Advanced Course on Simplicial Methods in Higher Categories

Notes of the Course

February 4 to 14, 2008

Centre de Recerca Matemàtica

Bellaterra (Spain)

Acknowledgements: The Advanced Course on Simplicial Methods in Higher Categories is supported by the AGAUR (ref. 2007/ARCS1/00104) and by the Spanish Topology Network (RET).

Contents

The notes contained in this booklet were printed directly from files supplied by the authors before the course.

Foreword

This booklet contains lecture notes for the Advanced Course on Simplicial Methods in Higher Categories that will take place at the CRM from February 4th to 14th, 2008. This course is one of the main activities of the 2007-2008 CRM Research Programme on Homotopy Theory and Higher Categories, which is organised by the CRM under the scientific responsibility of Carles Casacuberta (Universitat de Barcelona), André Joyal (Université du Québec à Montréal), Joachim Kock (Universitat Autònoma de Barcelona), Amnon Neeman (Australian National University, Canberra), and Frank Neumann (University of Leicester).

The main emphasis of the whole research programme is on categorical aspects of homotopy theory and applications to other areas of mathematics. The course is an expression of this emphasis, treating three hot topics within this general research direction, each featuring the interplay between model categories and higher category theory.

The lectures of André Joyal develop the theory of quasi-categories, which is a main higher-categorical alternative to model categories. This theory is fundamentally simplicial. The lectures of Ieke Moerdijk concern the theory of dendroidal sets, an extension of the theory of simplicial sets (and quasi-categories) designed for the homotopy theory of operads. Those of Bertrand Toën provide an introduction to derived geometry based on simplicial presheaves and using model categories as a main tool. This advanced course is addressed to young researchers with a basic knowledge of algebraic topology, category theory and algebraic geometry, and aim at providing them with the necessary background for doing research in the exciting new world of higher methods in homotopy theory.

We are indebted to the three lecturers for their very positive reaction to the conception of this event more than a year ago and for the great care that they took in the preparation of course materials. Thanks are also due to Mathieu Anel and Myles Tierney for offering preparatory lectures to the programme participants before the start of the course.

This activity is supported by the Generalitat de Catalunya and the Spanish Topology Network (RET). The Ingenio Mathematica (i-MATH) project and the Spanish Ministry of Education and Science have also provided funds through their overall contribution to the 2007-2008 CRM Research Programme on Homotopy Theory and Higher Categories. Our indebtedeness goes especially to the CRM Director for his help and guidance, and to the CRM Secretariat for their excellent, indispensible work.

The Scientific Organisers

Volume I

Dendroidal Sets

I. Moerdijk

Contents

Chapter 1

Introduction

These are the notes for my lectures in the Advanced Course on Simplicial Methods in Higher Categories at the CRM, February 2008. They are based on the following two papers [MW1, MW2] written with Ittay Weiss.

There is an intimate relation between simplicial sets and categories (and, more generally, between simplicial objects and enriched categories), which plays a fundamental role in many parts of homotopy theory. The goal of this paper is to introduce an extension of the category of simplicial sets, suitable for studying operads. We call the objects of this larger category "dendroidal sets", and denote the inclusion functor by

$$i_! : (\text{simplicial sets}) \to (\text{dendroidal sets}).$$

The pair of adjoint functors

$$\tau : (\text{simplicial sets}) \rightleftarrows (\text{categories}) : N$$

where N denotes the nerve and τ its left adjoint, will be seen to extend to a pair

$$\tau_d : (\text{dendroidal sets}) \rightleftarrows (\text{operads}) : N_d$$

having similar properties.

Many other properties and constructions of simplicial sets also extend to dendroidal sets. In particular, we will show that the cartesian closed monoidal structure on simplicial sets extends to a (non-cartesian!) closed monoidal structure on dendroidal sets. Here "extends" means that there is a natural isomorphism

$$i_!(X \times Y) \cong i_!(X) \otimes i_!(Y)$$

for any two simplicial sets X and Y. This tensor product of dendroidal sets is closely related to the Boardman-Vogt tensor product of operads. In fact, the latter can be defined in terms of the former by the isomorphism

$$\mathcal{P} \otimes_{BV} \mathcal{Q} \cong \tau_d(N_d\mathcal{P} \otimes N_d\mathcal{Q})$$

for any two operads $\mathcal{P}$ and $\mathcal{Q}$.

We will also define a notion of inner (or weak) Kan complex for dendroidal sets, extending the simplicial one in the sense that for any simplicial set X, one has that X is an inner Kan complex if, and only if, $i_!(X)$ is. The nerve of an operad always satisfies this dendroidal inner Kan condition, just like the nerve of a category satisfies the simplicial inner Kan condition. Moreover, this inner Kan condition has various basic properties related to the monoidal structure on dendroidal sets, the most significant one being that, under some conditions on a dendroidal set X, $\underline{Hom}(X, K)$ is an inner Kan complex whenever K is. The analogous property for simplicial sets was recently proved by Joyal, and forms one of the basic steps in the proof of the existence of the closed model structure on simplicial sets in which the inner Kan complexes are exactly the fibrant objects. Joyal calls these inner Kan complexes quasi-categories, and one might call a dendroidal set a *quasi-operad* if it satisfies our dendroidal version of the inner Kan condition. We expect that there is a closed model structure on dendroidal sets in which the quasi-operads are the fibrant objects. Dendroidal sets also seem to be useful in the theory of homotopy-$\mathcal{P}$-algebras for an operad $\mathcal{P}$ and weak maps between such algebras. As an illustration of this point, we will give an inductive definition of weak higher categories and weak functors between these, based on the theory of inner Kan complexes.

Chapter 2

Operads

In this paper, *operad* means *coloured symmetric operad*. (In the literature such operads are also referred to as symmetric multi-categories [Lei2].) We briefly recall the basic definitions, and refer to [BM2] for a more extensive discussion. An operad $\mathcal{P}$ is given by a set of colours C, and for each $n \geq 0$ and each sequence of colours $c_1, \cdots, c_n, c$ a set $\mathcal{P}(c_1, \cdots, c_n; c)$ (to be thought of as operations taking n inputs of colours $c_1, \cdots, c_n$ respectively to an output of colour c). Moreover, there are structure maps for units and composition. If we write $I = \{*\}$ for the one-point set, there is for each colour c a unit map

$$u : I \to \mathcal{P}(c; c)$$

taking $*$ to 1_c. The composition operations are maps

$$\mathcal{P}(c_1, \cdots, c_n; c) \times \mathcal{P}(d_1^1, \cdots, d_{k_1}^1; c_1) \times \cdots \times \mathcal{P}(d_1^n, \cdots, d_{k_n}^n; c_n) \to \mathcal{P}(d_1^1, \cdots, d_{k_n}^n; c)$$

which we denote $p, q_1, \cdots, q_n \mapsto p(q_1, \cdots, q_n)$. These operations should satisfy the usual associativity and unitary conditions. Furthermore, for each $\sigma \in \Sigma_n$ and colours $c_1, \cdots, c_n, c \in C$ there is a map $\sigma^* : \mathcal{P}(c_1, \cdots, c_n; c) \to \mathcal{P}(c_{\sigma(1)}, \cdots, c_{\sigma(n)}; c)$. These maps define a right action of Σ_n in the sense that $(\sigma\tau)^* = \tau^*\sigma^*$, and the composition operations should be equivariant in some natural sense. The definition can equivalently be cast in terms of the units and the "$\circ_i$-operations"

$$\mathcal{P}(c_1, \cdots, c_n; c) \times \mathcal{P}(d_1, \cdots, d_k; c_i) \xrightarrow{\circ_i} \mathcal{P}(c_1, \cdots, c_{i-1}, d_1, \cdots, d_k, c_{i+1}, \cdots, c_n; c).$$

A coloured operad $\mathcal{P}$ with set C of colours will also be referred to as an operad *coloured by* C, or an operad *over* C. For such an operad $\mathcal{P}$, a $\mathcal{P}$-algebra consists of a family of sets $\{A_c\}_{c \in C}$ together with arrows $\mathcal{P}(c_1, \cdots, c_n; c) \times A_{c_1} \times \cdots \times A_{c_n} \to A_c$, satisfying the usual compatibility conditions for unit, compositions, and symmetry.

The same definitions of operad and algebra still make sense if we replace *Set* by an arbitrary cocomplete symmetric monoidal category $\mathcal{E}$. In particular, the strong monoidal functor $Set \to \mathcal{E}$, which sends a set S to the S-fold coproduct of copies of the unit I of $\mathcal{E}$, maps every operad $\mathcal{P}$ over C in Set to an operad in $\mathcal{E}$, which we denote by $\mathcal{P}_{\mathcal{E}}$, or sometimes again by $\mathcal{P}$.

If $\mathcal{P}$ is an operad over C and $f : D \to C$ is a map of sets, then there is an evident induced operad $f^*(\mathcal{P})$ over D, given by

$$f^*(\mathcal{P})(d_1, \cdots, d_n; d) = \mathcal{P}(fd_1, \cdots, fd_n; fd).$$

If $\mathcal{P}$ and $\mathcal{Q}$ are operads, a *map* $\mathcal{Q} \xrightarrow{f} \mathcal{P}$ is given by a map of sets $f : D \to C$, and for each $d_1, \cdots, d_n, d$ a map

$$f_{d_1,\cdots,d_n,d} : \mathcal{Q}(d_1, \cdots, d_n; d) \to \mathcal{P}(f(d_1), \cdots, f(d_n); f(d))$$

which commutes with all the operations and the Σ_n-actions. If $D = C$ and $f : D \to C$ is the identity, we will call f a map of operads *over* C. For a fixed symmetric monoidal category $\mathcal{E}$, we denote by $Operad(\mathcal{E})$ the category of all coloured operads in $\mathcal{E}$. When $\mathcal{E} = Set$ we will simply write $Operad$ instead of $Operad(Set)$.

Example 2.0.1. Let $\mathcal{E}$ be a symmetric monoidal category. Then $\mathcal{E}$ gives rise to a coloured operad $\underline{\mathcal{E}}$, whose colours are the objects of $\mathcal{E}$. For a sequence $X_1,\cdots,X_n, X$ of such objects, $\underline{\mathcal{E}}(X_1, \cdots, X_n; X)$ is the set of arrows $X_1 \otimes \cdots \otimes X_n \to X$ in $\mathcal{E}$. If $\mathcal{E}$ is a symmetric *closed* monoidal category, then $\mathcal{E}$ may be viewed as an operad $\underline{\underline{\mathcal{E}}}$ in $\mathcal{E}$, with the objects of $\mathcal{E}$ as colours again, and with $\underline{\underline{\mathcal{E}}}(X_1, \cdots, X_n; X)$ the internal Hom-object $\underline{Hom}_{\mathcal{E}}(X_1 \otimes \cdots \otimes X_n, X)$.

Note that, in general, the objects of $\mathcal{E}$ form a proper class and not a set. However, in this paper, we will largely ignore such set-theoretic issues, and interpret "small" or "set" in terms of a suitable universe. In this context, let us point out that for any *set* S of objects of $\mathcal{E}$, there are operads $\underline{\mathcal{E}}_S$ and $\underline{\underline{\mathcal{E}}}_S$ obtained by restricting $\underline{\mathcal{E}}$ and $\underline{\underline{\mathcal{E}}}$ to the colours in S (If $i : S \to Objects(\mathcal{E})$ is the inclusion, then $\underline{\mathcal{E}}_S = i^*(\underline{\mathcal{E}})$, etc). In general, we will often identify a monoidal category with the corresponding operad, and simply write $\mathcal{E}$ for $\underline{\mathcal{E}}$ or $\underline{\underline{\mathcal{E}}}$.

Example 2.0.2. Any category $\mathcal{C}$ can be considered as an operad $\mathcal{P}_{\mathcal{C}}$ in the following way. The colours of $\mathcal{P}_{\mathcal{C}}$ are the objects of $\mathcal{C}$, and for any sequence of colours $c_1, \cdots, c_n, c$ we set

$$\mathcal{P}_{\mathcal{C}}(c_1, \cdots, c_n; c) = \begin{cases} \mathcal{C}(c_1, c), & \text{if } n = 1 \\ \phi, & \text{if } n \neq 1 \end{cases}$$

the compositions and units are as in $\mathcal{C}$ and the symmetric actions are all trivial. In this way we obtain a functor $j_! : Cat \to Operad$ from the category Cat of small categories to the category of operads. This functor has an evident right adjoint $j^* : Operad \to Cat$, sending an operad $\mathcal{P}$ to the category given by the colours and unary operations of $\mathcal{P}$. In exactly the same way, any $\mathcal{E}$-enriched category can be seen as an operad in $\mathcal{E}$ and we thus obtain adjoint functors $Cat(\mathcal{E}) \underset{j^*}{\overset{j_!}{\rightleftarrows}} Operad(\mathcal{E})$.

Remark 2.0.3. There is also the notion of a non-symmetric (also called planar) operad. A planar operad is exactly the same structure as an operad except that there are no symmetric actions involved. The resulting category of planar operads with their obvious notion of maps is denoted by $Operad_\pi(\mathcal{E})$. There is an evident forgetful functor $Operad(\mathcal{E}) \to Operad(\mathcal{E})_\pi$ which maps an operad to the same operad with the symmetric actions forgotten. This functor has a left adjoint $Symm : Operad(\mathcal{E})_\pi \to Operad(\mathcal{E})$, which we call the symmetrization functor. This functor is useful in the construction of operads, since sometimes it is easier to directly describe the non-symmetric operad whose algebras are the desired structures in a given context.

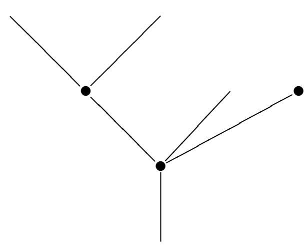
Example 2.0.4. Let S be a set. We describe now a planar operad $\mathcal{B}_S$ whose algebras are categories having S as set of objects. The set of colours of $\mathcal{B}_S$ is $S \times S$, and for any sequence of colours of the form $(s_1, s_2), (s_2, s_3), \cdots, (s_{n-1}, s_n)$ there is exactly one operation in $\mathcal{B}_S((s_1, s_2), \cdots, (s_{n-1}, s_n); (s_1, s_n))$. There are no other operations except those just given, which then completely determine the operadic structure. We thus have a planar operad in Set whose symmetrization we denote by $\mathcal{A}_S$. For any cocomplete monoidal category $\mathcal{E}$ we obtain an operad in $\mathcal{E}$ (still) denoted $\mathcal{A}_S$ which is the image of the original $\mathcal{A}_S$ under the functor $Operad(Set) \to Operad(\mathcal{E})$ described above. It is easy to verify that an $\mathcal{A}_S$-algebra in $\mathcal{E}$ is the same as an $\mathcal{E}$-enriched category having S as set of objects. For the special case where S is a one-point set, $\mathcal{A}_S$ is the familiar operad Ass.

We refer the reader to [BM2] for more examples of coloured operads.

Chapter 3

A category of trees

The trees we will consider are finite, non-empty (non-planar) trees with a designated root. As is common in the theory of operads [GK1, GK2, MSS] we allow some edges to have a vertex only on one side. These edges are called *outer* (or external) edges, while those having vertices on both sides are called *inner* (or internal) edges. By a designated root we mean a choice of one of the outer edges. The root defines an up-down direction in the tree (towards the root) and thus each vertex has a number of incoming edges (the number is the *valence* of the vertex) and one edge going out of it. We also allow vertices of valence 0. For example, the tree

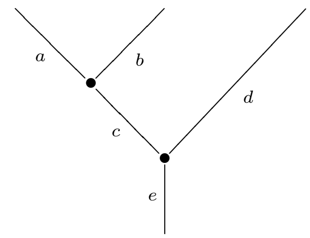


has three vertices, of valence 2, 3, and 0, and three input edges. A tree with no vertices

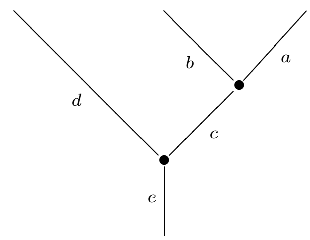
whose input edge (e say) coincides with its output edge will be denoted by η_e, or simply by η.

When we draw a tree we will always put the root at the bottom. One drawback of drawing a tree on the plane is that it immediately becomes a planar tree;

we thus have many different 'pictures' for the same tree. For instance the two trees

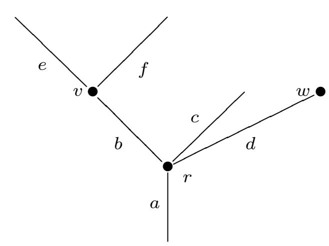


and



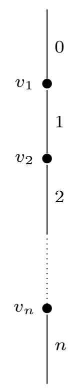
are different planar representations of the same tree.

Any tree T can be viewed as generating an operad $\Omega(T)$, whose colours are the edges of the tree, while the vertices of the tree are the generators of the operations. More explicitly, if we choose a planar representation of T then each vertex v with input edges $e_1, \cdots, e_n$ and output edge e defines an operation $v \in \Omega(T)(e_1, \cdots, e_n; e)$. The other operations are the unit operations and the operations obtained by compositions and by permutations, so as to obtain an operad in which every Hom set has at most one object. For example, in the same tree T pictured above, let us name the edges and vertices $a, b, \cdots, f$ and r, v, w.



Then $v \in \Omega(T)(e, f; b)$, $w \in \Omega(T)(; d)$ and $r \in \Omega(b, c, d; a))$ are the generators, while the other operations are the units $1_a, 1_b, 1_c \cdots 1_f$, the operations obtained by compositions $r \circ_1 v \in \Omega(T)(e, f, c, d; a)$, $r \circ_3 w \in \Omega(T)(b, c; a)$ and $r(v, 1_c, w) = (r \circ_1 v) \circ_4 w = (r \circ_3 w) \circ_1 v \in \Omega(T)(e, f, c; a)$, and permutations of these. This is a complete description of the operad $\Omega(T)$.

Viewing trees as coloured operads as above enables us to define the category Ω, whose objects are trees, and whose arrows $T \to T'$ are operad maps $\Omega(T) \to \Omega(T')$. The category Ω extends the simplicial category Δ. Indeed, any $n \geq 0$ defines a linear tree

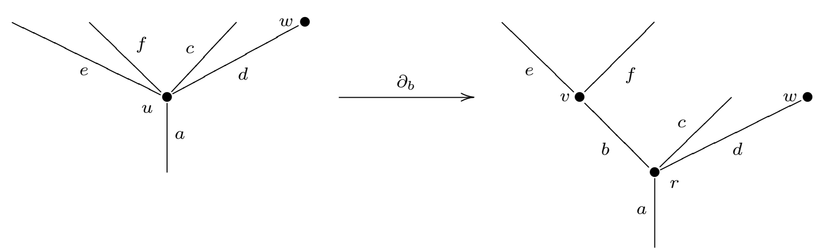


on $n+1$ edges and n vertices $v_1, \cdots, v_n$. We denote this tree by $[n]$. Any order preserving map $\{0, \cdots, n\} \to \{0, \cdots, m\}$ defines an arrow $[n] \to [m]$ in the category Ω. In this way, we obtain an embedding

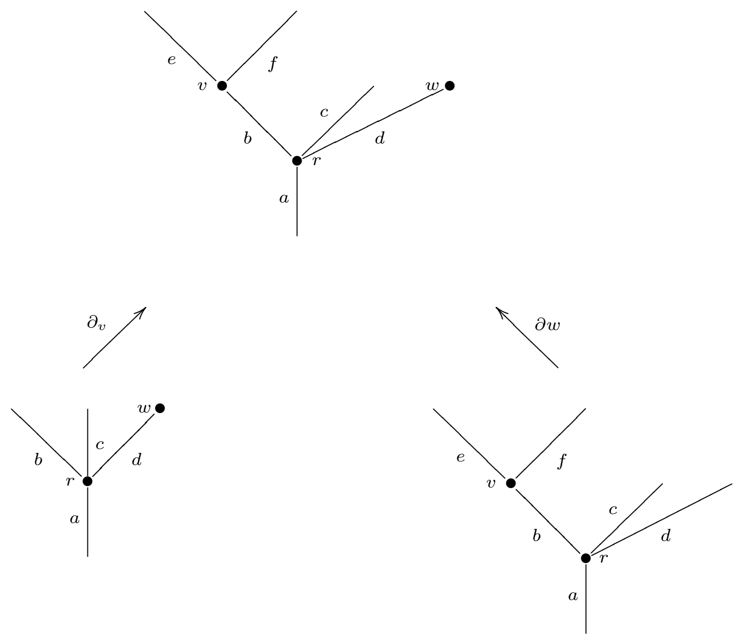
$$\Delta \xrightarrow{i} \Omega$$

This embedding is fully faithful. Moreover, it describes Δ as a sieve (or ideal) in Ω, in the sense that for any arrow $S \to T$ in Ω, if T is linear then so is S.

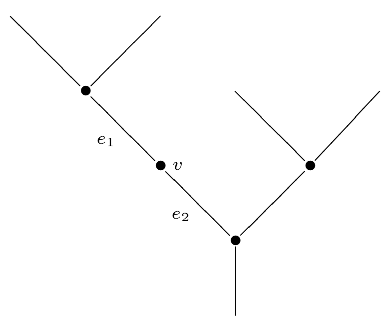
With a tree T one can associate certain maps in Ω as follows. If b is an inner edge in T, let T/b be the tree obtained from T by contracting b. Then there is a natural map $\partial_b : T/b \to T$ in Ω, called the *inner face* map associated with b, which locally in the tree looks like this:



Let v be a vertex in T with the property that all but one of the edges incident to v are outer. We call such a vertex an outer cluster. Let T/v be the tree obtained from T by removing the vertex v and all of the outer edges incident to it. Then there is a map $\partial_v : T/v \to T$ in Ω called the *outer face* associated with v. For example, the maps:



are two outer faces. We will use the term *face* map to refer to an inner or outer face map. One more type of map is a map that can be associated with a unary vertex v in T as follows. Let T/v be the tree obtained from T by removing the vertex v and merging the two edges incident to it into one edge e. Then there is a map $\sigma_v : T \to T/v$ in Ω called the *degeneracy map* associated with v, which sends the vertex v to the identity 1_e, and which can be pictured like this:



The following lemma is the generalization to Ω of the well known fact that in Δ each arrow can be written as a composition of degeneracy maps followed by face maps. We omit the proof.

Lemma 3.0.5. *Any arrow $f : A \to B$ in Ω decomposes as*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma"] & B \\ A' \arrow[r, "\varphi"] & B' \arrow[u, "\delta"'] \end{tikzcd}$$

where $\sigma : A \to A'$ is a composition of degeneracy maps, $\varphi : A' \to B'$ is an isomorphism, and $\delta : B' \to B$ is a composition of face maps.

Chapter 4

Dendroidal sets

We now define the category $dSet$ of dendroidal sets and discuss its relation to the category $sSet$ of simplicial sets.

Definition 4.0.6. A *dendroidal set* is a functor $\Omega^{op} \to Set$. A map between dendroidal sets is a natural transformation. The category of *dendroidal sets* thus defined is denoted *dSet.*

Thus, a dendroidal set X is given by a set X_T for each tree T, and a map $\alpha^* : X_T \to X_S$ for each map of trees (arrow in Ω) $\alpha : S \to T$; and these maps have to be functorial in α, in the sense that $id^* = id$ and $(\alpha\beta)^* = \beta^*\alpha^*$ for $R \xrightarrow{\beta} S \xrightarrow{\alpha} T$ in Ω. A morphism $Y \xrightarrow{f} X$ of dendroidal sets is given by maps (all denoted) $f : Y_T \to X_T$ for each tree T, commuting with the structure maps (i.e., $f(\alpha^* y) = \alpha^* f(y)$ for any $y \in Y_T$ and any $\alpha : S \to T$). An element of X_T is called a *dendrex* (plural *dendrices*) of *shape* T (This terminology is analogous to simplex, simplices). The dendrices of shape η will be referred to as *vertices.* As for simplicial sets, we call a dendrex $x \in X_T$ *degenerate* if there exists a degeneracy $\sigma : T \twoheadrightarrow S$ and a dendrex $y \in X_S$ with $\sigma^*(y) = x$.

Every tree T defines a representable dendroidal set $\Omega[T]$ as follows:

$$\Omega[T]_S = \Omega(S, T).$$

By the Yoneda Lemma each dendrex x of shape T in a dendroidal set X corresponds bijectively to a map $\hat{x} : \Omega[T] \to X$ of dendroidal sets. If $\partial_x : T \to R$ is a face map associated to an inner edge or an outer cluster x we use the same notation $\partial_x : \Omega[T] \to \Omega[R]$ for the induced map of dendroidal sets.

The inclusion functor $i : \Delta \to \Omega$ defines an obvious restriction functor

$$i^* : dSet \to sSet.$$

This functor has both a left adjoint $i_!$ and a right adjoint i_*, given by left and right Kan extension. The functor $i_! : sSet \to dSet$ is "extension by zero",

$$i_!(X)_T = \begin{cases} X_n, & \text{if T is linear with } n \text{ vertices} \\ \phi, & \text{otherwise} \end{cases}$$

(This is clear from the fact that $\Delta \subseteq \Omega$ is a sieve). It follows that $i_!$ is full and faithful, and that $i^* i_!$ is the identity functor on simplicial sets. The pair (i^*, i_*) defines a morphism of toposes $i : sSet \to dSet$, which is in fact an *open embedding.*

Example 4.0.7. If $\mathcal{P}$ is an operad, then the *dendroidal nerve* of $\mathcal{P}$ is the dendroidal set $N_d(\mathcal{P})$ given by

$$N_d(\mathcal{P})_T = Hom_{Operad}(\Omega(T), \mathcal{P}),$$

This construction defines a fully faithful functor

$$N_d : Operad \to dSet,$$

which has various nice properties as we will see. As already noted, any monoidal category $\mathcal{E}$ defines an operad $\underline{\mathcal{E}}$. The corresponding dendroidal set $N_d(\underline{\mathcal{E}})$ will simply be written $N_d(\mathcal{E})$ and will be called the *dendroidal nerve* of $\mathcal{E}$. Note that this extends the usual (simplicial) nerve of $\mathcal{E}$, in the sense that

$$i^*(N_d\mathcal{E}) = N(\mathcal{E}).$$

The functor $N_d : Operad \to dSet$ has a left adjoint

$$\tau_d : dSet \to Operad$$

defined by Kan extension. For a dendroidal set X, we refer to $\tau_d(X)$ as the *operad generated by* X. This functor τ_d extends the functor τ from simplicial sets to categories, left adjoint to $N : Cat \to sSet$. In particular, we obtain a diagram of functors

$$\begin{tikzcd} sSet \arrow[r, shift left, "i_!"] \arrow[d, shift right, "\tau"'] & dSet \arrow[l, shift left, "i^*"] \arrow[d, shift right, "\tau_d"'] \\ Cat \arrow[u, shift right, "N"'] \arrow[r, shift left, "j_!"] & Operad \arrow[l, shift left, "j^*"] \arrow[u, shift right, "N_d"'] \end{tikzcd}$$

(with left adjoints on the top or on the left), in which the following commutation relations hold up to natural isomorphisms

$$\tau N = id, \;\; \tau_d N_d = id, \;\; i^* i_! = id, \;\; j^* j_! = id$$

and

$$j_! \tau = \tau_d i_!, \;\; N j^* = i^* N_d, \;\; i_! N = N_d j_!.$$

The canonical map $\tau i^*(X) \to j^* \tau_d(X)$ is in general not an isomorphism. (For an example, consider the representable dendroidal set $\Omega[T]$ where T is the tree with three edges, one binary and one nullary vertex.)

Remark 4.0.8. For an arbitrary category $\mathcal{E}$, one can also consider the category $d\mathcal{E}$ of dendroidal objects in $\mathcal{E}$, i.e., contra-variant functors from Ω to $\mathcal{E}$. In particular, if one takes for $\mathcal{E}$ the category Top of compactly generated topological spaces, one obtains in this way the category $dTop$ of dendroidal spaces. Many constructions extend to this more general context. For example, if $\mathcal{P}$ is a topological operad, its dendroidal nerve $N_d(\mathcal{P})$ is naturally a dendroidal space, with the special property that its space $N_d(\mathcal{P})_\eta$ of vertices is discrete. Conversely, from such a dendroidal space X with this property, one can construct a topological operad, $\tau_d(X)$.

4.1 Diagrams of dendroidal sets

If $X : \mathbb{S}^{op} \to sSet$ is a diagram of simplicial sets (contravariantly) indexed by a small category $\mathbb{S}$, one can construct a "total" simplicial set $\int_{\mathbb{S}} X$ as follows. An n-simplex of $\int_{\mathbb{S}} X$ is a pair (s, x) where $s = (s_0 \xrightarrow{\alpha_1} s_1 \longrightarrow \cdots \xrightarrow{\alpha_n} s_n)$ is an n-simplex in the nerve of $\mathbb{S}$, and x is a function assigning to each map $u : [k] \to [n]$ in Δ a k-simplex x_k in $X(s_{u(0)})$, functorial in the following way. If $w = uv : [l] \xrightarrow{v} [k] \xrightarrow{u} [n]$, then $u(0) \leq w(0)$ so there is a composition of α_i's from $s_{u(0)}$ to $s_{w(0)}$ in $\mathbb{S}$, denoted $\alpha_{w,u} = \alpha_{w(0)} \circ \alpha_{w(0)-1} \circ \cdots \circ \alpha_{u(0)+1}$. Then the functorial condition on the x_α's is

$$\alpha^*_{w,u}(x_w) = v^*(x_u).$$

Here $\alpha^*_{w,u} : X(s_{w(0)}) \to X(s_{u(0)})$, and this is an identity between l-simplices in $X(s_{u(0)})$. Notice that in the special case where we start with a diagram $\mathbb{C} : \mathbb{S}^{op} \to Cat$ of small categories, the diagram $N(\mathbb{C}) : \mathbb{S}^{op} \to sSet$, obtained by composing with the nerve functor, satisfies the identity

$$\int_{\mathbb{S}} (N(\mathbb{C})) = N(\int_{\mathbb{S}} \mathbb{C}),$$

where $\int_{\mathbb{S}} \mathbb{C}$ on the right is the Grothendieck construction.

We shall now give a similar construction for diagrams of dendroidal sets. This construction will play a role in our definition of weak higher categories in Section 12. For this, we assume that the indexing category $\mathbb{S}$ has finite products. So, let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets. We define a dendroidal set $\int_{\mathbb{S}} X$ as follows. For a tree T, an element of $\int_{\mathbb{S}} X_T$ is again a pair (t, x). Here $t \in N_d(\mathbb{S})_T$ is an element of the dendroidal nerve of $\mathbb{S}$ (where $\mathbb{S}$ is viewed as an operad via the cartesian structure). Such an element determines an object $in(t) \in \mathbb{S}$, defined by $in(t) = t(e_1) \times \cdots \times t(e_n)$ where $e_1, \cdots, e_n$ are the input edges of T (in some fixed arbitrary order). Note that for any arrow $u : S \to T$ in Ω, the dendrex t determines a map $in(t) \to in(tu)$ in $\mathbb{S}$ (defined by projections, the maps given by t, and the coherence maps in $\mathbb{S}$). Now x is a function which assigns to each such u an element $x_u \in X(in(tu))_S$, functorial in the following way: if $w = u \circ v$ as in

$R \xrightarrow{v} S \xrightarrow{u} T$ then there is an induced map $in(tu) \xrightarrow{\alpha_{u,v}} in(tw)$ in $\mathbb{S}$, and we require

$$\alpha^*_{u,v}(x_v) = v^*(x_u)$$

The set $\int_{\mathbb{S}} X_T$ of such pairs (t, x) is contravariant in T, and defines the dendroidal set $\int_{\mathbb{S}} X$.

Note that this construction for dendroidal sets truly extends the one for simplicial sets, in the sense that for a diagram $X : \mathbb{S}^{op} \to sSet$ of simplicial sets where $\mathbb{S}$ is cartesian, there is a canonical isomorphism

$$i_! \int_{\mathbb{S}} X = \int_{\mathbb{S}} i_! X.$$

4.2 Skeletal filtration

As for any presheaf category, any dendroidal set X is a colimit of representables, of the form

$$X = \varinjlim \Omega[T]$$

(see [Mac]). We wish to refine this a little, in a way similar to the skeletal filtration for simplicial sets. To this end, call a dendrex $x \in X_T$ of shape T *degenerate* if there is a surjective map $T \xrightarrow{\alpha} T'$ in Ω (a composition of degeneracies) such that $x = \alpha^*(x')$ for some $x' \in X_{T'}$. Here α should not be an isomorphism of course, so that T' has strictly fewer vertices then T and α is a non-empty composition of degeneracies.

Given a dendroidal set X we denote by $Sk_n(X)$ the sub dendroidal set of X generated by all non-degenerate dendrices $x \in X_T$ where $|T| \leq n$. An arbitrary dendroidal set X is clearly the colimit (union) of the sequence

$$Sk_0(X) \subseteq Sk_1(X) \subseteq Sk_2(X) \subseteq \cdots \tag{1}$$

We call this the skeletal filtration of X. This filtration extends the skeletal filtration for simplicial sets in the precise sense that for any dendroidal set X and any simplicial set S, there are canonical isomorphisms

$$i^* Sk_n(X) = Sk_n(i^* X)$$

and

$$i_! Sk_n(S) = Sk_n(i_! S).$$

Consider now the following diagram:

$$\begin{tikzcd}
\coprod_{x,T} \partial\Omega[T] \arrow[r] \arrow[d, tail] & Sk_n(X) \arrow[d, tail] \\
\coprod_{x,T} \Omega[T] \arrow[r] & Sk_{n+1}(X)
\end{tikzcd}$$

where the sum is taken over isomorphism classes of pairs (x, T) in the category of elements of X where T is a tree with n vertices and $x \in X_T$ is non-degenerate, and $\partial\Omega[T]$ is the boundary of $\Omega[T]$, i.e., the union of its faces. We call the skeletal filtration of X *normal* if this square is a pushout for each $n > 0$.

Following Cisinski [C] we call a dendroidal set *normal* if for each non-degenerate dendrex $x \in X_T$, the only isomorphism fixing x is the identity. Cisinski (loc. cit.) proves that the normal dendroidal sets are precisely those whose skeletal filtrations are normal.

Example 4.2.1. If X is a simplicial set then $i_!(X)$ admits a normal skeletal filtration and in fact that skeletal filtration is isomorphic to the usual skeletal filtration of X. If $\mathcal{P}$ is a Σ-free operad then $N_d(\mathcal{P})$ is normal. In particular if $\mathcal{P}$ is the symmetrization of a planar operad then $N_d(\mathcal{P})$ is normal.

Chapter 5

The tensor product of dendroidal sets

Like any other category of presheaves of sets, the category $dSet$ has a closed cartesian structure. There is, however, another more interesting monoidal structure on the category of dendroidal sets, which we aim to describe in this section. To begin with, we will recall the tensor product for operads from [BV] (Definition 2.14, page 41).

