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THE L_2-LOCALIZATION OF $W(n)$

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ABSTRACT. In this paper we analyze the localization of $W(n)$, the fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$, with respect to $E(2)$. If four cells at the bottom of $D_p M^{2np-1}$, the pth extended power spectrum of the Moore spectrum, are collapsed to a point, then one obtains a spectrum C. Let $QM^{2np-1} \to QC$ be the James-Hopf map followed by the collapse map. Then we show that the secondary suspension map $BW(n) \to QM^{2np-1}$ has a lifting to the fiber of $QM^{2np-1} \to QC$ and this lifting is shown to be a v_2-periodic equivalence, hence an $E(2)$-equivalence.

1. INTRODUCTION

We begin by recalling the following construction from [24]. Consider the fiber sequence

$$F \longrightarrow QS^{2n+1} \xrightarrow{j_p} QD_pS^{2n+1}$$

where j_p is the James-Hopf map and D_pS^{2n+1} is the p^{th} extended power construction on the sphere. The stabilization map $S^{2n+1} \to QS^{2n+1}$ lifts to a map $S^{2n+1} \to F$, and in [24] it is shown that this lifting induces an isomorphism in complex K-theory. It follows that there is an equivalence $L_1S^{2n+1} \cong L_1F$ where L_1 stands for Bousfield localization with respect to K-theory on the category of spaces. This result enables one to get a handle on L_1S^{2n+1} since the functor L_1 is reasonably well behaved on fiber sequences, L_1 of an infinite loop space is something very close to the localization of the corresponding spectrum, and K-theory localization stably is well understood.

The aim of this paper is to explore an analogous construction for $L_2W(n)$. L_2 refers to Bousfield localization with respect to the p-local homology theory $E(2)$ with coefficients $E(2)_* = Z_{(p)}[v_1, v_2, v_2^{-1}]$ (for example see [28]). $W(n)$ is the homotopy fiber of the double suspension map $S^{2n-1} \to \Omega^2 S^{2n+1}$, localized at a prime p. For technical reasons which probably have to do with our method of proof more than anything else, we will assume $p \geq 5$. The analogue of the stabilization map is a 'secondary suspension map', which is a map $W(n) \to QM^{2np-2}$ that is degree one on the bottom Moore space. Here M^k denotes a mod p Moore space with top cell in dimension k. There are various constructions of maps such as this, for example see [8]. It will be more convenient to start with a delooped version of the secondary suspension. In [12] it is shown that there exists a delooping of $W(n)$,

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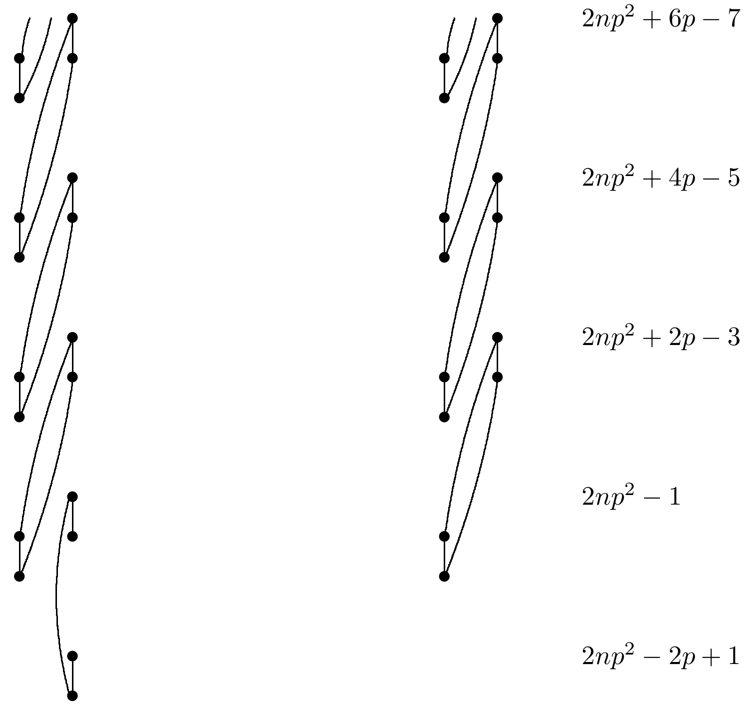

FIGURE 1. Cell diagram of D_pM^{2np-1} and $C = D_pM^{2np-1}/X$

denoted $BW(n)$. It follows from the construction of $BW(n)$ that there is a map $BW(n) \xrightarrow{\sigma} QM^{2np-1}$ which has degree one on the bottom cell. See [12] for details.

Consider the James-Hopf map

$$QM^{2np-1} \xrightarrow{j_p} QD_pM^{2np-1}.$$

The left side of Figure 1 gives a cell diagram for D_pM^{2np-1}. The short and long lines represent the actions of the Milnor primitives Q_0 and Q_1 respectively. Note the four cells near the bottom in dimensions $2np^2-1$, $2np^2-2$, $2np^2-2p+1$, and $2np^2-2p$. Denote this 4-cell complex by X. Since p is odd, X can be collapsed to a point. Let C denote the complex D_pM^{2np-1}/X which is pictured on the right side of Figure 1, and consider the fiber sequence

$$G^n \xrightarrow{i} QM^{2np-1} \longrightarrow QC, \tag{1.1}$$

where the second map is the James-Hopf map j_p composed with $Q(\pi)$ where π is the collapse map.

In the case of the sphere S^{2n+1}, the lifting of the stabilization map exists for purely dimensional reasons. Since $BW(n)$ is not finite dimensional, a secondary suspension map does not lift for such a simple reason.

