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v_n-ELEMENTS IN RING SPECTRA AND APPLICATIONS TO BORDISM THEORY

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Introduction. The work of Hopkins and Smith [HS] has shown that the stable homotopy category has layered periodic behavior. On the (p-local) sphere, the only non-nilpotent self-maps are multiplication by a power of p. But if we kill such a power to form the Moore space $M(p^k)$, then we get a new family of non-nilpotent self-maps, called the v_1 self-maps. Similarly, if we kill one of those, we get v_2 self-maps, and this behavior continues.

One of the great advantages of the Brown-Peterson spectrum BP is that the periodicities are not layered, but they all appear as homotopy classes $v_n \in \pi_{2(p^n-1)}BP$. Another great advantage of BP is that it is comparatively simple algebraically. Its coefficient ring is polynomial, and it is possible to calculate in the Adams-Novikov spectral sequence based on the operations in BP-homology. In fact, most of the spectra used by algebraic topologists are complex-oriented, in that they admit maps of ring spectra from BP. But there is one crucial example that does not, namely, real K-theory KO. Hopkins and Miller [HMi] have recently shown that KO is the tip of an iceberg of noncomplex-oriented theories that have interesting torsion.

It would be nice to have a bordism spectrum that did admit maps to the Hopkins-Miller theories EO_n. Recall that $MO\langle k\rangle$ is the Thom spectrum arising from the $k-1$-connected Postnikov cover $BO\langle k\rangle$ of BO. Similarly, $MU\langle k\rangle$ is the Thom spectrum arising from the $k-1$-connected Postnikov cover of BU. Note that $MO\langle 4\rangle = MSpin$ and $MU\langle 4\rangle = MSU$ both admit orientations to KO. We hope that this is also the beginning of a general phenomenon, and that the $MO\langle k\rangle$ and $MU\langle k\rangle$ will admit orientations to EO_n when k is sufficiently large.

Such an orientation may have some analytic meaning. Witten interprets a (conjectural) orientation from $MO\langle 8\rangle$ to elliptic cohomology, which should be EO_2, as the index of an S^1-equivariant Dirac operator on the free loop space of a manifold with $MO\langle 8\rangle$-structure. However, it will be hard to get at the algebraic meaning of such an orientation because we know so little about the $MO\langle k\rangle$.

This paper is an attempt to get some qualitative understanding of the $MO\langle k\rangle$ and $MU\langle k\rangle$. We try to find analogues of v_n in the homotopy of $MO\langle k\rangle$ and $MU\langle k\rangle$. It turns out to be easier to study the existence and properties of such analogues in general ring spectra. This is the content of the first section. This section owes much to the beautiful paper [HS], and discusses many of the same ideas. The highlight of this section is a sufficient condition for the existence of

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a v_n-element, defined below, in a ring spectrum. The idea here is to use the v_n-elements we already know exist in finite ring spectra. This condition takes a particularly nice form when the prime p is odd and when $n = 1$. Then if any of the homotopy classes $\alpha_t \in \pi_{2t(p-1)-1}S^0$ map to zero under the unit map to R, there is a v_1-element in R.

The second section is concerned with applications to $MO\langle k\rangle$ and $MU\langle k\rangle$. Because α_t is in the image of the J homomorphism, it is possible to determine when α_t goes to zero in $MO\langle k\rangle$ or $MU\langle k\rangle$. We therefore find v_1-elements in $MO\langle k\rangle$ and $MU\langle k\rangle$. Given a v_1-element v in any ring spectrum R, we can get the Bousfield localization of R at the Morava K-theory $K(1)$ by inverting v and completing at the prime p. On the other hand, it is also possible to calculate $L_{K(1)}MO\langle k\rangle$ in a different way by comparing $MO\langle k\rangle$ with $MSpin$. By comparing these two results, we can understand something about the homotopy of $MO\langle k\rangle$. We also show that, if the torsion in $\pi_*MO\langle k\rangle$ is bounded, then the natural map

$$MO\langle k\rangle_*(X) \otimes_{MO\langle k\rangle_*} KO_* \to KO_*(X)$$

is an isomorphism, extending the results of [HH].

All spectra are assumed to be p-local for some fixed prime p throughout this paper, unless explicitly stated otherwise.

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1. Analogues of v_n in general ring spectra

1.1. Notation. Given a ring spectrum R, we will use the standard notations μ for the multiplication map $R \wedge R \xrightarrow{\mu} R$ and η for the unit map $S^0 \xrightarrow{\eta} R$. We use μ_R and η_R if we have more than one ring spectrum around. Given a map $X \xrightarrow{f} Y$ of spectra, and another spectrum E, we will write $E_*(f)$ for the induced map on E-homology. On the other hand, if f is a homotopy class in a ring spectrum R, and E is a ring spectrum, we will also use $E_*(f)$ to denote the Hurewicz image $\eta_E \wedge f$ of f in $E_*(R)$. The context should make clear which notation we mean.

The reason for adopting such notation is the following lemma, which we would like to make explicit, even though it is well known and implicit in many papers on the subject. Given $v \in \pi_k R$ for a ring spectrum R, we can make a self-map $\hat{v}\colon \Sigma^k R \to R$ as the composition

$$\Sigma^k R \simeq S^k \wedge R \xrightarrow{v \wedge 1} R \wedge R \xrightarrow{\mu} R.$$

LEMMA 1.1.1. *Suppose E and R are ring spectra, and $v \in \pi_k(R)$. Then $E_*(\hat{v})$ is multiplication by $E_*(v)$ in the ring $E_*(R)$. Thus, since homology commutes with direct limits, we have*

$$E_*(v^{-1}R) \cong E_*(v)^{-1}(E_*R).$$

Proof. The map $E_*(\hat{v})$ is the effect on homotopy of the composite

$$E \wedge S^k \wedge R \xrightarrow{1 \wedge v \wedge 1} E \wedge R \wedge R \xrightarrow{1 \wedge \mu_R} E \wedge R.$$

This composite is the same as the composite

$$E \wedge S^k \wedge R \xrightarrow{1 \wedge \eta_E \wedge v \wedge 1} E \wedge E \wedge R \wedge R \xrightarrow{\mu_E \wedge \mu_R} E \wedge R.$$

We compare this with multiplication by E_*v in the ring spectrum E_*R by means of the following diagram.

$$\begin{tikzcd}
S^k \wedge E \wedge R \arrow[r, "\eta_E \wedge v \wedge 1 \wedge 1"] \arrow[d, "T \wedge 1"'] & E \wedge R \wedge E \wedge R \arrow[r, "1 \wedge T \wedge 1"] \arrow[d, "f \wedge 1"'] & E \wedge E \wedge R \wedge R \arrow[d, "\mu \wedge \mu"'] \\
E \wedge S^k \wedge R \arrow[r, "1 \wedge \eta_E \wedge v \wedge 1"] & E \wedge E \wedge R \wedge R \arrow[r, "\mu \wedge \mu"] & E \wedge R
\end{tikzcd}$$

Here T denotes the twist isomorphism, and f is the composite

$$E \wedge R \wedge E \xrightarrow{1 \wedge T} E \wedge E \wedge R \xrightarrow{T \wedge 1} E \wedge E \wedge R.$$

Multiplication by E_*v is the effect on homotopy of the top row and right column, so it suffices to show this diagram commutes. The left square commutes by inspection, but the right square will commute if and only if E is commutative. However, even if E is not commutative, the whole diagram will still commute, using the unit isomorphism for E. □

Note that in this lemma, if $E = R$, then $E_*(v)$ is the right unit

$$\eta_R \colon S^0 \wedge R \xrightarrow{\eta \wedge 1} R \wedge R$$

applied to v.

We will mostly be applying this lemma to the Morava K-theories $K(n)$. We will include $K(\infty) = H\mathbf{F}_p$ as a Morava K-theory unless otherwise stated. If R is a ring spectrum, note that $K(n)_*(\eta)$ is either injective or zero since $K(n)_*(S^0)$ is a graded field. We will denote $K(n)_*(\eta)(v_n^k) \in K(n)_*(R)$ by simply v_n^k, following [HS].

1.2. v_n-elements. We need to understand what we mean by an analog of v_n. We assume familiarity with the standard notions of [HS], such as the type of a finite spectrum and v_n-self-maps. First, we recall [HS, Definition 3.1].

Definition 1.2.1. Suppose $n > 0$. Given a finite ring spectrum R, and a homotopy class $v \in \pi_*(R)$, define v to be a v_n-element if $K(n)_*(v)$ is a unit and $K(i)_*(v)$ is nilpotent if $i \neq n$.

The advantage of this definition is that a v_n-element v induces a v_n-self-map $\hat{v}$ in the sense of Hopkins-Smith. However, there are several problems with this definition applied to infinite ring spectra. First of all, in a general ring spectrum, just knowing that $K(n)_*(v)$ is a unit will not be enough to conclude that $K(n)_*(v^k)$ is a power of v_n. The argument in [HS] relies on the finiteness of $K(n)_*(R) \otimes_{K(n)_*} \mathbf{F}_p$. But more importantly, $v_n \in BP_*$ is not a v_n-element using this definition, because $K(i)_*(v_n)$ is not nilpotent when $i < n$. These comments motivate the following definition.

Definition 1.2.2. Suppose $n \geqslant 0$. Given a ring spectrum R and $v \in \pi_*(R)$, define v to be a generalized v_n-element if $K(n)_*(v^j) = v_n^k$ for some j, k, and $K(i)_*(v)$ is nilpotent for $i > n$.

