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ISOGENIES OF FORMAL GROUP LAWS AND POWER OPERATIONS IN THE COHOMOLOGY THEORIES E_n

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1. Introduction

1.1. Overview. This paper has three parts. The first part is about finite isogenies or subgroups of formal groups as studied by Lubin [Lub67]. Its main result is the construction of coordinates on certain Lubin-Tate formal groups which are preserved under isogenies. It is written largely without reference to algebraic topology, to highlight the formal group-theoretic result (Theorem 4), which we hope is of independent interest.

The problem addressed by Theorem 4 arose in our study of cohomology operations in elliptic cohomology and the cohomology theories E_n, which comprises the second part of the paper. Recent work [DHS88, Hop87, Wit88, BT89] has shown these theories to be of considerable interest. They are all *Landweber exact*, which means that they can be constructed from the complex cobordism functor MU by means of a genus

$$MU^* \xrightarrow{t} E^*$$

such that the resulting functor

$$E^*(X) \overset{\text{def}}{=} E^* \underset{MU^*}{\otimes} MU^*(X) \tag{1.1.1}$$

is a cohomology theory on the category of finite complexes.

It would be very pleasant to have good geometric or analytic descriptions of these theories, along the lines of ordinary rational homology or K-theory (which are the initial cases E_0 and E_1). One expects that a good geometric description of a cohomology theory will provide a wealth of additional informaton. For example, one expects symmetries of the geometry to give rise to cohomology operations. We have turned this idea upside down and examined cohomology operations—in particular, power operations—in hopes of learning about the conjectural geometry of the theories E_n.

Our original aim was to imitate Atiyah's construction of "power operations" in K-theory [Ati66], using the work of Hopkins, Kuhn, and Ravenel [HKR92] in place of the representation theory of the symmetric group. The reason for singling

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out this method for investigation is that on the one hand, one can describe in purely algebraic terms what it is one would like to do: cohomology theories with power operations are called "H_∞ ring theories", and their axioms and properties have been developed in [BMMS86]. On the other hand, pushing this program through for a given cohomology theory has in the past been closely related to substantial geometric information about the cohomology theory. Some examples are mod-p cohomology, in which one obtains the Steenrod operations [SE62]; K-theory, where one obtains the exterior powers and Adams operations [Ati66]; and complex cobordism [tDi68], where Quillen uses power operations to prove his theorem about MU^* [Qui71].

In the absence of sufficient geometric intuition, we have used the power operations in MU to produce power operations in the cohomology theories E_n and elliptic cohomology; it turns out (Theorems 1 and 2) that they are parametrized by the *finite subgroups* of the formal group attached to the cohomology theory. In this paper we study the construction for E_n; elliptic cohomology involves different subtleties, and we shall address it in a separate paper. In either case, there is an obstruction to completing the construction. The obstruction depends on the complex orientation or, what is the same, the choice of coordinate on the formal group associated to E_n. The role of Theorem 4 is to provide a complete description of the orientations for which the obstruction vanishes.

Recently, M. Hopkins and H. Miller [HM93] have shown that E_n is an E_∞ ring spectrum; in particular, it has power operations in the sense of [BMMS86]. As an application of our methods, we show in Section 4 that the orientations of Theorem 4 are the only orientations which intertwine the power operations on MU and E_n. In particular, the power operations of this paper coincide with those coming from the E_∞ structure of Hopkins and Miller.

1.2. In more detail. Let Φ be the Honda formal group law over $\mathbb{F}_p$ (2.5.5), and let E_n denote Lubin and Tate's ring which represents lifts of Φ to complete local $\mathbb{Z}_p$-algebras. Let F be a formal group law over E_n which is a universal lift of Φ. It is classified by a homomorphism

$$MU^* \xrightarrow{t} E_n.$$

It is a consequence of Landweber's exact functor theorem ([Lan76]; see Section 3.1) that the functor

$$X \mapsto E_n \underset{t}{\otimes} MU^{2*}(X)$$

is the degree-zero part of a 2-periodic cohomology theory on finite complexes. We shall also use the notation E_n for this functor.

We shall use the abbreviation

$$x \underset{F}{+} y \overset{\text{def}}{=} F(x, y).$$

If x and y are elements of the maximal ideal of a complete local Noetherian E_n-algebra, then so is $x \underset{F}{+} y$; the maximal ideal with this group structure will be denoted $F(A)$. Its subgroup of points of order p^k will be denoted ${}_{p^k}F(A)$.

Let D_k be the ring extension of E_n obtained by adjoining the roots of the p^k-series of F; it turns out that ([LT65]; see section 2.4)

$$ {}_{p^k}F(D_k) \cong (\mathbb{Z}/p^k\mathbb{Z})^n. $$

If $H \subset F(D_k)$ is a finite subgroup, let $f_H(t)$ denote the power series

$$ f_H(t) = \prod_{h \in H} (h \underset{F}{+} t) \in D_k[\![t]\!]. $$

Lubin ([Lub67]; see Section 2.2) shows that there is a formal group law F/H over D_k such that f_H is a homomorphism of formal group laws

$$ F \xrightarrow{f_H} F/H. $$

The main result of Section 3 is the following.

THEOREM 1 (Theorem 3.4.4). *There is an unstable transformation of ring-valued functors*

$$ E_n(X) \xrightarrow{\Psi^H} D_k \underset{E_n}{\otimes} E_n(X). $$

If F is one of the formal group laws constructed by Theorem 4, then the effect of Ψ^H is described by the equation

$$ \Psi^H(eL) = f_H(eL) \in D_k \underset{E_n}{\otimes} E_n(X), $$

where eL is the Euler class of a line bundle $\begin{smallmatrix} L \\ \downarrow \\ X \end{smallmatrix}$.

These operations generalize the unstable Adams operations Ψ^{p^k} in K-theory, which are the cases $n = 1$ and

$$ H = {}_{p^k}F(D_k); $$

see Section 3.6. One problem is that the range of the Ψ^H has the enlarged ring of coefficients D_k. However D_k is Galois over E_n, with Galois group $G_k = \mathrm{Aut}[{}_{p^k}F(D_k)] \cong GL_n(\mathbb{Z}/p^k\mathbb{Z})$. G_k acts on the set of finite subgroups of ${}_{p^k}F(D_k)$. Given an expression of the form

$$ \rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} H, $$

where I is a finite set, $a_i \in E_n$, and the α_i are lists of finite subgroups, with possible repetitions, then we can form the operation

$$\Psi^\rho(x) = \sum_{i \in I} a_i \prod_{H \in \alpha_i} \Psi^H(x).$$

THEOREM 2 (Theorem 3.5.3). *If ρ is invariant under the action of G_k, then the range of the operation Ψ^ρ is just $E_n(X)$.*

For example, if H is the full group

$$H = {}_{p^k}F(D_k),$$

then we obtain (3.6.1) an operation

$$E_n(X) \xrightarrow{\Psi^{p^k}} E_n(X)$$

which is the identity on coefficients and whose effect on Euler classes is given by the p^k series

$$\Psi^{p^k}(eL) = [p^k]_F(eL).$$

We outline the construction here to explain why we have called these operations power operations. Traditionally, power operations arise from an "H_∞ ring spectrum" [BMMS86]. We have avoided this route by using the well-known fact [Qui71, tDi68, BMMS86] that the complex-cobordism spectrum MU is an H_∞^2 ring spectrum, and by taking advantage of the fact that the cohomology theories E_n are Landweber exact.

If π an abelian p-group of order r, denote by $D_\pi X$ the Borel construction

$$D_\pi X = E\pi \underset{\pi}{\times} X^\pi,$$

where

$$X^\pi = \mathrm{Map}[\pi, X]$$

is the left π-space obtained by using the right action of π on itself. tom Dieck and Quillen [tDi68, Qui71, see Section 3.2] show that there is a total power operation

$$MU^{2*}(X) \xrightarrow{P_\pi} MU^{2r*}(DX),$$

which factors the rth external power map

$$MU^{2*}(X) \xrightarrow{z \mapsto z^{\times r}} MU^{2r*}(X^\pi)$$

through the inclusion of the fiber $X^\pi \xrightarrow{i} D_\pi X$:

$$MU^{2r*}(D_\pi X) \xrightarrow{i^*} MU^{2r*}(X^\pi).$$

We associate (3.3.5) to a subgroup $H \subset {}_{p^k}F(D_k)$ a character map [HKR92]

$$E_n(D_{H^*}X) \xrightarrow{\chi^H} D_k \underset{E_n}{\otimes} E_n(X)$$

(when H is an finite abelian group, we denote by $H^* \overset{\text{def}}{=} \mathrm{Hom}[H, \mathbb{C}^\times]$ its character group). The composite

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{P_\pi} & MU^{2r*}(D_{H^*}X) \\ & & \big\downarrow t \\ & & E_n(D_{H^*}X) \xrightarrow{\chi^H} D_k \underset{E_n}{\otimes} E_n(X) \end{array}$$

is an operation we call Q^H (3.3.7). In contrast to the total power operation P_π, it is eminently computable: let F^{MU} denote the formal group law over MU, and let

$$F = t_* F^{MU}$$

be the formal group law over E_n induced by the orientation. If

$$e_{MU}L \in MU^2(X)$$

is the Euler class of a line bundle $\begin{smallmatrix} L \\ \downarrow \\ X \end{smallmatrix}$, then its E_n-Euler class is

$$eL = t(e_{MU}L).$$

THEOREM 3 (Theorems 3.3.8, 3.3.9). *Q^H is a ring homomorphism. Its effect on coefficients and on Euler classes is given by the equations*

$$Q^H_* F^{MU} = F/H$$

$$Q^H(e_{MU}L) = f_H(eL).$$

We construct Ψ^H by factoring the operation Q^H through the orientation

$$\begin{tikzcd} MU^*(X) \arrow[d, "t"'] \arrow[dr, "Q^H"] & \\ E_n(X) \arrow[r, "\Psi^H"] & D_k \otimes_{E_n} E_n(X). \end{tikzcd} \tag{1.2.1}$$

Since E_n is Landweber exact (1.1.1) and Q^H is a ring homomorphism, it is sufficient to construct a ring homomorphism

$$E_n(pt) \xrightarrow{\Psi^H(pt)} D_k$$

which factors (1.2.1) when X is the one-point space; then

$$\Psi^H = \Psi^H(pt) \otimes Q^H.$$

Since the equation

$$t_* F^{MU} = F$$

determines the effect of t on MU^*, and since by Theorem 3

$$Q^H_* F^{MU} = F/H,$$

the condition that Ψ^H must satisfy becomes

$$\Psi^H_* F = F/H. \tag{1.2.2}$$

It turns out that the condition (1.2.2) is tractable, because the ring E_n represents lifts of the formal group law Φ, and (2.5.1) F/H is in fact a lift of Φ. However, the condition is not trivial; this is best illustrated when $H = {}_pF(D_1)$ is the full subgroup of p-torsion of the formal group law F.

In that case Lubin's theory provides a homomorphism of formal group laws

$$F \xrightarrow{f_p} F/p$$

over E_n such that

$${}_pF(D_1) = \operatorname{Ker}[F(D_1) \xrightarrow{f_p} F/p(D_1)].$$

Lubin's formula for f_p is

$$f_p(t) = \prod_{\substack{v \in F(D_1) \\ [p]_F(v)=0}} (v \underset{F}{+} t).$$

On the other hand, the p-series of F is also a homomorphism of formal group laws

$$F \xrightarrow{[p]_F} F$$

with kernel

$$_pF(D_1) = \operatorname{Ker}[F(D_1) \xrightarrow{[p]_F} F(D_1)].$$

The fact that f_p and $[p]_F$ have the same kernel implies that there is an isomorphism of formal group laws

$$F/p \xrightarrow{g_p} F$$

such that

$$[p]_F = g_p \circ f_p.$$

It turns out (2.6.15) that a ring homomorphism Ψ^H satisfying (1.2.2) exists for all subgroups H if and only if g_p is the identity, i.e., such that

$$[p]_F(t) = f_p(t). \tag{1.2.3}$$

The condition (1.2.3) depends on the choice of formal group law F, or equivalently on the complex orientation

$$MU^*(X) \xrightarrow{t} E_n(X).$$

The main result of Section 2 is the following.

THEOREM 4 (Theorem 2.5.7). *In each $\star$-isomorphism class of universal lift of Φ to E_n, there is a unique formal group law F for which*

$$[p]_F(t) = f_p(t).$$

From one of these formal group laws, one obtains an orientation t and operations Ψ^H factoring the diagram (1.2.1).

Recently, Hopkins and Miller announced that E_n is an E_∞ ring spectrum. In particular, it has power operations

$$E_n(X) \xrightarrow{P_j^{hm}} E_n(D_jX).$$

Their results have not yet been published, but the spectrum E_n is so rigid that one can prove the following compatibility, using only the methods of this paper and general results about H_∞ ring spectra.

THEOREM 5. *The diagram*

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{P_j^{MU}} & MU^{2j*}(D_jX) \\ \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t} \\ E_n(X) & \xrightarrow{P_j^{hm}} & E_n(D_jX) \end{array} \tag{1.2.4}$$

commutes if and only if t is an orientation satisfying the condition of Theorem 4.

This is the E_n version of a theorem about the Atiyah-Bott-Shapiro orientation on K-theory: tom Dieck shows that if

$$MU^{2*}(X) \xrightarrow{t_{\mathrm{ABS}}} K(X)$$

is the Atiyah-Bott-Shapiro orientation, then the diagram

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{P_j^{MU}} & MU^{2j*}(D_\pi X) \\ \Big\downarrow{\scriptstyle t_{\mathrm{ABS}}} & & \Big\downarrow{\scriptstyle t_{\mathrm{ABS}}} \\ K(X) & \xrightarrow{P_j^{K}} & K(D_jX) \end{array} \tag{1.2.5}$$

commutes [tDi68].

