Modular equations for Lubin-Tate formal groups at chromatic level 2

Yifei Zhu

Northwestern University

AMS session on algebraic geometry 2016

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \text{Spaces} \to \text{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \text{Spaces} \to \text{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \text{Spaces} \to \text{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\text{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Steenrod algebra

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Adem relations $\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$, $0 < i < 2j$

Cartan formula $\mathrm{Sq}^i(xy) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k}(x)\,\mathrm{Sq}^k(y)$

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Adem relations $\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$, $0 < i < 2j$

Cartan formula $\mathrm{Sq}^i(xy) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k}(x)\,\mathrm{Sq}^k(y)$

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon\ h^*(-) \to h^{*+i}(-)$

Example (ordinary cohomology with $\mathbb{Z}/2$-coefficients)

Steenrod squares $\mathrm{Sq}^i\colon\ H^*(-;\mathbb{Z}/2) \to H^{*+i}(-;\mathbb{Z}/2)$

Power operation $\mathrm{Sq}^i(x) = x^2$ if $i = |x|$

Adem relations $\mathrm{Sq}^i\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\mathrm{Sq}^k$, $0 < i < 2j$

Cartan formula $\mathrm{Sq}^i(xy) = \sum_{k=0}^{i} \mathrm{Sq}^{i-k}(x)\,\mathrm{Sq}^k(y)$

Motivation: cohomology theories and their operations

Generalized cohomology theory $\{h^n\}\colon \ \mathrm{Spaces} \to \mathrm{AbGroups}$

Cup product $\rightsquigarrow h^*(X)$ a graded commutative algebra over $h^*(\mathrm{pt})$

Cohomology operation $Q^i\colon \ h^*(-) \to h^{*+i}(-)$

Example (complex K-theory)

Adams operations $\psi^i\colon \ K(-) \to K(-)$

Power operation $\psi^p(x) \equiv x^p \mod p$

$\psi^i\psi^j = \psi^{ij}$ $\qquad$ $\psi^i(xy) = \psi^i(x)\psi^j(y)$

J. F. Adams, *Vector fields on spheres*, Ann. of Math. (2) **75** (1962)

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

stable homotopy theory $\leftrightsquigarrow$ 1-dim formal group laws

complex-oriented $h^*(-)$ $\qquad$ $F(x,y)$ over $h^*(\mathrm{pt})$

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big)$$

Example

$$H^*(-;\mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x,y) = x+y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x+y-xy$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

stable homotopy theory $\leftrightsquigarrow$ 1-dim formal group laws

complex-oriented $h^*(-)$ $\quad$ $F(x, y)$ over $h^*(\mathrm{pt})$

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big)$$

Example

$$H^*(-; \mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x, y) = x + y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x, y) = x + y - xy$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

stable homotopy theory $\leftsquigarrow\!\!\rightsquigarrow$ 1-dim formal group laws

complex-oriented $h^*(-)$ $\qquad$ $F(x,y)$ over $h^*(\mathrm{pt})$

$$c_1(L_1 \otimes L_2) = F\big(c_1(L_1), c_1(L_2)\big)$$

Example

$$H^*(-;\mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x,y) = x + y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x + y - xy$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example

$$H^*(-;\mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x,y) = x + y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x + y - xy$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example

$$H^*(-;\mathbb{Z}/2) \leftrightsquigarrow \mathbb{G}_a(x,y) = x + y$$

$$K(-) \leftrightsquigarrow \mathbb{G}_m(x,y) = x + y - xy$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava 1978, Goerss-Hopkins-Miller 1990s–2004)

k = perfect field of characteristic $p > 0$
F = formal group over k of height $n < \infty$

