ABSTRACT HOMOTOPY. III

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1. *Introduction.*—In an earlier note[1] it was indicated how a homotopy theory may be developed for cubical complexes satisfying a certain extension axiom. In the same manner a homotopy theory may be developed for all c.s.s. complexes[2] which satisfy the following simplicial version of the extension axiom: *A c.s.s. complex K is said to satisfy the extension axiom if for every pair of integers (k, n) with $0 \leqslant k \leqslant n$ and for every n $(n - 1)$-simplices $\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \in K$ such that $\sigma_i\epsilon^{j-1} = \sigma_j\epsilon^i$ for $i < j$ and $i \neq k \neq j$, there exists an n-simplex $\sigma \in K$ such that $\sigma\epsilon^i = \sigma_i$ for $i = 0, \ldots, \hat{k}, \ldots, n$.* Let $\mathcal{S}$ denote the category of c.s.s. complexes, and let $\mathcal{S}_E$ be its full subcategory generated by the c.s.s. complexes which satisfy the extension axiom.

It is the purpose of this note to indicate how the homotopy theory on the category $\mathcal{S}_E$ mentioned above may be extended to a homotopy theory on the whole category $\mathcal{S}$. This is done by defining a functor[3] Ex^∞: $\mathcal{S} \to \mathcal{S}_E$. All homotopy notions defined on the category $\mathcal{S}_E$ then apply by composition with the functor Ex^∞ also to the category $\mathcal{S}$.

Our main tool will be what we call the *extension* $\mathrm{Ex}\, K$ of a c.s.s. complex K, which is in a certain sense the dual of the (barycentric) subdivision of K.

2. *The Standard Simplices and Their Subdivision.*—We shall use the notation of Eilenberg-Zilber,[2] except that the *standard n-simplex* in the category $\mathcal{S}$ will be denoted by Δ_n instead of by $K[n]$. For each integer $n \geqslant 0$, we define a c.s.s. complex Δ_n' (the *subdivision* of Δ_n) and a c.s.s. map d_n: $\Delta_n' \to \Delta_n$ as follows. A q-simplex of Δ_n' is a sequence $(\sigma_0, \ldots, \sigma_q)$, where σ_i is a *nondegenerate* simplex of Δ_n and σ_i lies on[2] σ_{i+1} for all i. For each map β: $[p] \to [q]$, we define

$$(\sigma_0, \ldots, \sigma_q)\beta = (\sigma_{\beta(0)}, \ldots, \sigma_{\beta(p)}).$$

Let $(\sigma_0, \ldots, \sigma_q) \in \Delta_n'$. Then $d_n(\sigma_0, \ldots, \sigma_q)$ is the q-simplex of Δ_n, i.e., the map $[q] \to [n]$ determined by

$$(d_n(\sigma_0, \ldots, \sigma_q))i = \sigma_i(\dim \sigma_i) \qquad \text{for } 0 \leqslant i \leqslant q.$$

For each map α: $[m] \to [n]$, we define c.s.s. maps Δ_α: $\Delta_m \to \Delta_n$ and Δ_α': $\Delta_m' \to \Delta_n'$ as follows. For each $\sigma \in \Delta_m$, $\Delta_\alpha\sigma$ is the composite map $\alpha\sigma$. The map Δ_α' is the map induced by Δ_α (the *subdivision* of Δ_α).

2.1 *For each map α: $[m] \to [n]$, commutativity holds in the following diagram:*

$$\begin{tikzcd}
\Delta_m \arrow[r, "\Delta_\alpha"] & \Delta_n \\
\Delta_m' \arrow[u, "d_m"] \arrow[r, "\Delta_\alpha'"'] & \Delta_n' \arrow[u, "d_n"']
\end{tikzcd}$$

3. *The Extension.*—For a c.s.s. complex K, its *extension* is the c.s.s. complex $\mathrm{Ex}\, K$ defined as follows. An n-simplex of $\mathrm{Ex}\, K$ is a c.s.s. map σ: $\Delta_n' \to K$. For

each map α: $[m] \to [n]$, $\sigma\alpha$ is the composite map $\sigma\Delta_\alpha'$. Similarly, for a c.s.s. map f: $K \to L$, a c.s.s. map Ex f: Ex $K \to$ Ex L is defined by $(\text{Ex } f)\sigma = f\sigma$ for $\sigma \in K$. Thus the resulting functor Ex: $\mathcal{S} \to \mathcal{S}$ is covariant. By Ex^n we shall mean the functor Ex applied n times.

