p-DIVISIBLE GROUPS IN STABLE HOMOTOPY THEORY

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Introduction

My research aims to elucidate connections between algebraic geometry and chromatic stable homotopy theory, using the technology of p-divisible groups. The moduli stack of formal groups is a well-known algebro-geometric approximation to the stable homotopy category, and I aim to show that certain moduli objects of p-divisible groups offer even better understanding of power operations as well as 'transchromatic' phenomena.

Background

Formal groups in stable homotopy theory. The story of algebraic topology is one in which complicated, topological categories like manifolds, spaces, and spectra are analyzed increasingly precise algebraic machinery. From the chromatic point of view, formal groups are one of the best ways into the stable homotopy category. This approach was initiated by Quillen in [11]. Recall that a cohomology theory is said to be **complex oriented** if it has a Thom class $u \in \tilde{E}^2(\mathbb{C}P^\infty)$ that restricts to $1 \in \tilde{E}^2(\mathbb{C}P^1) \cong E^0$. Such a class gives an identification

$$E^*(\mathbb{C}P^\infty) \cong E^*[[u]],$$

and the multiplication map $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ thus induces a map

$$F_E : E^*[[u]] \to E^*[[x, y]],$$

in other words, a power series in two variables over the coefficient ring E^*. As $\mathbb{C}P^\infty$ is actually a commutative group, up to homotopy, under this multiplication map, the power series F_E satisfies analogous coassociativity, cocommutativity, and counitality properties, which are summarized by saying that it is a (commutative, one-dimensional) **formal group law** over E^*. Any other Thom class u' would determine another formal group law F'_E, but we would also have $u' = f(u)$ for some one-variable power series $f(x) = x + \cdots$, and the two formal group laws would thus be related by

$$F'_E(f(x), f(y)) = f(F_E(x, y)).$$

This power series f is called a **strict isomorphism** of formal group laws.

Algebraically, formal group laws and their strict isomorphisms are well-understood, by the following theorem of Lazard.

Theorem ([10])**.** *Formal group laws over a ring R are represented by maps $L \to R$, where*

$$L \cong \mathbb{Z}[x_1, x_2, \cdots].$$

Strict isomorphisms of formal group laws over R are represented by maps $W \to R$, where

$$W \cong L[b_1, b_2, \cdots].$$

Quillen's theorem says that the presence of a complex orientation is precisely what is needed to realize the above topologically.

Theorem ([11]). *Complex orientations on a spectrum E are represented by maps $MU \to E$, and changes of orientation by maps $MU \wedge MU \to E$, where MU is the complex cobordism spectrum. Moreover, $\pi_* MU \cong L$, $MU_* MU \cong W$, and the induced maps on homotopy groups*

$$\pi_* MU \to \pi_* E$$
$$MU_* MU \to \pi_* E$$

are precisely the above maps representing formal group laws and their strict isomorphisms.

The moduli stack of formal groups. In modern language, the above theorem says that for any complex oriented spectrum E, the homotopy groups $\pi_* E$ (and likewise $E_* X$ for any spectrum X) form a comodule over the Hopf algebroid $(MU_*, MU_* MU) = (L, W)$. Better yet, this is a quasi-coherent sheaf on the associated stack, which by Lazard's theorem is the moduli stack $\mathcal{M}_{\mathrm{fg}}$ of formal groups and their isomorphisms. Since many important cohomology theories are complex oriented, many phenomena in the stable homotopy category, such as the E_2 page of the Adams-Novikov spectral sequence, can be recovered from the geometry of this stack, and the homotopy theory of its derived category of quasi-coherent sheaves.

The geometry of $\mathcal{M}_{\mathrm{fg}}$ is rich, and discussed at length in [6], but for our present purposes can be summarized as follows. The **height** of a formal group law F over a field is the exponent h appearing in the p-series $[p]_F(x) = ax^{p^h} + \cdots$ with a nonzero; this is invariant under isomorphism, so we can define heights for formal groups over schemes by checking them at each point. The stacks $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ parametrizing formal groups of height at most n are open substacks defining a filtration of $\mathcal{M}_{\mathrm{fg}}$. Taking the complement of $\mathcal{M}_{\mathrm{fg}}^{\leq n-1}$ in $\mathcal{M}_{\mathrm{fg}}^{\leq n}$ gives a locally closed substack $\mathcal{M}_{\mathrm{fg}}^{=n}$ which is known to have a single closed point, whose automorphisms (over $\overline{\mathbb{F}}_p$) are the Morava stabilizer group.

