SOME PROJECTS

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These projects are listed roughly in inverse order of the probability that I will work on them soon.

1. $K(n)$-HOMOLOGY OF FINITE GENERAL LINEAR GROUPS

- Different characteristics: HKR theory, Tanabe theory. Commutative algebra aspects.
- Equal characteristics: analogues of Harish-Chandra induction.
- Relationship with $K(\mathbb{F}_p)$ and $K(\mathbb{Q})$; hope for a more canonical view of the relationship between Adams operations and the Galois action.

Part of the HKR theory is written up, for the elliptic case, in my note on rational equivariant elliptic cohomology. I think I have some calculations for the Tanabe picture somewhere; I think that there are interesting things to say about the ring theory of $K(n)^*BGL_m(\mathbb{F}_q)$, but I don't know what they are.

For the equal characteristic case, I have a reasonably detailed analysis of the HKR picture for $GL_4(\mathbb{F}_3)$, and many pages of handwritten notes on related ideas and calculations. But I am not sure that it leads to anything enlightening.

Sam Marsh's thesis covers $K(n)^*BGL_d(\mathbb{F}_l)$ where l is a prime power with $v_p(l-1) = 1$ and $d \leq p$.

2. RATIONAL SPECTRA AND CHAIN COMPLEXES

Recently I have been working on a new project with Stefan Schwede. It is folklore that the category of rational spectra is closely related to the category of rational chain complexes. Ideally one would like a version of this statement that is strongly compatible with the relevant Quillen model structures and monoidal products, but the literature does not contain such a version. This problem has been most keenly felt in the work of Greenlees, Shipley and Barnes on algebraic models for various categories of rational equivariant spectra, and has been exacerbated by a mistake in the literature that has made some things look more complicated than they really are. Stefan and I independently constructed a new functor Φ_* from symmetric spectra to rational chain complexes that cures most of the problems and promises to simplify greatly projects such as those of Greenlees, Shipley and Barnes. We then combined forces, and we discovered that $\Phi_*(\Sigma^\infty_+ X)$ can be described as a kind of complex of simplicial de Rham currents, dual to Sullivan's complex $\Omega^*(X)$ of simplicial differential forms. It turns out to be a substantial job to set up this theory of currents and relate it to $\Phi_*(\Sigma^\infty_+ X)$, so we have written a paper of 25 pages that does only that. The current draft contains all the statements and proofs that it needs; it should be ready for submission after some expository improvements, addition of references, and checking of details. Later we will write a separate paper discussing Φ_* on more general symmetric spectra. We already have 25 pages or so of typed notes by Stefan or myself that could contribute to this second paper.

3. MULTICURVES AND EQUIVARIANT GENERALISED COHOMOLOGY

The idea here is to generalise the whole apparatus of formal group theory and complex-orientable ring spectra to an equivariant setting where the group of equivariance is finite and abelian. Most of what I know about this is in a recently submitted memoir. There remain a number of open questions, especially those related to the structure of the equivariant Lazard ring and its conjectural equivalence with the homotopy of the equivariant complex cobordism ring.

4. OPERATIONS IN E-THEORY

- Power operations for infinite loop spaces
- The small resolution
- Unstable operations

- Power operations for finite loop spaces

There is an interesting part of the theory of power operations that is quite formal and is essentially written up. There are also some conjectures due to Mike Hopkins that are highly nontrivial, and some very detailed but less conceptual conjectures of my own that follow on from these. I expect that these will be hard to prove. They connect in various ways with the project about multisets etc discussed below.

I also have notes on general unstable operations; here the basic idea is quite clear, but there are unpleasant technicalities about topologies on various rings, which may now have been resolved by John Hunton.

The theory for finite loop spaces is much less clear. There is recent work of McClure and Smith on ordinary homology operations on finite loop spaces, which may provide new insights. A good understanding of $E_*\Omega^k S^{n+k}$ (even for $k \leq 3$) would be helpful for Ravenel's approach to the telescope conjecture, and might also illuminate the proof of the nilpotence theorem.

