The Hecke algebra action on Morava E-theory of height 2

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Given a one-dimensional formal group of height 2, let E be the Morava E-theory spectrum associated to its universal deformation over the Lubin-Tate ring. By computing with moduli spaces of elliptic curves, we give an explicitation for an algebra of Hecke operators acting on E-cohomology. This leads to a vanishing result for Rezk's logarithmic cohomology operation on the units of E. It identifies a family of elements in the kernel with meromorphic modular forms whose Serre derivative is zero. Our calculation finds a connection to logarithms of modular units. In particular, we work out an action of Hecke operators on certain "logarithmic" q-series, in the sense of Knopp and Mason, that agrees with our vanishing result and extends the classical Hecke action on modular forms.

1 Introduction

In the context of elliptic cohomology, Hecke operators have been studied as cohomology operations by Baker [Baker1989, Baker1990], Ando [Ando1995], and Ganter [Ganter2009, Ganter2013a, Ganter2013b], among others. The various cohomology theories each have as coefficients a ring of modular forms of a particular type—modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with no condition at the cusp, p-adic modular forms, modular forms with level structure—or it is a closely related ring, e.g., a completion of the former at an ideal. These cohomology theories are topological realizations of the domain that Hecke operators act on.

The notion of an elliptic spectrum [Ando-Hopkins-Strickland2001, Definition 1.2] and the theory of multiplicative ring spectra, notably the theorem of Goerss, Hopkins, and Miller [Goerss-Hopkins2004, Corollary 7.6], enable studying this action of Hecke operators via power operations. These operations arise from the multiplicative ring structure on a spectrum, and they capture all the algebraic structure that naturally adheres to the cohomology theory represented by the spectrum [Bruner-May-McClure-Steinberger1986, Rezk2006a].

In particular, Morava E-theories are a family of cohomology theories whose power operations are better understood thanks to the work of Ando, Hopkins, Strickland, and Rezk [Ando-Hopkins-Strickland2004, Rezk2009, Rezk2014]. This family is organized by heights n and primes p. Here the theory of formal groups plays a key role: each E-theory spectrum E corresponds to a specific one-dimensional commutative formal group, whose finite flat subgroups, in turn, correspond precisely to power operations on E-cohomology. At height $n=2$, formal groups of elliptic curves provide concrete models for E-theories. The associated power operations can then be computed from moduli spaces that parametrize isogenies between elliptic curves.

In this paper, we study the action of Hecke operators on Morava E-theories at height 2, based on the explicit calculations of power operations in [Rezk2008, Zhu2014]. This is motivated by the interpretation—in terms of Hecke operators—for Rezk's logarithmic operations on E-cohomology at an arbitrary height (see [Rezk2006b, 1.12]). At height 2, these "logarithms" are critical in the work of Ando, Hopkins, and Rezk on rigidification of the string-bordism elliptic genus, or more precisely, on E_∞-string orientations of the spectrum of topological modular forms [Ando-Hopkins-Rezk2010]. For the space of such orientations, its set of components can be detected by elements in the kernel of a logarithm. Explicitly, these elements are identified with Eisenstein series $\mathcal{E}_k$, which are eigenforms of Hecke operators [Ando-Hopkins-Rezk2010, Theorem 12.3] (cf. [Sprang2014, Theorem 40]).

In Section 4, we give a different account of certain elements in the kernel of a logarithm at height 2: they are meromorphic modular forms with vanishing Serre derivative, including modular forms whose zeros and poles are located only at the cusps, such as the modular discriminant Δ. This is stated as Theorem 4.13.

The discrepancy between the above two sets of elements—Eisenstein series as opposed to the discriminant—results from the different domains of the logarithms. The former is the group of units in the zeroth cohomology of an even-dimensional sphere, while the latter is with respect to the infinite-dimensional complex projective space $\mathbb{CP}^\infty$, that is, the logarithms are defined on $E^0(S^{2k})^\times$ and $E^0(\mathbb{CP}^\infty)^\times$ respectively.

The finiteness of $E^0(S^{2k})$ as a module over E^0 leads to a simple formula for the logarithm. Specifically, the logarithm can be written as a combination of Hecke operators acting on $\log(1+f)$, where f is a generator of the truncated polynomial ring $E^0(S^{2k})$. Note that the formal power series expansion of $\log(1+f)$ simply equals f, because $f^2=0$ (see [Ando-Hopkins-Rezk2010, Proposition 4.8 and Example 4.9]).

In the latter case of $E^0(\mathbb{CP}^\infty)$, we calculate instead with $\log(g)$ for units g in E^0.

Through a generator u of the formal power series ring $E^0(\mathbb{CP}^\infty)$, certain g can be represented by meromorphic modular forms (cf. Proposition 2.8). Without nilpotency of g as in the case studied by Ando, Hopkins, and Rezk, a different set of tools from number theory is applied. In particular, it came as a surprise to the author that the formula of Rezk for the logarithms, which arose from a purely homotopy-theoretic construction, resembled the *logarithms of ratios of Siegel functions* studied by Katz in [Katz1976, Section 10.1] (see Remark 4.7). Indeed, Katz's approach to those logarithms inspired a final step in our proof of Theorem 4.13.

Section 5 provides a "feedback" to number theory. As mentioned above, various types of modular forms have been associated with elliptic cohomology theories. In our calculation of the logarithms on E-cohomology, the occurrence of $\log(g)$ indicates that Morava E-theories at height 2 witness a larger class of functions on elliptic curves than modular forms. In fact, for meromorphic modular forms g in a q-expansion, $\log(g)$ are *logarithmic* q-series studied by Knopp and Mason [Knopp-Mason2011]. A paradigm is $\log(\Delta)$, which we discuss in Example 5.1 and Remark 5.2. Motivated by the logarithmic operations from homotopy theory, we sketch how the domain of Hecke operators may extend to include certain logarithmic q-series. This is Definition 5.9, from which the statement for the case $f = \Delta$ in Theorem 4.13 falls out.

This interplay via Hecke operators between homotopy theory and number theory is bookended by the foundational material in Sections 2–3 and further discussions in Section 6 on Hecke operators as additive power operations.

Crucial to our understanding of the action of Hecke operators are the explicit computations for power operations. Building on the prime-2 case in [Rezk2008] and the prime-3 case in [Zhu2014], we present in Section 3.1 a general recipe for computing power operations on Morava E-theories at height 2 for all primes, and make the prime-5 calculations available as a working example throughout the paper (Examples 2.6, 3.4, 3.20, 4.8, 6.1, and 6.7). The explicit formulas show that the "topological" Hecke operators—constructed directly from power operations as in [Ando1995, Proposition 3.6.2]—do not agree with the classical Hecke operators acting on modular forms (Remark 3.19). This explicitation also shows that a topological Hecke operator may not commute with all additive power operations on E-cohomology (Theorem 6.8).

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1.2 Conventions

Throughout the paper, p will denote a prime, and N an integer prime to p with $N > 3$.

Let $\mathrm{MF}\big(\Gamma_1(N)\big)$ be the graded ring of modular forms of level $\Gamma_1(N)$ over $\mathbb{Z}[1/N]$ with no condition at the cusps (cf. [Katz1973, Section 1.2]). Complex-analytically, this corresponds to the graded complex vector space of weakly holomorphic modular forms on $\Gamma_1(N)$ (cf. [Ono2004, Definition 1.12]). For simplicity, we will refer to these as *modular forms of level* $\Gamma_1(N)$, or *modular forms* if the congruence subgroup $\Gamma_1(N)$ is clear from the context.

By a *meromorphic modular form*, we mean a complex-valued modular form that is meromorphic both at the cusps and over the upper half of the complex plane [Ono2004, Definition 1.8].

We use a list of symbols.

$\mathbb{F}_p$	a field with p elements
$\mathbb{Z}_p$	the ring of p-adic integers
$\mathbb{Q}_p$	the field of p-adic numbers
$\overline{k}$	a separable closure of a field k
$\mathbb{W}(k)$	the ring of p-typical Witt vectors over a field k of characteristic p
Σ_m	the symmetric group on m letters
$R^\times$	the multiplicative group of invertible elements in a ring R
$R_I^\wedge$	the completion of a ring R with respect to an ideal I
$R(\!(x)\!)$	the ring of formal Laurent series over a ring R in a variable x write $\sum_{n>-\infty} a_n x^n$ for a general element in this ring
C/R or C_R	a scheme C over a ring R
$[m]$	the multiplication-by-m map on a group scheme
$C[m]$	the kernel of $[m]$ on a group scheme C
$\widehat{C}$	the formal completion of an elliptic curve C at the identity
q	$e^{2\pi i z}$ for any $z \in \mathbb{C}$
$\sigma_s(m)$	$\sum_{\substack{d \mid m \\ d \geq 1}} d^s$ for any positive integer m, with $s \in \mathbb{Z}$

1.3 Correction to two cited formulas

For the reader's convenience, we list the following typographical corrections.

- [Katz1973, (1.11.0.4)] The last line should start with ℓ^{-k} instead of ℓ^{-1}.
- [Milnor-Stasheff1974, Problem 16-A] The left-hand side of Girard's formula should read $(-1)^n s_n/n$. The summation on the right-hand side is over $i_1 + 2\,i_2 + \cdots + k\,i_k = n$.

2 From rings of modular forms to homotopy groups of Morava E-theories

In this section, we explain how every modular form of level $\Gamma_1(N)$ gives an element in the zeroth coefficient ring of a Morava E-theory at height 2 and prime p. The connection comes from finding explicit models for the E-theory.

2.1 Models for an E-theory of height 2

Morava E-theory spectra can be viewed as topological realizations of Lubin-Tate rings. Specifically, given a formal group $\mathbb{G}_0$ of height $n < \infty$ over a perfect field k of characteristic p, the associated Morava E-theory (of height n at the prime p) is a complex-oriented cohomology theory E whose formal group $\operatorname{Spf} E^0\mathbb{CP}^\infty$ is the universal deformation of $\mathbb{G}_0$ over the Lubin-Tate ring

$$\mathbb{W}\left(\overline{\mathbb{F}}_p\right)[\![u_1, \ldots, u_{n-1}]\!] \cong E^0$$

(see [Lubin-Tate1966, Section 3] and [Goerss-Hopkins2004, Section 7]). For height $n = 2$, via the Serre-Tate theorem [Lubin-Serre-Tate1964] (cf. [Katz-Mazur1985, Theorem 2.9.1]), this universal deformation of a formal group can be obtained from a universal deformation of a supersingular elliptic curve. We construct the latter as the universal family of elliptic curves equipped with a level-$\Gamma_1(N)$ structure.

Write $\mathcal{P}_N$ for the representable moduli problem of smooth elliptic curves over $\mathbb{Z}[1/N]$ with a choice of a point P_0 of exact order N and a nonvanishing one-form ω. The following two universal families represent $\mathcal{P}_N$ for $N = 4$ and $N = 5$ respectively, and they suffice to model E-theories of height 2 at all primes.

Example 2.1 ([Zhu2014, Proposition 2.1]) The moduli problem $\mathcal{P}_4$ is represented by the curve

$$\mathcal{C}_4\colon\ y^2 + Axy + ABy = x^3 + Bx^2$$

over the graded ring

$$S_4 := \mathbb{Z}[1/4][A, B, \Delta^{-1}]$$

where $|A| = 1$, $|B| = 2$, and $\Delta = A^2B^4(A^2 - 16B)$. The chosen point is $P_0 = (0, 0)$ and the chosen one-form is $\omega = du$ with $u = x/y$.

Example 2.2 ([Behrens-Ormsby2015, Corollary 1.1.10]) The moduli problem $\mathcal{P}_5$ is represented by

$$\mathcal{C}_5\colon\ y^2 + Axy + B^2(A-B)y = x^3 + B(A-B)x^2$$

over the graded ring

$$S_5 := \mathbb{Z}[1/5][A, B, \Delta^{-1}]$$

where $|A| = |B| = 1$ and $\Delta = B^5(B-A)^5(A^2 + 9AB - 11B^2)$. Again the chosen point is $P_0 = (0, 0)$ and the chosen one-form is $\omega = du$ with $u = x/y$. Moreover, writing $\zeta := e^{2\pi i/5}$, we have $\Delta = B^5(B-A)^5\big(A - (5\zeta^4 + 5\zeta - 2)B\big)\big(A + (5\zeta^4 + 5\zeta + 7)B\big)$.

Remark 2.3 The moduli problem $\mathcal{P}_3$ is also representable [Mahowald-Rezk2009, Proposition 3.2], but $[\Gamma_1(3)]$ is not (cf. [Katz-Mazur1985, Corollary 2.7.4]). The beginning of [Behrens-Ormsby2015, Section 1] explains the relationship between these two moduli problems. For the prime 2, compare [Behrens-Ormsby2015, Corollary 1.1.11] and [Lawson-Naumann2014, Section 3.1] (see also Proposition 2.5 below).

