# WEIL PAIRINGS AND MORAVA $K$-THEORY

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## 1. Introduction

In [5], M. Hopkins and the present authors showed that for every elliptic spectrum $E$, there is a canonical map

$$\sigma_E\colon\ MU\langle 6\rangle \to E$$

of ring spectra, called the $\sigma$-orientation, which generalises the Witten genus [11] and gives an interesting interpretation of its modular invariance; for more background and motivation, see the introduction to [5]. A central part of the proof of the above result is the computation of the ring $E_0BU\langle 6\rangle$ (where $BU\langle 6\rangle$ is the 5-connected cover of $BU$) or equivalently the description of the scheme $\operatorname{spec}(E_0BU\langle 6\rangle)$. For this description we do not need $E$ to be an elliptic spectrum, but merely a commutative 2-periodic ring spectrum whose odd-dimensional homotopy groups are trivial. However, in order to cover this level of generality, it was necessary to give a rather computational proof.

In the present work, we give an alternative argument that is valid when $E$ is a two-periodic Morava $K$-theory spectrum (with associated formal group $G_K$ say). This proof is more conceptual, and it makes much closer contact with previous work by algebraic geometers on the theory of biextensions [8, 2]. In their language, the result is that $\operatorname{spec}(K_0BU\langle 6\rangle)$ is the scheme $C^3(G_K)$ of rigid cubical structures on the trivial $\mathbb{G}_m$-torsor over $G_K$; this is formulated more explicitly as Theorem 1.4.

Although the details of the argument given here were worked out and written up by the authors listed, we should emphasise that the conceptual basis is part of a large ongoing project which is joint work with M. Hopkins. We are grateful to him for allowing us to distribute this fragment of our enterprise.

**Notation.** Throughout this paper we fix a fixed prime number $p > 0$ and an integer $h > 0$. We fix as well a 2-periodic Morava $K$-theory $K$ with ground field $k \overset{\text{def}}{=} K^0(pt)$ of characteristic $p$ and formal group $G_K$ of height $h$. We let $n$ denote a (varying) power of $p$, and we write $P$ for $\mathbb{C}P^\infty$ and $P[n]$ for $B\mathbb{Z}/n$. We write $\mu = \mu_n$ for the multiplication map $H/n \wedge H/n \to H/n$ of Eilenberg-MacLane spectra, or for the map

$$P[n]^2 = K(\mathbb{Z}/n, 1) \times K(\mathbb{Z}/n, 1) \to K(\mathbb{Z}/n, 2)$$

derived from this. We also write $\beta = \beta_n$ for the map $H/n \to \Sigma H$ that induces the Bockstein operation $H^k(X;\mathbb{Z}/n) \to H^{k+1}(X;\mathbb{Z})$, or for the maps

$$P[n] = K(\mathbb{Z}/n, 1) \to P = K(\mathbb{Z}, 2)$$

$$K(\mathbb{Z}/n, 2) \to K(\mathbb{Z}, 3)$$

derived from this.

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By a *k-group* we shall mean an affine commutative group scheme over $k$. These form an abelian category (the opposite via spec of the category of abelian Hopf algebras over $k$). A map of $k$-groups

$$G = \operatorname{spec} A \xrightarrow{\operatorname{spec} f} \operatorname{spec} B = H$$

is a monomorphism if $f: B \to A$ is a surjective map of rings; it is an epimorphism if equivalently $f$ is faithfully flat or injective [4, III,§3, n. 7]. If $G$ is a $p$-divisible formal group over $k$, then its $n$-torsion is a $k$-group which we shall denote $G[n]$.

We will need to consider various torsors [4, III,§4] and biextensions [8, 2]; these will always be assumed to have the multiplicative group $\mathbb{G}_m$ as the fibre. We identify $\mathbb{G}_m$-torsors with line bundles in the usual way. If $\mathcal{L}$ and $\mathcal{M}$ are two torsors over $X$ then their product will be denoted $\mathcal{L} \otimes \mathcal{M}$. It will be convenient to make use of the following "punctual" notation: if $f: Y \to X$ is a morphism in some category with a notion of torsor, and $\mathcal{L} \to X$ is a torsor, then we shall describe the torsor $f^*\mathcal{L} \to Y$ as the torsor whose fibre at a generic point $y$ of $Y$ is

$$(f^*\mathcal{L})_y = \mathcal{L}_{f(y)}.$$

Similarly a map of torsors $g\colon \mathcal{M}/Y \to \mathcal{L}/X$ covering $f$ may be described as a map

$$\mathcal{M}_y \xrightarrow{g} \mathcal{L}_{f(y)}.$$

All formal groups will be assumed commutative. A formal group $G$ over a ring $R$ defines a functor from the category of pro-Artin $R$-algebras to groups, and a "point" of $G$ will mean a pro-Artin $R$-algebra $T$ and an element $g \in G(T)$. We will often omit the ring $T$ from the notation, and even write $g \in G$.

Recall that if $G$ and $H$ are formal groups, then a biextension structure on a torsor $\mathcal{B} \to G \times H$ equips the torsor $\mathcal{B}\,|_{g\times H}$ with the structure of a central extension of $H$ by $\mathbb{G}_m$; its identity element will be ${}_g1$. Similarly for $h \in H$ the torsor $\mathcal{B}\,|_{G\times h}$ becomes a central extension of $G$ by $\mathbb{G}_m$ whose identity will be denoted $1_h$. We use the notations $\underset{L}{\bullet}$ and $\underset{R}{\bullet}$ respectively for the maps

$$\mathcal{B}_{g_1,h} \otimes \mathcal{B}_{g_2,h} \xrightarrow{\underset{L}{\bullet}} \mathcal{B}_{g_1+g_2,h}$$
$$\mathcal{B}_{g,h_1} \otimes \mathcal{B}_{g,h_2} \xrightarrow{\underset{R}{\bullet}} \mathcal{B}_{g,h_1+h_2}$$

which give the multiplication in the two central extensions.

Now let $\tau\colon G \times G \to G \times G$ be the map exchanging the two factors. A *symmetric* biextension of $G$ is a biextension $\mathcal{B} \to G \times G$ equipped with a map of biextensions

$$\xi\colon \mathcal{B} \to \tau^*\mathcal{B},$$

whose restriction to the diagonal is the canonical isomorphism of torsors [2, p. 8].

Finally, if $\mathcal{L}$ is a torsor over $G$, let $\Lambda(\mathcal{L})$ be the torsor over $G \times G$ whose fibre at $(g,h)$ is

$$\Lambda(\mathcal{L})_{g,h} \overset{\text{def}}{=} \frac{\mathcal{L}_{g+h}}{\mathcal{L}_g \otimes \mathcal{L}_h}.$$

A *cubical structure* on $\mathcal{L}$ is a particular sort of symmetric biextension structure on $\Lambda(\mathcal{L})$ [2, 2.2].

A *rigid* torsor over $G$ is a torsor $\mathcal{L}$ equipped with a section of the fibre $\mathcal{L}_0$ over the identity. If $\mathcal{L} \to G$ is a rigid torsor, then a *rigid* cubical structure on $\mathcal{L}$ is a cubical structure whose various identity sections coincide with the sections produced from the rigid structure.

If $R$ is a $k$-algebra, then $C^3(G)(R)$ will denote the group of rigid cubical structures on the trivial torsor over $G_R$. Such a thing is just a map $f\colon G_R^3 \to \mathbb{G}_m$ such that

$$\begin{aligned} & f(0,0,0) = 1 \\ & f(g_1,g_2,g_3) \text{ is symmetric in } g_1, g_2, \text{ and } g_3 \\ & f(g_1,g_2,g_3) f(g_0+g_1,g_2,g_3)^{-1} f(g_0,g_1+g_2,g_3) f(g_0,g_1,g_3)^{-1} = 1. \end{aligned} \tag{1.1}$$

A choice of coordinate on $G$ determines an isomorphism $\mathcal{O}_{G_R} \cong R[\![x]\!]$ and a formal group law $F$ over $k$. In these terms, an element of $C^3(G)(R)$ amounts to a power series $u(x_1,x_2,x_3)$ with coefficients in $R$, such that

$$\begin{aligned} & u(0,0,0) = 1 \\ & u(x_1,x_2,x_3) \text{ is symmetric in } x_1,\ x_2, \text{ and } x_3 \\ & u(x_1,x_2,x_3) u(x_0 +_F x_1, x_2, x_3)^{-1} u(x_0, x_1 +_F x_2, x_3) u(x_0,x_1,x_3)^{-1} = 1. \end{aligned}$$

It is then not hard to check (see [5]) that there is an initial example of a ring equipped with such a power series, and so $C^3(G)$ is represented by a $k$-group, which we shall also denote by $C^3(G)$.

