

Norm coherence for descent of level structures on formal deformations

YIFEI ZHU

We give a formulation for descent of level structures on deformations of formal groups, and study the compatibility between the descent and a norm construction. Under this framework, we generalize Ando’s construction of H_∞ complex orientations for Morava E-theories associated to Honda formal group laws over \mathbb{F}_p . We show the existence and uniqueness of such an orientation for any Morava E-theory associated to a formal group law over an algebraic extension of \mathbb{F}_p and, in particular, orientations for a family of elliptic cohomology theories. These orientations correspond to coordinates on deformations of formal groups which are compatible with norm maps along descent.

1 Introduction

1.1 Algebraic motivations and statement of results Let R be a commutative ring with 1 and let A be an algebra over R . Suppose that, as an R -module, A is finitely generated and free. The norm of A is a map $A \rightarrow R$ which sends a to $\det(a \cdot)$, the determinant of multiplication by a as an R -linear transformation on A . It is multiplicative but not additive in general. Such norms appear as an important ingredient in various contexts: arithmetic moduli of elliptic curves [Katz-Mazur1985, § 1.8, § 7.7], actions of finite group schemes on abelian varieties [Mumford2008, § 12], isogenies of one-parameter formal Lie groups over p -adic integer rings [Lubin1967, § 1]. These norm maps are closely related to construction of *quotient objects*.

It is the purpose here to examine an interaction between norms and the corresponding *subobjects*—more precisely, a functorial interaction with chains of subobjects—in the context of Lubin and Tate’s formal deformations [Lubin-Tate1966]. The functoriality amounts to descent of “level structures” on deformations (see § 6 and § 8). In this paper, a level structure on a formal group is a choice of finite subgroup scheme, from which we obtain a quotient morphism of formal groups. A norm map between their rings of

functions then gets involved in making this quotient morphism into a homomorphism of formal group laws (2.3). This norm construction is compatible with successive quotient along a chain of subgroups.

On the other hand, given a deformation over a p -adic integer ring, there is a canonical (*i.e.*, coordinate-free) descent of level structures via Lubin and Tate’s universal deformations. Strickland studied the representability of this moduli problem [Strickland1997] so that the descent can be realized as canonical lifts of Frobenius morphisms (5.6).

Our main result shows the existence and uniqueness of deformations of formal group laws on which the canonical lifts of Frobenius coincide with quotient homomorphisms from the norm construction. We call these deformations *norm-coherent* (see §6, specifically Definition 6.17).

Theorem 1.2 (cf. Theorem 7.22 and Proposition 7.1) *Let k be an algebraic extension of \mathbb{F}_p , R a complete local ring with residue field containing k , G a formal group law over k of finite height, and F a deformation of G to R . There exists a unique formal group law F' over R , \star -isomorphic to F , which is norm-coherent. Moreover, when F is a Lubin-Tate universal deformation, F' is functorial under base change of G/k , under k -isogeny out of G , and under k -Galois descent.*

Remark 1.3 In the context of local class field theory via Lubin and Tate’s theory of complex multiplication [Lubin-Tate1965], Coleman’s norm operator [Coleman1979, Theorem 11] is used to compute norm groups. Walker observed its similarity to the norm construction above [Walker2008, Chapter 5]. Specifically, he reformulated the norm-coherence condition (for a special case) in terms of a particular way in which Coleman’s norm operator acts [*ibid.*, 5.0.10]. It would be interesting to have a conceptual understanding of this connection in the generality of Theorem 1.2.

1.4 Topological motivations and statement of results The relevance to topology (and, further, to geometry and mathematical physics) of this functorial interaction between norms and finite formal subgroup schemes lies, for instance, in having highly coherent multiplications for *genera*. These are cobordism invariants of manifolds; such multiplications refine the invariants by reflecting symmetries of the geometry (some known, some conjectural).

A prominent example is the Witten genus for string manifolds, which takes values in the ring of integral modular forms of level 1. Motivated by this, Hopkins and his collaborators developed highly-structured multiplicative *orientations* (*i.e.*,

genera of *families* of manifolds) for elliptic cohomology theories and for a universal theory of topological modular forms [Hopkins1995, Hopkins2002]. In particular, in [Ando-Hopkins-Strickland2004], they showed that their sigma orientation $MU\langle 6 \rangle \rightarrow E$ for any elliptic cohomology E is H_∞ , a commutativity condition on its multiplicative structure (2.9). Their analysis of this H_∞ -structure was based on [Ando1995, Ando1992] where the algebraic condition of norm coherence had made a first appearance.

Theorem 1.2 produces $H_\infty MU\langle 0 \rangle$ -orientations for a family of generalized cohomology theories called Morava E-theories (2.7) including those treated by Ando and by Ando, Hopkins, Strickland.

Theorem 1.5 (cf. Corollary 8.17 for a precise statement about uniqueness) *Let k and G be as in Theorem 1.2. For the form of Morava E-theory associated to G/k , there exists a unique $MU\langle 0 \rangle$ -orientation that is an H_∞ map.*

Remark 1.6 Rezk reminded us that the sigma orientations do not factor through these $H_\infty MU\langle 0 \rangle$ -orientations (8.1). On the other hand, the coefficient ring of an E-theory (of height 2) is a certain completion of a ring of modular forms. In [Zhu2015a], as a first step, we related its elements to certain quasimodular forms (and to mock modular forms) via Rezk’s logarithmic operations; see also [Rezk2016, remarks following Theorem 1.29]. Given Theorem 1.5, it would be interesting to acquire and analyze more exotic manifold invariants. In particular, we may investigate an analogue of the modular invariance of a sigma orientation [Ando-Hopkins-Strickland2001, 1.3] in view of the uniqueness above.

Remark 1.7 A natural question is whether there exist E_∞ complex orientations for Morava E-theories and, more specifically, whether the orientation in Theorem 1.5 rigidifies to be an E_∞ map. See [Hopkins-Lawson2016] for recent progress on E_∞ complex orientations, where the norm-coherence condition comes up.

Finally, the expositions in [Rezk2015] and [Rezk2016, esp. §4] provide some other perspectives. See also [Strickland, esp. §29].

1.8 Outline of the paper In §2, we recall some basic concepts from the theory of formal groups and homotopy theory, particularly quotient of formal groups (2.3), and set their notation.

In §3, following a suggestion of Rezk, we introduce an enlarged category of formal groups (cf. [Katz-Mazur1985, §4.1]). This viewpoint will be helpful in clarifying

deformations of Frobenius (5.2, 5.9), descent of level structures (6.7, 6.10), the norm-coherence condition (6.16), and functoriality of norm coherence (7.21).

§ 4 and § 5 give an account for the theorems of Lubin, Tate (4.6) and of Strickland (5.5) on deformations of formal groups. Our formulation follows Rezk’s (*e.g.*, in [Rezk2014, § 4]) but emphasizes formal group *laws*, as we are concerned with special coordinates. The purpose of these two sections is to provide a detailed exposition as well as a precise setup that is crucial for the notion of norm coherence to follow in desired generality.

In § 6, we introduce the central notion of this paper, norm coherence (6.15-6.19), building on Ando’s framework [Ando1995, § 2]. We then generalize his theorem and prove Theorem 1.2 in § 7. Our main results are Proposition 7.1 and Theorem 7.22, the latter stated in a form suggested by Rezk.

§ 8 discusses corresponding topological results for complex orientations, with (8.1) introducing further background on work of Ando, of Ando, Hopkins, Strickland, and of Ando, Strickland. In (8.3-8.15), we compare the setup for our results above with Ando, Hopkins, and Strickland’s descent data and norm maps [Ando-Hopkins-Strickland2004, Parts 1 and 3]. The purpose is to continue the exposition from § 5 while proving Theorem 1.5.

1.9 Acknowledgements This paper originated from a referee’s comment on the choice of coordinates in one of my earlier works. I thank the anonymous referee for their demand for precision on specifics.

I thank Anna Marie Bohmann, Paul Goerss, Fei Han, Michael Hill, Gerd Laures, Tyler Lawson, Niko Naumann, and Eric Peterson for helpful discussions. I thank Zhen Huan for the quick help with locating a reference.

I learned most of what I know about norm coherence and related questions from Charles Rezk. A good deal of the theory presented here was developed in discussions with him, including “norm-coherent.” The term is my choice over the synonym “Ando” and it is Matthew Ando who originally discovered this condition in algebra and applied it to topology.

I thank Eric Peterson for the feedback on a draft of this paper, and for explaining to me the results and methods of his joint work with Nathaniel Stapleton, which gives a different approach to questions considered here.

1.10 Conventions Unless otherwise indicated, a prime p is fixed throughout.

We often omit the symbol Spf and simply write R for $\mathrm{Spf} R$ when it appears as a base scheme. In particular, β^* means base change from R to S along $\beta: R \rightarrow S$, understood as $\beta: \mathrm{Spf} S \rightarrow \mathrm{Spf} R$.

We also write ψ^* for the pullback of functions along a morphism ψ of schemes.

Depending on the context, the symbol $/$ stands for “over” (indicating the structure morphism of a scheme) or “modulo” (indicating a quotient).

2 General notions

2.1 Formal groups, coordinates, and formal group laws Let R be a complete local ring with residue characteristic $p > 0$. A *formal group \mathcal{G} over R* is a group object in the category of formal R -schemes. In this paper, all formal groups will be commutative, one-dimensional, and affine. They can be viewed as covariant functors from the category of complete local R -algebras (and local homomorphisms) to the category of abelian groups.

A *coordinate x on \mathcal{G}* is a natural isomorphism $\mathcal{G} \xrightarrow{\sim} \hat{\mathbb{A}}_R^1$ of functors to pointed sets. It gives an isomorphism $\Gamma(\mathcal{G}, \mathcal{O}_{\mathcal{G}}) \cong R[[x]]$ of augmented R -algebras, as well as a trivialization of the ideal sheaf $\mathcal{I}_{\mathcal{G}}(\mathcal{O}) = \mathcal{O}_{\mathcal{G}}(-\mathcal{O})$ of functions on \mathcal{G} which vanish at the identity section \mathcal{O} . Here and throughout the paper, we remove the calligraphic effect of the notation for a formal group whenever it appears as a subscript. We will also simply write $\mathcal{O}_{\mathcal{G}}$ for the ring of global sections of $\mathcal{O}_{\mathcal{G}}$, and similarly for other sheaves.

A (*one-dimensional commutative*) *formal group law F over R* is a formal power series in two variables t_1 and t_2 with coefficients in R , often written $t_1 +_F t_2$, which satisfies a set of abelian-group-like axioms. In particular, the above data of \mathcal{G} and x determines a formal group law $G = G_x$ such that

$$x(P_1) +_G x(P_2) = x(P_1 + P_2)$$

for any R -points P_1 and P_2 on \mathcal{G} (where we identify an R -point on $\hat{\mathbb{A}}_R^1$ with an element in the maximal ideal of R). Conversely, given a formal group law F , it determines a formal group $\mathcal{F} = \mathrm{Spf} R[[x_F]]$ in a similar way.

2.2 Subgroups and isogenies By (*finite*) *subgroups* of a formal group over R , we mean finite flat closed subgroup schemes. Their points are often defined over an extension \tilde{R} of R .

An *isogeny* $\psi: \mathcal{G} \rightarrow \mathcal{G}'$ over R is a finite flat morphism of formal groups. Along ψ^* , $\mathcal{O}_{\mathcal{G}}$ becomes a free $\mathcal{O}_{\mathcal{G}'}$ -module of finite rank d , called the *degree* of ψ . Since the residue characteristic of R is p , d must be a power of p .

Suppose x and x' are coordinates on \mathcal{G} and \mathcal{G}' . Then ψ induces a homomorphism $G_x \rightarrow G'_{x'}$ of formal group laws, i.e., $h(t) \in t \cdot R[[t]]$ such that

$$h(t_1 +_{G_x} t_2) = h(t_1) +_{G'_{x'}} h(t_2)$$

In fact, $h(x) = \psi^*(x')$ and sometimes we will abuse notation by writing ψ for h . We will also denote this homomorphism by $\psi: G \rightarrow G'$ and say it is an isogeny of degree d (cf. [Lubin1967, 1.6]). By Weierstrass preparation, $h = mn$ with $m \in R[t]$ monic of degree d and $n \in R[[t]]$ invertible.

2.3 Kernels and quotients The notions of subgroups and of isogenies are connected as follows.

Given $\psi: \mathcal{G} \rightarrow \mathcal{G}'$ as above, its *kernel* \mathcal{K} is defined by $\mathcal{O}_{\mathcal{K}} = \mathcal{O}_{\mathcal{G}} \otimes_{\mathcal{O}_{\mathcal{G}'}} R$, where the tensor product is taken along ψ^* and the augmentation map of $\mathcal{O}_{\mathcal{G}'}$. It is naturally a subgroup of \mathcal{G} and has degree d as an effective Cartier divisor in \mathcal{G} .

Conversely, given a subgroup $\mathcal{H} \subset \mathcal{G}$ over \tilde{R} of degree p' , there is a corresponding isogeny $f_{\mathcal{H}}: \mathcal{G} \rightarrow \mathcal{G}/\mathcal{H}$ defined by an equalizer diagram

$$\mathcal{O}_{\mathcal{G}/\mathcal{H}} \xrightarrow{f_{\mathcal{H}}^*} \mathcal{O}_{\mathcal{G}} \xrightarrow[\pi^*]{\mu^*} \mathcal{O}_{\mathcal{G} \times \mathcal{H}}$$

where $\mu, \pi: \mathcal{G} \times \mathcal{H} \rightarrow \mathcal{G}$ are the multiplication, projection maps, and \mathcal{G}/\mathcal{H} is naturally a formal group over \tilde{R} . Moreover, given a coordinate x on \mathcal{G} ,

$$x_{\mathcal{H}} := \text{Norm}_{f_{\mathcal{H}}^*}(x)$$

is a coordinate on \mathcal{G}/\mathcal{H} , where $\text{Norm}_{f_{\mathcal{H}}^*}(x)$ equals the determinant of multiplication by x on $\mathcal{O}_{\mathcal{G}}$ as a finite free $\mathcal{O}_{\mathcal{G}/\mathcal{H}}$ -module via $f_{\mathcal{H}}^*$. Explicitly,

$$(2.4) \quad f_{\mathcal{H}}^*(x_{\mathcal{H}}) = \prod_{Q \in \mathcal{H}(\tilde{R})} (x +_G x(Q))$$

By writing $f_{\mathcal{H}}: G \rightarrow G/H$ as an isogeny of formal group laws, we will always intend the above compatibility between corresponding coordinates. Sometimes we write more specifically

$$f_{\mathcal{H}}^x: G_x \rightarrow G_x/H := (G/H)_{x_{\mathcal{H}}}$$

Note that over the residue field of R , (2.4) becomes

$$(2.5) \quad f_H^*(x_H) = x^{p^r}$$

as a formal group over a field of characteristic p has exactly one subgroup of degree p^r . Thus f_H is a lift of the relative p^r -power Frobenius isogeny.