5.1 The Boardman-Vogt tensor product

Let $\mathcal{P}$ be an operad in Set over C and $\mathcal{Q}$ one over D. Their tensor product $\mathcal{P} \otimes_{BV} \mathcal{Q}$ is an operad coloured by the product set $C \times D$. The operations in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ are generated by the following. Any $p \in \mathcal{P}(c_1, \cdots, c_n; c)$ and any $d \in D$ define an operation

$$p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \cdots, (c_n, d); (c, d)).$$

These operations compose in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ in a way to make $p \mapsto p \otimes d$ a map of operads. Similarly, each operation $q \in \mathcal{Q}(d_1, \cdots, d_n, d)$ and each $c \in C$ define an operation

$$c \otimes q \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \cdots (c, d_n); (c, d)),$$

and these compose as in $\mathcal{Q}$. Furthermore, the operations from $\mathcal{P}$ and $\mathcal{Q}$ distribute over each other, in the sense that for $p \in \mathcal{P}(c_1, \cdots, c_n; c)$ and $q \in \mathcal{Q}(d_1, \cdots, d_m; d)$,

$$\sigma^*_{n,m}((p \otimes d)(c_1 \otimes q, \cdots, c_n \otimes q)) = (c \otimes q)(p \otimes d_1, \cdots, p \otimes d_m)$$

where $\sigma_{n,m} \in \Sigma_{n \cdot m}$ is the permutation described as follows. Consider $\Sigma_{n \cdot m}$ as the set of bijections of the set $\{0, 1, \cdots, n \cdot m - 1\}$. Each number in this set can be written uniquely in the form $k \cdot n + j$ where $0 \leq k < m$ and $0 \leq j < n$ as well as in the form $k \cdot m + j$ where $0 \leq k < n$ and $0 \leq j < m$. The permutation $\sigma_{n,m}$ is then defined by $\sigma_{n,m}(k \cdot n + j) = j \cdot m + k$. This tensor product makes the category of operads into a symmetric monoidal category.

This Boardman-Vogt tensor product preserves colimits in each variable separately. In fact, there is a corresponding internal Hom, making the category *Operad* into a symmetric *closed* monoidal category. For two operads $\mathcal{P}$ and $\mathcal{Q}$ as above, $\underline{Hom}(\mathcal{P}, \mathcal{Q})$ is the operad whose colours are the maps $\mathcal{P} \to \mathcal{Q}$, and whose operations are suitably defined multi-natural transformations. (Explicitly, for $\alpha_1, \cdots, \alpha_n, \beta : \mathcal{P} \to \mathcal{Q}$, elements of $\underline{Hom}(\mathcal{P}, \mathcal{Q})(\alpha_1, \cdots, \alpha_n; \beta)$ are maps f assigning to each colour $c \in C$ of $\mathcal{P}$ an element $f_c \in \mathcal{Q}(\alpha_1 c, \cdots, \alpha_n c; \beta c)$. These f_c should be natural with respect to all operations in $\mathcal{P}$. For example, if $p \in \mathcal{P}(c_1, c_2; c)$ is a binary operation, then $\beta(p)(f_{c_1}, f_{c_2}) \in \mathcal{Q}(\alpha_1 c_1, \cdots, \alpha_n c_1, \alpha_1 c_2, \cdots, \alpha_n c_2; \beta c)$ is the image under a suitable permutation of $f_c(\alpha_1(p), \cdots, \alpha_n(p)) \in \mathcal{Q}(\alpha_1 c_1, \alpha_1 c_2, \cdots, \alpha_n c_1, \alpha_n c_2; \beta c))$.

For a symmetric monoidal category $\mathcal{E}$, the Boardman-Vogt tensor product of coloured operads in $\mathcal{E}$ still makes sense for *Hopf* operads $\mathcal{P}$ and $\mathcal{Q}$. For such operads, the categories $Alg_{\mathcal{E}}(\mathcal{P})$ and $Alg_{\mathcal{E}}(\mathcal{Q})$ are again symmetric monoidal, and a $(\mathcal{P} \otimes_{BV} \mathcal{Q})$-algebra in $\mathcal{E}$ is the same thing as a $\mathcal{P}$-algebra in $Alg_{\mathcal{E}}(\mathcal{Q})$, and is also the same thing as a $\mathcal{Q}$-algebra in $Alg_{\mathcal{E}}(\mathcal{P})$.

5.2 The tensor product of dendroidal sets

We now define a tensor product

$$\otimes : dSet \times dSet \to dSet$$

which is to preserve colimits in each variable separately. Since each dendroidal set is a colimit of representables, this tensor is completely determined by its effect on representable dendroidal sets $\Omega[S]$ and $\Omega[T]$, which we define as

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T)),$$

i.e., as the dendroidal nerve of the Boardman-Vogt tensor product of the operads $\Omega(S)$ and $\Omega(T)$. It follows by general category theory [Da, K] that there exists an internal Hom for this tensor, defined for two dendroidal sets X and Y and an object T of Ω by

$$\underline{Hom}(X, Y)_T = Hom_{dSet}(\Omega[T] \otimes X, Y)$$

We summarise this discussion in the following proposition:

Proposition 5.2.1. *There exists a unique (up to natural isomorphism) symmetric closed monoidal structure on dSet, with the property that there is a natural isomorphism* $\Omega[S] \otimes \Omega[T] \cong N_d(\Omega(S) \otimes_{BV} \Omega(T))$ *for any two objects* S, T *of* Ω.

More generally, for suitable symmetric monoidal categories $\mathcal{E}$, there is such a monoidal structure on the category $d\mathcal{E}$ of dendroidal objects. See the Appendix for a discussion of dendroidal objects.

We mention some basic properties of the tensor product on $dSet$, in relation to the tensor product of operads, and to the product of simplicial sets.

Proposition 5.2.2. *The following properties hold.*

(i) For any two dendroidal sets X and Y, there is a natural isomorphism

$$\tau_d(X \otimes Y) \cong \tau_d(X) \otimes_{BV} \tau_d(Y).$$

(ii) For any two operads $\mathcal{P}$ and $\mathcal{Q}$, there is a natural isomorphism

$$\tau_d(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})) \cong \mathcal{P} \otimes_{BV} \mathcal{Q}.$$

Proof. It suffices to check (i) for representable X and Y, in which case it follows from the identity $\tau_d N_d \cong id$. By the same identity, (ii) follows from (i). □

Proposition 5.2.3. *For any two simplicial sets X and Y, and any dendroidal set D, there are natural isomorphisms*

(i) $i_!(X \times Y) \cong i_!(X) \otimes i_!(Y)$,

(ii) $i^*\underline{Hom}(i_!(X), D) \cong i^*(D)^X$,

(iii) $i^*\underline{Hom}(i_!(X), i_!(Y)) \cong Y^X$.

Proof. The isomorphisms of type (ii) and (iii) are deduced from those of type (i), using the fact that $i_!$ is fully faithful. For (i), it suffices again to check this for representable simplicial sets $\Delta[n]$ and $\Delta[m]$. Observe first that, more generally, for any two small categories $\mathbb{C}$ and $\mathbb{B}$,

$$j_!(\mathbb{C} \times \mathbb{B}) \cong j_!(\mathbb{C}) \otimes_{BV} j_!(\mathbb{B}) \tag{1}$$

This holds in particular for the linear orders $[n]$ and $[m]$ viewed as categories, so

$$\begin{aligned} i_!(\Delta[n] \times \Delta[m]) &\cong i_!(N([n]) \times N([m])) \\ &\cong i_!(N([n] \times [m])) \\ &\cong N_d j_!([n] \times [m]) \\ &\cong N_d(j_![n] \otimes_{BV} j_![m]) \quad \text{(by (1))} \\ &\cong N_d(\Omega(n) \otimes_{BV} \Omega(m)) \\ &\cong \Omega[n] \otimes \Omega[m] \\ &\cong i_!(\Delta[n]) \otimes i_!(\Delta[m]). \end{aligned}$$

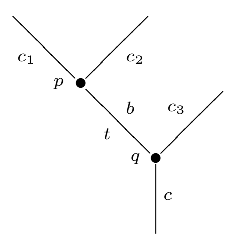
This shows that (i) holds for representables $\Delta[n]$ and $\Delta[m]$; as said, this completes the proof. □

Chapter 6

The homotopy coherent nerve

In this section we introduce the homotopy coherent dendroidal nerve of an operad $\mathcal{P}$. This construction plays a crucial role in the definition of homotopy $\mathcal{P}$-algebras and weak higher categories, in Section 12. We begin by recalling the Boardman-Vogt resolution of operads [BV] and its generalization [BM1].

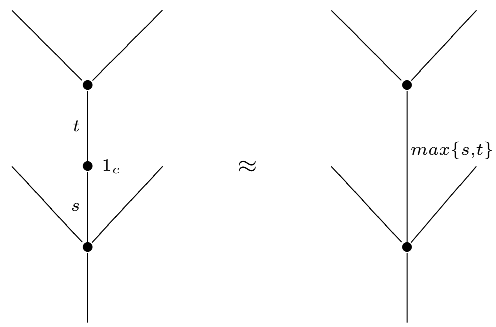
Let $\mathcal{P} = (C, P)$ be an operad in the category of compactly generated topological spaces, and let $H = [0, 1]$ be the unit interval. One can construct a (cofibrant) resolution $W(\mathcal{P}) \to \mathcal{P}$ as follows. $W(\mathcal{P})$ is again an operad coloured by C. The space $W(\mathcal{P})(c_1, \cdots, c_n; c)$ is a quotient of a space of labelled planar trees. The edges of such a tree are labelled by elements of C, where in particular the input edges are labelled by the given $c_1, \cdots, c_n$ and the output by c. Moreover, the inner edges carry a label $t \in H$ (a "length"), and each vertex v with input edges labelled $b_1, \cdots, b_n \in C$ (in the planar order) and output edge labelled $b \in C$, is labelled by an element $p \in \mathcal{P}(b_1, \cdots, b_n; b)$. For example,

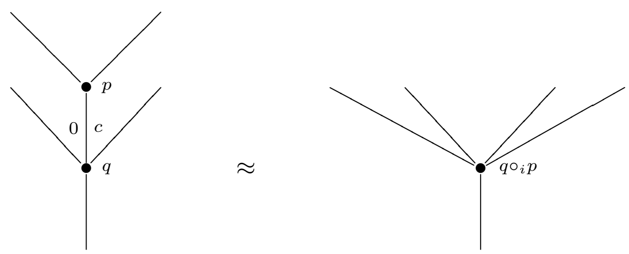


where $p \in \mathcal{P}(c_1, c_2; b)$, $q \in \mathcal{P}(b, c_3; c)$, $t \in [0, 1]$. There is a natural (product) topology on these trees, coming from the topology on $\mathcal{P}$ and that on H. The space $W(\mathcal{P})(c_1, \cdots, c_n; c)$ is now the quotient space, obtained (by identifying isomorphic planar trees with the same labelling and) the following two relations (illustrated by the pictures below):
(i) Vertices labelled by an identity can be deleted, taking the maximum of the two

adjacent lengths (or forgetting the lengths altogether if one of the adjacent edges is outer).
(ii) Edges of length zero can be contracted, using the $\circ_i$ form of the operad composition of $\mathcal{P}$.

The operad structure of $W(\mathcal{P})$ is given by grafting of trees, giving the newly arising inner edges length 1. The map $W(\mathcal{P}) \to \mathcal{P}$ is given by setting all lengths to zero (i.e., forget the lengths and compose in $\mathcal{P}$).





In [BM1], it is explained in detail how the above construction can be performed and studied in the more general context of operads in any symmetric monoidal category $(\mathcal{E}, \otimes, I)$, where $[0,1]$ is replaced by a suitable "interval" H in $\mathcal{E}$. This is an object H equipped with two "points" $0, 1 : I \rightrightarrows H$, an augmentation $\epsilon : H \to I$ satisfying $\epsilon 0 = id = \epsilon 1$, and a binary operation $\vee : H \otimes H \to H$ (playing the role of max) which is associative, and for which 0 is unital and 1 is absorbing ($0 \vee x = x = x \vee 0$ and $1 \vee x = 1 = x \vee 1$). This defines for any operad $\mathcal{P}$ in $\mathcal{E}$ a new operad $W_H(\mathcal{P})$ in $\mathcal{E}$ mapping to $\mathcal{P}$. The algebras for this operad are up-to-homotopy $\mathcal{P}$-algebras.

For example, one can take for $\mathcal{E}$ the category Cat of small categories which admits the following model category structure. The weak equivalences are categorical equivalences, the cofibrations are functors that are injective on objects, and the fibrations are those functors having the right lifting property with respect to the functor $0 \to H$, where H is the groupoid $0 \leftrightarrow 1$ with two objects and one isomorphism between them. The groupoid H also plays the role of the interval. We examine this possibility below, when we consider weak n-categories.

Example 6.0.4. Let $[n]$ be the linear tree, viewed as a (discrete) topological operad. So an $[n]$-algebra consists of a sequence of spaces $X_0, \cdots X_n$, together with maps $f_{ji} : X_i \to X_j$ for $i \leq j$, such that $f_{ii} = id$ and

$$f_{kj} \circ f_{ji} = f_{ki} \tag{6.0.1}$$

if $i \leq j \leq k$. A $W([n])$-algebra consists of such a sequence of spaces and maps, for which (6.0.1) holds only up to specified coherent higher homotopies. Since $W([n])$ is an operad with unary operations only, one can also think of it as a topological category: it has objects $0, 1, \cdots, n$, and an arrow $i \to j$ in $W[n]$ is a sequence of "times" $t_{i+1}, \cdots, t_{j-1}$ (each $t_k \in [0, 1]$). In other words, $W[n](i, j)$ is the cube $[0, 1]^{j-i-1}$ for $i + 1 \leq j$, a point for $i = j$, and the empty set for $i > j$. Composition is given by juxtaposing two such sequences, putting an extra time 1 in the middle: $(t_{i+1}, \cdots, t_{j-1}) : i \to j$ and $(t_{j+1}, \cdots, t_{k-1}) : j \to k$ compose to give $(t_{i+1}, \cdots, t_{j-1}, 1, t_{j+1}, \cdots, t_{k-1})$. If $\mathcal{C}$ is a category enriched in Top (i.e., a topological category with a discrete set of objects), then the sets of continuous functors

$$Top(W[n], \mathcal{C})$$

for varying n define a simplicial set, which is exactly the homotopy coherent nerve of $\mathcal{C}$, described in [V].

More generally, if $\mathcal{E}$ is a symmetric monoidal category with interval H, one can construct an $\mathcal{E}$-enriched category $W_H[n]$ with

$$W_H[n](i, j) = H^{\otimes_{j-i-1}}$$

and define for each $\mathcal{E}$-enriched category $\mathcal{C}$ its homotopy coherent nerve $hcN(\mathcal{C})$ as the simplicial set given by

$$hcN(\mathcal{C})_n = \mathcal{E}\text{-}Cat(W_H[n], \mathcal{C}),$$

that is, the set of all $\mathcal{E}$-enriched functors from $W_H[n]$ to $\mathcal{C}$. For example, if $\mathcal{E} = Cat$ and $H = 0 \leftrightarrow 1$ as above, then an element of $hcN(\mathcal{C})_2$ is given by a triangle

$$\begin{tikzcd} x_0 \arrow[r] \arrow[dr] & x_1 \arrow[d] \arrow[dl, phantom, "\simeq"] \\ & x_2 \end{tikzcd}$$

which composes up to a specified invertible 2-cell in $\mathcal{C}$.

The above generalizes in a completely straightforward way to operads. Suppose $\mathcal{E}$ and H are as above. Each tree T defines an operad $\Omega(T)$ in Set, which we can view as an operad in $\mathcal{E}$ (via the functor $Operad \to Operad(\mathcal{E})$). Applying the generalized Boardman-Vogt construction yields an operad $W_H(T)$ in $\mathcal{E}$. This construction produces a functor $\Omega \to Operad(\mathcal{E})$, which induces an adjunction

$$Operad(\mathcal{E}) \underset{|\cdot|_H}{\overset{hcN_d}{\rightleftarrows}} dSet$$

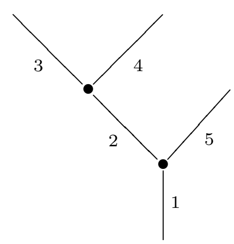
by Kan extension. For an operad $\mathcal{Q}$ in $\mathcal{E}$ the dendroidal set $hcN_d(\mathcal{Q})$ is called the *homotopy coherent dendroidal nerve* of $\mathcal{Q}$, and is given explicitly by

$$hcN_d(\mathcal{Q})_T = Hom_{Operad(\mathcal{E})}(W_H(T), \mathcal{Q}).$$

Remark 6.0.5. The functor $|-|_H$ is closely related to the W-construction for operads. In fact, if $\mathcal{P}$ is an operad in Set, then the Boardman-Vogt resolution $W_H(\mathcal{P}_{\mathcal{E}})$, of $\mathcal{P}$ viewed as an operad in $\mathcal{E}$, is isomorphic to the operad $|N_d(\mathcal{P})|_H$, as follows by direct inspection of the explicit construction of $W_H(\mathcal{P}_{\mathcal{E}})$ in [BM1]. In particular, for an operad $\mathcal{P}$ in Set and an operad $\mathcal{Q}$ in $\mathcal{E}$, there is a natural bijective correspondence

$$Hom_{Operad(\mathcal{E})}(W_H(\mathcal{P}_{\mathcal{E}}), \mathcal{Q}) = Hom_{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{Q})).$$

Remark 6.0.6. Consider the special case where $\mathcal{E}$ is the category Top of compactly generated spaces, and H is the unit interval. If $\mathcal{P}$ is a topological operad and T is a tree (an object of Ω), then the set $hcN_d(\mathcal{P})_T$ of maps of topological operads $W_H(\Omega(T)) \to \mathcal{P}$ has a natural topology, as a topological sum of generalized mapping fibrations. For example, for the tree T with edges numbered $1, \cdots, 5$,



$hcN_d(\mathcal{P})_T$ is the sum, over all 5-tuples $c_1, \cdots, c_5$ of colours of $\mathcal{P}$, of mapping fibrations of the maps

$$\mathcal{P}(c_3, c_4; c_2) \times \mathcal{P}(c_2, c_5; c_1) \to \mathcal{P}(c_3, c_4, c_5; c_1).$$

Let $(dTop)^\delta$ be the category of dendroidal spaces with discrete set of vertices. Then $hcN_d(\mathcal{P})$ with this topology defines a functor $hcN_d(-) : Operad(Top) \to$

$(dTop)^\delta$. This functor again has a left adjoint $|-|_H$,which relates to the Boardman-Vogt resolution of topological operads in the same way as above, by a natural isomorphism

$$W_H(\mathcal{P}) \cong |N_d(\mathcal{P})|_H,$$

where N_d and $|-|_H$ are now viewed as functors between the categories $Operad(Top)$ and $(dTop)^\delta$.

Chapter 7

Inner Kan complexes

We begin by introducing inner horns. For a tree T, each face map $\partial : T' \to T$ defines a monomorphism $\Omega[T'] \to \Omega[T]$ between (representable) dendroidal sets. The union (pushout) of these subobjects is the boundary of $\Omega[T]$, denoted

$$\partial\Omega[T] \rightarrowtail \Omega[T],$$

as above. If e is an inner edge of T, then the union of all the faces *except*

$$\partial_e : T/e \rightarrowtail T$$

defines a subobject of the boundary, denoted

$$\Lambda^e[T] \rightarrowtail \Omega[T]\ ,$$

and called the *inner horn* associated to e (and to T). This terminology and notation extends the one

$$\Lambda^k[n] \rightarrowtail \partial\Delta[n] \rightarrowtail \Delta[n]$$

for simplicial sets, in the sense that

$$i_!(\Lambda^k[n]) = \Lambda^k[i[n]]$$

$$i_!(\partial\Delta[n]) = \partial\Omega[i[n]]$$

as subobjects of $i_!(\Delta[n]) = \Omega[i[n]]$.

A dendroidal set K is said to be a (dendroidal) *inner Kan complex* if, for any tree T and any inner edge e in T, the map

$$K_T = Hom(\Omega[T], K) \to Hom(\Lambda^e[T], K)$$

is a surjection of sets. It is called a *strict* inner Kan complex if this map is a bijection (for any T and e as above). For example, we will see (Proposition 7.0.9 below) that the dendroidal nerve of an operad is always a strict inner Kan complex. This terminology is analogous to the one introduced by Boardman and Vogt, who say a simplicial set X satisfies the restricted Kan condition if, for any $0 < k < n$, the map $Hom(\Delta[n], X) \to Hom(\Lambda^k[n], X)$ is a surjection ([BV] Definition 4.8, page 102). In more recent work ([J1, J2]) Joyal develops the general theory of simplicial sets satisfying the restricted Kan condition. Joyal uses the terminology quasi-categories for such simplicial sets so as to stress the analogy with category theory. In fact a simplicial set X is a quasi-category iff $i_!(X)$ is a dendroidal inner Kan complex, and for any dendroidal inner Kan complex K, the restriction $i^*(K)$ is a quasi-category in the sense of Joyal.

Let us call a map $u : U \to V$ of dendroidal sets an *anodyne extension* if it can be obtained from the set of inner horn inclusions by coproducts, pushouts, compositions, and retracts (cf, [GZ], p. 60). Then obviously, the surjectivity property for inner Kan complexes extends to anodyne extensions, in the sense that the map of sets

$$u^* : Hom(V, K) \to Hom(U, K),$$

given by composition with u, is again surjective. Similarly, the map u^* is a bijection for any strict inner Kan complex.

For a tree T let $I(T)$ be the set of inner edges of T. For a non-empty subset $A \subseteq I(T)$ let $\Lambda^A[T]$ be the union of all faces of $\Omega[T]$ except those obtained by contracting an edge from A. Note that if $A = \{e\}$ then $\Lambda^A[T] = \Lambda^e[T]$.

Lemma 7.0.7. *For any non-empty $A \subseteq I(T)$ the inclusion $\Lambda^A[T] \to \Omega[T]$ is anodyne.*

Proof. By induction on $n = |A|$. If $n = 1$ then $\Lambda^A[T] \to \Omega[T]$ is an inner horn inclusion, thus anodyne. Assume the proposition holds for $n < k$ and suppose $|A| = k$. Choose an arbitrary $e \in A$ and put $B = A\backslash\{e\}$. The map $\Lambda^A[T] \to \Omega[T]$ factors as

$$\begin{tikzcd} \Lambda^A[T] \arrow[r] \arrow[dr] & \Lambda^B[T] \arrow[d] \\ & \Omega[T] \end{tikzcd}$$

The vertical map is anodyne by the induction hypothesis and it therefore suffices to prove that $\Lambda^A[T] \to \Lambda^B[T]$ is anodyne. The following diagram expresses that map as a pushout

$$\begin{tikzcd} \Lambda^B[T/e] \arrow[r] \arrow[d] & \Lambda^A[T] \arrow[d] \\ \Omega[T/e] \arrow[r] & \Lambda^B[T] \end{tikzcd}$$

and since the map $\Lambda^B[T/e] \to \Omega[T/e]$ is anodyne (by the induction hypothesis), the proof is complete. □

We denote by $\Lambda^I[T]$ the dendroidal set $\Lambda^A[T]$ where $A = I(T)$, that is $\Lambda^I[T]$ is the union of all outer faces of $\Omega[T]$. By the above proposition the inclusion $\Lambda^I[T] \to \Omega[T]$ is anodyne.

We now consider grafting of trees. For two trees T and S, and a leaf l of T, let $T \circ_l S$ be the tree obtained by grafting S onto T by identifying l with the root (output edge) of S. Then there are obvious inclusions $\Omega[S] \to \Omega[T \circ_l S]$ and $\Omega[T] \to \Omega[T \circ_l S]$, the pushout (union) of which we denote by $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$.

Lemma 7.0.8. *(Grafting) For any two trees T and S and any leaf l of T, the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[T \circ_l S]$ is anodyne.*

Proof. Let us write $R = T \circ_l S$. The case where $T = \eta$ or $S = \eta$ is trivial, we therefore assume that this is not the case. We proceed by induction on $n = |T|+|S|$, the sum of the degrees of T and S. The cases $n = 0$ or $n = 1$ are taken care of by our assumption that $T \neq \eta \neq S$. For the case $n = 2$ the same assumption implies that the inclusion $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ is an inner horn inclusion. In any case it is anodyne. Assume then that the result holds for $2 \leq n < k$ and suppose $|T| + |S| = k$.

Recall that $\Lambda^I[R]$ is the union of all the outer faces of $\Omega[R]$. First notice that $\Omega[T] \cup_l \Omega[S] \to \Omega[R]$ factors as

$$\begin{tikzcd} \Omega[T] \cup_l \Omega[S] \arrow[r] \arrow[dr] & \Lambda^I[R] \arrow[d] \\ & \Omega[R] \end{tikzcd}$$

and the vertical arrow is anodyne by a previous result. We now show that

$$\Omega[T] \cup_l \Omega[S] \to \Lambda^I[R]$$

is anodyne by exhibiting it as a pushout of an anodyne extension. Recall that an external cluster is a vertex v with the property that one of the edges adjacent to it is inner while all the other edges adjacent to it are outer. Let $Cl(T)$ (resp. $Cl(S)$) be the set of all external clusters in T (resp. S) which do not contain l (resp. the root of S). For each $C \in Cl(T)$ the face of $\Omega[R]$ corresponding to C is isomorphic to $\Omega[(T/C) \circ_l S]$ and the map $\Omega[T/C] \cup_l \Omega[S] \to \Omega[(T/C) \circ_l S]$ is anodyne by the induction hypothesis. Similarly for every $C \in Cl(S)$ the face of $\Omega[R]$ that corresponds to C is isomorphic to $\Omega[T \circ_l (S/C)]$ and the map $\Omega[T] \cup_l \Omega[S/C] \to \Omega[T \circ_l (S/C)]$ is anodyne by the induction hypothesis. The following diagram is a pushout

$$\begin{tikzcd}
\coprod_{C\in Cl(T)}(\Omega[T/C]\cup_l \Omega[S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T]\cup_l \Omega[S/C]) \arrow[r] \arrow[d] & \Omega[T]\cup_l \Omega[S] \arrow[d] \\
\coprod_{C\in Cl(T)}(\Omega[(T/C)\circ_l S]) \amalg \coprod_{C\in Cl(S)}(\Omega[T\circ_l (S/C)]) \arrow[r] & \Lambda^I[R]
\end{tikzcd}$$

where the map on the left is the coproduct of all of the anodyne extensions just mentioned. Since anodyne extensions are closed under coproducts, it follows that the map on the left of the pushout is anodyne and thus also the one on the right, which is what we set out to prove. This concludes the proof. □

We end this section with two remarks on strict inner Kan complexes.

Proposition 7.0.9. *The dendroidal nerve of any operad is a strict inner Kan complex.*

Proof. Let $\mathcal{P}$ be an operad. A dendrex $x \in N_d(\mathcal{P})_T$ is a map $x : \Omega[T] \to N_d(\mathcal{P})$ which is a map of operads $\Omega(T) \to P$. If we choose a planar representative for T then $\Omega(T)$ is specifically given in terms of generators and is a free operad. It follows that x is equivalent to a labeling of the (planar representative) T as follows. The edges are labeled by colours of $\mathcal{P}$ and the vertices are coloured by operations in $\mathcal{P}$ where the input of such an operation is the tuple of labels of the incoming edges to the vertex and the output is the label of the outgoing edge from the vertex. Any inner horn $\Lambda^e[T] \to N_d(\mathcal{P})$ is easily seen to be equivalent to such a labeling of the tree T and thus determines a unique filler. □

Proposition 7.0.10. *Any strict inner Kan complex is* 2*-coskeletal.*

Proof. Let X be a strict inner Kan complex. Let Y be any dendroidal set and assume $f : Sk_2Y \to Sk_2X$ is given. We first show that f can be extended to a dendroidal map $\hat{f} : Y \to X$. Suppose f was extended to a map $f_k : Sk_kY \to Sk_kX$ for $k \geq 2$. Let $y \in Sk_{k+1}(Y)$ be a non-degenerate dendrex and assume $y \notin Sk_k(Y)$. So $y \in Y_T$ and T has exactly $k+1$ vertices. Choose an inner horn $\Lambda^\alpha[T]$ (such an inner horn exist since $k \geq 2$). The set $\{\beta^*y\}_{\beta\neq\alpha}$ where β runs over all faces of T, defines a horn $\Lambda^\alpha[T] \to Y$. Since this horn factors through the k-skeleton of Y we obtain, by applying f_k, a horn $\Lambda^\alpha[T] \to X$ in X given by $\{f(\beta^*y)\}_{\beta\neq\alpha}$. Let $f_{k+1}(y) \in X_T$ be the unique filler of that horn. By construction we have for each $\beta \neq \alpha$ that

$$\beta^* f_{k+1}(y) = f(\beta^* y)$$

it thus remains to show the same for α. The dendrices $f(\alpha^*y)$ and $\alpha^* f_{k+1}(y)$ both have the same boundary and they are both of shape S where S has k vertices. Since $k \geq 2$, S has an inner face, but then it follows that both $f(\alpha^*y)$ and $\alpha^* f_{k+1}(y)$ are fillers for the same inner horn in X which proves that they are equal. By repeating the process for all dendrices in $Sk_{k+1}(Y)$ it follows that f_k can be extended to

$f_{k+1} : Sk_{k+1}(Y) \to Sk_{k+1}(X)$. This holds for all $k \geq 2$ which implies that f can be extended to $\hat{f} : Y \to X$. To show uniqueness of $\hat{f}$ assume that g is another extension of f. Suppose it has been shown that $\hat{f}$ and g agree on all dendrices of shape T where T has at most k vertices, and let $y \in X_S$ be a dendrex of shape S where S has $k+1$ vertices. But then the dendrices $\hat{f}(y)$ and $g(y)$ are dendrices in X that have the same boundary. Since $k \geq 2$ it follows that these dendrices are both fillers for the same inner horn and so are the same. This proves that $\hat{f} = g$. $\square$

Chapter 8

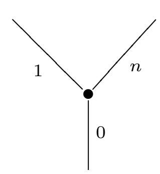
The operad generated by an inner Kan complex

We recall that $\tau_d : dSet \to Operad$ denotes the left adjoint to the dendroidal nerve functor N_d. In this section, we will give a more explicit description of the operad $\tau_d(X)$ in the case where X is an inner Kan complex. This description extends the one in [BV] of the category generated by a simplicial set satisfying the restricted Kan condition. It will lead to a proof of the following converse of Proposition 7.0.9.

Theorem 8.0.11. *For any strict inner Kan complex X, the canonical map $X \to N_d(\tau_d(X))$ is an isomorphism.*

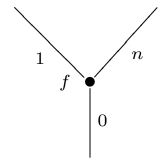
Proposition 7.0.9 and Theorem 8.0.11 together state that a dendroidal set is a strict inner Kan complex iff it is the nerve of an operad.

Consider an inner Kan complex X. For the description of $\tau_d(X)$, we first fix some notation. For each $n \geq 0$ let C_n be the n-corolla:



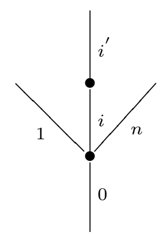
and for each $0 \leq i \leq n$ recall that $i : \eta \to C_n$ denotes the obvious (outer face) map in Ω that sends the unique edge of η to the edge i in C_n. An element $f \in X_{C_n}$

will be denoted by

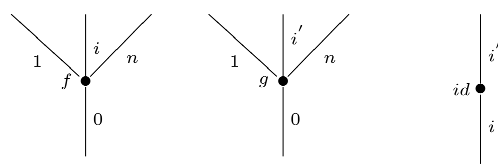


If $C_n^{'}$ is another n-corolla together with an isomorphism $\alpha : C_n^{'} \to C_n$ then we will usually write f again instead of $\alpha^*(f)$. We will use this convention quite often in the coming definitions and constructions, and in each case there will be an obvious choice for the isomorphism α given by the planar representation of the trees at question, which will usually not be mentioned.

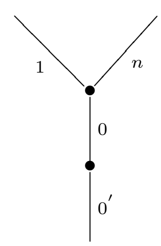
Definition 8.0.12. Let X be an inner Kan complex and let $f, g \in X_{C_n}$, $n \geq 0$. For $1 \leq i \leq n$ we say that f *is homotopic to* g *along the edge* i, and write $f \sim_i g$, if there is a dendrex H of shape



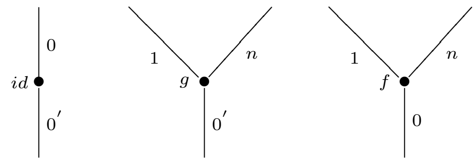
whose three faces are:



where the third one denotes a degeneracy. Similarly we will say that f is homotopic to g along the edge 0 and write $f \sim_0 g$ if there is a dendrex of shape



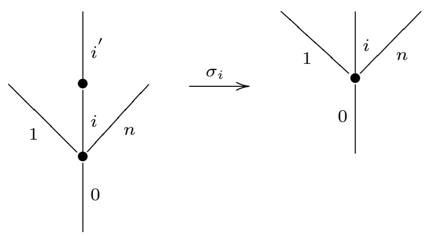
whose three faces are:



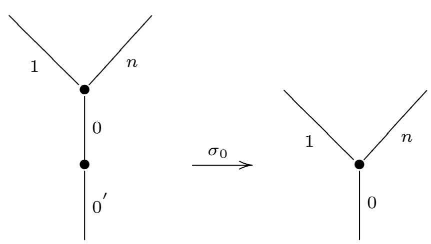
When $f \sim_i g$ for some $0 \leq i \leq n$ we will refer to the corresponding H as a *homotopy* from f to g along i and will sometimes write $H : f \sim_i g$.

Proposition 8.0.13. *Let X be an inner Kan complex. For each $0 \leq i \leq n$ the relation $\sim_i$ on the set X_{C_n} is an equivalence relation.*

Proof. First we prove reflexivity. For $1 \leq i \leq n$ let

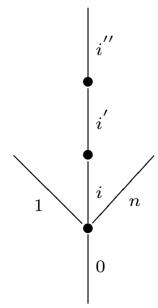


and for $i = 0$ let



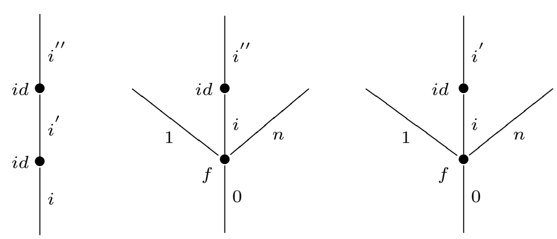
be the obvious degeneracies. It then follows that for any $f \in X_{C_n}$ the dendrex $\sigma_i^*(f)$ is a homotopy from f to f, thus $f \sim_i f$.

To prove symmetry assume $f \sim_i g$ for some $1 \leq i \leq n$ and let H_{fg} be a homotopy from f to g along i. Consider the tree T:

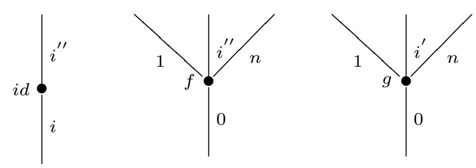


For the inner horn $\Lambda^i[T]$, corresponding to the edge i in the tree above, we now describe a map $\Lambda^i[T] \to X$. Such a map is given by specifying three dendrices in X of certain shapes such that their faces match in a suitable way. We describe this map by explicitly writing the mentioned dendrices and their faces:

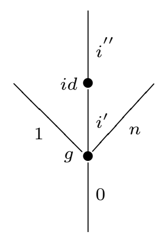
H_i $\quad$ H_f $\quad$ H_{fg}



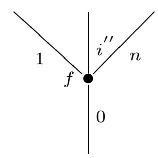
with inner faces of these dendrices:



where H_i is a double degeneracy of i, H_f is a homotopy from f to f (along the branch i) and H_{fg} is the given homotopy from f to g. It is easily checked that the faces indeed match so that we have a horn in X. Let x be a filler for that horn and consider $H_{gf} = \partial_i^*(x)$. This dendrex can be pictured as

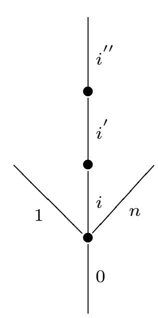


with inner face:



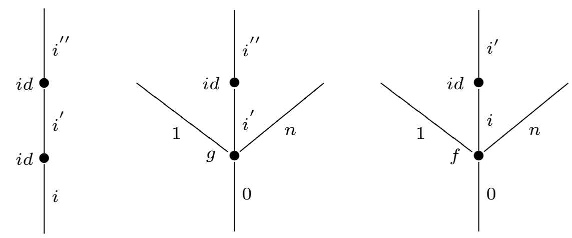
and is thus a homotopy from g to f along i, so that $g \sim_i f$. For $i = 0$ a similar proof works.