$BU\langle 6\rangle$ **and Cubical Structures.** Let $BU\langle 6\rangle$ be the 5-connected cover of $BU$. The map

$$P^3 \to BU$$

classifying the virtual bundle $\prod_i^3 (1-L_i)$ lifts to a map

$$P^3 \xrightarrow{\prod_i^3(1-L_i)} BU\langle 6\rangle.$$

From it we obtain a map

$$K_0(P^3) \to K_0 BU\langle 6\rangle \tag{1.2}$$

whose adjoint

$$\Pi_3 \in K^0(P^3) \hat{\otimes} K_0 BU\langle 6\rangle \cong \mathcal{O}_{(G_K^3)_{K_0 BU\langle 6\rangle}} \tag{1.3}$$

is a rigid cubical structure on the trivial torsor over $(G_K)_{K_0 BU\langle 6\rangle}$. It is classified by a map

$$\operatorname{spec} K_0 BU\langle 6\rangle \xrightarrow{\Pi_3} C^3(G_K),$$

which we also denote by $\Pi_3$. The purpose of this note is to give a geometric proof of

**Theorem 1.4.** $\Pi_3$ *induces an isomorphism of $k$-groups*

$$\operatorname{spec} K_0(BU\langle 6\rangle) \cong C^3(G_K).$$

In [5], M. Hopkins and the authors prove the analogue of Theorem 1.4 for any even-periodic ring spectrum. The purpose of this paper is to give a proof inspired by the study of cubical structures by algebraic geometers in for example [8] and [2].

The main ingredient is a description of the $K$-homology of the fibration

$$K(\mathbb{Z},3) \xrightarrow{\gamma} BU\langle 6\rangle \to BSU \tag{1.5}$$

in terms of the geometry of cubical structures. Before proceeding with the main text, we give a brief account of this model. Some of the results we cite are fairly involved, for example [2, Corollary 7.13]. However none of these more difficult results is required by the arguments in this paper; with the Atiyah-Hirzebruch-Serre spectral sequence and Ravanel and Wilson's calculation [10] of $K_0 K(\mathbb{Z},3)$ in hand, one may proceed in a more elementary fashion.

Mumford [8] shows that if $G$ is a $p$-divisible formal group over $k$, then the functor of isomorphism classes of symmetric biextensions of $G$ is represented by the scheme $W(G)$ of *Weil pairings* for $G$. Associating to a cubical structure the Weil pairing of its underlying symmetric biextension gives a map

$$C^3(G) \xrightarrow{e_*} W(G). \tag{1.6}$$

The kernel of $e_*$ consists of cubical structures whose underlying symmetric biextension is trivial. For a $k$-algebra $R$ let $C^2(G)(R)$ be the group of maps $f\colon G_R^2 \to \mathbb{G}_m$ such that

$$\begin{aligned}
&f(0,0) = 1\\
&f(g_1,g_2) = f(g_2,g_1)\\
&f(g_1,g_2)f(g_0+g_1,g_2)^{-1}f(g_0,g_1+g_2)f(g_0,g_1)^{-1} = 1.
\end{aligned} \tag{1.7}$$

To give such an $f$ is precisely to give the structure of a commutative central extension on the trivial torsor over $G_R$. Remarks similar to those for $C^3(G)$ show that $C^2(G)$ is represented by a $k$-group. There is a map of $k$-groups

$$C^2(G) \xrightarrow{\delta} C^3(G)$$

(see (7.1)), and Proposition 2.11 of [2] shows that

$$C^2(G) \xrightarrow{\delta} C^3(G) \xrightarrow{e_*} W(G) \tag{1.8}$$

is exact.

In fact, it is a short exact sequence. The kernel of $\delta$ consists of symmetric bilinear maps from $G$ to $\mathbb{G}_m$, and it is easy to see (Lemma 7.3) that there is only one. One way to see that $e_*$ is surjective in the category of $k$-groups is to remark that the map

$$G \xrightarrow{2} G$$

is an isogeny, and in that case Breen [2, 7.13] shows that every symmetric biextension of $G$ by $\mathbb{G}_m$ may be refined, locally in the flat topology, to a cubical structure. We shall not need to be more precise, as the surjectivity of $e_*$ is a pleasant consequence (Corollary 7.6) of our comparison to the topological situation.

In any case, the sequence (1.8) is our model for the fibration (1.5). We shall show that there are isomorphisms

$$\begin{aligned}
b_*\colon \operatorname{spec} K_0K(\mathbb{Z},3) &\cong W(G_K)\\
\Pi_2\colon \operatorname{spec} K_0BSU &\cong C^2(G_K)
\end{aligned}$$

such that the diagram

$$\begin{tikzcd}
\operatorname{spec} K_0BSU \arrow[r] \arrow[d, "\Pi_2"', "\cong"] & \operatorname{spec} K_0BU\langle 6\rangle \arrow[r, "\operatorname{spec} K_0\gamma"] \arrow[d, "\Pi_3"] & \operatorname{spec} K_0K(\mathbb{Z},3) \arrow[d, "\cong"', "b_*"] \\
C^2(G_K) \arrow[r, "\delta"] & C^3(G_K) \arrow[r, "e_*"] & W(G_K)
\end{tikzcd} \tag{1.9}$$

commutes (up to a possible sign). That $\Pi_3$ is an isomorphism follows by the five-lemma. See Remark 4.4 for discussion of the sign.

The paper is organised as follows. In Section 2 we recall Ravenel and Wilson's calculation of $K_0K(\mathbb{Z},3)$ [10]. This gives the identification $b_*$ between $\operatorname{spec} K_0K(\mathbb{Z},3)$ and $W(G_K)$. In Section 3 we give an explicit formula for the map $e_*$. This makes it possible to show in Section 5 that the right-hand square of (1.9) commutes. Section 6 describes the isomorphism $\Pi_2$, and shows that the left-hand square of (1.9) commutes. In Section 7 we show that $\delta$ is injective, and finish the proof of Theorem 1.4.

## 2. Ravenel and Wilson's calculation of $K_0K(\mathbb{Z},3)$

In this section we express some calculations of Ravenel and Wilson [10] in the language of schemes.

**Definition 2.1.** An $e_n$-pairing for $G$ over a $k$-algebra $R$ is a map

$$f\colon\ G[n]_R^2 \to (\mathbb{G}_m)_R$$

of schemes over $\mathrm{spec}(R)$ such that

$$\begin{aligned} f(g_1+g_2,h) &= f(g_1,h)f(g_2,h)\\ f(g,h_1+h_2) &= f(g,h_1)f(g,h_2)\\ f(g,g) &= 1. \end{aligned}$$

The group of $e_n$ pairings for $G$ over $R$ will be denoted $W_n(G)(R)$. This functor of $R$ is represented by a scheme $W_n(G)$ (as one sees easily after introducing a coordinate).

**Remark 2.2.** More generally, we say that a map $f$ satisfying the first two conditions is *biexponential*, and that $f$ is *alternating* if it satisfies the third condition. We say that $f$ is *weakly alternating* if $f(g,h)f(h,g)=1$; by considering $f(g+h,g+h)$ we see that alternating biexponential maps are weakly alternating. Conversely, if $f$ is weakly alternating then $f(g,g)^2=1$ so $f(2g,2g)=f(g,g)^4=1$. If $p$ is odd then the map $g\mapsto 2g$ is an isomorphism and we see that $f$ is alternating.

**Proposition 2.3.** *There is a canonical isomorphism* $W_n(G_K)=\mathrm{spec}(K_0K(\mathbb{Z}/n,2))$.

*Proof.* Let $\mu:\ P[n]^2 = K(\mathbb{Z}/n,1)^2 \to K(\mathbb{Z}/n,2)$ be the map induced by the cup product in ordinary cohomology. This gives a map $m:\ \mathrm{spf}(K^0P[n])^2 \to \mathrm{spf}(K^0K(\mathbb{Z}/n,2))$ of formal schemes. As the cup product $H^1\times H^1\to H^2$ is weakly alternating and additive in both variables, we see that $m$ is a weakly alternating, biadditive map of formal group schemes.

We claim that $m$ is actually alternating in the strong sense. We may assume that $p=2$, for otherwise there is nothing to prove. It is enough to check that the composite

$$g \overset{\text{def}}{=} (P[n] \xrightarrow{\text{diagonal}} P[n]^2 \xrightarrow{\mu} K(\mathbb{Z}/n,2))$$

induces the trivial map on $\widetilde{K}_0P[n]$. If we compose with the Bockstein map $\beta\colon K(\mathbb{Z}/n,2)\to K(\mathbb{Z},3)$, we get a map in $[K(\mathbb{Z}/n,1),K(\mathbb{Z},3)]=H^3B(\mathbb{Z}/n)=0$. Moreover, $\beta$ fits in a fibration

$$K(\mathbb{Z}/n,1)=P[n]\to P \xrightarrow{[n]} P \xrightarrow{\rho} K(\mathbb{Z}/n,2) \xrightarrow{\beta} K(\mathbb{Z},3).$$

We deduce that $g$ factors through $\rho$, so it is enough to check that $\rho$ is trivial in reduced $K$-homology, or that $[n]$ induces an epimorphism in $K$-homology, or that $[n]$ induces a monomorphism in $K$-cohomology. This is true by well-known calculation, because $[n]^*(x)=x^{n^h}$.