For more details, see [Lubin1967, § 1, esp. Theorems 1.4, 1.5], [Strickland1997, § 5, esp. Theorem 19] (cf. Remark 8.14 below), and [Ando1995, §§ 2.1-2.2].

2.6 Complex cobordism Let $MU\langle 0 \rangle$ be the Thom spectrum of the tautological (virtual) complex vector bundle over $\mathbb{Z} \times BU$. We have $\pi_* MU\langle 0 \rangle \cong \pi_* MU[\beta^{\pm 1}]$ with $|\beta| = 2$. More generally, let $MU\langle 2k \rangle$ be the Thom spectrum associated to the $(2k - 1)$ -connected cover $BU\langle 2k \rangle \rightarrow \mathbb{Z} \times BU$.

The spectrum $MU\langle 0 \rangle$ is often written MUP or MP for “periodic” (as can be seen from its homotopy groups). In fact,

$$MU\langle 0 \rangle = \bigvee_{m \in \mathbb{Z}} \Sigma^{2m} MU$$

so that $\pi_0 MU\langle 0 \rangle$ is the ring of cobordism classes of even-dimensional stably almost complex manifolds. This ring carries the universal formal group law of Lazard by [Quillen1969, Theorem 2].

The spectrum $MU\langle 2 \rangle = MU$. The homology of $MU\langle 2k \rangle$ is concentrated in even degrees if $0 \leq k \leq 3$.

2.7 Morava E-theories Let k be a perfect field of characteristic p , and \mathcal{G} be a formal group over k of finite height n . Associated to this data, there is a generalized cohomology theory, called a *Morava E-theory (of height n at the prime p)*. It is represented by a ring spectrum $E = E_n$. The formal scheme $\mathcal{G}_E := \mathrm{Spf} E^0 \mathbb{C}P^\infty$ is naturally a formal group over $E^0(\mathrm{pt}) = \pi_0 E$. The above association requires that \mathcal{G}_E be a Lubin-Tate universal deformation of \mathcal{G}/k (see § 4 below). We have

$$\pi_* E \cong \mathbb{W}k[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$$

where $|u_i| = 0$ and $|u| = 2$.¹

Thus a Morava E-theory spectrum is a topological realization of a Lubin-Tate ring. Strickland showed that $E^0 B\Sigma_{p^r} / I_{\mathrm{tr}}$ is a finite free module over $\pi_0 E$, where I_{tr} is the ideal generated by the images of transfers from proper subgroups of the symmetric

¹For some purposes, it is convenient to instead have $\mathbb{W}\bar{k}$ or $|u| = -2$ in $\pi_* E$.

group Σ_{p^r} on p^r letters. Moreover, this ring classifies degree- p^r subgroups of \mathcal{G}_E [Strickland1998, Theorem 1.1] (see §5). Ando, Hopkins, and Strickland then assembled these into a topological realization of descent data for level structures on \mathcal{G}_E in [Ando-Hopkins-Strickland2004, §3.2] (see §8).

When \mathcal{G} is the formal group of a supersingular elliptic curve, its corresponding E-theory (of height 2) is an *elliptic cohomology theory* [Ando-Hopkins-Strickland2001, Definition 1.2] via the Serre-Tate theorem.

2.8 Complex orientations for Morava E-theories A *complex orientation* for E is a coherent choice of Thom class in E -cohomology for every complex vector bundle. It amounts to the choice of a single class $\xi \in \tilde{E}^2\mathbb{CP}^\infty$ which restricts to 1 under the composite

$$\tilde{E}^2\mathbb{CP}^\infty \rightarrow \tilde{E}^2\mathbb{CP}^1 = \tilde{E}^2S^2 \cong E^0(\text{pt})$$

Given a coordinate $x \in \mathcal{O}_{G_E} = E^0\mathbb{CP}^\infty$, as a trivialization for $\mathcal{I}_{G_E}(O) = \tilde{E}^0\mathbb{CP}^\infty$, it corresponds to an invertible class $u_x \in \tilde{E}^0\mathbb{CP}^1 \cong \pi_2 E$. We then get a complex orientation for E from $x \cdot u_x^{-1} \in \tilde{E}^2\mathbb{CP}^\infty$. Conversely, we recover a coordinate on \mathcal{G}_E from a class ξ above and a generator for $\pi_2 E$.

An $MU\langle 0 \rangle$ -orientation for E is a map $MU\langle 0 \rangle \rightarrow E$ of homotopy commutative ring spectra. Consider the natural map

$$\mathbb{CP}_+^\infty \rightarrow (\mathbb{CP}^\infty)^\mathcal{L} \rightarrow \Sigma^2 MU \rightarrow MU\langle 0 \rangle$$

where \mathcal{L} is the tautological line bundle over \mathbb{CP}^∞ . Composing with this, each $MU\langle 0 \rangle$ -orientation gives a generator of $\tilde{E}^0\mathbb{CP}^\infty$ and thus a coordinate on \mathcal{G}_E . In fact, the correspondence is a bijection (see [Ando2000, Proposition 1.10(ii)] and [Ando-Hopkins-Strickland2001, Corollary 2.50]).

2.9 E_∞ and H_∞ structures Let Sp be a complete and cocomplete category of spectra, indexed over some universe, with an associative and commutative smash product \wedge (e.g., the category of \mathbb{L} -spectra in [Elmendorf-Kriz-Mandell-May1997, Chapter I]).

An E_∞ -ring spectrum is a commutative monoid in Sp . Equivalently, it is an algebra for the monad \mathbb{D} on Sp defined by

$$\mathbb{D}(-) := \bigvee_{m \geq 0} \mathbb{D}_m(-) := \bigvee_{m \geq 0} (-)^{\wedge m} / \Sigma_m$$

where Σ_m is the symmetric group on m letters acting on the m -fold smash product.

Weaker than being E_∞ , an H_∞ -ring spectrum is a commutative monoid in the homotopy category of Sp . It also has a description as an algebra for the monad which descends

from \mathbb{D} to the homotopy category. In particular, there are power operations D_m on the homotopy groups of such a spectrum (see [Bruner-May-McClure-Steinberger1986, Chapter I]).

Complex cobordism MU and its variants above are E_∞ -ring spectra [May1977, §IV.2]. Morava E -theories E are also E_∞ -ring spectra [Goerss-Hopkins2004, Corollary 7.6]. A morphism of E_∞ -ring (or H_∞ -ring) spectra is called an E_∞ (or H_∞) map.

3 Wide categories of formal groups

3.1 The category FG and its subcategories Consider FG whose objects are formal groups

$$\begin{array}{c} \mathcal{G} \\ \downarrow f \\ \mathrm{Spf} k \end{array}$$

of finite height over variable base fields of characteristic p , and whose morphisms are cartesian squares

$$(3.2) \quad \begin{array}{ccc} \mathcal{G} & \xrightarrow{\Psi} & \mathcal{G}' \\ f \downarrow & & \downarrow f' \\ \mathrm{Spf} k & \xrightarrow{\beta} & \mathrm{Spf} k' \end{array}$$

i.e., commutative squares such that the induced morphism of k -schemes

$$(3.3) \quad \mathcal{G} \xrightarrow{(\Psi, f)} \mathcal{G}' \times_{k'} k$$

is a homomorphism of formal groups over k . We also have subcategories $\mathrm{FG}_{\mathrm{isog}}$ and $\mathrm{FG}_{\mathrm{iso}}$ when (3.3) is restricted to be an isogeny or isomorphism. Write $\mathrm{FG}(k)$, $\mathrm{FG}_{\mathrm{isog}}(k)$, and $\mathrm{FG}_{\mathrm{iso}}(k)$ for the subcategories where the base field is fixed and $\beta = \mathrm{id}$ in (3.2).

We think of \mathbf{FG} , $\mathbf{FG}_{\text{isog}}$, and \mathbf{FG}_{iso} as “wide” categories given the factorization

$$(3.4) \quad \begin{array}{ccccc} \mathcal{G} & \longrightarrow & \mathcal{G}' \times_{k'} k & \longrightarrow & \mathcal{G}' \\ \downarrow f & & \downarrow \lrcorner & & \downarrow f' \\ \mathrm{Spf} k & \longrightarrow & \mathrm{Spf} k & \xrightarrow{\beta} & \mathrm{Spf} k' \end{array}$$

Example 3.5 For our purpose, a key example of morphisms in \mathbf{FG} is the following, where σ is the absolute p -power Frobenius and Frob is the relative one.

$$(3.6) \quad \begin{array}{ccccc} & & \sigma & & \\ & \searrow & \curvearrowright & \swarrow & \\ \mathcal{G} & \xrightarrow{\mathrm{Frob}} & \mathcal{G}^{(p)} & \longrightarrow & \mathcal{G} \\ \downarrow f & & \downarrow \lrcorner & & \downarrow f \\ \mathrm{Spf} k & \longrightarrow & \mathrm{Spf} k & \xrightarrow{\sigma} & \mathrm{Spf} k \end{array}$$

This is an endomorphism in $\mathbf{FG}_{\text{isog}}$ on the object \mathcal{G}/k . Denote it by Φ . It is not a morphism in $\mathbf{FG}_{\text{isog}}(k)$. The composite Φ^r corresponds to the p^r -power Frobenius.

3.7 Canonical factorization of Φ^r along an isogeny Given any $\psi: \mathcal{G} \rightarrow \mathcal{G}'$ in $\mathbf{FG}_{\text{isog}}(k)$, necessarily of degree p^r for some $r \geq 0$, there is a unique factorization in $\mathbf{FG}_{\text{isog}}$ of Φ^r along ψ as follows, where $\Phi^r = \Lambda_\psi \circ \psi$ with Λ_ψ in \mathbf{FG}_{iso} .

$$(3.8) \quad \begin{array}{ccccccc} & & & \Lambda_\psi & & & \\ & & & \curvearrowright & & & \\ \mathcal{G} & \xrightarrow{\psi} & \mathcal{G}' & \xrightarrow{\sim} & \mathcal{G}^{(p^r)} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \lrcorner & & \downarrow \\ \mathrm{Spf} k & \longrightarrow & \mathrm{Spf} k & \longrightarrow & \mathrm{Spf} k & \xrightarrow{\sigma^r} & \mathrm{Spf} k \end{array}$$

Correspondingly, between rings of functions on the formal groups, we have

$$(3.9) \quad \begin{array}{ccccc} \mathcal{O}_G & \xleftarrow{\psi^*} & \mathcal{O}_{G'} & \xleftarrow{\Lambda_\psi^*} & \mathcal{O}_G \\ & & & \searrow \sigma^r & \nearrow \end{array}$$

Lemma 3.10 *Given (3.8), the map Λ_ψ^* coincides with the norm of \mathcal{O}_G as a finite free module over $\mathcal{O}_{G'}$ along ψ^* ; that is, given any $a \in \mathcal{O}_G$, $\Lambda_\psi^*(a)$ equals the determinant of multiplication by a as an $\mathcal{O}_{G'}$ -linear transformation on \mathcal{O}_G .*

Proof First, consider the case of $\psi = \text{Frob}^r$. Let $\mathcal{O}_G = k[[x]]$ and $\mathcal{O}_{G'} = k[[y]]$ with $\psi^*(y) = x^{p^r}$. Write the norm map as $\text{Norm}_{\psi^*}: \mathcal{O}_G \rightarrow \mathcal{O}_{G'}$. We have

$$(3.11) \quad \text{Norm}_{\psi^*}(x) = \prod_{i=1}^{p^r} x_i = (-1)^{p^r+1} y = y$$

and $\text{Norm}_{\psi^*}(c) = c^{p^r}$, where $c \in k$ and x_i are the roots of the minimal polynomial of x over $\mathcal{O}_{G'}$. Note that in characteristic p , the norm map is additive and hence a local homomorphism. Thus composing with the k -linear map ψ^* , it becomes the absolute p^r -power Frobenius σ^r as follows, where $h^{(p^r)}$ is the series obtained by twisting the coefficients of h with the p^r -power Frobenius.

$$(3.12) \quad \begin{array}{ccccc} k[[x]] & \xleftarrow{\psi^*} & k[[y]] & \xleftarrow{\text{Norm}_{\psi^*}} & k[[x]] \\ h^{(p^r)}(x^{p^r}) & \xleftarrow{\quad} & h^{(p^r)}(y) & \xleftarrow{\quad} & h(x) \end{array}$$

The claim then follows by the uniqueness of the factorization (3.8).

In general, consider two such isogenies ψ_1 and ψ_2 out of \mathcal{G} , with $\psi_1 = g \circ \psi_2$ for a unique k -isomorphism g . We need only observe that

$$(3.13) \quad \psi_1^* \circ \text{Norm}_{\psi_1^*} = \psi_2^* \circ g^* \circ \text{Norm}_{\psi_1^*} = \psi_2^* \circ \text{Norm}_{\psi_2^*}$$

where the second equality is by extension of scalars along g^* for determinants. With $\psi_1 = \text{Frob}^r$ from above, the claim for ψ_2 follows. \square

4 Deformations of formal group laws

4.1 Set-up Let k be a field of characteristic $p > 0$, and G be a formal group law over k of height $n < \infty$. Let R be a complete local ring with maximal ideal \mathfrak{m} and residue field $R/\mathfrak{m} \supset k$, and let $\pi: R \rightarrow R/\mathfrak{m}$ be the natural projection.