This special case suggests a more philosophical remark.We embarked on this project because we hoped to obtain hints about the geometry and analysis behind the E_n's. While we have nothing concrete to offer, notice that the equation

$$f_p(t) = [p]_F(t)$$

asks that two power series with the same zeroes be equal. If the two power series were analytic functions, they would coincide up to a complex unit. In fact, exact equality holds when F is the multiplicative group $\mathbb{G}_m$ (Section 2.7). This is the case of K-theory (E_1), with the Atiyah-Bott-Shapiro orientation. For elliptic cohomology, whose formal group law is the (algebraic) addition law of an elliptic curve, the equality holds up to multiplication by a unit complex number [And92].

The problem is that over a $\mathbb{Z}_p$-algebra like E_n, the set of zeroes determines a power series only up to multiplication by a unit power series. Theorem 4 shows that the coordinate on the formal group over E_n can be chosen so that this unit power series is the identity, and in that sense we think of our orientations as spiritually analytic.

This paper grew out of a part of my Ph.D. thesis, which I completed under the supervision of Haynes Miller. I thank him and Mike Hopkins for their help. In particular, I thank Mike Hopkins for telling me repeatedly to study E_n in terms of

the functor it represents. Mike Hopkins and Neil Strickland taught me a good proof of Lemma 4.2.5. I also heartily thank Jim McClure for his interest and for saving me from at least one serious misstatement. Such misstatements as remain are mine alone.

2. Isogenies of formal groups

2.1. Notations for formal groups and coordinates. If R is a complete Noetherian local domain with residue characteristic $p > 0$, we denote by C_R the category of complete local Noetherian R-algebras. If F is a functor

$$C_R \xrightarrow{F} \text{sets},$$

then for $A \in C_R$, the elements of $F(A)$ will be called the *A-valued points* of F. For example, the set of A-valued points P of the functor

$$\mathbb{A}^1(A) \stackrel{\text{def}}{=} C_R[R[\![t]\!], A]$$

is naturally isomorphic to the maximal ideal of A, by the map which sends P to the image of t under P. One name for this element of A is $P(t)$, but it is customary to call $\mathbb{A}^1$ the formal affine line over R, and to call $R[\![t]\!]$ its ring of functions; in honor of this, one uses the notation $t(P)$ for this element of A. The element

$$0 \in \mathbb{A}^1(A)$$

lifts $\mathbb{A}^1$ to the category of pointed sets.

A commutative one-dimensional formal group ("formal group") over R is a functor

$$C_R \xrightarrow{F} \text{abelian groups}$$

which is isomorphic to $\mathbb{A}^1$ as a functor to pointed sets. If P, $Q \in F(A)$ are two A-valued points, then their sum will be written

$$P \underset{F}{+} Q.$$

A "coordinate" on F is a choice of isomorphism

$$F \underset{\cong}{\xrightarrow{x}} \mathbb{A}^1$$

of functors to pointed sets. A coordinate on F gives rise to a commutative, one-dimensional formal group *law*: there is a unique point

$$F_x(t_1, t_2) \in \mathbb{A}^1(R[\![t_1, t_2]\!])$$

such that

$$x(F(x^{-1}(t_1), x^{-1}(t_2))) = F_x(t_1, t_2).$$

We also write

$$t_1 \underset{F_x}{+} t_2$$

for power series $F_x(t_1, t_2)$. Of course, the power series F_x together with the coordinate x determine the functor F: explicitly, if $P, Q \in F(A)$, then

$$x(P \underset{F}{+} Q) = x(P) \underset{F_x}{+} x(Q).$$

We shall therefore use the notation F_x to denote the functor $\mathbb{A}^1$, considered as a functor to groups via the power series F_x.

A homomorphism of formal groups is a natural transformation

$$F \xrightarrow{h} G$$

of group-valued functors. In terms of coordinates,

$$F \xrightarrow{x} \mathbb{A}^1 \qquad \text{and} \qquad G \xrightarrow{y} \mathbb{A}^1$$

on F and G; the homomorphism is the natural transformation u making the diagram

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "u"] & \mathbb{A}^1 \\ F \arrow[u, "x"] \arrow[r, "h"'] & G \arrow[u, "y"'] \end{tikzcd}$$

commute. Explicitly, u is represented by the power series

$$u(t) = y(h(x^{-1}(t))) \in R[\![t]\!],$$

which is a homomorphism of formal group laws in the usual sense since

$$\begin{aligned} u(t_1 \underset{F_x}{+} t_2) &= yh(x^{-1}(t_1) \underset{F}{+} x^{-1}(t_2)) \\ &= y(h(x^{-1}(t_1)) \underset{G}{+} h(x^{-1}(t_2))) \\ &= y(y^{-1}u(t_1) \underset{G}{+} y^{-1}u(t_2)) \\ &= u(t_1) \underset{G_y}{+} u(t_2). \end{aligned}$$

When there is no possibility of confusion on the choice of coordinate, we shall drop it from the notation.

2.2. Lubin's theory of isogenies. Suppose that $A \in C_R$ is the ring of integers in a finite extension of the fraction field K of R. If

$$H \subset F(A)$$

is a subgroup, then the quotient

$$F/H(B) \stackrel{\text{def}}{=} F(B)/H$$

is a group-valued functor on C_A. Lubin shows that if H is a finite subgroup, then F/H is a formal group over A. More precisely, let

$$F \xrightarrow{x} \mathbb{A}^1$$

be a coordinate on F. If f_H^x denotes the power series

$$f_H^x(t) \stackrel{\text{def}}{=} \prod_{v \in H} (x(v) \underset{F_x}{+} t) \in A[[t]], \tag{2.2.1}$$

then

$$f_H^x(x(u)) = 0$$

for $u \in H$. In other words, f_H^x defines a natural transformation of functors

$$\mathbb{A}^1 \xrightarrow{f_H^x} \mathbb{A}^1$$

which vanishes on $x(H) \subset \mathbb{A}^1(A)$.

PROPOSITION 2.2.2 ([Lub67]). *The quotient F/H is a formal group over A. The coordinate on F determines a coordinate*

$$F/H \xrightarrow{x_H} \mathbb{A}^1$$

on F/H by requiring that the diagram

$$\begin{array}{ccc} F & \longrightarrow & F/H \\ {\scriptstyle x}\big\downarrow & & \big\downarrow{\scriptstyle x_H} \\ \mathbb{A}^1 & \xrightarrow{f_H^x} & \mathbb{A}^1 \end{array}$$

commutes. If H is stable under the action of $\operatorname{Aut}[\overline{K}/K]$, *then F/H and f_H^x can be defined over R.*

Because of this result, we denote by

$$F_x/H$$

the formal group law $(F/H)_{x_H}$ determined by the coordinate x_H.
For example, let

$${}_pF(A) \stackrel{\text{def}}{=} \{v \in F(A) | pv = 0\}$$

denote the subfunctor of p-torsion points of F. If F is a formal group of height $n < \infty$, then the group ${}_pF(A)$ is finite. If A is an extension of R, then by (2.2.2), the quotient

$$F/{}_pF(A)$$

is a formal group over A. If x is a coordinate on F, then there is a unique formal group law $F_x/{}_pF(A)$ over A such that the power series

$$f_p(t) = \prod_{\substack{v \in F(A) \\ pv=0}} (x(v) \underset{F_x}{+} t)$$

defines a homomorphism of formal group laws

$$F_x \to F_x/{}_pF(A).$$

A homomorphism of formal groups is called an isogeny. Nonzero isogenies are surjective in the following sense: suppose that $F \xrightarrow{f_1} G_1$ and $F \xrightarrow{f_2} G_2$ are isogenies of formal groups over R. Let $\mathcal{O}$ denote the ring of integers in the algebraic closure $\overline{K}$ of the fraction field K of R, and let

$$H_i = \mathrm{Ker}[F(\mathcal{O}) \xrightarrow{f_i} G_i(\mathcal{O})].$$

PROPOSITION 2.2.3 ([Lub67]). *If*

$$H_1 \subset H_2,$$

then there is a unique *isogeny*

$$G_1 \xrightarrow{g} G_2$$

defined over R such that $f_2 = g \circ f_1$.

Continuing with the example of p-torsion points, let x be a coordinate on F. The p series of the formal group law F_x is the power series

$$[p]_x(t) \overset{\text{def}}{=} \overbrace{t \underset{F_x}{+} t \underset{F_x}{+} \cdots \underset{F_x}{+} x}^{p \text{ terms}} \in R[\![t]\!]$$

which represents the homomorphism multiplication-by-p. The subfunctor ${}_pF_x$ of F_x is represented by the quotient

$$R[\![t]\!]/([p]_{F_x}(t)).$$

If F is a formal group law height $n < \infty$ over R, then its p-series has Weierstrass degree p^n, so there is a unique factorization

$$[p]_{F_x}(t) = g(t)\varepsilon(t),$$

where $g(t)$ is a monic polynomial of degree p^n and

$$\varepsilon(t) \in (R[\![t]\!])^{\times}.$$

Let $D_1 \in C_R$ be the extension of R obtained by adjoining all the roots of g, that is, all the roots of the p-series. It turns out (see (2.4.3)) that this ring is a complete local domain which is Galois over R with Galois group $\mathrm{Aut}[{}_pF(D_1)]$. We abbreviate

$$F/({}_pF(D_1))$$

as F/p; it and the isogeny

$$F_x \xrightarrow{f_p^x} F_x/p$$

are both defined over R. The isogenies f_p^x and $[p]_{F_x}$ are homomorphisms of group laws

$$F_x \xrightarrow{f_p^x} F_x/p \qquad \text{and}$$

$$F_x \xrightarrow{[p]_{F_x}} F_x,$$

with the same kernel, ${}_pF_x(D_1)$. In this situation Proposition 2.2.3 implies the following.

COROLLARY 2.2.4. *There is a* unique *isomorphism of formal group laws over R*

$$F_x/p \underset{\cong}{\xrightarrow{g_p^x}} F_x.$$

In a similar vein, if $H_1 \subset H_2 \subset F_x(\mathcal{O})$ are finite subgroups, then there are isogenies

$$F \to F/H_1,$$
$$F \to F/H_2, \qquad \text{and}$$
$$F/H_1 \xrightarrow{g} F/H_2. \tag{2.2.5}$$

The kernel of g is H_2/H_1. A coordinate x on F produces homomorphisms of formal group laws

$$F_x \xrightarrow{f_{H_1}} F_x/H_1,$$
$$F_x \xrightarrow{f_{H_2}} F_x/H_2, \quad \text{and}$$
$$F_x/H_1 \xrightarrow{f_{H_2/H_1}} (F_x/H_1)/(H_2/H_1).$$

An important property of Lubin's isogenies f_H is that they compose nicely.

PROPOSITION 2.2.6.

$$f_{H_2} = f_{H_2/H_1} \circ f_{H_1},$$

and so

$$(F_x/H_1)/(H_2/H_1) = F_x/H_2.$$

Proof. Let $S \subset H_2$ be a set of coset representatives of H_2/H_1. Then

$$\begin{aligned}
f_{H_2}(t) &= \prod_{v \in H_2} (x(v) \underset{F_x}{+} t) \\
&= \prod_{v \in S} \prod_{w \in H_1} (x(w) \underset{F_x}{+} x(v) \underset{F_x}{+} t) \\
&= \prod_{v \in S} f_{H_1}(x(v) \underset{F_x}{+} t) \\
&= \prod_{v \in S} (f_{H_1}(x(v)) \underset{F_x/H_1}{+} f_{H_1}(t)) \\
&= \prod_{\sigma \in H_2/H_1} (x_{H_1}(\sigma) \underset{F_x/H_1}{+} f_{H_1}(t)) \\
&= f_{H_2/H_1}(f_{H_1}(t)). \qquad \square
\end{aligned}$$

2.3. *The Lubin-Tate moduli problem.* The cohomology theory E_n is a ring theory whose coefficient ring $E_n(pt)$ (which we will also denote E_n) is the Lubin-Tate ring of lifts of a height-n formal group law Φ over $\mathbb{F}_p$; thus

$$E_n(pt) = \mathbb{Z}_p[\![u_1, \dots, u_{n-1}]\!].$$

The ring E_n represents a functor, which we now describe. Let R be a complete, Noetherian, local ring with residue field $\mathfrak{k} \supset \mathbb{F}_p$. Reduction to the residue field

$$\mathbb{Z}_p \to \mathfrak{k}$$

defines a restriction functor

$$C_{\mathfrak{k}} \xrightarrow{|_{\mathfrak{k}}} C_R.$$

A formal group F over R is a *lift* of Φ to R if

$$F|_{\mathfrak{k}} = \Phi\colon C_{\mathfrak{k}} \to \text{abelian groups}.$$

Two lifts F and F' of Φ to R are $\star$*-isomorphic* if there is an isomorphism

$$F \xrightarrow{f} F'$$

of formal groups over R which reduces modulo the maximal ideal to the identity, that is,

$$f|_{\mathfrak{k}} = \mathrm{id}_{\Phi}.$$

Now let

$$C_{\mathbb{Z}_p} \xrightarrow{\text{Lifts}} \text{sets}$$

be the functor which assigns to $R \in C_{\mathbb{Z}_p}$ the set

$$\text{Lifts}(R) \overset{\text{def}}{=} \left\{ \begin{matrix} \star\text{-isomorphism classes} \\ \text{of lifts of } \Phi \text{ to } R \end{matrix} \right\}.$$

Lubin and Tate show that the functor Lifts is represented by the ring E_n. More precisely, they show that there is a (by no means unique) formal group F over E_n such that the following theorem holds.

THEOREM 2.3.1 ([LT66]). *If $R \in C_{\mathbb{Z}_p}$ and $\mathcal{G} \in \text{Lifts}(R)$, then there is a unique homomorphism of local rings*

$$E_n \xrightarrow{\alpha} R$$

such that $\alpha_* F \in \mathcal{G}$. Moreover, for each formal group G in the $\star$-isomorphism class $\mathcal{G}$, the $\star$-isomorphism

$$G \to \alpha_* F$$

is uniquely determined by F and G.

