

THE ITERATION OF THE STEENROD SQUARES IN ALGEBRAIC TOPOLOGY

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In this work it is proved that the Steenrod squares on cohomology classes satisfy certain identities. These identities are applied to solve particular homotopy problems. Full details will appear elsewhere.

1. Let $H^q(K)$ be the q th cohomology group of a complex K with the integers mod 2 as coefficient group. The Steenrod squares,¹ denoted by Sq^i , ($i = 0, 1, \dots$), are invariant homomorphisms

$$Sq^i: H^q(K) \rightarrow H^{q+i}(K),$$

defined for any complex K , satisfying, among several others, the following properties: for any map f of one complex into another, $f^*Sq^i = Sq^if^*$; $Sq^0 = \text{identity}$; $Sq^qu = u \smile u$, (mod 2), if $q = \dim u$.

An *iterated* square is a composition of two or more of the squares, e.g., $Sq^iSq^jSq^k$. The following relations (mod 2) on iterated squares are well known

$$Sq^1Sq^k = \begin{cases} Sq^{k+1} & \text{when } k \text{ is even,} \\ 0 & \text{when } k \text{ is odd.} \end{cases}$$

A principal result of this paper is a system of additional relations as follows.

THEOREM 1.1. *For all $s > t$, the squares satisfy the mod 2 relations*

$$Sq^{2t}Sq^s = \sum_{j=0}^t \binom{s-t+j-1}{2j} Sq^{t+s+j} Sq^{t-j},$$

where $\binom{k}{j}$ is the mod 2 value of the binomial coefficient, with the convention: $\binom{k}{j} = 0$, if $j > k$.

From these relations we obtain

$$(1.2) \quad Sq^{2n(2r+1)} = Sq^{2n}Sq^{2^{n+1}r} + \sum_{p=0}^{n-1} Sq^{2^n(2r+1)-2^p}Sq^{2^p}.$$

As special cases of 1.2, we have

$$(1.3) \quad Sq^{2^n}Sq^{2^m} = Sq^{2^m+2^n} + \sum_{p=0}^{n-1} Sq^{2^m+2^n-2^p}Sq^{2^p}, \quad (m > n),$$

$$(1.4) \quad \text{Sq}^{2^n} \text{Sq}^{2^n} = \sum_{p=0}^{n-1} \text{Sq}^{2^{n+1}-2^p} \text{Sq}^{2^p}.$$

A direct consequence of 1.2 is

THEOREM 1.5. *The squares of the type Sq^{2^p} with $p = 0, 1, \dots$, form a base for the squaring operations. Precisely, the relations 1.2 can be used to write any square as a sum of iterated squares of exponents 2^p ; e.g., $\text{Sq}^6 = \text{Sq}^2 \text{Sq}^4 + \text{Sq}^1 \text{Sq}^4 \text{Sq}^1$.*

Remark: Using a different approach by means of Eilenberg-MacLane complexes, Serre² has proved that the iterated squares of the type $\text{Sq}^{i_1} \text{Sq}^{i_2} \dots \text{Sq}^{i_k}$, with $i_1 \geq 2i_2, \dots, i_{k-1} \geq 2i_k$, form a base for the iterated squares. Using the relations 1.1, a new proof of this fact can be given. It has the advantage that explicit formulae are obtained for expressing an arbitrary iterated square in terms of this base.

2. Let u be a cohomology class of a complex K , with coefficients in a group G . Using 1.2 we can write the following expression for the self-product of u .

If $q = \dim u = 2^n(2r + 1)$, $r > 0$, then

$$(2.1) \quad u \smile u = \text{Sq}^{2^n} \text{Sq}^{q-2^n} u + \sum_{p=0}^{n-1} \text{Sq}^{q-2^p} \text{Sq}^{2^p} u, \quad (\text{mod } 2).$$

If $q + 1 = \dim u = 2^n(2r + 1) + 1$, $r > 0$, $n > 0$, then

$$(2.2) \quad u \smile u = \delta^* \left[\text{Sq}^{2^n} \text{Sq}^{q-2^n} u + \sum_{p=0}^{n-1} \text{Sq}^{q-2^p} \text{Sq}^{2^p} u \right],$$

where δ^* is the coboundary homomorphism of the coefficient sequence $0 \rightarrow G \rightarrow G \rightarrow G/2G \rightarrow 0^3$. In this last case we suppose G has no elements of order two.

