Research statement

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The study of Morava E-theory at chromatic height 2 is like a raindrop in which all of modern homotopy theory—pervaded by the chromatic viewpoint in relation to the theory of one-dimensional formal groups, and by the study of multiplicative structures on chromatic spectra—is reflected. Hendrik Lenstra and Peter Stevenhagen wrote that "nothing can match the clarity of a formula when it comes to conveying a mathematical truth."[1] To understand E-theory in the specific case of height 2, our research has been focusing on its power operations and related structures. Here, contacts with algebraic geometry and number theory, particularly through the arithmetic moduli of elliptic curves, avail homotopy theorists effective tools to carry out explicit calculations.

We give an outline of our current research and future plans, with an emphasis on some specific aspects where algebraic topology, algebraic geometry, and number theory interact.

1 Elliptic curves: power operation structures at small primes

Cohomology operations have been a powerful tool central to algebraic topology. A classical example that has been extensively studied and widely applied is the Steenrod operations in ordinary cohomology with $\mathbb{Z}/p$-coefficients. When $p = 2$, for all integers $i \geq 0$ and $n \geq 0$, each Steenrod square takes the form

$$\mathrm{Sq}^i \colon\ H^n(X;\, \mathbb{Z}/2) \to H^{n+i}(X;\, \mathbb{Z}/2)\,,$$

natural in spaces X. Together they generate the mod-2 Steenrod algebra subject to a set of axioms, among which, notably, the Adem relations

$$\mathrm{Sq}^i\,\mathrm{Sq}^j = \sum_{k=0}^{[\frac{i}{2}]} \binom{j-k-1}{i-2k} \mathrm{Sq}^{i+j-k}\,\mathrm{Sq}^k \qquad\qquad 0 < i < 2j$$

In-depth study of the structure of the Steenrod algebra, and of analogous structures for other cohomology theories such as K-theory and motivic cohomology, has led to spectacular applications: Adams' solution to the problem of vector fields on spheres [Adams1962], and Voevodsky's proof of the Milnor conjecture [Voevodsky2003a, Voevodsky2003b], just to name two.

For Morava E-theory, after the foundational work of Ando [Ando1995], Strickland [Strickland1997, Strickland1998], and Ando-Hopkins-Strickland [Ando-Hopkins-Strickland2004], Rezk computed the first

[1] From their book review of *Solving the Pell equation*, Bull. Amer. Math. Soc. (N.S.) **52** (2015), no. 2, 345–351.

example of an explicit presentation for its algebra of power operations, in the case of height 2 at the prime 2 [Rezk2008].[2] This algebra is generated over $E^0(\text{point}) \cong \mathbb{W}\big(\overline{\mathbb{F}}_2\big)[\![h]\!]$ by

$$Q_i \colon\ E^0(X) \to E^0(X)\,,$$

$0 \le i \le 2$, which, in particular, satisfy "Adem relations"

$$\begin{aligned} Q_1Q_0 &= 2Q_2Q_1 - 2Q_0Q_2 \\ Q_2Q_0 &= Q_0Q_1 + hQ_0Q_2 - 2Q_1Q_2 \end{aligned}$$

Unlike their classical analogue, these formulas are computed from a specific moduli space of the universal formal deformation of a supersingular elliptic curve over $\overline{\mathbb{F}}_2$. Here, bridging algebraic topology and algebraic geometry are the work of Ando, Hopkins, Strickland, and Rezk [Ando-Hopkins-Strickland2004, Rezk2009] and the theorem of Serre and Tate [Lubin-Serre-Tate1964, Section 6].

We systematically studied and generalized Rezk's methods, and have obtained analogous results for E-theory at the primes 3 and 5.

Theorem 1.1 ([Zhu2014, Corollary 2.6 and Definition 3.8], [Zhu2015a, Examples 3.4 and 6.1]) *Let E be a Morava E-theory spectrum of height 2 at the prime p. There is a total power operation*

$$\begin{aligned} \psi^p \colon\ E^0(\text{point}) &\to E^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big) \end{aligned}$$

where I is an ideal of transfers.

