K(2)-local power operations in Lubin-Tate cohomology

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Contents

1 Introduction and statement of results.

In the article [Rav87] Ravenel compares homotopy theory with astronomy. On the one hand you observe the stars with telescopes and other instruments and the astronomers gain new insights by devising better tools for observation. On the other hand there are the homotopy groups of spheres. By inventing new methods of computation and instruments one gets a deeper understanding of the problem. Unfortunately it seems that knowing all homotopy groups of spheres is as impossibly complicated as knowing all stars in the universe.
Nevertheless, we have some methods to solve special cases. This allows us to progress deeper into the mathematical universe.

One part in the study of homotopy groups of spheres is the observation of permanent phenomenas in stable homotopy

$$\pi_k S \cong \lim_n \pi_{n+k} S^n.$$

This means that for some large enough n we are reaching a stable range in the homotopy groups of spheres. This result however will not spare us from the computation of homotopy groups. Hurewicz showed that for $k = 0$

$$\pi_0 S = \pi_n S^n \cong H_n S^n \cong \mathbb{Z}$$

a statement related to ordinary homology. This isomorphism does not hold for $k > 0$, and you need more precise instruments. Two of these are spectral sequences and generalized cohomology theories.

Novikov [Nov67] invented[1] a spectral sequence, the Adams-Novikov spectral sequence, that converges to $\pi_* S$.

$$E_2 = Ext_{MU_*MU}(MU_*, MU_*) \Longrightarrow \pi_* S$$

The main ingredient in this spectral sequence is the complex cobordism spectrum MU. This spectrum can also be seen as a universal object in the world of formal group laws. Quillen [Qui71] discovered that every formal group law is encoded by the canonical formal group law over the coefficient ring MU_*. Moreover we gain new (co)homology theories with formal group laws over Landweber exact rings E^* (see [Lan76]) by setting

$$E^*(X) := MU^*(X) \otimes_{MU_*} E^*.$$

[1] The perpetual question whether new knowledge in mathematics will be invented or discovered shall not discussed in this work.

Replacing complex cobordism in the Adams-Novikov spectral sequence with a cohomolgy theory constructed by Landweber exactness, it is possible to detect interesting families in the stable homotopy groups of the sphere spectrum. When using theories with higher multiplicative structure up to homotopy, so called H_∞-theories, we can compute some differentials in this spectral sequence via power operations. (see [BMMS86], Ch. 6). These operations are parametrized by the classifying space of the permutation group on n letters, Σ_n, and we get for a suitable H_∞-theory E, and an arbitrary space X a (total) power operation

$$P_n : E^*(X) \longrightarrow E^*(X \times B\Sigma_n) \cong E^*(X) \otimes_{E^*} E^*(B\Sigma_n).$$

Suitable means that $E^*B\Sigma_n$ is flat. For many spectra the power operations are well known. For the ordinary cohomology with coefficients in $\mathbb{Z}/2$ we have the Steenrod operations, for K-theory the Adams operations and for complex cobordism the tom Dieck-Steenrod operations – to name just a few. Cohomology theories of height n are of particular interest. Morava constructed $K(n)$, a generalized version of K-theory in the 1970s. These theories depend not only on the integer $n \geq 0$ but, if $n \geq 1$, also on a prime p. Furthermore they yield a filtration of the stable homotopy category by means of the Bousfield localization functor $L_n := L_{K(0)\vee\cdots\vee K(n)}$. Via the homotopy fiber product

$$L_nX \simeq L_{K(n)}X \times_{L_{n-1}L_{K(n)X}} L_{n-1}X$$

one can study a spectrum by understanding its monochromatic pieces. In this work we consider Lubin-Tate theories that are $K(n)$-local (see [HS99]) and therefore have height n.

In this thesis we consider the power operations of special Lubin-Tate cohomologies E_C, where the formal group law comes from an elliptic curve C. The only constraint for the choice of C is that it must be a lift from an elliptic curve C_0 over a finite field. The height of C_0 dictates whether E_C has height 1 or 2, since formal groups on elliptic curves admit only these two possibilities. These theories have all a higher multiplicative structure (see [GH04]) and the power operations have a geometric origin in the level structures (compare [And95, Rez09]) of the formal group.
In [Rez08] Charles Rezk presented some calculations of power operations for a Lubin-Tate theory related to an elliptic curve of the Weierstraß form $y^2 + a_1xy + y = x^3$, but he computed only the first few terms of these operations. In fact there is a series expansion, which is calculated in this thesis, for a more general elliptic curve.
If C has the form $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x$ then we have the following theorem for the associated Lubin-Tate theory E_C.

Theorem. *For the Lubin-Tate cohomology E associated to the elliptic curve C, the total power operation*

$$P : E^0(X) \longrightarrow E^0(X) \otimes_{E^*} E^0 B\Sigma_2/(\text{transfer})$$

on the complex orientation $u \in E^2\mathbb{C}P^\infty$ is given by

$$P(u) = -du + (d^2K + a_3) \sum_{k \geq 1} (-1)^{k+1} \frac{K^{k-1}d^k}{a_3^k} u^{k+1}$$

where $K := (a_1^2 a_4 - a_1 a_2 a_3 + a_3^2)d + (2a_1 a_4 - a_2 a_3)$.

In the formula above, the variable d is a generator of the E_C-cohomology of $B\Sigma_2$.

The only two requirements on the elliptic curve C are $a_6 = 0$ and C must be a lift of a height 2 elliptic curve. Knowing this we can immediately read off the total power operation from the coefficients of this elliptic curve. Moreover this improved formula gives us power operations in every degree. By expanding the operation one obtains relative operations Q_0, Q_1 and Q_2 that give a more detailed structure of the total operation P via

$$P(x) = Q_0(x) + Q_1(x)d + Q_2(x)d^2$$

Adem relations for these relative operations will also be computed.

This thesis in divided into five chapters.

The second chapter introduces the language of E_∞ and H_∞-spectra, the theoretical background for power operations. Some examples of well known operations are presented together with a short overview of the work of Strickland and Rezk, which is necessary for the calculations in Lubin-Tate theory.

The third chapter contains computations on the elliptic curve and for the Lubin isogeny, with special emphasis on the case where the coefficient a_6 in the Weierstraß equation is zero. There will also be a description for the computations on other elliptic curves.

In chapter 4 we compute the total and the relative $K(2)$-local power operations. In the last part of this chapter the Adem relations for the relative operations Q_i will be computed.

There a two examples in the last chapter.

In the first appendix one can find some computer programs to calculate power operations for Lubin-Tate theories linked to elliptic curves (again with $a_6 = 0$). For this one needs the programs Maple 17 and Macaulay 2 to do the calculations. Apart from the computational point of view it describes how to get the operations algorithmically. The second appendix contains for reference the first terms of the series expansions, fully reduced.

* * *

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2 Essentials

This section gives a short introduction into the world of E_∞-spectra, H_∞-spectra, power operations, Lubin-Tate cohomology, Frobenius deformations and elliptic curves. Furthermore we describe how these concepts interact.The main source for power operations is [BMMS86]. The other sources that are used for definitions and theorems are stated below in the corresponding point.

In the following we denote by $\mathcal{S}$ the Lewis-May-Steinberger category of coordinate free spectra, and the stable homotopy category by h$\mathcal{S}$. Definitions for these terms can be find in [LMSM86].

2.1 E_∞-spectra

A spectrum in $\mathcal{S}$ is indexed by finite dimensional subspaces of some countable inner product space $\mathcal{U}$. The space of linear isometries $\mathcal{L}(\mathcal{U}^n, \mathcal{U})$ with the usual topology is a free contractible Σ_n-space which we denote by $E\Sigma_n$. We define an endofunctor on $\mathcal{S}$ by setting $D_{\Sigma_n}X = E\Sigma_n \wedge_{\Sigma_n} X^{\wedge n}$. Here Σ_n acts on the second factor by permuting the factors.

Remark 2.1.1. We have an adequate functor if we work with unbased spaces U or based spaces B via

$$D_{\Sigma_n}U = E\Sigma_n \times_{\Sigma_n} U^{\times n} \qquad \text{resp.} \qquad D_{\Sigma_n}B = E\Sigma_{n+} \wedge_{\Sigma_n} B^{\wedge n}.$$

For an unbased space U we have a homeomorphism $(D_{\Sigma_n}U)_+ \cong D_{\Sigma_n}(U_+)$ Furthermore the endofunctor D_{Σ_n} commutes with the suspension functor Σ^∞_+ in the following way:

$$D_{\Sigma_n}\Sigma^\infty_+ U \cong \Sigma^\infty D_{\Sigma_n}(U_+) \cong \Sigma^\infty (D_{\Sigma_n U})_+ = \Sigma^\infty_+ D_{\Sigma_n}U.$$

Remark 2.1.2. The functor D_{Σ_n} is multiplicative but not additive. With [BMMS86] II.1.1 we get.

$$D_{\Sigma_n}(X \vee Y) = \bigvee_{i+j=n} (D_{\Sigma_i}X \wedge D_{\Sigma_j}Y)$$

Definition 2.1.3. One defines an endofunctor D on $\mathcal{S}$ with

$$DX = \bigvee_{n \geq 0} D_{\Sigma_n}X$$

Proposition 2.1.4. *[Rez98] There are natural transformations*

$$\mu : D^2 \longrightarrow D$$
$$\eta : id \longrightarrow D$$

that make D a monad on $\mathcal{S}$.

Remark 2.1.5. (see [ML98]) Let (T, μ, η) be a monad over a category $\mathcal{C}$. A T-algebra (x, h) consists of an object x of $\mathcal{C}$ and an arrow $h : Tx \to x$ of $\mathcal{C}$ such that the following diagrams commute:

$$\begin{tikzcd} T^2x \arrow[r, "Th"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "h"] \\ Tx \arrow[r, "h"'] & x \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\eta"] \arrow[dr, "id_x"'] & Tx \arrow[d, "h"] \\ & x \end{tikzcd}$$

This monad allows us to define a category which we refer as category of E_∞ ring spectra:

Definition 2.1.6. The category of E_∞ ring spectra is the category of D-algebras in $\mathcal{S}$.

Proposition 2.1.7. *[Noe09b] The monad D on $\mathcal{S}$ descends to a monad $\tilde{D}$ on $h\mathcal{S}$.*

Definition 2.1.8. The category of H_∞ ring spectra is the category of $\tilde{D}$-algebras in h$\mathcal{S}$.

Remark 2.1.9. Roughly speaking, the property of a spectrum to be an E_∞-spectrum or a H_∞-spectrum only differs by the category. Moreover one can show, that if we have an E_∞-spectrum we get an H_∞-structure in the homotopy category by the localization map. This is done by Justin Noel in [Noe09b]. The converse statement is wrong and were proofed again by Justin Noel in[Noe09a])

Note. In the following we will write D_n instead of D_{Σ_n}.

2.2 Power operations.

Now we are able to define power operations.

Definition 2.2.1. (compare with [BMMS86], I.4.1.) Let E be an H_∞-ring spectrum. For an abitrary spectrum X and an element $f \in E^0X$ we define $\tilde{P}_n(f) \in E^0(D_nX)$ as the composite

$$D_nX \xrightarrow{D_nf} D_nE \longrightarrow DE \longrightarrow E$$

The second map is the inclusion and the third map is the structure map which comes from the H_∞-structure.

Remark 2.2.2. The diagonal map $\Delta : X \to X^n$ induces a map

$$B\Sigma_{n_+} \wedge X \simeq E\Sigma_n \wedge_{\Sigma_n} X \xrightarrow{1\wedge\Delta} E\Sigma_n \wedge_{\Sigma_n} X^n = D_n X$$

at which for the component $E\Sigma_n \wedge_{\Sigma_n} X$ the group Σ acts trivially on X. From this we get a map induced in cohomology

$$E^0 X \xrightarrow{\tilde{P_n}} E^0 D_n X \xrightarrow{\Delta^*} E^0(X \wedge B\Sigma_n).$$

We denote the composite with P_n. The map P_n will be one of the main objects that we study for special cohomology theories. But this is done later.

We want to understand the power operations by asking the question: What is a power operation in general? If we take an arbitrary space X, the group Σ_n acts on the product X^n. For a multiplicative cohomology theory E, in the E_∞ sense, we construct with the methods above a total power operation P_n. The n-th power map is induced by the multiplicative structure of the cohomology theory:

$$E^* X \longrightarrow E^{n*} X, x \mapsto x^{\times n}.$$

The map P_n factors the n-th power map using a equivariant cohomology theory E_{Σ_n}. If we do not have a good candidate for such a theory it is often enough to take $E^*(D_n X)$, i.e. if

$$i^* : E^*_{\Sigma_n} X^n \longrightarrow E^* X^n$$

is the forgetful map, we want to find a filling for the diagram

$$\begin{tikzcd} E^* X \arrow[r, "x^{\times n}"] \arrow[dr, "P_n"'] & E^{n*}_{\Sigma_n} X^n \\ & E^*_{\Sigma_n} X^n \arrow[u, dotted] \end{tikzcd}$$

that would be arise if such an theory $E^*_{\Sigma_n}$ exists. One possible filling for the diagram is $E^*(D_n X)$ instead of $E^*_{\Sigma_n} X^n$.

