Calculating descent for 2-primary topological modular forms

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Abstract. We give an explicit presentation of the moduli stack of elliptic curves with full level three structures $\mathcal{M}(3)$, together with a description of the action of $GL_2(\mathbb{Z}/3)$ (due to Charles Rezk). Then we proceed to compute the cohomology ring $H^*(GL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)))$ with its full multiplicative structure including Massey products using several spectral sequences. The result is the E_2 page of a homotopy fixed point spectral sequence computing the homotopy groups of $Tmf[1/3]$.

1. Introduction

Topological modular forms exhibit some fascinating properties; for example, in [**Sto12**], the author showed that Tmf[1] is Anderson self dual (up to a shift) after inverting 2. The base step in that work consisted of constructing a ring spectrum $Tmf(2)$ of topological modular forms with level 2 structure, together with an action by the group $GL_2(\mathbb{Z}/2)$, and such that the homotopy fixed points $Tmf(2)^{hGL_2(\mathbb{Z}/2)}$ are naturally equivalent to $Tmf[1/2]$. The existence of such a spectrum does not automatically follow from the existence of the Goerss-Hopkins-Miller sheaf of E_∞ ring spectra on the étale site of the compactified moduli stack of elliptic curves $\mathcal{M}$, as $Tmf(2)$ lives over a *ramified* cover of $\mathcal{M}$. The construction of $Tmf(2)$ in [**Sto12**] imitated the construction of Tmf via obstruction theory, as in [**Beh**]. One reason that the author concentrated on the 3-primary phenomena in [**Sto12**] is to avoid the subtleties of 2-primary $K(1)$-local obstruction theory.

The 2-primary analogue relies on the construction of a ring spectrum $Tmf(3)$ equipped with a $GL_2(\mathbb{Z}/3)$ action, and such that the homotopy fixed point spectrum $Tmf(3)^{hGL_2(\mathbb{Z}/3)}$ is $Tmf[1/3]$. Such a spectrum does exist, and can be constructed by a more sophisticated version of the 2-primary obstruction theory. However, in work in progress, Mike Hill and Tyler Lawson extend the Goerss-Hopkins-Miller sheaf to the larger, so-called log-étale site of $\mathcal{M}$. The spectrum $Tmf(3)$, as well as $Tmf(n)$ for arbitrary level, lives over a ramified but log-étale cover of $\mathcal{M}$, so its existence is automatic after the Hill-Lawson sheaf is constructed.

Associated to the putative $Tmf(3)$ will be a homotopy fixed point spectral sequence

$$H^*(GL_2(\mathbb{Z}/3), \pi_*Tmf(3)) \Rightarrow \pi_*Tmf[1/3], \tag{1.1}$$

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[1] the non-connective and non-periodic version of topological modular forms

and the goal of the current paper is to compute the 2-completion of its E_2 page together with its complete multiplicative structure. The homotopy groups $\pi_* Tmf(3)$ will be computed via a degenerate descent spectral sequence

$$H^*(\mathcal{M}(3), \omega^*) \Rightarrow \pi_* Tmf(3),$$

where $\mathcal{M}(3)$ is the moduli scheme of generalized elliptic curves with (full) level 3 structure, and ω is its sheaf of invariant differentials.

In Section 4 we give an explicit presentation of the moduli scheme $\mathcal{M}(3)$ which allows for easy determination of the sheaf cohomology $H^*(\mathcal{M}(3), \omega^*)$, completed in Section 5. The ensuing sections are devoted to computing the group cohomology of $GL_2(\mathbb{Z}/3)$ with coefficients in $H^*(\mathcal{M}(3), \omega^*)$. As is to be expected, these computations are considerably more involved than their 3-primary counterparts.

The results of this paper will be used in subsequent work to deduce that Tmf is Anderson self dual at the prime 2, a result which glued together with the 3-primary duality of [**Sto12**] will imply an integral self-duality for Tmf.

In particular, the spectral sequence (1.1) will give an independent calculation of the homotopy groups of Tmf (at least after 2-completion). These calculations will be essentially the same as those of the homotopy fixed point spectral sequence

$$H^*(G_{48}, \pi_* E_2) \Rightarrow \pi_* E_2^{hG_{48}} \tag{1.2}$$

from [**Gut04, HM, Bau08**], where E_2 is the second Morava E-theory at the prime 2, and $G_{48} \cong GL_2(\mathbb{Z}/3)$ is a maximal subgroup of the (big) Morava stabilizer group $\mathbb{G}_2$, and the spectrum $E_2^{h\mathbb{G}_{48}}$ is the $K(2)$-localization of Tmf. The similarity is explained by looking at formal neighborhoods of the supersingular loci, $\mathcal{M}^{ss}$ and $\mathcal{M}(3)^{ss}$, of the respective moduli stacks and their maps to a formal neighborhood $\hat{\mathcal{H}}(2)$ of the height two locus of the moduli stack of formal groups. The moduli stack $\hat{\mathcal{H}}(2)$ has a $\mathbb{G}_2$-Galois cover by $\operatorname{Spf} E_2$ [**Goe**, Theorem 7.22]; pulling it back gives a G_{48}-Galois cover of $\mathcal{M}^{ss}$ by $\operatorname{Spf} E_2$, and pulling back further to $\mathcal{M}(3)^{ss}$ gives a trivial cover by $\operatorname{Spf} E_2$, i.e. $\mathcal{M}(3)^{ss} \simeq \operatorname{Spf} E_2$. This reflects the fact that the $K(2)$-localization of $Tmf(3)$ is a form of E_2, which can be deduced by our results in Section 4.

Nonetheless, one important advantage of using the homotopy fixed point spectral sequence (1.1) over (1.2) is that the former gives the homotopy of Tmf, not only its $K(2)$-localization. Moreover, the moduli approach is more geometric and avoids going through the action of $\mathbb{G}_2$ on E_2 which can be quite difficult to describe in an explicit manner. In particular, identifying an explicit maximal finite subgroup of $\mathbb{G}_2$ is a lengthy ad-hoc computation.

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2. Recalling elliptic curves

A curve over a base S is a map of schemes $p : E \to S$ which is flat, proper, has finite presentation and dimension one. An elliptic curve is a diagram

$$p : E \rightleftarrows S : e \tag{2.1}$$

where $p : E \to S$ is a curve of genus one whose geometric fibers are non-empty, connected, and smooth, and $e : S \to E$ is its section. An elliptic curve has a unique structure of an abelian group scheme with e as its identity [**KM85**, 2.1]. The object which classifies elliptic curves and isomorphisms between them is the moduli stack of elliptic curves, denoted[2] $\mathcal{M}^0$. Its compactification $\mathcal{M}$ classifies diagrams (2.1) where $p : E \to S$ is a curve of genus one whose geometric fibers can be smooth or have an isolated nodal singularity away from e [**DR73**, II.1.12]. The objects classified by $\mathcal{M}$ are called *generalized* elliptic curves.

Associated to an elliptic curve $p : E \to S$ is the sheaf $\omega_{E/S}$ of (translation-) invariant differentials, which can be described as the push-forward $p_*\Omega_{E/S}$ of the sheaf of relative differentials on E. Let $\mathcal{I}$ denote the ideal sheaf defining the identity section e; then by [**Har77**, II.8]

$$\omega_{E/S} = p_*\Omega_{E/S} = \mathcal{I}/\mathcal{I}^2. \tag{2.2}$$

The sheaf $\omega_{E/S}$ is invertible; if E is generalized (i.e. not necessarily smooth), $\omega_{E/S}$ can be defined as $\mathcal{I}/\mathcal{I}^2$, giving again an invertible line bundle. Consequently, the assignment

$$E/S \mapsto \omega_{E/S}$$

defines an invertible quasi-coherent sheaf on $\mathcal{M}$, which we denote by ω. The ring of (holomorphic) modular forms MF_* is defined to be the graded ring

$$H^0(\mathcal{M}, \omega^*) = \bigoplus_{n \geq 0} H^0(\mathcal{M}, \omega^{\otimes n}).$$

Locally, a choice of an $\mathcal{O}_S$-basis for $\omega_{E/S}$ gives rise to a Weierstrass equation for E as follows. Let $U = \operatorname{Spec} R$ be an open subset of S on which $\omega_{E/S}$ is trivializable, with η as a generator. Note that η is unique up to multiplication by a unit $u \in R^\times$. For any $n \in \mathbb{Z}$, η^n generates $\mathcal{I}^n/\mathcal{I}^{n+1}$, and for $n > 0$, the sheaf $p_*\mathcal{I}^{-n}$ is locally free of rank n [**KM85**, 2.2.5].