One obstruction to doing calculations using this picture is that height is not locally constant. Maps $\operatorname{Spec} R \to \mathcal{M}_{\mathrm{fg}}$ can 'spread out' over one of the open substacks $\mathcal{M}_{\mathrm{fg}}^{\leq n}$, so that base changing to a subscheme of R suddenly decreases the height. For instance, the Lubin-Tate formal group law of height h is a formal group law F over $W(\mathbb{F}_{p^h})[[u_1, \ldots, u_{h-1}]]$ with

$$[p]_F(x) = px +_F u_1 x^p +_F \cdots +_F u_{h-1} x^{p^{h-1}} +_F x^{p^h};$$

inverting $u_i, \ldots, u_{h-1}$, which is a base change to an open subscheme, lowers the height from h to i.

p-divisible groups. We can rigidify the theory of formal groups, removing this wrinkle, by introducing p-divisible groups. A **p-divisible group** over a scheme S is an ind-group scheme $\mathbb{G}$, presented as the colimit of commutative group schemes $\mathbb{G}_i$ over S, such that

- each $\mathbb{G}_i$ is finite and flat over S of order p^{ih} for some fixed h, called the **height** of $\mathbb{G}$, and
- each $\mathbb{G}_i$ is the p^i-torsion subgroup of all $\mathbb{G}_j$ for $j \geq i$.

Any formal group F over R (viewed as an ind-group scheme with underlying formal scheme $\operatorname{Spf} R[[x]]$) gives rise to a p-divisible group over $\operatorname{Spec} R$ with

$$\mathbb{G}_i = \operatorname{Spec} R[[x]]/[p^i]_F(x).$$

This construction is moreover realized topologically: after localizing at p,

$$\mathbb{C}P^\infty = BS^1 \simeq \operatorname{hocolim}_i B\mathbb{Z}/p^i$$

and taking E^*-cohomology gives the above p-divisible group. Unlike formal groups, though, p-divisible groups need not be connected: the constant formal group scheme $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ over any base is another example. In fact, every p-divisible group fits into a canonical exact sequence of ind-group schemes

$$0 \to \mathbb{G}^0 \to \mathbb{G} \to \mathbb{G}^{et} \to 0$$

with $\mathbb{G}^0$ 'formal' (that is, connected) and $\mathbb{G}^{et}$ ind-étale.

It's immediate from the definition that base change preserves height, so unlike with formal groups, we now have a separate stack $\mathcal{M}_p(n)$ parametrizing p-divisible groups of height n with formal part of dimension 1 for each n. Forgetting everything but the formal part gives a map $\mathcal{M}_p(n) \to \mathcal{M}_{\text{fg}}$ with image in $\mathcal{M}_{\text{fg}}^{\leq n}$, and by, for example, the Serre-Tate deformation theory [14], this is an isomorphism over $\mathcal{M}_{\text{fg}}^{=n}$. So the distinct stacks $\mathcal{M}_p(n)$ together form something like a blowup algebra for the stack $\mathcal{M}_{\text{fg}}$, and we can hope to understand a sheaf on $\mathcal{M}_{\text{fg}}$ supported on heights less than or equal to n (like $(E_n)_*X$ or $E(n)_*X$, for a spectrum X) by pulling it back to $\mathcal{M}_p(n)$.

Current work

It is generally agreed among homotopy theorists that formal groups are a sensitive but imperfect tool for studying the stable homotopy category, and the thesis of this work is that p-divisible groups are a yet more sensitive one. Two aspects of the moduli stacks $\mathcal{M}_p(n)$ invite particular attention. The first is the aforementioned rigidity of height, which has applications to transchromatic phenomena. The second is the possible presence of more structure than on $\mathcal{M}_{\text{fg}}$, which may help us understand power operations.

Transchromatic homotopy theory. Chromatic homotopy theory studies the stable homotopy category in terms of the $K(n)$-local categories, where $K(n)$ is the nth Morava K-theory. Transchromatic homotopy theory studies what happens as n varies. The first example is a transchromatic phenomenon is the chromatic fracture square, which is a homotopy pullback square

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X. \end{tikzcd}$$

Here L_n is localization with respect to the Morava E-theory E_n. Thus, this pullback square lets us inductively compute the homotopy of X from knowledge of the homotopy of $L_{K(n)}X$ for each n, so long as the bottom map in each diagram is understood. Taking $E(n)$-homology of this diagram gives a homotopy pullback square of sheaves on $\mathcal{M}_{\mathrm{fg}}$ (indeed, on $\mathcal{M}_{\mathrm{fg}}^{\leq n}$) of the form