Remark 2.2.3. The power operations that are defined in 2.2.1 are multiplicative but not additive. Indeed the sum decomposes by Lemma 2.1 in [BMMS86] p. 29 into

$$P_j(x + y) = P_j(x) + P_j(y) + Tr^*(xy).$$

If we want additive operation we have to divide out the ideal I generated by the transfer maps. Let $0 \leq j \leq m$. For Σ_m we have the inclusion $\Sigma_j \times \Sigma_{m-j} \to \Sigma_m$. With this we get a transfer map in cohomology:

$$E^0(B(\Sigma_j \times \Sigma_{m-j})) \overset{\text{transfer}}{\longrightarrow} E^0(B\Sigma_m).$$

The ideal I is now defined as the sum of the images of all possible transfer maps

$$I = \sum_{0 \leq m \leq j} \text{Image}(E^0(B(\Sigma_j \times \Sigma_{m-j})) \to E^0(B\Sigma_m))$$

We get by composition with a projection map

$$E^0 X \xrightarrow{P_n} E^0(X \times B\Sigma_m) \longrightarrow E^0(X \times B\Sigma_m)/I.$$

an additive operation that will be also denoted by P_n.

Before we give some examples of power operations we have a look at a special kind of cohomology theories. This sort of theories play an central role in the rest of this work. For this, let $S^2 \cong \mathbb{C}P^1 \longrightarrow \mathbb{C}P^\infty$ the trivial embedding. Then we define a complex orientable cohomology theory in the following way.

Definition 2.2.4. If E^* is a multipicative cohomology theory such that the map

$$E^2\mathbb{C}P^\infty \to E^2S^2$$

is surjective, then we call this cohomology theory complex orientable. A choice $u \in E^2\mathbb{C}P^\infty$ of the preimage of the generator in E^2S^2 is called a complex orientation.

Proposition 2.2.5. ([Ada74] part 2, Lemma 2.5)
If E is a complex oriented cohomology theory with orientation $u \in E^2\mathbb{C}P^\infty$, then we have an isomorphism

$$E^*\mathbb{C}P^\infty \cong E^*[[u]]$$

The multiplicative structure on $\mathbb{C}P^\infty$ gives an interesting structure on complex oriented cohomology theories. Given a line bundle $L \to X$. A line bundle is classified up to homotopy by a map $f : X \to \mathbb{C}P^\infty$. For the universal line bundle we consider the multiplication map m, we mean multiplication by $\otimes$,

$$m : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \longrightarrow \mathbb{C}P^\infty.$$

Considering the map in E-cohomology for an complex oriented cohomology theory yields

$$m^* : E^*\mathbb{C}P^\infty \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*\mathbb{C}P^\infty \otimes_{E^*} E^*\mathbb{C}P^\infty \cong E^*[[u_1, u_2]].$$

The image of u is a power series in two variables u_1 and u_2. By setting $c_1(L) := f^*(u)$, and similar for L_1 and L_2 , this are two complex line bundles over the same base, we see, that $c_1(L_1 \otimes L_2)$ is a power series in two variables $F(u_1, u_2) \in E^*[[u_1, u_2]]$. Independent of the choice of the complex oriented theory E this power series satisfies the conditions of a formal group law.

Definition 2.2.6. A power series $F(x, y)$ in two variables, written as $x +_F y$, over a ring R with the properties

- $(x +_F y) +_F z = x +_F (y +_F z)$ (associativity)
- $x +_F y = y +_F x$ (commutativity)
- $x +_F 0 = x$ (unitarity)

is called formal group law, more precisely a one-dimensional commutative formal group law.

Remark 2.2.7. We write $[m](x)$ for the m-fold formal sum instead of adding m-times:

$$x +_F x +_F \cdots +_F x.$$

Definition 2.2.8. A homomorphism $f : F \to G$ between to formal group laws F and G over a ring R is a power series $f(x) \in R[[x]]$ such that $f(0) = 0$ and $f(x +_F y) = f(x) +_G f(y)$.

Proposition 2.2.9. *If $f : F \to G$ is a homomorphism of formal group laws over a field k with $char(k) = p > 0$, and if $f(x) \neq 0$, then*

$$f(x) = g(x^{p^n})$$

for some $n \geq 0$ and some $g(x) \in k[[x]]$ with $g(0) = 0$ and $g'(0) \neq 0$.

Proof. (see [Rav86], Lemma A2.2.6.) *q.e.d.*

Definition 2.2.10. Let F be a formal group law over a field k with characteristic $p > 0$. The formal group law has height n if $[p]_F(x) = g(x^{p^n})$ for some $n \geq 0$ and some $g(x) \in k[[x]]$ with $g(0) = 0$ and $g'(0) \neq 0$.

Example 2.2.11. (The additive formal group law F_a)
In ordinary cohomology with coefficients in $\mathbb{Z}$ we have the additive formal group law

$$x +_{F_a} y = x + y.$$

Example 2.2.12. (The multiplicative formal group law F_m)
In K-theory we have the so called multiplicative formal group law

$$x +_{F_m} y = x + y - xy.$$

Example 2.2.13. (Honda formal group law)
Let k ba a field of characteristic $p > 0$. For a natural number n we define the n-th Honda formal group law H_n by

$$[p]_{H_n}(x) = x^{p^n}$$

Over a separably closed field a formal group law is characterized up to isomorphism by its height. Hence for a separably closed field we have always an isomorphism to the Honda formal group law.

In order to deal with power operations one should know something about the cohomology of finite groups. With our notations earlier in this chapter we are able to give a concise description of the E-cohomology of cyclic groups of order m.

Proposition 2.2.14. *Let E be an complex oriented cohomology theory and C_m the cyclic group of order m. Then*

$$E^*(BC_m) \cong E_*[[x]]/[m](x)$$

where x is the Euler class of the standard generator of the complex character ring of C_m.

Proof. We apply the Gysin sequence to the fibration $S^1 \to BC_m \to BS^1$, where $BS^1 \cong \mathbb{C}P^\infty$, we obtain a long exact sequence

$$\cdots \to E^n(BC_m) \to E^{n-1}(BS^1) \overset{\cdot[m](x)}{\to} E^{n+1}(BS^1) \to E^{n+1}(BC_m) \to \ldots$$

q.e.d.

Now we are well prepared to present some examples.

Example 2.2.15. (Power operations for the K-theory spectrum K.)
Let G be a finite group. Atiyah gives in the article [Ati61] a description how to deal with the classifying space BG in K-theory. More precisely he identifies the K-theory of BG_+ with the complex representation ring $RU(G)$. There is an isomorphism after completing the representation ring at its augmentation ideal I. For this example we need only $G = \Sigma_2$, which yields

$$K^0(B\Sigma_2) \cong RU(\Sigma_2)^\wedge_I.$$

With the trivial representation 1 and the sign representation s we get an isomorphism of rings $RU(\Sigma_2)^\wedge_I \cong \mathbb{Z}_2[s]/(s^2-1)$. The image of the transfer ideal is generated by $1-s$. Thus the total power operation on K-Theory is defined by

$$\begin{array}{ccc} K(X) & \longrightarrow & K(X) \otimes K(B\Sigma_2)/(\text{transfer}) \\ x & \mapsto & \sigma^2(x) - \lambda^2(x) =: \psi^2(x) \end{array}$$

Here σ^2 is the symmetric power and λ^2 is the exterior power which gives the second Adams operation ψ^2.

The K-theory example shows that it is not sufficient to know which algebraic structure is encoded in $E^*(B\Sigma_n)$. We also need some geometric information to make the the total power operation comprehensible.

Example 2.2.16. (Morava K-theory) The Morava K-theory spectrum $K(n)$ for $p=2$ is an example or counterexample depending on the height n. For $n=0$ we can identify $K(0)$ with rational cohomology $H\mathbb{Q}$. For $n=1$ we get p completed K-theory K_p and hence power operations from K-theory. For $n \geq 2$ there is no good geometric identification known, but there are non-commutative multiplications on this spectrum in two ways (see [Wür91] Theorem 1.3.).

2.3 Lubin-Tate cohomology

Now we focus our sight onto a special cohomology theory, the Lubin-Tate theory. This theory has all the useful properties, that are defined earlier in the text, i.e. it is complex oriented and E_∞. Our ambition are the power operations on a special species of this theory, where the formal group law comes from an elliptic curve. But we treat this later. At first we give the general constructions and the techniques that are important for us.

Let k be a field of characteristic $p>0$ and R a local ring with maximal ideal $\mathfrak{m}$ such that the map $R/\mathfrak{m} \xrightarrow{i} k$ is an isomorphism. Further let F be a formal group law over k of finite height n.

Definition 2.3.1. An deformation of (F,i) is a pair consisting of a formal group law G over R and an isomorphism i such that the formal group laws F and G are isomorphic on the quotient field via the isomorphism i resp. the canonical projection $\mathrm{proj} : R \to R/\mathfrak{m}$:

$$i^*F \cong \mathrm{proj}_* G.$$

We define a morphism of deformations $(G_1, i_1) \to (G_2, i_2)$ as morphism of formal group laws if $i_1 = i_2$. The isomorphim $f : G_1 \to G_2$ of formal group laws over R with the property

$$f(x) \equiv x \bmod \mathfrak{m}.$$

is called a $*$-isomorphism.

Now, we want to describe very briefly how to construct Lubin-Tate cohomology. For an more precise construction the first few chapters of the article [Rez98] written by Charles Rezk is a good source.
Let Lift(A) denote the set of $*$-isomorphism classes of lifts of the Honda formal group law H_n of height n to a $\mathbb{Z}_p$-algebra A. This is a functor from the category of $\mathbb{Z}_p$-algebras to the category of sets. Lubin and Tate [LT66] have shown that if we have the three following ingrediens, first a $\mathbb{Z}_p$-algebra A, second a formal group F on the polynomal ring over the Witt vectors $\mathbb{W}_{\mathbb{F}_p}[u_1, \dots u_{n-1}]$ and last an isomorphism class $\Gamma \in$Lift(A), then there is a unique homomorphism of local rings

$$\mathbb{W}_{\mathbb{F}_p}[u_1, \dots u_{n-1}] \xrightarrow{\alpha} A$$

such that $\alpha_* F \in \Gamma$. Moreover for each formal group G in the $*$-isomorphism class Γ a $*$-isomorphism

$$G \longrightarrow \alpha_* F$$

uniquely determined by F and G. In other words, the functor $A \mapsto$Lift(A) is represented by the ring $E_n := \mathbb{W}_{\mathbb{F}_p}[u_1, \dots u_{n-1}]$. With the Landweber exact functor theorem one can construct a cohomology theory

$$E_n^*(X) := E_n \otimes_{MU^*} MU^*(X).$$

Remark 2.3.2. By construction the cohomology theory E_n^* is complex orientable.

Remark 2.3.3. We can think of the Lubin-Tate cohomology E_n as classifying the deformations of formal group laws of height n.

Remark 2.3.4. Goerss and Hopkins have shown that the underlying spectrum of Lubin-Tate cohomology, which also denoted by E_n, is an E_∞-spectrum, thus we have power operations (see [GH04] Ch. 7). Furthermore, E_n is $K(n)$-local (see [HS99] Lemma 5.2).

2.4 Deformation of the Frobenius

In this subsection we write E instead of E_n. A more algebraic point of view is the construction of power operations as special deformations. These deformations coming from the Frobenius homomorphism $x \mapsto x^p$, that is indeed a homomorphism for every field k of characteristic p. We need this construction to get a more geometrical approach to the power operations in Lubin-Tate cohomology, since we ascertained in Example 2.2.15 that it is not enough to know the structure of the cohomology over the permutation group in n letters.
Let $\varphi : k \longrightarrow k$ with $\varphi(x) = x^p$ be the Frobenius homomorphism and Frob a map defined by Frob : $F \longrightarrow \varphi^* F$, the relative Frobenius.

Definition 2.4.1. Let (G, i) and (G', i') deformations over R of the formal group F. A morphism $(G, i) \longrightarrow (G', i')$ is called deformation of Frobenius such that the diagrams

$$\begin{tikzcd} \pi_* G \arrow[r, "\pi^*(f)"] \arrow[d, "\cong"'] & \pi_* G' \arrow[d, "\cong"] \\ i^* F \arrow[r, "i^*(Frob^r)"'] & i'^* F \end{tikzcd} \qquad \begin{tikzcd} k \arrow[r, "i'"] \arrow[d, "\varphi^r"'] & R/\mathfrak{m} \\ k \arrow[ur, "i"] & \end{tikzcd}$$

commute for some $r \geq 0$.

Remark 2.4.2. Deformations of Frobenius with domain (G, i) coincide with finite subgroup schemes of G, where $f \mapsto Ker(f) \subset G$.

We denote by Spf$(-)$ a formal spectrum. Let $G_E :=$ Spf$(E^* \mathbb{C}P^\infty)$ over $\pi_0 E$ (see [Str99] for the definitions). Now we have a look at the ring homomorphisms

$$s^* : E_0 \longrightarrow \pi_0 E \otimes E^0 B\Sigma_m / I$$
$$t^* : E_0 \longrightarrow \pi_0 E \otimes E^0 B\Sigma_m / I$$

where the first one induced by $B\Sigma_m \to *$ and the second one by a power operation $E^0(*) \to \pi_0 \otimes E_0 B\Sigma_m / I$ which produces out of

$$E^0(X) \stackrel{(tr \circ P_m)}{\longrightarrow} E^0(X) \otimes E^0(B\Sigma_m)/I$$

a homomorphism of formal groups

$$t^* G_E \stackrel{(tr \circ P_m)^*}{\longleftarrow} s^* G_E.$$

This map is a deformation of the Frobenius [Rez06] p. 62.