Note that this definition works well for $n = 0$ as well as positive n. Hopkins and Smith show in [HS, Lemma 3.2] that if v is a v_n-element, then there are j, k such that $K(n)_*(v^j) = v_n^k$, so that v is a generalized v_n-element. In order to show that Definition 1.2.2 is a reasonable generalization of Definition 1.2.1, we must show that every generalized v_n-element in a finite ring spectrum is in fact a v_n-element when $n > 0$. This will be the main goal of this section, but we need several lemmas before we can achieve it.

LEMMA 1.2.3. *Suppose R is a finite ring spectrum of type m. Then R has a v_n-element in the sense of* [HS] *if and only if $0 < n \leqslant m$.*

Proof. If $0 < n < m$, then zero is a v_n-element. If $0 < n = m$, then there are integers k and d and a v_m-self-map $f\colon \Sigma^d R \to R$ of R such that $K(m)_* f = v_m^k$ and $K(i)_* f = 0$ for $i \neq m$ by [HS, Theorem 9]. Let v be the composite

$$S^d \xrightarrow{\eta} \Sigma^d R \to R.$$

Then $K(i)_*(v) = K(i)_*(f)(1)$, so v is a v_m-element.

Conversely, suppose $n > m$, and v is a v_n-element. Then, using the methods of [HS, Lemma 3.2], we can find an N so that $K(i)_*(v^N) = 0$ for all $i \neq n$. Let Y denote the cofiber of $\widehat{v^N}$. Then Y is a finite spectrum and $K(m)_* Y$ is nonzero, but $K(n)_* Y = 0$. This is a contradiction to [R, Theorem 2.11]. $\square$

Another basic fact about generalized v_n-elements in finite ring spectra is that all the interesting ones have positive dimension.

LEMMA 1.2.4. *Suppose $v \in \pi_k R$ is a generalized v_n-element in a finite ring spectrum R of type m. Suppose that $m \leqslant n$ and $n > 0$. Then $k > 0$.*

Proof. If $k < 0$, then the map $\hat{v}$ would have to be nilpotent. Since $\hat{v}$ induces an isomorphism on $K(n)_* R \neq 0$, $\hat{v}$ is not nilpotent. Thus we must have $k \geqslant 0$.

Now suppose $k = 0$, and in addition $m = n$. In this case v is automatically a v_n-element in the sense of [HS]. By the preceding lemma and [HS, Theorem 9], there is a v_n-element w in $\pi_j R$ for a positive j. By [HS, Lemma 3.6], there are integers i and j such that $v^i = w^j$. For degree reasons, we must then have $j = 0$. This means that v is a unit in $\pi_* R$, and hence is both nilpotent and a unit in the nontrivial ring $K(n+1)_* R$. This is a contradiction, so we must have $k > 0$.

Now suppose $k = 0$ but $n > m$. Since the cofiber of $\hat{v}$ has type $n + 1 > m + 1$, $K(m+1)_* v$ is a unit. Since $K(m+1)_* R \otimes_{K(m+1)_*} \mathbf{F}_p$ is a finite ring, we can find an N such that $K(m+1)_*(v^N)$ is a power of v_{m+1}. Since $k = 0$, we must have $K(m+1)_*(v^N) = 1$. We can also assume, by taking a possibly larger N, that $K(n+1)_*(v^N) = 0$. Let Y denote the cofiber of $\widehat{v^N} - 1 = \widehat{v^N - 1}$. Then $K(m+1)_* Y$ is nonzero, but $K(n+1)_* Y = 0$, which is a contradiction to [R, Theorem 2.11]. Thus we must have $k > 0$. □

We also need the following lemma. The proof given here is due to Hal Sadofsky.

LEMMA 1.2.5. *Suppose X is a type-n finite spectrum and $f \colon \Sigma^k X \to X$ is a self-map with cofiber Y. Suppose $k \neq 0$. Then Y is either type n or type $n + 1$.*

Proof. Note that Y certainly has type at least n. Also, if $k < 0$, f is nilpotent. In that case, $K(n)_*(f)$ cannot be an isomorphism, so Y will have type n. So assume $k > 0$, and Y has some finite type greater than $n + 1$. Then $K(n)_*(f)$ and $K(n+1)_*(f)$ are both isomorphisms. This implies that $n > 0$. Indeed, if $n = 0$, then f induces an isomorphism of the finite-dimensional vector space $H\mathbf{Q}_*(X)$, which cannot happen unless $k = 0$.

Using the techniques of [HS], we can find a v_n-self-map g of X and an integer N such that $K(n)_*(f^N) = K(n)_*(g)$. Recall the method is to use the ring spectrum $R = DX \wedge X$. The map f corresponds to a homotopy class v in R such that $K(n)_*(v)$ is a unit. Since $K(n)_*(R) \otimes_{K(n)_*} \mathbf{F}_p$ is finite, there is an N such that $K(n)_*(v^N)$ is a power of v_n. Thus $K(n)_*(f^N)$ is also a power of v_n. Replacing N by a possibly larger value, we can find a v_n-self-map g which is multiplication by that same power of v_n. We can also assume $K(i)_*(g) = 0$ for $i \neq n$.

Let Z be the cofiber of $f^N - g$. Then Z is a finite spectrum such that $K(n)_*(Z) \neq 0$, but $K(n+1)_*(Z) = 0$. This contradicts [R, Theorem 2.11], so Y must have type $\leqslant n + 1$. □

Note that this lemma is in general false when $k = 0$. Indeed, one can simply take the wedge of a type-zero spectrum X and a type-2 spectrum Z with the self-map that is the identity on X and trivial on Z.

THEOREM 1.2.6. *If R is a finite ring spectrum and $n > 0$, every generalized v_n-element is, in fact, a v_n-element.*

Proof. Suppose R has type m, and $v \in \pi_k(R)$ is a generalized v_n-element. If $n \leqslant m$, then $K(i)_*(R) = 0$ for $i < n$, so v is a v_n-element. So assume $n > m$. By Lemma 1.2.4, we must have $k > 0$. But the cofiber of $\hat{v}$ has type $n+1$, which is a contradiction to Lemma 1.2.5. □

In view of Theorem 1.2.6, we will call our generalized v_n-elements simply v_n-elements in the remainder of the paper.

With this convention, Lemma 1.2.3 takes the following form.

COROLLARY 1.2.7. *Suppose R is a finite ring spectrum of type m. Then R has a v_n-element if and only if $n \leqslant m$.*

1.3. Properties of v_n-elements. The properties of v_n-elements that we need are summarized in the following theorem.

THEOREM 1.3.1. (a) *Suppose $R \xrightarrow{f} S$ is a map of ring spectra and $v \in \pi_* R$. If v is a v_n-element, so is $f_* v$.*

(b) *Suppose $R \xrightarrow{f} S$ is a map of ring spectra such that $K(i)_*(f)$ is injective for $i \geqslant n$. Suppose $v \in \pi_*(R)$. If $f_* v$ is a v_n-element, so is v.*

(c) *Suppose v, $w \in \pi_*(R)$ commute with each other. If v, $w \in \pi_* R$ are v_n-elements, so is vw.*

Proof. Note

$$K(i)_*(f)(K(i)_*(v)^k) = K(i)_*(f_* v)^k.$$

So if $K(i)_*(v)$ is nilpotent, so is $K(i)_*(f_* v)$. Conversely, if $K(i)_*(f)$ is injective and $K(i)_*(f_* v)$ is nilpotent, so is $K(i)_*(v)$. Also, since f commutes with the unit, if $K(n)_*(v^j) = v_n^k$, then

$$K(n)_*(f_* v^j) = K(n)_*(f)(v_n^k) = v_n^k.$$

Conversely, if $K(n)_*(f)$ is injective and $K(n)_*(f_* v^j) = v_n^k$, then

$$K(n)_*(f)(K(n)_*(v^j)) = K(n)_*(f)(v_n^k),$$

so $K(n)_*(v^j) = v_n^k$. This proves the first and second parts of the theorem. The third part follows from the fact that the Hurewicz map is a ring homomorphism. □

The main advantage of our definition of v_n-elements comes from the second part of Theorem 1.3.1, as expressed in the following corollary.

COROLLARY 1.3.2. *Suppose R is a ring spectrum $v \in \pi_*(R)$ and X is a finite ring spectrum of type m. Let $f: R \to R \wedge X$ be induced by the unit of X. If $f_* v$ is a v_n-element for some $n \geqslant m$, then so is v.*

Proof. The unit map $S^0 \xrightarrow{\eta} X$ is injective on $K(i)$-homology as long as $K(i)_*(X) \neq 0$. Since X is type m, this will happen when $i \geqslant m$. Since $K(i)_*$ is a field spectrum, and so has a Kunneth isomorphism,

$$K(i)_*(f) = K(i)_*(R) \otimes_{K(i)_*} K(i)_*(\eta)$$

is also injective when $i \geqslant m$. Part (b) of Theorem 1.3.1 completes the proof. □

1.4. Finding v_n-elements in ring spectra. We can now give our sufficient condition for the existence of a v_n-element in a ring spectrum R.

THEOREM 1.4.1. *Suppose we have a finite ring spectrum X, and let $g: X \to Y$ denote the cofiber of the unit $S^0 \xrightarrow{\eta_X} X$. Suppose X is type n, and $v \in \pi_k(X)$ is a v_n-element. Suppose R is a ring spectrum and suppose the composite*

$$S^k \xrightarrow{v} X \xrightarrow{g} Y \xrightarrow{\eta_R \wedge 1} R \wedge Y$$

is null. Then any lift of $(\eta_R \wedge 1)_* v$ to $\pi_k R$ is a v_n-element.

Note that the hypotheses of the theorem guarantee that a lift of $(\eta_R \wedge 1)_* v$ to $\pi_k R$ does exist.

