CHAPTER 20

Computing v_1-periodic Homotopy Groups of Spheres and some Compact Lie Groups

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1. Introduction

In this paper, we present an account of the principal methods which have been used to compute the v_1-periodic homotopy groups of spheres and many compact simple Lie groups. The two main tools have been J-homology and the unstable Novikov spectral sequence (UNSS), and we shall strive to present all requisite background on both of these.

The v_1-periodic homotopy groups of a space X, denoted $v_1^{-1}\pi_*(X;p)$, are a certain localization of the actual p-local homotopy groups $\pi_*(X)_{(p)}$. We shall drop the p from the notation except where it seems necessary. Very roughly, $v_1^{-1}\pi_*(X)$ is a periodic version of the portion of $\pi_*(X)$ detectable by real and complex K-theory and their operations. It is the first of a hierarchy of theories, $v_n^{-1}\pi_*(X)$, which should account for all of $\pi_*(X)$. Each group $v_1^{-1}\pi_i(X)$ is a direct summand of some actual group $\pi_{i+L}(X)$, at least if X has an H-space exponent and $v_1^{-1}\pi_i(X)$ is a finitely-generated abelian group, which is the case in all examples discussed here.

The v_1-periodic homotopy groups are important because for spaces such as spheres and compact simple Lie groups they give a significant portion of the actual homotopy groups, and yet are often completely calculable. The goal of this paper is to explain how those calculations can be made. One might hope that these methods can be adapted to learning about v_n-periodic homotopy groups for $n > 1$.

One application of v_1-periodic homotopy groups is to obtain lower bounds for the exponents of spaces. The p-exponent of X is the largest e such that some homotopy group of X contains an element of order p^e. We can frequently determine the largest p-torsion summand in $v_1^{-1}\pi_*(X)$. Such a summand must also exist in some $\pi_i(X)$, although we cannot usually specify *which* $\pi_i(X)$. Thus we obtain lower bounds for the p-exponents of spaces, which we conjecture to be sharp in many of the cases studied here. It is known to be sharp for S^{2n+1} if p is odd. See Corollaries 7.8 and 8.9 for estimates of the p-exponent of $\mathrm{SU}(n)$ when p is odd.

Another application, which we will not discuss in this paper, is to James numbers, which are an outgrowth of work on vector fields in the 1950's. It is proved in [23] that, for sufficiently large values of the parameters, the unstable James numbers equal the stable James numbers, and these equal a certain value which had been conjectured by a number of workers.

In Section 2, we present the definition and basic properties of the v_1-periodic homotopy groups. In Section 3, we describe the reduction of the calculation of the unstable groups $v_1^{-1}\pi_*(S^{2n+1})$ to the calculation of stable groups $v_1^{-1}\pi_*^s(B^{qn})$. Here B^{qn} is a space which will be defined in Section 3; when $p=2$, it is the real projective space P^{2n}. Here we have begun the use, continued throughout the paper, of q as $2p-2$. The original proof of this result, 3.1, appeared in [40] and [56]; it involved delicate arguments involving the lambda algebra. We present a new proof, due to Langsetmo and Thompson, which involves completely different techniques, primarily K-theoretic.

In Section 4, we explain how to compute $J_*(B^{qn})$, while in Section 5, we sketch the proof that if X is a spectrum, then $v_1^{-1}\pi_*(X) \approx v_1^{-1}J_*(X)$. Combining the results of Sections 3, 4, and 5 yields a nice complete result for $v_1^{-1}\pi_*(S^{2n+1})$, which can be summarized as

$$v_1^{-1}\pi_{2n+1+i}\big(S^{2n+1}\big) \approx v_1^{-1}\pi_i^s\big(B^{qn}\big) \approx v_1^{-1}J_i\big(B^{qn}\big)$$

and, if p is odd,

$$\approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \text{if } i = qa-2 \text{ or } qa-1, \\ 0 & \text{if } i \not\equiv -1 \text{ or } -2 \bmod q. \end{cases}$$

We will use $\mathbf{Z}/n$ and $\mathbf{Z}_n$ interchangeably, and let $\nu_p(n)$ denote the exponent of p in n. The subscript p of ν will sometimes be omitted if it is clear from the context. The final result for $v_1^{-1}\pi_*(S^{2n+1})$ when $p=2$ is more complicated; see Theorem 4.2.

In Section 6, we sketch the formation of the UNSS and its v_1-localization, and for S^{2n+1} we compute the entire v_1-localized UNSS and part of the unlocalized UNSS. In Section 7, we discuss the computation of the v_1-periodic UNSS and v_1-periodic homotopy groups in general for spherically resolved spaces and specifically for the special unitary groups $\mathrm{SU}(n)$. This is considerably easier at the odd primes than at the prime 2. The following key result of Bendersky ([4]) will be proved by observing that the homotopy-theoretic calculation and UNSS calculation agree for S^{2n+1}.

THEOREM 1.1. *If p is odd, and X is built by fibrations from finitely many odd-dimensional spheres, then $v_1^{-1}E_2^{s,t}(X)=0$ in the v_1-periodic UNSS unless $s=1$ or 2 and t is odd, in which case*

$$v_1^{-1}\pi_i(X;p) \approx \begin{cases} v_1^{-1}E_2^{1,i+1}(X) & \text{if } i \text{ is even,} \\ v_1^{-1}E_2^{2,i+2}(X) & \text{if } i \text{ is odd.} \end{cases}$$

In Section 7 we also review the computation of $E_2^1(\mathrm{SU}(n))$ in [6] and combine it with Theorem 1.1 to obtain the following result, which was the main result of [23].

DEFINITION 1.2. Let $\nu_p(m)$ denote the exponent of p in m, and define integers $a(k,j)$ and $e_p(k,n)$ by

$$(\mathrm{e}^x-1)^j = \sum_{k\geqslant j} a(k,j)\frac{x^k}{k!},$$

and $e_p(k,n)=\min\{\nu_p(a(k,j)) : n \leqslant j \leqslant k\}$.

THEOREM 1.3. *If p is odd, then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, and $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))$ is an abelian group of order $p^{e_p(k,n)}$, although not always cyclic.*

In Section 8, we illustrate the two principal methods used in computing v_1-periodic homotopy groups of the exceptional Lie groups, focusing on $v_1^{-1}\pi_*(G_2;5)$ for UNSS methods, and on $v_1^{-1}\pi_*(F_4/G_2;2)$ for homotopy (J-homology) methods. We also discuss the recent thesis of Yang ([58]), which gives formulas more tractable than that of Definition 1.2 for the numbers $e_p(k,n)$ which appear in Theorem 1.3, provided $n \leqslant p^2-p$.

2. Definition of v_1-periodic homotopy groups

In this section, we present the definition and basic properties of the v_1-periodic homotopy groups. We work toward the definition of $v_1^{-1}\pi_*(X)$ by recalling the definition of $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$. Let $M^n(k)$ denote the Moore space $S^{n-1}\cup_k e^n$. The mod k homotopy group $\pi_n(X;\mathbf{Z}/k)$ is defined to be the set of homotopy classes $[M^n(k),X]$. With the prime p implicit, and $q=2(p-1)$, let

$$s(e)=\begin{cases} p^{e-1}q & \text{if } p \text{ is odd,}\\ \max(8,2^{e-1}) & \text{if } p=2.\end{cases} \tag{2.1}$$

Let $A: M^{n+s(e)}(p^e)\to M^n(p^e)$ denote a map, as introduced by Adams in [1], which induces an isomorphism in K-theory. Such a map exists provided $n\geqslant 2e+3$ ([28, 2.11]). Then $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)$ is defined to be

$$\operatorname*{dirlim}_N \left[M^{i+Ns(e)}(p^e),X\right],$$

where the maps A are used to define the direct system. The map is what Hopkins and Smith would call a v_1-map, and they showed in [35] that any two v_n-maps of a finite complex which admits such maps become homotopic after a finite number of iterations (of suspensions of the same map), and hence $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ does not depend upon the choice of the map A. Note that although $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ is a theory yielding information about the unstable homotopy groups of X, the maps A which define the direct system may be assumed to be stable maps, since the direct limit only cares about large values of $i+Ns(e)$. Note also that the groups $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)$ are defined for all integers i and satisfy $v_1^{-1}\pi_i(X;\mathbf{Z}/p^e)\approx v_1^{-1}\pi_{i+s(e)}(X;\mathbf{Z}/p^e)$.

There is a canonical map $\rho: M^n(p^{e+1})\to M^n(p^e)$ which has degree p on the top cell, and degree 1 on the bottom cell. It satisfies the following compatibility with Adams maps.

LEMMA 2.1 ([34, p. 633]). *If $A: M^{n+s(e)}(p^e)\to M^n(p^e)$ and*

$$A': M^{n+s(e+1)}\left(p^{e+1}\right)\to M^n\left(p^{e+1}\right)$$

are v_1-maps, then there exists k so that the following diagram commutes.

$$\begin{tikzcd} M^{n+ks(e+1)}(p^{e+1}) \arrow[r, "\rho"] \arrow[d, "A'^k"] & M^{n+ks(e+1)}(p^e) \arrow[d, "A^{kp'}"] \\ M^n(p^{e+1}) \arrow[r, "\rho"] & M^n(p^e) \end{tikzcd}$$

Here $p'=p$ unless $p=2$ and $e<4$, in which case $p'=1$.

Thus, after sufficient iteration of the Adams maps, there are morphisms ρ^* between the direct systems used in defining $v_1^{-1}\pi_*(X;\mathbf{Z}/p^e)$ for varying e, and passing to direct limits, we obtain a direct system

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p^e) \xrightarrow{\rho^*} v_1^{-1}\pi_*(X;\mathbf{Z}/p^{e+1}) \xrightarrow{\rho^*} \cdots. \tag{2.2}$$

The following definition was given in [28], following less satisfactory definitions in [30] and [23].

DEFINITION 2.2. For any space X and any integer i,

$$v_1^{-1}\pi_i(X) = \operatorname*{dirlim}_e v_1^{-1}\pi_{i+1}(X;\mathbf{Z}/p^e),$$

using the direct system in (2.2).

The reason for the use of the $(i+1)$st mod-p^e periodic homotopy groups in defining the i-th (integral) periodic groups is that the maps ρ of Moore spaces have degree 1 on the bottom cells, but mod-p^e homotopy groups are indexed by the dimension of the top cell.

The mod-p^e periodic homotopy groups have received more attention in the literature, especially when $e = 1$. For spaces with H-space exponents, there is a close relationship between the integral periodic groups and the mod-p^e groups, which we recall after giving the relevant definition.

DEFINITION 2.3. A space X has H-space exponent p^e if for some positive integer L the p^e-power map $\Omega^L X \to \Omega^L X$ is null-homotopic.

By [20] and [36], spheres and compact Lie groups have H-space exponents.

PROPOSITION 2.4. (i) [28, 1.7] *If X has H-space exponent p^e, then there is a split short exact sequence*

$$0 \to v_1^{-1}\pi_i(X) \to v_1^{-1}\pi_i(X;\mathbf{Z}/p^e) \to v_1^{-1}\pi_{i-1}(X) \to 0.$$

(ii) *On the category of spaces with H-space exponents, there is a natural transformation $\pi_*(-)_{(p)} \to v_1^{-1}\pi_*(-;p)$.*

(iii) *If X has an H-space exponent, and $v_1^{-1}\pi_i(X)$ is a finitely generated abelian group, then $v_1^{-1}\pi_i(X)$ is a direct summand of $\pi_{i+L}(X)$ for some non-negative integer L.*

The proof of part ii utilizes the fibration

$$\operatorname{map}_*(M^{n+1}(p^e), X) \to \Omega^n X \xrightarrow{p^e} \Omega^n X,$$

where $\operatorname{map}_*(-,-)$ denotes the space of pointed maps. If the second map is null-homotopic, then the first map admits a section s. The natural transformation is induced by

$$\Omega^n X \xrightarrow{s} \operatorname{map}_*(M^{n+1}(p^e), X) \to \operatorname*{dirlim}_k \operatorname{map}_*(M^{n+1+ks(e)}(p^e), X).$$

Techniques of [28] imply naturality of this construction. To prove part (iii), we note that the map s allows $v_1^{-1}\pi_i(X)$ to be written as $\mathrm{dirlim}_k \pi_{i+ks(e)}(X)$, which is a direct summand of one of the groups in the direct system, provided the direct limit is finitely generated.

If X is a spectrum, then $v_1^{-1}\pi_*(X)$ is defined in exactly the same way as for spaces, that is, as in Definition 2.2. If X is a space, then stable groups, $v_1^{-1}\pi_*^s(X)$, can be defined either as $v_1^{-1}\pi_*(\Sigma^\infty X)$, where $\Sigma^\infty X$ denotes the suspension spectrum of X, or as $v_1^{-1}\pi_*(QX)$, where $QX = \Omega^\infty\Sigma^\infty X$ is the associated infinite loop space.

3. The isomorphism $v_1^{-1}\pi_*(S^{2n+1}) \approx v_1^{-1}\pi_{*-2n-1}^s(B^{qn})$

In this section, we sketch a new proof, due to Thompson and Langsetmo ([39, 4.2]), of the following crucial result.

THEOREM 3.1. *There is a map*

$$\Omega^{2n+1}S^{2n+1} \to QB^{qn} \tag{3.1}$$

which induces an isomorphism in $v_1^{-1}\pi_*(-)$.

Here $QX = \Omega^\infty\Sigma^\infty X$, and B^{qn} is the qn-skeleton of the p-localization of the classifying space $B\Sigma_p$ of the symmetric group Σ_p on p letters. Note that if $p = 2$, then B^{qn} is the real projective space RP^{2n}. The original proof, from [40] when $p = 2$ and [56] when p is odd, involved delicate arguments involving the lambda algebra and unstable Adams spectral sequences. We feel that the following argument, primarily K-theoretic, will speak to a broader cross-section of readers. The following elementary result shows that it is enough to show that the map (3.1) induces an iso in v_1-periodic mod p homotopy.

LEMMA 3.2. *If a map $X \to Y$ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$, then it induces an isomorphism in $v_1^{-1}\pi_*(-;p)$.*

PROOF. There are cofibrations of Moore spaces which induce natural exact sequences

$$\begin{aligned} &\to v_1^{-1}\pi_n(X;\mathbf{Z}/p^e) \xrightarrow{\rho^*} v_1^{-1}\pi_n(X;\mathbf{Z}/p^{e+1}) \to \\ &\quad v_1^{-1}\pi_n(X;\mathbf{Z}/p) \to v_1^{-1}\pi_{n-1}(X;\mathbf{Z}/p^e) \to . \end{aligned}$$

Induction on e using the 5-lemma implies that there are isomorphisms

$$v_1^{-1}\pi_*(X;\mathbf{Z}/p^e) \to v_1^{-1}\pi_*(Y;\mathbf{Z}/p^e)$$

for all e, compatible with the maps ρ^* which define the direct system (2.2). The desired isomorphism of the direct limits is immediate. □

The construction of the map (3.1) takes us far afield, and is not used elsewhere in the computations. For completeness, we wish to say something about it, but we will be extremely sketchy. The map is due to Snaith. Work of many mathematicians, especially Peter May, is important in the construction. However, we shall just refer the reader to [37], where the proof of naturality of these maps is given, along with references to the earlier work.

THEOREM 3.3. (i) *There are maps*

$$s_n : \Omega^{2n+1}S^{2n+1} \to QB^{qn}$$

which are compatible with respect to inclusion maps as n increases, and such that the adjoint map

$$\Sigma^\infty\Omega^{2n+1}S^{2n+1} \to \Sigma^\infty B^{qn}$$

is the projection onto a summand in a decomposition of $\Sigma^\infty\Omega^{2n+1}S^{2n+1}$ as a wedge of spectra.

(ii) *There is a map $QS^{2n+1} \xrightarrow{g} Q\Sigma^{2n+1}B_{(n+1)q-1}$ whose fiber, $\mathcal{F}$, satisfies*

$$v_1^{-1}\pi_*\big(\Omega^{2n+1}\mathcal{F};\mathbf{Z}/p\big) \approx v_1^{-1}\pi_*\big(QB^{qn};\mathbf{Z}/p\big).$$

SKETCH OF PROOF. Let $C_N(k)$ denote the space of ordered k-tuples of disjoint little cubes in I^N. If X is a based space, let C_NX denote the space of finite collections of disjoint little cubes of I^N labeled with points of X. More formally,

$$C_NX = \coprod_{k\geqslant 1} C_N(k) \times_{\Sigma_k} X^k/\sim,$$

where

$$\big[(c_1,\ldots,c_k),(x_1,\ldots,x_{k-1},*)\big] \sim \big[(c_1,\ldots,c_{k-1}),(x_1,\ldots,x_{k-1})\big].$$

There are natural maps, due to May,

$$C_NX \to \Omega^N\Sigma^NX, \tag{3.2}$$

which are weak equivalences if X is connected.

The space C_NX is filtered by defining $\mathcal{F}_m(C_NX)$ to be the subspace of m or fewer little labeled cubes. The successive quotients are defined by

$$D_{N,m}X = \mathcal{F}_m(C_NX)/\mathcal{F}_{m-1}(C_NX) \approx C_N(m)^+ \wedge_{\Sigma_m} X^{[m]},$$

where $X^{[m]}$ denotes the m-fold smash product. Snaith proved that if X is path-connected, there is a weak equivalence of suspension spectra

$$\Sigma^\infty C_NX \simeq \bigvee_{m\geqslant 1} \Sigma^\infty D_{N,m}X. \tag{3.3}$$

Let $N = 2n + 1$ and $X = S^0$. Stabilize the equivalence of (3.2), and project onto the summand of (3.3) with $m = p$ to get

$$\Sigma^\infty \Omega^{2n+1} S^{2n+1} \to \Sigma^\infty C_{2n+1}(p)/\Sigma_p.$$

The identification of $C_{2n+1}(p)/\Sigma_p$ as the nq-skeleton of $B\Sigma_p$ after localization at p was obtained by Fred Cohen in [19, p. 246].

Note that (3.2) and (3.3) yield a map $\Omega^N \Sigma^N X \to QD_{N,m}X$. The map g of ii) is obtained from the case $N = \infty$ and $X = S^{2n+1}$ as the composite

$$QS^{2n+1} \to QD_{\infty,p}S^{2n+1} = Q\big(B\Sigma_p^+ \wedge_{\Sigma_p} (S^{2n+1})^{[p]}\big) \simeq Q\Sigma^{2n+1} B_{(n+1)q-1}.$$

Here we have used results of [45] for the last equivalence.

The compatible maps s_n of 3.3(i) combine to yield a map $s' : QS^0 \to QB^\infty$, and one can show that the right square commutes in the diagram of fibrations below.

$$\begin{array}{ccccc}
\Omega^{2n+1}\mathcal{F} & \to & QS^0 & \xrightarrow{\Omega^{2n+1}g} & QB_{(n+1)q-1} \\
\downarrow{\scriptstyle t} & & \downarrow{\scriptstyle s'} & & \downarrow{\scriptstyle =} \\
QB^{qn} & \to & QB^\infty & \to & QB_{(n+1)q-1}
\end{array}$$

The map t then follows, and will induce an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$, as asserted in Theorem 3.3(ii), once we know that s' does. Kahn and Priddy showed that there is an infinite loop map $\lambda : QB^\infty \to QS^0$ such that $\lambda \circ s'$ induces an isomorphism in $\pi_j(-)$ for $j > 0$. It is easily verified using methods of the next two sections that the associated stable map $B^\infty \to S^0$ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$. Hence so does s'. □

Throughout this section, let $K_*(-)$ denote mod p K-homology, and M^k denote the mod p Moore space $M^k(p)$. The following two theorems, whose proofs occupy most of the rest of this section, imply that $j : S^{2n+1} \to \mathcal{F}$ induces an iso in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.

THEOREM 3.4 ([16, 14.4]). *If $k \geqslant 2$, and $\phi : X \to Y$ is a map of k-connected spaces such that $\Omega^k\phi$ is a K_*-equivalence, then ϕ induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.*

THEOREM 3.5 ([39]). *Let $\mathcal{F}$ be as in Theorem 3.3(ii). The map $S^{2n+1} \to QS^{2n+1}$ lifts to a map $j : S^{2n+1} \to \mathcal{F}$ such that $\Omega^2 j$ is a K_*-equivalence.*

The map j of Theorem 3.5 can be chosen so that $t \circ \Omega^{2n+1}j = s_n$, where t is the map in the proof of Theorem 3.3(ii) which induces the isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$. Theorem 3.1 is now proved by applying Lemma 3.2 to s_n.