4.2 Deformations and deformation structures A deformation of G to R is a triple (F, i, η) consisting of a formal group law F over R , an inclusion $i: k \hookrightarrow R/\mathfrak{m}$ of fields, and an isomorphism $\eta: \pi^*F \xrightarrow{\sim} i^*G$ of formal group laws over R/\mathfrak{m} , as in the following commutative diagram. The leftmost column is supposed to “deform” or “thicken” the rightmost column.

$$\begin{array}{ccccccc}
 F & \xleftarrow{\quad} & \pi^*F & \xrightarrow{\quad \eta \quad} & i^*G & \xrightarrow{\quad} & G \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spf } R & \xleftarrow{\quad \pi \quad} & \text{Spf } R/\mathfrak{m} & \xlongequal{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad i \quad} & \text{Spf } k
 \end{array}$$

We call the pair (i, η) a *deformation structure attached to F with respect to G/k* , and may simply call F a deformation of G to R if its deformation structure is understood. We also call the corresponding formal group \mathcal{F} a deformation of the formal group \mathcal{G} to R .

4.3 Base change of deformation structures Let $\beta: R \rightarrow S$ be a local homomorphism, and $\bar{\beta}: R/\mathfrak{m} \rightarrow S/\mathfrak{n}$ be the induced map between residue fields. Given a deformation (F, i, η) of G to R , there is a deformation (β^*F, i', η) to S by base change along β such that the following diagram commutes.² We write $\beta^*(i, \eta) := (i', \eta) = (\bar{\beta} \circ i, \eta)$.

$$\begin{array}{ccccccc}
 \beta^*F & \xleftarrow{\quad} & \bar{\beta}^*\pi^*F & \xrightarrow{\quad \eta \quad} & i'^*G & \xrightarrow{\quad} & G \\
 \swarrow & & \swarrow & & \swarrow & & \swarrow \\
 F & \xleftarrow{\quad} & \pi^*F & \xrightarrow{\quad \eta \quad} & i^*G & \xrightarrow{\quad} & G \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \text{Spf } S & \xleftarrow{\quad} & \text{Spf } S/\mathfrak{n} & \xlongequal{\quad} & \text{Spf } S/\mathfrak{n} & \xrightarrow{\quad i' \quad} & \text{Spf } k \\
 \swarrow \beta & & \swarrow \bar{\beta} & & \swarrow & & \swarrow \\
 \text{Spf } R & \xleftarrow{\quad \pi \quad} & \text{Spf } R/\mathfrak{m} & \xlongequal{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad i \quad} & \text{Spf } k
 \end{array}$$

4.4 \star -isomorphisms Let (F, i, η) and (F', i', η') be deformations of G/k to R . A \star -isomorphism $(F, i, \eta) \rightarrow (F', i', \eta')$ consists of an equality $i = i'$ and an isomorphism

²Here, as an abuse of notation, we also denote by η the pullback of η along $\bar{\beta}$ (i.e., applying $\bar{\beta}$ to the coefficients of the formal power series expression for η ; see (2.2)).

$\psi: F \xrightarrow{\sim} F'$ of formal group laws over R such that $\eta' \circ \pi^* \psi = \eta$, as in the following commutative diagram.

$$(4.5) \quad \begin{array}{ccccccc} & & F & \xleftarrow{\quad} & \pi^* F & \xrightarrow{\quad \eta \quad} & i^* G & \xrightarrow{\quad} & G \\ & \swarrow \psi & \downarrow & & \downarrow \pi^* \psi & & \downarrow & & \downarrow \\ F' & \xleftarrow{\quad} & \pi^* F' & \xrightarrow{\quad \eta' \quad} & i^* G & \xrightarrow{\quad} & G & & \\ \downarrow & & \downarrow \pi & & \downarrow & & \downarrow i & & \downarrow \\ & & \mathrm{Spf} R & \xleftarrow{\quad} & \mathrm{Spf} R/\mathfrak{m} & \xrightarrow{\quad} & \mathrm{Spf} R/\mathfrak{m} & \xrightarrow{\quad i \quad} & \mathrm{Spf} k \\ & \swarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spf} R & \xleftarrow{\quad \pi \quad} & \mathrm{Spf} R/\mathfrak{m} & \xrightarrow{\quad} & \mathrm{Spf} R/\mathfrak{m} & \xrightarrow{\quad i \quad} & \mathrm{Spf} k \end{array}$$

Continuing with the above definition, we simply call $\psi: F \rightarrow F'$ a \star -isomorphism if in addition $\eta = \eta'$ so that $\pi^* \psi = \mathrm{id}$. We use the symbol $\stackrel{*}{\simeq}$ for this equivalence relation. Clearly it is preserved under base change.

Proposition 4.6 (cf. [Rezk2014, Proposition 4.2]) *Let k, G, R be as in (4.1) and fix G/k . Then the functor*

$$R \mapsto \{\star\text{-isomorphism classes of deformations } (F, i, \eta) \text{ of } G \text{ to } R\}$$

from the category of complete local rings with residue field containing k to the category of sets is co-represented by the ring $E_n := \mathbb{W}k[[u_1, \dots, u_{n-1}]]$. Explicitly, there is a (by no means unique)³ deformation $(F_{\mathrm{univ}}, \mathrm{id}, \mathrm{id})$ to E_n satisfying the following universal property. Given any deformation (F, i, η) of G to R , there is a unique local homomorphism

$$\alpha: E_n \rightarrow R$$

such that it reduces to $i: k = E_n/I \hookrightarrow R/\mathfrak{m}$, with I and \mathfrak{m} the maximal ideals, and such that there is a unique \star -isomorphism

$$(4.7) \quad (F, i, \eta) \rightarrow (\alpha^* F_{\mathrm{univ}}, i, \mathrm{id})$$

Proof Let $\pi: R \rightarrow R/\mathfrak{m}$ and $\rho: E_n \rightarrow E_n/I$ be the natural projections. When η is allowed to be the identity only, this is [Lubin-Tate1966, Theorem 3.1] (cf. [Ando1995, Theorem 2.3.1]). More generally, the universal property claims that the following

³See (4.10) below.

diagram commutes, where α and ψ are both unique.

$$(4.8) \quad \begin{array}{ccccccc} & & F & \xleftarrow{\quad} & \pi^* F & \xrightarrow{\eta} & i^* G & \longrightarrow & G \\ & \searrow \psi & & & \searrow \pi^* \psi & & \parallel & & \parallel \\ & \alpha^* F_{\text{univ}} & \xleftarrow{\quad} & \pi^* \alpha^* F_{\text{univ}} & \xrightarrow{\quad} & i^* G & \longrightarrow & G \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \text{Spf } R & \xleftarrow{\pi} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{i} & \text{Spf } k \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F_{\text{univ}} & \xleftarrow{\quad} & \rho^* F_{\text{univ}} & \xrightarrow{\quad} & G & \longrightarrow & G & \longrightarrow & G \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \text{Spf } R & \xleftarrow{\pi} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{i} & \text{Spf } k \\ & \downarrow \alpha & & \downarrow i & & \downarrow i & & \downarrow \\ \text{Spf } E_n & \xleftarrow{\rho} & \text{Spf } E_n/I & \longrightarrow & \text{Spf } E_n/I & \longrightarrow & \text{Spf } k \end{array}$$

To show this, we refine a half of the diagram as follows, omitting the other half.

$$(4.9) \quad \begin{array}{ccccccc} & & F & \xleftarrow{\quad} & \pi^* F & \xrightarrow{\eta} & i^* G & \longrightarrow & G \\ & \searrow \phi & & & \searrow \eta & & \parallel & & \parallel \\ & \tilde{F} & \xleftarrow{\quad} & i^* G & \xrightarrow{\quad} & i^* G & \longrightarrow & G \\ & \downarrow \gamma & & \downarrow & & \downarrow & & \downarrow \\ & \alpha^* F_{\text{univ}} & \xleftarrow{\quad} & \pi^* \alpha^* F_{\text{univ}} & \xrightarrow{\quad} & i^* G & \longrightarrow & G \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \text{Spf } R & \xleftarrow{\pi} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{i} & \text{Spf } k \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & \text{Spf } R & \xleftarrow{\pi} & \text{Spf } R/\mathfrak{m} & \xrightarrow{\quad} & \text{Spf } R/\mathfrak{m} & \xrightarrow{i} & \text{Spf } k \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Spf } R & \xleftarrow{\pi} & \text{Spf } R/\mathfrak{m} & \longrightarrow & \text{Spf } R/\mathfrak{m} & \longrightarrow & \text{Spf } k \end{array}$$

Here $\phi: F \rightarrow \tilde{F}$ over R is any isomorphism lifting η ,⁴ so that \tilde{F} has deformation structure (i, id) . By [Lubin-Tate1966, Theorem 3.1], there is a unique local homomorphism $\alpha: E_n \rightarrow R$ such that it reduces to $i: k = E_n/I \hookrightarrow R/\mathfrak{m}$ and such that there is a unique \star -isomorphism $\gamma: \tilde{F} \rightarrow \alpha^* F_{\text{univ}}$. Thus (4.9) commutes and, consequently, so does (4.8) if we take $\psi = \gamma \circ \phi$.

Now, to show the uniqueness, suppose that α' and ψ' fit into (4.8) in place of α and

⁴Such lifts always exist because the ring co-representing (strict) isomorphisms between formal group laws over commutative rings is free polynomial. They are in fact unique by the uniqueness in [Lubin-Tate1966, Theorem 3.1].

ψ . Then α^*F_{univ} and α'^*F_{univ} are in the same \star -isomorphism class via $\psi' \circ \psi^{-1}$, so by the uniqueness *loc. cit.* we have $\alpha = \alpha'$. Moreover, $\psi' \circ \phi^{-1} = \gamma$ so that $\psi' = \gamma \circ \phi = \psi$. \square

4.10 Non-uniqueness of F_{univ} There can be F_{univ}/E_n and F'_{univ}/E_n , both satisfying the universal property. Namely, there exists a unique $\alpha_{\text{univ}}: E_n \rightarrow E_n$ with a unique $F'_{\text{univ}} \stackrel{*}{=} \alpha_{\text{univ}}^*F_{\text{univ}}$, and there exists a unique $\alpha'_{\text{univ}}: E_n \rightarrow E_n$ with a unique $F_{\text{univ}} \stackrel{*}{=} \alpha'_{\text{univ}}^*F'_{\text{univ}}$. Moreover, we have $\alpha'_{\text{univ}} \circ \alpha_{\text{univ}} = \alpha_{\text{univ}} \circ \alpha'_{\text{univ}} = \text{id}$.

Suppose that *a priori* we know $F_{\text{univ}} \stackrel{*}{=} F'_{\text{univ}}$. Then $\alpha_{\text{univ}} = \alpha'_{\text{univ}} = \text{id}$ and this \star -isomorphism is unique. Thus the classifying maps $\alpha: E_n \rightarrow R$ for F/R are independent of the choice between F_{univ} and F'_{univ} .

5 Deformations of Frobenius

The flexibility of having an isomorphism η in a deformation of a formal group law buys us a notion of pushforward of deformation structures along *any* isogeny, compatible with Frobenius in a precise way.

5.1 Pushforward of deformation structures along an isogeny Let (F, i, η) be a deformation of G to R . Let $\psi: F \rightarrow F'$ be an isogeny of formal group laws over R of degree p^r . Then F' can be endowed with a deformation structure (i', η') such that the following diagram commutes, where σ is the absolute p -power Frobenius and Frob is the relative one.

$$(5.2) \quad \begin{array}{ccccccc} F & \xleftarrow{\quad} & \pi^*F & \xrightarrow{\eta} & i^*G & \xrightarrow{\quad} & G \\ \psi \swarrow & & \pi^*\psi \swarrow & & i^*\text{Frob}^r \swarrow & & \text{Frob}^r \swarrow \\ F' & \xleftarrow{\quad} & \pi^*F' & \xrightarrow{\eta'} & i'^*G & \xrightarrow{\quad} & G \\ & \searrow \pi & \searrow \pi & & \searrow i & & \searrow i \\ & \text{Spf } R & \text{Spf } R/\mathfrak{m} & & \text{Spf } R/\mathfrak{m} & & \text{Spf } k \\ & \searrow \pi & \searrow \pi & & \searrow i & & \searrow \sigma^r \\ & \text{Spf } R & \text{Spf } R/\mathfrak{m} & \xrightarrow{i'} & \text{Spf } R/\mathfrak{m} & \xrightarrow{i} & \text{Spf } k \end{array}$$

We write $\psi_!(i, \eta) := (i', \eta')$ and call it the *pushforward of (i, η) along ψ* . Explicitly, the pair is determined by the equalities

$$i' = i \circ \sigma^r \quad \text{and} \quad \eta' \circ \pi^*\psi = i^*\text{Frob}^r \circ \eta$$

5.3 Categories of deformations Fix G/k . Let $\text{Def}_{\text{isog}}(R)$ be the category with objects deformations (F, i, η) of G to R and with morphisms $(F, i, \eta) \rightarrow (F', i', \eta')$, each consisting of an isogeny $\psi: F \rightarrow F'$ of formal group laws over R and an equality $(i', \eta') = \psi_!(i, \eta)$. The degree of ψ must be p^r for some $r \geq 0$. Note that the isomorphisms in $\text{Def}_{\text{isog}}(R)$ are precisely the \star -isomorphisms (cf. (4.5), when $r = 0$) and that the only automorphism of an object is the identity by the uniqueness in Proposition 4.6.

