Nilpotence and Stable Homotopy Theory II

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Contents

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Introduction

This paper is a continuation of [7]. Since so much time has lapsed since its publication a recasting of the context is probably in order.

In [15] Ravenel described a series of conjectures getting at the structure of stable homotopy theory in the large. The theory was organized around a family of "higher periodicities" generalizing Bott periodicity, and depended on being able to determine the nilpotent and non-nilpotent maps in the category of spectra. There are three senses in which a map of spectra can be nilpotent:

Definition 1.

i) A map of spectra

$$f : F \to X$$

is *smash nilpotent* if for $n \gg 0$ the map

$$f^{(n)} : F^{(n)} \to X^{(n)}$$

is null.

ii) A self map

$$f : \Sigma^k F \to F$$

is *nilpotent* if for $n \gg 0$ the map

$$f^n : \Sigma^{kn} F \to F$$

is null.

iii) A map

$$f : S^m \to R,$$

from the sphere spectrum to a ring spectrum, is *nilpotent* if it is nilpotent when regarded as an element of the ring $\pi_* R$.

The main result of [7] is

Theorem 2. *In each of the above situations, the map f is nilpotent if the spectrum F is finite, and if $MU_* f = 0$.*

In case the range of f is p-local, the condition $MU_* f = 0$ can be replaced with the condition $BP_* f = 0$.

The purpose of this paper is to refine this criterion and to produce some interesting non-nilpotent maps. Many of the results of this paper were conjectured by Ravenel in [15].

Let $K(n)$ be the n^{th} Morava K-theory at the prime p (see §1).

Theorem 3.

i) *Let R be a (p-local) ring spectrum. An element $\alpha \in \pi_* R$ is nilpotent if and only if for all $0 \le n \le \infty$, $K(n)_*(\alpha)$ is nilpotent.*

ii) *A self map $f : \Sigma^k F \to F$, of the p-localization of a finite spectrum, is nilpotent if and only if $K(n)_* f$ is nilpotent for all $0 \le n < \infty$.*

iii) *A map $f : F \to X$ from a finite spectrum to a p-local spectrum is smash nilpotent if and only if $K(n)_* f = 0$ for all $0 \le n \le \infty$.*

Of course, the hypothesis "p-local" can be dropped if the condition on the Morava K theory is checked at all primes.

At first, the criterion of this theorem seems less useful than the one provided by [7]. Using Theorem 3 to decide whether a map is nilpotent or not requires infinitely many computations. On the other hand, Morava K-theories are often easier to use than complex cobordism. Theorem 3 also determines which cohomology theories detect the nonnilpotent maps in the category of spectra.

Definition 4. A ring spectrum E is said to *detect nilpotence* if, equivalently,

i) for any ring spectrum R, the kernel of the Hurewicz homomorphism $E_* : \pi_* R \to E_* R$ consists of nilpotent elements;

ii) a map $f : F \to X$ from a finite spectrum F to any spectrum X is smash nilpotent if $1_E \wedge f : E \wedge F \to E \wedge X$ is null homotopic.

Corollary 5. *A ring spectrum E detects nilpotence if and only if*

$$K(n)_* E \neq 0$$

for all $0 \leq n \leq \infty$ and for all primes p.

Now let $\mathcal{C}_0$ be the homotopy category of p-local finite spectra and let $\mathcal{C}_n \subset \mathcal{C}_0$ be the full subcategory of $K(n-1)$-acyclics. The $\mathcal{C}_n$ fit into a sequence

$$\cdots \subset \mathcal{C}_{n+1} \subset \mathcal{C}_n \subset \cdots \subset \mathcal{C}_0.$$

This is a nontrivial fact. That there are inclusions $\mathcal{C}_{n+1} \subset \mathcal{C}_n$ is essentially the Invariant Prime Ideal Theorem. See [15]. That the inclusions are proper is a result of Steve Mitchell [12].

Definition 6. A full subcategory $\mathcal{C}$ of the category of spectra is said to be *thick* if it is closed under weak equivalences, cofibrations and retracts, ie;

i) An object weakly equivalent to an object of $\mathcal{C}$ is in $\mathcal{C}$.

ii) If $X \to Y \to Z$ is a cofibration, and two of $\{X, Y, Z\}$ are in $\mathcal{C}$ then so is the third.

iii) A retract of an object of $\mathcal{C}$ is in $\mathcal{C}$.

Theorem 7. *If $\mathcal{C} \subseteq \mathcal{C}_0$ is a thick subcategory, then $\mathcal{C} = \mathcal{C}_n$ for some n.*

Theorem 7 is in fact equivalent to the Nilpotence Theorem (the proof is sketched at the end of Section 4). It is often used in the following manner.

Call a property of p-local finite spectra *generic* if the full subcategory of $\mathcal{C}_0$ consisting of the objects with P is closed under cofibrations and retracts. To show that $X \in \mathcal{C}_n$ has a generic property P it suffices (by Theorem 7) to show that *any* object of $\mathcal{C}_n \setminus \mathcal{C}_{n+1}$ has P. The proofs of the next few results use this technique.

Theorem 3 limits the nonnilpotent maps in $\mathcal{C}_0$–they must be detected by some Morava K-theory. The simplest type is a v_n self-map.

Definition 8. Let X be a p-local finite spectrum, and $n > 0$. A self map $v : \Sigma^k X \to X$ is said to be a *v_n-self map* if

$$K(m)_* v \quad \text{is} \quad \begin{cases} \text{multiplication by a rational number} & \text{if } m = n = 0 \\ \text{an isomorphism} & \text{if } m = n \neq 0 \\ \text{nilpotent} & \text{if } m \neq n. \end{cases}$$

It turns out that the property of admitting a v_n self-map is generic.

Theorem 9. *A p local finite spectrum X admits a v_n self-map if and only if $X \in \mathcal{C}_n$. If X admits a v_n self-map, then for $N \gg 0$, X admits a v_n self-map*

$$v : \Sigma^{p^N 2(p^n-1)} X \to X$$

satisfying

$$(*) \qquad K(m)_* v = \begin{cases} v_n^{p^N} & \text{if } m = n. \\ 0 & \text{otherwise.} \end{cases}$$

The v_n self-maps turn out to be distinguished by another property.

Definition 10. A ring homomorphism

$$f : A \to B$$

is an *F-isomorphism* if

i) the kernel of f consists of nilpotent elements, and

ii) given $b \in B$, b^{p^n} is in the image of f for some n.

Two rings A and B are *F-isomorphic* ($A \approx_F B$) if there is an F-isomorphism between them.

Theorem 11. *Let $X \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. The $K(n)$-Hurewicz homomorphism gives rise to an F-isomorphism*

$$\text{center } [X,X]_* \approx_F \begin{cases} \mathbb{Z}_{(p)} & (n=0) \\ \mathbf{F}_p[v_n] & (n \neq 0). \end{cases} \tag{11.1}$$

The description of spectra as cell complexes encourages the intuition that the endomorphism rings of finite spectra approximate matrix algebras over the ring $\pi_* S^0$. This would suggest that the centers of these rings are generated by the maps obtained by smashing the identity map with a map between spheres – an impossibility by Theorem 11. A more accurate description might be that the 'Morita' equivalence classes of these rings are determined by the integer n of Theorem 11. This integer invariant can also be thought of as determining the 'birational' equivalence classes of finite spectra. For more on this analogy see [9].

There is a less metaphorical interpretation of the integer which occurs in Theorems 9 and 11.

Definition 12. Let X be a spectrum. The *Bousfield class* of X (denoted $\langle X \rangle$) is the collection of spectra Z for which $X \wedge Z$ is not contractible.

The Bousfield classes of spectra are naturally ordered by inclusion (though the relation is indicated with $\leq$, rather than $\subseteq$).

For a finite spectrum X, let $Cl(X) \subseteq \mathbf{N} \times \mathbf{P}$ denote the set of pairs (n,p) for which $K(n)_* X \neq 0$ at p. Here $\mathbf{N}$ is the set of nonnegative integers and $\mathbf{P}$ is the set of primes.

Theorem 13. *Let X and Y be finite spectra. Then $\langle X \rangle \leq \langle Y \rangle$ if and only if $Cl(X) \subseteq Cl(Y)$.*

Theorem 13 affirms Ravenel's Class Invariance conjecture ([15]).

Proof of Theorem 13: Since $\langle X \rangle \leq \langle Y \rangle$ if and only if $\langle X_{(p)} \rangle \leq \langle Y_{(p)} \rangle$ for all primes p, we may localize everything at a prime p. For a fixed Y, the property (of X)

$$\langle X \rangle \leq \langle Y \rangle$$

is a generic property. It follows that the class

$$\{\, X \mid \langle X \rangle \leq \langle Y \rangle \,\}$$

is equal to $\mathcal{C}_m$ for some m. Suppose that $Y \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$. We need to show that $m = n$. Since $\langle Y \rangle = \langle Y \rangle$, $m \leq n$. But if $X \notin \mathcal{C}_n$ then $\langle X \rangle \not\leq \langle Y \rangle$ (since $K(n+1) \notin \langle X \rangle$), so $m \not\leq n$. $\square$

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Notation and conventions

For the most part, we will work in the homotopy category of spectra. Of course, to form things like "the induced map of cofibers" requires choosing a diagram in a model for the category of spectra, introducing a certain ambiguity into the resulting map. This ambiguity plays only a very small role in this paper, and is dealt with each time it comes up.

The cofiber of a map $X \to Y$ will be written with the "cone coordinate" on the right

$$Y \cup_f CX = Y \coprod X \times I / \sim .$$

With this convention the cofiber of

$$Z \wedge X \to Z \wedge Y$$

is $Z \wedge (Y \cup_f CX)$ (modulo associativity of the smash product) not just isomorphic to it. With this convention, the cofiber of $Y \to Y \cup_f CX$ is $X \wedge S^1$, which is isomorphic, but not equal to ΣX. This avoids encountering the troublesome sign that can crop up when trying to relate the connecting homomorphism in a cofibration, with the connecting homomorphism in some suspension of the cofibration.

The assumption that a spectrum is finite is made several times. In contexts when the the category in mind is the category of p-local spectra, this term is used to refer to a spectrum which is weakly equivalent to the p-localization of a finite spectrum. The only property of finite spectra that is used is that the set of homotopy classes of maps from a finite spectrum to a directed colimit is the colimit of the maps.

$$[X, \varinjlim Y_\alpha] = \varinjlim [X, Y_\alpha].$$

In general, and object of a category with this property is said to be *small*. It can be shown that the small objects of the category of p-local spectra are precisely the objects which are weakly equivalent to the p-localizations of a finite spectrum.

A spectrum X is *connective* if $\pi_k X = 0$ for $k \ll 0$. It is *connected* if $\pi_k X = 0$ for $k < 0$. Thus "connected" and "(-1)-connected" are synonymous. Similarly, a graded abelian is *connective* if the homogeneous part of degree k is zero for $k \ll 0$. A graded abelian group is *connected* if the homogeneous component of degree k is zero for $k < 0$.

The Eilenberg-MacLane spectrum with coefficients in an abelian group A will be denoted HA. To be consistent with this, the homology of a spectrum X with coefficients in A will be denoted HA_*X.

Finally, the suspension of a map will always be labeled with the same symbol as the map.

1. Morava K–theories

1.1. Construction

The study of a ring is often simplified by passage to its quotients and localizations. The same is true of ring *spectra*, though constructing quotients and localizations can be difficult. In good cases the following constructions can be made:

Quotients

Suppose that E is a ring spectrum and that $\pi_* E = R$ is commutative. Given $x \in R$, define the spectrum $E/(x)$ by the cofibration

$$\Sigma^{|x|} E \xrightarrow{x\cdot} E \to E/(x).$$

If x is a non-zero divisor then $\pi_* E/(x)$ is isomorphic to the ring $R/(x)$. In good cases $E/(x)$ will still be a ring spectrum, and the map

$$E \to E/(x)$$

will be a map of ring spectra. Given a regular sequence

$$\{x_1, \ldots, x_n, \ldots\} \subset R$$

one can hope to iterate the above construction and form a ring spectrum $E/(x_1, \ldots, x_n, \ldots)$ with

$$\pi_* E/(x_1, \ldots, x_n, \ldots) \equiv R/(x_1, \ldots, x_n, \ldots),$$

and such that the natural map

$$E \to E/(x_1, \ldots, x_n, \ldots)$$

is a map of ring spectra.