Bousfield localization is involved in the proof of Theorem 3.4 and several other topics later in the paper, and so we review the necessary material. If E is a spectrum, then a space (or spectrum) X is *E_*-local* if every E_*-equivalence $Y \to W$ induces a bijection $[W,X] \to [Y,X]$. The *E_*-localization* of X is an E_*-local space (or spectrum) X_E together with an E_*-equivalence $X \to X_E$. The existence and uniqueness of these localizations were established in [18] and [17]. We will need the following result of

Bousfield, the proof of which is easily obtained using the ideas at the beginning of Section 5.

THEOREM 3.6 ([18, 4.8]). *A spectrum is K_*-local if and only if its mod p homotopy groups are periodic under the action of the Adams map.*

We will omit the proof of Theorem 3.4, as it requires many peripheral ideas. Instead, we will sketch a proof of the following result, which, although weaker than 3.4, has the same flavor, and predated it. An alternate proof of Theorem 3.1 can be given by using Theorem 3.7 and strengthening Theorem 3.5 to show that $K_*(\Omega^3 j)$ is bijective. As the calculations for Ω^3 seem significantly more difficult than for Ω^2, we omit that approach.

THEOREM 3.7 ([57]). *Let p be an odd prime, and let X and Y be 3-connected spaces. Suppose that $f : X \to Y$ is a map such that $K_*(\Omega^k f)$ is an isomorphism for $k = 0$, 1, 2, and 3. Then f induces an isomorphism in $v_1^{-1}\pi_*(-;\mathbf{Z}/p)$.*

SKETCH OF PROOF. If $F \to E \to B$ is a principal fibration, there is a bar spectral sequence converging to $K_*(B)$ with $E^2_{s,t} \approx \operatorname{Tor}^{K_*F}_{s,t}(K_*E, K_*)$. (See [55] for a discussion of this spectral sequence.)

Applied to the commutative diagram of principal fiber sequences

$$\begin{array}{ccccc} \Omega^3 X & \xrightarrow{p} & \Omega^3 X & \to & \operatorname{map}_*(M^3, X) \\ \downarrow{\scriptstyle \Omega^3 f} & & \downarrow{\scriptstyle \Omega^3 f} & & \downarrow{\scriptstyle f'} \\ \Omega^3 Y & \xrightarrow{p} & \Omega^3 Y & \to & \operatorname{map}_*(M^3, Y) \end{array},$$

the spectral sequence and the hypothesis of the theorem imply that f' induces an isomorphism in $K_*(-)$. Hence there is an equivalence of the K_*-localizations

$$\big(\operatorname{map}_*(M^3, X)\big)_K \xrightarrow{f'_K} \big(\operatorname{map}_*(M^3, Y)\big)_K.$$

Let $V(X)$ denote the mapping telescope of

$$\operatorname{map}_*(M^3, X) \xrightarrow{A^*} \operatorname{map}_*(M^{3+q}, X) \xrightarrow{A^*} \cdots.$$

Then $V(X)$ is K_*-local, since it is Ω^∞ of a periodic spectrum which is K_*-local by Theorem 3.6. This implies that there are maps i' making the following diagram commute.

$$\begin{array}{ccccc} \operatorname{map}_*(M^3, X) & \to & (\operatorname{map}_*(M^3, X))_K & \xrightarrow{i'} & V(X) \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f'_K} & & \downarrow{\scriptstyle V(f)} \\ \operatorname{map}_*(M^3, Y) & \to & (\operatorname{map}_*(M^3, Y))_K & \xrightarrow{i'} & V(Y) \end{array}$$

Since $v_1^{-1}\pi_*(X;\mathbf{Z}/p) \approx \pi_*(V(X);\mathbf{Z}/p)$, the desired isomorphism is a consequence of the following construction of an inverse to

$$V(f)_* : \pi_*\big(V(X);\mathbf{Z}/p\big) \to \pi_*\big(V(Y);\mathbf{Z}/p\big).$$

An element $\alpha \in \pi_k(V(Y); \mathbf{Z}/p)$ can be represented by a map

$$M^k \to \mathrm{map}_*\left(M^{3+qpj}, Y\right),$$

or, adjointing, by a map $M^{k+qpj} \to \mathrm{map}_*(M^3, Y)$. Here some care is required to see that we can switch the Moore space factor on which the map A is performed. The element which corresponds to α is the composite

$$\begin{aligned} M^{k+qpj} &\to \mathrm{map}_*\left(M^3, Y\right) \to \left(\mathrm{map}_*(M^3, Y)\right)_K \\ &\xrightarrow{(f'_K)^{-1}} \left(\mathrm{map}_*(M^3, X)\right)_K \xrightarrow{i'} V(X). \end{aligned}$$

□

The proof of Theorem 3.5 involves a good bit of delicate computation. The hardest part is the determination of $K_*(\Omega^2\mathcal{F})$ as a Hopf algebra. In order to conveniently obtain the coalgebra structure of $K_*(\Omega^2\mathcal{F})$, we proceed in two steps. We first calculate the algebra $K_*(\Omega^3\mathcal{F})$, using the bar spectral sequence associated to the principal fibration

$$\Omega^4 QS^{2n+1} \to \Omega^4 Q\Sigma^{2n+1} B_{q(n+1)-1} \to \Omega^3\mathcal{F}.$$

This spectral sequence is calculated in [38], obtaining an algebra isomorphism

$$K_*\left(\Omega^3\mathcal{F}\right) \approx P[y_1, y_2] \otimes E[z]. \tag{3.4}$$

Here y_i (resp. z) has bidegree $(1, 1)$ (resp. $(0, 1)$) in the spectral sequence, and hence even (resp. odd) degree in $K_*(\Omega^3\mathcal{F})$. The calculation of this spectral sequence requires some preliminary computation regarding the algebra structure of $K_*(\Omega^2\mathcal{F})$, and this requires major input from [44].

Now we calculate the bar spectral sequence associated to the principal fibration

$$\Omega^3\mathcal{F} \to * \to \Omega^2\mathcal{F}.$$

This spectral sequence, with $E_2 \approx \mathrm{Tor}^{P[y_1,y_2]\otimes E[z]}(K_*, K_*)$, collapses to yield an isomorphism of Hopf algebras

$$K_*\left(\Omega^2\mathcal{F}\right) \approx E[a_1, a_2] \otimes \Gamma[b], \tag{3.5}$$

where Γ denotes the divided polynomial algebra over $\mathbf{Z}_p$. The coproduct has a_1, a_2, and $\gamma_1(b)$ as the primitives, and

$$\psi\left(\gamma_i(b)\right) = \sum \gamma_j(b) \otimes \gamma_{i-j}(b).$$

Dualizing eq. (3.5) yields an isomorphism of algebras

$$K^*\left(\Omega^2\mathcal{F}\right) \approx E[\alpha_1, \alpha_2] \otimes P[\beta],$$

with $|\alpha_i|$ odd and $|\beta|$ even. This matches nicely with the following result from [52, 3.8].

PROPOSITION 3.8. *For any prime p, there is an isomorphism of algebras*

$$K^*(\Omega^2 S^{2n+1}) \approx E[u_0, u_1] \otimes P[w],$$

where E and P denote exterior and polynomial algebras over K_*.

We will use the Atiyah–Hirzebruch spectral sequence to show that the map

$$\Omega^2 S^{2n+1} \xrightarrow{\Omega^2 j} \Omega^2 \mathcal{F}$$

of Theorem 3.5 sends the generators of the isomorphic $K^*(-)$-algebras across. This will imply the second half of Theorem 3.5.

In [52], it is shown how the generators u_0, w, and u_1 of $K^*(\Omega^2 S^{2n+1})$ arise in the Atiyah–Hirzebruch spectral sequence whose E_2-term is $H^*(\Omega^2 S^{2n+1}; K_*)$. Indeed, they arise from the bottom three cohomology classes, of grading $2n-1$, $2pn-2$, and $2pn-1$, respectively. The map $\Omega^2 j$ induces an isomorphism in $H^i(-;\mathbf{Z}_p)$ for $i < 2pn - 1 + \min(q, 2n-2)$, and so it maps onto the three generators of $K^*(\Omega^2 S^{2n+1})$.

4. J-homology

In this section, we show how to compute $J_*(B^{qn})$. When combined with Theorems 3.1 and 5.1, this gives an explicit computation of $v_1^{-1}\pi_*(S^{2n+1})$, which we state at the end of this section as Theorem 4.2. This will be extremely important in our calculation of $v_1^{-1}\pi_*(Y)$ for other spaces Y. We begin with the case p odd, where the results are somewhat simpler to state. Historically it worked in the other order, with Mahowald's 2-primary results in [40] preceding Thompson's odd-primary work in [56].

Let p be an odd prime. We follow quite closely the exposition in [24] and [56, §3]. The spectrum $bu_{(p)}$ splits as a wedge of spectra $\Sigma^{2i}\ell$ satisfying $H^*(\ell;\mathbf{Z}_p) \approx A//E$, where A is the mod p Steenrod algebra, and E is the exterior subalgebra generated by $Q_0 = \beta$ and $Q_1 = P^1\beta - \beta P^1$. The spectrum ℓ is sometimes written $BP\langle 1\rangle$. Then $\ell_* = \pi_*(\ell)$ is calculated from the Adams spectral sequence (ASS) with

$$E_2^{s,t} \approx \mathrm{Ext}_A^{s,t}(H^*\ell, \mathbf{Z}_p) \approx \mathrm{Ext}_E^{s,t}(\mathbf{Z}_p, \mathbf{Z}_p) \approx \mathbf{Z}_p[a_0, a_1], \tag{4.1}$$

where a_i has bigrading $(1, iq+1)$. Here we have used the change-of-rings theorem in the middle step. There are no possible differentials in the spectral sequence, and since multiplication by a_0 corresponds to multiplication by p in homotopy, we find that $\pi_*(\ell)$ is a polynomial algebra over $\mathbf{Z}_{(p)}$ on a class of grading q. Using the ring structure of ℓ, one easily sees that there is a cofibration

$$\Sigma^q \ell \to \ell \to H\mathbf{Z}_{(p)}.$$

Let k be a $(p-1)$st root of unity mod p but not mod p^2, and let ψ^k denote the Adams operation. The map $\psi^k - 1 : \ell \to \ell$ lifts to a map $\theta : \ell \to \Sigma^q \ell$. The connective

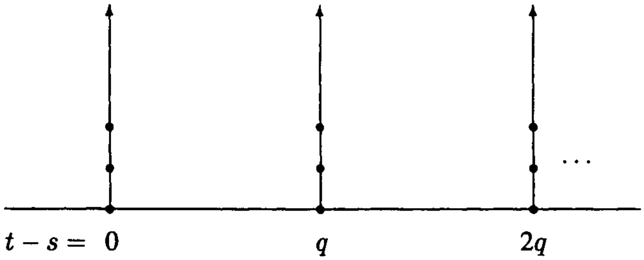


Figure 1. ASS for $\ell_*(R)$.

J-spectrum, J, is defined to be the fiber of θ. The homotopy exact sequence of θ easily implies that

$$\pi_i(J) \approx \begin{cases} \mathbf{Z}_{(p)} & \text{if } i = 0, \\ \mathbf{Z}/p^{\nu_p(j)+1} & \text{if } i = qj - 1 \text{ with } j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The image of the classical J-homomorphism is mapped isomorphically onto these groups by the map $S^0 \to J$; this is the reason for the name of the spectrum.

We now proceed toward the calculation of $J_*(B^{qn})$. We let B be the p-localization of the suspension spectrum of $B\Sigma_p$. Then, with coefficients always in $\mathbf{Z}_p$, the only nonzero groups $H^i(B)$ occur when $i \equiv 0$ or -1 mod q, and $i > 0$. These groups are cyclic of order p with generator x_i satisfying $Q_0 x_{aq-1} = x_{aq}$ and $Q_1 x_{aq-1} = x_{(a+1)q}$. We will work with the skeleta B^{qn} and the quotients $B_{q(n+1)-1} = B/B^{qn}$; these are suspension spectra of the spaces which appeared in Theorem 3.3.

There is a p-local map $B \to S^0$ constructed by Kahn and Priddy. If R denotes its cofiber, there is a filtration of the E-module H^*R with subquotients $\Sigma^{qi}E//E_0$ for $i \geqslant 0$. Here E_0 is the exterior subalgebra of E generated by Q_0. Since $\mathrm{Ext}_E(E//E_0) \approx \mathrm{Ext}_{E_0}(\mathbf{Z}_p) \approx \mathbf{Z}_p[a_0]$, we find that $\mathrm{Ext}_E(H^*R)$ has a "spike", consisting of the powers of a_0, for each non-negative value of $t - s$ which is a multiple of q. Here we have begun a practice of omitting $\mathbf{Z}_p$ from the second variable of $\mathrm{Ext}_B(-,-)$ if B is a subalgebra of the Steenrod algebra. The action of $\mathrm{Ext}_E(\mathbf{Z}_p)$ on $\mathrm{Ext}_E(H^*R)$ has a_1 always acting nontrivially.

We draw ASS pictures with coordinates $(t-s, s)$, so that horizontal component refers to homotopy group. A chart for the ASS of $R \wedge \ell$ is given in fig. 1.

The short exact sequence

$$0 \to H^*(\Sigma B) \to H^*(R) \to H^*(S^0) \to 0$$

induces an exact sequence

$$\begin{aligned} &\to \mathrm{Ext}_E^{s,t}(H^*S^0) \xrightarrow{i_*} \mathrm{Ext}_E^{s,t}(H^*R) \to \mathrm{Ext}_E^{s,t}(H^*(\Sigma B)) \\ &\to \mathrm{Ext}_E^{s+1,t}(H^*S^0) \to . \end{aligned} \tag{4.2}$$

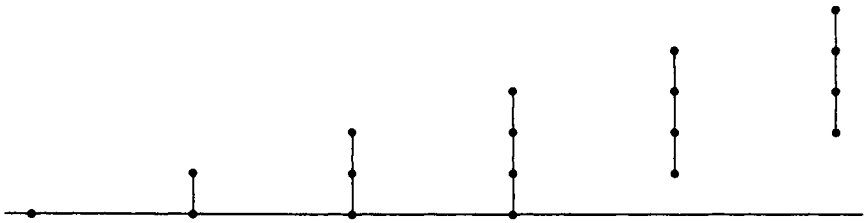


Figure 2. ASS for $\ell_*(B^{4q})$ for $t-s<6q$.

These morphisms are $\mathrm{Ext}_E(\mathbf{Z}_p)$-module maps, and the action of a_1 implies that i_* is injective. Thus there are elements $x_{iq-1}\in \mathrm{Ext}_E^{0,iq-1}(H^*B)$ for $i>0$ such that

$$\mathrm{Ext}_E^{s,t}(H^*B)=\begin{cases}\mathbf{Z}_p & \text{if } t-s=iq-1,\ i>0,\ 0\leqslant s<i,\\ 0 & \text{otherwise,}\end{cases}$$

with generators $a_0^s x_{iq-1}$. Hence

$$\ell_i(B)\approx\begin{cases}\mathbf{Z}/p^{(i+1)/q} & \text{if } i\equiv -1 \bmod q, \text{ and } i>0,\\ 0 & \text{otherwise.}\end{cases}$$

There is an isomorphism of E-modules $H^*(B_{q(n+1)-1})\approx \Sigma^{qn}H^*(B)$, and so

$$\ell_*(B_{q(n+1)-1})\approx \ell_*(\Sigma^{qn}B).$$

The Ext calculation easily implies that the morphism $\ell_*(B)\to\ell_*(B_{q(n+1)-1})$ induced by the collapse map is surjective, and so the exact sequence of the cofibration $B^{qn}\to B\to B_{q(n+1)-1}$ implies that

$$\ell_i(B^{qn})\approx\begin{cases}\mathbf{Z}/p^{\min((i+1)/q,n)} & \text{if } i\equiv -1 \bmod q, \text{ and } i>0,\\ 0 & \text{otherwise.}\end{cases}$$

The ASS chart for B^{4q} is illustrated in fig. 2.

The map $S^0\to R$ implies that $\theta_*:\ell_{qj}(X)\to\ell_{(q-1)j}(X)$ is multiplication by the same number for $X=R$ as it was for $X=S^0$. Thus it is multiplication by $p^{\nu_p(j)+1}$. Now the map $R\to\Sigma B$ implies that $\theta_*:\ell_{qj-1}(B)\to\ell_{(q-1)j-1}(B)$ is multiplication by $p^{\nu_p(j)+1}$. The maps $B^{qn}\to B\to B_{q(n+1)-1}$ imply that the same is true in B^{qn} and $B_{q(n+1)-1}$. We obtain

$$J_i(B^{qn})\approx\begin{cases}\mathbf{Z}/p^{\min(n,\nu_p(j)+1)} & \text{if } i=jq-1,\ j>0,\\ \mathbf{Z}/p^{\min(n,\nu_p(j)+1)} & \text{if } i=jq-2,\ j>n,\\ \mathbf{Z}/p^{\min(n-1,\nu_p(j))} & \text{if } i=jq-2,\ 0<j\leqslant n,\\ 0 & \text{otherwise.}\end{cases} \tag{4.3}$$

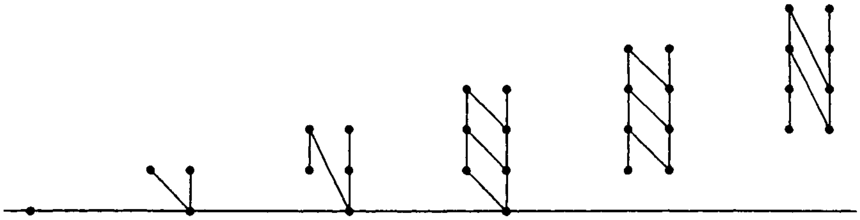


Figure 3. Beginning of chart for $J_*(B^{4q})$.

This is illustrated in fig. 3, which is not quite an ASS chart. It is a combination of the charts for $\ell_*(B^{qn})$ and $\ell_{*-q+1}(B^{qn})$ and the homomorphism θ_* between them, which is represented by lines of negative slope. The exact sequence

$$0 \to \text{coker}(\theta_{*+1}) \to J_*\big(B^{qn}\big) \to \ker(\theta_*) \to 0$$

says that elements which are not involved in these boundary morphisms comprise $J_*(B^{qn})$. There are several reasons for our having elevated the filtrations of $\ell_{*-q+1}(B^{qn})$ by 1 in this chart. One is that it makes all the boundary morphisms go up, so that it looks like an ASS chart. Another is that (by [40]) there is a resolution of $B^{qn} \wedge J$ (which is not an Adams resolution) for which the homotopy exact couple is depicted by this chart. A third is that if J_1 is defined to be the fiber of $J \to H\mathbf{Z}_2$, then the ASS chart for $B^{qn} \wedge J_1$ will agree with this chart in filtration greater than 1. See [13, §6] for an elaboration on this.

If X is a space or spectrum, then $v_1^{-1}J_i(X)$ is defined analogously to Definition 2.2 to be

$$\operatorname*{dirlim}_{e,k}\ \big[M^{i+1+ks(e)}(p^e), X \wedge J\big].$$

Since J is a stable object, we can S-dualize the Moore space, obtaining

$$\operatorname*{dirlim}_{e,k}\ J_{i+1+ks(e)}\big(X \wedge M(p^e)\big),$$

where the Moore spectrum $M(n)$ has cells of degree 0 and 1. The "+1" in this J-group is present due to the maps $M(p^e) \to M(p^{e+1})$ having degree 1 on the 1-cell.

One can often compute $v_1^{-1}J_*(X)$ directly from $J_*(X)$ without having to worry about the "$\wedge M(p^e)$". This can be done by extending the periodic behavior which occurs in positive filtration down into negative filtrations and negative stems. For example (cf. fig. 3), a chart for $v_1^{-1}J_*(B^{nq})$ has, for all integers a, adjacent towers of height n in $aq-2$ and $aq-1$ with $d_{\nu(a)+1}$-differential. If $\nu(a)+1 \geqslant n$, then the differential is 0. This interpretation of $v_1^{-1}J_*(-)$ can be justified using the following result.

PROPOSITION 4.1. *Let p be an odd prime, and let KU be the spectrum for periodic K-theory localized at p. Let k be a $(p-1)$st root of unity mod p but not mod p^2, and let* Ad *denote the fiber of* $KU \xrightarrow{\psi^k - 1} KU$. *Then* $v_1^{-1}J_*(-) \approx \mathrm{Ad}_*(-)$.

This follows from the fact that if $v_1^{-1}\ell_*(-)$ is defined as

$$\operatorname*{dirlim}_{e,k} \ell_{i+1+ks(e)}\big(X \wedge M(p^e)\big),$$

then $v_1^{-1}\ell_*(-) \approx KU_*(-)$, which is a consequence of the fact that $A \wedge \iota = v \wedge 1_M : \Sigma^q M \to M \wedge \ell$, where $A : \Sigma^q M \to M$ and $v : S^q \to \ell$.

