LECTURES ON POWER OPERATIONS (VERSION 0.11)

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Contents

1. Power operations in K-theory

1.1. **Equivariant K-theory.** Let X be a space (a finite CW-complex), and let $\mathrm{Vect}(X)$ denote the set of isomorphism classes of complex vector bundles over X. Then $K(X) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\mathrm{Vect}(X), \oplus)$.

More generally, if G is a compact Lie group, and X a G-space, we may let $\mathrm{Vect}_G(X)$ denote the set of G-equivariant vector bundles; then $K_G(X) \stackrel{\mathrm{def}}{=} \mathrm{Gr}(\mathrm{Vect}_G(X), \oplus)$.

There is another way to descibe the K-group; for details, I refer to [Ati89] or [Seg68]. The idea is to consider bounded complexes of bundles over X; the collection of such complexes, equipped with a suitable equivalence relation, gives the K-group.

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1.2. **The total power operation.** Consider a complex vector bundle $V \to X$; for the time being, I will assume that X is a *finite* CW-complex. Tensor product of vector spaces gives rise to a vector bundle over the product space $X^{\times m}$, which we will denote $V^{\boxtimes m}$, the **external tensor product**. Note that the bundle $V^{\boxtimes m}$ has an evident Σ_m-action, compatible with the evident action on $X^{\times m}$. Therefore, the bundle $V^{\boxtimes m} \to X^{\times m}$ is naturally a Σ_m-equivariant bundle.

In fact, this construction can be defined on complexes of vector bundles over X. This leads to a function

$$\mathcal{P}_m \colon K(X) \to K_{\Sigma_m}(X^{\times m})$$

which we call the **total mth power operation**.

Let $\delta \colon X \to X^{\times m}$ denote the diagonal inclusion. Composing with $\mathcal{P}_m$ gives a function

$$K(X) \xrightarrow{\mathcal{P}_m} K_{\Sigma_m}(X^{\times m}) \xrightarrow{\delta^*} K_{\Sigma_m}(X),$$

which we denote P_m. Since Σ_m acts trivially on the diagonal copy of X, there is a natural map

$$K(X) \otimes_{\mathbb{Z}} R\Sigma_m \to K_{\Sigma_m}(X)$$

which is an isomorphism.

Example 1.3. Let $m = 2$. Then $R\Sigma_2 \approx \mathbb{Z}[s]/(s^2 - 1)$, where s represents the sign representation. The function $P_2 \colon K(X) \to K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[s]/(s^2 - 1)$ decomposes as

$$P_2(x) = \sigma^2(x) \cdot 1 + \lambda^2(x) \cdot s,$$

where σ^2 is the second symmetric power, and λ^2 is the second exterior power.

In general, since $R\Sigma_m$ has a basis by irreducible representations V_π, we can write

$$P_m(x) = \sum \phi_\pi(x) \cdot [V_\pi].$$

In general, $\phi_1(x) = \sigma^m(x)$ and $\phi_{\mathrm{sgn}}(x) = \lambda^m(x)$.

More generally, if G is a finite group equipped with a homomorphism $G \to \Sigma_m$, we can define $\mathcal{P}_G \colon K(X) \to K_G(X^{\times m})$ as the composite

$$K(X) \xrightarrow{\mathcal{P}_m} K_{\Sigma_m}(X^{\times m}) \xrightarrow{\text{res}} K_G(X^{\times m}).$$

1.4. **Properties of the total power operation.** We now discuss some of the general properties of the operators $\mathcal{P}_m$ (which descend to properties of the P_m).

(a) Consider $\{e\} \to \Sigma_m$; then $\mathcal{P}_{\{e\}}(x) = x^{\boxtimes m}$. That is, the composite

$$K(X) \to K_{\Sigma_m}(X^{\times m}) \xrightarrow{\text{res}} K(X^{\times m})$$

is the external mth power map. Thus, the restriction of P_m to the trivial group is the ordinary mth power map.

(b) The operations $\mathcal{P}_m$ are multiplicative, in the sense that $\mathcal{P}_m(xy) = \mathcal{P}_m(x)\mathcal{P}_m(y)$. More generally, $\mathcal{P}_m$ commutes with the external Künneth product. (This is a consequence of the fact that for vector spaces V and W, there is a natural isomorphism $(V \otimes W)^{\otimes m} \approx V^{\otimes m} \otimes W^{\otimes m}$.

(c) $\mathcal{P}_m(1) = 1$.

1.5. **An equivariant generalization of the total power operation.** For the next property, I need to extend the definition a little. Given a group G, let $G \wr \Sigma_m$ denote the "wreath product", i.e., the group fitting in the split extension

$$1 \to G^m \to G \wr \Sigma_m \to \Sigma_m \to 1.$$

Note that the "diagonal" copy of G gives us a distinguished subgroup $G \times \Sigma_m \subset G \wr \Sigma_m$.

If G is equipped with a homomorphism $G \to \Sigma_n$, then there is a homomorphism $G \wr \Sigma_m \to \Sigma_{mn}$ obtained as follows. Since G acts on $\underline{n}$, we can have G^m act on $\underline{n} \times \underline{m}$, where the ith copy of G acts on $\underline{n} \times \{i\}$. Combining this with Σ_m acting on $\underline{n} \times \underline{m}$ through $\underline{m}$, we obtain an action by $G \wr \Sigma_m$.

We can define a power operation on G-*equivariant* K-theory, which takes

$$\mathcal{P}_m \colon K_G(X) \to K_{G \wr \Sigma_m}(X^{\times m}).$$

If we restrict to the diagonal copies of X and G, we can a map P_m:

$$K_G(X) \xrightarrow{\mathcal{P}_m} K_{G \wr \Sigma_m}(X^{\times m}) \xrightarrow{\text{res}} K_{G \times \Sigma_m}(X) \approx K_G(X) \otimes_{\mathbb{Z}} R\Sigma_m.$$

Remark 1.6. If you think about how all this works, you may see that all of this could be more cleanly stated using orbifolds. Thus, if we consider *orbifold K-theory*, and we have any orbifold M, then there should be power operations of the form

$$\mathcal{P}_m \colon K_{\text{orb}}(M) \to K_{\text{orb}}(M^{\times m}//\Sigma_m).$$

The point is that if $M = (X//G)$, then $M^{\times m}//\Sigma_m \approx (X^{\times m}//G \wr \Sigma_m)$.

Since I don't actually know what an orbifold is, I won't set things up this way.

1.7. **Properties of the total power operation, continued.**

(d) The diagram

$$\begin{tikzcd} K(X) \arrow[r, "\mathcal{P}_{mn}"] \arrow[d, "\mathcal{P}_n"'] & K_{\Sigma_{mn}(X^{mn})} \arrow[d, "\mathrm{res}^{\Sigma_{mn}}_{\Sigma_n \wr \Sigma_m}"] \\ K_{\Sigma_n}(X^n) \arrow[r, "\mathcal{P}_m"'] & K_{\Sigma_n \wr \Sigma_m}(X^{mn}) \end{tikzcd}$$

commutes. (This amounts to the observation that there is a natural isomorphism $(V^{\otimes n})^{\otimes m}$.)

1.8. **Transfer.** Given a finite covering $f \colon X \to Y$ of G-spaces, there is a "transfer" map:

$$f_! \colon K(X) \to K(Y);$$

if $V \to X$ is a G-bundle, we define a G-bundle $f_! V \to Y$ with

$$(f_! V)_y = \prod_{x \in f^{-1}(y)} V_x;$$

the group G-acts "diagonally", so that g sends $(v_x)_{x \in f^{-1}(y)} \mapsto (g v_{g^{-1}x'})_{x' \in f^{-1}(gy)}$. Consider $H \subset G$ a subgroup. Then there is a covering $G \times_H X \to X$, and the induced transfer

$$\text{ind}_H^G \colon K_H(X) \approx K_G(G \times_H X) \xrightarrow{f_!} K_G(X)$$

is also called an *induction map*. In particular, if $X = *$, then this is just the usual induction map $RH \to RG$.

Some properties of the transfer.

(i) Transfers are natural, in the sense that given a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of covers, the transfers satisfy $(gf)_! = g_! f_!$.

(ii) Given a cover $f\colon X \to Y$ of G-spaces, and an H-space Z, we have
$$(f \times 1_Z)_!(a \times c) = f_!(a) \times c,$$
where $a \in K_G(X)$ and $c \in K_H(Z)$.

(iii) Given a pullback square
$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "h"'] & Y \end{tikzcd}$$
where f and f' are coverings, we have
$$h^* f_! = (f')_! g^*.$$

(iv) Given a cover $f\colon X \to Y$, we have the formula
$$f_!(af^*(b)) = f_!(a)b$$
for $a \in K_G(X)$ and $b \in K_G(Y)$. (This property follows from the others.)

1.9. **Additivity property of the total power operation.** We now give an additivity property for the $\mathcal{P}_m$.

(e) We have
$$\mathcal{P}_m(0) = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m > 0. \end{cases}$$

(f) We have
$$\mathcal{P}_m(x+y) = \sum_{i+j=m} \operatorname{ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_m} (\mathcal{P}_i(x) \boxtimes \mathcal{P}_j(y)).$$

This is a consequence of the "distributive law" of tensor products:
$$(V \oplus W)^{\otimes m} \approx \bigoplus_{i+j=m} \binom{m}{i} V^{\otimes i} \otimes W^{\otimes j}.$$
As a Σ_m-representation, $\binom{m}{i} V^{\otimes i} \otimes W^{\otimes j} \approx \operatorname{ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_m} V^{\otimes i} \otimes W^{\otimes j}$.

The maps $\mu\colon K_{\Sigma_i}(X^i) \otimes K_{\Sigma_j}(X^j) \to K_{\Sigma_{i+j}}(X^{i+j})$ obtained by "induction" applied to the Künneth product fit together to make the product $\prod K_{\Sigma_m}(X^m)$ into a commutative ring. Showing this uses the properties of the transfer given above. For instance, associativity amounts to the fact that
$$\begin{tikzcd} K_{\Sigma_i}(X^i) \otimes K_{\Sigma_j}(X^j) \otimes K_{\Sigma_k}(X^k) \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"'] & K_{\Sigma_{i+j}(X^{i+j}) \otimes K_{\Sigma_k}(X^k)} \arrow[d, "\mu"] \\ K_{\Sigma_i}(X^i) \otimes K_{\Sigma_{j+k}}(X^{j+k}) \arrow[r, "\mu"'] & K_{\Sigma_{i+j+k}}(X^{i+j+k}) \end{tikzcd}$$

commutes, which can be proved using properties (i) and (ii).

Commutativity comes from the fact that the subgroups $\Sigma_i \times \Sigma_j$ and $\Sigma_j \times \Sigma_i$ are conjugate inside Σ_{i+j}. In general, given subgroups $H, K \subset G$ and an element $g \in G$ such that $g^{-1}Hg = K$, then

$$\begin{tikzcd} G \times_H X \arrow[r, "{[a,x] \mapsto [ag, g^{-1}x]}"] \arrow[d] & G \times_K X \arrow[d] \\ X \arrow[r, "\mathrm{id}"'] & X \end{tikzcd}$$

is a commutative square of G-spaces; since the horizontal maps are isomorphisms, it is a homotopy pullback. Applying G-equivariant K-theory gives a commutative square

$$\begin{tikzcd} K_H(X) \arrow[d, "\mathrm{ind}_H^G"'] & K_K(X) \arrow[l, "c^*"'] \arrow[d, "\mathrm{ind}_K^G"] \\ K_G(X) & K_G(X) \arrow[l, "\mathrm{id}"] \end{tikzcd}$$

It remains to observe that the map $c^*\colon K_K(X) \to K_H(X)$ is actually identical to the map induced by the group homomorphism $H \to K$ given by $h \mapsto g^{-1}hg$.

Properties (e) and (f) become the fact that

$$\mathcal{P} = (\mathcal{P}_m)\colon K(X) \to \prod K_{\Sigma_m}(X^{\times m})$$

satisfies $\mathcal{P}(0) = 1$ and $\mathcal{P}(x + y) = \mathcal{P}(x)\mathcal{P}(y)$.

We can pull back along diagonals to get

$$P = (P_m)\colon K(X) \to \prod K_{\Sigma_m}(X^m)$$

such that $P(0) = 1$ and $P(x + y) = P(x)P(y)$.

1.10. **Additive power operations and the transfer ideal.** Let $I_{\mathrm{tr}} \subseteq K_{\Sigma_m}(X^m)$ denote the subgroup

$$I_{\mathrm{tr}} = \sum_{\substack{i+j=m \\ 0<i,j<m}} \operatorname{Image}\left[\mathrm{ind}_{\Sigma_i\times\Sigma_j}^{\Sigma_m}\colon K_{\Sigma_i\times\Sigma_j}(X^m) \to K_{\Sigma_m}(X^m)\right].$$

Property (iv) of the transfer shows that I_{tr} is actually an ideal; I'll call it the *proper transfer ideal*.

Now properties (e) and (f) of the total power operation imply that the composite

$$K(X) \xrightarrow{\mathcal{P}_m} K_{\Sigma_m}(X^m) \to K_{\Sigma_m}(X^m)/I_{\mathrm{tr}}$$

is additive. Since $\mathcal{P}_m$ is already multiplicative by properties (b) and (c), we see that the composite is a ring homomorphism.

We can also form a transfer ideal after pullback to diagonal:

$$I_{\mathrm{tr}} = \sum_{\substack{i+j=m \\ 0<i,j<m}} \operatorname{Image}\left[\mathrm{ind}_{\Sigma_i\times\Sigma_j}^{\Sigma_m}\colon K(X) \otimes R(\Sigma_i \times \Sigma_j) \to K(X) \otimes R\Sigma_m\right].$$

In this case, since the transfer map operates only on the representation ring factors, this ideal has the form $K(X) \otimes I'_{\mathrm{tr}}$ where $I'_{\mathrm{tr}} \subset R\Sigma_m$. In the end, we get a ring homomorphism

$$K(X) \xrightarrow{P_m} K(X) \otimes R\Sigma_m \to K(X) \otimes (R\Sigma_m/I_{\mathrm{tr}}).$$

Example 1.11. Let $m = 2$. Then $R\Sigma_2 \approx \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot s$, where 1 and s are trivial and sign representation. The image of $R(\Sigma_1 \times \Sigma_1) \to R\Sigma_2$ is the ideal generated by the element $1+s$. Thus $R\Sigma_2/I_{\mathrm{tr}} \approx \mathbb{Z}$.