Localizations

Let E and R be as above, and suppose that $S \subset R$ is a multiplicatively closed subset. Since $S^{-1}R$ is a flat R–module, the functor

$$S^{-1}R \underset{R}{\otimes} E_*(\_\_)$$

is a homology theory, $S^{-1}E$. In good cases it is represented by a ring spectrum, and the localization map by a map of ring spectra

$$E \to S^{-1}E.$$

1.2. Spectra related to BP

When the ring spectrum in question is BP, the above constructions can always be made, using the Baas-Sullivan theory of bordism with singularities. See [4, 13, 17] for the details.

Recall that $BP_* \approx \mathbb{Z}_{(p)}[v_1, \ldots v_n \ldots]$ with $|v_n| = 2p^n - 2$. To fix notation, take the set $\{v_n\}$ to be the Hazewinkle generators [8]. For $0 < n < \infty$ the ring spectra $K(n)$ and $P(n)$ are defined by the isomorphisms

$$\begin{aligned} K(n)_* &\approx \mathbf{F}_p[v_n, v_n^{-1}] \\ P(n)_* &\approx \mathbb{Z}(p)[v_n, v_{n+1} \ldots], \end{aligned}$$

with the understanding that they are constructed from BP using a combination of the above methods. It is also useful to set

$$\begin{aligned} K(0) &= \mathbf{H}\mathbb{Q} \\ K(\infty) &= H\mathbf{F}_p. \end{aligned}$$

There are maps $P(n) \to P(n+1)$, and the limit

$$\varinjlim{}_n P(n)$$

is the Eilenberg-MacLane spectrum $H\mathbf{F}_p$.

Proposition 1.1. *The Bousfield classes of $K(n)$ and $P(n)$ are related by*

$$\langle P(n) \rangle = \langle K(n) \rangle \vee \langle P(n+1) \rangle.$$

Consequently,

$$\langle BP \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle.$$

Proof: The proposition follows from the next two results of Ravenel [15]. □

Proposition 1.2. *Let $v : \Sigma^k X \to X$ be a self map of a spectrum X. Let X/vX and $v^{-1}X$ denote the cofiber of v and the colimit of the sequence*

$$\ldots \Sigma^{-k|v|} X \xrightarrow{v} \Sigma^{-(k+1)|v|} X \to \ldots$$

respectively. Then there is an equality of Bousfield classes

$$\langle X \rangle = \langle X/vX \rangle \vee \langle v^{-1}X \rangle. \quad \square$$

Proposition 1.3. *There is an equality of Bousfield classes*

$$\langle v_n^{-1} P(n) \rangle = \langle K(n) \rangle. \quad \square$$

1.3. Fields in the category of spectra

The coefficient ring $K(n)_*$ is a graded field in the sense that all of its graded modules are free. This begets a host of special properties of the Morava K-theories.

Proposition 1.4. *For any spectrum X, $K(n) \wedge X$ has the homotopy type of a wedge of suspensions of $K(n)$.*

Proof: Choose a basis $\{e_i\}_{i \in I}$ of the free $K(n)_*$–module $K(n)_*X$, and represent it as a map

$$\bigvee_{i \in I} S^{|e_i|} \to K(n) \wedge X.$$

The composition

$$K(n) \wedge \bigvee_{i \in I} S^{|e_i|} \to K(n) \wedge K(n) \wedge X. \to K(n) \wedge X$$

is then a weak equivalence. □

Proposition 1.5. *For any two spectra X and Y, the natural map*

(1.5.1) $$K(n)_*X \otimes_{K(n)_*} K(n)_*Y \to K(n)_*X \wedge Y$$

is an equivalence.

Proof: Consider the map (1.5.1) as a transformation of functors of Y. The left side satisfies the Eilenberg-Steenrod axioms since $K(n)_*Y$ is a flat (in fact free) $K(n)_*$-module. The right side satisfies the Eilenberg-Steenrod axioms by definition. The transformation is an isomorphism when Y is the sphere, hence for all Y. □

Proof of Corollary 5: If for some n, $K(n)_*E = 0$, then E does not detect the nonnilpotent map

$$\iota : S^0 \to K(n).$$

If $K(n)_*E \neq 0$, then by Proposition 1.4

$$E_*\alpha = 0 \Rightarrow K(n)_*\alpha = 0,$$

so the result reduces to Theorem 3. □

Propositions 1.4 and 1.5 portray the Morava K-theories as being a lot like fields. One formulation of Theorem 3 is that they are the *prime fields* of the category of spectra.

A (skew) field is a ring, all of whose modules are free.

Definition 1.6. A ring spectrum E is a *field* if E_*X is a free E_*-module for all spectra X.

This property also admits a geometric expression.

Lemma 1.7. *If E is a field, then $E \wedge X$ has the homotopy type of a wedge of suspensions of E*

Proof: This is very similar to the proof of 1.7. □

Proposition 1.8. *Let E be a field. Then E has the homotopy type of a wedge of suspensions of $K(n)$ for some n.*

Proof: Since $1 \in \pi_*E$ is non–nilpotent, for some prime p and for some $n \leq \infty$,

$$K(n)_*E \neq 0.$$

Since $K(n)$ and E are both fields, it follows from Lemma 1.7 that $K(n) \wedge E$ is both a wedge of suspensions of $K(n)$ and a wedge of suspensions of E. In particular, E is a retract of a wedge of suspensions of $K(n)$. The result therefore follows from the next proposition. □

Proposition 1.9. *Let M have the homotopy type of a wedge of suspensions of $K(n)$ (fixed n). If E is a retract of M, then E itself has the homotopy type of a wedge of suspensions of $K(n)$.*

Lemma 1.10. *The homotopy homomorphism induced by the Hurewicz map*

$$\iota \wedge 1_M : M \approx S^0 \wedge M \to K(n) \wedge M$$

is a homomorphism of $K(n)_*$-modules.

Proof: The map in question is a wedge of suspensions of the map

$$\eta_R : K(n) \approx S^0 \wedge K(n) \to K(n) \wedge K(n),$$

so it suffices to prove the claim when M is $K(n)$. In this case the result is a consequence of the formula [16],

$$\eta_R(v_n) = v_n. \quad \square$$

Lemma 1.11. *Let $f : M \to N$ be a map of wedges of suspensions of $K(n)$. The homotopy homomorphism*

$$\pi_*f : \pi_*M \to \pi_*N$$

is a map of $K(n)_*$–modules.

Proof: Consider the following commutative diagram:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d] & N \arrow[d] \\ K(n)\wedge M \arrow[r, "1\wedge f"] & K(n)\wedge N. \end{tikzcd}$$

The right vertical arrow is the inclusion of a wedge summand since N admits the structure of a $K(n)$–module spectrum. It therefore suffices to prove that the composition induces a map of $K(n)_*$–modules. The left vertical arrow does by Lemma 1.10 and the bottom horizontal arrow is a map of $K(n)$–module spectra. □

Proof of Proposition 1.9: Since M has the homotopy type of a wedge of suspensions of $K(n)$'s, it can be given the structure of a $K(n)$–module spectrum. Let $i : E \to M$, and $p : M \to E$ be the inclusion and retraction mappings respectively. By Lemma 1.11, the composite $i \circ p$ induces a homomorphism of $K(n)_*$–modules

$$\pi_* M \to \pi_* M.$$

Choose a basis $\{e_i\}$ of the image of this map, and represent it by

$$\bigvee S^{|e_i|} \to M.$$

The map

$$N = K(n) \wedge \left(\bigvee S^{|e_i|}\right) \to K(n) \wedge M \to M$$

then gives rise to an isomorphism

$$\pi_* N \approx \text{image of } \pi_*(i \circ p),$$

since it sends the obvious basis of $\pi_* N$ to the basis $\{e_i\}$. The composite

$$N \to M \xrightarrow{p} E$$

is the desired homotopy equivalence. □

1.4. Morava K-theories and duality

We will often use the device of replacing a self map of a finite spectrum

$$f : \Sigma^n X \to X$$

with its Spanier-Whitehead dual

$$Df : S^n \to X \wedge DX,$$

a map from the n-sphere to the *ring spectrum* $X \wedge DX$. If V is the finite dimensional $K(n)_*$ vector space $K(n)_* X$, then the ring $K(n)_*(X \wedge DX)$ is naturally isomorphic to the ring

$$V \otimes V^* \approx \mathrm{End}(V).$$

The effect in Morava K-theory of the duality map

$$X \wedge DX \overset{\text{flip}}{\to} DX \wedge X \overset{duality}{\to} S^0$$

is to send an endomorphism to its trace (in the graded sense). Let $\{e_i\} \subset V$ be a basis of V, and $\{e_i^*\} \subset V^*$ the corresponding dual basis. The effect of the other duality map

$$S^0 \to X \wedge DX$$

is to send $1 \in K(n)_*$ to $\sum e_i \otimes e_i^* \in V \otimes V^*$. In particular,

Lemma 1.12. *The duality map $S^0 \to X \wedge DX$ induces a non-zero homomorphism in $K(n)$-homology if and only if $K(n)_* X \neq 0$.*

2. Proofs of Theorems 3 and 7

Some of the conditions in Theorem 3 require the case $n = \infty$, and some of them don't. When the target spectrum is finite, the case $n = \infty$ is superfluous.

Lemma 2.1. *Let X and Y be finite spectra. For $m \gg 0$*

i) $K(m)_*X \approx H\mathbf{F}_{p*}X \otimes K(m)_*$

ii) $K(m)_*Y \approx H\mathbf{F}_{p*}Y \otimes K(m)_*$

iii) $K(m)_*f = H\mathbf{F}_{p*}f \otimes 1_{K(m)_*}$ for every $f : X \to Y$.

Proof: This follows from the Atiyah–Hirzebruch spectral sequence, using the fact that $|v_m| \to \infty$ as $m \to \infty$. □

Corollary 2.2. *If f is either a self–map of a finite spectrum or an element in the homotopy of a finite ring spectrum, the following are equivalent:*

i) $K(m)_*f$ is nilpotent for $m \gg 0$

ii) $H\mathbf{F}_{p*}f$ is nilpotent.

If $|f| \neq 0$ then both of these conditions hold.

Proof: If $|f| \neq 0$ then, from dimensional considerations,

$$H\mathbf{F}_{p*}f^i = 0 \quad \text{for } i \gg 0.$$

It then follows from 2.1 that $K(m)_*f^i = 0$ for $i, m \gg 0$. When $|f| = 0$, part (3) of 2.1 applies to every power of f. The result follows easily from this. □

Let $f : S^0 \to X$ be a map of spectra. Consider the homotopy direct limit T of the sequence

$$S^0 \to X \to X \wedge X \to X \wedge X \wedge X \to \dots, \tag{2.2.1}$$

in which the map $X^{(n)} \to X^{(n+1)}$ is given by

$$f \wedge 1_{X^{(n)}} : X^{(n)} \approx S^0 \wedge X^{(n)} \to X^{(n+1)}.$$

The n-fold composition

$$S^0 \to \cdots \to X^{(n)}$$

is the iterated smash product

$$f^{(n)} = f \wedge \cdots \wedge f.$$

The map

$$f^{(\infty)} : S^0 \to T$$

can be thought of as the *infinite smash product* of f.