When $p = 2$, the results are a bit messier to state and picture. If bsp denotes the 2-local connected Ω-spectrum whose $(8k)$th space is $BSp[8k]$, then $\Sigma^4 bsp \simeq bo[4]$, the spectrum formed from bo by killing $\pi_i(-)$ for $i < 4$. The map $\psi^3 - 1 : bo \to bo$ lifts to a map $\theta : bo \to \Sigma^4 bsp$, and J is defined to be the fiber of θ.

Let A_1 denote the subalgebra of the mod 2 Steenrod algebra A generated by Sq^1 and Sq^2. Then $H^*bo \approx A//A_1$ and $H^*bsp \approx A \otimes_{A_1} N$, where $N = \langle 1, \mathrm{Sq}^2, \mathrm{Sq}^3 \rangle$. Hence, using the change of rings theorem, the E_2-term of the ASS converging to $\pi_*(bo)$ is $\mathrm{Ext}_{A_1}(\mathbf{Z}_2)$, while that for bsp is $\mathrm{Ext}_{A_1}(N)$. These are easily computed to begin as in fig. 4, with each chart acted on freely by an element in $(t-s, s) = (8,4)$. Positively sloping diagonal lines indicate the action of $h_1 \in \mathrm{Ext}_{A_1}^{1,2}(\mathbf{Z}_2)$. It corresponds to the Hopf map η in homotopy.

There are no possible differentials in these ASS's, and so we obtain

$$\pi_i(bo) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i \equiv 0 \bmod 4,\ i \geqslant 0, \\ \mathbf{Z}_2 & \text{if } i \equiv 1, 2 \bmod 8,\ i > 0, \\ 0 & \text{otherwise,} \end{cases}$$

in accordance with Bott periodicity.

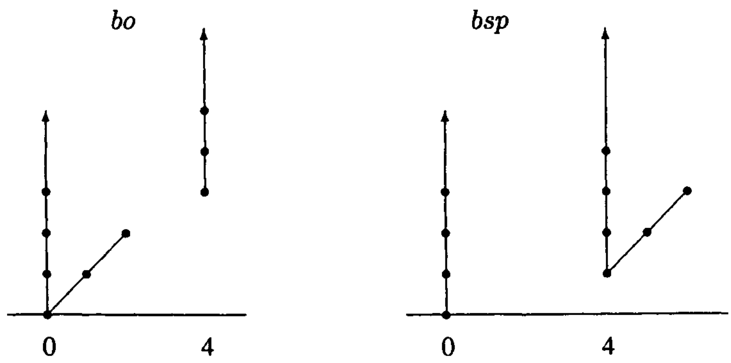


Figure 4. Part of ASS for bo and bsp.

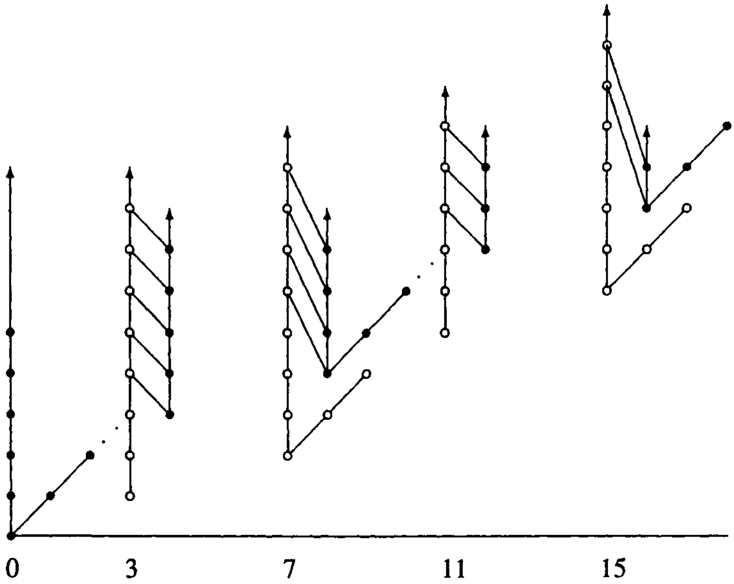


Figure 5. 2-primary $\pi_i(J)$, $i \leqslant 18$.

From Adams' work, we have $\theta_* : \pi_{4j}(bo) \to \pi_{4j}(\Sigma^4 bsp)$ hitting all multiples of $2^{\nu_2(j)+3}$, while θ_* is 0 on the $\mathbf{Z}_2$'s. This yields $\pi_i(J) = 0$ if $i < 0$, while for $i \geqslant 0$

$$\pi_i(J) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i = 0, \\ \mathbf{Z}/2^{\nu_2(i)+1} & \text{if } i \equiv 3 \bmod 4, \\ \mathbf{Z}_2 & \text{if } i \equiv 0, 2 \bmod 8,\ i > 0, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 1 \bmod 8,\ i > 1, \\ 0 & \text{if } i \equiv 4, 5, 6 \bmod 8. \end{cases} \tag{4.4}$$

From Adams' work and the confirmation of the Adams Conjecture, it is known that $\pi_*(S^0) \to \pi_*(J)$ sends the image of the classical J-homomorphism plus Adams' elements μ_j and $\eta\mu_j$ isomorphically onto $\pi_*(J)$. A chart for $\pi_i(J)$ with $i \leqslant 18$ is given in fig. 5. Here the elements coming from bo are indicated by •'s, while those from $\Sigma^4 bsp$ are indicated by o's.

The first dotted η-extension can be deduced from the fact that $\theta^* : H^4(\Sigma^4 bsp) \to H^4(bo)$ hits Sq^4, together with the relation $\eta^3 = 4\nu$. This η-action is then pushed along by periodicity. Another argument which is frequently useful for deducing η-extensions such as this involves Toda brackets. The generator of $\pi_{8i+4}(bo)$ is obtained from the element $\alpha \in \pi_{8i+2}(bo)$ as $\langle \alpha, \eta, 2 \rangle$. Clearly α pulls back to $\pi_*(J)$, and if $\alpha\eta$ were 0 here, then the bracket could also be formed in J. However, the boundary morphism on $\pi_{8i+4}(bo)$ implies that this bracket cannot be formed, and so $\alpha\eta$ must be nonzero in J. Moreover, it must be the element β such that 2β is $\delta(\langle \alpha, \eta, 2 \rangle)$.

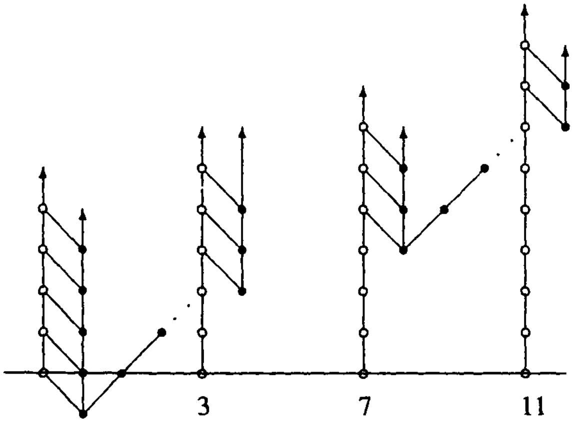


Figure 6. $\mathrm{Ext}_{A_1}(H^*P)$, from $\mathrm{Ext}_{A_1}(H^*S^0)$ and $\mathrm{Ext}_{A_1}(H^*R)$, $t-s<15$.

We will rename B^{qn} as P^{2n} when $p=2$, with P denoting the suspension spectrum of RP^∞. As in the odd primary case, there is a map $\lambda : P \to S^0$ with nontrivial cohomology operations in its mapping cone R. This 2-primary map λ can be viewed more geometrically than its odd-primary analogue, as an amalgamation of composites

$$P^n \to \mathrm{SO}(n+1) \xrightarrow{J} \Omega^n S^n.$$

With A_0 denoting the exterior subalgebra of A generated by Sq^1, H^*R can be filtered as an A_1-module with subquotients $\Sigma^{4i}A_1//A_0$ for $i \geqslant 0$, and so $\mathrm{Ext}_A(H^*(R\wedge bo))$ consists of h_0-spikes rising from each position $(t-s,s)=(4i,0)$ for $i\geqslant 0$. Here h_0 is the element of $\mathrm{Ext}_A^{1,1}(\mathbf{Z}_2)$ or $\mathrm{Ext}_{A_1}^{1,1}(\mathbf{Z}_2)$ corresponding to Sq^1 and to multiplication by 2 in homotopy. Also, we begin a practice of using without comment the relation

$$\mathrm{Ext}_A\big(H^*(X\wedge bo)\big) \approx \mathrm{Ext}_A\big(H^*X\otimes A//A_1\big) \approx \mathrm{Ext}_{A_1}\big(H^*X\big).$$

Thus the nonzero groups $bo_i(R)$ occur only when $i\equiv 0 \bmod 4$ and $i\geqslant 0$, and these groups are $\mathbf{Z}_{(2)}$. We also need

$$bsp_i(R) \approx bo_i\big(R\wedge\big(S^0\cup_\eta e^2\cup_2 e^3\big)\big) \approx \begin{cases} \mathbf{Z}_{(2)} & \text{if } i\equiv 0 \bmod 4,\ i\geqslant 0,\\ \mathbf{Z}_2 & \text{if } i\equiv 2 \bmod 4,\ i>0,\\ 0 & \text{otherwise.}\end{cases}$$

The $\mathbf{Z}_2$'s are obtained from the exact sequence in $bo_*(-)$ associated to the cofibration

$$R\wedge S^2 \to R\wedge\big(S^2\cup_2 e^3\big) \to R\wedge S^3 \xrightarrow{2}.$$

Analogous to (4.2) is an exact sequence which allows us to compute $\mathrm{Ext}_{A_1}(H^*P)$ from $\mathrm{Ext}_{A_1}(H^*S^0)$ and $\mathrm{Ext}_{A_1}(H^*R)$. This is most easily seen in the chart of fig. 6, in

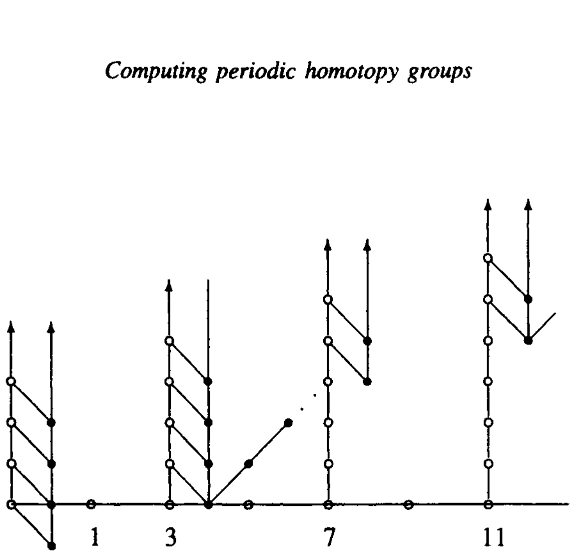


Figure 7. $bsp_*(P)$, from $bsp_*(S^0)$ and $bsp_*(R)$, $* \leqslant 11$.

which $\bullet$'s are from $\mathrm{Ext}_{A_1}(H^*S^0)$, and $\circ$'s are from $\mathrm{Ext}_{A_1}(H^*R)$.

The groups are read off from this as

$$bo_i(P) \approx \begin{cases} \mathbf{Z}/2^{4j+3}, & i = 8j+3,\ j \geqslant 0, \\ \mathbf{Z}/2^{4j}, & i = 8j-1,\ j > 0, \\ \mathbf{Z}_2, & i = 8j+1 \text{ or } 8j+2,\ j \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Next on the agenda is $bsp_*(P)$, which is computed from $bsp_*(S^0)$ (in $\bullet$) and $bsp_*(R)$ (in $\circ$) as in fig. 7.

Note that in positive filtration $bsp_*(P)$ looks like $bo_*(\Sigma^4 P)$ pushed up by 1 filtration. The explanation for this is the short exact sequence of A_1-modules

$$0 \to \Sigma^5\mathbf{Z}_2 \to A_1//A_0 \to N \to 0, \tag{4.5}$$

where $N = \langle 1, \mathrm{Sq}^2, \mathrm{Sq}^3\rangle$, as before. If this is tensored with any A_0-free A_1-module M, such as P, then the exact Ext_{A_1}-sequence reduces to isomorphisms

$$\mathrm{Ext}_{A_1}^{s-1,t}(\Sigma^5 M) \to \mathrm{Ext}_{A_1}^{s,t}(N \otimes M)$$

when $s > 1$. When $M = P$, an iso is also obtained when $s = 1$.

The isomorphism of A_1-modules $H^*(P_{4n+1}) \approx H^*(\Sigma^{4n}P)$ allows one to immediately obtain $bo_*(P_{4n+1})$ and $bsp_*(P_{4n+1})$ from the above calculations. One way of determining $bo_*(P_{4n+3})$ and $bsp_*(P_{4n+3})$ is from the short exact sequence of A_1-modules

$$0 \to H^*(\Sigma^{4n+4}\mathbf{Z}_2) \to H^*(P_{4n+3}) \to H^*(\Sigma^{4n+3}R) \to 0.$$

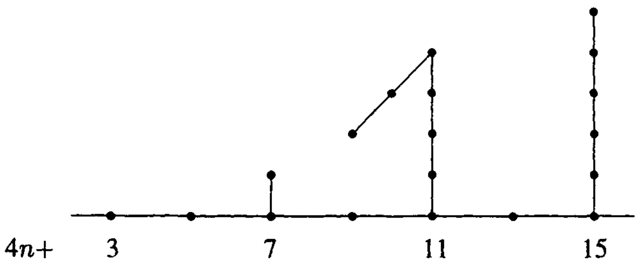


Figure 8. $bsp_*(P_{4n+3})$, $* \leqslant 15$.

This yields as $bsp_*(P_{4n+3})$ a chart which begins as in fig. 8, while $bo_*(P_{4n+7}) \approx bo_*(\Sigma^4 P_{4n+3})$ is obtained from this chart by deleting all classes in filtration 0.

Next we compute $bo_*(P^{2m})$ and $bsp_*(P^{2m})$ using the exact Ext_{A_1}-sequence corresponding to the short exact sequence

$$0 \to H^*(P_{2m+1}) \to H^*(P) \to H^*(P^{2m}) \to 0.$$

For example, this yields the calculation of $bo_*(P^{8n})$ indicated in fig. 9, where $bo_*(P)$ is in $\bullet$'s, while $bo_*(P_{8n+1})$ is in $\circ$'s.

Next we form $J_*(P^{2m})$ from $bo_*(P^{2m})$ and $(\Sigma^3 bsp)_*(P^{2m})$, with filtrations of the latter pushed up by 1, similarly to the odd primary case. The boundary morphism $bo_{4i-1}(P^{2m}) \to (\Sigma^4 bsp)_{4i-1}(P^{2m})$ is pictured by a differential in the chart, and, for the same reason as in the odd-primary case, its value is the same as in $bo_{4i-1}(S^0) \to (\Sigma^4 bsp)_{4i-1}(S^0)$, namely a nonzero $d_{\nu(i)+1}$ wherever possible. If $m \geqslant k$, then the charts for $J_*(P^{2m})$ and $J_*(P^{2k})$ are isomorphic through dimension $2k-1$. This is illustrated in fig. 10 for $k = 8$.

For $* \geqslant 2m-1$, the form of the chart for $J_*(P^{2m})$ depends upon the mod 4 value of m. The last of the filtration-1 $\mathbf{Z}_2$'s occurs in $* = 2m$ or $2m+2$. The chart near $* = 2m$ is indicated in fig. 11. Note how the bsp-part is like the bo-part shifted one unit left and two units down. Differentials d_r with $r > 1$ are omitted from this chart; they occur on the towers in $8n-1$ and $8n+7$.

To obtain $v_1^{-1}J_*(P^{2m})$ from $J_*(P^{2m})$, one removes the filtration-1 $\mathbf{Z}_2$'s, and extends into negative filtration the periodic behavior which is present in the towers to the right of dimension $2m$. The justification for this is similar to that in the odd-primary case, namely Proposition 4.1. For example, if i is any integer, $v_1^{-1}J_*(P^{8n+4})$ for $8i+6 \leqslant * \leqslant 8i+13$ looks like the portion of fig. 11 for P^{8n+4} between $8n+6$ and $8n+13$, with a $d_{\nu(4i+4)}$-differential on the tower in $8i+7$. One might find it easier to compute $v_1^{-1}J_*(P^{2m})$ directly without bothering to first compute the nonperiodic J; however, one sometimes needs the nonperiodic J-groups.

We combine the results of this section with Theorem 3.1 and Theorem 5.1 to obtain the following extremely important result, Theorem 4.2.

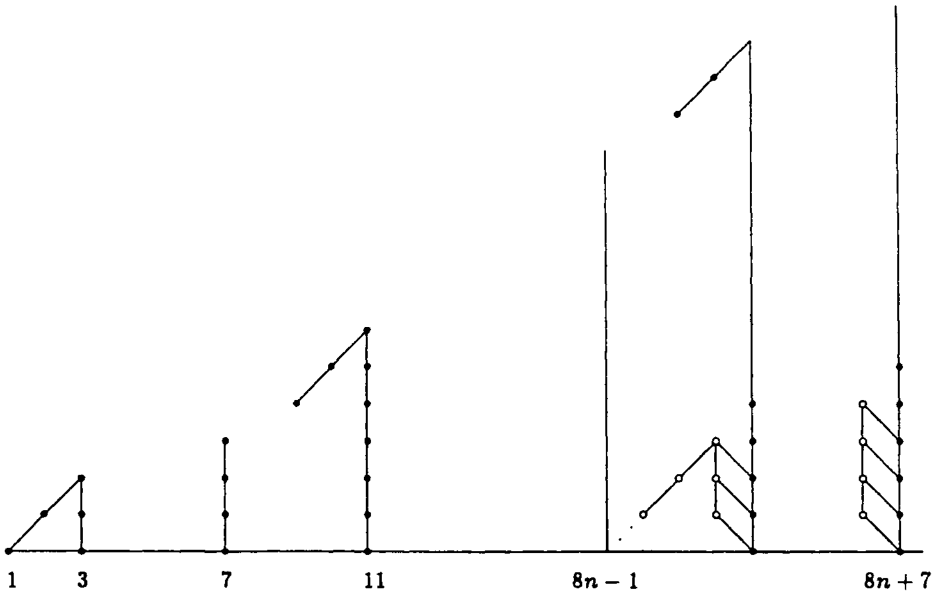


Figure 9. $bo_*(P^{8n})$, from $bo_*(P)$ and $bo_*(P_{8n+1})$.

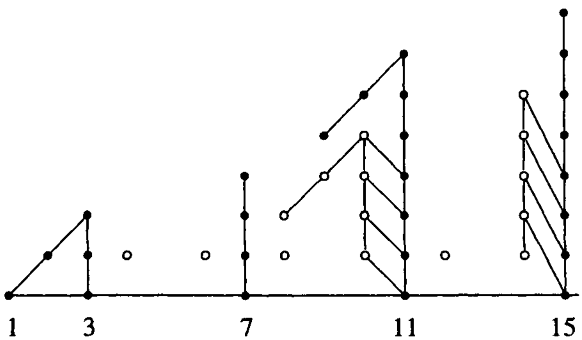


Figure 10. $J_*(P^{2m})$ in $* \leqslant 15$, provided $m \geqslant 8$.

THEOREM 4.2. *If p is odd, then*

$$v_1^{-1}\pi_{2n+1+i}\big(S^{2n+1};p\big) \approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \textit{if } i = qa-2 \textit{ or } qa-1, \\ 0 & \textit{if } i \not\equiv -1 \textit{ or } -2 \bmod q. \end{cases}$$

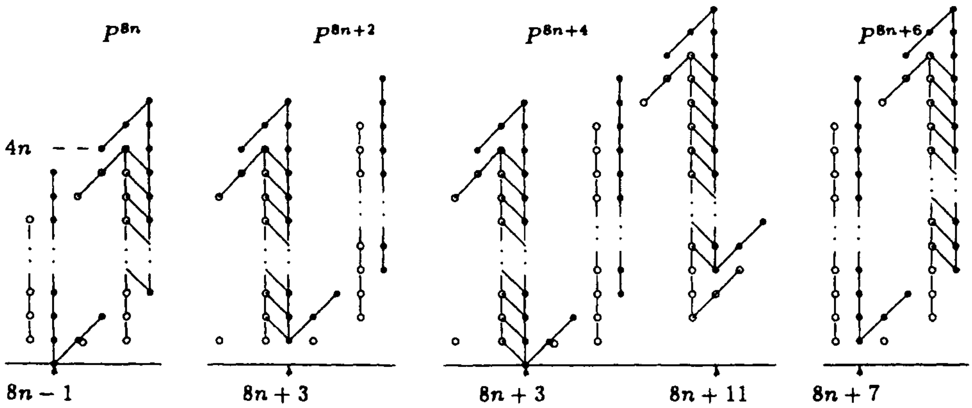


Figure 11. $J_*(P^{2m})$ where it starts to ascend.