The following theorem of Strickland sheds some light on the situation:

Theorem 2.4.3. *[Str98] The data $(s^*G_E, ker(tr \circ P_{p^r}))$ over $\pi_0 \otimes E_0 B\Sigma_{p^r}/I$ is the universal example of a pair (G, H) consisting of a deformation G of F and a finite subgroup scheme $H \subset G$ of rank p^r.*

The translation into the setting of formal groups says, that if we want to study power operations in the Lubin-Tate cohomology we have to examine finite subgroups of formal groups. These finite subgroups turn up as kernels of maps between formal groups.

2.5 Elliptic curves

The last ingredient that we need are elliptic curves, more precise elliptic curves in Weierstraß form. An elliptic curve is a special curve of genus 1 with a distinguished base point O. Let $R = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. We can write every elliptic curve as a locus in the projective space $\mathbb{P}^2(R)$ of a cubic equation with only one point on the line ∞. The equation has the form

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$$

We shall simultaneously use non-homogenous coordinates and substitute $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$. This simplify the notation to

$$C: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

More general we only need the following definitions.

Definition 2.5.1. Let R be a ring and $S := R[a_1, a_2, a_3, a_4, a_6]$. A Weiestraß equation over S is an equation of the form

$$C: \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2.1}$$

Definition 2.5.2. An elliptic curve over a ring R is a marked curve over spec(R) which locally is isomorphic to a Weierstraß curve.

Furthermore there is an addition defined on a elliptic curve. Denote the third intersection point between the line through P and Q and the elliptic curve with $P * Q$. Silverman and Tate ([ST92] p. 18f.) give this short and concise description of the operation on the elliptic curve:

> To add P and Q, take the third intersection point $P * Q$, join it to O, and then take the third intersection point $P + Q$. Thus by definition $P + Q = O * (P * Q)$.

This operation on elliptic curves is associative, we have a neutral element O, and for every point P on the curve we get with the line through the origin an inverse element $-P$. Thus every elliptic curve has an underlying formal group.
The attentive reader may ask, why does a third point always exists even if we do not work in an algebraically closed field? Let there two points on the elliptic curve. There is a line through the two points. The intersections between the curve and the line is given by an equation of degree three with coefficients in the base ring. By Vieta's theorem there is a coefficient in the equation that is the sum of all intersection points and therefore every intersection point lies in the base ring.
Moreover after changing coordinates in a such way that the origin O matches with the point $(0,0)$, Taylor expansion around the origin gives a (one dimensional) formal group law. (see [Sil09] p. 110 f.) We denote this procedure as completion of the elliptic curve. By substituting $u = \frac{x}{y}$ and $v = \frac{1}{y}$ the equation of the curve becomes

$$C : v + a_1uv + a_3v^2 = u^3 + a_2u^2v + a_4uv^2 + a_6v^3. \tag{2.2}$$

with the origin at $(0,0)$. Unfortunately we lose the Weierstraß form after applying this transformation.

Nowadays every mathematical object has its own kind of morphism, and there is no exception for elliptic curves.

Definition 2.5.3. Let (C,O) and (C',O') be elliptic curves with origin O resp. O'. An isogeny ϕ between C and C' is a map

$$\phi : C \longrightarrow C'$$

with $\phi(O) = O'$ and $\psi(C) \neq \{O'\}$.

Remark 2.5.4. It may be a bit surprising that the definition above says nothing about how the formal group structure behaves under an isogeny. This is however not necessary because for every isogeny we have $\psi(P +_C Q) = \psi(P) +_{C'} \psi(Q)$ (see [Sil09] p. 75 Theorem 4.8)

Example 2.5.5. Let Q a point of order 2, this means $Q +_C Q = O$. Let $\phi(P) = P - Q$ a translation by Q map, such that every point will be shifted with $-Q$. We define in coordinates an isogeny $\Psi : C \longrightarrow C'$ with $u(\Psi(P)) = u(P)u(\phi(P))$ and $v(\Psi(P)) = v(P)v(\phi(P))$. We have $\Psi(O) = O$ hence Ψ is an isogeny, which we call Lubin-isogeny.

We need some special quantities to simplify the equations in the following definitions. Let

$$\begin{aligned}
b_2 &:= a_1^2 + 4a_2 \\
b_4 &:= 2a_4 + a_1a_3 \\
b_6 &:= a_3^2 + 4a_6 \\
c_4 &:= b_2^2 - 24b_4 \\
\Delta &:= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6 \\
j &:= \frac{c_4^3}{\Delta}
\end{aligned}$$

Definition 2.5.6. We call Δ the discriminant and j the j-invariant.

Definition 2.5.7. An elliptic curve is called singular if $\Delta = 0$.

Remark 2.5.8. The discriminant gives some information about the smoothness of an elliptic curve. If an elliptic curve is not singular, its graph has neither a node nor a cusp.

As mentioned above, for every elliptic curve we get by coordinate change and completion a formal group law.

Definition 2.5.9. An elliptic curve has height n if the underlying formal group law has height n.

Remark 2.5.10. It follows from [Sil09] p. 137 Theorem 3.1. that an elliptic curve can only have height 1 or 2. If it has height 2 we call the elliptic curve supersingular which should not mixed up with the notion of singular.

Proposition 2.5.11. *(see [Sil09] p. 145 5.7) The only super singular elliptic curve over a field of characteristic* 2 *is the curve with* j*-invariant* 0.

Thus from [KM85] section 2.9 that the deformation theory for elliptic curves works in a similar way as for formal groups. Therefore we also can construct in this way a Lubin-Tate theory for elliptic curves. Thus if we consider a supersingluar elliptic curve and its deformations we obtain power operations in the Lubin-Tate theory with Theorem 2.4.3.

Let R be a local ring with quotient field $\mathbb{F}_2$. Let C_0 an elliptic curve over $\mathbb{F}_2$. After choosing a lift C over $R$the the formal completion of C at the identity gives a formal group law $\widehat{C}$. This gives a Lubin-Tate theory E_C with formal group law $\widehat{C}$. The connection to the usual Lubin-Tate theory lies in the choice of the coordinate. More precisely: If we take a p-typical coordinate u we get an isomorphism

$$E_C^*(\mathbb{C}P^\infty) \cong E_C^*(pt)[[u]]$$

In the next chapter we shall construct an isogeny, more precise a Lubin isogeny, whose the kernel contains all points of order 2. This is always possible (compare [Sil09] Proposition III.4.12.).

3 Calculations on elliptic curves.

With the last chapter we have enough tools to start with our calculations. The aim of this section a closed formula for the Lubin-isogeny on a elliptic curve. Unfortunately we can not compute such formula for an elliptic curve with the most general Weierstraß from. We have to drop the coefficient a_6 by the constraint $a_6 = 0$. Some calculations do not need this setting but when it is necessary there will be a hint. There also appear some fractions during the calculations because we are working in an affine chart of the elliptic curve. This does not matter because we can revert the substitution into a projective context, where no denominators emerge.
In this section we have the following setting. Let $R = \mathbb{Z}[a_1, a_2, a_3, a_4]$ and let C be an elliptic curve of the form

$$C : v + a_1uv + a_3v^2 = u^3 + a_2u^2v + a_4uv^2 + a_6v^3. \tag{3.1}$$

with $(0, 0)$ as the identity of the elliptic curve. We expand the ground ring to $S = R[\Delta^{-1}]$. This ensures that the curve is smooth.

3.1 Group structure on elliptic curves.

For our calculations we need the group law on an elliptic curve. In order to make things clearer we give explicit formulas. There are many sources where one can find formulas for formal group laws like [Sil09] or [Hus04], but they are not suitable for our purposes.
Let P_1 and P_2 be two point on C. The first step is to compute the group law $P_1 +_C P_2$. Let $L : v = \lambda u + m$ be the line through P_1 and P_2 or the tangent line if $P_1 = P_2$. Substituting the line into the equation of the elliptic curve, we get three roots u_1, u_2, u_3 where $v_i = \lambda u_i + b$ and $P_i = (u_i, v_i)$. The point P_3 has the property that $P_1 +_C P_2 +_C P_3 = O$ or equivalently $P_1 +_C P_2 = -_C P_3$. Comparing the coefficients of the factorized polynomial of degree 3 that has the roots u_1, u_2 and u_3:

$$c(u-u_1)(u-u_2)(u-u_3) = c(u^3+(-u_1-u_2-u_3)u^2+(u_1u_2+u_1u_3+u_2u_3)u-u_1u_2u_3)$$

with the polynomial that arise by inserting $v = \lambda u + m$ into the equation of the elliptic curve

$$\begin{aligned} u^3 \quad &+ \quad \frac{-3\lambda^2 ma_6 + \lambda^2 a_3 - 2\lambda ma_4 + \lambda a_1 - ma_2}{-\lambda^3 a_6 - \lambda^2 a_4 - \lambda a_2 - 1} u^2 \\ &+ \quad \frac{-3\lambda m^2 a_6 + 2\lambda ma_3 - m^2 a_4 + ma_1 + \lambda}{-\lambda^3 a_6 - \lambda^2 a_4 - \lambda a_2 - 1} u \quad + \quad \frac{-m^3 a_6 + m^2 a_3 + m}{-\lambda^3 a_6 - \lambda^2 a_4 - \lambda a_2 - 1} \end{aligned} \tag{3.2}$$

leads to (with $c=-1$). In the following proposition we deduce the coordinates (u_3, v_3) of the third point P_3.

Proposition 3.1.1. *Let* $P_1 = (u_1, v_1)$*,* $P_2 = (u_2, v_2)$ *and* $-(P_1+P_2) = (u_3, v_3)$*, then*

$$\begin{aligned} u_3 &= \frac{(-3ma_6+a_3)\lambda^2+(-2ma_4+a_1)\lambda-ma_2}{-\lambda^3a_6-\lambda^2a_4-\lambda a_2-1}-u_1-u_2 \\ v_3 &= \lambda u_3+m \end{aligned}$$

with

$$\begin{aligned} \lambda &= \frac{v_2-v_1}{u_2-u_1} \quad u_1 \neq u_2 \\ m &= \frac{v_1u_2-v_2u_1}{u_2-u_1} \quad u_1 \neq u_2. \end{aligned}$$

The computation of the inversion formulas is as follows. Take equation 3.2 with $P_2 = O$. The roots of this equation are the intersection of the curve and a line through the origin and the point $P=(u,v)$. The third root gives $-P$.

Proposition 3.1.2. *Let* $P=(u,v)$ *be a point on the curve* C *with origin* $(0,0)$*. The inverse regarding the formal group law in the curve* C *is given by*

$$-P = \left(-\frac{uv}{u^3+a_2u^2v+uv^2a_4+v^3a_6}, -\frac{v^2}{u^3+a_2u^2v+uv^2a_4+v^3a_6}\right)$$

Proof. We take equation 3.1 with $v=\lambda u$ (line through the origin) with $\lambda = \frac{v}{u}$ and solve it for u. The solutions are 0 and

$$\pm\frac{1}{2}\frac{\lambda^2a_3+\lambda\,a_1+\sqrt{\lambda^4{a_3}^2+4\,\lambda^4a_6+2\,\lambda^3a_1a_3+4\,\lambda^3a_4+\lambda^2{a_1}^2+4\,\lambda^2a_2+4\,\lambda}}{\lambda^3a_6+\lambda^2a_4+\lambda\,a_2+1}.$$

Substituting $\frac{1}{v}$ by $\frac{1}{2}\left(\frac{1}{v}-a_1\frac{u}{v}-a_3\right)$ (compare [Sil09] p. 46) we get

$$\frac{1}{2}\frac{\left(\lambda^2a_3+\lambda\,a_1-\lambda^2\left(\frac{2}{v}+\frac{a_1u}{v}+a_3\right)\right)}{(\lambda^3a_6+\lambda^2a_4+\lambda\,a_2+1)}$$

and hence

$$-\frac{uv}{u^3+a_2u^2v+a_4uv^2+a_6v^3}$$

for the first coordinate. The second coordinate follows from the equation of the line. *q.e.d.*

Now we get a closed form for the addition:

Corollary 3.1.3. Let $P_1 = (u_1, v_1)$ and $P_2 = (u_2, v_2)$. The addition is given by

$$P_1 + P_2 = \left(-\frac{u_3 v_3}{u_3^3 + a_2 u_3^2 v_3 + a_4 u_3 v_3^2 + a_6 v_3^3}, -\frac{v_3^2}{u_3^3 + a_2 u_3^2 v_3 + a_4 u_3 v_3^2 + a_6 v_3^3} \right)$$

with

$$\begin{aligned} u_3 &= \frac{-3a_6 m\lambda^2 + a_3\lambda^2 - (2ma_4 - a_1)\lambda - ma_2}{a_6\lambda^3 + a_4\lambda^2 + a_2\lambda + 1} - u_1 - u_2 \\ v_3 &= \lambda u_3 + m \\ \lambda &= \frac{v_2 - v_1}{u_2 - u_1} \quad u_1 \neq u_2 \\ m &= \frac{v_1 u_2 - v_2 u_1}{u_2 - u_1} \quad u_1 \neq u_2. \end{aligned}$$

3.2 Subgroup of order 2

In this next part we give an explicit form of the Lubin isogeny on the elliptic curve C. We defined this isogeny in Example 2.5.5. Let $P = (u, v)$ and $Q = (d, e)$ points on C where Q is a point of order 2, i.e. satisfying $Q = -_C Q$. To compute the isogeny, we need more information.