Proof. Since v is a v_n-element and $\eta_R \wedge 1$ is a map of ring spectra, $(\eta_R \wedge 1)_* v$ is also a v_n-element. Now apply Corollary 1.3.2 to complete the proof. □

Let us call v_n-elements that arise from v_n-elements in a finite ring spectrum in this way finitary v_n-elements.

CONJECTURE 1.4.2. *Every v_n-element is finitary.*

We will see one advantage of finitary v_n-elements in the next section. In practice, it can be difficult to tell whether a v_n-element is finitary. For example, showing that $v_1 \in BP_*$ is finitary at $p = 2$ requires finding a finite spectrum with a v_1-self-map of degree 2. This has been done by Davis and Mahowald [DM], but we do not know if their example is a ring spectrum.

The simplest case of Theorem 1.4.1 is when $n = 1$ and p is odd. Recall the elements $\alpha_t \in \pi_{qt-1} S^0$, where $q = 2p - 2$.

COROLLARY 1.4.3. *Let p be an odd prime. Suppose R is a ring spectrum and $\eta \circ \alpha_t$ is null. Then there is a v_1-element in $\pi_{qt} R$.*

Proof. Take the finite ring spectrum X in Theorem 1.4.1 to be the Moore spectrum $M(p)$. The definition of α_t is the composite of a v_1 element in degree qt with the pinch map to the top cell. □

We have a similar result at $p = 2$, which is complicated by the fact that $M(2)$ is not a ring spectrum. Here we need the elements $\alpha_{t/2} \in \pi_{8t-1}S^0$. There is not complete conformity on the name of this class; Ravenel calls it $\alpha_{4t/2}$ in [R1].

COROLLARY 1.4.4. *Let $p = 2$. Suppose R is a ring spectrum such that $\eta \circ \alpha_{t/2}$ is null. Then there is a v_1-element in $\pi_{8t}R$.*

Proof. This is precisely the same as the previous corollary except we must use $M(4)$. The relevant facts about $M(4)$ can be found in [DM]. □

To apply Theorem 1.4.1 more generally, we need specific examples of type-n finite ring spectra and v_n-elements. These are provided by the generalized Moore spectra $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ discussed in [HS, Proposition 5.12] and [MS]. These are type-n spectra such that

$$BP_*M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}) \cong BP_*/(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}}),$$

which exist (but are not unique) for a cofinal set of sequences $(i_0, \dots, i_{n-1})$. They can be chosen to be ring spectra, which will then have v_n-elements, for a smaller cofinal set of sequences by Devinatz [Dev]. Furthermore, Devinatz shows that we can assume these ring spectrum structures are coherent, in the sense that the natural map

$$M(p^{i_0}, \dots, v_{j-1}^{i_{j-1}}) \to M(p^{i_0}, \dots, v_j^{i_j})$$

is a map of ring spectra for all $j \leqslant n - 1$. Let us call sequences $(i_0, \dots, i_{n-1})$ for which $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ exists *realizable,* and sequences for which there is a ring spectrum $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ that is coherent in the above way *multiplicative.*

Note that $M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$ has 2^n cells, and the bottom 2^{n-1} cells and the top 2^{n-1} cells are both suspensions of $M(p^{i_0}, \dots, v_{n-2}^{i_{n-2}})$.

Given a realizable sequence, one can get generalized Greek letter elements in the homotopy of the sphere by the Adams construction: include the bottom cell, iterate the v_n-self-map some number of times, and pinch off to the top cell. One of the outstanding problems in homotopy theory is to understand these elements: when they exist, when they are nontrivial, and the extent to which they are all of the elements in the homotopy of the sphere.

We can use the idea behind Theorem 1.4.1 to get some partial results about these Greek letter elements.

COROLLARY 1.4.5. *Suppose $(i_0, \dots, i_{n-1})$ is multiplicative. Then the composite of including the bottom cell, iterating the v_n-self-map, and pinching off to the top 2^{n-1} cells is nontrivial.*

Proof. Because the sequence is multiplicative, the map

$$M(p^{i_0}, \dots, v_{n-2}^{i_{n-2}}) \to M(p^{i_0}, v_1^{i_1}, \dots, v_{n-1}^{i_{n-1}})$$

is a map of ring spectra. It is injective on $K(i)$-homology for $i \geqslant n$, because the maps used in making the $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ induce the evident maps on BP-homology. Hence, by part (b) of Theorem 1.3.1, if the composite in the statement of the corollary were null, we would get a v_n-element in the type-$n-1$ spectrum $M(p^{i_0}, \ldots, v_{n-2}^{i_{n-2}})$. This is a contradiction to Corollary 1.2.7. □

1.5. Generalized v_n-elements and localization. We need to recall some facts relating to the failure of the telescope conjecture. These can be found in [Hov] or [MS]. Recall that given a finite type-n spectrum X, its telescope $\mathrm{Tel}(X)$, defined as the homotopy-direct limit of any v_n-self-map, is well defined. Furthermore, the Bousfield class of $\mathrm{Tel}(X)$ is independent of the type-n finite spectrum X. We denote this Bousfield class by $\langle \mathrm{Tel}(n) \rangle$. Given a v_n-element v in a finite ring spectrum R, we have

$$L_{\mathrm{Tel}(n)}(R) = v^{-1}R.$$

In particular, $\mathrm{Tel}(n)_*(v)$ is a unit, no matter which spectrum we use for $\mathrm{Tel}(n)$. From this we get the following lemma.

LEMMA 1.5.1. *Suppose R is a ring spectrum and v is a finitary v_n-element. Then* $\mathrm{Tel}(n)_*(v)$ *is a unit.*

Proof. Since v is finitary, there is a finite type-n ring spectrum X and a v_n-element $w \in \pi_* X$ such that $(\eta_R \wedge 1)_* w = (1 \wedge \eta_X)_* v$. Since $\mathrm{Tel}(n)_*(w)$ is a unit, so is $\mathrm{Tel}(n)_*((1 \wedge \eta_X)_* v)$. Another way to say this is that the self-map $\hat{v}\colon \Sigma^k R \to R$ becomes a homotopy equivalence upon smashing with $\mathrm{Tel}(n) \wedge X$. We claim that $\langle \mathrm{Tel}(n) \wedge X \rangle = \langle \mathrm{Tel}(n) \rangle$, so that in fact $\hat{v}$ becomes a homotopy equivalence upon smashing with $\mathrm{Tel}(n)$, as required. Indeed, since we are only worried about Bousfield classes, we can use $w^{-1}X$ for $\mathrm{Tel}(n)$, and then note that $w^{-1}X \wedge X = w^{-1}X \wedge w^{-1}X$, which has the same Bousfield class as $w^{-1}X$. □

Thus, if v is a finitary v_n-element, the map $R \to v^{-1}R$ is a $\mathrm{Tel}(n)$-equivalence. Now $v^{-1}R$ is not $\mathrm{Tel}(n)$-local, but it is somewhat less local. Before describing this, we need a definition.

Definition 1.5.2. Given a spectrum R and an $n \geqslant 0$, say that R satisfies the telescope conjecture at n if

$$\langle R \wedge \mathrm{Tel}(n) \rangle = \langle R \wedge K(n) \rangle.$$

The usual telescope conjecture is that S^0 satisfies the telescope conjecture at all n, which we now know to be false for $n = 2$ [R2]. However, BP satisfies the telescope conjecture, as is proven in [Hov, Theorem 1.9], based on [R].

LEMMA 1.5.3. *Suppose R is a ring spectrum and v is a v_n-element in R_*. Then $v^{-1}R$ is* $\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)$*-local. If R satisfies the telescope conjecture,* *then $v^{-1}R$ is* $\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n-1) \vee K(n)$*-local. In particular, if v is a v_1-element, $v^{-1}R$ is K-local.*

Proof. First note that $K(i)_*(v^{-1}R) = 0$ for $i > n$. Thus, if X is a ring spectrum of type $n+1$, $K(i)_*(v^{-1}R \wedge X) = 0$ for all i. Since $v^{-1}R \wedge X$ is a ring spectrum, it must then be null by the nilpotence theorem [HS]. Then in fact $v^{-1}R \wedge Y$ is null for all finite spectra Y of type at least $n+1$, by a thick subcategory argument. In the terminology of [Hov], the finite acyclics of $v^{-1}R$ are $\mathscr{C}_{n+1}$. We showed in [Hov] that this means

$$\langle v^{-1}R \rangle \leqslant \langle \mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n) \rangle.$$

Since $v^{-1}R$ is a ring spectrum, it is self-local, so also $\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)$-local.

Now suppose R satisfies the telescope conjecture. Then

$$\langle v^{-1}R \rangle = \langle R \wedge v^{-1}R \rangle \leqslant \langle R \wedge (\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)) \rangle$$

$$\leqslant \langle \mathrm{Tel}(0) \vee \cdots \mathrm{Tel}(n-1) \vee K(n) \rangle. \qquad \square$$

This lemma includes [HH, Theorem 3] as a special case.

We now must recall more results from [Hov]. If X is a finite spectrum of type n, localization with respect to X is particularly simple. It is given by completion with respect to $p, v_1, \ldots, v_{n-1}$. That is, if Y is another spectrum,

$$L_X Y = \varprojlim Y \wedge M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) = Y_{p,\ldots,v_{n-1}}.$$

We show in [Hov] that if Y is already $\mathrm{Tel}(0) \vee \cdots \vee \mathrm{Tel}(n)$-local, then

$$L_{\mathrm{Tel}(n)} Y = Y_{p,\ldots,v_{n-1}}.$$

Similarly, if Y is $\mathrm{Tel}(0) \vee \cdots \mathrm{Tel}(n-1) \vee K(n)$-local, then

$$L_{K(n)} Y = Y_{p,\ldots,v_{n-1}}.$$

We have then proved the following theorem.