If $n \equiv 1$ *or* 2 mod 4, *then*

$$v_1^{-1}\pi_{2n+1+i}(S^{2n+1};2) \approx \begin{cases} \mathbf{Z}_2 & \text{if } i \equiv 0,5 \bmod 8, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 1,4 \bmod 8, \\ \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(3,n+1)} & \text{if } i \equiv 2,3 \bmod 8, \\ \mathbf{Z}/2^{\min(n-1,\nu_2(j)+4)} & \text{if } i = 8j-2 \text{ or } 8j-1. \end{cases}$$

If $n \equiv 0$ *or* 3 mod 4, *then*

$$v_1^{-1}\pi_{2n+1+i}(S^{2n+1};2) \approx \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } i \equiv 0,1 \bmod 8, \\ \mathbf{Z}_8 \oplus \mathbf{Z}_2 & \text{if } i \equiv 2 \bmod 8, \\ \mathbf{Z}_8 & \text{if } i \equiv 3 \bmod 8, \\ 0 & \text{if } i \equiv 4,5 \bmod 8, \\ \mathbf{Z}/2^{\min(n,\nu_2(j)+4)} & \text{if } i = 8j-2, \\ \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(n,\nu_2(j)+4)} & \text{if } i = 8j-1. \end{cases}$$

5. The v_1-periodic homotopy groups of spectra

In this section, we sketch three proofs of the following central result.

THEOREM 5.1. *If X is a spectrum, then* $v_1^{-1}\pi_*(X) \approx v_1^{-1}J_*(X)$.

This result was first stated, at least for mod p v_1-periodic homotopy groups, in [56].

Theorem 5.1 is a consequence of the following result, which is the special case where X is the mod p Moore spectrum $M = S^0 \cup_p e^1$.

THEOREM 5.2. *Let $v_1^{-1}M$ denote the mapping telescope of*

$$M \to \Sigma^{-s}M \to \Sigma^{-2s}M \to \cdots,$$

where $s = 8$ if $p = 2$ and $s = q$ if p is odd, and the maps are all suspensions of an Adams map A. Then the Hurewicz morphism

$$\pi_*(v_1^{-1}M) \to J_*(v_1^{-1}M)$$

is an isomorphism.

This theorem implies that for any spectrum X, the map

$$X \wedge v_1^{-1}M \to X \wedge v_1^{-1}M \wedge J$$

is an equivalence, which, after dualizing the Moore spectra, implies that Theorem 5.1 is true with mod p coefficients. The general case of the theorem then follows from Lemma 3.2.

As an aside, we note that these results are equivalent to the validity of Ravenel's Telescope Conjecture ([53]) when $n = 1$. This result, for which the analogue with $n = 2$ has been shown to be false, can be stated in the following way.

COROLLARY 5.3. *The v_1-telescope equals the K_*-localization, i.e. $v_1^{-1}M = M_K$.*

PROOF. Since the Adams maps induce isomorphisms in $K_*(-)$, the inclusion $M \to v_1^{-1}M$ is a K_*-equivalence. Since $v_1^{-1}M \simeq v_1^{-1}M \wedge J \simeq M \wedge v_1^{-1}J$, and, similarly to Proposition 4.1, there is a cofibration

$$v_1^{-1}J \to KO \to KO,$$

it follows readily that $v_1^{-1}M$ is K_*-local. □

The remainder of this section is concerned with proofs of Theorem 5.2. Three distinct proofs have been given, although each is too complicated to present in detail here. We sketch each, relegating details to the original papers.

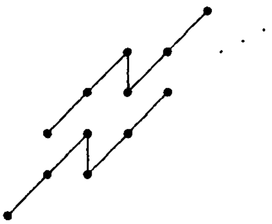
The first proof, when $p = 2$, was given by Mahowald in [40], although he was offering sketches of this proof as early as 1970. The odd-primary analogue was given in [24]. A sketch of Mahowald's proof, involving *bo*-resolutions, follows.

Using self-duality of M, it suffices to show that

$$\operatorname*{dirlim}_i\, [\Sigma^{k+8i}M, S^0] \to \operatorname*{dirlim}_i\, [\Sigma^{k+8i}M, J] \tag{5.1}$$

is an isomorphism. The target groups are easily determined by the methods of the preceding section to be given by two sequences of "lightning flashes" as in fig. 12. It is easily seen, for example from the upper edge of Adams spectral sequence, that these elements all come from actual stable homotopy classes, i.e. the morphism (5.1) is surjective.

The injectivity of the morphism (5.1) will be proved by showing that if $\Sigma^k M \xrightarrow{f} S^0 \to J$ is trivial, then for i sufficiently large, $\Sigma^{k+8i}M \xrightarrow{A^i} \Sigma^k M \xrightarrow{f} S^0$ is trivial. This will be done using *bo*-resolutions.



$k \equiv \ -2 \quad 0 \quad 2 \ \ 3 \quad \bmod 8$

Figure 12. $\operatorname{dirlim}_i[\Sigma^{k+8i}M, J]$.

Let $\overline{bo}$ denote the cofiber of the inclusion $S^0 \to bo$. There is a tower of (co)fibrations

$$\begin{tikzcd}
S^0 \arrow[d] & \Sigma^{-1}\overline{bo} \arrow[l] \arrow[d] & \Sigma^{-2}\overline{bo} \wedge \overline{bo} \arrow[l] \arrow[d] & \arrow[l] \\
bo & \Sigma^{-1}\overline{bo} \wedge bo & \Sigma^{-2}\overline{bo} \wedge \overline{bo} \wedge bo &
\end{tikzcd}$$

The homotopy exact couple of this tower gives the bo-ASS for S^0. It was proved in [25], following [41], that the E_2-term of this spectral sequence vanishes above a line of slope 1/5. That is, $E_2^{s,t}(S^0) = 0$ if $s > \frac{1}{5}(t-s) + 3$. One can show that $\overline{bo} \wedge bo \simeq \Sigma^4 bsp \vee W$, where W can be written explicitly, and the map

$$\Sigma^{-1}bo \xrightarrow{\delta} \Sigma^{-1}\overline{bo} \wedge bo \to \Sigma^3 bsp$$

may be used as the map whose fiber is J. Here δ induces the lowest d_1 in the bo-ASS, and the second map collapses W.

Let $E_s = (\Sigma^{-1}\overline{bo})^{\wedge s}$, the sth stage of the tower. By explicit calculation it can be shown that, if $s > 1$ or if $s = 1$ and the map is detected entirely in the W-part, a map $X \to E_s$ of Adams ($H\mathbf{Z}/2$) filtration greater than 1 can be varied so that its projection to E_{s-1} is unchanged, while the new map lifts to E_{s+1}. Originally it was thought that this was true for maps of Adams filtration greater than 0, but a complication was noted in [27].

Now suppose that $\Sigma^k M \xrightarrow{f} S^0 \to J$ is trivial. Then f lifts to a map into E_1 whose projection into $\Sigma^3 bsp$ is trivial. The Adams map A can be written as the composite of two maps, each of $H\mathbf{Z}/2$-Adams filtration greater than 1. Thus, by the result of the previous paragraph, $f \circ A^i$ lifts to E_{2i+1}. If i is chosen large enough that $2i+1 > \frac{1}{5}(k+8i+1)+3$, then this map $\Sigma^{k+8i}M \to S^0$ will have bo-filtration so large that all such maps are trivial by the vanishing line result, completing the proof.

The first proof of Theorem 5.2 for p odd was given by Haynes Miller. A proof analogous to his for $p = 2$ has not been achieved; [32] was a step in that direction. Miller's work did not involve the spectrum J. Instead, in [46], he defined a localized

ASS for M, and computed its E_2-term. Then, in [47], using a clever comparison with the BP-based Novikov spectral sequence, he computed the differentials in the ASS, obtaining the following result.

PROPOSITION 5.4. *If p is odd, then $\pi_*(v_1^{-1}M)$ is free over $\mathbf{Z}_p[v_1^{\pm 1}]$ on two classes, namely $[S^0 \hookrightarrow M \to v_1^{-1}M]$ and $[S^{q-1} \xrightarrow{\alpha_1} M \to v_1^{-1}M]$.*

The methods of Section 4 show easily that these map isomorphically to $v_1^{-1}J_*(M)$.

We provide a little more detail about Miller's calculations. In [46] he obtained as an E_2-term for the localized ASS

$$v_1^{-1}E[h_{i,0} : i \geqslant 1] \otimes P[b_{i,0} : i \geqslant 1], \tag{5.2}$$

where $h_{i,0}$ corresponds to $[\xi_i]$ and has bigrading $(1, 2(p^i - 1))$, while $b_{i,0}$ corresponds to

$$\sum \frac{1}{p}\binom{p}{j}[\xi_i^j|\xi_i^{p-j}]$$

and has bigrading $(2, 2p(p^i - 1))$. The first step in obtaining this is to use a change-of-rings theorem to write the E_2-term as $v_1^{-1}\mathrm{Cotor}_{A(1)_*}(\mathbf{Z}_p, \mathbf{Z}_p)$, where $A(1)_*$ is the quotient $A_*/(\tau_0)$. This is then shown to be isomorphic to

$$\mathbf{Z}_p[v_1^{\pm 1}] \otimes \mathrm{Cotor}_{P(1)}(\mathbf{Z}_p, \mathbf{Z}_p),$$

where $P(1) = \mathbf{Z}_p[\xi_1, \xi_2, \ldots]/(\xi_1^p, \xi_2^p, \ldots)$, and this yields eq. (5.2).

In [47], the differential $d_2(h_{i,0}) = v_1 b_{i-1,0}$ is established in the localized ASS. This leaves $\mathbf{Z}_p[v_1^{\pm 1}] \otimes E[h_{1,0}]$ as $E_3 = E_\infty$, and this is easily translated into Proposition 5.4. Miller first established this differential in an algebraic spectral sequence converging to the E_2-term of the BP-based Novikov spectral sequence, and then showed that this implies the desired differential in the ASS by a comparison theorem.

Somewhat later, Crabb and Knapp ([21]) gave a proof of Theorem 5.1 for finite spectra X which was much less computational than those just discussed. Their proof utilized the solution of the Adams conjecture, and some refinements thereof. They let $\mathrm{Ad}^*(-)$ be the generalized cohomology theory corresponding to the fiber of $\psi^k - 1 : KO \to KO$. By Proposition 4.1, this is just our $v_1^{-1}J^*$. They prove the following result about stable cohomotopy, which by S-duality is equivalent to Theorem 5.1 for finite spectra.

THEOREM 5.5. *If X is a finite spectrum, then the Hurewicz morphism*

$$v_1^{-1}\pi_s^*(X; \mathbf{Z}/p^e) \xrightarrow{h} \mathrm{Ad}^*(X; \mathbf{Z}/p^e)$$

is bijective.

Their main weapon is a result of May and Tornehave which says that if $A^*(-)$ is the connective theory associated to $\mathrm{Ad}^*(-)$, and j is the morphism given by a solution of the Adams conjecture, then the composite

$$A^0(X) \xrightarrow{j} \pi_s^0(X) \xrightarrow{h} A^0(X)$$

is bijective for a connected space X. This is used to show that, for k sufficiently large, there is a stable Adams map $\Sigma^{ks(e)}M(p^e) \xrightarrow{A_e} M(p^e)$ which is in the image under j from $A^{-ks(e)}(M(p^e);\mathbf{Z}/p^e)$. This is then used to show that for any element x of $\pi_s^n(X;\mathbf{Z}/p^e)$, for L sufficiently large, $A_e^L x$ is in the image of the morphism j, and this easily implies injectivity in Theorem 5.5. Care is required throughout in distinguishing stable maps from actual maps.

6. The v_1-periodic unstable Novikov spectral sequence for spheres

In this section, we review the basic properties of the unstable Novikov spectral sequence (UNSS) based on the Brown–Peterson spectrum BP, and sketch the determination of the 1- and 2-lines of this spectral sequence when applied to S^{2n+1}. Then we show how the v_1-periodic UNSS is defined, and compute it completely for S^{2n+1}.

The spectrum BP associated to the prime p is a commutative ring spectrum satisfying $BP_* = \pi_*(BP) = \mathbf{Z}_{(p)}[v_1, v_2, \ldots]$ and $BP_*(BP) = BP_*[h_1, h_2, \ldots]$, with $|v_i| = |h_i| = 2p^i - 2$. The generators v_i are those of Hazewinkel, while h_i is conjugate to Quillen's generator t_i. We shall often abbreviate BP_*BP as Γ.

We will make frequent use of the right unit $\eta_R : BP_* \to BP_*BP$.

PROPOSITION 6.1. *$\eta_R(v_1) = v_1 - ph_1$, and*

$$\eta_R(v_2) = v_2 - ph_2 + (p^{p-1} - 1)h_1^p v_1 + (p+1)v_1^p h_1 + \sum_{i=2}^{p} a_i v_1^{p+1-i} p^i h_1^i,$$

where $a_i \in \mathbf{Z}$.

In writing $h_1^p v_1$ here, we have begun the practice of writing $\eta_R(v)h$ as hv. Thus $h_1^p v_1 \neq v_1 h_1^p$. Proposition 6.1 is easily derived from formulas relating v_i to m_i, and for $\eta_R(m_i)$. See [14, 2.6], where the following formula for the comultiplication $\Delta : BP_*BP \to BP_*BP \otimes BP_*BP$ is also computed. All tensor products in this and subsequent sections are over BP_*.

PROPOSITION 6.2. *$\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1$, and*

$$\Delta(h_2) = h_2 \otimes 1 + 1 \otimes h_2 + \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} h_1^i \otimes h_1^{p-i} v_1 + h_1^p \otimes h_1.$$

Let $\mathbf{BP}_n$ denote the nth space in the Ω-spectrum for BP. If X is a space, then a space $BP(X)$ is defined as $\lim_n \Omega^n(\mathbf{BP}_n \wedge X)$. Define $D^1(X)$ to be the fiber of the unit map $X \to BP(X)$, and inductively define $D^s(X)$ to be the fiber of $D^{s-1}(X) \to D^{s-1}(BP(X))$. This gives rise to a tower of fibrations

$$\cdots \to D^2(X) \to D^1(X) \to X.$$

The homotopy exact couple of this tower is the UNSS of X; if X is simply connected, it converges to the localization at p of $\pi_*(X)$.

In general, computing this spectral sequence can be extremely difficult, but if BP_*X is free as a BP_*-module, and cofree as a coalgebra, then it becomes somewhat tractable. Indeed, in such a case

$$E_2^{s,t}(X) \approx \mathrm{Ext}^s_{\mathcal{U}}(A_t, P(BP_*X)), \tag{6.1}$$

where A_t denotes a free BP_*-module on a generator of degree t, $P(-)$ denotes the primitives in a coalgebra, and $\mathcal{U}$ denotes the category of unstable Γ-comodules. We sketch a definition of the category $\mathcal{U}$ and the proof of eq. (6.1), referring the reader to [6, p. 744] or [8, §7] for more details. If M is a free BP_*-module, then $U(M)$ is defined to be the BP_*-submodule of $\Gamma \otimes M$ spanned by all elements of the form $h^I \otimes m$ satisfying the unstable condition

$$2(i_1 + i_2 + \cdots) < |m|, \tag{6.2}$$

where $h^I = h_1^{i_1} h_2^{i_2} \cdots$. If M is not BP_*-free, then $U(M)$ is defined as $\mathrm{coker}(U(F_1) \to U(F_0))$, where F_0 and F_1 are free BP_*-modules with $M = \mathrm{coker}(F_1 \to F_0)$. We define $U^s(M)$ by iterating $U(-)$. The category $\mathcal{U}$ consists of BP_*-modules equipped with morphisms $M \xrightarrow{\psi} U(M)$, $U(M) \xrightarrow{\delta} U^2(M)$, and $U(M) \xrightarrow{\varepsilon} M$ satisfying certain properties. The unstable condition (6.2) is analogous to the one for unstable right modules over the Steenrod algebra, but its proof relies on deep work of Ravenel and Wilson in [54].

The category $\mathcal{U}$ is abelian. We abbreviate $\mathrm{Ext}^s_{\mathcal{U}}(A_t, N)$ to $\mathrm{Ext}^{s,t}_{\mathcal{U}}(N)$. These groups may be calculated as the homology groups of the unstable cobar complex $\overline{C}^{*,*}(N)$, defined by $\overline{C}^{s,t}(N) = U^s(N)_t$, with boundary $\overline{C}^s \xrightarrow{d} \overline{C}^{s+1}$ defined by

$$\begin{aligned} d[\gamma_1|\cdots|\gamma_s]m &= [1|\gamma_1|\cdots|\gamma_s]m \\ &\quad + \sum (-1)^j [\gamma_1|\cdots|\gamma_j'|\gamma_j''|\cdots|\gamma_s]m \\ &\quad + \sum (-1)^{s+1}[\gamma_1|\cdots|\gamma_s|\gamma']m'', \end{aligned}$$

where $\Delta(\gamma_j) = \sum \gamma_j' \otimes \gamma_j''$ and $\psi(m) = \sum \gamma' \otimes m''$.

We will use a reduced complex $C^{*,*}(N)$, which is chain equivalent to $\overline{C}^{*,*}(N)$. This is obtained from $\widetilde{U}(N) = \ker(U(N) \xrightarrow{\varepsilon} N)$ and its iterates $\widetilde{U}^s$ by $C^{s,t}(N) = \widetilde{U}^s(N)_t$. Finally, we illustrate how $d(v) = \eta_R(v) - v$ for $v \in BP_*$ comes into play. Suppose $\Delta(h) = h \otimes 1 + 1 \otimes h + \sum h' \otimes h''$ and $\psi(I) = 1 \otimes I$, and let $v, v' \in BP_*$. Then

$$\begin{aligned} d([vh]v'I) &= [1|vh]v'I - [vh|1]v'I - [v|h]v'I - \sum [vh'|h'']v'I + [vh|v']I \\ &= [\eta_R(v) - v|h]v'I - \sum [vh'|h'']v'I - [vh|\eta_R(v') - v']I. \end{aligned}$$

We abbreviate $C^{*,*}(BP_*(X))$ to $C^{*,*}(X)$.

The first result about the UNSS, both historically and pedagogically, is the following, which appeared in [8, 9.12]. We repeat their proof because it gives a good first example of working with the unstable cobar complex. Recall that $q = 2(p-1)$.

THEOREM 6.3. *Let p be an odd prime. If $k > 0$, then*

$$E_2^{1,2n+1+kq}\left(S^{2n+1}\right) \approx \mathbf{Z}/p^{\min(n,\nu_p(k)+1)}.$$

If $t \not\equiv 2n+1 \bmod q$, or if $t < 2n+1$, then $E_2^{s,t}(S^{2n+1}) = 0$.

PROOF. Since $|v_i|$ and $|h_i|$ are divisible by q, all nonzero elements in $BP_*(S^{2n+1})$ have degree congruent to $2n+1 \bmod q$, and so the only possible nonzero elements in $E_2^{s,t}(S^{2n+1})$ occur when $t \equiv 2n+1 \bmod q$, and $t \geqslant 2n+1$. There is an injective chain map

$$C^{*,*}\left(S^{2n-1}\right) \to C^{*,*+2}\left(S^{2n+1}\right)$$

defined by $A \otimes \iota_{2n-1} \mapsto A \otimes \iota_{2n+1}$, corresponding to the double suspension homomorphism of homotopy groups. Since the boundaries in $C^1(S^{2n-1})$ are sent bijectively to those in $C^1(S^{2n+1})$, the morphism $E_2^{1,t}(S^{2n-1}) \to E_2^{1,t+2}(S^{2n+1})$ is injective.

We quote a result, originally due to Novikov (but see [51, §5.3] for the proof), about the stable groups: if n is sufficiently large, then

$$E_2^{1,2n+1+kq}\left(S^{2n+1}\right) \approx \mathbf{Z}/p^{\nu_p(k)+1}$$

with generator $d(v_1^k)\iota_{2n+1}/p^{\nu_p(k)+1}$. We will prove Theorem 6.3 by showing that if $n \leqslant \nu_p(k)+1$, then $d(v_1^k)/p^n$ is defined on S^{2n+1}, but not on S^{2n-1}.

