MODULAR INVARIANCE OF VERTEX OPERATOR ALGEBRAS SATISFYING C_2-COFINITENESS

MASAHIKO MIYAMOTO

Abstract

We investigate trace functions of modules for vertex operator algebras (VOA) satisfying C_2-cofiniteness. For the modular invariance property, Zhu assumed two conditions in [Z]: (1) $A(V)$ is semisimple and (2) C_2-cofiniteness. We show that C_2-cofiniteness is enough to prove a modular invariance property. For example, if a VOA $V = \bigoplus_{m=0}^{\infty} V_m$ is C_2-cofinite, then the space spanned by generalized characters (pseudotrace functions of the vacuum element) of V-modules is a finite-dimensional $\mathrm{SL}_2(\mathbb{Z})$-invariant space and the central charge and conformal weights are all rational numbers. Namely, we show that C_2-cofiniteness implies "rational conformal field theory" in a sense as expected in Gaberdiel and Neitzke [GN]. Viewing a trace map as a symmetric linear map and using a result of symmetric algebras, we introduce "pseudotraces" and pseudotrace functions and then show that the space spanned by such pseudotrace functions has a modular invariance property. We also show that C_2-cofiniteness is equivalent to the condition that every weak module is an $\mathbb{N}$-graded weak module that is a direct sum of generalized eigenspaces of $L(0)$.

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1. Introduction

In this paper, we consider a vertex operator algebra $V = \coprod_{n=0}^{\infty} V_n$ with central charge c. One of the central concepts in conformal field theory (CFT) is *rationality*, a condition that is supposed to express a kind of finiteness of the theory. There exist various notions of finiteness. One of them is the complete reducibility of $\mathbb{N}$-graded weak modules, which is the condition called *rationality* by most researchers on vertex operator algebras. There is another important finiteness condition called *C_2-cofiniteness*. Complete reducibility of $\mathbb{N}$-graded weak modules is a condition for modules. In this case, Zhu algebra $A(V) = V/O(V)$ is a finite-dimensional semisimple algebra and V has only finitely many irreducible modules (see [DLM1] for a proof). On the other hand, C_2-cofiniteness is a property of V itself; that is, if $C_2(V) = \langle a(-2)b \mid a, b \in V\rangle$ is of finite codimension in V, then V is called *C_2-cofinite*. Here $Y(a,z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1}$ is the vertex operator of a. Then $A(V)$ is of finite dimension, and so V has only finitely many irreducible modules, say, $\{W^1, \dots, W^k\}$ (see [DLM3] for a proof). Under the two hypotheses that $A(V)$ is semisimple and V is C_2-cofinite (Condition C in his paper), Zhu proved in [Z] that the space spanned by trace functions

$$S^{W^i}(v,\tau) = \mathrm{tr}_{|W^i}\, o(v)q^{L(0)-c/24} \quad (q = e^{2\pi i\tau} \text{ and } \tau \in \mathscr{H})$$

has a modular-invariance (SL(2, $\mathbb{Z}$)-invariance) property, where c is the central charge of V, $\mathscr{H}$ is the upper half-plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$, and $o(v)$ is the grade-preserving operator of $v \in V$. Namely, for $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$, there is a $(k\times k)$-matrix (λ_{ij}) with $\lambda_{ij} \in \mathbb{C}$ such that

$$\frac{1}{(c\tau+d)^n}\left(S^{W^1}\left(u, \frac{a\tau+b}{c\tau+d}\right)\cdots S^{W^k}\left(u, \frac{a\tau+b}{c\tau+d}\right)\right) = \left(S^{W^1}(u,\tau)\cdots S^{W^k}(u,\tau)\right)\begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1k}\\ \vdots & \cdots & \vdots \\ \lambda_{k1} & \cdots & \lambda_{kk}\end{pmatrix}$$

for all $u \in V_{[n]}$ and $n = 0, 1, \dots$, where $V = \bigoplus_{n=0}^{\infty} V_{[n]}$ is the second grading on V introduced in [Z] (see Def. 2.3). Actually, Zhu assumed one more condition, but it is not necessary, as mentioned in [DLM3]. This is a fundamental result for modular invariance properties of vertex operator algebras and is extended by several authors (e.g., [DLM3], [Mi1]–[Mi3], and [Y]).

Since then it has become an important problem to study the relation between rationality and C_2-cofiniteness. C_2-cofiniteness was once conjectured to be equivalent to rationality, but recent research shows that C_2-cofiniteness does not imply complete reducibility (see [GK] and [M]).

On the other hand, C_2-cofiniteness was studied by Gaberdiel and Neitzke from a viewpoint of rationality in [GN], and they showed that if V is a C_2-cofinite VOA of CFT type, then V satisfies the most conditions required for a rational conformal field theory except for a modular invariance property, where V is called *CFT type* if $V = \bigoplus_{n=0}^{\infty} V_n$ and $\dim V_0 = 1$ (see also [Z], [DLM3], [L2]).

Although C_2-cofiniteness was introduced by Zhu as a technical assumption and it is a property of VOA itself, it is a natural condition to consider the characters of all (weak) modules. For in order to define $q^{L(0)}$ on a weak module W, W has to be a direct sum of generalized eigenspaces of $L(0)$, which is a condition equivalent to C_2-cofiniteness.

THEOREM 2.7

Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a vertex operator algebra. Then the following are equivalent.

(1) *V is C_2-cofinite.*

(2) *Every weak module is a direct sum of generalized eigenspaces of $L(0)$.*

(3) *Every weak module is an $\mathbb{N}$-graded weak module $W = \bigoplus_{n=0}^{\infty} W(n)$ such that $W(n)$ is a direct sum of generalized eigenspaces of $L(0)$.*

(4) *V is finitely generated and every weak module is an $\mathbb{N}$-graded weak module.*

Therefore if V is C_2-cofinite, then we can define several modules naturally. For example, for $\mathbb{N}$-graded weak V-modules U, the maximal weak V-submodule $D(U)$ of $\mathrm{Hom}(U, \mathbb{C})$ introduced in [L2] is an $\mathbb{N}$-graded weak module. Most results about C_2-cofininteness come from the existence of some spanning set (see, e.g., [GN], [L2], [Bu], [ABD]). We prove the existence of the following spanning set for a general VOA without negative weights.

LEMMA 2.4

Let A be a set of homogeneous elements of V such that $V = C_2(V) + \langle A \rangle$. Assume that V is C_2-cofinite, and assume that W is a weak module generated from w (by the action of the vertex operator). Then W is spanned by the following elements:

$$\{v^1(i_1)\cdots v^k(i_k)w \,|\, v^i \in A,\ i_1 < \cdots < i_k\}.$$

In particular, if we set

$$\mathscr{W}(m) = \langle v^1(i_1)\cdots v^k(i_k)w \,|\, v^i \in V,\ \deg(v^1(i_1)\cdots v^k(i_k)) = m \rangle,$$

then $\mathscr{W}(m) = 0$ *for* $m \ll 0$, *where* $\deg(v(i)\cdots u(j))$ *denotes the degree of* $v(i)\cdots u(j)$ *as an operator.*

The main purpose of this paper is to show that C_2-cofiniteness is enough for a modular invariance property without assuming that $A(V)$ is semisimple. As a result, the central charge c and the conformal weights are all rational numbers. Namely, C_2-cofiniteness provides the conditions required for a rational conformal field in a sense.

In the proof of the modular invariance property in [Z], Zhu introduced the space $\mathscr{C}_1(V)$ of one-point functions (see also [DLM3]). He proved the modular invariance property by showing that $\mathscr{C}_1(V)$ is spanned by trace functions $S^W(v,\tau) = \mathrm{tr}_W\, o(v)q^{L(0)-c/24}$. However, if $A(V)$ is not semisimple, $\mathscr{C}_1(V)$ may not be spanned by trace functions. One of our aims in this paper is to stuff suitable functions into a crevice. We introduce a new kind of trace map called a *pseudotrace map* (here *pseudotrace* has a different meaning than in an algebraic number field) on some kind of $\mathbb{N}$-graded weak V-modules with homogeneous spaces of finite dimension. One of the key steps in Zhu's proof is that $\mathscr{C}_1(V)$ is spanned by functions of the form

$$\sum_{i=0}^{N}\Big(\sum_{j=1}^{d}\Big(\sum_{k=0}^{\infty} C_{ij,k}(v)q^k\Big)q^{r_{ij}}\Big)\big(\ln(q)\big)^i$$

such that the coefficient $C_{ij,0}$ of the lowest degree satisfies the following conditions:

(1) $C_{ij,0}(O(V)) = 0$,

(2) $C_{ij,0}(ab-ba) = 0$ for all $a,b \in A(V)$, and

(3) $C_{ij,0}((\omega - c/24 - r_{ij})^{N-i+1}v) = 0$.

In particular, $A(V)/\mathrm{Rad}(C_{ij,0})$ is a symmetric algebra with a symmetric linear function $C_{ij,0}$, where $\mathrm{Rad}(\phi) \stackrel{\text{def}}{=} \{a \in A(V) \mid \phi(A(V)aA(V)) = 0\}$. We are not interested in symmetric algebras but symmetric linear functions. Originally, Nesbitt and Scott showed in [NS] that A is a symmetric algebra if and only if its basic algebra P is symmetric. Our strategy in this paper is as follows. We show that $C_{ij,n}$ is a symmetric linear map of nth Zhu algebra $A_n(V)$ (see §2.2), and so we have a symmetric algebra $A = A_n(V)/\mathrm{Rad}(C_{ij,n})$ and its symmetric basic algebra P with a symmetric linear function ϕ. We start from (P,ϕ) and construct a right P-module W (a generalized Verma module) such that the basic algebra of $R = \mathrm{End}_P(W)$ is P. We call such a module W *interlocked with* ϕ. Nesbitt and Scott's result tells us that R has a symmetric linear map tr^ϕ (we call it *pseudotrace*). Then we define a pseudotrace function

$$\mathrm{tr}^{\phi}_{W}\, o(v)q^{L(0)-c/24}$$

and show that $\mathscr{C}_1(V)$ is spanned by such pseudotrace functions.

For example,

$$P = \left\{ p = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \,\middle|\, a,b \in \mathbb{C} \right\}$$

is a basic symmetric algebra with a linear map $\phi(p) = b$. We note that $J(P) = \mathrm{soc}(P) = \left\{ \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \;\middle|\; b \in \mathbb{C} \right\}$. Consider a right P-module $T = \mathbb{C}^m \oplus \mathbb{C}^m$ given by $(\vec{u}, \vec{v}) \left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) = (a\vec{u}, b\vec{u} + a\vec{v})$. Then

$$\mathrm{End}_P(T) = \left\{ \alpha = \begin{pmatrix} A_\alpha & B_\alpha \\ 0 & A_\alpha \end{pmatrix} \;\middle|\; A_\alpha, B_\alpha \in M_{m,m}(\mathbb{C}) \right\}$$

and the basic algebra of $\mathrm{End}_P(T)$ is P. For any $\alpha \in \mathrm{End}_P(T)$, if we define $\mathrm{tr}_T^{\phi}(\alpha) = \mathrm{tr}(B_\alpha)$, then tr_T^{ϕ} is also a symmetric linear map.

We should note one more thing. Since we treat the general cases, $L(0)$ may not act on $W_T^{(n)}(m)$ as a scalar. However, since V is C_2-cofinite, every nth Zhu algebra $A_n(V)$ is finite-dimensional, as we will see, and so every generalized Verma V-module is a direct sum of modules $W = \bigoplus_{m=0}^{\infty} W(m)$ so that $L(0) - r - m$ acts on $W(m)$ as a nilpotent operator for some $r \in \mathbb{C}$, say, $(L(0) - r - m)^s = 0$ on $W(m)$. Let $L^s(0)$ denote the semisimple part of $L(0)$, that is, $m + r$ on $W(m)$. Then trace function $\mathrm{tr}_W^{\phi}\, o(v) q^{L(0)}$ on W is defined by

$$\mathrm{tr}_W^{\phi}\, o(v) q^{L(0)-c/24} = \mathrm{tr}_W^{\phi} \left\{ o(v) \sum_{i=0}^{s-1} \frac{(2\pi i \tau)^i (L(0) - L^s(0))^i}{i!} \right\} q^{L^s(0)-c/24}.$$

These modules W are called *logarithmic modules* (see [G], [M]), and Flohr introduced in [F] a concept of generalized characters to interpret the modular invariance property of characters of logarithmic modules. As we will see, $S^{W_T^{(n)}}(\mathbf{1}, \tau)$ is a linear combination of (ordinary) characters with coefficients in $\mathbb{C}[\tau]$ and plays the role of a generalized character.

Our main theorem is the following.

THEOREM 5.5

Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a C_2-cofinite VOA with central charge c. Then for $v \in V_{[m]}$, the set of pseudotrace functions

$$\left\langle \mathrm{tr}_W^{\phi}\, o(v) q^{L(0)-c/24} \;\middle|\; W \text{ is interlocked with a symmetric linear map } \phi \text{ of } A_n(V) \right\rangle$$

is invariant under the action of $\mathrm{SL}_2(\mathbb{Z})$ *with weight m. In particular,* $\dim \mathscr{C}_1(V) = \dim A_n(V)/[A_n(V), A_n(V)] - \dim A_{n-1}(V)/[A_{n-1}(V), A_{n-1}(V)]$ *for $n \gg 0$.*

In particular, the space spanned by generalized characters $\mathrm{tr}_W^{\phi}\, q^{L(0)-c/24}$ is a finite-dimensional $\mathrm{SL}_2(\mathbb{Z})$-invariant space. As corollaries, we obtain the following.

COROLLARIES 5.10 AND 5.11

If $V = \bigoplus_{m=0}^{\infty} V_m$ is a C_2-cofinite VOA, then the central charge and the conformal *weights are all rational numbers. Moreover, we have*

$$\tilde{c} \leq \frac{\dim V/C_2(V) - 1}{2},$$

where $\tilde{c} \overset{\text{def}}{=} c - h_{\min}$ *is the effective central charge of* V *and* $h_{\min}$ *is the smallest conformal weight.*

Gaberdiel and Neitzke showed in [GN] that C_2-cofiniteness implies C_m-cofiniteness for any $m = 1, 2, \dots$ if V is of CFT type. In this paper, we consider only a VOA without negative weights. However, if we consider C_{2+s}-cofiniteness when $V = \bigoplus_{n=-s}^{\infty} V_n$ has a negative weight, then it is not difficult to see that we have a result similar to that in Lemma 2.4 and the other results in this paper by replacing $C_2(V)$ by $C_{2+s}(V)$ and $\mathrm{wt}(v)$ by $\mathrm{wt}(v) - s$, respectively.

This paper is organized as follows. In Section 2, we explain the notation and fundamental results. In Section 3, we introduce a concept of modules interlocked with ϕ and define a pseudotrace map explicitly. In Section 4, we define a pseudotrace function for an $\mathbb{N}$-graded weak V-module interlocked with a symmetric function. In particular, we explain that if we take a sufficiently large integer n and ϕ is a symmetric linear map of $A_n(V)$, then we can construct a generalized Verma VOA-module $W_T^{(n)}$, which is interlocked with ϕ. In Section 5, we prove a modular invariance property.