Recall that the total power operation is given by

$$P_2(x) = \sigma^2(x) \cdot 1 + \lambda^2(x) \cdot s.$$

The image in $K(X) \otimes R\Sigma_2/I_{\mathrm{tr}}$ is thus

$$\sigma^2(x) - \lambda^2(x).$$

We leave it as an exercise to show that this is equal to the Adams operation $\psi^2(x)$.

Note that there is a map $\pi\colon K_{\Sigma_m}(X^m)/I_{\mathrm{tr}} \to K(X) \otimes (R\Sigma_m/I_{\mathrm{tr}})$. This is because there is a homotopy pullback diagram

$$\begin{tikzcd}
(\Sigma_m/\Sigma_i \times \Sigma_j) \times X \arrow[r] \arrow[d] & (\Sigma_m/\Sigma_i \times \Sigma_j) \times X^m \arrow[d] \\
X \arrow[r] & X^m
\end{tikzcd}$$

so that property (iii) of the transfer gives a commutative diagram

$$\begin{tikzcd}
K(X) \otimes R(\Sigma_i \times \Sigma_j) \arrow[d, "\mathrm{ind}"'] & K_{\Sigma_i \times \Sigma_j}(X^m) \arrow[l, "\delta^*"'] \arrow[d, "\mathrm{ind}"] \\
K(X) \otimes R\Sigma_m & K_{\Sigma_m}(X^m) \arrow[l, "\delta^*"]
\end{tikzcd}$$

and thus restriction along the diagonal sends the transfer ideal of $K_{\Sigma_m}(X^m)$ into that of $K(X) \otimes R\Sigma_m$.

It is natural to ask the question: how far is π from being an isomorphism. It is not always an isomorphism, but it comes close. For instance:

Proposition 1.12. *Let X be a finite discrete space. Then π is an isomorphism.*

Proof. Write $X^m \approx X \amalg (X^m - X)$; this is a decomposition as Σ_m-spaces. I claim that if $0 < i < m$, the projection map

$$p\colon \Big(\coprod_{0<i<m} \Sigma_m/\Sigma_i \times \Sigma_{m-i}\Big) \times (X^m - X) \to (X^m - X)$$

induces a surjective transfer map $\mathrm{ind}\colon K_{\Sigma_i \times \Sigma_j}(X^m - X) \to K_{\Sigma_m}(X^m - X)$. Given this, the result follows easily, since $K_{\Sigma_m}(X^m) \approx K_{\Sigma_m}(X) \oplus K_{\Sigma_m}(X^m - X)$.

I claim that there is a Σ_m-equivariant section of p. To state this, it is useful to interpret the coset space $\Sigma_m/\Sigma_i \times \Sigma_{m-i}$ as the set of *partitions* of $\underline{m} = \{1, \dots, m\}$ into two disjoint subsets of size i and $m - i$ respectively: if $\sigma \in \Sigma_m$, its coset corresponds to the partition

$(\{\sigma(1),\dots,\sigma(i)\},\{\sigma(i+1),\dots,\sigma(m)\}$. Then define $s\colon (X^m - X) \to (\Sigma_m/\Sigma_i \times \Sigma_{m-i}) \times (X^m - X)$ by

$$s(x) = ((A,B),x),$$

where $A = \{\, i \in \underline{m} \mid x_i = x_1 \,\}$, $B = \{\, i \in \underline{m} \mid x_i \neq x_1 \,\}$. It is clear that (A,B) are a partition of $\underline{m}$, and that neither A and B are empty, since the diagonal is excluded.

Now s is itself a covering map, so the identity $ps = \mathrm{id}_{X^m - X}$ and property (i) of the transfer tells us that $p_!$ is surjective. $\square$

2. Representation theory

Let RG denote the complex representation ring of the finite group G. Write

$$R_*G \overset{\text{def}}{=} \hom_{\mathbb{Z}}(RG,\mathbb{Z})$$

for the set of abelian group homomorphisms. Let $R_* = \bigoplus_{m\geq 0} R_*\Sigma_m$. Then R_* is a commutative ring, where the product $R_*\Sigma_k \otimes R_*\Sigma_l \to R_*\Sigma_{k+l}$ is given by the dual to the "restriction" maps $R\Sigma_{k+l} \to R(\Sigma_k \times \Sigma_l) \approx R\Sigma_k \otimes R\Sigma_l$.

Let $\sigma_m \in R_*\Sigma_m$ be the element corresponding to the homomorphism $R\Sigma_m \to \mathbb{Z}$ sending $[V] \mapsto \dim(V^{\Sigma_m})$. Let $\lambda_m \in R_*\Sigma_m$ be the element corresponding to the homomorphism $R\Sigma_m \to \mathbb{Z}$ sending $[V] \mapsto \dim(\hom(\mathrm{sgn}, V))$.

Proposition 2.1. *There is an isomorphism* $R_* \approx \mathbb{Z}[\sigma_1,\sigma_2,\sigma_3,\dots] \approx \mathbb{Z}[\lambda_1,\lambda_2,\lambda_3,\dots]$.

To prove this, we make use of another description of these groups. Let $\mathrm{Sym}_{m,r}[t] \subset \mathbb{Z}[t_1,\dots,t_r]$ denote the set of symmetric homogenous polynomials of degree m on a list of indeterminates $t_1,\dots,t_r$ with $r \geq m$. If $r \geq m$, then this group does not depend on r; in fact, the projection map

$$\mathrm{Sym}_{m,r+1}[t] \to \mathrm{Sym}_{m,r}[t]$$

obtained by sending $t_{r+1} \to 0$ is an isomorphism if $r \geq m$. We write $\mathrm{Sym}_m[t]$ for the limit as $r \to \infty$.

Let $\{V_\pi\}$ denote a complete set of irreducible representations of Σ_m; thus $\{[V_\pi]\}$ is a basis for $R\Sigma_m$. Note that we can write any Σ_m-representation W in the form

$$W \approx \bigoplus_\pi \hom_{\Sigma_m}(V_\pi, W) \otimes V_\pi.$$

We define a function

$$\Delta\colon R_*\Sigma_m \to \mathrm{Sym}_m[t]$$

as follows:

$$\Delta(\alpha) = \sum_\pi \mathrm{Trace}[\hom_{\Sigma_m}(V_\pi, T^{\otimes m})] \cdot \alpha([V_\pi]),$$

where $T = (t_1,\dots,t_r)$ represents a diagonal matrix over $\mathbb{C}$, and $T^{\otimes m}$ represents the induced linear endomorphism of $(\mathbb{C}^r)^{\otimes m}$.

The proof of the proposition follows from the following

Lemma 2.2. *The map* $\Delta\colon R_* \to \mathrm{Sym}[t] \approx \bigoplus \mathrm{Sym}_m[t]$ *is an isomorphism of rings, which sends* σ_m *to the symmetric homegeneous polynomial* $h_m = \sum_{m_1+\cdots+m_r=m} t_1^{m_1}\cdots t_r^{m_r}$*, i.e., the coefficient of* X^m *in* $\prod(1-t_iX)^{-1}$*, and sends* λ_m *to the the elementary symmetric polynomial* c_m*, i.e., the coefficient of* X^m *in* $\prod(1+t_iX)$.

Proof. To show that Δ is a ring homomorphism, I consider a slightly more general construction. Let $H \subseteq \Sigma_m$ be any subgroup, and define

$$\Delta_H \colon R_*H \to \mathrm{Sym}_m[t]$$

by

$$\Delta_H(\alpha) = \sum_\pi \mathrm{Trace}\left[\hom_H(V_\pi, T^{\otimes m})\right] \cdot \alpha([V_\pi]),$$

the sum being taken over irreducible H-representations.

I claim

(1) Let $\alpha \in R_*\Sigma_i$, $\beta \in R_*\Sigma_j$, and consider $\alpha \otimes \beta \in R_*\Sigma_i \otimes R_*\Sigma_j \approx R_*(\Sigma_i \times \Sigma_j)$. Then
$$\Delta_{\Sigma_i \times \Sigma_j}(\alpha \otimes \beta) = \Delta_{\Sigma_i}(\alpha)\Delta_{\Sigma_j}(\beta).$$

(2) If $\alpha \in R_*H$, and $\alpha' \in R_*\Sigma_m$ is the image of α under the map dual to restriction, then
$$\Delta_H(\alpha) = \Delta_{\Sigma_m}(\alpha').$$

Claim (1) follows from the fact that trace is multiplicative with respect to tensor products.

$$\begin{aligned}
\Delta_{\Sigma_i \times \Sigma_j}(\alpha \otimes \beta) &= \sum_{\pi,\rho} \mathrm{Trace}\left[\hom_{\Sigma_i \times \Sigma_j}(V_\pi \otimes V'_\rho, T^{\otimes m})\right] \alpha([V_\pi])\beta([V'_\rho]) \\
&= \sum_{\pi,\rho} \mathrm{Trace}\left[\hom_{\Sigma_i}(V_\pi, T^{\otimes i})\right] \mathrm{Trace}\left[\hom_{\Sigma_j}(V'_\rho, T^{\otimes j})\right] \alpha([V_\pi])\beta([V'_\rho]) \\
&= \Delta_{\Sigma_i}(\alpha)\Delta_{\Sigma_j}(\beta).
\end{aligned}$$

Here V_π and V'_ρ are respectively the irreps of Σ_i and Σ_j, and thus $\{V_\pi \otimes V'_\rho\}$ is the complete collection of irreps for $R(\Sigma_i \times \Sigma_j)$.

Claim (2) follows from Frobenius reciprocity. That is, if $\{V_\pi\}$ and $\{V'_\rho\}$ are irreps for H and G respectively, with $H \subseteq G$, and if

$$\mathrm{ind}_H^G V_\pi = \bigoplus_\rho n_{\pi\rho} V'_\rho,$$

then

$$\mathrm{res}_H^G V'_\rho = \bigoplus_\pi n_{\pi\rho} V_\pi.$$

(The trick is that $n_{\pi\rho} = \dim\left[\hom_H(V_\pi, \mathrm{res}_H^G V'_\rho)\right] = \dim\left[\hom_G(\mathrm{ind}_H^G V_\pi, V'_\rho)\right]$.)

So

$$\begin{aligned}
\Delta_H(\alpha) &= \sum_\pi \mathrm{Trace}\left[\hom_H(V_\pi, T^{\otimes m})\right] \alpha([V_\pi]) \\
&= \sum_\pi \mathrm{Trace}\left[\hom_{\Sigma_m}(\mathrm{ind}_H^{\Sigma_m} V_\pi, T^{\otimes m})\right] \alpha([V_\pi]) \\
&= \sum_{\pi,\rho} n_{\pi\rho}\ \mathrm{Trace}\left[\hom_{\Sigma_m}(V'_\rho, T^{\otimes m})\right] \alpha([V_\pi]) \\
&= \sum_\rho \mathrm{Trace}\left[\hom_{\Sigma_m}(V'_\rho, T^{\otimes m})\right] \alpha'([V'_\rho]) = \Delta_{\Sigma_m}(\alpha').
\end{aligned}$$

That $\Delta(\sigma_m) = h_m$ is a straightforward calculation:

$$\Delta(\sigma_m) = \mathrm{Trace}[\hom_{\Sigma_m}(1, T^{\otimes m})].$$

Likewise, $\Delta(\lambda_m) = c_m$ follows from

$$\Delta(\lambda_m) = \mathrm{Trace}[\hom_{\Sigma_m}(\mathrm{sgn}, T^{\otimes m})].$$

The standard theory of symmetric functions shows that Δ is surjective, and a dimension count gives the result: the ranks of $R\Sigma_m$ and $\mathrm{Sym}_m[t]$ are both equal to the number of integer partitions of m. □

Remark 2.3. The above proof also shows that

$$\sum \sigma_m X^m = \left(\sum \lambda_m(-X)^m\right)^{-1}.$$

The collection of elements $b_{\underline{m}} = \Delta^{-1}[(t_1^{m_1} \cdots t_r^{m_r}) + \cdots]$, each obtained by summing all distinct permutations of the monomial $t_1^{m_1} \cdots t_r^{m_r}$, is indexed by partitions of m, and forms a basis of $R_*\Sigma_m$. We will identify the dual basis in $R\Sigma_m$ below.

Let us study the Σ_m-representation $(\mathbb{C}^r)^{\otimes m}$; we assume $r \geq m$. For $\underline{m} = (m_1, \ldots, m_n)$ with $\sum m_i = m$, let $E_{\underline{m}}$ denote the eigenspace of the action of T on $(\mathbb{C}^r)^{\otimes m}$ corresponding to the eigenvalue $t_1^{m_1} \cdots t_r^{m_r}$. It is easy to see that $E_{\underline{m}}$ is spanned by the orbit under Σ_m of $e_1^{\otimes m_1} \otimes \cdots \otimes e_r^{\otimes m_r}$, where $e_i \in C^r$ are basis vectors, and thus $E_{\underline{m}} \approx \rho_{\underline{m}}$, where this is the induced representation $\rho_{\underline{m}} = \mathrm{ind}_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_r}}^{\Sigma_m}(1)$, and so $(\mathbb{C}^r)^{\otimes m} \approx \bigoplus E_{\underline{m}} \approx \bigoplus \rho_{\underline{m}}$, where the sum is over all sequences. Note that up to isomorphism $\rho_{\underline{m}}$ does not depend on the order of the entries $m_1, \ldots, m_r$, so these may be enumerated by partitions of m, i.e., sequences $m_1, \ldots, m_r$ such that $m_1 \geq m_2 \geq \cdots$ and $\sum m_i = m$.

Thus,

$$\begin{aligned}
\Delta(\alpha) &= \sum_{\pi} \sum_{m_1, \ldots, m_r} \mathrm{Trace}[\hom_{\Sigma_m}(V_\pi, T^{\otimes m} | E_{\underline{m}})] \cdot \alpha([V_\pi]) \\
&= \sum_{m_1, \ldots, m_r} t_1^{m_1} \cdots t_r^{m_n} \sum_{\pi} \dim[\hom_{\Sigma_m}(V_\pi, E_{\underline{m}})] \cdot \alpha([V_\pi]) \\
&= \sum_{m_1, \ldots, m_r} t_1^{m_1} \cdots t_r^{m_r} \cdot \alpha([E_{\underline{m}}]) \\
&= \sum_{m_1 \geq \cdots \geq m_r} \Delta(b_{\underline{m}}) \cdot \alpha([\rho_{\underline{m}}]).
\end{aligned}$$

Thus, the basis $b_{\underline{m}}$ of $R_*\Sigma_m$, is dual to the collection of $\rho_{\underline{m}}$ in $R\Sigma_m$. Thus, the latter form a basis over $\mathbb{Z}$ of $R\Sigma_m$.