We next claim that $\mathrm{spf}(K^0K(\mathbb{Z}/n,2))$ is the universal example of a formal group scheme equipped with an alternating biadditive map to it from $G_K[n]^2$. To see this, assume for the moment that $p>2$. In that case, Ravenel and Wilson show [10, 11.3] that $K_0(K(\mathbb{Z}/n,*))$ is the free Hopf ring generated over $k[\mathbb{F}_p]$ by the bicommutative Hopf algebra $K_0K(\mathbb{Z}/n,1)=K_0P[n]$. A Hopf ring is just a graded-commutative ring object in the category of cocommutative coalgebras over $k$, and this category is equivalent to that of formal schemes over $k$ by [3, p. 12]. Thus, $\mathrm{spf}(K^0K(\mathbb{Z}/n,*))$ is the free graded-commutative formal ring scheme generated over the constant scheme $\mathbb{F}_p$ by the formal group scheme $\mathrm{spf}(K^0K(\mathbb{Z}/n,1))=G_K[n]$. Moreover, such free objects are constructed in the obvious way: one can define colimits and tensor products of formal group schemes, so one can define $\Lambda^k(A)\overset{\text{def}}{=}A^{\otimes k}/\Sigma_k$ (where $\Sigma_k$ acts with signs) and these objects are the homogeneous pieces of the free ring scheme. These facts are implicit in the literature on Hopf rings, and an explicit treatment has recently been given by Hunton and Turner [6]. We conclude that $\mathrm{spf}(K^0K(\mathbb{Z}/n,2))=\Lambda^2G_K[n]$ as claimed.

For the case $p = 2$ we quote [7, Appendix]. They prove that $K_0K(\mathbb{Z}/n,*)$ is the free Hopf ring on $K_0K(\mathbb{Z}/n,1)$ modulo the relation that the squaring map $a \mapsto \sum a' \circ a''$ is trivial. Equivalently, we see that $\operatorname{spf}(K^0K(\mathbb{Z}/n,*))$ is the free graded-commutative formal ring scheme on $\operatorname{spf}(K^0K(\mathbb{Z}/n,1))$ modulo the relation that $\mu$ is strongly alternating. It follows again that $\operatorname{spf}(K^0K(\mathbb{Z}/n,2))$ has the required universal property.

We next use Cartier duality (see for example [3, p. 27]), which amounts to the fact that

$$\operatorname{spec}(K_0K(\mathbb{Z}/n,2)) = \operatorname{Hom}(\operatorname{spf}(K^0K(\mathbb{Z}/n,2)),\mathbb{G}_m).$$

Note that $K(\mathbb{Z}/n,2)$ is connected so the augmentation ideal in $K^0K(\mathbb{Z}/n,2)$ is topologically nilpotent, so $\operatorname{spf}(K^0K(\mathbb{Z}/n,2))$ is a connected formal neighbourhood of the identity element. Thus any map

$$\operatorname{spf}(K^0K(\mathbb{Z}/n,2)) \to \mathbb{G}_m$$

that sends 0 into $\widehat{\mathbb{G}}_m$ sends everything into $\widehat{\mathbb{G}}_m$, so $\operatorname{spec}(K_0K(\mathbb{Z}/n,2)) = \operatorname{Hom}(\operatorname{spf}(K^0K(\mathbb{Z}/n,2)),\widehat{\mathbb{G}}_m)$. Our description of $\operatorname{spf}(K^0K(\mathbb{Z}/n,2))$ immediately identifies this with $W_n(G_K)$. ☐

**Remark 2.4.** Ravenel and Wilson's more explicit calculations show that $K^0K(\mathbb{Z}/n,2)$ has finite dimension over $k$ so we have only used the simplest and most classical version of Cartier duality. In Lemma 6.2 we will use a version that applies in the infinite-dimensional case; this is treated in [3, p. 27].

Using Proposition 2.3, we have a map

$$b = b_n\colon \operatorname{spec} K_0(K(\mathbb{Z},3)) \xrightarrow{\operatorname{spec} K_0\beta_n} \operatorname{spec} K_0(K(\mathbb{Z}/n,2))) \cong W_n(G_K).$$

We also have a commutative diagram as follows:

$$\begin{tikzcd} K(\mathbb{Z}/pn,1)^2 \arrow[r, "\mu_{pm}"] \arrow[d] & K(\mathbb{Z}/pn,2) \arrow[r, "\beta_{pn}"] \arrow[d] & K(\mathbb{Z},3) \arrow[d, "p"] \\ K(\mathbb{Z}/n,1)^2 \arrow[r, "\mu_n"] & K(\mathbb{Z}/n,2) \arrow[r, "\beta_n"] & K(\mathbb{Z},3). \end{tikzcd} \tag{2.5}$$

When we apply the functor $\operatorname{spf}(K^0(-))$, the map $K(\mathbb{Z}/pn,1) \to K(\mathbb{Z}/n,1)$ becomes the map $p\colon G_K[pn] \to G_K[n]$. This implies that when $a$ is a point of $\operatorname{spec}(K_0K(\mathbb{Z},3))$ and $g$, $h$ are points of $G_K[pn]$ we have

$$b_{pn}(a)(g,h)^p = b_n(a)(pg,ph). \tag{2.6}$$

**Definition 2.7.** If $G$ is a formal group over $k$ and $R$ is a $k$-algebra, then a *Weil pairing on $G$ over $R$* is a collection $f_*$ of $e_n$-pairings

$$f_n \in W_n(G)(R)$$

such that

$$f_{pn}(g,h)^p = f_n(pg,ph).$$

For a $k$-algebra $R$, let $W(G)(R)$ be the group of Weil pairings on $G_R$. This is clearly represented by a closed subscheme $W(G) \subset \prod_n W_n(G)$.

**Remark 2.8.** Let $(f_n)$ be a collection of weakly alternating biexponential maps $G[n] \times G[n] \to \mathbb{G}_m$ such that $f_{pn}(x,y)^p = f_n(px,py)$. If $p > 2$ then any weakly alternating map is alternating. If $p = 2$ then $f_n(2x,2x) = f_{2n}(x,x)^2$ and this is 1 because $f_{2n}$ is weakly alternating. If $G$ has finite height then the map $2\colon G[2n] \to G[n]$ is faithfully flat and thus an epimorphism, so $f_n(y,y) = 1$ for all $y$, so $f$ is alternating. Thus, it does not matter whether we specify alternating or weakly alternating maps in the definition of a Weil pairing.

**Remark 2.9.** If $G$ is $p$-divisible and $f$ is an $e_{pn}$-pairing then it is not hard to check that there is a unique map $f'\colon G[n]^2 \to \mathbb{G}_m$ such that $f'(pg,ph) = f(g,h)^p$, and that this is an $e_n$-pairing. This construction gives a map $q\colon W_{pn}(G) \to W_n(G)$ and it is clear that $W(G) = \operatorname{invlim}_r W_{p^r}(G)$. One can also check that under our identification $\operatorname{spec}(K_0K(\mathbb{Z}/p^n,2)) = W_n(G_K)$, the Bockstein map $K(\mathbb{Z}/n,2) \xrightarrow{\beta'} K(\mathbb{Z}/pn,2)$ has $\operatorname{spec}(K_0\beta') = q$.

The maps $b_n$ clearly fit together to give a map $b_*\colon \operatorname{spec}(K_0K(\mathbb{Z},3)) \to W(G_K)$.

**Proposition 2.10.** *The map $b_*\colon \operatorname{spec}(K_0K(\mathbb{Z},3)) \to W(G_K)$ is an isomorphism.*

*Proof.* Ravenel and Wilson show that the map

$$K_0K(\mathbb{Z},3) \to \operatorname*{colim}_r K_0K(\mathbb{Z}/p^r,2).$$

is an isomorphism. □

## 3. The Weil pairing of a cubical structure

Mumford [8] shows that a cubical structure gives rise in a functorial way to a Weil pairing; see also [2]. Thus, there is a canonical map

$$C^3(G) \xrightarrow{e_*} W(G).$$

In this section we give an explicit formula for $e_*$. Lemmas 3.2 and 3.3 are standard facts about biextensions; see for example [2, Chapter 4].