From each of the above examples, restricting $\mathcal{C}_N$ over a closed point in the supersingular locus at p, we get a supersingular elliptic curve C_0 over $\overline{\mathbb{F}}_p$. By the Serre-Tate theorem [Lubin-Serre-Tate1964] (cf. [Katz-Mazur1985, Theorem 2.9.1]), the formal completion $\widehat{\mathcal{C}}_N$ of $\mathcal{C}_N$ at the identity then gives the universal deformation of the formal group $\widehat{C}_0$. Here $\widehat{C}_0/\overline{\mathbb{F}}_p$ is of height 2, and it models $\mathbb{G}_0/k$ for the E-theory we begin with.

Remark 2.4 For a fixed E-theory, the various models each involve a choice of N for the $\mathcal{P}_N$-structures, and a choice of an isomorphism class of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ equipped with a $\mathcal{P}_N$-structure. Over a separably closed field of characteristic p, any two formal group laws of the same height are isomorphic [Lazard1955, Théorème IV]. In view of this and the universal property in the Lubin-Tate theorem [Lubin-Tate1966, Theorem 3.1], we see that up to isomorphism the E-theory is independent of these choices.

Topologically, there is a $K(2)$-localization that corresponds to the above completion along the supersingular locus at p, as we now describe.

The graded rings S_N in Examples 2.1 and 2.2 can be identified as $\mathrm{MF}\big(\Gamma_1(N)\big)$. Their topological realizations are the periodic spectra $\mathrm{TMF}\big(\Gamma_1(N)\big)$ of topological modular forms of level $\Gamma_1(N)$ (see [Mahowald-Rezk2009, Section 2] and cf. [Hill-Lawson2015]). Following the convention that elements in algebraic degree k lie in topological degree $2k$, we have

$$\pi_*\mathrm{TMF}\big(\Gamma_1(4)\big) \cong \mathbb{Z}[1/4][A, B, \Delta^{-1}]$$

with $|A| = 2$, $|B| = 4$, and $\Delta = A^2B^4(A^2 - 16B)$, and

$$\pi_*\mathrm{TMF}\big(\Gamma_1(5)\big) \cong \mathbb{Z}[1/5][A, B, \Delta^{-1}]$$

with $|A| = |B| = 2$ and $\Delta = B^5(B - A)^5(A^2 + 9AB - 11B^2)$.

Proposition 2.5 (cf. [Behrens-Ormsby-Stapleton-Stojanoska2015, Section 3.5] and [Ormsby2012, Corollary on page 20]) *Let $K(2)$ be the Morava K-theory spectrum at height 2 and prime p, with $\pi_*K(2) \cong \overline{\mathbb{F}}_p[v_2^{\pm 1}]$, where $|v_2| = 2(p^2 - 1)$. Let $N > 3$ be an integer prime to p. Denote by $L_{K(2)}\mathrm{TMF}\big(\Gamma_1(N)\big)$ the Bousfield localization of $\mathrm{TMF}\big(\Gamma_1(N)\big)$ with respect to $K(2)$. Given a Morava E-theory spectrum E of height 2 at the prime p, there is a noncanonical isomorphism*

$$L_{K(2)}\mathrm{TMF}\big(\Gamma_1(N)\big) \cong \underbrace{E \times \cdots \times E}_{m \text{ copies}}$$

of E_∞-ring spectra, where m is the number of isomorphism classes of supersingular elliptic curves over $\overline{\mathbb{F}}_p$ equipped with a level-$\Gamma_1(N)$ structure.

Proof Let C_0 be a supersingular elliptic curve over $\overline{\mathbb{F}}_p$ equipped with a level $\Gamma_1(N)$-structure. Its formal group $\widehat{C}_0/\overline{\mathbb{F}}_p$ gives a model for the E-theory. By the Goerss-Hopkins-Miller theorem [Goerss-Hopkins2004, Corollary 7.6], the spectrum E admits an action of the automorphism group $\mathrm{Aut}(C_0/\overline{\mathbb{F}}_p)$.

Let G be the subgroup of $\mathrm{Aut}(C_0/\overline{\mathbb{F}}_p)$ consisting of automorphisms that preserve the $\Gamma_1(N)$-structure on C_0. In view of Remark 2.4, we then obtain $L_{K(2)}\mathrm{TMF}\big(\Gamma_1(N)\big)$ as taking homotopy fixed points for the action of G on E, one such copy for each closed point in the supersingular locus at p. By [Katz-Mazur1985, Corollary 2.7.4], the moduli problem $[\Gamma_1(N)]$ is rigid when $N > 3$, and thus G is trivial. Again by the Goerss-Hopkins-Miller theorem, we get the stated isomorphism as one between E_∞-ring spectra. $\square$

2.2 Homotopy groups of an E-theory at height 2

Thanks to the explicit models for E-theories, the global-to-local relationship between $\mathrm{TMF}\big(\Gamma_1(N)\big)$ and E in Proposition 2.5 can be spelled out on homotopy groups. The next example illustrates the passage from $\pi_*\mathrm{TMF}\big(\Gamma_1(N)\big)$ to

$$E_* \cong \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![u_1]\!][u^{\pm 1}]$$

where the "deformation" parameter u_1 in degree 0 comes from a Hasse invariant, and the 2-periodic unit u in degree -2 corresponds to a local uniformizer at the identity of the universal elliptic curve $\mathcal{C}_N$.

Example 2.6 Let $p = 5$ and $N = 4$.

By [Silverman2009, V.4.1a], the Hasse invariant of $\mathcal{C}_4$ at the prime 5 equals $A^4 - A^2B + B^2 \in \mathbb{F}_5[A,B]$. For a reason that will become clear in (3.5), we choose an integral lift of this Hasse invariant given by

$$H := A^4 - 16A^2B + 26B^2 \in S_4$$

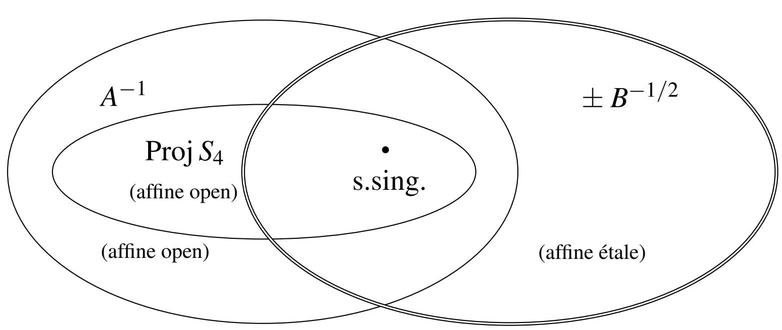
At the prime 5, the supersingular locus of $\mathcal{C}_4$ is then the closed subscheme of $\operatorname{Proj} S_4$ cut out by the ideal $(5, H)$. It consists of a single closed point, as H is irreducible over $\mathbb{F}_5$. Since $\Delta = A^2B^4(A^2 - 16B)$ gets inverted in S_4, the scheme $\operatorname{Proj} S_4$ is affine, and is contained in the affine open chart

$$\operatorname{Proj} \mathbb{Z}[1/4][A,B][A^{-1}]$$

for the weighted projective space $\operatorname{Proj} \mathbb{Z}[1/4][A,B]$. There is another affine chart

$$\operatorname{Proj} \mathbb{Z}[1/4][A,B][u]$$

with $u^2 = B^{-1}$ that is étale over $\operatorname{Proj} \mathbb{Z}[1/4][A,B]$. The supersingular point is contained in both charts, as illustrated in the following diagram.



We now pass to the homotopy groups of the corresponding E-theory spectrum E by a procedure of *dehomogenization*. Define elements

$$a := uA, \quad h := u^4 H = a^4 - 16a^2 + 26, \quad \text{and} \quad \delta := u^{12}\Delta = h - 26$$

Following the convention that elements in algebraic degree k lie in topological degree $2k$, we then have

$$E_* \cong \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![h]\!][u^{\pm 1}]$$

with $|h| = 0$ and $|u| = -2$. In particular, by Hensel's lemma, both a and δ are contained in $\big(E_0\big)^\times$. Moreover, u corresponds to a coordinate on $\widehat{\mathcal{C}}_4$ via a chosen isomorphism $\operatorname{Spf} E^0\mathbb{CP}^\infty \cong \widehat{\mathcal{C}}_4$ of formal groups over $\operatorname{Spf} E^0 \cong \operatorname{Spf}\Big(S4^\wedge_{(5,H)} \otimes_{\mathbb{W}(\mathbb{F}_5)} \mathbb{W}\big(\overline{\mathbb{F}}_5\big)\Big)$ (cf. [Ando-Hopkins-Strickland2001, Definition 1.2]).

Remark 2.7 As an abuse of notation, in Examples 2.1 and 2.2 we have written

$$u = \frac{x}{y}$$

where x and y are the affine coordinates in the Weierstrass equation for $\mathcal{C}_N$. The resulting algebraic degree -1 of u matches the topological degree -2 of u in E_*.

The process in the previous example applies to all pairs p and N. For certain pairs, the supersingular locus contains more than one closed point, e.g., the Hasse invariant of $\mathcal{C}_4$ at $p = 11$ factors as $(A^2+B)(A^8+3A^6B+4A^2B^3+B^4)$. In this case, H lifts one of the irreducible factors of the Hasse invariant. The corresponding closed point carries the supersingular elliptic curve whose formal group models $\mathbb{G}_0/k$ for the E-theory.

Recall that elements in $\mathrm{MF}\big(\Gamma_1(N)\big)$ are functions f on elliptic curves C/R equipped with a $\mathcal{P}_N$-structure (P_0, ω). Each value $f(C/R, P_0, \omega) \in R$ depends only on the R-isomorphism class of the triple $(C/R, P_0, \omega)$, and is subject to a modular transformation property that encodes the weight of f; moreover, its formation commutes with arbitrary base change (see, e.g., [Katz1973, Section 1.2]).

Proposition 2.8 (cf. [Behrens-Ormsby-Stapleton-Stojanoska2015, Lemma 6.3]) *Let E be a Morava E-theory of height 2 at the prime p. There is a composite*

$$\beta_N^{(p)}\colon\ \mathrm{MF}\big(\Gamma_1(N)\big) \hookrightarrow E^0 \times \mathbb{Z} \to E^0$$

of ring homomorphisms, where the first map is injective and the second map is the projection.

Proof In view of Remark 2.4, given a model for the E-theory as in Example 2.6, via dehomogenization a modular form f of weight k maps to $\big(u^k \cdot f(\mathcal{C}_N, P_0, du),\ k\big) \in E^0 \times \mathbb{Z}$. The injectivity of this map follows from the universal property of the triple $(\mathcal{C}_N, P_0, du)$. $\square$

We will drop the indices in "$\beta_N^{(p)}$" when there is no ambiguity.

3 From classical to topological Hecke operators

Morava E-theory spectra are E_∞-ring spectra [Goerss-Hopkins2004, Corollary 7.6], and thus they come equipped with power operations. The work of Ando, Hopkins, and Strickland [Ando-Hopkins-Strickland2004] (cf. [Rezk2009, Theorem B]) sets up a correspondence between power operations on an E-theory and deformations of Frobenius isogenies of formal groups.

At height 2, the Serre-Tate theorem [Lubin-Serre-Tate1964] (cf. [Katz-Mazur1985, Theorem 2.9.1]) sets up a second correspondence between isogenies of formal groups and isogenies of elliptic curves, in terms of their deformation theory.

Via these two bridges, the classical action of Hecke operators on modular forms then corresponds to an action of "topological" Hecke operators on E-cohomology [Rezk2006b, Section 14]. The purpose of this section is to give an explicit comparison between these two actions. In particular, we will explain why the ring homomorphism β in Proposition 2.8 *fails* to become a map of modules over a Hecke algebra.

3.1 Isogenies and power operations

Let $\mathcal{C}_N$ over S_N be the universal curve in Section 2.1 for the moduli problem $\mathcal{P}_N$. Denote by $\mathcal{G}_N^{(p)}$ the universal example of a degree-p subgroup of $\mathcal{C}_N$. It is defined over an extension ring $S_N^{(p)}$ that is free of rank $p+1$ as an S_N-module [Katz-Mazur1985, Theorem 6.6.1]. Explicitly,

$$S_N^{(p)} \cong S_N[\kappa]/\big(W(\kappa)\big)$$

where κ is a generator that satisfies $W(\kappa) = 0$ for a monic polynomial W of degree $p+1$. The roots $\kappa_0,\ \kappa_1,\ \ldots,\ \kappa_p$ of W each correspond to a degree-p subgroup of $\mathcal{C}_N$. In particular, let κ_0 correspond to the subgroup whose restriction over an ordinary point is the unique degree-p subgroup of $\widehat{\mathcal{C}}_N$.

We write

$$\Psi_N^{(p)}\colon\ \mathcal{C}_N \to \mathcal{C}_N/\mathcal{G}_N^{(p)}$$

for the universal degree-p isogeny over $S_N^{(p)}$, and write $\mathcal{C}_N^{(p)}$ for the quotient curve $\mathcal{C}_N/\mathcal{G}_N^{(p)}$. Following Lubin [Lubin1967, proof of Theorem 1.4], we construct $\Psi_N^{(p)}$ as a deformation of Frobenius, that is, over any closed point in the supersingular locus at p, $\Psi_N^{(p)}$ restricts as the p-power Frobenius endomorphism on the corresponding supersingular elliptic curve.