These formulae impose strong restrictions on self-products in the cohomology ring of a complex. As an example, 2.2 implies: *A complex K cannot exist with an integer 7-class u , such that $u \smile u \neq 0$, and $H^{11}(K; \text{mod } 2) = 0$.*

3. We call an even-dimensional manifold M , of type α , if it is connected, orientable, and its middle Betti number is 1. That is, $\dim M = 2k$ and $R_0 = R_k = R_{2k} = 1$. As is well known, a manifold of type α with k odd cannot exist. This is a direct consequence of Poincaré duality and the anticommutativity of the cup-product. By using Poincaré duality and formula 2.1 the following further restrictions can be given on the homology structure of manifolds.

THEOREM 3.1. *Let $\dim M = 2k = 2^{n+1}(2r + 1)$, $r > 0$. Then there does not exist a manifold M of type α such that $H^{2^n}(M; \text{mod } 2) = 0$ and $H^{k-2^p}(M; \text{mod } 2) = 0$, for $p = 1, \dots, n - 1$.*

A direct consequence of 3.1 is the following.

COROLLARY 3.2. *If there exists a $2k$ -dimensional manifold, without torsion, and with Poincaré polynomial $1 + x^k + x^{2k}$, then k is a power of two.*

Remark: Theorem 3.1 solves partially a problem, considered by Hirsch⁴ and Bassi⁵, about the existence of a 12-dimensional manifold with Poincaré polynomial $1 + x^6 + x^{12}$. It follows from 3.1 that such a manifold M does not exist with $H^2(M; \text{mod } 2) = 0$.

4. Given a map $f: S^{p+q-1} \rightarrow S^q$ (of a $(p+q-1)$ -sphere into a q -sphere), let K be the complex obtained by adjoining to S^q a $(p+q)$ -cell by means of f . The map is essential if $\text{Sq}^p: H^q(K) \rightarrow H^{p+q}(K)$ is different from zero. Since the intermediate cohomology groups of K are zero, it follows from 1.5 that Sq^p will be zero if p is not a power of two.

Hopf has constructed⁶ essential maps $f: S^{2n-1} \rightarrow S^n$ for all even n . He assigned to any such map an integer called the Hopf invariant $H(f)$ (an invariant of the homotopy class which is zero for inessential maps). For each even n , he exhibited maps f for any even $H(f)$. For $n = 1, 2, 4, 8$, he showed that any integer could be an $H(f)$. Recently G. Whitehead⁷ proved that $H(f)$ is even for any f when $n = 4k + 2$ ($k \geq 1$). In the problem of deciding for which n 's a map of Hopf invariant 1 exists, we will exclude many values of n with the following argument. The mod 2 value of $H(f)$ is 0 or 1 according as $\text{Sq}^n: H^n(K) \rightarrow H^{2n}(K)$ is 0 or not. Thus the result of the preceding paragraph proves the following.

THEOREM 4.1. *The Hopf invariant $H(f)$ of a map $f: S^{2n-1} \rightarrow S^n$ must be even if n is not a power of 2.*

The problem remains open of deciding whether or not there exists for each power of 2 a map of Hopf invariant 1.

It is well known that (4.1) has several implications; we will mention only the following.

COROLLARY 4.2. *A representation of S^m as a sphere bundle over S^n is possible only if $m = 2n - 1$ and n is a power of 2.*

This corollary holds for an arbitrary group of the bundle. For the rotation group R_n as group of the bundle, it was proved recently⁸ by N. E. Steenrod and J. H. C. Whitehead.

COROLLARY 4.3. *A map $f: S^n \times S^n \rightarrow S^n$ of type $(1, 1)$ can exist only if $n + 1$ is a power of 2.*

Hopf⁹ and Behrend¹⁰ studied the question of the existence of real division algebras in the euclidean n -space (i.e., a bilinear multiplication with a two-sided unity and without 0-divisors). They showed that this does not exist when n is not a power of 2. The preceding result enables us to improve on this result by the elimination of the condition of bilinearity.

5. By extending the process of adjoining cells to spheres, and by means of formulae 1.3 and 1.4, we prove the following general theorem about the compositions of suspensions of maps with Hopf invariant 1.

THEOREM 5.1. *Let $f: S^{2m-1} \rightarrow S^m$ and $g: S^{2n-1} \rightarrow S^n$ be maps with Hopf invariant 1. Suppose $m \geq n$. Let g' be the $(2m-1-n)$ -fold suspension of g . Then the composition fg' is always essential; moreover*

(1) *if $m = n$, the p -fold suspension of this composed map is essential for all p ,*

(2) *if $m > n$, the p -fold suspension of this composed map is essential for all $p < n$. All the essential elements constructed in this way are not divisible by 2.*

If, in the above, we take $f = g$ to be the map $S^3 \rightarrow S^2$ of Hopf invariant 1, then we obtain a new proof that $\pi_{n+2}(S^n) \neq 0$ for $n \geq 2$. Similarly, if $f = g$ is the map $S^7 \rightarrow S^4$ (respectively, $S^{15} \rightarrow S^8$) of Hopf invariant 1, we find that $\pi_{n+6}(S^n) \neq 0$ for $n \geq 4$ (respectively, $\pi_{n+14}(S^n) \neq 0$ for $n \geq 8$).