Lemma 2.3. *Let E be a ring spectrum with unit $\iota : S^0 \to E$. The following are equivalent:*

i) *$E \wedge T$ is contractible;*

ii) *$\iota \wedge f^{(\infty)} : S^0 \to E \wedge T$ is null;*

iii) *$\iota \wedge f^{(n)} : S^0 \to E \wedge X^{(n)}$ is null for $n \gg 0$;*

iv) *$1_E \wedge f^{(n)} : E \approx E \wedge S^0 \to E \wedge X^{(n)}$ is null for $n \gg 0$.*

Proof: i)⇒ii) and iv)⇒i) are immediate. Since

$$\varinjlim E \wedge X^{(n)} \approx E \wedge T,$$

and since homotopy groups commute with direct limits, a null homotopy of $S^0 \to E \wedge T$ must occur at some $S^0 \to E \wedge X^{(n)}$ for $n \gg 0$. This gives ii)⇒ iii). The implication iii)⇒iv) is the only one requiring E to be a ring spectrum. If $S^0 \to E \wedge X^{(n)}$ is null then so is the first map in the following factorization of $1_E \wedge f^{(n)}$:

$$E \wedge S^0 \to E \wedge E \wedge X^{(n)} \to E \wedge X^{(n)}$$

This completes the proof. □

Proof of Theorem 3: Part i) follows from part iii), since the iterated multiplication factors through iterated smashing. Part ii) follows from part i) since multiplication in the rings

$$\pi_* X \wedge DX \quad \text{and} \quad K(n)_* X \wedge DX$$

corresponds, under Spanier-Whitehead duality, to composition in

$$[X, X]_* \quad \text{and} \quad \mathrm{End}_{K(n)_*}(K(n)_*X).$$

Replacing

$$f : F \to X$$

with

$$Df : S^0 \to DF \wedge X$$

in part iii) if necessary, we may assume that $F = S^0$. The result reduces to Theorem 2 once it is shown that

$$1_{BP} \wedge f^{(m)}$$

is null for $m \gg 0$. From Lemma 2.3 (with the obvious notation) this is equivalent to showing that $BP \wedge T$ is contractible. In view of the Bousfield equivalence

$$\langle BP \rangle = \langle K(0) \rangle \vee \cdots \vee \langle K(n) \rangle \vee \langle P(n+1) \rangle.$$

it is enough to show that $P(n) \wedge T$ is contractible for $n \gg 0$. Again from 2.3 this equivalent to showing that

$$S^0 \to P(n) \wedge T$$

is null for $n \gg 0$. Now let n grow to infinity. Since

$$\varinjlim P(n) \approx H\mathbf{F}_p,$$

the map

$$S^0 \to \varinjlim P(n) \wedge T$$

is null by assumption. Since homotopy commutes with direct limits, the nullhomotopy arises at some

$$S^0 \to P(n) \wedge T.$$

This completes the proof of Theorem 3. $\square$

The proof of Theorem 7 requires a slight modification of the third assertion of Theorem 3, and a useful cofibration (2.6).

Corollary 2.4. *Let F and Z be finite spectra, E a ring spectrum, and X an arbitrary spectrum.*

i) *If a map $f : F \to X \wedge E$ satisfies*

$$K(n)_*(f) = 0 \quad \text{for all } 0 \le n \le \infty,$$

then for $m \gg 0$, the composite

$$F^{(m)} \xrightarrow{f^{(m)}} (X \wedge E)^{(m)} \approx X^{(m)} \wedge E^{(m)} \xrightarrow{1 \wedge \mu} X^{(m)} \wedge E$$

is null.

ii) *A map*

$$f : F \to X$$

has the property that

$$f^{(m)} \wedge 1_Z : F^{(m)} \wedge Z \to X^{(m)} \wedge Z$$

is null for $m \gg 0$ if and only if

$$K(n)_*(f \wedge 1_Z) = 0$$

for all $0 \le n \le \infty$.

Proof: In part i), the map $f^{(m)}$ is already null for $m \gg 0$ by part iii) of Theorem 3. The *only if* part of ii) is clear. Letting E be the ring spectrum $Z \wedge DZ$ and replacing

$$f \wedge 1_Z : F \wedge Z \to X \wedge Z$$

with its Spanier-Whitehead dual

$$F \to X \wedge Z \wedge DZ$$

reduces the *if* part to i). $\square$

Lemma 2.5. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of maps. The map $C_f \to C_{g \circ f}$ induced by g gives rise to a cofibration*

$$C_f \to C_{g \circ f} \to C_g.$$

Proof: Consider the following diagram in which the rows and columns are cofibrations:

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \longrightarrow & C_f \\
\| & & {\scriptstyle g}\downarrow & & \downarrow \\
X & \xrightarrow{g \circ f} & Z & \longrightarrow & C_{g \circ f} \\
 & & \downarrow & & \downarrow \\
 & & C_g & \longrightarrow & ?
\end{array}$$

The upper right square is a pushout. It follows that the bottom arrow is a homotopy equivalence. This completes the proof. □

Corollary 2.6. *Let $f : X \to Y$ and $g : Z \to W$ be two maps. There is a cofibration*

$$X \wedge C_g \to C_{f \wedge g} \to C_f \wedge G.$$

Proof: Apply the lemma to the factorization

$$f \wedge g = f \wedge 1_Y \circ 1_X \wedge g. \quad \square$$

Proof of Theorem 7: It suffices to establish

(2.6.1) *if $X \in \mathcal{C}$ and $X \in \mathcal{C}_n$ then $\mathcal{C}_n \subseteq \mathcal{C}$,*

for it then follows that $\mathcal{C} = \mathcal{C}_m$, where

$$m = \min\{\, n \mid \mathcal{C}_n \subseteq \mathcal{C}\,\}.$$

Since everything has been localized at p, set

$$Cl(X) = \{n \in \mathbf{N} \mid K(n)_*(X) \neq 0\}.$$

With this notation, (2.6.1) becomes:

(2.6.2) *if $X \in \mathcal{C}$ and $Cl(Y) \subset Cl(X)$, then $Y \in \mathcal{C}$.*

Suppose, then, that $X \in \mathcal{C}$. Then so is $Z \wedge X$ for any $Z \in \mathcal{C}_0$. Let $f : F \to S^0$ be the fiber of the duality map $S^0 \to X \wedge DX$. Then $Y \wedge C_f \in \mathcal{C}$. Setting $g = f^{(m-1)}$ in Corollary 2.6 and smashing with the identity map of Y gives a cofibration

$$Y \wedge F \wedge C_{f^{(m-1)}} \to Y \wedge C_{f^{(m)}} \to Y \wedge C_f \wedge F^{(m-1)}.$$

It follows that $Y \wedge C_{f^{(m)}} \in \mathcal{C}$ for all m.

By 1.12, $K(n)_* f \neq 0$ if and only if $n \notin Cl(X)$, so that

$$K(n)_*(1_Y \wedge f) = 0 \quad \text{for all } n,$$

since $Cl(Y) \subseteq Cl(X)$. Part ii) of Corollary 2.4 then gives that $1_Y \wedge f^{(m)}$ is null for $m \gg 0$. This means that

$$Y \wedge C_{f^{(m)}} \approx Y \vee (\Sigma Y \wedge F^{(m)}) \qquad \text{for m} \gg 0,$$

so $Y \in \mathcal{C}$. This completes the proof of Theorem 7. □

3. v_n self-maps

The purpose of the next two sections is to establish Theorem 9. The “only if” part, that $X \notin \mathcal{C}_n$ implies that X does not admit a v_n self-map is easy: if for some $j < n$ $K(j)_* X \neq 0$, and if v is a v_n self-map, then the cofiber Y of v is a finite spectrum satisfying

$$K(n)_* Y = 0$$
$$K(j)_* Y \neq 0,$$

contradicting the fact that $\mathcal{C}_n \subset \mathcal{C}_j$. The proof of the “if” part falls into two steps. In this section it is shown that the property of admitting a v_n self-map is generic. It then remains to construct for each n, a spectrum X_n with a v_n self-map. This is done in section 4.

For any spectrum X, the element $p \in [X, X]_*$ is a v_0 self-map satisfying condition $(*)$ of Theorem 9. We therefore need only consider v_n self-maps when $n \geq 1$. Because of this, unless otherwise mentioned, *in this and the next section, we will work entirely in the category $\mathcal{C}_1$.*

As mentioned in section 1.4 a self-map

$$\Sigma^k F \to F$$

of a finite spectrum corresponds, under Spanier-Whitehead duality, to a map from the k-sphere to the ring spectrum

$$R = F \wedge DF.$$

Definition 3.1. Let R be a finite ring spectrum, $n > 0$. An element

$$\alpha \in \pi_* R$$

is a v_n-*element* if

$$K(m)_*\alpha \quad \text{is} \quad \begin{cases} \text{a unit} & \text{if } m = n \\ \text{nilpotent} & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Let R be a finite ring spectrum, and $\alpha \in \pi_* R$ a v_n-element. There exist integers i and j such that*

$$K(m)_*\alpha^i \quad = \quad \begin{cases} 0 & \text{if } m \neq n \\ v_n^j & \text{if } m = n \end{cases}$$

Proof: It follows from lemma 2.2 that $H\mathbf{F}_{p_*}\alpha$ is nilpotent. Raising α to a power, if necessary, we may suppose that $H\mathbf{F}_{p_*}\alpha = 0$. It then follows from lemma 2.2 that $K(m)_*\alpha = 0$ for all but finitely many m. Raising α to a further power, if necessary, it can be arranged that $K(m)_*\alpha = 0$ for $m \neq n$.

The assertion $K(n)_*\alpha^i = v_n^j$ is equivalent to the assertion that $\alpha^i = 1 \in K(n)_*R/(v_n - 1)$. The ring $K(n)_*R/(v_n - 1)$ has a finite group of units, and so i can be taken to be the order of this group. $\square$

Corollary 3.3. *If $f : \Sigma^k F \to F$ is a v_n self-map, then there exist integers i, j with the property that*

$$K(m)_* f^i \quad = \quad \begin{cases} 0 & \text{if } m \neq n \\ \text{multiplication by } v_n^j & \text{if } m = n. \end{cases} \qquad \square$$

Lemma 3.4. *Suppose that x and y are commuting elements of a $\mathbb{Z}_{(p)}$-algebra. If $x - y$ is both torsion and nilpotent, then for $N \gg 0$,*

$$x^{p^N} = y^{p^N}.$$

Proof: Since we are working over a $\mathbb{Z}_{(p)}$-algebra it follows that

$$p^k(x - y) = 0$$

for some k. The result now follows by expanding

$$x^{p^N} = (y + (x - y))^{p^N}$$

using the binomial theorem. $\square$

Lemma 3.5. *Let R be a finite ring spectrum, and $\alpha \in \pi_k R$ a v_n-element. For some $i > 0$, α^i is in the center of $\pi_* R$.*

Proof: Raising α to a power, if necessary, we may assume that $K(m)_*\alpha$ is in the center of $K(m)_*R$ for all m. Let $l(\alpha)$ and $r(\alpha)$ be the elements of $\mathrm{End}(\pi_* R)$ given by left and right multiplication by α. Since $R \in \mathcal{C}_1$ the difference $l(\alpha) - r(\alpha)$ has finite order. Since

$$K(m)_*\left(l(\alpha) - r(\alpha)\right) = 0 \quad \text{for all } m,$$

$l(\alpha) - r(\alpha)$ is nilpotent by Theorem 3. The result now follows from 3.4. $\square$

Lemma 3.6. *Let $\alpha, \beta \in \pi_* R$ be v_n-elements. There exist integers i and j with $\alpha^i = \beta^j$.*

Proof: Raising α and β to powers if necessary, we may assume that $K(m)_*(\alpha - \beta) = 0$ for all m. The result follows, as above, from 3.4. $\square$

Corollary 3.7. *If f and g are two v_n self-maps of F, then f^i is homotopic to g^j for some i and j.* $\square$

Corollary 3.8. *Suppose X and Y have v_n self-maps v_X and v_Y. There are integers i and j so that for every Z and every*

$$f : Z \wedge X \to Y$$

the following diagram commutes:

$$\begin{tikzcd} Z \wedge X \arrow[r, "f"] \arrow[d, "1\wedge {v_X}^i"'] & Y \arrow[d, "{v_Y}^j"] \\ Z \wedge X \arrow[r, "f"] & Y. \end{tikzcd}$$

Proof: The spectrum $DX \wedge Y$ has two v_n self-maps: $Dv_X \wedge 1_Y$ and $1_{DX} \wedge v_Y$. By Corollary 3.7 there are integers i and j for which ${Dv_X}^i \wedge 1_Y$ is homotopic to $1_{DX} \wedge {v_Y}^j$. The result now follows from Spanier-Whitehead duality. $\square$

Corollary 3.9. *The full subcategory of $\mathcal{C}_1$ consisting of spectra admitting a v_n self-map is thick.*

Proof: Call the subcategory in question $\mathcal{C}$. Note that $X \in \mathcal{C}$ if and only if $\Sigma X \in \mathcal{C}$. To check that $\mathcal{C}$ is closed under cofibrations it therefore suffices to show that if

$$X \to Y \to Z \tag{3.9.1}$$

is a cofibration with X and Y in $\mathcal{C}$, then Z is in $\mathcal{C}$. Using Corollary 3.8 choose the v_n self-maps v_X and v_Y of X and Y so that

$$\begin{tikzcd} \Sigma^k X \arrow[r] \arrow[d, "v_X"'] & \Sigma^k Y \arrow[r] \arrow[d, "v_Y"'] & \Sigma^k Z \\ X \arrow[r] & Y \arrow[r] & Z. \end{tikzcd}$$

commutes. The induced map $v_Z : \Sigma^k Z \to Z$ is easily seen to be a v_n self-map.