Construction 3.1 (cf. [Lubin1967, proof of Theorem 1.4] and [Katz-Mazur1985, Section 7.7]) Let P be any point on $\mathcal{C}_N$. Write $u = x/y$ and $v = 1/y$, where x and y are the usual coordinates in an affine Weierstrass equation with the identity O at the infinity. We have $\big(u(O), v(O)\big) = (0,0)$, and u is a local uniformizer at O.

(i) Define $\Psi_N^{(p)}\colon\ \mathcal{C}_N \to \mathcal{C}_N^{(p)}$ by the formula

$$u\big(\Psi_N^{(p)}(P)\big) := \prod_{Q\in\mathcal{G}_N^{(p)}} u(P-Q)$$

(ii) Define $\kappa \in S_N^{(p)}$ in degree $-p+1$ as

$$\kappa := \prod_{Q\,\in\,\mathcal{G}_N^{(p)}\backslash\{O\}} u(Q)$$

We verify that the isogeny $\Psi_N^{(p)}$ has kernel precisely the subgroup $\mathcal{G}_N^{(p)}$. Moreover, it is a deformation of Frobenius, since at a supersingular point the p-divisible group is formal so that $Q = O$ for all $Q \in \mathcal{G}_N^{(p)}$.

Remark 3.2 The element κ in Construction 3.1 (ii) is the "norm" parameter for the moduli problem $[\Gamma_0(p)]$ as an "open arithmetic surface" (see [Katz-Mazur1985, Section 7.7]). As u is a local coordinate near the identity O, we note that by the above construction the cotangent map $\big(\Psi_N^{(p)}\big)^*$ at O sends du to κdu.

Via completion at a supersingular point, we see in Section 2.2 that the ring $S_N \cong \mathrm{MF}\big(\Gamma_1(N)\big)$ representing $\mathcal{P}_N$ is locally realized in homotopy theory as E^0. Here, given the ring $S_N^{(p)}$ representing the simultaneous moduli problem $\big(\mathcal{P}_N, [\Gamma_0(p)]\big)$, Strickland's theorem [Strickland1998, Theorem 1.1] identifies the completion of $S_N^{(p)}$ at the supersingular point as $E^0(B\Sigma_p)/I$, where I is a transfer ideal. In particular, $E^0(B\Sigma_p)/I$ is free over E^0 of rank $p+1$, isomorphic to $E^0[\alpha]/\big(w(\alpha)\big)$ for a monic polynomial w of degree $p+1$.

The universal degree-p isogeny $\Psi_N^{(p)}\colon\ \mathcal{C}_N \to \mathcal{C}_N^{(p)}$ is constructed above as a deformation of Frobenius. By [Rezk2009, Theorem B] it then corresponds to a *total power operation*

$$\psi^p\colon\ A^0 \to A^0(B\Sigma_p)/J \tag{3.3}$$

where A is any $K(2)$-local commutative E-algebra, and J is the corresponding transfer ideal. Since $E^0(B\Sigma_p)$ is free over E^0 of finite rank [Strickland1998, Theorem 3.2], we have $J \cong A^0 \otimes_{E^0} I$ and

$$A^0(B\Sigma_p)/J \cong \big(A^0 \otimes_{E^0} E^0(B\Sigma_p)\big)/J \cong A^0 \otimes_{E^0} \big(E^0(B\Sigma_p)/I\big) \cong A^0[\alpha]/\big(w(\alpha)\big)$$

Note that up to isomorphism, ψ^p is independent of the choice of N (see Remark 2.4). Moreover, taking quotient by the transfer ideal makes this operation additive and hence a local ring homomorphism.

Example 3.4 We continue Example 2.6 with $p = 5$ and $N = 4$.

The universal degree-5 isogeny $\Psi_4^{(5)}\colon\ \mathcal{C}_4 \to \mathcal{C}_4^{(5)}$ is defined over the graded ring $S_4^{(5)} \cong S_4[\kappa]/\big(W(\kappa)\big)$, where $|\kappa| = -4$ and

$$W(\kappa) = \kappa^6 - \frac{10}{B^2}\kappa^5 + \frac{35}{B^4}\kappa^4 - \frac{60}{B^6}\kappa^3 + \frac{55}{B^8}\kappa^2 - \frac{H}{B^{12}}\kappa + \frac{5}{B^{12}} \tag{3.5}$$

This polynomial is computed from the division polynomial ψ_5 in [Silverman2009, Exercise 3.7] for the curve $\mathcal{C}_4$. We first deduce from ψ_5 identities satisfied by the uv-coordinates of a universal example $Q \in \mathcal{G}_4^{(5)} \backslash \{O\}$ as in [Zhu2014, proof of Proposition 2.2]. We then compute an explicit formula for

$$\kappa = u(Q) \cdot u(-Q) \cdot u(2Q) \cdot u(-2Q)$$

using methods analogous to [Silverman2009, III.2.3]. Finally we solve for a monic degree-6 equation satisfied by κ and obtain the polynomial W. (We do not compute an equation for $\mathcal{C}_4^{(5)}$ as in [Zhu2014, Proposition 2.3].)

Passing to the corresponding power operation, we write

$$\alpha := u^{-4}\kappa_0 \tag{3.6}$$

(see Remark 2.7) where κ_0 corresponds to the subgroup of $\mathcal{C}_4$ whose restriction over an ordinary point is the unique degree-5 subgroup of $\widehat{\mathcal{C}}_4$. The total power operation $\psi^5\colon\ E^0 \to E^0(B\Sigma_5)/I$ then lands in

$$E^0(B\Sigma_5)/I \cong \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![h,\alpha]\!]/\big(w(\alpha)\big)$$

where $w(\alpha) = \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - h\alpha + 5$. To compute the effect of ψ^5 on the generator $h \in E^0$, we proceed as follows.

Consider a second universal degree-5 isogeny

$$\widetilde{\Psi}_4^{(5)}\colon\ \mathcal{C}_4^{(5)} \to \mathcal{C}_4^{(5)}/\widetilde{\mathcal{G}}_4^{(5)}$$

where $\widetilde{\mathcal{G}}_4^{(5)} = \mathcal{C}_4[5]/\mathcal{G}_4^{(5)}$. It is defined the same way as in Construction 3.1, with a parameter $\widetilde{\kappa} \in S_4^{(5)}$ in degree -20. Over $S_4^{(5)}$, the assignment

$$\Big(\big(\mathcal{C}_4, P_0, du\big),\ \mathcal{G}_4^{(5)}\Big) \mapsto \Big(\big(\mathcal{C}_4^{(5)}, \Psi_4^{(5)}(P_0), du\big),\ \widetilde{\mathcal{G}}_4^{(5)}\Big)$$

is an involution on the moduli problem $\big(\mathcal{P}_4, [\Gamma_0(5)]\big)$ (cf. [Katz-Mazur1985, 11.3.1]). In particular, by rigidity, we have the identity

$$\widetilde{\Psi}_4^{(5)} \circ \Psi_4^{(5)} = [5]$$

that lifts $\mathrm{Frob}^2 = [5]$ over the supersingular point, where Frob is the 5-power Frobenius endomorphism. Thus in view of Remark 3.2 we obtain a relation

$$\widetilde{\kappa} \cdot \kappa = \frac{5}{B^{12}}$$

in $S_4^{(5)}$ (cf. [Zhu2014, proof of Corollary 3.2]).

Correspondingly, there is an involution

$$(h, \alpha)\ \mapsto\ (\widetilde{h}, \widetilde{\alpha})$$

on $E^0(B\Sigma_5)/I$, coming from the Atkin-Lehner involution of modular forms on $\Gamma_0(5)$ (cf. [Atkin-Lehner1970, Lemmas 7–10]). In particular, the relation

$$\alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - h\alpha + 5 = 0 \tag{3.7}$$

has an analog

$$\widetilde{\alpha}^6 - 10\widetilde{\alpha}^5 + 35\widetilde{\alpha}^4 - 60\widetilde{\alpha}^3 + 55\widetilde{\alpha}^2 - \widetilde{h}\widetilde{\alpha} + 5 = 0 \tag{3.8}$$

and we have

$$\widetilde{\alpha} \cdot \alpha = 5 \tag{3.9}$$

Based on these, we compute that

$$
\begin{aligned}
\psi^5(h) &= \widetilde{h} \\
&= \widetilde{\alpha}^5 - 10\widetilde{\alpha}^4 + 35\widetilde{\alpha}^3 - 60\widetilde{\alpha}^2 + 55\widetilde{\alpha} + \alpha && \text{by (3.8) and (3.9)} \\
&= (-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^5 \\
&\quad - 10(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^4 \\
&\quad + 35(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^3 \\
&\quad - 60(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^2 \\
&\quad + 55(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h) \\
&\quad + \alpha && \text{by (3.9) and (3.7)} \\
&= h^5 - 10h^4 - 1065h^3 + 12690h^2 + 168930h \\
&\quad - 1462250 + (-55h^4 + 850h^3 + 39575h^2 \\
&\quad - 608700h - 1113524)\alpha + (60h^4 - 775h^3 \\
&\quad - 45400h^2 + 593900h + 2008800)\alpha^2 + (-35h^4 \\
&\quad + 400h^3 + 27125h^2 - 320900h - 1418300)\alpha^3 \\
&\quad + (10h^4 - 105h^3 - 7850h^2 + 86975h \\
&\quad + 445850)\alpha^4 + (-h^4 + 10h^3 + 790h^2 - 8440h \\
&\quad - 46680)\alpha^5 && \text{by (3.7)}
\end{aligned}
\tag{3.10}
$$

We also have $\psi^5(c) = Fc$ for $c \in \mathbb{W}\big(\overline{\mathbb{F}}_5\big)$, where F is the Frobenius automorphism. As ψ^5 is a local ring homomorphism, (3.10) then determines $\psi^5(x)$ for all $x \in E^0 \cong \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![h]\!]$.

The previous example illustrates a general recipe for computing power operations on a Morava E-theory at height 2 and prime p, with a model based on the moduli problem $\big(\mathcal{P}_N, [\Gamma_0(p)]\big)$. Crucial in this computation is an explicit expression for

$$
w(\alpha) = \alpha^{p+1} + w_p\alpha^p + \cdots + w_1\alpha + w_0 \in E^0[\alpha] \tag{3.11}
$$

Computing $w(\alpha)$ appears to be essentially computing a "modular equation" (see, e.g., [Milne2012, II.6] and [Zureick-Brown2010], and compare the *canonical modular polynomials* in Magma's Modular Polynomial Databases).

Remark 3.12 The coefficients in $w(\alpha)$ have the following properties.

- For $i \neq 1$, each w_i is divisible by p.

- The coefficient w_1 is a lift of the Hasse invariant at p, so we can always choose $h = -w_1$ and have $w(\alpha) \equiv \alpha(\alpha^p - h) \mod p$. The two factors in the mod-p reduction correspond to the unramified and ramified cusps of $\Gamma_0(p)$ respectively (see [Katz1973, Section 1.13]).

- We have

  $$w_0 = \widetilde{\alpha} \cdot \alpha = \mu p \tag{3.13}$$

  for some unit $\mu \in E^0$ that depends on the Frobenius endomorphism of the supersingular elliptic curve (cf. [Rezk2012, 3.8]).

3.2 Hecke operators

We first describe the classical action of Hecke operators on modular forms in terms of isogenies between elliptic curves. The p'th Hecke operator T_p that acts on $\mathrm{MF}\big(\Gamma_1(N)\big)$ can be built from universal isogenies as follows.

Construction 3.14 (cf. [Katz1973, (1.11.0.2)]) Let the notation be as in Section 3.1, with the subscripts "N" suppressed. Given any $f \in \mathrm{MF}\big(\Gamma_1(N)\big)$ of weight $k \geq 1$, $T_p f \in \mathrm{MF}\big(\Gamma_1(N)\big)$ is a modular form such that

$$T_p f(\mathcal{C}_S, P_0, du) := \frac{1}{p} \sum_{i=0}^{p} \kappa_i^k \cdot f\big(\mathcal{C}_{S^{(p)}}/\mathcal{G}_i^{(p)}, \Psi_i^{(p)}(P_0), du\big) \tag{3.15}$$

where each $\mathcal{G}_i^{(p)}$ denotes a degree-p subgroup of $\mathcal{C}$ over $S^{(p)}$, and $\Psi_i^{(p)}$ is the quotient map with kernel $\mathcal{G}_i^{(p)}$ as in Construction 3.1.