Our first result is the following:

Theorem 1.2. *Assume $p \geq 5$ and $n \geq 1$. There exists a map*

$$\sigma_1 : BW(n) \to \Omega^{2p} BW(n+1)$$

which is degree one on the bottom Moore space. The mapping telescope of the diagram

$$BW(n) \to \Omega^{2p} BW(n+1) \to \Omega^{4p} BW(n+2) \to \dots$$

is QM^{2np-1}. If we let $\sigma : BW(n) \to QM^{2np-1}$ denote the inclusion into the telescope, then there exists a map $\lambda : BW(n) \to G^n$ such that $i \circ \lambda = \sigma$.

This will be proved in section 2 by analyzing some properties of the James-Hopf maps. The hypothesis that $p \geq 5$ is required in order to use certain properties of Gray's delooping of $W(n)$ ([12]).

Our main result is the following:

Theorem 1.3. *Assume $p \geq 5$ and $2np - 2 - k$ is sufficiently large. Then*

$$\lambda : \Omega^k BW(n) \to \Omega^k G^n$$

induces an isomorphism in $E(2)_*$, hence

$$L_2 \Omega^k BW(n) \simeq L_2 \Omega^k G^n.$$

Just how large $2np - 2 - k$ must be for the theorem to hold is discussed below.

In [24] the K-theory isomorphism induced by the map $S^{2n+1} \to F$ is established by direct calculation of $K_*(F)$ relying on, among other things, the results of [27]. Techniques for calculating the $E(2)$-homology of spaces such as $\Omega^k BW(n)$ and $\Omega^k G^n$ are not in place yet, so Theorem 1.3 will be deduced from Theorem 1.5 stated below, via the following theorem of A. K. Bousfield [3]. In order to state this we recall some definitions.

For each $m \geq 1$, let V_{m-1} denote some finite cell complex which has type m, i.e. $K(i)_* V_{m-1} = 0$ if $i < m$ and $K(m)_* V_{m-1} \neq 0$, where $K(i)$ is the ith Morava K-theory spectrum (see [28]). Let $v : \Sigma^d V_{m-1} \to V_{m-1}$ be a v_m self map, i.e. a map inducing an isomorphism in $K(m)_*$ and inducing the zero map in $K(i)_*$ if $i \neq m$. Define the homotopy groups of a space Y with coefficients in V_{m-1} by

$$\pi_t(Y; V_{m-1}) = [\Sigma^t V_{m-1}, Y]$$

and define the v_m-periodic homotopy groups of Y, which we will denote by

$$v_m^{-1} \pi_t(Y; V_{m-1}),$$

as the colimit of the sequence

$$\pi_t(Y; V_{m-1}) \xrightarrow{v^*} \pi_{t+d}(Y; V_{m-1}) \xrightarrow{v^*} \dots$$

It can be shown that these periodic groups do not depend on the choice of v. They do depend on the choice of V_{m-1}, however if a map induces an isomorphism in $v_m^{-1}\pi_t(\,; V_{m-1})$ with one choice of V_{m-1}, then it also will with any other choice (Corollary 11.11, [3]). So for purposes of making statements about v_m-periodic isomorphisms, we are free to choose V_{n-1} as we like.

For each n, Bousfield defines an integer $c(n)$. The precise value of $c(n)$ is not known. Very roughly, $c(n)$ is bounded above by the dimension of the bottom cell of a minimally connected type n complex V_{n-1} which is a suspension. Also, $c(n)$ is bounded below by $n+1$. It is known that $c(0) = 1$ and $c(1) = 2$. Define a functor $\tilde{\Omega}$, going from the category of $c(n)$-connected spaces to itself, as the $c(n)$-connected cover of the loop space functor Ω. Let E_* be a homology theory. We say a map $f : X \to Y$ in the homotopy category of $c(n)$-connected spaces is a *durable* E_*-equivalence if $\tilde{\Omega}^k f : \tilde{\Omega}^k X \to \tilde{\Omega}^k Y$ is an E_*-equivalence for all $k \geq 0$.

The following is distilled from Bousfield [3].

Theorem 1.4 (Bousfield, 13.3 and 13.15 of [3])**.** *Let $f : X \to Y$ be a map in the homotopy category of $c(n)$-connected spaces. Then f induces an isomorphism in $v_m^{-1}\pi_t(\quad; V_{m-1})$ for all $0 \leq m \leq n$ if and only if f is a durable E_*-equivalence for all spectra E such that $E^*(V_n) = 0$.*

Such an equivalence is called a v_n-periodic equivalence. In particular, a v_n-periodic equivalence is always an $E(n)_*$-isomorphism.

The condition on n and k in Theorem 1.3 can be stated more precisely now: $2np - 2 - k$ is sufficiently large if $\Omega^k BW(n)$ is $c(2)$-connected.

Thus by using Bousfield's theorem we see that Theorem 1.3 follows from the following:

Theorem 1.5. *Assume that $p \geq 5$ and $n \geq 1$. The map $\lambda : BW(n) \to G^n$ induces an isomorphism in unstable v_m-periodic homotopy groups for $0 \leq m \leq 2$, i.e λ is a v_2-periodic equivalence.*

Theorem 1.5 will be proved in section 3. The proof is an adaptation to the present situation of the methods employed in [25], [23], [22], and [30]. In particular, Theorem 1.5 could be viewed as an odd primary analogue of the main result [25] which deals with the case $p = 2$. However there are two significant differences. The first is that in [25], we do not know if there is a map analogous to λ of Theorem 1.2. This means that the statement concerning v_2-periodic homotopy groups does not obviously translate into a result concerning homological localization. The second is that the lambda algebra calculations of [25] for $p = 2$ do not readily carry over to the odd primary case.