5.4 Deformations of Frobenius Given the diagram (5.2), we view a morphism $(F, i, \eta) \rightarrow (F', i', \eta')$ in $\text{Def}_{\text{isog}}(R)$ as a deformation to R of Φ^r in the wide category FG_{isog} (3.5).⁵ Thus, we call it a *deformation of Frobenius*, and simply call $\psi: F \rightarrow F'$ such if $\eta = \eta'$ (in the same sense of the footnote in (4.3)) so that $\pi^*\psi$ is a relative Frobenius (cf. [Rezk2009, 11.3]). Two deformations $(F_1, i_1, \eta_1) \rightarrow (F'_1, i'_1, \eta'_1)$ and $(F_2, i_2, \eta_2) \rightarrow (F'_2, i'_2, \eta'_2)$ of Frobenius are *isomorphic* if (F_1, i_1, η_1) and (F_2, i_2, η_2) are \star -isomorphic and if (F'_1, i'_1, η'_1) and (F'_2, i'_2, η'_2) are \star -isomorphic.

Proposition 5.5 (cf. [Rezk2014, Theorem 4.4]) *Let k, G, R, E_n be as in Proposition 4.6 and again fix G/k . Then, for each $r \geq 0$, the functor*

$$R \mapsto \{\text{isomorphism classes of deformations } (F, i, \eta) \rightarrow (F', i', \eta') \text{ of } \Phi^r \text{ to } R\}$$

from the category of complete local rings with residue field containing k to the category of sets is co-represented by a ring A_r , which is a bimodule over $A_0 = E_n$ with structure maps local homomorphisms $s_r, t_r: A_0 \rightarrow A_r$. Explicitly, there is a (by no means unique) deformation $(F_{\text{univ}}, \text{id}, \text{id})$ of G to A_0 satisfying the following universal property. Given any deformation $(F, i, \eta) \rightarrow (F', i', \eta')$ of Φ^r to R , there is a unique local homomorphism

$$\alpha_r: A_r \rightarrow R$$

such that $\alpha_r s_r, \alpha_r t_r: A_0 \rightarrow R$ reduce to $i, i': k = A_0/I \hookrightarrow R/\mathfrak{m}$ respectively, with I and \mathfrak{m} the maximal ideals, and such that there are unique \star -isomorphisms

$$(F, i, \eta) \rightarrow (\alpha_r^* s_r^* F_{\text{univ}}, i, \text{id}) \quad \text{and} \quad (F', i', \eta') \rightarrow (\alpha_r^* t_r^* F_{\text{univ}}, i', \text{id})$$

Proof Let $(F_{\text{univ}}, \text{id}, \text{id})$ be given from Proposition 4.6 and write $\mathcal{F}_{\text{univ}}$ for the formal group over E_n whose group law is F_{univ} as in (2.1). Clearly $A_0 = E_n$ with $s_0 = t_0 = \text{id}$. In general, for each $r \geq 0$, let $\text{Sub}_r(\mathcal{F}_{\text{univ}})$ be the affine formal scheme over A_0 which

⁵More precisely, with a corresponding wide category of formal group laws understood, it is a deformation of the endomorphism on G/k induced by Φ^r . We will also denote this endomorphism of formal group laws by Φ^r , consistent with (2.2).

classifies degree- p^r subgroups of $\mathcal{F}_{\text{univ}}$ [Strickland1997, Theorem 42] and let A_r be its ring of functions. We need only determine the maps s_r, t_r and show that $(F_{\text{univ}}, \text{id}, \text{id})$ satisfies the stronger universal property involving A_r as stated.

The structure morphism $A_0 \rightarrow A_r$ of $\text{Sub}_r(\mathcal{F}_{\text{univ}})/A_0$ reduces to the identity between residue fields (see [ibid., § 13]). Thus $F_{\text{univ}} \times_{A_0} A_r$ inherits the deformation structure (id, id) from F_{univ} along the base change. Let $\mathcal{H}_{\text{univ}}^{(p^r)} \subset \mathcal{F}_{\text{univ}} \times_{A_0} A_r$ be the subgroup of degree p^r classified by $\text{id}: A_r \rightarrow A_r$, and let $\mathcal{F}_{\text{univ}}^{(p^r)} := (\mathcal{F}_{\text{univ}} \times_{A_0} A_r)/\mathcal{H}_{\text{univ}}^{(p^r)}$ be the quotient group as in (2.3) with a particular group law $F_{\text{univ}}^{(p^r)}$. The quotient map of formal groups induces an isogeny

$$\psi_{\text{univ}}^{(p^r)}: F_{\text{univ}} \times_{A_0} A_r \rightarrow F_{\text{univ}}^{(p^r)}$$

of group laws over A_r . By (2.5) it is a deformation of Frobenius (5.4) and we have

$$\psi_{\text{univ}}^{(p^r)}(\text{id}, \text{id}) = (\sigma^r, \text{id})$$

In view of Proposition 4.6, let $s_r, t_r: A_0 \rightarrow A_r$ be the unique local homomorphisms which classify $(F_{\text{univ}} \times_{A_0} A_r, \text{id}, \text{id})$ and $(F_{\text{univ}}^{(p^r)}, \sigma^r, \text{id})$ respectively. Indeed, by uniqueness, s_r is the structure morphism of $\text{Sub}_r(\mathcal{F}_{\text{univ}})/A_0$.

It remains to verify the universal property. By Proposition 4.6, given any deformation $\psi \times \psi_!: (F, i, \eta) \rightarrow (F', i', \eta')$ of Φ^r to R , there are unique local homomorphisms

$$\alpha, \alpha': A_0 \rightarrow R$$

such that they reduce to $i, i': k = A_0/I \hookrightarrow R/\mathfrak{m}$ respectively, and such that there are unique \star -isomorphisms

$$(F, i, \eta) \rightarrow (\alpha^* F_{\text{univ}}, i, \text{id}) \quad \text{and} \quad (F', i', \eta') \rightarrow (\alpha'^* F_{\text{univ}}, i', \text{id})$$

Let $\mathcal{H} \subset \alpha^* \mathcal{F}_{\text{univ}}$ be the image of $\ker \psi \subset \mathcal{F}$ under the first \star -isomorphism.⁶ It is a subgroup of degree p^r . Then by [ibid., Theorem 42] (taking $X = \text{Spf } A_0$ and $Y = \text{Spf } R$) there is a unique local homomorphism

$$\alpha_r: A_r \rightarrow R$$

which classifies \mathcal{H} with $\alpha_r \circ s_r = \alpha$. Clearly $\alpha_r \circ s_r$ reduces to i and there is a unique

⁶In fact, since a \star -isomorphism of formal group laws can be thought of as a change of coordinates on a formal group, we may write $\mathcal{H} = \ker \psi$.

\star -isomorphism $(F, i, \eta) \rightarrow (\alpha_r^* s_r^* F_{\text{univ}}, i, \text{id})$ as above. On the other hand, we have

$$\begin{aligned}
 (F', i', \eta') &\stackrel{*}{=} (F/\ker \psi, i', \eta) && \text{cf. (5.2)} \\
 &\stackrel{*}{=} (\alpha^* F_{\text{univ}}/H, i', \text{id}) \\
 &= (\alpha_r^* (F_{\text{univ}} \times_{A_0} A_r) / \alpha_r^* H_{\text{univ}}^{(p')}, i', \text{id}) \\
 &= (\alpha_r^* F_{\text{univ}}^{(p')}, i', \text{id}) && \text{by [ibid., Theorem 19 (v)]} \\
 &\stackrel{*}{=} (\alpha_r^* t_r^* F_{\text{univ}}, i', \text{id})
 \end{aligned}$$

Therefore, $\alpha_r \circ t_r = \alpha'$, so $\alpha_r \circ t_r$ reduces to i' and there is a unique \star -isomorphism $(F', i', \eta') \rightarrow (\alpha_r^* t_r^* F_{\text{univ}}, i', \text{id})$. \square

5.6 Canonical lifts of Frobenius morphisms In summary of Proposition 5.5 and its proof, the ring A_r carries a universal example $\psi_{\text{univ}}^{(p')}$ of deformation of Φ^r to R as follows.⁷

$$(5.7) \quad \begin{array}{ccc} s_r^* F_{\text{univ}} = F_{\text{univ}} \times_{A_0} A_r & \xrightarrow{\psi_{\text{univ}}^{(p')}} & F_{\text{univ}}^{(p')} \stackrel{*}{=} t_r^* F_{\text{univ}} \\ & \searrow \quad \swarrow & \\ & \text{Spf } A_r & \end{array}$$

The central notion of norm coherence in this paper, introduced in the next section, concerns the question of when the \star -isomorphism in the above diagram is the identity. We write $\psi_{\text{univ}}^{(p')} \star$ for the composite of $\psi_{\text{univ}}^{(p')}$ with this \star -isomorphism.

Remark 5.8 Continuing with (4.10), we see from the proof of Proposition 5.5 that the maps α_r , s_r , t_r are independent of the choice between \star -isomorphic universal deformations.

5.9 Dependency of F_{univ} on G/k The choice of $F_{\text{univ}} = F_{\text{univ}}(G)$ as in (4.10, 5.8) can be made functorial with respect to morphisms in FG_{isog} . Specifically, for functoriality under base change, the right square in (3.4) deforms so that

$$F_{\text{univ}}(G' \times_{k'} k) = F_{\text{univ}}(G') \times_{E_n(G')} E_n(G' \times_{k'} k)$$

as formal group laws, where n is the height of G'/k' (invariant under base change) and $E_n(G') \rightarrow E_n(G' \times_{k'} k)$ sends each generator of the source to the corresponding one

⁷See [ibid., § 10, § 13] for more about the rings A_r . For an explicit example, see [Zhu2015b, Theorem 1.2] where $r = 1$ and G is of height 2 over $k = \overline{\mathbb{F}}_p$.

of the target. This identity follows from the construction of F_{univ} in [Lubin-Tate1966, Proposition 1.1] as a “generic group law” Γ .

Moving to the left square of (3.4), let $\psi: \mathcal{G} \rightarrow \mathcal{G}' \times_{k'} k$ be any isogeny of degree p^r and consider the following (cf. (3.8) and (5.2)).

$$\begin{array}{ccccccc}
 & & \text{Frob}^r & & & & \\
 & \swarrow & & \searrow & & & \\
 \mathcal{G} & \xrightarrow{\psi} & \mathcal{G}' \times_{k'} k & \xrightarrow{\sim} & \mathcal{G}^{(p^r)} & \xrightarrow{\quad} & \mathcal{G} \\
 \downarrow & & \downarrow & & \downarrow \lrcorner & & \downarrow \\
 \text{Spf } k & \xlongequal{\quad} & \text{Spf } k & \xlongequal{\quad} & \text{Spf } k & \xrightarrow{\sigma^r} & \text{Spf } k
 \end{array}$$

Note that Frob^r deforms to $\psi_{\text{univ}}^{(p^r)*}: s_r^* F_{\text{univ}}(G) \rightarrow t_r^* F_{\text{univ}}(G)$ over A_r as in (5.7). Moreover,

$$\begin{aligned}
 t_r^* F_{\text{univ}}(G) &= F_{\text{univ}}(G) \times_{A_0}^{t_r} A_r \\
 &= F_{\text{univ}}(G) \times_{E_n(G)} E_n(G^{(p^r)}) \times_{E_n(G^{(p^r)})} A_r \\
 &= F_{\text{univ}}(G^{(p^r)}) \times_{E_n(G^{(p^r)})} A_r
 \end{aligned}$$

where $E_n(G^{(p^r)}) = \mathbb{W}k[v_1, \dots, v_{n-1}]$ with each $v_i \mapsto t_r(u_i) \in A_r$.⁸ The construction of F_{univ} clearly respects isomorphisms so that

$$F_{\text{univ}}(G' \times_{k'} k) \times_{E_n(G' \times_{k'} k)} E_n(G^{(p^r)}) \cong F_{\text{univ}}(G^{(p^r)})$$

Thus, over A_r (omitting the base changes), ψ deforms to

$$F_{\text{univ}}(G) \xrightarrow{\psi_{\text{univ}}^{(p^r)*}} F_{\text{univ}}(G^{(p^r)}) \xrightarrow{\sim} F_{\text{univ}}(G' \times_{k'} k)$$

This shows the functoriality of F_{univ} under isogenies.

To summarize, given a morphism in FG_{isog} as above, we can choose the universal deformations of its source and target so that in terms of formal group laws (3.4)

⁸For an explicit example, see [Zhu2015b, Theorem 1.6(ii)] where $r = 1$ and G is of height $n = 2$ over $k = \overline{\mathbb{F}}_p$, with $u_1 = h$, $t_1(u_1) = \psi^p(h)$. Also, note that since E_n classifies \star -isomorphism classes of deformations, the \star -isomorphism in (5.7) does not affect the correspondence $v_i \mapsto t_r(u_i)$.

deforms over A_r as follows.

$$\begin{array}{ccccc}
 & & F_{\text{univ}}(G) & & F_{\text{univ}}(G' \times_{k'} k) \longrightarrow F_{\text{univ}}(G') \\
 & & \downarrow & & \downarrow \lrcorner \\
 & & \text{Spf } E_n(G) & \nearrow & F_{\text{univ}}(G^{(p^r)}) \\
 & & \downarrow & & \downarrow \\
 s_r^* F_{\text{univ}}(G) & \xrightarrow{\quad} & t_r^* F_{\text{univ}}(G) & \xrightarrow{\quad} & \text{Spf } E_n(G' \times_{k'} k) \longrightarrow \text{Spf } E_n(G') \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spf } A_r & \xrightarrow{\quad} & \text{Spf } A_r & \xrightarrow{\quad} & \text{Spf } E_n(G^{(p^r)})
 \end{array}$$

6 Norm-coherent deformations

6.1 Set-up Let k be an algebraic extension of \mathbb{F}_p (in particular, k is perfect) and G be a formal group law over k of finite height n . Let R be a complete local ring with maximal ideal \mathfrak{m} and residue field $R/\mathfrak{m} \supset k$. Let F/R be a deformation of G/k with deformation structure (i, id) as in (4.2).

Remark 6.2 Observe that, given any deformation (F, i, η) , there exists a unique deformation $(\tilde{F}, i, \text{id})$ such that the two are in the same \star -isomorphism class (cf. (4.9)). Without loss of generality, here we focus on the case of $\eta = \text{id}$.