$E_n(k,F)(-) \leftrightsquigarrow$ the Lubin-Tate universal deformation of F/k

$\pi_* E_n(k,F) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$ with $|u| = -2$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava 1978, Goerss-Hopkins-Miller 1990s–2004)

k = perfect field of characteristic $p > 0$
F = formal group over k of height $n < \infty$

$$E_n(k, F)(-) \leftrightsquigarrow \text{the Lubin-Tate universal deformation of } F/k$$

$$\pi_* E_n(k, F) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}] \text{ with } |u| = -2$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava 1978, Goerss-Hopkins-Miller 1990s–2004)

k = perfect field of characteristic $p > 0$
F = formal group over k of height $n < \infty$

$$E_n(k,F)(-) \leftrightsquigarrow \text{the Lubin-Tate universal deformation of } F/k$$

$$\pi_* E_n(k,F) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}] \text{ with } |u| = -2$$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava 1978, Goerss-Hopkins-Miller 1990s–2004)

k = perfect field of characteristic $p > 0$
F = formal group over k of height $n < \infty$

$E_n(k, F)(-) \leftrightsquigarrow$ the Lubin-Tate universal deformation of F/k

$\pi_* E_n(k, F) \cong \mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!][u^{\pm 1}]$ with $|u| = -2$

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2 =$ form of K-theory

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2 =$ form of K-theory

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k, F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2$ = form of K-theory

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{ccc} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2 =$ form of K-theory

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2 =$ form of K-theory

Motivation: chromatic homotopy theory

A connection between Alg. Top. and Alg. Geom. (Quillen 1969)

$$\begin{array}{rcl} \text{stable homotopy theory} & \leftrightsquigarrow & \text{1-dim formal group laws} \\ \text{complex-oriented } h^*(-) & & F(x,y) \text{ over } h^*(\mathrm{pt}) \\ c_1(L_1 \otimes L_2) & = & F\big(c_1(L_1), c_1(L_2)\big) \end{array}$$

Example (Morava E-theories cont.)

$\{E_n(k,F)\} \rightsquigarrow$ the *chromatic filtration* by heights and by primes

- $E_1(\mathbb{F}_p, \mathbb{G}_m) = K_p^\wedge$
- $E_2(\mathbb{F}_{p^2}, \widehat{C})$ with C an elliptic curve that is supersingular
- $K(1)$-localization $L_{K(1)}E_2 =$ form of K-theory

Results: power operations on E-theories of height 2

Theorem (Z.)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i \colon E(-) \to E(-)$, $0 \le i \le p$, and quadratic relations

$$Q_i Q_0 = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k \; - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $1 \le i \le p$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

Remark The first example, for $p = 2$, is computed by Rezk (2008).

Results: power operations on E-theories of height 2

Theorem (Z.)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i \colon E(-) \to E(-)$, $0 \le i \le p$, and quadratic relations

$$Q_i Q_j = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $i > pj \ge 0$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

Remark   The first example, for $p = 2$, is computed by Rezk (2008).

Theorem (Z.)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i \colon E(-) \to E(-)$, $0 \leq i \leq p$, and quadratic relations

$$Q_i Q_0 = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k \; - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $1 \leq i \leq p$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

Remark The first example, for $p = 2$, is computed by Rezk (2008).

Results: power operations on E-theories of height 2

Theorem (Z.)

Given any Morava E-theory E of height 2 at a prime p, there is an explicit presentation for its algebra of power operations, in terms of generators $Q_i\colon E(-) \to E(-)$, $0 \le i \le p$, and quadratic relations

$$Q_i Q_0 = -\sum_{k=1}^{p-i} w_0^k \, Q_{i+k} Q_k \; - \sum_{k=1}^{p} \sum_{m=0}^{k-1} w_0^m \, d_{i,\,k-m} \, Q_m Q_k$$

for $1 \le i \le p$, where the coefficients w_0 and $d_{i,\,k-m}$ arise from certain modular equations for elliptic curves.

<u>Remark</u> The first example, for $p = 2$, is computed by Rezk (2008).