2. Fundamental results

2.1. Vertex operator algebras

Definition 2.1

A *weak module* for VOA $(V, Y, \mathbf{1}, \omega)$ is a vector space M, equipped with a formal power series

$$Y^M(v, z) = \sum_{n \in \mathbf{Z}} v^M(n) z^{-n-1} \in \big(\mathrm{End}(M)\big)[[z, z^{-1}]]$$

(called the *module vertex operator* of v) for $v \in V$, satisfying the following:

(1) $v^M(n)w = 0$ for $n \gg 0$ where $v \in V$ and $w \in M$;

(2) $Y^M(\mathbf{1}, z) = 1_M$;

(3) $Y^M(\omega, z) = \sum_{n \in \mathbb{Z}} L^M(n) z^{-n-2}$, which satisfies

(3.a) the Virasoro algebra relations

$$\big[L^M(n), L^M(m)\big] = (n-m)L^M(n+m) + \delta_{n+m,0}\frac{n^3 - n}{12}c,$$

(3.b) the $L(-1)$-derivative property

$$Y^M\big(L(-1)v, z\big) = \frac{d}{dz} Y^M(v, z);$$

(4) "commutativity" holds:

$$[v^M(n), u^M(m)] = \sum_{i=0}^{\infty} \binom{n}{i} \big(v(i)u\big)^M (n+m-i);$$

and

(5) "associativity" holds:

$$\begin{aligned}&\big(v(n)u\big)^M(m)\\ &= \sum_{i=0}^{\infty}(-1)^i\binom{n}{i}\big\{v^M(n-i)u^M(m+i)-(-1)^n u^M(n+m-i)v^M(i)\big\},\end{aligned}$$

where $\binom{n}{i} = (n(n-1)\cdots(n-i+1))/i!$.

An $\mathbb{N}$-*graded weak module* is a weak V-module that carries an $\mathbb{N}$-grading, $M = \bigoplus_{n=0}^{\infty} M(n)$, such that

(1′) if $v \in V_r$, then $v^M(m)M(n) \subseteq M(n+r-m-1)$.

An ordinary module is an $\mathbb{N}$-graded weak V-module $W = \bigoplus_{n=0}^{\infty} W(n)$ such that $L(0)$ acts on $W(n)$ semisimply and $\dim W(n) < \infty$ for all n. For a simple ordinary module $W = \bigoplus_{i=0}^{\infty} W(i)$, $L(0)$ acts on $W(0)$ as a scalar, which is called a *conformal weight* of W.

The main object in this paper is not an ordinary module but an $\mathbb{N}$-graded weak module $W = \bigoplus_{m=0}^{\infty} W(m)$ with $\dim W(m) < \infty$. If $v \in V_m$, then $v^W(m-1)$ is a grade-preserving operator by (1′) and we denote it by $o(v)$ and extend it linearly.

Definition 2.2
V is called C_2-*cofinite* if the subspace $\langle v(-2)u : v, u \in V\rangle$ has a finite codimension in V.

Zhu has introduced the second vertex operator algebra $(V, Y[\,,\,], \mathbf{1}, \tilde{\omega})$ associated to V in [Z, Th. 4.2.1].

Definition 2.3
The *vertex operator* $Y[v,z] = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1}$ is defined for homogeneous v via the equality

$$Y[v,z] = Y(v, e^z - 1)e^{z|v|} \in \mathrm{End}(V)[[z, z^{-1}]],$$

and the *Virasoro element* $\tilde{\omega}$ is defined to be $\omega - (c/24)\mathbf{1}$, where $|v|$ denotes the weight of v.

Throughout this paper, we assume that $V=\bigoplus_{n=0}^{\infty}V_n$ is a C_2-cofinite VOA.

2.2. nth Zhu algebras

Following [FZ], V has a product

$$v*u=\mathrm{Res}_x\,\frac{(1+x)^{|v|}}{x}Y(v,x)u \tag{2.1}$$

for $v\in V_{|v|}$ and $u\in V$, where $|v|$ denotes the weight of v. Set

$$O(V)=\left\langle \mathrm{Res}_x\,\frac{(1+x)^{|v|}}{x^2}Y(v,x)u \,\middle|\, v,u\in V\right\rangle \tag{2.2}$$

and $A(V)=V/O(V)$. Then it is known (see [FZ, Th. 1.5.1]) that $A(V)$ is an associative algebra with a product $*$. We call it a *Zhu algebra*. Zhu has also shown that $\omega+O(V)$ is in the center of $A(V)$. From now on, abusing the notation, we use the same notation ω for $\omega+O(V)$. The essential property of Zhu algebra is that a top module $W(0)$ of an $\mathbb{N}$-graded weak $A(V)$-module $W=\bigoplus_{n=0}^{\infty}W(n)$ is an $A(V)$-module and every $A(V)$-module is a top module of some $\mathbb{N}$-graded weak V-module. This concept was naturally extended to the nth graded piece of $\mathbb{N}$-graded weak modules by Dong, Li, and Mason in [DLM2]. Set

$$O_n(V)=\left\langle \mathrm{Res}_x\,\frac{(x+1)^{|v|+n}}{x^{2+2n}}Y(v,x)u \,\middle|\, v,u\in V\right\rangle \tag{2.3}$$

and $A_n(V)=V/O_n(V)$. Like $A(V)$, $A_n(V)$ is an associative algebra with a product

$$v*_n u=\sum_{m=0}^{n}\binom{-n}{m}\mathrm{Res}_x\,Y(v,x)u\frac{(x+1)^{|v|+n}}{x^{n+m+1}},$$

and it has the property that an nth (and less) graded piece $W(n)$ of an $\mathbb{N}$-graded weak V-module $W=\bigoplus_{i=0}^{\infty}W(i)$ is an $A_n(V)$-module and every $A_n(V)$-module is an nth (or less) graded piece of an $\mathbb{N}$-graded weak V-module (see [DLM2, Th. 4.2]). $A_n(V)$ is called an *nth Zhu algebra*; in particular, the 0th Zhu algebra is the original Zhu algebra. It is easy to see that there is a natural homomorphism from $A_n(V)$ to $A_{n-1}(V)$. The product $*_n$ is characterized by the identity $o(v*_n u)=o(v)o(u)$ on $\bigoplus_{i=0}^{n}W(i)$ for every $\mathbb{N}$-graded weak V-module W, so that $A_n(V)$ is essentially the algebra of zero modes (grade-preserving operators) of fields. In particular, ω is a central element of $A_n(V)$ for any n. Viewing $A_n(V)$ as an algebra of zero modes, we use the following notation:

$$o(\alpha)=v(|v|-1+m)u(|u|-1-m)\quad\text{in }\ A_n(V),$$

which implies that $o(\alpha)w=v(|v|-1+m)u(|u|-1-m)w$ for any $\mathbb{N}$-graded weak module W and $w\in\bigoplus_{i=0}^{n}W(i)$.

We note that Garberdiel and Neitzke showed in [GN] that if V is a C_2-cofinite VOA of CFT type, then $A_n(V)$ is of finite dimension for any n. They did not mention this fact directly. Later, Buhl proved it for irreducible modules in [Bu] by using a spanning set of module. We would like to explain their results and prove the finiteness of dimension of $A_n(V)$ even if $\dim V_0 \neq 1$. First, by using a filtration, they showed that if $V = C_2(V) + \langle A \rangle$ for a set A of V, then V is spanned by elements of the form

$$v^1(-N_1)\cdots v^r(-N_r)\mathbf{1}$$

with $N_1 > \cdots > N_r > 0$ and $v^i \in A$ (see [GN, Prop. 8]). We first show the existence of such a spanning set for general VOAs without negative weights.

LEMMA 2.4

Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a C_2-cofinite VOA, and let A be a set of homogenous elements satisfying $V = C_2(V) + \langle A \rangle$. Let W be a weak module generated from w. Then W is spanned by the following elements:

$$\{v^1(i_1)\cdots v^k(i_k)w \mid v^i \in A,\ i_1 < \cdots < i_k\}.$$

In particular, if we set

$$\mathscr{W}(m) = \langle v^1(i_1)\cdots v^k(i_k)w \mid v^i \in V,\ \deg(v^1(i_1)\cdots v^k(i_k)) = m \rangle,$$

then $\mathscr{W}(m) = 0$ for $m \ll 0$, where $\deg(v(i)\cdots u(j))$ denotes the degree of $v(i)\cdots u(j)$ as an operator.

Proof

Define a filtration on W by

$$\mathscr{W}(n,m,r) = \left\langle v^1(i_1)\cdots v^k(i_k)w \;\middle|\; \begin{array}{l} v^i \in V,\ \sum_{i=1}^{k} \mathrm{wt}(v^i) \le n, \\ \deg(v^1(i_1)\cdots v^k(i_k)) = m,\ k \le r, \end{array} \right\rangle.$$

Clearly, $\mathscr{W}(n,m,r) \subseteq \mathscr{W}(n+1,m,r)$ and

$$W = \sum_{m\in\mathbb{Z}} \Big(\bigcup_{n=0}^{\infty} \bigcup_{r=0}^{\infty} \mathscr{W}(n,m,r) \Big).$$

We prove that $\mathscr{W}(n,m,r)$ is spanned by the desired elements contained in $\mathscr{W}(n,m,r)$ for each m. Suppose that this is false, and let (n,r) be a minimal counterexample with respect to lexicographical order. Let $U(n,m,r)$ be the subspace spanned by the desired elements contained in $\mathscr{W}(n,m,r)$; then there is a nonzero element $u \in \mathscr{W}(n,m,r) - U(n,m,r)$ and $\mathscr{W}(n-1,m,r) + \mathscr{W}(n,m,r-1) \subseteq U(n,m,r)$. We may assume

$$u = v^1(i_1)\cdots v^r(i_r)w.$$

Since $v^i \in V = \langle A\rangle + C_2(V)$, there are $u^i \in A$ and $a^{ij}, b^{ij} \in V$ such that $v^i = u^i + \sum a^{ij}(-2)b^{ij}$. Since $C_2(V)$ and $\langle A\rangle$ are direct sums of homogeneous spaces, we may assume $\mathrm{wt}(v^i) = \mathrm{wt}(u^i) = \mathrm{wt}(a^{ij}(-2)b^{ij}) = \mathrm{wt}(a^{ij}) + \mathrm{wt}(b^{ij}) - 1$. By

$$\big(a(-2)b\big)(s) = \sum_{i=0}^{\infty}(-1)^i\binom{-2}{i}\big\{a(-2-i)b(s+i) - b(-2+s-i)a(i)\big\},$$

we may assume

$$u = v^1(i_1)\cdots v^r(i_r)w \quad \text{ with } v^i \in A. \tag{2.4}$$

We choose $u \in \mathscr{W}(n,m,r) - U(n,m,r)$ so that u has a form (2.4) and $\min\{i_1,\ldots,i_k\}$ is minimal. The existence of minimal one follows from the next arguments. Since

$$v(i)u(j) = u(j)v(i) + [v(i),u(j)] = u(j)v(i) + \sum_{s=0}^{\infty}\binom{i}{s}\big(v(s)u\big)(i+j-s)$$

and $\mathrm{wt}(v(s)u) < \mathrm{wt}(v) + \mathrm{wt}(u)$ for $s \geq 0$, we may assume

$$u = v^1(i_1)\cdots v^r(i_r)w \quad \text{ with } v^r \in A \text{ and } i_1 \leq \cdots \leq i_k.$$

Since $v^i \in A$ and A is a finite set, there is an integer N such that $v(m)w = 0$ for $m > N$ and $v \in A$. Therefore we have $i_r \leq N$. In particular, the degree of $v^p(i_p)$ as an operator is bounded below. Since the total degree of $v^1(i_1)\cdots v^r(i_r)$ is m, the degree of $v^p(i_p)$ as an operator is bounded above, and so i_1 is bounded below. By induction on r, we may assume $i_2 < \cdots < i_r$. If $i_1 < i_2$ or $i_1 > i_2$, then $u \in \mathscr{W}(n,m,r)$ by the minimality of i_1. The remaining case is $i_1 = i_2$. Set $i = i_1 = i_2$. The expansion of $(v^1(-1)v^2)(2i+1)$ by associativity if $i \leq -1$ and that of $(v^2(-1)v^1)(2i+1)$ by associativity if $i \geq 0$ contain a nonzero term $v^1(i)v^2(i)$, and the other terms are $v^1(i-j)v^2(i+j)$ or $v^2(i-j)v^1(i+j)$ with $j \neq 0$. For example, if $i \leq -1$, then there are constants λ_j, μ_j such that

$$\begin{aligned} v^1(i)v^2(i) &= \big(v^1(-1)v^2\big)(2i+1) \\ &\quad + \sum_{j\neq 0}\big(\lambda_j v^1(i-j)v^2(i+j) + \mu_j v^2(i-j)v^1(i+j)\big). \end{aligned} \tag{2.5}$$

We substitute the right-hand side for $v^1(i_1)v^2(i_1)$ in u. Since $v^1(i-j)v^2(i+j)v^3(i_2)\cdots v^r(i_r)w$ has the form (2.4) and one of $i-j$ and $i+j$ is less than i, we may assume $u = (v^1(-1)v^2)(2i+1)v^2(i_3)\cdots w$, which is in $\mathscr{W}(n,m,r-1)$. Therefore we have a contradiction. Since $\dim\langle A\rangle < \infty$, there is an interger N such that $i_k < N$, and so $\deg(v^1(i_1)\cdots v^k(i_k)) \geq -kN + k(k-1)/2 > -N^2$ if $i_1 < \cdots < i_k < N$. Thus $\mathscr{W}(m) = 0$ if $m < -N^2$.

This completes the proof. □

Using this spanning set, we have the following theorem (see [GN, Lem. 3 and Th. 11] and [Bu, Cor. 5.5] if V is of CFT type).

THEOREM 2.5
If V is C_2-cofinite, then $A_n(V)$ are all finite-dimensional.

Proof
We fix n, and v and u denote homogeneous elements. Let $O_{(\infty^{2n+2})}$ be the subspace of V spanned by elements of the form

$$\big\langle v(-2-N-2n|v|)u \,\big|\, v,u \in V,\ |v| \geq 1,\ N \geq 0 \big\rangle.$$

We show that $O_{(\infty^{2n+2})}$ is of finite codimension. Since V is C_2-cofinite, we can choose a finite set A of homogenous elements such that V is spanned by $w_1(-N_1)\cdots w_r(-N_r)\mathbf{1}$ with $N_1 > \cdots > N_r$ and $w_i \in A$. Let t be the maximal weight of elements in A. If $N_1 \geq (2n)t+2$, then $w_{i_1}(-N_1)\cdots w_{i_r}(-N_r)\mathbf{1} \in O_{(\infty^{2n+2})}$. This leaves us only finitely many choices for the N_i, which gives a finite spanning set for $V/O_{(\infty^{2n+2})}$.