THEOREM 1.5.4. *If R is a ring spectrum and v is a finitary v_n-element, then*

$$L_{\mathrm{Tel}(n)} R = (v^{-1}R)_{p,\ldots,v_{n-1}}.$$

If R satisfies the telescope conjecture and v is an arbitrary v_n-element, then

$$L_{K(n)} R = (v^{-1}R)_{p,\ldots,v_{n-1}}.$$

COROLLARY 1.5.5. *Suppose R is a ring spectrum. Then $v \in \pi_* R$ is a v_1-element if and only if*

$$L_{K(1)} R = (v^{-1} R)_p.$$

Proof. Every spectrum satisfies the telescope conjecture when $n = 1$. □

2. The bordism spectra $MO\langle k\rangle$ and $MU\langle k\rangle$

2.1. Introduction. Recall that the bordism spectrum $MO\langle k\rangle$ is obtained by taking the Thom spectrum of the kth Postnikov cover $BO\langle k\rangle$ of BO. They are ring spectra, and the natural maps

$$BO\langle k+1\rangle \to BO\langle k\rangle$$

give maps of ring spectra

$$MO\langle k+1\rangle \to MO\langle k\rangle.$$

We have analogous statements for the complex analog $MU\langle k\rangle$.

PROPOSITION 2.1.1. *The map induced by the unit*

$$S^0 \to \varprojlim MO\langle k\rangle$$

is a homotopy equivalence. Similarly

$$S^0 \to \varprojlim MU\langle k\rangle$$

is a homotopy equivalence.

Proof. We will just prove the real case, as the complex case is similar. Note that the inclusion of the basepoint into $BO\langle k\rangle$ is an isomorphism on homotopy through dimension $k-1$. It is therefore also an isomorphism on homology through dimension $k-1$. If $k \geqslant 2$, we can apply the Thom isomorphism to see that $S^0 \to MO\langle k\rangle$ is an isomorphism on homology through dimension $k-1$. It is thus an isomorphism on homotopy through dimension $k-2$ and an epimorphism on homotopy in dimension $k-1$. □

This is one of the motivations for studying the $MO\langle k\rangle$ and the $MU\langle k\rangle$. But very little work has been done here. The spectra $MSU = MU\langle 4\rangle$ and $MSpin = MO\langle 4\rangle$ are reasonably well understood, though some open questions remain. References include [Pen], [Bo], [Ko] for MSU and [ABP], [GP] for $MSpin$, as well as the standard reference for cobordism [St1]. The spectrum $MO\langle 8\rangle$ has been studied at the prime 2 in [G] and [DM2]. Gorbunov and Mahowald have recently computed the first 50 or so homotopy groups of $MO\langle 8\rangle$ at $p=2$ (see [GM]). There are also some unpublished notes of Pengelley and Ravenel [PR] concerning $MO\langle 8\rangle$ at $p=3$ (where, incidentally, it is much simpler than at $p=2$) and at larger primes. After this paper was written, the author and Ravenel investigated $MO\langle 8\rangle$ at $p=3$ and $MU\langle 6\rangle$ at $p=2$ in [HR]. Even the structure of the homology of $MO\langle k\rangle$ and $MU\langle k\rangle$ as modules over the Steenrod algebra is not known, though the homology groups and partial information about the Steenrod algebra structure are given by Stong [St], Singer [Sin], and Giambalvo [G2]. Giambalvo shows in [G2] that the mod p cohomology of $MO\langle k\rangle$ is free over the reduced power part of the Steenrod algebra when $p \geqslant (k/2)+1$. The results of Singer [Sin] together with Giambalvo's proof show that $H^*(MU\langle k\rangle;\mathbf{F}_p)$ is also free over the reduced powers when $p \geqslant (k/2)+1$. The author and Ravenel gave different proofs of these results in [HR]. One would like to conclude from this that $MO\langle k\rangle$ and $MU\langle k\rangle$ split into a wedge of suspensions of BP when $p \geqslant (k/2)+1$. But there may be torsion in the homology of $MO\langle k\rangle$ when $k>8$ and in the homology of $MU\langle k\rangle$ when $k>6$. It is true that $MO\langle 8\rangle$ and $MU\langle 6\rangle$ split into a wedge of suspensions of BP when $p \geqslant 5$. However, neither splitting can be multiplicative, and the ring structure even of $MO\langle 8\rangle_*$ at $p=5$ is not known. More detailed information at $p=2$ for $MO\langle 8\rangle$ is given by Davis in [D].

Our primary motivation for studying the $MO\langle k\rangle$ and the $MU\langle k\rangle$ is that, if one had never heard of real K-theory, studying $MSpin$ would lead one to it. Similarly, we hope that studying $MO\langle k\rangle$ and $MU\langle k\rangle$ will lead us to higher analogs of real K-theory like the EO_n constructed by Hopkins and Miller in [HMi]. Those theories are both highly complete and nonconnective. We hope that there are uncompleted and connective versions of the EO_n that will split off of appropriate $MO\langle k\rangle$, at least at $p=2$. At odd primes, we expect to find analogues of the Pengelley spectrum BoP that splits off MSU and is a sort of amalgam of BP and ko.

Note that there is an important difference between the behavior of the $MO\langle k\rangle$ at odd primes and their behavior at $p=2$. At odd primes, $MSpin$ splits multiplicatively into a wedge of suspensions of BP. Thus there is a ring spectrum map $MO\langle k\rangle \to BP$ for $k \geqslant 4$. At $p=2$, on the other hand, Bahri and Mahowald [BM] show that the Thom class gives a map

$$A//A(r-1) \to H\mathbf{F}_2^*(MO\langle \phi(r)\rangle).$$

Here A is the mod 2 Steenrod algebra and $A(r-1)$ is the Hopf subalgebra generated by the Sq^{2^i} for $i \leqslant r-1$. Also, for $r=4a+b$ with $0 \leqslant b \leqslant 3$, $\phi(r)$ is $8a+2^b$. This would lead one to guess that $K(j)_*MO\langle\phi(r)\rangle = 0$ for $j \geqslant r$.

We make the following conjecture about the $MO\langle k\rangle$.

CONJECTURE 2.1.2. (1) *(Ravenel). The Bousfield class of $MU\langle k\rangle$ at all primes and of $MO\langle k\rangle$ at odd primes is the same as that of BP. At $p=2$, the*

Bousfield class of $MO\langle\phi(r)\rangle$ is the same as that of $BP\langle r-1\rangle$. In particular, $MO\langle k\rangle$ and $MU\langle k\rangle$ satisfy the telescope conjecture.

(2) *There are v_n-elements in $MO\langle k\rangle_*$ and $MU\langle k\rangle_*$ for all n.*

(3) *At $p=2$, $MO\langle\phi(r)\rangle$ admits an orientation to EO_{r-1}, and the natural map*

$$MO\langle\phi(r)\rangle_*(X)\otimes_{MO\langle\phi(r)\rangle_*} EO_{i_*}\to EO_{i_*}(X)$$

is an isomorphism.

Part of this conjecture is proved in [HR], written after this paper. Indeed, in [HR, Corollary 5.4], it is shown that $MU\langle k\rangle$ and, at odd primes, $MO\langle k\rangle$ have the same Bousfield class as BP and that, at $p=2$, $MO\langle\phi(r)\rangle$ has Bousfield class less than or equal to $BP\langle r-1\rangle$. In particular, $MO\langle k\rangle$ and $MU\langle k\rangle$ do satisfy the telescope conjecture for all n.

We expect $MO\langle k\rangle$ to admit orientations to certain of the EO_i at odd primes as well. The first case of this should be that $MO\langle 8\rangle$ admits an orientation to EO_2 at $p=3$. However, it is unlikely that there could be a tensor product theorem here, because $MO\langle 8\rangle$ probably has BP summands in it at $p=3$. The logical generalization of this would be that $MO\langle 2p+2\rangle$ admits an orientation to EO_{p-1}. We will see below that this is impossible for $p>3$. It is possible that $MO\langle 2p+2\rangle$ admits an orientation to $EO_{2(p-1)}$.

2.2. The image of J. The results of Section 1.4 say that to find v_1-elements in a ring spectrum R, we should determine which members of the α family in π_*S^0 map to zero under the unit map $\eta\colon S^0\to R$. This problem is made much simpler by the fact that most of the α family is in the image of the J-homomorphism. This is described in [R1].

Let us recall, from [Ad], for example, the J-homomorphism $J\colon\pi_*(SO)\to\pi_*(S^0)$. Consider a homotopy class $x\colon S^k\to SO$. The adjoint of x is a map $\tilde{x}\colon S^{k+1}\to BSO$. This defines an oriented vector bundle γ over S^{k+1}. The Thom spectrum $M(\gamma)$ of γ is a 2-cell complex with a cell in dimension zero and in dimension $k+1$. The attaching map of this complex is the J-homomorphism applied to x, as pointed out in [Ad]. So we have a cofiber sequence

$$S^k\xrightarrow{Jx}S^0\longrightarrow M(\gamma)\to S^{k+1}.$$

We also have a map of Thom spectra

$$M(\gamma)\to MSO$$

and the composite $S^0\to M(\gamma)\to MSO$ is the unit. Indeed, it is the Thom spectrum of the composite

$$*\to S^{k+1}\to BSO$$

that is the unit of the H-space BSO.