Since $\rho_{\underline{m}}$ is a permutation representation, its characters are integer valued. We conclude:

Proposition 2.4. *The characters of representations of symmetric groups are integer valued.*

This calculation also tells us that

$$R\Sigma_m / I_{\mathrm{tr}} \approx \mathbb{Z},$$

since every representation $\rho_{\underline{m}}$ is induced from a subgroup of a $\Sigma_i \times \Sigma_{m-i}$, *except* for the trivial representation $1 = \rho_{(m,0,\ldots,0)}$.

2.5. **Character maps.** Character theory gives special elements of R_*. Given a partition $\underline{m} = (m_1, \ldots, m_r)$ of m let $g_{\underline{m}} \in \Sigma_m$ denote an element which is a product of disjoint cycles of length m_1, m_2, etc. Let $\psi_{m_1,\ldots,m_r} \in R_*\Sigma_m$ be the function given by character evaluation:

$$\psi_{m_1,\ldots,m_r}([V]) = \chi(V, g_{m_1,\ldots,m_r}).$$

Proposition 2.6. *In the ring R_*, we have $\psi_{m_1,\ldots,m_r} = \psi_{m_1} \cdots \psi_{m_r}$.*

Proof. Up to conjugacy, we can identify $g_{\underline{m}}$ with the element $(g_{m_1}, \ldots, g_{m_r})$ of the subgroup $\Sigma_{m_1} \times \cdots \times \Sigma_{m_r} \subset \Sigma_m$.

$$\chi(V, g_{\underline{m}}) = \mathrm{Trace}((g_{m_1}, \ldots, g_{m_r})|V).$$

We need to show that this is equal to the product of traces $\prod_{i=1}^r \mathrm{Trace}(g_i|V)$. It suffices to check this equality for irreducible representations of $\Sigma_{m_1} \times \cdot \Sigma_{m_r}$. Such an irreducible representation has the form $W = W_1 \boxtimes \cdots \boxtimes W_r$, where W_i is an irrep of Σ_{m_i}. The formula follows from the multiplicativity of trace of tensor products. $\square$

Thus, the ring $R_* = \mathbb{Z}[\sigma_m, m \geq 1]$ contains a polynomial subring $\mathbb{Z}[\psi_m, m \geq 1]$ determined by characters. These two sets of elements are related by the following formula.

Proposition 2.7.

$$\sum_{m\geq 0} \sigma_m \cdot X^m = \exp\left[\sum_{m\geq 1} \psi_m \cdot \frac{X^m}{m}\right];$$

here X serves as a formal variable, so that this identity takes place in the ring $\prod_m R_*\Sigma_m \cdot X^m$. Also note $\sigma_0 = 1$.

Proof. Apply Δ to both sides:

$$\Delta\left[\sum_{m\geq 0} \sigma_m \cdot X^m\right] = \sum_{m\geq 0} h_m \cdot X^m = \prod_i (1 - t_i X)^{-1},$$

while, using the fact that $\Delta(\psi_m) = \sum_i t_i^m$, we get

$$\begin{aligned} \Delta \exp\left[\sum_{m\geq 1} \psi_m \cdot \frac{X^m}{m}\right] &= \exp\left[\sum_{m\geq 1}\sum_i t_i^m \cdot \frac{X^m}{m}\right] \\ &= \prod_i \exp\left[\sum_{m\geq 1} t_i^m \frac{X^m}{m}\right] \\ &= \prod_i \exp(-\log(1 - t_i X)) = \prod_i (1 - t_i X)^{-1}. \end{aligned}$$

$\square$

Remark 2.8. Formally, we can write

$$\sum_{m\geq 0} \sigma_m \cdot X^m = \prod_{m\geq 1} (1 - \theta_m X^m)^{-1}$$

for some elements $\theta_m \in R_*\Sigma_m$. It is clear that $\theta_m = \sigma_m + (\text{decomposables})$, so that $R_* \approx \mathbb{Z}[\theta_m, m \geq 1]$. A straightforward exercise shows that

$$\psi_m = \sum_{d|m} d\theta_d^{m/d}.$$

In particular, if p is a prime, $\psi_p = \theta_1^p + p\theta_p$.

2.9. **Operations and the algebraic structure of R_*.** Let's now connect the structure of $\bigoplus R_*\Sigma_m$ to operations. As we've already noted, an element $u \in R_*\Sigma_m$ gives an operation on $K(X)$, by

$$\mathrm{op}_u \colon K(X) \xrightarrow{P_m} K(X) \otimes R\Sigma_m \xrightarrow{\mathrm{id} \otimes u} K(X) \otimes \mathbb{Z} \approx K(X).$$

The elements σ_m and λ_m correspond to the operations of symmetric and exterior powers, respectively.

The map Δ encodes the "splitting principle", in the sense that

$$\mathrm{op}_\alpha(L_1 + \cdots + L_r) = \Delta(\alpha)(L_1, \ldots, L_r),$$

where L_i are line bundles. To prove this, note that instead of using $\mathbb{C}^r$, we could let the operator T act on the sum $V = L_1 \oplus \cdots \oplus L_r$; the eigenspace of $T^{\otimes m}|V^{\otimes m}$ associated to $t_{i_1} \cdots t_{i_m}$ is the line bundle $L_{i_1} \otimes \cdots \otimes L_{i_r}$. The argument is straightforward.

We have that

$$\mathrm{op}_{\alpha+\beta}(x) = \mathrm{op}_\alpha(x) + \mathrm{op}_\beta(x), \qquad \mathrm{op}_{\alpha\cdot\beta}(x) = \mathrm{op}_\alpha(x)\mathrm{op}_\beta(x).$$

The first equality is immediate. The second uses the multiplication map

$$R_*\Sigma_i \otimes R_*\Sigma_j \to R_*\Sigma_{i+j} \quad \text{dual to} \quad R\Sigma_{i+j} \xrightarrow{\mathrm{res}} R\Sigma_i \otimes R\Sigma_j.$$

There are maps

$$\Delta_+ \colon R_* \to R_* \otimes R_* \quad \text{and} \quad \epsilon_0 \colon R_* \to \mathbb{Z}$$

dual to

$$R\Sigma_m \xleftarrow{\mathrm{ind}} R\Sigma_i \otimes R\Sigma_j \quad \text{and} \quad R\Sigma_0 \xleftarrow{1} \mathbb{Z}.$$

These make R_* into a cocommutative coalgebra, and together with the ring structure on R_* make it into a commutative bialgebra. They encode the "additivity" formula, in the sense that

$$\mathrm{op}_\alpha(x+y) = \sum \mathrm{op}_{\alpha'_+}(x)\mathrm{op}_{\alpha''_+}(y) \quad \text{and} \quad \mathrm{op}_\alpha(0) = \epsilon_0(\alpha),$$

where $\Delta_+(\alpha) = \sum \alpha'_+ \otimes \alpha''_+$.

There are maps

$$\Delta_\times \colon R_* \to R_* \otimes R_* \quad \text{and} \quad \epsilon_1 \colon R_* \to \mathbb{Z}$$

dual to

$$R\Sigma_m \xleftarrow{\mu} R\Sigma_m \otimes R\Sigma_m \quad \text{and} \quad R\Sigma_m \xleftarrow{1} \mathbb{Z}.$$

These make R_* into a cocommutative coalgebra, and together with the ring structure on R_* make it into a commutative bialgebra. They encode a "multiplicativity" formula, in the sense that

$$\mathrm{op}_\alpha(xy) = \sum \mathrm{op}_{\alpha'_\times}(x)\mathrm{op}_{\alpha''_\times}(y) \quad \text{and} \quad \mathrm{op}_\alpha(1) = \epsilon_1(\alpha),$$

where $\Delta_\times(\alpha) = \sum \alpha'_\times \otimes \alpha''_\times$.

Example 2.10. We have that

$$\Delta_+(\psi_m) = \psi_m \otimes 1 + 1 \otimes \psi_m, \quad \text{and} \quad \epsilon_0(\psi_m) = 0.$$

Thus the ψ_m correspond to additive operations. Note that since rationally,

$$R_* \otimes \mathbb{Q} \approx \mathbb{Q}[\psi_1, \psi_2, \dots,]$$

we see that the primitives of the Δ_+-coalgebra structure in $R_* \otimes \mathbb{Q}$ are precisely the $\mathbb{Q}$-subspace spanned by the ψ_m's, and therefore the primitives in R_* are the $\mathbb{Z}$-subspace spanned by the ψ_m's. (Because Δ_+ preserves grading, the primitives must have a homogeneous basis, and thus ψ_m generates the primitives in $R_*\Sigma_m \otimes \mathbb{Q}$. The element ψ_m is not divisible in $R_*\Sigma_m$, since $\psi_m(1) = \chi(1, g_m) = 1$.)

Since

$$\Delta_\times(\psi_m) = \psi_m \otimes \psi_m, \quad \text{and} \quad \epsilon_1(\psi_m) = 1,$$

the operations $\psi^m = \mathrm{op}_{\psi_m}$ are ring homomorphisms.

Example 2.11. The symmetric power and exterior power operations, corresponding to σ_m and λ_m, are not additive. However, we can recover their additivity formulae by using the formula

$$\sum \sigma_m X^m = \exp(\sum \psi_m X^m/m).$$

Applying Δ_+ to this gives

$$\begin{aligned}\exp(\sum \psi_m \otimes 1 \cdot X^m/m) \exp(\sum 1 \otimes \psi_m \cdot X^m/m) &= (\sum \sigma_m \otimes 1 \cdot X^m)(\sum 1 \otimes \sigma_m \cdot X^m) \\ &= \sum_{i,j} \sigma_i \otimes \sigma_j \cdot X^{i+j}\end{aligned}$$

In other words,

$$\Delta_+(\sigma_m) = \sum_{i+j=m} \sigma_i \otimes \sigma_j.$$

Similarly,

$$\Delta_+(\lambda_m) = \sum_{i+j=m} \lambda_i \otimes \lambda_j.$$

2.12. **The total power operation in terms of Adams operations.** Let G be a finite group. Let

$$\langle -, - \rangle \colon RG \otimes RG \to \mathbb{Z}$$

denote the bilinear pairing given by the formula

$$\langle V, W \rangle = \tfrac{1}{|G|} \sum_{g \in G} \chi(g, V \otimes W).$$

Note that the right hand side is equal to the trace of the projector $\frac{1}{|G|} \sum_g g$, which is equal to the dimension of $(V \otimes W)^G$ and hence is an integer.

This pairing is non-degenerate; if V_π, V_ρ are irreducible representations, then $\langle V_\pi, \bar{V}_\rho \rangle = 0$ unless $\pi = \rho$, while $\langle V_\pi, \bar{V}_\pi \rangle = 1$. Thus we obtain a duality isomorphism $RG \to R_*G$ sending $V \in RG$ to the function $W \mapsto \langle V, W \rangle$.

Recall that we write $\psi_{m_1,\ldots,m_r} = \psi_{m_1}\cdots\psi_{m_r} \in R_*\Sigma_m$ for the element corresponding to evaluation of a character at the conjugacy class of $g_{\underline{m}}$. We obtain dual elements $(\psi_{m_1}\cdots\psi_{m_r})^* \in R\Sigma_m$.

Proposition 2.13.

$$\sum_{m\geq 0} P_m(x)\cdot X^m = \exp\left[\sum_{m\geq 1}\frac{1}{m}\psi_m^*\cdot\psi^m(x)\cdot X^m\right].$$

This takes place in the ring $\prod_m (K(X)\otimes R\Sigma_m)\cdot X^m$, where the product is given by the usual multiplication in $K(X)$, and by the induction product in $\prod R\Sigma_m$.

Lemma 2.14. (a) *Under the evaluation map $R_*\Sigma_m\otimes R\Sigma_m \to \mathbb{Z}$, we have*

$$(\psi_1^{d_1}\cdots\psi_r^{d_r})\otimes(\psi_1^{e_1}\cdots\psi_r^{e_r})^* \mapsto \begin{cases} c(d_1,\ldots,d_r) & \text{if } d_i = e_i \text{ for all } i,\\ 0 & \text{otherwise.}\end{cases}$$

Here $c(d_1,\ldots,d_r) = (1^{d_1})(2^{d_2})\cdots(r^{d_r})\cdot(d_1!)(d_2!)\cdots(d_r!)$.

(b) *We have*

$$(\psi_1^{d_1}\cdots\psi_r^{d_r})^* = (\psi_1^*)^{d_1}\cdots(\psi_r^*)^{d_r},$$

where the right-hand side is computed using the induction product on $\prod R\Sigma_m$.

Proof. Let $V = (\psi_1^{d_1}\cdots\psi_r^{d_r})^*$ in $R\Sigma_m$; then for any virtual representation W we have

$$\begin{aligned}\langle V,W\rangle &= \frac{1}{|\Sigma_m|}\sum_{\sigma\in\Sigma_m}\chi(\sigma;V)\chi(\sigma;W)\\ &= \chi(g;W)\end{aligned}$$

where g is a product of d_1 1-cycles, d_2 2-cycles, and so on. Thus $\chi(\sigma;V) = 0$ unless σ is conjugate to g, and $\frac{1}{|\Sigma_m|}\sum_\sigma \chi(\sigma;V) = 1$, where the sum is taken over elements of the conjugacy class of g. The number of such elements is $|\Sigma_m/C(g)|$, and so $\chi(g;V) = |\Sigma_m|/|\Sigma_m/C(g)| = |C(g)|$. It's not too hard to show that

$$C(g) \approx \prod_k \left((\mathbb{Z}/k)\wr\Sigma_{d_j}\right),$$

so $|C(g)| = c(\underline{d})$. This proves part (a).

Part (b) follows from "Frobenius reciprocity": if $H\subseteq G$ with $V\in RH$ and $W\in RG$, then

$$\langle \operatorname{ind}_H^G V, W\rangle = \langle V, \operatorname{res}_H^G W\rangle.$$

□

Proof of (2.13). We will use the notation $\psi^{\underline{d}} = \psi_1^{d_1}\cdots\psi_r^{d_r}$, and $c(\underline{d}) = c(d_1,\ldots,d_r)$, and we write $|\underline{d}| = d_1 + 2d_2 + \cdots + rd_r$. First, note that the lemma shows that the elements $(\psi^{\underline{d}})^*/c(\underline{d})$ are a basis of $\bigoplus R\Sigma_m\otimes\mathbb{Q}$ dual to the $\psi^{\underline{d}}$ in $\bigoplus R_*\Sigma_m\otimes\mathbb{Q}$. Thus, we if we evaluate the result $P_m(x)\in K(X)\otimes R\Sigma_m$ of the total power operation against the element $\sum_{|\underline{d}|=m}\frac{1}{c(\underline{d})}(\psi^{\underline{d}})^*\otimes\psi^{\underline{d}}\in R\Sigma_m\otimes R_*\Sigma_m$, we recover $P_m(x)$.