(i) *When $p = 3$, we have $w(h,\alpha) = \alpha^4 - 6\alpha^2 + (h-9)\alpha - 3$ and*

$$\psi^3(h) = h^3 - 27h^2 + 201h - 342 + (-6h^2 + 108h - 334)\alpha + (3h - 27)\alpha^2 + (h^2 - 18h + 57)\alpha^3$$

(ii) *When $p = 5$, we obtain analogous (but admittedly less readable) formulas.*

(iii) *These lead to presentations for the respective algebra of power operations on $K(2)$-local commutative E-algebras, in terms of explicit generators and quadratic relations.*

Question 1.2 Is there a presentation that applies to *all* primes for the algebra of power operations in Morava E-theory at height 2?

Rezk gave such a uniform presentation, *modulo p*, in [Rezk2012a, Section 4], which relies on explicit formulas for the mod-p reduction of a certain moduli space of elliptic curves from [Katz-Mazur1985, Theorem 13.4.7]. *Integrally*, we will provide an answer in Theorem 3.3 below.

We should emphasize that the arithmetic data extracted from the particular moduli of elliptic curves cannot be homotopy-theoretically meaningful without the aforementioned deep theorems of Ando-Hopkins-Strickland, of Rezk, and of Serre-Tate. In the program of understanding higher chromatic levels, these *computational* experiments supply tangible raw materials to studying *structural* properties of the homotopy category. Below are some directions that we have investigated and plan to explore further in.

[2] The chromatic viewpoint organizes cohomology theories by heights and by primes, according to their associated formal groups [Quillen1969, Ravenel1992, Hovey-Strickland1999, Lawson2009].

(i) Rezk shows that the algebra of power operations for Morava E-theory is "Koszul" at all heights and primes [Rezk2012b, Rezk2012a]. For height 2 and small primes, the power operation formulas give rise to explicit Koszul complexes. More generally, the homological algebra of these Koszul complexes has been applied, e.g., to studying Bousfield-Kuhn functors and unstable periodic homotopy groups by Behrens and Rezk (cf. [Rezk2008, Exercise 2.11], [Rezk2013, Theorem 10.1 and Example 2.13], and [Behrens-Rezk2015, Theorem 9.1 and Section 10]).

(ii) The power operations at height 2 "descend" to height 1 via $K(1)$-localization (see [Zhu2014, Section 4]). At small primes, we observed that the resulting formulas matched up numerically to the rings studied by Lubin which parametrize canonical subgroups of formal groups [Lubin1979, Definition above Theorem E]. These rings can be explicitly determined, one for each height and each prime. The pattern re-occurs in our study of Rezk's logarithms (see Section 2 below). These suggest a more precise relationship between the first and second chromatic layers from the perspective of power operations and subgroups of formal groups.

(iii) At height 2, the explicit formulas have led to a partial understanding of the center for the algebra of power operations in E-theory [Zhu2015a, Theorem 6.8]. This is related to the Hecke operators that we discuss next.

2 Modular forms: Hecke operators and Rezk's logarithms

In [Rezk2006], using Bousfield-Kuhn functors, Rezk constructs logarithmic cohomology operations that naturally act on the units of any strictly commutative ring spectrum. In particular, given a Morava E-theory spectrum E of height n at the prime p, he writes down a formula for its "logarithm" $\ell_{n,p}\colon\ E^0(X)^\times \to E^0(X)$ in terms of its power operations ψ_A [Rezk2006, Theorem 1.11] and he interprets this formula in terms of certain "Hecke operators" $T_{j,p}$ as follows:

$$\begin{aligned} \ell_{n,p}(x) &= \frac{1}{p}\log\left(\prod_{j=0}^{n}\prod_{\substack{A\subset(\mathbb{Q}_p/\mathbb{Z}_p)^n[p]\\ |A|=p^j}}\psi_A(x)^{(-1)^j p^{(j-1)(j-2)/2}}\right) \\ &= \sum_{j=0}^{n}(-1)^j p^{j(j-1)/2}\,T_{j,p}(\log x) \end{aligned} \tag{2.1}$$

These Hecke operators are cohomology operations constructed from power operations that were known to act on the E-cohomology of a space [Ando1995, Proposition 3.6.2]. Based on explicit calculations and a particular choice of parameters in the case of height 2, we revisited these operators to make a precise connection with Hecke operators acting on modular forms. In particular, we obtained the following.