Remark 3.16 The terms κ_i^k appear in the above formula so that T_p is independent of the choice of a basis for the cotangent space. Explicitly, we have

$$\big(\Psi_i^{(p)}\big)^* du = \kappa_i \cdot du \qquad \text{and} \qquad \big(\Psi_i^{(p)}\big)^* \big(\breve{\Psi}_i^{(p)}\big)^* du = p \cdot du$$

where each $\breve{\Psi}_i^{(p)} \colon \mathcal{C}/\mathcal{G}_i^{(p)} \to \mathcal{C}$ is a dual isogeny, and $\big(\breve{\Psi}_i^{(p)}\big)^* du$ is the choice in [Katz1973] for a nonvanishing one-form on a quotient curve (cf. Remark 3.2 and the discussion before (1.11.0.0) in [Katz1973, Section 1.11]). Thus we rewrite (3.15) as

$$\begin{aligned}
&\frac{1}{p}\sum_{i=0}^{p}\kappa_i^k\cdot f\big(\mathcal{C}_{S^{(p)}}/\mathcal{G}_i^{(p)},\Psi_i^{(p)}(P_0),du\big)\\
=&\frac{1}{p}\sum_{i=0}^{p}\kappa_i^k\cdot f\Big(\mathcal{C}_{S^{(p)}}/\mathcal{G}_i^{(p)},\Psi_i^{(p)}(P_0),(\kappa_i/p)\big(\breve{\Psi}^{(p)}\big)^*du\Big)\\
=&\frac{1}{p}\sum_{i=0}^{p}p^k\cdot f\Big(\mathcal{C}_{S^{(p)}}/\mathcal{G}_i^{(p)},\Psi_i^{(p)}(P_0),\big(\breve{\Psi}^{(p)}\big)^*du\Big)
\end{aligned}$$

and this agrees with [Katz1973, (1.11.0.2)].

Construction 3.17 (cf. [Rezk2006b, 1.12]) There is a *topological* Hecke operator $t_p\colon E^0\to p^{-1}E^0$ defined by

$$t_p(x):=\frac{1}{p}\sum_{i=0}^{p}\psi_i^p(x) \tag{3.18}$$

where ψ_i^p denotes the power operation $\psi^p\colon E^0\to E^0(B\Sigma_p)/I\cong E^0[\alpha]/\big(w(\alpha)\big)$ with the parameter α replaced by $\alpha_i=u^{-p+1}\kappa_i$ (cf. (3.6)).

Since the parameters α_i are the roots of $w(\alpha)\in E^0[\alpha]$, t_p indeed lands in $p^{-1}E^0$.

Remark 3.19 Given the ring homomorphism β in Proposition 2.8, the topological Hecke operator above does not coincide with the classical one on modular forms. Comparing (3.18) to (3.15), note that there are no terms α_i^k corresponding to κ_i^k. This is related to the fact that β is not injective: if we include α_i^k in the definition, for each $x\in E^0$ we need to determine a unique value of its "weight" k so that t_p is well-defined.

Note that the map β is not surjective either, so an element in E^0 may not come from any modular form. On the other hand, the total power operation ψ^p (and hence t_p) is defined on the entire E^0.

Example 3.20 Again, let $p=5$ and $N=4$. Recall from Example 2.6 that we have

$$\beta(\Delta)=\delta=h-26$$

In view of (3.7), we then compute from (3.10) that

$$\begin{aligned}
t_5(\delta) &= \frac{1}{5}\sum_{i=0}^{5}\psi_i^5(\delta)\\
&= \frac{1}{5}\sum_{i=0}^{5}\big(\psi_i^5(h) - 26\big)\\
&= \frac{1}{5}(h^5 - 10h^4 - 1340h^3 + 18440h^2 + 267430h - 3178396)\\
&= \frac{1}{5}(h^4 + 16h^3 - 924h^2 - 5584h + 122246)\cdot\delta
\end{aligned}$$

Given that the modular form Δ is of weight 12, we define and compute

$$\begin{aligned}
\breve{t}_5(\delta) &\coloneqq \frac{1}{5}\sum_{i=0}^{5}\alpha_i^{12}\cdot\psi_i^5(\delta)\\
&= 4830h - 125580\\
&= \tau(5)\cdot\delta
\end{aligned}$$

where $\tau\colon \mathbb{N}\to\mathbb{Z}$ is the Ramanujan tau-function, with $\tau(5) = 4830$. This recovers the action of T_5 on Δ.

More generally, in the theory of automorphic forms on $\Gamma_1(N)\subset \mathrm{SL}_n(\mathbb{Z})$, the above Hecke operator T_p can be renamed as $T_{1,p}$. It belongs to a family of operators $T_{i,p}$, $1\le i\le n$ that generate the p-primary Hecke algebra (see, e.g., [Shimura1994, Theorems 3.20 and 3.35]).

For the case $n = 2$, the other Hecke operator $T_{2,p}$ arises from the isogeny of multiplication by p, whose kernel is the degree-p^2 subgroup consisting of the p-torsion points. Explicitly (again with the subscripts "N" suppressed), given any $f\in \mathrm{MF}\big(\Gamma_1(N)\big)$ of weight $k\ge 2$, there exists $\lambda\in S^\times$ of degree $-p^2+1$ such that

$$\begin{aligned}
T_{2,p} f(\mathcal{C}, P_0, du) &\coloneqq \frac{1}{p^2}\Big((p\lambda)^k\cdot f\big(\mathcal{C}/\mathcal{C}[p],\ [p]P_0,\ du\big)\Big)\\
&= p^{k-2}\cdot f\big(\mathcal{C}/\mathcal{C}[p],\ [p]P_0,\ \lambda^{-1}du\big)\\
&= p^{k-2}\cdot f(\mathcal{C}, [p]P_0, du) && \text{via } \mathcal{C}/\mathcal{C}[p]\xrightarrow{\sim}\mathcal{C}
\end{aligned}\tag{3.21}$$

For example, when $N = 4$ with $p = 3$ or $p = 5$, we have $\lambda = B^{(1-p^2)/2}$ (cf. Example 2.1). Via the $S^{(p)}$-isomorphism

$$\mathcal{C}/\mathcal{C}[p]\cong \frac{\mathcal{C}/\mathcal{G}^{(p)}}{\mathcal{C}[p]/\mathcal{G}^{(p)}} \eqqcolon \frac{\mathcal{C}^{(p)}}{\widetilde{\mathcal{G}}^{(p)}}\tag{3.22}$$

the quotient curve $\mathcal{C}/\mathcal{C}[p]$ can be identified with the target in the composite

$$\mathcal{C} \xrightarrow{\Psi^{(p)}} \mathcal{C}^{(p)} \xrightarrow{\widetilde{\Psi}^{(p)}} \mathcal{C}^{(p)}/\widetilde{\mathcal{G}}^{(p)}$$

of deformations of Frobenius isogenies.

Correspondingly, given any $K(2)$-local commutative E-algebra A, there is a composite ϕ of total power operations $\psi^p \circ \psi^p \colon A^0 \to A^0$. In view of (3.3), note that ϕ lands in A^0, because up to the isomorphism (3.22) the composite $\widetilde{\Psi}^{(p)} \circ \Psi^{(p)}$ is an endomorphism on $\mathcal{C}$ over S. In particular, on E^0 or on $E^0(B\Sigma_p)/I$, the operation ϕ is the identity map, which is a manifest of the Atkin-Lehner involution (cf. Example 3.4).

Taking $A = E$, we define a *topological* Hecke operator $t_{2,p} \colon E^0 \to p^{-1}E^0$ by

$$t_{2,p}(x) := p^{-2}\phi(x) = p^{-2}x \tag{3.23}$$

(we will discuss the general case on A^0 in Section 6). Rewrite $t_{1,p} := t_p$. The ring $\mathbb{Z}[t_{1,p}, t_{2,p}]$ then acts on $p^{-1}E^0$. As we see in Remark 3.19, along the map $\beta \colon \mathrm{MF}\big(\Gamma_1(N)\big) \to E^0$, this action is not compatible with the action of the Hecke algebra $\mathbb{Z}[T_{1,p}, T_{2,p}]$ on modular forms. However, in certain instances, the two actions do interact well. The next section will exploit this connection.

4 Kernels of logarithmic cohomology operations at height 2

In Section 3.2, we give an explicitation for an algebra of topological Hecke operators acting on E^0, where E is a Morava E-theory of height 2. Based on this, we now use a number-theoretic calculation to understand a topological construction. This is Rezk's construction of logarithmic cohomology operations via Bousfield-Kuhn functors [Rezk2006b]. We first recall a formula of Rezk for computing these operations, which leads to a connection with Hecke operators.

4.1 Connecting to Hecke operators

Given any E_∞-ring spectrum R, let $\mathrm{gl}_1 R$ be the spectrum of units of R. Rezk constructs a family of operations that naturally acts on $\mathrm{gl}_1 R$ [Rezk2006b, Definition 3.6]. Specifically, given a positive integer n and a prime p, let $L_{K(n)}$ denote localization with respect to the n'th Morava K-theory at p, and let Φ_n be the corresponding Bousfield-Kuhn functor from the category of based topological spaces to the category of spectra. Consider the composite

$$\mathrm{gl}_1 R \to L_{K(n)}\mathrm{gl}_1 R \simeq \Phi_n \Omega^\infty \mathrm{gl}_1 R \xrightarrow{\sim} \Phi_n \Omega^\infty R \simeq L_{K(n)} R \tag{4.1}$$

Note that $\Omega^\infty \mathrm{gl}_1 R$ and $\Omega^\infty R$ have weakly equivalent basepoint components, but the standard inclusion $\Omega^\infty \mathrm{gl}_1 R \hookrightarrow \Omega^\infty R$ is not basepoint-preserving. The equivalence $\Phi_n \Omega^\infty \mathrm{gl}_1 R \to \Phi_n \Omega^\infty R$ thus involves a "basepoint shift" (see [Rezk2006b, 3.4]).

Let E be a Morava E-theory of height n at the prime p. With $R = E$, applying $\pi_0(-)$ to (4.1), we then obtain the logarithmic operation

$$\ell_{n,p} \colon \left(E^0\right)^\times \to E^0$$

which is a homomorphism from a multiplicative group to an additive group. More generally, let X be a space. Taking R to be the spectrum of functions from $\Sigma^\infty_+ X$ to E, we obtain the operation $\ell_{n,p}$ that acts on $E^0(X)^\times$, naturally both in X and in E.

The main theorem of [Rezk2006b] is a formula for this operation [Rezk2006b, Theorem 1.11]. In particular, for any $x \in \left(E^0\right)^\times$,

$$\ell_{n,p}(x) = \frac{1}{p} \log\left(1 + pM(x)\right) \tag{4.2}$$

where $M \colon \left(E^0\right)^\times \to E^0$ is a cohomology operation that can be expressed in terms of power operations ψ_A associated to certain subgroups A of $(\mathbb{Q}_p/\mathbb{Z}_p)^n$. Explicitly,

$$1 + pM(x) = \prod_{j=0}^{n} \prod_{\substack{A \subset (\mathbb{Q}_p/\mathbb{Z}_p)^n[p] \\ \#A = p^j}} \psi_A(x)^{(-1)^j p^{(j-1)(j-2)/2}}$$

Now let the E-theory be of height $n = 2$. In the presence of a model for E, the operations ψ_A above coincide with the power operations in Section 3. In particular,

$$\ell_{2,p}(x) = \frac{1}{p} \log\left(x^p \cdot \frac{1}{\psi_0^p(x) \cdots \psi_p^p(x)} \cdot \phi(x)\right) \tag{4.3}$$

Note that $\ell_{2,p}(x) = 0$ for any $x \in \mathbb{Z}_p \cap \left(E^0\right)^\times$.

Suppose that $x = \beta(f)$ for some modular form f of weight k (see Proposition 2.8). We next write (4.3) in terms of Hecke operators. We note the following.

- Recall from (3.11) that the parameters α_i in the operations ψ_i^p satisfy

  $$\alpha_0 \cdots \alpha_p = (-1)^{p+1} w_0 = (-1)^{p+1} \mu p$$

  for some $\mu \in \left(E^0\right)^\times$. In fact, by a theorem of Ando [Ando1995, Theorem 4], there exists a unique coordinate on the formal group of E such that its corresponding parameters α_i satisfy $\alpha_0 \cdots \alpha_p = p$. This is the coordinate $u = x/y$ when $p = 2$ with the $\mathcal{P}_3$-model [Rezk2008, Section 3] and when $p = 5$ with the $\mathcal{P}_4$-model (3.7). Henceforth we will choose this particular coordinate for all primes p and denote it by u.

- Recall that ψ_i^p, ϕ and β are ring homomorphisms. Multiplications by the terms κ_i^k in (3.15) and by p^k in (3.21) can be viewed as ring homomorphisms as well. More precisely, we allow the exponent k to vary, according to the weight of the modular form that immediately follows. We will denote these ring homomorphisms by $\kappa_i^\bullet$ and $p^\bullet$. The topological analog of $\kappa_i^\bullet$ on homotopy groups is $\alpha_i^\bullet$. In particular, given as a formal power series, the usual logarithm "log" commutes with these continuous ring homomorphisms.