Often the formal group Φ over $\mathfrak{k}$ comes with a preferred coordinate

$$\Phi \xrightarrow{y} \mathbb{A}^1_{\mathfrak{k}}.$$

In that case, a formal group F over R is a lift of Φ if and only if there is a coordinate

$$F \xrightarrow{x} \mathbb{A}^1_R$$

such that the diagram

$$\begin{tikzcd} F \arrow[r, "x"] & \mathbb{A}^1_R \\ \Phi \arrow[u, "|_{\mathfrak{k}}"] \arrow[r, "y"'] & \mathbb{A}^1_{\mathfrak{k}} \arrow[u, "|_{\mathfrak{k}}"'] \end{tikzcd}$$

commutes. Explicitly, a formal group law F_x is a lift of Φ_y if and only if

$$\mathfrak{k} \underset{R}{\otimes} F_x(t_1, t_2) = \Phi_y(t_1, t_2).$$

If F' is another lift of Φ with a lifted coordinate x', then F and F' are $\star$-isomorphic if and only if there is an isomorphism of formal group laws

$$F_x \xrightarrow{g} F'_{x'}$$

such that

$$\mathfrak{k} \underset{R}{\otimes} g(t) = t.$$

2.4. Subgroups of the Lubin-Tate formal groups, and the Drinfel'd moduli ring D_k, I. Once again, R is a complete, local domain with residue characteristic $p > 0$. (e.g., $R = E_n$ or $\mathbb{Z}_p$). Let F be a formal group law over R with height $n < \infty$. The subgroups of F which can occur are given by the following.

THEOREM 2.4.1 ([LT65]; see also Theorem 2.1.1 of [HKR92]). *Let $\mathcal{O}$ be the ring of integers in the algebraic closure of the fraction field of R. Then*

$$_{p^k}F(\mathcal{O}) \cong (\mathbb{Z}/p^k\mathbb{Z})^n$$

$$F(\mathcal{O})_{tors} \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n.$$

A monomorphism

$$(\mathbb{Z}/p^k\mathbb{Z})^n \xrightarrow{\phi} F(R)$$

is called a *level-p^k structure* on F. Let $\mathbf{D}_k$ denote the functor

$$C_{\mathbb{Z}_p} \xrightarrow{\mathbf{D}_k} \{sets\}$$

which assigns to a complete, local $\mathbb{Z}_p$-algebra R the set

$$\mathbf{D}_k(R) \overset{\text{def}}{=} \left\{ \begin{array}{c} \star\text{-isomorphism classes } \{F\} \text{ of lifts of } \Phi \\ \text{to } R, \text{ together with, for any representative } F, \\ \text{a level-}p^k \text{ structure} \\ (\mathbb{Z}/p^k\mathbb{Z})^n \xrightarrow{\phi} F(R) \end{array} \right\}.$$

Notice that there is a natural transformation

$$\mathbf{D}_k \to \text{Lifts} \tag{2.4.2}$$

which forgets the level-p^k structure, and that the group

$$GL_n(\mathbb{Z}/p^k\mathbb{Z})$$

acts naturally on the functor $\mathbf{D}_k$. These two structural features of $\mathbf{D}_k$ are connected by the following.

THEOREM 2.4.3 ([Dri73], [KM85], [Hop91]). *The functor $\mathbf{D}_k$ is represented by a complete local domain D_k. The map*

$$E_n \to D_k,$$

representing the forgetful transformation (2.4.2), is the inclusion of the invariants under the $GL_n(\mathbb{Z}/p^k\mathbb{Z})$ action. It gives D_k the structure of a finite, faithfully flat E_n-algebra.

Remark. One extends the functor $\mathbf{D}_k$ to cover the case when R is characteristic p by means of a "Drinfel'd" level structure. We shall not use these level structures explicitly, but they are essential for proving Theorem 2.4.3 as well as other properties of the ring D_k.

Remark. In [KM85], the case $n = 1$ is Theorem 1.12.7. The case $n = 2$ is part of Theorem 5.2.1, especially Theorem 5.3.2. All cases are worked out in [Hop91].

Remark. From our discussion, it seems elementary that the inclusions

$$(\mathbb{Z}/p^{k-1}\mathbb{Z})^n \hookrightarrow (\mathbb{Z}/p^k\mathbb{Z})^n$$

should induce forgetful transformations

$$\mathbf{D}_k \to \mathbf{D}_{k-1}$$

which are represented by ring homomorphisms

$$D_{k-1} \to D_k .$$

This is true, and we shall use that fact in the proof of (2.6.15), but by the time the definition of $\mathbf{D}_k$ has been extended to the notion of a "Drinfel'd level structure", it is not trivial [Hop91].

Remark. Let F be a universal lift of Φ to E_n; the inclusion

$$E_n \to D_k$$

makes F a formal group over D_k. The universal level-p^k structure is an isomorphism

$$(\mathbb{Z}/p^k\mathbb{Z})^n \xrightarrow{\phi_{\mathrm{univ}}} {}_{p^k}F(D_k).$$

2.5. *Two isogenies with the same kernel.* Let Φ be a formal group over $\mathbb{F}_p$, let F be a lift of Φ to E_n as in (2.3.1), and let H be any finite subgroup

$$H \subset F(D_k)$$

of order h (a power of p); by ϕ_{univ} it corresponds to a subgroup of $(\mathbb{Z}/p^k\mathbb{Z})^n$.

PROPOSITION 2.5.1. *The quotient formal group F/H is a lift of Φ to D_k.*

Proof. Let

$$F \xrightarrow{x} \mathbb{A}^1$$

be a coordinate on F. It induces a coordinate x on Φ and a coordinate x_H on F/H. According to (2.2.2), the coordinate x_H on F/H makes the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "x"'] & F/H \arrow[d, "x_H"] \\ \mathbb{A}^1 \arrow[r, "f_H"] & \mathbb{A}^1 \end{tikzcd} \tag{2.5.2}$$

commute. Since $x(v) \in \mathbb{A}^1(D_k) = \mathfrak{m}_{D_k}$, the power series

$$f_H(t) = \prod_{v \in H} (x(v) \underset{F_x}{+} t)$$

satisfies

$$f_H(t) \equiv t^h \qquad (\mathrm{mod}\ \mathfrak{m}_{D_k}),$$

so the diagram of formal group laws

$$\begin{tikzcd} F_x \arrow[r, "f_H"] & F_x/H \\ \Phi_x \arrow[u] \arrow[r, "t^h"] & \Phi_x \arrow[u] \end{tikzcd}$$

commutes, where the vertical arrows represent reduction to the residue field. The fact that $t^h = t^{p^r}$ (namely, the rth power of Frobenius) is an endomorphism of Φ follows from the requirement that Φ be defined over $\mathbb{F}_p$. This puts Φ_x in the lower right and completes the proof. □

By combining (2.5.1) and (2.3.1), we obtain the following.

PROPOSITION 2.5.3. *There is a unique ring homomorphism*

$$E_n \xrightarrow{\alpha^H} D_k$$

such that there is a $\star$-isomorphism

$$F/H \xrightarrow{g_H} \alpha^H_* F.$$

The $\star$-isomorphism g_H is uniquely determined by the subgroup H.

We wish to study the $\star$-isomorphism g_H in terms of formal group laws. A coordinate

$$F \xrightarrow{x} \mathbb{A}^1$$

on F induces a coordinate

$$F/H \xrightarrow{x_H} \mathbb{A}^1$$

on the quotient formal group law. The isomorphism g_H induces another coordinate on the quotient, namely

$$F/H \xrightarrow{g_H} \alpha^H_* F \xrightarrow{\alpha^H_* x} \mathbb{A}^1.$$

These two coordinates are connected by a unique $\star$-isomorphism of formal group laws

$$F_x/H \xrightarrow{g_H^x} \alpha_*^H F_x$$

making the diagram

$$\begin{tikzcd} F_x \arrow[r, "f_H^x"] & F_x/H \arrow[r, "g_H^x"] & \alpha_*^H F_x \\ F \arrow[u, "x"] \arrow[r] & F/H \arrow[u, "x_H"'] \arrow[r, "g_H"] & \alpha_*^H F \arrow[u, "\alpha_* x"'] \end{tikzcd}$$

commute.

The homomorphism of formal group laws

$$l_H^x(t) = g_H^x \circ f_H^x(t) \in D_k[\![t]\!]$$

is characterised by the following proposition.

PROPOSITION 2.5.4. *As a power series in $D_k[\![t]\!]$, l_H^x is uniquely characterised by the following three properties.*

i. *l_H^x is an isogeny of formal group laws with source F_x and with target of the form $\alpha_* F_x$ for some ring homomorphism*

$$E_n \xrightarrow{\alpha} D_k.$$

ii. *The kernel of l_H^x applied to $F_x(D_k)$ is $x(H)$.*

iii. *Reducing coefficients to the residue field yields the equation*

$$\mathbb{F}_p \otimes l_H(x) = x^h.$$

Proof. Let f be any isogeny

$$F_x \xrightarrow{f} F'$$

satisfying the properties listed. By (i), there is a homomorphism

$$E_n \xrightarrow{\alpha} D_k$$

such that

$$\alpha_* F_x = F'.$$

By (ii) and (2.2.3), there is a unique isomorphism

$$F' = \alpha_* F_x \underset{\cong}{\overset{g}{\to}} \alpha_*^H F,$$

i.e., so that

$$l_H = g \circ f.$$

By (iii), g is a $\star$-isomorphism, so by the uniqueness in (2.3.1),

$$\alpha = \alpha^H$$

and

$$g(t) = t. \qquad \square$$

We are going to show that for a nice choice of Φ which arises in algebraic topology, there is in each $\star$-isomorphism class of lift of Φ to E_n a *unique* choice of coordinate x such that

$$g_H^x(t) = t$$

for all finite subgroups $H \subset F(R)_{\text{tors}}$, i.e.,

$$l_H^x = f_H^x.$$

Thus for this coordinate, Lubin's isogeny f_H^x satisfies the conditions of (2.5.4). The particular formal group Φ over $\mathbb{F}_p$ which arises in algebraic topology is usually described in terms of a preferred coordinate: it is the formal group law over $\mathbb{F}_p$ classified by the map

$$\begin{aligned} BP^* \cong \mathbb{Z}_{(p)}[v_1, v_2, \ldots] &\to \mathbb{F}_p \\ v_i &\to \begin{cases} 0 & i \neq n \\ 1 & i = n, \end{cases} \end{aligned} \tag{2.5.5}$$

where the v_i are the Araki generators for BP^* (BP^* and the formal group defined by (2.5.5) are discussed, for example, in [Rav86, Appendix 1]). It has the special property that its p-series is given by

$$[p]_\Phi(t) = t^{p^n}. \tag{2.5.6}$$

THEOREM 2.5.7. *For each $\star$-isomorphism class of lift F of Φ to E_n, there is a unique choice of coordinate x on F, lifting the preferred coordinate on Φ, such* *that*

$$\alpha_*^H F_x = F_x/H, \tag{2.5.8}$$

or equivalently that

$$l_H^x = f_H^x,$$

for all finite subgroups H.

2.6. Proof of Theorem 2.5.7. As a first step, we investigate further the case in which

$$H = {}_pF(D_k) \cong (\mathbb{Z}/p\mathbb{Z})^n$$

is the full subgroup of points of order p. In this case we abbreviate g_H, f_H, F/H as g_p, f_p, F/p, etc.

The reason for the condition (2.5.6) on Φ is the following.

PROPOSITION 2.6.1. *Any coordinate*

$$F \xrightarrow{x} \mathbb{A}^1$$

on any universal lift F of the Φ given by (2.5.5) satisfies

$$l_p^x(t) = [p]_{F_x}(t),$$

and the homomorphism α^p determined by the subgroup ${}_pF(D_k)$ is the tautological inclusion

$$E_n \to D_k.$$

Proof. $[p]_{F_x}$ is an endomorphism of F_x, so it satisfies the conditions of (2.5.4) with α^p the inclusion map. In particular,

$$\mathbb{F}_p \otimes [p]_{F_x}(t) = [p]_\Phi(t) = t^{p^n}. \qquad \square$$

COROLLARY 2.6.2. *The condition*

$$\alpha_*^p F_x = F_x/p,$$

which is a special case of equation (2.5.8), is equivalent to the condition

$$f_p^x(t) = [p]_{F_x}(t). \tag{2.6.3}$$

The main step of the proof of (2.5.7) is to show that already the condition (2.6.3) on the p-series determines a unique coordinate. Let R be any complete local domain which is also a local $\mathbb{Z}_p$-algebra, and let F be a lift of Φ to R.

THEOREM 2.6.4. *There is exactly one coordinate x on F which lifts the coordinate on Φ and which satisfies*

$$f_p^x(t) = [p]_{F_x}(t).$$

Proof. Existence. First we reduce to the universal case. Let G be a universal lift of Φ to E_n: suppose we can construct a coordinate x on G which lifts the preferred coordinate on Φ and which satisfies (2.6.3). There is a unique homomorphism

$$E_n \xrightarrow{\beta} R$$

such that there is a $\star$-isomorphism

$$F \xrightarrow{g} \beta_* G.$$

We shall show that if G_x satisfies (2.6.3), then so does $\beta_* G_x$. Let $R_1 \supset R$ be an extension over which there is a level-p structure

$$(\mathbb{Z}/p\mathbb{Z})^n \xrightarrow{\phi} {}_pF(R_1).$$

By the definition of D_1, ϕ determines an extension of β to a ring homomorphism

$$D_1 \xrightarrow{\beta} R_1$$

such that the diagram

$$\begin{tikzcd}
(\mathbb{Z}/p\mathbb{Z})^n \arrow[r, equal] \arrow[d, "\phi_{\mathrm{univ}}"'] & (\mathbb{Z}/p\mathbb{Z})^n \arrow[d, "g\phi"] \\
{}_pG(D_1) \arrow[r, "\beta"] & {}_p\beta_* G(R_1)
\end{tikzcd}$$

commutes. Then

$$\begin{aligned}
[p]_{\beta_* G_x}(t) &= \beta_* \left[\prod_{c \in (\mathbb{Z}/p\mathbb{Z})^n} (x(\phi_{\mathrm{univ}}(c)) \underset{G_x}{+} t) \right] \\
&= \prod_{c \in (\mathbb{Z}/p\mathbb{Z})^n} (\beta_* x(g(\phi(c))) \underset{\beta_* G_x}{+} t), \qquad (2.6.5)
\end{aligned}$$

which is (2.6.3) for $\beta_* G_x$.