Now suppose that Y is a retract of X, and let $i : Y \to X$ and $p : X \to Y$ be the inclusion and retraction mappings respectively. Choose a v_n self-map v of X which commutes with $i \circ p$. The map

$$p \circ v \circ i$$

is easily checked to be a v_n self-map of Y. $\square$

Corollary 3.10. *The full subcategory of $\mathcal{C}_1$ consisting of spectra admitting a v_n self-map satisfying condition $(*)$ of Theorem 9 is thick.*

Proof: This is similar to 3.9, and involves checking that the integers which arise in 3.6–3.8 are powers of p. In fact, the only place where an integer which is not a power of p comes up is in using 3.7 to arrange that $K(m)_* v$ is in the center of $\mathrm{End}_{K(m)_*}(K(m)_* X)$. But this is guaranteed at the outset by condition $(*)$. $\square$

4. Construction of v_n self-maps

4.1. Preliminaries

The examples of self-maps needed for the proof of Theorem 9 are constructed using the Adams spectral sequence

$$\mathrm{Ext}_A^{s,t}[H^*Y, H^*X] \Rightarrow [X, Y]_{t-s}$$

which relates the mod p cohomology of X and Y as modules over the Steenrod algebra to $[X, Y]_*$. The spectral sequence is usually displayed in the $(t-s, s)$-plane, so that the groups lying in a given vertical line assemble to a single homotopy group. With this convention the "filtration jumps" are vertical in the sense that the difference between two maps representing the same class in

$$\mathrm{Ext}_A^{s,t}[H^*Y, H^*X]$$

represents a class in

$$\mathrm{Ext}_A^{s',t'}[H^*Y, H^*X],$$

with $s' > s$, and $t - s = t' - s'$.

There are many criteria for convergence of the Adams spectral sequence. A simple one, which is enough for the present purpose is [1]

Lemma 4.1. *If X a finite spectrum and Y is a connective spectrum with the property that each $\pi_k Y$ is a finite abelian p-group, then the Adams spectral sequence converges strongly to*

$$[X, Y]_*.$$

If $B \subseteq C$ are Hopf-algebras over a field k, the forgetful functor

$$C\text{-modules} \to B\text{-modules}$$

has both a left and a right adjoint. The left adjoint

$$M \mapsto C \underset{B}{\otimes} M$$

carries projectives to projectives, and so prolongs to a *change of rings isomorphism*

$$\operatorname{Ext}^*_C[C \underset{B}{\otimes} M, N] \approx \operatorname{Ext}^*_B[M, N]. \tag{4.1.1}$$

When M is a C-module this can be combined with the "shearing isomorphism"

$$\begin{aligned} C \underset{B}{\otimes} M &\to C//B \otimes M \qquad \left(C//B = C \underset{B}{\otimes} k \right) \\ c \otimes m &\mapsto \sum c_i' \otimes c_i'' m \\ \psi(c) &= \sum c_i' \otimes c_i'', \end{aligned}$$

to give another "change of rings" isomorphism

$$\operatorname{Ext}^*_C[C//B \otimes M, N] \approx \operatorname{Ext}^*_B[M, N].$$

The difference between Ext_C and Ext_B can therefore be measured by the augmentation ideal

$$\overline{C//B} = \ker \{\epsilon : C//B \to k\},$$

using the long exact sequence coming from

$$\overline{C//B} \otimes M \rightarrowtail C//B \otimes M \twoheadrightarrow M.$$

Recall that for $p = 2$, the dual Steenrod algebra is

$$\begin{aligned} A_* &= \mathbf{F}_2[\xi_1, \xi_2, \dots] \\ |\xi_i| &= 2^i - 1 \end{aligned}$$

and for p odd

$$\begin{aligned} A_* &= \Lambda[\tau_0, \tau_1, \dots] \otimes \mathbf{F}_p[\xi_1, \xi_2, \dots] \\ |\tau_i| &= 2p^i - 1 \\ |\xi_i| &= 2(p^i - 1). \end{aligned}$$

The subalgebra of the Steenrod algebra generated by

$$\begin{array}{cl} Sq^1, \dots, Sq^{2^n} & \text{when } p = 2 \\ \beta, \mathcal{P}^1, \dots, \mathcal{P}^{n-1} & \text{when } p \text{ is odd, and } n \neq 1 \\ \beta & \text{when } p \text{ is odd and } n = 0 \end{array}$$

is denoted A_n. It is the finite sub Hopf-algebra which is annihilated by the ideal

$$\begin{array}{cc} (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_{n+1}, \xi_{n+2}, \dots) & p = 2 \\ (\xi_1^{p^n}, \dots, \xi_n, \xi_{n+1}, \tau_{n+1}, \dots) & p \neq 2. \end{array}$$

The augmentation ideal of $A//A_n$ is $2p^n(p-1)$-connected. The fact that the connectivity goes to infinity with n plays an important role in the Approximation Lemma 4.5.

It is customary to give the dual Steenrod algebra the basis of monomials in the ξ's and τ's. With this convention, the *Adams-Margolis elements* are

$$P_t^s \text{ dual to } \quad \xi_t^{p^s} \quad (s < t)$$

$$Q_n \text{ dual to } \begin{cases} \tau_n & p \text{ odd} \\ \xi_{n+1} & p = 2. \end{cases}$$

Each Q_n is primitive, and together they generate an exterior sub Hopf-algebra of the Steenrod algebra. The P_t^s all satisfy

$$(P_t^s)^p = 0,$$

but are primitive only when $s=0$. The Adams-Margolis elements are naturally ordered by degree

$$\begin{aligned}|P^s_t| &= 2p^s(p^t-1)\\ |Q_n| &= 2p^n-1.\end{aligned}$$

4.2. Vanishing lines

Given an A-module M, and an Adams-Margolis element d, the *Margolis homology* of M, $H(M,d)$, is the homology of the complex (M_*, d_*), with

$$\begin{aligned}M_n &= M \quad n\in\mathbb{Z}\\ d_{2n} &= d\\ d_{2n+1} &= \begin{cases} d^{p-1} & \text{if } d=P^s_t\\ d & \text{if } d=Q_n.\end{cases}\end{aligned}$$

When X is a spectrum the symbol $H(X,d)$ will be used to denote $H(H^*X,d)$. The Margolis homology groups are periodic of period 1 if p even, or if $d=Q_n$, and are periodic of period 2 otherwise.

Definition 4.2. Let M be an A-module. A line

$$y=mx+b$$

is a *vanishing line* of

$$\operatorname{Ext}^{*,*}_A[M,\mathbf{F}_p]$$

if

$$\operatorname{Ext}^{s,t}_A[M,\mathbf{F}_p]=0 \quad \text{for} \quad s>m(t-s)+b.$$

The following result, due to Anderson-Davis [2] and to Miller-Wilkerson [10] relates vanishing lines to Margolis homology groups. It has not been stated in its strongest form.

Theorem 4.3. *If M is a connective A-module with*

$$H(M,d)=0 \quad \text{for} \quad |d|\le n,$$

then

$$\operatorname{Ext}^{*,*}_A[M,\mathbf{F}_p]$$

has a vanishing line of slope $1/n$. □

In general, there is no way to predict the intercept of the vanishing line, but there is the following:

Proposition 4.4. *Suppose that M is a connective A-module, and that*

$$y=mx+b$$

is a vanishing line for $\operatorname{Ext}^{*,*}_A[M,\mathbf{F}_p]$. If N is a $(c-1)$-connected A-module, then

$$y=m(x-c)+b$$

is a vanishing line for

$$\operatorname{Ext}^{*,*}_A[M\otimes N,\mathbf{F}_p].$$

Proof: Let N^k be the quotient of N by the elements of degree greater than k, and $N^k_j\subseteq N^k$ the submodule consisting of elements of degree $>j$. There is an exact sequence

$$N^k_j\to N^k\to N^j.$$

Since M is connective,

$$M\otimes N=\varinjlim_k M\otimes N^k$$

and

$$\operatorname{Ext}^{s,t}_A[M\otimes N,\mathbf{F}_p]=\varinjlim_k \operatorname{Ext}^{s,t}_A[M\otimes N^k,\mathbf{F}_p],$$

so it suffices to prove the result for each N^k. This is trivial for $k<c$, so suppose $k\ge c$, and by induction, that the result is true for $k'<k$. Suppose that (s,t) satisfies

$$s>m(t-s-c)+b$$

and consider the exact sequence

$$M \otimes N^k_{k-1} \to M \otimes N^k \to M \otimes N^{k-1}.$$

By induction,

$$\mathrm{Ext}^{s,t}_A[M \otimes N^{k-1}, \mathbf{F}_p] = 0.$$

The module N^k_{k-1} is just a sum of copies of $\Sigma^k \mathbf{F}_p$—the A-module which consists of $\mathbf{F}_p$ in degree k, and zero elsewhere. It follows that

$$\mathrm{Ext}^{s,t}_A[M \otimes N^k_{k-1}, \mathbf{F}_p]$$

is a product of copies of

$$\mathrm{Ext}^{s,t}_A[M \otimes \Sigma^k \mathbf{F}_p, \mathbf{F}_p] \approx \mathrm{Ext}^{s,t-k}_A[M, \mathbf{F}_p],$$

which is zero since

$$\begin{aligned} s &> m(t-s-c)+b \\ &> m((t-k)-s)+b. \quad \square \end{aligned}$$

Lemma 4.5 (Approximation lemma). *Let M be a connective A-module, and suppose that $\mathrm{Ext}^{*,*}_A[M, \mathbf{F}_p]$ has a vanishing line of slope m. Given b, for $n \gg 0$ the restriction map*

$$\mathrm{Ext}^{s,t}_A[M, \mathbf{F}_p] \to \mathrm{Ext}^{s,t}_{A_n}[M, \mathbf{F}_p]$$

is an isomorphism when

$$s \geq m(t-s)+b.$$

Proof: The result follows from the exact sequence

$$\overline{A//A_n} \otimes M \rightarrowtail A//A_n \otimes M \twoheadrightarrow M,$$

Proposition 4.4, and the fact that the connectivity of $\overline{A//A_n}$ can be made arbitrarily large by taking n to be large. $\square$

