A modular approach to the $K(2)$-local sphere at the prime 3

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Stable homotopy theory

Let S^0 denote the p-local sphere spectrum. The p-local stable homotopy groups of spheres are the groups

$$\pi_k S^0 = \pi_{k+n} S^n \otimes \mathbb{Z}_{(p)} \quad (\text{for } n \gg k \geqslant 0)$$

Remark

One strategy is to find other spectra whose homotopy groups approximate $\pi_* S^0$.

Examples

1. The spectrum TMF of topological modular forms helps account for $\pi_k S^0$ for $0 \leqslant k \leqslant 60$.
2. There exist spectra $\{L_{K(n)} S^0\}_{n=0,1,2,\ldots}$ that play an analogous role.

Bousfield localization

Theorem

Given a homology theory E, there exists a functor

$$L_E : \mathbf{S} \to \mathbf{S}$$

where $L_E X$ is "the part of X that E can see."

Remark

If $X \to Y$ induces $E_* X \cong E_* Y$, then $L_E X \simeq L_E Y$.

Examples

1. If $E = M$ is the mod p Moore spectrum, $L_E X = X_p^\wedge$.
2. If $E = H\mathbb{Q}$, $L_E X = X\mathbb{Q}$.
3. If $E = K$ is complex K-theory, $\pi_{-2} L_E S^0 = \mathbb{Q}/\mathbb{Z}$.

Homology theories

Theorem (Landweber exactness)

Given a module M over $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, the functor $BP_*(-) \otimes M$ is a homology theory iff for each $n > 0$, multiplication by v_n in $BP_*/I_n \otimes M$ is monic, where $I_n = (p, v_1, v_2, \ldots, v_{n-1})$.

We will need the following three theories:

1. Johnson-Wilson: $E(n)_* = \mathbb{Z}_{(p)}[v_1, \ldots, v_{n-1}, v_n^{\pm 1}]$
2. Lubin-Tate: $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$
3. Morava K-theories: $K(n)_* = \mathbb{Z}/(p)[v_n, v_n^{-1}]$

Remark

The Morava K-theories $K(n)$ are not Landweber exact.

Chromatic homotopy theory

Theorem (Chromatic convergence)

Let $L_n = L_{E(n)}$. *Then*

$$S^0 \simeq \mathrm{holim}(L_0 S^0 \leftarrow L_1 S^0 \leftarrow L_2 S^0 \leftarrow \cdots)$$

There is a homotopy pullback square

$$\begin{tikzcd} L_n S^0 \arrow[r] \arrow[d] & L_{K(n)} S^0 \arrow[d] \\ L_{n-1} S^0 \arrow[r] & L_{n-1} L_{K(n)} S^0 \end{tikzcd}$$

The building blocks of $\pi_* S^0$ are the spectra $L_{K(n)} S^0$, the $K(n)$-local spheres.

History

- **2002:** Shimomura and Yabe computed $\pi_* L_{K(2)} S^0$ at the prime 3. "The statement of their result is so complicated, it can only be parsed by an expert."
- **2005:** Goerss, Henn, Mahowald and Rezk showed that $L_{K(2)} S^0$ at $p = 3$ lies atop a 4-stage tower of fibrations analogous to

$$L_{K(1)} S^0 \to KO_2^\wedge \to KO_2^\wedge$$

at the prime 2.

- **2006:** Behrens (following Mahowald and Rezk) built a spectrum $Q(2)$ and proved

$$DQ(2) \to L_{K(2)} S^0 \to Q(2)$$

An elliptic curve

The supersingular elliptic curve

$$C_{/\mathbb{F}_9} : y^2 = x^3 - x$$

possesses a degree 2 isogeny $\psi : C \to C$ with kernel $H < C$ and inducing $\psi^\wedge : C^\wedge \cong C^\wedge$. We say C has a $\Gamma_0(2)$ structure.