To prove transitivity let $f \sim_i g$ and $g \sim_i h$ for $1 \leq i \leq n$. Let H_{fg} be a homotopy from f to g and let H_{gh} be a homotopy from g to h. We again consider the tree T:

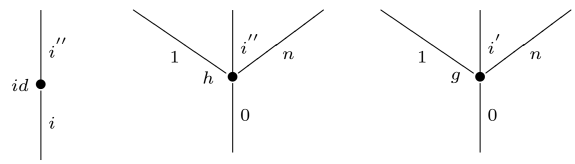


This time we look at $\Lambda^{i'}[T]$. The following describes a map $\Lambda^{i'}[T] \to X$ in X:

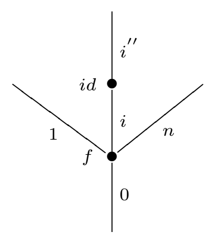
H_i H_{gh} H_{fg}



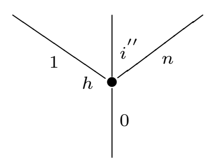
with inner faces being:



Let x be a filler for that horn and let $H_{fh} = \partial^*_{i'}(x)$. This dendrex can be pictured as follows:



with inner face:

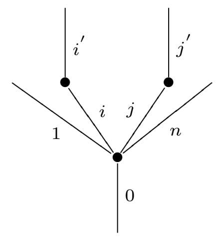


and is thus a homotopy from f to h so that $f \sim_i h$. The proof for $i = 0$ is similar. $\square$

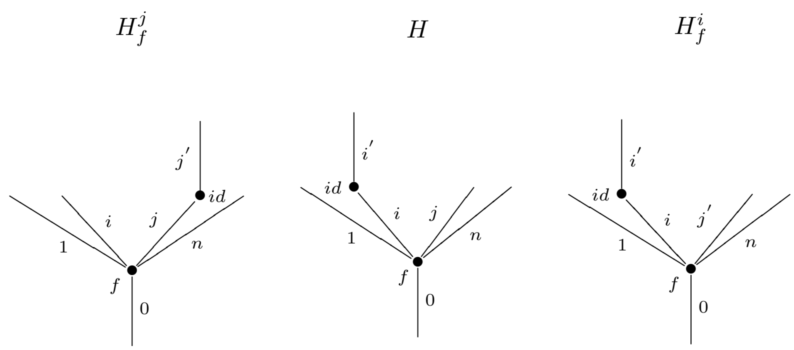
Lemma 8.0.14. *Let X be an inner Kan complex. The relations $\sim_0, \cdots, \sim_n$ on X_{C_n} are all equal.*

Remark 8.0.15. On the basis of this lemma, we will later just write $f \sim g$ instead of $f \sim_i g$.

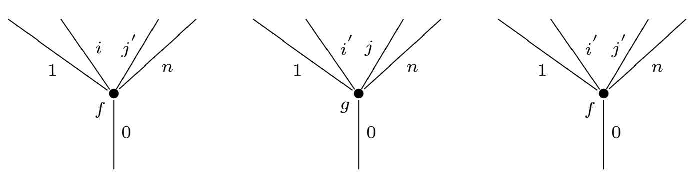
Proof. Suppose $H : f \sim_i g$ for $1 \leq i \leq n$ and let $1 \leq i < j \leq n$. We consider the tree T:



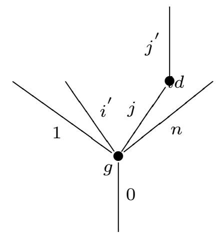
and the inner horn $\Lambda^i[T]$. The following then describes a map $\Lambda^i[T] \to X$:



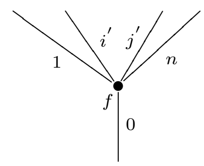
where $H^j_f : f \sim_j f$ and $H^i_f : f \sim_i f$. The inner faces of the three dendrices are



Let x be a filler for this horn, then $\partial_i^*(x)$ is the following dendrex



with inner face:



and is thus a homotopy from g to f along the j-th branch. Thus $g \sim_j f$ and so $f \sim_j g$ as well. The other cases to be considered follow in a similar way. $\square$

Given an inner Kan complex X and vertices $x_1, \cdots, x_n, x \in X_\eta$, let us write

$$X(x_1, \cdots, x_n; x) \subseteq X(C_n)$$

for the set of dendrices x of shape C_n with $0^*(x) = x$ and $i^*(x) = x_i$ for $i = 0, \cdots, n$. Here $i : \eta \to C_n$ denotes the map in Ω sending the unique edge of η to the one of C_n with name i. The equivalence relation $\sim$ on $X(C_n)$ given by the preceding lemma defines a quotient of $X(C_n)$ which we will denote by

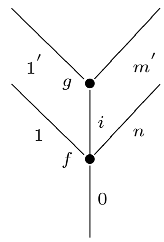
$$Ho(X)(x_1, \cdots, x_n; x) = X(x_1, \cdots, x_n; x)/\sim .$$

This defines a coloured collection $Ho(X)$, and a canonical quotient map of collections $Sk_1(X) \to Ho(X)$. We will now proceed to prove the following.

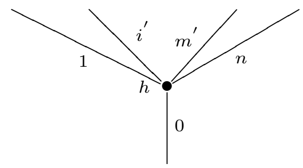
Proposition 8.0.16. *There is a unique structure of a (symmetric, coloured) operad on $Ho(X)$ for which the map of collections $Sk_1(X) \to Ho(X)$ extends to a map of dendroidal sets $X \to N_d(Ho(X))$. The latter map is an isomorphism whenever X is a* strict *inner Kan complex.*

To prepare for the proof of this proposition, we begin by defining the composition operations $\circ_i$ of the operad $Ho(X)$. Let X be an inner Kan complex and let $f \in X_{C_n}$ and $g \in X_{C_m}$ be two dendrices in X. We will say that a dendrex

$h \in X_{C_{n+m-1}}$ is a $\circ_i$-composition of f and g if there is a dendrex γ in X as follows:



with inner face

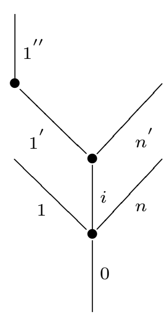


We will denote this situation by $h \sim f \circ_i g$ and call γ a *witness* for the composition.

Remark 8.0.17. Notice that for $1 \leq i \leq n$ we have by definition that $H : f \sim_i g$ iff H is a witness for the composition $g \sim f \circ_i id$. Similarly for $i = 0$ we have that $H : f \sim_0 g$ iff H is a witness for the composition $g \sim id \circ f$.

Lemma 8.0.18. *In an inner Kan complex X, if $h \sim f \circ_i g$ and $h' \sim f \circ_i g$ then $h \sim h'$.*

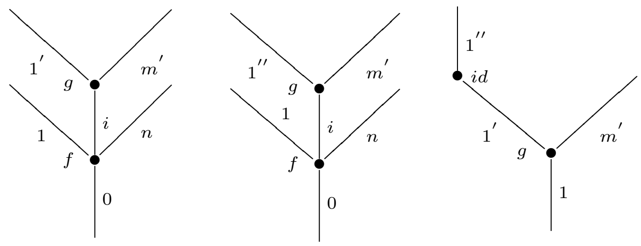
Proof. Let γ be a witness for the composition $h \sim f \circ_i g$ and γ' one for the composition $h' \sim f \circ_i g$. We consider the tree T:



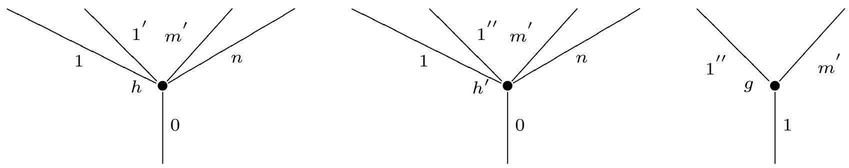
and the inner horn $\Lambda^i[T]$. Let $H_g : g \sim_i g$ and consider the following map $\Lambda^i[T] \rightarrow$

X

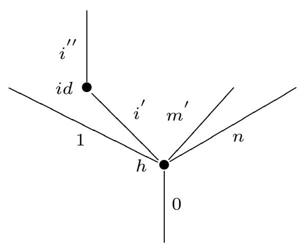
γ γ' H_g



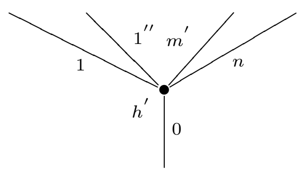
with inner faces



Let x be a filler for this horn. The face $\partial_i^* x$ is then the dendrex



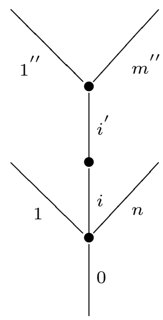
whose inner face is



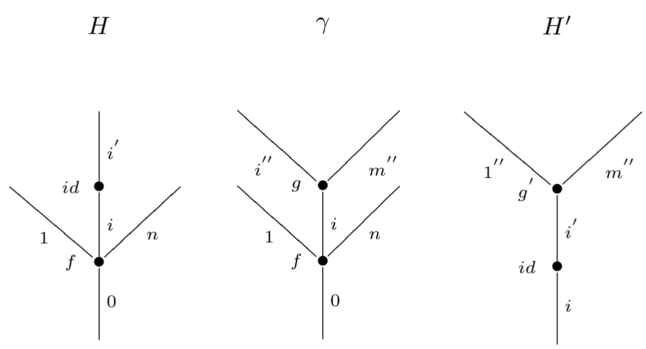
which proves that $h \sim h'$. $\square$

Lemma 8.0.19. *In an inner Kan complex X, let $f \sim f'$ and $g \sim g'$. If $h \sim f \circ_i g$ and $h' \sim f' \circ_i g'$ then $h \sim h'$.*

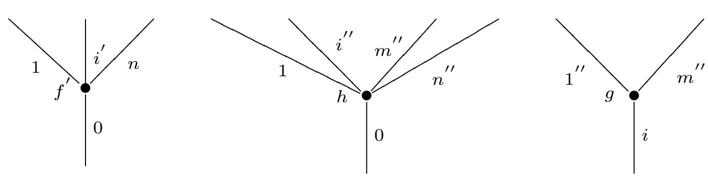
Proof. Let H be a homotopy from f to f' along the edge i, H' a homotopy from g' to g along the root, and γ a witness for the composition $h \sim f \circ_i g$. We now consider the tree T:



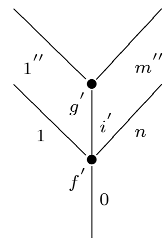
and the inner horn $\Lambda^i[T]$. The following is then a map $\Lambda^i[T] \to X$ in X:



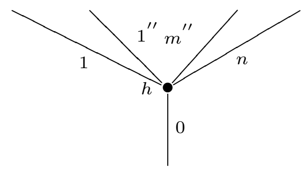
with inner faces:



The missing face of a filler for this horn is then:



with inner face



which proves that $h \sim f' \circ_i g'$, and thus by the previous result also that $h \sim h'$. $\square$

We now proceed to prove Proposition 8.0.16:

Proof. (of Proposition 8.0.16) Lemma 8.0.18 implies that for

$$[f] \in Ho(X)(x_1, \cdots, x_n; x)$$

and

$$[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$$

the assignment

$$[f] \circ_i [g] = [f \circ_i g]$$

is well-defined. This provides the $\circ_i$ operations in the operad $Ho(X)$. The Σ_n actions are defines as follows. Given a permutation $\sigma \in \Sigma_n$ let $\sigma : C_n \to C_n$ be the obvious induced map in Ω. The map $\sigma^* : X_{C_n} \to X_{C_n}$ restricts to a function

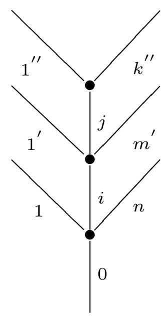
$$\sigma^* : X(x_1, \cdots, x_n; x) \to X(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x)$$

and it is trivial to verify that this map respects the homotopy relation. We thus obtain a map

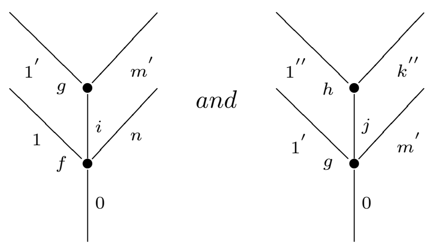
$$\sigma^* Ho(X)(x_1, \cdots, x_n; x) \to Ho(X)(x_{\sigma(1)}, \cdots, x_{\sigma(n)}; x).$$

We now need to show that these structure maps make the coloured collection $Ho(X)$ into an operad. The verification is simple and we exemplify it by proving associativity. Let $[f] \in Ho(X)(x_1, \cdots, x_n; x)$, $[g] \in Ho(X)(y_1, \cdots, y_m; x_i)$ and

$[h] \in Ho(X)(z_1, \cdots, z_k; y_m)$. We need to prove that $[f] \circ ([g] \circ [h]) = ([f] \circ [g]) \circ [h]$ (with the obvious choice for subscripts on the $\circ$) which is the same as showing that $f \circ (g \circ h) \sim (f \circ g) \circ h$ for any choice for compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$. Consider the tree T given by



and consider the anodyne extension $\Lambda^I[T] \to \Omega[T]$, cf. Lemma 7.0.7. The two given compositions $\psi \sim g \circ h$ and $\varphi \sim f \circ g$ define a map $\Lambda^I[T] \to X$ depicted by



whose inner faces are respectively ψ and φ. Let $x \in X_T$ be a dendrex extending this map. Let $c : C_m \to T$ be the map obtained by contracting both i and j and $\rho = c^*x$. It now follows that $\partial_i^* x$ is a witness for the composition $\rho \sim \psi \circ h$ and $\partial_j^* x$ is a witness for the composition $\rho \sim f \circ \varphi$. That proves associativity. The other axioms for an operad follow in a similar manner.

Next, let us show that the quotient map $q : Sk_1(X) \to Ho(X)$ extends to a map $q : X \to N_d(Ho(X))$ of dendroidal sets. Since we already know that $N_d(X)$ is 2-coskeletal, it suffices to give its values for dendrices $x \in X_T$ where T is a tree with two vertices. Let e be the inner edge of this tree. Then $\Lambda^e[T] \rightarrowtail \Omega[T] \xrightarrow{x} X$ factors through $Sk_1(X)$, so its composition $\Lambda^e[T] \to N_d(Ho(X))$ with q has a unique extension (Proposition 7.0.9), which we take to be $q(x) : \Omega[T] \to N_d(Ho(X))$. This defines $q : Sk_2(X) \to Sk_2(N_d(Ho(X)))$, and hence all of $q : X \to N_d(Ho(X))$ by 2-coskeletality, as said.

Finally, when X is itself a strict inner Kan complex, then the homotopy relation is the identity relation, so $Sk_1(X) \to Ho(X)$ is the identity map. Since X

and $N_d(Ho(X))$ are now both strict inner Kan complexes, the extension $q : X \to N_d(Ho(X))$ is an isomorphism. $\square$

The following Proposition, together with Proposition 8.0.16, now provide the proof of Theorem 8.0.11.

Proposition 8.0.20. *For any inner Kan complex X, the natural map $\tau_d(X) \to Ho(X)$ is an isomorphism of operads.*

Proof. It suffices to prove that the map $q : X \to N_d(Ho(X))$ of Proposition 8.0.16 has the universal property of the unit of the adjunction. This means that for any operad $\mathcal{P}$ and any map $\varphi : X \to N_d(\mathcal{P})$, there is a unique map of operads $\psi : Ho(X) \to \mathcal{P}$ for which $N_d(\psi)q = \varphi$. But φ induces a map $Ho(X) \to Ho(N_d(\mathcal{P}))$ for which

$$\begin{tikzcd} Sk_1(X) \arrow[r, "\varphi"] \arrow[d, "q_X"] & Sk_1 N_d(\mathcal{P}) \arrow[d, "q_{\mathcal{P}}"] \\ Ho(X) \arrow[r, "Ho(\varphi)"] & Ho(N_d(\mathcal{P})) \end{tikzcd}$$

commutes, and $Ho(N_d(\mathcal{P})) = \mathcal{P}$ while $q_{\mathcal{P}}$ is the identity as we have seen in (the proof of) Proposition 8.0.16. So $Ho(\varphi)$ in fact defines a map $\psi : Ho(X) \to \mathcal{P}$ of collections. It is easily seen that ψ is a map of operads. It is unique because q_X is surjective. $\square$

Chapter 9

Homotopy coherent nerves are inner Kan

In this section, we assume $\mathcal{E}$ is a monoidal model category with a cofibrant unit I. We also assume that $\mathcal{E}$ is equipped with an *interval* in the sense of [BM1]. Recall from 6 that such an interval is given by maps

$$I \underset{1}{\overset{0}{\rightrightarrows}} H \xrightarrow{\epsilon} I$$

and

$$H \otimes H \xrightarrow{\vee} H$$

satisfying certain conditions. In particular, H is an interval in Quillen's sense ([Q]), so 0 and 1 together define a cofibration $I \coprod I \to H$, and ϵ is a weak equivalence. In Section 6 we explained how such an interval H allows one to construct for each (coloured) operad $\mathcal{P}$ in $\mathcal{E}$ a "Boardman-Vogt" resolution $W_H(\mathcal{P}) \to \mathcal{P}$. Each operad in Set can be viewed as an operad in $\mathcal{E}$ (via the functor $Set \to \mathcal{E}$ which preserves coproducts and sends the one-point set to I), and hence has such a Boardman-Vogt resolution. When we apply this to the operads $\Omega(T)$, we obtain the *homotopy coherent dendroidal nerve* $hcN_d(\mathcal{P})$ of any operad $\mathcal{P}$ in $\mathcal{E}$, as the dendroidal set given by

$$hcN_d(\mathcal{P})_T = Hom(W_H(\Omega(T)), \mathcal{P})$$

where the Hom is that of operads in $\mathcal{E}$. Our goal here is to prove the following result.

Theorem 9.0.21. *Let $\mathcal{P}$ be an operad in $\mathcal{E}$, with the property that for each sequence $c_1, \cdots, c_n; c$ of colours of $\mathcal{P}$, the object $\mathcal{P}(c_1, \cdots, c_n; c)$ is fibrant. Then $hcN_d(\mathcal{P})$ is an inner Kan complex.*

Remark 9.0.22. As explained in Section 6, our construction of the dendroidal homotopy coherent nerve specializes to that of the homotopy coherent nerve of an $\mathcal{E}$-enriched category, and for the case where $\mathcal{E}$ is the category of topological spaces or simplicial sets, one recovers the classical definition ([CP]). In particular, as a special case of Theorem 9.0.21, one obtains that for an $\mathcal{E}$-enriched category with fibrant Hom objects (in other words, for a locally fibrant $\mathcal{E}$-enriched category), its homotopy coherent nerve is a quasi-category in the sense of Joyal. This result was proved, for the case where $\mathcal{E}$ is simplicial sets, by Cordier and Porter in [CP].

Before embarking on the proof of Theorem 9.0.21, we need to be a bit more explicit about the operads of the form $W_H\Omega(T)$ involved in the definition of the homotopy coherent nerve. Recall first of all the functor

$$Symm : Operad(\mathcal{E})_\pi \to Operad(\mathcal{E})$$

which is left adjoint to the forgetful functor from symmetric operads to non-symmetric (i.e., planar) ones. If T is an object in Ω and $\bar{T}$ is a chosen planar representative of T, then $\bar{T}$ naturally describes a planar operad $\Omega(\bar{T})$ for which $\Omega(T) = Symm(\Omega(\bar{T}))$. Since the W-construction commutes with symmetrization (as one readily verifies), it follows that

$$W_H(T) = Symm(W_H\Omega(\bar{T})).$$

This latter operad $W_H\Omega(\bar{T})$ is easily described explicitly. The colours of

$$W_H(\Omega(\bar{T}))$$

are the colours of $\Omega(\bar{T})$, i.e., the edges of T. By a *signature,* we mean a sequence $e_1, \cdots, e_n; e_0$ of edges. Given a signature $\sigma = (e_1, \cdots, e_n; e_0)$, we have that $W_H(\Omega(\bar{T}))(\sigma) = 0$ whenever $\Omega(\bar{T})(\sigma) = \phi$. And if $\Omega(\bar{T})(\sigma) \neq \phi$, there is a subtree T_σ of T (and a corresponding planar subtree $\bar{T}_\sigma$ of $\bar{T}$) whose leaves are $e_1, \cdots, e_n$, and whose root is e_0. Then

$$W_H\Omega(\bar{T})(e_1, \cdots, e_n; e_0) = \bigotimes_{f \in i(\sigma)} H,$$

where $i(\sigma)$ is the set of *inner* edges of T_σ (or of $\bar{T}_\sigma$). (This last tensor product is to be thought of as the "space" of assignments of lengths to inner edges in $\bar{T}_\sigma$; it is the unit if $i(\sigma)$ is empty.)

Remark 9.0.23. The composition operations in the operad $W_H\Omega(\bar{T})$ are given in terms of the $\circ_i$-operations as follows. For signatures $\sigma = (e_1, \cdots, e_n; e_0)$ and $\rho = (f_1, \cdots, f_m; e_i)$, the composition map

$$\begin{array}{c} \Omega(\bar{T})(e_1, \cdots, e_n; e_0) \otimes \Omega(\bar{T})(f_1, \cdots, f_m; e_i) \\ \downarrow \circ_i \\ \Omega(\bar{T})(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0) \end{array} \qquad (1)$$

is the following one. The trees $\bar{T}_\sigma$ and $\bar{T}_\rho$ can be grafted along e_i to form $\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho$, again a planar subtree of $\bar{T}$. In fact

$$\bar{T}_\sigma \circ_{e_i} \bar{T}_\rho = \bar{T}_{\sigma\circ_i\rho}$$

where $\sigma \circ_i \rho$ is the signature $(e_1, \cdots, e_{i-1}, f_1, \cdots, f_m, e_{i+1}, \cdots, e_n; e_0)$, and for the sets of inner edges we have

$$i(\sigma \circ_i \rho) = i(\sigma) \cup i(\rho) \cup \{e_i\}.$$

The composition map in (1) now is the map

$$\begin{tikzcd}
H^{\otimes i(\sigma)} \otimes H^{\otimes i(\rho)} \arrow[r, dotted] \arrow[d, "\cong"] & H^{\otimes i(\sigma\circ_i\rho)} \arrow[d, "\cong"] \\
H^{\otimes i(\sigma)\cup i(\rho)} \otimes I \arrow[r, "id\otimes 1"] & H^{\otimes i(\sigma)\cup i(\rho)} \otimes H
\end{tikzcd}$$

where $1 : I \to H$ is one of the "endpoints" of the interval H, as above.

This description of the operad $W_H\Omega(\bar{T})$ is functorial in the planar tree T. In particular, we note that for an inner edge e of T, the tree T/e inherits a planar structure $\overline{T/e}$ from $\bar{T}$, and $W_H\Omega(\overline{T/e}) \to W_H\Omega(\bar{T})$ is the natural map assigning length 0 to the edge e whenever it occurs (in a subtree given by a signature).

Proof. (Of Theorem 9.0.21) Consider a tree T and an inner edge e in T. We want to solve the extension problem

$$\begin{tikzcd}
\Lambda^e[T] \arrow[r, "\varphi"] \arrow[d, tail] & hcN_d(\mathcal{P}) \\
\Omega[T] \arrow[ur, dotted]
\end{tikzcd}$$

Fix a planar representative $\bar{T}$ of T. Then the desired map $\psi : \Omega[T] \to hcN_d(\mathcal{P})$ corresponds to a map of planar operads

$$\hat{\psi} : W_H\Omega(\bar{T}) \to \mathcal{P}.$$

Each face S of T inherits a planar structure $\bar{S}$ from $\bar{T}$, and the given map $\varphi : \Lambda^e[T] \to hcN_d(\mathcal{P})$ corresponds to a map of operads in $\mathcal{E}$,

$$\hat{\varphi} : W_H(\Lambda^e[T]) \to \mathcal{P},$$

where $W_H(\Lambda^e[T])$ denotes the colimit of operads in $\mathcal{E}$,

$$W_H(\Lambda^e[T]) = colim\, W(\Omega(\bar{S})) \qquad (2)$$

over all but one of the faces of T. In other words, φ corresponds to a compatible family of maps

$$\hat{\varphi}_S : W_H(\Omega(\bar{S})) \to \mathcal{P}.$$

Let us now show the existence of an operad map $\hat{\psi}$ extending the $\hat{\varphi}_S$ for all faces $S \neq T/e$. First, the colours of $\Omega(\bar{T})$ are the same as those of the colimit in (2), so we already have a map $\psi_0 = \varphi_0$ on colours:

$$\psi_0 : (\text{Edges of } T) \to (\text{Colours of } \mathcal{P}).$$

Next, if $\sigma = (e_1, \cdots, e_n; e_0)$ is a signature of T for which $W_H(\Omega(\bar{T})) \neq \phi$, and if $T_\sigma \subseteq T$ is not all of T, then T_σ is contained in an outer face S of T. So $W_H(\Omega(\bar{T}))(\sigma) = W_H(\Omega(\bar{T}_\sigma))(\sigma) = W_H(\Omega(\bar{S}))(\sigma)$, and we already have a map

$$\hat{\varphi}_S(\sigma) : W_H(\Omega(\bar{T}))(\sigma) \to \mathcal{P}(\sigma),$$

given by $\hat{\varphi}_S : W_H(\Omega(\bar{S})) \to \mathcal{P}$. Thus, the only part of the operad map $\hat{\psi} : W_H(\Omega(\bar{T})) \to \mathcal{P}$ not determined by φ is the one for the signature τ where $T_\tau = T$; i.e., $\tau = (e_1, \cdots, e_n; e_0)$ where $e_1, \cdots, e_n$ are all the input edges of $\bar{T}$ (in the planar order) and e_0 is the output edge. For this signature, $\hat{\psi}(\tau)$ is to be a map

$$\hat{\psi} : W_H(\Omega(\bar{T})(\tau) = H^{\otimes i(\tau)} \to \mathcal{P}(\tau)$$

which (i) is compatible with the $\hat{\psi}(\sigma) = \hat{\varphi}_S(\sigma)$ for other signatures σ; and (ii) together with these $\hat{\psi}(\sigma)$ respects operad composition. The first condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 0 on one of the tensor-factors marked by an edge e_i *other* than the given e. The second condition determines $\hat{\psi}(\tau)$ on the subobject of $H^{\otimes i(\tau)}$ which is given by a value 1 on one of the factors. Thus, if we write 1 for the map $I \rightarrowtail^{1} H$ and $\partial H \rightarrowtail H$ for the map $I \coprod I \to H$, and define $\partial H^{\otimes k} \rightarrowtail H^{\otimes k}$ by the Leibniz rule (i.e., $\partial(A \otimes B) = \partial(A) \otimes B \cup A \otimes \partial(b))$, then the problem of finding $\hat{\psi}(\tau)$ comes down to an extension problem of the form

$$\begin{tikzcd}
\partial(H^{\otimes i(\sigma)-\{e\}} \otimes H) \cup H^{\otimes i(\sigma)-\{e\}} \otimes I \arrow[r] \arrow[d, tail] & \mathcal{P}(\tau) \\
H^{\otimes i(\sigma)-\{e\}} \otimes H \arrow[r, "\cong"] & H^{\otimes i(\sigma)} \arrow[u, dotted, "\hat{\psi}(\sigma)"']
\end{tikzcd}$$

This extension problem has a solution, because $\mathcal{P}(\tau)$ is fibrant by assumption, and because the left hand map is a trivial cofibration (by repeated use of the push-out product axiom for monoidal model categories). This concludes the proof of the theorem. $\square$

Chapter 10

The Grothendieck construction preserves inner Kan complexes

Let $\mathbb{S}$ be a Cartesian category. A functor $X : \mathbb{S}^{op} \to dSet$ is called a *diagram* of dendroidal sets. In Section 4 a construction was given of the dendroidal set $\int_{\mathbb{S}} X$. This construction was then applied to the specific diagram of dendroidal sets $X : Set^{op} \to dSet$, where for a set A, $X(A)$ was the dendroidal set of weak n-categories having A as set of objects. The dendroidal set $\int_{\mathbb{S}} X$ was defined to be the dendroidal set of weak n-categories. Our aim in this section is to prove that for a given diagram of dendroidal sets $X : \mathbb{S}^{op} \to dSet$, if each $X(S)$ is an inner Kan complex then $\int_{\mathbb{S}} X$ is also an inner Kan complex. For the convenience of the reader we repeat here the definition of $\int_{\mathbb{S}} X$.

It will be convenient to consider dendroidal collections. A dendroidal collection is a collection of sets $X = \{X_T\}_{T\in\Omega}$. Each dendroidal set has an obvious underlying dendroidal collection. A map of dendroidal collections $X \to Y$ is a collection of functions $\{X_T \to Y_T\}_{T\in\Omega}$. Given a Cartesian category $\mathbb{S}$, consider the dendroidal nerve $N_d(\mathbb{S})$ where $\mathbb{S}$ is regarded as an operad via the Cartesian structure. There is a natural way of associating an object of $\mathbb{S}$ with each dendrex of $N_d(\mathbb{S})$. For a tree T in Ω, let $leaves(T)$ be the set of leaves of T, and for a leaf l, write $l : \eta \to T$ also for the map sending the unique edge in η to l in T. Then, since $\mathbb{S}$ is assumed to have finite products, each dendrex $t \in N_d(\mathbb{S})_T$ defines an object

$$in(t) = \prod_{l\in leaves(T)} l^*(t)$$

in $\mathbb{S}$. Notice that if $\alpha : S \to T$ is a composition of face maps, then by using the canonical symmetries and the projections in $\mathbb{S}$ there is a canonical arrow $in(\alpha) : in(t) \to in(\alpha^* t)$ for any $t \in X_T$.

Definition 10.0.24. Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets. The dendroidal set $\int_{\mathbb{S}} X$ is defined as follows. A dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ is a pair (t, x)

such that $t \in N_d(\mathbb{S})_T$ and x is a map of dendroidal collections

$$x : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$$

satisfying the following conditions. For each $r \in \Omega[T]_R$ (that is an arrow $r : R \to T$), we demand that $x(r) \in X(in(r^*t))$. Furthermore we demand the following compatibility conditions to hold. For any $r \in \Omega[T]_R$ and any map $\alpha : U \rightarrowtail R$ in Ω

$$\alpha^*(x(r)) = X(in(\alpha))x(\alpha^*(r)).$$

Theorem 10.0.25. *Let $X : \mathbb{S}^{op} \to dSet$ be a diagram of dendroidal sets. If for any $S \in ob(\mathbb{S})$ the dendroidal set $X(S)$ is a (strict) inner Kan complex then so is $\int_{\mathbb{S}} X$.*

Proof. Let T be a tree and e an inner edge. We consider the extension problem

$$\begin{tikzcd} \Lambda^e[T] \arrow[r] \arrow[d, tail] & \int_{\mathbb{S}} X \\ \Omega[T] \arrow[ur, dotted] & \end{tikzcd}$$

The horn $\Lambda^e[T] \to \int_{\mathbb{S}} X$ is given by a compatible collection $\{(r, x_R) : \Omega[R] \to \int_{\mathbb{S}} X\}_{R \neq T/e}$. We wish to construct a dendrex $(t, x_T) : \Omega[T] \to \int_{\mathbb{S}} X$ extending this family. First notice that the collection $\{r\}_{R \neq T/e}$ is an inner horn $\Lambda^e[T] \to N_d(\mathbb{S})$ (actually this horn is obtained by composing with the obvious projection $\int_{\mathbb{S}} X \to N_d(\mathbb{S})$ sending a dendrex (t, x) to t). We already know $N_d(\mathbb{S})$ to be an inner Kan complex (actually a strict inner Kan complex) and thus there is a (unique) filler $t \in N_d(\mathbb{S})_T$ for the horn $\{r\}_{R \neq T/e}$. We now wish to define a map of dendroidal collections $x_T : \Omega[T] \to \coprod_{S \in ob(\mathbb{S})} X(S)$ that will extend the given maps x_R for $R \neq T/e$. This condition already determines the value of x_T for any dendrex $r : U \to T$ other then $id : T \to T$ and $\alpha : T/e \to T$, since for each such r, the tree U factors through one of the faces $R \neq T/e$. To determine $x_T(id_T)$ and $x_T(\alpha)$ consider the family $\{y_R = x_R(id : R \to R)\}_{R \neq T/e}$. By definition we have that $y_R \in X(in(r))_R$. For each such R let $\alpha_R : R \to T$ be the corresponding face map in Ω. Since $\alpha^* t = r$ we obtain the map $in(\alpha_R) : in(r) \to in(t)$. We can now pull back the collection $\{y_R\}_{R \neq T/e}$ using $X(in(\alpha_R))$ to obtain a collection $\{z_R = X(in(\alpha_R))(y_R)\}_{R \neq T/e}$. This collection is now a horn $\Lambda^e[T] \to X(in(T))$ (this follows from the compatibility conditions in the definition of $\int_{\mathbb{S}} X$). Since $X(in(t))$ is inner Kan there is a filler $u \in X(in(t))_T$ for that horn. We now define $x_T(id : T \to T) = u$ and $x_T(\alpha : T/e \to T) = \alpha^*(u)$. Notice that since e is inner we have that $in(t) = in(\alpha)$ and thus the image of these dendrices are in the correct dendroidal set, namely $X(in(t))$. It follows from our construction that this makes (t, x_T) a dendrex $\Omega[T] \to \int_{\mathbb{S}} X$ which extends the given horn. This concludes the proof. $\square$

Chapter 11

The exponential property

Our aim in this section is to prove the following theorem concerning the closed monoidal structure of dendroidal sets.