Let $O_{\mathbf{u}}$ be the subspace of V spanned by elements of the form

$$v \diamond_M u = \mathrm{Res}_z\, Y(v,z)u \frac{(z+1)^{(n+1)|v|}}{z^{2n|v|+2+M}}$$

with $|v| \geq 1$ and $M \geq 0$. Since $2n(|v|-1) \geq n(|v|-1)$ and

$$v \diamond_M u = \mathrm{Res}_z\, Y(v,z)u \frac{(z+1)^{|v|+n+n(|v|-1)}}{z^{2+M+2n+2n(|v|-1)}},$$

$v \diamond_M u \in O_n(V)$ by [DLM2, Lem. 2.1], and so it is sufficient to show $\dim V/O_{\mathbf{u}} < \dim V/O_{(\infty^{2n+2})}$. We note that

$$v \diamond_M u = \sum_{i=0}^{(n+1)|v|} \binom{(n+1)|v|}{i} v(-2n|v|-2-M+i)u;$$

that is, the weights of all terms are less than or equal to the weight of $v(-2n|v|-2-M)u$. Let $\{v_1,\ldots,v_N\}$ be a set of representatives for $V/O_{(\infty^{2n+2})}$. Since $O_{(\infty^{2n+2})}$ is a direct sum of homogenous spaces, we may assume that v_i are all homogenous elements. We claim that $\langle v_1,\ldots,v_N\rangle + O_{\mathbf{u}} = V$, which offers the desired conclusion. Suppose that this is false, and let $u \in V - \{\langle v_1,\ldots,v_N\rangle + O_{\mathbf{u}}\}$ be a homogeneous element with minimal weight. By the choice of $\{v_i\}$, there are $a_r, b_j \in \mathbb{C}$ and homogeneous elements $v^r, u^r \in V$ and $N_r \in \mathbb{Z}_+$ such that

$$u = \sum_j b_j v_j + \sum_r a_r v^r(-N_r-2n|v|-2)u^r.$$

We may assume that the weights of all elements in the above equation are the same. But then

$$\hat{u} = u - \sum_j b_j v_j - \sum_r v^r \diamond_{N_r} u^r$$

is a linear combination of vectors whose weights are strictly smaller than that of u. By the minimality of u, $\hat{u}$ is contained in $\langle v_1, \ldots, v_n \rangle + O_{\mathbf{u}}$ and so is u.

This completes the proof of Theorem 2.5. □

2.3. Generalized Verma module

In this paper, we define a pseudotrace function for a generalized Verma module W^n_T constructed from an $A_n(V)$-module T. A generalized Verma module Verma(X) for an $A(V)$-module X was introduced in [L1] as an extension of the concept of a Verma module. It is the largest $\mathbb{N}$-graded weak V-module W which has X as a top module $W(0)$ and is generated from X. For an $A_n(V)$-module T, it is possible to consider the largest $\mathbb{N}$-graded weak V-module $W^{(n)}_T = \bigoplus_{i=0}^{\infty} W^{(n)}_T(i)$ which has T as its nth graded piece $W^{(n)}_T(n)$ and is generated from T. We should note that $W^{(n)}_T(0)$ might be zero if T is also an $A_{n-1}(V)$-module. However, in this paper, we treat only an $\mathbb{N}$-graded weak V-module $W^{(n)}_T$ constructed from an $A_n(V)$-module T which satisfies the following condition:

> For any $A_n(V)$-submodules $T^1 \subseteq T^2$ of T with an irreducible factor T^2/T^1, every V-module W with nth graded piece T^2/T^1 is irreducible. (2.6)

More precisely, we consider only an $\mathbb{N}$-graded weak V-module $W^{(n)}_T$ whose composition series has the same shape as does a composition series of $T = W^{(n)}_T(n)$. Therefore a V-module W with T as an nth graded piece is uniquely determined, and so a generalized Verma module coincides with $L_n(T)$, which is defined in [DLM2, Th. 4.2] and is the minimal one in a sense. So we do not need a concept of generalized Verma module, but in order to emphasize that this module is naturally constructed from an $A_n(V)$-module T, we call it a *generalized Verma module* and denote it by $W^{(n)}_T$. As Gaberdiel and Neitzke showed in [GN], we obtain the following.

LEMMA 2.6

If V is a C_2-cofinite VOA and T is a finite-dimensional $A_n(V)$-module, then $W_T(m)$ has a finite dimension for any $m = 0, 1, \ldots$.

2.4. Elliptic functions

We adopt the same notation from [Z]. The Eisenstein series $G_{2k}(\tau)$ $(k = 1, 2, \ldots)$ are series

$$G_{2k}(\tau) = \sum_{(m,n)\neq(0,0)} \frac{1}{(m\tau + n)^{2k}} \quad \text{for } k \geq 2$$

and

$$G_2(\tau) = \frac{\pi^2}{3} + \sum_{m\in\mathbb{Z}-\{0\}} \sum_{n\in\mathbb{Z}} \frac{1}{(m\tau + n)^2} \quad \text{for } k = 1.$$

They have the q-expansions

$$G_{2k}(\tau) = 2\xi(2k) + \frac{2(2\pi\iota)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \frac{n^{2k-1}q^n}{1-q^n},$$

where $\xi(2k) = \sum_{n=1}^{\infty} 1/n^{2k}$ and $q = e^{2\pi\iota\tau}$. We make use of the following normalized Eisenstein series:

$$E_k(\tau) = \frac{1}{(2\pi\iota)^k} G_k(\tau) \quad \text{for } k \geq 2.$$

2.5. C_2-cofiniteness

Li showed in [L2] that a C_2-cofinite VOA is finitely generated and that regularity implies rationality and C_2-cofiniteness. Conversely, Abe, Buhl, and Dong proved in [ABD] that regularity comes from C_2-cofiniteness and rationality. The regularilty means that every weak module is a direct sum of simple ordinary modules. This implies two conditions: every weak module is an $\mathbb{N}$-graded weak module and V is rational. In this section, we show that they are equivalent separately. Namely, C_2-cofiniteness implies that every weak module is an $\mathbb{N}$-graded weak module. We prove the following theorem.

THEOREM 2.7

Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a vertex operator algebra. Then the following are equivalent.

(1) *V is C_2-cofinite.*

(2) *Every weak module is a direct sum of generalized eigenspaces of $L(0)$.*

(3) *Every weak module is an $\mathbb{N}$-graded weak module $W = \bigoplus_{n=0}^{\infty} W(n)$ such that $W(n)$ is a direct sum of generalized eigenspaces of $L(0)$.*

(4) *V is finitely generated and every weak module is an $\mathbb{N}$-graded weak module.*

Proof

The proofs of (3) $\Rightarrow$ (2) and (1) + (3) $\Rightarrow$ (4) are clear.

Proof of (1) ⇒ (3). The proof is essentially the same as in [ABD]. Let W be a weak module. We note that V has only finitely many irreducible modules. Let $\{r_1, \dots, r_k\}$ be the set of conformal weights, and set $R = \bigcup_{i=1}^{k}(r_i + \mathbb{Z}_{\geq 0})$. For $r \in \mathbb{C}$, $W_r = \{w \in W \mid (L(0) - r)^N w = 0 \text{ for } N \gg 0\}$ denotes a generalized eigenspace of $L(0)$ with eigenvalue r. Clearly, $v(|v| - 1 + h)W_r \subseteq W_{r-h}$ for $v \in V$. We show $W = \bigoplus_{r\in R} W_r$, which implies (3). It is easy to see that there is a unique maximal $\mathbb{N}$-graded weak submodule $U = \bigoplus_{n=0}^{\infty} U(n)$ such that $U(n)$ is a direct sum of generalized eigenspaces of $L(0)$. Clearly, $U = \bigoplus_{r\in R} U \cap W_r$. Suppose $W/U \neq 0$. Since W/U is also a weak module, we may assume W/U is generated from $w \neq 0$. We use an argument similar to that in [ABD]. Applying Lemma 2.4 to W/U, there is an integer m such that $\mathscr{W}(m) \neq 0$, but $\mathscr{W}(t) = 0$ for $t < m$. Then $\mathscr{W}(m)$ is an $A(V)$-module. Since $A(V)$ is a finite-dimensional algebra, $\mathscr{W}(m)$ is a direct sum of generalized eigenspaces of $L(0)$, which contradicts the choice of U.

Proof of (2) ⇒ (1). As Li explained in [L2], $V^* = \mathrm{Hom}(V, \mathbb{C})$ contains a uniquely maximal weak submodule $W(V^*)$, and if $f(C_2(V)) = 0$, then $f \in W(V^*)$. By the assumption, $W(V^*)$ is a direct sum of generalized eigenspaces of $L(0)$. Suppose $\dim V/C_2(V) = \infty$. Let $S = \{i \mid V_i \neq (C_2(V))_i\}$. For $i \in S$, choose $v_i \in V_i - (C_2(V))_i$ and a hyperspace T_i of V_i containing $(C_2(V))_i$ such that $V_i = \mathbb{C}v_i + T_i$. Set $T_i = (C_2(V))_i$ for $i \notin S$. So V is spanned by $T = \bigoplus T_i$ and $\{v_i \mid i \in S\}$. Define $f \in V^*$ so that $f(T) = 0$ and $f(v_i) = 1$. Since $(L(0)f)(v_i) = f(L(0)v_i) = f(|v_i|v_i) = |v_i|$, a set $\{L(0)^i f \mid i = 0, 1, \dots\}$ is linearly independent. On the other hand, since $f \in W(V^*)$, f is a sum of finite elements in generalized eigenspaces of $L(0)$, and so $\langle L(0)^i f \mid i = 0, 1, \dots\rangle$ is of finite dimension, which is a contradiction.

Proof of (4) ⇒ (1). Suppose that V is not C_2-cofinite. Since V is finitely generated, $V/C_2(V)$ is a finitely generated abelian group (product is given by $u(-1)v$). Therefore there is a torsion-free element u; that is, $u(-1)^m\mathbf{1} \notin \mathbb{C}_2(V)$ for any $m \in \mathbb{N}$. Define $f \in V^*$ such that $f(C_2(V)) = 0$ and $f(u(-1)^m\mathbf{1}) = 1$ for any $m \in \mathbb{N}$. Then $v(2|v| - 1)^m f \neq 0$ for any $m \in \mathbb{N}$, which implies that f does not belong to an $\mathbb{N}$-graded weak module. Therefore V has a non-$\mathbb{N}$-graded weak module.

This completes the proof of Theorem 2.7. □

3. Symmetric algebras and pseudotrace maps

In this section, we always consider a finite-dimensional algebra over $\mathbb{C}$ with a unit 1. Let A be a ring, and let $L(a)$ and $R(a)$ denote the left and right regular representations of $a \in A$ given by a basis $\{v_i\}$ of A. A is called a *Frobenius algebra* if $L(a)$ and $R(a)$ are similar: $L(a) = Q^{-1}R(a)Q$ for some matrix Q. In particular, A is called a *symmetric algebra* when the matrix Q can be chosen as a symmetric matrix. It is also well known (cf. [CR]) that this is equivalent to A having a symmetric linear map $\phi : A \to \mathbb{C}$ such that $\mathrm{Rad}(\phi) \stackrel{\text{def}}{=} \{a \in A \mid \phi(AaA) = 0\}$ is zero and is also equivalent to A having a symmetric associative nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. A relation is given by $\phi(ab) = \langle a, b \rangle$, where a symmetric map implies $\phi(ab) = \phi(ba)$, and an associative bilinear form means $\langle ab, c \rangle = \langle a, bc \rangle$ for $a, b, c \in A$ (see [CR]). We denote a symmetric algebra A with a symmetric linear function ϕ by (A, ϕ).

Originally, Nesbitt and Scott showed in [NS] that A is a symmetric algebra if and only if its basic algebra P is symmetric. Later, Oshima gave a simpler proof of this equivalence (see [O]). Our strategy in this paper is as follows. Given a symmetric linear map ϕ of nth Zhu algebra $A_n(V)$, we have a symmetric algebra $A = A_n(V)/\mathrm{Rad}(\phi)$ and its symmetric basic algebra P with a symmetric linear function ϕ. We start from (P, ϕ). We next construct a right P-module W (a generalized Verma module) such that the basic algebra of $R = \mathrm{End}_P(W)$ is P. Nesbitt and Scott's result tells us that R has a symmetric linear map (we call it pseudotrace), and we define a pseudotrace function. In order to construct such right P-modules W, we introduce a concept of modules interlocked with ϕ, and the main purpose in this section is to construct a symmetric linear function tr_W^{ϕ} of $R = \mathrm{End}_P(W)$ explicitly.

Setting

Let P be a basic symmetric algebra; that is, $P/J(P)$ is a direct sum of the field of complex numbers and P has a symmetric linear function ϕ with $\mathrm{Rad}(\phi) = 0$, where $J(P)$ denotes the Jacobson radical of P. Set

$$P/J(P) = \mathbb{C}\bar{e}_1 \oplus \cdots \oplus \mathbb{C}\bar{e}_k. \tag{3.1}$$

The set of mutually orthogonal primitive idempotents is

$$\bigl\{e_i \in P \bigm| i = 1, \ldots, k\bigr\}, \tag{3.2}$$

and $\bar{e}_i$ is the image of e_i in $P/J(P)$. In particular, $1 = e_1 + \cdots + e_k$ is the identity of P. We also assume that

$$P \text{ contains a central element } \omega. \tag{3.3}$$

As we mentioned above, P has a symmetric associative nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ given by $\langle a, b \rangle = \phi(ab)$. Since $\mathrm{Rad}(\phi) = 0$, we obtain $\phi_{|M} \neq 0$ for any minimal ideal M. On the other hand, there is a symmetric linear map π of $P/J(P)$ such that $\pi(e_i) = \phi(e_i)$ for all i. Since such a map is a linear sum of ordinary trace maps, we may assume that we have started from $\phi - \pi$ and consider only the following case:

$$\phi(e_i) = 0 \quad \text{for all } i. \tag{3.4}$$

If P is decomposable as a ring, say, $P = P_1 \oplus P_2$, then (P, ϕ) is a sum of $(P_1, \phi_{|P_1})$ and $(P_2, \phi_{|P_2})$. Hence we assume that P is indecomposable. In particular, there is $r \in \mathbb{C}$ and $\mu(r) \in \mathbb{Z}_+$ such that

$$(\omega - r)^{\mu(r)} P = 0. \tag{3.5}$$

First we have the following.

LEMMA 3.1
We have $\mathrm{soc}(P) \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k$ *as* $(P \times P)$*-modules and* $\mathrm{soc}(P) \subseteq J(P) = \mathrm{soc}(P)^{\perp}$*, where* $\mathrm{soc}(P)$ *denotes the socle of* P*, that is, the sum of all minimal left ideals.*

Proof
Let M be a minimal ideal. If $e_i M e_j = M$ for $i \neq j$, then $\phi(m) = \phi(e_i m - m e_i) = 0$ for all $m \in M$, which contradicts $\mathrm{Rad}(\phi) = 0$. Hence M satisfies $e_i M e_i = M$ for some i. For $i = 1, \ldots, k$, $M_i = (\mathbb{C}e_1 + \cdots + \mathbb{C}e_{i-1} + \mathbb{C}e_{i+1} + \cdots + \mathbb{C}e_k + J(P))^{\perp}$ is a minimal ideal with $e_i M_i e_i = M_i$. If P has two minimal ideals M and N which are isomorphic to each other, then $L = \{(a, b) \in M \oplus N \mid \phi(a) + \phi(b) = 0\}$ is a nonzero ideal of P with $\phi(L) = 0$, which is a contradiction. Hence $\mathrm{soc}(P) \cong \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_k = J(P)^{\perp}$. Since P is indecomposable, $\mathrm{soc}(P) \subseteq J(P)$. □

Let $\{f_i \in \mathrm{soc}(P) \mid i = 1, \ldots, k\}$ be the dual basis of $\{e_i \mid i = 1, \ldots, k\}$. We note $\phi(f_i) = \langle 1, f_i \rangle = \langle e_i, f_i \rangle = 1$. Set $d_{ij} = \dim_{\mathbb{C}} e_i J(P) e_j / e_i \,\mathrm{soc}(P) e_j$. Then we can find the following basis of P.