Theorem (Goerss-Hopkins-Miller)

There is a functor FGL $\to$ Spectra *sending* $C^\wedge$ *to* E_2. *Since* $\psi^\wedge$ *is invariant under the action of* $\mathrm{Aut}_{/\mathbb{F}_3}(C, H) \cong D_8$, *it induces a map of spectra*

$$\psi_d : E_2^{hD_8} \to E_2^{hD_8}$$

where

$$E_2^{hD_8} \simeq TMF_0(2)$$

A semi-cosimplicial spectrum

We have $\mathrm{Aut}_{/\mathbb{F}_3}(C) \cong G_{24}$, and

$$E_2^{hG_{24}} \simeq TMF$$

Proposition (Behrens)

There is a semi-cosimplicial diagram of spectra

$$Q(2)^\bullet : \; TMF \rightrightarrows TMF \vee TMF_0(2) \underset{\psi_d}{\overset{\to}{\rightrightarrows}} TMF_0(2)$$

Definition

$$Q(2) := \mathrm{holim}\, Q(2)^\bullet$$

Dropping the veil

$Q(2)^\bullet$ realizes a semi-simplicial stack

$$\mathcal{M}_\bullet : \ \mathcal{M} \leftleftarrows \mathcal{M} \coprod \mathcal{M}_0(2) \overset{\psi_d}{\underset{\leftarrow}{\leftleftarrows}} \mathcal{M}_0(2)$$

where $\mathcal{M}$ is the moduli stack of non-singular elliptic curves over $\mathbb{Z}_{(3)}$:

$$\begin{tikzcd} \mathcal{M} \arrow[d] & \supset & \mathcal{M}_R \arrow[d] \\ \mathrm{Aff} & \ni & \operatorname{Spec} R \end{tikzcd}$$

and $\mathcal{M}_0(2)$ is the analogous stack of elliptic curves with a $\Gamma_0(2)$ structure.

Methods for calculating $\pi_* Q(2)$

1. The Bousfield-Kan spectral sequence for totalizations of cosimplicial spectra.

2. The chromatic spectral sequence

$$E_1^{s,t} = \pi_t(M_s Q(2)) \Rightarrow \pi_{t-s} Q(2)$$

where $M_s Q(2) \to L_s Q(2) \to L_{s-1} Q(2)$.

3. Compute the Adams-Novikov E_2-term for $Q(2)$ and add in the differentials using those from the ANSS for TMF.

The ANSS for TMF and $TMF_0(2)$

The Hopf algebroid

$$(B, \Gamma) = (\mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}], B[r]/(r^3 + q_2 r^2 + q_4 r))$$

represents the groupoid of elliptic curves of the form $y^2 = 4x(x^2 + q_2 x + q_4)$.

Theorem

The ANSS E_2-term for TMF is

$$\mathrm{Ext}^* := \mathrm{Ext}^*_\Gamma(B, B) = H^*(B \to \Gamma \to \Gamma^{\otimes 2} \to \Gamma^{\otimes 3} \to \cdots)$$

The ANSS for $TMF_0(2)$ is concentrated on the zero line, and gives

$$\pi_{2k} TMF_0(2) = B_k$$

The ANSS for $Q(2)$

Recall that

$$Q(2) = \mathrm{holim}(\,TMF \to TMF \vee TMF_0(2) \to TMF_0(2))$$

Proposition

The ANSS E_2-term for $Q(2)$ is the cohomology of the totalization of the double cochain complex $C^{*,*}$ given by

$$\begin{tikzcd}
\vdots & \vdots & & \\
\Gamma \otimes \Gamma \arrow[r] \arrow[u] & \Gamma \otimes \Gamma \arrow[r] \arrow[u] & 0 \arrow[r] & 0 \cdots \\
\Gamma \arrow[r] \arrow[u] & \Gamma \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \cdots \\
B \arrow[r] \arrow[u] & B \oplus B \arrow[r] \arrow[u] & B \arrow[r] \arrow[u] & 0 \cdots
\end{tikzcd}$$

The ANSS for $Q(2)$

Lemma

$$\mathrm{Ext}^0 = \mathbb{Z}_{(3)}[c_4, c_6, \Delta^{\pm 1}] =: MF$$

where $1728\Delta = c_4^3 - c_6^2$.

