-relative Hopf modules. Moreover, in terms of a unit preserving right $\mathcal{H}$-comodule

map $H \to M$, a left $\mathbf{N}$-module splitting of the equaliser of the $\mathcal{H}_R$-coaction on $M$, and the map $m \mapsto m \otimes_{\mathbf{R}} 1_{\mathbf{H}}$, can be constructed. This implies that the equaliser is preserved by the functor $M \otimes_{\mathbf{T}} -$, i.e. $(M \otimes_{\mathbf{T}} M)^{co\mathcal{H}_R} = M \otimes_{\mathbf{T}} N$. Thus the canonical map (4.11) is bijective by [19, Theorem 2.1].

Recall that a left module $V$ of a $k$-algebra $\mathbf{N}$ is $k$-relative projective if and only if the left action $N \otimes_k V \to V$ is a split epimorphism of left $\mathbf{N}$-modules. If $V$ has an additional structure of a right comodule for a $k$-coalgebra $C$, such that the $\mathbf{N}$-action is a right $C$-comodule map, then it can be asked if the action $N \otimes_k V \to V$ splits as a map of left $\mathbf{N}$-modules and right $C$-comodules too. In the case when it does, $V$ is said to be a $C$-equivariantly projective left $\mathbf{N}$-module. For a Galois extension $\mathbf{N} \subseteq \mathbf{M}$ by a $k$-Hopf algebra $H$, $H$-equivariant projectivity of the left $\mathbf{N}$-module $M$ was shown by Hajac to be equivalent to the existence of a strong connection [35]. Interpreting a Hopf Galois extension as a non-commutative principal bundle, this means its local triviality. In case of a Galois extension $\mathbf{N} \subseteq \mathbf{M}$ by a $k$-Hopf algebra $H$, equivalent conditions (i)-(iii) in Theorem 4.34 are known to imply $H$-equivariant projectivity of the left $\mathbf{N}$-module $M$. In order to obtain an analogous result for a Galois extension by a Hopf algebroid, slightly stronger assumptions are needed, see Theorem 4.36 below.

DEFINITION 4.35. Consider a Hopf algebroid $\mathcal{H}$ and consider a $\mathbf{T}$-ring $N$ for some algebra $\mathbf{T}$. Let $V$ be a left $\mathbf{N}$-module and right $\mathcal{H}$-comodule, such that the left $\mathbf{N}$-action on $V$ is a right $\mathcal{H}$-comodule map. $V$ is said to be a $\mathbf{T}$-*relative* $\mathcal{H}$-*equivariantly projective* left $\mathbf{N}$-module provided that the left action $N \otimes_{\mathbf{T}} V \to V$ is an epimorphism split by a left $\mathbf{N}$-module, right $\mathcal{H}$-comodule map.

THEOREM 4.36. *Consider a Hopf algebroid $\mathcal{H}$ with a bijective antipode. Let $M$ be a right $\mathcal{H}$-comodule algebra with $\mathcal{H}_R$-coinvariants $\mathbf{N}$. Let $\mathbf{T}$ be an algebra, such that $\mathbf{N}$ is a $\mathbf{T}$-ring. Assume that there exists a unit preserving right $\mathcal{H}$-comodule map $H \to M$ and that the lifted canonical map (4.16) is a split epimorphism of $\mathbf{L}$-bimodules. Then $\mathbf{N} \subseteq \mathbf{M}$ is a right $\mathcal{H}_R$-, and right $\mathcal{H}_L$-Galois extension and $M$ is a $\mathbf{T}$-relative $\mathcal{H}$-equivariantly projective left $\mathbf{N}$-module.*

Under the premises of Theorem 4.36, $\mathcal{H}_R$-Galois property holds by Theorem 4.34 and the $\mathcal{H}_L$-Galois property follows by Proposition 4.9 and Proposition 4.22. Proof of Theorem 4.36 is completed by constructing a required left $\mathbf{N}$-module right $\mathcal{H}$-comodule section of the action $N \otimes_{\mathbf{T}} M \to M$. The construction makes use of the relative Hopf module section of the lifted canonical map (4.16), on the existence of which it is concluded in the paragraph following Theorem 4.34. Examples of $\mathbf{L}$-relative $\mathcal{H}$-equivariantly projective Galois extensions are provided by cleft extensions by a Hopf algebroid $\mathcal{H}$ with a bijective antipode, over base algebras $\mathbf{L}$ and $\mathbf{R}$.