We deal with this second point by using the results of B. Gray concerning the odd primary lambda algebra [13] and [14]. Thus Theorem 1.5 is concerned with the application of the machinery of [13] and [14] to the unstable Adams spectral sequence. This was part of the original motivation for studying such subquotients of the lambda algebra. See [21], [15], [22], and [30].

Remark 1.6. If we localize with respect to $K(2)$ instead of $E(2)$ then we can say more. In [10] it is shown that Bousfield localization with respect to the Morava K-theory spectrum $K(n)$ preserves fiber sequences which are double loops except possibly in dimensions $n-1$, n, and $n+1$. Combining this with Theorem 1.3 yields the following corollary:

Corollary 1.7. *Let $p \geq 5$ and $2np - 4 > c(2)$. Then there is a map from $L_{K(2)}\Omega W(n)$ to the homotopy fiber of*

$$L_{K(2)}QM^{2np-3} \to L_{K(2)}Q\Sigma^{-2}C$$

which induces an isomorphism in homotopy groups except possibly in dimensions 1,2, and 3.

Furthermore, in [2] Bousfield proves that the localization of any infinite loop space $\Omega^\infty Z$ with respect to any spectrum E is again an infinite loop space. There is a certain localization functor associated to E on the category of (-1)-connected spectra, called the $E_*\Omega^\infty$ localization, and in [2] it is shown that the E-localization of the space $\Omega^\infty Z$ is Ω^∞ applied to the spectrum $E_*\Omega^\infty Z$. Thus Corollary 1.7 shows that the homotopy groups of $L_{K(2)}\Omega W(n)$ could in principle be computed from the LES associated to the $K(2)$-localization of (1.1), *if* one had explicit information about the $K(2)_*\Omega^\infty$ localization functor on connective spectra.

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2. James-Hopf maps

Theorem 1.2 follows from some basic properties of James-Hopf maps in conjunction with some properties of Gray's construction of $BW(n)$. We recall James-Hopf maps:

For nonnegative integers k and q (or k infinite) and each space X there are James-Hopf maps

$$j_q : \Omega^k\Sigma^k X \to QD_{k,q}X,$$

natural in X, where $D_{k,q}X$ is the extended power space $C_k(q)^+ \wedge_{\Sigma_q} X^{[q]}$. Here $C_k(q)$ is the space of ordered q-tuples of little cubes disjointly embedded in I^k. If k is infinite, we simply write D_qX. The maps j_q are defined in [6]. In [4] an important Cartan formula is proved for the James-Hopf maps, and in [18] various compatibility relations between the James-Hopf maps are established which are extremely useful.

Taking the wedge sum of the adjoints of the James-Hopf maps yields a map of spectra

$$J : \Sigma^\infty\Omega^k\Sigma^k X \to \bigvee_{q\geq 1} \Sigma^\infty D_{k,q}X \tag{2.1}$$

which is a stable equivalence. Such a stable splitting was first established in [17] for $k = \infty$ and [29] for finite k and then generalized in [6] and [4]. Such a splitting is not unique of course. Throughout this paper j_q will always refer to the James-Hopf maps of [6], [4], and the stable splitting of $\Omega^k\Sigma^k X$ will be the one in (2.1) induced by the maps j_q unless otherwise noted.

In [12] Gray shows that $W(n)$ is a loop space. More precisely, he shows that there exists a space $BW(n)$, together with a map $\Omega^2S^{2n+1} \xrightarrow{\nu} BW(n)$ such that the homotopy fiber of ν is S^{2n-1}. For p odd, $BW(n)$ is shown to be an H-space, and for $p \geq 5$, ν is an H-map. In what follows we need ν to be an H-map, hence the hypothesis in Theorem 1.2 that $p \geq 5$. Furthermore, in Proposition 7 of [12], it is shown that there is a splitting

$$\begin{aligned} \Sigma^2\Omega^2S^{2n+1} &\cong \Sigma^2(S^{2n-1} \times BW(n)) \\ &\cong \Sigma^2(S^{2n-1} \vee BW(n) \vee \Sigma^{2n-1}BW(n)). \end{aligned} \tag{2.2}$$

In [8], it is shown that the James-Hopf map admits a factorization

$$\Omega^2S^{2n+1} \to \Omega^{2p}\Sigma^{2p}M^{2np-1} \to QM^{2np-1} = QD_{2,p}(S^{2n-1}).$$

Definition 2.3. Let $s : \Sigma^2 BW(n) \to \Sigma^2\Omega^2 S^{2n+1}$ be the right inverse of $\Sigma^2\nu$ corresponding to (2.2). Let ${\sigma_1}' : BW(n) \to \Omega^{2p}\Sigma^{2p}M^{2np-1}$ be the adjoint of the composite

$$\Sigma^{2p}BW(n) \xrightarrow{\Sigma^{2p-2}s} \Sigma^{2p}\Omega^2 S^{2n+1} \xrightarrow{\tilde{j}_p} \Sigma^{2p}M^{2np-1}.$$

Finally, let $\sigma_1 : BW(n) \to \Omega^{2p}BW(n+1)$ be the composite

$$BW(n) \xrightarrow{{\sigma_1}'} \Omega^{2p}\Sigma^{2p}M^{2np-1} \to \Omega^{2p}BW(n+1)$$

where the second map is Ω^{2p} on the inclusion of the bottom cell.