6. We shall indicate how our relations on iterated squares can be used to construct some cohomology operations of the *second kind*. Let Z and Z_2 be the group of integers and integers mod 2, respectively, and suppose $\text{Sq}^2: H^q(K; Z) \rightarrow H^{q+2}(K; Z_2)$ defined with the natural pairing. Define $N^q(K) \subset H^q(K; Z)$ to be the kernel of Sq^2 . We shall construct an operation of the second kind

$$(6.1) \quad \Phi: N^q(K) \rightarrow H^{q+3}(K; Z_2)/\text{Sq}^2 H^{q+1}(K, Z).$$

First, consider $u \in H^q(K; Z)$ with $q \geq 2$, and let u_1 be a representative of u . With $n = 1$ in 1.4 we have the relation $\text{Sq}^2 \text{Sq}^2 u + \text{Sq}^3 \text{Sq}^1 u = 0$. The proof of our relations on iterated squares is by means of a cochain construction. For this particular relation we construct cochain mappings

$$E_j: C^p(K^4) \rightarrow C^{p-j}(K),$$

where $K^4 = K \times K \times K \times K$, such that mod 2

$$(u_1 \smile_i u_1) \smile_{i+2} (u_1 \smile_i u_1) + (u_1 \smile_{i+1} u_1) \smile_i (u_1 \smile_{i+1} u_1) = \delta E_{3i+3}(u_1^4),$$

where \smile_k is the cup- k product¹ and $i = q - 2$. If u is a class with integer coefficients, then $\text{Sq}^1 u = 0$, therefore $\text{Sq}^2 \text{Sq}^2 u = 0$. In this case a mod 2 cochain expression for this last relation is given by

$$(6.2) \quad (u_1 \smile_i u_1) \smile_{i+2} (u_1 \smile_i u_1) = \delta[E_{3i+3}(u_1^4) + \eta(u_1) \smile_{i-1} \eta(u_1) + \eta(u_1) \smile_i \delta \eta(u_1)],$$

where,

$$\eta(u_1) = \frac{1}{2}[u_1 \smile_{i+2} u_1 + u_1].$$

Now, suppose that $\text{Sq}^2 u = 0$, that is $u \in N^q(K)$. Then for some cochain b we have $u_1 \smile_i u_1 = \delta b$. It follows mod 2 that,

$$(6.3) \quad (u_1 \smile_i u_1) \smile_{i+2} (u_1 \smile_i u_1) = \delta[b \smile_{i+1} b + b \smile_{i+2} \delta b].$$

Therefore, in 6.2 and 6.3 we have the same expression being a coboundary for two different reasons. It follows from a general principle, already used by Steenrod on defining functional operations, that both expressions together will give rise to a new cohomology operation. By setting

$$w_1 = b \smile_{i+1} b + b \smile_{i+2} \delta b + E_{3i+3}(u_1^4) + \eta(u_1) \smile_{i-1} \eta(u_1) + \eta(u_1) \smile_i \delta \eta(u_1),$$

it follows from 6.2 and 6.3 that w_1 is a mod 2 cocycle and we define our operation of 6.1 by

$$\Phi u = \{w_1\} + \text{Sq}^2 H^{q+1}(K; Z).$$

The operation Φ can be proved to be a well-defined operation, independent of all the arbitrariness that may occur in our construction. Some of its properties are: (1) $f^* \Phi = \Phi f^*$, for any map f of one complex into another, therefore, it is an invariant operation; (2) the definition of Φ extends to the relative case and there Φ commutes with the coboundary operator; (3) Φ is a homomorphism; (4) Φ is an effectively computable operation. The fact that this operation may not be trivial is shown by the following.

According to the pattern introduced by Steenrod,¹¹ with the operation Φ and a map f we can define the functional operation Φ_f .

THEOREM 6.4. *Let $f: S^{n+2} \rightarrow S^n$ be a map of an $(n+2)$ -sphere into an n -sphere with $n \geq 2$. The functional operation*

$$\Phi_f: H^n(S^n; Z_2) \rightarrow H^{n+2}(S^{n+2}; Z_2),$$

is non-trivial if and only if the map f is essential.

In the above theorem a cochain expression can be given for Φ_f and that provides an effective method for deciding the homotopy class of a given simplicial map $f: S^{n+2} \rightarrow S^n$.