The first step will be the investigation of a subgroup of order 2.

Proposition 3.2.1. *Let (d, e) be a point of order 2, then the function*

$$f(u, v) = u^3 + a_2 u^2 v + uv^2 a_4 + v^3 a_6 + v$$

satisfies f(d,e)= 0.

Proof. A point Q of order 2 satisfies the property $Q = -Q$. With the inversion formula from above we get

$$u = -\frac{uv}{u^3 + a_2 u^2 v + uv^2 a_4 + v^3 a_6}.$$

and have $f(u, v)$. *q.e.d.*

Proposition 3.2.2. *Let C be the elliptic curve from (3.1) with a point (d, e) of order 2. Let*

$$\begin{aligned} f_d(u) = &\left(-a_1^3 a_6 + a_1^2 a_3 a_4 - a_1 a_2 a_3^2 + a_3^3\right) u^3 + \left(-6a_1^2 a_6 + 4a_1 a_3 a_4 - 2a_2 a_3^2\right) u^2 \\ &+ \left(-a_1 a_3^2 - 12 a_1 a_6 + 4 a_3 a_4\right) u - 2a_3^2 - 8a_6 \end{aligned}$$

and

$$f_e(u) = -\frac{ua_1 + 2}{a_3}.$$

Then d is a zero of the function f_d. For f_e we obtain $f_e(d) = e$.

Proof. We know from Proposition 3.2.1 that for a 2-torsion point (d, e) the function f vanishes, i.e.

$$f(d, e) = d^3 + a_2 d^2 e + de^2 a_4 + e^3 a_6 + e = 0$$

We get e with (3.1):

$$\begin{aligned} 0 = d^3 + a_2 d^2 e + de^2 a_4 + e^3 a_6 + e &= e + a_1 de + a_3 e^2 + e \\ &= a_1 de + a_3 e^2 + 2e \end{aligned}$$

and with a_3 invertible:

$$e = -\frac{a_1 d + 2}{a_3}$$

q.e.d.

Remark 3.2.3. Yifei Zhu [Zhu12] has computed the conditions for a point of order 3 for an elliptic curve of the form $y^2 + a_1 xy + a_1 a_2 y = x^3 + a_2 x^2$.

Remark 3.2.4. From now $a_6 = 0$ unless otherwise stated.

Corollary 3.2.5. A 2-torsion subgroup is generated by a point with coordinates

$$\left(d, -\frac{da_1 + 2}{a_3}\right).$$

Corollary 3.2.6. The universal example of a subgroup of oder 2 in C is defined over

$$R[d]/(f_d(d)) = R/((a_1^2 a_4 - a_1 a_2 a_3 + a_3^2)d^3 + (4a_1 a_4 - 2a_2 a_3)d^2 + (4a_4 - a_1 a_3)d - 2a_3)$$

Remark 3.2.7. With [And95] Theorem 2.4.1. every root d_i of $f_d(d)$ generates an order 2 subgroup of C.

3.3 The Lubin-isogeny

Now we are able to give an expression for the Lubin-isogeny. Let $\Psi : C \longrightarrow C'$ the Lubin isogeny (see Example 2.5.5) with

$$u(\Psi(P)) = u(P)u(P +_C Q)$$
$$v(\Psi(P)) = v(P)v(P +_C Q).$$

The kernel of this isogeny is a finite subgroup of the formal group on the elliptic curve consisting of the 2-torsion points of the formal group. We have studied a generator for a group of order 2 in Section 3.2.

The coordinate u is a uniformizer at the basepoint. And one gets

Proposition 3.3.1. *For the coordinate u we have*

$$u(\Psi(P)) = -du + (d^2K + a_3)\sum_{k\geq 1}(-1)^{k+1}\frac{K^{k-1}d^k}{a_3^k}u^{k+1}$$

where $K := (a_1^2a_4 - a_1a_2a_3 + a_3^2)d + (2a_1a_4 - a_2a_3)$.
For the coordinate v we get

$$v(\Psi(P)) = eu^3 + \ldots$$

with $e = -\frac{a_1d+2}{a_3}$.

For the proof we need

Lemma 3.3.2. *Let $P = (u, v)$ be an arbitrary point and $Q = (d, e)$ be a point of order 2 on C. Then $u(P +_C Q)$ is an rational function in u.*

Proof. (For the computational details see Appendix A.4) From 3.2.2 we know that $e = -\frac{a_1d+2}{a_3}$. Computing the formal sum of the two points one gets a rational function in u and v. Reducing with the equation of C gives a function of degree one in the variable v in the numerator and denominator. The function in the numerator look like $\alpha v+\beta$ and in the denominator $\gamma v+\delta$. Term manipulation obtains

$$\frac{\alpha v + \beta}{\gamma v + \delta} = \frac{\alpha}{\gamma} + \frac{\beta\gamma - \alpha\delta}{\gamma(\gamma v + \delta)}.$$

Modulo $f_d(d)$ the term $\beta\gamma - \alpha\delta$ vanishes and $\frac{\alpha}{\gamma}$ remains. *q.e.d.*

Remark 3.3.3. If we work in the ring $R[d]/(f_d(d))$ the first coordinate of the formal sum $u(P +_C Q)$ reduces to

$$\frac{-a_3du + a_3d^2}{((a_2a_3 - 2a_1a_4)d^2 + (a_1a_3 - 4a_4)d + 2a_3)u + a_3d}. \tag{3.3}$$

This is the output of Appendix A.4.

Proof of Proposition 3.3.1. With (3.3) we know that the coordinate

$$u(\Psi(P)) = u(P)u(P +_C Q)$$

of the isogeny Ψ decomposes in a rational function in u. We compute the series expansion of this function in u. We have

$$u(P)u(P +_C Q) = (-u)\frac{-a_3du + a_3d^2}{((a_2a_3 - 2a_1a_4)d^2 + (a_1a_3 - 4a_4)d + 2a_3)u + a_3d}.$$

Let $\alpha, \beta, \gamma, \delta$ the coefficients in the second factor. We expand the evident representation:

$$\begin{aligned}
\frac{\alpha u + \beta}{\gamma u + \delta} &= \frac{\alpha u + \beta}{\delta}\frac{1}{1 - (-\frac{\gamma}{\delta}u)} = \left(\frac{\beta}{\delta} + \frac{\alpha u}{\delta}\right)\sum_{k\geq 0}(-1)^k\left(\frac{\gamma}{\delta}\right)^k u^k \\
&= \frac{\beta}{\delta}\sum_{k\geq 0}(-1)^k\left(\frac{\gamma}{\delta}\right)^k u^k + \frac{\alpha}{\delta}\sum_{k\geq 0}(-1)^k\left(\frac{\gamma}{\delta}\right)^k u^{k+1} \\
&= \frac{\beta}{\delta} + \frac{\beta}{\delta}\sum_{k\geq 1}(-1)^k\left(\frac{\gamma}{\delta}\right)^k u^k + \frac{\alpha}{\delta}\sum_{k\geq 1}(-1)^{k-1}\left(\frac{\gamma}{\delta}\right)^{k-1} u^k \\
&= \frac{\beta}{\delta} + \sum_{k\geq 1}\left(\frac{\beta}{\delta}(-1)^k\left(\frac{\gamma}{\delta}\right)^k + \frac{\alpha}{\delta}(-1)^{k-1}\left(\frac{\gamma}{\delta}\right)^{k-1}\right)u^k \\
&= \frac{\beta}{\delta} + \frac{1}{\delta}\sum_{k\geq 1}\frac{(-1)^k\beta\gamma^k + (-1)^{k-1}\alpha\delta\gamma^{k-1}}{\delta^k}u^k \\
&= \frac{\beta}{\delta} + \frac{1}{\delta}\sum_{k\geq 1}(-1)^k\frac{\beta\gamma^k - \alpha\delta\gamma^{k-1}}{\delta^k}u^k \\
&= \frac{\beta}{\delta} + \frac{1}{\delta}\sum_{k\geq 1}(-1)^k\frac{(\beta\gamma - \alpha\delta)\gamma^{k-1}}{\delta^k}u^k \\
&= \frac{\beta}{\delta} + \frac{\beta\gamma - \alpha\delta}{\delta}\sum_{k\geq 1}(-1)^k\frac{\gamma^{k-1}}{\delta^k}u^k
\end{aligned}$$

Observe that

$$\begin{aligned}\gamma &= (a_2a_3 - 2a_1a_4)d^2 + (a_1a_3 - 4a_4)d + 2a_3 \\ &= (a_1^2a_4 - a_1a_2a_3 + a_3^2)d^3 + (2a_1a_4 - a_2a_3)d^2\end{aligned}$$

in the quotient ring $R[d]/(f_d(d))$, hence we get for the sum

$$\sum_{k\geq 1}(-1)^k\frac{((a_1^2a_4 - a_1a_2a_3 + a_3^2)d + (2a_1a_4 - a_2a_3))^{k-1}d^{k-2}}{a_3^k}u^k$$

If we set $K := (a_1^2 - a_1a_2a_3 + a_3^2)d + (2a_1a_4 - a_2a_3)$ then

$$\begin{aligned}\frac{\beta\gamma - \alpha\delta}{\delta} &= (a_2a_3 - 2a_1a_4)d^3 + (a_1a_3 - 4a_4)d^2 + 3a_3d \\ &= (Kd^2 + a_3)d\end{aligned}$$

and $\frac{\beta}{\gamma} = d$. We obtain the formula

$$d + (d^2K + a_3)\sum_{k\geq 1}(-1)^k\frac{K^{k-1}d^k}{a_3^k}u^k$$

Multiplying with $(-u)$ gives the solution of the first statement. For the second statement we refer to the series expansion in Appendix B. *q.e.d.*

Remark 3.3.4. In B.2 more terms of the power series in Proposition 3.3.1 are listed.

Proposition 3.3.5. *Let $\Psi : C \longrightarrow C'$ the Lubin isogeny. Then the coefficients $b_1, \ldots, b_4$ have the form:*

$$\begin{aligned}b_1 &= \frac{1}{a_3^2}(-a_1^3a_3a_4 + a_1^2a_2a_3^2 + 2a_1^2a_4^2 - 2a_1a_2a_3a_4 - a_1a_3^3 + 2a_3^2a_4)d^2 \\ &\quad +(-a_1^2a_3a_4 - a_1a_2a_3^2 + 8a_1a_4^2 - 4a_2a_3a_4 + 3a_3^3)d \\ &\quad +(a_1^2a_3^2 + 2a_1a_3a_4 - 4a_2a_3^2 + 8a_4^2) \\ b_2 &= \frac{1}{a_3^4}(6a_1^3a_3a_4^3 - 10a_1^2a_2a_3^2a_4^2 + 4a_1a_2^2a_3^3a_4 + 8a_1^2a_4^4 - 8a_1a_2a_3a_4^3 \\ &\quad +6a_1a_3^3a_4^2 - 4a_2a_3^4a_4 + 8a_3^2a_4^3)d^2 \\ &\quad +(2a_1^3a_3^2a_4^2 - 3a_1^2a_2a_3^3a_4 + a_1a_2^2a_3^4 + 28a_1^2a_3a_4^3 - 32a_1a_2a_3^2a_4^2 + 2a_1a_3^4a_4 \\ &\quad +8a_2^2a_3^3a_4 - a_2a_3^5 + 32a_1a_4^4 - 16a_2a_3a_4^3 + 4a_3^3a_4^2)d \\ &\quad +4a_1^2a_4^2a_3^2 - 6a_1a_2a_3^3a_4 + a_2^2a_3^4 + 32a_1a_4^3a_3 - 24a_2a_3^2a_4^2 + 6a_3^4a_4 + 32a_4^4\end{aligned}$$