The natural inclusion $\mathcal{O}_C \cong \mathcal{I}^0 \to \mathcal{I}^{-n}$ defines a generator 1 of $\mathcal{I}^{-n}$ for any $n \geq 0$. Let x be a generator of $\mathcal{I}^{-2}$ which reduces to η^{-2} in $\mathcal{I}^{-2}/\mathcal{I}^{-1}$, and let y be a generator of $\mathcal{I}^{-3}$ which reduces to η^{-3} in $\mathcal{I}^{-3}/\mathcal{I}^{-2}$. Then $\{1, x\}$ is a basis for $\mathcal{I}^{-2}$, and $\{1, x, y\}$ is a basis for $\mathcal{I}^{-3}$. Note x and y are uniquely determined up to a change of variables

$$\begin{aligned} x &\mapsto u^{-2}x + r, \\ y &\mapsto u^{-3}y + u^{-2}sx + t, \end{aligned} \tag{2.3}$$

where u is a unit, and r, s, t are arbitrary elements of R. Continuing in this fashion, we find that $p_*\mathcal{I}^{-4}$ is freely generated by $1, x, y, x^2$, and $p_*\mathcal{I}^{-5}$ by $1, x, y, x^2, xy$. Next, $p_*\mathcal{I}^{-6}$ is freely generated on either $1, x, y, x^2, xy, y^2$ or $1, x, y, x^2, xy, x^3$, where

[2] as in [**DR73**].

$y^2 - x^3$ is in fact an element of $\mathcal{I}^{-5}$, as x^3 and y^2 both reduce to η^{-6}. Therefore, a relation called a Weierstrass equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{2.4}$$

must hold, for some $a_i \in R$. In fact, the map

$$\phi = [x, y, 1] : E_U \to \mathbb{P}^2_U$$

identifies E_U with the locus of vanishing of (2.4), with the identity section e mapping to the point at infinity $[0:1:0]$ in $\mathbb{P}^2_U$][**Sil86**, III.3],[**KM85**, 2.2.5]. The differential form η is expressed as

$$\eta = \frac{dx}{2y + a_1x + a_3} = \frac{dy}{3x^2 + 2a_2x + a_4 - a_1y}. \tag{2.5}$$

Conversely, any curve given by an equation (2.4) which is smooth or has at most a nodal singularity is a generalized elliptic curve. Consequently, the moduli stack of generalized elliptic curves $\mathcal{M}$ is an open substack of the stack represented by the Hopf algebroid (A, Γ), where $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$ and $\Gamma = A[u^{\pm 1}, r, s, t]$, while the structure map $\psi : A \to \Gamma$ is deduced from the change of variables (2.3). The formulas can be found in [**Sil86**, Table 1.2], [**Del75**], or [**Bau08**, Section 3].

This presentation eases the computation of the ring of modular forms; one has [**Del75**]

$$MF_* = \mathbb{Z}[c_4, c_6, \Delta]/(12^3\Delta = c_4^3 - c_6^2),$$

where c_n is a global section of $\omega^{\otimes n}$, and Δ is the discriminant of the equation (2.4). The global sections of $\omega^{\otimes 12}$ c_4^3 and Δ do not simultaneously vanish (as that would indicate a cusp singularity), and they define a map

$$j = [c_4^3 : \Delta] : \mathcal{M} \to \mathbb{P}^1 \tag{2.6}$$

called the j-invariant (its target usually called the projective j-line), which classifies the line bundle $\omega^{\otimes 12}$ and which restricts to $j : \mathcal{M}^0 \to \mathbb{A}^1 \overset{i}{\hookrightarrow} \mathbb{P}^1$, where i includes the complement of the point $[1:0]$ in $\mathbb{P}^1$. The j invariant can be used to describe the compactification $\mathcal{M}$ of $\mathcal{M}^0$ as the normalization in the field of functions of $\mathcal{M}^0$ of the projective j-line [**DR73**].

The embedding $\phi : E_U \to \mathbb{P}^2_U$ also gives a geometric way to describe the group law on E_U by interpreting Abel's theorem [**KM85**, 2.1.2] as explained in [**Sil86**, III.2]. A line in $\mathbb{P}^2_U$ intersects E_U at exactly three points (counted with multipilicites) since the defining equation (2.4) has degree three. Then the sum $P + Q + R$ of three (not necessarily distinct) points of E is the identity if and only if they are collinear in $\mathbb{P}^2_U$.

3. Recalling level structures

Let n be a positive integer and let S be a scheme over $\mathbb{Z}[1/n]$; then multiplication by n on a smooth elliptic curve E/S is a finite map of degree n^2 whose kernel $E[n]$ is étale locally isomorphic to $(\mathbb{Z}/n)^2$. Specification of one such isomorphism is called a (full) level n structure on E. For $i = 1, 2$, let $\big(E_i, \varphi_i : (\mathbb{Z}/n)^2 \to E_i\big)$ be two elliptic curves (both over S) with level n structures; an isomorphism

$$f : (E_1, \varphi_1) \to (E_2, \varphi_2)$$

is a commutative diagram

$$\begin{tikzcd}
(\mathbb{Z}/n)^2 \arrow[r, "\varphi_1"] \arrow[d, equal] & E_1 \arrow[d, "\psi"] \\
(\mathbb{Z}/n)^2 \arrow[r, "\varphi_2"] & E_2
\end{tikzcd}$$

where ψ is an isomorphism of elliptic curves. We denote by $\mathcal{M}(n)^0$ the moduli stack classifying elliptic curves with level n structure and isomorphisms between them. In fact, $\mathcal{M}(n)^0$ is a scheme whenever $n \geq 3$ [**DR73**, IV.2.7]. Forgetting the level structure gives a covering map $f : \mathcal{M}(n)^0 \to \mathcal{M}^0[1/n]$, hence also a j-invariant $j : \mathcal{M}(n)^0 \to \mathbb{A}^1[1/n]$ by composition.

The finite group $E[n]$ of n-torsion points in E is equipped with a non-degenerate alternating form

$$e_n : E[n] \times E[n] \to \mu_n,$$

called the Weil pairing [**KM85**, 2.8] into the group of n-th roots of unity. While at first we have $\mathcal{M}(n)^0$ as a scheme over $\mathbb{Z}[\frac{1}{n}]$, the Weil pairing gives a map $\mathcal{M}(n)^0 \to \operatorname{Spec}\mathbb{Z}[\zeta_n, \frac{1}{n}]$ by sending (E, φ) to $e_n(\varphi(1,0), \varphi(0,1)) = \zeta_n$.

Now the compactification $\mathcal{M}(n)$ of $\mathcal{M}(n)^0$ can be described as the normalization in the field of functions of $\mathcal{M}(n)^0$ of the projective j-line over $\mathbb{Z}[\zeta_n]$ [**DR73**]. Deligne-Rapoport [**DR73**] develop insightful and important modular description of these compactifications using so-called Néron polygons, but for the purposes of this work we will stick to the approach via normalization.

The automorphism group $GL_2(\mathbb{Z}/n)$ of $(\mathbb{Z}/n)^2$ acts on the right on $\mathcal{M}(n)^0$ by precomposition; namely, $g \in GL_2(\mathbb{Z}/n)$ maps (E, φ) to $(E, \varphi \circ g)$. This action is in fact free and transitive, making the forgetful map $f : \mathcal{M}(n)^0 \to \mathcal{M}^0[1/n]$ a torsor for the group $GL_2(\mathbb{Z}/n)$. Moreover, the action extends over $\mathcal{M}(n)$, but the stabilizers of the cusps are non-trivial; they are conjugates of the subgroup

$$\pm U := \left\{ \pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset GL_2(\mathbb{Z}/n),$$

as described in [**DR73**, IV.5].