6.3 Quotient by the p -torsion subgroup As in (2.1) write \mathcal{F} for the formal group over R whose group law is F (upon choosing a coordinate) and write $\mathcal{F}[p]$ for its subgroup scheme of p -torsions. This is defined over an extension \tilde{R} of R obtained by adjoining the roots of the p -series of F . Let $\mathcal{F}/\mathcal{F}[p] := (\mathcal{F} \times_R \tilde{R})/\mathcal{F}[p]$ be the quotient group as in (2.3) with a particular group law $F/F[p]$ so that the isogeny

$$f_p: F \rightarrow F/F[p]$$

induced by the quotient morphism of formal groups is a deformation of Frobenius (5.4). Note that $\mathcal{F}[p](\tilde{R})$ is stable under the action of $\text{Aut}(\tilde{R}/R)$. Thus f_p can be defined over R (cf. [Lubin1967, Theorem 1.4]).

Remark 6.4 The restriction of f_p on the special fiber is the relative p^n -power Frobenius. It is not an endomorphism unless $k \subset \mathbb{F}_{p^n}$ (cf. [Ando1995, proof of Proposition 2.5.1]).

6.5 The isogeny l_p By Proposition 5.5 there is a unique local homomorphism $\alpha_n: A_n \rightarrow R$ together with a unique \star -isomorphism $(F/F[p], i \circ \sigma^n, \text{id}) \rightarrow (\alpha_n^* t_n^* F_{\text{univ}}, i \circ \sigma^n, \text{id})$. Write

$$g_p: F/F[p] \rightarrow \alpha_n^* t_n^* F_{\text{univ}}$$

for the corresponding \star -isomorphism of formal group laws. Let

$$l_p: F \rightarrow \alpha_n^* t_n^* F_{\text{univ}}$$

be the composite $g_p \circ f_p$.

Remark 6.6 The isogeny l_p of formal group laws over R is uniquely characterized by the following properties (cf. [ibid., Proposition 2.5.4], the proof here being completely analogous).

- (i) It has source F and target of the form $\alpha^* t_n^* F_{\text{univ}}$ for some local homomorphism $\alpha: A_n \rightarrow R$.
- (ii) The kernel of l_p applied to \mathcal{F} is $\mathcal{F}[p]$.
- (iii) Over the residue field, l_p reduces to the relative p^n -power Frobenius.

Explicitly, with notation as in (5.2), f_p and l_p fit into the following commutative diagram. Their restrictions on the special fiber are highlighted with corresponding decorations, which are in fact identical.

$$(6.7) \quad \begin{array}{ccccccc} & & F & \xleftarrow{\quad} & \pi^* F & \xlongequal{\quad} & i^* G \xrightarrow{\quad} G \\ & \swarrow l_p & \downarrow f_p & \swarrow \pi^* f_p & \downarrow i^* \text{Frob}^n & \searrow \text{Frob}^n & \\ & & F/F[p] & \xleftarrow{\quad} & \pi^*(F/F[p]) & \xlongequal{\quad} & i^* G^{(p^n)} \xrightarrow{\quad} G^{(p^n)} \\ & \swarrow g_p & \downarrow \pi^* g_p & \swarrow & \downarrow & \searrow & \\ & & \alpha_n^* t_n^* F_{\text{univ}} & \xleftarrow{\quad} & \pi^* \alpha_n^* t_n^* F_{\text{univ}} & \xlongequal{\quad} & i^* G \xrightarrow{\quad} G \end{array}$$

Example 6.8 Let $k = \mathbb{F}_p$ and G be the Honda formal group law given by [ibid., 2.5.5] so that $[p]_G(t) = t^{p^n}$. Then the relative Frobenius Frob^n coincides with the absolute Frobenius automorphism on G and so $l_p = [p]_F$ (cf. [ibid., Proposition 2.6.1]).

6.9 The isogenies l_H More generally, let $\mathcal{H} \subset \mathcal{F}$ be a subgroup of degree p^r , $\psi_H: F \rightarrow F'$ be any isogeny with kernel \mathcal{H} , and $\psi_H \times \psi_{H!}: (F, i, \text{id}) \rightarrow (F', i', \eta')$ be the corresponding deformation of Frobenius. The diagram (6.7) generalizes as follows.

(6.10)

$$\begin{array}{ccccccc}
 & & F & \xleftarrow{\quad} & \pi^* F & \xrightarrow{\eta = \text{id}} & i^* G \xrightarrow{\quad} G \\
 & \swarrow l_H & \downarrow \psi_H & & \downarrow \pi^* \psi_H & \downarrow i^* \text{Frob}^r & \downarrow \text{Frob}^r \\
 & & F' & \xleftarrow{\quad} & \pi^* F' & \xrightarrow{\eta'} & i^* G^{(p^r)} \xrightarrow{\quad} G^{(p^r)} \\
 & \swarrow l_H & \downarrow \psi_H & & \downarrow \pi^* \psi_H & \downarrow i^* \text{Frob}^r & \downarrow \text{Frob}^r \\
 \alpha_r^* t_r^* F_{\text{univ}} & \xleftarrow{\quad} & \pi^* \alpha_r^* t_r^* F_{\text{univ}} & \xrightarrow{\quad} & i^* G & \xrightarrow{\quad} & G
 \end{array}$$

In particular, when $\psi_H = f_H: F \rightarrow F/H$ is the deformation of Frobenius with kernel \mathcal{H} , we have the following commutative diagram.

(6.11)

$$\begin{array}{ccc}
 F & \xrightarrow{f_H} & F/H \\
 & \searrow l_H & \downarrow g_H \\
 & & \alpha_r^* t_r^* F_{\text{univ}}
 \end{array}$$

Remark 6.12 This construction of l_H is functorial under base change and under quotient, due to the functoriality of f_H and g_H (see [Strickland1997, Theorem 19 (v)], [Ando1995, Proposition 2.2.6], Proposition 5.5, and (5.9)). To be precise, given any local homomorphism $\beta: R \rightarrow R'$ and any finite subgroups $\mathcal{H}_1 \subset \mathcal{H}_2$ of \mathcal{F} , we have

$$l_{\beta^* H} = \beta^* l_H \quad \text{and} \quad l_{H_2/H_1} \circ l_{H_1} = l_{H_2}$$

where the composition is taken up to a \star -isomorphism, as shown in the following commutative diagrams.

(6.13)

$$\begin{array}{ccc}
 F & \xleftarrow{\quad} & \beta^* F \\
 f_H \downarrow & & \downarrow \beta^* f_H \quad \searrow f_{\beta^* H} \\
 F/H & \xleftarrow{\quad} & \beta^*(F/H) = \beta^* F / \beta^* H \\
 g_H \downarrow & & \downarrow \beta^* g_H \quad \swarrow g_{\beta^* H} \\
 \alpha_r^* t_r^* F_{\text{univ}} & \xleftarrow{\quad} & \beta^* \alpha_r^* t_r^* F_{\text{univ}}
 \end{array}$$

$$(6.14) \quad \begin{array}{ccccc} F & \xrightarrow{f_{H_2}} & F/H_2 & \xrightarrow{g_{H_2}} & \alpha_{r_2}^* t_{r_2}^* F_{\text{univ}} \\ \downarrow f_{H_1} & & \parallel & & \nearrow \tilde{g}_{H_2/H_1} \\ F/H_1 & \xrightarrow{f_{H_2/H_1}} & F/H_1/H_2/H_1 & & \\ \downarrow g_{H_1} & & \downarrow \tilde{g}_{H_1} & & \\ \alpha_{r_1}^* t_{r_1}^* F_{\text{univ}} & \xrightarrow{\tilde{f}_{H_2/H_1}} & \alpha_{r_1}^* t_{r_1}^* F_{\text{univ}}/H_{\text{univ}}^{(p^{r_2-r_1})} & & \end{array}$$

6.15 Definition of norm coherence Recall from the proof of Lemma 3.10 that $\text{Norm}_{\psi^*}: \mathcal{O}_G \rightarrow \mathcal{O}_{G'}$ sends a coordinate x_G on \mathcal{G} to the coordinate on $\mathcal{G}' = \mathcal{G}^{(p^r)}$ which pulls back along $\psi = \text{Frob}^r: \mathcal{G} \rightarrow \mathcal{G}^{(p^r)}$ to $x_G^{p^r}$. In other words, the norm map agrees with pushing forward a coordinate along the Frobenius isogeny.

This agreement on x_G over k may not extend to R for an arbitrary coordinate x on \mathcal{F} lifting x_G . On one hand, given a subgroup $\mathcal{H} \subset \mathcal{F}$ of degree p^r , the isogeny $f_H: \mathcal{F} \rightarrow \mathcal{F}/\mathcal{H}$ lifts the norm map in the sense that

$$(6.16) \quad \begin{aligned} x_H(f_H(P)) &= \prod_{Q \in \mathcal{H}(\tilde{R})} (x(P) +_F x(Q)) && \text{by (2.4)} \\ &= \prod_{\sigma \in \text{Aut}(\mathcal{O}_F/\mathcal{O}_{F/H})} \sigma \cdot x(P) \\ &= f_H^* \text{Norm}_{f_H^*}(x)(P) && \text{cf. (3.11)} \\ &= \text{Norm}_{f_H^*}(x)(f_H(P)) \end{aligned}$$

where x_H is the coordinate corresponding to the group law F_x/H , P is any R -point on \mathcal{F} , and \tilde{R} is an extension of R to define the p^r points of \mathcal{H} .⁹ On the other hand, the

⁹There is an analogue of this in the context of Galois theory, where the finite free extension $f_H^*: \mathcal{O}_{F/H} \rightarrow \mathcal{O}_F$ of rings is replaced by a finite Galois extension (see, e.g., [Rotman2010, pp. 916-920, esp. Corollary 10.87]). Moreover, consider a coordinate on \mathcal{F} as a map $\mathcal{F} \rightarrow \hat{\mathbb{A}}^1$ (2.1). We then have

$$\mathcal{F} \xrightarrow{f_H} \mathcal{F}/\mathcal{H} \rightarrow \hat{\mathbb{A}}^1$$

and $\text{Norm}_{f_H^*}$ gives

$$\mathcal{F} \xrightarrow{x} \hat{\mathbb{A}}^1 \quad \mapsto \quad \mathcal{F}/\mathcal{H} \xrightarrow{x_H} \hat{\mathbb{A}}^1$$

which is analogous to a *norm map* as a piece of structure in a Tambara functor [Tambara1993, 3.1]. This last notion of a norm map has been packaged into equivariant stable homotopy theory and turned out as a key ingredient in recent advances in the field [Brun2007, Hill-Hopkins2016].

isogeny $l_H = g_H \circ f_H$ lifts Frob^r canonically with respect to \mathcal{H} ; that is, if f'_H is another lift with kernel \mathcal{H} and classifying \star -isomorphism g'_H , then $g'_H \circ f'_H = l_H$ (Remark 6.6).

Definition 6.17 Let (F, i, id) be a deformation of G to R as in (6.1). We say that it is *norm-coherent* if given any finite subgroup \mathcal{H} of \mathcal{F} , the identity

$$(6.18) \quad l_H = f_H$$

holds. In other words, the condition is that the \star -isomorphism $g_H = \text{id}$.

More generally, given any deformation (F, i, η) of G to R , let $(\tilde{F}, i, \text{id})$ be the unique deformation associated to it (Remark 6.2). We say that (F, i, η) is norm-coherent if $(\tilde{F}, i, \text{id})$ is.

With the deformation structure understood, we also call the formal group law F , as well as its corresponding coordinate x_F on \mathcal{F} , norm-coherent.

Remark 6.19 Explicitly, in terms of a norm-coherent coordinate x , (6.18) boils down to the identity

$$h^{(p^r)}(l_H(x)) = \prod_{c \in \mathcal{H}(\tilde{R})} h(x(c) \underset{F}{+} x)$$

for all $h(x) \in R[[x]]$, where $h^{(p^r)}$ is the series obtained by twisting the coefficients of h with the automorphism on R which lifts the absolute p^r -power Frobenius (cf. (3.12)).

A more conceptual form of norm coherence is as follows. Let $\psi \times \psi_! : (F, i, \eta) \rightarrow (F', i', \eta')$ be any deformation of Φ^r to R (5.4) and write $\mathcal{H} := \ker \psi$. Let $x_{F, i, \eta} := x_F$ be the coordinate on \mathcal{F} corresponding to F , the latter equipped with deformation structure (i, η) . Consider the identity

$$(6.20) \quad x_{F', \psi_!(i, \eta)} = \text{Norm}_{\psi^*}(x_{F, i, \eta})$$

in $\mathcal{O}_{F'}$, which specializes to (6.16) if $\psi = f_H$; (6.18) is equivalent to the condition that this identity hold for *any* ψ . Indeed, let us turn to the universal case. The pushforward $\psi_!(i, \eta)$ of deformation structure (5.2) indicates a change of coordinates on \mathcal{F}' (cf. (4.9)) under which the left-hand side of (6.20) corresponds to the formal group law $\alpha_r^* t_r^* F_{\text{univ}}$, i.e., the target of l_H , as in (6.10). Meanwhile, by functoriality of norm maps (i.e., extension of scalars for determinants), the right-hand side changes to $\text{Norm}_{l_H^*}(x_{F, i, \eta})$ (cf. (3.13)). Thus, in the universal case, (6.20) becomes

$$x_{\alpha_r^* t_r^* F_{\text{univ}}, i \circ \sigma^r, \eta} = \text{Norm}_{l_H^*}(x_{F, i, \eta})$$

Pulling this back along l_H^* to \mathcal{O}_F , we see that it holds if and only if (6.18) does. We will return to this formulation of norm coherence toward the end of Section 8.

6.21 Functoriality of norm coherence Recall from (4.3) and (5.1) the operations of base change and pushforward of deformation structures. The notion of norm coherence in Definition 6.17 is preserved under both as follows.