4.6. **Alternative notions.** In the literature there is an accord that the right generalisation of a bialgebra to the case of a non-commutative base algebra is a bialgebroid. On the contrary, there is some discussion about the structure to replace a Hopf algebra. In current final section we revisit and compare the various suggestions.

4.6.1. *Lu's Hopf algebroid.* In Definition 4.1 the antipode axioms are formulated for a compatible pair of a left and a right bialgebroid. In following Definition 4.37, quoted from [46], only a left bialgebroid is used. While the first one of the antipode axioms in Definition 4.1 (iv) is easily formulated also in this case, in order to formulate the second one some additional assumption is needed.

DEFINITION 4.37. Consider a left bialgebroid $\mathcal{B}$, over a $k$-algebra $\mathbf{L}$, with structure maps denoted as in Definition 3.3. $\mathcal{B}$ is a *Lu's Hopf algebroid* provided that there exists an anti-algebra map $S : \mathbf{B} \to \mathbf{B}$, and a $k$-module section $\xi$ of the canonical epimorphism $B \otimes_k B \to B \otimes_{\mathbf{L}} B$, such that the following axioms are satisfied.

(i) $S \circ t = s$,
(ii) $\mu_B \circ (S \otimes_{\mathbf{L}} B) \circ \Delta = t \circ \epsilon \circ S$,
(iii) $\mu_{\mathbf{B}} \circ (B \otimes_k S) \circ \xi \circ \Delta = s \circ \epsilon$,

where $\mu_B$ denotes multiplication in the $\mathbf{L}$-ring $(B, s)$ and $\mu_{\mathbf{B}}$ denotes multiplication in the underlying $k$-algebra $\mathbf{B}$.

None of the notions of a Hopf algebroid in Definition 4.1, or in Definition 4.37, seems to be more general than the other one. Indeed, a Hopf algebroid in the sense of Definition 4.1, which does not satisfy the axioms in Definition 4.37, is constructed as follows. Let $k$ be a commutative ring in which 2 is invertible. For the order 2 cyclic group $Z_2$, consider the group bialgebra $kZ_2$ as a left bialgebroid over $k$. Equip it with the twisted (bijective) antipode $S$, mapping the order 2 generator $t$ of $Z_2$ to $S(t) := -t$. Together with the unique right bialgebroid, determined by the requirement that $S$ is a bialgebroid anti-isomorphism in the sense of Proposition 4.4, they constitute a Hopf algebroid as in Definition 4.1. However, for this Hopf algebroid there exists no section $\xi$ as in Definition 4.37.

4.6.2. *A $\times_R$-Hopf algebra.* The coinvariants of the (left or right) regular comodule of a bialgebra $H$, over a commutative ring $k$, are precisely the multiples of the unit element $1_{\mathbf{H}}$. $H$ is known to be a Hopf algebra if and only if $\mathbf{H}$ is an $H$-Galois extension of $k$. Indeed, the hom-tensor relation $_H\mathrm{Hom}^H(H \otimes_k H, H \otimes_k H) \cong \mathrm{Hom}_k(H, H)$ relates the inverse of the canonical map to the antipode. Motivated by this characterisation of a Hopf algebra, in [62] Schauenburg proposed the following definition.

DEFINITION 4.38. Let $\mathcal{B}$ be a left bialgebroid over an algebra $\mathbf{L}$, with structure maps denoted as in Definition 3.3. Consider the left regular $\mathcal{B}$-comodule, whose coinvariant subalgebra is $t(\mathbf{L}^{op})$. $\mathcal{B}$ is said to be a *$\times_L$-Hopf algebra* provided that the algebra extension $t : \mathbf{L}^{op} \to \mathbf{B}$ is left $\mathcal{B}$-Galois.