We turn to the universal case. The proof is inductive, on powers of the maximal ideal I of D_1. Let F be a universal lift of Φ to E_n, and denote by H the subgroup

$$H = {}_pF(D_1).$$

Let y be any coordinate on F which gives a group law F_y lifting the preferred coordinate on Φ. Define a power series $a(t) \in E_n[\![t]\!]$ by

$$g_p^y(t) = t + a(t).$$

Since g_p^y is a $\star$-isomorphism, we get automatically the case $r = 2$ of the inductive hypothesis that

$$a(t) = \sum_{j \geqslant 1} a_j t^j, \qquad \text{with } a_j \in I^{r-1}.$$

Let $\delta(t)$ be the power series

$$\delta(t) = t - a(t).$$

Since g_p^y is defined over E_n, the coordinate

$$x = \delta(y)$$

on F yields a formal group law F_x over E_n, such that δ is a $\star$-isomorphism

$$F_y \xrightarrow{\delta} F_x.$$

We shall show that this choice of coordinate x produces the equation

$$g_p^x(t) \equiv t \pmod{I^r}. \tag{2.6.6}$$

Since the formal group laws F_x and F_y will coincide modulo I^{r-1}, one obtains by induction and Krull's theorem a formal group law F_z such that $g_p^z(t) = t$.

As a homomorphism of formal group laws, δ makes the diagram

$$\begin{tikzcd} F_y \arrow[r, "\delta"] \arrow[d, "{[p]_{F_y}}"'] & F_x \arrow[d, "{[p]_{F_x}}"] \\ F_y \arrow[r, "\delta"] & F_x \end{tikzcd}$$

commute; in other words, we have

$$\delta([p]_{F_y}(t)) = [p]_{F_x}(\delta(t)). \tag{2.6.7}$$

By (2.6.1), we have

$$[p]_{F_x}(t) = g_p^x(f_p^x(t))$$

and similarly for the coordinate y. Substituting these equations into equation (2.6.7) yields

$$\delta[g_p^y(f_p^y(t))] = g_p^x[f_p^x(\delta(t))]. \tag{2.6.8}$$

Notice that

$$\begin{aligned} f_p^x(\delta(t)) &= \prod_{c \in H} (x(c) \underset{F_x}{+} \delta(t)) \\ &= \prod_{c \in H} \delta(\delta^{-1}(x(c)) \underset{F_y}{+} t) \\ &= \prod_{c \in H} \delta(y(c) \underset{F_y}{+} t), \end{aligned}$$

so (2.6.8) becomes

$$\delta[g_p^y(f_p^y(t))] = g_p^x\left[\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t)\right]. \tag{2.6.9}$$

We can evaluate the left side modulo I^r very easily:

$$\begin{aligned} \delta(g_p^y(f_p^y(t))) &= \delta(f_p^y(t) + a(f_p^y(t))) \\ &\equiv \delta(f_p^y(t) + a(t^{p^n})) \\ &\equiv f_p^y(t) + a(t^{p^n}) - a(f_p^y(t) + a(t^{p^n})) \\ &\equiv f_p^y(t) + a(t^{p^n}) - a(f_p^y(t)) \\ &\equiv f_p^y(t), \end{aligned} \tag{2.6.10}$$

where the equivalences are all modulo I^r. The first equivalence uses the fact that $f_p^y(t) \equiv t^{p^n}$ (mod I) plus the fact that $a(t)$ is a power series with constant term zero and coefficients in I^{r-1} (with $r \geqslant 2$!).The third equivalence follows from this description of $a(t)$, and the last equivalence is like the first.

All we know *a priori* about g_p^x is that it is a power series of the form

$$g_p^x(t) = t + b(t),$$

where

$$b(t) = \sum_{j \geqslant 1} b_j t^j, \qquad b_j \in I, \tag{2.6.11}$$

so we start by evaluating the right-hand side of (2.6.9) modulo I^2. We can compute $\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t)$ modulo I^r:

$$\begin{aligned}
\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t) &= \prod_{c \in H} [(y(c) \underset{F_y}{+} t) - a(y(c) \underset{F_y}{+} t)] \\
&\equiv \prod_c (y(c) \underset{F_y}{+} t) - \sum_c \left[\prod_{d \neq c} (y(d) \underset{F_y}{+} t) \right] a(y(c) \underset{F_y}{+} t) \\
&\equiv \prod_c (y(c) \underset{F_y}{+} t) - \sum_c y^{p^n - 1} a(t) \qquad (2.6.12) \\
&\equiv \prod_c (y(c) \underset{F_y}{+} t) - p^n t^{p^n - 1} a(t) \\
&\equiv \prod_c (y(c) \underset{F_y}{+} t) \\
&\equiv f_p^y(t). \qquad (2.6.13)
\end{aligned}$$

(recall that $p \in I$ is contained in the maximal ideal). Continuing (2.6.12) modulo I shows that

$$\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t) \equiv t^{p^n} \qquad (\text{mod } I).$$

Equipped with these observations, we compute that if

$$b(t) \equiv 0 \qquad (\text{mod } I^{j-1}), \qquad j \geqslant 2,$$

then for $j \leqslant r$,

$$\begin{aligned}
g_p^x\left(\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t) \right) &= \prod_{c \in H} \delta(y(c) \underset{F_y}{+} t) + b\left(\prod_{c \in H} \delta(y(c) \underset{F_y}{+} t) \right) \\
&\equiv f_p^y(t) + b(t^{p^n}) \qquad (\text{mod } I^j). \qquad (2.6.14)
\end{aligned}$$

Comparing (2.6.10) and (2.6.14), it follows in view of (2.6.11) that

$$b(t) \equiv 0 \qquad (\text{mod } I^j),$$

and inductively that

$$b(t) \equiv 0 \qquad (\text{mod } I^r).$$

Uniqueness. Suppose that F_y satisfies (2.6.3) and that

$$F_y \xrightarrow{\delta} F_x$$

is a $\star$-isomorphism

$$\delta(t) = t + \sum_{j \geqslant 1} a_j t^j.$$

Let $\mathfrak{m}$ be the maximal ideal of R, and let r be as large as possible so that

$$a_j \in \mathfrak{m}^r$$

for all j. Note that $r \geqslant 1$, and that if $r = \infty$ then $F_x = F_y$, so we may suppose that r is finite. Working modulo $\mathfrak{m}^{r+1}$, we have

$$\begin{aligned}
[p]_{F_x}(t) &= \delta([p]_{F_y}(\delta^{-1}(t))) \\
&= \delta\left(\prod_{c \in H} (y(c) \underset{F_y}{+} \delta^{-1}(t))\right) \\
&= \delta\left(\prod_{c \in H} \delta^{-1}(\delta(y(c)) \underset{F_x}{+} t)\right) \\
&\equiv \prod_{c \in H} \delta^{-1}(x(c) \underset{F_x}{+} t) + a(t^{p^n}) \\
&\equiv f_p^x(t) - \sum_{c \in H} a(x(c) \underset{F_x}{+} t) \prod_{d \neq c} (x(c) \underset{F_x}{+} t) + a(t^{p^n}) \\
&\equiv f_p^x(t) - p^n a(t) t^{p^n - 1} + a(t^{p^n}) \\
&\equiv f_p^x(t) + a(t^{p^n}),
\end{aligned}$$

so F_x fails to satisfy (2.6.3). $\square$

The proof of Theorem 2.5.7 is completed by the following proposition.

PROPOSITION 2.6.15. *Let F be a universal lift of Φ to E_n.The coordinate x on F constructed in (2.6.4) satisfies the stronger condition asserted in Theorem 2.5.7.*

Proof. Let $H \subset {}_{p^k}F_x(D_k)$ be a finite subgroup. We must show that

$$\alpha_H^* F_x = F_x/H.$$

By the reduction to the universal case in the proof of (2.6.4), the formal group law $\alpha_H^* F_x$ satisfies (2.6.3). By the uniqueness part of (2.6.4), it is the unique formal group law in its $\star$-isomorphism class which does. By construction (see (2.5.3)), g_H^x is a $\star$-isomorphism

$$F_x/H \overset{g_H^x}{\to} \alpha_*^H F_x.$$

We will show that F_x/H satisfies (2.6.3); it follows that F_x/H and $\alpha_*^H F_x$ coincide.
The level-p^k-structure

$$(\mathbb{Z}/p^k\mathbb{Z})^n \to F_x(D_k)$$

induces a level-$(\mathbb{Z}/p^k\mathbb{Z})^n/H$ structure

$$(\mathbb{Z}/p^k\mathbb{Z})^n/H \to F_x/H(D_k).$$

Consider H as a subgroup of $(\mathbb{Z}/p^{k+1}\mathbb{Z})^n$ by the usual inclusion

$$(\mathbb{Z}/p^k\mathbb{Z}) \to (\mathbb{Z}/p^{k+1}\mathbb{Z}),$$

and denote by Λ' the subgroup

$$\Lambda' = {}_p((\mathbb{Z}/p^{k++1}\mathbb{Z})^n/H).$$

Any choice of isomorphism

$$(\mathbb{Z}/p\mathbb{Z})^n \overset{\cong}{\to} \Lambda'$$

determines a level-p structure

$$\phi\colon (\mathbb{Z}/p\mathbb{Z})^n \overset{\cong}{\to} \Lambda' \to F_x/H(D_{k+1}).$$

Here we have used the fact (see the remarks after Theorem 2.4.3) that there is a ring homomorphism

$$D_k \to D_{k+1}$$

representing the forgetful map

$$\mathbf{D}_{k+1} \to \mathbf{D}_k.$$

Let $Z(t)$ be the product

$$Z(t) = \prod_{c \in (\mathbf{Z}/p\mathbf{Z})^n} (\phi(c) \underset{F_x/H}{+} t) \in D_{k+1}[\![x]\!];$$

$Z(t)$ is the Lubin isogeny

$$Z(t) = f_{p^{-1}H/H}(t)$$

for the group law F_x/H, where $p^{-1}H$ is the subgroup

$$p^{-1}H = \{c \mid pc \in H\} \subset F_x(D_{k+1}).$$

We need to show that

$$Z(t) = [p]_{F_x/H}(t).$$

The functional equation

$$\begin{tikzcd} F_x \arrow[r, "f_H^x"] \arrow[d, "{[p]_{F_x}}"'] & F_x/H \arrow[d, "{[p]_{F_x/H}}"] \\ F_x \arrow[r, "f_H^x"] & F_x/H \end{tikzcd} \tag{2.6.16}$$

determines the p-series $[p]_{F_x/H}(t)$ because D_{k+1} is a domain. We shall show that $Z(t)$ satisfies this functional equation. Proposition 2.2.6 shows that

$$Z(f_H(t)) = f_{p^{-1}H/H}(f_H(t)) = f_{p^{-1}H}(t).$$

On the other hand, x is constructed so that

$$[p]_{F_x}(t) = f_p(t),$$

so

$$\begin{aligned} f_H([p]_{F_x}(t)) &= f_H(f_p(t)) \\ &= f_{p^{-1}H}(t). \qquad \square \end{aligned}$$

2.7. Example: The multiplicative group. Over E_1 the multiplicative group law

$$x \underset{\mathbb{G}_m}{+} y = x + y - xy$$

is a universal lift of Φ. It arises in topology as the formal group law for K-theory, where the Euler class of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ is

$$eL = 1 - L,$$

since

$$e(L_1 \otimes L_2) = 1 - L_1 L_2 = (1 - L_1) \underset{\mathbb{G}_m}{+} (1 - L_2).$$

The roots of the p-series of $\mathbb{G}_m$ are

$$(1 - \zeta^j), \qquad 0 \leqslant j < p,$$

where $\zeta = e^{2\pi i/p}$.

Suppose $p = 2$.Then

$$[2](x) = 2x - x^2 = 1 - L^2.$$

On the other hand,

$$\begin{aligned} f_2(x) &= (1 - L)[(1 - L) \underset{\mathbb{G}_m}{+} 2] \\ &= (1 - L)(1 - L + 2 - 2 + 2L) \\ &= (1 - L^2) \\ &= [2](x). \end{aligned}$$

Similarly, for p odd, we have

$$[p](x) = (1 - L^p),$$

while

$$\begin{aligned} f_p(x) &= \prod_{j=0}^{p-1} [(1 - L) \underset{\mathbb{G}_m}{+} (1 - \zeta^j)] \\ &= \prod_{j=0}^{p-1} (1 - \zeta^j L) \end{aligned}$$

$$= \zeta^{p(p-1)/2} \prod_{j=0}^{p-1} (\zeta^{-j} - L)$$

$$= (1 - L^p)$$

$$= [p](x). \tag{2.7.1}$$

It is amusing that for p odd, the parametrization

$$x \underset{\mathbb{G}'_m}{+} y = x + y + xy$$

also produces an equality

$$f_p(x) = [p]_{\mathbb{G}'_m}(x):$$

these two parametrizations are related by the isomorphism

$$x \mapsto -x$$

which is *not a* $\star$*-isomorphism.* However, when $p = 2$, the isomorphism

$$x \mapsto -x$$

is a $\star$-isomorphism, but

$$f_2(x) = -[2]_{\mathbb{G}'_m}(x),$$

so we are rescued from a contradiction.

3. Power operations in the cohomology theory E_n

3.1. The cohomology theory E_n and a remark about grading. Let E_n^* be the graded ring

$$E_n^* = \mathbb{Z}_p[\![u_1, \ldots, u_{n-1}]\!][u, u^{-1}],$$

where $|u_i| = 0$ and $|u| = -2$. E_n^* maps to Lubin and Tate's ring E_n studied in the last section by

$$u \mapsto 1.$$

In terms of this map, one universal lift of the formal group law Φ given by (2.5.5) is the genus

$$MU^* \to BP^* \to E_n^*,$$

which in terms of the Araki generators is given (see Appendix 1 of [Rav86]) by

$$BP^* \to E_n^*$$

$$v_i \to \begin{cases} 0 & i > n \\ u^{p^n} & i = n \\ u_i u^{p^i} & i < n. \end{cases} \tag{3.1.1}$$

The next proposition follows from the exact functor theorem of Landweber [Lan76].