Proposition 6.22 *Let (F, i, η) be a norm-coherent deformation of G to R .*

- (i) *Given any local homomorphism $\beta: R \rightarrow R'$, the deformation $(\beta^*F, \beta^*(i, \eta))$ is norm-coherent.*
- (ii) *Given any isogeny $\psi: F \rightarrow F'$ over R , the deformation $(F', \psi_!(i, \eta))$ is norm-coherent. In particular, given any finite subgroup $\mathcal{H} \subset \mathcal{F}$ of degree p^r , the deformation $(F/H, i \circ \sigma^r, \eta)$ is norm-coherent.*

Proof For (i), in view of (6.2), first note that

$$\begin{aligned} (\widetilde{F}, i, \text{id}) &\stackrel{*}{=} (F, i, \eta) \\ \implies (\beta^*\widetilde{F}, \beta^*(i, \text{id})) &\stackrel{*}{=} (\beta^*F, \beta^*(i, \eta)) \\ \implies \beta^*\widetilde{F} &= \widetilde{\beta^*F} \end{aligned}$$

To see that $\beta^*\widetilde{F}$ is norm-coherent, we have from (6.13)

$$l_{\beta^*H} = \beta^*l_H = \beta^*f_H = f_{\beta^*H}$$

For (ii), suppose that ψ is of degree p^r and let $i' = i \circ \sigma^r$. In view of

$$(F', \psi_!(i, \eta)) \stackrel{*}{=} (F/H, i', \eta) \stackrel{*}{=} (\widetilde{F/H}, i', \text{id}) = (\widetilde{F}/H, i', \text{id})$$

we are reduced to the special case of

$$(F, i, \text{id}) \xrightarrow{f_H \times f_{H^1}} (F/H, i \circ \sigma^r, \text{id})$$

Since the source is norm-coherent, we have from (6.14)

$$l_H = f_H \quad \text{and} \quad l_{K/H} \circ l_H = l_K = f_K = f_{K/H} \circ f_H$$

where K is any finite subgroup of \mathcal{F} containing \mathcal{H} , and the first composition is on-the-nose because of the first identity in the display. Given that $g_{K/H}$ is an isomorphism, we then deduce from these

$$l_{K/H} = f_{K/H}$$

which shows the norm coherence of $(F/H, i \circ \sigma^r, \text{id})$. \square

7 Existence and uniqueness of norm-coherent deformations

The following generalizes a result of Ando's.

Proposition 7.1 (cf. [Ando1995, Theorem 2.5.7]) *Let k, G, R, F be as in (6.1) and fix G/k . There exists a unique formal group law F' over R , \star -isomorphic to F , that is norm-coherent. In other words, given any coordinate x_G on the formal group \mathcal{G} and a coordinate x_F on \mathcal{F} that lifts x_G , there exists a unique norm-coherent coordinate on \mathcal{F} whose corresponding formal group law is \star -isomorphic to that of x_F .*

To show this, we will follow Ando's proof of his theorem, making alterations for greater generality whenever necessary (most significantly in (7.6)). The argument breaks into two parts, the first focusing on norm coherence for the p -torsion subgroup $\mathcal{F}[p]$ and the second showing functoriality for all finite subgroups. We begin with the following key lemma.

Lemma 7.2 (cf. [ibid., Theorem 2.6.4]) *Given any coordinate x_F on \mathcal{F} that lifts x_G , there exists a unique coordinate on \mathcal{F} whose corresponding formal group law is \star -isomorphic to that of x_F and satisfies*

$$(7.3) \quad l_p = f_p$$

Proof Existence First we reduce the proof to the universal case. Let F_{univ} be a universal deformation of G/k to E_n as in Proposition 4.6, so that there is a unique local homomorphism

$$\alpha: E_n \rightarrow R$$

together with a unique \star -isomorphism

$$g: F \rightarrow \alpha^* F_{\text{univ}}$$

Suppose that we can construct a coordinate x on $\mathcal{F}_{\text{univ}}$ whose corresponding formal group law F'_{univ} satisfies (7.3) and is \star -isomorphic to F_{univ} . Taking $\mathcal{H} = \mathcal{F}_{\text{univ}}[p]$ in the proof of Proposition 6.22(i), we then see that $\alpha^* F'_{\text{univ}}$ satisfies (7.3) and is \star -isomorphic to F .

We turn to the universal case. The proof is inductive, on powers of the maximal ideal I of E_n . Let y be the coordinate corresponding to F_{univ} from above, so we may write $F_y := F_{\text{univ}}$. With respect to y , given that $g_p^y: F_y/F[p] \rightarrow \alpha_n^* t_n^* F_y$ is defined over E_n as in (6.3), let $a(t) \in E_n[[t]]$ be such that

$$(7.4) \quad g_p^y(t) = t + a(t)$$

We shall construct a desired coordinate x on the universal formal group \mathcal{F} by inductively modifying the coordinate y so that $a(t) \equiv 0 \pmod{I^r}$ for increasing r .

Let the inductive hypothesis be

$$(7.5) \quad a(t) = \sum_{j \geq 1} a_j t^j \quad \text{with } a_j \in I^{r-1}$$

Since g_p^y is a \star -isomorphism, we get the case of $r = 2$. Let $\delta(t)$ be the power series

$$(7.6) \quad \delta(t) = t - a^{(-p^n)}(t)$$

where $a^{(-p^n)}(t)$ is the series obtained by twisting the coefficients a_j with the inverse of the local automorphism $\alpha_n t_n$ on $A_0 = E_n$,¹⁰ and has its coefficients in I^{r-1} as well. The coordinate

$$(7.7) \quad z := \delta(y)$$

on \mathcal{F} then yields a formal group law F_z over E_n such that $\delta: F_y \rightarrow F_z$ is a \star -isomorphism. With respect to z , let $b(t) \in E_n[[t]]$ be such that

$$(7.8) \quad g_p^z(t) = t + b(t)$$

We will show that this choice of coordinate z gives

$$(7.9) \quad b(t) = \sum_{j \geq 1} b_j t^j \quad \text{with } b_j \in I^r$$

and in particular produces the equation

$$g_p^z(t) \equiv t \pmod{I^r}$$

Note that the formal group laws F_y and F_z coincide modulo I^{r-1} . Thus, by induction and Krull's intersection theorem, we will then obtain in the limit a coordinate x such that $g_p^x(t) = t$, or $l_p^x(t) = f_p^x(t)$, as desired.

¹⁰ This automorphism lifts the p^n -power Frobenius σ^n on k to $\mathbb{W}k$ and fixes the generators u_i . Indeed, by [Lubin1967, Theorem 1.5], the isogeny f_p differs by an E_n -isomorphism from the endomorphism $[p]_F$; moreover, their targets can be equipped with deformation structures so that they are \star -isomorphic as deformations (cf. (6.7)). Since the generators u_i parametrize \star -isomorphism classes of deformations as in Proposition 4.6, they remain unchanged under $\alpha_n t_n$. Cf. [Rezk2013, 4.15-4.16].

Consider the diagram

$$(7.10) \quad \begin{array}{ccc} F_y & \xrightarrow{\delta} & F_z \\ \downarrow l_p^y & & \downarrow l_p^z \\ \alpha_n^* t_n^* F_y & \xrightarrow{\tilde{\delta}} & \alpha_n^* t_n^* F_z \end{array}$$

where $\tilde{\delta} := \alpha_n^* t_n^* \delta$ is a \star -isomorphism.¹¹ By the unique characterization of l_p in Remark 6.6, we have $\tilde{\delta} \circ l_p^y \circ \delta^{-1} = l_p^z$. Thus the diagram commutes and we get $\tilde{\delta} \circ l_p^y(t) = l_p^z \circ \delta(t)$, or

$$(7.11) \quad \tilde{\delta} \circ g_p^y \circ f_p^y(t) = g_p^z \circ f_p^z \circ \delta(t)$$

We shall compare the two sides of (7.11) modulo I' to show (7.9) and thus complete the induction.

The left-hand side of (7.11) can be evaluated modulo I' as follows.

$$(7.12) \quad \begin{aligned} \tilde{\delta} \circ g_p^y \circ f_p^y(t) &= \tilde{\delta}(f_p^y(t) + a \circ f_p^y(t)) && \text{by (7.4)} \\ &\equiv \tilde{\delta}(f_p^y(t) + a(t^{p^n})) && \text{by (2.5) and (7.5)} \\ &= f_p^y(t) + a(t^{p^n}) - a(f_p^y(t) + a(t^{p^n})) && \text{by (7.6)} \\ &\equiv f_p^y(t) + a(t^{p^n}) - a(f_p^y(t)) && \text{by (7.5)} \\ &\equiv f_p^y(t) + a(t^{p^n}) - a(t^{p^n}) && \text{by (2.5) and (7.5)} \\ &= f_p^y(t) \end{aligned}$$

¹¹The classifying maps for $F_y/F[p]$ and $F_z/F[p]$ are both $\alpha_n t_n$ because F_y and F_z are \star -isomorphic (Remark 5.8).

For the right-hand side of (7.11), first note that modulo I^r we have

$$\begin{aligned}
 (7.13) \quad f_p^z \circ \delta(t) &= \prod_{c \in \mathcal{F}[p](\tilde{E}_n)} (z(c) +_{F_z} \delta(t)) && \text{by (2.4)} \\
 &= \prod_c \delta(y(c) +_{F_y} t) && \text{by (7.7)} \\
 &= \prod_c [(y(c) +_{F_y} t) - a^{(-p^n)}(y(c) +_{F_y} t)] && \text{by (7.6)} \\
 &\equiv \prod_c (y(c) +_{F_y} t) \\
 &\quad - \sum_c a^{(-p^n)}(y(c) +_{F_y} t) \prod_{d \neq c} (y(d) +_{F_y} t) && \text{by (7.5)} \\
 &\equiv f_p^y(t) - \sum_c a^{(-p^n)}(t) t^{p^n-1} && \text{by (2.4), (7.5) and (2.5)} \\
 &= f_p^y(t) - p^n a^{(-p^n)}(t) t^{p^n-1} \\
 &\equiv f_p^y(t) && \text{by (7.5) and since } p \in I
 \end{aligned}$$

In particular, by (2.5), this gives

$$(7.14) \quad f_p^z \circ \delta(t) \equiv t^{p^n} \pmod{I}$$

Thus, given $k \geq 2$, if in (7.8) we have

$$b(t) = \sum_{j \geq 1} b_j t^j \quad \text{with } b_j \in I^{k-1}$$

then for $k \leq r$, on the right-hand side of (7.11) we have

$$\begin{aligned}
 g_p^z \circ f_p^z \circ \delta(t) &= f_p^z \circ \delta(t) + b(f_p^z \circ \delta(t)) && \text{by (7.8)} \\
 &\equiv f_p^y(t) + b(t^{p^n}) \pmod{I^k} && \text{by (7.13) and (7.14)}
 \end{aligned}$$

Comparing this to (7.12), we get

$$b(t) \equiv 0 \pmod{I^k}$$

Since g_p^z in (7.8) is a \star -isomorphism, we can proceed by induction on k and obtain

$$b(t) \equiv 0 \pmod{I^r}$$

which implies (7.9).

Uniqueness Let \mathcal{F}/R be a deformation of \mathcal{G}/k . Let x and y be two coordinates on \mathcal{F} , both lifting x_G on \mathcal{G} and both satisfying (7.3). Suppose F_x and F_y are in the

same \star -isomorphism class so that there is a \star -isomorphism $\delta: F_x \rightarrow F_y$ fitting into a commutative diagram analogous to (7.10).

$$\begin{array}{ccc} F_x & \xrightarrow{\delta} & F_y \\ \downarrow l_p^x = f_p^x & & \downarrow f_p^y = l_p^y \\ F_x/F[p] & \xrightarrow{\tilde{\delta}} & F_y/F[p] \end{array}$$

Let \mathfrak{m} be the maximal ideal of R . Let $c(t) \in R[[t]]$ be such that

$$(7.15) \quad \delta(t) = t + c(t)$$

where

$$c(t) = \sum_{j \geq 1} c_j t^j \quad \text{with } c_j \in \mathfrak{m}$$

Since x and y are distinct, there exists $r_0 \geq 2$ such that it is the largest r satisfying

$$(7.16) \quad c_j \in \mathfrak{m}^{r-1} \quad \text{for all } j$$

Modulo \mathfrak{m}^{r_0} we then have

$$\begin{aligned} f_p^y(t) &= \tilde{\delta} \circ f_p^x \circ \delta^{-1}(t) \\ &= f_p^x \circ \delta^{-1}(t) + c^{(p^n)} \circ f_p^x \circ \delta^{-1}(t) && \text{by (7.15)} \\ &\equiv f_p^x \circ \delta^{-1}(t) + c^{(p^n)}(t^{p^n}) && \text{by (2.5) and (7.16)} \\ &\equiv f_p^y(t) + c^{(p^n)}(t^{p^n}) && \text{analogous to (7.13)} \end{aligned}$$

which is a contradiction. \square

Proof of Proposition 7.1 (cf. [Ando1995, proof of Proposition 2.6.15]) We need only show that the coordinate x on \mathcal{F} constructed in Lemma 7.2 satisfies the stronger condition $l_H = f_H$ for any finite $\mathcal{H} \subset \mathcal{F}$. As in the proof of existence there, we are reduced to the universal case with $F_{\text{univ}} =: F = F_x$ over E_n .

Given any $\mathcal{H} \subset \mathcal{F}$ of degree p^r , we will show that the \star -isomorphism

$$(7.17) \quad g_H: F/H \rightarrow \alpha_r^* t_r^* F$$

is the identity by the uniqueness from Lemma 7.2. Namely, the source and target are in the same \star -isomorphism class, and we show that both of them satisfy (7.3). That the target does is clear from the proof of Proposition 6.22 (i). For the source of (7.17), let

$p^{-1}\mathcal{H} := \{c \in \mathcal{F} \mid pc \in \mathcal{H}\}$.¹² It contains both \mathcal{H} and $\mathcal{F}[p]$ as subgroups. We need to show

$$(7.18) \quad l_{p^{-1}H/H}^{x_H} = f_{p^{-1}H/H}^{x_H}$$

where x_H is the coordinate corresponding to the formal group law F_x/H .