The proof that QM^{2np-1} is the mapping telescope of σ_1 is the same as that in [8]. Note that the map $BW(n) \xrightarrow{\sigma} QM^{2np-1}$ is just

$$BW(n) \xrightarrow{{\sigma_1}'} \Omega^{2p}\Sigma^{2p}M^{2np-1} \to QM^{2np-1}$$

where the second map is the inclusion.

For the last statement in Theorem 1.2 we need several lemmas.

The following lemma is a variation of Lemma 3.6 of [20]. The difference is that the secondary suspension map α defined in Lemma 3.6 of [20] is not *a priori* the same as the map σ defined here. One can conclude after the fact that α and σ are the same since $BW(n)$ splits off of $\Omega^2 S^{2n+1}$ stably.

Lemma 2.4. *There exists a factorization up to homotopy of the James-Hopf map:*

$$\begin{tikzcd} \Omega^2 S^{2n+1} \arrow[r, "\nu"] \arrow[d, "="'] & BW(n) \arrow[d, "\sigma"] \\ \Omega^2 S^{2n+1} \arrow[r, "j_p"] & QM^{2np-1} \end{tikzcd}$$

Proof. The mod p homology algebra of $\Omega^2 S^{2n+1}$ for p odd is given by ([5])

$$E(\iota, Q_1\iota, Q_1^2\iota, \dots) \otimes P(\beta Q_1\iota, \beta Q_1^2\iota, \dots).$$

If we assign weights to the monomials by $\mathrm{wt}(Q_1^j\iota) = \mathrm{wt}(\beta Q_1^j\iota) = p^j$ and $\mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y)$ then the homology of $D_{2,j}S^{2n-1}$ is the vector space of monomials of weight j. It follows that $\Omega^2 S^{2n+1}$, localized at p, splits stably into a wedge $\bigvee_{j=1}^{\infty} D_{2,j}S^{2n-1}$ where $j \equiv 0$ or $1 \pmod p$. Let J^{-1} stand for the homotopy equivalence which is inverse to the stable splitting of (2.1) given by the James-Hopf maps.

It can be verified by an easy calculation in homology that the composite

$$\Sigma^\infty BW(n) \xrightarrow{\Sigma^\infty s} \Sigma^\infty\Omega^2 S^{2n+1} \xrightarrow{J} \bigvee_{j\equiv 0 \pmod p} \Sigma^\infty D_{2,j}S^{2n-1} \tag{2.5}$$

is a homotopy equivalence. Thus we have a stable splitting

$$\Sigma^\infty BW(n) \vee \Big(\bigvee_{j\equiv 1 \pmod p} \Sigma^\infty D_{2,j}S^{2n-1}\Big) \xrightarrow[\cong]{\Sigma^\infty s \vee J^{-1}} \Sigma^\infty\Omega^2 S^{2n+1}$$

Consider the adjoint of the diagram in Lemma 2.4. It is immediate that the adjoint diagram commutes when restricted to the piece $\Sigma^\infty BW(n)$. To show that the diagram commutes on the other piece first note that $\tilde{j}_p : \Sigma^\infty\Omega^2 S^{2n+1} \to \Sigma^\infty M^{2np-1}$, is null homotopic on the pieces of the splitting where $j \equiv 1 \pmod p$. Thus the proof of 2.4 is completed by the following lemma. $\square$

Lemma 2.6. *The composite map*

$$\bigvee_{j\equiv 1 \pmod p} D_{2,j}S^{2n-1} \xrightarrow{J^{-1}} \Sigma^\infty\Omega^2 S^{2n+1} \xrightarrow{\Sigma^\infty\nu} \Sigma^\infty BW(n)$$

is null homotopic.

Proof. This makes use of the Cartan formula for James-Hopf maps given in [4] and the fact that $BW(n)$ is an H-space [12]. There are pairings $D_{k,j}X \wedge D_{k,r}X \to D_{k,j+r}X$ induced by the inclusion $\Sigma_j \times \Sigma_r \subset \Sigma_{j+r}$ and the Cartan formula for James-Hopf maps says that these pairings are compatible, via the stable splitting, with the stabilization of the H-space multiplication on $\Omega^k\Sigma^k X$. In the following diagram we will abbreviate $D_{2,j}S^{2n-1}$ to D_j. We will suppress the symbol Σ^∞ but the diagram is to be understood as being stable.

$$\begin{tikzcd}
S^{2n-1}\wedge D_{pk} \arrow[r] \arrow[d, "="'] & S^{2n-1}\wedge\Omega^2 S^{2n+1} \arrow[r] \arrow[d, "i\wedge 1"'] & * \arrow[d] \\
D_1\wedge D_{pk} \arrow[r, "J^{-1}"] \arrow[d, "\cong"'] & \Omega^2 S^{2n+1}\wedge\Omega^2 S^{2n+1} \arrow[r, "\nu\wedge\nu"] \arrow[d, "m"'] & BW(n)\wedge BW(n) \arrow[d, "m"'] \\
D_{pk+1} \arrow[r, "J^{-1}"] & \Omega^2 S^{2n+1} \arrow[r, "\nu"] & BW(n)
\end{tikzcd}$$

The upper left vertical map is an equivalence because $D_1 = S^{2n-1}$. The lower left vertical map induces an isomorphism in homology hence is an equivalence. The lower middle vertical map is the Hopf construction on the H-space multiplication on Ω^2S^{2n+1}. The right hand lower vertical map is the Hopf construction on the H-space multiplication on $BW(n)$. Since ν is an H-map, the lower right hand square commutes. The upper right square commutes since $S^{2n-1} \to \Omega^2S^{2n+1} \xrightarrow{\nu} BW(n)$ is null. This completes the proof of 2.6. □

Before completing the proof of Theorem 1.2 we recall a result from [19] concerning the composite of two James-Hopf maps:

Theorem 2.7 (part of 5.2 of [19])**.** *For $k, n, r, q \geq 1$ let $f^n_{r,q}$ be the composite map*

$$\Sigma^\infty D_{k,n}X \hookrightarrow \Sigma^\infty\Omega^k\Sigma^k X \xrightarrow{\Sigma^\infty j_q} \Sigma^\infty QD_{k,q}X \to \Sigma^\infty D_r D_{k,q}X$$

Then $f^n_{r,q}$ is null homotopic if $n > rq$.

