Mackey Functors In Equivariant Homotopy and Cohomology Theory

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June 18, 2011

Equivariant Cohomology Theories

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An equivariant cohomology theory is a sequence of contravariant functors H^n_G : G-complexes $\rightarrow Ab$

Equivariant Cohomology Theories

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The orbit category O_G is the category consisting of objects G/H for $H \leq G$ and morphisms $G/H \to G/K$ whenever $g^{-1}Hg \subset K$ for some $g \in G$.

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Since G-complexes are built from the orbits G/H using equivariant maps $G/H \to G/K$, any ECT should include groups $H^n(G/H)$ and homomorphisms $H^n(G/K) \to H^n(G/H)$.

Coefficient Systems

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A coefficient system $\underline{M}$ is a contravariant functor from O_G, the orbit category, to Ab.
The collection of coefficient systems forms a category $\mathcal{C}_G$.

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In equivariant <u>ordinary</u> cohomology:

$$H^*(G/H; \underline{M}) = H^0(G/H; \underline{M}) = \underline{M}(G/H)$$

for any $\underline{M} \in \mathcal{C}_G$

Coefficient Systems

Example

A Mackey functor $\underline{M}$ is a pair of functors

$$M^* : O_G^{op} \to Ab \text{ and } M_* : O_G \to Ab$$

such that

$$M^*(X) = M_*(X) = \underline{M}(X)$$

and which send disoint unions to direct sums and satisfy certain commutativity relations.

Notation: For $f : G/H \to G/K$ we call $M^*(f)$ a restriction and $M_*(f)$ a transfer.

Bredon Cohomology

If X is a G-complex, define the chains on X by:

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and so we define Bredon cohomology to be

$$H^n_G(X; \underline{M}) = H^n(C^*_G(X; \underline{M}))$$

(Nonstable) Equivariant Homotopy Groups

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Definition

Let X be a G-space. For each $H \leq G$ the equivariant homotopy groups of X are given by

$$\pi_n^H(X) = [S^n \wedge G/H_+, X]_G$$

Stable Equivariant Homotopy Groups

Definition

A G-spectrum X is a collection of G-spaces X_k together with equivariant maps $\Sigma X_k \to X_{k+1}$ (or equivalently $X_k \to \Omega X_{k+1}$)

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Note: These homotopy groups form a Mackey functor:

$$\underline{\pi_n(X)}(G/H) = [\Sigma^\infty S^n \wedge G/H_+, X]_G = \pi_n^H(X)$$

Equivariant Homotopy Group Mackey Functor

$$\begin{aligned}
\underline{\pi_k(X)}(G/H) &= [G/H_+ \wedge S^k, X]_G \\
&= [G_+ \wedge_H S^k, X]_G \\
&= [S^k, X]_H \\
&= \pi_k(X^H)
\end{aligned}$$

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Restriction Map $\qquad \pi_k(X^K) \to \pi_k(X^H)$

Induced from inclusion of fixed points $X^K \to X^H$

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Restriction Map $\quad \pi_k(X^K) \to \pi_k(X^H)$

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Transfer Map $\quad \pi_k(X^H) \to \pi_k(X^K)$

Induced from

$$X^H \to X^K$$
$$x \to \sum_{gH \in K/H} g \cdot x$$

Cohomology Theories from G-Spectra

Let X be a G-space and Y be a G-spectrum.

The groups $[\Sigma^{k-n}X, Y_k]_G$ form a direct system:

$$[\Sigma^{k-n}X, Y_k]_G \to [\Sigma^{k-n+1}X, \Sigma Y_k]_G \to [\Sigma^{k-n+1}X, Y_{k+1}]_G$$

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So we can define cohomology:

$$\begin{aligned} \widetilde{Y}^n_G(X) &= \operatorname*{colim}_k [\Sigma^{k-n}X, Y_k]_G \\ &= \operatorname*{colim}_k \pi_{k-n}(F(X, Y_k)^G) \\ &= \pi_{-n}(\underline{F}(X, Y)^G) \end{aligned}$$

Equivalence of Categories

From which G-spectrum do we obtain Bredon cohomology?

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Proposition

There is an equivalence of categories between the category of Mackey functors and the homotopy category consisting of G-spectra X such that $\underline{\pi_i(X)} = 0$ *for* $i \neq 0$.

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Proposition

There is an equivalence of categories between the category of Mackey functors and the homotopy category consisting of G-spectra X such that $\underline{\pi_i(X)} = 0$ for $i \neq 0$.