Consider the following commutative diagram, where x_p denotes the coordinate corresponding to $F_x/F[p]$, and $\tilde{f}_H^x := (\alpha_n^x)^*(t_n^x)^*f_H^x$. The upper rectangle commutes due to the functoriality of the isogeny f under quotient [*ibid.*, Proposition 2.2.6]. The lower rectangle commutes due to the functoriality from Proposition 5.5 of the \star -isomorphisms g under quotient.¹³

$$\begin{array}{ccc}
 F_x & \xrightarrow{f_H^x} & F_x/H \\
 \downarrow f_p^x & \searrow f_{p^{-1}H}^x & \downarrow f_{p^{-1}H/H}^{x_H} \\
 F_x/F[p] & \xrightarrow{f_{p^{-1}H/F[p]}^{x_p}} & F_x/p^{-1}H \\
 \parallel g_p^x & & \downarrow g_{p^{-1}H/H}^{x_H} \\
 (\alpha_n^x)^*(t_n^x)^*F_x & \xrightarrow{\tilde{f}_H^x} & (\alpha_n^x)^*(t_n^x)^*F_x/(\alpha_n^x)^*(t_n^x)^*H = (\alpha_n^{x_H})^*(t_n^{x_H})^*F_{x_H}
 \end{array}$$

Note that $p^{-1}\mathcal{H}/\mathcal{F}[p] \cong \mathcal{H}$. In the lower rectangle, $g_p^x = \text{id}$ and hence $f_{p^{-1}H/F[p]}^{x_p} = \tilde{f}_H^x$. Moreover, $\alpha_n^{x_H} t_n^{x_H} = \alpha_n^{x_H} t_n^{x_H}$ by the uniqueness from Proposition 4.6. This forces the \star -isomorphism $g_{p^{-1}H/H}^{x_H}$ to be the identity, and (7.18) follows. \square

Remark 7.19 In [Zhu2015a, §3.3], for the purpose of studying Hecke operators in elliptic cohomology, we showed the existence of an analogue of Ando's coordinate. It is conceptually different from the norm-coherent coordinates here. Note that there the base change is not along a *local* homomorphism (see [Zhu2014, §4, footnote] and cf. (7.21) below).

Example 7.20 Let $k = \mathbb{F}_{p^2}$ and G be the formal group law of a supersingular elliptic curve over k . We choose a curve such that its p^2 -power Frobenius endomorphism coincides with the map of multiplication by $(-1)^{p-1}p$ as in [Zhu2015a, 3.24]. We then have $l_p = [-p]$, if $p = 2$, in view of Remark 6.6 and Footnote 10.

¹²The notation $c \in \mathcal{F}$ means $[c] \subset \mathcal{F}$, where $[c]$ is the effective Cartier divisor defined by a section. To be precise, this set-theoretic description defines the subgroup scheme $p^{-1}\mathcal{H}$ of \mathcal{F} as a sum of effective Cartier divisors.

¹³Here we view $F_x/H = F_{x_H}$ as a universal deformation with structure (σ^r, id) (5.9).

Let E be the Morava E-theory associated to \mathcal{G}/k as in (2.7) and choose a *preferred* \mathcal{P}_N -model for E in the sense of [ibid., Definition 3.29]. In particular, there is a chosen coordinate u on the universal deformation of \mathcal{G}/k . Given [ibid., 3.28], the cotangent map along f_p^u is multiplication by p . Thus, by the criterion (7.3), u cannot be norm-coherent if $p = 2$.

More explicitly, let us consider the supersingular elliptic curve $C_0/\mathbb{F}_2: y^2 + y = x^3$. A direct calculation shows that $\text{Frob}^2 = [-2]$ on C_0 . In [Rezk2008], Rezk chooses the coordinate $u := x/y$ for its formal group \widehat{C}_0 and for its universal formal deformation \widehat{C} over $\mathbb{Z}_2[[a]]$, with $C: y^2 + axy + y = x^3$. Let Q be the universal example of a point of exact order 2 in \widehat{C} . Rezk then chooses $d := u(Q)$ as a parameter for the modular curve $X_0(2)$ near the supersingular locus, and computes its equation as

$$d^3 - ad - 2 = 0$$

Taking $\mathcal{H} = \widehat{C}[2]$ in (2.4), we see that the cotangent map along f_2^u is multiplication by $d_1 d_2 d_3 = 2$, with each d_i a root of the modular equation, whereas $l_2^u = [-2]$. If we instead choose $\tilde{u} := -x/y$ as a coordinate for \widehat{C} , it agrees with u over \mathbb{F}_2 while satisfying $f_2^{\tilde{u}} = l_2^{\tilde{u}}$ by rigidity (cf. [Rezk2013, Remark 4.16]).

7.21 More functoriality of norm coherence We continue the discussion in (6.21) with varying \mathcal{G}/k . Let $\text{Coord}: \text{FG}_{\text{isog}} \rightarrow \text{Set}$ be the “wide functor”

$$\mathcal{G}/k \mapsto \{\text{coordinates on } \mathcal{G}\} \subset \mathcal{O}_{\mathcal{G}}$$

in the following sense: given diagram (3.4), Coord is contravariant along the right square and covariant along the left square. More specifically, Coord is contravariant with respect to base change $\text{Spf } k \rightarrow \text{Spf } k'$ and pullback along an isomorphism over k , hence contravariant with respect to any morphism in the subcategory FG_{iso} . On the other hand, given an isogeny $\mathcal{G} \rightarrow \mathcal{G}'$ over k of degree p^r , any coordinate x on \mathcal{G} determines a unique coordinate on $\mathcal{G}^{(p^r)}$ which pulls back along Frob^r to x^{p^r} ; this coordinate on $\mathcal{G}^{(p^r)}$ then corresponds to one on \mathcal{G}' via the isomorphism between the two formal groups. Thus Coord is also covariant with respect to any morphism in the subcategory $\text{FG}_{\text{isog}}(k)$.

Let $\text{NCoh}: \text{FG}_{\text{isog}} \rightarrow \text{Set}$ be the wide functor

$$\mathcal{G}/k \mapsto \{\text{norm-coherent coordinates on } \mathcal{F}_{\text{univ}}(\mathcal{G})/E_n\}$$

where $\mathcal{F}_{\text{univ}}(\mathcal{G})$ is a choice of universal deformation of \mathcal{G} as in (5.9). Its “wideness,” in the same sense as above, follows from Proposition 6.22 and the discussion in (5.9).

Theorem 7.22 *The natural transformation $\rho: \text{NCoh} \rightarrow \text{Coord}$ of wide functors by restricting a coordinate on $\mathcal{F}_{\text{univ}}(\mathcal{G})$ to \mathcal{G} is an isomorphism. Moreover, it satisfies Galois descent: given \mathcal{G}/k in FG_{isog} and a Galois extension K/k , the following commutes, where the vertical maps take fixed points under the Galois action.*

$$\begin{array}{ccc} \text{NCoh}(\mathcal{G} \times_k K) & \xrightarrow{\sim} & \text{Coord}(\mathcal{G} \times_k K) \\ \downarrow & & \downarrow \\ \text{NCoh}(\mathcal{G}) & \xrightarrow{\sim} & \text{Coord}(\mathcal{G}) \end{array}$$

Moreover, this diagram is natural in \mathcal{G}/k and K/k .

Proof On each object in FG_{isog} , the natural transformation ρ is an isomorphism by Proposition 7.1, and the descent is clear since the condition (6.18) of norm coherence is stable under Galois actions. Each of the naturality properties is straightforward to check. \square

8 Norm coherence and H_∞ complex orientations

8.1 Introduction Given a Morava E-theory spectrum E , consider its complex orientations, or, more precisely, homotopy multiplicative maps $MU\langle 0 \rangle \rightarrow E$. A necessary and sufficient condition for such an orientation to be H_∞ (2.9) is that its corresponding coordinate on the formal group of E is norm-coherent. Ando showed this for E-theories associated to the Honda formal groups over \mathbb{F}_p [Ando1995, Theorem 4.1.1]. There, the norm-coherence condition (6.18) boils down to the identity $[p] = f_p$ (cf. (7.3) and (6.8)). Moreover, he established the existence and uniqueness of coordinates, hence orientations, with the desired property [ibid., Theorem 2.6.4].

In fact, to show that norm coherence is necessary and sufficient for H_∞ orientations, Ando's proof does not depend on the choice of the formal groups being the Honda formal groups (see [ibid., Lemma 4.4.4]). However, his setup does require them be defined over \mathbb{F}_p so that the relative p^r -power Frobenius is an endomorphism for every $r \geq 0$ (cf. [ibid., Proposition 2.5.1] and Remark 6.4).

With results in sufficient generality about level structures on formal groups from [Strickland1997], Ando, Hopkins, and Strickland extended the applicability of the

above condition for H_∞ orientations: $MU\langle 0 \rangle$ generalizes to $MU\langle 2k \rangle$, $k \leq 3$, and E generalizes to any even periodic H_∞ -ring spectrum whose zeroth homotopy is a p -regular admissible local ring with perfect residue field of characteristic p and whose formal group is of finite height [Ando-Hopkins-Strickland2004, Proposition 6.1]. They did this by first reformulating Ando's condition so that in particular it applies to E-theories associated to formal groups over any perfect field of positive characteristic [*ibid.*, Proposition 4.13].

Based on this general condition, they established the existence and uniqueness of $H_\infty MU\langle 6 \rangle$ -orientations for H_∞ elliptic spectra, called the sigma orientations, from corresponding norm-coherent cubical structures of elliptic curves [*ibid.*, Proposition 16.5]. However, when the elliptic spectrum represents an E-theory associated to the formal group of a supersingular elliptic curve, such an orientation does not factor through $MU\langle 4 \rangle$ due to obstruction from Weil pairings (see [Ando-Strickland2001, proof of Theorem 1.4]). Thus, in this case, we cannot deduce the existence and uniqueness of $H_\infty MU\langle 2k \rangle$ -orientations for $0 \leq k \leq 2$ from the sigma orientation.

8.2 Set-up Let E be a Morava E-theory spectrum, with $\mathcal{G}_E = \mathcal{F}_{\text{univ}}(\mathcal{G})$ for some \mathcal{G}/k whose group law is as in (6.1). We will show the existence and uniqueness of $H_\infty MU\langle 0 \rangle$ -orientations for E by combining Proposition 7.1 with Ando, Hopkins, and Strickland's condition for H_∞ orientations. Specifically, we need only check that their criterion [Ando-Hopkins-Strickland2004, 4.14] and our definition (6.18) for norm coherence agree.

8.3 Descent for level structures on formal deformations We carry out the needed comparison by recalling the canonical descent data for level structures on $\mathcal{G}_E = \mathcal{F}_{\text{univ}}(\mathcal{G})$ from [*ibid.*, Part 3]. Since \mathcal{G} is over k of characteristic p , the finite subgroups \mathcal{H} of \mathcal{G}_E must be of degree p^r for some $r \geq 0$.

Definition 8.4 (cf. [*ibid.*, Definitions 3.1, 9.9, Proposition 10.10 (i), 12.5]) Let A be an “abstract” finite abelian group of order p^r . Let $S_E = \text{Spf } \pi_0 E$ and $T = \text{Spf } R$ with R as in (6.1). Let $i: T \rightarrow S_E$ be a morphism of formal schemes, faithfully flat and locally of finite presentation, which classifies a deformation of \mathcal{G}/k to R . Write \mathcal{A}_T for the constant formal group scheme of A over T . A morphism

$$\ell: \mathcal{A}_T \rightarrow i^* \mathcal{G}_E$$

of formal groups over T , equivalent to a group homomorphism $\phi_\ell: A \rightarrow i^* \mathcal{G}_E(T)$, is a *level A -structure on \mathcal{G}_E* if the effective Cartier divisor $\mathcal{D}_\ell := \sum_{a \in A} [\phi_\ell(a)]$ of degree p^r is a subgroup of $i^* \mathcal{G}_E$.

Remark 8.5 Note that a level A -structure ℓ on \mathcal{G}_E uniquely corresponds to a finite subgroup $\mathcal{H} = \mathcal{D}_\ell$, which is different from the scheme-theoretic image of \mathcal{A}_T under ℓ (the latter automatically a subgroup, but possibly of smaller degree). Automorphisms of A correspond to automorphisms of \mathcal{H} (cf. [ibid., Definition 3.1 (3)]).

Definition 8.6 (cf. [ibid., Definition 3.9, Remark 3.12]) Let $\ell: \mathcal{A}_{\mathrm{Spf} R} \rightarrow i^* \mathcal{G}_E$ be a level A -structure on \mathcal{G}_E as above. Define $\psi_\ell^E: \pi_0 E \rightarrow R$ to be the composite

$$\pi_0 E \xrightarrow{D_{p^r}} \pi_0 E^{(B\Sigma_{p^r})_+} \rightarrow \pi_0 E^{(B\Sigma_{p^r})_+} / I_{\mathrm{tr}} \xrightarrow{\alpha_r} R$$

where the power operation D_{p^r} arises from the H_∞ -ring structure of E (2.9), I_{tr} is the ideal generated by the images of transfers from proper subgroups of Σ_{p^r} , and α_r classifies the subgroup of $i^* \mathcal{G}_E$ corresponding to ℓ (8.5, 2.7).

Remark 8.7 In the presence of a level structure as in Definition 8.6, the structure morphism i of T over S_E in Definition 8.4 is given by the classifying map

$$\alpha: A_0 \xrightarrow{s_r} A_r \xrightarrow{\alpha_r} R$$

from Propositions 4.6 and 5.5, while ψ_ℓ^E is precisely the classifying map

$$\alpha': A_0 \xrightarrow{t_r} A_r \xrightarrow{\alpha_r} R$$

(cf. [Rez2009, Theorem B] for the identification with t_r).