In Section 3.2, we give a comparison between classical and topological Hecke operators, particularly as illustrated in Example 3.20. In view of this comparison and the two observations above, we can now rewrite (4.3) as follows (cf. [Rezk2006b, 1.12]). Given the element $x = \beta(f)$,

$$\begin{aligned}\ell_{2,p}(x) &= \frac{1}{p}\log\frac{x^p\cdot p^\bullet\phi(x)}{\prod_{i=0}^p \alpha_i^\bullet\psi_i^p(x)}\\ &= \left(1-\frac{1}{p}\sum_{i=0}^p \alpha_i^\bullet\psi_i^p + p\cdot\frac{1}{p^2}p^\bullet\phi\right)\log x\\ &= \beta\left((1-T_{1,p}+p\cdot T_{2,p})\log f\right)\end{aligned}\tag{4.4}$$

Following Rezk [Rezk2006b, 1.12], we write

$$F_X \coloneqq 1 - T_{1,p}\cdot X + pT_{2,p}\cdot X^2 \in \mathbb{Z}[T_{1,p},T_{2,p}][X]\tag{4.5}$$

In particular, with $X = 1$, (4.4) becomes

$$\ell_{2,p}(x) = \beta\big(F_1(\log f)\big)\tag{4.6}$$

Remark 4.7 Let $x = \beta(f) \in \left(E^0\right)^\times$ for some unit f in $\mathrm{MF}\big(\Gamma_1(N)\big)$ of weight k (cf. [Kubert-Lang1981]). The formula (4.3) expresses $p\ell_{2,p}(x)$ as the logarithm of a ratio: the numerator, via β, corresponds to a modular form of weight $pk+p^2k$, and the denominator corresponds to one having weight $(p+1)pk = pk+p^2k$. Thus $p\ell_{2,p}(x)$ corresponds to a p-adic modular form of weight 0 in view of (4.2) (cf. [Katz1976, Section 10.1]). We note the similarity between the logarithm in [Katz1976, 10.2.7] and the one following Theorem 1.9 in [Rezk2006b].

Example 4.8 We revisit the case $p = 5$, with the model for the E-theory given by the moduli problem $\mathcal{P}_4$. Consider $\delta = \beta(\Delta) \in \left(E^0\right)^\times$. As in Example 3.20, we calculate from (3.10) and (3.7) that

$$
\begin{aligned}
\ell_{2,5}(\delta) &= \frac{1}{5}\log\left(\delta^5 \cdot \frac{1}{\psi_0^p(\delta)\cdots\psi_p^p(\delta)} \cdot \phi(\delta)\right) \\
&= \frac{1}{5}\log\left(\delta^5 \cdot \frac{1}{\delta^6} \cdot \delta\right) \\
&= \frac{1}{5}\log 1 \\
&= 0
\end{aligned}
$$

We compare this to the prime-2 case with the $\mathcal{P}_3$-model. In [Mahowald-Rezk2009, Proposition 3.2], Mahowald and Rezk show that $\mathcal{P}_3$ is represented by

$$y^2 + Axy + By = x^3$$

over $\mathbb{Z}[1/3][A,B,\Delta^{-1}]$ with $|A| = 1$, $|B| = 3$, and $\Delta = B^3(A^3 - 27B)$. In [Rezk2008, 2.8], Rezk computes that

$$\ell_{2,2}\big(\beta_3^{(2)}(\Delta)\big) = \frac{1}{2}\log(-1) = 0$$

We interpret the "log" here as the 2-adic logarithm (see, e.g., [Koblitz1984, §IV.1] and cf. [Katz1976, 10.2.16]). Again, the modular form Δ produces an element in the kernel of a logarithmic operation.

Moreover, since $\Delta = B^3(A^3 - 27B)$ in this case, we have

$$\beta_3^{(2)}(\Delta) = (a-3)(a^2+3a+9)$$

with $E^0 \cong \mathbb{W}\big(\overline{\mathbb{F}}_2\big)[\![a]\!]$ (cf. [Rezk2008, Section 4]). Using Rezk's formula for the total power operation on E^0, we compute that

$$\ell_{2,2}(a-3) = \frac{1}{2}\log(-1) = 0$$

$$\ell_{2,2}(a^2+3a+9) = \frac{1}{2}\log 1 = 0$$

Thus, as $\ell_{2,2}$ is a group homomorphism, the formal power series

$$\frac{1}{a-3} = -\sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty}(-2)^j\right)^{i+1} a^i \ \in \mathbb{W}\big(\overline{\mathbb{F}}_2\big)[\![a]\!]^\times$$

is also contained in the kernel of $\ell_{2,2}$.

The above turn out to be instances of a general vanishing result for the logarithms that we discuss next.

4.2 Detecting the kernels

In this section, via the formula (4.4) for a logarithmic operation in terms of Hecke operators, we detect a family of elements contained in the kernel of this operation. It includes the elements computed in Example 4.8.

We first review some preliminaries about differential structures on rings of modular forms (see, e.g., [Zagier2008, §5] and [Ono2004, Section 2.3]). In connection to Morava E-theories, we have been considering the p-local behavior of integral modular forms of level $\Gamma_1(N)$, with p not dividing N (which embed into the ring of p-adic modular forms). Thus for our purpose these modular forms can equivalently be viewed as over $\mathbb{C}$. Henceforth we will freely interchange between these two perspectives.

Recall that there is a differential operator D acting on meromorphic elliptic functions over $\mathbb{C}$. Specifically, given any meromorphic modular form f, it has a q-expansion

$$f(z) = \sum_{j > -\infty} a_j\, q^j$$

with $a_j \in \mathbb{C}$, where $q = e^{2\pi i z}$ as usual. We then have

$$Df := \frac{1}{2\pi i}\frac{df}{dz} = q\frac{df}{dq} \tag{4.9}$$

If the q-expansion of f has coefficients in $\mathbb{Z}$, so does the q-series for Df.

In general, the function Df is no longer modular. There is another derivation ϑ that preserves modularity. If f has weight k, its *Serre derivative* is defined by

$$\vartheta f := Df - \frac{k}{12}\mathcal{E}_2 \cdot f \tag{4.10}$$

where $\mathcal{E}_2(z) = 1 - 24\sum_{j=1}^{\infty} \sigma_1(j)\, q^j$ is the quasimodular Eisenstein series of weight 2. This is a meromorphic modular form of weight $k+2$ (see [Ono2004, Proposition 2.11], and cf. [Serre1973, Théorème 5(a)] for the action of D on p-adic modular forms).

Example 4.11 Consider the modular discriminant Δ, a cusp form of weight 12. The product expansion

$$\Delta = q\prod_{n=1}^{\infty}(1 - q^n)^{24}$$

implies that

$$\begin{aligned}\log\Delta &= \log q + 24\sum_{n=1}^{\infty}\log(1-q^n)\\ &= \log q + 24\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{(-q^n)^k}{k}\\ &= \log q - 24\sum_{m=1}^{\infty}\sigma_{-1}(m)\,q^m\end{aligned} \tag{4.12}$$

Thus $D\log\Delta = \mathcal{E}_2$, and hence $\vartheta\Delta = 0$.

Theorem 4.13 *Let E be a Morava E-theory of height 2 at the prime p, and let $N > 3$ be any integer prime to p. Let*

$$\ell_{2,p}\colon\ \big(E^0\big)^{\times} \to E^0$$

be Rezk's logarithmic cohomology operation. Let

$$\beta\colon\ \mathrm{MF}\big(\Gamma_1(N)\big) \to E^0$$

be the ring homomorphism in Proposition 2.8. Suppose that $f \in \mathrm{MF}\big(\Gamma_1(N)\big)^{\times}$ has trivial Nebentypus character. If the Serre derivative $\vartheta f = 0$, then $\beta(f)$ is contained in the kernel of $\ell_{2,p}$.

Our proof consists of two parts. In the first part (Lemma 4.14 below), we show that $\ell_{2,p}\big(\beta(f)\big)$ is constant, i.e., it is the image of a constant modular form under β. This is based on an interplay between the differential structures and the action of Hecke operators on modular forms. Looking at q-expansions, we then show in the second part that this constant equals zero. It boils down to an analysis of the behavior of Tate curves under isogenies.

Lemma 4.14 *Given any $f \in \mathrm{MF}\big(\Gamma_1(N)\big)^{\times}$ such that $\vartheta f = 0$, the elliptic function $F_1(\log f)$ is constant, where $F_1 = 1 - T_{1,p} + pT_{2,p}$ is the operator defined in (4.5).*

Proof Suppose that f is of weight k. By (4.10), since $\vartheta f = 0$, we have

$$D\log f = \frac{Df}{f} = \frac{k\mathcal{E}_2}{12}$$

Note that $\mathcal{E}_2$ is a weight-2 eigenform for every Hecke operator $T_m = T_{1,m}$, $m \geq 1$ with eigenvalue $\sigma_1(m)$.

By comparing the effects on q-expansions, we have

$$D \circ T_{i,p} = \frac{1}{p^i} \cdot T_{i,p} \circ D \tag{4.15}$$

for $i = 1$ and $i = 2$. We then compute that

$$\begin{aligned}
D\big(F_1(\log f)\big) &= D\big((1 - T_{1,p} + pT_{2,p}) \log f\big) \\
&= \left(1 - \frac{1}{p} \cdot T_{1,p} + \frac{1}{p^2} \cdot pT_{2,p}\right)(D \log f) \\
&= \left(1 - \frac{1}{p} \cdot T_{1,p} + \frac{1}{p^2} \cdot pT_{2,p}\right)\frac{k\mathcal{E}_2}{12} \\
&= \left(1 - \frac{1}{p} \cdot (1+p) + \frac{1}{p^2} \cdot p\Big(\frac{1}{p^2} \cdot p^2\Big)\right)\frac{k\mathcal{E}_2}{12} \\
&= 0
\end{aligned}$$

Thus as a function of the complex variable z, $F_1(\log f)$ is constant. □

Proof of Theorem 4.13 Let $x := \beta(f)$. Since β is a ring homomorphism, $x \in \left(E^0\right)^\times$. Recall from (4.4) that

$$\ell_{2,p}(x) = \frac{1}{p} \log \frac{x^p \cdot p^\bullet \phi(x)}{\prod_{i=0}^p \alpha_i^\bullet \psi_i^p(x)} = \beta\big(F_1(\log f)\big) \tag{4.16}$$

By Lemma 4.14, the above equals an element in $\mathbb{W}\big(\overline{\mathbb{F}}_p\big)^\times$, which we denote by c_f.

Suppose that f has weight k. Via the correspondence between power operations and deformations of Frobenius in [Rezk2009, Theorem B], the ratio of values of power operations on x in (4.16) equals a ratio of values of f on the corresponding universal elliptic curves. Explicitly, with notation as in (3.15) and (3.21), we have

$$\frac{x^p \cdot p^\bullet \phi(x)}{\prod_{i=0}^p \alpha_i^\bullet \psi_i^p(x)} = \frac{f(\mathcal{C}_S, P_0, du)^p \cdot p^k f(\mathcal{C}_S, [p]P_0, du)}{\prod_{i=0}^p \kappa_i^k f\big(\mathcal{C}_{S^{(p)}}/\mathcal{G}_i^{(p)}, \Psi_i^{(p)}(P_0), du\big)} \tag{4.17}$$

To determine c_f, we need only inspect the constant term in a q-expansion for the right-hand side.

Over a punctured formal neighborhood of each cusp, the universal curve $\mathcal{C}$ is isomorphic to the Tate curve $\mathrm{Tate}(q^N)$ with the level-$\Gamma_1(N)$ structure corresponding to that cusp. The universal degree-p isogeny on $\mathrm{Tate}(q^N)$ is defined over the ring $\mathbb{Z}[1/pN, \zeta_p]((q^{1/p}))$, where ζ_p is a primitive p'th root of unity (see [Katz1973, Sections 1.2, 1.4, and 1.11]). In particular, the $(p+1)$ subgroups of order p are

$$G_0^{(p)}, \text{ generated by } \zeta_p, \qquad \text{and} \qquad G_i^{(p)},\ 1 \le i \le p, \text{ generated by } (\zeta_p^i q^{1/p})^N$$

Let $\sum_{j=m}^{\infty} a_j\, q^j$ be a q-expansion of f. We now compare as follows the lowest powers of q at the denominator and the numerator of (4.17).

By [Katz1973, (1.11.0.3) and (1.11.0.4)] (cf. Section 1.3), including a normalizing factor p^k in place of κ_i^k (see Remark 3.16), we have at the denominator a leading term as the product of

$$p^k a_m (q^p)^m \qquad \text{and} \qquad p^k \left(p^{-k} a_m \left(\zeta_p^i q^{1/p} \right)^m \right)$$

where i runs from 1 to p. Note that since the Nebentypus character of f is trivial, the coefficient a_m is independent of where the level-$\Gamma_1(N)$ structure goes under each degree-p isogeny.

At the numerator, for the first factor, we have a leading term $(a_m q^m)^p$. For the second factor, we have a leading term $p^k a_m q^m$, again under the assumption that f has trivial Nebentypus character.

Combining these, we see that the ratio has a leading constant term

$$\frac{(a_m q^m)^p \cdot p^k a_m q^m}{p^k a_m (q^p)^m \cdot \prod_{i=1}^{p} p^k \left(p^{-k} a_m \left(\zeta_p^i q^{1/p} \right)^m \right)} = \zeta_p^{-m(1+p)p/2} = \begin{cases} (-1)^m & \text{if } p = 2 \\ 1 & \text{if } p \text{ is odd} \end{cases}$$

Thus applying the p-adic logarithm (cf. Example 4.8) we get $c_f = 0$. □

Remark 4.18 Let f be any nonzero meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$. Bruinier, Kohnen, and Ono give an explicit formula for ϑf as a multiple of $-f$ by a certain function f_Θ [Bruinier-Kohnen-Ono2004, Theorem 1]. The function f_Θ encodes a sequence of modular functions $j_m(z)$, $m \geq 0$ defined by applying Hecke operators to the usual j-invariant, whose values are of arithmetic and combinatorial significance.