Thus, it suffices to check the identity

$$\sum_{m\geq 0}\left[\sum_{|\underline{d}|=m}\frac{1}{c(\underline{d})}(\psi^{\underline{d}})^*\otimes\psi^{\underline{d}}\right]\cdot X^m=\exp\left[\sum_{m>0}\frac{1}{m}\psi_m^*\otimes\psi_m\cdot X^m\right]$$

in the ring $\prod(R\Sigma_m\otimes R_*\Sigma_m)\cdot X^m$. By part (b) of the lemma we can obtain this from the formal identity

$$\sum_{\underline{d}}\frac{1}{c(\underline{d})}U^{\underline{d}}=\exp\left[\sum_{m>0}\frac{1}{m}U_m\right]\quad\text{in}\quad\mathbb{Q}[U_1,U_2,\dots],$$

which is easily checked. □

3. λ-RINGS

3.1. **Definition of λ-ring.** A **λ-ring** is a commutative ring A equipped with functions $\lambda_m\colon A\to A$, for $m\geq 0$, satisfying the following. (Let x, y represent arbitrary elements of A.)

(1) $\lambda^0(x)=1$.
(2) $\lambda^1(x)=x$.
(3) $\lambda^m(1)=0$ for $m>1$.
(4) $\lambda^m(x+y)=\sum_{i+j=m}\lambda^i(x)\lambda^j(y)$.
(5) $\lambda^m(xy)=F_m(\lambda^1(x),\dots,\lambda^m(x);\lambda^1(y),\dots,\lambda^m(y))$.
(6) $\lambda^m(\lambda^n(x))=G_{m,n}(\lambda^1(x),\dots,\lambda^{mn}(x))$.

Here $F_m\in\mathbb{Z}[X_1,\dots,X_m,Y_1,\dots,Y_m]$ and $G_{m,n}\in\mathbb{Z}[X_1,\dots,X_{mn}]$ are certain polynomials, defined by

$$\sum_m F_m(c_1(\underline{t}),\dots,c_m(\underline{t}),c_1(\underline{u}),\dots,c_m(\underline{u}))\cdot X^m=\prod_{i,j}(1+t_iu_jX),$$

and

$$\sum_m G_{m,n}(c_1(\underline{t}),\dots,c_{mn}(\underline{t}))\cdot X^m=\prod_{i_1<\dots<i_n}(1+t_{i_1}\dots t_{i_n}X),$$

where $c_i(\underline{t})$ is the ith elementary symmetric polynomial in the t_js. It is now straightforward to show that power operations give $K(X)$ the structure of a λ-ring. (This definition is from [AT69], where these are called *special λ-rings*.)

3.2. **Free λ-ring.** The ring R_* is itself a λ-ring, and is in fact the free λ-ring on one generator. I'll sketch the proof that R_* is a λ-ring.

Consider $R(S^1)^r\approx\mathbb{Z}[t_1^{\pm},\dots,t_r^{\pm}]$, which is a λ-ring. The polynomial subring $A_r\overset{\text{def}}{=}\mathbb{Z}[t_1,\dots,t_r]\subset R(S^1)^r$ is seen to be a sub-λ-ring, since λ_m applied to a sum of lines is still a sum of lines. Let A denote the inverse limit of the tower of maps $A_{r+1}\to A_r$ obtained by sending $t_{r+1}\mapsto 0$. The ring $\mathrm{Sym}[t]$ of the previous section is a subring of A, and is clearly a λ-subring. Since $\Delta\colon R_*\to\mathrm{Sym}[t]$ is an isomorphism, we get a λ-ring structure on R_*.

Note that if we set $x=\lambda_1\in R_*$, then $\lambda^m(x)=\lambda_m$. In particular, R_* is generated as a λ-ring by the one element x.

To show that R_* is the free λ-ring on one generator, we must show that for any λ-ring A and element $a\in A$, there is a unique λ-ring homomorphism $\phi\colon R_*\to A$ sending $x\mapsto a$. The only candidate for such a homomorphism is the commutative ring homomorphism sending $\lambda_m = \lambda^m(x) \mapsto \lambda^m(a)$; since R_* is polynomial in the λ_m's, this is well-defined as a map of commutative rings.

To check that ϕ is a λ-ring homomorphism, we must check that for each $y \in R_*$, that $\lambda^m(\phi(y)) = \phi(\lambda^m(y))$. It is clear using the axioms (1)–(6), that for any polynomial $P(X_1, \dots, X_r)$ with integer coefficients, there exists a polynomial $Q(X_1, \dots, X_s)$ such that for any λ-ring A and any $a \in A$,

$$\lambda^m P(\lambda^1(a), \dots, \lambda^r(a)) = Q(\lambda^1(a), \dots, \lambda^s(a)).$$

Thus, if $y = P(\lambda_1, \dots, \lambda_r) = P(\lambda^1(x), \dots, \lambda^r(x)) \in R_*$, we have

$$\begin{aligned}
\lambda^m(\phi(y)) &= \lambda^m P(\lambda^1(a), \dots, \lambda^r(a)) \\
&= Q(\lambda^1(a), \dots, \lambda^s(a)) \\
&= \phi(Q(\lambda^1(x), \dots, \lambda^s(x))) \\
&= \phi(\lambda^m(P(\lambda^1(x), \dots, \lambda^r(x)))) = \phi(\lambda^m(y)).
\end{aligned}$$

3.3. **Wilkerson criterion.** To talk about λ-rings, we must use the λ^m-operations; however, it is much more convenient to use the Adams operations. These are ring homomorphisms $\psi^m \colon A \to A$, for $m \geq 1$, with the following properties

(1) $\psi^1(x) = x$.
(2) $\psi^m(\psi^n(x)) = \psi^{mn}(x)$.
(3) $\psi^p(x) \equiv x^p \mod p$ for primes p.

Note that if A is a torsion free λ-ring, then the λ^m operations are completely determined by the ψ^m's; this is because $R_* \otimes \mathbb{Q} \approx \mathbb{Q}[\psi^m, m \geq 1]$. Conversely, one may wonder whether, given a torsion free ring A with ring homomorphisms $\psi^m \colon A \to A$, one can extend this to the structure of a λ-ring. Conditions (1) and (2) above are clearly necessary; the question is what additional "congruence" conditions such as (3) are also necessary. The answer was provided in [Wil82].

Theorem 3.4 (Wilkerson)**.** *Let A be a torsion free ring equipped with ring homomorphisms $\psi^m \colon A \to A$ satisfying (1)–(3) above. Then there is a unique structure of λ-ring on A compatible with these.*

Proof. First, note that if $\mathbb{Q} \subset A$, then we can define operations λ^m using the fact that every $\lambda_m \in R_*$ can be written as a polynomial in the ψ^m's with rational coefficients. It is then "straightforward" to check that A is a λ-ring.

In general, if A is torsion free then $A \subset A \otimes \mathbb{Q}$, and the problem reduces to showing that A is closed under the λ-operations. We'll make use of the fact that $R_* \approx \mathbb{Z}[\theta_m, m \geq 0]$, and actually show that A is closed under the θ-operations.

Given $a \in A$, we will show that $\theta^m(a) \in A$ by induction on m. It will be enough to show that $\theta^m(a) \in A \otimes \mathbb{Z}_{(p)}$ for all primes p.

Recall that

$$\begin{aligned}
\psi^m(a) &= \sum_{d|m} d\theta^d(a)^{m/d} \\
&= \theta^1(a)^m + \cdots + m\theta^m(a).
\end{aligned}$$

If $(p,m)=1$ it follows that $\theta^m(a)\in A\otimes\mathbb{Z}_{(p)}$, so suppose $m=np^k$, $k>0$. Separate the factors d of m into those for which $d|(m/p)$ and those for which $p^k|d$:

$$\begin{aligned}
\psi^m(a) &= \sum_{d|(m/p)} d\,\theta_d(a)^{p\frac{m}{pd}} + \sum_{e|n} p^k e\,\theta_{p^k e}(a)^{n/e} \\
&\equiv \sum_{d|(m/p)} d\,\psi^p(\theta_d(a))^{\frac{m}{pd}} + \sum_{e|n} p^k e\,\theta_{p^k e}(a)^{n/e} \mod p^k A_{(p)} \\
&= \psi^p\left[\sum_{d|(m/p)} d\,\theta_d(a)^{\frac{m}{pd}}\right] + \sum_{e|n} p^k e\,\theta_{p^k e}(a)^{n/e} \\
&= \psi^p\psi^{m/p}(a) + \sum_{e|n} p^k e\,\theta_{p^k e}(a)^{n/e}.
\end{aligned}$$

The congruence of the second line follows from the fact that $x^p\equiv\psi^p(x) \mod pA_{(p)}$, and therefore that $p^i\,x^{p^j}\equiv p^i\,\psi^p(x)^{p^{j-1}} \mod p^{i+j}A_{(p)}$ if $j>0$.

Since $\psi^m(a)$ and $\psi^p(\psi^{m/p}(a))$ are in A, the sum term must lie in $p^kA_{(p)}$, and therefore

$$\sum_{e|n} e\,\theta_{p^k e}(a)^{n/e}\in A_{(p)}.$$

By induction, all of these terms except $\theta_{p^k n}(a)^{n/n}=\theta_m(a)$ are contained in A, from which we conclude that $\theta_m(a)\in A_{(p)}$, as desired. □

4. Power operations in non-zero degrees

4.1. **The relative total power operation.** Given a pair (X,A) of spaces, we can define a total power operation on relative k-theory:

$$\mathcal{P}_m\colon K(X,A)\to K_{\Sigma_m}((X,A)^m).$$

Here $(X,A)^m=(X^m,Y)$, where Y is the space of tuples $(x_1,\dots,x_m)\in X^m$ such that at least one $x_i\in A$. We can write this in terms of reduced K-theory:

$$\mathcal{P}_m\colon \widetilde{K}(X/A)\to\widetilde{K}_{\Sigma_m}((X/A)^{\wedge m}).$$

4.2. **Power operations on elements of odd degree.** Let $q\geq 0$, and let X be an unbased space. The composite

$$K^{-q}(X)\approx\widetilde{K}^0(S^q\wedge X_+)\xrightarrow{\mathcal{P}_m}\widetilde{K}^0_{\Sigma_m}((S^q)^{\wedge m}\wedge(X^m)_+)\xrightarrow{\delta^*}\widetilde{K}^0_{\Sigma_m}((S^q)^{\wedge m}\wedge X_+)$$

gives a total power operation on elements of degree q; note that the

We would like to understand what the target of this map is. To do so, we must first calculate $\widetilde{K}^*_{\Sigma_m}((S^q)^{\wedge m})$, where Σ_m permutes the sphere factors. I'll work this out in the case of $m=2$, $q=1$.

The Σ_2-equivariant space $S^1\wedge S^1$ contains a diagonal copy of the circle, which I'll denote by S^1_Δ, which is fixed by Σ_2. There is a pushout diagram

$$\begin{tikzcd} S^1\vee S^1 \arrow[r] \arrow[d] & D^2\vee D^2 \arrow[d] \\ S^1_\Delta \arrow[r] & S^1\wedge S^1 \end{tikzcd}$$

In the spaces in the top row, Σ_2 acts by permuting the two wedge factors; in general, $\widetilde{K}^*_{\Sigma_2}(X\vee X)\approx\widetilde{K}^*(X)$ when Σ_2 acts this way. In particular, $\widetilde{K}^*(D^2\vee D^2)\approx 0$. Since S^1_Δ is fixed by Σ_2, we know that the evident map $\widetilde{K}^*(S^1)\otimes_{\mathbb{Z}} R\Sigma_2\to\widetilde{K}^*_{\Sigma_2}(S^1_\Delta)$ is an isomorphism. Thus the Mayer-Vietoris sequence has the form

$$\widetilde{K}^{-1}_{\Sigma_2}(S^1\wedge S^1)\to\widetilde{K}^{-1}(S^1)\otimes R\Sigma_2\to\widetilde{K}^{-1}(S^1)\to\widetilde{K}^0_{\Sigma_2}(S^1\wedge S^1)\to\widetilde{K}^0(S^1)\otimes R\Sigma_2\to\widetilde{K}^0(S^1)\to$$

which becomes

$$\to\widetilde{K}^{-1}_{\Sigma_2}(S^1\wedge S^1)\to R\Sigma_2\xrightarrow{d}\mathbb{Z}\to\widetilde{K}^0_{\Sigma_2}(S^1\wedge S^1)\to 0\to 0\to,$$

where $d\colon R\Sigma_2\to\mathbb{Z}$ is the map induced by the inclusion $\{e\}\subset\Sigma_2$, i.e., the map sending a virtual representation to its dimension. Since d is surjective, we get

$$\widetilde{K}^q_{\Sigma_2}(S^1\wedge S^1)\approx\begin{cases}0 & \text{if } q \text{ even}\\ \mathbb{Z} & \text{if } q \text{ odd.}\end{cases}$$

In particular, $\widetilde{K}^*_{\Sigma_2}(S^1\wedge S^1)$ is free of rank one (on a generator of odd degree) as a module over K_*. Thus

$$\widetilde{K}^*_{\Sigma_2}(S^1\wedge S^1)\otimes_{K_*}K^*(X)\to\widetilde{K}^*_{\Sigma_2}(S^1\wedge S^1\wedge X^+)$$

is an isomorphism, so

$$\widetilde{K}^q_{\Sigma_2}(S^1\wedge S^1\wedge X^+)\approx K^{q-1}(X).$$

It is interesting to compare the total power square with the cup square. Consider the commutative diagram

$$\begin{tikzcd} & \widetilde{K}^0_{\Sigma_2}(S^1\wedge S^1\wedge(X^2)_+) \arrow[r, "\delta^*"] \arrow[d] & \widetilde{K}^0_{\Sigma_2}(S^1\wedge S^1\wedge X_+) \arrow[d, "\mathrm{res}"] \\ \widetilde{K}^0(S^1\wedge X_+) \arrow[r] \arrow[ur, "\mathcal{P}_2"] & \widetilde{K}^0(S^1\wedge S^1\wedge(X^2)_+) \arrow[r, "\delta^*"'] & \widetilde{K}^0(S^1\wedge S^1\wedge X_+) \end{tikzcd}$$

The bottom line is the cup square; using the suspension isomorphisms, it is the map $K^{-1}(X)\to K^{-2}(X)$ sending $x\mapsto x^2$. Of course, we must have $2x^2=0$ by the usual argument.