We begin by observing

$$\begin{aligned} d\left(v_1^k\right)/p^n &= \left(\left(\eta_R(v_1)\right)^k - v_1^k\right)/p^n = \left((v_1 - ph_1)^k - v_1^k\right)/p^n \\ &= \sum_{j=1}^{k} (-1)^j \binom{k}{j} p^{j-n} v_1^{k-j} h_1^j. \end{aligned} \tag{6.3}$$

Note that the coefficients $\binom{k}{j}p^{j-n}$ have non-negative powers of p, since

$$\nu_p\left(\binom{k}{j}\right) + j \geqslant \nu_p(k) + 1 \geqslant n$$

for $j \geqslant 1$. Now we work mod terms that are defined on S^{2n-1}. This allows us to ignore terms in the sum (6.3) for $j < n$. For the other terms, we write $p^{j-n}h_1^j$ as $(v_1 - \eta_R v_1)^{j-n} h_1^n$, and note that when this is expanded by the binomial theorem, all terms except $v_1^{j-n}h_1^n$ may be ignored, since $(\eta_R v_1)^i h_1^n \iota_{2n-1} = h_1^n v_1^i \iota_{2n-1}$ satisfies eq. (6.2) when $i > 0$. Thus the sum (6.3) reduces to

$$\sum_{j=n}^{k} (-1)^j \binom{k}{j} v_1^{k-n} h_1^n = -\sum_{j=0}^{n-1} (-1)^j \binom{k}{j} v_1^{k-n} h_1^n,$$

since

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}=0.$$

If $j>0$ in the right-hand sum, then $\binom{k}{j}$ is divisible by p, and then ph_1^n can be written as $v_1h_1^{n-1}-h_1^{n-1}v_1$, so that the term is defined on S^{2n-1}. Thus, mod S^{2n-1}, (6.3) reduces to $-v_1^{k-n}h_1^n$. This class is not defined on S^{2n-1}. □

If $p=2$, a similar argument establishes the following result.

THEOREM 6.4. *If $p=2$, then, for $u>0$,*

$$E_2^{1,2n+1+u}(S^{2n+1})\approx\begin{cases}0, & u \text{ odd},\\ \mathbf{Z}/2, & \nu_2(u)=1,\\ \mathbf{Z}/4, & u=4,\\ \mathbf{Z}/2^{\min(n,\nu_2(u)+1)}, & u\equiv 0 \bmod 4, \text{ and } u>4.\end{cases}$$

If $u=2k$ in the three nonzero cases, then the generators are, respectively, $d(v_1^k)/2$, $d(v_1^2)/4$, and $d(v_1^k+2^{\nu_2(k)+1}v_1^{k-3}v_2)/2^{\nu_2(k)+2}$.

When p is odd, the element

$$-\left(d\left(v_1^k\right)/p^j\right)\iota_{2n+1}\in E_2^{1,2n+1+kq}\left(S^{2n+1}\right)$$

is denoted $\alpha_{k/j}$. If $j=1$, this will frequently be shortened to α_k. We note the following from the proof of Theorem 6.3.

PROPOSITION 6.5. *If $n\geqslant j$, then $\alpha_{k/j}\iota_{2n+1}=v_1^{k-j}h_1^j\iota_{2n+1}$ mod terms defined on S^{2j-1}.*

Next we cull from [5] information about unstable elements in $E_2^{2,*}(S^{2n+1})$, which form a subgroup which we shall denote by $\widetilde{E}_2^{2,*}(S^{2n+1})$. By "unstable", we mean an element in the kernel of the iterated suspension. The main theorem of [5] is the following.

THEOREM 6.6. *Let p be an odd prime, and let $t=\nu_p(a)$. Then*

$$\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})\approx\begin{cases}\mathbf{Z}/p^n & \text{if } n\leqslant t+1,\\ \mathbf{Z}/p^{t+1} & \text{if } t+1\leqslant n<a-t,\\ \mathbf{Z}/p^{a-n} & \text{if } a-t-1\leqslant n<a.\end{cases}$$

The homomorphism $\widetilde{E}_2^{2,qa+2n-1}(S^{2n-1})\xrightarrow{\Sigma^2}\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})$ is

$$\begin{cases}\text{injective} & \text{if } n\leqslant t+1,\\ \cdot p & \text{if } t+1<n<a-t,\\ \text{surjective} & \text{if } a-t\leqslant n<a.\end{cases}$$

Let $m = \min(n, a - t - 1)$. *Then* p^j *times the generator of* $\widetilde{E}_2^{2,qa+2n+1}(S^{2n+1})$ *is* $h_1 \otimes v_1^{a-m+j-1} h_1^{m-j} \iota_{2n+1}$ *mod terms defined on* $S^{2(m-j)-1}$.

This is illustrated in the chart below, where we list just leading terms, an element connected to one just below it by a vertical line is p times that element, and elements at the same horizontal level are related by the iterated double suspension homomorphism. We omit the subscript from h_1 and v_1, and the $\otimes$.

$$
\begin{array}{ccccc}
S^3 & S^5 & & S^{2t+3} & S^{2t+5} \\
\hline
hv^{a-2}h & hv^{a-2}h & \cdots & hv^{a-2}h & \\
 & | & & | & \\
 & hv^{a-3}h^2 & & hv^{a-3}h^2 & hv^{a-3}h^2 \\
 & & & | & | \\
 & & & \vdots & \vdots \\
 & & & | & | \\
 & & & hv^{a-t-2}h^{t+1} & hv^{a-t-2}h^{t+1} \\
 & & & & | \\
 & & & & hv^{a-t-3}h^{t+2} \\
 & & & & & \ddots
\end{array}
$$

$$
\begin{array}{cccc}
S^{2(a-t)-1} & S^{2(a-t)+1} & & S^{2a-1} \\
\hline
hv^{2t}h^{a-2t-1} & & & \\
| & & & \\
hv^{2t-1}h^{a-2t} & hv^{2t-1}h^{a-2t} & & \\
| & | & & \\
\vdots & \vdots & \ddots & \\
hv^{t}h^{a-t-1} & hv^{t}h^{a-t-1} & \cdots & hv^{t}h^{a-t-1}
\end{array}
$$

The proof of Theorem 6.6 requires results about Hopf invariants which we will address shortly. We begin by describing a plausibility argument for it using only elementary ideas about the unstable cobar complex. We continue to omit the subscript of h_1 and v_1.

(i) The lead term $h \otimes v^{a-n+j-1}h^{n-j}$ does not pull back to $S^{2(n-j)-1}$ because $h^{n-j}\iota_{2(n-j)-1}$ does not satisfy the unstable condition.

(ii) By Proposition 6.1, $ph = v - \eta_R(v)$, a fact which we will begin to use frequently. It implies that

$$p \cdot h \otimes v^{a-n+j-1}h^{n-j} = h \otimes v^{a-n+j}h^{n-j-1} - h \otimes v^{a-n+j-1}h^{n-j-1}v.$$

The second term desuspends below $S^{2(n-j)-1}$, and the first term is the next term up the unstable tower.

(iii) We show that Σ^2 applied to the element of order p on S^{2n+1} is a boundary when $n \geqslant t+1$. To do this, we give a more precise description of this element of order p as $d(v^{a-n-1}h^{n+1})\iota_{2n+1}$. This clearly double suspends to the boundary $d(v^{a-n-1}h^{n+1}\iota_{2n+3})$. Note how we had to wait until S^{2n+3} in order to put the ι inside the $d(-)$, since $h_{n+1}\iota_{2n+1}$ does not satisfy the unstable condition. It remains to show that the lead term is correct, which is the content of the following proposition.

PROPOSITION 6.7. *If $t = \nu_p(a)$, then*

$$d\left(v^{a-n-1}h^{n+1}\right) \equiv h \otimes v^{a+t-n-1}h^{n-t}$$

mod terms defined on $S^{2(n-t-1)+1}$.

PROOF. Replacing v by $ph + \eta_R(v)$ implies

$$v^{a-n-1}h^{n+1} \equiv p^{a-n-1}h^a \quad \left(\operatorname{mod} S^{2n+1}\right).$$

Since boundaries on S^{2n+1} desuspend to $S^{2(n-t)-1}$, we obtain

$$d\left(v^{a-n-1}h^{n+1}\right) \equiv p^{a-n-1}d\left(h^a\right)$$

mod the indeterminacy stated in the proposition. Now

$$d(h^a) = \sum \binom{a}{j} h^j \otimes h^{a-j},$$

and, since

$$\nu_p\left(\binom{a}{j}\right) \geqslant t+2-j \quad \text{for } j>1,$$

we find

$$p^{a-n-1}d\left(h^a\right) \equiv sp^{a-n-1+t}h \otimes h^{a-1} \equiv sh \otimes v^{a+t-n-1}h^{n-t}$$

mod terms defined on $S^{2(n-t-1)+1}$. Here $a = sp^t$ with s not a multiple of p, and we have freely replaced ph by $v - \eta_R(v)$. □

The main detail in the proof of Theorem 6.6 which is lacking in the plausibility argument above is an argument for why these are the only unstable elements on the 2-line. For this, we need the following result, which will also be useful in other contexts.

THEOREM 6.8. (i) *There is an unstable Γ-comodule $W(n)$ and an exact sequence*

$$\xrightarrow{P_2} E_2^{s,t-1}\left(S^{2n-1}\right) \xrightarrow{\Sigma^2} E_2^{s,t+1}\left(S^{2n+1}\right) \xrightarrow{H_2} \mathrm{Ext}_{\mathcal{U}}^{s-1,t-1}\left(W(n)\right)$$
$$\xrightarrow{P_2} E_2^{s+1,t-1}\left(S^{2n-1}\right). \tag{6.4}$$

(ii) *$W(n)$ is a free module over BP_*/p on classes x_{2p^in-1} for $i > 0$ with coaction*

$$\psi(x_{2p^kn-1}) = \sum_i p^{k-i} h_{k-i}^{np^i} \otimes x_{2p^in-1}.$$

(iii) $\mathrm{Ext}_{\mathcal{U}}^0(W(n)) \approx \mathbf{Z}_p[v_1]x_{2pn-1}$.
(iv) *If $z \in \mathrm{Ext}_{\mathcal{U}}(W(n))$ is represented by $\sum \gamma_k \otimes x_{2p^kn-1}$, then*

$$P_2(z) = d\left(\sum \gamma_k \otimes p^{k-1} h_k^n\right) \otimes \iota_{2n-1}.$$

(v) *Every element $x \in E_2^s(S^{2n+1})$ may be represented, mod terms which desuspend to S^{2n-1}, by a cycle of the form*

$$\sum \gamma_k \otimes p^{k-1} h_k^n \otimes \iota_{2n+1},$$

with $\gamma_k \in C^*(A_{2p^kn-1} \otimes \mathbf{Z}_p)$. Then

$$H_2(x) = \sum \gamma_k \otimes x_{2p^kn-1}.$$

Recall in (v) that A_t is the free BP_*-module on a generator of degree t, and that $C^*(-)$ denotes the reduced unstable cobar complex.

We provide a bare outline of the proof, beginning with the construction from [9]. There is a nonabelian category G of unstable Γ-coalgebras, and a notion of Ext_G such that if BP_*X is a free BP_*-module of finite type, then $E_2(X)$ (of the UNSS) is $\mathrm{Ext}_G(BP_*X)$. Letting $P_G(-)$ denote the primitives in G, one finds that if M is an object of G, then $P_G(M)$ is in the category $\mathcal{U}$. By considering an appropriate double complex, one can construct a composite functor spectral sequence converging to $\mathrm{Ext}_G(M)$ with

$$E_2^{p,q} \approx \mathrm{Ext}_{\mathcal{U}}^p\left(R^qP_G(M)\right).$$

Here R^qP_G denotes the qth right derived functor of P_G. If M satisfies $R^qP_GM = 0$ for $q > 1$, then the spectral sequence has only two nonzero columns, and reduces to an exact sequence

$$\to \mathrm{Ext}_{\mathcal{U}}^s(P_GM) \to \mathrm{Ext}_G^s(M) \to \mathrm{Ext}_{\mathcal{U}}^{s-1}\left(R^1P_GM\right) \to . \tag{6.5}$$

This will be the case when $M = BP_*(\Omega S^{2n+1})$. One verifies that $P_GBP_*(\Omega S^{2n+1}) \approx A_{2n}$, and that $R^1P_GBP_*(\Omega S^{2n+1})$ is the comodule $W(n)$ described in Theorem 6.8, and the exact sequence (6.5) reduces to the sequence (6.4) in this case. The descriptions of the morphisms in (iv) and (v) are obtained in [3] using an alternate construction of the exact sequence. Part (iii) is proved in [5, p. 535] by studying explicit cycles.

Now we complete our observations on the proof of Theorem 6.6. If x is any nonzero unstable element on the 2-line, then there must be k and n so that

$$\Sigma^{2k}x \neq 0 \in \ker\left(E_2^{2,*}\left(S^{2n-1}\right) \xrightarrow{\Sigma^2} E_2^{2,*+2}\left(S^{2n+1}\right)\right).$$

Then by Theorem 6.8, there must be an element $v_1^e x_{2pn-1} \in \mathrm{Ext}^0_{\mathcal{U}}(W(n))$ such that

$$P_2(v_1^e x_{2pn-1}) = d(v_1^e h_1^n)\iota_{2n-1} = \Sigma^{2k}x.$$

But this is exactly the description of the unstable elements on the 2-line which was given in the third part of our plausibility argument for Theorem 6.6.

When $p = 2$, the discussion above about the unstable elements on the 2-line goes through almost without change, as described in [10, pp. 482–484]. The result is that if $n \leqslant a - \nu_2(a) - 2$, then

$$\widetilde{E}_2^{2,2n+1+2a}\left(S^{2n+1}\right) \approx \begin{cases} \mathbf{Z}/2 & \text{if } a \text{ is odd,} \\ \mathbf{Z}/2^{\min(\nu_2(a)+2,n)} & \text{if } a \text{ is even.} \end{cases}$$

The orders when a is even are 1 larger than in the odd primary case because v_2 can be used to obtain 1 additional desuspension.

Now we construct the v_1-periodic UNSS, following [4] for the most part. In [7] a UNSS converging to $\mathrm{map}_*(Y,X)$ was constructed. If $Y = M^n(p^e)$, the E_2-term is the homology of $C^*(P(BP_*(X))) \otimes \mathbf{Z}/p^e$. The Adams map A induces

$$\mathrm{UNSS}\left(\mathrm{map}_*\left(M^n(p^e), X\right)\right) \xrightarrow{A^*} \mathrm{UNSS}\left(\mathrm{map}_*\left(M^{n+s(e)}(p^e), X\right)\right),$$

where $s(e)$ is as in eq. (2.1). By [35], on E_2 this is just multiplication by a power of v_1 after iterating sufficiently. As in our definition of $v_1^{-1}\pi_*(-)$, we define the v_1-periodic UNSS of X by

$$v_1^{-1}E_r^{*,*}(X) = \operatorname*{dirlim}_{e,k} E_r^{*,*+1}\left(\mathrm{map}_*\left(M^{ks(e)}(p^e), X\right)\right).$$

The direct system over e utilizes the maps $\rho : M^n(p^{e+1}) \to M^n(p^e)$ used in Section 1, and the shift of one dimension is done for the same reason as in our definition of $v_1^{-1}\pi_*(-)$. Similarly to Proposition 2.4, we have

PROPOSITION 6.9. *On the category of spaces with H-space exponents, there is a natural transformation from the UNSS to the v_1-periodic UNSS.*

One of the main theorems of [4] is the following determination of the v_1-periodic UNSS of S^{2n+1}.

THEOREM 6.10. *Let p be an odd prime. The v_1-periodic UNSS of S^{2n+1} collapses from E_2 and satisfies*

$$v_1^{-1}E_2^{s,2n+1+u}\left(S^{2n+1}\right) \approx \begin{cases} \mathbf{Z}/p^{\min(n,\nu_p(a)+1)} & \text{if } s = 1 \text{ or } 2, \text{ and } u = qa, \\ 0 & \text{otherwise.} \end{cases}$$

The morphism $E_2^{s,t}(S^{2n+1}) \to v_1^{-1}E_2^{s,t}(S^{2n+1})$ is an isomorphism if $s = 1$ and $t > 2n+1$, while for $s = 2$ it sends the unstable towers injectively, and bijectively unless $n \geqslant a - \nu_p(a) - 1$, where $t = 2n+1+qa$.

PROOF. It is readily verified that the elements on the 1- and 2-lines described in Theorems 6.3 and 6.6 form v_1-periodic families. The main content of this theorem is that there is nothing else which is v_1-periodic.

In order to prove this, we use a v_1-periodic version of the double suspension sequence (6.4). It is proved in [4] that the morphisms of (6.4) behave nicely with respect to v_1-action, yielding an exact sequence

$$\begin{aligned} &\xrightarrow{P_2} v_1^{-1}E_2^{s,t-1}\left(S^{2n-1}\right) \xrightarrow{\Sigma^2} v_1^{-1}E_2^{s,t+1}\left(S^{2n+1}\right) \\ &\xrightarrow{H_2} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s-1,t-1}\left(W(n)\right) \xrightarrow{P_2} . \end{aligned} \tag{6.6}$$

By Theorem 6.8(ii), there is a spectral sequence converging to $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(W(n))$ with

$$E_1^{s,t} \approx \bigoplus_{i\geqslant 1} v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}(A_{2p^in-1} \otimes \mathbf{Z}_p). \tag{6.7}$$

There is a short exact sequence given by the universal coefficient theorem

$$\begin{aligned} 0 &\to v_1^{-1}E_2^{s,t}\left(S^n\right) \otimes \mathbf{Z}_p \to v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}(A_n \otimes \mathbf{Z}_p) \\ &\to \mathrm{Tor}\left(v_1^{-1}E_2^{s-1,t}\left(S^n\right), \mathbf{Z}_p\right) \to 0. \end{aligned} \tag{6.8}$$

We will use eqs. (6.6), (6.7), and (6.8) to show inductively that there are no unexpected elements in $v_1^{-1}E_2(S^{2n+1})$. But first we show how the known elements fit into this framework. By (6.8), each summand in $v_1^{-1}E_2^1(S^n)$ gives two $\mathbf{Z}_p$'s, called stable, in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(A_n \otimes \mathbf{Z}_p)$, and similarly each summand in $v_1^{-1}E_2^2(S^n)$ gives two summands in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}(A_n \otimes \mathbf{Z}_p)$, called unstable. We claim that

$$v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t}\left(W(n)\right) \approx \begin{cases} \mathbf{Z}_p & \text{if } s = 0 \text{ or } 1, \text{ and } t \equiv 2n-1 \bmod q, \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

In [4, p. 57], the relationship between (6.9) and (6.7) is discussed: in the spectral sequence (6.7), stable classes from the $(i+1)$-summand hit unstable classes from the i-summand, yielding in $v_1^{-1}E_\infty$ only the stable classes from the 1-summand. These are the elements described in eq. (6.9). On the other hand, in the exact sequence (6.6), let $t = 2n + kq$ with $e = \nu_p(k)$. If $e < n-1$, then Σ^2 is $\mathbf{Z}/p^e \xrightarrow{\cdot p} \mathbf{Z}/p^e$ when $s = 2$, yielding the elements in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t-1}(W(n))$ for $s = 0$ and 1, while if $e \geqslant n-1$, then for $s = 1$ and 2, Σ^2 is $\mathbf{Z}/p^{n-1} \hookrightarrow \mathbf{Z}/p^n$, also yielding elements in $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{s,t-1}(W(n))$ for $s = 0$ and 1.

It seems useful for this proof to have one more bit of input, namely the result for the v_1-periodic stable Novikov spectral sequence, which can be defined as a direct limit over e of stable Novikov spectral sequences of $M(p^e)$.