Theorem 11.0.26. *Let K and X be dendroidal sets, and assume X is normal. If K is a (strict) inner Kan complex, then so is $\underline{Hom}_{dSet}(X, K)$.*

The internal *Hom* here is defined by the universal property, giving a bijective correspondence between maps $Y \otimes X \to K$ and $Y \to \underline{Hom}(X, K)$ for any dendroidal set Y, and natural in Y. We recall that $\otimes$ is defined in terms of the Boardman-Vogt tensor product of operads. We remind the reader that for two (coloured) operads $\mathcal{P}$ and $\mathcal{Q}$ with respective sets of colours C and D, this tensor product operad $\mathcal{P} \otimes_{BV} \mathcal{Q}$ has the product $C \times D$ as its set of colours, and is described in terms of generators and relations as follows. The operations in $\mathcal{P} \otimes_{BV} \mathcal{Q}$ are generated by the operations

$$p \otimes d \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d), \cdots, (c_n, d); (c, d))$$

for any $p \in \mathcal{P}(c_1, \cdots, c_n; c)$ and any $d \in D$, and

$$c \otimes q \in \mathcal{P} \otimes_{BV} \mathcal{Q}((c, d_1), \cdots, (c, d_m); (c, d))$$

for any $q \in \mathcal{Q}(d_1, \cdots, d_m; d)$ and any $c \in C$. The *relations* between these state, first of all, that for fixed $c \in C$ and $d \in D$, the maps $p \mapsto p \otimes d$ and $c \mapsto c \otimes q$ are maps of operads. Secondly, there is an *interchange law* stating that, for p and q as above, the composition $p \otimes d(c \otimes q, \cdots, c \otimes q)$ in

$$\mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d_1), \cdots, (c_1, d_m), \cdots, (c_n, d_1), \cdots, (c_n, d_m); (c, d))$$

and $c \otimes q(p \otimes d, \cdots, p \otimes d)$ in

$$\mathcal{P} \otimes_{BV} \mathcal{Q}((c_1, d_1), \cdots, (c_n, d_1), \cdots, (c_1, d_m), \cdots, (c_n, d_m); (c, d))$$

are mapped to each other by the obvious permutation $\tau \in \Sigma_{n\times m}$ which puts the two sequences of input colours in the same order. The tensor product of dendroidal sets is then uniquely determined (up to isomorphism) by the fact that it preserves colimits in each variable separately, together with the identity

$$\Omega[S] \otimes \Omega[T] = N_d(\Omega(S) \otimes_{BV} \Omega(T))$$

stated in Section 5.1, which gives the tensor product of two representable dendroidal sets.

First of all, let us prove that Theorem 11.0.26 follows by a standard argument from the following proposition.

Proposition 11.0.27. *For any two objects S and T of Ω, and any inner edge e in S, the map*

$$\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T] \rightarrowtail \Omega[S] \otimes \Omega[T]$$

is an anodyne extension.

In the proposition above, the union is that of subobjects of the codomain, which is the same as the pushout over the intersection $\Lambda^e[S] \otimes \partial\Omega[T]$.

Proof. (of Theorem 11.0.26 from Proposition 11.0.27) The theorem states that for any tree S and any inner edge $e \in S$, any map of dendroidal sets

$$\varphi : \Lambda^e[S] \otimes X \to K$$

extends to some map (uniquely in the strict case)

$$\psi : \Omega[S] \otimes X \to K.$$

By writing X as the union of its skeleta,

$$X = \varinjlim Sk_n(X)$$

as in Section 4, and using the fact that the skeletal filtration is normal, we can build this extension ψ by induction on n. For $n = 0$, $Sk_0(X)$ is a sum of copies of $\Omega[\eta]$, the unit for the tensor product, so obviously the restriction $\varphi_0 : \Lambda^e[S] \otimes Sk_0(X) \to K$ extends to a map

$$\psi_0 : \Omega[S] \otimes Sk_0(X) \to K.$$

Suppose now that we have found an extension $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ of the restriction $\varphi_n : \Lambda^e[S] \otimes Sk_n(X) \to K$. Consider the following diagram:

$$\begin{tikzcd}
\coprod \Lambda^e[S] \otimes \partial\Omega[T] \arrow[rr] \arrow[dr] \arrow[dd, tail] & & \coprod \Lambda^e[S] \otimes \Omega[T] \arrow[dr] \arrow[dd, tail] & \\
& \Lambda^e[S] \otimes Sk_n(X) \arrow[rr] \arrow[dd, tail] & & \Lambda^e[S] \otimes Sk_{n+1}(X) \arrow[dd, tail] \\
\coprod \Omega[S] \otimes \partial\Omega[T] \arrow[rr] \arrow[dr] & & \coprod \Omega[S] \otimes \Omega[T] \arrow[dr] & \\
& \Omega[S] \otimes Sk_n(X) \arrow[rr] & & \Omega[S] \otimes Sk_{n+1}(X)
\end{tikzcd}$$

In this diagram, the top and bottom faces are pushouts given by the normal skeletal filtration of X. Now inscribe the pushouts U and V in the back and front face, fitting into a square

$$\begin{tikzcd}
U \arrow[r] \arrow[d, tail] & \coprod \Omega[S] \otimes \Omega[T] \arrow[d, tail] \\
V \arrow[r] & \Omega[S] \otimes Sk_{n+1}(X)
\end{tikzcd}$$

The maps $\psi_n : \Omega[S] \otimes Sk_n(X) \to K$ and $\varphi_{n+1} : \Lambda^e[S] \otimes Sk_{n+1}(X) \to K$ together define a map $V \to K$. So, to find ψ_{n+1}, it suffices to prove that

$$V \rightarrowtail \Omega[S] \otimes Sk_{n+1}(X)$$

is anodyne. But, by a diagram chase argument, the square above is a pushout, so in fact, it suffices to prove that $U \rightarrowtail \coprod \Omega[S] \otimes \Omega[T]$ is anodyne. The latter map is a sum of copies of anodyne extensions as in the statement of the proposition. $\square$

Corollary 11.0.28. *The monoidal structure on the category of coloured operads given by the Boardman-Vogt tensor product is closed. It is related to the closed monoidal structure on dendroidal sets by two natural isomorphisms*

$$\tau_d(N_d\mathcal{P} \otimes N_d\mathcal{Q}) = \mathcal{P} \otimes_{BV} \mathcal{Q}$$

and

$$N_d(\underline{Hom}(\mathcal{Q}, \mathcal{R})) = \underline{Hom}(N_d\mathcal{Q}, N_d\mathcal{R})$$

for any operads $\mathcal{P}, \mathcal{Q}$ and $\mathcal{R}$.

Proof. The first isomorphism was proved in Proposition 5.2.2. The second isomorphism follows from the first one together with (the strict version of) Theorem 11.0.26, Theorem 8.0.11, and the fact that N_d is fully faithful. $\square$

In the rest of this section, we will be concerned with the proof of Proposition 11.0.27, and we fix S, T and e as in the statement of the proposition from now on. Our strategy will be as follows. First, let us write

$$A_0 \subseteq \Omega[S] \otimes \Omega[T]$$

for the dendroidal set given by the image of $\Lambda^e[S] \otimes \Omega[T] \cup \Omega[S] \otimes \partial\Omega[T]$. We are going to construct a sequence of dendroidal subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_N = \Omega[S] \otimes \Omega[T]$$

such that each inclusion is an anodyne extension. This will be done by writing $\Omega[S] \otimes \Omega[T]$ as a union of representables, as follows. We will explicitly describe a sequence of trees

$$T_1, T_2, \cdots, T_N$$

together with canonical monomorphisms (all called)

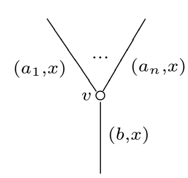
$$m : \Omega[T_i] \rightarrowtail \Omega[S] \otimes \Omega[T] \ ,$$

and we will write $m(T_i) \subseteq \Omega[S] \otimes \Omega[T]$ for the dendroidal subset given by the image of this monomorphism. We will then define

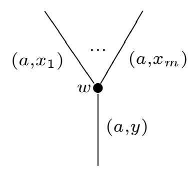
$$A_{i+1} = A_i \cup m(T_{i+1}) \qquad\qquad (i = 0, \cdots, N-1)$$

and prove that each $A_i \rightarrowtail A_{i+1}$ thus constructed is anodyne. For the rest of this section, we will fix planar structures on the trees S and T. These will then induce a natural planar structure on each of the trees T_i, and avoid unnecessary discussion involving automorphisms in the category Ω.

To define the T_i, let us think of the vertices of S as *white* (drawn $\circ$) and those of T as *black* (drawn $\bullet$). The edges of T_i are (labelled by) pairs (a, x) where a is an edge of S and x one of T. We refer to a as the *S-colour* of this edge (a, x), and to x as its *T-colour*. There are two kinds of vertices in T_i (corresponding to the generators for $\Omega[S] \otimes \Omega[T]$ coming from vertices of S or of T). There are *white* vertices in T_i labelled

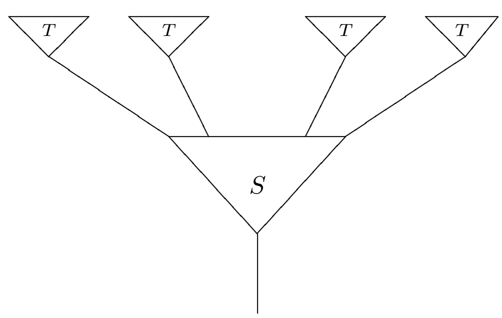


where v is a vertex in S with input edges $a_1, \cdots, a_n$ and output edge b, while x is an edge of T; and there are *black* vertices in T_i labelled

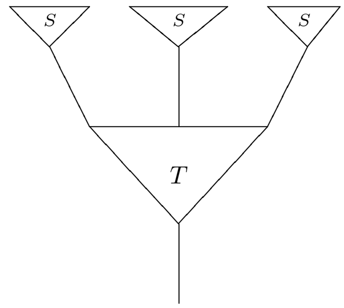


where w is a vertex in T with input edges $x_1, \cdots, x_m$ and output edge y, while a is an edge in S. Moreover, each such tree T_i is *maximal,* in the sense that its output (root) edge is labelled (r_S, r_T) where r_S and r_T are the roots of S and T, and its input edges are labelled by all pairs (a, x) where a is an input edge of S and x one of T.

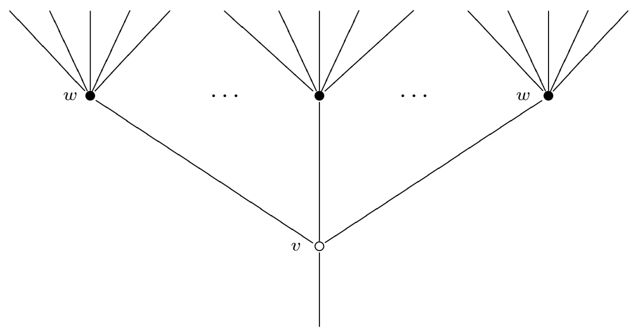
All the possible such trees T_i come in a natural (partial) order. The minimal tree T_1 in the poset is the one obtained by stacking a copy of the black tree T on top of each of the input edges of the white tree S. Or, more precisely, on the bottom of T_1 there is a copy $S \otimes r_T$ of the tree S all whose edges are renamed (a, r_T) where r_T is the output edge at the root of T. For each input edge b of S, a copy of T is grafted on the edge (b, r) of $S \otimes r$, with edges x in T renamed (b, x). The maximal tree T_N in the poset is the similar tree with copies of the white tree S grafted on each of the input edges of the black tree. Pictorially T_1 looks like



and T_N looks like

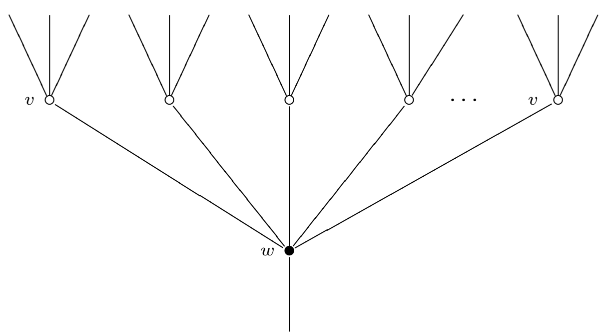


The intermediate trees T_k $(1 < k < N)$ are obtained by letting the black vertices in T_1 slowly percolate in all possible ways towards the root of the tree. Each T_k is obtained from an earlier T_l by replacing a configuration



(A)

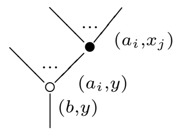
in T_l by



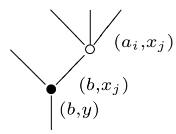
(B)

in T_k. More explicitly, if v and w are vertices in S and T,

then the edges in (A) are named

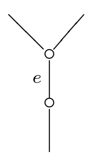
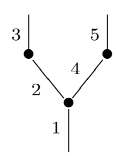


and those in (B) are named

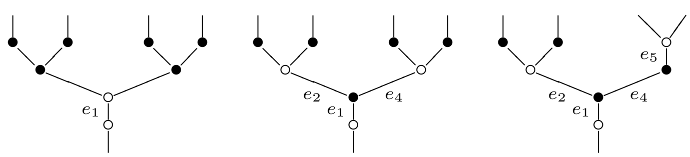


We will refer to these trees T_i as the *percolation schemes* for S and T, and if T_k is obtained from T_l by replacing (A) by (B), then we will say that T_l is obtained by a *single percolation step.*

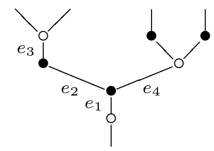
Example 11.0.29. Many of the typical phenomena that we will encounter already occur for the following two trees S and T; here, we have singled out one particular edge e in S, we've numbered the edges of T as $1, \cdots, 5$, and denoted the colour (e, i) in T_i by e_i.

$S =$  $T =$ 

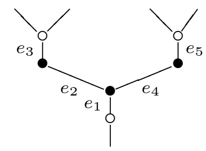
There are 14 percolation schemes $T_1, \cdots, T_{14}$ in this case. Here is the complete list of them:



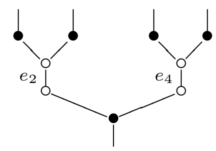
T_1 T_2 T_3



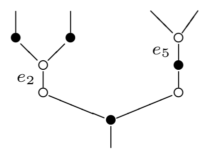
T_4



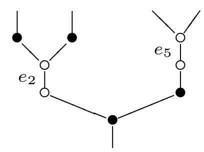
T_5



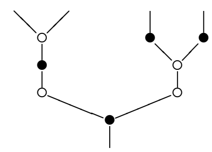
T_6



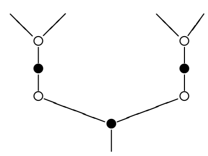
T_7



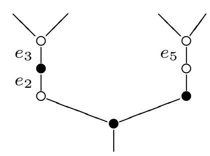
T_8



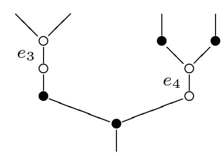
T_9



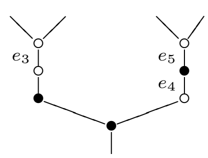
T_{10}



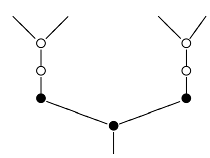
T_{11}



T_{12}

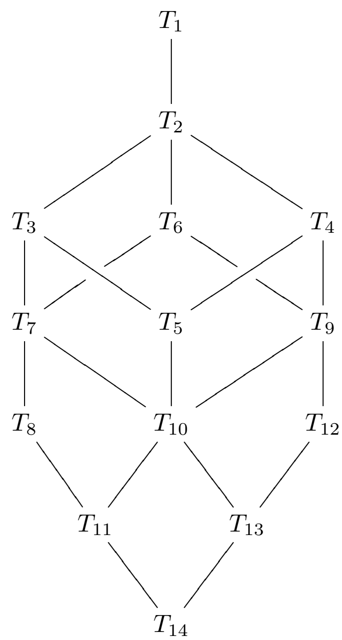


T_{13}



T_{14}

As claimed, there is a partial order on the percolation schemes $T_1, \cdots, T_N$ for $S \otimes T$, in which T_1 (copies of T on top of S) is the minimal element and T_N (copies of S on top of T) the maximal one. The partial order is given by defining $T \leq T'$ whenever the percolation scheme T' can be obtained from the percolation scheme T by a sequence of percolations. For example, the poset structure on the percolation trees above is:



The planar structures of S and T provide a way to refine this partial order by a linear order. It is not important exactly how this is done, but we shall from now on assume that the percolation schemes for S and T are ordered $T_1, \cdots, T_N$ where T_i comes before T_j only if $T_i \leq T_j$ in the partial order.

Lemma 11.0.30. *(and notation) Each percolation scheme T_i is equipped with a canonical monomorphism*

$$m : \Omega[T_i] \rightarrowtail \Omega[S] \otimes \Omega[T] .$$

The dendroidal subset given by the image of this monomorphism will be denoted

$$m(T_i) \subseteq \Omega[S] \otimes \Omega[T].$$

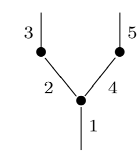
Proof. The vertices of the dendroidal set $\Omega[T_i]$ are the edges of the tree T_i. The map m is completely determined by asking it to map an edge named (a, x) in T_i to the vertex with the same name in $\Omega[S] \otimes \Omega[T]$. This map is a monomorphism. In fact, any map

$$\Omega[R] \to X,$$

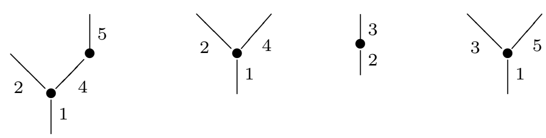
from a representable dendroidal set to an arbitrary one, is a monomorphism as soon as the map $\Omega[R]_\eta \to X_\eta$ on vertices is. $\square$

Before we continue, we need to introduce a bit of terminology for trees, i.e., for objects of Ω. Let R be such a tree. A map $R' \to R$ which is a composition of basic face maps (maps of type (*ii*) or (*iii*) in Section 3) will also be referred to as a *face* of R, just like for simplicial sets. If it is a composition of *inner* faces (resp. *outer* faces), the map $R' \rightarrowtail R$ will be called an *inner face* (resp. *outer face)* of R. A *top face* of R is an outer face map $\partial_v : R' \to R$ where R' is obtained by deleting a top vertex from R. An *initial segment* $R' \rightarrowtail R$ is a composition of top faces (it is a special kind of outer face of R). If v is the vertex above the root of R and e is an input edge of v, then R contains a subtree R' whose root is e. We'll refer to an inclusion of this kind as a *bottom* face of R (it is again a special kind of outer face). In all these cases, we'll often leave the monomorphism $R' \rightarrowtail R$ implicit, and apply the same terminology not only to the map $R' \rightarrowtail R$ but also to the tree R'.

For example, for the tree T constructed above

$T =$ 

The following sub-trees are examples of, respectively, a top face, an initial segment, a bottom face, and an inner face:



Remark 11.0.31. We observe the following simple properties, which we will repeatedly use in the proofs of the lemmas below. In stating these properties and below, we denote by $m(R)$ the image of the composition of the inclusion $\Omega[R] \rightarrowtail \Omega[T_i]$ given by a subtree (a face) R of T_i and the canonical monomorphism

$$m : \Omega[T_i] \rightarrowtail \Omega[S] \otimes \Omega[T] \ .$$

(i) Let R be a subtree of T_i. If $m(R) \subseteq A_0$ then R misses a T-colour, or an S-colour other than e, or a stump of either S or T. Here, a stump is a top vertex of valence zero (i.e., without input edges). We say that R "misses" such a stump $v \in S$, for example, if $m(R) \subseteq \partial_v[S] \otimes \Omega[T]$. The tree R is a sub-tree of

T_i, where edges are coloured by pairs (a, x), where a is an S-colour and x a T-colour. By saying that R "misses" a T-colour y, we mean that none of the colours (a, x) occurring in R has $x = y$ as second coordinate. "Missing an S-colour" is interpreted similarly.

(ii) This implies in particular that for any bottom face $R \longrightarrow T_i$ of any percolation scheme T_i the dendroidal set $m(R)$ is contained in A_0, because it must miss either the root colour r_S (in case the root of T_i is white), or the root colour r_T (in case the root of T_i is black), and $r_S \neq e$ because e is assumed inner.

(iii) If F, G are faces of T_i then F is a face of G iff $m(F) \subseteq m(G)$. (This is clear from the fact that the map from $\Omega[R]$ onto its image $m(R)$ is an isomorphism of dendroidal sets.)

(iv) Let Q and R be initial segments of T_i, and let F be an inner face of Q. If $m(F) \subseteq m(R)$ then also $m(Q) \subseteq m(R)$ (and hence Q is a face of R, by (iii)). In fact, let $Inn(Q)$ denote the set of all inner edges of Q and $Q/Inn(Q) \rightarrowtail Q$ the inner face of Q given by contracting all these. Then if $m(Q/Inn(Q)) \subseteq m(R/Inn(R))$, it follows by comparing labels of input edges of Q and R that Q is a face of R.

These remarks prepare the ground for the following lemma. Recall that $A_k = A_0 \cup m(T_1) \cup \cdots \cup m(T_k)$, where $m(T_i)$ is the image in $\Omega[S] \otimes \Omega[T]$ of the dendroidal set $\Omega[T_i]$.

Lemma 11.0.32. *Let $R, Q_1, \cdots, Q_p$ be a family of initial segments in T_{k+1} and write $B = m(Q_1) \cup \cdots \cup m(Q_p) \subseteq \Omega[S] \otimes \Omega[T]$. Suppose*

(i) For every top face F of R, $m(F) \subseteq A_k \cup B$.

(ii) There exists an edge ξ in R such that for every inner face $F \rightarrowtail R$, if $m(F)$ is not contained in $A_k \cup B$ then neither is $m(F/(\xi))$.

Then the inclusion $A_k \cup B \rightarrowtail A_k \cup B \cup m(R)$ is anodyne.

We call ξ a characteristic edge of R with respect to $Q_1, \cdots, Q_p$.

Proof. If $m(R) \subseteq A_k \cup B$ there is nothing to prove. If not, then by (ii), $m(R/(\xi))$ is not contained in $A_k \cup B$. Let

$$\xi = \xi_0, \xi_1, \cdots, \xi_n$$

be all the inner edges in R such that the dendroidal set $m(R/(\xi_i))$ is not contained in $A_k \cup B$. For a sub-sequence $\xi_{i_1}, \cdots, \xi_{i_p}$ of these $\xi_0, \cdots, \xi_n$, we have the dendroidal subset of $\Omega[S] \otimes \Omega[T]$,

$$m(R/(\xi_{i_1}, \cdots, \xi_{i_p})), \tag{1}$$

obtained by contracting each of $\xi_{i_1}, \cdots, \xi_{i_p}$ and composing with $m : \Omega[T_{k+1}] \to \Omega[S] \otimes \Omega[T]$. We are going to consider a sequence of anodyne extensions

$$A_k \cup B = B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{2^n} = A_k \cup B \cup m(R)$$

by considering images of faces of $\Omega[R]$ of this type (1).

Consider first

$$R_{(0)} = m(R/(\xi_1, \cdots, \xi_n)).$$

If $m(R_{(0)})$ is contained in $A_k \cup B$, let $B_1 = B_0 = A_k \cup B$. Otherwise, let B_1 be the pushout

$$\begin{tikzcd}
m(\Lambda^{\xi_0} R_{(0)}) \arrow[r] \arrow[d] & B_0 \arrow[d] \\
m(R_{(0)}) \arrow[r] & B_1
\end{tikzcd}$$

Notice that $m(\Lambda^{\xi_0} R_{(0)})$ is indeed contained in $B_0 = A_k \cup B$. For, any outer face F of $R_{(0)}$ is a face of an outer face G of R

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & R/(\xi_1, \cdots, \xi_n) = R_{(0)} \arrow[d] \\
G \arrow[r] & R
\end{tikzcd}$$

if G is a top face, then $m(G) \subseteq A_k \cup B$ by assumption (i); and if G is a bottom face, it already factors through $A_0 \subseteq A_k$ (cf Remark 11.0.31 (ii) before the lemma). On the other hand, if $F \subseteq R_{(0)}$ is an inner face of $R_{(0)}$ given by contracting an edge ζ in $R/(\xi_1, \cdots, \xi_n)$, then F is a face of $R/(\zeta)$. So if $m(F) \nsubseteq B_0$ then $m(R/(\zeta))$ wouldn't be contained in B_0 either, and hence ζ must be one of $\xi_0, \cdots, \xi_n$. But $\xi_1, \cdots, \xi_n$ are no longer edges in $R/(\xi_1, \cdots, \xi_n)$, so ζ must be ξ_0. This shows that for any inner face F of $R_{(0)}$ other then $R_{(0)}/(\xi_0)$, the dendroidal set $m(F)$ is contained in B_0, as claimed.

Next, consider all sub-sequences $(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_n)$, and the faces

$$R_{(i)} = R/(\xi_1, \cdots, \hat{\xi}_i, \cdots \xi_n) \quad i = 1, \cdots, n$$

We will define

$$B_2, \cdots, B_{n+1}$$

by considering these $R_{(1)}, \cdots, R_{(n)}$. Suppose $B_1, \cdots, B_i$ have been defined. Consider $R_{(i)}$ to form B_{i+1}. If its image $m(R_{(i)})$ is contained in B_i, let $B_{i+1} = B_i$. Otherwise, $m(R_{(i)}) \to \Omega[S] \otimes \Omega[T]$ does not factor through B_i, and a fortiori doesn't factor through $A_k \cup B = B_0$ either. So by assumption (ii), we have that $m(R_{(i)}/(\xi_0)) \nsubseteq A_k \cup B$. But then $m(R_i/(\xi_0))$ is not contained in B_i either, because by Remark 11.0.31(iv), if $m(R_i/(\xi_0))$ would be contained in one of $m(R_{(0)}), \cdots, m(R_{(i-1)})$, then $R_i/(\xi_0)$ would be a face of one of $R_{(0)}, \cdots, R_{(i-1)}$, which is obviously not the case. On the other hand, ξ_0 is the *only* edge of $R_{(i)}$ for which $m(R_{(i)}/(\xi_0))$ is not contained in B_i (indeed, the only other candidate would be ξ_i, but $R_{(i)}/\xi_i = R_{(0)}$ and $m(R_{(0)}) \subseteq B_1$). So, we can form the pushout

$$\begin{tikzcd} m(\Lambda^{\xi_0} R_{(i)}) \arrow[r] \arrow[d, tail] & B_i \arrow[d, tail] \\ m(R_{(i)}) \arrow[r] & B_{i+1} \end{tikzcd}$$

Next, consider for each $i < j$ the tree

$$R_{(ij)} = R/(\xi_1, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_n),$$

and order these lexicographically, say as

$$R_1^2, \cdots, R_u^2, \quad (u = \binom{n}{2}).$$

We are going to form anodyne extensions of B_{n+1} by using these trees,

$$B_{n+1} \rightarrowtail B_{n+2} \rightarrowtail \cdots \rightarrowtail B_{n+1+u}\ ,$$

treating R_p^2 in the step to form $B_{n+p} \rightarrowtail B_{n+p+1}$ (for each $p = 1, \cdots, u$). Suppose B_{n+p} has been formed, and consider $R_p^2 = R_{(ij)}$ say. If $m(R_p^2) \subseteq B_{n+p}$ then let $B_{n+p+1} = B_{n+p}$. If not, then surely $m(R_p^2) \nsubseteq A_k \cup B$, so by assumption (ii) $m(R_p^2/(\xi_0)) = m(R/(\xi_0, \xi_1, \cdots, \hat{\xi}_i, \cdots, \hat{\xi}_j, \cdots, \xi_n))$ is not contained in $A_k \cup B$. On the other hand, Remark 11.0.31(iv) implies that $m(R_p^2/(\xi_0))$ cannot be contained in any of $m(R_1), \cdots, m(R_n), m(R_1^2), \cdots, m(R_{p-1}^2)$ either. So $m(R_p^2/(\xi_0))$ is not contained in B_{n+p}. As before, ξ_0 is the *only* inner edge ζ for which $m(R_p^2/(\zeta))$ is not contained in B_{n+p}. So we can form the pushout

$$\begin{tikzcd} m(\Lambda^{\xi_0}(R_{(p)}^2)) \arrow[r] \arrow[d, tail] & B_{n+p} \arrow[d, tail] \\ m(R_{(p)}^2) \arrow[r] & B_{n+p+1} \end{tikzcd}$$

Next consider for each $i_1 < i_2 < i_3$ the tree

$$R_{(i_1 i_2 i_3)} = R/(\xi_1, \cdots, \hat{\xi}_{i_1}, \cdots, \hat{\xi}_{i_2}, \cdots, \hat{\xi}_{i_3}, \cdots, \xi_n)$$

and adjoin the pushout along

$$m(\Lambda^{\xi_0} R_{(i_1 i_2 i_3)}) \rightarrowtail m(R_{(i_1 i_2 i_3)})$$

if necessary. Continuing in this way for all $l = 0, 1, \cdots, n-1$ and all sub-sequences $i_1 < \cdots < i_l$ and corresponding trees

$$R/(\xi_1, \cdots, \hat{\xi}_{i_1}, \cdots, \hat{\xi}_{i_l}, \cdots, \xi_n),$$

we end up with a sequence of anodyne extensions

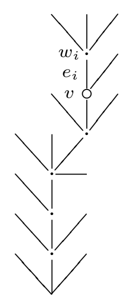
$$B_1 \rightarrowtail \cdots \rightarrowtail B_q$$

where $q = \Sigma_{l=0}^{n-1} \binom{n}{l} = 2^n - 1$, and where $m(R/(\xi_i))$ is contained in B_q for each $i = 1, \cdots, n$. In the final step, and exactly as before, we let $B_{2^n} = B_{2^n-1}$ if $m(R) \subseteq B_{2^n-1}$; and if not, we form the pushout

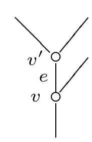
$$\begin{tikzcd} m(\Lambda^{\xi_0}(R)) \arrow[r] \arrow[d, tail] & B_{2^n-1} \arrow[d, tail] \\ m(R) \arrow[r] & B_{2^n} \end{tikzcd}$$

Then B_{2^n} is the pushout of $A_0 \cup B$ and $m(R)$ over $(A_0 \cup B) \cap m(R)$ (because every face F of R for which $m(F)$ is contained in $A_0 \cup B$ occurs in some corner of the pushouts taken in the construction of the B_i). This proves the lemma. $\square$

Consider the tree T_{k+1}, and look at all lowest occurrences of the S-colour e (Recall e is the fixed edge in S, occurring in the statement of Proposition 11.0.27). More precisely, let $e_i = (e, x_i)$ for $i = 1, \cdots, t$ be all the edges in T_{k+1} whose S-colour is e, while the S-colour of the edge immediately below it isn't. This means that (e, x_i) is an edge having a white vertex at its bottom. Let β_i be the branch in T_{k+1} from the root to and including this edge e_i. Each such β_i is an initial segment in T_{k+1}, to which we will refer as the *spine* through e_i. For example, this is a picture of a spine in T_{k+1},

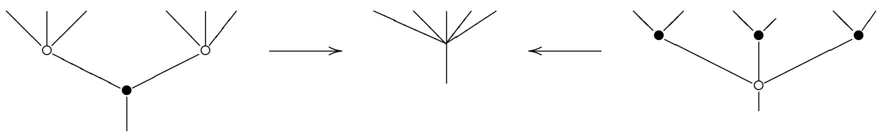


corresponding to the edge e in S



Lemma 11.0.33. *Let $R, Q_1, \cdots, Q_p$ be initial segments in T_{k+1}, as in the preceding lemma, and suppose condition (i) of that lemma is satisfied. Then for any spine β_i contained in R, the edge $e_i \in \beta_i$ is characteristic for R with respect to $Q_1, \cdots, Q_p$.*

Proof. We have to check condition (ii) of Lemma 11.0.32. So, suppose F is an inner face of R, and suppose $m(F/(e_i))$ is contained in $A_k \cup B = A_0 \cup m(T_1) \cup \cdots \cup m(T_k) \cup m(Q_1) \cup \cdots \cup m(Q_p) \subseteq m(T_{k+1})$. Since $m(F/(e_i))$ is isomorphic to the representable dendroidal set $\Omega[F/(e_i)]$, it must be contained in one of the dendroidal sets constituting this union. But, if $m(F/(e_i))$ is contained in A_0, then by Remark 11.0.31 (ii) $m(F)$ is also contained in A_0. (the only colour occurring in F but possibly not in $F/(e_i)$ is the S-colour e). And, if $m(F/(e_i))$ is contained in $m(T_j)$ for some $j \leq k$, then there must be a tensor product relation applying to the image of $F/(e_i)$, which allows a black vertex to move up so as to get into an earlier T_j, as in:



where the left tree is in T_{k+1}, the right tree is in T_j, and the middle one in $F/(e_i)$.

But then the same relation must apply to the image of F, because the edge e_i, having a white vertex at its root, cannot contribute to this relation. Finally, if $m(F/(e_i))$ is contained in $m(Q_l)$ for some $l \leq p$, then by Remark 11.0.31 (iv), we have $m(R) \subseteq m(Q_l)$. So a fortiori, $m(F)$ is contained in $m(Q_i)$. This proves the lemma. □

Recall that our aim is to prove for $A_k = A_0 \cup m(T_1) \cup \cdots \cup m(T_k)$ that each inclusion

$$A_k \rightarrowtail A_{k+1}$$

is anodyne. Consider the tree T_{k+1}, and let $\beta_1, \cdots, \beta_t$ be all the spines contained in it. We shall prove by induction that $A_k \rightarrowtail A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne, for any family $R_1, \cdots, R_q$ of initial segments each of which contains at least one such spine. The induction will be on the number of such initial segments as well as on their size. When applied to the maximal initial segment T_{k+1} itself, this will show that $A_k \rightarrowtail A_k \cup m(T_{k+1}) = A_{k+1}$ is anodyne, as claimed. The precise form of induction is given by the following lemma.

Lemma 11.0.34. *Fix l with $0 \leq l \leq t$. Let $Q_1, \cdots, Q_p$ be a family of initial segments in T_{k+1}, each containing at least one and at most l spines. Let $R_1, \cdots, R_q$ be initial segments, each of which contains $l+1$ spines. Then the inclusion*

$$A_k \rightarrowtail A_k \cup B \cup C$$

for $B = m(Q_1) \cup \cdots \cup m(Q_p)$ and $C = m(R_1) \cup \cdots \cup m(R_q)$, is anodyne.

Proof. We can measure the size of each of the initial segments R_j by counting the number of vertices in R_j which are not on one of the $l+1$ spines. If this number is not bigger than u, we say that R_j has size at most u, and write $size(R_j) \leq u$. Let $\Lambda(l,u)$ be the assertion that the lemma holds for l, for any families $\{Q_i\}$ and $\{R_j\}$ where the R_j all have $size(R_j) \leq u$. We will prove $\Lambda(l,u)$ by induction, first on l and then on u.

Case $l = 0$: This is the case where there are no Q's, i.e., $p = 0$. For $l = 0$, first consider the case where $u = 0$ also. Then each R_i is itself a spine, say β_i, with top inner edge e_i running from a copy of v to a copy w_i of w. We will prove that each of the inclusions

$$A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \rightarrowtail A_k \cup m(R_1) \cup \cdots \cup m(R_i)$$

for $i = 0, \cdots, q$, is anodyne. If $R_i = \beta_i$ coincides with one of the earlier spines R_j, $j < i$, then there is nothing to prove. If R_i is a different spine, then its outer top face is contained in A_0 because it misses the vertex v' which is above e in S. So condition (i) of Lemma 11.0.32 is satisfied, where $R_i, R_1, \cdots, R_{i-1}$ take the role of $R, Q_1, \cdots, Q_p$ in that lemma. By Lemma 11.0.33, the edge $e_i \in R_i$ is characteristic so Lemma 11.0.32 gives that $A_k \cup m(R_1) \cup \cdots \cup m(R_{i-1}) \rightarrowtail A_k \cup m(R_1) \cup \cdots \cup m(R_i)$ is anodyne, as claimed. The composition of these inclusions will then be anodyne also, which proves the statement $\Lambda(0,0)$.