$$
\begin{aligned}
b_3 \;=\;& \frac{1}{a_3^6}(a_1^5a_3^3a_4^3 - 3a_1^4a_2a_3^4a_4^2 + 3a_1^3a_2^2a_3^5a_4 - a_1^2a_2^3a_3^6 + 18a_1^4a_3^2a_4^4 - 46a_1^3a_2a_3^3a_4^3 \\
& +3a_1^3a_3^5a_4^2 + 38a_1^2a_2^2a_3^4a_4^2 - 6a_1^2a_2a_3^6a_4 - 10a_1a_2^3a_3^5a_4 + 3a_1a_2^2a_3^7 \\
& +48a_1^3a_3a_4^5 - 88a_1^2a_2a_3^2a_4^4 + 26a_1^2a_3^4a_4^3 + 40a_1a_2^2a_3^3a_4^3 - 36a_1a_2a_3^5a_4^2 \\
& +2a_1a_3^7a_4 + 10a_2^2a_3^6a_4 - 2a_2a_3^8 + 32a_1^2a_4^6 - 32a_1a_2a_3a_4^5 + 48a_1a_3^3a_4^4 \\
& -40a_2a_3^4a_4^3 + 8a_3^6a_4^2 + 32a_3^2a_4^5)d^2 \\
& +(9a_1^4a_3^3a_4^3 - 22a_1^3a_2a_3^4a_4^2 + 17a_1^2a_2^2a_3^5a_4 - 4a_1a_2^3a_3^6 + 92a_1^3a_3^2a_4^4 \\
& -184a_1^2a_2a_3^3a_4^3 + 16a_1^2a_3^5a_4^2 + 112a_1a_2^2a_3^4a_4^2 - 22a_1a_2a_3^6a_4 - 20a_2^3a_3^5a_4 \\
& +6a_2^2a_3^7 + 208a_1^2a_3a_4^5 - 272a_1a_2a_3^2a_4^4 + 52a_1a_3^4a_4^3 + 80a_2^2a_3^3a_4^3 \\
& -32a_2a_3^5a_4^2 + 3a_3^7a_4 + 128a_1a_4^6 - 64a_2a_3a_4^5 + 16a_3^3a_4^4)d \\
& +14a_1^3a_3^3a_4^3 - 32a_1^2a_2a_3^4a_4^2 + 22a_1a_2^2a_3^5a_4 - 4a_3^6a_2^3 + 112a_1^2a_3^2a_4^4 \\
& -184a_1a_2a_3^3a_4^3 + 20a_1a_3^5a_4^2 + 72a_4^2a_3^4a_2^2 - 20a_4a_3^6a_2 + a_3^8 + 224a_1a_3a_4^5 \\
& -192a_4^4a_3^2a_2 + 40a_4^3a_3^4 + 128a_4^6
\end{aligned}
$$

$$
\begin{aligned}
b_4 \;=\;& \frac{a_4}{a_3^8}((a_1^6a_3^4a_4^3 - 3a_1^5a_2a_3^5a_4^2 + 3a_1^4a_2^2a_3^6a_4 - a_1^3a_2^3a_3^7 + 25a_1^5a_3^3a_4^4 \\
& -69a_1^4a_2a_3^4a_4^3 + 3a_1^4a_3^6a_4^2 + 65a_1^3a_2^2a_3^5a_4^2 - 6a_1^3a_2a_3^7a_4 - 23a_1^2a_2^3a_3^6a_4 \\
& +3a_1^2a_2^2a_3^8 + 2a_1a_2^4a_3^7 + 104a_1^4a_3^2a_4^5 - 236a_1^3a_2a_3^3a_4^4 + 47a_1^3a_3^5a_4^3 \\
& +168a_1^2a_2^2a_3^4a_4^3 - 80a_1^2a_2a_3^6a_4^2 + 3a_1^2a_3^8a_4 - 36a_1a_2^3a_3^5a_4^2 + 35a_1a_2^2a_3^7a_4 \\
& -3a_1a_2a_3^9 - 2a_2^3a_3^8 + 144a_1^3a_3a_4^6 - 240a_1^2a_2a_3^2a_4^5 + 132a_1^2a_3^4a_4^4 \\
& +96a_1a_2^2a_3^3a_4^4 - 160a_1a_2a_3^5a_4^3 + 22a_1a_3^7a_4^2 + 36a_2^2a_3^6a_4^2 - 14a_2a_3^8a_4 + a_3^{10} \\
& +64a_1^2a_4^7 - 64a_1a_2a_3a_4^6 + 144a_1a_3^3a_4^5 - 96a_2a_3^4a_4^4 + 28a_3^6a_4^3 + 64a_3^2a_4^6)d^2 \\
& +(10a_1^5a_3^4a_4^3 - 25a_1^4a_2a_3^5a_4^2 + 20a_1^3a_2^2a_3^6a_4 - 5a_1^2a_2^3a_3^7 + 138a_1^4a_3^3a_4^4 \\
& -308a_1^3a_2a_3^4a_4^3 + 20a_1^3a_3^6a_4^2 + 226a_1^2a_2^2a_3^5a_4^2 - 30a_1^2a_2a_3^7a_4 - 60a_1a_2^3a_3^6a_4 \\
& +10a_1a_2^2a_3^8 + 4a_2^4a_3^7 + 480a_1^3a_3^2a_4^5 - 840a_1^2a_2a_3^3a_4^4 + 138a_1^2a_3^5a_4^3 \\
& +448a_1a_2^2a_3^4a_4^3 - 156a_1a_2a_3^6a_4^2 + 10a_1a_3^8a_4 - 72a_2^3a_3^5a_4^2 + 38a_2^2a_3^7a_4 \\
& -5a_2a_3^9 + 608a_1^2a_3a_4^6 - 704a_1a_2a_3^2a_4^5 + 176a_1a_3^4a_4^4 + 192a_2^2a_3^3a_4^4 \\
& -96a_2a_3^5a_4^3 + 12a_3^7a_4^2 + 256a_1a_4^7 - 128a_2a_3a_4^6 + 32a_3^3a_4^5)d \\
& +16a_1^4a_3^4a_4^3 - 38a_1^3a_2a_3^5a_4^2 + 28a_1^2a_2^2a_3^6a_4 - 6a_1a_2^3a_3^7 + 176a_1^3a_3^3a_4^4 \\
& -340a_1^2a_2a_3^4a_4^3 + 28a_1^2a_3^6a_4^2 + 192a_1a_2^2a_3^5a_4^2 - 36a_1a_2a_3^7a_4 - 28a_2^3a_3^6a_4 \\
& +8a_2^2a_3^8 + 544a_1^2a_3^2a_4^5 - 736a_1a_2a_3^3a_4^4 + 136a_1a_3^5a_4^3 + 224a_2^2a_3^4a_4^3 \\
& -96a_2a_3^6a_4^2 + 9a_3^8a_4 + 640a_1a_3a_4^6 - 448a_2a_3^2a_4^5 + 128a_3^4a_4^4 + 256a_4^7)
\end{aligned}
$$

Proof. For the values of the images of the coefficients under the Lubin isogeny we use the series expansion from Proposition 3.3.1. This describes the isogeny

$$\Psi : C \longrightarrow C'$$

with

$$C' : v' + b_1 u'v' + b_3 v'^2 = u'^3 + b_2 u'^2 v' + b_4 u'v'^2$$

where

$$u' := u(\Psi(P)) \qquad \text{and} \qquad v' := -\frac{d^3}{e} v(\Psi(P)).$$

The factor $-\frac{d^3}{e}$ in the definition of v' appears since we work with the Weierstraßform of C' Remember that $a_6 = 0$. We have a series expansion of C' starting with

$$v' = u'^3 - b_1 u'^4 + (b_1^2 + b_2)u'^5 + (-b_1^3 - 2b_1 b_2 - b_3)u'^6 + (b_1^4 + 3b_1^2 b_2 + 3b_1 b_3 + b_2^2 + b_4)u'^7$$

Comparing the coefficients of the Weierstraßequation of E with the series expansion of C' gives the solution. For the computational part see AppendixA.6.

q.e.d.

Remark 3.3.6. For clarity we write $\Psi(a_i)$ for the b_i.

Example 3.3.7. If we set $a_1 = a$, $a_3 = 1$ and $a_2 = a_4 = 0$ we get a curve C of the form

$$v + auv + v^2 = u^3.$$

The operation Ψ corresponds to the operation of Rezk [Rez08]. For C' we obtain the equation

$$v + (-ad^2 + 3d + a^2)uv + v^2 = u^3.$$

Example 3.3.8. Let C be of the form

$$v + v^2 = u^3,$$

then C' has the form

$$v + (3d)uv + v^2 = u^3.$$

Example 3.3.9. For the curve

$$C : v + auv + v^2 = u^3 + u^2 v$$

the target curve is described by the equation

$$\begin{aligned} & v + ((a^2 - a)d^2 + (-a + 3)d + a^2 - 4)uv \\ & +((-a + 3a - 2)d^2 + (-4a + 6)d - 3)v^2 \\ = \; & u^3 + ((a - 1)d + 1)u^2 v. \end{aligned}$$

4 $K(2)$-local power operations.

In this part we want to use the formulas from Chapter 3 to study $K(2)$-local power operations. First we have another look a the Lubin-Tate theory, especially to the polynomial generators. Then we define the total power operation and deduce the action on a complex orientation of this theory. Later on we examine the reduced operations and calculate some relations, i.e. Cartan formulas and Adem relations.

4.1 Lubin-Tate cohomology revisited

In Section 2.3 we saw how a Lubin-Tate theory arises from the deformations of a formal group law. Here we work with an elliptic curve C_0 given by $y^2 + y = x^3$ over $\mathbb{F}_2$. From Proposition 2.5.11 we know that this elliptic curve has height 2 since its j-invariant is 0. If we study the lifts of C_0 to $\mathbb{Z}_2[a_1, a_2, a_3, a_4]$ where C is the lift, we obtain a cohomology theory E_C with formal group $\widehat{C}$. This Lubin-Tate theory is $K(2)$-local since it is constructed from the lift of a height 2 formal group law.

Remark 4.1.1. For the rest of this section we write E instead of E_C.

Proposition 4.1.2. *Let E be as above. The Hazewinkel generators of the formal group $\widehat{C}$ over $\mathbb{Z}_2[a_1, \dots a_4]$ are given by*

$$v_1 = a_1 \text{ and } v_2 = a_3 - a_1 a_2.$$

Proof. (see [Lau04]) *q.e.d.*

Remark 4.1.3. A formal group law that is generated by the additive structure on an elliptic curve can only have height 1 or 2. The coefficients of the Lubin-Tate cohomology theory associated to the deformations of this formal group law are given by $\mathbb{W}_{\mathbb{F}_p}[t^{\pm 1}]$ in the height 1 case and by $\mathbb{W}_{\mathbb{F}_p}[[u_1]][t^{\pm 1}]$ if the height is 2 (see Chapter 2.3). For the prime 2 and height 1, the first Hazewinkel generator v_1 correspond to u. In the second case the we have relation

$$u_1 = t^{-1} v_1 \text{ and } t^3 = v_2.$$

Remark 4.1.4. In literature the element t is often denoted with u. Unfortunately the complex orientation is already denoted by u in our setting.

From now let $u \in E^*\mathbb{C}P^\infty \cong E^*[|u|]$ be a complex orientation of E. For the next proposition we need the ideal tr that is generated by the stable transfer $Tr_1^{\Sigma_2}(1) = \frac{[2]_{\widehat{C}}(u)}{u}$ and the ideal $(f_d(d))$ generated by the function from Lemma 3.2.2. Both ideals lying in $E^0 B\Sigma_2$.

Proposition 4.1.5. *For the ideal $(f_d(d))$ and* tr *the following relation holds:*

$$tr = (f_d(u))$$

Proof. First of all, we know that $uf_d(u) \in ([2]_{\widehat{C}}(u))$ because $[2]_{\widehat{C}}(u)$ and $f_d(u)$ vanishes for 0 or a point of order 2. Hence

$$f_d(u) \in \left(\frac{[2]_{\widehat{C}}(u)}{u}\right) = (\text{transfer}).$$

On the other hand by the formal Weierstraß preparation theorem we have $uf_d(u) = \epsilon(u)[2]_{\widehat{C}}(u)$ where $\epsilon(u)$ is a invertible power series. Thus $[2]_{\widehat{C}}(u) \in (f_d(u))$ *q.e.d.*

Proposition 4.1.6. *We get a total power operation*

$$P : E^0(X) \longrightarrow E^0(X) \otimes E^0 B\Sigma_2/(\text{transfer})$$

with

$$P(x) = Q_0(x) + Q_1(x)d + Q_2(x)d^2$$

where the variable d keeps record of the reduced operations.

Proof. From Section 2 we know that such an operation exists, because we have an E_∞-structure on E. The transfer ideal is generated by the polynomial f_d of degree 3 and $E^*B\Sigma_2$ is a free E^*-module with one generator d. *q.e.d.*

Remark 4.1.7. Comparing our results with [Rez08] we obtain a more general operation. Let F be a $K(2)$-local E algebra. Then we have a total operation $P : \pi_0 F \longrightarrow \pi_0 F \otimes E^0 B\Sigma_2/(\text{transfer})$

Remark 4.1.8. Initially the power operation P is only defined in degree zero. In [Rez09], part 3.27 Rezk argues that for arbitrary degrees the power operations works in the same way as for degree 0.

Let $X = \mathbb{C}P^\infty$, we have the following result.

Proposition 4.1.9. *Let $u \in E^*\mathbb{C}P^\infty$ be a complex orientation. Then*

$$P(u) = u(\Psi(P)).$$

Proof. The idea behind Ψ was to construct an isogeny whose kernel consists of points of order 2. With Theorem 2.4.3 the operation Ψ is a lift of the Frobenius, hence a total power operation P. Looking at the projection to $\mathbb{F}_2$ the only term in the series $u(\Psi(P))$ (Proposition 3.3.1) that remains is u^2. *q.e.d.*

In Remark 3.3.6 we explained $\Psi(a_i)$.

Proposition 4.1.10. *Let $x \in E^0(X)$. For $1 \leq i \leq 4$ we have for P the following relations:*

$$P(a_i x) \;=\; \Psi(a_i)x.$$

Proof. The operation Ψ describes an isogeny from an elliptic curve C to a target curve C'. The coefficients of the target are the images of the coefficients which are computed in Proposition 3.3.5. *q.e.d.*

4.2 Reduced operations

The total power operation P is by construction a ring homomorphism. For this reason we can deduce some relations for Q_i, i.e. Cartan formulas.