An optimal treatment of power operations will rely on an improved version of the deformation theory of formal groups that works for deformations over arbitrary formal schemes. Part of this is written up. It would also be good to revisit the theory of Dieudonné modules for the relevant kind of formal groups, and its relation to Gross-Hopkins duality and the Picard group. In particular, I conjecture that $\mathrm{THH}(F(K(\mathbb{Z}, n+1)_+, L_{K(n)}S^0))$ should be an invertible $K(n)$-local spectrum whose image in the algebraic Picard group should correspond to the reduced norm character of the Morava stabiliser group. If true, this might lead to a more geometric approach to Gross-Hopkins duality.

5. Multisets, symmetric powers and so on

I have an old project to clarify the relationships between a long list of different things:

- Symmetric powers of (stable or unstable) spheres.
- Partition complexes.
- Buildings.
- Universal spaces for families.
- Multisets.
- Steinberg idempotents.
- Steinberg pieces of $GL(V)$-spectra.
- Mitchell's complexes with $\mathcal{A}(n)$-free cohomology.
- Arone's complexes with $\mathcal{A}(n)$-free cohomology.
- The spectra $L(n)$ and $M(n)$.
- The Whitehead conjecture.
- The classical Steenrod algebra.
- The classical Dyer-Lashof algebra.
- The classical Lambda algebra.
- Power operations in Morava E-theory.
- The Hopkins resolution.
- The bar resolution $B(\Sigma^\infty, Q, H)$.
- The Bousfield-Kan resolution $Q^{\bullet+1}X$.
- The Goodwillie functors $P_n\mathrm{Id}$ and $D_n\mathrm{Id}$.
- The EHP sequence and (super)stable versions.
- The Adams Ext groups.
- The Bousfield-Kuhn functor.
- Thick subcategories in the nilpotence theorem.
- Ravenel's approach to $\pi_*X(n)$ and π_*S^0.

I wrote an elaborate talk for the Abel symposium surveying these ideas. Other documents related to this project are as follows:

- A 31 page note explaining a variety of things from the above list.
- Nine pages of notes on various related phenomena in ordinary mod p (co)homology, including some interesting new insights into the rings $H^*(B(\mathbb{Z}/p)^n)^{GL_n(\mathbb{Z}/p)}$ at odd primes.
- A fifteen page note on Hecke algebras, the Bruhat decomposition and so on. Much of this is exposition of known material in a form that is convenient for our applications. This is more or less finished.

- Eight more pages on finer properties of the Hecke algebra. Unfortunately these expose an error in earlier work of Kuhn, on which we hoped to rely: a certain formula that he established in characteristic p does not in fact lift to characteristic zero, contrary to his later assertion. This makes it much less likely that Kuhn's computational proof of the formula can easily be replaced by something more conceptual.
- A 33 page note explaining various relevant operads, and a little general theory of operads. This is incomplete in a number of ways, partly because I spent a long time trying to prove things about the Stasheff operad that now seem unlikely to be true; I have a further nine pages of notes explaining negative results related to this.

6. Equivariant Bousfield classes

- Formalities about families etc.
- The Segal conjecture reformulated in this language.
- Chromatic theory.
- Which height functions are realisable?

I have a preprint about this, mostly covering formalities and the chromatic theory. Currently I am working on reformulating the Segal conjecture, which seems to be both harder and more interesting than previously anticipated. The Bousfield lattice maps naturally to the lattice of "admissible height functions"; the final problem is to determine the image of this map. This also seems likely to be hard. The most interesting result obtained so far involves the Greenlees-May theory of generalised Tate cohomology, which also plays a large role in the Segal conjecture. It is possible that a better understanding of the Segal conjecture will also lead to further progress on the realisation question.