Suppose now that $\Lambda(0,u)$ has been proved, and consider families $R_1, \cdots, R_q$ of initial segments which are each of size not bigger than $u+1$. Suppose that among these, $R_1, \cdots, R_{q'}$ actually have size not bigger than u, while $R_{q'+1}, \cdots R_q$ have size $u+1$. We shall prove that

$$A_k \rightarrowtail A_k \cup m(R_1) \cup \cdots \cup m(R_q)$$

is anodyne, by induction on the number $r = q - q'$ of initial segments that have size $u+1$. If $r = 0$, this holds by $\Lambda(0,u)$. Suppose we have proved this for *any* family with not more than r initial segments of size $u+1$, and consider such a family $R_1, \cdots, R_q$ where $q - q' = r+1$. Write β_q for the spine contained in R_q (there is only one such because we are still in the case $l = 0$). For a top outer face $\partial_x(R_q)$ of R_q, *either* $x \neq w_q$ so that $\partial_x(R_q)$ still contains β_q but has size at most u, *or* $x = w_q$ so that $m(\partial_x(R_i))$ is contained in A_0 because it misses the vertex v' immediately above e in S. Thus, if we let

$$P = m(R_1) \cup \cdots \cup m(R_{q-1}) \cup \bigcup_x m(\partial_x(R_q))$$

where x ranges over all the top vertices in R_q, then by the fact that $\Lambda(0, u+1)$ is assumed to hold for $r = (q-1) - q'$,

$$A_k \rightarrowtail A_k \cup P \tag{1}$$

is anodyne. To prove that $A_k \cup P \rightarrowtail A_k \cup P \cup m(R_q) = A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne as well, we can now apply Lemma 11.0.32. Indeed, the family of initial segments containing P is made to contain the images of all the top faces of R_q, and $e_q \in R_q$ is characteristic by Lemma 11.0.33. This proves that $A_k \cup P \rightarrowtail A_k \cup P \cup m(R_q)$ is anodyne, as claimed. When composed with (1), we find that $A_k \rightarrowtail A_k \cup m(R_1) \cup \cdots \cup m(R_q)$ is anodyne. This proves $\Lambda(0, u+1)$ and completes the inductive proof of $\Lambda(l, u)$ for $l = 0$ and all u.

Suppose now that we have proved $\Lambda(l', u)$ for all $l' \leq l$ and all u. We will now prove $\Lambda(l+1, u)$ by induction on u.

Case $u = 0$: This is the assertion that for any given initial segments

$$Q_1, \cdots, Q_p, R_1, \cdots, R_q$$

of T_{k+1}, where the Q_j contain at most l spines while each R_i is made up out of exactly $l+1$ spines (and no other vertices), the inclusion

$$A_k \rightarrowtail A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q) \tag{2}$$

is anodyne. We shall prove by induction on q that this holds for all p. For $q = 0$, the conclusion follows by the inductive assumption that $\Lambda(l, u)$ holds. Suppose the assertion holds for $q - 1$, and consider R_q. Each top vertex of R_q lies at the end of a spine, so $\partial^x(R_q)$ contains at most l spines. Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x \partial_x(R_q)$$

where x ranges over the top vertices of R_q. Then, by the assumption for $q - 1$,

$$A_k \rightarrowtail A_k \cup D \cup m(R_1) \cup \cdots \cup m(R_{q-1}) \tag{3}$$

is anodyne. To prove that (2) is anodyne, it then suffices to apply Lemma 11.0.32, and show that R_q has a characteristic edge with respect to the family of initial segments containing the union $D \cup m(R_1) \cup \cdots \cup m(R_{q-1})$ in (3). But by Lemma 11.0.33, any top edge e_q of R_q is characteristic. This proves $\Lambda(l+1, u)$, for $u = 0$.

Case $u+1$: Suppose now $\Lambda(l+1, u)$ holds. To prove $\Lambda(l+1, u+1)$, consider families

$$Q_1, \cdots, Q_p, R_1, \cdots, R_{q'}, R_{q'+1}, \cdots, R_q \tag{4}$$

of initial segments in T_{k+1}, where the Q_i contain at most l spines, the R_i contain exactly $l+1$ spines, the $R_1, \cdots, R_{q'}$ are of size not more than u, and $R_{q'+1}, \cdots, R_q$ are of size exactly $u+1$. We will show by induction on the last number $r = q - q'$ that for any such family, the inclusion

$$A_k \rightarrowtail A_k \cup m(Q_1) \cup \cdots \cup m(Q_p) \cup m(R_1) \cup \cdots \cup m(R_q) \tag{5}$$

is anodyne. For $r = q - q' = 0$ there is nothing to prove, because this is the case covered by $\Lambda(l+1, u)$. Suppose we have proved that (5) is anodyne for *any* family (4) with $q - q' \leq r$, and consider such a family with $q - q' = r + 1$. The initial segment R_q has two kinds of top outer faces, namely the $\partial_x(R_q)$ which remove the top of a spine, and the $\partial_x(R_q)$ where x does not lie on a spine. Outer faces of the first kind contain l spines only, and outer faces of the second kind are of size not more than u. Let

$$D = m(Q_1) \cup \cdots \cup m(Q_p) \cup \bigcup_x m(\partial_x R_q)$$

where x ranges over the top vertices of R_q which are on a spine. Let

$$E = m(R_1) \cup \cdots \cup m(R_{q'}) \cup \bigcup_x m(\partial_x(R_q))$$

where x ranges over the top vertices of R_q which are not on a spine. Then, by the assumption that $\Lambda(l+1, u+1)$ has been established for families (4) where $q - q' \leq r$, we see that

$$A_k \rightarrowtail A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_{q-1} \tag{6}$$

is anodyne. The union $D \cup E$ is made to contain all the images $\partial_x(R_q)$ of top faces $\partial_x(R_q)$ of R_q, and by Lemma 11.0.33, any edge e_q on the top of a spine β_q in R_q is characteristic with respect to the family of initial segments making up the union on the right-hand-side of (6). So by Lemma 11.0.32, the map

$$A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_{q-1} \rightarrowtail A_k \cup D \cup E \cup R_{q'+1} \cup \cdots \cup R_q$$

is anodyne. When composed with (6), this gives (5), and proves the case $u + 1$.

This established $\Lambda(l+1, u+1)$ and completes, for $l+1$, the induction on u, thus completing the proof. $\square$

Chapter 12

Applications and further developments

In this last, somewhat speculative section, we would like to point out some possible further developments of the theory of dendroidal sets, related to "weak" maps between up-to-homotopy algebras, to enriched and weak higher categories, and to Quillen model categories.

To begin with, let $\mathcal{P}$ be an operad in Set. If $\mathcal{E}$ is a symmetric monoidal model category with a suitable interval H, then $W_H(\mathcal{P})$ is an operad in $\mathcal{E}$ whose algebras are homotopy $\mathcal{P}$-algebras (as mentioned in Section 6 above). The maps of $W_H(\mathcal{P})$-algebras are maps of homotopy $\mathcal{P}$-algebras which strictly commute with all higher homotopies, and this is a notion of map which for many purposes is too restrictive. It is possible to define a weaker notion of map between homotopy $\mathcal{P}$-algebras, but then the question arises to what extent these weak maps form a category.

Boardman and Vogt [BV] construct a "quasi-category" of weak maps in the context of topological spaces; in [BM1], Theorem 6.9, a kind of Segal category of weak maps is constructed in the context of left proper monoidal model categories; in [HPS] this question is approached via bimodules. The theory of dendroidal sets is relevant here. Indeed, $W_H(\mathcal{P})$-algebras in $\mathcal{E}$ are the same thing as operad maps $W_H(\mathcal{P}) \to \mathcal{E}$, or equivalently, as maps of dendroidal sets $N_d(\mathcal{P}) \to hcN_d(\mathcal{E})$ (see Remark 6.0.5 above). They thus arise as the vertices of the dendroidal set

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}), hcN_d(\mathcal{E})). \tag{12.1}$$

Dendrices of shape $i[1]$ (where $i : \Delta \to \Omega$) encode a suitable notion of weak map, and such weak maps can be composed (in an up-to-homotopy way) whenever this dendroidal set (12.1) is an inner Kan complex. This is the case, for example, when $\mathcal{P}$ is Σ-free and every object in $\mathcal{E}$ is fibrant, *c.f.* Theorem 9.0.21.

Notice that, more generally, one might consider (weak) $\mathcal{P}$-algebras with values in any dendroidal set X, as vertices of the dendroidal Hom-set

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}), X).$$

If $\mathcal{P}$ is Σ-free then this dendroidal set is an inner Kan complex whenever X is (Theorem 11.0.26), in which case maps between $\mathcal{P}$-algebras (again defined as dendrices of shape $i[1]$) can be composed. The case $X = hcN_d(\mathcal{E})$ is the one discussed before. It is also possible to iterate this construction, and consider for another operad $\mathcal{Q}$ the dendroidal set

$$\underline{Hom}_{dSet}(N_d\mathcal{Q}, \underline{Hom}_{dSet}(N_d\mathcal{P}, X))$$

which is of course isomorphic to

$$\underline{Hom}_{dSet}(N_d(\mathcal{P}) \otimes N_d(\mathcal{Q}), X).$$

This dendroidal set admits a map from

$$\underline{Hom}_{dSet}(N_d(\mathcal{P} \otimes_{BV} \mathcal{Q}), X)$$

but is in general not isomorphic to it, unless X is (the dendroidal nerve of) an operad. In particular, for the case $X = hcN_d(\mathcal{E})$, one has a map

$$\underline{Hom}(W_H(\mathcal{P} \otimes_{BV} \mathcal{Q}), \mathcal{E}) \rightarrow \underline{Hom}(|N_d(\mathcal{P}) \otimes N_d(\mathcal{Q})|_H, \mathcal{E})$$

which gives different but related notions of iterated weak algebras in $\mathcal{E}$. It would be interesting to compare this to the work of Dunn, Fiedorowicz, and Vogt on the tensor product of operads (see e.g., [Du, Fi]) . (In this context, we should point out that, up to now, $\mathcal{P}$ and $\mathcal{Q}$ have been operads in *Set*, but the same applies to topological operads. Indeed, for the category Top of compactly generated spaces, the homotopy coherent dendroidal nerve $hcN_d(Top)$ with respect to the usual unit interval is naturally a (large) dendroidal space. If $\mathcal{P}$ is an operad in Top, then homotopy $\mathcal{P}$-algebras in Top are the vertices of the dendroidal space $\underline{Hom}(N_d(\mathcal{P}), hcN_d(Top))$, etc. We expect that (under suitable cofibrancy conditions on $\mathcal{P}$) this dendroidal space satisfies the inner Kan condition.

We would like to consider the special case of the operad $\mathcal{A}_S$ whose algebras are categories with a given set S as objects (Example 2.0.4). Note that this operad is Σ-free (like any operad obtained by symmetrization, cf. Remark 2.0.3). For a fixed dendroidal set X, one can consider the dendroidal set

$$\underline{Hom}(N_d(\mathcal{A}_S), X).$$

By definition, we call its vertices X-enriched categories over S. Its dendrices of shape $i[1]$ provide an interpretation of the notion of "functor" between X-enriched categories over S. By varying S, one obtains a *Set*-indexed diagram of dendroidal sets, which the dendroidal Grothendieck construction (see 4.1) assembles into a single dendroidal set

$$\underline{Cat}(X) := \int_{Set} \underline{Hom}(N_d(\mathcal{A}_\mathcal{S}), X).$$

By definition, its vertices are categories enriched in X, while its dendrices of shape $i[1]$ are functors between such categories. In this context, it is relevant to observe that by Theorem 11.0.26 and Theorem 10.0.25, $\underline{Cat}(X)$ is a dendroidal inner Kan complex whenever X is, so that a composition of functors between X-enriched categories exists. We also note that the construction can be iterated, so as to form the dendroidal inner Kan complex

$$\underline{Cat}^2(X) = \underline{Cat}(\underline{Cat}(X))$$

of X-enriched bicategories, and so on.

Let us consider a few special cases of this construction. First of all, if $\mathcal{E}$ is a symmetric monoidal category, one can construct its dendroidal nerve $N_d(\mathcal{E})$. The dendroidal set $\underline{Cat}(N_d(\mathcal{E}))$ then captures the *usual* notion of $\mathcal{E}$-enriched categories and functors. More precisely, it is isomorphic to the dendroidal nerve of the usual monoidal category $Cat(\mathcal{E})$ of $\mathcal{E}$-enriched categories,

$$\underline{Cat}(N_d(\mathcal{E})) \cong N_d(Cat(\mathcal{E})),$$

where $Cat(\mathcal{E})$ is considered as an operad via the usual tensor product of enriched categories. As a particular case, consider the category $\underline{Cat}$ of small categories with its cartesian monoidal structure. Then the dendroidal set $Cat^n(N_d(\underline{Cat}))$, obtained by iterating the construction n times, is the dendroidal nerve of the category of strict $(n+1)$-categories. It also encodes all higher structure of functors, natural transformations, modifications, and so on.

If $\mathcal{E}$ is a monoidal model category with a suitable interval H, one can consider categories enriched in the homotopy coherent nerve $hcN_d(\mathcal{E})$ (defined in terms of H). For example, if $\mathcal{E}$ is the category of chain complexes over a ring R (with the projective model structure and the usual interval H of normalized chains on the standard 1-simplex), then $\underline{Cat}(hcN_d(\mathcal{E}))$ is a dendroidal inner Kan complex whose vertices are precisely A_∞-categories ([Fu, Lef, Ly]). As another example, let $\mathcal{E} = Top$ with the unit interval, and consider for the one-point set $*$ the operad $Ass = \mathcal{A}_*$ and the dendroidal inner Kan complex

$$A_\infty = \underline{Hom}(N_d(Ass), hcN_d(\mathcal{E})).$$

The vertices of this dendroidal set are precisely A_∞-spaces, while dendrices of more general shapes encode operations between A_∞-spaces. Again, the construction can be iterated to form dendroidal inner Kan complexes $A_\infty^{(1)} = A_\infty$ and

$$A_\infty^{(n+1)} = \underline{Hom}(N_d(Ass), A_\infty^{(n)}).$$

It would be interesting to study the relation between $A_\infty^{(n)}$ and n-fold loop spaces in topology [Du, May]. In this context, it is important to note that, although $Ass \otimes_{BV} Ass = Comm$, the dendroidal tensor product $N_d(Ass) \otimes N_d(Ass)$ is considerably larger than $N_d(Comm)$, and in fact cannot be the nerve of an operad.

Finally, the category $\underline{Cat}$ of small categories itself is a monoidal model category with interval H as in Section 6 above, and

$$\underline{Hom}(N_d(\mathcal{A}_S), hcN_d(\underline{Cat}))$$

is a dendroidal inner Kan complex capturing the notion of a *bicategory* with S as set of objects [B] (or rather the notion of an unbiased bicategory [Lei2]). This construction can again be iterated. For example, the dendroidal inner Kan complex $\underline{Hom}(N_d(Ass), \underline{Hom}(N_d(Ass), hcN_d(\underline{Cat})))$ captures braided monoidal categories (and all higher maps between them). The above construction of categories enriched in $\mathcal{E}$ yields, by considering $\mathcal{E} = \underline{Cat}$, an inductive definition of weak n-categories. More precisely, let $WCat_1 = \underline{Cat}$ and for $n > 1$ let

$$WCat_n = \underline{Cat}^{n-1}(hcN_d(\underline{Cat})).$$

For each $n \geq 1$, $WCat_n$ is a dendroidal inner Kan complex. Its vertices are weak n-categories of a special kind. (They have an underlying strict category of 1-cells, and for any two objects x and y, the same is true at level $n-1$ for the dendroidal set $Hom(x, y)$). There are many alternative notions of weak n-categories in the literature (see [Lei1] for a survey of 10 such definitions and [BD] for a more general discussion of weak n-categories), and we expect that for any reasonable notion, a weak n-category can be "strictified" to a weak n-category in our sense.

Finally, we would like to say a few words about possible Quillen model structures on dendroidal Sets. Recall from [J2, Lu] that there is a Quillen model structure on simplicial sets, in which the inner Kan complexes are exactly the fibrant objects. This model structure is related to the "folk" monoidal model structure on $\underline{Cat}$ already mentioned above, in which the weak equivalences are the equivalences of categories and the cofibrations are the functors which are injective on objects. Indeed, a map $X \to Y$ between simplicial sets is a weak equivalence in Joyal's model structure if, and only if, for every simplicial inner Kan complex K, the map $\tau(K^Y) \to \tau(K^X)$ is an equivalence of categories (here $\tau : sSet \to Cat$ is the functor discussed in Section 4). The analog of Theorem 11.0.26 for simplicial sets, which states that K^X and K^Y are again inner Kan complexes, plays an important role in Joyal's model structure.

The folk model structure on $\underline{Cat}$ generalizes without much effort to one on (coloured) operads, in which a map $f : \mathcal{Q} \to \mathcal{P}$, from an operad $\mathcal{Q}$ on D to an operad $\mathcal{P}$ on C (as in Section 2) is a weak equivalence if, and only if, $j^*(f) : j^*\mathcal{Q} \to j^*\mathcal{P}$ is an equivalence of categories, and moreover f induces a bijection

$$\mathcal{Q}(d_1, \cdots, d_n; d) \to \mathcal{P}(fd_1, \cdots, fd_n; fd)$$

for any sequence $d_1, \cdots, d_n, d$ of colours in D. We conjecture that the inner Kan complexes are the fibrant objects in a model structure on dendroidal sets in which a map $X \to Y$ is a weak equivalence if, and only if, for any dendroidal inner Kan complex K, the map

$$\tau_d(\underline{Hom}_{dSet}(Y, K)) \to \tau_d(\underline{Hom}_{dSet}(X, K))$$

is a weak equivalence of operads. Theorem 11.0.26 should be a substantial step towards a proof of this conjecture.

Chapter 13

Appendix: The tensor product of dendroidal objects

Let $\mathcal{E}$ be a symmetric monoidal category. The category of dendroidal objects in $\mathcal{E}$ is the functor category $\mathcal{E}^{\Omega^{op}}$, which we denote by $d\mathcal{E}$. This category has a Boardman-Vogt style tensor product, and a corresponding internal $\underline{Hom}$ whenever $\mathcal{E}$ itself is closed. The construction and its basic properties are explained most easily after recalling some basic facts about "enriched Kan extensions", so we'll do that first. None of the material in 13.1 is really new, and we refer the reader to [K] for more background.

13.1 Enriched Kan extensions

We begin by developing a bit of formalism similar to the language of rings and bimodules. Let $\mathcal{E}$ be a symmetric monoidal category, and let $\mathcal{S}$ be any $\mathcal{E}$-enriched category. Suppose $\mathcal{S}$ is *tensored* over $\mathcal{E}$. (This means that one can construct an object $E \otimes S$ in $\mathcal{S}$ for E in $\mathcal{E}$ and S in $\mathcal{S}$, with the property that there is a natural $\underline{Hom}$-tensor correspondence between maps $E \otimes S \to T$ in $\mathcal{S}$ and $E \to \underline{Hom}(S, T)$ in $\mathcal{E}$; see [K] for a formal definition. For small categories $\mathbb{A}$ and $\mathbb{B}$ (in Set), we write

$$_{\mathbb{A}}\mathcal{E}_{\mathbb{B}} = \mathcal{E}^{\mathbb{B}^{op} \times \mathbb{A}}$$

for the category of functors $\mathbb{B}^{op} \times \mathbb{A} \to \mathcal{E}$. For objects $X \in_{\mathbb{A}} \mathcal{E}_{\mathbb{B}}$ and A in $\mathbb{A}$, B in $\mathbb{B}$, we write

$$_{A}X_{B} = X(B, A) \in \mathcal{E}$$

for the value at (B, A). Also, if $\mathbb{A}$ or $\mathbb{B}$ is the trivial category $\star$ we delete it from the notation. So

$$_{\mathbb{A}}\mathcal{E}_{\star} = {}_{\mathbb{A}}\mathcal{E} = \mathcal{E}^{\mathbb{A}}, \;\; _{\star}\mathcal{E}_{\mathbb{B}} = \mathcal{E}_{\mathbb{B}} = \mathcal{E}^{\mathbb{B}^{op}}.$$

Now assume $\mathcal{E}$ has small limits and $\mathcal{S}$ has small colimits. There is a tensor product functor

$$\otimes_{\mathbb{B}} = {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}} \times {}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}} \tag{13.1}$$

defined for E in ${}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}}$ and S in ${}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}}$, by the usual coequalizer

$$_C(E \otimes_{\mathbb{B}} \mathcal{S})_A \leftarrow \textstyle\coprod_B ({}_C E_B) \otimes ({}_B S_A) \leftleftarrows \coprod_{B \to B'} ({}_C E_{B'}) \otimes ({}_B S_A)$$

for any two objects $C \in \mathbb{C}$, $A \in \mathbb{A}$. This tensor product has a corresponding internal $\underline{Hom}$,

$$\underline{Hom}_{\mathbb{A}} : {}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}} \times {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}}, \tag{13.2}$$

satisfying the usual adjunction property stating a bijective correspondence between maps

$$E \otimes_{\mathbb{B}} S \to T \quad \text{in } {}_{\mathbb{C}}\mathcal{S}_{\mathbb{A}}$$

and maps

$$E \to \underline{Hom}_{\mathbb{A}}(S, T) \quad \text{in } {}_{\mathbb{C}}\mathcal{E}_{\mathbb{B}}.$$

We point out two special cases of this Hom-tensor correspondence. First, if F is an element of ${}_{\mathbb{B}}\mathcal{S}_{\mathbb{A}}$, i.e., $F : \mathbb{B} \to \mathcal{S}_{\mathbb{A}}$, then we obtain adjoint functors

$$f_! : \mathcal{E}_{\mathbb{B}} \rightleftarrows \mathcal{S}_{\mathbb{A}} : f^*$$

defined in terms of the previous functors for the special case $\star = \mathbb{C}$, by

$$f_!(E) = E \otimes_{\mathbb{B}} F \quad f^*(S) = \underline{Hom}_{\mathbb{A}}(F, S) = \underline{Hom}(F, S).$$

These functors f^* and $f_!$ are the right and left Kan extensions along F. Secondly, there are "external" tensor and Hom functors

$$\underline{\otimes} : \mathcal{E}_{\mathbb{C}} \times \mathcal{S}_{\mathbb{A}} \to \mathcal{S}_{\mathbb{C} \times \mathbb{A}} \tag{13.3}$$

$$\underline{Hom}_{\mathbb{A}} : \mathcal{S}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{C} \times \mathbb{A}} \to \mathcal{E}_{\mathbb{C}} \tag{13.4}$$

for which there is a natural correspondence between maps

$$X \underline{\otimes} Y \to Z \quad \text{in } \mathcal{S}_{\mathbb{C} \times \mathbb{A}}$$

and

$$X \to \underline{Hom}_{\mathbb{A}}(Y, Z) \quad \text{in } \mathcal{E}_{\mathbb{C}}$$

Indeed, this is the special case where $\mathbb{B} = \star$ while $\mathbb{C}$ is replaces by $\mathbb{C}^{op}$, so that (13.1) and (13.2) can be rewritten as $\otimes : {}_{\mathbb{C}^{op}}\mathcal{E} \times \mathcal{E}_{\mathbb{A}} \to {}_{\mathbb{C}^{op}}\mathcal{E}_{\mathbb{A}}$ and $\underline{Hom}_{\mathbb{A}} : \mathcal{S}_{\mathbb{A}} \times {}_{\mathbb{C}^{op}}\mathcal{S}_{\mathbb{A}} \to {}_{\mathbb{C}^{op}}\mathcal{E}$, defining (13.3) and (13.4).

Now consider a functor $F : \mathbb{A} \times \mathbb{A} \to \mathcal{S}_{\mathbb{A}}$, i.e., $F \in_{\mathbb{A}\times\mathbb{A}} \mathcal{S}_{\mathbb{A}}$. Then by Kan extension we have a functor

$$\mathcal{E}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{A}} \xrightarrow{\underline{\otimes}} \mathcal{S}_{\mathbb{A}\times\mathbb{A}} \xrightarrow{f_!} \mathcal{S}_{\mathbb{A}}$$

which we write as $\otimes^{(F)}$; so

$$E \otimes^{(F)} S = f_!(E\underline{\otimes}S) = (E\underline{\otimes}S) \otimes_{\mathbb{A}\times\mathbb{A}} F.$$

The above discussion also yields a corresponding Hom-functor, denoted

$$\underline{Hom}^{(F)} : \mathcal{S}_{\mathbb{A}} \times \mathcal{S}_{\mathbb{A}} \to \mathcal{E}_{\mathbb{A}},$$

for which there is a bijective correspondence between maps

$$E \otimes^{(F)} S \to T \quad (\text{in } \mathcal{S}_{\mathbb{A}})$$

and maps

$$E \to \underline{Hom}^{(F)}(S, T) \quad (\text{in } \mathcal{E}_{\mathbb{A}}).$$

Indeed, one can simply define $\underline{Hom}^{(F)}$ in terms of the earlier $\underline{Hom}$ and the right adjoint f^*, as

$$\underline{Hom}^{(F)}(S, T) = \underline{Hom}_{\mathcal{A}}(S, f^*T)$$

13.2 Monoidal closed structure of $d\mathcal{E}$

Let us now consider a complete and cocomplete symmetric closed monoidal category $\mathcal{E}$, and the category $d\mathcal{E}$ of dendroidal object in $\mathcal{E}$. Let $\mathbb{A} = \Omega$, let $\mathcal{S} = \mathcal{E}$, and let $F = BV$ be the Boardman-Vogt tensor product of Hopf operads in $\mathcal{E}$, restricted to operads coming from Ω:

$$BV : \Omega \times \Omega \to Operad(\mathcal{E}) \xrightarrow{N_d} d\mathcal{E}.$$

Then the last construction of 13.1 yields a functor

$$\otimes^{(BV)} : d\mathcal{E} \times d\mathcal{E} \to d\mathcal{E}$$

and a corresponding Hom-functor

$$\underline{Hom}^{(BV)} : d\mathcal{E} \times d\mathcal{E} \to d\mathcal{E}$$

satisfying the usual properties, and *making $d\mathcal{E}$ into a closed symmetric monoidal category.*

Bibliography

[BD] J. Baez, J. Dolan, n-categories, School on Category Theory and Applications Textos Mat. Ser. B, 21 Coimbra, Coimbra, 1999.

[B] J. Bénabou, Introduction to bicategories, in J. Bénabou *et al.* (eds.), Reports of the Midwest Category Seminar, Lecture Notes in Mathematics 47, Springer, 1967.

[BM1] C. Berger, I. Moerdijk, The Boardman-Vogt resolution of operads in monoidal model categories, Topology 45, 2006, 807-849.

[BM2] C. Berger, I. Moerdijk, Resolution of coloured operads and rectification of homotopy algebras, preprint, math.AT/0512576.

[BV] J.M. Boardman, R.M. Vogt, Homotoy Invariant Algebraic Structures on Topological Spaces, Lecture Notes in Mathematics, Vol. 347, Springer, Berlin, 1973.

[C] D.C. Cisinski, Les préfaisceaux comme modèles des types d'homotopie, Astérisque 308, Société mathématique de France, 2006.

[CP] J.M Cordier, T. Porter, Vogt's theorem on categories of homotopy coherent diagrams, Math. Proc. Cambridge Philos. Soc. 100, 1986, 65-90.

[Da] B.J. Day, On Closed Categories of Functors, Report of the Midwest Category Seminar IV, Lecture Notes in Mathematics, Vol. 304, Springer, Berlin, 1972.

[Du] G. Dunn, Tensor product of operads and iterated loop spaces, Journal of Pure and Applied Algebra 50, 1988, no. 3, 237–258.

[Fi] Z. Fiedorowicz, Construction of E_n-operads, preprint, math.AT/9808089.

[Fu] K. Fukaya, Morse homotopy, A^∞-category, and Floer homologies, The Proceedings of the 1993 GARC Workshop on Geometry and Topology (H.J. Kim, ed.), Lecture Notes series, vol. 18, Seoul National University, 1993, 1-102.

[GZ] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergebnisse der Mathematik und Ihrer Grenzgebiete, Band 35 Springer-Verlag New York, Inc, New York 1967.

[GK1] E. Getzler, M. Kapranov, Cyclic operads and cyclic homology, Geometry, topology, and physics, 167-201, Conf. Proc. Lecture Notes Geom. Topology, IV Int. Press, Cambridge, MA, 1995.

[GK2] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Math. J. 76 1994, no. 1, 203-272.

[HPS] K. Hess, P.E. Parent, J. Scott, Co-rings over operads characterize morphisms, preprint, math.AT/0505559.

[H] M. Hovey, Model Categories, AMS Mathematical Survey and Monographs, Vol. 63, American Mathematical Society, 1999.

[J1] A. Joyal, Quasi-categories and Kan complexes, Journal of Pure and Applied Algebra Vol. 175, 2002.

[J2] A. Joyal, Theory of Quasi-Categories, in preparation.

[K] G.M. Kelly, Basic Concepts of Enriched Category Theory, London Mathematical Society Lecture Notes Series, Vol. 64, Cambridge University Press, Cambridge, 1982.

[Lef] K. Lefèvre-Hasegawa, Sur les A-infini catégories, thesis, available at: http://www.math.jussieu.fr/ keller/lefevre/publ.html

[Lei1] T. Leinster, A survey of definitions of n-category. Theory and Applicatoins of Category Theory 10, 2002, 1–70.

[Lei2] T. Leinster, Higher Operads, Higher Categories, London Mathematical Society Lecture Note Series 298, Cambridge University Press, 2004.

[Lu] J. Lurie, Higher topos theory, preprint math.CT/0608040.

[Ly] V. Lyubashenko, Category of A_∞-categories, Homology, Homotopy and Applications 5, 2003, 1-48.

[Mac] S. Mac Lane, Categories for the Working Mathematician, Graduate Text in Mathematics, Vol. 5, Springer, Berlin, 1971.

[MM] S. Mac Lane, I. Moerdijk, Sheaves in Geometry and Logic. A first introduction to topos theory. Springer-Verlag, New York, 1992.

[MSS] M. Markl, S. Shnider, J. Stasheff, Operads in Algebra, Topology and Physics, Mathematical Surveys and monographs, 96. American Mathematical society, Providence, RI, 2002.

[May] J.P. May, The Geometry of Iterated Loop Spaces, Lectures Notes in Mathematics 271, 1972.

[N] J. Nichols-Barrer, On Quasi-Categories as a Foundation for Higher Algebraic Stacks, 2007, MIT PhD thesis.

[MW1] I. Moerdijk, I. Weiss, Dendroidal Sets, Algebraic and Geometric Topology 7, 2007, 1441-1470.

[MW2] I. Moerdijk, I. Weiss, On inner Kan complexes in the category of dendroidal sets, preprint, math.AT/0701295.

[Q] D. Quillen, Homotopical Algebra, Lecture Notes in Mathematics, Vol. 43, Springer, Berlin, 1967.

[V] R.M. Vogt, Homotopy limits and colimits, Math. Z. 134, 1973, 11-52.

Simplicial Presheaves and Derived Algebraic Geometry

Bertrand Toën

Contents

Chapter 1

Motivations and objectives

The purpose of this first lecture is to present some motivations for derived algebraic geometry, and to present the objectives of the series of lectures. I will start by a brief review of the notion of moduli problems and moduli spaces. In a second part I will present the particular example of the moduli problem of linear representations of a discrete group. The study of this example will show the importance of two constructions useful to produce and understand moduli spaces: intersections (or more generally fiber products), and group quotients (or more generally quotient by groupoids). As many algebraic constructions these are not *exact* in some sense and possess derived versions. This will provides motivations for derived algebraic geometry, which is a geometrico-algebraic setting in which these derived versions exist and are well behaved.

We warn the reader that this section is highly informal and that several notions and ideas will be explained more formally later during the lectures.

1.1 The notion of moduli spaces

The main object studied in algebraic geometry are schemes (or more generally algebraic spaces, these notions will be redefined later). They often appear as solutions to *moduli functors* (or equivalently *moduli problems*), which intuitively means that their points classify certain geometico-algebraic objects (e.g. a scheme whose points are in one-to-one correspondence with algebraic subvarieties of the projective space $\mathbb{P}^n$). More precisely, we often are given a *moduli functor*

$$F : Comm \longrightarrow Set,$$

from the category of commutative rings to the category of sets. The set $F(A)$ has to be thought as the set of families of objects parametrized by the scheme $Spec\, A$ (we will see later many examples). When it exists, a scheme X is then called a *moduli space for* F (or a *solution to the moduli problem* F, we also say that *the scheme* X *represents* F or that F *is representable by* X) if there are functorial bijections

$$F(A) \simeq Hom(Spec\,A, X).$$

An important special case is when X is an affine scheme, say $Spec\,B$. Then, X represents F if and only if there are functorial isomorphisms

$$F(A) \simeq Hom(B, A),$$

where $Hom(B, A)$ is the set of ring morpihsms from B to A.

The notion of moduli space is extremelly important for at least two reasons:

1. A good geometric understanding of the moduli space of a given moduli problem can be considered as a step towards a solution to the corresponding classification problem. For instance, a good enough understanding of the moduli space of algebraic curves could be understood as a solution to the problem of classifiying algebraic curves.

2. The notion of moduli problems is a rich source to construct new and interesting schemes. Indeed, the fact that a given scheme X is the solution to a moduli problem often makes its geometry rather rich. Typically the scheme will have interesting subschemes corresponding to objects satisfying certain additional properties.

1.2 Construction of moduli spaces: one example

For a given moduli problem $F : Comm \longrightarrow Set$ the question of the existence of a moduli space is never an easy question. There are two general strategies to prove the existence of such a moduli space, either by applying the so-called *Artin's representability theorem*, or by a more direct approach consisting of constructing the moduli space explicitly. The first approach is the most powerful to prove the existence, but the second one is often needed to have a better understanding of the moduli space itself (e.g. to prove that it satisfies some further properties). In this paragraph we will study the particular example of the moduli problem of linear representations of a discrete group, and will try to construct the corresponding moduli space by a direct approach. This example is chosen so that the moduli space does not exist, which is most often the case, but still the approach to the construction we present is right, at leats when it will be done in the context of derived algebraic geometry (we will of course come back to this fundamental example later on when the techniques of derived stacks will be at our disposal).

So let Γ be a group that will be assumed to be finitely presented. We want to study finite dimensional linear representations of Γ and for this we are looking for a moduli space of those. We start to define a moduli functor

$$R(\Gamma) : Comm \longrightarrow Set,$$

sending a commutative ring A to the set of isomorphism classes of $A[\Gamma]$-modules whose underlying A-module is projective and of finite type over A. As projective A-modules of finite type correspond to vector bundles on the scheme $Spec\, A$, $R(\Gamma)(A)$ can also be identified with the set of isomorphism classes of vector bundles on $Spec\, A$ endowed with an action of Γ. For a morphism of commutative rings $A \to A'$, we have a base change functor $-\otimes_A A'$ from $A[\Gamma]$-modules to $A'[\Gamma]$-modules, which induces a morphism

$$R(\Gamma)(A) \longrightarrow R(\Gamma)(A').$$

This defines the moduli functor $R(\Gamma)$.

The strategy to try to construct a solution to this moduli problem is to start to study a *framed* (or *rigidified*) version of it. We introduce for any integer n an auxiliary moduli problem $R'_n(\Gamma)$, whose values at a commutative ring A is the set of group morphisms $\Gamma \longrightarrow Gl_n(A)$. We set $R'(\Gamma) := \coprod_n R'_n(\Gamma)$, and we define a morphism (i.e. a natural transformation of functors)

$$\pi : R'(\Gamma) \longrightarrow R(\Gamma),$$

sending a morphism $\rho : \Gamma \longrightarrow Gl_n(A)$ to the A-module A^n together with the action of Γ defined by ρ. At this point we would like to argue on two steps:

1. The moduli functor $R'(\Gamma)$ is representable.
2. The moduli functor $R(\Gamma)$ is the disjoint union (for all n) of the quotients of the schemes $R'_n(\Gamma)$ by the group schemes Gl_n.