Let $K \in \mathcal{S}$. For each n-simplex $\sigma \in K$, let φ_σ: $\Delta_n \to K$ denote the c.s.s. map defined by $\varphi_\sigma\alpha = \sigma\alpha$ for each $\alpha \in \Delta_n$. We then define a monomorphism (i.e., isomorphism into) e_K: $K \to$ Ex K by $e_k\sigma = \varphi_\sigma d_n$ for $\sigma \in K$, dim $\sigma = n$. Clearly e_K is natural, i.e., $(\text{Ex } f)e_K = e_{\text{Ex } L} f$ for every c.s.s. map f: $K \to L$. We shall denote by $e_K{}^n$ the composite monomorphism $e_K{}^n$: $K \to \text{Ex}^n K$.

The functor Ex $\mathcal{S} \to \mathcal{S}$ has the following properties.

3.1. *The functor* Ex: $\mathcal{S} \to \mathcal{S}$ *maps homotopic maps into homotopic maps.*

3.2. *The map* e_K: $K \to$ Ex K *induces isomorphisms of the homology groups, i.e.,* e_{K*}: $H_*(K) \approx H_*(\text{Ex } K)$.

3.3. *If* $K \in \mathcal{S}_E$, *then* Ex $K \in \mathcal{S}_E$ *and* e_K: $K \to$ Ex K *is a homotopy equivalence.*

3.4. *For every pair of integers* (k, n) *with* $0 \leqslant k \leqslant n$ *and for every* n $(n - 1)$-*simplices* $\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \in$ Ex K *such that* $\sigma_i\epsilon^{j-1} = \sigma_j\epsilon^i$ *for* $i < j$ *and* $i \neq k \neq j$, *there exists an* n-*simplex* $\rho \in \text{Ex}^2 K$ *such that* $\rho\epsilon^i = e_{\text{Ex } K}{}^{\sigma_i}$ *for* $i = 0, \ldots, \hat{k}, \ldots, n$.

A c.s.s. map f: $K \to L$ is called a *fiber map* if for each pair of integers (k, n) with $0 \leqslant k \leqslant n$, for every n $(n - 1)$-simplices $\sigma_0, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \in K$ such that $\sigma_i\epsilon^{j-1} = \sigma_j\epsilon^i$ for $i < j$ and $i \neq k \neq j$, and for every n-simplex $\tau \in L$ such that $\tau\epsilon^i = f\sigma_i$ for $i = 0, \ldots, \hat{k}, \ldots, n$, there exists an n-simplex $\sigma \in K$ such that $f\sigma = \tau$ and $\sigma\epsilon^i = \sigma_i$ for $i = 0, \ldots, \hat{k}, \ldots, n$. The *fiber of* f *over a* 0-*simplex* $\varphi \in L$ is the counterimage of φ and its degeneracies. It is denoted by $F(f, \varphi)$. We now may state one more property of the functor Ex.

3.5. *If* f: $K \to L$ *is a fiber map and* φ *is a* 0-*simplex of* L, *then* Ex f: Ex $K \to$ Ex L *is a fiber map and* Ex $F(f, \varphi) = F(\text{Ex } f, e_L\varphi)$.

4. *The Functor* Ex^∞.—Consider the sequence

$$K \xrightarrow{e_K} \text{Ex } K \xrightarrow{e_{\text{Ex } K}} \text{Ex}^2 K \longrightarrow .$$

Let $\text{Ex}^\infty K$ be the direct limit of this sequence, and let Ex^∞ denote the resulting covariant functor. Similarly, let $e_K{}^\infty$: $K \to \text{Ex}^\infty K$ denote the (natural) limit monomorphism. The following properties of the functor Ex^∞ then follow from properties 3.1–3.5.

4.1. *The functor* Ex^∞ *maps homotopic maps into homotopic maps.*

4.2. *The map* $e_K{}^\infty$: $K \to \text{Ex}^\infty K$ *induces isomorphisms of the homology groups, i.e.,* $e_{K*}{}^\infty$: $H_*(K) \approx H_*(\text{Ex}^\infty K)$.

4.3. $\text{Ex}^\infty K \in \mathcal{S}_E$ *for all* $K \in \mathcal{S}$, *i.e.,* Ex^∞ *is a functor* Ex_∞: $\mathcal{S} \to \mathcal{S}_E$.

4.4. *If* $K \in \mathcal{S}_E$, *then* $e_K{}^\infty$: $K \to \text{Ex}^\infty K$ *is a homotopy equivalence.*

4.5. *If* f: $K \to L$ *is a fiber map and* φ *is a* 0-*simplex of* L, *then* $\text{Ex}^\infty f$: $\text{Ex}^\infty K \to \text{Ex}^\infty L$ *is a fiber map and* $\text{Ex}^\infty F(f, \varphi) = F(\text{Ex}^\infty f, e_L{}^\infty\varphi)$.