Proposition 4.2.1.

$$\begin{aligned}
Q_i(x+y) &= Q_i(x) + Q_i(y) \\
Q_0(xy) &= Q_0(x)Q_0(y) + \frac{2a_3}{N}(Q_1(x)Q_2(y) + Q_2(x)Q_1(y)) \\
&\quad + \frac{-4a_3(2a_1a_4 - a_2a_3)}{N^2}Q_2(x)Q_2(y) \\
Q_1(xy) &= Q_0(x)Q_1(y) + Q_1(x)Q_0(y) + \frac{a_1a_3 - 4a_4}{N}(Q_1(x)Q_2(y) \\
&\quad + Q_2(x)Q_1(y)) + \frac{1}{N}(-2a_1^2a_3a_4 + 16a_1a_4^2 \\
&\quad - 8a_2a_3a_4 + 2a_3^3)Q_2(x)Q_2(y) \\
Q_2(xy) &= Q_0(x)Q_2(y) + Q_1(x)Q_1(y) + Q_2(x)Q_0(y) \\
&\quad + \frac{-2(2a_1a_4 - a_2a_3)}{N}(Q_1(x)Q_2(y) + Q_2(x)Q_1(y)) \\
&\quad + \frac{1}{N^2}(a_1^3a_3a_4 - a_1^2a_2a_3^2 + 12a_1^2a_4^2 - 12a_1a_2a_3a_4 + a_1a_3^3 \\
&\quad + 4a_2^2a_3^2 - 4a_3^2a_4)Q_2(x)Q_2(y)
\end{aligned}$$

with $N := a_1^2a_4 - a_1a_2a_3 + a_3^2$.

Proof. The Q_i's are additive since the total operation is. The multiplicative relations follow from the fact that $P(xy) = P(x)P(y)$, i.e. we have

$$
\begin{aligned}
P(x)P(y) &= (Q_0(x)+Q_1(x)d+Q_2(x)d^2)(Q_0(y)+Q_1(y)d+Q_2(y)d^2) \\
&= Q_0(x)Q_0(y)+Q_0(x)Q_1(y)d+Q_0(x)Q_2(y)d^2+Q_1(x)Q_0(y)d \\
&\quad +Q_1(x)Q_1(y)d^2+Q_1(x)Q_2(y)d^3+Q_2(x)Q_0(y)d^2 \\
&\quad +Q_2(x)Q_1(y)d^3+Q_2(x)Q_2(x)d^4
\end{aligned}
$$

After reduction modulo $f_d(d)$ one can read of the relations. *q.e.d.*

Remark 4.2.2. The formulas above contains powers of the leading coefficient N of $f_d(d)$ in the denominator. By the construction of $f_d(d)$ it is not necessary that this coefficient be invertible. The equations should be seen as a formal statement. In the case that N is not invertible the formulas became true after multiplying N or N^2.

Proposition 4.2.3. *For $a_2 = a_4 = 0$ and $Q_i(u) = \sum_{n\geq 1} q_i^{(n)} u^n$ we have for $n \geq 3$*

$$
\begin{aligned}
q_0^{(n+1)} &= 2q_1^{(n)} \\
q_1^{(n+1)} &= a_1 q_1^{(n)} + 2q_2^{(n)} \\
q_2^{(n+1)} &= a_1 q_2^{(n)} - a_3 q_0^{(n)}
\end{aligned}
$$

Proof. We start with the n-th coefficient of $P(u)$ that decomposes in

$$q_0^{(n)} + q_1^{(n)} d + q_2^{(n)} d^2.$$

By multiplying with $(-a_3 d^2)$ one gets, after reduction by $a_3 d^3 = a_1 d + 2$:

$$-(a_3 d^2)(q_0^{(n)} + q_1^{(n)} d + q_2^{(n)} d^2) = 2q_1^{(n)} + (a_1 q_1^{(n)} + 2q_2^{(n)})d + q_2^{(n)} - a_3 q_0^{(n)} d^2.$$

With Proposition 4.1.6 this is the $(n+1)$-st coefficient of $P(u)$. *q.e.d.*

We can read off the action of the Q_i in this coordinate with Proposition 3.3.1 and B.2.

Corollary 4.2.4. With $N := (a_1^2a_4 - a_1a_2a_3 + a_3^2)$ we have

$$\begin{aligned}
Q_0(u) &= 3u^2 + \frac{1}{N}(-2(a_1^3a_4 - a_1^2a_2a_3 + a_1a_3^2 + a_2^2a_3 - 4a_3a_4))u^3 \\
&+\frac{1}{N^2}(2(a_1^6a_4^2 - 2a_1^5a_2a_3a_4 + a_1^4a_2^2a_3^2 + 2a_1^4a_2a_4^2 + 2a_1^4a_3^2a_4 \\
&-a_1^3a_2^2a_3a_4 - 2a_1^3a_2a_3^3 - a_1^2a_2^3a_3^2 - 12a_1^3a_3a_4^2 + 4a_1^2a_2^2a_4^2 \\
&+16a_1^2a_2a_3^2a_4 + a_1^2a_3^4 - 4a_1a_2^3a_3a_4 - a_1a_2^2a_3^3 + 2a_2^4a_3^2 - 16a_1^2a_4^3 \\
&+16a_1a_2a_3a_4^2 - 12a_1a_3^3a_4 - 12a_2^2a_3^2a_4 + 2a_2a_3^4 + 16a_3^2a_4^2))u^4 + \ldots \\
Q_1(u) &= -u + \frac{a_1a_3 - 4a_4}{a_3}u^2 \\
&-\frac{1}{a_3N}((a_1^4a_3a_4 - a_1^3a_2a_3^2 - 2a_1^3a_4^2 + a_1^2a_2a_3a_4 + a_1^2a_3^3)u^3 \\
&+\frac{1}{a_3N^2}(a_1^7a_3a_4^2 - 2a_1^6a_2a_3^2a_4 + a_1^5a_2^2a_3^3 - 2a_1^6a_4^3 + 2a_1^5a_2a_3a_4^2 \\
&+2a_1^5a_3^3a_4 + 5a_1^4a_2^2a_3^2a_4 - 2a_1^4a_2a_3^4 - 5a_1^3a_2^3a_3^3 - 8a_1^4a_2a_4^3 \\
&-10a_1^4a_3^2a_4^2 + 4a_1^3a_2^2a_3a_4^2 + a_1^3a_3^5 + 6a_1^2a_2^3a_3^2a_4 + 13a_1^2a_2^2a_3^4 \\
&+48a_1^3a_3a_4^3 - 16a_1^2a_2^2a_4^3 - 72a_1^2a_2a_3^2a_4^2 - 2a_1^2a_3^4a_4 + 16a_1a_2^3a_3a_4^2 \\
&-4a_1a_2^2a_3^3a_4 - 14a_1a_2a_3^5 - 8a_2^4a_3^2a_4 + 2a_2^3a_3^4 + 64a_1^2a_4^4 - 64a_1a_2a_3a_4^3 \\
&+80a_1a_3^3a_4^2 + 48a_2^2a_3^2a_4^2 - 16a_2a_3^4a_4 + 6a_3^6 - 64a_3^2a_4^3)u^4 + \ldots \\
Q_2(u) &= \frac{2a_1a_4 - a_2a_3}{a_3}u^2 + \frac{1}{a_3N}(a_1^4a_4^2 - a_1^2a_2^2a_3^2 - 2a_1^2a_3^2a_4 + 4a_1a_2^2a_3a_4 \\
&+4a_1a_2a_3^3 - 2a_2^3a_3^2 - 16a_1a_3a_4^2 + 8a_2a_3^2a_4 - 3a_3^4)u^3 \\
&+\frac{1}{a_3N^2}(a_1^7a_4^3 - 3a_1^5a_2^2a_3^2a_4 + 2a_1^4a_2^3a_3^3 + 4a_1^5a_2a_4^3 - 3a_1^5a_3^2a_4^2 \\
&-a_1^4a_2^2a_3a_4^2 + 12a_1^4a_2a_3^3a_4 - 3a_1^3a_2^3a_3^2a_4 - 9a_1^3a_2^2a_3^4 - 36a_1^4a_3a_4^3 \\
&+8a_1^3a_2^2a_4^3 + 52a_1^3a_2a_3^2a_4^2 - 9a_1^3a_3^4a_4 - 12a_1^2a_2^3a_3a_4^2 - 16a_1^2a_2^2a_3^3a_4 \\
&+12a_1^2a_2a_3^5 + 12a_1a_2^4a_3^2a_4 + 7a_1a_2^3a_3^4 - 4a_2^5a_3^3 - 32a_1^3a_4^4 + 48a_1^2a_2a_3a_4^3 \\
&-16a_1^2a_3^3a_4^2 - 72a_1a_2^2a_3^2a_4^2 - 8a_1a_2a_3^4a_4 - 5a_1a_3^6 + 28a_2^3a_3^3a_4 \\
&-7a_2^2a_3^5 + 96a_1a_3^2a_4^3 - 48a_2a_3^3a_4^2 + 20a_3^5a_4)u^4 + \ldots
\end{aligned}$$

Remark 4.2.5. The operations produced by Rezk in [Rez08] do not see the coefficient of u^2 in Q_2.

4.3 Adem relations

In this part of the work we compute Adem relations for the reduced operations Q_i. One way to obtain such relations is described in [Rez08]. We follow the lines of [Zhu12] chapter 3.2, since Zhu's approach (for the prime 3) can be adapted to work at the prime 2.

Proposition 4.3.1. *The Adem relations are*

$$\begin{aligned}
Q_0Q_0(u) &= \frac{1}{N^2}((-a_1^3a_3a_4 + a_1^2a_2a_3^2 + 4a_1^2a_4^2 - 4a_1a_2a_3a_4 \\
&\quad -a_1a_3^3 + 4a_3^2a_4)Q_1Q_0(u) + (2a_1^2a_3a_4 - 2a_1a_2a_3^2 + 2a_3^3)Q_1Q_1(u) \\
&\quad +(-a_1^2a_3^2 + 4a_2a_3^2 - 16a_4^2)Q_2Q_0(u) \\
&\quad +(2a_1a_3^2 - 8a_3a_4)Q_2(u)Q_1(u) - 4a_3^2Q_2Q_2(u)) \\
Q_1Q_0(u) &= \frac{1}{N^2}((4a_1^3a_4^2 - 6a_1^2a_2a_3a_4 + 2a_1a_2^2a_3^2 + 4a_1a_3^2a_4 - 2a_2a_3^3)Q_0Q_1(u) \\
&\quad +(2a_1^2a_3a_4 - 2a_1a_2a_3^2 + 2a_3^3)Q_2Q_1(u) \\
&\quad +(2a_1^2a_3a_4 - 16a_1a_4^2 + 8a_2a_3a_4 - 2a_3^3)Q_0Q_2(u) \\
&\quad +(-8a_1a_3a_4 + 4a_2a_3^2)Q_1Q_2(u)) \\
Q_2Q_0(u) &= \frac{1}{N^2}(NQ_0Q_1(u) + (a_1a_3 - 4a_4)Q_0Q_2(u) - 2a_3Q_1Q_2(u))
\end{aligned}$$

where $N := (a_1^2a_4 - a_1a_2a_3 + a_3^2)$.

To prove this we have to examine $P(P(u))$. With Proposition 4.1.6 we get a map

$$\begin{aligned}
E^0(X) &\xrightarrow{P} E^0(X) \otimes E^0B\Sigma_2/(\text{transfer}) \\
&\xrightarrow{P} (E^0(X) \otimes E^0B\Sigma_2/(\text{transfer})) \otimes E^0B\Sigma_2/(\text{transfer}) \\
&\cong (E^0(X) \otimes E^0[d']/(\tilde{f}_d(d'))) \otimes E^0[d]/(f_d(d)).
\end{aligned}$$

with $P(d) = d'$ and hence $P(f_d(d)) = \tilde{f}_d(d')$.

Remark 4.3.2. The construction of [Zhu12] contains the treatment of the cotangent space of the elliptic curve at the identity. Every non-constant map $\varphi : C \longrightarrow C'$ on elliptic curves induces a map on differentials $\varphi^* : \Omega_{C'} \longrightarrow \Omega_C$ via

$$\varphi^*(f du) = \varphi^* f d(\varphi^* u).$$

For a functorial definition of Ω_C over local rings see [Har97] II.8.

Remark 4.3.3. To abuse confusion with the variable d we denote the 1-from du with brackets: (du)

Lemma 4.3.4. *Let $\Psi : C \longrightarrow C'$ the isogeny from Section 3.3. The induced map Ψ^* on the cotangent space of C' at the identity sends (du) to $(-d)(du)$.*

Proof. The claim follows immediately from (Proposition 3.3.1) the fact that

$$u(\Psi(P)) = -d \cdot u + \dots$$

q.e.d.