THEOREM 6.11 ([13, §2]). *There is a v_1-periodic stable Novikov spectral sequence for S^0, satisfying*

$$v_1^{-1}E_r^{s,t}(S^0) = \operatorname*{dirlim}_n v_1^{-1}E_r^{s,t+2n+1}(S^{2n+1}),$$

and

$$v_1^{-1}E_2^{s,t}(S^0) \approx \begin{cases} \mathbf{Z}/p^{\nu_p(t)+1} & \text{if } s = 1 \text{ and } t \equiv 0 \bmod q, \\ 0 & \text{otherwise.} \end{cases}$$

We prove by induction on s that, for all n, $v_1^{-1}E_2^s(S^{2n+1})$, $v_1^{-1}E_2^{s+1}(S^{2n+1})$, and $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^s(W(n))$ contain only the elements described in Theorem 6.10 and eq. (6.9). This is easily seen to be true when $s = 0$, where we know all groups completely. Assume it is true for all $s < \sigma$. Then $v_1^{-1}E_2^\sigma(S^{2n+1})$ contains no unexpected elements because $\sigma = (\sigma - 1) + 1$. If $v_1^{-1}E_2^{\sigma+1}(S^{2n+1})$ contains an unexpected element, then some $v_1^{-1}E_2^{\sigma+1}(S^{2L+1})$ must contain one in $\ker(\Sigma^2)$ by Theorem 6.11. Such an element must be $P_2(x)$, where x is an unexpected element of $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^{\sigma-1}(W(n))$, but no such element exists by our induction hypothesis. Finally, $v_1^{-1}\mathrm{Ext}_{\mathcal{U}}^\sigma(W(n))$ contains no unexpected elements by (6.7) and (6.8) since, as just established, $v_1^{-1}E_2^\sigma(S^{2m+1})$ and $v_1^{-1}E_2^{\sigma+1}(S^{2m+1})$ contain no unexpected elements for any m. □

The UNSS and v_1-periodic UNSS are considerably more complicated at the prime 2 than at the odd primes, but the v_1-periodic UNSS of S^{2n+1} is still completely understood. We shall not discuss it in detail because most of our applications in this paper will be at the odd primes. The reader desiring more detail is referred to [4], which gives a chart with UNSS names of the elements.

We reproduce in figs. 13 and 14 the charts from [10, p. 488] of the v_1-periodic UNSS of S^{2n+1} at the prime 2. Here "3" means $\mathbf{Z}/2^3$, and "ν" means $\mathbf{Z}/2^\nu$, where

$$\nu = \min\left(\nu_2(8k+8) + 1, n\right).$$

Differentials emanating from a summand of order greater than 2 are nonzero only on a generator of the summand. Note how $\mathbf{Z}/8$ in periodic homotopy is obtained as an extension by the $\mathbf{Z}_2$ in filtration 3 of the elements in the 1-line group which are divisible by 2 in a $\mathbf{Z}/8$.

Figure 14 applies to S^{2n+1} when $n \equiv 1$ or $2 \bmod 4$, with $n > 2$. The reader is referred to [10, p. 487] for the minor changes required when $n \leqslant 2$. In fig. 14, the dotted differential is present if and only if $\nu = n$. In both charts, the left η-action on the $\mathbf{Z}/2^\nu$ on line 1, which is usually indicated by positively sloping solid lines, is indicated by the dotted line if $n < \nu(8k+8) + 1$.

The argument establishing these charts appears in [4]. Note that $v_1^{-1}E_4^{s,*}(S^{2n+1}) = 0$ if $s > 4$, and hence no higher differentials are possible.

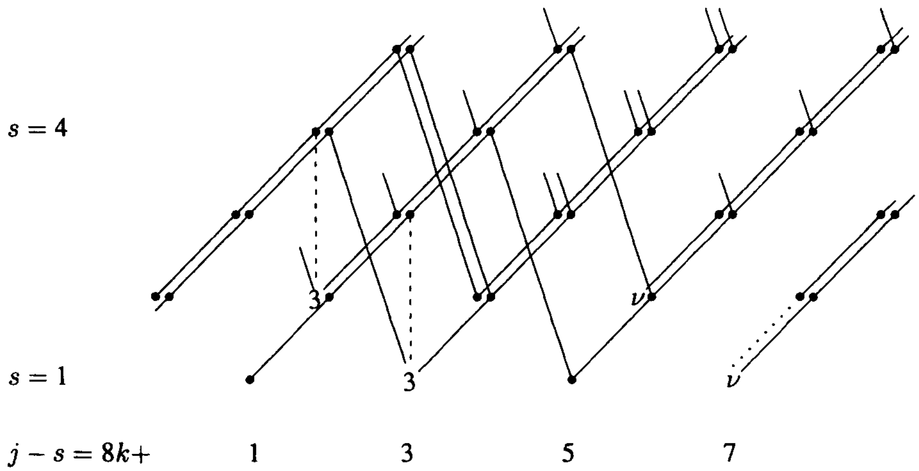


Figure 13. $v_1^{-1}E_2^{s,2n+1+j}(S^{2n+1})$, $n \equiv 0, 3 \bmod 4$, $p = 2$.

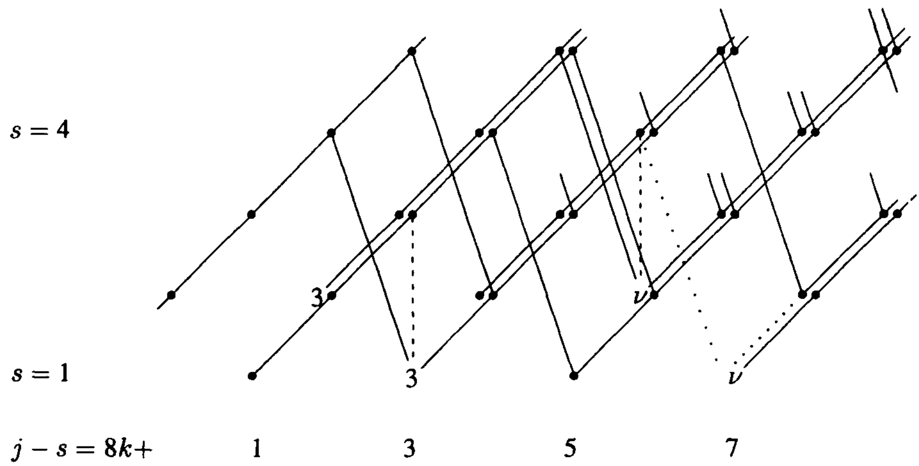


Figure 14. $v_1^{-1}E_2^{s,2n+1+j}(S^{2n+1})$, $n \equiv 1, 2 \bmod 4$, $n > 2$, $p = 2$.

7. v_1-periodic homotopy groups of $\mathrm{SU}(n)$

In this section we show how the v_1-periodic UNSS determines the v_1-periodic homotopy groups of spherically resolved spaces. This relationship is particular nice when localized at an odd prime, and it is this case on which we focus most of our attention. At the end of the section, we discuss the changes required when $p = 2$. After proving the general result for spherically resolved spaces, we specialize to $\mathrm{SU}(n)$, where the result is, in some sense, explicit.

It is not clear that the v_1-periodic UNSS of a space X must converge to the p-primary v_1-periodic homotopy groups of X. There might be periodic homotopy classes which are not detected in the periodic UNSS because multiplication by v_1 repeatedly increases BP filtration. It is also possible that a v_1-periodic family in E_2 might support arbitrarily long differentials in the unlocalized spectral sequence, in which case it would exist in all $v_1^{-1}E_r$, but would not represent an element of periodic homotopy. We now show that neither of these anomalies can occur for a spherically resolved space, essentially because they cannot happen for an odd sphere, where the v_1-periodic homotopy groups are known to agree with the $v_1^{-1}E_2$-term.

DEFINITION 7.1. A space X is *spherically resolved* if there are spaces $X_0, \ldots, X_L$, with $X_0 = *$, $X_L = X$, and fibrations

$$X_{i-1} \to X_i \to S^{n_i} \tag{7.1}$$

with n_i odd, and algebra isomorphisms

$$H^*(X_i) \approx H^*(X_{i-1}) \otimes H^*(S^{n_i}).$$

The following result was stated as Theorem 1.1. It was proved in [4].

THEOREM 7.2. *If p is odd, and X is spherically resolved, then $v_1^{-1}E_2^{s,t}(X) = 0$ unless $s = 1$ or 2, and t is odd. The v_1-periodic UNSS collapses to the isomorphisms*

$$v_1^{-1}\pi_i(X) \approx \begin{cases} v_1^{-1}E_2^{1,i+1}(X) & \text{if } i \text{ is even,} \\ v_1^{-1}E_2^{2,i+2}(X) & \text{if } i \text{ is odd.} \end{cases}$$

PROOF. Each algebra $BP^*(X_i)$ is free, and so eq. (6.1) applies to give $E_2(X_i) \approx \mathrm{Ext}_{\mathcal{U}}(M(x_{n_1}, \ldots, x_{n_i}))$, where $M(-)$ denotes a free BP_*-module on the indicated generators. There are short exact sequences in $\mathcal{U}$

$$0 \to M(x_{n_1}, \ldots, x_{n_{i-1}}) \to M(x_{n_1}, \ldots, x_{n_i}) \to M(x_{n_i}) \to 0,$$

and hence long exact sequences

$$\to E_2^{s,t}(X_{i-1}) \to E_2^{s,t}(X_i) \to E_2^{s,t}(S^{n_i}) \to E_2^{s+1,t}(X_{i-1}) \to . \tag{7.2}$$

These exact sequences are compatible with the direct system of v_1-maps whose limit is the v_1-periodic groups. Thus there is a v_1-periodic version of (7.2), and hence by Theorem 6.10 and induction on i, $v_1^{-1}E_2^{s,t}(X) = 0$ unless $s = 1$ or 2, and t is odd. Thus $v_1^{-1}E_2(X) \approx v_1^{-1}E_\infty(X)$.

If $u > \max\{n_i\}$, $s = 1$ or 2, and $s + u$ is odd, there are natural edge morphisms $\pi_u(X) \to E_2^{s,s+u}(X)$, and these are compatible with the direct system of v_1-maps, giving morphisms $v_1^{-1}\pi_u(X) \to v_1^{-1}E_2^{s,s+u}(X)$. These yield a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
0 \longrightarrow v_1^{-1}\pi_{2k}(X_{i-1}) & \longrightarrow & v_1^{-1}\pi_{2k}(X_i) & \longrightarrow & v_1^{-1}\pi_{2k}(S^{n_i}) & \longrightarrow \\
\downarrow{\scriptstyle \phi_{i-1}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \psi} & \\
0 \longrightarrow v_1^{-1}E_2^{1,2k+1}(X_{i-1}) & \longrightarrow & v_1^{-1}E_2^{1,2k+1}(X_i) & \longrightarrow & v_1^{-1}E_2^{1,2k+1}(S^{n_i}) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccc}
\longrightarrow v_1^{-1}\pi_{2k-1}(X_{i-1}) & \longrightarrow & v_1^{-1}\pi_{2k-1}(X_i) & \longrightarrow & v_1^{-1}\pi_{2k-1}(S^{n_i}) & \longrightarrow 0 \\
\downarrow{\scriptstyle \phi'_{i-1}} & & \downarrow{\scriptstyle \phi'_i} & & \downarrow{\scriptstyle \psi'} & \\
\longrightarrow v_1^{-1}E_2^{2,2k+1}(X_{i-1}) & \longrightarrow & v_1^{-1}E_2^{2,2k+1}(X_i) & \longrightarrow & v_1^{-1}E_2^{2,2k+1}(S^{n_i}) & \longrightarrow 0.
\end{array} \tag{7.3}$$

The zeros at the ends of the $v_1^{-1}E_2$-sequence follow from the previous paragraph. The zero morphism coming into $v_1^{-1}\pi_{2k}(X_{i-1})$ follows from $v_1^{-1}E_2^0(S^{n_i}) = 0$ and ϕ_{i-1} being an isomorphism, which is inductively known. The zero morphism coming out from $v_1^{-1}\pi_{2k-1}(S^{n_i})$ follows since anything in the image must have filtration $\geqslant 2$, but, by induction, $v_1^{-1}\pi_{2k-2}(X_{i-1})$ is 0 above filtration 1.

Comparison of Theorems 4.2 and 6.10 shows that the groups related by ψ and by ψ' are isomorphic, and it is easy to see that ψ and ψ' induce the isomorphisms. (See [23] for reasons.) Since X_1 is a sphere, this comparison also shows that ϕ_1 and ϕ'_1 are isomorphisms, which starts the induction. Thus all ϕ_i and ϕ'_i are isomorphisms by induction and the 5-lemma. □

Theorem 7.2 is a nice result, but it still leaves the formidable task of calculating $v_1^{-1}E_2(X)$. In [6], Bendersky proved the following seminal result, whose proof we will discuss throughout much of the remainder of this section.

THEOREM 7.3. *If $k \geqslant n$, then in the UNSS $E_2^{1,2k+1}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, where $e_p(k,n)$ is as defined in Definition 1.2, and p is any prime.*

This allows us to easily deduce Theorem 1.3, now demoted to corollary status, which we restate for the convenience of the reader.

COROLLARY 7.4. *If p is odd, then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n)) \approx \mathbf{Z}/p^{e_p(k,n)}$, and $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))$ is an abelian group of the same order.*

PROOF OF COROLLARY. The first part of the corollary is a straightforward application of Theorems 7.2 and 7.3, once we know that the groups in Theorem 7.3 are v_1-periodic. This can be seen by observing how they arise, from exact sequences built from spheres, where the classes are all v_1-periodic. In [23], a slightly different proof of this part of the corollary was given, before the v_1-periodic UNSS had been hatched.

The exact sequence like the top row of (7.3) for the fibration

$$\mathrm{SU}(n-1) \to \mathrm{SU}(n) \to S^{2n-1} \tag{7.4}$$

implies, by induction on n, that $|v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n))| = |v_1^{-1}\pi_{2k}(\mathrm{SU}(n))|$. Indeed, the orders are equal for S^{2n-1} by Theorem 6.10, and so if they are equal for $\mathrm{SU}(n-1)$, then they will be equal for $\mathrm{SU}(n)$, since the alternating sum of the exponents of p in an exact sequence is 0. The fact that $\mathrm{SU}(2) = S^3$ starts the induction. □

In [23], an example of a noncyclic group $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n);3)$ was given, and in [13] it was shown that $v_1^{-1}\pi_{2k-1}(\mathrm{SU}(n);2)$ will often have many summands (in addition to a regular pattern of $\mathbf{Z}_2$'s).

In order to prove Theorem 7.3, it is convenient to work with the UNSS based on MU, rather than BP. This allows us to work with the ordinary exponential series, rather than its p-typical analogue. The facts about MU that we need are summarized in the following result.

PROPOSITION 7.5. (i) *$MU_*(MU)$ is a polynomial algebra over MU_* with generators B_i of grading $2i$ for $i > 0$. There are elements $\beta_i \in MU_{2i}(CP^\infty)$ which form a basis for $MU_*(CP^\infty)$ as an MU_*-module.*

(ii) *Let B denote the formal sum $1 + \sum_{i>0} B_i$. The coaction*

$$MU_*(CP^\infty) \xrightarrow{\psi} MU_*MU \otimes_{MU_*} MU_*(CP^\infty)$$

satisfies

$$\psi(\beta_n) = \sum_j (B^j)_{n-j} \otimes \beta_j.$$

Here $(B^j)_{n-j}$ denotes the component in grading $2(n-j)$ of the jth power of the formal sum B.

(iii) *There is a ring homomorphism $\bar{e} : MU_*(MU) \otimes \mathbf{Q} \to \mathbf{Q}$ satisfying*

- *$\bar{e}(B_i) = 1/(i+1)$.*
- *$\bar{e}(\eta_R(a)) = 0$ if $a \in MU_i$ with $i > 0$.*
- *$\bar{e}$ induces an injection $E_2^{1,2n+1+2k}(S^{2n+1}) \to \mathbf{Q}/\mathbf{Z}$.*

(iv) *The BP-based UNSS is the p-localization of the MU-based UNSS.*

PROOF. Part (i) is standard (e.g., [2]), while part (ii) is [2, 11.4]. Part (iii) is from [6]. There are elements $m_i \in MU_{2i} \otimes \mathbf{Q}$ such that $MU_*MU \otimes \mathbf{Q}$ is a polynomial algebra over $\mathbf{Q}$ on all m_i and $\eta_R(m_i)$. One defines $\bar{e}$ to be the ring homomorphism which sends m_i to $1/(i+1)$ and $\eta_R(m_i)$ to 0. The second property in (iii) is clear, and the first follows by conjugating [2, 9.4] to obtain

$$m_n = \sum \eta_R(m_i)\left(B^{i+1}\right)_{n-i},$$

and then applying $\bar{e}$ to obtain $\bar{e}(m_n) = \bar{e}(B_n)$.

One way to see the third property is to localize at p and pass to BP. Then $m_{p^i-1}^{MU}$ passes to m_i^{BP}, and so our $\bar{e}$ passes to that of [6, 4.3]. It is shown on [6, p. 751] that $\bar{e}_{BP}$ sends the p-local 1-line injectively, and this works for all p. □

Now we can state a general theorem which incorporates most of the work in proving Theorem 7.3. This theorem was stated without proof in [11, 3.10], where it was applied to $X = Sp(n)$, $p = 2$. We will outline the proof, which is a direct generalization of [6], later in this section.

THEOREM 7.6. *Suppose X is spherically resolved as in Definition 7.1 with $n_1 < n_2 < \cdots$, and L possibly infinite. Then $MU_*(X_k)$ is an exterior algebra over MU_* on classes $y_1, \dots, y_k$, with $|y_i| = n_i$. Let*

$$\overline{E}^{1,t}(X_k) = \ker\left(E_2^{1,t}(X_k) \to E_2^{1,t}(X_L)\right).$$

Let $\gamma_{k,j} \in MU_*(MU)$ be defined in terms of the coaction in $MU_*(X_i)$ by

$$\psi(y_k) = \sum_{j=1}^{k} \gamma_{k,j} \otimes y_j, \tag{7.5}$$

and let $b_{k,j} = \overline{e}(\gamma_{k,j}) \in \mathbf{Q}$. Then the matrix $B = (b_{k,j})$ is lower triangular with 1's on the diagonal. Let $C = (c_{k,j})$ be the inverse of B, and let

$$\omega_k(m) = \text{l.c.m.}\left\{\text{den}(c_{k,j}) : \ m \leqslant j \leqslant k\right\}.$$

Then $\operatorname{coker}(\overline{E}^{1,n_k}(X_{m-1}) \to \overline{E}^{1,n_k}(X_{k-1}))$ *is cyclic of order $\omega_k(m)$.*

Now we specialize to $\mathrm{SU}(n)$, where we need the following result.

PROPOSITION 7.7. *In the UNSS*

(i) $E_2^{s,t}(\mathrm{SU}) = 0$ *if* $s > 0$.
(ii) $E_2^{1,2k+1}(\mathrm{SU}(n)) = 0$ *if* $n > k$.
(iii) $E_2^{1,2k+1}(\mathrm{SU}(k)) \approx \mathbf{Z}/k!$.
(iv) *If $i < j \leqslant k$, the inclusion* $\mathrm{SU}(i) \to \mathrm{SU}(j)$ *induces an injection in* $E_2^{1,2k+1}$.
(v) SU *is spherically resolved as in Theorem 7.6 with* $X_k = \mathrm{SU}(k+1)$, $n_k = 2k+1$, *and if the MU-coaction on* SU *is as in eq. (7.5), then*

$$\sum_{k\geqslant j} \overline{e}(\gamma_{k,j})x^k = \left(-\log(1-x)\right)^j. \tag{7.6}$$

PROOF. Part (i) is a nontrivial consequence of the fact that SU is an H-space with torsion-free homology and homotopy. See [6, 3.1]. The coalgebra $MU_*(\mathrm{SU}(n))$ is cofree with primitives isomorphic to $MU_*(\Sigma CP^{n-1})$. Hence by (6.1)

$$E_2(\mathrm{SU}(n)) \approx \mathrm{Ext}_{\mathcal{U}}\left(MU_*\left(\Sigma CP^{n-1}\right)\right),$$

and so the fibrations (7.4) induce exact sequences in E_2. Part ii follows from part i and the exact sequence, and part iv is also immediate from the exact sequence.

Let $y_{2k+1} \in MU_{2k+1}(\mathrm{SU}(n))$ be the generator corresponding to

$$\Sigma\beta_k \in MU_{2k+1}(\Sigma CP^{n-1})$$

for $k < n$. Part (iii) is proved on [6, p. 748] by showing that the generator of $E_2^{0,2k+1}(\mathrm{SU})$ is of the form $k!y_{2k+1}+$ lower terms, so that

$$E_2^{0,2k+1}(\mathrm{SU}(k+1)) \to E_2^{0,2k+1}(S^{2k+1})$$

sends the generator of one $\mathbf{Z}$ to $k!$ times the generator of the other $\mathbf{Z}$. This implies part (iii).