The notion of a $\times_L$-Hopf algebra in Definition 4.38 is more general than that of a Hopf algebroid in Definition 4.1. Indeed, consider a Hopf algebroid $(\mathcal{H}_L, \mathcal{H}_R, S)$, over the base algebras $\mathbf{L}$ and $\mathbf{R}$, with structure maps denoted as in Definition 4.1. The canonical map $H \otimes_{\mathbf{L}^{op}} H \to H \otimes_{\mathbf{L}} H$, $h \otimes h' \mapsto h_{(1)} \otimes h_{(2)}h'$ is bijective, with inverse $h \otimes h' \mapsto h^{(1)} \otimes S(h^{(2)})h'$. Hence $\mathcal{H}_L$ is a $\times_L$-Hopf algebra. Although it is believed that not every a $\times_L$-Hopf algebra is a constituent left bialgebroid in a Hopf algebroid, we do not know about any counterexample.

Extending a result [59] by Schauenburg about Hopf algebras, the following proposition was proven in [38].

PROPOSITION 4.39. *Consider a left bialgebroid $\mathcal{B}$ over a base algebra $\mathbf{L}$. If there is a left $\mathcal{B}$-Galois extension $\mathbf{N} \subseteq \mathbf{M}$, such that $M$ is a faithfully flat left $\mathbf{L}$-module, then $\mathcal{B}$ is a $\times_L$-Hopf algebra.*

Indeed, the canonical maps can : $M \otimes_{\mathbf{N}} M \to B \otimes_{\mathbf{L}} M$ and $\vartheta : B \otimes_{\mathbf{L}^{op}} B \to B \otimes_{\mathbf{L}} B$ satisfy the pentagonal identity $(B \otimes_{\mathbf{L}} \mathrm{can}) \circ (\mathrm{can} \otimes_{\mathbf{N}} M) = (\vartheta \otimes_{\mathbf{L}} M) \circ \mathrm{can}_{13} \circ (M \otimes_{\mathbf{N}} \mathrm{can})$, where the (well defined) map $\mathrm{can}_{13} : M \otimes_{\mathbf{N}} (B \otimes_{\mathbf{L}} M) \to (B \otimes_{\mathbf{L}^{op}} B) \otimes_{\mathbf{L}} M$ is obtained by applying can in the first and third factors.

In Theorem 3.13 bialgebroids were characterised via strict monoidality of a forgetful functor. A characterisation of a similar flavour of $\times_L$-Hopf algebras was given in [62, Theorem and Definition 3.5]. Recall that a monoidal category $(\mathcal{M}, \otimes, U)$ is said to be *right closed* if the endofunctor $- \otimes X$ on $\mathcal{M}$ possesses a right adjoint, denoted by $\mathrm{hom}(X, -)$, for any object $X$ in $\mathcal{M}$. The monoidal category of bimodules of an algebra $\mathbf{L}$ is right closed with $\mathrm{hom}(X, Y) = \mathrm{Hom}_{\mathbf{L}}(X, Y)$. It is slightly more involved to see that so is the category of left $\mathbf{B}$-modules, for a left $\mathbf{L}$-bialgebroid $\mathcal{B}$, with $\mathrm{hom}(X, Y) = _{\mathbf{B}}\mathrm{Hom}(B \otimes_{\mathbf{L}} X, Y)$ (where, for a left $\mathbf{B}$-module $X$, $B \otimes_{\mathbf{L}} X$ is a left $\mathbf{B}$-module via the diagonal action and a right $\mathbf{B}$-module via the first factor). A strict monoidal functor $\mathbb{F} : \mathcal{M} \to \mathcal{M}'$ between right closed categories is called *strong right closed* provided that a canonical morphism $\mathbb{F}(\mathrm{hom}(X, Y)) \to \mathrm{hom}(\mathbb{F}(X), \mathbb{F}(Y))$ is an isomorphism, for all objects $X, Y \in \mathcal{M}$.