The vertical maps are basically induced by restriction of groups. In the right-hand column, both ends of the restricition map are cohomology theories of X, and so the map is the degree 0 part of

$$\widetilde{K}^*_{\Sigma_2}(S^1\wedge S^1)\otimes_{K_*}K^*(X)\xrightarrow{\mathrm{res}\otimes\mathrm{id}}\widetilde{K}^*(S^1\wedge S^1)\otimes_{K_*}K^*(X).$$

The map

$$\mathrm{res}\colon\widetilde{K}^*_{\Sigma_2}(S^1\wedge S^1)\to\widetilde{K}^*(S^1\wedge S^1)$$

is equal to zero, since either the source or target is 0 in any given degree. Thus we have proved the surprising fact

Proposition 4.3. *If $x \in K^{-1}(X)$, then $x^2 = 0$.*

I first learned about this fact from Jim McClure.

5. Algebraic theories

A **theory** is a category T with object set $\operatorname{obj} T = \{T^0, T^1, T^2, \dots\}$, together with **projection maps** $\pi_i \colon T^n \to T^1$ for all $n \geq 0$, $1 \leq i \leq n$, such that $T(T^k, T^n) \xrightarrow{\pi_i} \prod_{i=1}^n T(T^k, T^1)$ is a bijection for all k and n. That is, T^n is isomorphic to the n-fold product of T^1's.

A **model** (or **algebra**) of the theory T is a functor $A \colon F \to \mathrm{Set}$ which preserves finite products.

Example 5.1. Let F be the full subcategory of the category Grp of groups having as objects $\{F_0, F_1, F_2, \dots\}$, where $F_n = \langle x_1, \dots, x_n \rangle$ is a free group on n generators, and let $T = F^{\mathrm{op}}$. Then T is a theory (the theory of groups). A model for T amounts to a group. A group G determines a model A by $A(T^n) = \mathrm{Grp}(F_n, G) \approx G^n$. A model A determines a group G by $G = A(T^1)$; the identity of G is the image of the map $* = A(T^0) \to A(T^1)$, and the product on G is given by $A(T^1) \times A(T^1) \approx A(T^2) \to A(T^1)$, where the morphism $T^2 \to T^1$ corresponds to the homomorphism $F_1 \to F_2$ sending $x_1 \mapsto x_1 x_2$.

Similarly, theories can be constructed for the categories of abelian groups, associative rings, commutative rings, lie algebras, etc.

Example 5.2. Let R be a commutative ring. Write C_R for the theory of commutative R-algebras. Thus, in the notation of the previous example, $F_n \approx R[x_1, \dots, x_n]$.

Write Model_T for the category of models for the theory T.

A model A for a theory T can be thought of as the set $X = A(T^1)$, together with operations $\phi_f \colon X^{\times m} \to X^{\times n}$ corresponding to each $f \in T(T^m, T^n)$. Thus, X is the **underlying set** of the model A. We write $U_T \colon \mathrm{Model}_T \to \mathrm{Set}$ for the functor associating to a model its underlying set.

A **free model** on n-generators is the model $F_T(n)$ defined by

$$F_T(n)(T^k) \approx T(T^n, T^k).$$

It has the property that

$$\mathrm{Model}_T(F_T(n), A) \approx A(T^n) \approx \mathrm{Set}(\underline{n}, U_T(A)).$$

Note that

$$\mathrm{Model}_T(F_T(n), F_T(m)) \approx T(T^m, T^n).$$

To each projection map $\pi_i \colon T^n \to T^1$, $i = 1, \dots, n$, there is a corresponding element $x_i \in F_T(n)$; the elements $x_1, \dots, x_n$ are the **generators** of the free algebra $F_T(n)$.

More generally, given any set S, define $F_T(S)$ to be the functor $T \to \mathrm{Set}$ obtained by taking the colimit of $F_T(K)$ over all finite subsets $K \subseteq S$. Since directed colimits of sets commute with finite products, $F_T(S)$ is a model. We have an adjoint pair

$$F_T \colon \mathrm{Set} \rightleftarrows \mathrm{Model}_T : U_T.$$

Lemma 5.3. *Given models A and B of a theory T, there is a bijective correspondence between* $\mathrm{Model}_T(A,B)$ *and the set of maps* $f \in \mathrm{Set}(U_TA, U_TB)$ *making the diagram*

$$\begin{tikzcd} U_TF_TU_TA \arrow[r, "U_TF_Tf"] \arrow[d, "U_T\epsilon A"'] & U_TF_TU_TB \arrow[d, "U_T\epsilon B"] \\ U_TA \arrow[r, "f"'] & U_TB \end{tikzcd}$$

commute.

Proof. The correspondence sends $g \in \mathrm{Model}_T(A,B)$ to $f = U_Tg$. A morphism of models is determined uniquely by the function it induces on the underlying set, so the correspondence is injective. Thus, it remains to show that given $f\colon U_TA \to U_TB$ making the diagram commute, we can lift it to a morphism of models. Set $g(T^n)\colon A(T^n) \to B(T^n)$ to be $(f,\dots,f)$ using the identifications $A(T^n) \approx (U_TA)^{\times n}$ and $B(T^n) \approx (U_TB)^{\times n}$. I will show that the maps g give a natural transformation of functors $T \to \mathrm{Set}$. Thus, given $t \in T(T^m, T^n)$ and $a \in A(T^m)$, we want to show that $g(t(a)) = t(g(a))$ in $B(T^n)$. Let $\alpha\colon m \to U_TA$ represent a; then $f \circ \alpha\colon m \to U_TB$ represents $g(a)$. Let $\tau\colon n \to U_TF_T(m)$ represent t. In the commutative diagram

$$\begin{tikzcd} n \arrow[r, "\tau"] \arrow[dr, "\omega"'] & U_TF_T(m) \arrow[d, "U_TF_T\alpha"] \arrow[dr, "U_TF_T(f\alpha)"] & \\ & U_TF_TU_TA \arrow[r, "U_TF_Tf"] \arrow[d, "U_T\epsilon A"'] & U_TF_TU_TB \arrow[d, "U_T\epsilon B"] \\ & U_TA \arrow[r, "f"'] & U_TB \end{tikzcd}$$

the map ω represents $t(a)$. Following the diagram gives the desired result. $\square$

Proposition 5.4. *The category* Model_T *is complete and cocomplete. Limits, filtered colimits, and reflexive coequalizers are calculated pointwise.*

Proof. Since limits, filtered colimits, and reflexive coequalizers in the category of sets all commute with finite products, the second statement is clear. It thus remains to construct arbitrary colimits in Model_T. Given a functor $A\colon I \to \mathrm{Model}_T$, where I is a small category, consider the reflexive pair

$$F_T(\mathrm{colim}_{i\in I} U_TF_TU_TA_i) \begin{smallmatrix}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{smallmatrix} F_T(\mathrm{colim}_{i\in I} U_TA_i)$$

where the top arrow is obtained by applying $F_T \operatorname{colim}_I U_T$ to the counit of the adjunction $F_TU_TA_i \to A_i$; the middle arrow is obtained by applying $F_T \operatorname{colim}_I$ to the unit of the adjunction $U_TA_i \to U_TF_T(U_TA_i)$; the bottom arrow is adjoint to the map of sets $\operatorname{colim}_I U_TF_TA_i \to U_TF_T(\operatorname{colim}_I U_TA_i)$ obtained from the family of maps you get by applying U_TF_T to the family of maps $U_TA_i \to \operatorname{colim}_I U_TA_i$. The coequalizer exists in Model_T, and it is straightforward (using the lemma) to check that it is the colimit of the diagram $A\colon I \to \mathrm{Model}_T$. $\square$

The functor $M = U_T F_T \colon \mathrm{Set} \to \mathrm{Set}$ is a *monad* on Set, i.e., a monoid object in the monoidal category of endofunctors of Set. There is an associated category $\mathrm{Algebra}_M$ of M-algebras, objects of which are pairs (X, ϕ) consisting of a set X and a map $\phi \colon MX \to X$ making suitable diagrams commute.

Corollary 5.5. *The functor $i \colon \mathrm{Model}_T \to \mathrm{Algebra}_M$ sending A to $(U_T A, M(U_T A) = U_T F_T U_T A \xrightarrow{U_T \epsilon A} U_T A)$ is an equivalence of categories.*

Proof. That i is fully faithful is the content of (5.3). Thus it suffices to show that i is essentially surjective. Let (X, ϕ) be an M-algebra. Consider the diagram of T-models

$$F_T U_T F_T X \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} F_T X$$

where the two right-facing maps are $\epsilon_{F_T X}$ and $F_T \phi$ respectively, and the left-facing map is $F_T \eta_X$. This is a reflexive pair, and thus by (5.4) has a cokernel C. Furthermore, after applying U_T, we get a coequalizer of sets

$$U_T F_T U_T F_T X \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} U_T F_T X \longrightarrow U_T C$$

On the other hand, the map $\phi \colon U_T F_T X \to X$ is evidently part of a split coequalizer of sets. Therefore, we have a canonical identification of X with $U_T C$. It is then a straightforward exercise to show that X and $U_T C$ are isomorphic as M-algebras. $\square$

5.6. **Graded theories.** There is a variant of the notion of a theory, called **graded theories** (or, after Boardman, **theories with colors**). Let C be some fixed set of **colors**. Let $\mathbb{N}[C]$ denote the free commutative monoid on the set C. A C-graded theory T is a category with object set $\{T^d\}_{d \in \mathbb{N}[C]}$, together with, for each $d = \sum_{c \in C} d_c[c] \in \mathbb{N}[C]$, a specified identification of T^d with the categorical product $\prod (T^{[c]})^{\times d_c}$. A **model** for T is a product preserving functor $A \colon T \to \mathrm{Set}$.

Instead of a single underlying set, a T-model A has an underlying *graded* set, graded by C; for $c \in C$, the cth graded piece of the model is $A(T^{[c]})$. There are adjoint functors

$$F_T \colon \mathrm{Set}^C \rightleftarrows \mathrm{Model}_T : U_T,$$

as before, and Model_T is equivalent to the category of algebras of the evident monad on Set^C. Usually I'll omit the forgetful functor, and I'll write A_c for the cth graded piece of the model.

Let $[c]$ denote the graded set with one element in grading $c \in C$, and empty in other gradings. Then an element of

$$T(T^{[c_1]+\cdots+[c_m]}, T^{[d]}) = F_T([c_1] \amalg \cdots \amalg [c_m])_d$$

gives rise to a natural function $A_{c_1} \times \cdots \times A_{c_m} \to A_d$ on T-models A.

Example 5.7. Let R_* be a graded commutative ring. Let $T = C_{R_*}$ denote the theory of graded commutative R_*-algebras. This is a $\mathbb{Z}$-graded theory. The free object $F_T([c_1] + \cdots + [c_m]) \approx R_*[x_1, \ldots, x_m]$, where generator x_i is in degree d_i.

Example 5.8. Fix a commutative ring R. Let $T = O_{HR}$ be the following theory (with colors $\mathbb{Z}$).

$$T(O_{HR}^{[c_1]+\cdots+[c_m]}, O_{HR}^{[d_1]+\cdots+[d_n]}) = [K(R, c_1) \times \cdots \times K(R, c_m), K(R, d_1) \times \cdots \times K(R, d_n)],$$

where we use homotopy classes of maps; use the convention $K(R, c) = *$ for $c < 0$.

If $R = \mathbb{F}_p$, then $\mathrm{Model}_{O_{H\mathbb{F}_p}}$ is the category of unstable algebras over the mod-p Steenrod algebra.

It is a useful exercise to determine what $\mathrm{Model}_{O_{H\mathbb{Q}}}$ looks like. (It is closely related, but not equal, to the theory of graded commutative $\mathbb{Q}$-algebras.)

Example 5.9. More generally, any spectrum E gives rise to a $\mathbb{Z}$-graded theory $T = O_E$, with

$$T(T^{[c_1]+\cdots+[c_m]}, T^{[d_1]+\cdots+[d_n]}) = [\Omega^\infty\Sigma^{c_1}E \times \cdots \times \Omega^\infty\Sigma^{c_m}E, \Omega^\infty\Sigma^{d_1}E \times \cdots \times \Omega^\infty\Sigma^{d_n}E].$$

6. Commutative operation theories

A **morphism of theories** is a functor $\phi\colon T \to T'$ such that $\phi(T^k) = {T'}^k$ and such that projection maps are sent to the corresponding projection maps.

Given a morphism of theories, there is a corresponding functor $\phi^*\colon \mathrm{Model}_{T'} \to \mathrm{Model}_T$ given by $A \mapsto A \circ \phi$. The functor ϕ^* preserves the underlying set, in the sense that there is a natural isomorphism $U_{T'} \approx U_T\phi^*$.

Example 6.1. Let S be the theory with $S(S^k, S^n) = \mathrm{Set}(n, k)$, so that $\mathrm{Model}_S \approx \mathrm{Set}$. This is the initial theory, in the sense that for every theory T there is a unique morphism of theories $\phi\colon S \to T$. The functor $\mathrm{Model}_T \to \mathrm{Model}_S \approx \mathrm{Set}$ is equivalent to U_T.

An **commutative operation theory** is a triple (T, R, ϕ) consisting of a theory T, a commutative ring R, and a morphism $\phi\colon C_R \to T$ of theories (where C_R is the theory of commutative R-algebras), such that the functor $\phi^*\colon \mathrm{Model}_T \to \mathrm{Model}_{C_R}$ commutes with finite coproducts.

In other words, every T-model has an "underlying" structure of a commutative R-algebra, and coproducts in Model_T are computed by tensor products over R.

Remark 6.2. It is an immediate consequence of (5.4) that if ϕ^* preserves finite coproducts, then in fact it preserves all colimits (as well as all limits). In particular, the initial object in Model_T has "underlying" ring R.

We will write $R\{x_1, \dots, x_n\}$ for the generic example of a free T-model on n-generators. By the above, we have that g

$$R\{x_1, \dots, x_n\} \approx R\{x_1\} \otimes_R \cdots \otimes_R R\{x_n\}.$$

Example 6.3. Clearly, the theory C_R admits a trivial structure of a COT, with $\phi\colon C_R \to C_R$ being the identity functor.