Since $\mathcal{M}(n)$ is a stack (scheme when $n > 2$) over $\mathbb{Z}[\zeta_n, \frac{1}{n}]$, after a finite étale extension, it splits as a disjoint union of stacks $\mathcal{M}(n)^\zeta$ indexed by the primitive n-th roots of unity. To be more precise, let k be a ring in which n is invertible and which contains a primitive n-th root of unity; then $k[\zeta_n] := k[x]/\phi_n(x)$, where ϕ_n is the n-th cyclotomic polynomial, splits as a product $k \times \cdots \times k$, and therefore any scheme or stack X over $k[\zeta_n]$ splits as a disjoint union indexed over $\mu_n^\times$. In particular, this happens for $\mathcal{M}(n)$ and we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}(n)^\zeta \arrow[r] \arrow[d] & \coprod_{\mu_n^\times} \mathcal{M}(n)^\zeta \arrow[r] \arrow[d] & \mathcal{M}(n) \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\zeta_n] \arrow[r] & \operatorname{Spec}\mathbb{Z}[\zeta_n, \frac{1}{n}] \arrow[r] & \operatorname{Spec}\mathbb{Z}[\frac{1}{n}],
\end{tikzcd} \tag{3.1}$$

in which the middle and left squares are pullbacks. The moduli stack $\mathcal{M}(n)^\zeta$ (seen as an object over $\operatorname{Spec} k$) has action by the subgroup $SL_2(\mathbb{Z}/n)$, and in fact the composed map

$$\mathcal{M}(n)^{\zeta} \to \mathcal{M} \underset{\mathbb{Z}[\frac{1}{n}]}{\otimes} k$$

is an $SL_2(\mathbb{Z}/n)$-torsor away from the cusps which have $\pm U \subset SL_2(\mathbb{Z}/n)$ as stabilizers.

4. Level-3-structures made explicit

In this section we describe an explicit presentation of the moduli stack $\mathcal{M}(3)$, due to Charles Rezk. From this point on, 3 will be assumed to be invertible everywhere.

Let E/S be a generalized elliptic curve (over a scheme on which 3 is invertible); by completing the cube in the Weierstrass equation (2.4), we get that locally E is isomorphic to a Weierstrass curve of the form

$$y^2 + a_1xy + a_3y = x^3 + a_4x + a_6, \tag{4.1}$$

with discriminant $\Delta = (a_1^3 - 27a_3)a_3^3$. The points of order three are the inflection points of E. Choose $P = (r, t)$ to be such a point; applying the transformation $(x, y) \mapsto (x + r, y + t)$ puts E in the form

$$y^2 + a_1xy + a_3y = x^3, \tag{4.2}$$

where now P has coordinates $(0, 0)$. To see why in the equation (4.2) a_4 and a_6 are zero, let $f(x, y)$ denote the polynomial

$$f(x, y) = x^3 + a_4x + a_6 - y^2 - a_1xy - a_3y;$$

then a_6 in (4.2) is precisely $f(r, t)$ which is zero as P is a point on E, and a_4 in (4.2) is $\frac{\partial f}{\partial x}(r, s)$ which is zero as P is an inflection point. Moreover, the inversion map $[-1] : E \to E$ is now given by $[-1](x, y) = (x, y - a_1x - a_3)$. Thus $[-1]P = (0, -a_3)$, and the tangent line to $[-1]P$ is $y = -a_1x - a_3$.

4.1. The nonsingular case. If the curve E is smooth, choose $Q = (e_2, e_3)$ to be another point of order three which is different from $\pm P$. There are exactly three points on E with x-coordinate equal to zero ($\pm P$ and the point at infinity), hence $e_2 = x(Q)$ is invertible. From (4.2) it follows then that $e_3 = y(Q)$ is also invertible. If $y = b_1x + b_3$ is the tangent line to E at Q, we have (as Q is an inflection point)

$$x^3 - (x - e_2)^3 = (b_1x + b_3)^2 + a_1x(b_1x + b_3) + a_3(b_1x + b_3),$$

which yields

$$\begin{aligned} 3e_2 &= b_1^2 + a_1b_1 \\ -3e_2^2 &= 2b_1b_3 + a_1b_3 + b_1a_3 \\ e_2^3 &= b_3^2 + a_3b_3, \end{aligned}$$

whence b_1, b_3, as well as $e_3 - b_3 = b_1e_2$ must be invertible. In particular, the quotient e_3/b_3 cannot be 1. However,

$$\begin{aligned} \frac{e_3^3}{b_3^3} &= \frac{(b_1e_2 + b_3)^3}{b_3^3} \\ &= \frac{(b_3^2 + a_3b_3)b_1^3 - (2b_1b_3 + a_1b_3 + b_1a_3)b_1^2b_3 + (b_1^2 + a_1b_1)b_1b_3^2 + b_3^3}{b_3^3} = 1, \end{aligned}$$

hence $\frac{e_3}{b_3}$ must be a primitive third root of 1. Set $\zeta = e_3/b_3$, and denote $\gamma_1 = b_1$ and $\gamma_2 = a_1 + b_1$. We have the following formulas.

$$\begin{aligned} a_1 &= \gamma_2 - \gamma_1 \\ e_2 &= \frac{1}{3}\gamma_1\gamma_2 \\ b_3 &= -\frac{1}{9}(1-\zeta^2)\gamma_1^2\gamma_2 \\ e_3 &= \frac{1}{9}(1-\zeta)\gamma_1^2\gamma_2 \\ a_3 &= \frac{1}{9}(1-\zeta^2)\gamma_1\gamma_2(\gamma_1+\zeta\gamma_2) \\ a_1^3 - 27a_3 &= (\gamma_2 - \zeta\gamma_1)^3. \end{aligned}$$

Let us also record that the modular forms associated to the curve (4.2) are

$$\begin{aligned} c_4 &= a_1^4 - 24a_1a_3 \\ c_6 &= a_1^6 + 36a_1^3a_3 - 216a_3^2 \\ \Delta &= a_3^3(a_1^3 - 27a_3). \end{aligned} \tag{4.3}$$

4.2. A presentation. Let $\Gamma = \mathbb{Z}[1/3,\zeta][\gamma_1,\gamma_2]$ be the graded ring with γ_i in degree 1.[3] The above discussion shows that the locus $\mathcal{M}^0(3)^\zeta$ of smooth curves in $\mathcal{M}(n)^\zeta$ is

$$\left(\operatorname{Spec}\Gamma[\Delta^{-1}]\right)//\mathbb{G}_m.$$

Consequently, the compactification $\mathcal{M}(3)^\zeta$ must be $\operatorname{Proj}\Gamma$.

4.3. The action of $GL_2(\mathbb{Z}/3)$. Fix an elliptic curve E and its Weierstrass equation adapted to the level structure (P,Q) as above, and think of a_1, b_1, ζ as functions of the level structure (P,Q). To determine the action of $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL_2(\mathbb{Z}/3)$ on $\mathcal{M}(3)$, we need to determine the Weierstrass equation associated to E with the level structure $(P,Q)A = (\alpha P + \gamma Q, \beta P + \delta Q)$ (giving $a_1((P,Q)A)$), the slope $b_1((P,Q)A)$ of the tangent line at $\beta P + \delta Q$, as well as $\zeta = \frac{y(\beta P + \delta Q)}{b_3((P,Q)A)}$.

Note that a_1 is in fact only a function of the first point of order three. We already saw that when P is at $(0,0)$, then $-P$ has coordinates $(0,-a_3)$ and a tangent line $y = -a_1x - a_3$. The transformation

$$\begin{aligned} x &\mapsto x \\ y &\mapsto y - a_1x - a_3 \end{aligned}$$

moves $-P$ to $(0,0)$, putting E in the form

$$y^2 - a_1xy - a_3 = x^3.$$

In particular $a_1(-P) = -a_1$.