Let K be the cell complex constructed by adjoining a cell E^{n+3} to S^n , by means of the map $f: S^{n+2} \rightarrow S^n$. It follows from 6.4 that $\Phi: H^n(K; Z_2) \rightarrow H^{n+3}(K; Z_2)$ will map the generator of $H^n(K; Z_2)$ onto the generator of $H^{n+3}(K; Z_2)$, if and only if the given map f is essential.

The main application of the new operation is the computation of the third obstruction.

THEOREM 6.5. *Let $f: K^n \rightarrow S^n$ be a map from the n -skeleton of a complex K into an n -sphere. Let s^n be a generator of $H^n(S^n)$. Suppose that the first and second obstruction for the extension of this map vanish, i.e., $\delta f^* s^n = 0$ and $\text{Sq}^2 f^* s^n = 0$. Then the third obstruction is given by*

$$\Phi f^* s^n \in H^{n+3}(K; \pi_{n+2}(S^n)) / \text{Sq}^2 H^{n+1}(K; \pi_{n+1}(S^n)).$$

In this last theorem the pairings for the different coefficient groups are the natural ones. By following the general pattern given by Steenrod,¹ the maps of an $(n+2)$ -complex into an n -sphere can be classified. In particu-

lar, we can compute the cohomotopy groups $\pi^n(K^{n+2})$ up to group extensions.

The construction of our operation Φ corresponds to the relation $Sq^2Sq^2 = Sq^4Sq^1$. By using other relations on iterated squares we can generalize the operation Φ in an obvious way. In particular, with the relations 1.3 and 1.4 new operations can be constructed, and with them the essential maps obtained in 5.1 can be computed.

7. We shall indicate here the method used in proving our relations on iterated squares.

The procedure devised by Steenrod¹² for constructing the squaring operations is by means of homomorphisms of chains of K into those of $K \times K$, which neighbor the diagonal. He uses the cyclic group of order two, operating in $K \times K$ by permutation of the factors. In treating iterated squares we must consider similar homomorphisms from K to $K^4 = K \times K \times K \times K$.

A rough description of our procedure is to parallel the construction of the squares, replacing K^2 by K^4 and taking as group operating in K^4 a group of order four, contained in the dihedral group of order eight. We obtain in this form a new doubly indexed set of invariant operations, $\left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right]$, each of which is a homomorphism

$$\left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right]: H^q(K) \rightarrow H^{4q-i}(K).$$

It is then shown, that $\left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right]$ coincides with a sum of iterated squares $\sum Sq^r Sq^s$.

This is achieved by an explicit calculation.

An automorphism of the full dihedral group is used to prove that the $\left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right]$ contains a symmetry, which is expressed by the relation $\left[\begin{smallmatrix} i \\ k \end{smallmatrix} \right] = \left[\begin{smallmatrix} i \\ i-k \end{smallmatrix} \right]$. Thus, for each i, k we obtain that one sum of iterated squares equals another. These are the basic identities.

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LOCAL TRANSFORMATIONS WITH FIXED POINTS ON COMPLEX SPACES WITH SINGULARITIES

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We will supplement a recent paper of ours¹ and we first of all describe the nature of the complex space there introduced. Let S be the unit sphere in Euclidean space of k complex variables z_1, \dots, z_k , and over S we consider a system of polynomials in ζ 's,

$$\psi_\nu(z_1, \dots, z_k, \zeta_1, \dots, \zeta_n) = 0, \nu = 1, \dots, n \quad (1)$$

with coefficients which are analytic in S . This defines for us a topological space which will be denoted by U . If R is any subset of S then $U(R)$ will denote those points of U whose base points are in R . Now there is a function $D(z)$ analytic and not identically zero in S such that for E denoting the points of S where D vanishes the subset $U(E)$ are possible singularities on U . However, $U - U(E) = U(S - E)$ shall be a complex analytic coordinate space.

If m_ν is the degree of ψ_ν in the variable ζ_ν and if $m = m_1 \dots m_n$ then the system (1) has m sets of solutions

$$\{\zeta_1^{(\mu)}(z), \dots, \zeta_n^{(\mu)}(z)\}, \mu = 1, \dots, m. \quad (2)$$

At each point of $S - E$ these sets are assumed distinct, and they shall form n continuous functional elements in a neighborhood of every point of $S - E$.

A function $F(z, \zeta)$ was called *holomorphic* in U if it is defined throughout U , is bounded in $U(R)$ for every compact subset R of S , and is given in $U(S - E)$ by an expression of the form

$$F(z, \zeta) = \frac{1}{\Delta^2(z)} \sum P_{\lambda_1 \dots \lambda_n}(z) \zeta_1^{\lambda_1} \dots \zeta_n^{\lambda_n} \quad (3)$$

where the P 's are analytic in S . In (3) the summation is taken over the range