PROPOSITION 3.1.2. *The functor*

$$X \mapsto E_n^*(X) \stackrel{\text{def}}{=} E_n^* \underset{MU^*}{\otimes} MU^*(X)$$

is a cohomology theory on finite complexes.

The discussion and results of Section 2 carry over to the case of formal groups over graded rings: one replaces the category C_R with the category of graded noetherian local algebras over a graded complete noetherian local ring R, and insists that a formal group is a Hopf algebra in the graded sense which is isomorphic to $R[\![x]\!]$ as an R-algebra, with $|x| = 2$. The degree is taken to be 2 so that the rings remain commutative in the ungraded sense. This is the approach of [And92].

Notice, however, that E_n is 2-periodic. We feel that, by analogy with K-theory, it is simpler to deal with the functor

$$E_n(X) \stackrel{\text{def}}{=} E_n^0(X).$$

This has the advantage that $E_n(pt)$ becomes the ring E_n of the last section. We hope that this choice makes our results more perspicuous to those familiar with the usually ungraded literature of formal groups. We hope that topologists, accustomed as they are to moving back and forth between the 2-periodic $K^*(X)$ and its 0-component $K(X)$, will not find this convention too annoying.

3.2. Power operations in MU. tom Dieck and Quillen constructed a total power operation

$$MU^{2*}X \xrightarrow{P_r^{MU}} MU^{2r*}(D_r X)$$

for complex cobordism [tDi68], [Qui71]. Here we have badly amended our notation: when r is a natural number we define $D_r X$ to be

$$D_r X \stackrel{\text{def}}{=} ES_r \underset{S_r}{\times} X^r,$$

where S_r is the symmetric group on r things. If π is an abelian p-group of order r, then by ordering the elements of π, we obtain a map

$$D_\pi X \to D_r X$$

in terms of which P_r^{MU} determines the operation P_π^{MU} mentioned in the introduction.

Briefly, the construction is as follows. If a map

$$M^k \xrightarrow{f} X^l$$

of (k- and l-dimensional) manifolds is complex-oriented, then there is a Gysin homomorphism

$$MU^*M \xrightarrow{f_*} MU^{*+l-k}X.$$

In particular,

$$f_* 1 \in MU^{l-k}X.$$

Thom's theorem [Tho54] shows that every class in MU^*X can be obtained in this way.

Now suppose that

$$M \xrightarrow{f} X$$

is a complex-oriented map of even dimension $2d$. Then

$$D_r M \xrightarrow{D_r f} D_r X$$

inherits a complex orientation, and by definition

$$P_r^{MU}(f_* 1) = (D_r f)_* 1. \tag{3.2.1}$$

From this construction, one can quickly check the following properties of P_r^{MU} and P_r^{MU}, which justify calling P_r^{MU} a "power operation". When there is no possibility of confusion, we abbreviate P_r^{MU} as P_r.

LEMMA 3.2.2 ([tDi68]). (i) *Let*

$$X^r \xrightarrow{i} D_r X$$

denote the inclusion of the fiber; then

$$i^* P_r z = z^{\times r}.$$

(ii) *If $X \xrightarrow{f} Y$, then the diagram*

$$\begin{array}{ccc} MU^{2*}Y & \xrightarrow{P_r} & MU^{2*r}(D_rY) \\ \Big\downarrow{\scriptstyle f^*} & & \Big\downarrow{\scriptstyle (D_rf)^*} \\ MU^{2*}X & \xrightarrow{P_r} & MU^{2*r}(D_rX) \end{array} \tag{3.2.3}$$

commutes.

LEMMA 3.2.4 ([tDi68]). *P_r is multiplicative: for $x \in MU^{2k}X$ and $y \in MU^{2l}X$,*

$$P_r(xy) = P_r(x)P_r(y) \in MU^{2r(k+l)}D_rX.$$

One of the features of power operations that makes the subject difficult is that they are not additive. However, the failure of P_r to be additive can be expressed as a sum of terms which are transfers. Let

$$Tr_{j,r}^{MU}: MU^*(ES_r \underset{S_j \times S_{r-j}}{\times} X^r) \to MU^*(D_rX) \tag{3.2.5}$$

be the MU-transfer associated to the fibration

$$S_r/(S_j \times S_{r-j}) \to ES_r \underset{S_j \times S_{r-j}}{\times} X^r \to D_rX, \tag{3.2.6}$$

and let d denote the map

$$ES_r \underset{S_j \times S_{r-j}}{\times} X^r \xrightarrow{d} D_jX \times D_{r-j}X.$$

The demonstration (3.3.9) depends on the following description of the additivity of P_r.

LEMMA 3.2.7 (compare [BMMS86, p. 25])

$$P_r(x+y) = \sum_{j=0}^{r} Tr_{j,r}^{MU} d^*(P_jx \times P_{r-j}y).$$

Proof. Represent x by a map

$$U \xrightarrow{f} X$$

and y by a map

$$V \xrightarrow{g} X.$$

Then $P_r(x+y)$ is represented by

$$D_r(U \amalg V) \xrightarrow{D_r(f \amalg g)} D_r X.$$

Now

$$D_r(U \amalg V) \cong \coprod_{j=0}^{r} ES_r \underset{S_j \times S_{r-j}}{\times} (U^j \times V^{r-j}), \tag{3.2.8}$$

and on the j factor, the map $D_r(f \amalg g)$ restricts to

$$\begin{tikzcd} ES_r \underset{S_j \times S_{r-j}}{\times} (U^j \times V^{r-j}) \arrow[rr, "ES_r \underset{S_j \times S_{r-j}}{\times} (f^j \times g^{r-j})"] \arrow[d] & & ES_r \underset{S_j \times S_{r-j}}{\times} X^r \arrow[d] \\ D_r(U \amalg V) \arrow[rr, "D_r(f \amalg g)"] & & D_r X \end{tikzcd},$$

where the vertical maps are projections. The counterclockwise composite represents the j summand of $P_r(x+y)$ coming from the decomposition (3.2.8); the clockwise composite represents the class

$$Tr_{j,r}^{MU} d^*(P_j x \times P_{r-j} y). \qquad \square$$

P_π^{MU} *on Euler classes.* Once again let π be an abelian p-group of order r. If $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ is a complex vector bundle of rank k, let

$$eV \in MU^{2k}X$$

be its MU Euler class. The group π acts on the product $\begin{matrix} V^\pi \\ \downarrow \\ X^\pi \end{matrix}$, and the resulting Borel construction $\begin{matrix} D_\pi V \\ \downarrow \\ D_\pi X \end{matrix}$ is a complex vector bundle over $D_\pi X$ with rank kr.

PROPOSITION 3.2.9 ([tDi68]).

$$P_\pi(eV) = e \begin{bmatrix} D_\pi V \\ \downarrow \\ D_\pi X \end{bmatrix}.$$

Proof. Let $\zeta\colon X \to V$ be the zero-section. Then the Thom class of $\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ is the image of 1 under the push-forward

$$MU^*X \xrightarrow{\zeta_*} MU^{*+2k}X^V.$$

By definition, the Euler class eV is the pull-back of the Thom class by ζ; in other words,

$$eV = \zeta^*\zeta_* 1.$$

We have

$$\begin{aligned} P_\pi(eV) &= (D_\pi\zeta)^* P_\pi(\zeta_* 1) \\ &= (D_\pi\zeta)^*(D_\pi\zeta)_* 1 \\ &= e\begin{bmatrix} D_\pi V \\ \downarrow \\ D_\pi X \end{bmatrix}, \end{aligned}$$

where the first equality is the naturality of P_π with respect to pull-backs (3.2.3), the second is the definition of P_π (3.2.1), and the last is the definition of the Euler class. □

$P_\pi^{MU}(eL)$ when L is a line bundle. Now let $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ be a complex line bundle, and $eL \in MU^2X$ its Euler class. Let Δ denote the "diagonal map"

$$B\pi \times X \xrightarrow{\Delta} D_\pi X.$$

The *sine qua non* of this paper is the similarity between the expression for $\Delta^* P_\pi(eL)$, which we now give, and Lubin's homomorphism $f_H(t)$ from a formal group law to its quotient by a finite subgroup (2.2.1). Let

$$\pi^* = \mathrm{Hom}[\pi, \mathbb{C}^\times]$$

be the group of (continuous) characters of π.

PROPOSITION 3.2.10 (compare [Qui71, p. 42]).

$$\Delta^* P_\pi(eL) = \prod_{u \in \pi^*} \left[e \left(\begin{array}{c} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ B\pi \end{array} \right) \underset{MU}{+} eL \right] \in MU^{2r}(B\pi \times X), \tag{3.2.11}$$

where $x \underset{MU}{+} y$ *denotes the formal sum with respect to the formal group law of* MU, *and we omit symbols for the pull-backs under the projections* $B\pi \times X \to X$ *and* $B\pi \times X \to B\pi$.

Proof. By (3.2.9),

$$\begin{aligned} \Delta^* P_\pi x &= e \left(\begin{array}{c} \Delta^*(E\pi \underset{\pi}{\times} L^\pi) \\ \downarrow \\ B\pi \times X \end{array} \right) \\ &= e \left(\begin{array}{c} Reg_\pi \otimes L \\ \downarrow \\ B\pi \times X \end{array} \right) \\ &= e \left[\bigoplus_{u \in \pi^*} \left(\begin{array}{c} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ B\pi \end{array} \right) \otimes L \right] \\ &= \prod_{u \in \pi^*} \left[e \left(\begin{array}{c} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ B\pi \end{array} \right) \underset{MU}{+} eL \right]. \qquad \square \end{aligned}$$

The bridge between Lubin's formula (2.2.1) and the expression (3.2.11) is built from the subject of the next section.

3.3. The character theory of Hopkins-Kuhn-Ravenel and the ring D_k, II. In Lubin's formula (2.2.1)

$$f_H(x) = \prod_{v \in H} (x(v) \underset{F_x}{+} x),$$

the indeterminate x represents a choice of coordinate on the formal group F. The choice is equivalent to the choice in topology of an Euler class

$$x = eL$$

for the tautological line bundle L over $\mathbb{C}P^\infty$; in particular, such a choice determines an isomorphism

$$E_n(\mathbb{C}P^\infty) \cong E_n[\![x]\!]$$

between $E_n(\mathbb{C}P^\infty)$ and the formal affine line over E_n. The $x(v)$'s are points of finite order in the group

$$F_x(R)$$

of R-valued points of F_x. The corresponding terms in (3.2.11) are the Euler classes of line bundles over BA. The relationship between these is well known, but is expressed particularly nicely by Hopkins-Kuhn-Ravenel. Let

$$A^* \xrightarrow{\phi} F_x(E_n(BA))$$

be the group homomorphism

$$u \overset{\phi}{\mapsto} e \begin{bmatrix} EA \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BA \end{bmatrix} \in E_n(BA).$$

LEMMA 3.3.1 (Proposition 2.4.1 of [HKR92]). *The natural transformation*

$$\mathrm{Hom}_{C_{E_n}}[E_n(BA), R] \to \mathrm{Hom}_{gps}[A^*, F_x(R)]$$

given by

$$f \mapsto f \circ \phi$$

is an equivalence of functors of pairs (A, R), where A is a finite abelian group, and $R \in C_{E_n}$.

In particular, let Λ_∞ be an abelian group isomorphic to $\mathbb{Z}_p^n$, so

$$\Lambda_\infty^* \cong (\mathbb{Q}_p/\mathbb{Z}_p)^n,$$

although no explicit isomorphism has been chosen. Let $\Lambda_k^* = {}_{p^k}\Lambda_\infty^*$ be the subgroup of elements of order p^k; Λ_k^* is precisely the dual of

$$\Lambda_k = \Lambda_\infty/p^k\Lambda_\infty.$$

Then the image of the map

$$\Lambda_k^* \xrightarrow{\phi} F_x(E_n(B\Lambda_k))$$

is the set of E_n-Euler classes of line bundles over $B\Lambda_k$. Let $S \subset E_n(B\Lambda_k)$ be the multiplicative set generated by the set

$$\{\phi(u) | u \neq 0\}$$

of Euler classes of *nontrivial* line bundles.

Define the ring D_k to be the image

$$D_k = \mathrm{Im}[E_n(B\Lambda_k) \to S^{-1}E_n(B\Lambda_k)].$$

This notation is justified by the next theorem.

THEOREM 3.3.2 (See Proposition 2.4.5 of [HKR92]). *D_k is isomorphic to the ring D_k described in Theorem 2.4.3, any isomorphism being determined by a choice of isomorphism*

$$(\mathbb{Z}/p^k\mathbb{Z})^n \underset{\cong}{\to} \Lambda_k^*,$$

which determines a level structure

$$(\mathbb{Z}/p^k\mathbb{Z})^n \underset{\cong}{\longrightarrow} \Lambda_k^* \xrightarrow{\phi_{\mathrm{univ}}} F_x(E_n(B\Lambda_k)) \longrightarrow {}_{p^k}F_x(D_k).$$

Sketch of proof. We outline the proof here, because it highlights the interaction between the theory of level structures on formal groups and $E_n(B\Lambda_k)$ which is the foundation of our work. According to [KM85, Section 1.10], D_k is the initial (complete local E_n-) algebra with a homomorphism

$$\Lambda_k^* \xrightarrow{\phi} F_x(D_k)$$

such that the corresponding map of Hopf algebras

$$D_k[\![x]\!] \Big/ \Big(\prod_{u \in \Lambda_k^*} (x - \phi(u)) \Big) \to \prod_{u \in \Lambda_k^*} D_k[\![x]\!]/(x - \phi(u))$$

is a monomorphism. Hopkins-Kuhn-Ravenel prove the following.