If $\mathcal{B} \to G \times H$ is any biextension, there is a canonical isomorphism

$$(n \times 1)^*\mathcal{B} \cong (1 \times n)^*\mathcal{B} \tag{3.1}$$

of torsors over $G \times H$, given by

$$(n \times 1)^*\mathcal{B} \xleftarrow{[n]_L} \mathcal{B}^n \xrightarrow{[n]_R} (1 \times n)^*\mathcal{B},$$

where $\mathcal{B}^n$ denotes the torsor over $G \times H$ whose fibre at $(g,h)$ is the $n$-fold product

$$\mathcal{B}_{g,h} \otimes \mathcal{B}_{g,h} \otimes \ldots \otimes \mathcal{B}_{g,h}.$$

**Lemma 3.2.** *The isomorphism* (3.1) *is an isomorphism of biextensions.* □

**Lemma 3.3.** *Any biextension $\mathcal{B} \to G \times H$ has a canonical trivialisation when restricted to $0 \times H$ or $G \times 0$, given by*

$$G \ni g \mapsto {}_g1$$
$$H \ni h \mapsto 1_h.$$

□

Combining Lemmas 3.2 and 3.3 gives an automorphism

$$\underline{1} \cong (n \times 1)^*\mathcal{B} \cong (1 \times n)^*\mathcal{B} \cong \underline{1}$$

of the trivial biextension over $G[n] \times H[n]$, or equivalently a biexponential map

$$G[n] \times H[n] \to \mathbb{G}_m.$$

Now suppose that $G = H$; that $\mathcal{L} \to G$ is a torsor; and $t$ is a trivialisation of $\mathcal{L}$. Let

$$s(g,h) \overset{\text{def}}{=} \frac{t(g+h)}{t(g)t(h)}$$

be the resulting trivialisation of $\Lambda(\mathcal{L})$. A cubical structure on $\mathcal{L}$ gives rise to a function

$$u\colon G^3 \to \mathbb{G}_m$$

satisfying the equations (1.1) and such that the biextension structure on $\Lambda(\mathcal{L})$ is given by the formulae

$$s(g_1, g_3) \underset{L}{\bullet} s(g_2, g_3) = u(g_1, g_2, g_3) s(g_1 + g_2, g_3)$$
$$s(g_1, g_2) \underset{R}{\bullet} s(g_1, g_3) = u(g_1, g_2, g_3) s(g_1, g_2 + g_3).$$

**Proposition 3.4.** *The $e_n$ pairing associated to $u$ is given by the explicit formula*

$$e_n(g, h) = e_n(u)(g, h) = \prod_{j=1}^{n-1} \frac{u(g, jg, h)}{u(g, jh, h)}. \tag{3.5}$$

*Proof.* The isomorphism

$$\mathcal{B}^n \cong (n \times 1)^* \mathcal{B}$$

over $G \times G$ is given by the formula

$$\underbrace{s(g, h) \dots s(g, h)}_{n \text{ terms}} \mapsto \left( \prod_{j=1}^{n-1} u(g, jg, h) \right) s\big(ng, h\big), \tag{3.6}$$

as we see by induction on $n$. If $ng = 0$ then

$$s(ng, h) = s(0, h) = 1_h.$$

Following the recipe for $e_n$ given above yields formula (3.5). □

**Remark 3.7.** Note that the formula (3.5) makes sense on $G^2$, and so $e_n(u)$ may be regarded as a function $G^2 \to \mathbb{G}_m$. Proposition (3.4) implies that this function gives a root of unity when evaluated on $G[n]^2$.

It is particularly transparent in formula (3.6) that

**Lemma 3.8.**

$$e_n(g, g) = 1$$

$$e_n(g, h) = e_n(h, g)^{-1}.$$

□

Formula (3.5) also shows directly that

**Lemma 3.9.**

$$e_{pn}(g, h)^p = e_n(pg, ph). \tag{3.10}$$

*Proof.* Consider the section $s(g, h)^{np^2}$ of $\mathcal{B}^{np^2}$. The axioms of a biextension guarantee that all ways of multiplying these to an element of $\mathcal{B}$ over $(npg, ph)$ are the same. It follows that there is an equation

$$\left[ \prod_{j=1}^{np-1} u(g, jg, h) \right]^p \left[ \prod_{j=1}^{p-1} u(npg, jh, h) \right] = \left[ \prod_{j=1}^{p-1} u(g, jg, h) \right]^{np} \left[ \prod_{j=1}^{p-1} u(pg, jh, h) \right]^p \left[ \prod_{j=1}^{n-1} u(pg, jpg, ph) \right]$$

Similarly, all ways of multiplying these to an element of $\mathcal{B}$ over $(pg, nph)$ are the same, so there is an equation

$$\left[ \prod_{j=1}^{np-1} u(g, jh, h) \right]^p \left[ \prod_{j=1}^{p-1} u(g, jg, nph) \right] = \left[ \prod_{j=1}^{p-1} u(g, jg, h) \right]^{np} \left[ \prod_{j=1}^{p-1} u(pg, jh, h) \right]^p \left[ \prod_{j=1}^{n-1} u(pg, jph, ph) \right].$$

Dividing the first by the second, we get

$$\left[\prod_{j=1}^{np-1} \frac{u(g,jg,h)}{u(g,jh,h)}\right]^p \left[\prod_{j=1}^{p-1} \frac{u(npg,jh,h)}{u(g,jg,nph)}\right] = \prod_{j=1}^{n-1} \frac{u(pg,jpg,ph)}{u(pg,jph,ph)}$$

Setting $npg = 0 = nph$ gives the result. □

From Lemmas 3.8 and 3.9, one has

**Proposition 3.11.** *There is a unique map*

$$e_*\colon\ C^3(G) \to W(G)$$

*whose image in $W_n(G)$ is $e_n$.* □

## 4. The bundle associated to the $e_n$-pairing

Let $V$ be a virtual complex bundle over a space $X$, with a chosen lifting $v\colon\ X \to BU\langle 6\rangle$ of the classifying map $X \to \mathbb{Z} \times BU$. Given a class $c \in K^0 BU\langle 6\rangle$ we obtain an element $v^*c \in K^0X$ which we can think of as a characteristic class of $v$ (or by abuse, of $V$). Alternatively, the map $v_*\colon\ K_0X \to K_0BU\langle 6\rangle$ can be regarded as an element $u \in K^0X\widehat{\otimes}K_0BU\langle 6\rangle$, which we call the *total characteristic class* of $V$.

Note that if $L_1$, $L_2$ and $L_3$ are line bundles over $X$ then each virtual bundle $1 - L_i$ has a unique lifting to $BU$, which is the second space in the connective complex $K$-theory spectrum $ku$, which is a ring spectrum. This gives a natural lift of $\prod_i^3(1-L_i)$ to the sixth space of the $ku$ spectrum, which is $BU\langle 6\rangle$. Thus, if $V$ is an arbitrary virtual bundle, then an expression for $V$ as a sum of virtual bundles of the form $\prod_i^3(1-L_i)$ gives rise to a lifting of $V$ to $BU\langle 6\rangle$.

Motivated by Proposition 3.4, we make the following definition.

**Definition 4.1.** If $L_1$, $L_2$ are line bundles over a space $X$, we put

$$d_n(L_1,L_2) \overset{\text{def}}{=} \sum_{j=1}^{n-1}(1-L_1)(1-L_1^j)(1-L_2) - \sum_{j=1}^{n-1}(1-L_1)(1-L_2^j)(1-L_2) \in ku^6X = [X, BU\langle 6\rangle].$$

Note that if we forget about liftings to $BU\langle 6\rangle$ and just work with virtual bundles we have

$$\begin{aligned} d_n(L_1,L_2) &= (1-L_1)(1-L_2)\left(\sum_{j=1}^{n-1} L_1^j - L_2^j\right) \\ &= (1-L_2)(1-L_1^n) - (1-L_1)(1-L_2^n) \\ &= -L_2 - L_1^n + L_1^nL_2 + L_1 + L_2^n - L_1L_2^n. \end{aligned}$$

The main ingredient in the proof of Theorem 1.4 is the following compatibility.

**Theorem 4.2.** *Let $L_1$ and $L_2$ be the obvious line bundles over $P^2$. Then for each $n$, the following diagram commutes (up to a possible sign).*

$$\begin{tikzcd} P[n]^2 \arrow[d, "\beta\mu"'] \arrow[dr, "d_n(L_1,L_2)"] & \\ K(\mathbb{Z},3) \arrow[r, "\gamma"'] & BU\langle 6\rangle \end{tikzcd} \tag{4.3}$$

**Remark 4.4.** Let us say that a sequence of spectra

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

is a $\pm$-fibration if either $(f, g, h)$ is a fibration, or $(f, g, -h)$ is a fibration. This definition is independent of all conventions such as which end of the unit interval we take as the basepoint, which side we write cone and suspension coordinates and so on. Cofibrations are $\pm$-fibrations, and $\pm$-fibrations are preserved by shifting, suspending and dualising. Suppose we have a diagram as follows, in which the square commutes up to sign and the rows are $\pm$-fibrations.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\
{\scriptstyle u}\downarrow & & & & {\scriptstyle w}\downarrow & & {\scriptstyle \Sigma u}\downarrow \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & \Sigma X'
\end{array}$$

Then there is a map $v\colon Y \to Y'$ making everything commute up to sign; moreover, if $[Z, Y'] = 0$ then $w$ is unique up to sign. These facts are easily deduced from the corresponding ones for genuine fibrations. The sign ambiguity in Theorem 4.2 could be resolved by a careful analysis of conventions to determine whether certain sequences are fibrations, or whether a sign needs to be changed to make this true. We have not felt it worthwhile to pursue these questions.

At the end of this section we give a short proof of a weaker result (Lemma 4.6); the next section constitutes the rather harder proof of the theorem itself. First, however, we draw an important corollary.