7. Tambara theory, Dieudonné modules, and the Nilpotence Theorem

- I have written a 17 page note on Tambara functors [9], which encode the algebraic structure of the zeroth homotopy group of an equivariant, strictly commutative ring spectrum. This note presents a new perspective on the definition of a Tambara functor, which allows one to see much more clearly that Tambara functors are to rings what Mackey functors are to abelian groups. My note also has an analysis of rational Tambara functors, which I believe to be new. Moreover, it discusses a link between Witt rings (in the generalised sense of Dress and Siebeneicher [3]) and free Tambara functors; this extends and corrects a paper of Morten Brun [1], and makes contact with Jesse Elliott's perspective [5] on the Dress-Siebeneicher construction. These Witt rings (for cyclic groups of equivariance) are related to topology in two different ways. On the one hand, for any (-1)-connected strictly commutative ring spectrum A, there is a topological Hochschild homology spectrum $THH(A)$ with a circle action, whose equivariant homotopy groups in degree zero can be described in terms of Witt rings. On the other hand, if we fix a prime p, we can fit together Witt rings in a certain way to form a "Dieudonné module" for any bicommutative Hopf algebra over $\mathbb{Z}/p$. Paul Goerss has shown that this construction is relevant to the generalised homology of the important space $\Omega^2 S^3$; this is a striking result, but the details are less complete, direct and canonical than one would like. My work on Tambara functors arose from an attempt to improve this; I may at some point return to that problem.
- My thoughts about Dieudonné modules arose in turn from an attempt to revisit the proof of Devinatz, Hopkins and Smith's seminal Nilpotence Theorem [2], using the tools of derived algebraic geometry, and in particular the duality apparatus of Dwyer, Greenlees and Iyengar [4]. The spectrum $R = \Sigma_+^\infty \Omega^2 S^3$ plays a central role in the original proof, and the spectrum $Q = F(\Sigma_+^\infty \Omega S^3, S^0)$ appears implicitly. Naively one would expect the duality theory to give a close relationship between R-modules and Q-modules, but this breaks down because certain finiteness assumptions are not satisfied. One can attempt to fix this by replacing Q by a certain "uncompleted version" Q', which arises as the reduced topological Hochschild homology of $F(S_+^3, S^0)$. It is not clear whether this approach can be made to work (although I have some interesting small results on peripherally related questions). Certainly, it does not offer the gains in simplicity for which I originally hoped. I have

documents about these questions totalling 27 pages, but a significant fraction of this may be a dead end.

8. The unstable $K(n)$-local category

- The functors $\phi, \theta, \Sigma^\infty, \Omega^\infty$ and their formal properties.
- $K(n)$-localisation preserves fibrations up to a small error term.
- The low-dimensional and high-dimensional part of the unstable $K(n)$-local category, and the stable category: find a good categorical formulation.
- Fibrations with Wilson spaces.
- How to recognize $L_{K(n)} \bigvee_i \Sigma^{d_i} E(n)$.
- The Rezk logarithm.
- The general theory of the adjunction between $\Sigma^\infty \Omega^\infty$ and gl_1. Twisted versions for infinite loop Thom spectra. Relation with THH.
- Computation of $\phi \colon [\Omega^\infty E, \Omega^\infty F] \to [E, F]$ when E and F are complex-orientable.
- Surgery spectra as examples.
- Work of Mahowald et al on v_n-periodic homotopy: does this compute $\phi(S^n)$?
- Relation with the EHP spectral sequence and the Goodwillie tower of the identity.
- Interface between rational and chromatic theory.

I have handwritten notes on some of this, not very detailed. I am not up to speed on what Bousfield and/or Kuhn have done recently in these directions. The technology level is too high for a PhD.

9. Stable homotopy of comodules

- Modified model structure.
- Tannaka theory.
- Stable homotopy axioms.
- Quasicoherent sheaves over regular schemes?