Example 6.4. Let R be a commutative ring, and let G be a monoid. Consider the category having as objects: commutative R-algebras A equipped with a G-action; and morphisms: homomorphisms of R-algebras which commute with the G-action. This category is a category of models over a COT (T, R, ϕ); the free T-algebra on n generators is isomorphic to the polynomial algebra

$$R[x_i^g; \quad i = 1, \dots, n, \quad g \in G],$$

where the G action is given by $g(x_i^h) = x_i^{gh}$.

Example 6.5. Let R be a commutative ring. Consider the category having as objects: commutative R-algebras A equipped with an R-derivation, i.e., with an R-module map $\partial\colon A \to A$ such that

$$\partial(xy) = \partial(x)\, y + x\, \partial(y).$$

This category is a category of models over a COT; the free T-model on one generator is isomorphic to the polynomial algebra

$$R[x, \partial(x), \partial^2(x), \dots].$$

Example 6.6. Let (T, R, ϕ) be a COT, and let A be a T-model. The category $A\backslash\mathrm{Model}_T$ of T-models under T is equivalent to a category Model_{T_A} of models of a theory T_A, and there is a COT (T_A, A, ϕ'). We have

$$F_{T_A}(n) \approx F_T(n) \otimes_R A \approx R\{x_1, \dots, x_n\} \otimes_R A.$$

Given an element $f \in F_T(n)$, a T-model A, and elements $a_1, \dots, a_n \in A$, let $f{\propto}(a_1, \dots, a_n)$ denote the image of f under the map $F_T(n) \to A$ sending x_i to a_i. We call the function $f{\propto}\colon A^{\times n} \to A$ the **operation** associated to the element f.

In a COT, the isomorphism

$$F_T(n) \approx F_T(1) \otimes_R \cdots \otimes_R F_T(1)$$

allows us to write an arbitrary element $f \in F_T(n)$ in the form

$$f = \sum_j f_j^{(1)} \otimes \cdots \otimes f_j^{(n)}$$

for some elements $f_j^{(i)} \in F_T(1)$. Therefore, we can always write

$$f{\propto}(a_1, \dots, a_n) = \sum_j (f_j^{(1)}{\propto}a_1) \cdots (f_j^{(n)}{\propto}a_n).$$

In other words, *the $n$ary operations on a T-model can be written in terms of unary operations.*

The unitary operations satisfy certain rules:

$$\begin{aligned} x{\propto}a &= a \\ (f{\propto}g){\propto}a &= f{\propto}(g{\propto}a) \\ (f+g){\propto}a &= f{\propto}a + g{\propto}a \\ (fg){\propto}a &= (f{\propto}a)(g{\propto}a) \\ r{\propto}a &= r \end{aligned}$$

for $r \in R \subseteq R\{x\}$. That is, the binary operation $\propto\colon R\{x\} \times A \to A$ is an R-algebra homomorphism with respect to the left-hand input, and defines an associative monoid structure on $R\{x\}$ together with an action on any T-model A.

6.7. **Additive and multiplicative elements.** An element $f \in R\{x\}$ is **additive** if

$$f{\propto}(a+b) = f{\propto}a + f{\propto}b \quad \text{and} \quad f{\propto}0 = 0$$

for any T-model A and all $a, b \in A$.

Proposition 6.8. *There is a coassociative and cocommutative coalgebra structure (the* ***additive coalgebra****) on $R\{x\}$ with comultiplication given by the T-model map $\Delta_+\colon R\{x\} \to R\{x_1, x_2\}$ sending $x \mapsto x_1 + x_2$ and counit given by the T-model map $\epsilon_0\colon R\{x\} \to R$ sending $x \mapsto 0$. The primitives for this coalgebra structure are precisely the additive elements.*

Proof. The statement about the coalgebra is clear. If $f \in R\{x\}$, then it is straightforward to show that for any T-model A, and $a_1, a_2 \in A$,

$$f\propto(a_1 + a_2) = \sum (f'_+\propto a_1)(f''_+\propto a_2)$$

where $\Delta_+ f = \sum f'_+ \otimes f''_+$, and

$$f\propto 0 = \epsilon_0 f.$$

□

An element $f \in R\{x\}$ is **multiplicative** if

$$f\propto(ab) = (f\propto a)(f\propto b) \quad \text{and} \quad f\propto 1 = 1$$

for any T-model A and all $a, b \in A$.

Proposition 6.9. *There is a coassociative and cocommutative coalgebra structure (the* ***multiplicative coalgebra****) on $R\{x\}$ with comultiplication given by the T-model map $\Delta_\times\colon \mathbb{R}\{x\} \to R\{x_1, x_2\}$ sending $x \mapsto x_1x_2$ and counit given by the T-model map $\epsilon_1\colon R\{x\} \to R$ sending $x \mapsto 1$. The grouplike elements for this coalgebra structure are precisely the multiplicative elements.*

Proof. The statement about the coalgebra is clear. If $f \in R\{x\}$, then it is straightforward to show that for any T-model A, and $a_1, a_2 \in A$,

$$f\propto(a_1a_2) = \sum (f'_\times\propto a_1)(f''_\times\propto a_2)$$

where $\Delta_\times f = \sum f'_\times \otimes f''_\times$, and

$$f\propto 1 = \epsilon_1 f.$$

□

6.10. **Ideal theory.** Let A be a T-model, and $I \subseteq A$ an ideal. We say that I is an **invariant ideal** if

$$f\propto u - f\propto 0 \in I \qquad \text{for all } f \in R\{x\},\ u \in I.$$

Proposition 6.11. *If $\phi\colon A \to B$ is a homomorphism of T model, then $\ker\phi$ is an invariant ideal. Conversely, if $I \subseteq A$ is an invariant ideal, then A/I has a unique structure of T-model making $A \to A/I$ a map of T-models.*

Proof. The first part is immediate. To prove the second part, we need to show that for any $f \in R\{x_1, \dots, x_n\}$, the induced function $f\propto\colon A^{\times n} \to A$ descends to a well-defined function $(A/I)^{\times n} \to A/I$ on quotients. Since

$$f\propto(a_1, \dots, a_n) = \sum_j (f_j^{(1)}\propto a_1)\cdots(f_j^{(n)}\propto a_n)$$

it is enough to show that each $f \in R\{x\}$ the induced function $f\propto A \to A$ descends to $A/I \to A/I$.

Given $a \in A$ and $u \in I$, we have

$$\begin{aligned} f{\propto}(a+u) - f{\propto}a &= f{\propto}(a+u) - f{\propto}(a+0) \\ &= \sum (f'_+{\propto}a)(f''_+{\propto}u) - (f'_+{\propto}a)(f''_+{\propto}0) \\ &= \sum (f'_+{\propto}a)(f''_+{\propto}u - f''_+{\propto}0) \in I, \end{aligned}$$

using that $\Delta_+ f = \sum f'_+ \otimes f''_+$. □

Proposition 6.12. *If $I, J \subseteq A$ are invariant ideals, then so are $I + J, IJ \subseteq A$.*

Proof. First, I prove two claims. For $u, v \in A$, and $I, J \subseteq A$ ideals such that $f{\propto}u - f{\propto}0 \in I$ and $f{\propto}v - f{\propto}0 \in J$, then

(1) $f{\propto}(u+v) - f{\propto}0 \in I + J$.
(2) $f{\propto}(uv) - f{\propto}0 \in IJ$.

Claim (1) follows from

$$\begin{aligned} &[f{\propto}(u+v) - f{\propto}0] - [f{\propto}u - f{\propto}0] - [f{\propto}v - f{\propto}0] \\ &\qquad = f{\propto}(u+v) - f{\propto}u - f{\propto}v + f{\propto}0 \\ &\qquad = f{\propto}(u+v) - f{\propto}(u+0) - f{\propto}(0+v) + f{\propto}(0+0) \\ &\qquad = \sum (f'_+{\propto}u - f'_+{\propto}0)(f''_+{\propto}v - f''_+{\propto}0) \in IJ \end{aligned}$$

whence $f{\propto}(u+v) - f{\propto}0 \in I + J$. Claim (2) follows from

$$\begin{aligned} f{\propto}(uv) - f{\propto}(0) &= f{\propto}(uv) - f{\propto}(u0) - f{\propto}(0v) + f{\propto}(0) \\ &= \sum (f'_\times{\propto}u - f'_\times{\propto}0)(f''_\times{\propto}v - f''_\times{\propto}0) \in IJ. \end{aligned}$$

The result follows easily. □

6.13. **Graded COTs.** As a variant, we can consider graded COTs. This is a triple (T, R_*, ϕ) consisting of a $\mathbb{Z}$-graded theory T, a graded commutative ring R_*, and $\phi \colon C_{R_*} \to T$ such that $\phi^* \colon \mathrm{Model}_T \to \mathrm{Model}_{C_{R_*}}$ preserves colimits.

Example 6.14. Consider the example (5.8), the theory $O_{H\mathbb{F}}$ where $\mathbb{F}$ is a field. This is a COT by the Künneth theorem: we have

$$\begin{aligned} F_T([c_1] + \cdots + [c_m]) &\approx H^*(K(\mathbb{F}, c_1) \times \cdots \times K(\mathbb{F}, c_m); \mathbb{F}) \\ &\approx H^*(K(\mathbb{F}, c_1); \mathbb{F}) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} H^*(K(\mathbb{F}, c_m); \mathbb{F}) \\ &\approx F_T([c_1]) \otimes_{\mathbb{F}} \cdots \otimes_{\mathbb{F}} F_T([c_m]). \end{aligned}$$

7. Algebra of additive operations

We return to the case of an ordinary COT (T, R, ϕ).

Let $\mathcal{A}$ denote the set of additive elements in $R\{x\}$. By the above remarks, it becomes an associtive ring with respect to a product given by $\propto$. The unit element of $\mathcal{A}$ is the generator $x \in R\{x\}$, which represents the identity operation.

Example 7.1. Let R be a commutative ring, and let $T = C_R$, as in (6.3). If R is torsion free, then $\mathcal{A} \approx R$. If R is a field of characteristic p, then $\mathcal{A} \approx R\langle\phi\rangle$, with relation $\phi r = r^p \phi$ for all $r \in R$; the element ϕ corresponds to the pth power operation on algebras of characteristic p.

Example 7.2. Let R be a commutative ring and G a monoid, and consider the COT of (6.4). If R is torsion free, then $\mathcal{A} \approx R[G]$, the monoid ring on R.

Example 7.3. Let R be a commutative ring and consider the COT of (6.5). If R is torsion free, then $\mathcal{A} \approx R[\partial]$, a polynomial algebra on one generator ∂.

Example 7.4. Let $(T, \mathbb{Z}, \phi)$ be the theory of λ-rings. Then

$$\mathcal{A} \approx \bigoplus_{m \geq 1} \mathbb{Z}\psi^m \approx \mathbb{Z}[\psi^p,\, p \text{ prime}].$$

There is a ring homomorphism $R \to \mathcal{A}$, sending r to $rx \in \mathcal{A} \subset R\{x\}$. The image of R need not be central in $\mathcal{A}$, as (7.1) shows.

Let $\mathcal{A}_k$ denote the set of **k-multilinear elements**; that is, the subset of $R\{x_1, \dots, x_k\}$ consisting of elements $f \in R\{x_1, \dots, x_k\}$ such that

$$\begin{aligned} f \propto (\dots, a_i + b_i, \dots) &= f \propto (\dots, a_i, \dots) + f \propto (\dots, b_i, \dots) \\ f \propto (\dots, 0, \dots) &= 0. \end{aligned}$$

There are evident homomorphisms

$$\mathcal{A}^l \underset{R}{\otimes}{}^l \cdots {}^l \underset{R}{\otimes}{}^l \mathcal{A} \to \mathcal{A}_k,$$

where the notation means that we take the tensor product using the left R-module structures on $\mathcal{A}$. Note that $\mathcal{A}_0 \approx R$. We want to consider the special case when these homomorphisms are *isomorphisms*. For instance, we have the following.

Lemma 7.5. *If $R\{x\}$ and $\mathcal{A}$ are flat as left R-modules, then $\mathcal{A}^l \underset{R}{\otimes}{}^l \cdots {}^l \underset{R}{\otimes}{}^l \mathcal{A} \to \mathcal{A}_k$ are isomorphisms.*

Proof. First, note that if $f \in R\{x\}$ satisfies $\Delta_+ f = f \otimes 1 + 1 \otimes f$, then $\epsilon_0(f) = 0$ automatically; if $f \propto (a + b) = f \propto a + f \propto b$, then $f \propto (0 + 0) = f \propto 0 + f \propto 0$.

Thus, the sequence

$$0 \to \mathcal{A} \to R\{x\} \xrightarrow{d} R\{x_1, x_2\}$$

is an exact sequence of left R-modules, where $d(f) = \Delta_+(f) - f \otimes 1 - 1 \otimes f$.

Now tensor two of these sequences together. We get a diagram of exact sequences

$$\begin{tikzcd} \mathcal{A} \otimes_R \mathcal{A} \arrow[r, tail] \arrow[d, tail] & \mathcal{A} \otimes_R R\{x\} \arrow[r] \arrow[d, tail] & \mathcal{A} \otimes_R R\{x_1, x_2\} \arrow[d, tail] \\ R\{y\} \otimes_R \mathcal{A} \arrow[r, tail] \arrow[d] & R\{y\} \otimes_R R\{x\} \arrow[r, "\mathrm{id} \otimes \Delta"] \arrow[d, "\Delta \otimes \mathrm{id}"] & R\{y\} \otimes_R R\{x_1, x_2\} \arrow[d] \\ R\{y_1, y_2\} \otimes_R \mathcal{A} \arrow[r, tail] & R\{y_1, y_2\} \otimes_R R\{x\} \arrow[r] & R\{y_1, y_2\} \otimes_R R\{x_1, x_2\} \end{tikzcd}$$

so that in particular the upper left hand square is a pullback. By definition, $\mathcal{A}_2$ is the intersection of the kernels of $\mathrm{id} \otimes \Delta$ and $\Delta \otimes \mathrm{id}$, so this must be isomorphic to $\mathcal{A} \otimes_R \mathcal{A}$. A similar proof identifies $\mathcal{A}_k$ with a k-fold tensor product. $\square$

The coproduct $\Delta_\times$ sits in a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{A}_2 \arrow[d] \\ R\{x\} \arrow[r] & R\{x_1, x_2\} \end{tikzcd}$$

and thus we see that under our special hypothesis, $\mathcal{A}$ obtains a coalgebra structure; I'll write $\Delta\colon \mathcal{A} \to \mathcal{A}^l \underset{R}{\otimes} {}^l\mathcal{A}$ for this structure map. The map $\epsilon_1\colon R\{x\} \to R$ restricts to a counit map $\epsilon\colon \mathcal{A} \to R$. Finally, write $\eta\colon R \to \mathcal{A}$ for the unit map obtained by sending $r \in R$ to $rx \in \mathcal{A}$; it is clear that η is a ring homomorphism.