LEMMA 3.3.3 ([HKR92, Lemma 2.4.3]). *For any complete local E_n-algebra R with a homomorphism*

$$\Lambda_k^* \xrightarrow{\phi} F_x(R),$$

the map

$$R[\![x]\!] \Big/ \Big(\prod_{u \in \Lambda_k^*} (x - \phi(u)) \Big) \to \prod_{u \in \Lambda_k^*} R[\![x]\!]/(x - \phi(u))$$

is a monomorphism if and only if

$$\phi(u) \text{ is not a zero divisor for } 0 \neq u \in \Lambda_k^*. \tag{3.3.4}$$

By (3.3.1), any ϕ is represented by a homomorphism

$$E_n(B\Lambda_k) \to R.$$

The image of $E_n(B\Lambda_k)$ in its localization $S^{-1}E_n(B\Lambda_k)$ is the initial object in C_{E_n} with a level structure satisfying (3.3.4). □

Remark. Because (2.4.3) D_k is flat over $E_n(pt)$, the functor

$$X \mapsto D_k(X) \stackrel{\text{def}}{=} D_k \underset{E_n}{\otimes} E_n(X)$$

is again a cohomology theory for finite complexes.

The character map associated to a subgroup. The rings D_k appear in the work of Hopkins-Kuhn-Ravenel as the natural range of the character map. If G is a finite group and

$$\Lambda_k \xrightarrow{\alpha} G$$

is a homomorphism, the character map associated to α is the ring homomorphism

$$E_n(BG) \xrightarrow{\chi_\alpha} D_k$$

given by

$$E_n(BG) \xrightarrow{\alpha^*} E_n(B\Lambda_k) \to D_k.$$

Under a choice of isomorphism

$$\Lambda_k \cong (\mathbb{Z}/p^k\mathbb{Z})^n,$$

a homomorphism α corresponds to an n-tuple of commuting elements of G of order dividing p^k, and χ_α corresponds to evaluation of characters on this n-tuple.

Fix a finite subgroup

$$H \subset \Lambda_k^*$$

of order r. Via the isomorphism

$$\Lambda_k^* \xrightarrow{\phi_{\text{univ}}} {}_{p^k}F_x(D_k)$$

such H correspond to subgroups of ${}_{p^k}F_x(D_k)$ of order r. Let $G = H^*$ be its dual, that is,

$$\Lambda_k \xrightarrow{\pi} G = \Lambda_k/\text{Ann}[H].$$

Definition 3.3.5. The *character map associated to H* is the ring homomorphism

$$\chi^H\colon E_n(D_G X) \xrightarrow{\Delta^*} E_n(BG) \underset{E_n}{\otimes} E_n(X) \xrightarrow{\chi_\pi \otimes 1} D_k(X).$$

In terms of (3.3.1), it represents the homomorphism

$$G^* = H \subset \Lambda_k^* \xrightarrow{\phi_{\text{univ}}} F_x(D_k);$$

equivalently, for $u \in H$ we have the equation

$$\chi^H \left[e \begin{bmatrix} EG \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BG \end{bmatrix} \right] = \phi_{\text{univ}}(u). \tag{3.3.6}$$

The choice of generator $x \in E_n(\mathbb{C}P^\infty)$ determines a complex orientation

$$MU^*(X) \xrightarrow{t_x} E_n(X)$$

and in particular a formal group law

$$F_x = t_x F^{MU}$$

over E_n. We also obtain some cohomology operations.

Definition 3.3.7. *The operation*

$$MU^{2*}(X) \xrightarrow{Q^H} D_k(X)$$

associated to a subgroup $H \subset \Lambda_k^*$ and an orientation

$$MU^{2*}(X) \xrightarrow{t_x} E_n(X)$$

is the natural transformation

$$MU^{2*}(X) \xrightarrow{P_G^{MU}} MU^{2r*}(D_G(X)) \xrightarrow{\Delta^*} MU^{2r*}(BG \times X) \xrightarrow{t_x} E_n(BG \times X)$$
$$\xrightarrow{\cong} E_n(BG) \underset{E_n}{\otimes} E_n(X) \xrightarrow{\chi^H \otimes 1} D_k(X).$$

The significance of the operations Q^H is given by the following two theorems.

THEOREM 3.3.8. *Q^H is a ring homomorphism.*

In particular, the ring homomorphism

$$MU^* \xrightarrow{Q^H} D_k$$

classifies a formal group law $Q_*^H F^{MU}$ over D_k. To identify this group law, we will calculate the effect of Q^H on Euler classes of complex line bundles. If $L \to X$ is a complex line bundle, then we denote by

$$e_x L = t_x(e_{MU} L)$$

its Euler class in the orientation t_x.

THEOREM 3.3.9. *The effect of Q^H on Euler classes is given by the equation*

$$Q^H(e_{MU}L) = f_H^x(e_x L) \in D_k(X). \tag{3.3.10}$$

Its effect on coefficients is given by the equation

$$Q_*^H F^{MU} = F_x/H. \tag{3.3.11}$$

The proof of the additivity of Q^H in Theorem 3.3.8 imitates Atiyah's proof [Ati66] of the additivity of the Adams operations in K-theory. The basic ingredient is the formula of Hopkins-Kuhn-Ravenel for induced characters. To state this formula, let G be a finite group and let $S \subset G$ be a subgroup. Denote by

$$Tr^E\colon E_n(BS) \to E_n(BG)$$

the transfer in E-cohomology associated to the fibration

$$G/S \to BS \to BG.$$

If

$$\Lambda_k \xrightarrow{f} G$$

is a homomorphism, then we denote by

$$(G/S)^{\mathrm{Im}\, f}$$

the cosets gS which are fixed under left multiplication by the image of f. Note that a coset gS is fixed by the image of f if and only if

$$g^{-1}\,\mathrm{Im}\, fg \subset S.$$

THEOREM 3.3.12 ([HKR92, Theorem C]). *For $u \in E_n(BS)$ and $\Lambda_k \xrightarrow{f} G$ one has the formula*

$$\chi_f(Tr^E u) = \sum_{gS \in (G/S)^{\mathrm{Im} f}} \chi_{g^{-1}fg}(u).$$

Proof of Theorem 3.3.8. In (3.3.7), all the maps are ring homomorphisms except P_G. By Lemma 3.2.4, P_G is multiplicative, and its failure to be additive is measured by (3.2.7). Suppose we order the elements of G, which determines an isomorphism

$$\mathrm{Aut}_{\mathrm{sets}}\, G \xrightarrow{\cong} S_r,$$

where S_r is the symmetric group on r letters. Letting G act on itself by multiplication on the right provides a homomorphism

$$\Lambda_k \xrightarrow{\pi} G \xrightarrow{\omega} S_r.$$

Then one obtains a character map

$$E_n(D_r X) \xrightarrow{\chi_\omega} D_k(X)$$

such that the diagram

$$\begin{tikzcd} MU^{2*}X \arrow[r, "P_r"] \arrow[d, "Q^H"'] & MU^{2r*}(D_r X) \arrow[d, "t"] \\ D_k(X) & E_n(D_r X) \arrow[l, "\chi_\omega"'] \end{tikzcd}$$

commutes. By (3.2.7), P_r is given on sums by

$$P_r(x+y) = \sum_{j=0}^{r} Tr^{MU}_{j,r}(P_j x \times P_{r-j} y),$$

so

$$Q^H(x+y) = \sum_{j=0}^{r} \chi_\omega Tr^{E}_{j,r} t(P_j x \times P_{r-j} y),$$

since t commutes with the transfer. By the formula for induced characters (3.3.12),

$$\chi_\omega Tr^E_{j,r} u = \sum_{gS_j \times S_{r-j} \in (S_r/S_j \times S_{r-j})^{\mathrm{Im}\,\omega}} \chi_{g^{-1}\omega g} u.$$

Since the image of G in S_r is transitive, the sum is empty unless $j = 0$ or $j = r$. So

$$\begin{aligned} Q^H(x+y) &= \chi_\omega t(P_r x) + \chi_\omega t(P_r y) \\ &= Q^H(x) + Q^H(y). \end{aligned}$$

Thus Q^H is additive. □

Proof of Theorem 3.3.9. First we prove (3.3.10). By definition,

$$\begin{aligned} Q^H(e_{MU}L) &= (\chi^H \otimes 1) t \Delta^* P_G e_{MU} L \\ &= (\chi^H \otimes 1) t \prod_{u \in G^* = H} \left[e_{MU} \begin{pmatrix} EG \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BG \end{pmatrix} \underset{MU}{+} e_{MU} L \right] \\ &= (\chi^H \otimes 1) \prod_{u \in H} \left[e_{E_n} \begin{pmatrix} EG \underset{u}{\times} \mathbb{C} \\ \downarrow \\ BG \end{pmatrix} \underset{F_x}{+} e_E L \right] \\ &= \prod_{u \in H} [\phi_{\mathrm{univ}}(u) \underset{F_x}{+} e_E L] \\ &= f^x_H(e_E L). \end{aligned}$$

Here the second equation is Lemma 3.2.10 and the fourth is (3.3.6).

It remains to examine the effect of Q^H on coefficients (3.3.11). Since D_k is a domain, the formal group law F_x/H is uniquely characterised by the functional equation

$$f_H^x F_x(t_1, t_2) = F_x/H(f_H^x(t_1), f_H^x(t_2)).$$

Let L_1 and L_2 be the two tautological bundles over $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$, and let t_1 and t_2 be their E_n Euler classes. By (3.3.10),

$$\begin{aligned} Q^H(e_{MU}L_1 \underset{MU}{+} e_{MU}L_2) &= Q^H \left\{ e_{MU} \begin{bmatrix} L_1 \otimes L_2 \\ \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{bmatrix} \right\} \\ &= f_H^x \left\{ e_{E_n} \begin{bmatrix} L_1 \otimes L_2 \\ \downarrow \\ \mathbb{C}P^\infty \times \mathbb{C}P^\infty \end{bmatrix} \right\} \\ &= f_H^x(t_1 \underset{F_x}{+} t_2). \end{aligned} \tag{3.3.13}$$

On the other hand, we know that Q^H is a ring homomorphism

$$MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \xrightarrow{Q^H} D_k(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$$

(recall that $MU^*(pt)$ and $MU^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ are concentrated in even degrees) so

$$\begin{aligned} Q^H(e_{MU}L_1 \underset{MU}{+} e_{MU}L_2) &= Q^H(e_{MU}L_1) \underset{Q_*^H F^{MU}}{+} Q^H(e_{MU}L_2) \\ &= f_H^x(t_1) \underset{Q_*^H F^{MU}}{+} f_H^x(t_2). \end{aligned} \tag{3.3.14}$$

Comparing (3.3.13) and (3.3.14), we see that

$$f_H^x F_x(t_1, t_2) = Q_*^H F^{MU}(f(t_1), f(t_2)),$$

so $Q_*^H F^{MU} = F_x/H$. □

As operations on E_n, the operations Q^H have two unsatisfactory features: they have the wrong source, MU, and the wrong target, D_k. We approach the second problem in Section 3.5; now we turn to the first problem.

3.4. Factoring the operations Q^H through E_n. We would like to produce an operation

$$E_n(X) \xrightarrow{\Psi^H} D_k(X)$$

filling in the diagram

$$\begin{tikzcd} MU^*(X) \arrow[d, "t_X"'] \arrow[dr, "Q^H"] & \\ E_n(X) \arrow[r, "\Psi^H"] & D_k \underset{E_n}{\otimes} E_n(X). \end{tikzcd}$$

Since (3.1.2) E_n is *Landweber exact*, that is,

$$E_n(X) \cong E_n \underset{MU^*}{\otimes} MU^{2*}(X),$$

and since (3.3.8) Q^H is a homomorphism of rings, the next lemma follows by the universal property of the tensor product.

LEMMA 3.4.1. *The operation Ψ^H exists if and only if there is a ring homomorphism*

$$E_n(pt) \xrightarrow{\alpha^H} D_k$$

such that

$$\begin{tikzcd} MU^* \arrow[d, "t_X"'] \arrow[dr, "Q^H"] & \\ E_n \arrow[r, "\alpha^H"] & D_k \end{tikzcd}$$

commutes.

Proof.

$$\Psi^H = \alpha^H \underset{t}{\otimes} Q^H$$

does the job. $\square$

Now the homomorphism

$$MU^* \xrightarrow{Q^H} D_k$$

was computed by (3.3.9):

$$Q_*^H F^{MU} = F_x/H.$$

By construction, we have

$$t_{x*}F^{MU} = F_x.$$

These two equations combine with Lemma 3.4.1 to prove the next theorem.

THEOREM 3.4.2. *The operation Ψ^H exists if and only if there is a ring homomorphism*

$$E_n(pt) \xrightarrow{\alpha^H} D_k$$

such that

$$\alpha_*^H F_x = F_x/H. \tag{3.4.3}$$

The condition (3.4.3) is exactly the condition (2.5.8) of Theorem 2.5.7, so we conclude the following.

THEOREM 3.4.4. *There is a unique orientation*

$$MU^{2*}(X) \xrightarrow{t_P} E_n(X)$$

which is $\star$-isomorphic to the usual one (3.1.1) and through which the operations

$$MU^{2*}(X) \xrightarrow{Q^H} D_k(X)$$

factor to yield ring operations

$$E_n(X) \xrightarrow{\Psi^H} D_k(X)$$

for all subgroups $H \subset \Lambda_k^* \cong {}_{p^k}F_P(D_k)$; here

$$F_P \overset{\text{def}}{=} t_{P*}F^{MU}$$

is the formal group law induced by t_P.

Remark. Recall (2.6.15) that the condition (3.4.3) is satisfied for all H if and only if it is satisfied in the special case that $H = {}_pF_x(D_k)$ is the subgroup of points of order p, and that in that case (3.4.3) is equivalent by Corollary 2.6.2 to the equation

$$[p]_{F_x}(x) = f_p(x).$$

Thus the condition (3.4.3) is really a condition on the p-series.

In Chapter VIII, Section 7 of [BMMS86]), McClure follows a similar line of reasoning to ours to give a sufficient condition on the p-series of the formal group law over BP for BP to be an H_∞ ring spectrum. In our situation the problem becomes tractable because the issue can conveniently be expressed in terms of the functor represented by E_n. McClure considers only the complex orientation on BP coming from the Quillen idempotent; our work shows that one must plan on adjusting the orientation in order to get power operations from the ones on MU. The relationship between E_n and BP makes us wonder what relationship there may be between the orientations of Theorem 4 as n varies, especially, as n "goes to ∞."