LEMMA 3.2
P has a basis

$$\Omega = \left\{\rho_0^{ii},\ \rho_{d_{ii}+1}^{ii},\ \rho_{s_{ij}}^{ij} \,\middle|\, i, j = 1, \ldots, k,\ s_{ij} = 1, \ldots, d_{ij}\right\}$$

satisfying

(1) $\rho_0^{ii} = e_i$, $\rho_{d_{ii}+1}^{ii} = f_i$,

(2) $e_i \rho_s^{ij} e_j = \rho_s^{ij}$ *for all* i, j, s,

(3) $\langle \rho_s^{ij}, \rho_{d_{ab}-t}^{ab} \rangle = \delta_{ib}\delta_{ja}\delta_{s,t}$,

(4) $\rho_s^{ij} \rho_{d_{ji}-s}^{ji} = f_i$, *and*

(5) $\langle \rho_s^{ij}, \rho_{s+1}^{ij}, \ldots, \rho_{t_{ij}}^{ij} \rangle$ *is* $e_i P e_i$*-invariant, where* $t_{ij} = s_{ij}$ *if* $i \neq j$ *and* $t_{ii} = d_{ii} + 1$.

Proof
We first choose $\rho_0^{ii} = e_i$ and $\rho_{d_{ii}+1}^{ii} = f_i$, which satisfy (2), (3), (4) and (5). As we

showed, $J(P)^{\perp} = \mathrm{soc}(P) \subseteq J(P)$ and so $J(P)/\mathrm{soc}(P)$ have a nondegenerate symmetric bilinear form and so do $e_iJ(P)e_i/e_i\,\mathrm{soc}(P)e_i$ and $e_iJ(P)e_j/e_i\,\mathrm{soc}(P)e_j + e_jJ(P)e_i/e_j\,\mathrm{soc}(P)e_i$. Since $J(P)$ is nilpotent, there is a sequence of ideals $\mathfrak{A}_i$ of e_iPe_i,

$$e_iPe_i = \mathfrak{A}_0 \supsetneq J(P)\cap e_iPe_i = \mathfrak{A}_1 \supsetneq \cdots \supsetneq \mathfrak{A}_{d_{ii}+1} = \mathrm{soc}(P)\cap e_iPe_i \supsetneq \mathfrak{A}_{d_{ii}+2} = 0,$$

such that $\mathfrak{A}_s/\mathfrak{A}_{s+1}$ is a simple P-module and $\mathfrak{A}_s\mathfrak{A}_{d-s+1} \subseteq e_i\,\mathrm{soc}(P)e_i$. In particular, we may choose $\mathfrak{A}_s$ so that $\mathfrak{A}_s^{\perp} = \mathfrak{A}_{d_{ii}+2-s}$ for all s. Choosing a base ρ_s^{ii} in $\mathfrak{A}_s - \mathfrak{A}_{s+1}$ and its dual base $\rho^{ii}_{d_{ii}-s+1} = (\rho^{ij})^*$ in $\mathfrak{A}_{s+1}^{\perp}$ for $s \leq \mathfrak{A}(d_{ii})$ so that $\langle \rho_s^{ii}, \rho_t^{ii}\rangle = \delta_{s+t,d_{ii}+1}$ inductively, we have the desired set for e_iPe_i. For $i \neq j$, we first note that $d_{ij} = d_{ji}$. Then there are sequences of e_iPe_i-invariant subspaces $\mathfrak{B}_s$ of e_iPe_j- and e_jPe_j-invariant subspaces $\mathfrak{C}_s$ of e_jPe_i,

$$\begin{aligned} e_iPe_j &= \mathfrak{B}_1 \supsetneq \mathfrak{B}_2 \supsetneq \cdots \supsetneq \mathfrak{B}_{d_{ij}} \supsetneq 0,\\ e_jPe_i &= \mathfrak{C}_1 \supsetneq \mathfrak{C}_2 \supsetneq \cdots \supsetneq \mathfrak{C}_{d_{ij}} \supsetneq 0, \end{aligned}$$

such that $\mathfrak{B}_s\mathfrak{C}_{d_{ij}-s+1} = \mathbb{C}f_i$ and $\mathfrak{C}_{d_{ij}-s+1}\mathfrak{B}_s = \mathbb{C}f_j$. So we can take $\rho_s^{ij} \in \mathfrak{B}_s - \mathfrak{B}_{s+1}$ and choose suitable $\rho^{ij}_{d_{ij}-s+1} \in \mathfrak{C}_{d_{ij}-s+1} - \mathfrak{C}_{d_{ij}-s}$ so that they satisfy (3), (4), and (5). It is easy to see that the set of all ρ_s^{ij} satisfies the desired conditions. $\square$

For $\rho \in \Omega$, ρ^* denotes its dual, that is, a unique element $\rho^* \in \Omega$ such that $\langle \rho, \rho^*\rangle = 1$.

LEMMA 3.3
Set $P^0 = (\mathbb{C}e_1 + \cdots + \mathbb{C}e_k)^{\perp}$. Then for $\rho, \mu \in \Omega$, if $\mu \neq \rho^*$, then $\rho\mu \in P^0$.

Proof
Assume $e_j\rho = \rho$. Then we have $\langle e_i, \rho\mu\rangle = \langle e_i\rho, \mu\rangle = \langle \delta_{ij}\rho, \mu\rangle = \delta_{ij}\langle \rho, \mu\rangle = 0$ for all i. $\square$

We assume that P has a basis satisfying the conditions in Lemma 3.2. As we explain, we would like to call a right P-module W *interlocked with ϕ* if a basic algebra of $\mathrm{End}_P(W)$ is P. However, we need a symmetric linear map tr^{ϕ} explicitly, and so we just introduce a sufficient condition for that.

Definition 3.4
Let (P, ϕ) be a basic symmetric algebra with a symmetric linear map ϕ satisfying (3.4), and let W be a finite-dimensional right P-module. We say that W is *interlocked with ϕ* if $\mathrm{Ker}(f_i) \stackrel{\mathrm{def}}{=} \{w \in W \mid wf_i = 0\}$ is equal to $\sum_{\rho\in\Omega-\{e_i\}} W\rho$ for each $i = 1, \ldots, k$, where $f_i = \rho^{ii}_{d_{ii}+1}$ is a dual base of e_i.

Set $R = \mathrm{End}_P(W)$. Then $W\alpha$ is an R-module for each $\alpha \in P$ and the condition above implies that $Wf_i \cong W/\{w \in W \mid wf_i = 0\} = W/\sum_{\alpha\in\Omega-\{e_i\}} W\alpha \cong W/\big(\sum_{i\neq j} We_j + WJ(P)\big)$ as R-modules. Namely, the definition says that the socle part $\bigoplus_{i=1}^k Wf_i$ and the semisimple part $W/WJ(P)$ are isomorphic to each other. Not only the top and the bottom, but we also have the following.

LEMMA 3.5
$We_i/WJ(P)e_i \to We_i\rho e_j/WJ(P)e_i\rho e_j$ is also an R-isomorphism for any $e_i\rho e_j \in \Omega$.

Proof
Since the composition map

$$(e_i\rho e_j)^*(e_i\rho e_j) : We_i/\sum_{\mu<e_i} W\mu \xrightarrow{e_i\rho e_j} We_i\rho e_j/\sum_{\mu<e_i} W\mu e_i\rho e_j$$
$$\xrightarrow{(e_i\rho e_j)^*} We_i\rho e_j e_j\rho^* e_i = Wf_i$$

is an R-isomorphism, so is

$$e_i\rho e_j : We_i/\sum_{\mu<e_i} W\mu \to We_i\rho e_j/\sum_{\mu<e_i} W\mu e_i\rho e_j$$

for any $e_i\rho e_j \in \Omega$, where $\mu < e_i$ implies $W\mu \subsetneq We_i$. □

Therefore it is easy to see that W is interlocked with ϕ if and only if there are vector spaces T_p ($\cong Wf_p$) for each p such that

$$W \cong \bigoplus_{e_p\rho e_s\in\Omega} T_p \otimes e_p\rho e_s$$

as right P-modules. Now we define a symmetric linear map tr_W^ϕ of $R = \mathrm{End}_P(W)$. Let $\{v_i^{e_p} \mid i = 1, \ldots, \dim T_p\}$ be a basis of T_p.

Definition 3.6
Let (P, ϕ) be a basic symmetric algebra, and let W be a right P-module interlocked with ϕ. For $\alpha \in R = \mathrm{End}_P(W)$, we have a $\big(\big(\sum_{p=1}^k \dim T_p\big) \times \big(\sum_{e_p\rho e_s\in\Omega} \dim T_p\big)\big)$-matrix $(\alpha_{ji}^{e_p\rho e_s})$ such that

$$\alpha(v_j^{e_s} \otimes e_s) = \sum_{e_p\rho e_s\in\Omega} \Big(\sum_{i=1}^{\dim T_p} \alpha_{ji}^{e_p\rho e_s} v_i^{e_p} \otimes e_p\rho e_s \Big)$$

for $v_j^{e_s} \otimes e_s$. Then we define a pseudotrace map $\mathrm{tr}_W^{\phi}(\alpha)$ by the sum of traces of matrices $(\alpha_{ji}^{f_s})_{ji}$ for $s = 1, \ldots, k$. Namely, we define

$$\mathrm{tr}_W^{\phi}(\alpha) = \sum_{s=1}^{k} \mathrm{tr}(\alpha_{ji}^{f_s}) = \sum_{s=1}^{k} \sum_{i=1}^{\dim T_s} \alpha_{ii}^{f_s}.$$

Namely, we take the trace of $\mathrm{Hom}(W/WJ(P) \to W\,\mathrm{soc}(P))$ as $\mathrm{tr}_T^{\phi}(a) = \mathrm{tr}(B\alpha)$ in the introduction.

PROPOSITION 3.7
We have that tr_W^{ϕ} is a symmetric linear map.

Proof
It is easy to see that tr_W^{ϕ} does not depend on the choice of bases of T_p's. Let $\alpha, \beta \in R = \mathrm{End}_P(W)$. Then there are $\alpha_{ji}^{e_p\rho e_s}, \beta_{ji}^{e_p\rho e_s} \in \mathbb{C}$ such that

$$\alpha(v_j^{e_s} \otimes e_s) = \sum_{e_p\rho e_s \in \Omega} \Big(\sum_i \alpha_{ji}^{e_p\rho e_s} v_i^p \otimes e_p\rho e_s \Big)$$

and

$$\beta(v_i^{e_p} \otimes e_p) = \sum_{e_t\mu e_p \in \Omega} \Big(\sum_h \beta_{ih}^{e_t\mu e_p} v_h^{e_t} \otimes e_t\mu e_p \Big).$$

By direct calculation, we obtain

$$\begin{aligned}
\beta\alpha(v_j^{e_s} \otimes e_s) =& \beta\Bigg(\sum_{e_p\rho e_s \in \Omega} \Big(\sum_i \alpha_{ji}^{e_p\rho e_s} v_i^{e_p} \otimes e_p\rho e_s \Big)\Bigg) \\
=& \sum_{e_p\rho e_s \in \Omega} \Big(\sum_i \alpha_{ji}^{e_p\rho e_s} \beta(v_i^{e_p} \otimes e_p) e_p\rho e_s \Big) \quad \text{since } \beta \in \mathrm{End}_P(W), \\
=& \sum_{e_p\rho e_s \in \Omega} \Bigg(\sum_i \alpha_{ji}^{e_p\rho e_s} \Big(\sum_{t,\mu} \Big(\sum_h \beta_{ih}^{e_t\mu e_p} v_h^{e_t} \otimes e_t\mu e_p \Big)\Big) e_p\rho e_s \Bigg) \\
=& \sum_{e_p\rho e_s \in \Omega} \sum_i \sum_{t,\mu} \sum_h \alpha_{ji}^{e_p\rho e_s} \beta_{ih}^{e_t\mu e_p} v_h^{e_t} \otimes e_t\mu e_p e_p\rho e_s.
\end{aligned}$$

By Lemma 3.3,

$$e_t\mu e_p e_p\rho e_s \in P^0 = \big\langle \Omega - \{f_1, \ldots, f_k\} \big\rangle$$

except for $e_t\mu e_p = (e_p\rho e_s)^*$, and so we have

$$\mathrm{tr}_W^{\phi}(\beta\alpha) = \sum_{s=1}^{k} \mathrm{tr}(\alpha_{ji}^{e_p\rho e_s})(\beta_{ih}^{e_s\rho^* e_p}).$$

Hence we obtain

$$\mathrm{tr}_W^{\phi}(\beta\alpha)=\sum_{s=1}^{k}\mathrm{tr}(\alpha_{ji}^{e_p\rho e_s})(\beta_{ih}^{e_s\rho^* e_p})=\sum_{s=1}^{k}\mathrm{tr}(\beta_{ih}^{e_s\rho^* e_p})(\alpha_{ji}^{e_p\rho e_s})=\mathrm{tr}_W^{\phi}(\alpha\beta),$$

as desired. □

We call tr_W^{ϕ} a *pseudotrace map*. We should note that we treat an ordinary trace map as a pseudotrace map, although ϕ does not satisfy (3.4). For if a basic symmetric algebra (P,ϕ) is $(\mathbb{C},1)$, then tr_W^{1} coincides with the ordinary trace map. We next investigate the properties of pseudotrace maps. Let ω be a central element of P such that $(\omega-r)^s P=0$ and $(\omega-r)^{s-1}P\neq 0$ for $r\in\mathbb{C}$. Set $\mathfrak{N}=\{a\in P\mid(\omega-r)a=0\}$. It is easy to see that if we define ϕ' by

$$\phi'(a)=\phi\big((\omega-r)a\big),$$

then ϕ' is also a symmetric linear function of $P/\mathfrak{N}$. We denote it by $(\omega-r)\phi$, and we denote the right action of $\omega-r$ by $(\omega-r)_P$.

PROPOSITION 3.8
If W is a right P-module interlocked with ϕ, then $W/W\mathfrak{N}$ is a right $P/\mathfrak{N}$-module interlocked with $(\omega-r)\phi$ and

$$\mathrm{tr}_W^{\phi}\left(g(\omega-r)_P\right)=\mathrm{tr}_{W/W\mathfrak{N}}^{(\omega-r)\phi}\,g \tag{3.6}$$

for $g\in\mathrm{End}_P(W)$. (We also consider $g\in\mathrm{End}_{P/\mathfrak{N}}(W/W\mathfrak{N})$.)