For the point (1), we write a presentation of Γ by generators and relations

$$\Gamma \simeq < g_1, \dots, g_m > / < r_1, \dots, r_p > .$$

From this presentation, we deduce the existence of a cartesian square of moduli functors

$$\begin{tikzcd} R'_n(\Gamma) \arrow[r] \arrow[d] & Gl_n^m \arrow[d] \\ \{1\} \arrow[r] & Gl_n^p, \end{tikzcd}$$

where Gl_n is the functor $A \mapsto Gl_n(A)$. The functor Gl_n is representable by an affine scheme. Indeed, if we set

$$C_n : \mathbb{Z}[T_{i,j}][Det(T_{i,j})^{-1}],$$

where $T_{i,j}$ are formal variables with $1 \leq i, j \leq n$, then the affine scheme $Spec\, C_n$ represents the functor Gl_n. This implies that Gl_n^r is also representable by an affine scheme for any integer r, precisely $Spec\,(C_n^{\otimes r})$. And finally, we see that $R'_n(\Gamma)$ is representable by the affine scheme

$$Spec\,(C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}).$$

This sounds good, but an important observation here is that in general $C_n^{\otimes m}$ is not a flat $C_r^{\otimes p}$-algebra, and thus that the tensor product $C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}$ is not well behaved from the point of view of homological algebra. Geometrically this is related to the fact that $R'_n(\Gamma)$ is the intersection of two subschemes in $Gl_n^m \times Gl_n^p$, namely the graph of the morphism $Gl_n^m \longrightarrow Gl_n^p$ and $Gl_n^m \times \{1\}$, and that these two subschemes are not in general position. Direct consequences of this is the fact that the scheme $R'_n(\Gamma)$ can be badly singular at certain points, precisely the points for which the above intersection is not transversal. Another bad consequence is that the tangent complex (a derived version of the tangent space, we will review this notion later in the course) is not easy to compute for the scheme $R'_n(\Gamma)$. The main philosophy of derived algebraic geometry is that the tensor product $C_n^{\otimes m} \otimes_{C_r^{\otimes p}} \mathbb{Z}$ should be replaced by its derived version $C_n^{\otimes m} \otimes^{\mathbb{L}}_{C_r^{\otimes p}} \mathbb{Z}$ which also encodes the higher Tor's $Tor_*^{C_r^{\otimes p}}(C_n^{\otimes m}, \mathbb{Z})$, for instance by considering simplicial commutative rings. Of course $C_n^{\otimes m} \otimes^{\mathbb{L}}_{C_r^{\otimes p}} \mathbb{Z}$ is no longer a commutative ring and thus the notion of schemes should be extended in order to be able to consider object of the form $''Spec\, A''$, where A is now a simplicial commutative ring.

We now consider the point (2). The functor $Gl_n : Comm \longrightarrow Set$ sending A to $Gl_n(A)$ is a group object (in the category of functors) and it acts naturally on $R'_n(\Gamma)$. For a given $A \in Comm$, the action of $Gl_n(A)$ on $R'_n(\Gamma)(A) = Hom(\Gamma, Gl_n(A))$ is the one induced by the conjugaison action of $Gl_n(A)$ on itself. The morphism $R'_n(\Gamma) \longrightarrow R(\Gamma)$ is equivariant for this action and thus factorizes as a morphism

$$R'_n(\Gamma)/Gl_n \longrightarrow R(\Gamma).$$

We thus obtain a morphism of functors

$$\coprod_n R'_n(\Gamma)/Gl_n \longrightarrow R(\Gamma).$$

Intuitively this morphism should be an isomorphism and in fact it is close to be. It is a monomorphism but it is not an epimorphism because not every projective A-module of finite type is free. However, up to a localization for the Zariski topology on $Spec\, A$ this is the case, and therefore we see that the above morphism is an epimorphism in the sense of sheaf theory. In other words, this morphism is an isomorphism if the left hand side is understood as the *quotient sheaf* with respect to the Zariski topology on the category $Comm^{op}$. This sounds like a good situation as both functors $R'_n(\Gamma)$ and Gl_n are representable by affine schemes. However, the quotient sheaf of an affine scheme by the action of an affine group scheme is in general not a scheme when the action has fixed points. It is for instance not so hard to see that the quotient sheaf $\mathbb{A}^1/(\mathbb{Z}/2)$ is not representable by a scheme (here $\mathbb{A}^1 = Spec\,\mathbb{Z}[T]$ is the affine line and the action is induced by the involution $T \mapsto -T$). In our situation the action of Gl_n on $R'_n(\Gamma)$ has many fixed points, as for a given $A \in Comm$, the stabilizer of a given morphism $\Gamma \longrightarrow Gl_n(A)$ is precisely the group of automorphisms of the corresponding $A[\Gamma]$-module. We see here that the reason for the non-representability of the quotients $R'_n(\Gamma)/Gl_n$ is the existence of non trivial automorphism groups. Here the philosophy is the same of for the previous point, the quotient construction is not exact in some sense and should be derived. The derived quotient of a group G acting on a set X is the groupoid $[X/G]$, whose objects are the points of x and whose morphisms from x to y are the elements $g \in G$ such that $g.x = y$. The set of isomorphism classes of objects in $[X/G]$ is the usual quotient X/G, but the derived quotient $[X/G]$ also remembers the stabilizers of the action in the automorphism groups of $[X/G]$. This suggests that the right think to do is to replace $\coprod_n R'_n(\Gamma)/Gl_n$ by the more evolved construction $\coprod_n [R'_n(\Gamma)/Gl_n]$, which is now a functor from $Comm$ to the catgeory of groupoids rather than the category of sets. In the same way, this suggests that the functor $R(\Gamma)$ should rather be replaced by $\underline{R}(\Gamma)$, sending a commutative ring A to the whole groupoid of $A[\Gamma]$-modules whose underlying A-module is projective and of finite type. We see here that we again need to extend the notion of schemes in order to be able to find a geometric object representing $\underline{R}(\Gamma)$, as the functor represented by a scheme is always set valued by definition.

1.3 Conclusions

We arrive at the conclusions of this first lecture. The fundamental objects of algebraic geometry are functors

$$Comm \longrightarrow Set.$$

However we have seen that certain constructions on rings (tensor products), or on sets (quotients), are not exact and should rather be derived in order to be better behaved. Deriving the tensor product for commutative rings forces us to introduce simplicial commutative rings, an deriving quotients forces us to introduce groupoids (when it is a quotient by a group) and more generally simplicial sets (when it is a more complicate quotient). The starting point of derived algebraic geometry is that its fundamental objects are functors

$$sComm \longrightarrow SSet,$$

from the category of simplicial commutative rings to the category of simplicial sets. The main objective of the series of lectures is to explain how the basic notions of algebraic geometry (schemes, algebraic spaces, flat, smooth and étale morphisms ...) can be extended to this derived setting, and how this is useful for the study of moduli problems.

We will proceed in two steps. We will first explain how to do half of the job and to allow derived quotients but not derived tensor products (i.e. considering functors $Comm \longrightarrow SSet$). In other words we will start to explain formally how the quotient problem (point (2) of the discussion of the last paragraph) can be solved. This will be done by introducing the notions of *stacks* and *algebraic stacks*, which is based on the well known homotopy theory of simplicial presheaves of Joyal-Jardine. Later on we will explain how to incorporate derived tensor products and simplicial commutative rings to the picture.

Chapter 2

Simplicial presheaves and stacks

The purpose of this second lecture is to present the homotopy theory of simplicial presheaves on a Grothendieck site, and explain how these are models for stacks. In the next lecture, simplicial presheaves will be used to produce models for (higher) stacks in the context of algebraic geometry and will allow us to define the notion of algebraic n-stacks, a far reaching generalization of the notion of schemes for which all quotients by reasonable equivalence relations exists.

2.1 Review of the model category of simplicial presheaves

We let (C, τ) be a Grothendieck site. Recall that this means that we are given a category C, together a Grothendieck topology τ on C. The Grothendieck topology τ is the data for any object $X \in C$ of a family $cov(X)$ of sieves over X (i.e. sub-functors of the representable functor $h_X := Hom(-, X)$) satisfying the following three conditions.

1. For any $X \in C$, we have $h_X \in cov(x)$.
2. For any morphism $f : Y \to X$ in C, and any $u \in cov(X)$, we have $f^*(u) := u \times_{h_X} h_Y \in cov(Y)$.
3. Let $X \in C$, $u \in cov(X)$, and v be any sieve on X. If for all $Y \in C$ and any $f \in u(Y) \subset Hom(Y, X)$ we have $f^*(v) \in cov(Y)$, then $v \in cov(X)$.

Recall that for such a Grothendieck site we have its associated category of presheaves $Pr(C)$, which by definition is the category of all functors from C^{op} to the category of sets. The full sub-category of sheaves $Sh(C)$ is defined to be the sub-category of presheaves $F : C^{op} \longrightarrow Set$ such that for any $X \in C$ and any

$u \in cov(X)$, the natural morphism

$$F(X) \simeq Hom_{Pr(C)}(h_X, F) \longrightarrow Hom_{Pr(C)}(u, F)$$

is bijective.

A standard result from sheaf theory states that the inclusion functor

$$i : Sh(C) \hookrightarrow Pr(C)$$

has an exact (i.e. commutes with finite limits) left adjoint

$$a : Pr(C) \longrightarrow Sh(C)$$

called the associated sheaf functor.

We now let $SPr(C)$ be the category of simplicial objects in $Pr(C)$. We start to endow the category $SPr(C)$ with a levelwise model category structure defined as follows.

Definition 2.1.1. *Let $f : F \longrightarrow F$ be a morphism in $Pr(C)$.*

1. *The morphism f is a* global fibration *if for any $X \in C$ the induced morphism*

$$F(X) \longrightarrow F'(X)$$

is a fibration of simplicial sets (for the standard model category structure, i.e. is a Kan fibration).

2. *The morphism f is an* global equivalence *if for any $X \in C$ the induced morphism*

$$F(X) \longrightarrow F'(X)$$

is an equivalence[1] *of simplicial sets (again for the standard model category structure on simplicial sets).*

3. *The morphism f is a* global cofibration *if it has the right lifting property with respect to every fibration which is also an equivalence.*

It is well known that the above definitions endow the category $SPr(C)$ with a cofibrantly generated model category structure. This model category is moreover proper and cellular (in the sense of [Hi]). This model structure will be refered to the *global model structure*. There is a small set theory problem here when the category C is not *small*. This problem can be easily solved by fixing universes and will be simply neglected in the sequel.

We now take into account the Grothendieck topology τ on C in order to rafine the global model structure. This is an important step as when the quotient of

[1] In these notes the expression *equivalence* always refers to *weak equivalence*.

group action on a scheme exists, the presheaf represented y the quotient scheme is certainly not the quotient presheaf. However, for free actions the sheaf represented y the quotient scheme is the quotient sheaf.

We start by introducing the so-called homotopy sheaves of a simplicial presheaf $F : C^{op} \longrightarrow SSet$. We define a presheaf

$$\pi_0^{pr}(F) : C^{op} \longrightarrow Set$$

simply by sending $X \in C$ to $\pi_0(F(X))$. In the same way, for any $X \in C$ and any 0-simplex $s \in F(X)_0$ we define presheaves of groups on C/X

$$\pi_i^{pr}(F, s) : (C/X)^{op} \longrightarrow Gp$$

sending $f : Y \to X$ to $\pi_i(F(Y), f^*(s))$. Here, $F(Y)$ is the simplicial set of values of F over Y, $f^*(s) \in F(Y)_0$ is the inverse image of the base point s, and finally $\pi_i(F(Y), f^*(s))$ denotes the *correct* homotopy groups of the simplicial set $F(Y)$ based at $f^*(s)$. By *correct* we mean either the simplicial (or combinatorial) homotopy groups of a fibrant model for $F(Y)$, or more easily the topological homotopy groups of the the geometric realization $|F(Y)|$.

The associated sheaves to the presheaves $\pi_0^{pr}(F)$ and $\pi_i^{pr}(F, s)$ will be denoted by $\pi_0(F)$ and $\pi_i(F, s)$. These are called the *homotopy sheaves of* F. They are functorial in F.

Definition 2.1.2. *Let $f : F \longrightarrow F'$ be a morphism of simplicial presheaves.*

1. *The morphism f is a* local equivalence *if it satisfies the following two conditions*
   - *(a) The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an isomorphism of sheaves.*
   - *(b) For any $X \in C$, any $s \in F(X)_0$ and any $i > 0$ the induced morphism $\pi_i(F, s) \longrightarrow \pi_i(F', f(s))$ is an isomorphism of sheaves on C/X.*
2. *The morphism f is a* local cofibration *if it is a global cofibration in the sense of definition 2.1.1.*
3. *The morphism f is a* local fibration *if it has the left lifting property with respect to every local cofibration which is also a local equivalence.*

For simplicity, we will use the expressions equivalence, fibrations *and* cofibration *in order to refer to* local equivalence, local fibration *and* local cofibration.

It is also well known that the above definition endows the category $SPr(C)$ with a model category structure, but this is a much harder result than the existence of the global model structure. This result, as well as several small modifications, is due to Joyal (for simplicial sheaves) and Jardine (for simplicial presheaves), and we refer to [Bl, DHI] for recent references. Unless the contrary is specified we will always assume that the category $SPr(C)$ is endowed with this model category structure, which will be called the *local model structure*.

A nice result proved in [DHI] is the following characterization of fibrant objects in $SPr(C)$ (for the local model structure). Recall first that a hypercovering of an object $X \in C$ is the data of a simplicial presheaf F together with a morphism $H \longrightarrow X$ and satisfying the following two conditions.

1. For any integer n the presheaf H_n is a disjoint union of representable presheaves.

2. For any $n \geq 0$ the morphism of presheaves
$$H_n \simeq Hom(\Delta^n, H) \longrightarrow Hom(\partial\Delta^n, H) \times_{Hom(\partial\Delta^n, X)} Hom(\Delta^n, X)$$
induces an epimorphism on the associated sheaves.

Here Δ^n denotes the simplicial simplex of dimension n as well as the corresponding constant simplicial presheaf. In the same way $\partial\Delta^n$ is the $(n-1)$-skeleton of Δ^n and is considered here as a constant simplicial presheaf. Finally, Hom denote here the presheaves of morphisms between two simplicial presheaves. This second condition can equivalently be stated by saying that for any $Y \in C$, and any commutative square of simplicial sets

$$\begin{tikzcd} \partial\Delta^n \arrow[rr] \arrow[d] & & H(Y) \arrow[d] \\ \Delta^n \arrow[r] & X(Y) = Hom(Y,X), & \end{tikzcd}$$

there exists a covering sieve $u \in cov(Y)$ such that for any $f : U \to Y$ in the sieve u there exists a morphism $\Delta^n \longrightarrow H(U)$ making the following square

$$\begin{tikzcd} \partial\Delta^n \arrow[rr] \arrow[d] & & H(U) \arrow[d] \\ \Delta^n \arrow[r] \arrow[urr] & X(U) = Hom(U,X). & \end{tikzcd}$$

This property is also called the *local lifting property*, and is the local analog of the lifting property characterizing trivial fibrations of simplicial sets. For any $X \in C$ and any hypercovering H of X, we can define an augmented cosimplicial diagram of simplicial sets
$$F(X) \longrightarrow ([n] \mapsto F(H_n)).$$
Here, each H_n is a coproduct of representables, say $H_n = \coprod_i H_{n,i}$, and by definition we set $F(H_n) = \prod_i F(H_{n,i})$.

With these notions and notations it is possible to proove (see [DHI]) that an object $F \in SPr(C)$ is fibrant (for the local model structure) if and only if it satisfies the following two conditions.

1. For any $X \in C$ the simplicial set $F(X)$ is fibrant.
2. For any $X \in C$ and any hypercovering $H \longrightarrow X$ the natural morphism
$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$
is an equivalence of simplicial sets.

The above first condition is rather anodyne and the second condition is of course the important one. It is a homotopy analog of the sheaf conditions, in the sense that when F is a presheaf of sets, considered as a simplicial presheaf constant in the simplicial direction, then this second condition for F is equivalent to the fact that F is a sheaf (this is because the homotopy limits is then simply a usual limits in the category of sets and the condition becomes the usual descent conditions for sheaves).

Definition 2.1.3. *1. An object $F \in SPr(C)$ is called a* stack *if for any $X \in C$ and any hypercovering $H \longrightarrow X$ the natural morphism*
$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$
is an equivalence of simplicial sets.

2. The homotopy category $Ho(SPr(C))$ will be called the homotopy category of stacks on the site (C,τ) *(or simply the* category of stacks*). Most often objects in $Ho(SPr(C))$ will simply be called* stacks*. The expressions* morphism of stacks *and* isomorphism of stacks*, will refer to morphisms and isomorphisms in $Ho(SPr(C))$. The set of morphisms of stacks from F to F' will be denoted by $[F,F']$.*

In the sequel, we will often use the following terminology and notations.

- For a diagram of stacks $F \longrightarrow H \longleftarrow G$, we denote by $F \times^h_H G$ the corresponding homotopy fiber product of simplicial presheaves (note that this constuction is not functorially defined on $Ho(SPr(C))$ and requires some lift of the diagrams to $SPr(C)$).
- A morphism of stacks $F \longrightarrow Ho(SPr(C))$ is an *epimorphism* if the induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is a sheaf epimorphism.
- A morphism of stacks $F \longrightarrow F'$ is a *monomorphism* if the diagonal morphism $F \longrightarrow F \times^h_{F'} F$ is an isomorphism.

Equivalently, a morphism of stacks $F \longrightarrow F'$ is a monomorphism if and only if it satisfies the following two conditions

1. The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is a monomorphism.
2. For all $X \in C$ and all $s \in F(X)$ the induced morphisms $\pi_i(F,s) \longrightarrow \pi_i(F', f(s))$ are isomorphisms for all $i > 0$.

2.2 Basic examples

To finish this section we present some basic and general examples of stacks. These are very general examples and we will see more specific examples in the context of algebraic geometry in the next lectures.

Sheaves: We start by noticing that there is a full embedding

$$Sh(C) \longrightarrow Ho(SPr(C))$$

from the category of sheaves (of sets) to the homotopy category of stacks, simply by considering a sheaf of sets as a simplicial presheaf (constant in the simplicial direction). This inclusion functor has a left adjoint, which sends a simplicial presheaf F to the sheaf $\pi_0(F)$. This will allow us to consider any sheaf as a stack, and in the sequel we will do this implicitely. In this way, the category of stacks $Ho(SPr(C))$ is an extension of the category of sheaves. Moreover, any objects in $Ho(SPr(C))$ is isomorphic to a homotopy colimit of sheaves (this is because any simplicial set X is naturally equivalent to the homotopy colimit of the diagram $[n] \mapsto X_n$), which shows that stacks are obtained from sheaves by taking derived quotients.

Truncations and n-stacks: A stack $F \in Ho(SPr(C))$ is *n-truncated*, or en *n-stack*, if for any $X \in C$ and any $s \in F(X)_0$ we have $\pi_i(F,s) = 0$ for all $i > n$. The full sub-category of n-stacks will be denoted by $Ho(SPr_{\leq n}(C))$. We note that $Ho(SPr_{\leq 0}(C))$ is the essential image of the inclusion morphism $Sh(C) \longrightarrow Ho(SPr(C))$, and thus that there exists an equivalence of categories $Sh(C) \simeq Ho(SPr_{\leq 0}(C))$.

The inclusion functor $Ho(SPr_{\leq n}(C)) \hookrightarrow Ho(SPr(C))$ admits a left adjoint

$$t_{\leq n} : Ho(SPr(C)) \longrightarrow Ho(SPr_{\leq n}(C))$$

called the *truncation functors*. We have $t_{\leq 0} \simeq \pi_0$, and in general $t_{\leq n}$ is obtained by applying levelwise the usual truncation functor for simplicial sets. Another possible understanding of this situation is by intoducing the left Bousfield localization (in the sense of [Hi]) of the model category $SPr(C)$, by inverting all the morphisms $\partial\Delta^{n+2} \times X \longrightarrow X$, for all $X \in C$. The fibrant objects for this localized model structure are precisely the n-truncated fibrant simplicial presheaves, and its homotopy category can be naturally identified with $Ho(SPr_{\leq n}(C))$. The functor $t_{\leq n}$ is then the localization functor for this left Bousfield localization.

For any stack F, there exists a tower of stacks

$$F \longrightarrow \cdots \longrightarrow t_{\leq n}(F) \longrightarrow t_{\leq n-1}(F) \longrightarrow \cdots \longrightarrow t_{\leq 0}(F) = \pi_0(F),$$

called the Postnikov tower for F. A new feature here is that this tower does not converge in general, or in other words the natural morphism

$$F \longrightarrow Holim_n t_{\leq n}(F)$$

is not an equivalence in general. It is the case under some rather strong boundness conditions on the cohomological dimension of the sheaves of groups $\pi_i(F)$.

Classifying stacks: Let G be a group object in $SPr(C)$, that is a presheaf of simplicial groups. From it we construct a simplicial presheaf BG by applying levelwise the classifying space construction. More explicitely BG is the simplicial presheaf whose presheaf of n-simplices is G_n^n, and whose face and degeneracies are defined using the composition and units in G as well as the face and degeneracies of the underlying simplicial set of G. The simplicial presheaf BG has a natural global point $*$, and by construction we have

$$\pi_i(BG,*) \simeq \pi_{i-1}(G,e).$$

When G is abelian, the simplicial presheaf BG is again an abelian group object in $SPr(C)$, and the construction can then be iterated.

When A is a sheaf of abelian groups on C we let

$$K(A,n) := \underbrace{B\dots B}_{n\ times}(A).$$

By construction $K(A,n)$ is a pointed simplicial presheaf such that

$$\pi_i(K(A,n),*) \simeq 0\ if\ i \neq n \qquad \pi_n(K(A,n),*) \simeq A,$$

and this property characterizes $K(A,n)$ uniquely up to an isomorphism in $Ho(SPr(C))$. By construction $K(A,n)$ is a n-truncated, and it can be proved that for any $X \in C$ there is a natural isomorphism

$$[X, K(A,n)] \simeq H^n(X,A),$$

where the left hand side is the set of morphisms in $Ho(SPr(C))$ and the right hand side denotes sheaf cohomology. This shows that as defined above the simplicial presheaf $K(A,n)$ is a stack in the sense of definition 2.1.3 (1) if and only if $H^i(X,A) = 0$ for any $X \in C$ and any $i > n$.

Internal Hom: An important property of the category $Ho(SPr(C))$ is that it admits internal Homs (i.e. is cartesian closed). Explicitely, if F and F' are two stacks, we define a simplicial presheaf

$$\mathbb{R}\underline{\mathcal{H}om}(F,F') : C^{op} \longrightarrow SSet,$$

by

$$\mathbb{R}\underline{\mathcal{H}om}(F,F')(X) := \underline{Hom}(X \times Q(F), R(F')),$$

where $\underline{Hom}$ denotes the natural simplicial enrichement of the category $SPr(C)$, $Q(F)$ is a cofibrant model for F and $R(F')$ is a fibrant model for F'. When the object $\mathbb{R}\underline{\mathcal{H}om}(F,F')$ is considered in $Ho(SPr(C))$ it is possible to show that we have functorial isomorphisms

$$[F'', \mathbb{R}\underline{\mathcal{H}om}(F,F')] \simeq [F'' \times F, F']$$

for any $F'' \in Ho(SPr(C))$. The stack $\mathbb{R}\underline{\mathcal{H}om}(F,F')$ is called the stack of morphisms from F to F'.

Twisted forms: We let $F \in Ho(SPr(C))$ be a stack. We consider the following presheaf of simplicial monoids

$$\mathbb{R}\underline{\mathcal{E}nd}(F) : X \mapsto \underline{Hom}(X \times QR(F), QR(F)),$$

where $QR(F)$ is a fibrant and cofibrant model for F. The monoid structure on this presheaf is simply induced by composing endomorphisms. We define another presheaf of simplicial monoids by the following homotopy pull-back square

$$\begin{tikzcd}
\mathbb{R}\underline{\mathcal{A}ut}(F) \arrow[r] \arrow[d] & \mathbb{R}\underline{\mathcal{E}nd}(F) \arrow[d] \\
\pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))^{inv} \arrow[r] & \pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F)),
\end{tikzcd}$$

where $\pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))^{inv}$ denotes the subsheaf of invertible elements in the sheaf of monoids $\pi_0(\mathbb{R}\underline{\mathcal{E}nd}(F))$.

The stack $\mathbb{R}\underline{\mathcal{A}ut}(F)$ is called the stack of auto-equivalence of F. It is a represnted by a presheaf in simplicial monoids for which all elements are invertible up to homotopy. Even if this is not strictly speaking a presheaf of simplicial groups we can apply the classifying space construction to get a new stack $B\mathbb{R}\underline{\mathcal{A}ut}(F)$. The interesting property of the stack $B\mathbb{R}\underline{\mathcal{A}ut}(F)$ is that it classifies twisted forms of the stack F in the following sense. For any $X \in C$, it is possible to show that the set $[X, B\mathbb{R}\underline{\mathcal{A}ut}(F)]$ is in a natural bijection with the set of isomorphism classes of objects in $Ho(SPr(C)/X)$ which are locally equivalent to F. Here an object $F' \longrightarrow X$ is locally equivalent to F if there is a covering sieve u of X, such that for any $U \to X$ in u the two objects $F' \times_X U$ and $F \times U$ are isomorphic in $Ho(SPr(C)/U)$.

An important example is when $F = K(A,n)$, as twisted forms of F are sometimes refered to n-gerbes with coefficients in A. It can be shown that the stack $B\mathbb{R}\underline{\mathcal{A}ut}(F)$ has a two steps Postnikov tower

$$\begin{tikzcd}
K(A,n+1) \arrow[r] & B\mathbb{R}\underline{\mathcal{A}ut}(F) \arrow[d, "p"] \\
 & K(aut(A),1),
\end{tikzcd}$$

and that the projection p has a section (in $Ho(SPr(C))$). As a consequence we see that the set of equivalence classes of n-gerbes on X with coefficients in A is in bijection with the set of pairs (ρ, α), where ρ is a $aut(A)$-torsor over X, and $\alpha \in H^{n+1}(X, A_\rho)$, where A_ρ is the twisted form of the sheaf A determined by ρ.

Chapter 3

Algebraic stacks

In the previous lecture we have introduced the notion of stacks over some site. We will now consider the more specific case of stacks over the étale site of affine schemes and introduce an important class of stacks called *algebraic stacks*. These are generalizations of schemes and algebraic spaces for which quotients by smooth actions always exist.

All along this lecture we will consider $Comm$ the category of commutative rings and set $Aff := Comm^{op}$. For $A \in Comm$ we denote by $Spec\, A$ the corresponding object in Aff (therefore "*Spec*" stands for a formal notation here). We endow Aff with the étale topology defined as follows. Recall that a morphism of commutative rings $A \longrightarrow B$ is *étale* if it satisfies the following three conditions:

1. B is flat as an A-module.
2. B is finitely presented as a commutative A-algebra (i.e. of the form $A[T_1, \ldots, T_n]/(P_1, \ldots, P_r)$).
3. B is flat as a $B \otimes_A B$-module.

There exists several equivalent characterizations of étale morphisms (see e.g. [SGA1]) , for instance the third condition can be equivalently replaced by the condition $\Omega^1_{B/A} = 0$, where $\Omega^1_{B/A}$ is the B-module of relative Kahler derivations (corepresenting the functor sending a B-module M to the set of A-linear derivations on B with coefficients in M). Etale morphisms are stable by base change and composition in Aff, i.e. by cobase change and composition in $Comm$. Geometrically an étale morphism $A \longrightarrow B$ should be thought as a "local isomorphism" of schemes $Spec\, B \longrightarrow Spec\, A$, though here *local* should not be understood in the sense of the Zariski topology.

Now, a family of morphisms $\{A \longrightarrow A_i\}_{i \in I}$ is an étale covering if each morphism $A \longrightarrow A_i$ is étale and if the family of base change functors

$$- \otimes_A A_i : A - Mod \longrightarrow A_i - Mod$$

is conservative. This defines a topology on Aff by defining that a sieve on $Spec\, A$ is a covering sieve if it is generated by an étale covering family.

Finally, a morphism $Spec\, B \longrightarrow Spec\, A$ is a Zariski open immersion if it is étale and a monomorphism (this is equivalent to say that the natural morphism $B \otimes_A B \longrightarrow B$ is an isomorphism, or equivalently that the forgetful functor $B - Mod \longrightarrow A - Mod$ is fully faithful).

3.1 Schemes and algebraic n-stacks

We start by the definition of schemes and then define algebraic n-stacks as certain succesive quotients of schemes.

For $Spec\, A \in Aff$ we can consider the presheaf represented by $Spec\, A$

$$Spec\, A : Aff^{op} = Comm \longrightarrow Set,$$

by setting $(Spec\, A)(B) = Hom(A, B)$. A standard result of commutative algebra (faithfully flat descent) states that the presheaf $Spec\, A$ is always a sheaf. We thus consider $Spec\, A$ has a stack and as an object in $Ho(SPr(Aff))$. This defines a fully faithful functor

$$Aff \longrightarrow Ho(SPr(Aff)).$$

Any objects in $Ho(SPr(Aff))$ isomorphic to a sheaf of the form $Spec\, A$ will be called an *affine scheme*. The full sub-category of $Ho(SPr(Aff))$ consisting of affine schemes is equivalent to $Aff = Comm^{op}$, and these two categories will be implicitly identified.

Definition 3.1.1. 1. *Let $Spec\, A$ be an affine scheme, F a stack and $i : F \longrightarrow Spec\, A$ a morphism. We say that i is a* Zariski open immersion *(or simply an* open immersion*) if it satisfies the following two conditions.*

   (a) The stack F is a sheaf (i.e. 0-truncated) and the morphism i is a monomorphism of sheaves.

   (b) There exists a family of Zariski open immersions $\{A \longrightarrow A_i\}_i$ such that F is the image of the morphism of sheaves

$$\coprod_i Spec\, A_i \longrightarrow Spec\, A.$$

2. *A morphism of stacks $F \longrightarrow F'$ is a* Zariski open immersion *(or simply an* open immersion*) if for any affine scheme $Spec\, A$ and any morphism $Spec\, A \longrightarrow F'$, the induced morphism*

$$F \times^h_{F'} Spec\, A \longrightarrow Spec\, A$$

is a Zariski open immersion in the sense above.

3. *A stack F is a* scheme *if there exists a family of affine schemes $\{Spec\, A_i\}_i$ and Zariski open immersions $Spec\, A_i \longrightarrow F$, such that the induced morphism of sheaves*

$$\coprod_i Spec\, A_i \longrightarrow F$$

is an epimorphism. Such a family of morphisms $\{Spec\, A_i \longrightarrow F\}$ will be called a Zariski altas for F.

It is easy to see that any Zariski open immersion $F \longrightarrow F'$ is a monomorphism (in the sense we explained at the end of section §2.1). In particular, this implies that a scheme F is always 0-truncated, and thus equivalent to a sheaf.

We now pass to the definition of algebraic stacks. These are stacks obtained by gluing schemes along smooth quotient, and we first need to recall the notion of smooth morphisms of schemes.

Recall that a morphism of commutative rings $A \longrightarrow B$ is *smooth*, if it is flat of finite presentation and if moreover B is of finite Tor dimension as a $B \otimes_A B$-module. Smooth morphisms are the algebraic analog of submersions, and there exists equivalent definitions making this analogy more clear (see [SGA1]). Smooth morphisms are stable by compositions and base change in Aff. The notion of smooth morphisms can be extended to a notion for all schemes by the following way. We say that a morphism of schemes $X \longrightarrow Y$ is smooth if there exists Zariski atlas $\{Spec\, A_i \longrightarrow X\}$ and $\{Spec\, A_j \longrightarrow Y\}$ together with commutative squares

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \uparrow & & \uparrow \\ Spec\, A_i & \longrightarrow & Spec\, A_j, \end{array}$$

with $Spec\, A_i \longrightarrow Spec\, A_j$ a smooth morphism (here j depends on i). Again, smooth morphisms of schemes are stable by composition and base change.

We are now ready to define the notion of algebraic stack. The definition is by induction on an algebraicity index n representing the number of successive smooth quotients we take. This index will be forgetten after the definition is acheived.

Definition 3.1.2. 1. *A stack F is* 0-algebraic *if it is a scheme.*

2. *A morphism of stacks $F \longrightarrow F'$ is* 0-algebraic *(or* 0-representable*) if for any scheme X and any morphism $X \longrightarrow F'$ the stack $F \times^h_{F'} X$ is 0-algebraic (i.e. a scheme).*

3. *A 0-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if for any scheme X and any morphism $X \longrightarrow F'$ the morphism of schemes $F \times^h_{F'} X \longrightarrow X$ is smooth.*

4. *We now let $n > 0$, and we assume that the notions of $(n-1)$-algebraic stack, $(n-1)$-algebraic morphisms and smooth $(n-1)$-algebraic morphisms have been defined.*

   (a) *A stack F is* n-algebraic *if there exists a scheme X together with smooth $(n-1)$-algebraic morphisms $X \longrightarrow F$ which is an epimorphism. Such a morphism $X \longrightarrow F$ is called a* smooth n-atlas for F.

   (b) *A morphism of stacks $F \longrightarrow F'$ is* n-algebraic *(or* n-representable*) if for any scheme X and any morphism $X \longrightarrow F'$ the stack $F \times_{F'} X$ is n-algebraic.*

   (c) *An n-algebraic morphism of stacks $F \longrightarrow F'$ is smooth if for any scheme X and any morphism $X \longrightarrow F'$ there exists a smooth n-atlas $Y \longrightarrow F \times^h_{F'} X$ such that each morphism $Y \longrightarrow X$ is a smooth morphism of schemes.*

5. *An* algebraic stack *is a stack which is n-algebraic for some integer n. An* algebraic n-stack *is an algebraic stack which is also an n-stack. An* algebraic space *is an algebraic* 0*-stack.*

6. *A morphism of stacks $F \longrightarrow F'$ is* algebraic *(or* representable*) if it is n-algebraic for some n.*

7. *A morphism of stacks $F \longrightarrow F'$ is* smooth *if it is n-algebraic and smooth for some integer n.*

Long, but formal arguments show that algebraic stacks satisfy the following properties.

- Algebraic stacks are stable by finite homotopy limits (i.e. by homotopy pullbacks).
- Algebraic stacks are stable by disjoint union.
- Algebraic morphisms of stacks are stable by composition and base change.
- Algebraic stacks are stable by smooth quotients. To be more precise, if $F \longrightarrow F'$ is a smooth epimorphism of stacks, then F' is algebraic if and only if F is so.

The standard finiteness properties of schemes can be extended to algebraic stacks in the following way.

- An algebraic morphism $F \longrightarrow F'$ is *locally of finite presentation* if for any scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \to F \times^h_{F'} X$ such that the induced morphism $Y \longrightarrow X$ is locally of finite presentation.

- An algebraic morphism $F \longrightarrow F'$ is *quasi-compact*, if for any affine scheme X and any morphism $X \longrightarrow F'$, there exists a smooth atlas $Y \longrightarrow F \times^h_{F'} X$ with Y an affine scheme.