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{\wedge} \arrow[d] \\ Ri_*i^*\mathcal{F} \arrow[r] & Ri_*i^*(\mathcal{F}^{\wedge}), \end{tikzcd}$$

where i is the inclusion $\mathcal{M}_{\mathrm{fg}}^{\leq n-1} \to \mathcal{M}_{\mathrm{fg}}^{\leq n}$ and completion means derived completion at the point $\mathcal{M}_{\mathrm{fg}}^{=n}$. But since the map $\mathcal{M}_p(n)^{=n} \to \mathcal{M}_{\mathrm{fg}}^{=n}$ is an isomorphism, we can compute the bottom map by using the maps $\mathcal{M}_{\mathrm{fg}}^{\leq n-1} \leftarrow \mathcal{M}_p(n)^{\leq n-1} \to \mathcal{M}_p(n)$. In essence, to understand what happens when we base change a formal group, we should first base change it as a p-divisible group, which is fairly simple, and then forget its étale part, which is slightly less simple but still tractable. In this case, the étale part is height 1 and thus (over an algebraically closed field) of the form $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$, which has $\mathbb{Z}_p^\times$ as its automorphism group; thus, the cohomology of $\mathbb{Z}_p^\times$ will interfere in the homological fracture square in an essential way.

Transchromatic phenomena are not limited to fracture squares connecting adjacent heights. In [8], in order to investigate Morava E-homology of finite groups, Hopkins, Kuhn, and Ravenel constructed character maps from E_n to its rationalization. As Stapleton points out, this can be viewed as a map from height n to height 0; in [13], he constructs generalized character maps from height n to all lower heights. Again, the geometry of p-divisible groups appears in an essential way: we are now using the inclusions of stacks $\mathcal{M}_p(n)^{\leq t} \to \mathcal{M}_p(n)$, which have even more complicated homology than above. Better understanding these generalized character maps requires some nontrivial computations with the moduli p-divisible groups.

Power operations. A ring spectrum E forces a ring structure on the cohomology groups E^*X. If E is actually an E_∞-ring spectrum, then by [5] among others, then this ring additionally carries **power operations**, which are maps

$$P_m : E^0X \to E^0(B\Sigma_m \times X)$$

with the property that the composition

$$E^0X \to E^0(B\Sigma_m \times X) \to E^0X,$$

the second map being induced by the inclusion $* \times X \to B\Sigma_m \times X$, is just the map $x \mapsto x^m$. Familiar examples include the Steenrod operations on mod p ordinary cohomology and the Adams operations on real and complex K-theory. More recently, Ando [1] described power operations on the Morava E-theories E_n.

The analogue of power operations like the Adams operations in the algebraic setting is a structure called a **λ-ring**. A λ-structure on a torsion-free ring R can be given by a ring endomorphism ψ_p of R for each p such that

$$\psi_p(x) \equiv x^p \pmod{p}.$$

Borger has shown [3, 4] that λ-rings are well-behaved from the point of view of algebraic geometry, and proposes them as a model for $\mathbb{F}_1$-algebras, where $\mathbb{F}_1$ is the supposed 'field with one element.' The ramifications of this proposal in number theory do not concern us, but the following theorem is useful for studying λ-rings and power operations in the abstract.

Theorem ([4])**.** *There is a map of topoi*

$$v^* : \operatorname{Spec}\mathbb{F}_1 \leftrightarrow \operatorname{Spec}\mathbb{Z} : v_*$$

where $\operatorname{Spec}\mathbb{Z}$ *is the big étale topos over* $\mathbb{Z}$*,* $\operatorname{Spec}\mathbb{F}_1$ *is the topos of étale spaces over* $\mathbb{Z}$ *with λ-structure, and the left adjoint v^* forgets the λ-structure.*

In [12], Rezk describes the rings-with-power-operations $(E_n)_*X$, where X is a $K(n)$-local spectrum, as algebras over a certain noncommutative ring subject to a congruence condition. Borger's λ-algebraic geometry and the theory of p-divisible groups together offer a considerably streamlined approach to this difficult theorem, which may be outlined as follows. The stacks $\mathcal{M}_p(n)$ are λ-stacks, which is to say that they are pulled back from 'underlying stacks' $\widetilde{\mathcal{M}_p(n)}$ over $\operatorname{Spec}\mathbb{F}_1$. Sheaves on $\mathcal{M}_{\mathrm{fg}}$ of the form $(E_n)_*X$ may be pulled back to quasicoherent algebra sheaves on $\mathcal{M}_p(n)$, and it is believed that these satisfy the right conditions to descend to $\widetilde{\mathcal{M}_p(n)}$. But this forces them to be sheaves of $\mathbb{F}_1$-algebras, meaning that they carry a λ-structure; by the Goerss-Hopkins-Miller theorem, this will induce power operations on the spectra E_n, which should be precisely those defined by Ando. The following question is then immediate (though consequences to topology may be limited, due to the scope of the Goerss-Hopkins-Miller theorem).

Question. Do other sheaves on $\mathcal{M}_{\mathrm{fg}}$ acquire power operations in this way?