Recall E-theory at height n and prime p has an underlying model

$$\begin{array}{ccc} F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] & \leftrightsquigarrow & E_n(F,k) \\ \circlearrowleft & & \circlearrowleft \\ \text{Frobenius isogenies} & & \text{power operations} \end{array}$$

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$\begin{tikzcd} F/k & \Gamma/\mathbb{W}(k)[\![u_1,\ldots,u_{n-1}]\!] \arrow[l, "\text{univ defo}"'] \arrow[loop below] & \rightsquigarrow & E_n(F,k) \arrow[loop below] \end{tikzcd}$$

Frobenius isogenies — power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \rightsquigarrow \quad E_n(F, k)$$

Frobenius isogenies   power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F, k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \rightsquigarrow \quad E_n(F,k)$$

Frobenius isogenies power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \rightsquigarrow \quad E_n(F,k)$$

Frobenius isogenies   power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \leftrightsquigarrow \quad E_n(F, k)$$

Frobenius isogenies power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F, k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \leftrightsquigarrow \quad E_n(F, k)$$

$$\circlearrowleft \qquad\qquad\qquad\qquad \circlearrowleft$$

Frobenius isogenies power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F, k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Recall E-theory at height n and prime p has an underlying model

$$\begin{array}{ccc} F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] & \leftrightsquigarrow & E_n(F,k) \\ \circlearrowleft & & \circlearrowleft \\ \text{Frobenius isogenies} & & \text{power operations} \end{array}$$

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$F/k \xleftarrow{\text{univ defo}} \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] \quad \leftrightsquigarrow \quad E_n(F,k)$$

Frobenius isogenies　　　　power operations

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and algebras of power operations

Recall E-theory at height n and prime p has an underlying model

$$\begin{array}{ccccc} F/k & \xleftarrow{\text{univ defo}} & \Gamma/\mathbb{W}(k)[\![u_1, \ldots, u_{n-1}]\!] & \rightsquigarrow & E_n(F,k) \\ & & \circlearrowleft & & \circlearrowleft \\ & & \text{Frobenius isogenies} & & \text{power operations} \end{array}$$

An equivalence of cats (Ando-Hopkins-Strickland '04, Rezk '09)

$$\left\{ \begin{array}{c} \text{qcoh sheaves of grd comm algs} \\ \text{over the moduli problem of} \\ \text{defos of } F/k \text{ and Frob isogs} \end{array} \right\} \cong \left\{ \begin{array}{c} \text{grd comm algs over} \\ \text{a } \textit{Dyer-Lashof algebra} \\ \text{for } E_n(F,k) \end{array} \right\}$$

Goal Compute one side explicitly to get the other side.

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of an ec $C \cong$ defo thy of its p-divisible gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$

Future directions Can we get an explicit presentation for the Dyer-Lashof algebra of Morava E-theory at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of an ec $C \cong$ defo thy of its p-divisible gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$(\alpha - 1)^p(\alpha - p) - \big((-1)^p + (-1)^{p-1}(-p)p + h\big)\alpha = 0$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

Future directions: Can we get an explicit presentation for the Dyer-Lashof algebra of Morava E-theory at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open* *arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation
$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

<u>Future directions</u> Can we get an explicit presentation for the Dyer-Lashof algebra of Morava E-theory at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha=0$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha \equiv \alpha(\alpha^p-h) \mod p$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha \equiv \alpha(\tilde{j}^p-j) \mod p$$

Future directions Can we get an explicit presentation for
the Dyer-Lashof algebra of Morava E-theory
at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha \equiv \alpha(\alpha^p-h) \mod p$$

<u>Future directions</u> Can we get an explicit presentation for the Dyer-Lashof algebra of Morava E-theory at a **higher** chromatic level?

Moduli of elliptic curves and D.-L. algebras at height 2

Moduli of formal groups and moduli of ell. curves (Serre-Tate 1964)
p-adically, defo thy of a *supersingular* $C \cong$ defo thy of its formal gp

$[\Gamma_0(p)]$ as an *open arithmetic surface* (Katz-Mazur 1985)
parameters for its local ring at a supersingular point

Theorem (Z.)

A choice of such parameters, h and α, satisfies the equation

$$(\alpha-1)^p(\alpha-p)-\big((-1)^p+(-1)^{p-1}(-p)p+h\big)\alpha \equiv \alpha(\alpha^p-h) \mod p$$

<u>Future directions</u> How much (local) structure of the moduli carries on to formal groups of higher height (or higher-dimensional abelian varieties)?

Thank you.