**Corollary 4.5.** *The diagram*

$$\begin{array}{ccc}
\operatorname{spec} K_0(BU\langle 6\rangle) & \xrightarrow{\operatorname{spec} K_0(\gamma)} & \operatorname{spec} K_0(K(\mathbb{Z}, 3)) \\
{\scriptstyle \Pi_3}\downarrow & & \downarrow{\scriptstyle b_*} \\
C^3(G_K) & \xrightarrow{e_*} & W(G_K)
\end{array}$$

*commutes (up to a possible sign).*

*Proof.* By the definition of $W$, it suffices to check that that $b_n \operatorname{spec} K_0(\gamma) = (e_n \Pi_3)^\epsilon$ in $W_n(G_K)$ for arbitrary $n$, where $\epsilon = \pm 1$ and is independent of $n$. In fact the independence of $n$ is automatic: if $v_*$ and $w_*$ are two Weil pairings, and

$$v_n(g, h) = w_n(g, h)^{\epsilon(n)},$$

then

$$v_{pn}(pg, ph) = v_n(g, h)^p = w_n(g, h)^{p\epsilon(n)} = w_{pn}(pg, ph)^{\epsilon(n)};$$

since $p\colon G_K \to G_K$ is surjective, it follows that $\epsilon(pn) = \epsilon(n)$.

Moreover, $W_n(G_K)$ is a subscheme of the scheme of maps $G_K[n]^2 \to \mathbb{G}_m$, so it suffices to check that the two adjoint maps $G_K[n]^2 \times \operatorname{spec}(K_0 BU\langle 6\rangle) \to \mathbb{G}_m$ are the same.

From now on we fix $n$ and write $\beta$ for $\beta_n$ and so on.

Note that any element $z \in kU^6 Z = [Z, BU\langle 6\rangle]$ gives rise to a map $z_*\colon K_0 Z \to K_0 BU\langle 6\rangle$ or equivalently an element $\hat{z} \in K^0 Z \widehat{\otimes} K_0 BU\langle 6\rangle$ which may be viewed as a map

$$\hat{z}\colon \operatorname{spf}(K^0 Z) \times \operatorname{spec}(K_0 BU\langle 6\rangle) \to \mathbb{G}_m.$$

This construction converts sums to products and is natural in $Z$. If $z = \prod_i^3 (1 - L_i) \in ku^6 P^3$ then $\hat{z}:$ $G_K^3 \times \operatorname{spec}(K_0 BU\langle 6\rangle) \to \mathbb{G}_m$ is just the composite

$$G_K^3 \times \operatorname{spec}(K_0 BU\langle 6\rangle) \to G_K^3 \times C^3(G_K) \xrightarrow{E} \mathbb{G}_m,$$

where $E(g_1, g_2, g_3, f) = f(g_1, g_2, g_3)$. It follows by naturality that if $z = (1 - L_1)(1 - L_1^k)(1 - L_2) \in ku^6 P^2$ then $\hat{z}$ corresponds to the map $(g_1, g_2, f) \mapsto f(g_1, kg_1, g_2)$, and thus that $d_n(L_1, L_2)^\wedge$ corresponds to the map

$$(g_0, g_1, f) \mapsto \prod_{k=1}^{n-1} \frac{f(g_0, kg_0, g_1)}{f(g_0, kg_1, g_1)} = e_n(f)(g_0, g_1).$$

This is what we get by going around the bottom left of the square in the statement of the corollary.

A similar procedure converts elements $w \in H^3 Z$ to maps $\hat{w}\colon\ \mathrm{spf}(K^0 Z) \times \mathrm{spec}(K_0 K(\mathbb{Z}, 3)) \to \mathbb{G}_m$. If $w = \beta(a_1 a_2) \in H^3 P[n]^2$ then

$$\hat{w}\colon\ G[n]^2 \times \mathrm{spec}(K_0 K(\mathbb{Z}, 3)) \to \mathbb{G}_m$$

is adjoint to

$$bm\colon\ G[n]^2 \to \mathrm{spf}(K^0 K(\mathbb{Z}, 3)),$$

essentially by the definition of $b$. By an obvious naturality statement we see that $\hat{w} \circ \mathrm{spec}(K_0(\gamma))$ is adjoint to $\widehat{\gamma_* w}$, but in the case $w = \beta(a_1 a_2)$ the theorem gives $\gamma_* w = \pm d_n(L_1, L_2)$. This implies that $\widehat{\gamma^* w} = (\pm d_n(L_1, L_2))^\wedge$, which is adjoint to $(e_n \Pi_3)^{\pm 1}$. ☐

We next give the promised crude version of Theorem 4.2.

**Lemma 4.6.** *There is a unique $\lambda = \lambda_n \in \mathbb{Z}/n$ such that the following diagram commutes.*

$$\begin{tikzcd}
P[n]^2 \arrow[d, "\lambda\beta\mu"'] \arrow[dr, "d_n(L_1,L_2)"] & \\
K(\mathbb{Z}, 3) \arrow[r, "i"'] & BU\langle 6\rangle
\end{tikzcd}$$

*Proof.* For brevity we put $d \stackrel{\mathrm{def}}{=} d_n(L_1, L_2)\colon\ P[n]^2 \to BU\langle 6\rangle$. As bundles we have

$$d = (1 - L_2)(1 - L_1^n) - (1 - L_1)(1 - L_2^n),$$

and we work over $P[n]^2$ so $L_1^n = L_2^n = 1$ so $d = 0$. This means that the map $P[n]^2 \xrightarrow{d} BU\langle 6\rangle \xrightarrow{f} BSU \xrightarrow{g} BU$ is null. The fibre of $g$ is $S^1 = K(\mathbb{Z}, 1)$ and $[P[n]^2, K(\mathbb{Z}, 1)] = H^1 P[n]^2 = 0$ so we see that $fd = 0$. The fibre of $f$ is $K(\mathbb{Z}, 3) \xrightarrow{\gamma} BU\langle 6\rangle$ so there is a map $d'\colon\ P[n]^2 \to K(\mathbb{Z}, 3)$ with $d = \gamma d'$. The fibre of $\gamma$ is $SU$, which is a retract of $U = S^1 \times SU$, and $[P[n]^2, U] = KU^1 P[n]^2 = 0$ by well-known calculations, so $d'$ is unique. Further easy calculations show that $[P[n]^2, K(\mathbb{Z}, 3)] = H^3(P[n]^2) = \mathbb{Z}/n$, generated by $\beta\mu$. Thus there is a unique $\lambda \in \mathbb{Z}/n$ such that $d' = \lambda\beta\mu$ as claimed. ☐

## 5. The proof of Theorem 4.2

We shall obtain the commutativity of (4.3) by looping down the corresponding diagram of spectra, which we shall see commutes. So let

$$\begin{aligned}
H &\stackrel{\mathrm{def}}{=} \text{the integral Eilenberg-MacLane spectrum} \\
H/n &\stackrel{\mathrm{def}}{=} \text{the cofibre of } H \xrightarrow{n} H \\
ku &\stackrel{\mathrm{def}}{=} \text{the connective complex } K\text{-theory spectrum} \\
B &\stackrel{\mathrm{def}}{=} \text{the sphere bundle of } L^{\otimes n} \text{ over } \mathbb{C}P^\infty \\
P &\stackrel{\mathrm{def}}{=} \text{the disk bundle of } L^{\otimes n} \text{ over } \mathbb{C}P^\infty
\end{aligned}$$

Let us write $j$ for the inclusion

$$j\colon B \to P,$$

which can be identified up to homotopy with the map $P[n] \to P$ or the map $\beta\colon K(\mathbb{Z}/n,1) \to K(\mathbb{Z},2)$. Let us label as $q$ and $\delta$ the maps in the cofibration sequence

$$B \xrightarrow{j} P \xrightarrow{q} T = P/B \xrightarrow{\delta} \Sigma B.$$

Then $T$ is the Thom space of $L^{\otimes m}$ over $P$.

Let us use the notations

$$\begin{aligned} y&\colon P \to \Sigma^2 H \\ x&\colon P \to \Sigma^2 ku \\ v&\colon S^2 \to ku \\ \sigma&\colon ku \to H \\ \rho&\colon H \to H/n \end{aligned}$$

to denote

i. the ordinary Euler class of $L$
ii. the $ku$ Euler class corresponding to the map $(1-L)\colon P \to BU$
iii. the $ku$ class corresponding to the reduced Hopf bundle $(1-H)$ over $S^2$.
iv. the ring map such that $\sigma x = y\colon P \to \Sigma^2 H$.
v. the reduction map

Note that there are $\pm$-fibration sequences

$$\Sigma^2 ku \xrightarrow{v} ku \xrightarrow{\sigma} H \xrightarrow{\gamma} \Sigma^3 ku$$
$$H \xrightarrow{n} H \xrightarrow{\rho} H/n \xrightarrow{\beta} \Sigma H.$$

The fibration 1.5 is obtained by looping down the first of these.

The classes $x$ and $y$ determine complex orientations for $ku$ and $H$. Let $u \in ku^2(T)$ and $w \in H^2(T)$ be the resulting Thom classes.