3.5. Making operations that land in E_n: Galois theory. In this section we use the Galois group $\mathrm{Aut}[\Lambda_k^*]$ of D_k over E_n (2.4.3) to produce operations whose target is E_n rather than D_k. The group $\mathrm{Aut}[\Lambda_k^*]$ acts on the set

$$\mathscr{S}\mathscr{G}_k = \left\{ \begin{matrix} \text{subgroups of} \\ \Lambda_k^* \end{matrix} \right\}.$$

Let $E_n[\mathscr{S}\mathscr{G}_k]$ be the polynomial ring on the set $\mathscr{S}\mathscr{G}_k$. The action of $\mathrm{Aut}[\Lambda_k^*]$ on $\mathscr{S}\mathscr{G}_k$ induces an action of $\mathrm{Aut}[\Lambda_k^*]$ on $E_n[\mathscr{S}\mathscr{G}_k]$. We define the set Op^k by

$$Op^k = E_n[\mathscr{S}\mathscr{G}_k]^{\mathrm{Aut}[\Lambda_k^*]}.$$

Note that $Op^k \subset Op^{k+1}$, and we define

$$Op = \operatorname*{colim}_k Op^k.$$

An element ρ of Op can be written by

$$\rho = \sum_{i \in I} a_i \prod_{H \in \alpha_i} H,$$

where I is a finite set, $a_i \in E_n$, and the α_i are lists of elements of $\mathscr{S}\mathscr{G}_k$ for k sufficiently large, with possible repetitions. For such a $\rho \in Op$, let Q^ρ be the operation

$$MU^{2*}(X) \xrightarrow{Q^\rho} D_k(X)$$

given by

$$Q^\rho(z) = \sum_{i \in I} a_i \prod_{H \in \alpha_i} Q^H(z). \tag{3.5.1}$$

Similarly, we define

$$E_n(X) \xrightarrow{\Psi^\rho} D_k(X)$$

by

$$\Psi^\rho(z) = \sum_{i \in I} a_i \prod_{H \in \alpha_i} \Psi^H(z).$$

In this situation, we have the following.

THEOREM 3.5.2. *For an element $\rho \in Op$, the operation Q^ρ factors through E_n to define an operation*

$$MU^{2*}X \xrightarrow{Q^\rho} E_n(X).$$

Proof. A subgroup $H \subset \Lambda_k^*$ isomorphic to π is given by a monomorphism

$$\pi \overset{\alpha_H}{\hookrightarrow} \Lambda_k^*$$

which is dual to a map

$$\Lambda_k \xrightarrow{\alpha_H^*} \pi.$$

We can rewrite Q^H (compare (3.3.7)) as

$$Q^H = (\chi_{\alpha_H^*} \otimes 1) t \Delta^* P_\pi.$$

If $a \in \mathrm{Aut}[\Lambda_k^*]$ is an automorphism, then we get a new injection

$$\pi \overset{\alpha_H}{\hookrightarrow} \Lambda_k^* \xrightarrow{a} \Lambda_k^*$$

in terms of which

$$Q^{aH} = (\chi_{(a\alpha_H)^*} \otimes 1) t \Delta^* P_\pi = ((a\chi_{\alpha_H^*}) \otimes 1) t \Delta^* P_\pi.$$

It follows that the operation Q^ρ on $MU^{2*}(X)$ really lands in

$$D_k^{\mathrm{Aut}[\Lambda_k^*]} \otimes_{E_n} E_n(X) = E_n(X). \qquad \square$$

Corollary 3.5.3. For an element $\rho \in Op^r$, the operation Ψ^ρ factors through E_n to give an operation

$$E_n(X) \xrightarrow{\Psi^\rho} E_n(X).$$

3.6. Example: Adams operations and Hecke operations. For example, we obtain the unstable "Adams operations" in the following way.

PROPOSITION 3.6.1. *The full subgroup ${}_{p^k}F(D_k)$ of points of order p^k defines an operation*

$$E_n(X) \xrightarrow{\Psi^{p^k}} E_n(X)$$

which is a natural transformation of rings. It induces the identity on coefficients, and its effect on the Euler class eL of a line bundle $\begin{matrix} L \\ \downarrow \\ X \end{matrix}$ *is*

$$\Psi^{p^k}(eL) = [p^k]_{F_P}(eL).$$

Proof. The fact that the image of Ψ^{p^k} is $E_n(X)$ follows immediately from (3.5.3). It is a ring homomorphism by (3.4.4). Its effect on coefficients and on Euler classes follows from (3.3.9) and (2.6.1). □

Remark. By standard methods (e.g., [Wil82]) one can construct a *stable* operation

$$E_n^*(X) \xrightarrow{\psi^p} \frac{1}{p} E_n^*(X)$$

whose effect on Euler classes is given by

$$\psi^p(eL) = \frac{[p]_F(eL)}{p}$$

and whose effect on coefficients is

$$\psi^p(m) = p^{-|m|/2} m.$$

With much greater sophistication, Wilson shows in [Wil82] using Hopf rings that the unstable operation which on $2i$-dimensional classes is given by $p^i\psi^p$ in fact lands in $E_n^{2i}(X)$; that is,

$$E_n^{2i}(X) \xrightarrow{p^i\psi^p} E_n^{2i}(X).$$

The same method shows that this unstable operation coincides with our Ψ^p. Our point of view in terms of power operations gives a direct construction of the integral unstable Ψ^p which is analogous to the construction of Ψ^p for K-theory in [Ati66].

As another consequence of (3.5.3) we get the next proposition.

PROPOSITION 3.6.2 ("Hecke operations"). *The operation*

$$T_r = \sum_{\substack{H \subset \Lambda_k^* \\ |H|=r}} \Psi^H$$

is an additive operation

$$E_n X \xrightarrow{T_r} E_n X.$$

In elliptic cohomology, we have sketched how the operation $(1/r)T_r$ produces an operation whose effect on coefficients is the classical Hecke operator T_r [And92]. These operations have been constructed by Andrew Baker [Bak90]. We shall return to that case in the near future.

3.7. The character-theoretic description of the total power operation. In this section we calculate the total MU power operation in terms of the operations Q^H and the character theory of [HKR92]. Let π be an abelian group, and let k be an integer greater than or equal to the p-exponent of π. Hopkins-Kuhn-Ravenel give a description of the cohomology of the full Borel construction $D_\pi X$ in terms of a total character map

$$E_n(D_\pi X) \xrightarrow{\chi} \prod_{\Lambda_k \xrightarrow{\alpha} \pi} D_k(X^{\pi/(\alpha)}).$$

For a map $\Lambda_k \xrightarrow{\alpha} \pi$, let $G = \operatorname{Im} \alpha$ be the image of α. The isomorphism of G-sets

$$\pi \underset{G}{\cong} G \times (\pi/G)$$

gives an isomorphism

$$\operatorname{Fixed}_G[\operatorname{Map}[\pi, X]] \cong \operatorname{Map}[\pi/G, X]$$

between the fixed point set of X^π by the action of G and $X^{\pi/G}$. The component of χ corresponding to a map α is

$$E_m(D_\pi X) \to E_n(BG \times X^{\pi/G}) \xrightarrow{\chi_\alpha} D_k(X^{\pi/G}). \tag{3.7.1}$$

Let Q be the natural transformation

$$MU^{2*}(X) \xrightarrow{P_\pi} MU^{2s*}(D_\pi X) \xrightarrow{t} E_n(D_\pi X), \tag{3.7.2}$$

where s is the order of π.

PROPOSITION 3.7.3. *For a map* $\Lambda_k \xrightarrow{\alpha} \pi$, *let* $G = \operatorname{Im} \alpha$ *be the image of* α *and let* $H = \operatorname{Im} \alpha^*$ *be the image of the dual homomorphism*

$$\pi^* \xrightarrow{\alpha^*} \Lambda_\infty^*.$$

Let $r = [\pi : G]$. *The image of* Q *under the character map of Hopkins-Kuhn-Ravenel is given by the equation*

$$\chi_\alpha Q(z) = Q^H(z^{\times r}) \in D_k(X^k).$$

Proof. Comparison of (3.3.7) and (3.7.1) shows that if α is surjective, then

$$\chi_\alpha Q = Q^H.$$

For general α, the result follows from the case of surjective α, the definition of Q^H (3.3.7), and the next lemma.

LEMMA 3.7.4. *In the situation of the proposition, the diagram*

$$\begin{tikzcd} MU^{2*}(X) \arrow[r, "P_\pi"] \arrow[d, "x \mapsto x^{\times r}"'] & MU^{2s*}(D_\pi X) \arrow[d] \\ MU^{2r*}(X^{\pi/G}) \arrow[r, "P_G"] & MU^{2s*}(D_G(X^{\pi/G})) \end{tikzcd}$$

commutes.

Proof. Represent a class $z \in MU^{2*}X$ as f_*1 where f is a complex-oriented map

$$Z \xrightarrow{f} X.$$

Then $P_\pi z$ is represented by the map

$$D_\pi Z \xrightarrow{D_\pi f} D_\pi X.$$

The isomorphism of G-sets

$$\pi \underset{G}{\cong} G \times \pi/G$$

provides the vertical isomorphisms in the commutative diagram

$$\begin{tikzcd} EG \underset{G}{\times} Z^\pi \arrow[r, "EG \underset{G}{\times} f^\pi"] \arrow[d, "\cong"'] & EG \underset{G}{\times} X^\pi \arrow[d, "\cong"] \\ D_G(Z^{\pi/G}) \arrow[r, "D_G(f^{\pi/G})"] & D_G(X^{\pi/G}). \end{tikzcd}$$

The bottom row of the diagram represents $P_G(z^{\times k})$ by definition. The top row represents the composite

$$MU^{2*}(X) \xrightarrow{P_\pi} MU^{2s*}(D_\pi X) \to MU^{2s*}(D_G X)$$

because it fits into the pull-back diagram

$$\begin{tikzcd} EG \underset{G}{\times} Z^\pi \arrow[r] \arrow[d, "EG \underset{G}{\times} f^\pi"'] & D_\pi Z \arrow[d, "D_\pi f"] \\ EG \underset{G}{\times} X^\pi \arrow[r] & D_\pi X. \end{tikzcd} \qquad \square$$

This completes the proof of Proposition 3.7.3. $\square$

Remarks. (1) For example, the component of Q corresponding to the trivial map $\Lambda_\infty \xrightarrow{0} \pi$ is the sth external power, which we knew already since P_π is a total power operation and evaluation at 0 corresponds to the pull-back by the inclusion of the fiber

$$X^\pi \to D_\pi X.$$

(2) Also, a computation similar to (3.2.10) shows that for

$$\Lambda_\infty \xrightarrow{\alpha} H \xhookrightarrow{i} G,$$

we have

$$\chi_\alpha (Bi)^* P_G eL = (f_H(eL))^{|G/H|},$$

which is a special case of Proposition 3.7.3 in view of the multiplicativity of Q^H.

4. H_∞ orientations for the cohomology theory E_n

4.1. Introduction. Mike Hopkins and Haynes Miller [HM93] have shown that the spectrum E_n is an E_∞ ring spectrum; in particular, it has power operations in the sense of [BMMS86]. The remainder of this paper is devoted to proving that the orientations produced by Theorem 4 are the only orientations on E_n which intertwine the power operations on MU and E_n.

THEOREM 4.1.1. *The diagram*

$$\begin{array}{ccc} MU^{2*}(X) & \xrightarrow{P_j^{MU}} & MU^{2j*}(D_jX) \\ \Big\downarrow{\scriptstyle t} & & \Big\downarrow{\scriptstyle t} \\ E_n(X) & \xrightarrow{P_j^{hm}} & E_n(D_jX) \end{array} \tag{4.1.2}$$

commutes if and only if t is an orientation satisfying the condition of Theorem 4.

Since Hopkins and Miller show that the E_∞ structure on E_n is unique up to E_∞ automorphisms (isomorphic to the stablizer group), it is pleasant that the proof of Theorem 4.1.1 depends only on the axioms of an H_∞ structure given in [BMMS86]. Thus it is possible to give the proof here, before the appearance of [HM93]. We apologize to those who might have hoped to learn about their technique from this paper.

In Section 4.2, we compute the effect of P^{hm} on coefficients and on Euler classes. In Section 4.3 we show that the power operations produced by the two ways of going around (4.1.2) commute as far as can be detected by the character theory of Hopkins-Kuhn-Ravenel, if and only if t comes from Theorem 4. In Section 4.4, we show that this is sufficient to guarantee the commutativity of (4.1.2).

4.2. Euler classes and the E_∞-structure of Hopkins-Miller. Let

$$MU \xrightarrow{t} E_n$$

be an orientation such that the resulting formal group law F is a universal lift of the group law Φ of (2.5.5). Let $H \subset \Lambda_k^*$. Recall from Section 2.5 that there are two homomorphisms of formal group laws over D_k,

$$F \xrightarrow{f_H} F/H \quad \text{and}$$

$$F \xrightarrow{l_H} \alpha_*^H F,$$

with kernel H. As in Secrtion 3.3, let $G = H^* \subset \Lambda_k$ be the complex dual of H. Recall that the operation Q^H is the composite

$$MU^{2*}(X) \xrightarrow{P_G^{MU}} MU^{2r*}(D_G(X)) \xrightarrow{t} E_n(D_G(X)) \xrightarrow{\chi^H} D_k(X).$$

Its effect on the Euler class of a line bundle L is

$$e_{MU}L \xmapsto{Q^H} f_H(te_{MU}L).$$

PROPOSITION 4.2.1. *The composite*

$$E_nX \xrightarrow{P^{hm}_G} E_n(D_GX \xrightarrow{\chi^H} D_k \underset{E_n}{\otimes} E_nX \tag{4.2.2}$$

sends the Euler class $te_{MU}L$ to the isogeny $l_H(te_{MU}L)$.

Proof. The next three lemmas verify the conditions listed in Proposition 2.5.4. Let C_H denote the natural transformation defined by equation (4.2.2).