As seen in Section 3.2, we have three different subgroups of order 2. Let G_0, G_1 and G_2 generators for the respective subgroup and for $i = 0, 1$ and 2 let d_i a root of $f_d(d)$. Then we have the equation

$$d_0 d_1 d_2 = -\frac{2a_3}{N}.$$

Lemma 4.3.5. *For d and d' we have the relation*

$$dd' = \frac{-2a_3}{N}.$$

Proof. Define on C an isogeny Φ via

$$\begin{aligned} u(\Phi(P)) &= u(P)u(P - G_0)u(P - G_1)u(P - G_2) \\ v(\Phi(P)) &= v(P)v(P - G_0)v(P - G_1)v(P - G_2). \end{aligned}$$

We see that the map Φ and the 2-series $[2]_{\widehat{C}}$ on C have the same kernel, so

$$[2]^*_{\widehat{C}}(du) = d_1 d_2 d_3 \cdot (du) = -\frac{2a_3}{N} \cdot (du) \tag{4.1}$$

Let $\Psi : C \longrightarrow C'$ the isogeny studied before with kernel G. Denote with C[2] the 2-divisible subgroup, i.e. the kernel of $[2]_{\widehat{C}}$. Recall that G is the universal 2-subgroup of C. Then we define an isogeny Ψ' with source C' and the subgroup $G' := C[2]/G$ as kernel. This is also a lift of the Frobenius since the kernel is a 2-subgroup. Consider the diagram

$$\begin{tikzcd} C \arrow[rr, "\Psi"] \arrow[dr, "{[2]_{\widehat{C}}}"'] & & C/G = C' \arrow[dl, "\Psi'"] \\ & C/C[2] \cong C'/G' & \end{tikzcd}$$

where the lower isomorphim arises from

$$C/C[2] \cong (C/G)/(C[2]/G) = C'/G'.$$

This diagram commutes because both maps Ψ and Ψ' reduce to the Frobenius endomorphism Ψ_0 on the base curve C_0 over $\mathbb{F}_2$. With [KM85] Theorem 2.4.2, Theorem 2.6.3 and the fact that the trace of Ψ_0 is zero, gives for reduced $\Psi \circ \Psi'$ the identification $\Psi_0 \circ \Psi_0 = [2]$ on $\mathbb{F}_2$. On the 1-form (du) we get

$$\begin{tikzcd} dd' \cdot (du) & & d' \cdot (du) \arrow[ll, mapsto, "\Psi^*"'] \\ & (du) \arrow[ul, mapsto, "{[2]^*_{\widehat{C}}}"] \arrow[ur, mapsto, "(\Psi')^*"'] & \end{tikzcd}$$

Thus the image of (du) under $[2]^*_{\widehat{C}}$ is $dd' \cdot (du)$, which proves together with (4.1) the Lemma. *q.e.d.*

Proof of Proposition 4.3.1. Let us again look at

$$PP : E^0(X) \longrightarrow (E^0(X) \otimes E^0[d']/(\tilde{f}_d(d'))) \otimes E^0[d]/(f_d(d)).$$

For $P(P(u))$ we have

$$\begin{aligned} P(P(u)) &= P(Q_0(u) + Q_1(u)d + Q_2(u)d^2) \\ &= P(Q_0(u)) + P(Q_1(u))P(d) + P(Q_2(u))P(d)^2 \\ &= Q_0Q_0(u) + Q_1Q_0(u)d + Q_2Q_0(u)d^2 \\ &\quad +(Q_0Q_1(u) + Q_1Q_1(u)d + Q_2Q_1(u)d^2)d' \\ &\quad +(Q_0Q_2(u) + Q_1Q_2(u)d + Q_2Q_2(u)d^2)d'^2 \end{aligned}$$

For the map

$$E^0(X) \xrightarrow{PP} (E^0(X) \otimes E^0[d']/(\tilde{f}_d(d'))) \otimes E^0[d]/(f_d(d)) \xrightarrow{r*} E^0(X) \otimes E^0[d]/(f_d(d))$$

with $r(d) = d$ and $r(d') = -\frac{2a_3}{dN}$ we get the Adem relations by setting the coefficients, seen as polynomiasl in d, equals to zero. Remember that

$$-\frac{2a_3}{dN} = -Nd^2 - (4a_1a_4 - 2a_2a_3)d - (-a_1a_3 + 4a_4).$$

For further details on the computation see Appendix A.8. *q.e.d.*

5 Examples in E_C cohomology

Now we want to compute some examples in the E_C cohomology with the power operations form the last chapter. First of all we have a look at the Hopf invariant and then we say something about the structure of BU. In this chapter we again suppress the elliptic curve C from the notation and write E for E_C.

5.1 Hopf invariant

To say something about the Hopf invariant we use the Hopf fibrations

$$\begin{aligned} S^1 \longrightarrow S^3 \xrightarrow{\eta} S^2, \\ S^3 \longrightarrow S^7 \xrightarrow{\nu} S^4.. \end{aligned}$$

Since the mapping cones of these bundles are $\mathbb{C}P^2$ and $\mathbb{H}P^2$, respectively, we consider for η the E-cohomology $E^*\mathbb{C}P^2$. This is isomorphic to $E_* \langle 1, u, u^2 \rangle$, where u is a complex orientation. For a usable elliptic curve C we have with Corollary 4.2.4 the reduced operations

$$\begin{aligned} Q_0(u) &= 3u^2 \\ Q_1(u) &= -u + \frac{a_1 a_3 - 4a_4}{a_3} u^2 \\ Q_2(u) &= \frac{2a_1 a_4 - a_2 a_3}{a_3} u^2 \end{aligned}$$

We see that for any choice of the a_i's the coefficient of u^2 do not disappear. For Q_1 and Q_2 there are choices of a_i's thinkable that the coefficient of u^2 vanishes.

For the Hopf map ν we use the embedding

$$\mathbb{C}P^2 \xrightarrow{i} \mathbb{H}P^2$$

where $\mathbb{H}P^2 \simeq C(\nu)$. The E-cohomology of $\mathbb{H}P^2$ is generated by a class h in degree 4 which maps to u^2 under the homomorphism induced by the embedding. The embedding induces an map in E-cohomology where $h \mapsto u^2$. For the operation Q_0 we get with the product relations in 4.2.1:

$$i^* Q_0(h) = Q_0(u^2) = 9u^4 + (\text{higher Terms}) = i^*(9h^2) + (\text{higher Terms}),$$

hence $Q_0(h) = 9h^2 + \dots$.

5.2 E cohomology of BU.

Since E is complex oriented, the cohomology E^*BU is isomorphic to the power series ring $E_*[[c_1, c_2, \dots]]$ where the c_i denotes the Chern classes. For the computation of the cohomology one embeds the maximal torus in $U(n)$ and considers the classifying spaces of these groups in cohomology. This induces a map

$$E^*BU(n) \longrightarrow E^*\mathbb{C}P^\infty \cong E^*[u_1, \dots, u_n]$$

where the image is contained in $E_*[u_1, \dots, u_n]^{\Sigma_n}$. This ring of invariants is given by the Chern classes, where i-th Chern class is the i-th elementary symmetric function on $u_1, \dots u_i$. Taking the inverse limit

$$\lim_{\leftarrow} E^*BU(n)$$

gives the cohomology E^*BU. There is no $\lim^1$ term because the maps $E^*BU(n) \longrightarrow E^*BU(n-1)$ are surjective.

The effect of the total operation is not surprising. Taking the k-th Chern class that can be written as

$$c_k = \sum_{M \subset \{1,\dots,n\}} \prod_{i \in M} u_i,$$

where $|M| = k$, the total operation is compatible with the additive and multiplicative structure

$$P(c_k) = \sum_{M \subset \{1,\dots,n\}} \prod_{i \in M} P(u_i)$$

A Program code

In this part we show how to compute the total power operation in $E_C^*\mathbb{C}P^\infty$ via the isogeny on the elliptic curve C with a computer algebra system. in fact, we are use two packages. Maple 17 helps with term manipulation and series expansions. The second one is Macaulay 2 which is very helpful for computing a small representation in a factor ring. We start with Macaulay, because every result that we get with Maple 17 should be reduced with Macaulay. The environment for this always the same.

A.1 Macaulay 2

Start Macaulay2 with the following setting:

```
R=ZZ[a_1,a_2,a_3,a_4,d,u,v]
I=((a_4*a_1^2-a_1*a_2*a_3+a_3^2)*d^3+(4*a_4*a_1-2*a_2*a_3)*d^2
    +(-a_1*a_3+4*a_4)*d-2*a_3)
S=R/I
use(S)
```

This code reduces everything modulo the ideal I generated by $f_d(d)$ (compare Corollary 3.2.6). Now one can insert the results of Maple 17. For compatibility the results should transformed into strings, command `convert(-,string);`.

Next, we give the program code for the computations in Maple 17.

A.2 Formal group law on an elliptic curve

This program computes the formal group law for an elliptic curve of the form

$$v + a_1uv + a_3v^2 = u^3 + a_2u^2v + a_4uv^2 + a_6v^3$$

with identity at the point $(0,0)$.

```
lambda := (v_2 - v_1)/(u_2 - u_1);
m := (v_1*u_2 -v_2*u_1)/(u_2-u_1);
koeff:=(-3*lambda^2*m*a_6+lambda^2*a_3-(2*m*a_4-a_1)*lambda -m*a_2)/
       (lambda^3*a_6+lambda^2*a_4+lambda*a_2+1);
fgl_u:=koeff - u_2 -u_1;
```

A.3 Series expansion of the elliptic curve.

The following program code computes the series expansion of the elliptic curve in one variable. The accuracy depends on the variable 'deg', i.e. the command 'uexpansion(15)' gives a polynomial of degree 15 which contains the first 15 exact coefficients of the series expansion.

```
uexpansion := proc(deg)
local i, it, itv0, curve;
curve := (u^3 - a_1*u*v - a_3*v^2 + a_2*u^2*v + a_4*u*v^2 + a_6*v^3 );
it := (u^3 - a_1*u*v - a_3*v^2 + a_2*u^2*v + a_4*u*v^2 + a_6*v^3 );
for i from 1 to deg+1 by 1 do
it := rem(rem(simplify(eval(it, v=curve)),v^(deg+1),v),u^(deg+1),u);
end do:
itv0:= eval(it,v=0);
return itv0;
end proc:
uex:=uexpansion(10);
```

A.4 Proof of Lemma 3.3.2

Here we give the computational background of the proof of Lemma 3.3.2. The first two definitions are the inverse of a point $P = (u, v)$ for the formal group law from above. The value u_{poly} is the the u-coordinate of the formal sum $P +_C Q$ where $Q = (d, \frac{-(a_1 d+2)}{a_3})$ is the universal point of order two. In 'step1' and 'step2' the value u_{poly} gets reduced by the equation of the elliptic curve and $f_d(d) = 0$. In the line marked with 'required' one can see that $\beta\gamma - \alpha\delta$ vanishes as stated in the proof of Lemma 3.3.2. The output u_{polyr} is the reduced form of u_{poly}. Remember that for the most computations $a_6 = 0$.

```
u_in:=-(u*v)/(u^2*v*a_2+u*v^2a_4+v^3*a_6+u^3);
v_in:=-(v^2)/(u^2*v*a_2+u*v^2a_4+v^3*a_6+u^3);
u_poly:= eval(fgl_u,{u_1=u_in,v_1=v_in,u_2=d,v_2=-(d*a_1+2)/a_3}):

polyn := eval(simplify(expand(u_poly)),{a_6 = 0}):
```

```
step1 := simplify(rem(numer(polyn),
                  -u^2*v*a_2-u*v^2*a_4-u^3+u*v*a_1+v^2*a_3+v, v)/
                  rem(denom(polyn),
                  -u^2*v*a_2-u*v^2*a_4-u^3+u*v*a_1+v^2*a_3+v, v)
                  ):

np := numer(step1): dp := denom(step1):
red := (a_1^2*a_4-a_1*a_2*a_3+a_3^2)*d^3+(4*a_1*a_4-2*a_2*a_3)*d^2
      +(-a_1*a_3+4*a_4)*d-2*a_3:
rem(coeff(np,v,0)*coeff(dp,v,1)-coeff(np,v,1)*coeff(dp,v,0),red,d);
#required

step2 := collect(simplify(rem(coeff(numer(step1),v,1),red,d)/
                          rem(coeff(denom(step1),v,1), red,d)),u):
u_polyr:=step2:
```

Now you have to insert the transformed value of u_{polyr} in Macaulay 2 and the output coincides with Remark 3.3.3.

A.5 The image of $\Psi(u)$ and $\Psi(v)$ in terms of u.

Now we compute $\Psi(u)$ and $\Psi(v)$. The value im_u is the reduced form of u_{poly} multiplied with $(-u)$. On can also use u_{poly} instead of the reduced form. The value im_v is the v-coordinate of the formal sum $P +_C Q$.

```
a_6:=0:

im_u:= (-u)*(-d*a_3+u*a_3)/
       (-d^2*u*a_1^2*a_4+d^2*u*a_1*a_2*a_3-d^2*u*a_3^2
       -2*d*u*a_1*a_4+d*u*a_2*a_3-a_3):
series_u:=convert(simplify(series(im_u,u,9)),polynom);

v_poly:= eval((lambda*fgl_u+m),
          {u_1=u_in,v_1=v_in,u_2=d,v_2=-(d*a_1+2)/a_3}):
im_v:=eval((v*v_poly),v=uex):
series_v:=convert(simplify(series(im_v,u,9),{red=0}),polynom);
```

Now it is possible to evaluate the operation with certain coefficients. We get with

```
eval(series_u,{a_1=a,a_3=1,a_2=0,a_4=0,a_6=0});
```

the example of Rezk.