Similarly, the transformation

$$\begin{aligned} x &\mapsto x + e_2 \\ y &\mapsto y + b_1x + e_3 \end{aligned}$$

[3]Note that this is the algebraic degree; the topological degree is twice the algebraic, and we will not use it in this work other than in the depiction of spectral sequences.

moves Q to $(0,0)$ and gives that $a_1(Q) = a_1 + 2b_1$. This transformation also moves P to $(-e_2, -e_3)$, giving that

$$b_1(Q,P) = -b_1, \qquad b_3(Q,P) = -e_3,$$
$$e_3(Q,P) = -b_3, \qquad \zeta(Q,P) = \zeta^{-1}.$$

The line through P and Q is $y = \frac{e_3}{e_2}x$, the other point of E which lies on this line is

$$R = \left(-\frac{a_3e_3}{e_2^2}, -\frac{a_3e_3^2}{e_2^3}\right),$$

and $R = -P - Q$. The tangent line at R is given by

$$y = -\zeta b_1 x - \frac{b_1^2}{9}((\zeta^2 - 1)a_1 + (\zeta - 1)b_1).$$

Consequently,

$$b_1(Q, -P-Q) = \zeta^2 b_1, \qquad b_3(Q, -P-Q) = -\frac{b_1^2}{9}((\zeta - 1)a_1 + 2\zeta b_1),$$
$$e_3(Q, -P-Q) = \frac{b_1^2}{9}((2\zeta + 1)a_1 - 3\zeta^2 b_1), \qquad \zeta(Q, -P-Q) = \zeta.$$

Putting all of the above together, we deduce that the (left) action of $GL_2(\mathbb{Z}/3)$ on Γ is the ring action determined by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} : \begin{matrix} \gamma_1 \mapsto -\gamma_1, \\ \gamma_2 \mapsto -\gamma_2, \end{matrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \begin{matrix} \gamma_1 \mapsto -\gamma_2, \\ \gamma_2 \mapsto \gamma_1, \end{matrix}$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \begin{matrix} \gamma_1 \mapsto -\gamma_1, \\ \gamma_2 \mapsto \gamma_2, \end{matrix} \qquad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} : \begin{matrix} \gamma_1 \mapsto \zeta^2\gamma_1, \\ \gamma_2 \mapsto \gamma_2 - \zeta\gamma_1. \end{matrix}$$

The elements of $SL_2(\mathbb{Z}/3)$ preserve ζ, while the rest map ζ to $\zeta^{-1} = \zeta^2$.

Fix the following choice of generators of $SL_2(\mathbb{Z}/3)$

$$x = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), \quad y = \left(\begin{smallmatrix} -1 & -1 \\ -1 & 1 \end{smallmatrix}\right), \quad z = \left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right);$$

we will refer to these same generators again in Section 7. The above analysis gives us the following summary of the action of $SL_2(\mathbb{Z}/3)$.

Proposition 4.1. *The group $SL_2(\mathbb{Z}/3)$ acts on $\mathcal{M}(3)^\zeta = \operatorname{Proj}\Gamma$ by the map*

$$\chi : SL_2(\mathbb{Z}/3) \to PGL_2(\mathbb{Z}[1/3, \zeta]) = \operatorname{Aut}(\operatorname{Proj}\Gamma)$$

given by

$$x \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -\zeta^2 & \zeta \\ \zeta & \zeta^2 \end{pmatrix}, \quad z \mapsto \begin{pmatrix} \zeta^2 & 0 \\ -\zeta & 1 \end{pmatrix}. \tag{4.4}$$

Remark 4.2. Note that the modular forms c_4, c_6, Δ from (4.3) form an $SL_2(\mathbb{Z}/3)$-invariant subring $MF_* = \mathbb{Z}[1/3, \zeta][c_4, c_6, \Delta]/(2^6\Delta - c_4^3 - c_6^2)$ of Γ. We will see later that MF_* consists of *all* such invariants.

Remark 4.3. The modular form Δ has a cube root in Γ, it is

$$d = \sqrt[3]{\Delta} = \frac{1 - \zeta^2}{9}\gamma_1\gamma_2(\gamma_1 + \zeta\gamma_2)(\gamma_2 - \zeta\gamma_1).$$

The element d is preserved by $x, y \in SL_2(\mathbb{Z}/3)$, whereas $z \cdot d = \zeta^2 d$.

5. Serre Duality on $\mathcal{M}(3)^\zeta$

Since $\mathcal{M}(3)^\zeta = \operatorname{Proj}\Gamma$ is a projective line, it has Serre duality, and its dualizing sheaf is the invertible sheaf of differentials $\Omega = \Omega_{\mathcal{M}(3)^\zeta}$. Note that line bundles on $\mathcal{M}(3)^\zeta$ are in bijection with shifts of Γ as a module over itself; namely, for any integer k, $\mathcal{O}(k)$ denotes the line bundle corresponding to the graded module $\Gamma[k]$ which in degree t is the $(t+k)$-graded part of Γ. We have $\mathcal{O}(k)\otimes\mathcal{O}(n) = \mathcal{O}(k+n)$.

Now the differential form $\gamma_1 d\gamma_2 - \gamma_2 d\gamma_1$ is a nowhere vanishing differential form of degree two, hence it is a trivializing global section of $\mathcal{O}(2)\otimes\Omega$. We conclude that $\Omega \cong \mathcal{O}(-2)$.

On the other hand, the sheaf ω is a line bundle locally generated by the invariant differential $\eta = \frac{dx}{2y+a_1x+a_3}$ which is of degree 1, so $\omega \cong \mathcal{O}(1)$. Consequently, $\Omega \cong \omega^{-2}$.

The cohomology $H^*(\mathcal{M}(3)^\zeta, \Omega)$ is zero in degrees other than 1, and is $\mathbb{Z}[1/3,\zeta]$ in degree 1. The group $SL_2(\mathbb{Z}/3)$ acts on $H^1(\mathcal{M}(3)^\zeta,\Omega) =: \mathbb{Z}_\zeta$ via the determinant of the image of χ in $PGL_2(\mathbb{Z}[1/3,\zeta])$ (4.4). Hence x and y act trivially, and z acts as multiplication by ζ^2.

There is an $SL_2(\mathbb{Z}/3)$-equivariant Serre duality pairing [**Har77**, III.7.1]

$$H^0(\mathcal{M}(3)^\zeta,\omega^*)\otimes H^1(\mathcal{M}(3)^\zeta,\omega^{-*-2}) \to \mathbb{Z}_\zeta,$$

hence $H^1(\mathcal{M}(3)^\zeta,\omega^{-*-2}) \cong \operatorname{Hom}(\Gamma,\mathbb{Z}_\zeta)[-2] =: \Gamma_\zeta^\vee[-2]$, where $\Gamma_\zeta = \Gamma\otimes\mathbb{Z}_\zeta$. We will now proceed to compute the cohomology

$$H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^\zeta,\omega^*)) = H^*(SL_2(\mathbb{Z}/3), \Gamma\oplus\Gamma_\zeta^\vee[-2]).$$

6. Quaternion group cohomology

Having determined the action of $GL_2(\mathbb{Z}/3)$ on Γ, we will proceed to compute the cohomology ring $H^*(GL_2(\mathbb{Z}/3),\Gamma)$. We will make repeated use of Lyndon-Hochschild-Serre (LHSSS) and Bockstein spectral sequences (BSS), and we will keep track of Massey products, which in particular will be useful in identifying hidden extensions in the E_∞-pages of the various spectral sequences.

The quaternion group Q_8, which has a presentation

$$\langle x,y | xyx = y, x^2 = y^2, x^4 = 1\rangle$$

is a subgroup of $GL_2(\mathbb{Z}/3)$, and a Sylow 2-subgroup of $SL_2(\mathbb{Z}/3)$. As a preliminary calculation, we will determine $H^*(Q_8,\mathbb{F}_4)$, where Q_8 acts trivially on $\mathbb{F}_4$, and consequently the cohomology of Q_8 with trivial coefficients for any extension of $\mathbb{F}_4$.

The Lyndon-Hochschild-Serre spectral sequence for

$$1 \to C_2\times C_2 \to Q_8 \to C_2 \to 1$$

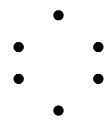
looks as

$$\mathbb{F}_4[a,b,c] = H^*(C_2, H^*(C_2\times C_2,\mathbb{F}_4)) \Rightarrow H^*(Q_8,\mathbb{F}_4),$$

where $a,b\in H^1(C_2\times C_2,\mathbb{F}_4)$. Then $d_2(c) = a^2+ab+b^2$, and $d_3(c^2) = a^2b+ab^2$ [**AM04**, IV.2.10]. Consequently,

$$H^*(Q_8,\mathbb{F}_4) = \mathbb{F}_4[P][a,b]/(a^2+ab+b^2, a^3, b^3, a^2b+ab^2),$$

where P is the class of c^4. We can visualize this as the pattern



tensored with $\mathbb{F}_4[P]$, where each dot represents an $\mathbb{F}_4$-generator, and they are arranged so that the cohomological degree is mapped along the vertical axis (not displayed).