Proof
Set $R=\mathrm{End}_P(W)$. The first assertion is clear. Since $\omega-r\in Z(P)$, we obtain $\omega-r\in Z(R)$, where $Z(A)$ denotes the center of A. Let $\mathfrak{D}$ be an ideal of R such that $\mathfrak{D}/R\mathfrak{N}=\mathrm{soc}(R/R\mathfrak{N})$. Then $(\omega-r)\mathfrak{D}\subseteq\mathrm{soc}(R)\cap\mathrm{Im}(\omega-r)$ and $(\omega-r)\mathfrak{D}\cong\mathfrak{D}/\mathfrak{N}$. On the other hand, $\mathfrak{E}=\{a\in R\mid(\omega-r)a\in\mathrm{soc}(R)\}$ satisfies $\mathfrak{E}/R\mathfrak{N}\subseteq\mathrm{soc}(R/R\mathfrak{N})$. Hence $(\omega-r)$ is an isomorphism from $\mathrm{soc}(R/R\mathfrak{N})$ to $\mathrm{soc}(R)\cap\mathrm{Im}(\omega-r)$. From the definition of pseudotrace maps, tr_W^{ϕ} is given by the traces of $\mathrm{Hom}_P(R/J(R),\mathrm{soc}(R))$, and so $\mathrm{tr}_W^{\phi}(\omega-r)$ is given by the traces of $\mathrm{Hom}_P(R/J(R),(\omega-r)^{-1}\,\mathrm{soc}(R)/R\mathfrak{N})$, which equal the traces of $\mathrm{Hom}_P(R/J(R),\mathrm{soc}(R/\mathfrak{N}))$. Hence we have $\mathrm{tr}_W^{\phi}(g(\omega-r)_P)=\mathrm{tr}_{W/W\mathfrak{N}}^{(\omega-r)\phi}\,g$, as desired. □

Let A be a finite-dimensional symmetric algebra with a symmetric linear map ϕ. Let $A/J(A)=A_1\oplus\cdots\oplus A_k$ be the decomposition into the direct sum of simple components. Let $\{e_i\mid i=1,\dots,k\}$ be a set of orthogonal idempotents such that

$\bar{e}_i = e_i + J(A)$ is a primitive idempotent of A_i for each i. Set $e = \sum_{i=1}^{k} e_i$. Then eAe is a basic algebra and eAe has a symmetric map ϕ (we use the same notation ϕ) with zero radical. Viewing Ae as a right eAe-module, it is easy to check that Ae is interlocked with ϕ, and so we can define a pseudotrace map tr^{ϕ}_{Ae} of $A \subseteq \mathrm{End}_{eAe}(Ae)$ on Ae. From the definition of tr^{ϕ}_{Ae} and the previous arguments, it is easy to see the following.

LEMMA 3.9
We have $\mathrm{tr}^{\phi}_{Ae}(a) = \phi(a)$ *for all* $a \in \mathrm{soc}(A)$.

So we have the following theorem, which we need later.

THEOREM 3.10
Let A be a finite-dimensional associative algebra over $\mathbb{C}$, and let ϕ be a linear function of A satisfying $\phi(ab) = \phi(ba)$ for every $a, b \in A$. Let ω be in the center of A, and assume that $\phi((\omega - r)^{\mu(r)}a) = 0$ for every $a \in A$. Then there are linear symmetric functions ϕ_i of A $(i = 1, \dots, s)$ and basic symmetric subalgebras P_i of factor rings $A/\mathfrak{N}_i$ with symmetric linear functions ϕ_i and $(A \times P_i)$-modules $M^i = (A/\mathfrak{N}_i)e$ satisfying $(\omega - r)^{\mu(r)} M^i = 0$ such that

$$\phi(b) = \sum_{i=1}^{s} \mathrm{tr}^{\phi_i}_{M^i}(b)$$

for every $b \in A$, where $\mathfrak{N}_i = \mathrm{Rad}(\phi_i)$.

Proof
We use induction on $\dim A$. If ϕ has a nonzero radical M, then $\phi : A/M \to \mathbb{C}$ satisfies the same condition, but $\dim A/M < \dim A$. So we may assume that $\mathrm{Rad}(\phi)$ is zero and A is a symmetric algebra with a symmetric linear map ϕ. We may also assume that A is indecomposable. By Lemma 3.9, there is an $(A \times eAe)$-module Ae and a symmetric linear map ψ such that $\mathrm{tr}^{\psi}_{Ae}(a) = \phi(a)$ for all $a \in \mathrm{soc}(A)$. Set $\phi' = \phi - \mathrm{tr}^{\psi}_{Ae}$. Then ϕ' satisfies the same condition and $\mathrm{Rad}(\phi') \supseteq \mathrm{soc}(A)$. By induction on $\dim A$, ϕ' is a sum of pseudotrace maps and so is ϕ. □

4. Pseudotrace functions

Let $\{W^1, \dots, W^k\}$ be the set of all irreducible V-modules, and set $W^i = \bigoplus_{m=0}^{\infty} W^i(m)$ with $W^i(0) \neq 0$ and $L(0)_{|W^i(0)} = r$. It may happen that $W^i(h) = 0$ for some $h > 0$ and $W^i(h-1) \neq 0$. Then, since $\mathrm{sl}_2(\mathbb{C}) \cong \langle L(-1), 2L(0), L(1)\rangle$ acts on $\bigoplus_{m=0}^{h-1} W^i(m)$, $T = \{2r, 2r+2, \dots, 2r+2h-2\}$ is a union of eigenvalues of $2L(0)$ in finite-dimensional $\mathrm{sl}_2(\mathbb{C})$-modules. Hence if $s \in T$, then $-s \in T$, which implies

$2r+2r+2h-2=0$ and the conformal weight r of W^i is $(1-h)/2$. Therefore there is an integer l such that $W^i(m)\neq 0$ for any $m>l$ and i.

In this section, we assume the following:

$$\begin{array}{l}A_n(V) \text{ has a symmetric linear map } \phi \text{ satisfying } \phi((\omega-r)^s * v)=0\\ \text{for some } r\in\mathbb{C} \text{ and } s\in\mathbb{N}, \text{ and the real part } \mathrm{Re}(r) \text{ of } r \text{ is greater than}\\ \text{the real parts of conformal weights of all irreducible modules by } l.\end{array} \tag{4.1}$$

Set $A=A_n(V)/\operatorname{Rad}(\phi)$, which is a symmetric algebra with a symmetric linear map ϕ. Let $A/J(A)=A_1\oplus\cdots\oplus A_k$ be the decomposition into the direct sum of simple components A_i. Let $\{e_i \mid i=1,\dots,k\}$ be a set of mutually orthogonal primitive idempotents of A such that $\bar{e}_i=e_i+J(A)$ is a primitive idempotent of A_i. Set $e=e_1+\cdots+e_k$. Then $P=eAe$ is a basic symmetric algebra with a symmetric linear map ϕ. As we showed in Section 3, $T=Ae$ is interlocked with ϕ.

Let $W_T^{(n)}$ denote a generalized Verma V-module generated from an $A_n(V)$-module T. We assert that

$$W_T^{(n)} \text{ is interlocked with } \phi.$$

We first note that $W_T^{(n)}(m)$ has a finite dimension for every m and $L(0)-r$ acts on $W_T^{(n)}(n)\cong T$ as a nilpotent operator. By the definition of generalized Verma module $W_T^{(n)}$, $W_T^{(n)}$ is a right P-module by the action

$$\Big(\sum v^1(i_1)\cdots v^s(i_s)x\Big)g=\sum v^1(i_1)\cdots v^s(i_s)(xg)$$

for $v^i\in V$, $x\in T$, and $g\in P$. Since the real part $\mathrm{Re}(r)$ of r is greater than the real parts of conformal weights of any modules by l, every nonzero submodule of $W_T^{(n)}$ has a nonzero intersection with $W_T^{(n)}(n)$, and so all irreducible factors W/U of composition series of $W_T^{(n)}$ with isomorphic nth graded pieces as $A_n(V)$-modules are isomorphic to each other as V-modules. In particular, the semisimple part $W_T^{(n)}/W_T^{(n)}J(P)$ and the socle part $W_T^{(n)}\operatorname{soc}(P)$ are isomorphic to each other as $(V\times P)$-modules. Thus $W_T^{(n)}$ is interlocked with (P,ϕ).

We note that $W_T^{(n)}(0)$ may be zero if T is also an $A_{n-1}(V)$-module. We should also note that we have defined a pseudotrace map for a finite-dimensional vector space W, and so we have to say that for each N, $\bigoplus_{h=0}^N W_T^{(n)}(h)$ is interlocked with ϕ. However, by the definition of pseudotrace map, tr_W^ϕ on $W=\bigoplus_{h=0}^N W_T^{(n)}(h)$ does not depend on the choice of N, and so it is uniquely defined on any $W_T^{(n)}(h)$ as is an ordinary trace map. We also note that

$$\mathrm{tr}^\phi_{W_T^{(n)}(n)}(a)=\mathrm{tr}_T^\phi(a)=\phi(a)$$

for $a \in \operatorname{soc}(A)$ by definition.

4.1. Logarithmic modules

In this subsection, ω denotes Virasoro element and we fix $r \in \mathbb{C}$ and $s \in \mathbb{N}$. Assume $(\omega - n - r)^s T = 0$ and $(\omega - n - r)T \neq 0$; thus $L(0) = o(\omega)$ does not act on the graded piece $W_T^{(n)}(m)$ semisimply. However, since $(L(0) - m - r)^s W_T^{(n)}(m) = 0$, we are able to see that

$$e^{2\pi i L(0)\tau} = e^{2\pi i (m+r)\tau}\Big(\sum_{j=0}^{s-1} \frac{1}{j!}\big(2\pi i \tau (L(0) - m - r)\big)^j\Big) \quad \text{on } W_T^{(n)}(m)$$

and define $q^{L(0)}$ on $W_T^{(n)}(m)$ by

$$q^{m+r}\Big(\sum_{j=0}^{s-1} \frac{1}{j!}\big(2\pi i \tau (L(0) - m - r)\big)^j\Big).$$

Such a module is called a *logarithmic module*. We note that the left action $L(0)-m-r$ on $W_T^{(n)}(m)$ is equal to the right action of $\omega - n - r \in P$ on $W(m)$, and we denote it by $(\omega - n - r)_P$. Set $\mathfrak{N}_i = \{a \in P \mid (\omega - n - r)^i a = 0\}$, and let $L^s(0)$ be a degree operator that acts on $W(m)$ as $m + r$; that is, the semisimple part of $L(0)$ and $L(0) - L^s(0)$ is nilpotent. By Proposition 3.8, we have the following.

LEMMA 4.1

We have

$$\operatorname{tr}_W^{\phi} \big(L(0) - L^s(0)\big)^i g = \operatorname{tr}_W^{\phi} g(\omega - n - r)_P^i = \operatorname{tr}_{W/W\mathfrak{N}_i}^{(\omega-n-r)^i\phi} g.$$

Set

$$S^W(v, \tau) = \operatorname{tr}_W^{\phi} o(v) q^{L(0)-c/24} \quad (q = e^{2\pi i \tau})$$

for a generalized Verma module $W = W_T^{(n)}$ interlocked with ϕ. Although we are studying a general (nonsemisimple) operator $L(0)$ and a pseudotrace function, they satisfy the following properties, as does the action of $L(0)$ on modules and trace map:

(1) $\operatorname{tr}_W^{\phi}$ is a symmetric function,

(2) $[L(0), v(m)] = (|v| - m - 1)v(m)$, and

(3) $\operatorname{tr}_W^{\phi} o(\omega) o(v) q^{L(0)} = \frac{1}{2\pi i}\frac{d}{d\tau} \operatorname{tr}_W^{\phi} o(v) q^{L(0)}$,

which are the properties that Zhu used in the proof for ordinary trace functions. Therefore we have the following results by exactly the same arguments as in [Z].

PROPOSITION 4.2

For any $v, u \in V$ and a generalized Verma module W interlocked with (P, ϕ), we *have*

$$\begin{aligned}
&\mathrm{tr}_W^{\phi}\, v(|v|-1-k)u(|u|-1+k)q^{L(0)} \\
&\qquad = \mathrm{tr}_W^{\phi}\, \frac{-q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1-k}{s} o(v(s)u)q^{L(0)}, \\
&\mathrm{tr}_W^{\phi}\, u(|u|-1-k)v(|v|-1+k)q^{L(0)} \\
&\qquad = \mathrm{tr}_W^{\phi}\, \frac{q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s} o(v(s)u)q^{L(0)}, \\
&\mathrm{tr}_W^{\phi}\, o(v[0]u)q^{L(0)} = 0, \\
&\mathrm{tr}_W^{\phi}\, o(v)o(u)q^{L(0)} = \mathrm{tr}_W^{\phi}\, o(v[-1]u)q^{L(0)} - \sum_{k=1}^{\infty} E_{2k}(\tau)\,\mathrm{tr}_W^{\phi}\, o(v[2k-1]u)q^{L(0)}, \\
&\mathrm{tr}_W^{\phi}\, o(v[-2]u)q^{L(0)} + \sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau)\mathrm{tr}^{\phi}|_W o(v[2k-2]u)q^{L(0)} = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{tr}_W^{\phi}\, o(L[-2]u)q^{L(0)-c/24} - \sum_{k=1}^{\infty} E_{2k}(\tau)\,\mathrm{tr}_W^{\phi}\, o(L[2k-2]u)q^{L(0)-c/24} \\
&\qquad = \frac{1}{2\pi i}\frac{d}{d\tau}\big(\mathrm{tr}_W^{\phi}\, o(u)q^{L(0)-c/24}\big).
\end{aligned}$$

4.2. One-point functions

By the investigation of trace functions in [Z], Zhu showed

$$S^M\Big(v[-2]u + \sum_{k=1}^{\infty}(2k-1)E_{2k}(\tau)v[2k-2]u, \tau\Big) = 0,$$

$$S^M\Big(L[-2]u + \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u, \tau\Big) = \frac{1}{2\pi i}\frac{d}{d\tau}S^M(u,\tau),$$

and

$$S^M(v[0]u, \tau) = 0 \tag{4.2}$$

for a V-module M. Since $[o(v), o(u)] = o(v[0]u)$, (4.2) implies that $S_W(*, \tau)$ is a symmetric linear function on $\langle o(v) \mid v \in V\rangle$ in a sense.

Consider

$$V[E_4(q), E_6(q)] \subseteq V[[q]].$$

$O_q(V)$ is the submodule of $V[E_4(q), E_6(q)]$ generated by elements of the type

$$v[0]u$$

and

$$v[-2]u + \sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau)\otimes v[2k-2]u, \quad v, u \in V.$$

We first prove the following lemma.

LEMMA 4.3
For $\alpha \in V$ and a fixed integer $n > l$, there are v^i and $u^i \in V$ $(i = 1, \dots, p)$ such that

$$o(\alpha) = \sum_{i=1}^{p} v^i(|v^i| - 1 + n)u^i(|u^i| - 1 - n) \quad \text{in } A(V),$$

where the statement "$o(\alpha) = \sum_{i=1}^{p} v^i(|v^i| - 1 + n)u^i(|u^i| - 1 - n)$ in $A(V)$" implies $o(\alpha)u = \sum_{i=0}^{p} v^i(|v^i| - 1 + n)u^i(|u^i| - 1 - n)u$ for any $\mathbb{N}$-graded weak V-module W and $u \in W(0)$.

Proof
We first note the following fact, which is a natural consequence of associativity. For any $v, u \in V$ and $r, s \in \mathbb{Z}$ and $m \in \mathbb{N}$, there is an element β of V which is a linear combination $\beta = \sum \lambda_i v(i)u + \sum \mu_i u(i)v$ of $v(i)u$ and $u(i)v$ such that

$$\begin{aligned}\beta(|\beta| - 1 + r - s) = v(|v| - 1 - s)u(|u| - 1 + r) &\\ + \sum_{i\geq m} a_i v(|v| - 1 - s + r - i)u(|u| - 1 + i) &\\ + \sum_{i\geq m} b_i u(|u| - 1 - s + r - i)v(|v| - 1 + i) &\end{aligned}$$

with $a_i, b_i \in \mathbb{C}$. In particular, when $s = r = 0$, then $\beta = v *_m u$ is the product in $A_m(V)$. We use this argument in several places.