Proof. (of Theorem 1.2)

In order to get a lifting $BW(n) \xrightarrow{\lambda} G$ we need to know that the composite $BW(n) \xrightarrow{\sigma} QM^{2np-1} \to QC$ is null homotopic. Gray's map $\Omega^2S^{2n+1} \xrightarrow{\nu} BW(n)$ has a right inverse stably (Theorem 8(e) [12]), so by Lemma 2.4 it suffices to show that $\Omega^2S^{2n+1} \xrightarrow{j_p} QM^{2np-1} \to QC$ is null homotopic. This is equivalent to a factorization of $j_p \circ j_p$ through QX, where X is the four cell complex at the bottom of D_pM^{2np-1} defined in Section 1. See the diagram below. Notice that X is homotopy equivalent to $D_{2,p^2}(S^{2n-1})$, so Theorem 1.2 is proved once we know that the following square commutes up to homotopy:

$$\begin{array}{ccccc} QM^{2np-1} & \xrightarrow{j_p} & QD_pM^{2np-1} & & \\ {\scriptstyle j_p}\uparrow & & \uparrow & & \\ \Omega^2S^{2n+1} & \xrightarrow{j_{p^2}} & QD_{2,p^2}(S^{2n-1}) & = & QX \end{array}$$

Equivalently, we consider the adjoint diagram and check that it commutes on each piece of the stable splitting of Ω^2S^{2n+1}:

$$\begin{array}{ccc} \Sigma^\infty QM^{2np-1} & \xrightarrow{\tilde{j}_p} & \Sigma^\infty D_pM^{2np-1} \\ {\scriptstyle \Sigma^\infty j_p}\uparrow & & \uparrow \\ \Sigma^\infty\Omega^2S^{2n+1} & \xrightarrow{\tilde{j}_{p^2}} & \Sigma^\infty D_{2,p^2}(S^{2n-1}) \end{array} \tag{2.8}$$

The right hand vertical map is a sort of transfer, defined as the composite

$$\Sigma^\infty D_{2,p^2}(S^{2n-1}) \hookrightarrow \Sigma^\infty\Omega^2S^{2n+1} \xrightarrow{\Sigma^\infty j_p} \Sigma^\infty QM^{2np-1} \xrightarrow{\tilde{j}_p} \Sigma^\infty D_pM^{2np-1}.$$

Thus the square (2.8) commutes on the p^2 piece of the splitting by definition. The bottom horizontal map is null on $\Sigma^\infty D_{2,m}(S^{2n-1})$ for each $m \neq p^2$. The composite $\tilde{j}_p \circ \Sigma^\infty j_p$ is null on $\Sigma^\infty D_{2,m}(S^{2n-1})$ for $m < p^2$ for purely dimensional reasons. Finally, $\tilde{j}_p \circ \Sigma^\infty j_p$ is null on $\Sigma^\infty D_{2,m}(S^{2n-1})$ for $m > p^2$ by Theorem 2.7. □

3. Unstable v_2-periodic homotopy groups

In this section we will prove Theorem 1.5. To start, we have

Lemma 3.1. *$\lambda : BW(n) \to G^n$ induces an isomorphism in $v_0^{-1}\pi_*(\quad)$ and $v_1^{-1}\pi_*(\quad)$.*

Proof. $v_0^{-1}\pi_*(\quad)$ is just rational homotopy and both spaces are torsion. The map $\sigma : BW(n) \to QM^{2np-1}$ induces an isomorphism in $v_1^{-1}\pi_*(\quad)$ by [30]. To see that $v_1^{-1}\pi_*(QC) = 0$, use the fact that C has a filtration with subquotients $V(1)$, and so $K(1)_*V(1) = 0$. By the telescope theorem for $n = 1$ (Theorem 4.11 of [1]), we have that stably $v_1^{-1}\pi_*(V(1)) = 0$. □

We will define unstable v_2-periodic homotopy groups by taking V_1 to be the Smith-Toda complex $V(1)$, which we will denote simply by V. Since $p \geq 5$, V has a v_2-self map $v_2 : \Sigma^{|v_2|}V \to V$. Using a p-local version of the Freudenthal suspension theorem (see [11]) we see that this v_2-self map is defined unstably as long as V is at least $d-1$-connected, where $d = \frac{2p^2+1}{p-1} + 3$.

Consider the map of pairs

$$(QM^{2np-1}, BW(n)) \to (QM^{2np-1}, G^n) \to (QC, *).$$

It suffices to show that this induces an isomorphism

$$v_2^{-1}\pi_*(QM^{2np-1}, BW(n); V) \xrightarrow{\cong} v_2^{-1}\pi_*(QC; V). \tag{3.2}$$

The proof of this is based on the modified unstable Adams spectral sequence techniques of [22], [23], [25], [30]. This machinery takes as input certain calculations involving subquotients of the lambda algebra. See [15] and [21]. In the present case, the relevant lambda algebra calculations are provided by [13] and [14] so we will use that framework. We recall the construction.