Notice that $\tilde{x}$ obviously lifts to

$$S^{k+1} \xrightarrow{y} BO\langle k+1\rangle.$$

Thus we get a lift of the Thom class

$$M(\gamma) \to MO\langle k+1\rangle.$$

If we want to consider $MU\langle k+1\rangle$ instead, we need to consider the complex J-homomorphism J^c, and start with a class $x \in \pi_k U$. In that case, we get a Thom class

$$M(\gamma) \to MU\langle k+1\rangle.$$

The results of [Bott] show that $\pi_k U \to \pi_k SO$ is an isomorphism for $k \equiv 3 \pmod 8$ but is multiplication by two for $k \equiv 7 \pmod 8$. Thus $\operatorname{Im} J^c$ has index 2 in $\operatorname{Im} J$ in dimensions congruent to -1, 0, and 1 (mod 8) and is all of $\operatorname{Im} J$ in dimensions congruent to 3 (mod 8).

THEOREM 2.2.1. *The composite*

$$\operatorname{Im} J \to \pi_* S^0 \to \pi_* MO\langle k\rangle$$

is injective in dimensions $\leqslant k-2$ *and zero in dimensions* $\geqslant k-1$. *Similarly, the composite*

$$\operatorname{Im} J^c \to \pi_* S^0 \to \pi_* MU\langle k\rangle$$

is injective in dimensions $\leqslant k-2$ *and zero in dimensions* $\geqslant k-1$.

Proof. The proof is exactly the same in the real and complex cases, so we will just prove the real case. The unit map $S^0 \to MO\langle k\rangle$ is a $k-2$-equivalence, as pointed out in Proposition 2.1.1, so the injectivity in those dimensions is clear. Given a class $x \in \operatorname{Im} J$ in dimension $n \geqslant k-1$, we saw above that the unit map factors through the cofiber of $x\colon S^n \to S^0$, so the theorem follows in the real case. □

As an example, consider $MU\langle 8\rangle$. We have

$$(H\mathbf{F}_2)_0 MU\langle 8\rangle = (H\mathbf{F}_2)_8 MU\langle 8\rangle = \mathbf{F}_2,$$

and these are the only nonzero groups in dimensions $\leqslant 8$. The Steenrod algebra acts trivially through dimension 8, because the Thom class pulls back to $(H\mathbf{F}_2)^0 MU$, and the map

$$(H\mathbf{F}_2)^8 MU \to (H\mathbf{F}_2)^8 MU\langle 8\rangle,$$

is trivial. We use the standard notation for elements in the E_2 term of the Adams spectral sequence at $p = 2$ (see [R1]). So in the Adams spectral sequence for $\pi_* MU\langle 8\rangle$, the E_2 term will have $h_0^i h_3$ in filtration $i+1$ and topological degree 7, for $i \leqslant 3$, just as in the Adams spectral sequence for $\pi_* S^0$. Each such element is a permanent cycle, since it is a permanent cycle in the Adams spectral sequence for the sphere. No differential can hit h_3 since it is in filtration 1, so it survives to $\pi_7 MU\langle 8\rangle$, despite the fact that it corresponds to a class in Im J. The class $h_0 h_3$, on the other hand, corresponds to a class in the image of J^c, so must be hit by a differential by Theorem 2.2.1. There is a tower beginning in filtration zero on a class x in topological degree 8, and so we must have $d_2(x) = h_0 h_3$. Thus we have a homotopy class v in filtration 4 that exists because $h_0^4 h_3 = 0$. This class v is a v_1-element.

From Theorem 2.2.1 and the results of Section 1, we get the following theorem. Recall that $q = 2(p-1)$ for odd primes p, and let $q = 8$ at $p = 2$. Also note that $\alpha_{t/2}$ is in the image of the complex (in fact the quaternionic) J-homomorphism at $p = 2$.

THEOREM 2.2.2. *There are v_1-elements in $\pi_{qt} MO\langle k\rangle$ and $\pi_{qt} MU\langle k\rangle$ if and only if $qt \geqslant k$. Furthermore, if $k \geqslant 4$, any such element maps to a unit under the orientation to KO.*

Proof. The only thing we have not already shown is that a v_1-element maps to a unit in (the p-localization of) KO. To see this, note that the image must be a v_1-element in KO, and any such is a unit. □

Consider for a moment the manifold M corresponding to a v_1-element v constructed as above. Because $v \in \pi_{qt} MO\langle k\rangle$ can be lifted to $\pi_{qt} MO\langle qt\rangle$ simply by lifting the Thom class, the classifying map of the tangent bundle of M can be lifted to $BO\langle qt\rangle$. So the tangent bundle is trivial on the $qt-1$-skeleton. Such manifolds are called *almost parallelizable* and have been studied, for example, by Milnor-Kervaire in [MK]. Note that M is not parallelizable, since we just showed above that v maps to a (p-local) unit in KO_*, so the $\hat{A}$ genus of M is nontrivial. The author's original approach to this problem involved using these Milnor-Kervaire manifolds directly, but the above approach (suggested by Hopkins) using the image of J gives more information. We still need the Milnor-Kervaire manifolds for a lemma later.

We would like to extend this result to higher periodicities. We restate a part of Conjecture 2.1.2 and make it more specific.

CONJECTURE 2.2.3. *There are v_n-elements in $\pi_* MO\langle k\rangle$ and $\pi_* MU\langle k\rangle$ for all n. In $MO\langle 8\rangle$, there is a v_2-element in dimension* 144 *at $p = 3$, and in dimension* 192 *at $p = 2$.*

We give an outline of a possible proof of this conjecture. Consider a generalized Moore spectrum $M(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}})$ for a multiplicative sequence, and let Y denote the cofiber of the unit map, as in Section 1.5. Given a v_n-element v, let

$f: S^k \to Y$ denote its image in Y, and let C denote its cofiber. The key fact that allows us to handle the case $n = 1$ is that C, or a suspension of it, is a Thom spectrum. In general, no suspension of C can be a Thom spectrum of an oriented bundle, because the Thom class would have to be in degree zero, and there will be a nontrivial (secondary) Bockstein on the Thom class in C. However, we make the following conjecture.

CONJECTURE 2.2.4. *A suspension of the Spanier-Whitehead dual of C is a Thom spectrum.*

If this conjecture were true, then it is easy to see that the base space, which should probably be a suspension of the dual of Y, has to be highly connected, so we could repeat the argument above to find v_n-elements in $MO\langle k\rangle$.

The problem of finding the lowest dimension in which a v_n-element can occur is extremely intricate. Note that there is an orientation from $MO\langle 8\rangle$ to $K(n)$ at odd primes, so any v_n-element must map to an integral power of v_n. Using the method of this paper, one would have to know which sequences are multiplicative. At $p = 3$, $M(p, v_1)$ exists but is not a ring spectrum. $M(p, v_1^2)$ is a ring spectrum [Oka], but it is not known what power of v_2 occurs. There is some reason to think it is v_2^3, which is in dimension 48 [OT]. But the computation of EO_2 at $p = 3$, together with computations of the author in a joint work with Ravenel, suggests that the first v_2-element in $\pi_* MO\langle 8\rangle$ is v_2^9 in dimension 144. The situation at $p = 2$ is even less well understood. The results of Davis and Mahowald [DM] suggest that $M(4, v_1^4)$, which is a ring spectrum, should have a v_2^8-self-map. But their results were based on an error in the calculations of $\pi_* S^0$, which has recently been discovered by Kochman and Mahowald [KM]. The corrected version suggests it is v_2^{32} that occurs in dimension 192.

2.3. $K(1)$-localizations. From the previous section, we know that $L_{K(1)} MO\langle k\rangle$ and $L_{K(1)} MU\langle k\rangle$ can be obtained by inverting a v_1-element and completing at p. However, it is also possible to give a different description of the $K(1)$-localization.

PROPOSITION 2.3.1. *For any prime p and any $k \geqslant 4$, the natural map*

$$MO\langle k\rangle \to MSpin$$

is a $K(1)$-equivalence. Similarly, the natural map

$$MU\langle k\rangle \to MSU$$

is a $K(1)$-equivalence for $k \geqslant 2$.

Proof. It suffices to show that the natural maps

$$MO\langle k\rangle \to MO\langle k-1\rangle$$

and

$$MU\langle k\rangle \to MU\langle k-1\rangle$$

are $K(1)$-equivalences. As the complex case is similar to the real case, we concentrate on the real case. Consider the fibration

$$K(\pi_{k-1}BO, k-2) \to BO\langle k\rangle \to BO\langle k-1\rangle \tag{1}$$

for $k > 4$. (When $k = 4$, $MO\langle k\rangle = MSpin$ so there is nothing to prove.) Ravenel and Wilson have calculated the Morava K-theories of Eilenberg-MacLane spaces in [RW]. Their paper is restricted to the odd prime case, but their results also hold for $p = 2$ [JW]. In particular, $K(1)_*(K(Z, j)) = K(1)_*$ for $j \geqslant 3$ and $K(1)_*(K(Z/2Z, j) = K(1)_*$ for $j \geqslant 2$.

Consider the (convergent) Serre spectral sequence generalized by Atiyah and Hirzebruch, of the fibration (1):

$$E_2 = H_*(BO\langle k-1\rangle; K(1)_*) \Rightarrow K(1)_*BO\langle k\rangle.$$

There is a natural map of the fibration (1) to the trivial fibration

$$* \to BO\langle k-1\rangle \to BO\langle k-1\rangle.$$

The resulting map of spectral sequences is an isomorphism on E_2. Thus the map

$$BO\langle k\rangle \to BO\langle k-1\rangle$$

induces an isomorphism on $K(1)$ homology.