- An algebraic stack F is *strongly quasi-compact* if for all integer n the induced morphism
$$F \longrightarrow \mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F)$$
is quasi-compact.

- An algebraic stack morphism $F \longrightarrow F'$ is *strongly of finite presentation* if for any affine scheme X and any morphism $X \longrightarrow F'$ the stack $F \times^h_{F'} X$ is locally of finite presentation and strongly quasi-compact.

Note that when $n = 0$ we have $\mathbb{R}\underline{\mathcal{H}om}(\partial\Delta^n, F) \simeq F \times F$, and the condition of strongly quasi-compactness implies in particular that the diagonal morphism $F \longrightarrow F \times F$ is quasi-compact. In general, being strongly quasi-compact involves quasi-compactness conditions for all the "higher diagonals".

3.2 Some examples

Groupoid quotients: We start here by the standard way to construct algebraic stacks using quotients by smooth groupoid actions. We start by a simplicial objects in $SPr(C)$
$$F_* : \Delta^{op} \longrightarrow SPr(Aff).$$
We will say that F_* is a *Segal groupoid* if it satisfies the following two conditions

1. For any $n > 1$, the natural morphism
$$F_n \longrightarrow F_1 \times^h_{F_0} F_1 \cdots \times^h_{F_0} F_1,$$
induced by the morphism $[1] \to [n]$ sending 0 to i and 1 to $i+1$ (for $0 \leq i < n$), is an isomorphism of stacks.

2. The natural morphism
$$F_2 \longrightarrow F_1 \times^h_{F_0} F_1$$
induced by the morphism $[1] \to [2]$ sending 0 to 0 and 1 to 1 or 2, is an isomorphism of stacks.

We now assume that F_* is a Segal groupoid and moreover that all the face morphisms $F_1 \longrightarrow F_0$ are smooth morphism between algebraic stacks. We consider the homotopy colimit of the diagram $[n] \mapsto F_n$, and denote it by $|F_*| \in Ho(SPr(Aff))$. The stack $|F_*|$ is called the *quotient stack of the Segal groupoid* F_*. It can been proved that $|F_*|$ is again an algebraic stack. Moreover if each F_i is an algebraic n-stack then $|F_*|$ is an algebraic $(n+1)$-stack. This is a formal way to produce higher algebraic stacks starting say from schemes, but this is often not the way stacks arise in practice.

An important very special case of the quotient stack constuction is the case of a smooth group scheme G acting on a scheme X. In this case we form the groupoid object $B(X,G)$ whose values in degre n is $X \times G^n$, and whose transition morphisms are given by the action of G on X. This is a groupoid object in schemes and thus can be considered as a groupoid objects in sheaves and therefore as a very special kind of Segal groupoid. The quotient stack of this Segal groupoid is denoted by $[X/G]$ and is called the quotient stack of X by G. It is an algebraic 1-stack for which a natural smooth atlas is the natural projection $X \longrightarrow [X/G]$. It can be characterize by a universal property: morphisms of stacks $[X/G] \longrightarrow F$ are in one-to-one correspondence with morphisms of G-equivariant stacks $X \longrightarrow F$ (here we need to use a model category $G - SPr(Aff)$ of G-equivariant simplicial presheaves in order to have the correct homotopy category of G-equivariant stacks).

Simplicial presentation: Algebraic stacks can also be characterized as the simplicial presheaves having being represented by certain kind of simplicial schemes. For this we let X_* be a simplicial object in the category of schemes. For any finite simplicial set K (finite here means generated by a finite number of cells) we can form X_*^K, which is the scheme of morphism from K to X_*. It is, by definition the equalizer of the two natural morphisms

$$\prod_n X_n^{K_n} \rightrightarrows \prod_{[p]\to[q]} X_p^{K_q}.$$

This equalizer exists as a scheme when K is finite (because it then only invovles finite limits).

A simplicial scheme X_* is then called a *weak smooth groupoid* if for any $0 \leq k \leq n$, the natural morphism

$$X_n = X_*^{\Delta^n} \longrightarrow X_*^{\Lambda^{n,k}}$$

is a smooth and surjective morphism of schemes (surjective here has to be understood pointwise, but as the morphism is smooth this is equivalent to say that it induces an epimorphism on the corresponding sheaves). A weak smooth groupoid X_* is moreover *n-truncated* if for any $k > n+1$ the natural morphism

$$X_k = X^{\Delta^k} \longrightarrow X_*^{\partial\Delta^k}$$

is an isomorphism.

It is then possible to prove that a stack F is an algebraic n-stack if there exists an n-truncated weak smooth groupoid X_* and an isomorphism in $Ho(SPr(Aff))$ $F \simeq X_*$.

Some famous algebraic 1-stacks: We review here two famous examples of algebraic 1-stacks, the stack of smooth and proper curves and the stack of vector bundles on curve. We refer to [La-Mo] for more details.

For $X \in Aff$ an affine scheme we let $\mathcal{M}_g(X)$ be the full sub-groupoid of sheaves F on Aff/X such that the corresponding morphism of sheaves $F \longrightarrow X$ is representable by a smooth and proper curve of genus g over X (i.e. F is itself a scheme, the morphism $F \longrightarrow X$ is smooth, proper, with geometric fibers being connected curves of genus g). For $Y \longrightarrow X$ in Aff, we have a restriction functor from sheaves on Aff/X to sheaves on Aff/Y, and this defines a natural functor of groupoids

$$\mathcal{M}_g(X) \longrightarrow \mathcal{M}_g(Y).$$

This defines a presheaf in groupoids on Aff, and taking the nerve of these groupoids gives a simplicial presheaf denoted by $\mathcal{M}_g$. The stack $\mathcal{M}_g$ is called the *stack of smooth curves of genus g*. It is such that for $X \in Aff$, $\mathcal{M}_g(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of smooth proper curves of genus g over X, and whose π_1 at a given curve is its automorphism group. It is a well known theorem that $\mathcal{M}_g$ is an algebraic 1-stack which is smooth and of finite presentation over $Spec\,\mathbb{Z}$. This stack is even Deligne-Mumford, that is the diagonal morphism $\mathcal{M}_g \longrightarrow \mathcal{M}_g \times \mathcal{M}_g$ is unramified (i.e. locally for the étale topology an closed immersion). Equivalently this means that there exists an atlas $X \longrightarrow \mathcal{M}_g$ which is étale rather than only smooth.

Another very important and famous example of an algebraic 1-stack is the stack of G-bundles on some smooth projective curve C (say over some base field k). Let G be a smooth affine algebraic group over k. We start by consider the stack BG, which is a stack over $Spec\,k$. It is the quotient stack $[Spec\,k/G]$ for the trivial action of G on $Spec\,k$. As G is a smooth algebraic group this stack is an algebraic 1-stack. When C is a smooth and proper curve over $Spec\,k$ we can consider the stack of morphisms (of stacks over $Spec\,k$)

$$Bun_G(C) := \mathbb{R}\underline{\mathcal{H}om}_{Aff/Spec\,k}(C, BG),$$

which by definition is the stack of principal G-bundles on C. By definition, for $X \in Aff$, $Bun_G(C)(X)$ is a 1-truncated simplicial set whose π_0 is the set of isomorphism classes of principal G-bundles on C and whose π_1 at a given bundle is its automorphism group. It is also a well known theorem that the stack $Bun_G(C)$ is an algebraic 1-stack, which is smooth and locally of finite presentation over $Spec\,k$. However, this stack is not quasi-compact and is only a countable union of quasi-compact open substack.

Higher linear stacks: Let $X = Spec\,A$ be an affine scheme and E be positively graded cochain complex of A-modules. We assume that E is perfect, i.e. it is quasi-isomorphic to a bounded complex of projective A-modules of finite type. We define a stack $\mathbb{V}(E)$ over X in the following way. To any commutative A-algebra B we

set
$$\mathbb{V}(E)(B) := Map(E, B),$$
where Map denotes the mapping spaces of the model category of complexes of A-modules. More explicitly, $\mathbb{V}(E)(B)$ is the simplicial set whose set of n-simplicies is the set $Hom(Q(E) \otimes_A C_*(\Delta^n, A), B)$. Here $Q(E)$ is a cofibrant resolution of E in the model category of complexes A-modules (for the projective model structure for which equivalences are quasi-isomorphisms and fibrations are epimorphisms), $C_*(\Delta^n, A)$ is the homology complexe of the simplicial set Δ^n with coefficients in A, and the Hom is taken in the category of complexes of A-modules. In other words, $\mathbb{V}(E)(B)$ is the simplicial set obtained from the complex $\underline{Hom}^*(Q(E), B)$ by the Dold-Kan correspondence. When B varies in the category of commutative A-algebras this defines a simplicial presheaf $\mathbb{V}(E)$ together with a morphism $\mathbb{V}(E) \longrightarrow X = Spec\, A$. For any commutative A-algebras we have
$$\pi_i(\mathbb{V}(E)(B)) \simeq Ext^{-i}(E, B).$$

It can be shown that the stack $\mathbb{V}(E)$ is an algebraic n-stack strongly of finite presentation over X, where n is such that $H^i(E) = 0$ for all $i > n$, and that $\mathbb{V}(E)$ is smooth if and only if the Tor amplitude of E is non negative (i.e. E is quasi-isomorphic to a complex of projective A-modules of finite type which is moreover concentrated in non negative degrees). For this, we can first assume that E is a bounded complex of projective modules of finite type. We then set $K = E^{\leq 0}$ the part of E which is concentrated in non positive degrees, and we have a natural morphism of complexes $E \longrightarrow K$. This morphism induces a morpihsm of stacks
$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E).$$
Now, by definition $\mathbb{V}(K)$ is naturally equivalent to the affine scheme $Spec\, A[H^0(K)]$, with $A[H^0(K)]$ the free commutative A-algebra generated by the A-module $H^0(K)$. It is well known that $\mathbb{V}(H^0(k))$ is a smooth over $Spec\, A$ if and only if $H^0(K)$ is projective and of finite type. This is equivalent to say that E has non negative Tor amplitude. The only thing to check is then that the natural morphism
$$\mathbb{V}(K) \longrightarrow \mathbb{V}(E),$$
is $(n-1)$-algebraic and smooth. But this follows by induction on n as this morphism is locally on $\mathbb{V}(E)$ of the form $Y \times \mathbb{V}(L) \longrightarrow Y$, for L the homotopy cofiber (i.e. the cone) of the morphism $E \longrightarrow K$. This homotopy cofiber is itself quasi-isomorphic to $E^{>0}[1]$, and thus is a perfect complex of non negative Tor amplitude with $H^i(L) = 0$ for $i > n-1$.

The algebraic 2-stack of abelian categories: This is a non trivial example of an algebraic 2-stack. The material is taken from [An]. For a commutative ring A we consider $Ab(A)$ the following category. Its objects are abelian A-linear categories which are equivalent to $R - Mod$, the category of left R-modules for some associative A-algebra R which is projective and of finite type as an A-module. The morphism in $Ab(A)$ are the A-linear equivalences of categories. For a morphism of commutative rings $A \longrightarrow B$ we have a functor

$$Ab(A) \longrightarrow Ab(B)$$

sending an abelian category $\mathcal{C}$ to $\mathcal{C}^{B/A}$, the category of B-modules in $\mathcal{C}$. Precisely $\mathcal{C}^{B/A}$ can be taken to be the category of all A-linear functors from BB, the A-linear category with a unique object and B as its A-algebra of endomorphisms, to $\mathcal{C}$. This defines a presheaf of categories $A \mapsto Ab(A)$ on Aff. Taking the nerves of these categories we obtain a simplicial presheaf $\mathbf{Ab} \in SPr(Aff)$. The simplicial presheaf $\mathbf{Ab}$ is not a stack, but we still consider it as an object in $Ho(SPr(Aff))$. Of the main result of [An] states that $\mathbf{Ab}$ is an algebraic 2-stack which is locally of finite presentation.

The algebraic n-stack of $[n,0]$-perfect complexes: For an commutative ring A we consider a category $P(A)$ defined as follows. Its objetcs are the cofibrant complexes of A-modules (for the projective model structures) which are perfect (i.e. quasi-isomorphic to a bounded complex of projective modules of finite type). The morphisms in $P(A)$ are the quasi-isomorphisms of complexes of A-modules. For a morphism of commutative ring $A \longrightarrow B$ we have a base change functor

$$- \otimes_A B : P(A) \longrightarrow P(B).$$

This does not however define stricly speaking a presheaf of categories, as the base change functors are only compatible with composition up to a natural isomorphism. In other words, $A \mapsto P(A)$ is only a weak functor from $Comm$ to the 2-category of categories. Hopfully there exists a standard procedure to replace any weak functor by an equivalent strict functor: it consists of replacing P by the presheaf of cartesian sections of the Grothendieck construction $\int P \longrightarrow Comm$ (see [SGA1]). In few words, we define a new category $P'(A)$ whose objects consist of the following data.

1. For any morphism of commutative A-algebra B an object $E_B \in P(B)$.
2. For any commutative A-algebra B and any commutative B-algebra C, an isomorphism in $P'(C)$

$$\phi_{B,C} : E_B \otimes_B C \simeq E_C.$$

We require moreover that for any commutative A-algebra B, any commutative B-algebra C and any commutative C-algebra D, the two possible isomorphisms

$$\phi_{C,D} \circ (\phi_{B,C} \otimes_C D) : (E_B \otimes_B C) \otimes_C D) \simeq E_B \otimes_B D \longrightarrow E_D$$

$$\phi_{B,D} : E_B \otimes_B D \longrightarrow E_D$$

are equal. The morphisms in $P'(A)$ are simply taken to be families of morphisms $E_B \longrightarrow E'_B$ which commute with the $\phi_{B,C}$'s and the $\phi'_{B,C}$.

With these definitions $A \mapsto P'(A)$ is a functor $Comm \longrightarrow Cat$, and there is moreover an equivalence of lax funcors $P' \longrightarrow P$. We compose the functor P' with the nerve construction and we get a simplicial presheaf **Perf** on Aff. It can be proved that the simplicial presheaf **Perf** is a stack in the sense of definition 2.1.3 (1). This is not an abvious result (see for instance [H-S] for a proof), and can be reduced to a the well known flat cohomological descent for quasi-coherent complexes. It can also be proved that for $Spec\, A \in Aff$, the simplicial set $\mathbf{Perf}(X)$ satisfies the following properties.

1. The set $\pi_0(\mathbf{Perf}(X))$ is in a natural bijection with the set of quasi-isomorphism classes of perfect complexes of A-modules.

2. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_1(\mathbf{Perf}(X), x) \simeq Aut(E),$$

   where the automorphism group is taken in the derived category $D(A)$ of the ring A.

3. For $x \in \mathbf{Perf}(X)$ corresponding to a perfect complex E, we have

$$\pi_i(\mathbf{Perf}(X), x) \simeq Ext^{1-i}(E, E)$$

   for any $i > 1$. Again, these ext groups are computed in the triangle category $D(A)$.

For any $n \geq 0$ and $a \leq b$ with $b - a = n$ we can define a subsimplicial presheaf $\mathbf{Perf}^{[a,b]} \subset \mathbf{Perf}$ which consists of all perfect complexes of Tor amplitude contained in the interval $[a, b]$ (i.e. complexes quasi-isomorphic to a complex of projective modules of finite type concentrated in degrees $[a, b]$). It can be proved that the substacks $\mathbf{Perf}^{[a,b]}$ form an open covering of **Perf**. Moreover, $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack which is locally of finite presentation. This way, even thought **Perf** is not stricly speaking an algebraic stack (because it is not an n-stack for any n), it is an increasing union of open algebraic substacks. We say that **Perf** is *locally algebraic*. The fact that $\mathbf{Perf}^{[a,b]}$ is an algebraic $(n+1)$-stack is also not easy. We refer to [To-Va] for a complete proof.

Chapter 4

Simplicial commutative algebras

In this lecture we review the homotopy theory of simplicial commutative rings. It will be used all along the next lectures in order to define and study the notion of derived schemes and derived stacks.

4.1 Quick review of the model category of commutative simplicial algebras and modules

We let $sComm$ be the category of simplicial objects in $Comm$, that is of simplicial commutative algebras. For $A \in sComm$ a simplicial commutative algebra, we let $sA - Mod$ be the category of simplicial A-modules. Recall that an object in $sA - Mod$ is the data of a simplicial abelian group M_n, together with A_n-module structures on M_n in a way that the transition morphisms $M_n \longrightarrow M_m$ are morphisms of A_n-modules (for the A_n-module structure on M_m induced by the morphism $A_n \longrightarrow A_m$). We will say that a morphism in $sComm$ or in $sA - Mod$ is an equivalence (resp. a fibration), if it the morphism induced on the underlying simplicial sets is so. It is well known that this defines model category structures on $sComm$ and $sA - Mod$. These model category are cofibrantly generated, proper and cellular.

To any simplicial commutative algebras A let $\pi_*(A) := \oplus_n \pi_n(A)$. We do not specify base points as the underlying simplicial sets of a simplicial algebra is a simplicial abelian group, and thus its homotopy groups do not depend on the base point (by convention we will take 0 as base point). The graded abelian group $\pi_*(A)$ has a natural structure of a graded commutative (in the graded sense) algebra. The multiplication of two elements $a \in \pi_n(A)$ and $b \in \pi_m(A)$ is defined as follows. We represent a and b by morphisms of pointed simplicial sets

$$a : S^n := (S^1)^{\wedge n} \longrightarrow A \qquad b : S^m := (S^1)^{\wedge n} \longrightarrow A,$$

where S^1 is a model for the pointed simplicial circle. We then consider the induced morphism

$$a \otimes b : S^n \times S^m \longrightarrow A \times A \longrightarrow A \otimes A.$$

Composing with the multiplication in A we get a morphism of simplicial set

$$a.b : S^n \times S^m \longrightarrow A.$$

This last morphism sends $S^n \times *$ and $* \times S^m$ to the base point $0 \in A$. Therefore, it factorizes as a morphism

$$S^n \wedge S^m \simeq S^{n+m} \longrightarrow A.$$

As the left hand side has the homotopy type of S^{n+m} we obtain a morphism

$$S^{n+m} \longrightarrow A$$

which gives an element $ab \in \pi_{n+m}(A)$. This multiplication is associative, unital and graded commutative. In the same way, if A is a simplicial commutative algebra and M a simplicial A-module, $\pi_*(M) = \oplus_n \pi_n(M)$ has a natural structure of a graded $\pi_*(A)$-module.

For a morphism of simplicial commutative rings $f : A \longrightarrow B$ we have an adjunction

$$- \otimes_A B : sA - Mod \longrightarrow sB - Mod \qquad sA - Mod \longleftarrow sB - Mod : f^*,$$

where the right adjoint f^* is the forgetful functor. This adjunction is a Quillen adjunction which is moreover a Quillen equivalence if f is an equivalence of simplicial algebras. The left derived functor of $- \otimes_A B$ will be denoted by

$$- \otimes_A^{\mathbb{L}} B : Ho(sA - Mod) \longrightarrow Ho(sB - Mod).$$

Finally, a (non simplicial) commutative ring will always be considered as a constant simplicial commutative ring and thus as an object in $sComm$. This induces a fully faithful functor $Comm \longrightarrow sComm$ which induces a fully faithful embedding on the level of the homotopy category

$$Comm \longrightarrow Ho(sComm).$$

This last functor possesses a left adjoint

$$\pi_0 : Ho(sComm) \longrightarrow Comm.$$

In the same manner, if $A \in sComm$ any (non simplicial) $\pi_0(A)$-module can be considered as a constant simplicial A-module, and thus as an object in $sA - Mod$. This also defines a full embedding

$$\pi_0(A) - Mod \longrightarrow Ho(sA - Mod)$$

which admits a left adjoint

$$\pi_0 : Ho(sA - Mod) \longrightarrow \pi_0(A) - Mod.$$

4.2 Cotangent complexes

We start to recall tha notion of trivial square zero extension of a commutative ring by a module. For any commutative ring A and any A-module M we define another commutative ring $A \oplus M$. The underlying abelian group of $A \oplus M$ is the direct sum of A and M, and the multiplication is defined by the following formula

$$(a, m).(a', m') := (aa', am' + a'm).$$

The commutative ring $A \oplus M$ is called the *trivial square zero extension of A by M*. It is an augmented A-algebra by the natural morphisms

$$A \longrightarrow A \oplus M \longrightarrow A,$$

send respectively a to $(a, 0)$ and (a, m) to a. The main property of the ring $A \oplus M$ is that the set of sections (as morphisms or rings) of the projection $A \oplus M \longrightarrow A$ is in natural bijection with the set $Der(A, M)$ of derivations from A to M (this can be taken as a definition of $Der(A, M)$). A standard result states that the functor

$$A - Mod \longrightarrow Set$$

sending M to $Der(A, M)$ is corepresented by an A-module Ω^1_A, the A-module of Kahler differentials on A.

The generalization of the above notion to the context of simplicial commutative rings leads to the notion of cotangent complexes and André-Quillen homology (and cohomology). We let A be a simplicial commutative ring and $M \in sA - Mod$ be a simplicial A-module. By applying the construction of the trivial square zero extension levelwise for each A_n and each M_n we obtain a new simplicial commutative ring $A \oplus M$, together with two morphisms

$$A \longrightarrow A \oplus M \longrightarrow A.$$

The model category $sComm$ is a simplicial model category and we will denote by $\underline{Hom}$ its simplicial Hom sets, and by $\mathbb{R}\underline{Hom}$ its derived version (i.e. $\mathbb{R}\underline{Hom}(A, B) := \underline{Hom}(Q(A), B)$ where $Q(A)$ is a cofibrant model for A). The simplicial set $\mathbb{R}Der(A, M)$, of derived derivation from A to M is by definition the homotopy fiber of the natural morphism

$$\mathbb{R}\underline{Hom}(A, A \oplus M) \longrightarrow \mathbb{R}\underline{Hom}(A, A)$$

taken at the identity. In another way we have

$$\mathbb{R}Der(A, M) \simeq \mathbb{R}\underline{Hom}_{/A}(A, A \oplus M),$$

where now $\underline{Hom}_{/A}$ denotes the simplicial Hom sets of the model category $sComm/A$ of commutative simplicial rings over A. It is a well known result that the functor

$$\mathbb{R}Der(A, -) : Ho(sA - Mod) \longrightarrow Ho(SSet)$$

is corepresented by a simplicial A-module $\mathbb{L}_A$ called the *cotangent complex of A* (see for example [Q, Go-Ho]). One possible construction of $\mathbb{L}_A$ is has follows. We start by considering a cofibrant replacement $Q(A) \longrightarrow A$ for A. We then apply the construction of Kahler differentials levelwise for $Q(A)$ to get a simplicial $Q(A)$-module $\Omega^1_{Q(A)}$. We then set

$$\mathbb{L}_A := \Omega^1_{Q(A)} \otimes^{\mathbb{L}}_{Q(A)} A \in Ho(sA - Mod).$$

In this way, $A \mapsto \mathbb{L}_A$ is the left derived functor of $A \mapsto \Omega^1_A$. We note that by adjunction we always have

$$\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)}.$$

The cotangent complex is functorial in A. Therefore for any morphism of simplicial commutative rings $A \longrightarrow B$ we have a morphism $\mathbb{L}_A \longrightarrow \mathbb{L}_B$ in $Ho(sA-Mod)$. By adjunction this morphism can also be considered as a morphism in $Ho(sB - Mod)$

$$\mathbb{L}_A \otimes^{\mathbb{L}}_A B \longrightarrow \mathbb{L}_B.$$

The homotopy cofiber of this morphism will be denoted by $\mathbb{L}_{B/A}$, is called the relative cotangent complex of B over A.

An important fact concerning cotangent complexes is that it can be used in order to describe Postnikov invariants of commutative rings as follows. A simplicial commutative ring A is said to be n-truncated if $\pi_i(A) = 0$ for all $i > n$. The inclusion functor of the full subcategory $Ho(sComm_{\leq n})$ of n-truncated simplicial commutative rings has a left adjoint

$$\tau_{\leq n} : Ho(sComm) \longrightarrow Ho(sComm_{\leq n}),$$

called the n-th truncation functor. These functors can be easily obtained by applying the general mahcinery of left Bousfield localizations to $sComm$. They are the localization functors associated to the left Bousfield localizations of $sComm$ with respect to the morphism $S^{n+1} \otimes \mathbb{Z}[T] \longrightarrow \mathbb{Z}[T]$. For any $A \in Ho(sComm)$ we then have a Postnikov tower

$$A \longrightarrow \cdots \qquad \tau_{\leq n}(A) \longrightarrow \tau_{n-1}(A) \longrightarrow \cdots \longrightarrow \tau_{\leq 0}(A) = \pi_0(A).$$

It can be proved that for any $n > 0$ there is a homotopy pull-back square

$$\begin{tikzcd}
\tau_{\leq n}(A) \arrow[r] \arrow[d] & \tau_{\leq n-1}(A) \arrow[d, "0"] \\
\tau_{\leq n-1}(A) \arrow[r, "k_n"'] & \tau_{\leq n-1}(A) \oplus \pi_n(A)[n+1],
\end{tikzcd}$$

where $\pi_n(A)[n+1]$ is the simplicial A-module $S^{n+1} \otimes \pi_n(A)$ (i.e. the $(n+1)$-suspension of $\pi_n(A)$), 0 stands for the trivial derivation and k_n is a certain derived derivation with values in $\pi_n(A)[n+1]$. This derivation is an element in

$[\mathbb{L}_{\tau_{\leq n-1}(A)}, \pi_n(A)[n+1]]$ which is by definition the n-Postnikov invariant of A. This element completely determines the simplicial commutative ring $\tau_{\leq n}(A)$ from $\tau_{\leq n-1}(A)$ and the $\pi_0(A)$-module $\pi_n(A)$. It is non-zero precisely when the projection $\tau_{\leq n}(A) \longrightarrow \tau_{\leq n-1}(A)$ has no sections (in $Ho(sComm)$). It is zero precisely when $\tau_{\leq n}(A)$ is equivalent (as an object over $\tau_{\leq n-1}(A)$) to $\tau_{\leq n-1}(A) \oplus \pi_n(A)[n]$.

4.3 Flat, smooth and étale morphisms

We arrive at the three fundamental notions of flat, smooth and étale morphisms of commutative simplicial rings. The material of this paragraph is less standard that the one of the previous paragraph and thus refer to [HAGII, §2.2.2] for the details.

Definition 4.3.1. *Let $f : A \longrightarrow B$ be a morphism of simplicial commutative rings.*

1. *The morphism f is* homotopically of finite presentation *if for any filtered system of commutative simplicial A-algebras $\{C_\alpha\}$ the natural morphism*
$$Colim_\alpha \mathbb{R}\underline{Hom}_{A/sComm}(C_\alpha, B) \longrightarrow \mathbb{R}\underline{Hom}_{A/sComm}(Colim_\alpha C_\alpha, B)$$
is an equivalence.

2. *The morphism f* flat *is the base change functor*
$$- \otimes^{\mathbb{L}}_A B : Ho(sA - Mod) \longrightarrow Ho(sB - Mod)$$
commutes with homotopy pull-backs.

3. *The morphism f is* formally étale *if $\mathbb{L}_{B/A} \simeq 0$.*

4. *The morphism f is* formally smooth *if for any simplicial B-module M with $\pi_0(M) = 0$ we have $[\mathbb{L}_{B/A}, M] = 0$.*

5. *The morphism f is* smooth *if it is formally smooth and homotopically of finite presentation.*

6. *The morphism f is* étale *if it is formally étale and homotopically of finite presentation.*

7. *The morphism f is a* Zariski open immersion *if it is flat, homotopically of finite presentation and if moreover the natural morphism $B \otimes^{\mathbb{L}}_A B \longrightarrow B$ is an equivalence.*

All the above notions of morphisms are stable by composition in $Ho(sComm)$. They are also stable by homotopy cobase change in the sense that if a morphism $f : A \longrightarrow B$ is homotopically of finite presentation (resp. flat, resp. formally étale ...) , then for any $A \longrightarrow A'$ the induced morphism $A' \longrightarrow A' \otimes^{\mathbb{L}}_A B$ is again homotopically of finite presentation (resp. flat, resp. formally étale ...).

Here follows a sample of standard results concerning the above notions.

- A Zariski open immersion is étale, an étale morphism smooth and a smooth morphism is flat.

- A morphism $A \longrightarrow B$ is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) if and only if it satisfies the two following properties.

  1. The induced morphism of rings $\pi_0(A) \longrightarrow \pi_0(B)$ is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) in the usual sense.
  2. For all $i > 0$ the induced morphism
  $$\pi_i(A) \otimes_{\pi_0(A)} \pi_0(B) \longrightarrow \pi_i(B)$$
  is bijective.

- An important direct consequence of the last point is that a morphism of (non simplicial) commutative rings is flat (resp. étale, resp. smooth, resp. a Zariski open immersion) in the usual sense if and only if it is so in the sense of definition 4.3.1.

- A morphism $A \longrightarrow B$ is homotopically of finite presentation if and only if B is equivalent to a retract of a finite cellular A-algebra. Recall here that a finite cellular A-algebra is a commutative A-algebra B' such that there exists a finite sequence
$$A = B'_0 \longrightarrow B'_1 \ldots \longrightarrow B'_n = B',$$
such that for any i there exists a cocartesian square of commutativr simplicial rings
$$\begin{tikzcd} B'_i \arrow[r] & B'_{i+1} \\ \partial\Delta^{n_i} \otimes A \arrow[u] \arrow[r] & \Delta^{n_i} \otimes A. \arrow[u] \end{tikzcd}$$
In particular, if $A \longrightarrow B$ is a morphism of commutative rings, being homotopically finitely presented is a much stronger condition than being finitely presented in the usual sense. For instance, if B is a a commutative k-algebra of finite type (for k a field) which admits a singularity which is not a local complete intersection, then B is not homotopically finitely presented over k. Intuitively, B is homotopically finitely presented over A when it admits (up to a retract) a finite resolution by free A-algebras of finite type.

- For a given $A \in sComm$, there exists a functor
$$\pi_0 : Ho(A/sComm) \longrightarrow \pi_0(A)/Comm$$
from the homotopy category of commutative simplicial A-algebras to the category of commutative $\pi_0(A)$-algebras. This functor induces an equivalence on the full subcategories of étale morphisms Zariski open immersions. The corresponding fact for smooth and flat morphisms is not true.

Chapter 5

Derived stacks and derived algebraic stacks

We arrive at the notion of derived stacks and derived algebraic stacks. We first present the homotopy theory of derived stacks, which is very similar to the homotopy theory of simplicial presheaves presented in §2, but where the category $Comm$ must be replaced by the more complicated model category $sComm$. The new feature here is to incorporated correctly the model category structure of $sComm$ which makes the definitions a bit more technical.

5.1 Derived stacks

We set $dAff := sComm^{op}$, which by definition is the category of derived affine schemes. It is endowed with the dual model category structure of $sComm$. An object in $dAff$ corresponding to $A \in sComm$ will be denoted formally by $Spec\,A$. This $Spec$ has only a formal meaning, and we will define another, more interesting, Spec functor that will be denoted by $\mathbb{R}\underline{Spec}$.

We consider $SPr(dAff)$, the category of simplicial presheaves on $dAff$. We will define three different model category structures on $SPr(dAff)$, each one being a left Bousfield localization of the previous one. In order to avoid confusion we will use different notations for these three model categories (contrary to what we have done in the sections §2 and §3), even thought the underlying categories are identical. They will be denoted by $SPr(dAff)$, $dAff^{\wedge}$ and $dAff^{\sim}$.

The first model structure is the projective levelwise model category structure on $SPr(dAff)$, for which equivalences and fibrations are defined levelwise. We do not give any specific name to this model category. We consider the Yoneda embedding

$$\begin{array}{cccc} h: & dAff & \longrightarrow & SPr(dAff) \\ & X & \mapsto & h_X = Hom(-,X). \end{array}$$

Here h_X is a presheaf of sets and is consider as a simplicial presheaf constant in the simplicial direction. For any equivalence $X \longrightarrow Y$ in $dAff$ we deduce a morphism $h_X \longrightarrow h_Y$ in $SPr(dAff)$. By definition the model category $dAff^\wedge$ is the left Bousfield localization of the model category $SPr(dAff)$ with respect to the set of all morphisms $h_X \longrightarrow h_Y$ obtained from equivalences $X \longrightarrow Y$ in $dAff$. The model category $dAff^\wedge$ is called the *model category of prestacks over* $dAff$.

By definition, the fibrant objects in $dAff^\wedge$ are the simplicial presheaves $F : dAff^{op} \longrightarrow SSet$ satisfying the following two conditions.

1. For any $X \in dAff$ the simplicial set $F(X)$ is fibrant.
2. For any equivalence $X \longrightarrow Y$ in $dAff$, the induced morphism $F(Y) \longrightarrow F(X)$ is an equivalence of simplicial sets.

The first condition above is anodyne, but the second one is not. This second condition is called the *prestack* condition. This is the essential new features of derived stack theory compare with stack theory for which this condition did not appear (simply because there is no notion of equivalence conisdered in $Comm$ except the trivial one: the notion of isomorphism). The standard results about left Bousfield localizations imply that $Ho(dAff^\wedge)$ is naturally equivalent to the full subcategory of $Ho(SPr(dAff))$ consisting of all simplicial presheaves satisfying condition (2) above. We will implicitely identify these two categories. Moreover, the inclusion functor

$$Ho(dAff^\wedge) \longrightarrow Ho(SPr(dAff))$$

has a left adjoint which simply consists of sending a simplicial presheaf F to its fibrant model. We necessary this functor will be denoted by $F \mapsto F^\wedge$.

We come back to the Yoneda functor

$$h : dAff \longrightarrow SPr(dAff) = dAff^\wedge.$$

We compose it with the natural functor $dAff^\wedge \longrightarrow Ho(dAff^\wedge)$ and we obtain a functor

$$h : dAff \longrightarrow Ho(dAff^\wedge).$$

By construction this functor sends equivalences in $dAff$ to isomorphisms in $Ho(dAff^\wedge)$. Therefore it induces a well defined functor

$$Ho(h) : Ho(dAff) \longrightarrow Ho(dAff^\wedge).$$

A general result, called the Yoneda lemma for model categories (see [HAGI]), states two properties concerning $Ho(h)$.

1. The functor $Ho(h)$ is fully faithful. This is the mdoel category version of the Yoneda lemma for categories.

2. For $X \in dAff$, the object $Ho(h)(X) \in Ho(SPr(dAff))$ can be described as follows. We take RX a fibrant model for X in $dAff$ (i.e. if $X = Spec\,A$, $RX = Spec\,, Q(A)$ for $Q(A)$ a cofibrant model for A in $sComm$). We consider the simplicial presheaf $\underline{h}_{RX}$ $Y \mapsto \underline{Hom}(Y, RX)$ where $\underline{Hom}$ denotes the simplicial Hom sets of $dAff$. When $X = Spec\,A$, this simplicial presheaf is also $Spec\,B \mapsto \underline{Hom}(Q(A), B)$. Then, the simplicial presheaf $Ho(h)(X)$ is equivalent to $\underline{h}_{RX}$. When $X = Spec\,A$ we will also use the following notation

$$\mathbb{R}\underline{Spec}\,A := Ho(h)(X) \simeq \underline{h}_{RX} \simeq \underline{Hom}(Q(A), -).$$

In an equivalent way the Yoneda lemma in this setting states that the functor $A \mapsto \underline{Hom}(Q(A), -)$ induces a fully faithful functor

$$Ho(Comm)^{op} \longrightarrow Ho(dAff^{\wedge}) \subset Ho(SPr(dAff)).$$

We now introduce the notion of local equivalences for morphisms in $dAff^{\wedge}$ which will be our equivalences for the final model structure. For this we endow the category $Ho(dAff)$ with a Grothendieck topology as follows. We say that a family of morphisms $\{A \longrightarrow A_i\}_i$ is an *étale covering* if each of the morphisms $A \longrightarrow A_i$ is étale in the sense of definition 4.3.1 and if the family of functors

$$\{- \otimes^{\mathbb{L}}_A A_i : Ho(sA - Mod) \longrightarrow Ho(sA_i - Mod)\}_i$$

is conservative. By definition, the étale topology on $Ho(dAff)$ is the topology for which covering sieves are the one generated by étale covering families. In the same way, for any fixed $X \in dAff$ we define an étale topology on $Ho(dAff/X)$.