The periodicity element P can be written as a Massey product

$$P = \langle a+b, ab, a+b, ab\rangle,$$

as $d_3(c^2) = a^2b + ab^2$, and P is the class of c^4. This is not a Massey product that we will use since it involves elements $(ab, a+b)$ which are not invariant under larger subgroups of $GL_2(\mathbb{Z}/3)$. However, a crucial Massey product is closely related to this one and is described in Proposition 7.4 below.

7. Using Bockstein spectral sequences

Before continuing, let us summarize the structure of the group $SL_2(\mathbb{Z}/3)$. The summary will also serve as a guideline for the method we will use to execute the computations.

The group $G = SL_2(\mathbb{Z}/3)$ has a presentation

$$SL_2(\mathbb{Z}/3) = \langle x, y, z | x^2 = y^2, x^4 = 1 = z^3, xyx = y, xz = zy^3, zyx = yz\rangle,$$

where the elements x, y, z are as in Proposition 4.1. The elements x and y generate a normal subgroup isomorphic to Q_8, and there is an exact sequence

$$1 \to Q_8 \to G \to C_3 \to 1.$$

This implies that if M is any G-module on which 3 is invertible, we have that

$$H^*(G, M) = H^*(Q_8, M)^{C_3}.$$

The action of G on $\Gamma = \mathbb{Z}[1/3, \zeta][\gamma_1, \gamma_2]$ is described in Proposition 4.1; explicitly,

$$\begin{aligned} x :&\gamma_1 \mapsto -\gamma_2 & \qquad y :&\gamma_1 \mapsto -\zeta^2\gamma_1 + \zeta\gamma_2 \\ &\gamma_2 \mapsto \gamma_1 & &\gamma_2 \mapsto \zeta\gamma_1 + \zeta^2\gamma_2 \\ z :&\gamma_1 \mapsto \zeta^2\gamma_1 & & \\ &\gamma_2 \mapsto \gamma_2 - \zeta\gamma_1. & & \end{aligned}$$

This group action preserves the ideals $I_0 = (2)$ and $I_1 = (2, \gamma_1 + \gamma_2)$, and we will compute cohomology using the corresponding Bockstein spectral sequences.

First, $\Gamma_1 = \Gamma/I_1 = \mathbb{F}_4[\bar{\gamma}]$, where $\bar{\gamma}$ is the class of γ_i. The elements x and y of G act trivially on Γ_1, while z maps $\bar{\gamma}$ to $\zeta^2\bar{\gamma}$. We have, first of all,

$$H^*(Q_8, \Gamma_1) = H^*(Q_8, \mathbb{F}_4) \otimes_{\mathbb{F}_4} \Gamma_1 = \mathbb{F}_4[\bar{\gamma}][P][a,b]/(a^2 + ab + b^2, a^2b + ab^2, a^3, b^3),$$

which is a ring bigraded by the cohomological degree and the internal algebraic degree of Γ_1. The periodicity class P is in bidegree $(4, 0)$, a and b are in bidegree $(1, 0)$, and $\bar{\gamma}$ is in bidegree $(0, 1)$. From the conjugation action of z on Q_8, i.e. the relations in G

$$zxz^{-1} = xy \qquad zyz^{-1} = x^3,$$

we get that z acts on a and b as

$$z : a \mapsto a + b, \qquad b \mapsto a.$$

The element z preserves the periodicity element P since its action on $H^4(Q_8, \mathbb{F}_4) \cong \mathbb{F}_4 P$ is $\mathbb{F}_4$-linear.

In bidegree $(*, 0)$, we have $\mathbb{F}_4[P][a, b]/(a^2+ab+b^2, a^2b+ab^2, a^3, b^3)$, and the only C_3-invariants are the powers of P. In bidegree $(*, 1)$ we have $\bar{\gamma}\mathbb{F}_4[P][a, b]/(a^2+ab+b^2, a^2b+ab^2, a^3, b^3)]$, and here $\bar{\gamma}P^k$ is acted on by multiplication by ζ^2. However, the elements $\bar{\gamma}(a + \zeta^2 b)P^k$ are invariant, as are $\bar{\gamma}(a^2 + \zeta^2 b^2)P^k$, for any $k \geq 0$. Similarly, in bidegree $(*, 2)$ the invariants are the elements $\bar{\gamma}^2(a + \zeta b)P^k$ as well as $\bar{\gamma}^2(a^2 + \zeta b^2)P^k$. Thus we obtain

$$\begin{aligned}
&H^*(G, \Gamma_1) = H^*(Q_8, \Gamma_1)^{C_3} = \\
&\mathbb{F}_4[\bar{\gamma}^3, P] \left\langle 1, a^2b = ab^2, (a + \zeta^2 b)\bar{\gamma}, (a^2 + \zeta^2 b^2)\bar{\gamma}, (a + \zeta b)\bar{\gamma}^2, (a^2 + \zeta b^2)\bar{\gamma}^2 \right\rangle,
\end{aligned}$$

with all the above relations and thus obtained multiplicative structure.

Denote suggestively

$$\begin{aligned}
h_1 &= [(a + \zeta^2 b)\bar{\gamma}] \\
h_2 &= [(a + \zeta b)\bar{\gamma}^2] \\
e_0 &= [(a^2 + \zeta^2 b^2)\bar{\gamma}] \\
e_3 &= [a^2 b] = [ab^2] \\
\bar{a}_3 &= [\bar{\gamma}^3],
\end{aligned}$$

the elements in bidegrees $(1, 1), (1, 2), (2, 1), (3, 0), (0, 3)$ respectively. Then $H^*(G, \Gamma_1) = \mathbb{F}_4[\bar{a}_3, P, h_1, h_2, e_3]/(\sim)$, the relations being described as

$$\begin{aligned}
h_1^3 &= e_3 \bar{a}_3 \\
h_2^3 &= h_1^3 \bar{a}_3 \\
e_i^2 &= 0.
\end{aligned}$$

Pictorially, we have the pattern in Figure 1, where the cohomological degree is along the vertical axis, while the horizontal axis depicts the "topological" degree $2t - s$. Each dot represents an $\mathbb{F}_4$, and multiplications by h_1 and h_2 are depicted as connecting line segments.

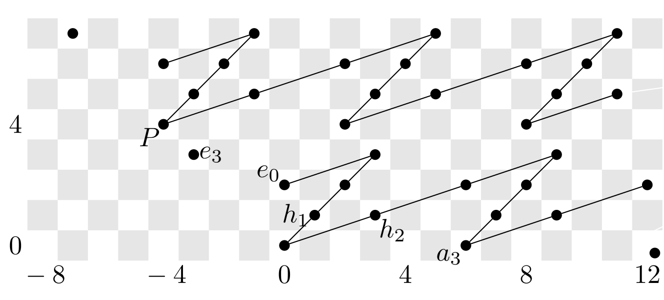


FIGURE 1. $H^*(SL_2(\mathbb{Z}/3), \Gamma_1)$

The relation $h_2^3 = a_3 h_1^3$ holds since

$$h_2^3 - a_3 h_1^3 = [(\zeta^2 + \zeta)(a^2 b + ab^2)\bar{\gamma}^6] = 2[e_3][\bar{\gamma}^6] = 0,$$

and similarly

$$e_0 h_2 - h_1^3 = (ab^2 + a^2 b)\bar{\gamma}^3 = 0.$$

7.1. The a_1-Bockstein spectral sequence. We proceed to compute the a_1-BSS

$$H^*(G, \Gamma_1)[a_1] \Rightarrow H^*(G, \Gamma_0).$$

Recall that $a_1 = \gamma_1 + \gamma_2$; to simplify the notation, $x \doteq y$ will mean that x and y are equal up to multiplication by a unit.