THEOREM 4.40. *A left bialgebroid $\mathcal{B}$ over a base algebra $\mathbf{L}$ is a $\times_L$-Hopf algebra if and only if the (strict monoidal) forgetful functor $_{\mathbf{B}}\mathcal{M} \to {_{\mathbf{L}}}\mathcal{M}_{\mathbf{L}}$ is strong right closed.*

For a Hopf algebra $H$ over a commutative ring $k$, those (left or right) $H$-modules, which are finitely generated and projective $k$-modules, possess (right or left) duals in the monoidal category of (left or right) $H$-modules. This property extends to $\times_L$-Hopf algebras (hence to Hopf algebroids!) as follows.

PROPOSITION 4.41. *Let $\mathcal{B}$ be a $\times_{\mathbf{L}}$-Hopf algebra over a base algebra $\mathbf{L}$. Denote its structure maps as in Definition 3.3. For a left $\mathbf{B}$-module $M$, the dual $M^* := \mathrm{Hom}_{\mathbf{L}}(M, L)$ is a left $\mathbf{B}$-module, via the action*

$$(b \cdot \phi)(m) := \epsilon\big(b^{\langle 1\rangle} t(\phi(b^{\langle 2\rangle} \cdot m))\big), \qquad \text{for } b \in B, \ \phi \in M^*, \ m \in M,$$

*where for the inverse of the (right $\mathbf{B}$-linear) canonical map $B \otimes_{\mathbf{L}^{op}} B \to B \otimes_{\mathbf{L}} B$ the index notation $b \otimes b' \mapsto b^{\langle 1\rangle} \otimes b^{\langle 2\rangle} b'$ is used. Furthermore, if $M$ is a finitely generated and projective right $\mathbf{L}$-module then $M^*$ is a right dual of $M$ in the monoidal category ${}_{\mathbf{B}}\mathcal{M}$.*

4.6.3. *Hopf monad.* In Theorem 3.5 bialgebroids were related to those bimonads on a bimodule category which possess a right adjoint. In the paper [15] special bimonads, so called Hopf monads, on autonomous categories were studied. Recall that a monoidal category is said to be left (resp. right) autonomous provided that every object possesses a left (resp. right) dual. In particular, a category of finitely generated and projective bimodules is autonomous. Instead of a somewhat technical definition in [15, 3.3], we adopt an equivalent description in [15, Theorem 3.8] as a definition.

DEFINITION 4.42. A *left (resp. right) Hopf monad* is a bimonad $\mathbb{B}$ on a left (resp. right) autonomous monoidal category $\mathcal{M}$, such that the left (resp. right) autonomous structure of $\mathcal{M}$ lifts to the category of $\mathbb{B}$-algebras.

The reader should be warned that, although the same term 'Hopf monad' is used in the papers [15] and [49], they have different meanings (and a further totally different meaning of the same term is used in [48]). Also, the notions of a comodule and a corresponding (co)integral in [15] are different from the notions used in these notes.

4.6.4. *A $^*$-autonomous structure on a strong monoidal special opmorphism between pseudomonoids in a monoidal bicategory.* In the paper [27], strong monoidal special opmorphisms $h$ in monoidal bicategories, from a canonical pseudomonoid $R^{op} \otimes R$ to some pseudomonoid $B$, were studied. The opmorphism $h$ was called *Hopf* if in addition there is a $^*$-autonomous structure on $B$ and $h$ is strong $^*$-autonomous (where $R^{op} \otimes R$ is meant to be $^*$-autonomous in a canonical way). In [27, Section 3] a bialgebroid was described as a strong monoidal special opmorphism $h$ of pseudomonoids in the monoidal bicategory of `[Algebras; Bimodules; Bimodule maps]`. This opmorphism $h$ is strong $^*$-autonomous if and only if the corresponding bialgebroid constitutes a Hopf algebroid with a bijective antipode, see [13, Section 4.2].

## ACKNOWLEDGEMENT

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Research Institute for Particle and Nuclear Physics, Budapest, H-1525 Budapest 114, P.O.B.49, Hungary

*E-mail address*: `G.Bohm@rmki.kfki.hu`

*URL*: `http://www.rmki.kfki.hu/∼bgabr`