Combination of the naturality of e^∞ with property 4.4 yields the result that on the category $\mathcal{S}_E$ the homotopy notions induced by the functor Ex^∞ coincide with the original ones, i.e., the homotopy theory on the whole category $\mathcal{S}$ induced by the functor Ex^∞ is an *extension* of the original homotopy theory on the category $\mathcal{S}_E$.

5. *Geometrical Realization.*—By the *geometrical realization* $|K|$ of a c.s.s. complex K we mean the CW-complex of which the n-cells are in one-to-one correspondence with the *nondegenerate* n-simplices of K. For a c.s.s. map $f: K \to L$, let $|f|: |K| \to |L|$ denote the induced continuous map. Let S be the functor which assigns to every topological space X its simplicial singular complex SX. We now want to compare the complexes $\mathrm{Ex}^\infty K$ and $S|K|$.

5.1. $S|e_K|: S|K| \to S|\mathrm{Ex}\, K|$ *is a homotopy equivalence for all* $K \in \mathcal{S}$.

Let $jK: K \to S|K|$ denote the natural monomorphism defined by $(jK)\sigma = |\varphi_\sigma|$ for $\sigma \in K$. Then[4]

5.2. *If* $K \in \mathcal{S}_E$, *then* $jK: K \to S|K|$ *is a homotopy equivalence.*

Now consider the diagram

$$\begin{tikzcd}
K \arrow[r, "e_K"] \arrow[d, "jK"] & \mathrm{Ex}\, K \arrow[r, "e_{\mathrm{Ex}\, K}"] \arrow[d, "j\, \mathrm{Ex}\, K"] & \mathrm{Ex}^2 K \arrow[r] \arrow[d, "j\, \mathrm{Ex}^2 K"] & \dots \arrow[r] & \mathrm{Ex}^\infty K \arrow[d, "j\, \mathrm{Ex}^\infty K"] \\
S|K| \arrow[r, "S|e_K|"] & S|\mathrm{Ex}\, K| \arrow[r, "S|e_{\mathrm{Ex}\, K}|"] & S|\mathrm{Ex}^2 K| \arrow[r] & \dots \arrow[r] & S|\mathrm{Ex}^\infty K|
\end{tikzcd}$$

Then properties 4.3 and 5.2 imply that $j\, \mathrm{Ex}^\infty K$ is a homotopy equivalence, and it follows from property 5.1 that $S|e_K^\infty|$ is also a homotopy equivalence. Thus we have, for each $K \in \mathcal{S}$,

5.3. $S|K|$ *and* $\mathrm{Ex}^\infty K$ *have the same homotopy type.*

6. *Subdivision and Extension.*—It is possible to generalize the usual definition of the (barycentric) subdivision of simplicial complexes[5] to c.s.s. complexes. Let $\mathrm{Sd}\, K$ denote the subdivision of a c.s.s. complex K, and let d_K: $\mathrm{Sd}\, K \to K$ be the natural epimorphism (i.e., c.s.s. map onto) which for $K = \Delta_n$ is the map d_n: $\Delta_n' \to \Delta_n$. Write $\mathrm{Sd}^n K$ for the n-fold subdivision of K, and let d_K^n: $\mathrm{Sd}^n K \to K$ denote the corresponding composite epimorphism. The duality between subdivision and extension then may be illustrated by the following lemma.

6.1. *Let* K, $L \in \mathcal{S}$. *For every integer* $n > 0$, *there exists (in a natural way) a one-to-one correspondence between the c.s.s. maps* $\mathrm{Sd}^n K \to L$ *and the c.s.s. maps* $K \to \mathrm{Ex}^n L$.

A c.s.s. complex K is called *finite* if it has a finite number of nondegenerate simplices. It now follows from property 5.3 that

6.2. *If* K, $L \in \mathcal{S}$ *and* K *is finite, then for every continuous map* $f: |K| \to |L|$, *there exists an integer* $n > 0$ *and a c.s.s. map* $g: K \to \mathrm{Ex}^n L$ *such that* $|g| \simeq |e_L^n| f$.

Combining Lemma 6.2 with Lemma 6.1, we get the following version of the simplicial approximation theorem.

6.3. *If* K, $L \in \mathcal{S}$ *and* K *is finite, then for every continuous map* $f: |K| \to |L|$, *there exists an integer* $n > 0$ *and a c.s.s. map* h: $\mathrm{Sd}^n K \to L$ *such that* $|h| \simeq f|d_K^n|$.

[1] Cf. D. M. Kan, these PROCEEDINGS, **41**, 1092–1096, 1955.

[2] Cf. S. Eilenberg and J. A. Zilber, *Ann. Math.*, **51**, 499–513, 1950, and *Am. J. Math.*, **75**, 200–204, 1953.

[3] Another such functor was found by A. Heller.

[4] This result is due to J. Milnor.

[5] Cf. S. Lefschetz, *Introduction to Topology*, p. 112.