In [13] it is shown that there are spaces $\{W^n_{(0)}\}_{n\geq 0}$ and maps

$$\Omega W^{2n-1}_{(0)} \to \Omega^3 W^{2n+1}_{(0)} \to \Omega^5 W^{2n+3}_{(0)} \to \Omega^7 W^{2n+5}_{(0)} \to \dots. \tag{3.3}$$

The two cell complex at the bottom of $\Omega^{2k+1}W^{2(n+k)-1}_{(0)}$ is M^{2n-2} and each of the above maps is degree one on this bottom Moore space. The homotopy colimit of this sequence is QM^{2n-2}. The spaces $W^{2n-1}_{(0)}$ are defined as follows:

$$W^{2n-1}_{(0)} = \text{fiber}(\pi_n : \Omega^2 S^{2n+1} \to S^{2n-1})$$

where

$$\pi_n = \begin{cases} \pi_n & \text{from [7] if } (n,p)=1, \\ \phi_m & \text{from [12] if } n = pm. \end{cases} \tag{3.4}$$

Thus $\Omega W^{2np-1}_{(0)} = W(n)$.

We need to prove that there is an isomorphism

$$v_2^{-1}\pi_*(QM^{2np-2}, \Omega W^{2np-1}_{(0)}; V) \xrightarrow{\cong} v_2^{-1}\pi_*(Q\Sigma^{-1}C; V). \tag{3.5}$$

Even though the map $\Omega W^{2np-1}_{(0)} \to QM^{2np-2}$ defined by (3.3) is not necessarily the same as $\Omega\sigma : W(n) \to QM^{2np-2}$, we will nevertheless see that the proof of (3.5) leads to the proof of (3.2).

In [14] certain subquotients of Λ, the odd primary lambda algebra, are defined. These are denoted by $\Lambda_{(m)}(n)$, $m \geq -1$, $n \geq 0$. There are SES's

$$0 \to \Lambda_{(m)}(2n-1) \to \Lambda_{(m)}(2n) \to \Lambda_{(m)}(2np-1) \to 0$$

and

$$0 \to \Lambda_{(m)}(2n) \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m)}(2np+2p^{m+1}-1) \to 0$$

which yield EHP sequences in homology and a SES

$$0 \to \Lambda_{(m)}(2n-1) \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m+1)}(2np-1) \to 0 \tag{3.6}$$

which yields the double suspension sequence.

We have

$$\Lambda_{(m)} = \bigcup_{n=1}^{\infty} \Lambda_{(m)}(n) = E(\tau_0, \dots, \tau_m)\tilde{\otimes}\Lambda$$

where $E(\tau_0, \dots, \tau_m)$ is the exterior subalgebra of the dual Steenrod algebra A_*. In those cases where $V(m)$ exists we have $E(\tau_0, \dots, \tau_m) = H_*V(m)$ and

$$H_*(E(\tau_0, \dots, \tau_m)\tilde{\otimes}\Lambda) = \text{Ext}_{A_*}(H_*V(m)).$$

The chain complex $\Lambda_{(m+1)}(k)$ has a splitting given by the SES's

$$0 \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m+1)}(2n+1) \to \Lambda_{(m)}(2n+2p^{m+1}+1) \to 0$$

and

$$0 \to \Lambda_{(m)}(2n+1) \to \Lambda_{(m+1)}(2n) \to \Lambda_{(m)}(2n+2p^{m+1}-1) \to 0$$

There are v_m-self maps

$$v_m : \Lambda_{(m-1)}(2n+2p^m-1) \to \Lambda_{(m-1)}(2n-1)$$

and isomorphisms

$$v_m^{-1}\Lambda_{(m-1)}(2n-1) \cong v_m^{-1}\Lambda_{(m-1)}(2n+1) \cong v_m^{-1}(E(\tau_0,\dots,\tau_{m-1})\tilde{\otimes}\Lambda).$$

Recall from [22] and [30] that a resolution of a space X is a tower of fibrations,

$$\begin{array}{ccccccc} F_0 & & F_1 & & F_2 & & \\ \big\downarrow{=} & & \big\downarrow & & \big\downarrow & & \\ X_0 & \longleftarrow & X_1 & \longleftarrow & X_2 & \longleftarrow & \dots \end{array}$$

with each fiber F_s being a GEM, and compatible maps $f_s : X \to X_s$, with $f_\infty : X \to X_\infty$ being the p-completion. Given a resolution of a space, there is the usual homotopy spectral sequence.

Lemma 3.7. (1) *There is a resolution of* $W_{(0)}^{2n-1}$ *with*

$$E_2^{s,t} \cong H_*(\Lambda_{(0)}(2n-1)).$$

(2) *The map* $\Omega W_{(0)}^{2n-1} \to \Omega^3 W_{(0)}^{2n+1}$ *is covered by a map of resolutions, and the induced map of* E_2*-terms is* $H_*(\Lambda_{(0)}(2n-1)) \to H_*(\Lambda_{(0)}(2n+1))$ *from* (3.6).

(3) *Let*

$$\Omega^2 W_{(1)}^{2np-1} \to \Omega W_{(0)}^{2n-1} \to \Omega^3 W_{(0)}^{2n+1}$$

be the homotopy fiber sequence of [13]. *Then there is a resolution of* $\Omega^2 W_{(1)}^{2np-1}$ *with* $E_2^{s,t} \cong H_*(\Lambda_{(1)}(2np-1))$.