The étale topology on $Ho(dAff)$ can be used in order to define homotopy sheaves for objects $F \in Ho(dAff^{\wedge})$. We start to define the homotopy presheaves as follows. Let $F : dAff^{op} \longrightarrow SSet$ be an object in $Ho(dAff^{\wedge})$, so in particular we assume that F sends equivalences in $dAff$ to equivalences in $SSet$. We consider the presheaf of sets $X \mapsto \pi_0(F(X))$. This presheaf sends equivalences in $dAff$ to isomorphisms in Set and thus factorizes as a functor $\pi_0^{pr}(F) : Ho(dAff)^{op} \longrightarrow Set$. In the same way, for $X \in dAff$ et $s \in F(X)$, we define a presheaf of groups on $dAff/X$ wich sends $f : Y \longrightarrow X$ to $pi_i(F(Y), f^*(s))$. Again this presheaf sends equivalences to isomorphisms and thus induces a functor $pi_i^{pr}(F, s) : Ho(dAff/X)^{op} \longrightarrow Set$. With these notations, the associated sheaves (for the étale topology defined above) to $\pi_0^{pr}(F)$ and $\pi_i^{pr}(F, s)$ are denoted by $\pi_0(F)$ and $\pi_i(F, s)$ and are called the *homotopy sheaves of* F. These are defined for $F : dAff^{op} \longrightarrow SSet$ sending equivalences to equivalences. Now, for a general simplicial presheaf we set

$$\pi_0(F) := \pi_0(F^{\wedge}) \qquad \pi_i(F, s) := \pi_i(F^{\wedge}, s)$$

where $F^{\wedge}$ is a fibrant model for F in $dAff^{\wedge}$.

Definition 5.1.1. *Let $f : F \longrightarrow F'$ be a morphism of simplicial presheaves on $dAff$.*

1. *The morphism f is a* local equivalence *if it satisfies the following two conditions*

   (a) *The induced morphism $\pi_0(F) \longrightarrow \pi_0(F')$ is an isomorphism of sheaves on $Ho(dAff)$.*

   (b) *For any $X \in dAff$, any $s \in F(X)_0$ and any $i > 0$ the induced morphism $\pi_i(F, s) \longrightarrow \pi_i(F', f(s))$ is an isomorphism of sheaves on $dAff/X$.*

2. *The morphism f is a* local cofibration *if it is a cofibration in $dAff^\wedge$ (or equivalently in $SPr(dAff)$).*

3. *The morphism f is a* local fibration *if it has the left lifting property with respect to every local cofibration which is also a local equivalence.*

For simplicity, we will use the expressions equivalence, fibrations *and* cofibration *in order to refer to* local equivalence, local fibration *and* local cofibration.

It can be proved (see [HAGI]) that these notions of equivalences, fibrations and cofibrations define a model category structure on $SPr(dAff)$. This model category will be denoted by $dAff^\sim$. As for the case of simplicial presheaves it is possible to characterize fibrant objects in $dAff^\sim$ as functors $F : dAff^{op} \longrightarrow SSet$ satisfying the following three conditions (we de not precise the definition of étale hypercovering in this context, it is very similar to the one we gave for simplicial presheaves in §2.1).

1. For any $X \in dAff$ the simplicial set $F(X)$ is fibrant.

2. For any equivalence $X \longrightarrow Y$ in $dAff$ the induced morphism $F(Y) \longrightarrow F(X)$ is an equivalence of simplicial sets.

3. For any $X \in dAff$ and any étale hypercovering $H \longrightarrow X$ the natural morphism
$$F(X) \longrightarrow Holim_{[n]\in\Delta} F(H_n)$$
is an equivalence of simplicial sets.

Definition 5.1.2. 1. *An object $F \in SPr(dAff)$ is called a* derived stack *if it satisfies the conditions* (2) *and* (3) *above.*

2. *The homotopy category $Ho(dAff^\sim)$ will be called the* homotopy category of derived stacks. *Most often objects in $Ho(dAff^\sim)$ will simply be called* derived stacks. *The expressions* morphism of derived stacks *and* isomorphism of derived stacks, *will refer to morphisms and isomorphisms in $Ho(dAff^\sim)$. The set of morphisms of derived stacks from F to F' will be denoted by $[F, F']$.*

To finish this first paragraph we mention how stacks and derived stacks are compared. For this we consider the functor $Comm \longrightarrow sComm$ which consists of considering a commutative ring as a constant simplicial commutative ring. This induces a functor $i : Aff \longrightarrow dAff$. Pulling back along this functor induces a functor

$$i^* : dAff^{\sim} \longrightarrow SPr(Aff).$$

This functor is seen to be right Quillen whose left adjoint is denoted by

$$i_! : SPr(Aff) \longrightarrow dAff^{\sim}.$$

The derived Quillen adjunction is denoted by

$$j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim}) \qquad Ho(SPr(Aff)) \longleftarrow Ho(dAff^{\sim}) : h^0.$$

The functor j is fully faithful, as this follows from the fact that the functor $Comm \longrightarrow Ho(sComm)$ is fully faithful and compatible with the étale topologies on both sides. Therefore, any stack can be considered as a derived stacks. The functor h^0 is called the *classical part functor*, and remembers only the part related to non simplicial commutative rings of a given derived stack. Using the functor j we will see any stack as a derived stack.

Definition 5.1.3. *Given a stack $F \in Ho(SPr(Aff))$, a* derived extension of F *is the data of a derived stack $\widetilde{F} \in Ho(dAff^{\sim})$ together with an isomorphism of stacks $F \simeq h^0(\widetilde{F})$.*

The existence of the full embedding j implies that any stack admits a derived extension $j(F)$, but this extension is somehow the trivial one. The striking fact about derived algebraic geometry is that most (if not all) of the moduli problems admits natural derived extensions, and these are not the trivial one in general. We will see many such examples in the next lecture.

5.2 Algebraic derived n-stacks

We now mimick the definitions of schemes and algebraic stacks given in §3 for our new context of derived stacks.

We start by considering the Yoneda embedding

$$Ho(dAff) \longrightarrow Ho(dAff^{\wedge}).$$

The fathfully flat descent stays true in the derived setting and this embedding induces a fully faithful functor

$$Ho(dAff) \longrightarrow Ho(dAff^{\sim}).$$

Equivalently, this means that for any $A \in sComm$, the prestack $\mathbb{R}\underline{Spec}\, A$, sending B to $\underline{Hom}(Q(A), B)$, satisties the descent condition for étale hypercoverings (i.e. is a derived stack). Objects in the essential image of this functor will be called *derived affine schemes*, and the full subcategory of $Ho(dAff^{\sim})$ consisting of derived affine schemes will be implicitly identified with $Ho(dAff)$.

One of the major difference between stacks and derived stacks is that derived affine schemes are not 0-truncated. The definition of Zariski open immersion given in definition 3.1.1 has therefore to be slightly modified.

Definition 5.2.1. *1. A morphism of derived stacks* $F \longrightarrow F'$ *is a* monomorphism *if the induced morphism* $F \longrightarrow F \times^h_{F'} F$ *is an equivalence.*

2. *A morphism of derived stacks* $F \longrightarrow F'$ *is an* epimorphism *if the induced morphism* $\pi_0(F) \longrightarrow \pi_0(F')$ *is an epimorphism of sheaves.*

3. *Let* $X = \mathbb{R}\underline{Spec}\, A$ *be a derived affine scheme,* F *a derived stack and* $i : F \longrightarrow \mathbb{R}\underline{Spec}\, A$ *a morphism. We say that* i *is a* Zariski open immersion *(or simply an* open immersion*) if it satisfies the following two conditions.*

   (a) *The morphism* i *is a monomorphism.*

   (b) *There exists a family of Zariski open immersions* $\{A \longrightarrow A_i\}_i$ *such that the morphism* $\mathbb{R}\underline{Spec}\, A_i \longrightarrow \mathbb{R}\underline{Spec}\, A$ *all factor throught* F *in a way that the resulting morphism*

   $$\coprod_i \mathbb{R}\underline{Spec}\, A_i \longrightarrow \mathbb{R}\underline{Spec}\, A$$

   is an epimorphism.

4. *A morphism of derived stacks* $F \longrightarrow F'$ *is a* Zariski open immersion *(or simply an* open immersion*) if for any derived affine scheme* X *and any morphism* $X \longrightarrow F'$*, the induced morphism*

   $$F \times^h_{F'} X \longrightarrow X$$

   is a Zariski open immersion in the sense above.

5. *A derived stack* F *is a* derived scheme *if there exists a family of derived affine schemes* $\{\mathbb{R}\underline{Spec}\, A_i\}_i$ *and Zariski open immersions* $\mathbb{R}\underline{Spec}\, A_i \longrightarrow F$*, such that the induced morphism of sheaves*

   $$\coprod_i \mathbb{R}\underline{Spec}\, A_i \longrightarrow F$$

   is an epimorphism. Such a family of morphisms $\{\mathbb{R}\underline{Spec}\, A_i \longrightarrow F\}$ *will be called a* Zariski altas for F.

We say that a morphism of derived schemes $X \longrightarrow Y$ is smooth (resp. flat, resp. étale) if there exists Zariski atlas $\{\mathbb{R}\underline{Spec}\,A_i \longrightarrow X\}$ and $\{\mathbb{R}\underline{Spec}\,A_j \longrightarrow Y\}$ together with commutative squares (in $Ho(dAff^{\sim})$)

$$\begin{tikzcd} X \arrow[r] & Y \\ \mathbb{R}\underline{Spec}\,A_i \arrow[u] \arrow[r] & \mathbb{R}\underline{Spec}\,A_j, \arrow[u] \end{tikzcd}$$

with $\mathbb{R}\underline{Spec}\,A_i \longrightarrow \mathbb{R}\underline{Spec}\,A_j$ a smooth (resp. flat, resp. étale) morphism (here j depends on i). Smooth morphisms of derived schemes are stable by compositions and homotopy base change.

The following is the main definition of this series of lectures.

Definition 5.2.2. *1. A derived stack F is* 0-algebraic *if it is a derived scheme.*

2. *A morphism of derived stacks $F \longrightarrow F'$ is* 0-algebraic *(or* 0-representable*) if for any derived scheme X and any morphism $X \longrightarrow F'$ the derived stack $F \times^h_{F'} X$ is 0-algebraic (i.e. a derived scheme).*

3. *A 0-algebraic morphism of derived stacks $F \longrightarrow F'$ is smooth if for any derived scheme X and any morphism $X \longrightarrow F'$ the morphism of derived schemes $F \times^h_{F'} X \longrightarrow X$ is smooth.*

4. *We now let $n > 0$, and we assume that the notions of $(n-1)$-algebraic derived stack, $(n-1)$-algebraic morphisms and smooth $(n-1)$-algebraic morphisms have been defined.*

   (a) A derived stack F is n-algebraic *if there exists a derived scheme X together with smooth $(n-1)$-algebraic morphisms $X \longrightarrow F$ which is an epimorphism. Such a morphism $X \longrightarrow F$ is called a* smooth n-atlas for F.

   (b) A morphism of derived stacks $F \longrightarrow F'$ is n-algebraic *(or* n-representable*) if for any derived scheme X and any morphism $X \longrightarrow F'$ the derived stack $F \times_{F'} X$ is n-algebraic.*

   (c) An n-algebraic morphism of derived stacks $F \longrightarrow F'$ is smooth *(resp.* flat*, resp.* étale*) if for any derived scheme X and any morphism $X \longrightarrow F'$ there exists a smooth n-atlas $Y \longrightarrow F \times^h_{F'} X$ such that each morphism $Y \longrightarrow X$ is a smooth (resp. flat, resp. étale) morphism of derived schemes.*

5. *An* derived algebraic stack *is a derived stack which is n-algebraic for some integer n.*

6. *A morphism of derived stacks $F \longrightarrow F'$ is* algebraic *(or* representable*) if it is n-algebraic for some n.*

7. *A morphism of derived stacks $F \longrightarrow F'$ is* smooth *(resp.* flat*, resp.* étale*) if it is n-algebraic and smooth (resp. flat, resp. étale) for some integer n.*

We finish this part by some basic properties of derived algebraic stacks, and in particular with a comparison between the notions of algebraic stacks and derived algebraic stacks.

- Derived algebraic stacks are stable by finite homotopy limits (i.e. homotopy pull-backs).
- Derived algebraic stacks are stable by disjoint union.
- Algebraic morphisms of derived stacks are stable by composition and homotopy base change.
- Derived algebraic stacks are stable by smooth quotients. To be more precise, if $F \longrightarrow F'$ is a smooth epimorphism of derived stacks, then F' is algebraic if and only if F is so.
- A (non derived) stack F is algebraic if and only if the derived stack $j(F)$ is algebraic.
- If F is an algebraic derived stack then the stack $h^0(F)$ is an algebraic stack. When $h^0(F)$ is an algebraic n-stack we say that F is a derived algebraic n-stack (thought it is not n-truncated as a simplicial presheaf on $dAff$).
- A derived algebraic space is a derived algebraic stack F such that $h^0(F)$ is an algebraic space. In other words a derived algebraic space is a derived algebraic 0-stack
- If F is an algebraic derived n-stack, and A is an m-truncated commutative simplicial ring then $F(A)$ is an $(n+m)$-truncated simplicial set.
- If $f : F \longrightarrow F'$ is a flat morphism of derived algebraic stack, and if F' is an algebraic stack (i.e. of the form $j(F'')$ for an algebraic stack F''), then F is itself an algebraic stack.

We see that the formal properties of derived algebraic stacks are the same as the formal properties of non derived algebraic stacks. However, we would like to make the important comment here that the inclusion functor $j : Ho(SPr(Aff)) \hookrightarrow Ho(dAff^{\sim})$ from stacks to derived stacks does not commute with homotopy pull-backs. In other words, if $F \longleftarrow H \longrightarrow G$ is a diagram of stacks then the natural morphism

$$j(F \times^h_H G) \longrightarrow j(F) \times^h_{j(H)} j(G)$$

is not an isomorphism in general. As this morphism induces an isomorphism on h^0, this is an example of a non trivial derived extension of a stack as a derived stack. Each time a stack is presented as a certain finite homotopy limit of other stacks it has a natural, and in general non trivial, derived extension by considering the same homotopy limit in the bigger category of derived stacks.

5.3 Cotangent complexes

To finish this lecture we now explain the notion of cotangent complexes of a derived stack at given point. We let F be an algebraic derived stack and $X = Spec\, A$ be a (non derived) affine scheme. We fix a point (i.e. a morphism of stacks)

$$x : X \longrightarrow F.$$

We let $D^{\leq 0}(A)$ be the non positive derived category of cochain complexes of A-modules. By the Dold-Kan correspondence we will also identify $D^{\leq 0}(A)$ with $Ho(sA{-}Mod)$ the homotopy category of simplicial A-modules. We define a functor

$$\mathbb{D}er_x(F, -) : D^{\leq 0}(A) \longrightarrow Ho(SSet)$$

by the following way. For $M \in D^{\leq 0}(A)$ we form $A \oplus M$, which is now a commutative simplicial ring (here we consider M as a simplicial A-module), and we set $X[M] := \mathbb{R}\underline{Spec}\, A \oplus M$. The natural projection $A \oplus M$ induces a morphism of derived schemes $X \longrightarrow X[M]$. By definition, the simplicial set $\mathbb{D}er_x(F, M)$ is the homotopy fiber of the natural morphism

$$F(X[M]) \longrightarrow F(X)$$

taken at the point x (here we use the Yoneda lemma stating that $\pi_0(F(X)) \simeq [X, F]$). The simplicial set $\mathbb{D}er_x(F, M)$ is called the simplicial set of derivations of F at the point x with coefficients in M. This is functorial in M and thus defines a functor

$$\mathbb{D}er_x(F, -) : D^{\leq 0}(A) \longrightarrow Ho(SSet).$$

It can be proved that this functor is corepresentable by a complex of A-modules. More precisely, there exists a complex of A-modules (a priori not concentrated in non positive degrees anymore) $\mathbb{L}_{F,x}$, called the cotangent complex of F at x, and such that there exist natural isomorphisms in $Ho(SSet)$

$$\mathbb{D}er_x(F, M) \simeq Map(\mathbb{L}_{F,x}, M),$$

where Map are the mapping spaces of the model category of (unbounded) complexes of A-modules. When the derived stack F is affine this is a reformulation of the existence of a cotangent complex as recalled in §4. In general one reduces the statement to the affine case by a long a tedious induction (on n proving the result for algebraic derived n-stacks). Finally, with a bit of care we can show that $\mathbb{L}_{F,x}$ is unique and functorial (but this requires to state a rafined universal property, see [HAGII]).

Definition 5.3.1. *With the notation above the complex* $\mathbb{L}_{F,x}$ *is called the* cotangent complex of F at the point x. *Its dual* $\mathbb{T}_{F,x} := \mathbb{R}\underline{Hom}(\mathbb{L}_{F,x}, A)$ *is called the* tangent complex of F at x. *The cohomology groups*

$$T^i_{F,x} := H^i(\mathbb{T}_{F,x})$$

are called the higher tangent spaces of F at x.

For a derived algebraic n-stack F and $x : X = Spec\, A \longrightarrow F$ a point, the cotangent complex $\mathbb{L}_{F,x}$ belongs to $D^{\leq n}(A)$ the derived category of complexes concentrated in degrees $]-\infty, n]$. The part of $\mathbb{L}_{F,x}$ concentrated in negative degree is the one related to the *derived part of* F (i.e. the one making the difference between commutative rings and commutative simplicial rings), and the non negative part is related to the *stacky part* of F (i.e. the part related to the higher homotopy sheaves of $h^0(F)$). For instance, when F is a derived scheme its stacky part is trivial and thus $\mathbb{L}_{F,x}$ belongs to $D^{\leq 0}(A)$. On the other hand, when F is a smooth algebraic n-stack (say over $Spec\, \mathbb{Z}$ to simplify), then $\mathbb{L}_{F,x}$ is concentrated in degrees $[0, n]$ (even more is true, it is of Tor-amplitude concentrated in degrees $[0, n]$).

We will in the next lecture several examples of derived stacks which will show that the tangent complexes contain interesting cohomological information. Also, tangent complexes are very useful to provide smoothness and étaleness criterion, which are in general easy to check in practice. For this reason, proving smoothness is in general much more easy in the context of derived algebraic geometry than in the usual context of algebraic geometry. Here is for instance a smoothness criterion (see [HAGII] for details).

Let $f : F \longrightarrow F'$ be a morphism of algebraic derived stack. For any affine scheme $X := Spec\, A$ and any point $x : X \longrightarrow F$ we consider the homotopy fiber of the natural morphism

$$\mathbb{L}_{F,x} \longrightarrow \mathbb{L}_{F',x},$$

which is called the relative cotangent complex of f at x and is denoted by $\mathbb{L}_{f,x}$. We assume that f is locally homotopically of finite presentation (i.e. F and F' admits atlases compatible with f so that the induced morphism on the atlases is homotopically of finite presentation). Then f is smooth if and only if for any affine scheme $X = Spec\, A$ and any $x : X \longrightarrow F$, the complex $\mathbb{L}_{f,x}$ is of non negative Tor amplitude (i.e. for all $M \in D^{\leq -1}(A)$ we have $[\mathbb{L}_{f,x}, M] = 0$).

Chapter 6

Two examples of derived algebraic stacks

In this last lecture we present two examples of derived algebraic stacks.

6.1 The derived moduli space of local systems

We come back to the example we presented in the first lecture, the moduli problem of linear representations of a discrete group. We will now reconsider it from the point of view of derived algebraic geometry. We will try to treat this example with some details as we think it is a rather simple, but interesting, example of a derived algebraic stack.

A linear representation of group G can also be interpreted as a local system on the space BG. We will therefore study the moduli problem from this topological point of view. We fix a finite CW complex X and we are going to define a derived stack $\mathbb{R}Loc(X)$, classifying local systems on X. We will see that this stack is an algebraic derived 1-stack and we will describe its higher tangent spaces in termes of cohomology groups of X. When $X = BG$ for a discrete group G the derived algebraic stack $\mathbb{R}Loc(X)$ is the *correct moduli space* of linear representations of G.

We start to consider the non derived algebraic 1-stack **Vect** classifying projective modules of finite type. By definition, **Vect** sends a commutative ring A to the nerve of the groupoid of projective A-modules of finite type. The stack **Vect** is a 1-stack. It is easy to see that **Vect** is an algebraic 1-stack. Indeed, we have a decomposition

$$\mathbf{Vect} \simeq \coprod_n \mathbf{Vect}_n,$$

where $\mathbf{Vect}_n \subset \mathbf{Vect}$ is the substack of projective modules of rank n (recall that a projective A-module of finite type M is of rank n is for any field K and any morphism $A \longrightarrow K$ the K-vector space $M \otimes_A K$ is of dimension n). It is therefore enough to prove that $\mathbf{Vect}_n$ is an algebraic 1-stack. This last statement will itsefl follow from the identification

$$\mathbf{Vect}_n \simeq [*/Gl_n] = BGl_n,$$

where Gl_n is the affine group scheme sending A to $Gl_n(A)$. In order to prove that $\mathbf{Vect}_n \simeq BGl_n$ we construct a morphism of simplicial presheaves

$$BGl_n \longrightarrow \mathbf{Vect}_n$$

by sending the base point of BGl_n to the trivial projective module of rank n: for a given commutative ring A the morphism

$$BGl_n(A) \longrightarrow \mathbf{Vect}_n(A)$$

sends the base point to A^n and identifies $Gl_n(A)$ with the automorphism group of A^n. The claim is that the morphism $BGl_n \longrightarrow \mathbf{Vect}_n$ is a local equivalence of simplicial presheaves. As by construction this morphism induces isomorphisms on all higher homotopy sheaves it only remains to show that it induces an isomorphism on the sheaves π_0. But this in turn follows from the fact that $\pi_0(\mathbf{Vect}_n) \simeq *$, because any projective A-module of finite type is locally free for the Zariski topology on $Spec\, A$.

The algebraic stack **Vect** is now considered as an algebraic derived stack using the inclusion functor $j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim})$. We consider $F \in dAff^{\sim}$, a fibrant model for $j(\mathbf{Vect})$, and we define a new simplicial presheaf

$$\mathbb{R}Loc(X) : dAff^{op} \longrightarrow SSet$$

which sends $A \in sComm$ to $Map(X, |F(A)|)$ the simplicial set of continuous maps from X to $|F(A)|$.

Definition 6.1.1. *The derived stack* $\mathbb{R}Loc(X)$ *defined above is called the* derived moduli stack of local systems on X*.*

We will now describe some basic properties of the derived stack $\mathbb{R}Loc(X)$. We start by a description of its classical part $h^0(\mathbb{R}Loc(X))$, which will show that it does classify local systems on X. We will then show that $\mathbb{R}Loc(X)$ is an algebraic derived stack locally of finite presentation over $Spec\,\mathbb{Z}$, and that it can be written as

$$\mathbb{R}Loc(X) \simeq \coprod_n \mathbb{R}Loc_n(X)$$

where $\mathbb{R}Loc_n(X)$ is the part classifiying local systems of rank n and is itself strongly of finite type. Finally, we will compute its tangent spaces in terms of the cohomology of X.

For $A \in Comm$, $h^0(\mathbb{R}Loc(X))(A)$ is by definition the simplicial set $Map(X, |F(A)|)$. Now, $F(A)$ is a fibrant model for $j(\mathbf{Vect})(A) \simeq \mathbf{Vect}(A)$, so is equivalent to the nerve of the groupoid of projective A-modules of finite rank. The simplicial set $Map(X, |F(A)|)$ is then naturally equivalent to the nerve of the groupoid of functors $Fun(\Pi_1(X), F(A))$, from the fundamental groupoid of X to $F(A)$. This last groupoid is in turn equivalent to the groupoid of local systems of projective A-modules of finite type on the space X. Thus, we see that $h^0(\mathbb{R}Loc(X))(A)$ is naturally equivalent to the nerve of the groupoid of local systems of projective A-modules of finite type on the space X. We thus have the following properties.

1. The set $\pi_0(h^0(\mathbb{R}Loc(X))(A))$ is functorially in bijection with the set of isomorphism classes of local systems of projective A-modules of finite type on X. In particular, when A is a field this is also the set of local systems of finite dimensional vector spaces over X.

2. For a local system $E \in \pi_0(h^0(\mathbb{R}Loc(X))(A))$ we have
$$\pi_1(h^0(\mathbb{R}Loc(X))(A), E) = Aut(E),$$
the automorphism group of E as a sheaf of A-modules on X.

3. For all $i > 1$ and all $E \in \pi_0(h^0(\mathbb{R}Loc(X))(A))$ we have $\pi_i(h^0(\mathbb{R}Loc(X))(A), E) = 0$.

Let us explain now why the derived stack $\mathbb{R}Loc(X)$ is algebraic. We start by the trivial case where X is a contractible space. Then by definition we have $\mathbb{R}Loc(X) \simeq \mathbb{R}Loc(*) \simeq j(\mathbf{Vect})$. As we already know that $j(\mathbf{Vect})$ is an algebraic stack this implies that $\mathbb{R}Loc(X)$ is an algebraic derived stack when X is contractible.

The next step is to prove that $\mathbb{R}Loc(S^n)$ is algebraic for any $n \geq 0$. This can be seen by induction on n. The case $n = 0$ is obvious. Moreover, for any $n > 0$ we have a homotopy push-out of topological spaces

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & D^n \arrow[d] \\ D^n \arrow[r] & S^n, \end{tikzcd}$$

where D^n is the n-dimensional ball. This implies the existence of a homotopy pull-back diagram of derived stack

$$\begin{tikzcd} \mathbb{R}Loc(S^n) \arrow[r] \arrow[d] & \mathbb{R}Loc(D^n) \arrow[d] \\ \mathbb{R}Loc(D^n) \arrow[r] & \mathbb{R}Loc(S^{n-1}). \end{tikzcd}$$

By induction on n and by what have just seen before the derived stacks $\mathbb{R}Loc(D^n)$ and $\mathbb{R}Loc(S^{n-1})$ are algebraic. By the stability of algebraic derived stacks by homotopy pull-backs we deduce that $\mathbb{R}Loc(S^n)$ is an algebraic derived stack.

We are now ready to show that $\mathbb{R}Loc(X)$ is algebraic. We write X_k the k-th skeleton of X. As X is a finite CW complex there is an n such that $X = X_n$. Moreover, for any k there exists a homotopy push-out diagram of topological spaces

$$\begin{tikzcd} X_{k-1} \arrow[r] \arrow[d] & X_k \arrow[d] \\ \coprod S^{k-1} \arrow[r] & \coprod D^k, \end{tikzcd}$$

where the disjoint unions are finite. This implies that we have a homotopy pull-back square of derived stacks

$$\begin{tikzcd} \mathbb{R}Loc(X_k) \arrow[r] \arrow[d] & \mathbb{R}Loc(X_{k-1}) \arrow[d] \\ \prod^h \mathbb{R}Loc(S^{k-1}) \arrow[r] & \prod^h \mathbb{R}Loc(S^k). \end{tikzcd}$$

By the stability of algebraic derived stacks by finite homotopy limits we deduce that $\mathbb{R}Loc(X_k)$ is algebraic by induction on k (the case $k = 0$ being clear as $\mathbb{R}Loc(X_0)$ is a finite product of $\mathbb{R}Loc(*)$).

To finish the study of this example we will compute the higher tangent spaces of the derived stack $\mathbb{R}Loc(X)$. We let A be a commutative algebra and we consider the natural morphism

$$\mathbb{R}Loc(*)(A \oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A).$$

This morphism This morphism has a natural section and its homotopy fiber at an A-module E is equivalent to $K(End(E), i+1)$. It is therefore naturally equivalent to

$$[K(End(-), i+1)/Vect(A)] \longrightarrow N(Vect(A)),$$

where $Vect(A)$ is the groupoid of projective A-modules of finite type, $N(Vect(A))$ is its nerve, and $[K(End(-), i+1)/Vect(A)]$ is the homotopy colimit of the simplicial presheaf $Vect(A) \longrightarrow SSet$ sending E to $K(End(E), i+1)$ (this is a general fact, for any simplicial presheaf $F : I \longrightarrow SSet$ we have a natural morphism $Hocolim_I F \longrightarrow N(I) \simeq Hocolim_I *$). We consider the geometric realization of this morphism to get a morphism of topological spaces

$$|[K(End(-), i+1)/Vect(A)]| \longrightarrow |N(Vect(A))|,$$

which is equivalent to the geometric realization of

$$\mathbb{R}Loc(*)(A \oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A).$$

We take the image of this morphism by $Map(X,-)$ to get

$$\mathbb{R}Loc(X)(A\oplus A[i])\simeq Map(X,\mathbb{R}Loc(*)(A\oplus A[i]))\longrightarrow \mathbb{R}Loc(X)(A)\simeq Map(X,\mathbb{R}Loc(*)(A)).$$

This implies that the morphism

$$\mathbb{R}Loc(*)(A\oplus A[i]) \longrightarrow \mathbb{R}Loc(*)(A)$$

is equivalent to the morphism

$$Map(X,|[K(End(-),i+1)/Vect(A)]|) \longrightarrow Map(X,|N(Vect(A))|).$$

A morphism $X \longrightarrow |N(Vect(A))|$ correspond to a local system E of projective A-module of finite type on X. The homotopy fiber of the above morphism at E is then equivalent to the simplicial set of homotopy lifts of $X \longrightarrow |N(Vect(A))|$ to a morphism $X \longrightarrow |[K(End(-),i+1)/Vect(A)]|$. This simplicial set is in turn naturally equivalent to $DK(C^*(X,End(E))[i+1])$, the simplicial set obtained from the complex $C^*(X,End(E))[i+1]$ by the Dold-Kan construction. Here $C^*(X,End(E))$ denotes the complex of cohomology of X with coefficients in the local system $End(E)$. We therefore have the following formula for the higher tangent complexes

$$T^i_E\mathbb{R}Loc(X) \simeq H^0(C^*(X,End(E))[i+1]) \simeq H^{i+1}(X,End(E)).$$

More generally, it is possible to prove that there is an isomorphism in $D(A)$

$$\mathbb{T}_E\mathbb{R}Loc(X) \simeq C^*(X,End(E))[1].$$

6.2 The derived moduli of maps

As for non derived stacks, the homotopy category of derived stacks $Ho(dAff^{\sim})$ is cartesian closed. The corresponding internal Homs will be denoted by $\mathbb{R}\underline{\mathcal{H}om}$. Note that even thought we use the same notations for the internal Homs of stacks and derived stacks the inclusion functor

$$j : Ho(SPr(Aff)) \longrightarrow Ho(dAff^{\sim})$$

does not commute avec them. However, we always have

$$h^0(\mathbb{R}\underline{\mathcal{H}om}(F,F')) \simeq \mathbb{R}\underline{\mathcal{H}om}(h^0(F),h^0(F'))$$

for any derived stacks F and F'. The situation is therefore very similar to the case of homotopy pull-backs.

We have just seen an example of a derived stack constructed as an internal Hom between two stacks. Indeed, if we use again the notations of the last example we have

$$\mathbb{R}Loc(X) \simeq \mathbb{R}\underline{\mathcal{H}om}(K,\mathbf{Vect}),$$

where $K := S_*(X)$ is the singular simplicial set of X.

We now consider another example. Let X and Y be two schemes, and assume that X is flat and proper (say over $Spec\, k$ for some base ring k), and that Y is smooth over k. It is possible to prove that the derived stack $\mathbb{R}\underline{\mathcal{H}om}_{dAff/Speck}(X,Y)$ is a derived scheme which is homotopically finitely presented over $Spec\, k$. We will not sketch the argument here which is out of the scope of these lectures, and we refer to [HAGII] for more details. The derived scheme $\mathbb{R}\underline{\mathcal{H}om}(X,Y)$ is called the derived moduli space of maps from X to Y. Its classical part $h^0(\mathbb{R}\underline{\mathcal{H}om}(X,Y))$ is the usual moduli scheme of maps from X to Y, and for such a map we have

$$\mathbb{T}_f\mathbb{R}\underline{\mathcal{H}om}_{dAff/Spec\, k}(X,Y) \simeq C^*(X, f^*(\mathbb{T}_Y),$$

(where all these tangent complexes are relative to $Spec\, k$).

We mention here that these derived mapping space of maps can also be used in order to construct the so-called derived moduli of stable maps to an algebraic variety, by letting X varies in the moduli space of stable curves. We refer to [To] for more details about this construction, and for some explanations of how Gromov-Witten theory can be extracted from this derived stack of stable maps.

Bibliography

[An] M. Anel, *Moduli of linear and abelian categories*, preprint math.AG/0607385.

[Bl] B. Blander, *Local projective model structure on simpicial presheaves*, K-theory **24** (2001) No. 3, 283-301.

[Du1] D. Dugger *Universal homotopy theories*, Adv. in Math. **164** (2001), 144-176.

[DHI] D. Dugger, S. Hollander, D. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Cambridge Philos. Soc. **136** (2004), 9-51.

[Go-Ho] P. Goerss, M. Hopkins, *André-Quillen (co)homology for simplicial algebras over simplicial operads*, Une Dégustation Topologique [Topological Morsels]: Homotopy Theory in the Swiss Alps (D. Arlettaz and K. Hess, eds.), Contemp. Math. 265, Amer. Math. Soc., Providence, RI, 2000, pp. 41-85.

[Hi] P. Hirschhorn, *Model categories and their localizations*, Math. Surveys and Monographs Series 99, AMS, Providence, 2003.

[H-S] A. Hirschowitz, C. Simpson, *Descente pour les n-champs*, preprint math.AG/9807049.

[Ho] M. Hovey, *Model categories*, Mathematical surveys and monographs, Vol. **63**, Amer. Math. Soc., Providence 1998.

[La-Mo] G. Laumon and L. Moret-Bailly, *Champs algébriques*, A series of Modern Surveys in Mathematics vol. **39**, Springer-Verlag 2000.

[Q] D. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol XVII, New York, 1964), 65-87. Amer. Math. Soc., Providence, P.I.

[SGA1] M. Artin, A. Grothendieck, *Revêtements étales et groupe fondamental*, Lecture notes in Math. **224**, Springer-Verlag 1971.

[To] B. Toën, *Higher and derived stacks: A global overview* preprint math.AG/0604504.

[To-Va] B. Toën, M. Vaquié, *Moduli of objects in dg-categories*, pré-publication arXiv math.AG/0503269, to appear in Annales de l'ENS.

[HAGI] B. Toën, G. Vezzosi, *Homotopical algebraic geometry I: Topos theory*, Adv. in Math. **193** (2005), 257-372.

[HAGII] B. Toën, G. Vezzosi, *Homotopical algebraic geometry II: Geometric stacks and applications*, to appear in Memoires of the AMS.