Given a space $X$, a spectrum $E$ and a class $z \in E^k X = [\Sigma^\infty X, \Sigma^k E]$, we write $\Omega^\infty(z)$ for the adjoint map $X \to \Omega^\infty \Sigma^k E$ of spaces; this is of course a mild abuse.

We will need the following simple lemma.

**Lemma 5.1.** *Suppose we have a diagram as follows, in which the left and right squares commute up to sign, the rows are $\pm$-fibrations, and $[Z, Y'] = 0$.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r, "g"] \arrow[d, "v"'] & Z \arrow[r, "h"] \arrow[d, "w"'] & \Sigma X \arrow[d, "\Sigma u"'] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & \Sigma X' \end{tikzcd}$$

*Then the central square also commutes up to sign.*

*Proof.* As in Remark 4.4, there is a map $v'\colon Y \to Y'$ making the whole diagram commute up to sign. After changing the sign of $v$ in necessary, we have $vf = v'f = \pm f'u$. Thus $(v - v')f = 0$, so $v - v' = rg$ for some $r\colon Z \to Y'$, but $[Z, Y'] = 0$ so $v = v'$. Thus, the central square comutes up to sign. ∎

**Modelling $\beta\mu$.** The obvious generator $a \in H^1(B;\mathbb{Z}/n)$ corresponds to a map $a$ making the diagram

$$\begin{tikzcd} B \arrow[r, "j"] \arrow[d, "a"'] & P \arrow[r, "q"] \arrow[d, "y"] & T \arrow[r, "\delta"] \arrow[d, "w"] & \Sigma B \arrow[d, "a"] \\ \Sigma H/n \arrow[r, "\beta_n"] & \Sigma^2 H \arrow[r, "n"] & \Sigma^2 H \arrow[r, "\rho"] & \Sigma^2 H/n \end{tikzcd} \tag{5.2}$$

commute up to sign. (The first two squares commute by well-known calculations, and $[\Sigma B, \Sigma^2 H] = H^1 B = 0$ so the third square commutes up to sign by Lemma 5.1.)

Let $\beta(a_1 a_2)$ be the map making the diagram

$$\begin{tikzcd} B^{(2)} \arrow[r, "a \wedge a"] \arrow[d, "\beta(a_1 a_2)"'] & (\Sigma H/n)^{(2)} \arrow[d, "\mu"] \\ \Sigma^3 H & \Sigma^2 H/n \arrow[l, "\beta"'] \end{tikzcd}$$

commute. The map $\Omega^\infty \beta(a_1 a_2)$ is a model for $\beta\mu$.

**Modelling $d_n(L_1, L_2)$.** For each $k \geq 0$ let $[k](z)$ be the polynomial

$$[k](z) \overset{\text{def}}{=} \frac{1-(1-vz)^k}{v} \in ku^*[z].$$

As Thom classes restrict to Euler classes on the zero-section, we have

$$q^* w = [n](x) \in ku^2(P).$$

Let $x_i \in ku^2(P^{(2)})$ be the Euler class of $L_i$ for $i = 1, 2$. If

$$d \overset{\text{def}}{=} x_1 x_2 \sum_{k=1}^{n-1} \big([k](x_1) - [k](x_2)\big) \in ku^6(P^{(2)}) < ku^6(P^2),$$

then from Definition 4.1 it is clear that $\Omega^\infty d = d_n(L_1, L_2)$.

Combining these definitions and observations yields

**Lemma 5.3.** *To prove Theorem 4.2, it suffices to show that the diagram*

$$\begin{tikzcd} B^{(2)} \arrow[r, "j \wedge j"] \arrow[d, "\beta(a_1 a_2)"'] & P^{(2)} \arrow[d, "d"] \\ \Sigma^3 H \arrow[r, "\gamma"] & \Sigma^6 ku \end{tikzcd} \tag{5.4}$$

*commutes up to sign.* □

Let us write $r$ and $\Delta$ for the indicated maps in the cofibrations

$$B^{(2)} \xrightarrow{j \wedge j} P^{(2)} \xrightarrow{r} P^{(2)}/B^{(2)} \xrightarrow{\Delta} \Sigma B^{(2)}.$$

**Lemma 5.5.** *There is a commutative diagram of the form*

$$
\begin{tikzcd}
\Sigma B^{(2)} & B \wedge T \arrow[l, "B\wedge\delta"'] \arrow[d, "B_1"'] \arrow[dr, "j\wedge T"] & \\
T \wedge B \arrow[u, "\delta\wedge B"] \arrow[r, "B_2"'] \arrow[dr, "T\wedge j"'] & P^{(2)}/B^{(2)} \arrow[ul, "\Delta"] \arrow[r, "A_2"'] \arrow[d, "A_1"'] & P \wedge T \\
& T \wedge P & P^{(2)} \arrow[l, "q\wedge P"] \arrow[ul, "r"] \arrow[u, "P\wedge q"']
\end{tikzcd}
\tag{5.6}
$$

*in which the linear sequences are cofibrations.*

*Proof.* The map $A_1$ is just the projection

$$\frac{P \wedge P}{B \wedge B} \to \frac{P \wedge P}{B \wedge P} = \frac{P}{B} \wedge P = T \wedge P,$$

and similarly for $A_2$. The map $B_1$ is obtained from the inclusion $B \wedge P \to P \wedge P$ by collapsing out the subcomplex $B \wedge B$ to get a map

$$B \wedge T = \frac{B \wedge P}{B \wedge B} \to \frac{P \wedge P}{B \wedge B},$$

and similarly for $B_2$. From these definitions we see directly that all parts of the diagram not involving $\Delta$ commute, and that the middle row and column are cofibrations. To see that $\Delta B_1 = B \wedge \delta$, recall the naturality of connecting maps and examine the following diagram. (Equivalently, one can think of this as an instance of the octahedral axiom.)

$$
\begin{tikzcd}
B^{(2)} \arrow[r, "B\wedge j"] \arrow[d, "="'] & B \wedge P \arrow[r] \arrow[d, "j\wedge B"] & B \wedge T \arrow[r, "B\wedge\delta"] \arrow[d, "B_1"] & \Sigma B^{(2)} \arrow[d, "="] \\
B^{(2)} \arrow[r, "j\wedge j"] \arrow[d] & P^{(2)} \arrow[r] \arrow[d] & P^{(2)}/B^{(2)} \arrow[r, "\Delta"] \arrow[d, "A_1"] & \Sigma B^{(2)} \\
* \arrow[r] & T \wedge P \arrow[r, equal] & T \wedge P &
\end{tikzcd}
$$

∎

Now let $f\colon P^{(2)}/B^{(2)} \to \Sigma^4 ku$ be the map

$$f \overset{\text{def}}{=} \mu(u \wedge x)A_1 - \mu(x \wedge u)A_2$$

where $\mu\colon ku \wedge ku \to ku$ is the multiplication.

**Lemma 5.7.** *To prove Theorem 4.2, it suffices to show that the diagram*

$$
\begin{tikzcd}
P^{(2)}/B^{(2)} \arrow[r, "\Delta"] \arrow[d, "f"'] & \Sigma B^{(2)} \arrow[d, "\beta(a_1 a_2)"] \\
\Sigma^4 ku \arrow[r, "\sigma"] & \Sigma^4 H
\end{tikzcd}
\tag{5.8}
$$

*commutes up to sign.* ∎

*Proof.* Consider the following diagram.

$$\begin{tikzcd}
P^{(2)}/B^{(2)} \arrow[r, "\Delta"] \arrow[d, "f"'] & \Sigma B^{(2)} \arrow[r, "j\wedge j"] \arrow[d, "\beta(a_1a_2)"] & \Sigma P^{(2)} \arrow[r, "r"] \arrow[d, "d"] & \Sigma^2 P^{(2)}/B^{(2)} \arrow[d, "f"] \\
\Sigma^4 ku \arrow[r, "\sigma"] & \Sigma^4 H \arrow[r, "\gamma"] & \Sigma^7 ku \arrow[r, "v"] & \Sigma^5 ku
\end{tikzcd} \tag{5.9}$$

We first show that the right-hand square commutes. One composite is

$$\begin{aligned}
fr &= \mu(u\wedge x)A_1 r - \mu(x\wedge u)A_2 r \\
&= \mu(u\wedge x)(q\wedge P) - \mu(x\wedge u)(P\wedge q) && \text{by (5.6)} \\
&= [n](x_1)x_2 - x_1[n](x_2).
\end{aligned}$$

The other composite is

$$\begin{aligned}
vd &= x_1x_2\sum_{k=1}^{n-1}(1-vx_1)^k - (1-vx_2)^k \\
&= x_1x_2\left(\frac{1-(1-vx_1)^n}{vx_1} - \frac{1-(1-vx_2)^n}{vx_2}\right) \\
&= x_2[n](x_1) - x_1[n](x_2),
\end{aligned}$$

as required.