Proposition 7.1. *The differentials in the a_1-BSS are determined by*

$$d_i(a_1) = 0 \qquad d_1(a_3) \doteq a_1 h_2 \qquad d_1(e_0) \doteq a_1 e_3 \qquad d_2(a_3^2) \doteq a_1^2 h_1 \bar{\gamma}^3.$$

Proof. For $d_1(a_3)$ and $d_2(a_3^2)$, one checks that a_3 is not invariant in $\Gamma_0/(\gamma_1 + \gamma_2)^2$ and a_3^2 is not invariant in $\Gamma_0/(\gamma_1+\gamma_2)^3$, so they need to support the specified non-trivial differentials.

We can directly compute that $H^*(SL_2(\mathbb{Z}/3), (\Gamma_0)_1)$ (internal algebraic degree one part of Γ_0) is $\mathbb{F}_4$ in cohomological degrees congruent to 1 modulo 4, and zero otherwise; the only differential making this work is $d_1(e_0) \doteq a_1 e_3$. $\square$

The following are the resulting charts, in which a $\bullet$ denotes an element in the cohomology of Γ_1, a $\circ$ is an a_1-multiple of such an element, a $\square$ is an a_1^2-multiple of such an element, and for clarity we have omitted drawing all higher powers of a_1. The elements which support a differential or are hit by one are grayed out. The entire pattern is P-periodic and the result is also a_3^4-periodic.

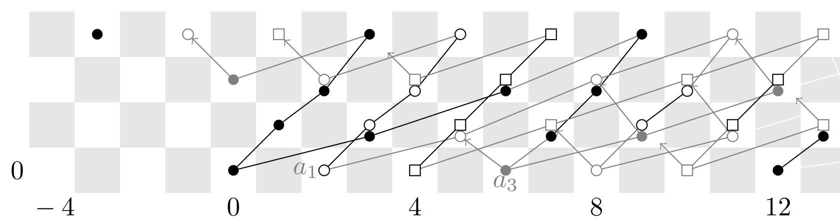


Figure 2. d_1 differential in the a_1-BSS

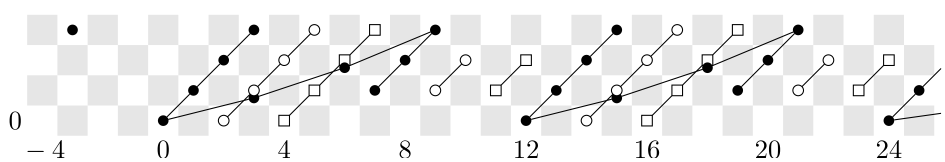


Figure 3. E_2-page of a_1-BSS

The resulting E_∞-page is in Figure 5, where we have stopped distinguishing between the elements which have different a_1-divisibility, and for the sake of readability we have omitted all multiples of a_1^3. The dotted lines denote hidden extensions which we prove in the following few results.

Lemma 7.2. *We have Massey products*

$$x := [a_3 h_1] = \langle a_1, h_2, h_1 \rangle$$
$$y := [a_3^2 h_2] = \langle x, a_1^2, h_2 \rangle = \langle a_1 x, a_1, h_2 \rangle.$$

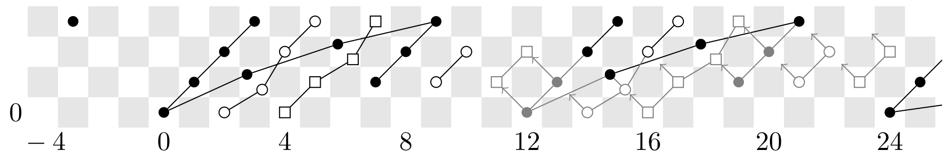


FIGURE 4. d_2 differential in the a_1-BSS

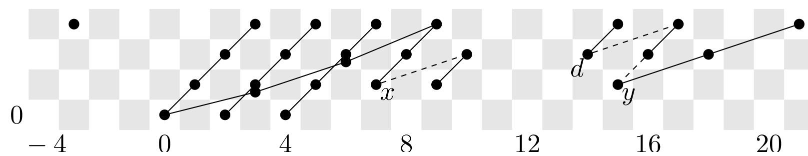


FIGURE 5. E_∞-page of a_1-BSS

PROOF. Since $d_1(a_3) = a_1h_2$, and $h_2h_1h_1 = 0$, we conclude that $\langle a_1, h_2, h_1\rangle = [a_3h_1] = x$. For the next, we have that $d_2(a_3^2) = a_1^2$, $d_1(a_1a_3) = a_1^2h_2$, and $d_1(a_3) = a_1h_2$, hence

$$\langle x, a_1^2, h_2\rangle = [a_3^2h_2 + a_1a_3x] = [a_3^2h_2] = y, \text{ and}$$
$$\langle a_1x, a_1, h_2\rangle = [a_3^2h_2 + a_1a_3x] = y.$$

□

COROLLARY 7.3. *Let* $d = [a_3^2h_1^2]$. *We have the following multiplications in* $H^*(G, \Gamma_0)$

$$xh_2 = a_1xh_1, \qquad yh_1 = a_1d \qquad dh_2 = yh_1^2.$$

PROOF. For the first equality, we note that xh_2 is represented by the class

$$(a^2 + ab + b^2)\gamma_1^2(\gamma_1 + \zeta\gamma_2)\gamma_1\gamma_2(\gamma_1 + \zeta^2\gamma_2)$$

and a_1xh_1 is represented by the class

$$\zeta^2(a^2 + \zeta b^2)\gamma_1^2(\gamma_1 + \gamma_2)\gamma_1\gamma_2(\gamma_1 + \zeta^2\gamma_2).$$

Hence the sum $xh_2+a_1xh_1$ is represented by the sum of the representatives, namely

$$\gamma_1^3\gamma_2(\gamma_1 + \zeta^2\gamma_2)(a + \zeta^2b)(\zeta^2a\gamma_1 + b\gamma_2).$$

But this element reduces to $\zeta a_3h_1h_2$, which is zero. Therefore $xh_2 = a_1xh_1$.

For the rest, we use Lemma 7.2, simple shuffling, and that $xh_2 = a_1xh_1$. We have

$$yh_1 = \langle a_1x, a_1, h_2\rangle h_1 = a_1x\langle a_1, h_2, h_1\rangle = a_1x[a_3h_1] = a_1d$$
$$yh_1^2 = \langle a_1x, a_1, h_2\rangle h_1^2 = a_1x\langle a_1, h_2, h_1^2\rangle = a_1x[a_3h_1^2] = [xh_2a_3h_1] = dh_2.$$

□

PROPOSITION 7.4. *There is an extension* $h_1^4 = a_1^4P$

PROOF. To show this we will need a description of (a periodic multiple of) P as a Massey product in the cohomology of G (and not just its subgroup Q_8).

Recall that in $H^*(Q_8,\mathbb{F}_4)$, $P = \langle a+b, ab, a+b, ab\rangle$. However, we also have a Massey product in $H^*(Q_8,\Gamma_1)$

$$P\bar{\gamma}^{12} = \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & h_1^2\bar{\gamma}^3 \\ h_1^2\bar{\gamma}^3 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ h_1^2\bar{\gamma}^3 \end{pmatrix} \right\rangle;$$

following from the fact that $h_2^3 + h_1^3\bar{\gamma}^3 = (ab^2+a^2b)\bar{\gamma}^6 = d_3(c^2\bar{\gamma}^6)$, and sine all the other products in the Massey product are represented by zero.