Remark

After taking cohomology with respect to the vertical arrows, $C^{*,*}$ becomes

$$\begin{tikzcd}
\vdots & \vdots & & \\
\mathrm{Ext}^1 \arrow[r, "0"] & \mathrm{Ext}^1 \arrow[r, "0"] & 0 \arrow[r] & 0 \cdots \\
MF \arrow[u] \arrow[r, "\Phi"] & B \oplus MF \arrow[u] \arrow[r, "\Psi"] & B \arrow[u] \arrow[r] & 0 \cdots
\end{tikzcd}$$

The ANSS for $Q(2)$

In the cochain complex

$$C^{0,*} : MF \xrightarrow{\Phi} B \oplus MF \xrightarrow{\Psi} B$$

the maps Φ and Ψ are sums of $\mathbb{Z}_{(3)}$-module maps corresponding to the maps of spectra in $Q(2)^{\bullet}$.

Example

The map $\psi_d : TMF_0(2) \to TMF_0(2)$ corresponds to $\psi_d^* : B \to B$ defined by

$$q_2 \mapsto -2q_2, \quad q_4 \mapsto -4q_4 + q_2^2$$

and $\Psi = (\psi_d^* + 1) \oplus g$ for $g : MF \to B$.

Dropping the veil again

As a map of stacks,

$$\psi_d : \mathcal{M}_0(2) \to \mathcal{M}_0(2)$$

and on the level of R-points,

$$(C, H) \mapsto (C/H, \widehat{H})$$

where $\widehat{H}$ is the kernel of the dual isogeny $\widehat{\psi} : C/H \to C$.

Remark

The formula for $\psi_d^* : B \to B$ comes from studying the effect of above map on Weierstrass equations.

Progress

Theorem (L.)

$H^0(C^{0,*}) = \mathbb{Z}_{(3)}$, while $H^1(C^{0,*})$ and $H^2(C^{0,*})$ are infinite direct sums of cyclic modules, entirely torsion in positive degrees and torsion-free in degree zero.

Remarks

- The only other possibly nontrivial differential is

$$d_2 : \mathrm{Ext}^1 \to \operatorname{coker} \Psi$$

- The next step is to use the ANSS differentials for TMF to complete the calculation of $\pi_* Q(2)$.

Greek letter elements

For any prime p, there exist spectra $Q(N)$ built from degree N isogenies of elliptic curves, as long as N is a topological generator of $\mathbb{Z}_p^\times$.

Theorem (Behrens 2009)

Let $p \geqslant 5$. There is a 1-1 correspondence that associates to each additive generator

$$\beta_{i/j,k} \in \mathrm{Ext}^{2,*}_{BP_*BP}(BP_*, BP_*)$$

a modular form $f_{i/j,k} \in MF_{2i(p^2-1)}$ satisfying certain congruence conditions.

Conjecture

The above theorem holds at $p = 3$.

Greek letter elements

Theorem (Behrens 2009)

Let $p \geqslant 5$. The spectrum $Q(N)$ is $E(2)$-local, and the images of the homotopy elements $\alpha_{i,j}$ and $\beta_{i/j,k}$ under the homomorphism

$$\pi_* L_2 S^0 \to \pi_* Q(N)$$

are non-trivial.

Conjecture

The homotopy Greek letter elements $\beta^h_{i/j,k}$ are detected by the spectra $Q(N)$ at the primes 2 and 3.

The $K(2)$-local sphere

Theorem (Behrens 2006)

At $p = 3$, there is a cofiber sequence

$$D_{K(2)}Q(2) \to L_{K(2)}S^0 \to Q(2)$$

and the same is true at $p = 5$.

Open questions

1. Does this theorem for all p and N? If so, is there a uniform proof?
2. Describe the connecting map

$$Q(N) \to \Sigma D_{K(2)}Q(N)$$

The End

Thank you!