Other directions

Moduli of p-divisible groups. The moduli stack of formal groups has the advantage of the nice Hopf algebroid presentation (π_*MU, MU_*MU). For the purposes of making calculations, it would be nice to have an equally good presentation of $\mathcal{M}_p(n)$. Some headway has been made by Lawson [9], using Zink's theory of displays [15]. Lawson's Hopf algebroids, though, are large, and only present moduli of p-divisible groups with formal part of height at least 2. The problem is that Zink's theory is only set up for formal p-divisible groups, and Lawson attacks the 1-dimensional p-divisible groups of interest to homotopy theory by dualizing them to get formal p-divisible groups. Thus, with an expansion of the theory of displays to the non-formal case, the desired presentation could potentially be obtained.

Realization theorems. Though most of algebraic topology consists of turning topological information into algebraic information, we are occasionally able to reverse this arrow and realize algebraic information as a spectrum. The first such theorem is arguably Brown representability, which says that any functor on spaces worth calling a cohomology theory is in fact realized in the stable homotopy category. In the chromatic world, the Goerss-Hopkins-Miller theorem constructs E_∞ ring spectra from certain sheaves of algebras on $\mathcal{M}_{\mathrm{fg}}$, namely the structure sheaves of the formal neighborhoods of its closed points. Most recently, Lurie has proved a generalization of the Goerss-Hopkins-Miller theorem using moduli of p-divisible groups, one version of which (from [7]) runs as follows.

Theorem (Lurie)**.** *Suppose given a Deligne-Mumford stack $\mathcal{M}$ with a formally étale map to $\mathcal{M}_p(n)$. Then there is a sheaf of spectra $\mathcal{O}$ over $\mathcal{M}_{\mathrm{fg}}$ such that*

$$\pi_k\mathcal{O} = \begin{cases} \omega^{\otimes t} & k = 2t, \\ 0 & k = 2t+1 \end{cases}$$

where ω is the sheaf of invariant differentials for the composition $\mathcal{M} \to \mathcal{M}_p(n) \to \mathcal{M}_{\mathrm{fg}}$.

This theorem was most powerfully used in [2] to construct the topological automorphic forms spectra, but its applications have not fully been explored. In particular, to establish étaleness of the map $\mathcal{M} \to \mathcal{M}_p(n)$ in their case of interest – $\mathcal{M}$ being a certain Shimura variety – Behrens and Lawson use a local criterion based on the Serre-Tate deformation theory of p-divisible groups. It would be interesting, particularly in connection with the above work on power operations, to establish other étaleness criteria for maps to $\mathcal{M}_p(n)$, and thus use Lurie's theorem to realize new highly structured spectra.

References

[1] Matthew Ando. Isogenies of formal group laws and power operations in the cohomology theories E_n. *Duke Mathematical Journal*, 79(2):423–485, 1995.

[2] Mark Behrens and Tyler Lawson. Topological automorphic forms. *Memoirs of the American Mathematical Society*, 204(958), 2010.

[3] James Borger. The basic geometry of Witt vectors, I: The affine case. *arXiv preprint arXiv:0801.1691*, 2008.

[4] James Borger. The basic geometry of Witt vectors, II: Spaces. *Mathematische Annalen*, 351(4):877–933, 2011.

[5] RR Bruner, JP May, JE McClure, and M Steinberger. *H_∞ Ring Spectra and their Applications*, volume 1176 of *Lecture Notes in Mathematics*. 1986.

[6] Paul G Goerss. Quasi-coherent sheaves on the moduli stack of formal groups. *arXiv preprint arXiv:0802.0996*, 2008.

[7] Paul G Goerss. Topological modular forms (after Hopkins, Miller, and Lurie). *Astérisque*, 332, 2010.

[8] Michael Hopkins, Nicholas Kuhn, and Douglas Ravenel. Generalized group characters and complex oriented cohomology theories. *Journal of the American Mathematical Society*, 13(3):553–594, 2000.
[9] Tyler Lawson. Structured ring spectra and displays. *arXiv preprint arXiv:0912.5094*, 2009.
[10] Michel Lazard. *Commutative formal groups.* Berlin, 1975.
[11] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc*, 75(6):1293–1298, 1969.
[12] Charles Rezk. The congruence criterion for power operations in Morava E-theory. *Homology, Homotopy and Applications*, 11(2):327–379, 2009.
[13] Nathaniel Stapleton. Transchromatic generalized character maps. *Algebraic & Geometric Topology*, 13:171–203, 2013.
[14] John Tate. p-Divisible groups. In *Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin*, pages 158–183, 1967.
[15] Thomas Zink. The display of a formal p-divisible group. *Astérisque*, pages 127–248, 2002.