Theorem 2.2 ([Zhu2015a, Proposition 2.8 and Theorem 4.13]) *Let E be a Morava E-theory spectrum of height 2 at the prime p, and let $N>3$ be any integer prime to p.*

(i) *There is a ring homomorphism $\beta_p^N\colon\ \mathrm{MF}\big(\Gamma_1(N)\big)\to E^0(\text{point})$, where $\mathrm{MF}\big(\Gamma_1(N)\big)$ is the graded ring of weakly holomorphic modular forms on $\Gamma_1(N)$.*

(ii) *Given $f \in \mathrm{MF}\big(\Gamma_1(N)\big)^\times$ with trivial Nebentypus, if its Serre derivative $\vartheta f = 0$, then $\beta_p^N(f)$ is contained in the kernel of the logarithmic operation $\ell_{2,p}\colon E^0(\mathrm{point})^\times \to E^0(\mathrm{point})$.*

The logarithms in E-theory at height 2 are critical in the work of Ando, Hopkins, and Rezk on the rigidification of the string-bordism elliptic genus [Ando-Hopkins-Rezk2010, Theorem 12.3]. Roughly, in their setting, the kernel of a logarithm contains the desired genera, which they identified with certain Eisenstein series.

Question 2.3 With our result in Theorem 2.2 about the kernel of a logarithmic operation, can we develop an analysis of E_∞-orientations analogous to the work of Ando, Hopkins, and Rezk?

The logarithm of a meromorphic modular form (on which Hecke operators act) appears in Rezk's formula (2.1). Serre's differential operator ϑ appears in Theorem 2.2. In view of these, we ask the following.

Question 2.4 Do these specific pieces of number theory enter homotopy theory in a *structural* way? For example, do Rezk's logarithmic operations bring in a wider class of automorphic functions to homotopy theory? What is present at chromatic level higher than 2?

In [Zhu2015a, Section 5], we have started to investigate certain aspects of the aforementioned type of elliptic functions, not totally modular, in the framework of "logarithmic q-series" originally studied by Knopp and Mason [Knopp-Mason2011]. It has a curious relationship to mock modular forms, as observed in [Zhu2015a, Remark 5.2].

3 Formal groups: modular equations for Lubin-Tate formal groups

Classically, the Kronecker congruence

$$\big(j' - j^p\big)\big((j')^p - j\big) \equiv 0 \mod p$$

gives an equation, reduced modulo p, for the curve that represents $[\Gamma_0(p)]$, the moduli problem of finite flat subgroup schemes of rank p for elliptic curves. Indeed, this is precisely the formula that underlies Rezk's uniform presentation for the mod-p reduction of the power operation algebra (see Section 1).

Strickland studied various moduli problems for formal groups of finite height [Strickland1997] and he applied them to the study of power operations in Morava E-theory [Strickland1998]. At height 2, we have obtained an integral lift of the congruence above, with a different pair of parameters.

Theorem 3.1 ([Zhu2015b]) *Let $\mathbb{G}_0$ be a formal group of height 2 over $\overline{\mathbb{F}}_p$, and let $\mathbb{G}$ be its universal deformation. Write A_m for the ring $\mathcal{O}_{\mathrm{Sub}_m(\mathbb{G})}$ studied in [Strickland1997], which classifies degree-p^m subgroups of the formal group $\mathbb{G}$. In particular, write $A_0 \cong \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!]$ according to the Lubin-Tate theorem [Lubin-Tate1966, Section 3]. Then the ring $A_1 \cong \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big)$ is determined by the polynomial*

$$w(h,\alpha) = (\alpha - p)\big(\alpha + (-1)^p\big)^p - \big(h - p^2 + (-1)^p\big)\alpha \tag{3.2}$$

which reduces to $\alpha(\alpha^p - h)$ modulo p.