As all the classes in this Massey product expression for $P\bar{\gamma}^{12}$ are G-invariant, by naturality of Massey products, we conclude that the same relation holds in $H^*(G,\Gamma_1)$. Consequently, we get

$$\begin{aligned} a_1^4a_3^4P = a_1^4\bar{\gamma}^{12}P &= \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} h_2 & a_3h_1^2 \\ a_3h_1^2 & h_2 \end{pmatrix}, \begin{pmatrix} h_2^2 & h_1 \\ h_1 & h_2^2 \end{pmatrix}, \begin{pmatrix} h_2 \\ a_3h_1^2 \end{pmatrix} \right\rangle a_1^4 \\ &\subseteq \left\langle \begin{pmatrix} h_2^2 & h_1 \end{pmatrix}, \begin{pmatrix} 0 & a_1a_3h_1^2 \\ a_1a_3h_1^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & a_1h_1 \\ a_1h_1 & 0 \end{pmatrix}, \begin{pmatrix} h_2 \\ a_3h_1^2 \end{pmatrix} \right\rangle a_1^2 \\ &= \langle h_1, a_1a_3h_1^2, a_1h_1, a_3h_1^2\rangle a_1^2 + \langle h_2^2, a_1a_3h_1^2, a_1h_1, h_2\rangle a_1^2 = L + R. \end{aligned}$$

By shuffling, we get

$$R \subseteq \langle a_1^2, h_2^2, a_1a_3h_1^2, a_1h_1\rangle h_2 = 0, \qquad L = h_1^4a_3^4.$$

But multiplication by a_3^4 is injective, so the result follows. $\square$

7.2. The 2-Bockstein spectral sequence. Finally, we compute the 2-Bockstein spectral sequence which converges to the G-cohomology of the 2-completion of Γ.

Proposition 7.5. *In the 2-BSS we have the following differentials, which determine all the rest.*

$$\begin{aligned} &d_1(a_1) \doteq 2h_1 && d_1(x) = 2h_2^2 && d_1(y) = 2d \\ &d_2(a_1^2) \doteq 4h_2 \\ &d_3(e_3) = 8P. \end{aligned}$$

Proof. For the differentials on powers of a_1, we just check that a_1 is invariant mod 2 but not mod 4, and a_1^2 is invariant mod 4 but not mod 8.

The cohomology $H^*(SL_2(\mathbb{Z}/3),\mathbb{Z}) = \mathbb{Z}[P]/(8P)$, which gives that $d_3(e_3) = 8P$.

Since $d_1(a_1) = 2h_1$, and $d_1(h_1) = 0 = d_1(h_2)$ we get $d_1(x) = d_1(\langle a_1, h_2, h_1\rangle) = 2\langle h_1, h_2, h_1\rangle = 2h_2^2$.

$d_1(yh_1) = h_1d_1(y) = d_1(a_1d) = 2h_1d$, so $d_1(y) = 2d$. $\square$

The chart is displayed in Figure 6; the d_1 differentials are dotted, the d_2 differentials are dashed, and the d_3 differentials are the solid curved lines.

The resulting E_∞-page is in Figure 7; a bullet denotes an $\mathbb{F}_4$, mod 2 extensions are depicted as circles around the bullet, and a box denotes a copy of the 2-completion of $\mathbb{Z}[1/3,\zeta]$, which is the ring of Witt vectors $\mathbb{W}(\mathbb{F}_4)$.

Note that surviving non-torsion generators are cohomologous to elements in the ring of modular forms $(MF_*)\hat{_2} = \mathbb{W}(\mathbb{F}_4)[c_4,c_6,\Delta]/(2^6\Delta - c_4^3 - c_6^2)$, implying (because of Remark 4.2) that

$$H^0(G,\Gamma\hat{_2}) = (MF_*)\hat{_2}.$$

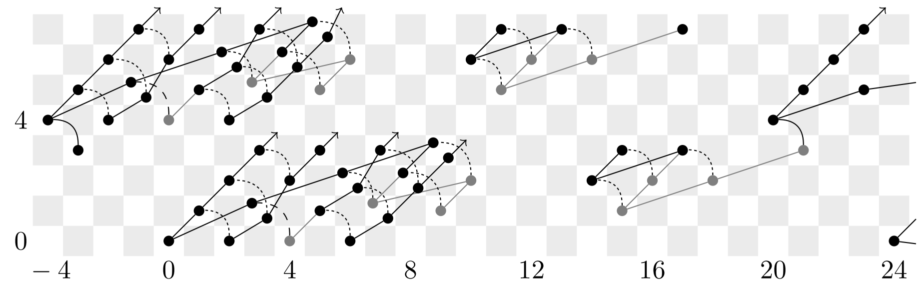


FIGURE 6. The mod 2-Bockstein spectral sequence

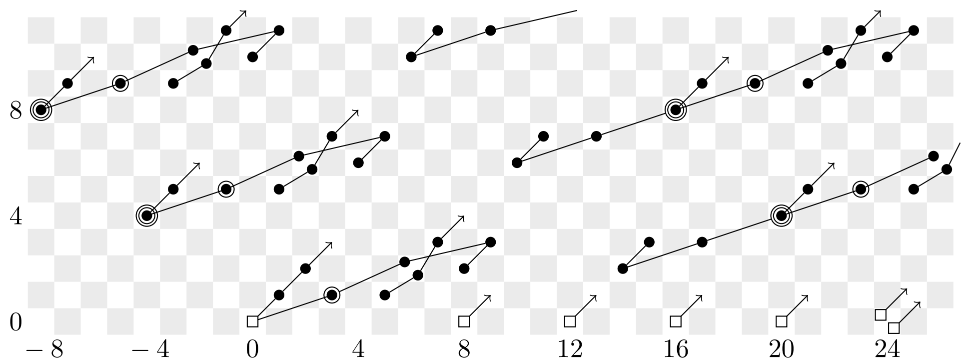


FIGURE 7. $H^*(G, \Gamma_2^{\hat{}})$

8. The cohomology $H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^\zeta))$

Completely analogously to the above calculations, one can obtain the group cohomology of the (2-completion of the) twisted module $\Gamma_\zeta = \Gamma \otimes \mathbb{Z}_\zeta$, where $\mathbb{Z}_\zeta = H^1(\mathcal{M}(3)^\zeta, \Omega)$. *For better readability, assume everything is 2-completed for the remainder of the paper.* As we have seen in Section 5, $\mathbb{Z}_\zeta$[4] is the module on which Q_8 acts trivially, and the element $z \in SL_2(\mathbb{Z}/3) = G$ of order three acts as multiplication by ζ^2. The resulting pattern is displayed in Figure 8; again there are periodicity operators P of degree $(-4, 4)$ and a_3^4 of degree $(24, 0)$.

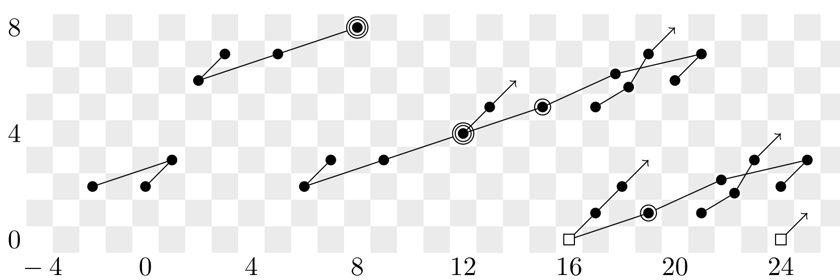


FIGURE 8. $H^*(SL_2(\mathbb{Z}/3), \Gamma_\zeta)$

[4]Recall that after 2-completion, $\mathbb{Z}_\zeta$ is the Witt vectors $\mathbb{W}(\mathbb{F}_4)$.

To obtain the cohomology of the dual module $\Gamma_\zeta^\vee$, we proceed as in Section 10 of [**Sto12**]. Namely, since Γ_ζ is torsion-free, we get a degenerate spectral sequence

$$\mathrm{Ext}^p_{\mathbb{W}(\mathbb{F}_4)}(H_q(G,(\Gamma_\zeta)),\mathbb{W}(\mathbb{F}_4)) \Rightarrow H^{p+q}(G,\Gamma_\zeta^\vee), \tag{8.1}$$

which we can use because the G-homology of Γ_ζ is easily computed by periodicity. Specifically, for $q>0$, cupping with P^k gives an isomorphism

$$H_q(G,\Gamma_\zeta) = \hat{H}^{-q-1}(G,\Gamma_\zeta) \cong H^{-q-1+4k}(G,\Gamma_\zeta),$$

where the second expression is Tate cohomology, and k is any integer greater than $\frac{q+1}{4}$. This gives the cohomology groups $H^*(G,\Gamma_\zeta^\vee)$ for $*>1$.