Now, the canonical vector bundles over $BO\langle k\rangle$ and $BO\langle k-1\rangle$ are both spin bundles. Spin bundles are KO-oriented, so in particular are $K(1)$-oriented. The resulting Thom isomorphism completes the proof. □

COROLLARY 2.3.2. *Suppose X is a torsion spectrum and $v \in \pi_*MO\langle k\rangle$ is a v_1-element where $k \geqslant 4$. Then the induced map*

$$v^{-1}MO\langle k\rangle_*X \to v^{-1}MSpin_*X$$

is an isomorphism. Similarly, if $v \in \pi_*MU\langle k\rangle$ is a v_1-element where $k \geqslant 2$, then the induced map

$$v^{-1}MU\langle k\rangle_*X \to v^{-1}MSU_*X$$

is an isomorphism.

Proof. We only prove the real case, as the complex case is similar. From Proposition 2.3.1, the map $v^{-1}MO\langle k\rangle \xrightarrow{f} v^{-1}MSpin$ is a $K(1)$-equivalence. On the other hand, Lemma 1.5.3 shows that $v^{-1}MO\langle k\rangle$ and $v^{-1}MSpin$ are K-local. It follows that the fiber F of f is rational, and hence that $F \wedge X$ is trivial. □

The method used to prove Proposition 2.3.1 applies more generally as well.

PROPOSITION 2.3.3. *Suppose p is an odd prime and r is an integer. Then, for all $k \geqslant r+3$, the map $MO\langle k\rangle \to MO\langle r+3\rangle$ is a $K(r)$-equivalence. Similarly, if p is arbitrary, then the map $MU\langle k\rangle \to MU\langle r+3\rangle$ is a $K(r)$-equivalence.*

The proof of this proposition is exactly the same as the proof of Proposition 2.3.1. The Thom isomorphism in $K(r)$-homology applies because the bundles involved are complex-oriented. This proposition may not be true for the $MO\langle k\rangle$ at $p = 2$. However, as in that case, we do not know if the Thom isomorphism applies.

We now use Proposition 2.3.3 to show that EO_{p-1} is not $MO\langle k\rangle$-oriented for any k when $p > 3$.

PROPOSITION 2.3.4. *Suppose $p > 3$ and k is arbitrary. Then there is no map of ring spectra*

$$MO\langle k\rangle \to EO_{p-1}.$$

Proof. There are two facts we need about EO_{p-1} from [HMi]. First, EO_{p-1} is $K(p-1)$-local, and second, the image of the unit map on homotopy contains $\alpha_1 \in \pi_{q-1}EO_{p-1}$. If we had an orientation $MO\langle k\rangle \to EO_{p-1}$, it would factor through $L_{K(p-1)}MO\langle k\rangle$. Proposition 2.3.3 shows that, if $k \geqslant p+2$,

$$L_{K(p-1)}MO\langle k\rangle \simeq L_{K(p-1)}MO\langle p+2\rangle.$$

Thus we would have an orientation $MO\langle k\rangle \to EO_{p-1}$ for some $k \leqslant p+2$. Thus $\alpha_1 \in \pi_{q-1}S^0$ must be mapped nontrivially to $\pi_{q-1}MO\langle p+2\rangle$. Since α_1 is in the image of J, by Theorem 2.2.1, we must have $q-1 \leqslant p$, that is, $p \leqslant 3$. □

Note that the above argument also shows that if any $MO\langle k\rangle$ admits an orientation to $EO_{2(p-1)}$, then $MO\langle 2p+2\rangle$ must admit such an orientation. This is no longer a contradiction, since $MO\langle 2p+2\rangle$ is the first $MO\langle k\rangle$ to detect α_1 at $p > 3$.

At odd primes both MSU and $MSpin$ are wedges of suspensions of BP. Therefore $L_{K(1)}MO\langle k\rangle$ and $L_{K(1)}MU\langle k\rangle$ are wedges of suspensions of $L_{K(1)}BP$, whose homotopy groups are completely known. To understand $L_{K(1)}MSpin$ and $L_{K(1)}MSU$ at $p = 2$, we must do a little work. Recall that Anderson, Brown, and Peterson [ABP] construct a 2-local splitting of $MSpin$

$$MSpin \xrightarrow{\simeq} \bigvee_{1 \notin J, n(J) \text{ even}} \Sigma^{4n(J)}ko \vee \bigvee_{1 \notin J, n(J) \text{ odd}} \Sigma^{4n(J)-4}ko\langle 2\rangle \vee \bigvee \Sigma^{?}H\mathbf{F}_2.$$

Here $ko\langle 2\rangle$ denotes the 1-connected cover of ko, which is in turn the connective cover of KO. J denotes a (possibly empty) partition, and $n(J)$ denotes the integer of which J is a partition, that is, the sum of the elements of J. Using this description, we can obtain the homotopy type of $L_{K(1)}MSpin$.

LEMMA 2.3.5. *The natural map*

$$ko\langle n\rangle \to KO$$

is a $K(1)$-equivalence. Thus

$$L_{K(1)}(ko\langle n\rangle) \simeq (KO)_2.$$

Proof. Denote the periodicity element in $\pi_8 ko$ by β. Note that the fiber of

$$\Sigma^8 ko\langle n\rangle \xrightarrow{\times\beta} ko\langle n\rangle$$

has a finite Postnikov tower. It is therefore $K(j)$-acyclic for all positive j. Hence $\times v$ is a $K(1)$-equivalence, and so

$$ko\langle n\rangle \to KO$$

is also. The $K(1)$-localization of any K-local spectrum is given by its p-completion, as is explained in [Hov, Section 2]. □

PROPOSITION 2.3.6. *We have*

$$L_{K(1)}MSpin \simeq \left(\bigvee_{1\notin J} KO_J\right)_2.$$

Proof. Note that, if $n(J)$ is even, $\Sigma^{4n(J)}ko \simeq ko\langle 4n(J)\rangle$. Similarly, if $n(J)$ is odd, $\Sigma^{4n(J)-4}ko\langle 2\rangle \simeq ko\langle 4n(J)-2\rangle$. Indeed,

$$\Sigma^{4n(J)}ko \xrightarrow{\times\beta^{n(J)/2}} ko$$

lifts to $ko\langle 4n(J)\rangle$ and is easily seen to be a homotopy equivalence. The proof is similar when $n(J)$ is odd.

Hence, from the Anderson-Brown-Peterson splitting,

$$MSpin \to \bigvee_{1\notin J} KO_J$$

is a $K(1)$-equivalence. Here the subscript J just serves to distinguish the different copies of KO. Now $\bigvee_{1 \notin J} KO_J$ is a KO-module spectrum, and thus is K-local. Hence, we obtain its $K(1)$-localization by completing it at 2. □

In [HH, Section 3], it was hinted that this splitting should hold.

Note that this means that we understand $L_{K(1)}MSpin$ at $p = 2$ in terms of simpler spectra, but at the moment we cannot write $L_{K(1)}MSpin$ at odd primes in a similar way. Nor do we have a similar result for MSU at $p = 2$. This leads us to make the following conjecture.

CONJECTURE 2.3.7. *Let*

$$L_{K(n)}BP \simeq \left(\bigvee E(n)\right)_{p,v_1,\dots,v_{n-1}}.$$

Similarly, at $p = 2$,

$$L_{K(1)}BoP \simeq \left(\bigvee KO \vee \bigvee K\right)_2.$$

Recall that $L_{K(n)}BP$ is the completion of $v_n^{-1}BP$ at the ideal $I_n = (p, v_1, \dots, v_{n-1})$. Baker and Würgler [BW] give a splitting of the Artinian completion of $v_n^{-1}BP$, but that is a much more drastic completion.

Recall Ochanine's result [Och] that

$$MSU_*(X) \otimes_{MSU_*} KO_* \to KO_*(X)$$

is not an isomorphism. If this conjecture were true, it would indicate that one could not fix this by replacing MSU with BoP. There would be summands of K in $(v^{-1}BoP)_2$ indicating that such an isomorphism is unlikely.

We now have two descriptions of $L_{K(1)}MO\langle k\rangle$ and $L_{K(1)}MU\langle k\rangle$. We can compare them to get some information about the homotopy of $MO\langle k\rangle$ and $MU\langle k\rangle$. Let v denote any v_1-element in $MSpin_*$ or MSU_* of positive Adams filtration. All the v_1-elements constructed above do have positive Adams filtration. First we need to know the homotopy of $v^{-1}MSpin$ and $v^{-1}MSU$.

LEMMA 2.3.8. *$v^{-1}MSpin_*$ is a free $\mathbf{Z}_{(p)}$-module on countably many generators in dimensions $4k$. It is, if $p = 2$, an $\mathbf{F}_2$-vector space with countably infinite basis in dimensions $8k+1$ and $8k+2$. It is zero in other dimensions. $v^{-1}MSU_*$ is a free $\mathbf{Z}_{(p)}$-module on countably many generators in even degrees not congruent to 2 modulo 8. In dimensions congruent to 2 modulo 8, it is the direct sum of such a free module with an $\mathbf{F}_2$-vector space of countably infinite dimension. In dimensions congruent to 1 modulo 8, it is an $\mathbf{F}_2$-vector space of countably infinite dimension.*

Proof. We begin with $MSpin$. If p is odd, $MSpin_*$ is a polynomial ring with one generator in each dimension $4k$ for $k > 0$. v is not a multiple of p, so it is easy to see that $v^{-1}MSpin_*$ has the required form. At $p = 2$, we must use the results of [ABP]. Recall that there are maps

$$\pi^J\colon MSpin \to KO,$$

one for each partition J. The ideal I_* of elements x in $MSpin_*$, such that $\pi^J(x) = 0$ for all J, is killed by v. Indeed, I_* maps monomorphically under the Hurewicz homomorphism to mod 2 homology, and v has trivial Hurewicz image. Thus $v^{-1}MSpin_* = v^{-1}(MSpin_*/I_*)$. Now $MSpin_*/I_*$ is free in dimensions $4k$ on finitely many generators and is an $\mathbf{F}_2$-vector space in dimensions $8k+1$ and $8k+2$. We will show that

$$MSpin_{4k}/I_{4k} \xrightarrow{\times v} MSpin_{4k+|v|}/I_{4k*+|v|}$$

is the inclusion of a summand, and the lemma will follow.