4.3. Morava K-theories and the Adams spectral sequence

We need to be able to examine the $K(n)$-Hurewicz homomorphism from the point of view of the Adams spectral sequence. This can be done, but it is a little easier to work with the connected cover $k(n)$ of $K(n)$. The spectrum $k(n)$ is a ring spectrum, with

$$k(n)_* = \mathbf{F}_p[v_n] \subset K(n)_* = \mathbf{F}_p[v_n, v_n^{-1}].$$

Lemma 4.6. *The transformation $k(n)_*X \to K(n)_*X$ extends to a natural isomorphism*

$$v_n^{-1} k(n)_* X \approx K(n)_* X.$$

Proof: Since localization is exact, both sides satisfy the exactness properties of a homology theory. They agree when X is the sphere, hence for all X. $\square$

Corollary 4.7. *If $k(n)_*X$ is finite then $K(n)_*X = 0$.*

Proof: If $k(n)_*X$ is finite, then for $j \gg 0$, $k(n)_j X = 0$. This means that for each $x \in k(n)_*X$, $v_n^m x = 0$ for $m \gg 0$. The result then follows from lemma 4.6. $\square$

Since $k(n)$ is a ring spectrum, the mod p cohomology $H^*k(n)$ is a coalgebra over the Steenrod algebra. It has been calculated by Baas and Madsen [5]

Proposition 4.8. *As a coalgebra over the Steenrod algebra,*

$$H^*k(n) \approx A//E[Q_n].$$

It follows that the E_2-term of the Adams spectral sequence for $\pi_* k(n) \wedge X$ is isomorphic to

$$\mathrm{Ext}^{s,t}_{E[Q_n]}[H^*X, \mathbf{F}_p],$$

and that the map of E_2-terms induced by the Hurewicz homomorphism is the natural restriction. $\square$

Corollary 4.9. *If X is a finite spectrum and $H(X, Q_n) = 0$, then $K(n)_* X = 0$.*

Proof: The group

$$\mathrm{Ext}^{*,*}_{E[Q_n]}[H^*(X), \mathbf{F}_p]$$

is the cohomology of the complex

$$H^*X \xrightarrow{Q_n} H^*X \xrightarrow{Q_n} H^*X \xrightarrow{Q_n} \dots \quad .$$

This means that for $s > 0$, the graded abelian group

$$\mathrm{Ext}^{s,*}_{E[Q_n]}[H^*X, \mathbf{F}_p]$$

is isomorphic to the Margolis homology group $H(X, Q_n)$. The vanishing of these groups implies that

$$\mathrm{Ext}^{*,*}_{E[Q_n]}[H^*X, \mathbf{F}_p] \approx \mathrm{Ext}^{*,0}_{E[Q_n]}[H^*X, \mathbf{F}_p] \subseteq H^*X$$

is finite, and hence that $k(n)_*X$ is finite. The result then follows from Corollary 4.7. $\square$

4.4. Examples of self maps

The key to constructing self-maps is the following result of the second author [19]. An account appears in [14].

Theorem 4.10 (Smith). *For each $n = 1, 2, \dots$ there is a finite spectrum X_n satisfying*

i) *The Adams spectral sequence*

$$\mathrm{Ext}^{s,t}_{E[Q_n]}[H^*X_n \wedge DX_n, \mathbf{F}_p] \Rightarrow k(n)_*X_n \wedge DX_n$$

collapses;

ii) *The Margolis homology groups $H(X_n \wedge DX_n, d)$ are zero if*

$$|d| < |Q_n|.$$

Theorem 4.11. *The spectrum X_n is in $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$ and has a v_n self-map satisfying $(*)$ of Theorem 9.*

The proof of Theorem 4.11 uses the Adams spectral sequence and the following consequence of the results of Wilkerson [20]. The proof is in the appendix to this section.

Theorem 4.12. *Suppose that $B \subset C$ are finite, connected, graded, cocommutative Hopf-algebras over a field k of characteristic $p > 0$. If*

$$b \in \mathrm{Ext}^{*,*}_B[k, k],$$

then for $N \gg 0$, b^{p^N} is in the image of the restriction map

$$\mathrm{Ext}^{*,*}_C[k, k] \to \mathrm{Ext}^{*,*}_B[k, k]. \quad \square$$

Proof of Theorem 4.11: That X_n is in $\mathcal{C}_n \setminus \mathcal{C}_{n-1}$ follows from Corollary 4.9.

For the construction of the self-map it is slightly cleaner to work from the point of view of finite ring spectra. Thus let R be the finite ring spectrum $X_n \wedge DX_n$. The ring π_*R is an algebra over π_*S^0, and the image of π_*S^0 in π_*R is in the center (in the graded sense). Similarly, if $B \subseteq A$ is a sub Hopf-algebra, the ring

$$\mathrm{Ext}^{*,*}_B[H^*R, \mathbf{F}_p]$$

is a central algebra over $\mathrm{Ext}^{*,*}_B[\mathbf{F}_p, \mathbf{F}_p]$.

To show that X_n admits a v_n self-map satisfying condition $(*)$ of Theorem 9 it suffices to exhibit an element

$$v \in \pi_*R$$

satisfying

(4.12.1) $\quad k(n)_*v^{p^M} = v_n^{p^N} \cdot 1, \qquad$ for some $M, N > 0$;

(4.12.2) $\quad$ the map $k(m)_*v$ is nilpotent when $m \neq n$.

Step 1: First to find an approximation to a v_n self-map in the E_2-term of the Adams spectral sequence. Let

$$v_n \in \operatorname{Ext}^{1,2p^n-1}_{E[Q_n]}[\mathbf{F}_p, \mathbf{F}_p]$$

be the element represented by $v_n \in k(n)_*$. We need to find a

$$w \in \operatorname{Ext}^{p^N, p^N(2p^n-1)}_A[H^*R, \mathbf{F}_p]$$

restricting to $v_n^{p^N} \cdot 1$, for $N \gg 0$. By 4.3, the bigraded group

$$\operatorname{Ext}^{*,*}_A[H^*R, \mathbf{F}_p]$$

has a vanishing line of slope $1/2(p^n-1)$. Using the approximation lemma, an integer n can be chosen for which the restriction map

$$\operatorname{Ext}^{s,t}_A[H^*R, \mathbf{F}_p] \to \operatorname{Ext}^{s,t}_{A_n}[H^*R, \mathbf{F}_p] \tag{4.12.3}$$

is an isomorphism if

$$s > \frac{1}{2(p^n-1)}(t-s).$$

By Theorem 4.12 there is an element

$$\tilde{w} \in \operatorname{Ext}^{*,*}_{A_n}[\mathbf{F}_p, \mathbf{F}_p]$$

restricting to $v_n^{p^N} \in \operatorname{Ext}^{*,*}_{E[Q_n]}[\mathbf{F}_p, \mathbf{F}_p]$. The class w can be taken to be the image of $\tilde{w} \cdot 1$ under the isomorphism (4.12.3).

Step 2: This construction of the class w actually gives something more. Since

$$\operatorname{Ext}^{*,*}_{A_n}[\mathbf{F}_p, \mathbf{F}_p]$$

is in the center (in the graded sense) of

$$\operatorname{Ext}^{*,*}_{A_n}[H^*R, \mathbf{F}_p],$$

the class w commutes with every

$$\alpha \in \operatorname{Ext}^{s,t}_A[H^*R, \mathbf{F}_p]$$

with

$$s \geq \frac{1}{2(p^n-1)}(t-s). \tag{4.12.4}$$

Step 3: Now to choose a power of w which survives the Adams spectral sequence. The differentials in the Adams spectral sequence are derivations, and the values of $d_r w$ lie in the region (4.12.4). This means that

$$d_{r-1}w = 0 \Rightarrow d_r w^p = 0.$$

Since $d_1 w = 0$ it follows that $d_b w^{p^b} = 0$. The possible values of $d_r w^{p^b}$ for $r > b$ lie in the region

$$s > \frac{1}{2(p^n-1)}(t-s),$$

which is above the vanishing line. This means that the class w^{p^b} is a permanent cycle.

Step 4: For simplicity, replace w with w^{p^b}, and adjust the integer N so that w restricts to $v_n^{p^N} \cdot 1$. Let

$$v \in \pi_* R$$

be a representative of w. We will see that this is the desired class.

The difference $k(n)_*(v - v_n^{p^N})$ is represented by a class in

$$\operatorname{Ext}^{s,t}_{E[Q_n]}[H^*R, \mathbf{F}_p]$$

with $s > 1/2(p^n-1)\,(t-s)$. Some power of $k(n)_*(v - v_n^{p^N b})$ is therefore represented by a class above the vanishing line of

$$\operatorname{Ext}^{*,*}_{E[Q_n]}[H^*R, \mathbf{F}_p]$$

(which has slope $1/2(p^n-1)$), and hence is zero. Lemma 3.4 then gives that that

$$k(n)_* v^{p^M} = v_n^{p^{MN}} \qquad M \gg 0.$$

This proves property (4.12.1)

Property (4.12.2) is trivial when $m < n$, since $R \in \mathcal{C}_n$. When $m > n$, it is a consequence of the fact that the Adams spectral sequence $k(m)_*R$ has a vanishing line of slope $1/2(p^m-1)$, and that the powers of v are represented by classes lying on the line

$$s = \frac{1}{2(p^n-1)}(t-s)$$

which has a larger slope. This completes the proof. $\square$

4.5. Proof that Theorem 7 implies the Nilpotence Theorem

This subsection is included to satiate any curiosity aroused by the claim made after the statement of Theorem 7. Since the argument is not necessary for establishing any of the results of this paper, it is included only as a sketch.

In [7, Section 1] the Nilpotence Theorem is reduced to showing that if R is a connective, associative ring spectrum, and

$$\alpha \in \pi_* R$$

is in the kernel of the MU-Hurewicz homomorphism then α is nilpotent. This in turn is easily reduced to the case when R is localized at p and MU is replaced with BP. The case $|\alpha| \leq 0$ is easy, so it may be assumed that $|\alpha| > 0$.

Let $\overline{\alpha}$ be the map

$$\Sigma^{|\alpha|} R \overset{\alpha \wedge 1}{\to} R \wedge R \to R. \tag{4.12.5}$$

The map (4.12.5) induces multiplication by $BP_* \alpha = 0$ in BP homology. This means that from the point of view of the Adams-Novikov spectral sequence, composition withe $\overline{\alpha}$ moves the homotopy to the right along a line of positive slope.

Step 1: The construction used to produce the spectra X_n of this section can be used to construct finite torsion free spectra Y_n with the property that $H^* Y_n$, as a module over A_n is free over $A_n // E$, where E is the sub-Hopf-algebra

$$\Lambda[Q_0, \ldots, Q_n].$$

See [19].

Step 2: Use the spectral sequence of [16, Theorem 4.4.3] to show that

$$\mathrm{Ext}^{s,t}_{BP_* BP}[BP_*, BP_* R \wedge Y_n]$$

has a vanishing line with slope tending to zero as $n \to \infty$.

Step 3: It follows from the vanishing line that for $n \gg 0$ the spectrum

$$Y_n \wedge \alpha^{-1} R$$

is contractible.

Step 4: Now use Theorem 13 to conclude that Y_n is Bousfield equivalent to the sphere, hence that $\alpha^{-1} R$ is contractible, hence that α is nilpotent.

5. Endomorphisms, up to nilpotents

5.1. N-endomorphisms

The v_n self maps form an endomorphism (up to nilpotent elements) of the category $\mathcal{C}_n$. It turns out that these are the only endomorphisms of this kind that can occur in the category of finite spectra.