A.6 Image of the coefficients

Here, give the code for the computation of the coefficients in the image of the isogeny. This is just the implementation of the proof of Proposition 3.3.5.

```
e := -(a_1*d+2)/a_3:
left:=simplify(-d^3/e*series_v,{red=0}):
right:=eval(eval(uex,{a_1=b_1,a_2=b_2,a_3=b_3,a_4=b_4,a_6=0}),
       u=series_u):
simplify(coeff(left,u,3)-coeff(right,u,3), {red = 0});
erga1:=simplify(solve(coeff(left,u,4)=coeff(right,u,4), b_1),
       {red = 0});
erga2:=simplify(solve(eval(coeff(left,u,5)=coeff(right,u,5),
{b_1 = erga1}), b_2), {red = 0});
erga3:=simplify(solve(eval(coeff(left,u,6)=coeff(right,u,6),
{b_1 = erga1, b_2 = erga2}), b_3), {red = 0});
erga4:=simplify(solve(eval(coeff(left,u,7)=coeff(right,u,7),
{b_1 = erga1, b_2 = erga2, b_3 = erga3}), b_4), {red = 0});
```

To get the desired result without any d's in the denominator it is necessary to substitute 2 with $f_d(d)$:

```
um2 := ((a_1^2*a_4-a_1*a_2*a_3+a_3^2)*d^3+(4*a_1*a_4-2*a_2*a_3)*d^2
       +(-a_1*a_3+4*a_4)*d)/a_3
```

Then we can reproduce the formulas in Proposition 3.3.5.

A.7 Product relations of $Q_i(xy)$

This is the computation in the proof of Proposition 4.2.1. For the product relations one has only to reduce the product of

$$(Q_0(x) + Q_1(x)d + Q_2(x)d^2)(Q_0(y) + Q_1(y)d + Q_2(y)d^2)$$

modulo the usual ideal generated by $f_d(d)$. The output lists the coefficients of every combination $Q_i(x)Q_j(y)$. When $a_6 := 0$ is commented out one gets the relation for the most general Weierstraß form.

```
a_6:=0;
red := (-a_1^3*a_6+a_1^2*a_3*a_4-a_1*a_2*a_3^2+a_3^3)*d^3
+(-6*a_1^2*a_6+4*a_1*a_3*a_4-2*a_2*a_3^2)*d^2
+(-a_1*a_3^2-12*a_1*a_6+4*a_3*a_4)*d-2*a_3^2-8*a_6;
printf("Caption: c[1] = Q_0Q_0, c[2] = Q_0Q_1, c[3] = Q_0Q_2,
c[4] = Q_1Q_0, c[5] = Q_1Q_1, c[6] = Q_1Q_2, c[7] = Q_2Q_0,
c[8] = Q_2Q_1, c[9] = Q_2Q_2.");
prod := collect(simplify(rem(d^4*c[9]+d^3*c[6]+d^3*c[8]+d^2*c[3]
+d^2*c[5]+d^2*c[7]+d*c[2]+d*c[4]+c[1], red, d)), d);

printf("Coefficients for Q_0(xy):");
for i to 9 do
c[i], factor(coeff(coeff(prod, d, 0), c[i]))_
end do;
printf("Coefficients for Q_1(xy):");
for i to 9 do
c[i], factor(coeff(coeff(prod, d, 1), c[i])):
end do;
printf("Coefficients for Q_2(xy):");
for i to 9 do
c[i], factor(coeff(coeff(prod, d, 2), c[i])):
end do;
```

A.8 Adem relations

The difficult part of the Adem relations is dealt with in Lemma 4.3.5. Once we have the relation of in the Lemma, we have to build $P(P(u))$ at first. Then substitute d', in the program code denoted by 'delta', and reduce with $f_d(d)$.

```
a_6:=0;
red:=(a_1^2*a_4-a_1*a_2*a_3+a_3^2)*d^3
+(4*a_1*a_4-2*a_2*a_3)*d^2+(-a_1*a_3+4*a_4)*d-2*a_3;
N:= coeff(red,d,3);
printf("Caption: A = Q_0Q_0, B = Q_0Q_1, C = Q_0Q_2,
D = Q_1Q_0, E = Q_1Q_1, F = Q_1Q_2, G = Q_2Q_0,
H = Q_2Q_1, J = Q_2Q_2.");
ergadem := collect(factor(simplify(eval(J*d^2*delta^2+F*d^2*delta
+H*d*delta^2+C*d^2+E*d*delta+G*delta^2+B*d+D*delta+A,
delta = -um2a3/(d*N)), {red = 0})), d);
Q_00:=collect(simplify(solve(coeff(ergadem, d, 0), A)),
      {A, B, C, D, E, F, G, H, J});
Q_10:=collect(simplify(solve(coeff(ergadem, d, 1), B)),
      {D, F, G, H, J});
Q_20:=collect(simplify(solve(coeff(ergadem, d, 2), C)),
      {D, G, H, J});
```

B Series expansion.

B.1 Series expansion of the elliptic curve.

The series expansion of the elliptic curve up to degree 10:

$$\begin{aligned} E(u) &= u^3 - a_1u^4 + (a_1^2 + a_2)u^5 + (-a_1^3 - 2a_1a_2 - a_3)u^6 \\ &\quad +(a_1^4 + 3a_1^2a_2 + 3a_1a_3 + a_2^2 + a_4)u^7 \\ &\quad +(-a_1^5 - 4a_1^3a_2 - 6a_1^2a_3 - 3a_1a_2^2 - 3a_1a_4 - 3a_2a_3)u^8 \\ &\quad +(a_1^6 + 5a_1^4a_2 + 10a_1^3a_3 + 6a_1^2a_2^2 + 6a_1^2a_4 + 12a_1a_2a_3 + a_2^3 + 3a_2a_4 \\ &\quad +2a_3^2 + a_6)u^9 + (-a_1^7 - 6a_1^5a_2 - 15a_1^4a_3 - 10a_1^3a_2^2 - 10a_1^3a_4 \\ &\quad -30a_1^2a_2a_3 - 4a_1a_2^3 - 12a_1a_2a_4 - 10a_1a_3^2 - 6a_2^2a_3 \\ &\quad -4a_1a_6 - 4a_3a_4)u^{10} + \dots \end{aligned}$$

B.2 The image of P via the map Ψ in terms of u.

We give an idea what the first few terms of the series expansion for a general elliptic curve ($a_6 \neq 0$) are. The programs above should be a bit modified for this. The images of $u(P)$ and $v(P)$ in terms of u have the form

$$\begin{aligned} u(\Psi(P)) &= b_0 + b_1u + b_2u^2 + b_3u^3 + b_4u^4 + \dots \\ v(\Psi(P)) &= c_0 + c_1u + c_2u^2 + c_3u^3 + c_4u^4 + \dots \end{aligned}$$

modulo the ideal generated by

$$f_d(d) = (a_1^2a_4 - a_1a_2a_3 + a_3^2)d^3 + (4a_1a_4 - 2a_2a_3)d^2 + (4a_4 - a_1a_3)d - 2a_3$$

Let $N := a_1^2a_4 - a_1a_2a_3 + a_3^2$.For the first series we have:

$$\begin{aligned} b_0 &= 0 \\ b_1 &= -d \\ b_2 &= \frac{-(2a_1a_4 - a_2a_3)}{a_3}d^2 + \frac{(a_1a_3 - 4a_4)}{a_3} + 3 \end{aligned}$$

$$
\begin{aligned}
b_3 &= \frac{1}{a_3 N}((a_1^4 a_4^2 - a_1^2 a_2^2 a_3^2 - 2a_1^2 a_3^2 a_4 + 4a_1 a_2^2 a_3 a_4 + 4a_1 a_2 a_3^3 - 2a_2^3 a_3^2 \\
&\quad -16a_1 a_3 a_4^2 + 8a_2 a_3^2 a_4 - 3a_3^4)d^2 + (-a_1^4 a_3 a_4 + a_1^3 a_2 a_3^2 + 2a_1^3 a_4^2 \\
&\quad -a_1^2 a_2 a_3 a_4 - a_1^2 a_3^3 - 2a_1 a_2^2 a_3^2 + 6a_1 a_3^2 a_4 + 4a_2^2 a_3 a_4 + a_2 a_3^3 - 16a_3 a_4^2)d \\
&\quad -a_3(2a_1^3 a_3 a_4 + 2a_1^2 a_2 a_3^2 - 2a_1 a_3^3 - 2a_2^2 a_3^2 + 8a_3^2 a_4)) \\
b_4 &= \frac{1}{a_3 N^2}((-a_1^7 a_4^3 + 3a_1^5 a_2^2 a_3^2 a_4 - 2a_1^4 a_2^3 a_3^3 - 4a_1^5 a_2 a_4^3 + 3a_1^5 a_3^2 a_4^2 + a_1^4 a_2^2 a_3 a_4^2 \\
&\quad -12a_1^4 a_2 a_3^3 a_4 + 3a_1^3 a_2^3 a_3^2 a_4 + 9a_1^3 a_2^2 a_3^4 + 36a_1^4 a_3 a_4^3 - 8a_1^3 a_2^2 a_4^3 - 52a_1^3 a_2 a_3^2 a_4^2 \\
&\quad +9a_1^3 a_3^4 a_4 + 12a_1^2 a_2^3 a_3 a_4^2 + 16a_1^2 a_2^2 a_3^3 a_4 - 12a_1^2 a_2 a_3^5 - 12a_1 a_2^4 a_3^2 a_4 \\
&\quad -7a_1 a_2^3 a_3^4 + 4a_2^5 a_3^3 + 32a_1^3 a_4^4 - 48a_1^2 a_2 a_3 a_4^3 + 16a_1^2 a_3^3 a_4^2 + 72a_1 a_2^2 a_3^2 a_4^2 \\
&\quad +8a_1 a_2 a_3^4 a_4 + 5a_1 a_3^6 - 28a_2^3 a_3^3 a_4 + 7a_2^2 a_3^5 - 96a_1 a_3^2 a_4^3 + 48a_2 a_3^3 a_4^2 \\
&\quad -20a_3^5 a_4)d^2 + (a_1^7 a_3 a_4^2 - 2a_1^6 a_2 a_3^2 a_4 + a_1^5 a_2^2 a_3^3 - 2a_1^6 a_4^3 + 2a_1^5 a_2 a_3 a_4^2 \\
&\quad +2a_1^5 a_3^3 a_4 + 5a_1^4 a_2^2 a_3^2 a_4 - 2a_1^4 a_2 a_3^4 - 5a_1^3 a_2^3 a_3^3 - 8a_1^4 a_2 a_4^3 - 10a_1^4 a_3^2 a_4^2 \\
&\quad +4a_1^3 a_2^2 a_3 a_4^2 + a_1^3 a_3^5 + 6a_1^2 a_2^3 a_3^2 a_4 + 13a_1^2 a_2^2 a_3^4 + 48a_1^3 a_3 a_4^3 - 16a_1^2 a_2^2 a_4^3 \\
&\quad -72a_1^2 a_2 a_3^2 a_4^2 - 2a_1^2 a_3^4 a_4 + 16a_1 a_2^3 a_3 a_4^2 - 4a_1 a_2^2 a_3^3 a_4 - 14a_1 a_2 a_3^5 - 8a_2^4 a_3^2 a_4 \\
&\quad +2a_2^3 a_3^4 + 64a_1^2 a_4^4 - 64a_1 a_2 a_3 a_4^3 + 80a_1 a_3^3 a_4^2 + 48a_2^2 a_3^2 a_4^2 - 16a_2 a_3^4 a_4 + 6a_3^6 \\
&\quad -64a_3^2 a_4^3)d + a_3(2a_1^6 a_3 a_4^2 - 4a_1^5 a_2 a_3^2 a_4 + 2a_1^4 a_2^2 a_3^3 + 4a_1^4 a_2 a_3 a_4^2 + 4a_1^4 a_3^3 a_4 \\
&\quad -2a_1^3 a_2^2 a_3^2 a_4 - 4a_1^3 a_2 a_3^4 - 2a_1^2 a_2^3 a_3^3 - 24a_1^3 a_3^2 a_4^2 + 8a_1^2 a_2^2 a_3 a_4^2 + 32a_1^2 a_2 a_3^3 a_4 \\
&\quad +2a_1^2 a_3^5 - 8a_1 a_2^3 a_3^2 a_4 - 2a_1 a_2^2 a_3^4 + 4a_2^4 a_3^3 - 32a_1^2 a_3 a_4^3 + 32a_1 a_2 a_3^2 a_4^2 \\
&\quad -24a_1 a_3^4 a_4 - 24a_2^2 a_3^3 a_4 + 4a_2 a_3^5 + 32a_3^3 a_4^2))
\end{aligned}
$$

$$
\begin{aligned}
c_0 &= 0 \\
c_1 &= 0 \\
c_2 &= 0 \\
c_3 &= \frac{-a_1 d - 2}{a_3} \\
c_4 &= \frac{1}{a_3^2}((a_1^2 a_4 - 2a_1 a_2 a_3 + 3a_3^2)d^2 + (2a_1^2 a_3 + 4a_1 a_4 - 4a_2 a_3)d + 4a_1 a_3 + 4a_4)
\end{aligned}
$$

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