We also have $[\Sigma P^{(2)}, \Sigma^4 H] = H^3 P^{(2)} = 0$. Thus, if we have $\beta(a_1a_2)\circ\Delta = \pm\sigma f$ we can apply Lemma 5.1 to see that $e\circ(j\wedge j) = \pm\gamma d$, and the claim then follows by Lemma 5.3. $\square$

**Lemma 5.10.** *We have*

$$n\sigma f = 0 = n\beta(a_1a_2)\Delta$$

*in* $H^4(P^{(2)}/B^{(2)})$.

*Proof.* It is clear that $n\beta(a_1a_2) = 0$, so it suffices to check that $n\sigma f = 0$. Now by definition,

$$\sigma f = \mu(w\wedge y)A_1 - \mu(y\wedge w)A_2.$$

The commutative diagram

$$\begin{tikzcd}
P \arrow[r, "q"] \arrow[d, "y"'] & T \arrow[d, "w"] \\
\Sigma^2 H \arrow[r, "n"] & \Sigma^2 H
\end{tikzcd}$$

gives the commutative triangles in the diagram

$$\begin{tikzcd}
P^{(2)}/B^{(2)} \arrow[r, "B_1"] \arrow[d, "B_2"'] & T\wedge P \arrow[d, "T\wedge q"] \arrow[ddr, "w\wedge ny"] & \\
P\wedge T \arrow[r, "q\wedge T"] \arrow[drr, "ny\wedge w"'] & T\wedge T \arrow[dr, "w\wedge w"] & \\
& & \Sigma^4 H
\end{tikzcd}$$

commute. The square commutes because either composite is just the projection

$$\frac{P\wedge P}{B\wedge B} \to \frac{P\wedge P}{B\wedge P\cup P\wedge B}.$$

The difference between the two outermost composites is $n\sigma f$. $\square$

**Lemma 5.11.** *The maps*

$$B_1^*\colon\ H^4(P^{(2)}/B^{(2)}) \to H^4(B \wedge T)$$
$$B_2^*\colon\ H^4(P^{(2)}/B^{(2)}) \to H^4(T \wedge B)$$

*are such that* $\mathrm{Ker}(B_1^*) \cap \mathrm{Ker}(B_2^*)$ *is torsion-free.*

*Proof.* In the diagram

$$\begin{tikzcd}
B^{(2)} \arrow[r] \arrow[d, "="'] & (B \wedge P) \cup (P \wedge B) \arrow[r] \arrow[d] & (B \wedge T) \cup (T \wedge B) \arrow[d, "B_1 \cup B_2"] \\
B^{(2)} \arrow[r] \arrow[d] & P^{(2)} \arrow[r] \arrow[d] & P^{(2)}/B^{(2)} \arrow[d] \\
* \arrow[r] & T^{(2)} \arrow[r, equal] & T^{(2)},
\end{tikzcd}$$

the rows and columns are cofibrations, and so we have an exact sequence

$$H^3(B \wedge T) \oplus H^3(T \wedge B) \to H^4(T^{(2)}) \to H^4(P^{(2)}/B^{(2)}) \xrightarrow{B_1^* \oplus B_2^*} H^4(B \wedge T) \oplus H^4(T \wedge B).$$

It is easy to check that the first term is torsion, and the second term is $\mathbb{Z}$. Thus, the first map must be zero, and the kernel of the third map must be $\mathbb{Z}$. $\square$

*Proof of Theorem 4.2.* By Lemma 5.7, it suffices to check that the element

$$z \overset{\text{def}}{=} \sigma f + \beta(a_1 a_2)\Delta \in H^4 P^{(2)}/B^{(2)}$$

is zero. By Lemma 5.10 we have $nz = 0$, so by Lemma 5.11, it suffices to check that $B_1^* z = B_2^* z = 0$.

For $B_1^* z$ we have

$$\begin{aligned}
\beta(a_1 a_2)\Delta B_1 &= \beta(a_1 a_2)(B \wedge \delta) && \text{by Lemma 5.5} \\
&= \beta\mu(a \wedge a)(B \wedge \delta) \\
&= \beta\mu(a \wedge a\delta) \\
&= \beta\mu(a \wedge \rho w) && \text{by diagram (5.2)} \\
&= \beta(a_1 \rho w_2) \\
&= (y_1 j) w_2 && \text{by diagram (5.2).}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sigma f B_1 &= \mu(w \wedge y) A_1 B_1 - \mu(y \wedge w) A_2 B_1 \\
&= \mu(y \wedge w)(T \wedge j) && \text{by Lemma 5.5} \\
&= -(y_1 j) w_2.
\end{aligned}$$

Thus $B_1^* z = \beta(a_1 a_2)\Delta B_1 + \sigma f B_1 = 0$ as claimed. The case of $B_2^* z$ is similar.

$\square$

## 6. Morava $K$-theory of $BSU$

The Morava $K$-theory of $BSU$ is accessible through the fibration

$$BSU \to BU \xrightarrow{B\det} P. \tag{6.1}$$

Note that the inclusion $i\colon S^1 = U(1) \to U$ gives $Bi\colon P \to BU$ which is a non-additive splitting of $B\det$, and the sum of $Bi$ with the inclusion of $BSU$ gives an equivalence $P \times BSU \cong BU$.

**Lemma 6.2.** *The sequence*

$$\operatorname{spec} K_0P \xrightarrow{\operatorname{spec} K_0 B\det} \operatorname{spec} K_0BU \to \operatorname{spec} K_0BSU$$

*is a short exact sequence of $k$-groups.*

*Proof.* The splitting gives a short exact sequence of formal group schemes

$$0 \to \operatorname{spf}(K^0BSU) \to \operatorname{spf}(K^0BU) \to \operatorname{spf}(K^0P) \to 0,$$

which splits nonadditively. Cartier duality is exact, so we have a short exact sequence as claimed. (Note however that Cartier duality is only functorial for homomorphisms, so $\operatorname{spf}(K^0Bi)$ does not induce a splitting of this sequence.) □

The adjoint of the map

$$K_0P \xrightarrow{K_0(1-L)} K_0BU$$

is an element $\Pi_1$ of $\mathcal{O}_{(G_K)_{K_0BU}}$ whose value at the origin is 1. Let $C^1(G_K)$ denote the scheme of such functions; $\Pi_1$ is classified by a map

$$\operatorname{spec} K_0BU \xrightarrow{\Pi_1} C^1(G_K).$$

**Lemma 6.3.** $\Pi_1$ *is an isomorphism of $k$-groups. Moreover there is an isomorphism*

$$\operatorname{spec} K_0P \cong \operatorname{Hom}(G_K, \mathbb{G}_m)$$

*such that the diagram*

$$\begin{array}{ccc} \operatorname{spec} K_0P & \xrightarrow[\cong]{} & \operatorname{Hom}(G_K, \mathbb{G}_m) \\ {\scriptstyle \operatorname{spec} K_0B\det}\downarrow & & \downarrow{\scriptstyle \textit{natural inclusion}} \\ \operatorname{spec} K_0BU & \xrightarrow[\cong]{\Pi_1} & C^1(G_K) \end{array}$$

*commutes.*

*Proof.* We leave it to the reader to check that this is a coordinate-free version of the usual descriptions of the Hopf algebras $K_0BU$ and $K_0P$, as in for example [1] and [9, 3.4] □

Recall from (1.7) that for a $k$-algebra $R$, $C^2(G_K)(R)$ is defined to be the group of symmetric 2-cocycles on $(G_K)_R$ with values in the multiplicative group $\mathbb{G}_m$. (Such maps necessarily land in $\widehat{\mathbb{G}}_m \subset \mathbb{G}_m$, so $\mathbb{G}_m$ may be replaced by $\widehat{\mathbb{G}}_m$ if desired.) Moreover as a functor of $R$ it is represented by a $k$-group which we denote $C^2(G_K)$.

The adjoint of the map

$$K_0P^2 \xrightarrow{K_0 \prod_i^2 (1-L_i)} K_0BSU$$

is naturally an element

$$\Pi_2 \in C^2(G_K)(K_0 BSU) \tag{6.4}$$

which is classified by a map

$$\operatorname{spec} K_0 BSU \xrightarrow{\Pi_2} C^2(G_K).$$

The purpose of this section is to give a proof of the following. In fact the analogous statement is true for any even periodic ring spectrum [5].

**Proposition 6.5.** $\Pi_2$ *is an isomorphism of $k$-groups.*

There is a natural map

$$\delta : C^1(G_K) \to C^2(G_K)$$

given by the formula

$$(\delta f)(g_1, g_2) \overset{\text{def}}{=} \frac{f(g_1)f(g_2)}{f(g_1+g_2)}.$$

**Lemma 6.6.** *The diagram*

$$\begin{array}{ccc} \operatorname{spec} K_0 BU & \longrightarrow & \operatorname{spec} K_0 BSU \\ {\scriptstyle \Pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \Pi_2} \\ C^1(G_K) & \xrightarrow{\delta} & C^2(G_K) \end{array}$$

*commutes.*

*Proof.* Let $\mu$, $\pi_1$, and $\pi_2$ denote the product and two projection maps

$$P^2 \to P.$$

Let $L$ denote the tautological line bundle over $P$, let

$$L_i \overset{\text{def}}{=} \pi_i^* L.$$

In view of the descriptions of the maps $\Pi_1$ and $\Pi_2$, the lemma boils down to the equation

$$(1-L_1)(1-L_2) = \pi_1^*(1-L) + \pi_2^*(1-L) - \mu^*(1-L).$$

□

**Proposition 6.7.** *The map of $k$-groups*

$$C^1(G_K) \xrightarrow{\delta} C^2(G_K)$$

*is an epimorphism with kernel* $\operatorname{Hom}(G_K, \mathbb{G}_m)$.