In particular, the formula of Bruinier, Kohnen, and Ono immediately shows that a nonzero meromorphic modular form f has vanishing Serre derivative precisely when its zeros and poles are located only at the cusp (cf. [Dumas-Royer2014, Proposition 6]). The functions in Example 4.8, and in fact any unit in $\mathrm{MF}\big(\Gamma_1(N)\big)$, all have the latter property, though they are associated to $\Gamma_1(N)$ instead of $\mathrm{SL}_2(\mathbb{Z})$.

Their theorem has been generalized by Ahlgren to $\Gamma_0(p)$ with $p \in \{2, 3, 5, 7, 13\}$ [Ahlgren2003, Theorem 2] and further by Choi to $\Gamma_0(n)$ for any n that is square-free [Choi2006, Theorem 3.4]. In view of the assumption on Nebentypus character in Theorem 4.13, we note that modular forms of level $\Gamma_1(N)$ with trivial Nebentypus character are precisely those of level $\Gamma_0(N)$.

5 Extending the action of Hecke operators onto logarithmic q-series

The purpose of this section is to give an account of elliptic functions of the form $\log f$, where f is a meromorphic modular form. As we have seen in Section 4, such functions arise in computations for logarithmic cohomology operations on Morava E-theories at height 2, e.g., in (4.4).

Example 5.1 Consider the function $\log \Delta$ in Example 4.11. In the final form of (4.12), the second summand is a convergent q-series. We may call it an *Eisenstein series of weight* 0, by analogy to q-expansions for the usual Eisenstein series of higher weight (also cf. the real analytic Eisenstein series of weight 0 discussed in [Funke2007, Sections 3.3 and 4.1]).

The first summand, $\log q$, never shows up in the q-expansion of a meromorphic modular form. It is indeed this term that we will address, given the importance of Δ in the context of logarithmic operations (see Theorem 4.13). Specifically, with motivations from homotopy theory, we aim to extend the classical action of Hecke operators on modular forms to incorporate series such as (4.12). We will then apply the extended action in Example 5.11.

Remark 5.2 Given a cusp form $f = \sum_{n=1}^{\infty} a_n q^n$ of weight k, its Eichler integral $\widetilde{f} = \sum_{n=1}^{\infty} n^{-k+1} a_n q^n$ is a mock modular form of weight $2-k$ (see the end of [Zagier2009, §6]). Recall from Example 4.11 that $D \log \Delta = \mathcal{E}_2$. Since $D^{k-1}\widetilde{f} = f$, we may then view $\log \Delta$ as a generalized Eichler integral, "generalized" in that $\mathcal{E}_2$ is not a cusp form. We may even approach proving Lemma 4.14 from this viewpoint.

In [Knopp-Mason2011], given a representation $\rho\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{C})$, Knopp and Mason consider n-dimensional vector-valued modular forms associated to ρ. In particular, they show in [Knopp-Mason2011, Theorem 2.2] that the components of certain vector-valued modular forms are functions

$$f(z) = \sum_{j=0}^{t} (\log q)^j h_j(z) \tag{5.3}$$

where $t \geq 0$ is an integer, and each h_j is a convergent q-series with at worst real exponents (cf. [Knopp-Mason2011, (7), (13), and Sections 3.2–3.3]). They remark that q-expansions of this form occur in logarithmic conformal field theory (e.g., cf. [Zhu1996, (5.3.9)] and [Dong-Li-Mason2000, (6.12)]). The one in (4.12) gives another example. Following Knopp and Mason, we call the series in (5.3) a *logarithmic* q-series.

Proposition 5.4 *The action of the Hecke operator T_p on modular forms of level $\Gamma_0(N)$ extends so that*

$$T_p(\log q) = \left(p^{-1} + p^{-2}\right) \log q$$

Proof Consider modular forms in $\mathrm{MF}\big(\Gamma_1(N)\big)$ with trivial Nebentypus character, i.e., those on $\Gamma_0(N)$. We follow the modular description in [Katz1973, Section 1.11] for Hecke operators in the presence of the Tate curve $\mathrm{Tate}(q^N)$ over $\mathbb{Z}[1/pN, \zeta_p](\!(q^{1/p})\!)$, where ζ_p is a primitive p'th root of unity.

Write $\mathcal{F} := \log q$. Let ω_{can} be the canonical differential on $\mathrm{Tate}(q^N)$. Let $\mathcal{M}_N := \operatorname{Proj} S_N$ be the scheme over $\mathbb{Z}[1/N]$ representing the moduli problem $\mathcal{P}_N$ (see Examples 2.1, 2.2, and 2.6). Denote by $\underline{\omega} := \mathrm{pr}_* \Omega^1_{\mathcal{C}_N/\mathcal{M}_N}$ the pushforward along the structure morphism $\mathrm{pr}\colon \mathcal{C}_N \to \mathcal{M}_N$ of the relative cotangent sheaf $\Omega^1_{\mathcal{C}_N/\mathcal{M}_N}$.

By [Katz-Mazur1985, Theorem 10.13.11], the isomorphism

$$\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{M}_N\big/\mathbb{Z}[1/N]}$$

on $\mathcal{M}_N$ extends to an isomorphism

$$\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\overline{\mathcal{M}}_N\big/\mathbb{Z}[1/N]}(\log \text{cusps})$$

on the compactification $\overline{\mathcal{M}}_N$, where the target is the invertible sheaf of one-forms with at worst simple poles along the cusps. In particular, over the cusps, ω^2_{can} corresponds to $N \cdot d\mathcal{F}$ under this isomorphism (cf. [Katz1973, Section 1.5]). Since $\mathcal{F} = \log q = 2\pi i z$ is linear in z, we have

$$\mathcal{F}\big(\mathrm{Tate}(q^N),\ P_0,\ p \cdot \omega_{\mathrm{can}}\big) = p^2 \cdot \mathcal{F}\big(\mathrm{Tate}(q^N),\ P_0,\ \omega_{\mathrm{can}}\big) \tag{5.5}$$

By [Katz1973, (1.11.0.3) and (1.11.0.4)] (cf. Section 1.3), given the assumption of trivial Nebentypus character, we then calculate that

$$\begin{aligned}
T_p(\log q) &= \frac{1}{p} \cdot p^k \left(\log(q^p) + \sum_{i=1}^{p} p^{-k} \log(\zeta_p^i q^{1/p}) \right) \\
&= p^{k-1} \left(p \log q + p^{-k} \sum_{i=1}^{p} (\log \zeta_p^i + p^{-1} \log q) \right) \\
&= p^{k-1} \left(p \log q + p^{-k} \sum_{i=1}^{p} p^{-1} \log q \right) \\
&= p^{k-1} \left(p + p^{-k} \right) \log q
\end{aligned} \tag{5.6}$$

where we interpret "log" as the p-adic logarithm so that $\log \zeta_p^i = 0$ (cf. Example 4.8). Setting $k = -2$ in view of (5.5), we obtain the stated identity. □

Remark 5.7 More generally, if we view $\log q$ as generalizing modular forms in $\mathrm{MF}\big(\Gamma_1(N)\big)$ with Nebentypus character χ, we compute as in (5.6) and obtain

$$T_p(\log q) = \big(p^{-1} + \chi(p)\,p^{-2}\big)\log q$$

For the rest of this section, we focus on the case when χ is trivial (cf. Theorem 4.13).

Let K be a number field. In [Bruinier-Ono2003], Bruinier and Ono study meromorphic modular forms g on $\mathrm{SL}_2(\mathbb{Z})$ with q-expansion

$$g(z) = q^m \left(1 + \sum_{n=1}^{\infty} a_n q^n\right) \tag{5.8}$$

where $m \in \mathbb{Z}$ and $a_n \in \mathcal{O}_K$ (the ring of integers in K). They show that if g satisfies a certain condition with respect to a prime p, its logarithmic derivative $D\log(g)$ is a p-adic modular form of weight 2 [Bruinier-Ono2003, Theorem 1] (the differential operator D is defined in (4.9)).

This theorem has been generalized to meromorphic modular forms on $\Gamma_0(p)$ for $p \geq 5$ [Getz2004, Theorem 4]. Examples include $\mathcal{E}_{p-1}$ at each $p \geq 5$ and, for all p, meromorphic modular forms whose zeros and poles are located only at the cusps (cf. [Bruinier-Ono2003, Definition 3.1]). In particular, when $g = \Delta$, we have $D\log\Delta = \mathcal{E}_2$ (cf. the discussion before [Serre1973, Théorème 5]).

Given any g as in (5.8), note that $\log(g)$ is a logarithmic q-series. The following definition is based on Proposition 5.4 (specifically (5.5) and (5.6)) and the theorem of Bruinier and Ono above.

Definition 5.9

(i) Given any integer $j \geq 0$, define the *weight* of $(\log q)^j$ to be $-2j$.

(ii) Let g be a meromorphic modular form such that $D\log(g)$ is a p-adic modular form of weight 2 for all p. Define the *weight* of $\log(g)$ to be 0.

(iii) Given any prime p and any logarithmic q-series

$$f = \sum_{j=0}^{t} (\log q)^j h_j$$

of weight k, define $T_p f$ as follows. For each j, if

$$h_j = \sum_{m>-\infty} a_m q^m$$

(the index m and the coefficients a_m depend on j), define

$$T_p\big((\log q)^j h_j\big) := (\log q)^j \sum_{m>-\infty} b_m q^m$$

where the coefficients

$$b_m = p^{j+k-1} a_{m/p} + p^{-j} a_{pm}$$

with the convention that $a_{m/p} = 0$ unless $p|m$. We then define

$$T_p f := \sum_{j=0}^{t} T_p\big((\log q)^j h_j\big)$$

Remark 5.10

(i) The definitions of weight above are compatible with the action of the differential operator D. Specifically, we have

$$D \log q = q\frac{d}{dq}(\log q) = 1$$

Thus applying D to $\log q$ increases the weight by 2, which agrees with [Serre1973, Théorème 5 (a)]. More generally, for $j \geq 0$, since $D(\log q)^j = j(\log q)^{j-1}$, the same compatibility holds.

(ii) The definition for $T_p\big((\log q)^j h_j\big)$ extends [Katz1973, Formula 1.11.1] by a computation analogous to (5.6) under the assumption of trivial Nebentypus character (see Remark 5.7). Moreover, the following identities for operators acting on modular forms extend to the series $(\log q)^j h_j$:

$$D \circ T_p = \frac{1}{p} \cdot T_p \circ D \qquad \text{cf. (4.15)}$$

$$T_\ell \circ T_p = T_p \circ T_\ell \qquad \text{for primes } \ell \text{ and } p$$

(assuming that Hecke operators preserve weight). We can also define the Hecke operators T_m acting on $(\log q)^j h_j$ for any positive integer m as in the remark after [Serre1973, Théorème 4].

Example 5.11 We now return to Example 5.1. By Definition 5.9 (ii), $\log \Delta$ is a logarithmic q-series of weight 0. In view of (4.12), we then compute by Definition 5.9 (iii) that

$$T_p(\log \Delta) = \sigma_{-1}(p) \log \Delta$$

Thus in (4.6) we have

$$F_1(\log \Delta) = \big(1 - \sigma_{-1}(p) + p^{-1}\big) \log \Delta = 0$$

This gives a second proof of Theorem 4.13 in the case $f = \Delta$.

6 Topological Hecke operators in terms of individual power operations

Constructed from total power operations, the topological Hecke operators in (3.18) and (3.23) can be defined more generally on E^0X for any space X. In this section, we examine their role in the algebra of additive E-cohomology operations, by expressing them in terms of generators of this algebra.

Let E be a Morava E-theory of height n at the prime p. There is an algebraic theory—in the sense of Lawvere [Lawvere1963]—constructed from the extended power functors on the category of E-modules (cf. [Rezk2006a, Section 9] and [Rezk2009, Section 4]). It describes all homotopy operations on $K(n)$-local commutative E-algebras.

Rezk shows that under a certain congruence condition, a model for this algebraic theory amounts to a p-torsion-free graded commutative algebra over an associative ring Γ [Rezk2009, Theorem A]. Following Rezk, we call Γ the *Dyer-Lashof algebra* of the E-theory. It is constructed in [Rezk2009, 6.2] as a direct sum of the E^0-linear duals of the rings $E^0(B\Sigma_{p^k})/I$, for all $k \geq 0$ (cf. (3.3)).

Example 6.1 Using the formulas for $\widetilde{h}$ and $\widetilde{\alpha}$ in (3.10), we compute as in [Zhu2014, Proposition 3.6] and obtain a presentation for the Dyer-Lashof algebra Γ at height 2 and prime 5. Specifically, Γ is the following graded twisted bialgebra over $E_0 \cong \mathbb{W}\big(\overline{\mathbb{F}}_5\big)[\![h]\!]$ with generators Q_i, $0 \leq i \leq 5$.