Proof. Proposition 6.3 of [25] states that if we are given a map of spaces $f : X \to Y$, and resolutions of X and Y, then there is a map of resolutions covering f if the largest dimensional homotopy class in $\pi_* F_s$, for the target space Y, is in the range through which f_s, for the source space X, is surjective in cohomology. This was used in [22] (Proposition 4.10) and [30] (Theorem 2.27) to produce a map of resolutions covering a secondary suspension map $W(n) \to \Omega^{2p} W(n+1)$. The proof of Proposition 6.3 of [25] is the same as the proof of Proposition 4.10 of [22]. If we replace the resolution of the target space Y by the same tower starting in degree i, then we have the result that there is a filtration i map of resolutions covering f if the largest dimensional homotopy class in $\pi_* F_{s+i}$, for the target, is in the range through which f_s, for the source, is surjective in cohomology.

We apply this to the map $\pi_n : \Omega^2 S^{2n+1} \to S^{2n-1}$ of (3.4). As usual, take the Adams resolution for S^{2n-1} with $\Lambda(2n-1)$ as E_1-term, and for $\Omega^2 S^{2n+1}$ take double loops on the Adams resolution for S^{2n+1}. The map of resolutions needs to be a filtration one map. As in [30], the dimension of a class in $\pi_* F_{s+1}$, for S^{2n-1}, is at most

$$q(n-1)[1+p+\cdots+p^s] + 2n-1.$$

This is less than $(2n-1)p^{s+1} + (p-2)p^s$, which is the range through which $f_s^* : H^*\Omega^2 S^{2n+1} \leftarrow H^* X_s$ is onto in the resolution of $\Omega^2 S^{2n+1}$.

Proposition 3.3 of [22] (see also 2.20 of [30]) states that if we are given a map of resolutions covering a given map f, then there is a resolution of the fiber of f, and a long exact sequence of E_2-terms. It is implicit in [22] that one of the maps in the LES is induced by the map f. This last fact is proved explicitly in [23].

For our resolution of $W_{(0)}^{2n-1}$ we take the resolution of the fiber corresponding to the map of resolutions covering $\pi_n : \Omega^2 S^{2n+1} \to S^{2n-1}$ constructed above.

The statement regarding the E_2-term follows from the LES of E_2-terms, once we check that the map of resolutions induces the homomorphism $v_0 : \Lambda_{(0)}(2n+1) \to \Lambda_{(0)}(2n-1)$ at least on E_2. Following the proof of Proposition 2.32 and Lemma 2.29 of [30], let P'_* be a chain complex of free unstable A-modules corresponding to the resolution of $\Omega^2 S^{2n+1}$, and P''_* a chain complex of free unstable A-modules for the resolution of S^{2n-1}. Let $\epsilon : P''_* \to P'_*$ denote the difference between the chain map induced by the map of resolutions constructed above, and the given map v_0. Since P''_* is acyclic (the resolution of S^{2n-1} is an Adams resolution), the composite $P''_* \xrightarrow{\epsilon} P'_* \xrightarrow{\sigma} P''_*$ is chain homotopically trivial, where σ is double suspension. Thus there is a lifting $P''_* \to \ker\sigma$. Now $\ker\sigma$ is a chain complex of free unstable A-modules corresponding to a resolution of $W(n)$. This lifting is zero since $\ker\sigma$ is acyclic in the range of dimensions in which $\mathrm{Hom}_A(P''_*, Z/p)$ is nonzero, which is easy to check by the calculations of section two of [30].

Part 3) follows immediately from part 2) by using the resolution of the fiber. Part 2) uses the same argument as Theorem 2.27 and Lemma 2.29 of [30]. Actually no new calculations are needed as the estimates given in [30] yield part 2) directly. □

The 4-cell complex at the bottom of $\Omega^2 W_{(1)}^{2np^2-1}$ is $V = V(1)$ with the bottom cell in dimension $2np^2-4$. Checking the p-local Freudenthal suspension condition, we see that as long as $n \geq 1$ this V at the bottom is the target of the self map v_2.

By [16] there is an exponent k such that $v_2^k \wedge 1$ is the same as $1 \wedge v_2^k$ as a stable self map of $V \wedge V$. As in [25] we consider the following diagram of pointed mapping spaces. For brevity, denote $\Omega^2 W_{(1)}^{2np^2-1}$ by W, v_2^k by v, and set $j = |v_2^k|$.

$$\begin{array}{ccccccc}
\mathrm{map}_*(V,W) & \to & \mathrm{map}_*(\Sigma^j V, W) & \to & \mathrm{map}_*(\Sigma^{2j} V, W) & \to & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathrm{map}_*(V,V) & \to & \mathrm{map}_*(\Sigma^j V, V) & \to & \mathrm{map}_*(\Sigma^{2j} V, V) & \to & \dots \\
 & \searrow & \uparrow & & \uparrow & & \\
 & & \mathrm{map}_*(\Sigma^j V, \Sigma^j V) & \to & \mathrm{map}_*(\Sigma^{2j} V, \Sigma^j V) & \to & \dots \\
 & & & \searrow & \uparrow & & \\
 & & & & \mathrm{map}_*(\Sigma^{2j} V, \Sigma^{2j} V) & \to & \\
 & & & & & \searrow & \dots
\end{array}$$