Definition 8.8 (cf. [Ando-Hopkins-Strickland2004, 3.13-3.15]) Let $F := E^{X_+}$ and $f: E \rightarrow F$ be the natural map of H_∞ -ring spectra. Given any level A -structure $\ell: \mathcal{A}_T \rightarrow i^* \mathcal{G}_E$ on \mathcal{G}_E , let ℓ' be the unique level A -structure on \mathcal{G}_F induced by f ,¹⁴ so that the following diagram commutes (with all but the front-left and back-left squares cartesian).

$$\begin{array}{ccccccc}
 & & \mathcal{A}_{T'} & \xrightarrow{\ell'} & i'^* \mathcal{G}_F & \longrightarrow & \mathcal{G}_F \\
 & \swarrow & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{A}_T & \xrightarrow{\ell} & i^* \mathcal{G}_E & \longrightarrow & \mathcal{G}_E & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 T' & \xrightarrow{\quad} & i^* S_F & \xrightarrow{i'} & S_F & & \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 T & \xrightarrow{\quad} & T & \xrightarrow{i} & S_E & &
 \end{array}$$

¹⁴Level A -structures on \mathcal{G}_F are defined analogously to those on \mathcal{G}_E as in Definition 8.4.

Let $\psi_{\ell'}^F: T' \rightarrow S_F$ be the morphism analogous to ψ_{ℓ}^E in Definition 8.6, obtained by the naturality of power operations on $E^0(X)$. Define $\psi_{\ell}^{F/E}$ to be the unique T -morphism which fits into the following commutative diagram.

$$(8.9) \quad \begin{array}{ccccc} & & \psi_{\ell'}^F & & \\ & \searrow & \text{---} & \nearrow & \\ S_F & \xleftarrow{\quad} & i^* S_F & \xrightarrow{\psi_{\ell}^{F/E}} & (\psi_{\ell}^E)^* S_F & \xrightarrow{\quad} & S_F \\ & \downarrow \scriptstyle S_f & \downarrow \scriptstyle \lrcorner & & \downarrow \scriptstyle \lrcorner & & \downarrow \scriptstyle S_f \\ S_E & \xleftarrow{\quad i \quad} & T & \xlongequal{\quad} & T & \xrightarrow{\psi_{\ell}^E} & S_E \end{array}$$

In particular, when $F = E^{(\mathbb{CP}^\infty)_+}$, write $\psi_{\ell}^{F/E}$ as

$$\psi_{\ell}^{G/E}: i^* \mathcal{G}_E \rightarrow (\psi_{\ell}^E)^* \mathcal{G}_E$$

Remark 8.10 Let $F = E^{(\mathbb{CP}^\infty)_+}$. When $A = \mathbb{Z}/p$, the diagram (8.9) lifts (3.6). More generally, let $\mathcal{H} \subset i^* \mathcal{G}_E$ correspond to ℓ as in Remark 8.5. Comparing (8.9) to the universal example (5.7) and Remark 8.7, we see that $\psi_{\ell}^{G/E}$ is precisely the isogeny $l_{\mathcal{H}}$ from (6.9) if we assume without loss of generality that the \star -isomorphism (4.7) is the identity.

8.11 Norm maps In view of [Ando-Hopkins-Strickland2004, Theorem 3.25], we have compared above the ingredients that constitute descent data for level structures on \mathcal{G}_E —level structures ℓ , classifying maps i and ψ_{ℓ}^E , isogenies $\psi_{\ell}^{G/E}$ —with corresponding terms from the earlier sections of this paper. There is one more and key ingredient that goes into the condition [ibid., 4.14] for $H_\infty MU\langle 0 \rangle$ -orientations.

Definition 8.12 (cf. [ibid., Definitions 10.1, 10.9]) Let $\psi: \mathcal{G} \rightarrow \mathcal{G}'$ be an isogeny of formal groups with kernel \mathcal{K} . Let $\mu, \pi: \mathcal{G} \times \mathcal{K} \rightarrow \mathcal{G}$ be the multiplication, projection maps, and $q: \mathcal{G} \rightarrow \mathcal{G}/\mathcal{K}$ be the quotient map, as in (2.3). Define $N_\psi: \mathcal{O}_{\mathcal{G}} \rightarrow \mathcal{O}_{\mathcal{G}'}$ to

be the horizontal composite

$$(8.13) \quad \begin{array}{ccccc} \mathcal{O}_G & \dashrightarrow & \mathcal{O}_{G/K} & \xrightarrow{\sim} & \mathcal{O}_{G'} \\ & \searrow \mu^* & \downarrow q^* & & \\ & \mathcal{O}_{G \times K} & \downarrow \text{Norm}_{\pi^*} & & \\ & & \mathcal{O}_G & & \\ & & \downarrow \mu^* \quad \downarrow \pi^* & & \\ & & \mathcal{O}_{G \times K} & & \end{array}$$

where the vertical maps exhibit $\mathcal{O}_{G/K}$ as an equalizer, Norm_{π^*} sends a to the determinant of multiplication by a on $\mathcal{O}_{G \times K}$ as a finite free \mathcal{O}_G -module via π^* , and the factorization through $\mathcal{O}_{G/K}$ was shown, *e.g.*, in [Strickland1997, Theorem 19].

Remark 8.14 Since $q \circ \mu = q \circ \pi$, we have $\text{Norm}_{\pi^*} \circ \mu^* = q^* \circ \text{Norm}_{q^*}$ (by an argument similar to the proof of the factorization mentioned above). Thus the dashed arrow in (8.13) is Norm_{q^*} by uniqueness from the universal property of an equalizer.

Suppose that the isogeny ψ is over a field k of characteristic p , and is hence of degree p^r for some $r \geq 0$. Comparing [ibid., Theorem 19 (i)] and Lemma 3.10, we see that N_ψ is precisely the map $\Lambda_\psi^* = \text{Norm}_{\psi^*}$ in (3.9).

Remark 8.15 Let ψ be the isogeny $\psi_\ell^{G/E}: i^* \mathcal{G}_E \rightarrow (\psi_\ell^E)^* \mathcal{G}_E$ over R from Definition 8.8. Let x be any coordinate on $i^* \mathcal{G}_E$. In view of Remark 8.5, we have from [Ando-Hopkins-Strickland2004, 10.11] that

$$\psi^* N_\psi(x) = q^* N_q(x) = \prod_{a \in A} T_a^*(x) = \prod_{Q \in \mathcal{H}(R)} (x + x(Q))$$

where $T_a: i^* \mathcal{G}_E \rightarrow i^* \mathcal{G}_E$ translates any R -point P on $i^* \mathcal{G}_E$ to $P + Q$, with $Q = \phi_\ell(a)$ (cf. (3.13) for the first equality). Comparing this to (6.16), with $\mathcal{F} = i^* \mathcal{G}_E$, we see that

$$\psi^* N_\psi(x) = f_H^*(x_H)$$

Now, given any coordinate s on \mathcal{G}_E , the condition [ibid., 4.14] states that

$$(\psi_\ell^E)^* s = N_{\psi_\ell^{G/E}}(i^* s)$$

Pulling this back along $\psi_\ell^{G/E}$ and writing $x := i^* s$, we get an equivalent identity

$$(8.16) \quad l_H^*(x') = f_H^*(x_H)$$

where $l_H = \psi_\ell^{G/E}$ from Remark 8.10, and $x' := (\psi_\ell^E)^* s = \alpha_r^* t_r^* s$ from Remark 8.7. In view of (6.10, 6.11), we see that (8.16) is equivalent to (6.18). This shows that [ibid., 4.14] and our norm-coherence condition agree (cf. (6.20)).

Corollary 8.17 *Let E , \mathcal{G}_E , and \mathcal{G} be as in (8.2). Given any coordinate $x_{\mathcal{G}}$ on \mathcal{G} , there exists a unique coordinate x on \mathcal{G}_E lifting $x_{\mathcal{G}}$ such that its corresponding $MU\langle 0 \rangle$ -orientation for E is H_{∞} .*

Proof In view of Remark 8.15, the corollary follows from Proposition 7.1. In particular, as p is not a zero-divisor in $\pi_0 E$, we may apply [Ando-Hopkins-Strickland2004, Proposition 6.1] for $H_{\infty} MU\langle 2k \rangle$ -orientations with $k = 0$ (cf. [ibid., discussion following 1.6]). \square

References

- [Ando1992] Matthew Ando, *Operations in complex-oriented cohomology theories related to subgroups of formal groups*, ProQuest LLC, Ann Arbor, MI, 1992, Thesis (Ph.D.)—Massachusetts Institute of Technology. [MR2716371](#)
- [Ando1995] Matthew Ando, *Isogenies of formal group laws and power operations in the cohomology theories E_n* , Duke Math. J. **79** (1995), no. 2, 423–485. [MR1344767](#)
- [Ando2000] Matthew Ando, *Power operations in elliptic cohomology and representations of loop groups*, Trans. Amer. Math. Soc. **352** (2000), no. 12, 5619–5666. [MR1637129](#)
- [Ando-Hopkins-Strickland2001] M. Ando, M. J. Hopkins, and N. P. Strickland, *Elliptic spectra, the Witten genus and the theorem of the cube*, Invent. Math. **146** (2001), no. 3, 595–687. [MR1869850](#)
- [Ando-Hopkins-Strickland2004] Matthew Ando, Michael J. Hopkins, and Neil P. Strickland, *The sigma orientation is an H_{∞} map*, Amer. J. Math. **126** (2004), no. 2, 247–334. [MR2045503](#)
- [Ando-Strickland2001] M. Ando and N. P. Strickland, *Weil pairings and Morava K -theory*, Topology **40** (2001), no. 1, 127–156. [MR1791270](#)
- [Brun2007] M. Brun, *Witt vectors and equivariant ring spectra applied to cobordism*, Proc. Lond. Math. Soc. (3) **94** (2007), no. 2, 351–385. [MR2308231](#)
- [Bruner-May-McClure-Steinberger1986] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_{∞} ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. [MR836132](#)
- [Coleman1979] Robert F. Coleman, *Division values in local fields*, Invent. Math. **53** (1979), no. 2, 91–116. [MR560409](#)
- [Elmendorf-Kriz-Mandell-May1997] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. [MR1417719](#)

- [Goerss-Hopkins2004] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. [MR2125040](#)
- [Hill-Hopkins2016] Michael A. Hill and Michael J. Hopkins, *Equivariant symmetric monoidal structures*. [arXiv:1610.03114](#)
- [Hopkins1995] Michael J. Hopkins, *Topological modular forms, the Witten genus, and the theorem of the cube*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 554–565. [MR1403956](#)
- [Hopkins2002] M. J. Hopkins, *Algebraic topology and modular forms*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 291–317. [MR1989190](#)
- [Hopkins-Lawson2016] Michael J. Hopkins and Tyler Lawson, *Strictly commutative complex orientation theory*. [arXiv:1603.00047](#)
- [Katz-Mazur1985] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. [MR772569](#)
- [Lubin1967] Jonathan Lubin, *Finite subgroups and isogenies of one-parameter formal Lie groups*, Ann. of Math. (2) **85** (1967), 296–302. [MR0209287](#)
- [Lubin-Tate1965] Jonathan Lubin and John Tate, *Formal complex multiplication in local fields*, Ann. of Math. (2) **81** (1965), 380–387. [MR0172878](#)
- [Lubin-Tate1966] Jonathan Lubin and John Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49–59. [MR0238854](#)
- [May1977] J. Peter May, *E_∞ ring spaces and E_∞ ring spectra*, Lecture Notes in Mathematics, Vol. 577, Springer-Verlag, Berlin-New York, 1977, With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave. [MR0494077](#)
- [Mumford2008] David Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay; by Hindustan Book Agency, New Delhi, 2008, With appendices by C. P. Ramanujam and Yuri Manin, Corrected reprint of the second (1974) edition. [MR2514037](#)
- [Quillen1969] Daniel Quillen, *On the formal group laws of unoriented and complex cobordism theory*, Bull. Amer. Math. Soc. **75** (1969), 1293–1298. [MR0253350](#)
- [Rezk2008] Charles Rezk, *Power operations for Morava E -theory of height 2 at the prime 2*. [arXiv:0812.1320](#)
- [Rezk2009] Charles Rezk, *The congruence criterion for power operations in Morava E -theory*, Homology Homotopy Appl. **11** (2009), no. 2, 327–379. [MR2591924](#)
- [Rezk2013] Charles Rezk, *Power operations in Morava E -theory: structure and calculations (Draft)*, available at <http://www.math.uiuc.edu/~rezk/power-ops-ht-2.pdf> (version: 2013-09-13).

- [Rezk2014] Charles Rezk, *Isogenies, power operations, and homotopy theory*, Proceedings of the ICM, Seoul 2014, vol. 2 (2014), 1125–1146.
- [Rezk2015] Charles Rezk, *Elliptic cohomology and elliptic curves (Part 4)*, available at https://www.youtube.com/watch?v=r_7SsIoU9No.
- [Rezk2016] Charles Rezk, *Elliptic cohomology and elliptic curves*, available at <http://www.math.uiuc.edu/~rezk/felix-klein-lectures-notes.pdf> (version: 2016-06-06).
- [Rotman2010] Joseph J. Rotman, *Advanced modern algebra*, Graduate Studies in Mathematics, vol. 114, American Mathematical Society, Providence, RI, 2010, Second edition [of MR2043445]. [MR2674831](#)
- [Strickland1997] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208. [MR1473889](#)
- [Strickland1998] N. P. Strickland, *Morava E -theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779. [MR1607736](#)
- [Strickland] Neil P. Strickland, *Functorial philosophy for formal phenomena*, available at <http://hopf.math.purdue.edu/Strickland/fpfp.pdf>.
- [Tambara1993] D. Tambara, *On multiplicative transfer*, Comm. Algebra **21** (1993), no. 4, 1393–1420. [MR1209937](#)
- [Walker2008] Barry John Walker, *Multiplicative orientations of K -Theory and p -adic analysis*, ProQuest LLC, Ann Arbor, MI, 2008, Thesis (Ph.D.)—University of Illinois at Urbana-Champaign. [MR2712595](#)
- [Zhu2014] Yifei Zhu, *The power operation structure on Morava E -theory of height 2 at the prime 3*, Algebr. Geom. Topol. **14** (2014), no. 2, 953–977. [MR3160608](#)
- [Zhu2015a] Yifei Zhu, *The Hecke algebra action on Morava E -theory of height 2*, available at <https://yifeizhu.github.io/ho.pdf>.
- [Zhu2015b] Yifei Zhu, *Modular equations for Lubin-Tate formal groups at chromatic level 2*, available at <https://yifeizhu.github.io/me.pdf>.