LEMMA 4.2.3. *C_H is a ring homomorphism.*

Proof. The argument is exactly as in (3.3.8). C_H is multiplicative because each of the arrows in (4.2.2) is multiplicative. Additivity follows because the failure of P^{hm} to be additive is measured by tranfers [BMMS86, p. 25] which are killed by χ_H. □

In particular, when X is a point, we get a ring homomrphism

$$E_n(pt) \xrightarrow{C_H(pt)} D_k.$$

LEMMA 4.2.4. *The power series*

$$c(t) = C_H(t) \in E_n[\![t]\!],$$

which describes the effect of C_H on the Euler class $t \in E_n\mathbb{C}P^\infty$ of the tautological bundle, is an isogeny of formal group laws

$$F \xrightarrow{c} C_H^*F$$

over D_k. The coefficients of $c(t)$ reduced modulo the maximal ideal $\mathfrak{m}_D$ are given by

$$\bar{c}(t) = t^h.$$

Proof. If t_1 and $t_2 \in E_n(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ are the pull-backs of the Euler classes of the tautological bundles over each copy of $\mathbb{C}P^\infty$, then

$$C_H(t_1 \underset{F}{+} t_2) = c(t_1) \underset{C_H^*F}{+} c(t_2)$$

since C_H is a ring homomorphism. On the other hand, if

$$\mathbb{C}P^\infty \times \mathbb{C}P^\infty \xrightarrow{m} \mathbb{C}P^\infty$$

is the multiplication map, then

$$t_1 \underset{F}{+} t_2 = m^*t,$$

so

$$\begin{aligned} C_H(t_1 \underset{F}{+} t_2) &= C_H(m^*t) \\ &= m^*C_H(t) \\ &= m^*c(t) \\ &= c(t_1 \underset{F}{+} t_2). \end{aligned}$$

Now for the part about reducing coefficients to the residue field. Recall that the character map

$$E_n(D_G X) \xrightarrow{\chi_H} D_k \otimes_{E_n} E_n X$$

is given by pulling back by the diagonal followed by a character map on $E_n(BG)$, which we also call χ_H:

$$E_n(D_G X) \xrightarrow{\Delta^*} E_n(BG) \otimes_{E_n} E_n(X) \xrightarrow{\chi_H \otimes 1} D_k \otimes_{E_n} E_n X.$$

The image of any Euler class of a nontrivial line bundle over BG under the character map

$$E_n(BG) \xrightarrow{\chi_H} D_k$$

is an element of the maximal ideal $\mathfrak{m}_D$. It follows that the diagram

$$\begin{tikzcd} E_n(D_G X) \arrow[d, "\Delta^*"] & \\ E_n(BG) \otimes_{E_n} E_n(X) \arrow[r, "\chi_H \otimes 1"] \arrow[d, "\varepsilon \otimes 1"] & D_k \otimes_{E_n} E_n X \arrow[d] \\ E_n(X) \arrow[r] & \mathfrak{k} \otimes_{E_n} E_n(X) \end{tikzcd}$$

commutes. But if we now attach P_G^{hm} to this diagram

$$\begin{tikzcd}
E_nX \arrow[d, "P_G^{hm}"'] & \\
E_n(D_GX) \arrow[d, "\Delta^*"'] & \\
E_n(BG)\otimes_{E_n}E_n(X) \arrow[r, "\chi_H\otimes 1"] \arrow[d, "\varepsilon\otimes 1"'] & D_k\otimes_{E_n}E_nX \arrow[d] \\
E_n(X) \arrow[r] & \mathfrak{k}\otimes_{E_n}E_n(X)
\end{tikzcd}$$

then the left vertical composite is raising to the h power. □

Finally, we have to prove the following lemma.

LEMMA 4.2.5. *The kernel of the isogeny $C_H(t)$ applied to $F(D_k)$ is H.*

Proof. Mike Hopkins and Neil Strickland taught me this proof, which is much simpler than my first proof. The Euler class t of the tautological bundle is also the Thom class of the tautological bundle, via the zero section

$$\mathbb{C}P^\infty \underset{\cong}{\overset{\zeta}{\to}} \mathbb{C}P^{\infty L}$$

which is a homeomorphism. Consider the diagram

$$\begin{tikzcd}
E_n(\mathbb{C}P^\infty_+) \arrow[d, "P_G"'] & E_n(\mathbb{C}P^\infty) \arrow[l] & E_n(\mathbb{C}P^{\infty L}) \arrow[l, "\zeta"', "\cong"] \arrow[d, "P_G"] \\
E_n(D_G\mathbb{C}P^\infty_+) \arrow[d, "\Delta"'] & & E_n(D_G\mathbb{C}P^{\infty D_GL}) \arrow[ll, "\zeta"'] \arrow[d, "T(\Delta)"] \\
E_n(BG\times\mathbb{C}P^\infty_+) & & E_n(BG\times\mathbb{C}P^{\infty\,\mathrm{Reg}_G\otimes L}). \arrow[ll, "\zeta"']
\end{tikzcd}$$

Here $T(\Delta)$ is map of Thom spaces induced by

$$\begin{tikzcd}
\mathrm{Reg}_G\otimes L \arrow[r, "\Delta"] \arrow[d] & D_GL \arrow[d] \\
BG\times\mathbb{C}P^\infty \arrow[r, "\Delta"] & D_G\mathbb{C}P^\infty .
\end{tikzcd}$$

By Lemma 4.2.6, to be proved next,

$$P_G(t) \in E_n(D_G \mathbb{C}P^{\infty D_G L})$$

generates

$$E_n(D_G \mathbb{C}P^{\infty D_G L})$$

as a module over $E_n(D_G \mathbb{C}P^\infty)$, and so

$$T(\Delta)^* P_G(t)$$

generates

$$E_n(BG \times \mathbb{C}P^{\infty\, \mathrm{Reg}_G \otimes L})$$

as a module over $E_n(BG \times \mathbb{C}P^\infty)$. Hence it differs from any other Thom class for this bundle by multiplication by a unit in $E_n(BG \times \mathbb{C}P^\infty)$, and likewise for the corresponding Euler classes.

The Euler class of

$$\begin{array}{c} \mathrm{Reg}_G \otimes L \\ \downarrow \\ BG \times \mathbb{C}P^\infty \end{array}$$

is

$$\prod_{h \in H} \left[e \begin{pmatrix} E\pi \underset{u}{\times} \mathbb{C} \\ \downarrow \\ B\pi \end{pmatrix} \underset{F}{+} t \right],$$

which maps to $f_H(t)$ in $D_k(\mathbb{C}P^\infty)$. In particular it has kernel H. Thus

$$\begin{aligned} C_H(t) &= \Delta^* P_G(\zeta^* t) \\ &= \zeta^* T(\Delta)^* P_G(t) \\ &= \varepsilon(t) f_H(t), \end{aligned}$$

where $\varepsilon(t)$ is a unit in $E_n(BG \times \mathbb{C}P^\infty)$ and so certainly maps to a unit in $D_k(\mathbb{C}P^\infty)$. $\square$

In the course of the proof, we used the following fact.

LEMMA 4.2.6. *If X is connected, and*

$$u \in E_n(X^V)$$

is the Thom class of a bundle

$$\begin{array}{c} V \\ \downarrow \\ X \end{array},$$

then

$$P(u) \in E_n(EG_+ \wedge_G (X^V)^{\wedge G}) = E_n(D_G X^{D_G V})$$

is a Thom class for the bundle

$$\begin{array}{c} V^G \\ \downarrow \\ X^G \end{array},$$

in the sense that $P(u)$ generates $E_n(D_G X^{D_G V})$ as a module over $E_n(D_G X)$.

Proof. That u is a Thom class means that it restricts to a generator of $E_n(S^F)$, where $S \in X$ is a point and F is the fiber over V, so that the map of Thom spaces

$$S^F \to X^V$$

is the inclusion of the bottom cell.

The inclusion of the bottom cell of the Thom space of the Borel construction

$$(S^F)^{\wedge G} \xrightarrow{j} D_G X^{D_G V}$$

is induced by the maps

$$\begin{tikzcd} V^G \arrow[r, "i"] \arrow[d] & \mathrm{Reg}_G \otimes V \arrow[r] \arrow[d] & D_G V \arrow[d] \\ S \arrow[r, "i"] & D_G S = BG \arrow[r] & D_G X, \end{tikzcd}$$

so

$$j^*P_G(u) = u^{\wedge G}$$

is a generator of $E_n((S^F)^{\wedge G})$. □

This completes the proof of Proposition 4.2.1.

4.3. Comparison of the total power operations. The character map χ_H associated to a finite subgroup

$$H \subset \Lambda_k^* \cong {}_{p^k}F(D_k)$$

induces two ring operations

$$MU^{2*}X \to D_k \otimes_{E_n} E_n(X) \overset{\text{def}}{=} D_k(X).$$

They are produced from the two ways of going around the square in the diagram

$$\begin{tikzcd} MU^{2*}X \arrow[r, "P_G^{MU}"] \arrow[d, "t"] & MU^{2h*}(D_G(X)) \arrow[d, "t"] \\ E_nX \arrow[r, "P_G^{hm}"] & E_n(D_G(X)) \arrow[d, "\chi_H"] \\ & D_k(X) \end{tikzcd}$$

The clockwise direction is Q^H. The counterclockwise direction, which is $C_H \circ t$, I shall call R^H.

An immediate corollary of Proposition 4.2.1 is

COROLLARY 4.3.1. *The operations Q^H and R^H agree on $MU^{2*}(pt)$ and $MU^{2*}(\mathbb{C}P^\infty)$ if and only if the orientation t is chosen so that, for all H, l_H and f_H coincide.*

In fact, this is sufficient to prove the following proposition.

PROPOSITION 4.3.2. *If the orientation t is chosen so that, for all H, l_H, and f_H coincide, then Q^H and R^H are the same operation.*

Proof. The proof imitates the argument of [Wil82], and is based on the computation by [RW77] of the Hopf ring for $D_{k*}(\mathbf{MU}_*)$, where

$$\mathbf{MU}_k \overset{\text{def}}{=} \Omega^\infty\Sigma^k MU.$$

To check that Q^H and R^H coincide as unstable operations

$$MU^{2r}(X) \to D_k(X),$$

it suffices to check that they coincide as elements of

$$\mathrm{Hom}_{MU_*}[MU_*\mathbf{MU}_{2r}, D_k].$$

By [RW77], this amounts to checking that the operations (which are ring operations) coincide on $\mathbb{C}P^\infty$ and on the 1-point space. □

4.4. Proof of Theorem 4.1.1.

LEMMA 4.4.1 (McClure, Proposition VIII 7.2 of [BMMS86]). *The diagram (4.1.2) commutes for all X and all j if and only if the diagram*

$$\begin{tikzcd} MU^{2r}(\mathbf{MU}_{2d}) \arrow[r, "P_\pi^{MU}"] \arrow[d, "t"] & MU^{2rp}(D_\pi \mathbf{MU}_{2r}) \arrow[d, "t"] \\ E_n(\mathbf{MU}_{2r}) \arrow[r, "P_\pi^{hm}"] & E_n(D_\pi \mathbf{MU}_{2r}) \end{tikzcd}$$

commutes, where π is the cyclic group of order p.

LEMMA 4.4.2 (McClure, Proposition VIII 7.3 of [BMMS86]). *The map*

$$\bar{E}_n^*(E\pi_+ \underset{\pi}{\wedge} \mathbf{MU}_{2r}^{\wedge p}) \xrightarrow{i^*\oplus\Delta^*} \bar{E}_n^*(\mathbf{MU}_{2r}^{\wedge p}) \oplus \bar{E}_n^*(B\pi_+ \wedge \mathbf{MU}_{2r})$$

is injective.

Proof. McClure's result applies to this situation in view of Wilson's theorem that $\mathbf{MU}_{2r}$ is a torsion-free spectrum of finite type whose cells are concentrated in even dimensions. □

COROLLARY 4.4.3. *For $n \geqslant 1$, $E_n(D_p\mathbf{MU}_{2r})$ is p-torsion free, and the odd degree part $E_n^1(D_p\mathbf{MU}_{2r})$ is zero.*

With Lemma 4.4.1 in hand, Theorem 4.1.1 reduces to the following.

LEMMA 4.4.4. *Let π be a cyclic group of order p, and suppose once again that the orientation t has been chosen so that $l_H = f_H$ for all H. Then the diagram*

$$\begin{tikzcd} MU^{2*}X \arrow[r, "P_\pi^{MU}"] \arrow[d, "t"] & MU^{2p*}D_\pi X \arrow[d, "t"] \\ E_n X \arrow[r, "P_\pi^{hm}"] & E_n D_\pi X \end{tikzcd} \tag{4.4.5}$$

commutes.

Proof. Let us use the abbreviations Q and R to denote the two ways of going around the diagram (4.4.5). The total character map for $E_n(D_\pi X)$ of [HKR92] can be written

$$E_n(D_\pi X) \xrightarrow{\prod \chi_\alpha} \prod_{\Lambda_1 \xrightarrow{\alpha} \pi} D_1(X^{\pi/(\alpha)}). \tag{4.4.6}$$

We computed the projections $\chi_\alpha Q$ in Proposition 3.7.3. If

$$\Lambda_1 \xrightarrow{0} \pi$$

is the zero map, and $z \in MU^{2d}(X)$, then

$$\chi_0 Q(z) = t(z)^{\times p} = \chi_0 R(z).$$

If

$$\Lambda_1 \xrightarrow{\alpha} \pi$$

is nonzero, then it is surjective, and

$$\chi_\alpha Q = Q^{(\alpha)},$$

where $Q^{(\alpha)}$ is the operation corresponding to the subgroup

$$(\alpha^*) \subset \Lambda_1^*,$$

which is the image of the homomorphism

$$\pi^* \xrightarrow{\alpha^*} \Lambda_1^*$$

dual to α. Similarly,

$$\chi_\alpha R = R^{(\alpha)}$$

is the operation $R^{(\alpha)}$.

Now suppose that orientation t is chosen as in Theorem 2.5.7. By Proposition 4.3.2, it follows that Q and R coincide as far as the character map (4.4.6) is able to detect. Moreover, according to [HKR92], the character map for $E_n(D_\pi \mathbf{MU}_{2r})$ is injective: the character map is always injective after inverting p, and $E_n(D_\pi \mathbf{MU}_{2r})$ is p-torsion free by Corollary 4.4.3. □

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