The idea would be to introduce algebraic axioms on a category (satisfied by comodule categories over a suitable Hopf algebroid, and possibly also by categories of quasicoherent sheaves over regular schemes) allowing one to give the category a "cellular" model structure. This would make it a stable model category, whose homotopy category would be stable in the Hovey-Palmieri-Strickland sense; in particular, the generators would be dualisable. I think that all this could be done by assembling work of myself, Mark Hovey, and Mark's student Jim Gillespie; the main issue would just be to tidy it up.

10. Secondary operations and representations of categories

Consider the abelian category $\mathcal{A}$ of additive functors from the Spanier-Whitehead category $\mathcal{F}$ to the category of Abelian groups. The category $\mathcal{H}$ of homology theories (or spectra mod phantoms) embeds in $\mathcal{A}$. The smash product and extended power functors can be extended to $\mathcal{A}$.

Any functor in $\mathcal{A}$ can be expressed as the image (or kernel, or cokernel) of a morphism of homology theories. A higher-order homology operation is the same thing as a morphism in $\mathcal{A}$. The project would be to reexamine a lot of classical results about such operations, and their relationship with differentials in the Adams spectral sequence, in terms of this point of view.

Given a module over the Steenrod algebra (or equivalent for other cohomology theories) there is a spectral sequence converging to the homotopy groups of the space of realizations, which seems closely related to the Adams spectral sequence. It would also be good to make contact with this picture.

Partly in the same vein, it would be good to have sharper algebraic models for various categories of low-dimensional spectra. I have a note that gives a precise model for Moore spectra. It should be possible to do something with connective j-theory to handle p-local spectra of dimension at most $2p - 2$ or so. I also have notes on work of Baues and Drozd on the classification of 5-dimensional torsion-free spectra.

11. Conformal field theory and elliptic cohomology

- Parabolic series
- The Tate curve: θ-functions, the Edwards model, the Neron model.
- Subgroups and isogenies for the Tate curve
- Representations of loop groups and their central extensions
- Line bundles on powers of elliptic curves
- Modular operads
- Explicit CFT's for tori, simple Lie groups, and 3-manifolds

12. Moduli spaces of stable curves, and quantum cohomology

- Symmetric group action
- Generalities about modular operads
- Operadic computations
- Formulation in terms of formal groups

This would of course follow on from Daniel Singh's PhD.

13. String topology and the cactus operad

I have a note about the combinatorial structure of the cactus operad, which is incomplete. It might be a plausible PhD project to complete it, relate it to the Cohen-Jones picture, and so on. However, there might not be enough here that is not already known. I have been corresponding with Craig Westerland about this.

14. Unstable homotopy

I have notes on

- Concrete representatives of homotopy groups of $SO(n)$ for small n.
- Anick spaces and the odd primary EHP fibrations.
- Attempts at a sharper analysis of Hopf-invariant maps and the Hilton-Milnor theorem.
- Clarifying the concept of genealogy for homotopy elements (at the prime 2).

Probably all of this is well-known to the experts.

15. Complex orientable cohomology of various manifolds

- Kummer manifolds
- Fermat hypersurfaces
- Springer varieties

In each case, the ordinary cohomology is known, at least up to some kind of filtration. The challenge would be to give a natural description, without indeterminacy or arbitrary choices. I think that in all three cases, a really good answer could lead to some interesting ideas, and new techniques for dealing with Frobenius algebras. I'd be happy to get a first year PhD student to look at these examples, but I would want to have an escape route in mind if they did not seem to be leading to anything usable. Philip Eve has been looking at Springer varieties.

16. New foundations for equivariant stable homotopy

The project would be to set up a version of (globally) equivariant stable homotopy theory using a model structure on S-modules built from the "orbit cells" $S(\mathcal{U}_G)/G$. The first task would be to set up reprove some basic results (such as the Adams isomorphism and the tom Dieck splitting) from this point of view. One would then prove things like $[S(\mathcal{U}_G)/G, S(\mathcal{U}_H)/H] = A(G, H)$.