Set

$$D_n = \big\langle \alpha \in V \,\big|\, \alpha = v(|v| - 1 + n)u(|u| - 1 - n) \text{ in } A(V) \text{ for some } v, u \in V \big\rangle.$$

Clearly, D_n contains $O(V)$ and $D_n/O(V)$ is an ideal of $A(V)$. If W is an irreducible V-module and $0 \neq w \in W(0)$, then there is an element $v \in V$ such that $0 \neq v(|v| - 1 - n)w \in W(n)$ since $W(n) \neq 0$. There is also an element $u \in V$ such that $u(|u| - 1 + n)v(|v| - 1 - n)w \neq 0$ (see [DM]). Hence D_n covers $\mathrm{End}(W(0))$ for all simple modules W, and so $A(V) = D_n + J(A(V))$, which means $D_n = V$. □

The purpose in this subsection is to prove the following three propositions, which we use in the next section.

PROPOSITION 4.4
If $S(v,\tau) = \sum_{i=0}^{\infty} S_i(v)q^i \in \mathbb{C}[[q]]$ *satisfies* $S(\alpha,\tau) = 0$ *for all* $\alpha \in O_q(V)$*, then* $S(v,\tau) \in \mathbb{C}[[q]]q^{n+1}$ *for all* $v \in O_n(V)$*.*

In particular, S_i is a symmetric linear function of $A_i(V)$.

PROPOSITION 4.5
Assume that $S(v,\tau) = \sum_{j=0}^{N} \big(\sum_{i=0}^{\infty} S_{ji}(v)q^{i+r}\big)(2\pi i\tau)^j \in \mathbb{C}[[q]]q^r[\tau]$ *satisfies* $S(\alpha,\tau) = 0$ *for all* $\alpha \in O_q(V)$*, and assume that*

$$S\Big(L[-2]v - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]v, \tau\Big) = \frac{1}{2\pi i}\frac{d}{d\tau}S(v,\tau)$$

for all $v \in V$*. Then* $S_{jm}((\omega - r - c/24 - m)^{N-j+1} *_m v) = 0$ *for any* m *and* j*.*

PROPOSITION 4.6
Assume that $S(v,\tau) = \sum_{i=0}^{\infty} S_i(v)q^i \in \mathbb{C}[[q]]$ *satisfies* $S(\alpha,\tau) = 0$ *for all* $\alpha \in O_q(V)$*, and assume that* $S_m((\omega - r - c/24 - m)^s *_m v) = 0$ *for all* $v \in V$ *and* m*. If* $S_n = 0$ *for some* $n > l + r$*, then* $S_0 = 0$*, where* l *is given in the beginning of this section.*

Proof of Propositions 4.4, 4.5, and 4.6
In order to prove the three propositions above at once, we review the proof of [Z, Props. 4.3.2 and 4.3.3]. Zhu first obtained

$$\begin{aligned}
&\mathrm{tr}_{|M}\, v(|v|-1-k)u(|u|-1+k)q^{L(0)} \\
&\qquad = \mathrm{tr}_{|M}\, \frac{-q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1-k}{s} o\big(v(s)u\big)q^{L(0)}, \\
&\mathrm{tr}_{|M}\, u(|u|-1-k)v(|v|-1+k)q^{L(0)} \\
&\qquad = \mathrm{tr}_{|M}\, \frac{q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s} o\big(v(s)u\big)q^{L(0)},
\end{aligned}$$

for V-modules M and $k \neq 0$ (see the proof of [Z, Prop. 4.3.2]).

Using the equations above, Zhu denoted

$$\mathrm{tr}_{|M}\, w^{|v|}z^{|u|}Y(v,w)Y(u,z)q^{L(0)} \qquad \text{and} \qquad \mathrm{tr}_{|M}\, z^{|u|}w^{|v|}Y(u,z)Y(v,w)q^{L(0)}$$

as infinite linear combinations of $o(v)o(u)$, $o(v[0]u) = o(v)o(u) - o(u)o(v)$, and the following two trace forms:

$$\mathrm{tr}_{|M}\, \frac{-q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1-k}{s} o\big(v(s)u\big)q^{L(0)}\Big(\frac{w}{z}\Big)^k,$$

and

$$\mathrm{tr}_{|M} \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} o\big(v(s)u\big) q^{L(0)} \Big(\frac{z}{w}\Big)^k.$$

Then substituting them into the normal product

$$Y\big(v(i)u, z\big) = \mathrm{Res}_w \big\{(w-z)^i Y(v,w)Y(u,z) - (-z+w)^i Y(u,z)Y(v,w)\big\}$$

and using an expansion $v[-1]u = \sum_{i \geq -1} c_i v(i)u$ with $c_i \in \mathbb{C}$, he expressed the term $o(v[-1]u)$ between $\mathrm{tr}_{|M}$ and $q^{L(0)}$ as a linear combination of $o(v)o(u)$, $o(v[0]u)$, $\frac{-q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1-k}{s} o(v(s)u)$, and $\frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} o(v(s)u)$.

The next step is the crucial part of Zhu's paper. He changed the shape of the above expression of $o(v[-1]u)$ into

$$\underbrace{o(v)o(u) - o(v[0]u)}_{o(u)o(v)} + \sum_{k=1}^{\infty} E_{2k}(q) o(v[2k-1]u)$$

using the equations above (cf. [Z, (4.3.8)–(4.3.11) and Prop. 4.3.2]). Namely, he proved

$$\mathrm{tr}_{|M} \Big\{o(v[-1]u) - o(v)o(u) + o(v[0]u) - \sum_{k=1}^{\infty} E_{2k}(q) o(v[2k-1]u)\Big\} q^{L(0)} = 0.$$

He obtained two important equations from it. The first is, by substituting $\tilde{\omega}$ for v,

$$\mathrm{tr}_{|M} \Big\{o(L[-2]u) - \sum_{k=1}^{\infty} E_{2k}(\tau) o(L[2k-2]u)\Big\} q^{L(0)-c/24} = \frac{1}{2\pi i} \frac{d}{d\tau} \mathrm{tr}_{|M}\, o(u) q^{L(0)-c/24}.$$

The second is, by substituting $L[-1]v$ for v,

$$\mathrm{tr}_{|M} \Big\{o(v[-2]u) + \sum_{k=2}^{\infty} (2k-1) o(v[2k-2]u) E_{2k}(q)\Big\} q^{L(0)} = 0.$$

These equations are so beautiful that we can see the modular invariance property of $O_q(V)$ clearly. Once we know the modular invariance property of $O_q(V)$, we do not need these forms. We are interested in the structure of $O_q(V)$ from the viewpoint of ordinary vertex operators $Y(v,z) = \sum_{i \in \mathbb{Z}} v(i) z^{-i-1}$. We go back to the former form of

$$o(u)o(v) + \sum_{k=1}^{\infty} E_{2k}(\tau) o(v[2k-1]u).$$

Namely, we change an expansion of $o(v[-1]u)$:

$$
\begin{aligned}
o(v[-1]u) &= \sum_{i\geq -1} c_i o(v(i)u) \\
&= \sum_{i\geq -1} c_i \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \bigl\{ v(i-j)u(|v|+|u|-i-2+j) \\
&\qquad\qquad - (-1)^i u(|v|+|u|-2-j)v(j) \bigr\}
\end{aligned}
$$

by replacing $v(|v|-1-k)u(|u|-1+k)$ and $u(|u|-1-k)v(|v|-1+k)$ by $\frac{-q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1-k}{s}v(s)u$ and $\frac{q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s}v(s)u$, respectively, for $k\neq 0$. We note that $c_{-1}=1$.

To simplify the arguments, we express this process by the notation θ; that is, for $\alpha=\sum_i a_i v(i)u$, we first develop $\sum_i a_i o(v(i)u)$ as a linear sum

$$
ao(v)o(u) + bo(u)o(v) + \sum_{k\neq 0} \lambda_k v(|v|-1-k)u(|u|-1+k) \\
+ \sum_{k\neq 0} \mu_k u(|u|-1-k)v(|v|-1+k)
$$

with $a, b, \lambda_i, \mu_i \in \mathbb{C}$ by using associativity, and then we define

$$
\begin{aligned}
&\theta\Bigl(\sum_i a_i v(i)u - ao(v)o(u) - bo(u)o(v)\Bigr) \\
&\qquad = \sum_{k\neq 0} \lambda_k \frac{-q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1-k}{s} v(s)u \\
&\qquad\quad + \sum_{k\neq 0} \mu_k \frac{q^k}{1-q^k} \sum_{s=0}^{\infty} \binom{|v|-1+k}{s} v(s)u.
\end{aligned}
$$

What Zhu has obtained are the following:

$$
v[-1]u - \theta\bigl(v[-1]u - o(u)o(v)\bigr) = v[-1]u - \sum_{k=1}^{\infty} E_{2k}(\tau) v[2k-1]u \tag{4.3}
$$

and

$$
v[-2]u - \theta(v[-2]u) = v[-2]u + \sum_{k=1}^{\infty} (2k-1)E_{2k}(\tau) v[2k-2]u \in O_q(V). \tag{4.4}
$$

On the other hand, if $i \geq 0$, then we have

$$\begin{aligned} o\big(v(i)u\big) &= \sum_{j=0}^{i}(-1)^j\binom{i}{j}\big\{v(i-j)u(|v|+|u|-i-2+j) \\ &\qquad\qquad - (-1)^i u(|v|+|u|-2-j)v(j)\big\} \\ &= \sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\big\{v(j)u(|v|+|u|-2-j) \\ &\qquad\qquad - u(|v|+|u|-2-j)v(j)\big\}. \end{aligned}$$

If we replace $v(|v|-1+k)u(|u|-1-k) - u(|u|-1-k)v(|v|-1+k)$ by

$$\begin{aligned} \frac{1}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s}v(s)u - \frac{q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s}v(s)u \\ = \sum_{s=0}^{\infty}\binom{|v|-1-k}{s}v(s)u \end{aligned}$$

for each $k \neq 0$, then

$$\begin{aligned} &o\big(v(i)u\big) \\ &\quad = \sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\big\{v(j)u(|v|+|u|-2-j) - u(|v|+|u|-2-j)v(j)\big\} \end{aligned}$$

is replaced by

$$\sum_{j=0}^{i}(-1)^{i+j}\binom{i}{j}\sum_{s=0}^{\infty}\binom{j}{s}v(s)u = v(i)u.$$

Namely, $\theta(v(i)u) = v(i)u$ for $i \geq 0$. By canceling $v(i)u$ $(i \geq 0)$ from both sides of $v[-1]u$ and $\theta(v[-1]u)$ in (4.3) and also from both sides of $v[-2]u$ and $\theta(v[-2]u)$ in (4.4), we obtain

$$v(-1)u - \theta\big(v(-1)u - o(u)o(v)\big) = v[-1]u - \sum_{k=1}^{\infty}E_{2k}(\tau)v[2k-1]u, \tag{4.5}$$

$$\omega(-1)u - \frac{c}{24}u - \theta\big(\omega(-1)u - o(u)o(\omega)\big) = L[-2]u - \sum_{k=1}^{\infty}E_{2k}(\tau)L[2k-2]u, \tag{4.6}$$

and

$$v(-2)u + |v|v(-1)u - \theta\big(v(-2)u + |v|v(-1)u\big)$$
$$= v[-2]u + \sum_{k=1}^{\infty}(2k-1)E_{2k}(\tau)v[2k-2]u \in O_q(V) \quad (4.7)$$

since $\theta(\mathbf{1}(-1)u - o(u)) = 0$. Substituting $L(-1)^m v$ into v of (4.7), we have

$$v(-2-m)u + \frac{|v|m}{m+1}v(-1-m)u - \theta\Big(v(-2-m)u + \frac{|v|m}{m+1}v(-1-m)u\Big) \in O_q(V).$$

If $a \in O_n(V)$, then a is a linear combination of

$$v \circ_n u = \mathrm{Res}_z\, Y(v,z)u\frac{(1+z)^{|v|+n}}{z^{2+2n}}$$

for some $v, u \in V$ and the expansion of $v \circ_n u$ by associativity is an infinite linear sum of $v(|v|-1-k)u(|u|-1+k)$ and $u(|u|-1-k)v(|v|-1+k)$ with $k \geq n+1$. Since, for $k \geq n+1$,

$$\theta\big(v(|v|-1-k)u(|u|-1+k)\big) = \frac{-q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1-k}{s}v(s)u$$

and

$$\theta\big(u(|u|-1-k)v(|v|-1+k)\big) = \frac{q^k}{1-q^k}\sum_{s=0}^{\infty}\binom{|v|-1+k}{s}v(s)u$$

are contained in $V[[q]]q^{n+1}$, we obtain $S(a,q) \in V[[q]]q^{n+1}$, which proves Proposition 4.4.

By the same arguments, if

$$S(v,\tau) = \sum_{j=0}^{N} S_j(v,\tau)q^r(2\pi i\tau)^j = \sum_{j=0}^{N}\Big(\sum_{i=0}^{\infty}S_{ji}(v)q^i\Big)q^r(2\pi i\tau)^j$$

satisfies $S(\alpha,\tau) = 0$ for all $\alpha \in O_q(V)$ and

$$S\Big(L[-2]v - \sum_{k=1}^{\infty}E_{2k}(\tau)L[2k-2]v, \tau\Big) = \frac{1}{2\pi i}\frac{d}{d\tau}S(v,\tau)$$

for all $v \in V$, then

$$S\Big(\sum_{i>-M}\lambda_i\omega(i)u, \tau\Big) = s\Big(\frac{1}{2\pi i}\frac{d}{d\tau}S(u,\tau) + \frac{c}{24}S(u,\tau)\Big)$$
$$+ S\Big(\theta\Big(\sum_{i>-M}\lambda_i\omega(i)u - so(\omega)o(u)\Big), \tau\Big)$$

for $\sum_{i>-M}\lambda_i\omega(i)u$, where s is given by

$$o\Big(\sum_{i>-M}\lambda_i\omega(i)u\Big)=s\,o(\omega)o(u)+\sum_{j>0}\Big(a_j\omega(1-j)u(|u|-1+j)\\ +b_ju(|u|-1-j)\omega(1+j)\Big)$$

with $s,a_j,b_j\in\mathbb{C}$.