Definition 5.1. Let $\mathcal{C}$ be a full subcategory of $\mathcal{C}_0$ which is closed under suspensions. A collection v, of self–maps

$$v_X : \Sigma^{k_X} X \to X \qquad X \in \mathcal{C}$$

satisfying is an *N–endomorphism* of $\mathcal{C}$ if

i) The map $v_{\Sigma X}$ is the composite

$$\begin{array}{ccc} \Sigma^k \Sigma X & \xrightarrow{\text{flip}} & \Sigma^k X \wedge S^1 \\ & & \downarrow{\scriptstyle v_X \wedge 1_{S^1}} \\ & & X \wedge S^1 \xrightarrow{\text{flip}} \Sigma X \end{array}$$

ii) for each $f : X \to Y$ in $\mathcal{C}$ there are integers i and j with $ik_X = jk_Y$, such that the following diagram commutes:

$$\begin{array}{ccc} \Sigma^N X & \xrightarrow{f} & \Sigma^N Y \\ {\scriptstyle v_X{}^i}\downarrow & & \downarrow{\scriptstyle v_Y{}^j} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

An N–endomorphism is an F*–endomorphism* if the integers i and j can be taken to be powers of p.

Two N–endomorphisms v and v' will be identified if for each $X \in \mathcal{C}$ there are integers i and j with ${v_X}^i = {v'_X}^j$. Two F–endomorphisms v and v' will be identified if for each $X \in \mathcal{C}$ there are integers i and j with ${v_X}^{p^i} = {v'_X}^{p^j}$.

Remark 5.2.

(1) If v is an N-endomorphism of a category $\mathcal{C}$, and $f : X \to Y$ an isomorphism with $X \in C$, then defining v_Y to be

$$\begin{tikzcd} \Sigma^k Y \arrow[r, "\Sigma^k f"] & \Sigma^k X \arrow[d, "v_X"] & \\ & X \arrow[r, "f^{-1}"'] & Y \end{tikzcd}$$

extends v to the full subcategory obtained from C by adjoining the suspensions of Y. Because of property ii), the resulting N–endomorphism is independent of the choice of isomorphism f. In this way an N–endomorphism can always be extended uniquely to a full subcategory which is closed under suspensions and isomorphisms. This procedure will be used without comment, so once an N–endomorphism has been defined on a subcategory C of finite spectra, it will be taken to be extended to the smallest full subcategory containing C, which is closed under suspensions and isomorphisms. Among other things, this means that if v_X is defined, so is $v_{X\wedge S^1}$ and

$$v_{X\wedge S^1} = v_X \wedge 1_{S^1}.$$

(2) An N–endomorphism is of *degree zero* if all of the integers k_X are zero. If an N–endomorphism is not of degree zero, then none of the integers k_X is zero, and the maps v_X can all be chosen to have finite order. Given two spectra $X, Y \in \mathcal{C}$, the maps v_X and v_Y can be chosen in such a way that the integers k_X and k_Y coincide. With this arrangement, given a map

$$f \in [X, Y]_*,$$

if there are integers i and j for which

$$v_Y^i \circ f = f \circ v_X^j,$$

then it must be the case that $i = j$. This same discussion applies to any finite collection of elements of $\mathcal{C}$.

Example 5.3.

(1) Taking v_X to be nilpotent defines an F–endomorphism.

(2) Taking each v_X to be a multiple of the identity defines an actual endomorphism .

(3) Suppose $\mathcal{C} \subseteq \mathcal{C}_n$. Taking v_X to be a v_n self–map defines an N–endomorphism . Taking v_X to be a v_n self–map satisfying condition $(*)$ of Theorem 9 defines an F–endomorphism .

5.2. Classification of N-endomorphisms

The above list of examples turns out to be complete.

Theorem 5.4. *Suppose that v is an N–endomorphism of a full subcategory $C \subseteq \mathcal{C}_0$ which is closed under suspensions. Then v_X is nilpotent for every X, some power of v_X is a multiple of the identity, or $C \subseteq \mathcal{C}_n$ for some n and v_X is a v_n self–map.*

Of course, these possibilities aren't exclusive. If $X \in \mathcal{C}_{n+1} \subseteq \mathcal{C}_n$ any v_n self–map of X is nilpotent.

Corollary 5.5. *Suppose the $X \in \mathcal{C}_0$, and that $v \in [X, X]_*$ is in the center. Then v is nilpotent, a power of v is a multiple of the identity, or v is a v_n self–map.*

Proof: Let $\mathcal{C}$ be the full subcategory of $\mathcal{C}_0$ consisting of the suspensions of X. The map v determines an N–endomorphism of $\mathcal{C}$, so the result follows from Theorem 5.4. $\square$

Theorem 11 is an immediate consequence of Corollary 5.5 and Theorem 9.

The proof of Theorem 5.4 falls into two parts. First it is shown that an N–endomorphism extends uniquely to a thick subcategory. It then suffices to construct, for each n, a spectrum $X_n \in \mathcal{C}_n \setminus \mathcal{C}_{n+1}$ whose only non–nilpotent self–map is a v_n self–map.

First to dispense with the N–endomorphisms of degree zero.

Proposition 5.6. *If X is in $\mathcal{C}_0$, and $v : X \to X$ is in the center of $[X, X]_* = \pi_* X \wedge DX$ then there are integers m and n for which*

$$v^n = \text{ multiplication by } m.$$

Proof: Since

$$\pi_* X \wedge DX \otimes \mathbb{Q} \approx \mathbf{H}\mathbb{Q}_* X \wedge DX \approx \operatorname{End} \mathbf{H}\mathbb{Q}_* X,$$

The map $\mathbf{H}\mathbb{Q}_* v$ must be in the center of $\operatorname{End} \mathbf{H}\mathbb{Q}_* X$ which consists of the endomorphism "multiplication by a constant". Since the Hurewicz map $\mathbf{H}\mathbb{Q}_*$ factors through $\mathbf{H}\mathbb{Z}_*$, this constant must be an integer k. The map $w = v - k$ then has finite order.

Since all of the eigenvalues of $H\mathbf{F}_{p*} w^{p-1}$ are equal to 0 or 1, the map $H\mathbf{F}_{p*} w^{(p-1)p^N}$ is an idempotent for $N \gg 0$. Replace w with $w^{(p-1)p^N}$. The map w still has finite order, and is in the center of $[X, X]_*$. Define connective spectra A_1 and A_2 by

$$A_1 = w^{-1} X$$
$$A_2 = (1 - w)^{-1} X$$

The map

$$X \to A_1 \vee A_2$$

induces an isomorphism on both mod p and rational homology, hence on homology with coefficients in $\mathbb{Z}_{(p)}$. It is therefore a homotopy equivalence, and in particular A_1 and A_2 are finite.

The ring of self–maps $[X, X]_*$ can be written as a ring of 2×2 matrices, in which the ij–entry is in $[A_j, A_i]$. The map w is represented by the matrix

$$\begin{pmatrix} w|_{A_1} & 0 \\ 0 & 0 \end{pmatrix},$$

whose $(1,1)$ entry is an equivalence. Given a map $f : \Sigma^k A_2 \to A_1$, let $\tilde{f}$ be the map

$$\begin{pmatrix} 0 & f \\ 0 & 0 \end{pmatrix}.$$

Then

$$\operatorname{ad}(w)\tilde{f} = \begin{pmatrix} w|_{A_1} f & 0 \\ 0 & 0 \end{pmatrix}.$$

Since w is central, and $w|_{A_1}$ is an equivalence this means that f is null. By Lemma 5.7 below, it follows that one of A_1 and A_2 is contractible. If A_1 is contractible, then w is nilpotent, and the result follows from Lemma 3.4. If A_2 is contractible, then $1 - w$ is nilpotent, $\mathbf{H}\mathbb{Q}_* X = 0$, and we may assume that the integer k is 0, so that $w = v^{(p-1)p^N}$. It then follows from Lemma 3.4 that $v^{(p-1)p^M} = 1$ for $M \gg 0$. This completes the proof. □

We have used

Lemma 5.7. *If A and B are non–contractible p–local finite spectra, then $[A, B]_* \neq 0$.*

Proof: Since

$$H\mathbf{F}_{p*} DA \wedge B = \hom[H\mathbf{F}_{p*} A, H\mathbf{F}_{p*} B] \neq 0,$$

the spectrum $DA \wedge B$ is not–contractible. It therefore has a non–zero homotopy group. Now use the isomorphism

$$\pi_* DA \wedge B \approx [A, B]_*. \quad \square$$

5.3. Some technical tools

The next few results are a bit technical, but they come up several times.

Lemma 5.8. *Suppose that M is a bimodule over the ring $\mathbb{Z}_{(p)}[v]$, and for $m \in M$ let*

$$\operatorname{ad}(v)m = vm - mv.$$

If there are integers i, j and k, for which

i) $k \operatorname{ad}(v^i)m = 0$, *and*

ii) $\operatorname{ad}(v^j)\big(\operatorname{ad}(v^i)m\big) = 0$,

then

$$\operatorname{ad}\left(v^{ijk}\right) m = 0.$$

Clearly, k can be taken to be a power of p, so that if i and j are powers of p, then so is ijk.

Lemma 5.9. *Suppose M is a bimodule over the ring $\mathbb{Z}[v]$. Let $\operatorname{ad}(v) : M \to M$ be the operator $\operatorname{ad}(v)m = vm - mv$. Then there is a formula*

$$\operatorname{ad}(v^n)m = \sum_{i<n} \binom{n}{i} \operatorname{ad}^i(v)m \cdot v^{n-i}. \tag{5.9.1}$$

Proof: Let $l(v)$ and $r(v)$ be the operators of left and right multiplication by v respectively. Then

$$\operatorname{ad}(v) = l(v) - r(v),$$

and the operators $\operatorname{ad}(v)$, $l(v)$, and $r(v)$ all commute. Now take the equation

$$l(v) = \operatorname{ad}(v) + r(v),$$

raise both sides to the power n, and use the binomial theorem. □

Proof of 5.8: Replacing j with ij and v with v^i, we may suppose that $i = 1$. Since the operators $\operatorname{ad}(v^j)$ and $\operatorname{ad}(v)$ commute, it follows from (5.9.1) that

$$\operatorname{ad}(v^j)\big(\operatorname{ad}(v^j)m\big) = 0.$$

Replacing v with v^j, we may therefore also assume that $j = 1$, and hence that $\operatorname{ad}^2(v)m = 0$. But then, again from (5.9.1),

$$\operatorname{ad}(v^k)m = k \operatorname{ad}(v)m = 0.$$

This completes the proof. □

5.4. N-endomorphisms and thick subcategories

Lemma 5.10. *Suppose that v and v' are N–endomorphisms of a thick subcategory $\mathcal{C} \subseteq \mathcal{C}_0$. The full subcategory $\mathcal{D} \subseteq \mathcal{C}$ consisting of objects for which v and v' coincide, is thick.*

Proof: The case in which the degree of v is zero follows from Proposition 5.6, so we may assume that the degree of v is not zero. In this case remark 5.2 applies.

Suppose that $i : Y \to X$ is the inclusion of a retract, with $X \in \mathcal{D}$ and $Y \in C$. Choose the maps v_X, v_Y, v'_X and v'_Y, so that

(1) $v_X i = i v_Y$, $v'_X i = i v'_Y$, and

(2) $v_X = v'_X$.

Then $i v_Y = i v'_Y$, so $v_Y = v'_Y$ since i is a monomorphism.