What remains in order to get H^0 and H^1 is to compute the coinvariants $H_0(G,\Gamma_\zeta)$. As in Lemma 10.4 of [**Sto12**], we use the exact sequence

$$0 \to \hat{H}^{-1}(G,\Gamma_\zeta) \to H_0(G,\Gamma_\zeta) \xrightarrow{Norm} H^0(G,\Gamma_\zeta) \xrightarrow{\pi} \hat{H}^0(G,\Gamma_\zeta) \to 0. \tag{8.2}$$

To use this sequence, it remains to explicitly compute the invariants of Γ_ζ; recall from Remark 4.3 that the cube root d of Δ (which is an element of Γ) is preserved by Q_8 and multiplied by ζ^2 by the action of z. As in Proposition 10.3 of [**Sto12**], we conclude that

$$H^0(G,\Gamma_\zeta) = (\Gamma\otimes\mathbb{Z}_\zeta)^G = d^2MF_*.$$

Let $\mathcal{I}$ be the ideal $(8,c_4,2c_6)$ in MF_*; then $d\mathcal{I}$ is the kernel of the map π. (This can be seen by computing the image of the norm, or more easily, by periodicity.) Finally, again by periodicity, $\hat{H}^{-1}(G,\Gamma_\zeta)\cong H^3(G,\Gamma_\zeta)$, which is torsion. Consequently, the exact sequence (8.2) implies

$$H_0(G,\Gamma_\zeta) = d^2\mathcal{I}\oplus H^3(G,\Gamma_\zeta).$$

Now (8.1) gives $H^1(G,\Gamma_\zeta^\vee)$ (as the Pontryagin dual of $H^3(G,\Gamma_\zeta)$) and also

$$H^0(G,\Gamma_\zeta^\vee) = (d^2\mathcal{I})^\vee. \tag{8.3}$$

Putting everything together, we obtain the familiar pattern depicted in Figure 9.

9. $GL_2(\mathbb{Z}/3)$ cohomology

The cohomology ring $H^*(SL_2(\mathbb{Z}/3),H^*(\mathcal{M}(3)^\zeta,\omega^*)_{\hat{2}})$, which is a $\mathbb{W}(\mathbb{F}_4)$-module, has an action by the quotient group

$$C_2 = GL_2(\mathbb{Z}/3)/SL_2(\mathbb{Z}/3).$$

From the decomposition of the moduli stack $\mathcal{M}(3)$ in (3.1) for $k=\mathbb{W}(\mathbb{F}_4)$, we conclude that this C_2 action is the Galois action $\zeta\mapsto\zeta^2$. By the additive form of Hilbert's theorem 90 [**Ser79**, X.1], it follows that the higher C_2 cohomology vanishes and

$$H^*(GL_2(\mathbb{Z}/2),H^*(\mathcal{M}(3)^\zeta,\omega^*)_{\hat{2}}) = \left(H^*(SL_2(\mathbb{Z}/3),H^*(\mathcal{M}(3)^\zeta,\omega^*)_{\hat{2}})\right)^{C_2}. \tag{9.1}$$

Taking these C_2 invariants simply amounts to reinterpreting the symbols in Figure 9; now dots mean $\mathbb{F}_2$ and boxes mean $\mathbb{Z}_2$.

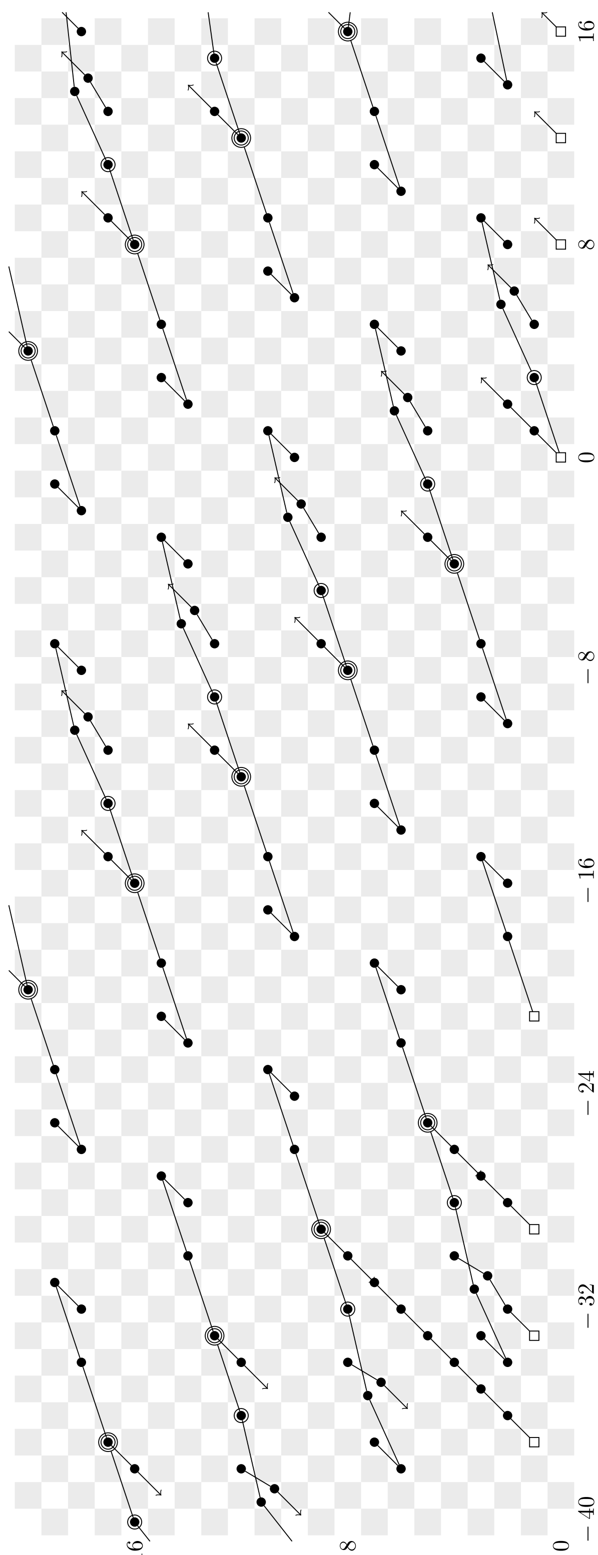


FIGURE 9. $H^*(SL_2(\mathbb{Z}/3), H^*(\mathcal{M}(3)^\zeta))$

References

[AM04] Alejandro Adem and R. James Milgram, *Cohomology of finite groups*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 309, Springer-Verlag, Berlin, 2004. MR 2035696 (2004k:20109)

[Bau08] Tilman Bauer, *Computation of the homotopy of the spectrum* `tmf`, Groups, homotopy and configuration spaces, Geom. Topol. Monogr., vol. 13, Geom. Topol. Publ., Coventry, 2008, pp. 11–40. MR 2508200 (2010h:55008)

[Beh] Mark Behrens, *Notes on the contruction of tmf*, To appear in proceedings of 2007 Talbot Workshop, http://math.mit.edu/˜mbehrens/papers/buildTMF.pdf.

[Del75] P. Deligne, *Courbes elliptiques: formulaire d'après J. Tate*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 53–73. Lecture Notes in Math., Vol. 476. MR 0387292 (52 #8135)

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1973, pp. 143–316. Lecture Notes in Math., Vol. 349. MR 0337993 (49 #2762)

[Goe] Paul Goerss, *Quasi-coherent sheaves on the moduli stack of formal groups*, http://www.math.northwestern.edu/˜pgoerss/papers/modfg.pdf.

[Gut04] Gary Gutman, *Finite group actions in elliptic cohomology*, Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, Contemp. Math., vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 239–252. MR 2066501 (2005e:55008)

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR MR0463157 (57 #3116)

[HM] Michael Hopkins and Mark Mahowald, *From elliptic curves to homotopy theory*, http://hopf.math.purdue.edu/Hopkins-Mahowald/eo2homotopy.pdf.

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR MR772569 (86i:11024)

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979, Translated from the French by Marvin Jay Greenberg. MR 554237 (82e:12016)

[Sil86] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1986. MR MR817210 (87g:11070)

[Sto12] Vesna Stojanoska, *Duality for topological modular forms*, Doc. Math. **17** (2012), 271–311. MR 2946825

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