It follows from the definition of $*_n$ that $o(\omega *_n u)\in\omega(1)o(u)+O_n(V)$, and so

$$S(\omega *_n u,\tau)=\frac{1}{2\pi i}\frac{d}{d\tau}S(u,\tau)+\frac{c}{24}S(u,\tau)+S\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big).$$

Therefore we obtain

$$\begin{aligned}\sum_{j=0}^{N}\sum_{i=0}^{\infty}&S_{ji}(\omega *_n u)q^{i+r}(2\pi i\tau)^j=\sum_{j=0}^{N}S_j(\omega *_n u,\tau)q^r(2\pi i\tau)^j=S(\omega *_n u,\tau)\\ &=\frac{1}{2\pi i}\frac{d}{d\tau}S(u,\tau)+\frac{c}{24}S(u,\tau)+S\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big)\\ &=\frac{1}{2\pi i}\frac{d}{d\tau}\Big(\sum_{j=0}^{N}\sum_{i=0}^{\infty}S_{ji}(u)q^{i+r}(2\pi i\tau)^j\Big)+\frac{c}{24}\sum_{j=0}^{N}\sum_{i=0}^{\infty}S_{ji}(u)q^{i+r}(2\pi i\tau)^j\\ &\quad+\sum_{j=0}^{N}S_j\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big)(2\pi i\tau)^j\\ &=\sum_{j=0}^{N}\sum_{i=0}^{\infty}S_{ji}(u)(i+r)q^{i+r}(2\pi i\tau)^j+\sum_{j=0}^{N}\sum_{i=0}^{\infty}jS_{ji}(u)q^{i+r}(2\pi i\tau)^{j-1}\\ &\quad+\frac{c}{24}\sum_{j=0}^{N}\sum_{i=0}^{\infty}S_{ji}(u)q^{i+r}(2\pi i\tau)^j+\sum_{j=0}^{N}S_j\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big)(2\pi i\tau)^j\\ &=\sum_{j=0}^{N}\sum_{i=0}^{\infty}S_{ij}\big((i+r)u\big)q^{i+r}(2\pi i\tau)^j+\sum_{j=0}^{N}\sum_{i=0}^{\infty}jS_{ji}(u)q^{i+r}(2\pi i\tau)^{j-1}\\ &\quad+\sum_{j=0}^{N}S_{ij}\Big(\frac{c}{24}u\Big)q^{i+r}(2\pi i\tau)^j+\sum_{j=0}^{N}S_j\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big)(2\pi i\tau)^j,\end{aligned}$$

and so

$$\sum_{i=0}^{\infty}S_{j,i}(\omega *_n u)q^{i+r}=\sum_{i=0}^{\infty}S_{j,i}\bigg(\Big(i+r+\frac{c}{24}\Big)u\bigg)q^{i+r}+\sum_{i=0}^{\infty}(j+1)S_{j+1,i}(u)q^{i+r}\\ +S_j\big(\theta(\omega *_n u-o(\omega)o(u)),\tau\big)$$

for each j. Thus we have

$$S_{j,i}\left(\left(\omega - i - r - \frac{c}{24}\right) *_n u\right) = (j+1)S_{j+1,i}(u)$$

for $i \le n$ since $\theta(\omega *_n u - o(\omega)o(u)) \in V[[q]]q^{n+1}$. It follows from $S_{N+1,n}(u) = 0$ that $S_{j,n}((\omega - n - r - c/24)^{N-j+1} *_n u) = 0$, which proves Proposition 4.5.

Suppose that Proposition 4.6 is false. Namely, there is an integer $n > l + r$ such that $S_n = 0$ and $S_0(\alpha) \neq 0$ for some $\alpha \in V$. By Proposition 4.4, S_0 is a (symmetric) linear map of $A(V) = V/O(V)$. Set $A = A(V)/\operatorname{Rad}(S_0)$. We note that $(\omega - r - c/24)^s A = 0$. By Lemma 4.3, there are $v^i, u^i \in V$ such that $\sum_{i=1}^{p} v^i(|v^i| - 1 + n)u^i(|u^i| - 1 - n) = o(\alpha)$ in $A(V)$. By the choice of α, we may assume $v(|v| - 1 + n)u(|u| - 1 - n) = o(\alpha)$ in $A(V)$ for some $v, u \in V$.

As we mentioned in the proof of Lemma 4.3, there is an element $\beta \in V$ such that

$$\begin{aligned} o(\beta) = u(|u| - 1 - n)v(|v| - 1 + n) &+ \sum_{i>n} a_i v(|v| - 1 - i)u(|u| - 1 + i) \\ &+ \sum_{i>n} b_i u(|u| - 1 - i)v(|v| - 1 + i) \end{aligned}$$

for some $a_i, b_i \in \mathbb{C}$. Then we obtain

$$\begin{aligned} o(\alpha) &= v(|v| - 1 + n)u(|u| - 1 - n) = \big[v(|v| - 1 + n), u(|u| - 1 - n)\big] \\ &= \sum_{i=0}^{\infty} \binom{|v| - 1 + n}{i} o\big(v(i)u\big) \end{aligned}$$

on $W(0)$ for any $\mathbb{N}$-graded weak V-modules W, and so $\alpha = \sum_{i=0}^{\infty} \binom{|v|-1+n}{i} v(i)u$ in $A(V)$. On the other hand, since $n > 1$, we have $\beta \in O(V)$ and

$$\begin{aligned} S(\beta, \tau) &= S(\theta(\beta), \tau) \\ &\in \frac{q^n}{1 - q^n} S\left(\sum_{i=0}^{\infty} \binom{|v| - 1 + n}{i} v(i)u, \tau\right) + q^{n+1}\mathbb{C}[[q]] \\ &= S\left(\sum_{i=0}^{\infty} \binom{|v| - 1 + n}{i} v(i)u, \tau\right) q^n + q^{n+1}\mathbb{C}[[q]] \\ &= S(\alpha, \tau)q^n + q^{n+1}\mathbb{C}[[q]]. \end{aligned}$$

Since coefficients of $S(\beta, \tau)$ at q^n are always zero and the constant term of $S(\alpha, \tau)$ is nonzero, we have a contradiction.

This completes the proofs of the three propositions. □

5. The space of one-point functions on the torus

In this section, we just follow the proofs in [Z] and [DLM3] with suitable modification since we use pseudotrace functions that satisfy the same properties as do the ordinary trace functions, and so we skip most of the proof. The difference between our case and Zhu's case (and also the case in [DLM3]) is that we treat $A_n(V)$ as not a semisimple algebra; for example, ω might not act on $A_n(V)$ semisimply. However, since $A_n(V)$ is a finite-dimensional algebra, there are $r_i \in \mathbb{C}$ and $\mu(r_i) \in \mathbb{Z}$ such that $\prod_{i=1}^{s}(\omega - c/24 - r_i - n)^{\mu(r_i)} A_n(V) = 0$.

Let us recall the following notation from [Z] and [DLM3]. Consider $V[E_4(q), E_6(q)] \subseteq V[[q]]$. $O_q(V)$ is the submodule of $V[E_4(q), E_6(q)]$ generated by elements of the type

$$v[-2]u + \sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau) \otimes v[2k-2]u \quad \text{with } v, u \in V$$

and

$$v[0]u \quad \text{with } v, u \in V.$$

We also recall the definition of the space $\mathscr{C}_1(V) = \mathscr{C}(1,1)$ of one-point functions with trivial automorphisms from [DLM3].

Definition 5.1

We define the space $\mathscr{C}_1(V)$ of one-point functions on V to be the $\mathbb{C}$-linear space consisting of functions

$$S : V[E_4(q), E_6(q)] \otimes \mathscr{H} \to \mathbb{C}$$

satisfying the following conditions.

(C1) For $u \in V(\Gamma(1))$, $S(u, \tau)$ is holomorphic in $\tau \in \mathscr{H}$.

(C2) $S\big(\sum f_i(\tau) \otimes u_i, \tau\big) = \sum_i f_i(\tau) S(u_i, \tau)$ for $f_i(\tau) \in \mathbb{C}[E_4(q), E_6(q)]$ and $u_i \in V$.

(C3) For $u \in O_q(V)$, $S(u, \tau) = 0$.

(C4) For $u \in V$,

$$S\big(L[-2]u, \tau\big) = \frac{1}{2\pi i}\frac{d}{d\tau} S(u, \tau) + \sum_{k=1}^{\infty} E_{2k}(\tau) S(L[2k-2]u, \tau).$$

By exactly the same proof, we have the following modular invariance property of $\mathscr{C}_1(V)$ (see [Z, Th. 5.1.1] and [DLM3, Th. 5.4]).

THEOREM 5.2 (Modular-invariance)
For $S \in \mathscr{C}_1(V)$ *and* $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$, *define*

$$S|\gamma(v, \tau) = \frac{1}{(c\tau + d)^h} S\big(v, \gamma(\tau)\big)$$

for $v \in V_{[h]}$ *and extend it linearly. Then* $S|\gamma \in \mathscr{C}_1(V)$.

Set

$$S^W(u, \tau) = \mathrm{tr}_W^{\phi}\, o(u) q^{L(0)-c/24}$$

for a generalized Verma module W interlocked with a symmetric linear function ϕ of $A_n(V)$ and an element $u \in V$, where c is the central charge of V. Then extend it linearly for $V[E_4(q), E_6(q)]$.

We next prove that $S^W(u, \tau) \in \mathscr{C}_1(V)$. What we have to do is to show that $S^W(u, \tau)$ is holomorphic on the upper half-plane. In order to prove this fact, we show that S^W and all $S \in \mathscr{C}_1(V)$ satisfy differential equations. The proof is essentially the same as the arguments [Z, (4.4.11)] and [DLM3, Lem. 6.1].

THEOREM 5.3
Assume that V *is* C_2*-cofinite. If* $S(v, \tau) = 0$ *for* $v \in O_q(V)$ *and*

$$S\Big(L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau) L[2k-2]u, \tau\Big) = \frac{1}{2\pi i \tau} \frac{d}{d\tau} S(u, \tau)$$

for $u \in V$, *then there are* $m \in \mathbb{N}$ *and* $r_i(\tau) \in \mathbb{C}[E_4(q), E_6(q)]$, $0 \le i \le m-1$, *such that*

$$S(L[-2]^m v, \tau) + \sum_{i=0}^{m-1} r_i(\tau) S(L[-2]^i v, \tau) = 0.$$

In particular, $S(u, \tau)$ *converges absolutely and uniformly in every closed subset of the domain* $\{q \mid |q| < 1\}$ *for every* $u \in V$, *and the limit function can be written as a linear sum of* $q^h f(q)$, *where* $f(q)$ *is some analytic function in* $\{q \mid |q| < 1\}$. *In particular,* $S \in \mathscr{C}_1(V)$.

As a corollary, we obtain the following.

COROLLARY 5.4
$S^W(u, \tau)$ *is holomorphic on the upper half-plane for* $u \in V$ *and* $S^W(*, \tau) \in \mathscr{C}_1(V)$.

Spanning set of $\mathscr{C}_1(V)$
It remains to show that $\mathscr{C}_1(V)$ is spanned by pseudotrace functions $S^W(*,\tau)$. So we prove the following main theorem, which covers a nonsemisimple version of [Z, Th. 5.3.1].

THEOREM 5.5
Suppose that $V=\bigoplus_{m=0}^{\infty}V_m$ is a C_2-cofinite VOA with central charge c. Take an integer n sufficiently large, and let $\{W^1,\dots,W^m\}$ be the set of nth generalized Verma V-modules W^i interlocked with some symmetric linear function ϕ^i of $A_n(V)$. Then $\mathscr{C}_1(V)$ is spanned by

$$\{S^{W^1}(\cdot,\tau),\dots,S^{W^m}(\cdot,\tau)\}.$$

In particular,

$$\dim\mathscr{C}_1(V)=\dim A_n(V)/[A_n(V),A_n(V)]-\dim A_{n-1}(V)/[A_{n-1}(V),A_{n-1}(V)].$$

We note that the dimension of $A_n(V)$ is finite, and so $\mathscr{C}_1(V)$ is also of finite dimension.

Proof
Let $S\in\mathscr{C}_1(V)$. We prove that S is a sum of pseudotrace functions. By the same arguments as in [Z], it follows from Theorem 5.3 that there are integers $d,N_1,\dots,N_d$ which do not depend on v such that

$$S(v,\tau)=\sum_{s=0}^{d}S_s(v,\tau)q^{r_s},$$

and each $S_s(v,\tau)$ can be further decomposed as

$$S_s(v,\tau)=\sum_{j=0}^{N_s}S_{sj}(v,\tau)(2\pi i\tau)^j,$$

where $r_1,\dots,r_d$ are complex numbers independent of v, $r_{s_1}-r_{s_2}\notin\mathbb{Z}$ for $s_1\neq s_2$ and $S_{sj}(v,\tau)$ has a q-expansion $S_{sj}(v,\tau)=\sum_{i=0}^{\infty}C_{sji}(v)q^k$ with $C_{sji}(v)\in\mathbb{C}$ and for each j there is s such that $C_{sj0}\neq 0$. Since $r_{s_1}-r_{s_2}\notin\mathbb{Z}$ for $s_1\neq s_2$, each $S_s(v,\tau)q^{r_s}$ satisfies (C2)$\sim$(C4) and also (C1) by Theorem 5.3. Hence we may assume $d=1$ and

$$S(v,\tau)=\sum_{j=0}^{N}S_j(v,\tau)(2\pi i\tau)^j=\sum_{j=0}^{N}\Big(\sum_{i=0}^{\infty}C_{ji}(v)q^{i+r}\Big)(2\pi i\tau)^j \tag{5.1}$$

with $r\in\mathbb{C}$. In the case where $A(V)$ is semisimple, it was proved in [Z] and [DLM3] that $N=0$ and $\sum_{i=0}^{\infty}S_i(v)q^{i+r}$ is a sum of trace functions. However, if $A_n(V)$ is not semisimple, we may have nonzero N since we consider logarithmic modules, too. In order to continue the proof, we need the following two lemmas.

LEMMA 5.6
Let $W = W_T^{(n)}$ be an nth generalized Verma module interlocked with (P, ϕ) satisfying $(\omega - r - n - c/24)^{N+1} W_T^{(n)}(n) = 0$ for some $r \in \mathbb{C}$ and $N \in \mathbb{N}$. Then there are constants $b_1, \dots, b_{s-1}$ such that

$$\mathrm{tr}_W^{\phi}\, o(v) q^{L^s(0)-c/24} = S^W(v,\tau) - \sum_{i=1}^{N} b_i S^{W/W\mathfrak{N}^i}(v,\tau)(2\pi i\tau)^i,$$

where $\mathfrak{N}^i = \{a \in P \mid (\omega - r - n - c/24)^i a = 0\}$ and $L^s(0)$ is a semisimple part of $L(0)$ which acts on $W(m)$ as $m + r + c/24$ and $(L(0) - L^s(0))^{N+1} = 0$ on W.

Proof
We first note that there are constants $b_0 = 1, b_1, \dots$ such that $e^{2\pi i\alpha} = 1 + b_1\alpha e^{2\pi i\alpha} + b_2\alpha^2 e^{2\pi i\alpha} + \cdots + b_{s-1}\alpha^{s-1}e^{2\pi i\alpha} + \cdots$. Hence we obtain

$$\begin{aligned}
S^W(v,\tau) &= \mathrm{tr}^{\phi}\, o(v) q^{L(0)-c/24} = \mathrm{tr}^{\phi}\, o(v) q^{(L^s(0)-c/24)} q^{(L(0)-L^s(0))} \\
&= \mathrm{tr}_W^{\phi}\, o(v) q^{L^s(0)-c/24} \\
&\quad + \sum_{i=1}^{N} \mathrm{tr}_W^{\phi}\, o(v) b_i \big(L(0) - L^s(0)\big)^i (2\pi i\tau)^i q^{L^s(0)-c/24} \\
&= \mathrm{tr}_W^{\phi}\, o(v) q^{L^s(0)-c/24} \\
&\quad + \sum_{i=1}^{N} \mathrm{tr}_W^{\phi}\, o(v) b_i \Big(\omega - n - \frac{c}{24} - r\Big)_P^i q^{L(0)-c/24}(2\pi i\tau)^i \\
&= \mathrm{tr}_W^{\phi}\, o(v) q^{L^s(0)-c/24} \\
&\quad + \sum_{i=1}^{N} b_i \big(\mathrm{tr}_{W/W\mathfrak{N}^i}^{(\omega-n-c/24-r)^i\phi}\, o(v) q^{L(0)-c/24}(2\pi i\tau)^i\big) \qquad \text{(by (3.6))} \\
&= S^W(v,\tau) + \sum_{i=1}^{N} b_i S^{W/W\mathfrak{N}^i}(v,\tau)(2\pi i\tau)^i. \qquad \square
\end{aligned}$$

We prove that $S_0(v,\tau) = \sum_{i=0}^{\infty} C_{0i}(v) q^{i+r}$ is a linear sum of pseudotrace functions with semisimple grading operator $L^s(0)$, say,

$$S_0(v,\tau) = \sum_p a_p\, \mathrm{tr}_{W^p}^{\phi^p}\, o(v) q^{L^s(0)-c/24}.$$

Then

$$\tilde{S}(v,\tau)=S(v,\tau)-\sum_p a_p\Big(S^{W^p}(v,\tau)-\sum_{i=1}^{s-1}b_i S^{W/W\mathfrak{N}^i}(v,\tau)(2\pi i\tau)^i\Big)\in\mathscr{C}_1(V),$$

but if we express it by

$$\tilde{S}(v,\tau)=\sum_{s=0}^{N}\sum_{j=0}^{\infty}\tilde{S}_{sj}(v)q^{j+r}(2\pi i\tau)^s,$$

then $\tilde{S}_{0j}(v)=0$ for all j and $v\in V$. However, since

$$\tilde{S}\Big(L[-2]u-\sum_{k=1}^{\infty}E_{2k}(\tau)L[2k-2]u,\tau\Big)=\frac{1}{2\pi i}\frac{d}{d\tau}\tilde{S}(u,\tau),$$

$S_{0,j}=0$ implies $S_{1,j}=0$, and so on. Hence we have $\tilde{S}(v,\tau)=0$ for all $v\in V$, as desired.