17. Revisiting the image of J

I am dissatisfied with the traditional treatment of the image of J. I would prefer to see something like this:

- One should be able to write down a homotopy-cartesian cube of strictly commutative ring spectra, one corner of which is the K-local sphere, which we denote by J.
- The (non-cartesian) square

$$
\begin{array}{ccc}
J & \longrightarrow & K \\
\downarrow & & \downarrow \\
S\mathbb{Q} & \longrightarrow & K\mathbb{Q}
\end{array}
$$

should relate well to the cube.
- The cube should provide a proof that the corner is indeed J.
- The cube should provide a calculation of $\pi_* J$. The Bernoulli numbers should enter the calculation in the natural way, as coefficients of the series $\log(t/(1 - e^{-t}))$, rather than as the gcd of expressions like $k^N(k^n - 1)$.
- If we let j denote the connective cover $J[0, \infty)$, and write Ij for its Brown-Comenetz dual, then we have $J/j = \Sigma^{-2} Ij$. Ideally, this should be visible from the cube. Note also that $(L_K kO)/kO = \Sigma^{-6} IkO$ and $(L_K kU)/kU = \Sigma^{-4} IkU$.
- The stable J-homomorphism $k \to \mathrm{bglp}_1(S; K)$ should enter naturally into the picture.
- Everything should be done rationally or integrally, without separation of primes.
- Ideally, the quadratic and symmetric L-theory spectra should fit into the picture, as well as $MSTop$ and possibly $K(\mathbb{Z})$.
- For a number field F it is known (by work of Borel) that $K_*(\mathcal{O}_F) \otimes \mathbb{Q}$ is abstractly isomorphic to $\mathbb{Q} \oplus \Sigma((K_*(\mathbb{R} \otimes F)/\mathbb{Z}) \otimes \mathbb{Q})$. Similar things seem to work if we complete at an odd regular prime rather than rationalising. It would be good to have a more structured approach to all this. If I understand correctly, one can deduce that that the map $K(\mathcal{O}_F) \to kU$ actually factors through j (even integrally).

I have various notes related to all this. It is conceivable that there is a connection with the idea of John Rognes that $K(kU)$ might be related to tmf.

18. Geometry of spectra

I have a large body of old notes about geometric categories of spectra, which never got published because I was unable to prove the technical results necessary for a satisfying theory. Related documents include the following:

- A 23 page note on compactly generated weak Hausdorff spaces, including a useful supply of counterexamples, and technical results on the interaction of various kinds of limits and colimits.
- An 8 page expository introduction to the finite stable category.
- An 18 page compendium of miscellaneous results on the interaction between linear algebra and homotopy theory.
- A 48 page document giving a new formulation of the EKMM category of S-modules, and proving many technical results about it.
- A three page note giving a new construction of the geometric fixed point functor for equivariant spectra.
- An eight page note giving axioms (but not a construction) for a good equivariant version of the functor

$$
gl_1 \colon \{ \text{ commutative ring spectra } \} \to \{ \text{ spectra } \},
$$

and drawing consequences from those axioms.

19. Surgery for Frobenius algebras

The theory of Poincaré complexes is a homotopical version of the theory of manifolds, formulated in the category of spaces. It includes "surgical" constructions, like gluing Poincaré complexes along a common

boundary component, or cutting out a tubular neighbourhood of a subcomplex. It should be possible to set up a similar theory in many other model categories. In particular, given a strictly commutative ring spectrum k, we let $\mathcal{X}_k$ denote the opposite of the category of commutative k-algebras. There is a functor from spaces to $\mathcal{X}_k$ given by $X \mapsto F(X_+, k)$, which preserves a great deal of structure. Thus, if X is a manifold (or more generally, a Poincaré complex), then $F(X_+, k)$ is the analogous thing in $\mathcal{X}_k$, which is a Frobenius algebra. We therefore expect to be able to do surgery with Frobenius algebras, by mimicking the standard constructions with Poincaré complexes.