By Proposition 7.5(ii) and the relationship of $MU_*(\mathrm{SU}(n))$ with $MU_*(\Sigma CP^{n-1})$ noted above, we find

$$\psi(y_{2k+1}) = \sum \left(B^j\right)_{k-j} \otimes y_{2j+1}.$$

By Proposition 7.5(iii), we have $\sum \overline{e}(B_i)x^{i+1} = -\log(1-x)$. These facts yield part (v). $\square$

Now we can prove Theorem 7.3. If $f(x)$ is a power series with constant term 0, let $[f(x)]$ denote the infinite matrix whose entries $a_{k,j}$ satisfy

$$f(x)^j = \sum_k a_{k,j}x^k.$$

One easily verifies that $[g(x)][f(x)] = [f(g(x))]$. Hence the inverse of the matrix $[-\log(1-x)]$ is $[1-e^{-x}]$. This observation, with Theorem 7.6 and Proposition 7.7, implies that there is a short exact sequence

$$0 \to E_2^{1,2k+1}(\mathrm{SU}(n)) \to E_2^{1,2k+1}(\mathrm{SU}(k)) \to \mathbf{Z}/\omega_k(n) \to 0$$

with middle group $\mathbf{Z}/k!$ and

$$\omega_k(n) = \text{l.c.m.}\left\{\operatorname{den}\left(\operatorname{coef}\left(x^k, (1-\mathrm{e}^{-x})^j\right)\right):\ n \leqslant j \leqslant k\right\}.$$

Thus $E_2^{1,2k+1}(\mathrm{SU}(n))$ is cyclic of order

$$\begin{aligned} &k!\ /\ \text{l.c.m.}\left\{\operatorname{den}\left(\operatorname{coef}\left(x^k, (\mathrm{e}^x-1)^j\right)\right):\ n \leqslant j \leqslant k\right\} \\ &\quad = \gcd\left\{\operatorname{coef}\left(\frac{x^k}{k!}, (\mathrm{e}^x-1)^j\right):\ n \leqslant j \leqslant k\right\}. \end{aligned}$$

Looking at exponents of p yields Theorem 7.3 by Proposition 7.5(iv).

It remains to prove Theorem 7.6, the notation of which we employ without comment. Define $b_{k,j}(m)$ for $m \leqslant k$ recursively by $b_{k,j}(k) = b_{k,j}$ and

$$b_{k,j}(m) = b_{k,j}(m+1) - b_{k,m}(m+1)b_{m,j}. \tag{7.7}$$

We begin by noting that if row reduction is performed on $(B|I)$ so as to get at each step one more diagonal of 0's below the main diagonal of B, we find that the entries $c_{k,j}$ of B^{-1} satisfy

$$c_{k,j} = \begin{cases} 0 & \text{if } j > k, \\ 1 & \text{if } j = k, \\ -b_{k,j}(j+1) & \text{if } j < k. \end{cases}$$

Then

$$\omega_k(m) = \max\bigl(\operatorname{ord}\bigl(b_{k,j}(j+1)\bigr):\ m \leqslant\ j \leqslant k\bigr). \tag{7.8}$$

Here and throughout this proof, ord$(-)$ refers to order in $\mathbf{Q}/\mathbf{Z}$.

Fix k, and let

$$\tau(m) = \bigl|\operatorname{coker}\bigl(\overline{E}^{1,n_k}(X_{m-1}) \to \overline{E}^{1,n_k}(X_m)\bigr)\bigr|.$$

We drop the subscript k from eqs. (7.7) and (7.8). The fibration $X_{k-1} \to X_k \to S^{n_k}$ implies that the generator $g(k-1)$ of $\overline{E}^{1,n_k}(X_{k-1})$ is

$$d(y_k) = \sum_{j<k} \gamma_{k,j} \otimes y_j.$$

Let $a_j(k-1) = \overline{e}(\gamma_{k,j}) = b_{k,j}$. By Proposition 7.5(iii), $\tau(k-1) = \operatorname{ord}(a_{k-1}(k-1))$. Then there is $\alpha \in MU_*$ so that $\tau(k-1)g(k-1) + d(\alpha y_{k-1})$ pulls back to a generator of $\overline{E}^{1,n_k}(X_{k-2})$. Write this generator as $\sum \gamma_j(k-2)y_j$. Then

$$\begin{aligned} a_j(k-2) &:= \overline{e}\bigl(\gamma_j(k-2)\bigr) = \tau(k-1)a_j(k-1) + \overline{e}(\alpha)b_{k-1,j} \\ &= \tau(k-1)\bigl(a_j(k-1) - a_{k-1}(k-1)b_{k-1,j}\bigr). \end{aligned}$$

Here we have used that $\overline{e}(\alpha) = -\overline{e}(d(\alpha))$, which follows from Proposition 7.5(iii).

This procedure can be continued until we obtain a generator $\sum \gamma_j(m-1)y_j$ of $\overline{E}^{1,n_k}(X_{m-1})$ with $a_j(m-1) := \overline{e}(\gamma_j(m-1))$ satisfying

$$a_j(m-1) = \tau(m)\bigl(a_j(m) - a_m(m)b_{m,j}\bigr)$$

and $\tau(m) = \operatorname{ord}(a_m(m))$. Now we prove by downward induction on m that $a_j(m-1) = \omega(m)b_j(m)$, the case $m = k$ being trivial.

$$\begin{aligned} a_j(m-1) &= \tau(m)\ \bigl(a_j(m) - a_m(m)b_{m,j}\bigr) \\ &= \tau(m)\ \omega(m+1)\bigl(b_j(m+1) - b_m(m+1)b_{m,j}\bigr) \\ &= \operatorname{ord}(a_m(m))\omega(m+1)b_j(m) \\ &= \operatorname{ord}\bigl(\omega(m+1)b_m(m+1)\bigr)\omega(m+1)b_j(m) \\ &= \max\bigl(\omega(m+1), \operatorname{ord}\bigl(b_m(m+1)\bigr)\bigr)b_j(m) \\ &= \omega(m)b_j(m) \end{aligned}$$

The portion of this string after the first line and not including the last factor shows that $\tau(m)\omega(m+1) = \omega(m)$, and hence $\omega(m) = \tau(m)\cdots\tau(k-1)$, which is a restatement of the desired conclusion of Theorem 7.6. □

Since each $v_1^{-1}\pi_i(\mathrm{SU}(n))$ occurs as a direct summand of some actual homotopy group of $\mathrm{SU}(n)$, the following result about the p-exponent of $\mathrm{SU}(n)$ is immediate from Theorem 1.3.

COROLLARY 7.8. *Let* $\exp_p(X)$ *denote the largest* e *such that, for some* i, $\pi_i(X)$ *has an element of order* p^e. *Then, if* p *is odd,*

$$\exp_p\big(\mathrm{SU}(n)\big) \geqslant e_p(n),$$

where $e_p(n) = \max\{e_p(k,n) : k \geqslant n\}$.

We conjecture that this bound is sharp, the main evidence being that the analogous statement is true for odd spheres, by [20]. The numbers $e_p(k,n)$ are explicit in Definition 1.2, and this formula can be used with a computer for specific calculations. However, this formula is not very tractable. In Section 8, we sketch how simple formulas for $v_1^{-1}\pi_{2k}(\mathrm{SU}(n))$ can be obtained for $n \leqslant p^2 - p$ without using Theorem 1.3, but rather by studying the exact sequences of UNSS E_2-terms.

Theorem 7.6 is valid when $p = 2$, but Theorems 7.2 and 1.3 must be modified, due to the more complicated form of the v_1-periodic UNSS for S^{2n+1} when $p = 2$, as discussed at the end of Section 6. We state below the 2-primary version of Theorem 7.2 which was proved in [4], but we refer the reader to [10] for the 2-primary analogue of Theorem 1.3.

THEOREM 7.9 ([6]). *Suppose* X *is spherically resolved and* $p = 2$. *Then*

- $v_1^{-1}E_2(X)$ *is generated as an* η*-module by elements with* $s = 1$ *or* 2. *Here* η *has* $(s,t) = (1,2)$.
- η *acts freely on elements with* $s > 2$.
- $v_1^{-1}E_4(X) = v_1^{-1}E_\infty(X)$, *and* $v_1^{-1}E_4^s(X) = 0$ *if* $s > 4$.
- *If the groups* $v_1^{-1}E_2^{1,t}(X)$ *are cyclic, then the* v_1*-periodic UNSS converges to* $v_1^{-1}\pi_*(X)$.

8. v_1-periodic homotopy groups of some Lie groups

In this section we focus on two examples. One uses UNSS methods to determine $v_1^{-1}\pi_*(G_2;5)$, while the other uses ASS methods to determine $v_1^{-1}\pi_*(F_4/G_2;2)$. Here G_2 and F_4 are the two simplest exceptional Lie groups. The first example is just one of many discussed in [14]. We also discuss how these UNSS methods can be used to give tractable formulas for $v_1^{-1}\pi_*(\mathrm{SU}(n);p)$ when p is odd and $n \leqslant p^2 - p$. We close by summarizing the status of the program, initially proposed by Mimura, of computing the v_1-periodic homotopy groups of all compact simple Lie groups.

Our first theorem concerns the v_1-periodic homotopy groups of certain sphere bundles over spheres, which appear frequently as direct factors of compact simple Lie groups localized at p, according to the decompositions given in [49].

THEOREM 8.1. *Let p be an odd prime, and let $B_1(p)$ denote an S^3-bundle over S^{2p+1} with attaching map α_1. Then the only nonzero v_1-periodic homotopy groups of $B_1(p)$ are*

$$v_1^{-1}\pi_{2p+qm-1}\big(B_1(p)\big) \approx v_1^{-1}\pi_{2p+qm}\big(B_1(p)\big) \approx \mathbf{Z}/p^{\min(p+1,1+\nu_p(m-p^{p-1}))}.$$

This is the case $k=1$ of [14, 2.1]. The spaces $B_1(p)$ were called $B(3,2p+1)$ in [14]. The following result follows immediately from the 5-local equivalence $G_2 \simeq B_1(5)$.

COROLLARY 8.2.

$$v_1^{-1}\pi_i(G_2;5) \approx \begin{cases} \mathbf{Z}/5^{\min(6,1+\nu_5(i-5009))} & \textit{if } i \equiv 1 \bmod 8, \\ \mathbf{Z}/5^{\min(6,1+\nu_5(i-5010))} & \textit{if } i \equiv 2 \bmod 8, \\ 0 & \textit{otherwise.} \end{cases}$$

The following result is the central part of the proof of Theorem 8.1. Indeed, this theorem, 8.3, along with Theorem 6.3, gives the order of each group $E_2^{1,t}(B_1(p))$, and Theorem 8.5 shows the group is cyclic. Then Theorem 7.2 shows that this gives $v_1^{-1}\pi_*(B_1(p))$ when $*$ is even, and the proof of Corollary 7.4 shows that

$$\big|v_1^{-1}\pi_{2k-1}\big(B_1(p)\big)\big| = \big|v_1^{-1}\pi_{2k}\big(B_1(p)\big)\big|.$$

Finally, $v_1^{-1}\pi_*(B_1(p))$ is shown to be cyclic when $*$ is odd in Theorem 8.6.

THEOREM 8.3. *In the exact sequence*

$$\begin{aligned} 0 \to E_2^{1,qm+2p+1}\big(S^3\big) &\xrightarrow{i_*} E_2^{1,qm+2p+1}\big(B_1(p)\big) \\ &\xrightarrow{j_*} E_2^{1,qm+2p+1}\big(S^{2p+1}\big) \xrightarrow{\partial} E_2^{2,qm+2p+1}\big(S^3\big), \end{aligned}$$

the morphism ∂ is a surjection to $\mathbf{Z}/p$ unless

$$\nu_p(m) \geqslant p-1 \quad \textit{and} \quad m/p^{p-1} \equiv 1 \bmod p,$$

in which case it is 0.

We will use the double suspension Hopf invariant H_2 discussed in Theorem 6.8. We denote by H' the morphism

$$H' : E_2^2\big(S^{2n+1}\big) \xrightarrow{H_2} \mathrm{Ext}^1_{\mathcal{U}}\big(W(n)\big) \to E_2^1(M),$$

obtained by following H_2 by the stabilization. Here

$$E_2^s(M) \approx \mathrm{Ext}^s_{BP_*BP}(BP_*, BP_*/p)$$

denotes the E_2-term of the stable NSS for the mod p Moore spectrum M. We will eventually need the following facts about $E_2(M)$.

LEMMA 8.4. (i) $E_2(M)$ *is commutative.*

(ii) $v_1^k h_1 \neq 0 \in E_2^1(M)$.

(iii) *If* $x \in E_2(M)$, *then* $v_2 x = x v_2 = 0$.

(iv) $v_1 h_1^{k(p-1)+1} = v_1^{k(p-1)+1} h_1$ *in* $E_2(M)$.

(v) *If* $s \not\equiv 0 \bmod p$, *then* $\alpha_{sp^{e-1}/e} = -s v_1^{sp^{e-1}-1} h_1$ *in* $E_2(M)$.

PROOF. Part (i) is well known, part ii is [51, p 157], and part (iii) follows from [48, 2.10]. Part (iv) follows from Proposition 6.1, part (iii) of this lemma, and the fact that $ph_1 = 0 \in E_2(M)$. To prove part (v), we use 6.1 to expand $v_1^{sp^{e-1}}$, obtaining

$$\begin{aligned}\alpha_{sp^{e-1}/e} &= \frac{1}{p^e}\Big(\eta_R\big(v_1^{sp^{e-1}}\big) - \big(ph_1 + \eta_R(v_1)\big)^{sp^{e-1}}\Big)\\ &= -\sum_{j=1}^{sp^{e-1}} \binom{sp^{e-1}}{j} p^{j-e} h_1^j v_1^{sp^{e-1}-j}.\end{aligned}$$

All terms except $j = 1$ are divisible by p, and hence are 0. To insure that terms with j large are p times an admissible element, write $p^{j-e}h_1^j$ as $p(v_1 - \eta_R(v_1))^{j-e-1}h^{e+1}$. □

Now we begin the proof of Theorem 8.3. We begin with the case $\nu(m) < p - 1$. In this case,

$$\partial(\text{gen}) = \alpha_{m/\nu(m)+1} \otimes \alpha_1 \iota_3 \equiv -\alpha_{m/\nu(m)+1} \otimes h_1 \iota_3, \tag{8.1}$$

mod terms that desuspend to S^1. Here we have used [6, 4.9] and Proposition 6.5. By Proposition 6.5, the assumption that $\nu(m) < p-1$ implies that $\alpha_{m/\nu(m)+1}$ is defined on S^{2p-1}, and hence Theorem 6.8(v) implies that

$$H'\big(\partial(\text{gen})\big) = -\alpha_{m/\nu(m)+1} \neq 0,$$

where the last step uses parts v and ii of Lemma 8.4. Thus $\partial \neq 0$ in this case, as claimed.

Now we complete the proof of Theorem 8.3 by considering the case $\nu(m) \geqslant p - 1$. We let $s = m/p^{\nu(m)}$ and

$$\varepsilon = \begin{cases} 0 & \text{if } \nu(m) > p-1,\\ 1 & \text{if } \nu(m) = p-1.\end{cases}$$

We will establish the following string of equations in the next paragraph, and then we will further analyze whether these terms are 0 by studying their Hopf invariant. The following string is valid mod terms which desuspend to S^1.

$$\begin{aligned}\partial(\text{gen}) &= \alpha_{m/p} \otimes \alpha_1 \iota_3\\ &= p^{-p}\big(\eta_R(v_1^m) - \big(ph_1 + \eta_R(v_1)\big)^m\big) \otimes \alpha_1 \iota_3 \qquad (8.2)\\ &= -\sum_{j=1}^{m} \binom{m}{j} p^{j-p} h_1^j \otimes v_1^{m-j} \alpha_1 \iota_3\end{aligned}$$

$$= -p^{m-p}h_1^m \otimes \alpha_1\iota_3 - \varepsilon s h_1 \otimes v_1^{m-1}\alpha_1\iota_3 \tag{8.3}$$

$$= -v_1^{m-p}h_1^p \otimes \alpha_1\iota_3 - v_1^{m-p-1}h_1^p \otimes v_1h_1\iota_3 + \varepsilon s h_1 \otimes v_1^{m-1}h_1\iota_3 \tag{8.4}$$

$$= A + B + C, \tag{8.5}$$

where A, B, and C denote the three terms in the preceding line.

Line (8.2) follows from Propositions 6.5 and 6.1. Line (8.3) has been obtained by observing that in the sum all terms desuspend to S^1 except $j = m$ and, if $\nu(m) = p-1$, $j = 1$. To see this, we observe that we need to have a p to make $\alpha_1\iota_3$ desuspend. This factor will be present unless $j = 1$ and $\nu(m) = p-1$. The requirement that j be $\leqslant \frac{1}{2}$ times the degree of the symbols following h_1^j will only be a problem for large values of j. When j is large, write the term as

$$\binom{m}{j}p\big(v_1 - \eta_R(v_1)\big)^{j-p-1}h_1^{p+1} \otimes v_1^{m-j}\alpha_1\iota_3.$$

Since p times anything which is defined on S^3 desuspends to S^1, this desuspends to S^1 provided $p+1 \leqslant (p-1)(m-j+1)+1$, which simplifies to $1 \leqslant (p-1)(m-j)$, i.e. $j < m$. To obtain (8.4), we have rewritten the first term of eq. (8.3) as

$$-p\big(v_1 - \eta_R(v_1)\big)^{m-p-1}h_1^{p+1} \otimes \alpha_1\iota_3,$$

observed that when this is expanded, all terms except the first desuspend, and in that first term we write $ph_1 = v_1 - \eta_R(v_1)$.

We note first that, by Proposition 6.2, A is $d(h_2)$ mod S^1, and so $H'(A) = 0$. We can evaluate the Hopf invariant of B and C by Theorem 6.8(v); using Lemma 8.4, we obtain

$$H'(B+C) = (-1+\varepsilon s)v_1^{m-1}h_1.$$

Hence $H'(\partial(\text{gen})) = 0$ if and only if $-1+\varepsilon s \equiv 0 \bmod p$. Since H' is injective on $E_2^2(S^3)$, this completes the proof of Theorem 8.3. □

Now we settle the extension in the exact sequence of Theorem 8.3.

THEOREM 8.5. *The groups* $E_2^{1,qm+2p+1}(B_1(p))$ *in Theorem* 8.3 *are cyclic.*

PROOF. We will show that whenever $\ker(\partial) \neq 0$ in the exact sequence of 8.3, there is an element $z \in E_2^{1,qm+2p+1}(B_1(p))$ such that $j_*(z) = \alpha_m\iota_{2p+1}$, the element of order p, and $pz = i_*(\text{gen})$. Since $\partial(\alpha_m\iota_{2p+1}) = 0$ in these cases, there is $w\iota_3 \in C^{1,qm+2p+1}(S^3)$ such that $d(w\iota_3) = \alpha_m \otimes \alpha_1\iota_3$. Let

$$z = \alpha_m\iota_{2p+1} - w\iota_3.$$

Then z is a cycle, since $d(z) = \alpha_m \otimes \alpha_1\iota_3 - \alpha_m \otimes \alpha_1\iota_3$, and clearly $j_*(z)$ is as required. Since $p\alpha_m = d(v_1^m)$, we have

$$\begin{aligned} pz - d\big(v_1^m\iota_{2p+1}\big) &= d\big(v_1^m\big)\iota_{2p+1} - pw\iota_3 - d\big(v_1^m\big)\iota_{2p+1} + v_1^m\alpha_1\iota_3 \\ &= i_*\big((v_1^m\alpha_1 - pw)\iota_3\big). \end{aligned}$$

We show $(v_1^m\alpha_1 - pw)\iota_3 \neq 0 \in E_2^{1,qm+2p+1}(S^3)$ by noting from [7, §7] that the Hopf invariant

$$H_2 : E_2^1(S^3) \to \operatorname{Ext}^0(W(1))$$

factors through the mod p reduction of the unstable cobar complex. Thus

$$H_2\big((v_1^m\alpha_1 - pw)\iota_3\big) = H_2\big(v_1^m h_1\iota_3\big) = v_1^m \neq 0.$$

The second "=" uses Theorem 6.8(v), and the "$\neq$" uses 6.8(iii). □

The following result completes the proof of Theorem 8.1 according to the outline given after Corollary 8.2.