To do this it suffices to show that the quotient is torsion-free or, equivalently, that if $x \in MSpin_*/I_*$ is not divisible by 2^k, neither is vx. Since x is not divisible by 2^k, one of the $\pi^J(x)$ is not divisible by 2^k either, because some of the π^J give a 2-local splitting of $MSpin_*/I_*$. (See the next section or [ABP] for more details.) Recall that $n(J)$ denotes the sum of the elements of J. Choose a J so that $\pi^J(x)$ is not divisible by 2^k but all the $\pi^K(x)$ with $n(K) < n(J)$ are divisible by 2^k. There is a formula to calculate $\pi^J(vx)$, involving $\pi^0(v)\pi^J(x)$ and terms including $\pi^K(x)$ with $n(K) < n(J)$. All of the lower-order terms will be divisible by 2^k, but, since $\pi^0(v)$ is a power of the periodicity element by choice of v, $\pi^0(v)\pi^J(x)$ is not. Thus vx cannot be divisible by 2^k.

Proving the theorem for MSU is the same for p odd. For $p = 2$ we replace the results of [ABP] by those of Kochman [Ko] and Botvinnik [Bo], which show that multiplication by v_1^t includes MSU_j in MSU_{j+8t} as a summand. $\square$

Now take a v_1-element v in $MO\langle k\rangle_*$ with positive Adams filtration. We will denote the image of v in $MSpin_*$ by v as well. Consider the cofiber sequence

$$F \xrightarrow{f} v^{-1}MO\langle k\rangle \xrightarrow{g} v^{-1}MSpin.$$

Because g becomes a homotopy equivalence upon p-completion, F is necessarily rational, so is a wedge of copies of $H\mathbf{Q}$. Thus the image of f_* in $\pi_*(v^{-1}MO\langle k\rangle)$ is divisible. Since the map

$$MO\langle k\rangle \to MSpin$$

becomes the inclusion of a summand rationally, the same is true for

$$v^{-1}MO\langle k\rangle \to v^{-1}MSpin.$$

So the image of f_* must be a direct sum of copies of $\mathbf{Q}/\mathbf{Z}$. In particular, f_* cannot be injective. Similar comments hold when $MU\langle k\rangle$ and MSU replace $MO\langle k\rangle$ and $MSpin$. We therefore get the following corollary.

COROLLARY 2.3.9. *$v^{-1}MO\langle k\rangle_*$ is a free $\mathbf{Z}_{(p)}$-module in dimensions $4k$. If $p=2$, it is an $\mathbf{F}_2$-vector space on countably many generators in dimensions $8k+1$ and $8k+2$. It is a direct sum of copies of $\mathbf{Q}/\mathbf{Z}$ in dimensions $4k-1$. It is zero in all other dimensions. Similarly, $v^{-1}MU\langle k\rangle_*$ is a free $\mathbf{Z}_{(p)}$-module in even dimensions not congruent to 2 modulo 8. In dimensions congruent to 2 modulo 8, it is the direct sum of such a module with an $\mathbf{F}_2$-vector space of countably infinite dimension. In dimensions congruent to 1 modulo 8, it is the direct sum of such a vector space and copies of $\mathbf{Q}/\mathbf{Z}$. In all other dimensions, it is a direct sum of copies of $\mathbf{Q}/\mathbf{Z}$.*

This corollary does give some limited information about $MO\langle k\rangle_*$. It says, for example, that multiplication by v acts nilpotently in dimensions $8k+5$ and $8k+6$. It would be better if we knew that the divisible summands do not actually appear.

CONJECTURE 2.3.10. *There are no divisible summands in $v^{-1}MO\langle k\rangle_*$ or $v^{-1}MU\langle k\rangle_*$. Equivalently, the torsion in $v^{-1}MO\langle k\rangle_*$ and $v^{-1}MU\langle k\rangle_*$ is bounded.*

This conjecture would be true if the torsion in $MO\langle k\rangle_*$ and $MU\langle k\rangle_*$ were bounded. This conjecture was proved for $MO\langle 8\rangle$ at $p=3$ and for $MU\langle 6\rangle$ at $p=2$ in [HR].

2.4. Tensor products. We show that Conjecture 2.3.8 implies that the natural map

$$MO\langle k\rangle_*(X)\otimes_{MO\langle k\rangle_*}KO_*\to KO_*(X)$$

is an isomorphism, and that the natural map

$$MU\langle k\rangle_*(X)\otimes_{MU\langle k\rangle_*}KO_*\to KO_*(X)$$

is an isomorphism after inverting two. Recall that v is a v_1-element of positive Adams filtration in $MO\langle k\rangle_*$ or $MU\langle k\rangle_*$. We use the homotopy equivalences

$$(v^{-1}MO\langle k\rangle)_p\to(v^{-1}MSpin)_p \text{ and } (v^{-1}MU\langle k\rangle)_p\to(v^{-1}MSU)_p$$

proved above, together with the facts that (i)

$$MSpin_*(X)\otimes_{MSpin_*}KO_*\to KO_*(X)$$

is an isomorphism [HH], and (ii)

$$MSU_*(X)\otimes_{MSU_*}KO_*\to KO_*(X)$$

is an isomorphism away from 2. Inverting v is fine, since v maps to a unit of KO_*, but the completion is another story. The importance of Conjecture 2.3.8 is that, if it is true, passing to the completion does not lose any information. It is still somewhat technically complicated to prove our theorem though, as readers of [HH] and [Hov2] will recall.

We begin with the following lemma.

LEMMA 2.4.1. *Let v be a v_1-element in $\pi_{qt}MU\langle k\rangle$, where $q = 2p-2$ if p is odd and $q = 8$ if $p = 2$. Then the induced map of ring spectra*

$$v^{-1}MU\langle k\rangle \to KO$$

induces a surjection of homotopy groups. Therefore the map

$$v^{-1}MO\langle k\rangle \to KO$$

also induces a surjection of homotopy groups.

Proof. Recall that

$$KO_* \cong \mathbf{Z}_{(p)}[v_1, v_1^{-1}][a]/(a^{(p-1)/2} - v_1)$$

if p is odd, and

$$KO_* \cong \mathbf{Z}_{(2)}[v_1, v_1^{-1}][a, \alpha]/(a^2 - 4v_1, a\alpha, 2\alpha, \alpha^3)$$

if $p = 2$. Here we have written the periodicity element at $p = 2$ as v_1 for notational convenience. The image of a v_1-element in $MU\langle k\rangle_*$ is a v_1-element in KO_*, so must be a unit multiple of v_1^s for some t. By Theorem 2.2.2, there are v_1-elements in $\pi_{qs}MU\langle k\rangle$ for all sufficiently large s. It follows that the image of $v^{-1}MU\langle k\rangle_*$ in KO_* contains the subring $\mathbf{Z}_{(p)}[v_1, v_1^{-1}]$. In addition, the torsion classes in KO_* at $p = 2$ are in the image of the unit map, and so also in the image of $MU\langle k\rangle_*$. To complete the proof, we will show that $v_1^s a$ is in the image of $MU\langle k\rangle_*$ for all sufficiently large s.

For this we need the Milnor-Kervaire manifolds of [MK]. These are almost-parallelizable manifolds M^{4n} whose $\hat{A}$ genus is given by the formula

$$\hat{A}(M^{4n}) = -a_n \times \text{Numer}\left(\frac{(-1)^{n-1}B_{2n}}{4n}\right).$$

Here a_n is 1 if n is even and 2 if n is odd, B_{2n} is the $2n$th Bernoulli number, and Numer is the function which takes the numerator of a fraction when expressed in lowest terms. Since M^{4n} is almost parallelizable, the classifying map of its tangent bundle factors through S^{4n}, so M^{4n} defines a homotopy class in

$\pi_{4n}MO\langle k\rangle$ as long as $4n \geqslant k$. Furthermore, if n is odd, the map $\pi_{4n}BU \to \pi_{4n}BO$ is an isomorphism, so we get homotopy classes in $\pi_{4n}MU\langle k\rangle$.

We will show that if $4n = qj + 4$, then $\hat{A}(M^{4n})$ is not divisible by p if p is odd and is not divisible by 4 if $p = 2$. This will show that the image of M^{4n} in KO_* is a (p-local) generator and will complete the proof, since we can choose n to be odd and large enough to get a homotopy class in $MU\langle k\rangle$.