Now suppose that

$$X \to Y \to Z$$

is a cofibration in $\mathcal{C}$ with X and Y in $\mathcal{D}$. Choose the maps v to have finite order, with $v_X = v'_X$, $v_Y = v'_Y$, and so that

$$\begin{tikzcd} \Sigma^N Y \arrow[r] \arrow[d, "v_Y"'] & \Sigma^N Z \arrow[r, "\delta"] \arrow[d, "v_Z"'] & \Sigma^N X \wedge S^1 \arrow[d, "v_{X \wedge S^1}"'] \\ Y \arrow[r] & Z \arrow[r, "\delta"'] & X \wedge S^1 \end{tikzcd}$$

and

$$\begin{tikzcd} \Sigma^N Y \arrow[r] \arrow[d, "v'_Y"'] & \Sigma^N Z \arrow[r, "\delta"] \arrow[d, "v'_Z"'] & \Sigma^N X \wedge S^1 \arrow[d, "v'_{X \wedge S^1}"'] \\ Y \arrow[r] & Z \arrow[r, "\delta"'] & X \wedge S^1. \end{tikzcd}$$

commute. Make $[Z,Z]_*$ into a $\mathbb{Z}_{(p)}[v]$–bimodule by

(5.10.1) $$vm = v_Z \circ m$$

(5.10.2) $$mv = m \circ v'_Z$$

The goal is to show that $\operatorname{ad}(v^k)1_Z$ is zero, for some k. We know that $\operatorname{ad}(v^k)1_Z$ has finite order, and that it factors through a map $\Sigma^N X \wedge S^1 \to Z$. In general, if a map $f \in [Z,Z]_*$ factors through some $W \in \mathcal{D}$ then $\operatorname{ad}(v^j)f = 0$ for some j. It follows that

$$\operatorname{ad}(v^j)\,(\operatorname{ad}(v)1_Z) = 0$$

for some j. The result then follows from Lemma 5.8. □

Lemma 5.11. *Let v be an N–endomorphism of a full subcategory $\mathcal{C}$ of $\mathcal{C}_0$ which is closed under suspensions. If $\mathcal{D}$ is the largest full subcategory of $\mathcal{C}_0$ to which v extends, then $\mathcal{D}$ is thick.*

Proof: If the degree of v is zero, then the result follows from 5.6. We may therefore assume that the maps v all have finite order, and for the finitely many spectra that come up in the proof, that they all have the same degree. Suppose that $X \in \mathcal{D}$ and that

$$i : Y \to X$$
$$p : X \to Y$$

satisfy $p \circ i = 1_Y$. Choose the map v_X so that it commutes with the idempotent $i \circ p$, and set

$$v_Y = p \circ v_X \circ i.$$

If it can be shown that this map v_Y extends v to the full subcategory $\mathcal{D} \cup \{\Sigma^k Y\}_{k \in Z}$, it will follow that $Y \in \mathcal{D}$ by maximality.

To check this, let $W \in \mathcal{D} \cup \{\Sigma^k Y\}_{k \in Z}$ and suppose at first that $W \in \mathcal{D}$. Given a map $f : W \to Y$, choose an integer j so that the outer rectangle in the following diagram commutes:

$$\begin{tikzcd}
\Sigma^N W \arrow[r, "f"] \arrow[d, "v_W^j"'] & \Sigma^N Y \arrow[r, "i"] \arrow[d, "v_Y^j"'] & \Sigma^N X \arrow[d, "v_X^j"'] \\
W \arrow[r, "f"] & Y \arrow[r, "i"] & X.
\end{tikzcd}$$

The right square commutes by construction, so the left must also, since the map i is the inclusion of a wedge summand. The argument for dealing with a map $Y \to W$ is similar. Finally, given

$$f : \Sigma^k Y \to Y,$$

using the above, find and integer k so that

$$i \circ v_Y^k \circ f = v_X^k \circ i \circ f = i \circ f \circ v_Y^k.$$

Then, again, since $i_* : [Y,Y]_* \to [Y,X]_*$ is a monomorphism,

$$v_Y^k \circ f = f \circ v_Y^k.$$

Now suppose that

$$X \to Y \to Z$$

is a cofibration with X and Y in $\mathcal{D}$. Choose the maps v_X and v_Y so that the left square in the following diagram commutes:

$$\begin{tikzcd}
\Sigma^N X \arrow[r] \arrow[d, "v_X"'] & \Sigma^N Y \arrow[r] \arrow[d, "v_Y"'] & \Sigma^N Z \arrow[d, "v_Z"'] \\
X \arrow[r] & Y \arrow[r] & Z
\end{tikzcd},$$

Let v_Z be any map of finite order making the right square commute. If it can be shown that this map v_Z extends v to the category

$$\mathcal{D} \cup \{\Sigma^k Z\}_{k \in \mathbb{Z}},$$

it will follow that $Z \in \mathcal{D}$ by maximality.

To check this, suppose that $f : Z \to W$ is a map with W an object of $\mathcal{D} \cup \{\Sigma^k Z\}_{k \in \mathbb{Z}}$. If necessary raise that maps v_W, v_X, v_Y, and v_Z to powers so that they are all of finite order, and that they all have the same degree.

Case 1: *$W \in \mathcal{D}$, and f factors through $\delta : Z \to X \wedge S^1$.* Write

$$f = g \circ \delta,$$

and choose an integer i so that right square in the following diagram commutes:

$$\begin{tikzcd}
\Sigma^N Z \arrow[r, "\delta"] \arrow[d, "v_Z^i"'] & \Sigma^N X \wedge S^1 \arrow[r, "g"] \arrow[d, "v_X^i"'] & \Sigma^N W \arrow[d, "v_W^i"] \\
Z \arrow[r, "\delta"'] & \Sigma X \arrow[r, "g"'] & W.
\end{tikzcd}$$

The left hand square commutes by definition of v_Z, so the whole diagram commutes.

Case 2: *The spectrum W is in $\mathcal{D}$.* Make the graded abelian group $[Z,W]_*$ into a bimodule over $\mathbb{Z}_{(p)}[v]$ by

$$\begin{aligned} v\cdot m &= v_W\circ m\\ m\cdot v &= m\circ v_Z. \end{aligned}$$

For some i, $\mathrm{ad}(v^i)f$ vanishes on Y and so factors through $X\wedge S^1$. By Case 1,

$$\mathrm{ad}(v^j)\left(\mathrm{ad}(v^i)f\right)=0 \qquad \text{some } j,$$

and the result follows from Lemma 5.8.

Case 3: $W{=}\Sigma^k Z$. By Case 2, the maps v can be chosen to commute in addition with all elements of $[Z,\Sigma^k X\wedge S^1]$. It then follows that $\mathrm{ad}(v)f$ factors through $\Sigma^k Y$, and so by case 1,

$$\mathrm{ad}(v^j)\,(\mathrm{ad}(v)f)=0 \quad \text{some } j.$$

The result then follows from Lemma 5.8.

The case of maps $W\to Z$ is handled similarly. This completes the proof. □

5.5. A spectrum with few nonnilpotent self maps

Now to construct, for each n a spectrum $X_n\in\mathcal{C}_n\setminus\mathcal{C}_{n-1}$ whose only non–nilpotent self–maps are roots of the identity, or v_n self–maps. The spectrum X_0 can be taken to be the sphere.

Proposition 5.12. *For each $n>0$ there exists a sequence*

$$\underline{k}=(k_0,\ldots,k_{n-1}),$$

and a finite spectrum $M(\underline{k})\in\mathcal{C}_n\setminus\mathcal{C}_{n-1}$, satisfying:

i) $BP_*M(\underline{k})=BP_*/(v_0^{p^{k_0}},\ldots,v_{n-1}^{p^{k_{n-1}}})$ $\qquad (v_0=p)$*;*

ii) *If*

$$v:\Sigma^j M(\underline{k})\to M(\underline{k})$$

is a non–nilpotent self–map, then some power of v is the identity map, or v is a v_n self–map.

Proof: Suppose by induction on n that a sequence

$$\underline{k}=(k_0,\ldots,k_{n-1})$$

and a spectrum

$$M=M(\underline{k})$$

have been found, satisfying condition (1). When $n=1$ the sequence can be taken to be (1), and the spectrum M, the mod p Moore spectrum

$$S^0\cup_p e^1.$$

Let $I(\underline{k})\subset BP_*$ be the ideal

$$(v_0^{p^{k_0}},\ldots,v_{n-1}^{p^{k_{n-1}}}).$$

If v is a non-nilpotent self-map of $M(\underline{k})$ then the BP–Hurewicz image

$$BP_*v:BP_*/I(\underline{k})\to BP_*/I(\underline{k})$$

must be non–nilpotent. The map BP_*v must also be a map of BP_*–modules, and of BP_*BP–comodules, and so is an element of

$$\mathrm{Hom}_{BP_*BP}[BP_*/I(\underline{k}),BP_*/I(\underline{k})]\subset BP_*/I(\underline{k}).$$

Modulo the ideal

$$(p,v_1,\ldots,v_{n-1}) \tag{5.12.1}$$

this group is just [16, Theorem 4.3.2]

$$\mathbf{F}_p[v_n].$$

Since (5.12.1) is nilpotent modulo $I(\underline{k})$

$$BP_*v\equiv\lambda v_n^k \mod I(\underline{k}) \qquad k\in\mathbb{Z},\lambda\in\mathbf{F}_p.$$

It then follows from Lemma 3.4 that

$$BP_* v^{(p-1)p^N} = v_n^{k(p-1)p^N} \qquad N \gg 0.$$

Replace v with $v^{(p-1)p^N}$. If $k = 0$, then $BP_*(v-1) = 0$, and so $v-1$ is nilpotent. It then follows from Lemma 3.4 that

$$v^{p^N} = 1_M \qquad N \gg 0.$$

Suppose then that $k \neq 0$, and let w be a v_n self–map of M. By the above discussion applied to w, there are integers i and j, for which $BP_* v^i = BP_* w^j$. But this means that $v^i - w^j$ is nilpotent, so by Lemma 3.4 some power of v is homotopic to some power of w, and v is a v_n self–map. This proves (2). For the rest of the induction step, let

$$w : \Sigma^{2(p^n-1)P^N} M \to M$$

be a v_n self–map satisfying condition $(*)$ of Theorem 9. The integer k_n can be taken to be N, and

$$M(k_0, \ldots k_n),$$

the cofiber of the map w. $\square$

5.6. Proof of Theorem 5.4

Let v be an N–endomorphism of $\mathcal{C} \subseteq \mathcal{C}_0$. Then v extends uniquely to the smallest thick subcategory $\mathcal{C}_n \subset \mathcal{C}_0$ containing $\mathcal{C}$. Let

$$\underline{k} = (k_0, \ldots, k_{n-1})$$

and $M = M(\underline{k})$ be as in Lemma 5.12, and let $\mathcal{D}$ be the full subcategory of $\mathcal{C}_n$ consisting of the suspensions of M. Then v is also uniquely determined by its restriction to $\mathcal{D}$, ie by the map v_M. By Proposition 5.12, there are three possibilities for v_M, and these are the restrictions of the nilpotent, identity, and v_n self–map N–endomorphisms. This completes the proof of Theorem 5.4. $\square$

A. Proof of Theorem 4.12

The purpose of this appendix is to prove (rather, deduce from [20]) Theorem 4.12. All of the techniques used here can be found in [20].

Throughout this appendix, all Hopf-algebras will be over a field of characteristic $p > 0$. They will be connected, graded, cocommutative, and finite dimensional. The dual of a Hopf-algebra will be graded in such a way that the dual of the homogeneous component of degree k has degree $-k$. This convention enables the co-action map (A.0.3) to preserve degrees.

The action of a Hopf-algebra B on a module M can be expressed as an "action"

(A.0.2) $$B \otimes M \to M$$

or as a "coaction"

(A.0.3) $$M \to B^* \otimes M.$$

A module M which happens to be an algebra is an *algebra over* B if the multiplication map

$$M \otimes M \to M$$

is a map of B-modules. This is equivalent to the requirement that the coaction map (A.0.3) be multiplicative. All algebras over Hopf-algebras in this appendix will be graded and connected.

If $B \subseteq C$ is normal, and M is a C-module, then the sub-module of elements invariant under B,

$$M^B = \hom_B[k, M],$$

inherits an action of the quotient Hopf-algebra $C//B$. In fact, so do all of the derived functors

(A.0.4) $$\mathrm{Ext}^*_B[k, M].$$

If M is an algebra over B, then (A.0.4) becomes an algebra over $C//B$ [18].