- The "twists" are given in terms of *commutation relations*:

$$
\begin{aligned}
Q_i c &= (Fc) Q_i \quad \text{for } c \in \mathbb{W}\big(\overline{\mathbb{F}}_5\big) \text{ and all } i, \text{ with } F \text{ the Frobenius automorphism} \\
Q_0 h &= (h^5 - 10h^4 - 1065h^3 + 12690h^2 + 168930h - 1462250)Q_0 + (5h^4 \\
&\quad - 50h^3 - 3950h^2 + 42200h + 233400)Q_1 + (25h^3 - 250h^2 - 12875h \\
&\quad + 104750)Q_2 + (125h^2 - 1250h - 30000)Q_3 + (625h - 6250)Q_4 \\
&\quad + 3120Q_5 \\
Q_1 h &= (-55h^4 + 850h^3 + 39575h^2 - 608700h - 1113524)Q_0 + (-275h^3 \\
&\quad + 4250h^2 + 122250h - 1462250)Q_1 + (-1375h^2 + 21250h \\
&\quad + 233400)Q_2 + (-6875h + 104750)Q_3 - 30000Q_4 + (h - 6250)Q_5 \\
Q_2 h &= (60h^4 - 775h^3 - 45400h^2 + 593900h + 2008800)Q_0 + (300h^3 \\
&\quad - 3875h^2 - 144500h + 1453876)Q_1 + (1500h^2 - 19375h \\
&\quad - 310000)Q_2 + (7500h - 96600)Q_3 + 36000Q_4 + 4320Q_5
\end{aligned}
$$

$$\begin{aligned}
Q_3h = {} & (-35h^4+400h^3+27125h^2-320900h-1418300)Q_0+(-175h^3\\
&+2000h^2+87500h-792000)Q_1+(-875h^2+10000h+196876)Q_2\\
&+(-4375h+50000)Q_3-21600Q_4-1440Q_5\\
Q_4h = {} & (10h^4-105h^3-7850h^2+86975h+445850)Q_0+(50h^3-525h^2\\
&-25500h+215500)Q_1+(250h^2-2625h-58750)Q_2+(1250h\\
&-13124)Q_3+6250Q_4+240Q_5\\
Q_5h = {} & (-h^4+10h^3+790h^2-8440h-46680)Q_0+(-5h^3+50h^2+2575h\\
&-20950)Q_1+(-25h^2+250h+6000)Q_2+(-125h+1250)Q_3\\
&-624Q_4+10Q_5
\end{aligned}$$

- The product structure is subject to *Adem relations*:

$$\begin{aligned}
Q_1Q_0 = {} & 55Q_0Q_1+(55h-300)Q_0Q_2+(55h^2-300h-14250)Q_0Q_3+(55h^3\\
&-300h^2-29375h+163750)Q_0Q_4+(55h^4-300h^3-44500h^2\\
&+328750h+3228750)Q_0Q_5+275Q_1Q_2+(275h-1500)Q_1Q_3\\
&+(275h^2-1500h-71250)Q_1Q_4+(275h^3-1500h^2-146875h\\
&+818750)Q_1Q_5-5Q_2Q_1+1375Q_2Q_3+(1375h-7500)Q_2Q_4\\
&+(1375h^2-7500h-356250)Q_2Q_5-25Q_3Q_2+6875Q_3Q_4\\
&+(6875h-37500)Q_3Q_5-125Q_4Q_3+34375Q_4Q_5-625Q_5Q_4\\
Q_2Q_0 = {} & -60Q_0Q_1+(-60h+175)Q_0Q_2+(-60h^2+175h+16250)Q_0Q_3\\
&+(-60h^3+175h^2+32750h-138000)Q_0Q_4+(-60h^4+175h^3\\
&+49250h^2-276125h-3943750)Q_0Q_5-300Q_1Q_2+(-300h\\
&+875)Q_1Q_3+(-300h^2+875h+81250)Q_1Q_4+(-300h^3+875h^2\\
&+163750h-690000)Q_1Q_5-1500Q_2Q_3+(-1500h+4375)Q_2Q_4\\
&+(-1500h^2+4375h+406250)Q_2Q_5-5Q_3Q_1-7500Q_3Q_4\\
&+(-7500h+21875)Q_3Q_5-25Q_4Q_2-37500Q_4Q_5-125Q_5Q_3\\
Q_3Q_0 = {} & 35Q_0Q_1+(35h-50)Q_0Q_2+(35h^2-50h-9600)Q_0Q_3+(35h^3\\
&-50h^2-19225h+66250)Q_0Q_4+(35h^4-50h^3-28850h^2\\
&+132500h+2411875)Q_0Q_5+175Q_1Q_2+(175h-250)Q_1Q_3\\
&+(175h^2-250h-48000)Q_1Q_4+(175h^3-250h^2-96125h\\
&+331250)Q_1Q_5+875Q_2Q_3+(875h-1250)Q_2Q_4+(875h^2\\
&-1250h-240000)Q_2Q_5+4375Q_3Q_4+(4375h-6250)Q_3Q_5\\
&-5Q_4Q_1+21875Q_4Q_5-25Q_5Q_2\\
Q_4Q_0 = {} & -10Q_0Q_1+(-10h+5)Q_0Q_2+(-10h^2+5h+2750)Q_0Q_3
\end{aligned}$$

$$
\begin{aligned}
&+(-10h^3+5h^2+5500h-16375)Q_0Q_4+(-10h^4+5h^3+8250h^2\\
&-32750h-705000)Q_0Q_5-50Q_1Q_2+(-50h+25)Q_1Q_3+(-50h^2\\
&+25h+13750)Q_1Q_4+(-50h^3+25h^2+27500h-81875)Q_1Q_5\\
&-250Q_2Q_3+(-250h+125)Q_2Q_4+(-250h^2+125h\\
&+68750)Q_2Q_5-1250Q_3Q_4+(-1250h+625)Q_3Q_5-6250Q_4Q_5\\
&-5Q_5Q_1
\end{aligned}
$$

$$
\begin{aligned}
Q_5Q_0 = {} & Q_0Q_1+hQ_0Q_2+(h^2-275)Q_0Q_3+(h^3-550h+1500)Q_0Q_4+(h^4\\
&-825h^2+3000h+71250)Q_0Q_5+5Q_1Q_2+5hQ_1Q_3+(5h^2\\
&-1375)Q_1Q_4+(5h^3-2750h+7500)Q_1Q_5+25Q_2Q_3+25hQ_2Q_4\\
&+(25h^2-6875)Q_2Q_5+125Q_3Q_4+125hQ_3Q_5+625Q_4Q_5
\end{aligned}
$$

- The coproduct structure is given by *Cartan formulas*:

$$
\begin{aligned}
Q_0(xy) = {} & Q_0(x)Q_0(y)-5\big(Q_1(x)Q_5(y)+Q_2(x)Q_4(y)+Q_3(x)Q_3(y)\\
&+Q_4(x)Q_2(y)+Q_5(x)Q_1(y)\big)-50\big(Q_2(x)Q_5(y)+Q_3(x)Q_4(y)\\
&+Q_4(x)Q_3(y)+Q_5(x)Q_2(y)\big)-325\big(Q_3(x)Q_5(y)+Q_4(x)Q_4(y)\\
&+Q_5(x)Q_3(y)\big)-1800\big(Q_4(x)Q_5(y)+Q_5(x)Q_4(y)\big)\\
&-9350Q_5(x)Q_5(y)\\
Q_1(xy) = {} & \big(Q_0(x)Q_1(y)+Q_1(x)Q_0(y)\big)+h\big(Q_1(x)Q_5(y)+Q_2(x)Q_4(y)\\
&+Q_3(x)Q_3(y)+Q_4(x)Q_2(y)+Q_5(x)Q_1(y)\big)+(10h-5)\big(Q_2(x)Q_5(y)\\
&+Q_3(x)Q_4(y)+Q_4(x)Q_3(y)+Q_5(x)Q_2(y)\big)+(65h\\
&-50)\big(Q_3(x)Q_5(y)+Q_4(x)Q_4(y)+Q_5(x)Q_3(y)\big)+(360h\\
&-325)\big(Q_4(x)Q_5(y)+Q_5(x)Q_4(y)\big)+(1870h-1800)Q_5(x)Q_5(y)\\
Q_2(xy) = {} & \big(Q_0(x)Q_2(y)+Q_1(x)Q_1(y)+Q_2(x)Q_0(y)\big)-55\big(Q_1(x)Q_5(y)\\
&+Q_2(x)Q_4(y)+Q_3(x)Q_3(y)+Q_4(x)Q_2(y)+Q_5(x)Q_1(y)\big)+(h\\
&-550)\big(Q_2(x)Q_5(y)+Q_3(x)Q_4(y)+Q_4(x)Q_3(y)+Q_5(x)Q_2(y)\big)\\
&+(10h-3580)\big(Q_3(x)Q_5(y)+Q_4(x)Q_4(y)+Q_5(x)Q_3(y)\big)+(65h\\
&-19850)\big(Q_4(x)Q_5(y)+Q_5(x)Q_4(y)\big)+(360h-103175)Q_5(x)Q_5(y)\\
Q_3(xy) = {} & \big(Q_0(x)Q_3(y)+Q_1(x)Q_2(y)+Q_2(x)Q_1(y)+Q_3(x)Q_0(y)\big)\\
&+60\big(Q_1(x)Q_5(y)+Q_2(x)Q_4(y)+Q_3(x)Q_3(y)+Q_4(x)Q_2(y)\\
&+Q_5(x)Q_1(y)\big)+545\big(Q_2(x)Q_5(y)+Q_3(x)Q_4(y)+Q_4(x)Q_3(y)\\
&+Q_5(x)Q_2(y)\big)+(h+3350)\big(Q_3(x)Q_5(y)+Q_4(x)Q_4(y)\\
&+Q_5(x)Q_3(y)\big)+(10h+18020)\big(Q_4(x)Q_5(y)+Q_5(x)Q_4(y)\big)+(65h\\
&+92350)Q_5(x)Q_5(y)
\end{aligned}
$$

$$\begin{aligned}
Q_4(xy) = {} & \big(Q_0(x)Q_4(y) + Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y) \\
& + Q_4(x)Q_0(y)\big) - 35\big(Q_1(x)Q_5(y) + Q_2(x)Q_4(y) + Q_3(x)Q_3(y) \\
& + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)\big) - 290\big(Q_2(x)Q_5(y) + Q_3(x)Q_4(y) \\
& + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)\big) - 1730\big(Q_3(x)Q_5(y) + Q_4(x)Q_4(y) \\
& + Q_5(x)Q_3(y)\big) + (h - 9250)\big(Q_4(x)Q_5(y) + Q_5(x)Q_4(y)\big) + \big(10h \\
& - 47430\big)Q_5(x)Q_5(y) \\
Q_5(xy) = {} & \big(Q_0(x)Q_5(y) + Q_1(x)Q_4(y) + Q_2(x)Q_3(y) + Q_3(x)Q_2(y) \\
& + Q_4(x)Q_1(y) + Q_5(x)Q_0(y)\big) + 10\big(Q_1(x)Q_5(y) + Q_2(x)Q_4(y) \\
& + Q_3(x)Q_3(y) + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)\big) + 65\big(Q_2(x)Q_5(y) \\
& + Q_3(x)Q_4(y) + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)\big) + 360\big(Q_3(x)Q_5(y) \\
& + Q_4(x)Q_4(y) + Q_5(x)Q_3(y)\big) + 1870\big(Q_4(x)Q_5(y) + Q_5(x)Q_4(y)\big) \\
& + (h + 9450)Q_5(x)Q_5(y)
\end{aligned}$$

- Subject to the above commutation and Adem relations, Γ becomes a free left module over E_0. A basis consists of monomials in the generators Q_i which are in the form $Q_0^m Q_{i_1} \cdots Q_{i_n}$, with $m \geq 0$, $n \geq 0$ ($n = 0$ corresponds to Q_0^m), and $1 \leq i_k \leq 5$. The grading on Γ refers to the sum of the exponents in each of these monomials.

Let E be a Morava E-theory of height n at the prime p. Given any $K(n)$-local commutative E-algebra A, the elements Q_i above are examples of additive power operations having A^0 as both domain and range. Such operations arise from the total power operation

$$\psi^p\colon A^0 \to A^0(B\Sigma_p)/J \cong A^0[\alpha]/\big(w(\alpha)\big) \tag{6.2}$$

where J is a transfer ideal, and $w(\alpha) \in E^0[\alpha]$ is a monic polynomial of degree $r := 1 + p + p^2 + \cdots + p^{n-1}$ (cf. (3.3)). We define *individual power operations* $Q_i\colon A^0 \to A^0$ by the formula

$$\psi^p(x) = \sum_{i=0}^{r-1} Q_i(x)\,\alpha^i \tag{6.3}$$

In particular, given a space X, each Q_i acts on E^0X if we take A to be the spectrum of functions from $\Sigma^\infty_+ X$ to E. These individual power operations Q_i generate the Dyer-Lashof algebra of the E-theory (cf. [Zhu2014, proof of Theorem 3.10]).