THEOREM 8.6. *In Theorem* 8.1, *the group* $v_1^{-1}\pi_{qm+2p-1}(B_1(p))$ *is cyclic.*

PROOF. We use the exact sequence in $v_1^{-1}\pi_*(-)$ of the fibration which defines $B_1(p)$. The cyclicity follows from that of $v_1^{-1}\pi_{qm+2p-1}(S^{2p+1})$ unless

$$\partial = 0 : v_1^{-1}\pi_{qm+2p}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+2p-1}\big(S^3\big). \tag{8.6}$$

If eq. (8.6) is satisfied, then

$$\circ\alpha_1 \neq 0 : v_1^{-1}\pi_{qm+2}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+2p-1}\big(S^{2p+1}\big)$$

by [23, 6.2], and

$$\partial\colon\ v_1^{-1}\pi_{qm+2}\big(S^{2p+1}\big) \to v_1^{-1}\pi_{qm+1}\big(S^3\big)$$

is an isomorphism of $\mathbf{Z}/p$'s by Theorem 8.3. Let G denote a generator of $v_1^{-1}\pi_{qm+2}(S^{2p+1})$, and let

$$Y \in v_1^{-1}\pi_{qm+2p-1}\big(B_1(p)\big)$$

project to $G \circ \alpha_1$. By [50, 2.1],

$$pY = i_*\big(\langle\partial G, \alpha_1, p\rangle\big) = \partial(G) \circ v_1 \neq 0.$$

□

It is shown in [49] that $B_1(p)$ is a direct factor of $\mathrm{SU}(n)_{(p)}$ if $p < n < 2p$, and hence $v_1^{-1}\pi_i(\mathrm{SU}(n);p)$ is given by Theorem 8.1 if $p < n < 2p$ and $i \equiv 1$ or $2 \bmod q$. This yields the following number theoretic result.

COROLLARY 8.7. *If p is an odd prime, $k \equiv 1 \bmod p-1$, and $p < n < 2p$, then the number $e_p(k,n)$ defined in* 1.2 *equals* $\min(p, \nu_p(k - p - p^p + p^{p-1})) + 1$.

The author has been unable to prove this result without the UNSS. In fact, the only tractable result for $e_p(k,n)$ which follows easily from Definition 1.2 seems to be that if

$n \leqslant p$ and $k \equiv n-1 \bmod p-1$, then $e_p(k,n) \geqslant \min(n-1, \nu_p(k-n+1)+1)$, which is proved using the Little Fermat Theorem as in [22, p. 792].

Using methods similar to those in our proof of Theorem 8.1, Yang ([58]) has proved the following tractable result for $v_1^{-1}\pi_*(\mathrm{SU}(n);p)$ when p is odd and $n \leqslant p^2-p$. Of course this can also be interpreted as a theorem about the numbers $e_p(k,n)$. We emphasize that the proof of Theorem 8.8 does not involve the use of Theorem 1.3.

THEOREM 8.8. *Suppose p is odd, $k = c+(p-1)d$ with $1 \leqslant c < p$, and*

$$c+(p-1)b+1 \leqslant n \leqslant c+(p-1)(b+1)$$

with $0 \leqslant b \leqslant p-1$. Define j by $1 \leqslant j \leqslant p$ and $d \equiv j \bmod p$. Then $v_1^{-1}\pi_{2k}(\mathrm{SU}(n);p)$ is cyclic of order p^e, with

$$e = \begin{cases} \min(c+(p-1)j,\ b+\nu(d-j)+1) \\ \qquad \text{if } b < c \text{ and } 1 \leqslant j \leqslant b, \\ \min(c+(p-1)j+1,\ b+\nu(d-j+(-1)^j j\binom{c}{j}p^{j(p-1)})) \\ \qquad \text{if } c \leqslant b \text{ and } 1 \leqslant j \leqslant b, \\ \min(c,\ b+1+\nu(d)) \\ \qquad \text{if } b < c \text{ and } b < j \leqslant p, \\ b \qquad \text{if } c \leqslant b \text{ and } b < j \leqslant p. \end{cases}$$

One can easily read off from Theorem 8.8 the precise value of the numbers $e_p(n)$ which appeared in Corollary 7.8, yielding the following result for the p-exponent of the space $\mathrm{SU}(n)$.

COROLLARY 8.9. *If p is odd and $n \leqslant p^2-p$, then*

$$\exp_p(\mathrm{SU}(n)) \geqslant e_p(n) = \begin{cases} n & \text{if } i(p-1)+2 \leqslant n \leqslant ip+1 \text{ for some } i, \\ n-1 & \text{otherwise.} \end{cases}$$

When $p = 2$, UNSS methods of computing $v_1^{-1}\pi_*(-;p)$ become more complicated because of the η-towers. Then ASS techniques become more useful, as they did for $v_1^{-1}\pi_*(G_2;2)$ in [29]. Here we show how to use ASS methods to determine $v_1^{-1}\pi_*(F_4/G_2;2)$. It is hoped that the result of the calculations of $v_1^{-1}\pi_*(G_2;2)$ and $v_1^{-1}\pi_*(F_4/G_2;2)$ might be combined to yield $v_1^{-1}\pi_*(F_4;2)$, but this involves one difficulty not yet resolved. Our main reason for including this example is to give a new illustration of this method.

The proof of the following theorem will consume most of the remainder of this paper. If G denotes an abelian group, then mG denotes the direct sum of m copies of G.

THEOREM 8.10. *Let $G(n)$ denote some group of order n.*

$$v_1^{-1}\pi_i(F_4/G_2;2) \approx \begin{cases} 4\mathbf{Z}_2 & i \equiv 0 \bmod 8,\\ \mathbf{Z}_8 \oplus \mathbf{Z}_8 \oplus \mathbf{Z}_2 & i \equiv 1 \bmod 8,\\ \mathbf{Z}/64 & i \equiv 2 \bmod 8,\\ 0 & i \equiv 3 \text{ or } 4 \bmod 8,\\ G(2^{\min(15,\nu(i-21)+4)}) & i \equiv 5 \bmod 8,\\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}/2^{\min(15,\nu(i-22)+4)} & i \equiv 6 \bmod 8,\\ 5\mathbf{Z}_2 & i \equiv 7 \bmod 8. \end{cases}$$

There is a fibration

$$S^{15} \xrightarrow{i} F_4/G_2 \xrightarrow{p} S^{23}, \tag{8.7}$$

derived in [29, 1.1]. Here and throughout this proof all spaces and spectra are localized at 2.

In [31], it was shown that for any spherically resolved space Y, there is a finite torsion spectrum X satisfying $v_1^{-1}\pi_*(Y) \approx v_1^{-1}J_*(X)$. In our case, we have

PROPOSITION 8.11. *There is a spectrum X such that*

(i) $v_1^{-1}J_*(X) \approx v_1^{-1}\pi_*(F_4/G_2)$, *and*
(ii) *there is a cofibration*

$$\Sigma^{15}P^{14} \to X \to \Sigma^{23+L}P^{22}, \tag{8.8}$$

where L equals 0 *or a large* 2*-power.*

We present in fig. 15 a chart which depicts an initial part of the ASS for $v_1^{-1}J_*(X)$ if X is as in Proposition 8.11 and $L = 0$. It depicts the direct sum of the spectral sequences for $v_1^{-1}J_*(\Sigma^{15}P^{14})$ and $v_1^{-1}J_*(\Sigma^{23}P^{22})$, together with one differential, which will be established in Proposition 8.13. The •'s are elements from P^{14}, while the ∘'s are from P^{22}. Charts such as these for $v_1^{-1}J_*(P^n)$ were derived in Section 4.

To see that fig. 15 is also valid when L in Proposition 8.11 is a large 2-power, we use the following result.

LEMMA 8.12. *If L is a large* 2*-power, then the attaching map*

$$\Sigma^{22+L}P^{22} \to v_1^{-1}\Sigma^{15}P^{14}$$

in Proposition 8.11 *has filtration $L/2 + 1$.*

Using results of [40], this implies that a resolution of $v_1^{-1}X$ can be formed from $v_1^{-1}\Sigma^{15}P^{14}$ and $\phi^{L/2}v_1^{-1}\Sigma^{23+L}P^{22}$, where ϕ^j increases filtrations by j, and this yields fig. 15.

PROOF OF LEMMA 8.12. Under S-duality, the generator corresponds to an element of $v_1^{-1}J_{L+7}(P^{14} \wedge D(P^{22}))$. This group is isomorphic to

$$v_1^{-1}J_{L+6}(P^{14} \wedge P_{-23}^{-2}). \tag{8.9}$$

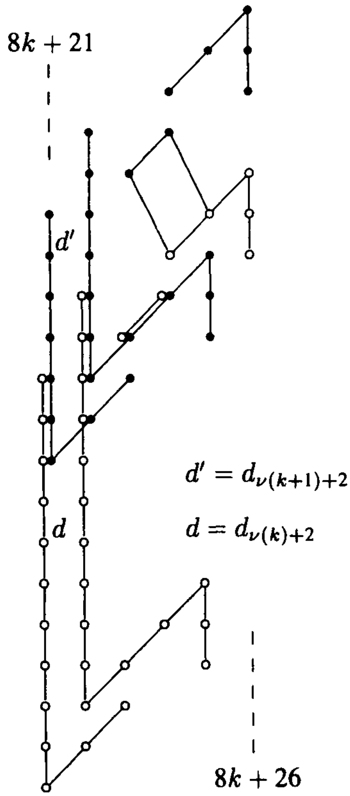


Figure 15. Initial chart for $v_1^{-1}\pi_*(F_4/G_2)$.

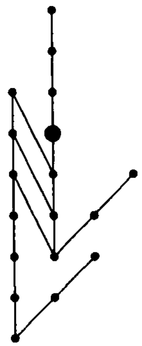


Figure 16. The generator of $v_1^{-1}J_{L+6}(P^{14} \wedge P_{-23}^{-2})$.

Using methods of [26], one can show that the relevant chart is as in fig. 16, where the class indicated with a bigger $\bullet$ is the generator of (8.9), and has filtration $L/2+1$.

Borel ([15]) showed that $\mathrm{Sq}^8(x_{15}) = x_{23}$ in $H^*(F_4;\mathbf{Z}_2)$. This implies that the attaching map in F_4/G_2 and in X is the Hopf map σ, and hence corresponds to the generator of (8.9). Thus the attaching map has filtration $L/2+1$. □

We can now establish the d_2-differentials in fig. 15.

PROPOSITION 8.13. *There are d_2-differentials as indicated in fig.* 15.

PROOF. This follows from the σ attaching map just observed, together with the observation that the first of the pair of elements that are related by the differential in fig. 15 are v_1-periodic versions of η_{23} and $\eta\sigma_{15}$. □

This d_2-differential implies there is a nontrivial extension in $v_1^{-1}\pi_{8k+26}(F_4/G_2)$ as follows.

PROPOSITION 8.14. $v_1^{-1}\pi_{8k+26}(F_4/G_2) \approx \mathbf{Z}/64$.

PROOF. This follows from fig. 15 and a standard Toda bracket argument ([50, 2.1]), which in this situation says the following. Let A be the element supporting the higher of the two d_2-differentials, and let D be the lowest $\bullet$ in $8k+26$. Let ∂ denote the boundary morphism in the exact sequence in $v_1^{-1}\pi_*(-)$ associated to the fibration (8.7). Then D lies in the Toda bracket $\langle \partial(A), \eta, 2\rangle$, and so there exists an element $E \in v_1^{-1}\pi_{8k+26}(F_4/G_2)$ such that $p_*(E) = A \circ \eta$ and $i_*(D) = 2E$. □

As indicated in fig. 15, there are $d_{\nu(k)+2}$-differentials between $\circ$-towers in $8k+22$ and $8k+21$, and there are $d_{\nu(k+1)+2}$-differentials between $\bullet$-towers in $8k+22$ and $8k+21$. This follows just from standard $J_*(-)$-considerations. But there may also be differentials from the $\circ$-tower in $8k+22$ to the $\bullet$-tower in $8k+21$. These differentials from $\circ$ to $\bullet$ are determined from the homomorphism

$$v_1^{-1}\pi_{8k+22}\left(S^{23}\right) \xrightarrow{\partial} v_1^{-1}\pi_{8k+21}\left(S^{15}\right), \tag{8.10}$$

which is evaluated in the following result.

PROPOSITION 8.15. *The image of the homomorphism* (8.10) *consists of all multiples of* 8 *if $\nu(k) \leqslant 7$, and is* 0 *if $\nu(k) > 7$.*

The following result plays a central role in the proof of Proposition 8.15.

PROPOSITION 8.16. *Let $(S^{15})_K$ denote the K_*-localization as constructed in* [42]. *There is a commutative diagram*

$$\begin{array}{ccc} \Omega S^{23} & \xrightarrow{\partial} & S^{15} \\ \downarrow{\scriptstyle \ell} & & \downarrow{\scriptstyle e} \\ \Omega^\infty(\Sigma^{15}P^{14}\wedge J) & \xrightarrow{h} & (S^{15})_K \end{array}$$

in which ∂ is obtained from the fiber sequence (8.7), *e is the localization, and h induces an isomorphism in $\pi_j(-)$ for $j = 22$ and $j \geqslant 28$.*

PROOF. The map h is constructed as in [29, pp. 669–670], using results of [42]. It induces an isomorphism in $\pi_j(-)$ for many other small values of j, but we only care about values of j which are positive multiples of 22. The obstructions to its being an isomorphism for all small values of j are $\mathbf{Z}_2$-classes in filtration 1 in $J_j(\Sigma^{15}P^{14})$ for $j = 19$, 23, and 27. The map ℓ is obtained by obstruction theory, since ΩS^{23} has cells only in dimensions which are positive multiples of 22. □

Now we prove Proposition 8.15. Let ℓ be as in Proposition 8.16. The morphism $\pi_*(\ell)$ can be factored as

$$\pi_*(\Omega S^{23}) \to \pi_*^s(\Omega S^{23}) \to J_*(\Sigma^{15}P^{14}).$$

There is a splitting

$$\pi_*^s(\Omega S^{23}) \approx \bigoplus_{i>0} \pi_*^s(S^{22i}).$$

We will use the method of [43] to deduce that

$$\pi_{8k+21}^s(S^{22}) \to J_{8k+21}(\Sigma^{15}P^{14})$$

sends the v_1-periodic generator ρ_k to 8 times the generator. Indeed, the stable map $S^{22} \to \Sigma^{15}P^{14} \wedge J$ which induces the morphism factors through $\Sigma^{15}P^8 \wedge J$, from which it projects nontrivially to $\Sigma^{15}P_7^8 \wedge J$. We then use [43, 2.8] to deduce that ρ_k goes to the nonzero element of $J_{8k+21}(\Sigma^{15}P_7^8)$. This implies that its image in $J_{8k+21}(\Sigma^{15}P^8)$ is the generator, and this maps to 8 times the generator of $J_{8k+21}(\Sigma^{15}P^{14})$.

The composite of v_1-periodic summands of

$$\pi_{8k+21}(\Omega S^{23}) \to \pi_{8k+21}^s(\Omega S^{23}) \to \pi_{8k+21}^s(S^{22}) \tag{8.11}$$

is bijective if $\nu(k) \leqslant 7$, but is not surjective if $\nu(k) > 7$. Thus when the composite (8.11) is followed into $J_{8k+21}(\Sigma^{15}P^{14})$, the image of a v_1-periodic generator is 8 times the generator if $\nu(k) \leqslant 7$ and $0 \in \mathbf{Z}/16$ if $\nu(k) > 7$. Once we observe that, in the diagram of Proposition 8.16, h induces an isomorphism in $\pi_{8k+21}(-)$ and e sends the v_1-periodic summand isomorphically, we obtain the desired conclusion of Proposition 8.15. □

The differentials implied by Proposition 8.15 have an interesting and unexpected implication about fig. 15. Since d_r from the $\circ$-tower in $8k+22$ hits the top $\circ$ in $8k+21$ and the $\bullet$ just above it with the same r, and since d_r respects the action of h_0, there must be an h_0-extension between these classes in $8k+21$. If $L = 0$ in (8.8), then this extension can only be accounted for by a failure of the map (8.8) to induce a split short exact sequence of A_1-modules in cohomology. Indeed, we have

PROPOSITION 8.17. *If $L = 0$ in* (8.8), *then there is a splitting of A_1-modules*

$$H^*X \approx H^*(\Sigma^{15}P^{30}) \oplus H^*(\Sigma^{23}P^6).$$

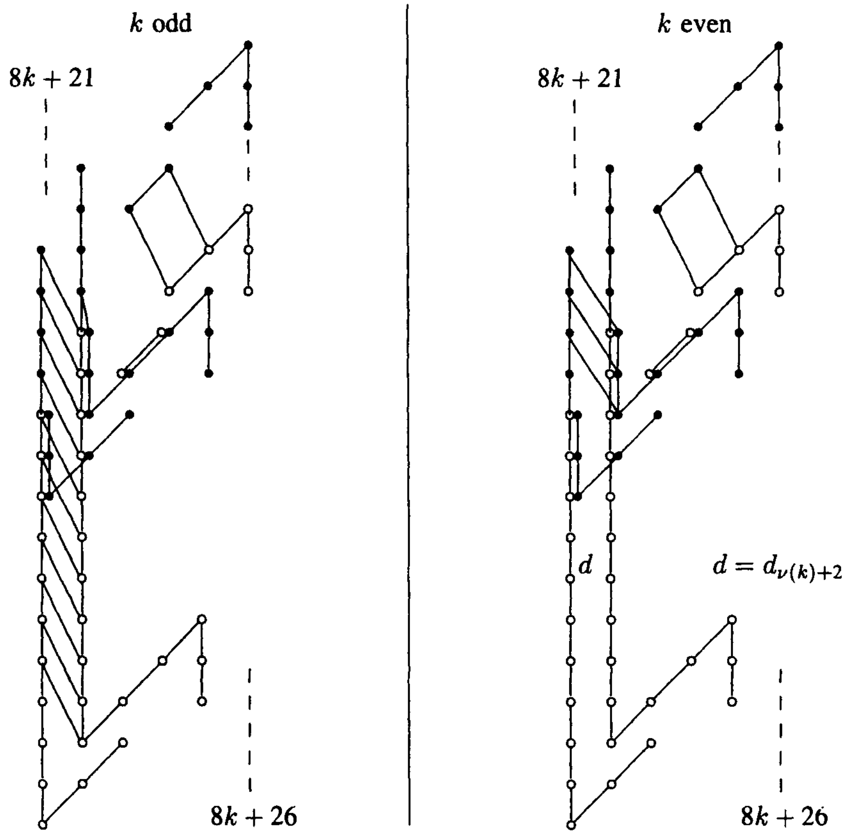


Figure 17. Final chart for $v_1^{-1}\pi_*(F_4/G_2)$.

This splitting is caused by having $\mathrm{Sq}^2 \neq 0$ on the class in H^*X corresponding to the top cell of $H^*(\Sigma^{15}P^{14})$, i.e.

$$\mathrm{Sq}^2 : H^{29}X \to H^{31}X$$

is an isomorphism. This is the only way to account for the h_0-extension in fig. 15. Proposition 8.17 implies that the h_0-extensions are present in the chart for values of k ($\nu(k) > 7$) where they cannot be deduced from differentials. It also implies that there is an h_0-extension on the top $\circ$ in $8k+22$. If $L > 0$ in (8.8), the same conclusion about the charts can be deduced from a more complicated analysis.

It causes fig. 15 to take the form of fig. 17.

We can read off almost all of Theorem 8.10 from fig. 17. We must show that d_6 is 0 on the $\circ$'s near the bottom in $8k+23$ and $8k+24$. This is done by the argument used to prove Proposition 8.15. The classes involved are present in all spaces in the diagram in

Proposition 8.16, but they are not mapped across by ℓ_*, since it factors through $\pi_*^s(\Omega S^{23})$.

All that remains is the verification of the abelian group structure. Most of the extensions are trivial due to the relation $2\eta = 0$. The extension in $8k+22$ when k is odd was present before the exotic extension was deduced, and remains true. The cyclicity of this 2^7 summand can also be deduced by consideration of the kernel of the homomorphism in the fibration which defines J, but that seems unnecessary. Note that no claim is made about the group structure in $8k+21$. □

In 1989, Mimura suggested to the author that he try to calculate $v_1^{-1}\pi_*(G)$ for all compact simple Lie groups G. If p is odd, and $G = \mathrm{Sp}(n)$ or $\mathrm{SO}(n)$, then the result follows from Theorem 1.3 and [33]. With great effort, $v_1^{-1}\pi_*(\mathrm{Sp}(n);2)$ was calculated in [13]. The result involves a surprising pattern of differentials among $\mathbf{Z}_2$'s from the various spheres which build $\mathrm{Sp}(n)$, resulting in $[\log_2(4n/3)]$ copies of $\mathbf{Z}_2$ in certain $v_1^{-1}\pi_i(\mathrm{Sp}(n))$. Of the classical groups, only $v_1^{-1}\pi_*(\mathrm{SO}(n);2)$ remains. All torsion-free exceptional Lie groups were handled in [14], using the UNSS. In [29] and [12], the torsion cases $(G_2,2)$, $(F_4,3)$, and $(E_6,3)$ were handled. Remaining then are seven cases of (G,p) yet to be calculated. At least a few of these should lend themselves to the methods of this paper.

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