This gives an explicit description of $[\Gamma_0(p)]$ at a supersingular point (cf. [Katz-Mazur1985, Section 7.7]). It is *not* an equation for the modular curve over $\operatorname{Spec}\mathbb{Z}$, which, as hinted in [Rezk2014a], might lead to power operations for a "globally equivariant" elliptic cohomology. It would be interesting to explore this local-global relationship, intertwined by the actions of the Morava stabilizer groups and of the modular group, which unites the chromatic and equivariant perspectives on homotopy theory. We have started to investigate related functorial constructions, such as Witt ring schemes [Hazewinkel1978, Section 25], plethories [Borger-Wieland2005], and topological modular forms with level structure [Hill-Lawson2015].

Put in the context of homotopy theory, Theorem 3.1 yields the following (cf. Theorem 1.1).

Theorem 3.3 ([Zhu2015b]) *Let E be a Morava E-theory spectrum of height 2 at the prime p. There is a total power operation*

$$\begin{aligned} \psi^p\colon\ E^0(\text{point}) &\to E^0(B\Sigma_p)/I \\ \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h]\!] &\to \mathbb{W}\big(\overline{\mathbb{F}}_p\big)[\![h,\alpha]\!]/\big(w(h,\alpha)\big) \end{aligned}$$

where I is an ideal of transfers.

(i) *The polynomial*

$$w(h,\alpha) = w_{p+1}\alpha^{p+1} + \cdots + w_1\alpha + w_0 \qquad\qquad w_i \in E^0(\text{point})$$

can be given as (3.2) from Theorem 3.1. In particular, $w_{p+1} = 1$, $w_1 = -h$, $w_0 = (-1)^{p+1}p$, and the remaining coefficients

$$w_i = (-1)^{p\,(p-i+1)}\left[\binom{p}{i-1} + (-1)^{p+1}\,p\binom{p}{i}\right]$$

(ii) *The image $\psi^p(h) = \sum_{i=0}^{p} Q_i(h)\,\alpha^i$ is then given by*

$$\psi^p(h) = \alpha + \sum_{i=0}^{p}\alpha^i\sum_{\tau=1}^{p} w_{\tau+1}\,d_{i,\tau}$$

where

$$d_{i,\tau} = \sum_{n=0}^{\tau-1}(-1)^{\tau-n}\,w_0^n \sum_{\substack{m_1+\cdots+m_{\tau-n}=\tau+i \\ 1\le m_s\le m_{s+1}\le p+1}} w_{m_1}\cdots w_{m_{\tau-n}}$$

In particular, $Q_0(h) \equiv h^p \mod p$.

(iii) *These lead to a presentation for the algebra of power operations on $K(2)$-local commutative E-algebras. In particular, the generators $Q_i\colon\ E^0(X)\to E^0(X)$ satisfy quadratic relations*

$$Q_kQ_0 = -\sum_{j=1}^{p-k} w_0^j\,Q_{k+j}Q_j - \sum_{j=1}^{p}\sum_{i=0}^{j-1} w_0^i\,d_{k,j-i}\,Q_iQ_j \qquad\qquad 1\le k\le p$$

where the first summation is vacuous if $k = p$.

To prove this theorem, we followed a recipe for computing $\psi^p(h)$, as given in [Zhu2015a, Example 3.4], and generalized the proof of [Zhu2015a, Proposition 6.4]. Both methods particularly use the Atkin-Lehner involution, which is a manifest of the autoduality of elliptic curves.

Question 3.4 Without the autoduality of an elliptic curve, to study power operations in Morava E-theory at height greater than 2, can we generalize Theorem 3.1 for the ring A_2 (cf. [Rezk2014b, Section 4.6] and [Rezk2013, Section 7])?

A difficulty for such a generalization lies in the current methods for proving Theorem 3.1: we argue with the q-expansions of certain Hauptmoduln and their Hecke translates, as in [Choi2006, Example 2.4] and [Katz1973, Section 1.11], which are specific to heights 1 and 2. Moreover, we are in much need of results from concrete "computational experiments" at higher chromatic levels, indeed, at height 3 (see [Lawson2015, Meier2014]). It would be interesting to work on this.

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