Proposition 6.4 *Let E be a Morava E-theory of height $n = 2$ at the prime p. Given any $K(2)$-local commutative E-algebra A, let ψ^p be the total power operation in* (6.2) *with the polynomial*

$$w(\alpha) = w_{p+1}\alpha^{p+1} + \cdots + w_1\alpha + w_0 \qquad w_{p+1} = 1 \tag{6.5}$$

and let Q_i, $0 \leq i \leq p$ *be the corresponding individual power operations in* (6.3). *For* $\mu = 1$ *and* $\mu = 2$, *let*

$$t_{\mu,p} \colon A^0 \to p^{-1}A^0$$

be the topological Hecke operators that are defined as in (3.18) *and* (3.23) *from the above total power operation* ψ^p *on* A^0. *We have the following identities:*

$$t_{1,p} = \frac{1}{p}\sum_{i=0}^{p} c_i\, Q_i \qquad \text{and} \qquad t_{2,p} = \frac{1}{p^2}\sum_{j=0}^{p}\sum_{i=0}^{j} w_0^i\, d_{j-i}\, Q_i Q_j$$

where

$$c_i = \begin{cases} p+1 & i = 0 \\ -\sum_{k=0}^{i-1} w_{p+1+k-i}\, c_k + (p+1-i)\, w_{p+1-i} & 1 \leq i \leq p \end{cases}$$

$$d_\tau = \begin{cases} 1 & \tau = 0 \\ -\sum_{k=0}^{\tau-1} w_0^{\tau-k-1} w_{\tau-k}\, d_k & 1 \leq \tau \leq p \end{cases}$$

In fact, for $i \geq 1$ *and* $\tau \geq 1$,

$$c_i = i \sum_{\substack{m_1+2m_2+\cdots+vm_v=i \\ m_s\geq 1}} (-1)^{m_1+\cdots+m_v} \frac{(m_1+\cdots+m_v-1)!}{m_1!\cdots m_v!}\, w_{p+1-1}^{m_1}\cdots w_{p+1-v}^{m_v}$$

$$d_\tau = \sum_{n=0}^{\tau-1}(-1)^{\tau-n}\, w_0^n \sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau \\ 1\leq m_s\leq m_{s+1}\leq p}} w_{m_1}\cdots w_{m_{\tau-n}}$$

Proof By definitions (3.18) and (6.3),

$$\begin{aligned} t_{1,p}(x) &= \frac{1}{p}\sum_{j=0}^{p}\psi_j^p(x) \\ &= \frac{1}{p}\sum_{j=0}^{p}\sum_{i=0}^{p} Q_i(x)\alpha_j^i \\ &= \frac{1}{p}\sum_{i=0}^{p}\left(\sum_{j=0}^{p}\alpha_j^i\right) Q_i(x) \end{aligned}$$

Since the parameters α_j are the roots of $w(\alpha)$ in (6.5), the formulas for $c_i = \sum_{j=0}^{p}\alpha_j^i$ then follow from Newton's and Girard's formulas relating power sums and elementary symmetric functions (see, e.g., [Milnor-Stasheff1974, Problem 16-A] and Section 1.3).

For $t_{2,p}(x)$, we first write by (3.23) that

$$\begin{aligned} t_{2,p}(x) &= \frac{1}{p^2}\psi^p\big(\psi^p(x)\big) \\ &= \frac{1}{p^2}\psi^p\sum_{j=0}^{p} Q_j(x)\alpha^j \\ &= \frac{1}{p^2}\sum_{j=0}^{p}\psi^p\big(Q_j(x)\big)\psi^p(\alpha)^j \\ &= \frac{1}{p^2}\sum_{j=0}^{p}\left(\sum_{i=0}^{p} Q_iQ_j(x)\alpha^i\right)\widetilde{\alpha}^j \end{aligned}$$

Since the target of $t_{2,p}$ is $p^{-1}A^0$, the above identity should simplify to contain neither α nor $\widetilde{\alpha}$. Thus in view of (3.13), we rewrite

$$t_{2,p}(x) = \frac{1}{p^2}\sum_{j=0}^{p}\sum_{i=0}^{j} w_0^i\, Q_iQ_j(x)\,\widetilde{\alpha}^{j-i}$$

For $0 \le \tau \le p$, we next express each $\widetilde{\alpha}^\tau$ as a polynomial in α of degree at most p with coefficients in E^0, and verify that the constant term of this polynomial is d_τ as stated in the proposition.

The case $\tau = 0$ is clear. For $1 \le \tau \le p$, we have

$$\begin{aligned} \widetilde{\alpha}^\tau &= \left(\frac{w_0}{\alpha}\right)^\tau && \text{by (3.13)} \\ &= \frac{w_0^{\tau-1}(-w_{p+1}\alpha^{p+1} - \cdots - w_1\alpha)}{\alpha^\tau} && \text{by (6.5)} \\ &= w_0^{\tau-1}(-w_{p+1}\alpha^{p+1-\tau} - \cdots - w_{\tau+1}\alpha) \\ &\quad - w_0^{\tau-1}w_\tau - \frac{w_0^{\tau-1}w_{\tau-1}}{\alpha} - \cdots - \frac{w_0^{\tau-1}w_1}{\alpha^{\tau-1}} \\ &= w_0^{\tau-1}(-w_{p+1}\alpha^{p+1-\tau} - \cdots - w_{\tau+1}\alpha) \\ &\quad - w_0^{\tau-1}w_\tau - w_0^{\tau-2}w_{\tau-1}\widetilde{\alpha} - \cdots - w_1\widetilde{\alpha}^{\tau-1} && \text{by (3.13)} \end{aligned}$$

and thus

$$\begin{aligned} d_\tau &= - w_0^{\tau-1}w_\tau - w_0^{\tau-2}w_{\tau-1}d_1 - \cdots - w_1 d_{\tau-1} \\ &= -\sum_{k=0}^{\tau-1} w_0^{\tau-k-1}w_{\tau-k}\, d_k \end{aligned}$$

This gives the first identity for d_τ stated in the proposition. From this relation, we show the second stated identity for d_τ by induction on τ. The base case $\tau = 1$, with

$d_1 = -w_1$, can be checked directly. For $\tau \geq 2$, we first rewrite

$$d_\tau = -w_0^{\tau-1} w_\tau - \sum_{r=1}^{\tau-1} w_0^{r-1} w_r \, d_{\tau-r}$$

Expanding $d_{\tau-r}$ for $1 \leq r \leq \tau - 1$ by the induction hypothesis, we then arrange $\sum_{r=1}^{\tau-1} w_0^{r-1} w_r \, d_{\tau-r}$ above into a sum indexed by n with $0 \leq n \leq \tau - 2$, where we collect terms that contain exactly an n-power of w_0:

$$\begin{aligned} d_\tau =& - w_0^{\tau-1} w_\tau - \sum_{n=0}^{\tau-2} \Bigg(\sum_{r=1}^{n+1} w_0^{r-1} w_r \, (-1)^{\tau-r-(n-r+1)} w_0^{n-r+1} \\ & \qquad \sum_{\substack{m_{1,r}+\cdots+m_{\tau-r-(n-r+1),r}=\tau-r \\ 1 \leq m_{s,r} \leq m_{s+1,r} \leq p}} w_{m_{1,r}} \cdots w_{m_{\tau-r-(n-r+1),r}} \Bigg) \\ =& - w_0^{\tau-1} w_\tau - \sum_{n=0}^{\tau-2} (-1)^{\tau-n-1} w_0^n \sum_{r=1}^{n+1} w_r \sum_{\substack{m_{1,r}+\cdots+m_{\tau-n-1,r}=\tau-r \\ 1 \leq m_{s,r} \leq m_{s+1,r} \leq p}} w_{m_{1,r}} \cdots w_{m_{\tau-n-1,r}} \\ =& - w_0^{\tau-1} w_\tau + \sum_{n=0}^{\tau-2} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau \\ 1 \leq m_s \leq m_{s+1} \leq p}} w_{m_1} \cdots w_{m_{\tau-n}} \\ =& \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau \\ 1 \leq m_s \leq m_{s+1} \leq p}} w_{m_1} \cdots w_{m_{\tau-n}} \end{aligned}$$

□

Remark 6.6 In the proof above, the method for computing $t_{2,\,p}$ applies more generally. It enables us to find formulas for Adem relations—one for each $Q_i Q_0$, $1 \leq i \leq p$—in terms of the coefficients w_j of $w(\alpha)$ (see Example 6.1 and cf. [Zhu2014, proof of Proposition 3.6 (iv)]). We can also express Cartan formulas using these coefficients. However, commutation relations are determined by both the terms w_j and $\psi^p(w_j)$.

Example 6.7 For $p = 5$, by Proposition 6.4, we compute from (3.7) that

$$\begin{aligned} t_{1,\,5} =& \frac{6}{5} Q_0 + 2Q_1 + 6Q_2 + 26Q_3 + 126Q_4 + (h + 600)Q_5 \\ t_{2,\,5} =& \frac{1}{25} \big(Q_0Q_0 + hQ_0Q_1 + (h^2 - 275)Q_0Q_2 + (h^3 - 550h + 1500)Q_0Q_3 \\ & + (h^4 - 825h^2 + 3000h + 71250)Q_0Q_4 + (h^5 - 1100h^3 + 4500h^2 \end{aligned}$$

$$+ 218125h - 818750)Q_0Q_5 + 5Q_1Q_1 + 5hQ_1Q_2 + (5h^2 - 1375)Q_1Q_3 \\ + (5h^3 - 2750h + 7500)Q_1Q_4 + (5h^4 - 4125h^2 + 15000h \\ + 356250)Q_1Q_5 + 25Q_2Q_2 + 25hQ_2Q_3 + (25h^2 - 6875)Q_2Q_4 + (25h^3 \\ - 13750h + 37500)Q_2Q_5 + 125Q_3Q_3 + 125hQ_3Q_4 + (125h^2 \\ - 34375)Q_3Q_5 + 625Q_4Q_4 + 625hQ_4Q_5 + 3125Q_5Q_5\big)$$

Theorem 6.8 *Let E be a Morava E-theory of height 2 at the prime p, and let Γ be its Dyer-Lashof algebra. Define $\tilde{t}_{\mu,p} := p^{\mu} t_{\mu,p}$ for $\mu = 1$ and $\mu = 2$, where $t_{\mu,p}$ are the topological Hecke operators in Proposition 6.4. Then $\tilde{t}_{2,p}$ lies in the center of Γ but $\tilde{t}_{1,p}$ does not.*

Proof By Proposition 6.4, both $\tilde{t}_{1,p}$ and $\tilde{t}_{2,p}$ are contained in Γ. In fact, $\tilde{t}_{2,p} = \psi^p \circ \psi^p$, and is thus a ring homomorphism. Note that $\tilde{t}_{2,p}(\alpha) = \alpha$ as a result of the relation (3.13). Since $\psi^p = Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p$, we can then write the three-fold composite $\psi^p \circ \psi^p \circ \psi^p$ in two ways:

$$\begin{aligned} \tilde{t}_{2,p}(Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p) &= \tilde{t}_{2,p}Q_0 + (\tilde{t}_{2,p}Q_1)(\tilde{t}_{2,p}\alpha) + \cdots + (\tilde{t}_{2,p}Q_p)(\tilde{t}_{2,p}\alpha)^p \\ &= \tilde{t}_{2,p}Q_0 + (\tilde{t}_{2,p}Q_1)\alpha + \cdots + (\tilde{t}_{2,p}Q_p)\alpha^p \\ (Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p)\tilde{t}_{2,p} &= Q_0\tilde{t}_{2,p} + (Q_1\tilde{t}_{2,p})\alpha + \cdots + (Q_p\tilde{t}_{2,p})\alpha^p \end{aligned}$$

For each $0 \le i \le p$, comparing the coefficients for α^i, we see that $\tilde{t}_{2,p}$ commutes with Q_i. Thus $\tilde{t}_{2,p}$ lies in the center of Γ.

It remains to show that the other operation $\tilde{t}_{1,p}$ is not central in Γ. Analogous to the proof of Proposition 6.4, we find that the term Q_0Q_1 has coefficient w_2 in the Adem relation for Q_1Q_0 (see Remark 6.6). Since $p|w_i$ for $2 \le i \le p$, we then compute by Proposition 6.4 that

$$\begin{aligned} Q_1\tilde{t}_{1,p} &= Q_1\big((p+1)Q_0 - w_pQ_1 + c_2Q_2 + \cdots + c_pQ_p\big) \\ &\equiv Q_1Q_0 + Q_1(c_2Q_2 + \cdots + c_pQ_p) \mod p \end{aligned}$$

Thus the reduction of $Q_1\tilde{t}_{1,p}$ modulo p does not contain Q_0Q_1. On the other hand, the term Q_0Q_1 in

$$\tilde{t}_{1,p}Q_1 = \big((p+1)Q_0 - w_pQ_1 + c_2Q_2 + \cdots + c_pQ_p\big)Q_1$$

has coefficient $p+1$. Therefore $\tilde{t}_{1,p}Q_1 \neq Q_1\tilde{t}_{1,p}$. □

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