So it is sufficient to prove that $S_0(v,\tau)=\sum_{i=0}^{\infty}C_{0i}(v)q^{i+r}$ is the coefficient of the $(2\pi i\tau)^0$-term of a linear sum of pseudotrace functions $S^{W^p}(v,\tau)$.

Since $S(L[-2]u-\sum_{k=1}^{\infty}E_{2k}(\tau)L[2k-2]u,\tau)=\frac{1}{2\pi i}\frac{d}{d\tau}S(v,\tau)$, we obtain the following lemma from Propositions 4.4 and 4.5.

LEMMA 5.7
We have

$$C_{0,n}(v*_n u)=C_{0,n}(u*_n v),$$
$$C_{0,n}(v)=0\quad \textit{for } v\in O_{n+1}(V),$$
$$C_{0,n}\Big(\Big(\omega-\frac{c}{24}-r_{sj}-n\Big)^{N+1}*_n v\Big)=0.$$

In particular, $A_n(V)$ has a symmetric linear map $C_{0,n}$. Let $A_1\oplus\cdots\oplus A_k$ be the decomposition of $A_n(V)/J(A_n(V))$ into the direct sum of simple algebras A_i, and let $\{e_i\mid i=1,\ldots,k\}$ be a set of mutually orthogonal primitive idempotents of $A_n(V)$ such that $e_i+J(A_n(V))\in A_i$. Set $e=\sum e_i$. By Theorem 3.10, there are symmetric linear functions ϕ_p of $A_n(V)$ and $(A_n(V)\times eA_n(V)e)$-modules $(A_n(V)/\mathfrak{N}_p)e$ such that $(\omega-c/24-r-n)^{N+1}\in\mathfrak{N}_p$ and $C_{0,n}$ is a sum of pseudotrace maps; that is,

$$C_{0,n}=\sum_p a_p\,\mathrm{tr}^{\phi_p}_{(A_n(V)/\mathfrak{N}_p)e},$$

where $\mathfrak{N}_p=\mathrm{Rad}(\phi_p)$ and $(A_n(V)/I_p)e$ are all indecomposable $A_n(V)$-modules.

We have assumed that n is large enough so that there are no conformal weights greater than $r+n-l$. Construct generalized Verma modules

$$W^p = W^{(n)}_{T_p}$$

from $A_n(V)$-module $T_p = (A_n(V)/\mathfrak{N}_p)\bar{e}$. As we showed, W^p is interlocked with ϕ_p and $L(0)-c/24-r-m$ acts on $W^p(m)$ as a nilpotent operator. Define a pseudotrace function

$$S^{W^p}(v,\tau) = \mathrm{tr}^{\phi_p}\, o(v)q^{L(0)-c/24}.$$

Then

$$\tilde{S}(v,\tau) = S(v,\tau) - \sum_p a_p S^{W^p}(v,\tau) = \sum_{s=0}^{N}\Big(\sum_{i=0}^{\infty}\tilde{C}_{si}(v)q^{i+r}\Big)(2\pi i\tau)^s$$

satisfies the same properties, but

$$\tilde{C}_{0n}(v,\tau) = 0 \quad \text{for all } v \in V.$$

Then by Proposition 4.5, we have $\tilde{C}_{00} = 0$. Since

$$\tilde{S}\Big(L[-2]u - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]u, \tau\Big) = \frac{1}{2\pi i}\frac{d}{d\tau}\tilde{S}(u,\tau),$$

$\tilde{C}_{s0}(u) = 0$ for all $u \in V$ and $s = 0,1,\ldots,N$. Namely, the lowest weight $\tilde{r}$ of $\tilde{S}(*,\tau)$ is greater than that of $S(*,\tau)$. Repeating these steps, we obtain the desired result since V has only finitely many lowest weights of pseudotrace functions. This completes the proof of Theorem 5.5. □

We next consider the case where $v = \mathbf{1}$. Then by Theorem 5.5,

$$\big\langle \mathrm{tr}^{\phi}_W\, q^{L(0)-c/24} \,\big|\, W \text{ is interlocked with } \phi \text{ of } A_n(V)\big\rangle$$

is $\mathrm{SL}_2(\mathbb{Z})$-invariant. We have that $\mathrm{tr}^{\phi}_W\, q^{L(0)-c/24}$ plays the role of a generalized character introduced in [F], and so we call it a *generalized character* of W. Let us study generalized characters for a while. If $\phi(1) = 0$ and $(L(0) - L^s(0))W = 0$, then $\mathrm{tr}^{\phi}_W\, 1q^{L(0)} = 0$, and so we have the following.

THEOREM 5.8
Let $V = \bigoplus_{m=0}^{\infty} V_m$ be a C_2-cofinite VOA. Then the space spanned by generalized characters is $\mathrm{SL}_2(\mathbb{Z})$*-invariant. In particular, if there is no logarithmic module, then the space spanned by the set of all (ordinary) characters is* $\mathrm{SL}_2(\mathbb{Z})$*-invariant.*

We may assume $\phi(1) = 0$ and $(L(0) - L^s(0))^m W = 0$. Let r be a conformal weight of W. Then

$$\begin{aligned}\mathrm{tr}_W^{\phi}\, q^{L(0)-c/24} &= \mathrm{tr}_W^{\phi} \sum_{j=0}^{m} \frac{1}{j!}\big(L(0) - L^s(0)\big)^j q^{L^s(0)-c/24}(2\pi i\tau)^j \\ &\in \mathbb{C}[[q]]q^{r-c/24}[\tau]\tau.\end{aligned}$$

By Lemma 4.1, $\mathrm{tr}_W^{\phi}(L(0) - L^s(0))^j q^{L^s(0)-c/24} = \mathrm{tr}_{W/W\mathfrak{N}_j}^{(\omega-n-r)^j\phi}\, q^{L^s(0)-c/24}$ is a linear combination of characters. Therefore we obtain the following.

PROPOSITION 5.9
A generalized character is a linear combination of characters with coefficients in $\mathbb{C}[\tau]$.

As an application of Theorem 5.5, $\langle \mathrm{ch}_W(\tau)^{\mathrm{SL}_2(\mathbb{Z})}\rangle$ is of finite dimension for an irreducible V-module W and $\mathrm{ch}_W(\frac{-1}{\tau}) \in \sum_{i=1}^{k} \mathbb{C}[[q]][\tau]q^{r_i-c/24}$. Therefore we can apply the same arguments as in the proofs of [AM, Prop. 3] and [DLM3, Th. 11.3] with suitable modifications (q-powers should be replaced by elements in $\mathbb{C}[[q]][\tau]$), and so we obtain the following corollary.

COROLLARY 5.10
If $V = \bigoplus_{m=0}^{\infty} V_m$ *is a* C_2*-cofinite VOA, then the central charge and the conformal weights are all rational numbers.*

We next prove a bound of the effective central charge $\tilde{c} = c - 24h_{\min}$, where $h_{\min}$ is the smallest conformal weight.

COROLLARY 5.11
Let V *be a* C_2*-cofinite VOA. Then* $\tilde{c} \le \big(\dim(V/C_2(V)) - 1\big)/2$.

Proof
The proof is essentially the same as in [GN] with slight modifications. Set $k = \dim V/C_2(V) - 1$, and define $f_2(q) = \sqrt{2}q^{1/24}\prod_{n=1}^{\infty}(1 + q^n)$. By using a spanning set of irreducible module W with a conformal weight r given in Lemma 2.4, there is a polynomial $g(q) = \sum_{i=0}^{s} g_i q^i \in \mathbb{C}[q]$ such that $g_i \ge 0$ and

$$\mathrm{ch}_W(\tau) \le 2^{-k/2}q^{-k/24} f_2(q)^k g(q) q^r.$$

Here and in the following we always assume that $0 < q < 1$. As we showed,

$$\mathrm{ch}_W(-1/\tau) = \sum_X a_X^W(\tau)\mathrm{ch}_X(\tau),$$

where $\mathrm{ch}_X(\tau)$ runs over the set of distinct characters and $a_X^W(\tau) \in \mathbb{C}[\tau]$. Hence

$$\Big|\sum a_X(\tau)\mathrm{ch}_X(\tau)\Big| \leq \tilde{q}^{-(k+c)/24}2^{-k/2} f_2(\tilde{q})^k g(\tilde{q}),$$

where $\tilde{q} = e^{-2\pi i/\tau}$ and $f_4(q) = q^{-1/48}\prod_{n=1}^{\infty}(1-q^{n-1/2})$. In the limit $\tau \to i\infty$ $(q \to 0, \tilde{q} \to 1)$,

$$\mathrm{ch}_W(\tilde{q}) = |\tau|^m q^{h-c/24}\big(a+o(1)\big)g(1)$$

for some integer m and constants a, where h is a minimal one among conformal weights that appear in $\sum_X a_X^W(\tau)\mathrm{ch}_X(\tau)$. Since $\tau \to -1/\tau$ is an involution, there is an irreducible V-module W such that a character with a minimal conformal weight $h_{\min}$ appears in $\mathrm{ch}_W(\tilde{q})$. Hence there is a constant C such that

$$|\tau|^m q^{h_{\min}-c/24}| \leq q^{-k/48}\big(C+O(q)\big),$$

and so we have $h_{\min} - c/24 \geq -k/48$, as desired. □

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References

[ABD] T. ABE, G. BUHL, and C. DONG, *Rationality, regularity, and C_2-cofiniteness*, preprint, arXiv:math.QA/0204021, to appear in Trans. Amer. Math. Soc.

[AM] G. ANDERSON and G. MOORE, *Rationality in conformal field theory*, Comm. Math. Phys. **117** (1988), 441 – 450. MR 0953832

[B] R. E. BORCHERDS, *Vertex algebras, Kac-Moody algebras, and the Monster*, Proc. Nat. Acad. Sci. U.S.A. **83** (1986), 3068 – 3071. MR 0843307

[Bu] G. BUHL, *A spanning set for VOA modules*, J. Algebra **254** (2002), 125 – 151. MR 1927435

[CR] C. W. CURTIS and I. REINER, *Methods of Representation Theory, Vol. 1*, Wiley Classics Lib., Wiley, New York, 1990. MR 1038525

[DLM1] C. DONG, H. LI, and G. MASON, *Twisted representations of vertex operator algebras*, Math. Ann. **310** (1998), 571 – 600. MR 1615132

[DLM2] ———, *Vertex operator algebras and associative algebras*, J. Algebra **206** (1998), 67 – 96. MR 1637252

[DLM3] ———, *Modular-invariance of trace functions in orbifold theory and generalized Moonshine*, Comm. Math. Phys. **214** (2000), 1 – 56. MR 1794264

[DM] C. DONG and G. MASON, *On quantum Galois theory*, Duke Math. J. **86** (1997), 305 – 321. MR 1430435

[F] M. A. I. FLOHR, *On modular invariant partition functions of conformal field theories with logarithmic operators*, Internat. J. Modern Phys. A **11** (1996), 4147 – 4172. MR 1403683

[FLM] I. FRENKEL, J. LEPOWSKY, and A. MEURMAN, *Vertex Operator Algebras and the Monster*, Pure Appl. Math. **134**, Academic Press, Boston, 1988. MR 0996026

[FZ] I. B. FRENKEL and Y. ZHU, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123 – 168. MR 1159433

[G] M. R. GABERDIEL, *An algebraic approach to logarithmic conformal field theory*, Internat. J. Modern Phys. A **18** (2003), 4593 – 4638.

[GK] M. R. GABERDIEL and H. G. KAUSCH, *A rational logarithmic conformal field theory*, Phys. Lett. B **386** (1996), 131 – 137. MR 1411388

[GN] M. R. GABERDIEL and A. NEITZKE, *Rationality, quasirationality and finite W-algebra*, Comm. Math. Phys. **238** (2003), 305 – 331. MR 1990879

[L1] H. LI, *Determining fusion rules by $A(V)$-modules and bimodules*, J. Algebra **212** (1999), 515 – 556. MR 1676853

[L2] ———, *Some finiteness properties of regular vertex operator algebras*, J. Algebra **212** (1999), 495 – 514. MR 1676852

[M] A. MILAS, “Weak modules and logarithmic intertwining operators for vertex operator algebras” in *Recent Developments in Infinite-Dimensional Lie Algebras and Conformal Field Theory (Charlottesville, Va., 2000)*, Contemp. Math. **297**, Amer. Math. Soc., Providence, 2002, 201 – 225. MR 1919819

[Mi1] M. MIYAMOTO, *A modular invariance on the theta functions defined on vertex operator algebras*, Duke Math. J. **101** (2000), 221 – 236. MR 1738180

[Mi2] ———, “Modular invariance of trace functions on VOAs in many variables” in *Proceeding on Moonshine and Related Topics (Montréal, 1999)*, CRM Proc. Lecture Notes **30**, Amer. Math. Soc., Providence, 2001, 131 – 137. MR 1877763

[Mi3] ———, *Intertwining operators and modular invariance*, preprint, arXiv:math.QA/0010180

[NT] K. NAGATOMO and A. TSUCHIYA, *Conformal field theories associated to regular chiral vertex operator algebras, I: Theories over the projective line*, preprint, arXiv:math.QA/0206223

[NS] C. NESBITT and W. M. SCOTT, *Some remarks on algebras over an algebraically closed field*, Ann. of Math. (2) **44** (1943), 534 – 553. MR 0009024

[O] M. OSIMA, *A note on symmetric algebras*, Proc. Japan Acad. **28** (1952), 1 – 4. MR 0049877

[Y] H. YAMAUCHI, *Orbifold Zhu theory associated to intertwining operators*, J. Algebra **265** (2003), 513 – 538. MR 1987015

[Z] Y. ZHU, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), 237 – 302. MR 1317233

Institute of Mathematics, University of Tsukuba, Tsukuba 305, Japan;
miyamoto@math.tsukuba.ac.jp