(Note here however, that the functor $X \mapsto F(X_+, k)$ does not have good monoidal properties when $\pi_1(X) \neq 0$; this is related to non-convergence of Eilenberg-Moore spectral sequences.)

Now let k be Morava E-theory. It is then known that $F(BG_+, k)$ is a Frobenius algebra over k, for any finite group G. In many cases the ring $k^* BG = \pi_* F(BG_+, k)$ cannot be described in a civilised way using generators and relations. We hope to show that there are much more tractable descriptions using surgical language.

20. Gysin maps and local (co)homology

- Cohomology of Grassmann bundles and local homology.
- Gysin maps and residues.
- Toric examples. Possible connections with mirror symmetry for toric Landau-Ginzburg models.
- Relation with the local homology spectral sequence.
- General theory of cobordism of Frobenius algebras.
- Relationship with geometric quantization and symplectic reduction.
- Can we obtain the profinite completion of S^0 by a Tate-like procedure?

I have substantial notes about toric things generally, but not about Gysin maps, or the detailed relationship with local cohomology. My multicurves paper has a detailed analysis of Gysin maps for projective bundles, but not for Grassman bundles. The bit about geometric quantization is speculative. The papers [6–8] may be relevant.

21. Generalised cohomology of $BSO(n)$

I have some notes about this, largely following an approach of Inoue and Yagita. Probably the best plan would be to incorporate these into the bestiary (see Section 28).

22. Morava K-theory of extraspecial p-groups

I have written up the case of the nonabelian group of order 27 and exponent 3, in a rather terse way. This could be published with a little extra expository material. The Chern approximation is not exact in this example.

23. Arithmetic equivariant elliptic cohomology

This project is probably too stale now. As part of it, I wrote notes on étale extensions of ring spectra. There has been some interest in these for their expository value; I may publish them in the proceedings of the Banff meeting. Some other notes may also be useful.

24. Symmetric cocycles and ring spectra

There are some interesting things left over from my old σ-orientation project with Ando and Hopkins. There we proved that $\operatorname{spec}(E_0 BU\langle 6\rangle) = C^3(G; G_m)$. I still have not published the stronger fact that $\operatorname{spf}(E^0 BU\langle 6\rangle) = C_3(G)$. It should be possible to understand the homotopy and generalised homology of $MSO[1/2]$, $M\operatorname{Spin}[1/2]$ and $M\operatorname{String}[1/6]$ along similar lines. With more work it should also be possible to do some of this without inverting 2 or 6.

25. Immersions

Given a manifold M, one can ask for the smallest d such that there exists an immersion $M \to \mathbb{R}^d$. Gonzalez and Zarate have studied this, in the case where M is a lens space S^{2n+1}/C_{2^r}, using BP theory. My feeling is that it should be possible to clarify and improve their results by using more theory of frmal groups. I have some notes on this, but they do not get very far.

26. Jacobians

I have some notes on formal Jacobians of curves and their splittings, which could be used to produce new formal groups and thus new ring spectra. Tyler Lawson and Mark Behrends have now gone far beyond where I got to, so this is probably no longer of interest.

27. Linking systems

I have a 7 page note on the theory of linking systems, as used by Broto, Levi and Oliver in their work on p-finite groups. I like my formulation much better than theirs, but I cannot really do anything with it until I have found the right way to translate Alperin's Fusion Theorem, which currently mystifies me.

28. The bestiary

This is an account of various interesting examples in algebraic topology. It would be good to make a hyperlinked, wikified version, ideally including machine-processable descriptions of the main algebraic structures. It would also be good to add more examples, and possibly integrate material from some lecture notes etc. The bestiary directory also contains a compendium of examples on spectral sequences.

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