We record the following properties:

(a) The maps Δ and ϵ make $\mathcal{A}$ into a coassociative and cocommutative coalgebra over R (in particular, Δ and ϵ are maps of left R-modules).
(b) The map η is a homomorphism of coalgebras over R.
(c) We have that
$$\Delta(ab) = \sum_{i,j} a_i' b_j' \otimes a_i'' b_j'',$$
where $\Delta(a) = \sum_i a_i' \otimes a_i''$ and $\Delta(b) = \sum_j b_j' \otimes b_j''$.
(d) We have that $\epsilon(ab) = \epsilon(a \cdot \eta\epsilon(b))$.

Note that if $\eta(R)$ lies in the center of $\mathcal{A}$, then $\mathcal{A}$ is just a cocommutative bialgebra. Observe that (b) implies that $\Delta(\eta(r)) = r\Delta(1) = \eta(r) \otimes 1 = 1 \otimes \eta(r)$, so that

(e)
$$\Delta(a \cdot \eta(r)) = \sum_i a_i' \eta(r) \otimes a_i'' = \sum_i a_i' \otimes a_i'' \eta(r).$$

This implies that the image of $\Delta\colon \mathcal{A} \to \mathcal{A}^l \underset{R}{\otimes} {}^l\mathcal{A}$ lands in the subset consisting of elements $\sum_i a_i \otimes b_i$ such that $\sum_i a_i \eta(r) \otimes b_i = \sum_i a_i \otimes b_i \eta(r)$ for all $r \in R$.

Let $\mathrm{Mod}_\mathcal{A}$ denote the category of left $\mathcal{A}$-modules. There is an evident forgetful functor $\mathrm{Mod}_\mathcal{A} \to \mathrm{Mod}_R$, by restricting the action along $\eta\colon R \to \mathcal{A}$.

(i) The ring R is equipped with a standard $\mathcal{A}$-module structure by the rule $a \cdot r \overset{\text{def}}{=} \epsilon(a \cdot \eta(r))$. (Properties (b) and (d) imply that this is in fact a module structure.
(ii) Given $M, N \in \mathrm{Mod}_\mathcal{A}$, we make $M \otimes_R N$ into a left $\mathcal{A}$-module by the rule
$$a(m \otimes n) \overset{\text{def}}{=} (a'm) \otimes (a''n)$$
where $\Delta a = \sum a' \otimes a''$. That $a(rm \otimes n) = a(m \otimes rn)$ follows from (e). That this really makes $M \otimes_R N$ into a module follows from (c).

Proposition 7.6. *Given the special hypothesis above, the above structure makes* $\mathrm{Mod}_\mathcal{A}$ *into a symmetric monoidal category, which with respect to the forgetful functors* $\mathrm{Model}_T \to \mathrm{Mod}_\mathcal{A} \to \mathrm{Mod}_R$ *agrees with coproduct of T-models, and tensor product of R-modules.*

7.7. **Additive operations for graded COTs.** For a graded COT (T, R_*, ϕ), let $\mathcal{A}(c,d)$ be the set of elements $f \in R\{x\}_d$ (with $|x| = c$), which are primitive under the comultiplication

$$R\{x\} \xrightarrow{x \mapsto x_1 + x_2} R\{x_1, x_2\}.$$

Such $f \in \mathcal{A}(c,d)$ give rise to additive functions $f\propto\colon B_c \to B_d$ natural in the model B. The element $x \in R\{x\}_c$ corresponds to the identity map on B_c. Thus we obtain a *category* $\mathcal{A}$ of additive operations, the objects of which are $\mathbb{Z}$. The category $\mathcal{A}$ is a preadditive category; sometimes such a thing is called a **ringoid**.

We also can think of the graded ring R_* as a category, where $R_*(c,d) \approx R_{d-c}$. Then there is a functor $\eta\colon R_* \to \mathcal{A}$.

I'll continue to use ring theoretic language when discussing $\mathcal{A}$. A left $\mathcal{A}$-module is an additive functor from $\mathcal{A}$ to abelian groups. In particular, a module $M = \{M_n\}_{n\in\mathbb{Z}}$ comes with maps

$$\mathcal{A}(c,d) \otimes M_c \to M_d.$$

By restriction, an $\mathcal{A}$-module is an R_*-module.

The object R_* is an $\mathcal{A}$-module.

Example 7.8. Consider the example $O_{H\mathbb{F}_p}$. Here $\mathcal{A}(c,d)$ is the set of additive operations $H^c(-;\mathbb{F}_p) \to H^d(-;\mathbb{F}_p)$. For instance, if $p = 2$, then all additive operations $H^c(-;\mathbb{F}_2) \to H^*(-;\mathbb{F}_p)$ are linear combinations of monomials $\mathrm{Sq}^I = \mathrm{Sq}^{i_1} \cdots \mathrm{Sq}^{i_r}$ of Steenrod operations, which are admissible ($i_j \geq 2i_{j+1}$), and have $e(I) \leq c$, where $e(I) \overset{\text{def}}{=} i_1 - i_2 - \cdots - i_r$ is the **excess**. (Other admissible monomials act trivially on elements in degree c.) Thus, the category of $\mathcal{A}$-modules is precisely the category of unstable modules over the Steenrod algebra.

Note that $\mathcal{A}(c,-)$ is a left R_*-module. If each $\mathcal{A}(c,-)$ and each $R\{x\}$ with $|x| = c$ are flat left R_*-modules, then we can define maps

$$\Delta\colon \mathcal{A}(c,d) \to \bigoplus_{d_1+d_2=d} \mathcal{A}(c,d_1)^l \underset{R_*}{\otimes} {}^l\mathcal{A}(c,d_2)$$

and

$$\epsilon\colon \mathcal{A}(0,d) \to R_*(0,d) = R_d,$$

which make the category of $\mathcal{A}$-modules into a symmetric monoidal category, in such a way that

$$\mathrm{Model}_T \to \mathrm{Mod}_A \to \mathrm{Mod}_{R_*}$$

preserve the monoidal structure.

8. Abelian group objects

Fix a COT (T, R, ϕ), and a model A. An **abelian group object** over A is an object $p\colon B \to A$ in Model_T/A equipped with maps

$$e\colon A \to B, \quad i\colon B \to B, \quad \sigma\colon B \times_A B \to B$$

in Model_T/A satisfying the axioms for abelian group. Note that using $e\colon A \to B$ we can think of this as an object of $A\backslash\mathrm{Model}_T/A \approx \mathrm{Model}_{T_A}/A$, in which case i and p make it into an abelian group object in this category. Thus without losing any generality, we can just consider the case when $A = R$. Write $(\mathrm{Model}_T/R)^{\mathrm{ab}}$ for the category of abelian group objects over R.

Proposition 8.1. *Let $R \xrightarrow{e} B \xrightarrow{p} R$ be an object of Model_T/R. Let $I = \mathrm{Ker}(p)$. The following are equivalent.*

(1) $I^2 = 0$.
(2) *B admits a structure of abelian group object compatible with e and p.*

If these hold, then B admits a unique *structure of abelian group object.*

Furthermore, $(\mathrm{Model}_T/R)^{\mathrm{ab}}$ is equivalent to the full subcategory of Model_T/R consisting of $p\colon B \to R$ with $\mathrm{Ker}(p)^2 = 0$.

Proof. First, note that in general $R \times I \xrightarrow{\sim} B$ by $(r, u) \mapsto r + u$, and $R \times I \times I \xrightarrow{\sim} B \times_R B$ by $(r, u, v) \mapsto (r + u, r + v)$.

Suppose that $\sigma\colon B \times_R B \to B$ is part of an abelian group structure on B. Let $\tau\colon I \times I \to I$ be defined by $\tau(u, v) = \sigma(0+u, 0+v)$; we can recover σ from τ by $\sigma(r+u, r+v) = r+\tau(u, v)$. Clearly, τ defines a group law on the set I, with identity element 0. On the other hand, τ is a homomorphism with respect to the "usual" abelian group structure on I. Thus, we have

$$\tau(u, v) = \tau((u, 0) + (0, v)) = \tau(u, 0) + \tau(0, v) = u + v,$$

so that τ must coincide with the usual group law on I. In particular, B can admit at most one abelian group structure, which must be given by $\sigma(r + u, r + v) = r + u + v$.

We have

$$(1 + u)(1 + v) = \sigma(1 + u, 1)\sigma(1, 1 + v) = \sigma((1 + u, 1)(1, 1 + v)) = \sigma(1 + u, 1 + v) = 1 + u + v;$$

thus σ can define a group structure if and only if $uv = 0$ for all $u, v \in I$. $\square$

Note that we obtain adjoint functors

$$\mathrm{Model}_T/R \rightleftarrows (\mathrm{Model}_T/R)^{\mathrm{ab}},$$

where the left adjoint sends $p\colon B \to R$ to $B/\mathrm{Ker}(p)^2 \to R$.

Given a set S, we can form a "free" abelian group object F_S, by taking $F_S = R\{S\}/J^2$, where $J = \mathrm{Ker}[R\{S\} \to R]$. Then

$$(\mathrm{Model}_T/R)^{\mathrm{ab}}(F_S, B) \approx \mathrm{Model}_T/R(R\{S\}, B) \approx \prod_S I,$$

where $B \approx R \times I$.

Let $\mathcal{D}$ be the endomorphism ring of the free abelian group object on one generator. That is,

$$\begin{aligned} \mathcal{D} &= (\mathrm{Model}_T/R)^{\mathrm{ab}}(F_1, F_1) \\ &\approx \mathrm{Model}_T/R(R\{x\}, R\{x\}/J^2) \\ &\approx J/J^2. \end{aligned}$$

The product in $\mathcal{D}$ is induced by the composition pairing $\propto\colon R\{x\} \times R\{x\} \to R\{x\}$, which restricts to a map $J \times J \to J$ and gives rise to a well defined product on $\mathcal{D}$. There is a ring homomorphism $R \to \mathcal{D}$ corresponding to $r \mapsto rx \in J$. Again, this homomorphism need not be central.

The composite map $\mathcal{A} \to J \to \mathcal{D}$ is a ring homomorphism.

Example 8.2. Let R be a commutative ring. Let $T = C_R$. Then $\mathcal{D} \approx R$. If R is torsion free, then $\mathcal{A} \to \mathcal{D}$ is an isomorphism. If $R \supseteq \mathbb{F}_p$, then $\mathcal{A} \to \mathcal{D}$ is surjective, sending ϕ to 0.

Example 8.3. Let $(T, \mathbb{Z}, \phi)$ be the theory of λ-rings. Then

$$\mathcal{D} \approx \bigoplus_{m \geq 1} \mathbb{Z}\theta^m \approx \mathbb{Z}[\theta_p,\ p \text{ prime}].$$

The natural map $\mathcal{A} \to \mathcal{D}$ maps ψ^m to $m\theta^m$.

There is a functor

$$U \colon (\mathrm{Model}_T/R)^{\mathrm{ab}} \to \mathrm{Mod}_{\mathcal{D}},$$

given by $U(B) = (\mathrm{Model}_T/R)^{\mathrm{ab}}(F_1, B) \approx \mathrm{Ker}[B \to R]$.

Proposition 8.4. *The functor* $U \colon (\mathrm{Model}_T/R)^{\mathrm{ab}} \to \mathrm{Mod}_{\mathcal{D}}$ *is an equivalence of categories.*

Proof. Let $C = (\mathrm{Model}_T/R)^{\mathrm{ab}}$.

(1) First, note that colimits are easy to compute in C. In fact, finite sums are given by:
$$(R \times I) \amalg^C (R \times I') \approx (R \times I) \otimes_R (R \times I')/J^2 \approx R \times I \times I'.$$
Reflexive coequalizers and filtered colimits are computed setwise, since this is what holds in Model_T/R.

(2) In particular, we see that $U \colon C \to \mathrm{Mod}_{\mathcal{D}}$ preserves limits and colimits.

(3) For free objects in C, we have
$$U(F_S) \approx \mathcal{D}S \approx \bigoplus_S \mathcal{D}.$$
Furthermore, we easily compute that
$$U \colon C(F_S, B) \to \mathrm{Mod}_{\mathcal{D}}(U(F_S), U(B))$$
is a bijection for all abelian group objects B.

(4) Given $M \in \mathrm{Mod}_{\mathcal{D}}$, consider a free resolution
$$\mathcal{D}S_1 \xrightarrow{d} \mathcal{D}S_0 \to M \to 0.$$
Lift this to a pair in C, and let B denote the coequalizer:
$$F_{S_1} \rightrightarrows F_{S_0} \to B.$$
Then there is an evident map $M \to U(B)$ which is seen to be an isomorphism, since U preserves colimits. Thus, U is essentially surjective.

(5) Given $B \in C$, let $M = U(B)$, form a free resolution of M in $\mathrm{Mod}_{\mathcal{D}}$ and lift to C, as above. We get a coequalizer diagram
$$F_{S_1} \rightrightarrows F_{S_0} \to B'$$
in C, which after applying U gives a free resolution for M. Using (3) we can lift the map $\mathcal{D}S_0 \to M \approx U(B)$ to a map $F_{S_0} \to B$ in C which equalizes the pair of arrows, and thus extends to a unique map $B' \to B$, which is necessarily an isomorphism. Thus, every object in C admits a free resolution.

(6) Finally, use (5) with (3) to show that U is fully faithful, by using a free resolution of B to compute maps in C out of B and to compare this to maps in $\mathrm{Mod}_{\mathcal{D}}$ out of $U(B)$.