In particular, for any Mackey functor $\underline{M}$, we have an associated Eilenberg-MacLane spectrum $H\underline{M}$ satisfying:

$$\underline{\pi_k(H\underline{M})} = \begin{cases} \underline{M} & \text{k=0} \\ 0 & \text{otherwise} \end{cases}$$

Bredon Cohomology

Now for any Mackey functor $\underline{M}$, we may obtain Bredon Cohomology from $H\underline{M}$ as follows:

$$\begin{aligned}
\widetilde{H}^n_G(X;\underline{M}) &= \operatorname*{colim}_k [\Sigma^{k-n}X, (H\underline{M})_k]_G \\
&= [\Sigma^{-n}X, H\underline{M}]_G \\
&= \pi_{-n}(F(X, H\underline{M})^G) \\
&= \pi^G_{-n}(F(X, H\underline{M})) \\
&= \underline{\pi_{-n}(F(X, H\underline{M}))}(G/G)
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\end{aligned}$$

Note: For a group G, Bredon Cohomology is the image of G/G under a Mackey functor.

RO(G)-grading

In working with equivariant theories, we want to consider spheres with nontrivial G-action. In particular, we will look at linear spheres arising from representations of G.

Definition

For a group G and a vector space V, we will say a representation of G is a homomorphism $\rho : G \to O(V)$

Definition

For a representation space V we will write S^V to denote the one-point compactification of V

RO(G)-graded Homotopy Groups

If $V \in RO(G)$ then it is also an H-representation for any $H \leq G$ so we have RO(G)-graded homotopy groups:

$$\pi_V^H(X) = [S^V, X]_H = [G_+ \wedge_H S^V, X]_G$$

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Note: Our original $\mathbb{Z}$-graded homotopy groups $\pi_n^H(X)$ are the homotopy groups associated to the trivial representation $n \in RO(G)$ where n stands for $\mathbb{R}^n$.

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So extending the usual suspension axiom

$$\sigma_n : H^n(X) \to H^{n+1}(\Sigma X)$$

we obtain RO(G)-graded cohomology groups:

$$H^{\alpha}_G(X) \cong H^{\alpha+V}_G(\Sigma^V X)$$

for $\alpha, V \in RO(G)$

Why is the RO(G)-grading important?

A few examples:

- (Lewis) Let X be a $\mathbb{Z}/p$-complex constructed from even dimensional unit disks of real G-representations. The $H_G^*(X)$ is a free $RO(G)$-graded module over the equivariant ordinary cohomology of a point.

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- (Lewis) Let V be a complex G-representation and P(V) the associated complex projective space. Then all generators of $H_G^*(P(V))$ live in dimensions corresponding to nontrivial representations of G.

Why is the RO(G)-grading important?

A few examples:

- (Lewis) Let X be a $\mathbb{Z}/p$-complex constructed from even dimensional unit disks of real G-representations. The $H^*_G(X)$ is a free $RO(G)$-graded module over the equivariant ordinary cohomology of a point.
- (Lewis) Let V be a complex G-representation and P(V) the associated complex projective space. Then all generators of $H^*_G(P(V))$ live in dimensions corresponding to nontrivial representations of G.
- (tom Dieck) RO(G)-graded cohomology theories admit important splitting theorems.

When can we extend?

In the $RO(G)$-graded setting we have transfer maps

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These induce transfer homomorphisms

$$\begin{array}{c} \widetilde{H}_H^n(X; \underline{M}) \cong \widetilde{H}_G^{V+n}(\Sigma^V(G/H_+ \wedge X); \underline{M}) \\ \big\downarrow \\ \widetilde{H}_G^{V+n}(\Sigma^V X; \underline{M}) \cong \widetilde{H}_G^n(X; \underline{M}) \end{array}$$

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If $n = 0$ and $X = S^0$ we get a transfer map

$$\underline{M}(G/H) \to \underline{M}(G/G)$$

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Additionally it can be shown that this necessary condition is also sufficient:

Theorem

(May, Waner) The ordinary $\mathbb{Z}$-graded cohomology theory $\widetilde{H}^*_G(-;\underline{M})$ extends to an RO(G)-graded theory if and only if $\underline{M}$ extends to a Mackey functor.

Mackey Functor Valued Cohomology

We may additionally think of our Equivariant Cohomology Theory as being Mackey functor valued:

$$H^{\alpha}_{G}(X; \underline{M}) = \underline{\pi_{-k}(X)}(G/G)$$

and

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In general we have

$$\underline{H^{\alpha}_{G}(X;\underline{M})}(G/H) = H^{\alpha}_{G}(G/H_{+} \wedge X;\underline{M})$$

An Example

$$
\begin{tikzcd}
\underline{H^{\alpha}_{C_p}(X;\underline{M})}(C_p/C_p) = H^{\alpha}_{C_p}(X;\underline{M}) \arrow[d, bend right, "\pi^*"'] \\
\underline{H^{\alpha}_{C_p}(X;\underline{M})}(C_p/e) = H^{\alpha}_{C_p}(C_p \times X;\underline{M}) \arrow[u, bend right, "\pi_!"']
\end{tikzcd}
$$

π^* is induced from the projection $\pi : C_p \times X \to X$

$\pi_!$ is the transfer map arising from regarding the projection π as a covering space.

Note: $H^{\alpha}_{G}(G \times X;\underline{M}) \cong H^{|\alpha|}(X;\underline{M}(G/e))$