To do this, we need to know about the divisibility of Bernoulli numbers. This subject has been much studied ever since Kummer showed that it was relevant to Fermat's last theorem. Many facts about Bernoulli numbers, with very simple proofs, can be found in [Joh]. The simplest one is Von Staudt's theorem, found also in [Lang, p. 49]. It is easy to use this to show that the numerator of B_{2n} is always odd. This completes the proof at $p = 2$. At $p = 3$ there is nothing further to prove. For $p > 3$, we find in [Joh] that, if $p - 1$ does not divide r, B_r/r is a p-local integer, and that

$$\frac{B_r}{r} \equiv \frac{B_{r+p-1}}{r+p-1} \pmod{p}.$$

In particular,

$$\frac{B_{(p-1)j+2}}{(p-1)j+2} \equiv \frac{B_2}{2} = \frac{1}{12} \not\equiv 0 \pmod{p}.$$

Since the $\hat{A}$ genus of M^{qj+4} is a unit multiple of the numerator of $B_{(p-1)j+2}/((p-1)j+2)$, it is not divisible by p either. $\square$

Now, we know from [HH] that

$$v^{-1}MSpin_*(X) \otimes_{v^{-1}MSpin_*} KO_* \to KO_*(X)$$

is an isomorphism. In particular, from the preceding lemma,

$$v^{-1}MSpin_*(X) \to KO_*(X)$$

is surjective. If X is a finite torsion spectrum, then

$$v^{-1}MO\langle k\rangle_*(X) = v^{-1}MSpin_*(X)$$

by Corollary 2.3.2. Similarly, when p is odd, we know that

$$v^{-1}MSU_*(X) \otimes_{v^{-1}MSU_*} KO_* \to KO_*(X)$$

is an isomorphism, so for X a finite torsion spectrum,

$$v^{-1}MU\langle k\rangle_*(X) \to KO_*(X)$$

is surjective. From now on, let R denote either $v^{-1}MO\langle k\rangle$ or $v^{-1}MU\langle k\rangle$. When R is $v^{-1}MU\langle k\rangle$, we assume that p is odd. We apply the following proposition.

PROPOSITION 2.4.2. *Suppose we have a map $E \to M$ of (p-local) spectra such that, for any finite torsion spectrum X, $E_*(X) \to M_*(X)$ is surjective. Suppose as well that the torsion in E_* is bounded and that M_* is a finitely generated $\mathbf{Z}_{(p)}$-module in each dimension. Then $E_*(X) \to M_*(X)$ is surjective for all finite X.*

Before proving this proposition, we need an elementary lemma.

LEMMA 2.4.3. *Suppose $f: G \to H$ is a homomorphism of (p-local) abelian groups and that H is finitely generated. Suppose the composite*

$$G \xrightarrow{f} H \to H/p^nH$$

is surjective for some n. Then f is surjective.

Proof of Lemma 2.4.3. Consider the diagram of right exact sequences below.

$$\begin{array}{ccccc} G & \xrightarrow{\times p^n} & G & \longrightarrow & G/p^nG \\ {\scriptstyle f}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow \\ H & \xrightarrow{\times p^n} & H & \longrightarrow & H/p^nH \end{array}$$

The proof of the snake lemma shows that the times p^n map on the cokernel of f is surjective. Thus the cokernel of f is a finitely generated p-local, p-divisible group. Thus it must be zero. □

Proof of Proposition 2.4.2. It is easy to see that, if X is finite, $M_*(X)$ must be finitely generated in each dimension. Our plan is to show that there is a value of k such that

$$E_*(X) \to M_*(X)/(p^kM_*(X))$$

is surjective. We then apply the preceding lemma in each dimension to deduce that

$$E_*(X) \to M_*(X)$$

is surjective.

We are assuming that E_* has bounded torsion. It follows from [HH, Lemma 6] that for X finite, $E_*(X)$ also has bounded torsion. Choose k so large that p^k kills the torsion in $E_*(X)$. Consider the following diagram of short exact sequences, which comes from the defining cofibration for $M(p^k)$, the mod p^k Moore spectrum.

$$\begin{array}{ccccc}
E_*(X)/(p^k) & \xrightarrow{i_k} & E_*(X \wedge M(p^k)) & \xrightarrow{j_k} & \operatorname{Tor}(E_{*-1}(X)) \\
{\scriptstyle f_k}\downarrow & & {\scriptstyle g_k}\downarrow & & \downarrow \\
M_*(X)/(p^k) & \xrightarrow{i_k} & M_*(X \wedge M(p^k)) & \xrightarrow{j_k} & \operatorname{Tor}(M_{*-1}(X), \mathbf{Z}/p^k)
\end{array}$$

We have a similar diagram where k is replaced by $2k$, and a map between the diagram for $2k$ and the diagram for k, which we will call r for reduction. The key point is that the map r between the Tor terms is multiplication by p^k, which is the zero map on the top rows.

Now choose $x \in M_*(X)/(p^k)$. Choose a lift $y \in M_*(X)/(p^{2k})$, so that $r(y) = x$. There is a class

$$z \in E_*(X \wedge M(p^{2k}))$$

such that $g_{2k}(z) = i_{2k}(y)$. Then

$$g_k(r(z)) = r(g_{2k}(z)) = ri_{2k}(y) = i_k r(y) = i_k(x),$$

but also $j_k(r(z)) = r(j_{2k}(z)) = 0$, so there is a

$$w \in E_*(X)/(p^k)$$

such that $i_k(w) = r(z)$. Thus $f_k(w) = x$, and we see that

$$E_*(X) \to M_*(X)/(p^k)$$

is surjective. □

We have thus proved that, if Conjecture 2.3.8 is valid, then

$$v^{-1}R_*(X) \to KO_*(X)$$

is surjective for all finite X, and therefore that

$$R_*(X) \otimes_{R_*} KO_* \to KO_*(X)$$

is surjective for all finite X.

To prove injectivity, we follow the outline of [HH]. The argument on page 194 of that paper proves the following theorem.

THEOREM 2.4.4. *Suppose E is a (p-local) ring spectrum and M an E-module spectrum equipped with an E-module map $f: E \to M$. Suppose they satisfy the following conditions.*

(1) *The natural map*

$$(E \wedge M(p^k))_*(X) \otimes_{(E \wedge M(p^k))_*} (M \wedge M(p^k))_* \to (M \wedge M(p^k))_*(X)$$

is an isomorphism for all sufficiently large k and for all finite X.

(2) *$f_*: E_* \to M_*$ is surjective.*

(3) *$E_*(X) \otimes_{E_*} M_*$ has no infinitely p-divisible elements for finite X.*

(4) *$E_*(X)$ has no infinitely p-divisible elements for finite X.*

(5) *E_* has bounded torsion.*

Then the natural map

$$E_*(X) \otimes_{E_*} M_* \to M_*(X)$$

is injective for all finite X.

We claim that, assuming Conjecture 2.3.8, these conditions are satisfied with p arbitrary and $E = v^{-1}MO\langle k\rangle$, or p odd and $E = v^{-1}MU\langle k\rangle$ and $M = KO$. Note that we have already proved the second condition in Lemma 2.4.1. We have assumed the fifth condition. Under this assumption, both the third and fourth conditions follow from [Hov2, Lemma 1]. (Take $M = KO$ or $M = R$ in that lemma, and use Conjecture 2.3.8.)

We are left with verifying the first condition, that

$$(R \wedge M(p^k))_*(X) \otimes_{(R \wedge M(p^k))_*} (KO \wedge M(p^k))_* \to (KO \wedge M(p^k))_*(X)$$

is an isomorphism. But, by Corollary 2.3.2, this is the same as checking this statement for $R = MSpin$ and for $R = MSU$ and p odd. We did that for $p = 2$ in [HH]. For odd primes, we will just work with *MSpin*. The proof is exactly the same for MSU. Note that

$$(MSpin \wedge M(p^k))_* \cong MSpin_*/(p^k).$$

It is easy to check that, in general, if R is a ring and I is an ideal, and A and B are R/I-modules, then

$$A \otimes_R B \cong A \otimes_{R/I} B.$$

Thus

$$\begin{aligned} &MSpin \wedge M(p^k)_*(X) \otimes_{MSpin \wedge M(p^k)_*} KO \wedge M(p^k)_* \\ &\quad \cong MSpin \wedge M(p^k)_*(X) \otimes_{MSpin_*} KO_*/(p^k) \\ &\quad \cong MSpin_*(X \wedge M(p^k)) \otimes_{MSpin_*} KO_* \otimes_{KO_*} KO_*/(p^k) \\ &\quad \cong KO_*(X \wedge M(p^k)) \otimes_{KO_*} KO_*/(p^k) \cong KO_*(X \wedge M(p^k)). \end{aligned}$$

We have therefore proved the following theorem.

THEOREM 2.4.5. *If $v^{-1}MO\langle k\rangle_*$ has bounded torsion, in particular if $MO\langle k\rangle_*$ has bounded torsion, the natural map*

$$MO\langle k\rangle_*(X) \otimes_{MO\langle k\rangle_*} KO_* \to KO_*(X)$$

is an isomorphism for all X. Similarly, if $v^{-1}MU\langle k\rangle_*$ has bounded torsion away from 2, the natural map

$$MU\langle k\rangle_*(X) \otimes_{MU\langle k\rangle_*} KO_* \to KO_*(X)$$

is an isomorphism upon inverting 2.

Note that this statement can be interpreted both locally and globally. If all the torsion in $MO\langle k\rangle_*$ is bounded, we get a global statement. If we only know the p-torsion in $MO\langle k\rangle_*$ is bounded, we still get that

$$MO\langle k\rangle_*(X) \otimes_{MO\langle k\rangle_*} KO_* \to KO_*(X)$$

is an isomorphism after p-localization. In particular, $MO\langle 8\rangle_*$ has no p-torsion at all if $p > 3$, and has bounded torsion at $p = 3$ [HR], so we find that

$$MO\langle 8\rangle_*(X) \otimes_{MO\langle 8\rangle_*} KO_* \to KO_*(X)$$

is an isomorphism upon inverting 2. For $p > 3$, this is not very surprising, since the p-localization of $MO\langle 8\rangle$ splits into a wedge of copies of BP when $p > 3$. But this splitting is not multiplicative, and in fact the ring structure of $MO\langle 8\rangle_*$ is not known even at such large primes. It is definitely not polynomial [PR].

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