**Remark 6.8.** The argument which follows has the virtue of brevity, but its tone is not really in keeping with the coordinate-free arguments in the rest of the paper. There is a more "natural" proof in [5], to give which would however require a substantial detour in the present paper.

The proof of the proposition uses the Artin-Hasse exponential

$$\begin{aligned} A(x) &\overset{\text{def}}{=} \exp\left\{\sum_{k\ge 0} \frac{x^{p^k}}{p^k}\right\} \\ &= 1 + x + o[2] \in \mathbb{Z}_{(p)}[\![x]\!] \end{aligned}$$

and the following "symmetric 2-cocycle lemma". Let $c_r \in \mathbb{Z}[x,y]$ be the polynomial

$$c_r(x,y) \stackrel{\text{def}}{=} \frac{1}{d(r)}(x^r + y^r - (x+y)^r)$$

where

$$d(r) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } r \text{ is not a prime power} \\ q & \text{if } r \text{ is a power of a prime } q. \end{cases}$$

**Lemma 6.9.** *If $A$ is an abelian group and $f \in A[x,y]$ is a homogeneous polynomial of degree $r$ satisfying the equation*

$$f(x,y) - f(w+x,y) + f(w,x+y) - f(w,x) = 0$$

*then $f$ is a scalar multiple of $c_r$.*

*Proof.* See for example Adams [1]. □

*Proof of Proposition 6.7.* The statement about the kernel is clear. According to Lemma 6.10, to show that $\delta$ is an epimorphism, we must show that if $R$ is a $k$-algebra and $f \in C^2(G_K)(R)$ is a 2-cocycle, then there is a faithfully flat $R$-algebra $R'$ and an element $f' \in C^1(G_K)(R')$ such that $f = \delta f'$.

Choose a coordinate $x$ on $G_K$ so that

$$C^1(G_K)(R) \cong 1 + xR[\![x]\!].$$

Then $f \in C^2(G_K)(R)$ can be expressed as a power series in two variables $x$ and $y$, and there is a smallest integer $t \geq 1$ such that $f(x,y)$ is of the form

$$f(x,y) = 1 + ac_t(x,y) + o[t+1]$$

with $a \neq 0$. If $t$ is not a power of $p$ then let $S = R$ and let

$$g(x) = 1 + ax^t \in C^1(G_K, \mathbb{G}_m)(R).$$

If $t = p^s$ with $s > 1$, then let $S$ be a faithfully flat extension of $R$ containing a solution $b$ to the equation

$$\begin{aligned} b + b^p &= a \qquad && \text{if } h = 1 \\ b^p &= a \qquad && \text{if } h > 1. \end{aligned}$$

Here $h$ is the height of $G_K$ so $K = K(h)$. Let $g$ be the specialisation to $S$ of the reduction modulo $p$ of

$$A(bx^{p^{s-1}}) \in \mathbb{Z}_{(p)}[b][\![x]\!].$$

Using the formula

$$x \underset{F}{\bullet} y = x + y + c_{p^h}(x,y) + o[p^h+1],$$

it is easy to check that

$$\delta g(x,y) = 1 + ac_{p^s}(x,y) + o[p^s+1].$$

So there is a faithfully flat extension $S$ of $R$ and a $g \in C^1(G_K, \mathbb{G}_m)(S)$ such that

$$\frac{f}{\delta g} = 1 + o[t+1].$$

By induction, one obtains $R'$ and $f'$, and concludes that $\delta$ is surjective. □

In the proof of the preceding proposition we used the following algebraic result.

**Lemma 6.10.** *A map $f : H \to G$ of $k$-groups is an epimorphism if and only if for every $k$-algebra $R$ and every $R$-valued point $a \in G(R)$, there is a faithfully flat $R$-algebra $S$ and a point $b \in H(S)$ such that $f(b) = a_S$.*

*Proof.* Since an epimorphism of $k$-groups is a map of $k$-groups which is faithfully flat, the only if direction is clear. For the other direction, consider the case $R = \mathcal{O}_G$ and $S = \mathcal{O}_H$. The hypothesis is that there is a faithfully flat $R$-algebra $T$ and a factorisation

$$\begin{tikzcd} & S \arrow[dl, dashed] \\ T & R. \arrow[l] \arrow[u, "f^*"'] \end{tikzcd}$$

In particular, $f^*$ is injective; it follows [4, III,§3, n. 7] that it is faithfully flat. □

*Proof of Proposition 6.5.* From Lemmas 6.2, 6.3, and 6.6 and Proposition 6.7, the diagram

$$\begin{tikzcd} \operatorname{spec} K_0 P \arrow[r] \arrow[d, "\cong"'] & \operatorname{spec} K_0 BU \arrow[r] \arrow[d, "\cong"'] & \operatorname{spec} K_0 BSU \arrow[d, "\Pi_2"] \\ \operatorname{Hom}(G_K, \mathbb{G}_m) \arrow[r] & C^1(G_K) \arrow[r, "\delta"] & C^2(G_K) \end{tikzcd}$$

commutes, the rows are short exact sequences of $k$-groups, and the first two vertical arrows are isomorphisms. □

## 7. Proof of Theorem 1.4

There is a natural map

$$C^2(G_K) \xrightarrow{\delta} C^3(G_K)$$

representing the natural transformation

$$(\delta f)(g_1, g_2, g_3) \stackrel{\text{def}}{=} \frac{f(g_1 + g_2, g_3)}{f(g_1, g_3) f(g_2, g_3)}. \tag{7.1}$$

**Lemma 7.2.** *The diagram*

$$\begin{tikzcd} \operatorname{spec} K_0 BSU \arrow[r] \arrow[d, "\Pi_2"'] & \operatorname{spec} K_0 BU\langle 6\rangle \arrow[d, "\Pi_3"] \\ C^2(G_K) \arrow[r, "\delta"] & C^3(G_K) \end{tikzcd}$$

*commutes.*

*Proof.* The same as the proof of Lemma 6.6. □

In contrast with the case of $C^1 \to C^2$, we have

**Lemma 7.3.**

$$C^2(G_K) \xrightarrow{\delta} C^3(G_K)$$

*is injective in the category $k$-groups.*

*Proof.* The kernel of $\delta$ consists of symmetric, biexponential maps from $G_K^2$ to $\mathbb{G}_m$. Let $f$ be such a map. On $G_K[p^r] \times G_K$ we have $f(g, p^r h) = f(p^r g, h) = f(0, h) = 1$, but $G_K$ is $p$-divisible so the map $h \mapsto p^r h$ is an epimorphism (and remains so after taking the product with $G_K[p^r]$), so $f(g, h) = 1$ on $G_K[p^r] \times G_K$. Another consequence of the $p$-divisibility of $G_K$ is that $G_K = \operatorname{colim}_r G_K[p^r]$ (and again this colimit is preserved by products) so we conclude that $f = 1$. ∎

**Lemma 7.4.** *The fibration*

$$K(\mathbb{Z}, 3) \to BU\langle 6\rangle \to BSU \tag{7.5}$$

*gives rise to a short exact sequence of abelian Hopf algebras*

$$K_0 K(\mathbb{Z}, 3) \to K_0 BU\langle 6\rangle \to K_0 BSU.$$

*Proof.* The Atiyah-Hirzebruch spectral sequences for the $K$-homology of $BSU$ and for the $K$-homology of the fibration (7.5) collapse, because they start in even bidegrees. ∎

*Proof of Theorem 1.4.* According to Lemma 7.2 and Corollary 4.5, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{spec} K_0 BSU \arrow[r] \arrow[d, "\Pi_2"', "\cong"] & \operatorname{spec} K_0 BU\langle 6\rangle \arrow[r] \arrow[d, "\Pi_3"'] & \operatorname{spec} K_0 K(\mathbb{Z}, 3) \arrow[d, "b_*"', "\cong"] \\
C^2(G_K) \arrow[r, "\delta"] & C^3(G_K) \arrow[r, "e_*"] & W.
\end{tikzcd}$$

Proposition 6.5 and Proposition 2.10 give that the marked vertical arrows are isomorphisms. The top row is a short exact sequence of $k$-groups by Lemma 7.4, and $\delta$ is injective by Lemma 7.3. It follows that $e_*$ is surjective and $\Pi_3$ is an isomorphism. ∎

In the course of the proof we obtained the following result. This result follows from Corollary 7.13 of [2], which is proved by methods quite different from the topological methods of this paper.

**Corollary 7.6.** *The map of $k$-groups $e_* \colon C^3(G_K) \to W$ is surjective.* ∎

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