In case $B \subseteq C$ is normal, the relationship between the cohomologies of B and C is given by the Lyndon-Hochschild-Serre spectral sequence

$$\mathrm{Ext}^*_{C//B}[k, \mathrm{Ext}^*_B[k, M]] \Rightarrow \mathrm{Ext}^*_C[k, M].$$

The main result of this appendix is

Theorem A.1. *Suppose that R is a Noetherian C-algebra, and that $B \subseteq C$ is normal. Then*

i) *The algebra $\mathrm{Ext}_C^*[k,R]$ is Noetherian, hence finitely generated.*

ii) *The Lyndon-Hochschild-Serre spectral sequence*

$$\mathrm{Ext}^*_{C//B}\,[k, \mathrm{Ext}^*_B[k,R]] \Rightarrow \mathrm{Ext}^*_C[k,R].$$

terminates at a finite stage in the sense that there is an integer N with the property that all of the differentials d_r are zero, if $r > N$.

iii) *There is an integer N with the property that $d_r x^{p^N}$ is zero, for all x and all r.*

iv) *The Lyndon-Hochschild-Serre spectral sequence is a spectral sequence of finitely generated modules over some connected, graded, Noetherian ring T.*

The parts of this theorem are closely related.

Lemma A.2. *In Theorem A.1, parts* i), ii), *and* iii) *follow from* iv). *Given* i), *parts* ii), iii), *and* iv) *are equivalent.*

Proof: Suppose first that iv) holds. Then part ii) follows from Lemma A.3 below. Part iii) follows from ii) since the differentials are derivations. That iv)⇒i) follows from the fact that if a ring is complete with respect to an exhaustive filtration, and if the associated graded ring is Noetherian, then so is the original ring (see [6, 3.2.9 and Corollary 2 to Proposition 12] or [3, Corollary 10.25]).

Now suppose that part i) holds. Then the the E_2-term of the Lyndon-Hochschild-Serre spectral sequence is Noetherian, hence finitely generated over k. Given ii), part iii) follows as above. Given part iii), the algebra T in part iv) can be taken to be the algebra of $\left(p^N\right)^{\mathrm{th}}$-powers in E_2. The implication iv)⇒ii) was established in the preceding paragraph. This completes the proof. □

We have used:

Lemma A.3. *Let $\{E_r, d_r\}$ be a spectral sequence of finitely generated modules over a Noetherian ring T. There is an integer N with the property that all of the differentials d_r are zero, if $r > N$.*

Proof: The modules E_r are sub-quotients of E_2. Define

$$B_{r+1} \subseteq B_r \subseteq \cdots \subseteq Z_r \subseteq Z_{r+1} \cdots \subseteq E_2$$

with the property that

$$E_{r+1} = Z_r / B_r.$$

The graded T-modules Z_r and B_r can be thought of as the kernel and image of d_r respectively. By the ascending chain condition, there is an integer N with the property that $B_r = B_N$ if $r \geq N$. But this implies, for $r \geq N+1$ that $E_r \subseteq E_{N+1}$, so image of d_r is zero. □

Wilkerson [20] has proved a special case of Theorem A.1.

Theorem A.4 (Wilkerson).

i) *Suppose that $B \subseteq C$ is in the center, and that the action of C on R is trivial. Then i)–iv) of Theorem A.1 hold.*

ii) *If $B \subseteq C$ is normal, the map*

$$\mathrm{Ext}^*_C[k,k] \to \mathrm{Ext}^*_B[k,k]$$

is finite.

The requirement that C act trivially on R turns out not to be much of a restriction.

Lemma A.5. *Suppose a Hopf-algebra A acts on a graded commutative ring R. Given an element $x \in R$, for $N \gg 0$, the element x^{p^N} is invariant under A. In particular, if R is Noetherian, then*

$$R^A \hookrightarrow R$$

is finite.

Proof: This is easiest to verify from the point of view of the coaction. By assumption, the coaction is given by

$$\psi(x) = 1 \otimes x + \sum a_i \otimes x_i,$$

where $|x_i| \neq 0$. Since A is finite dimensional, there is an N with the property that $a^{p^N} = 0$ for all $a \in A^*$ with $|a| \neq 0$. But then

$$\begin{aligned} \psi(x^{p^N}) &= 1 \otimes x^{p^N} + \sum a_i^{p^N} \otimes x_i^{p^N} \\ &= 1 \otimes x^{p^N}. \end{aligned}$$

This completes the proof. $\square$

Lemma A.6. *If A is a finite Hopf-algebra and $R \to S$ is a finite map of Noetherian A-algebras, then*

$$\operatorname{Ext}^*_A[k, R] \to \operatorname{Ext}^*_A[k, S]$$

is finite.

Corollary A.7. *If C is a Hopf-algebra, and R is a Noetherian C-module, then the cohomology algebra*

$$\operatorname{Ext}^*_C[k, R]$$

is Noetherian.

Proof: By the above result, the map $R^C \to R$ is finite. Again by the above result,

$$R^C \otimes \operatorname{Ext}^*_C[k, k] \approx \operatorname{Ext}^*_C[k, R^C] \to \operatorname{Ext}^*_C[k, R]$$

is finite. The result now follows from A.4. $\square$

Corollary A.8. *It suffices to prove Theorem A.1 when $R = k$.*

Proof: It is enough to deduce part iv). Suppose that the Lyndon-Hochschild-Serre spectral sequence

$$\operatorname{Ext}^*_{C//B}[k, \operatorname{Ext}^*_B[k, k]] \Rightarrow \operatorname{Ext}^*_C[k, k]$$

consists of finitely generated modules over the Noetherian ring T. Then the spectral sequence

$$\operatorname{Ext}^*_{C//B}[k, \operatorname{Ext}^*_B[k, R^C]] \Rightarrow \operatorname{Ext}^*_C[k, R^C]$$

consists of finitely generated modules over the Noetherian ring $R^C \otimes T$. By Lemma A.5, the map $R^C \to R$ is finite. It follows from Lemma A.6 that the map

$$\operatorname{Ext}^*_{C//B}[k, \operatorname{Ext}^*_B[k, R^C]] \to \operatorname{Ext}^*_{C//B}[k, \operatorname{Ext}^*_B[k, R]]$$

is finite, so the spectral sequence

$$\operatorname{Ext}^*_{C//B}[k, \operatorname{Ext}^*_B[k, R]] \Rightarrow \operatorname{Ext}^*_C[k, R]$$

is also a spectral sequence of finite modules over $R^C \otimes T$. $\square$

The proof of A.8 is built out of a few special cases.

Lemma A.9. *Suppose that E is a Hopf-algebra of the form $E[x]$, where*

$$E[x] = \begin{cases} k[x]/x^2 & \text{if } |x| \text{ is odd} \\ k[x]/x^p & \text{if } |x| \text{ is even,} \end{cases} \tag{A.9.1}$$

and $R \to S$ is a finite map of Noetherian E-algebras. If the action of E on R is trivial, then

$$\operatorname{Ext}^*_E[k, R] \to \operatorname{Ext}^*_E[k, S]$$

is finite.

Proof: Let's take the case in which $E = k[x]/x^p$ with $|x|$ even. The others are similar. If M is an E-module, the cohomology

$$\operatorname{Ext}^*_E[k, M]$$

is the cohomology of the complex

$$M \otimes \Lambda[a] \otimes k[b]$$

with differential

$$d(m \otimes b^k) = x^{p-1} m \otimes a \otimes b^k$$
$$d(m \otimes a \otimes b^k) = xm \otimes b^{k+1}.$$

When the action of E on M is trivial, the differential d is zero. The result now follows since the complex for calculating $\mathrm{Ext}^*_E[k, S]$ is already a finite module over the $\mathrm{Ext}^*_E[k, R]$. □

Lemma A.10. *Suppose A is a Hopf-algebra and $R \to S$ is a finite map of Noetherian A-algebras. If the action of A on R is trivial, then*

$$\mathrm{Ext}^*_A[k, R] \to \mathrm{Ext}^*_A[k, S]$$

is finite.

Proof: The proof is by induction on the dimension of A, the case in which the dimension of A is 1 being a tautology. Suppose that the dimension of A is greater than 1, and that the result is known to be true for Hopf-algebras of dimension less than that of A.

Let $E \subseteq A$ be a central sub-Hopf-algebra of the form (A.9.1), and let

$$\{E_r\}, \quad \text{and } \{E'_r\}$$

be the associated Lyndon-Hochschild-Serre spectral sequences with coefficients in R and S, respectively. The spectral sequence $\{E_r\}$ is just the tensor product of R with the Lyndon-Hochschild-Serre spectral sequence with coefficients in k. By Theorem A.4 it is a spectral sequence of finite modules over a Noetherian ring of the form $R \otimes T$. It follows that the map

$$E_\infty \to E'_\infty$$

is finite, and so the map

$$\mathrm{Ext}^*_A[k, R] \to \mathrm{Ext}^*_A[k, S]$$

is finite by [3, Proposition 10.24]. □

Proof of Lemma A.6: Since S is finite over R and R is finite over R^A by Lemma A.5, S is finite over R^A. It follows from Lemma A.10 that

$$\mathrm{Ext}^*_A[k, R^A] \to \mathrm{Ext}^*_A[k, S]$$

is finite, so a fortiori

$$\mathrm{Ext}^*_A[k, R] \to \mathrm{Ext}^*_A[k, S]$$

is finite. This completes the proof. □

Proof of Theorem A.1: By Corollary A.8 we may assume that $R = k$. Choose an integer N with the property that x^{p^N} is invariant under the action of $C//B$ for every $x \in \mathrm{Ext}_B[k, k]$, and let

$$S \subset \mathrm{Ext}^*_B[k, k]$$

be the sub-algebra consisting of the $(p^N)^{\text{th}}$ powers of the elements in the image of $\mathrm{Ext}_C *[k, k]$. Then the maps

$$S \to \mathrm{Ext}^*_B[k, k], \quad \text{and}$$
$$\mathrm{Ext}^*_{C//B}[k, S] \to \mathrm{Ext}^*_{C//B}[k, \mathrm{Ext}^*_B[k, k]]$$

are finite by A.5 and A.6. But

$$\mathrm{Ext}^*_{C//B}[k, S] = \mathrm{Ext}^*_{C//B}[k, k] \otimes S$$

is Noetherian, and consists of permanent cycles. Taking T to be this algebra establishes part iv), and completes the proof. □

To deduce Theorem 4.12 requires

Lemma A.11. *Suppose that $B \subseteq C$ is an inclusion of finite Hopf-algebras. There is a sequence*

$$B = C_0 \lhd C_1 \cdots \lhd C_n = C$$

with each $C_i \lhd C_{i+1}$ normal, and with the property that the Hopf-algebra $C_{i+1}//C_i$ is of the form (A.9.1).

Proof: It suffices, by induction on $\dim_k C//B$, to show that if $B \neq C$ then there is a surjective map of Hopf-algebras $C \to E[x]$ with the property that the composite

$$B \to C \to E[x]$$

is trivial (which means that it is the augmentation followed by the inclusion of the degree zero part). Since $B \neq C$ the map of dual algebras

$$C^* \to B^*$$

is not a monomorphism. It follows from [11, Proposition 3.9] that the map of primitives is not injective. Let D be a primitive in the kernel. The element D can by thought of as a derivation from A to k with the property that $D(b) = 0$ when $b \in B$.

Give x the degree $-|D|$. The map to $E[x]$ is then given by Taylor's formula:

$$a \mapsto \begin{cases} \sum_{n=0}^{p-1} D^n a \frac{x^n}{n!} & |D| \text{ even} \\ D^0 a + D(a)\, x & |D| \text{ odd} \end{cases}$$

The powers of D are taken in the algebra C^*. In particular, D^0, being the unit of C^*, is the augmentation. $\square$

Proof of Theorem 4.12: It suffices, by Lemma A.11 to deal with the case in which $B \subseteq C$ is normal. Let

$$b \in \mathrm{Ext}^*_B[k, k]$$

be a cohomology class. By Lemma A.5 there is an integer M with the property that b^{p^M} is invariant under $C//B$. This gives a class in the E_2–term of the Lyndon–Hochschild–Serre spectral sequence. For convenience, replace b^{p^M} with b. By Theorem A.1 there is an integer N with the property that $d_r b^{p^N} = 0$ for all r. The class in $\mathrm{Ext}^*_C[k, k]$ represented by b^{p^N} is then the desired class. $\square$

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