As in [25], this yields a commutative diagram of abelian groups after applying π_*. This produces a homomorphism

$$\pi_*^{\mathrm{S}}(V;V) \to v_2^{-1}\pi_*(W;V)$$

which extends to give a homomorphism

$$v_2^{-1}\pi_*^{\mathrm{S}}(V;V) \xrightarrow{\phi} v_2^{-1}\pi_*(W;V).$$

Theorem 3.8. *The homomorphism ϕ is an isomorphism.*

Proof. As in [25], we have a corresponding diagram of E_2-terms

$$\begin{array}{ccccccc}
E_2^{s,*}(W;V) & \to & E_2^{s+1,*}(W;V) & \to & E_2^{s+2,*}(W;V) & \to & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
E_2^{s,*}(V;V) & \to & E_2^{s+1,*}(V;V) & \to & E_2^{s+2,*}(V;V) & \to & \dots \\
 & \searrow & \uparrow & & \uparrow & & \\
 & & E_2^{s,*}(\Sigma^j V;V) & \to & E_2^{s+1,*}(\Sigma^j V;V) & \to & \dots \\
 & & & \searrow & \uparrow & & \\
 & & & & E_2^{s,*}(\Sigma^{2j} V;V) & \to & \\
 & & & & & \searrow & \dots
\end{array}$$

which gives a homomorphism

$$\mathrm{Ext}_A^{s,t}(H_*V, H_*V) \to v_2^{-1}E_2^{s,t}(W;V)$$

which extends to a homomorphism

$$v_2^{-1}\mathrm{Ext}_A^{s,t}(H_*V, H_*V) \xrightarrow{\psi} v_2^{-1}E_2^{s,t}(W;V).$$

In [14] it shown that there is an isomorphism

$$v_2^{-1}E_2^{s,t}(W) \xrightarrow{\theta} v_2^{-1}\mathrm{Ext}_A^{s,t}(H_*V).$$

Reducing mod V, we get an isomorphism

$$v_2^{-1}E_2^{s,t}(W;V) \xrightarrow{\theta} v_2^{-1}\mathrm{Ext}_A^{s,t}(H_*V, H_*V).$$

Now the argument of [9], Theorem 3.10, shows that the composite

$$v_2^{-1}\mathrm{Ext}_A^{s,t}(H_*V, H_*V) \xrightarrow{\psi} v_2^{-1}E_2^{s,t}(W;V) \xrightarrow{\theta} v_2^{-1}\mathrm{Ext}_A^{s,t}(H_*V, H_*V)$$

is an isomorphism, and this proves Theorem 3.8. □

Returning to the proof of the isomorphism in (3.5), consider the tower of fibrations

$$\begin{array}{ccccccc}
* & \to & (\Omega^3 W_{(0)}^{2np+1}, \Omega W_{(0)}^{2np-1}) & \to & (\Omega^5 W_{(0)}^{2np+3}, \Omega W_{(0)}^{2np-1}) & \to & \dots \\
\uparrow & & \uparrow & & \uparrow & & \\
\Omega^2 W_{(1)}^{2np^2-1} & & \Omega^4 W_{(1)}^{2np^2+2p-1} & & \Omega^6 W_{(1)}^{2np+4p-1} & &
\end{array}$$

The homotopy colimit of this tower is the pair $(QM^{2np-2}, \Omega W_{(0)}^{2np-1})$. By applying the functor $v_2^{-1}\pi_*(\quad; V)$ to this tower we get a spectral sequence which converges to $v_2^{-1}\pi_*(QM^{2np-2}, \Omega W_{(0)}^{2np-1}; V)$.

The complex C has a filtration with subquotients copies of V, (see Figure 1), and this filtration is compatible with the above tower. This gives a map of spectral sequences, with v_2 inverted. Theorem 3.8 says this map of spectral sequences is an isomorphism on E_2-terms and (3.5) follows.

Now consider (3.2). First note that if we "speed up the filtration" of the pair $(QM^{2np-2}, \Omega W_{(0)}^{2np-1})$ we get

$$* \to (\Omega^{2p+1} W_{(0)}^{2(n+1)p-1}, \Omega W_{(0)}^{2np-1}) \to (\Omega^{4p+1} W_{(0)}^{2(n+2)p-1}, \Omega W_{(0)}^{2np-1}) \to \dots.$$

The fiber at each stage is a space F whose bottom $4p$ cells is a complex A_1, whose cohomology is $A(1)$, the subalgebra of the Steenrod algebra generated by β and $\mathcal{P}^1$. Note that A_1 consists of p copies of $V(1)$ attached together.

In [26] it shown that there is a v_2-self map $\Sigma^{|v_2|}A_1 \to A_1$. Again, by the p-local Freudenthal suspension condition, this map desuspends to a map of spaces, as long as A_1 is at least $d-1$-connected, where $d = \frac{2p^2+1}{p-1} + 2p + 1$. The dimension of the bottom cell of the first A_1 is $2np^2 - 4$, and so it is the target of the self map v_2.

Lemma 2.27 of [30] shows that the map

$$W(n) \xrightarrow{\Omega\sigma_1} \Omega^{2p} W(n+1)$$

is covered by a map of resolutions and Lemma 2.29 of [30] shows that the induced map of E_2-terms is the same as that of the argument above. Thus we have an isomorphism

$$v_2^{-1}E_2^{s,t}(F;V) \xrightarrow{\theta} v_2^{-1}\mathrm{Ext}_A^{s,t}(A_1, H_*V)$$

and the v_2-periodic homotopy of F is the stable v_2-periodic homotopy of A_1.

Now the proof of (3.2) proceeds exactly as above with W replaced by F and $\mathrm{map}_*(V, V)$ replaced by $\mathrm{map}_*(V, A_1)$.

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