□

8.5. **Abelian group objects for graded COTs.** Given a graded COT (T, R_*, ϕ), there is an associated category $(\mathrm{Model}_T/R_*)^{\mathrm{ab}}$ of abelian group objects. Given a $\mathbb{Z}$-graded set S_*, we can form a free abelian group object $F_{S_*} \overset{\mathrm{def}}{=} R\{S\}/J^2$. We define $\mathcal{D}$ to be the ringoid with object set $\mathbb{Z}$ and with

$$\mathcal{D}(c,d) \overset{\mathrm{def}}{=} (\mathrm{Model}_T/R_*)^{\mathrm{ab}}(R_*\{x_d\}/J^2, R_*\{x_c\}/J^2),$$

where the notation $R_*\{x_c\}/J^2$ denotes the quotient of the free model on a generator in dimension c by the square of the ideal $J = \mathrm{Ker}[R_*\{x_c\} \to R_*]$. Explicitly, this means

$$\mathcal{D}(c,d) \approx (J/J^2)_d \quad \text{where} \quad J = \mathrm{Ker}[R_*\{x_c\} \to R_*].$$

Example 8.6. Consider the graded theory $O_{H\mathbb{F}_2}$ of algebras over the mod p Steenrod algebra. We calculate that

$$\mathcal{D}(c,*) \approx \widetilde{H}^*(K(\mathbb{F}_2,c))/\widetilde{H}^*(K(\mathbb{F}_2,c))^2,$$

which is graded vector space spanned by admissible monomials Sq^I which have $e(I) < c$.

The natural map of ringoids $\mathcal{A} \to \mathcal{D}$ is surjective, but is not an isomorphism. The kernel is spanned by admissible monomials $\mathrm{Sq}^I \colon H^c \to H^{c+|I|}$ with $e(I) = c$; these are precisely the monomials of the form $\mathrm{Sq}^{2^{j-1}c}\mathrm{Sq}^{2^{j-2}c} \cdots \mathrm{Sq}^{2c}\mathrm{Sq}^c$, which is just the 2^jth power map.

As before, we can define a functor $U \colon (\mathrm{Model}_T/R)^{\mathrm{ab}} \to \mathrm{Mod}_{\mathcal{D}}$, and we can prove the following.

Proposition 8.7. *The functor $U \colon (\mathrm{Model}_T/R)^{\mathrm{ab}} \to \mathrm{Mod}_{\mathcal{D}}$ is an equivalence of categories.*

9. Theories of Dyer-Lashof algebras

Let Alg_S denote a convenient category of commutative S-algebras (say, the category of EKMM). I'll write Mod_S for the category of S-modules, which is a model for the usual category of spectra. I write $h\mathrm{Alg}_S$ and $h\mathrm{Mod}_S$ for the associated homotopy categories. There are adjoint functors

$$\mathbb{P} \colon \mathrm{Mod}_S \rightleftarrows \mathrm{Alg}_S : U,$$

where U is the forgetful functor and $\mathbb{P}$ is the free algebra functor, defined by

$$\mathbb{P}(X) = \bigvee_{m \geq 0} \mathbb{P}^m(X) \overset{\mathrm{def}}{=} \bigvee_{m \geq 0} X^{\wedge m}/\Sigma_m.$$

Both these functors descend to homotopy categories, giving adjoint pairs

$$\mathbb{P} \colon h\mathrm{Mod}_S \rightleftarrows h\mathrm{Alg}_S : U.$$

Usually, I'll be talking about these functors on the level of homotopy categories. For a commutative S-algebra A we have

$$\pi_q A \approx h\mathrm{Mod}_S(S^q, A) \approx h\mathrm{Alg}_S(\mathbb{P}(S^q), A).$$

We have that

$$\mathbb{P}^m(\Sigma^\infty T) \approx \Sigma^\infty(T^{\wedge m}_{h\Sigma_m}).$$

In particular,

$$\mathbb{P}^m(S^0) \approx \mathbb{P}^m(\Sigma^\infty S^0) \approx \Sigma^\infty_+ B\Sigma^m.$$

For $d \geq 0$, we have

$$\mathbb{P}^m(S^d) \approx \mathbb{P}^m(\Sigma^\infty S^d) \approx (S^d)^{\wedge m}_{h\Sigma_m} \approx B\Sigma_m^{dV_m},$$

where $V_m = \mathbb{R}^m$ equipped with the Σ_m-action given by permuting coordinates. This extends to arbitrary $d \in \mathbb{Z}$ using the theory of Thom spectra of virtual bundles, so that we can write

$$\mathbb{P}_m(S^d) \approx B\Sigma_m^{dV_m}.$$

Given a commutative S-algebra R, we write Mod_R for the category of R-modules, and Alg_R for the category of commutative R-algebras. We have adjoint functors

$$\mathbb{P}_R\colon \mathrm{Mod}_R \rightleftarrows \mathrm{Alg}_R : U^R,$$

as above. There are also adjoint functors

$$\mathrm{Mod}_S \rightleftarrows \mathrm{Mod}_R \quad \text{and} \quad \mathrm{Alg}_S \rightleftarrows \mathrm{Alg}_R,$$

where the right adjoints are forgetful functors, and the left adjoints are given by $R \wedge_S -$.

For an R-algebra A, we have

$$\pi_q A \approx h\mathrm{Mod}_S(S^q, A) \approx h\mathrm{Mod}_R(R \wedge_S S^q, A) \approx h\mathrm{Alg}_S(\mathbb{P}(S^q), A) \approx h\mathrm{Alg}_R(\mathbb{P}_R(R \wedge_S S^q), A).$$

9.1. **Dyer-Lashof theory.** Given a commutative S-algebra R, let DL_R denote the $\mathbb{Z}$-graded theory T defined by

$$T(T^{[c_1]+\cdots+[c_m]}, T^{[d_1]+\cdots+[d_n]}) \approx h\mathrm{Alg}_R(\mathbb{P}_R(R \wedge (S^{d_1} \vee \cdots \vee \wedge S^{d_n})), \mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee \wedge S^{c_m}))).$$

This is a theory, because $\mathcal{P}_R(M \vee N) \approx \mathcal{P}_R(M) \wedge_R \mathcal{P}_R(N)$ is a coproduct in the homotopy category of commutative R-algebras. Note that free theories are described by

$$F([c_1] + \cdots + [c_m])_{[d]} \approx \pi_d \mathbb{P}_R(R \wedge (S^{c_1} \vee \cdots \vee S^{c_m})) \approx \pi_d[R \wedge (\mathbb{P}(S^{c_1}) \wedge \cdots \wedge \mathbb{P}(S^{c_m}))].$$

In particular, if $\pi_* R \wedge \mathcal{P}(S^c)$ are flat as $\pi_* R$-modules, we see that

$$F([c_1] + \cdots + [c_m])_* \approx F([c_1]) \otimes_{R_*} \cdots \otimes_{R_*} F([c_m]),$$

in which case DL_R is a COT.

In particular, we see that taking homotopy groups defines a functor

$$\pi_*\colon h\mathrm{Alg}_R \to \mathrm{Model}_{\mathrm{DL}_R}.$$

Thus, DL_R describes all homotopy operations on commutative R-algebras.

We note some particular instances of this.

(1) If A is a commutative S-algebra, then $R \wedge_S A$ is a commutative R-algebra. Thus we have a composite functor $A \mapsto \pi_* R \wedge_S A \approx R_* A$, and hence a functor

$$R_*\colon h\mathrm{Alg}_S \to \mathrm{Model}_{\mathrm{DL}_R}.$$

Thus, DL_R describes homology operations on commutative S-algebras.

(2) If X is a spectrum, then $\Omega^\infty X$ is an infinite loop space, and thus in particular an E_∞-space. It can be shown that $\Sigma^\infty_+\Omega^\infty X$ is a commutative S-algebra. Thus we have a functor
$$R_*\Omega^\infty\colon h\mathrm{Mod}_S \to \mathrm{Model}_{\mathrm{DL}_R}.$$
The R-homology of an ∞-loop space admits the structure of a DL_R-model.

(3) If T is a space, then there is a commutative R-algebra
$$R^T \overset{\text{def}}{=} \hom(\Sigma^\infty_+ T, R).$$
Thus we get a contravariant functor
$$R^*\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Model}_{\mathrm{DL}_R},$$
the R-cohomology of a space is a DL_R-model.

9.2. **Dyer-Lashof algebra for ordinary rational homology.** Let R a commutative $\mathbb{Q}$-algebra. Then
$$\pi_* HR \wedge \mathbb{P}(S^c) \approx \begin{cases} R[x_c] & \text{if } c \text{ even,} \\ R[x_c]/(x_c)^2 & \text{if } c \text{ odd.} \end{cases}$$
Thus, DL_{HR} is a COT, and is equal to the theory of graded commutative R-algebras.

To prove this, note that we must compute
$$\pi_* HR \wedge \mathbb{P}_m(S^c) \approx H_*((S^c)^{\wedge m}_{h\Sigma_m}; R),$$
and there is a spectral sequence
$$E_2^{*,*} \approx H_*(B\Sigma_m; \underline{H^*((S^c)^{\wedge m}; R)}) \Longrightarrow H_*((S^c)^{\wedge m}_{h\Sigma_m}; R),$$
where the coefficients are a local system over $B\Sigma_m$. Since $R \supset \mathbb{Q}$, we have
$$E_2^{*,*} = E_2^{0,*} \approx H_0(B\Sigma_m; \underline{H^*((S^c)^{\wedge m}; R)}).$$
If c is even, the local system is the trivial representation, and so this becomes a copy of R in bidegree $(0, mc)$. If c is odd, the local system is the sign representation, and so this is zero unless $m = 0$ or $m = 1$.

10. Dyer-Lashof for ordinary homology

If R is a ring containing $\mathbb{F}_p$, there is a complete description of DL_{HR}, which is a COT. I'll concentrate on the characteristic 2 case.

Suppose $R \supset \mathbb{F}_2$. Then DL_{HR} is a COT, since
$$\pi_* HR \wedge \mathbb{P}(S^c) \approx \bigoplus_m HR_* B\Sigma_m^{cV_m}$$
is a free graded R-module.

The following describes the category of DL_{HR}-models. A DL_{HR}-model is a graded commutative R-algebra A_*, equipped with functions
$$Q^s\colon A_c \to A_{c+s} \qquad \text{for all } s, c \in \mathbb{Z},$$
such that

(1) the Q^s are additive homomorphisms,
(2) $Q^s(a) = 0$ if $s < |a|$,

(3) $Q^s(a) = a^2$ if $s = |a|$,
(4) $Q^s(r) = 0$ for $s \neq 0$, $r \in R$,
(5) (Cartan formula)
$$Q^s(ab) = \sum_{i+j=s} Q^i(a) \cdot Q^j(b),$$
(6) (Adem relations)
$$Q^r Q^s = \sum_{i+j=r+s} \binom{j-s-1}{2j-r} Q^i Q^j$$
for $r > 2s$.

Note that the above defines a theory, which I'll call T. On choosing a grading c, we see that the free T-model on one generator in degree c has the form
$$R_*\{x\} \approx R[Q^I(x)],$$
where the $Q^I = Q^{i_1} \cdots Q^{i_r}$ range over all monomials I with $i_j \leq 2i_{j+1}$ (admissibility), and $e(I) = i_1 - i_2 - \cdots - i_r > c$ (excess). (Note that if $e(I) = c$, then $Q^I(x) = x^{2^r}$.)

Remark 10.1. The operations Q^s are not R-linear. Rather, we have
$$Q^s(ra) = r^p Q^s(a)$$
for $a \in A_*$ and $r \in R$. This follows from (3), (4), and (5), since
$$Q^s(ra) = \sum_{i+j=s} Q^i(r) Q^j(a) = Q^0(r) Q^s(a) = r^p Q^s(a).$$
If $R = \mathbb{F}_2$, this is the same as being $\mathbb{F}_2$-linear, but not generally.

Example 10.2. If T is a space, then $R^*T \approx \pi_{-*}R^T$. Thus, the Q^s determine operations
$$Q^s \colon R^q T \to R^{q-s} T,$$
which are non-zero only if $-s \leq q$, and such that $Q^s(a) = a^2$ if $|a| = -s$. If we set $\mathrm{Sq}^s = Q^{-s}$, these look like (and are) the usual Steenrod operations on the mod 2 cohomology of T.

Note that the above axioms include many, but not all of the axioms for the Steenrod operations.

Let T denote the theory whose models are described by the above axioms. I want to show that T is isomorphic to DL_R. To do this, I'll first construct a map $T \to \mathrm{DL}_R$, essentially by constructing the operations Q^s and showing they have the right properties. Then we show that the map is an isomorphism, by showing that the induced functor $\mathrm{Model}_{\mathrm{DL}_R} \to \mathrm{Model}_T$ sends free models to free models; this is an argument due to McClure.

The operation $Q^s \colon A_c \to A_{c+s}$ corresponds to an element $e_s \in \pi_{c+s}\mathbb{P}^2_R(R \wedge S^c)$. First consider the case $R = H\mathbb{F}_2$. Since
$$\mathbb{P}^2_{H\mathbb{F}_2}(H\mathbb{F}_2 \wedge S^c) \approx H\mathbb{F}_2 \wedge B\Sigma_2^{cV_2} \approx H\mathbb{F}_2 \wedge \Sigma^c \mathbb{RP}^\infty_c,$$
we have that
$$\pi_{s+c}\mathbb{P}^2_{H\mathbb{F}_2}(H\mathbb{F}_2 \wedge S^c) \approx H_{s+c}(\Sigma^c \mathbb{RP}^\infty_c; \mathbb{F}_2) \approx H_s(\mathbb{RP}^\infty_c; \mathbb{F}_2) \approx \begin{cases} 0 & \text{if } s < c, \\ \mathbb{F}_2 & \text{if } s \geq c, \end{cases}$$

using the Thom isomorphism for mod 2 homology. Set e_s to be the non-trivial element of this group if $s \geq c$, otherwise set $e_s = 0$ if $s < c$. A general R is an $\mathbb{F}_2$-algebra, and we let $e_s \in \pi_{c+s}\mathbb{P}^2_R(R \wedge S^c) \approx \pi_{c+s}R \wedge_{H\mathbb{F}_2} \mathbb{P}^2_{H\mathbb{F}_2}(H\mathbb{F}_2 \wedge S^c)$ be the image of the class just defined.

Axiom (2) is now clear by construction.

To prove axiom (3), observe that given an R-algebra A and an element $a \in \pi_c A$, to compute the cup square of this element we can consider the diagram of R-modules

$$\begin{tikzcd} \mathbb{P}_R(S^c \wedge S^c) \arrow[r, "\tilde{f}"] & \mathbb{P}_R(S^c) \arrow[r, "a"] & A \\ R \wedge (S^c \wedge S^c) \arrow[u] \arrow[r, "f"'] & \mathbb{P}^2_R(S^c) \arrow[u, "i"'] & \end{tikzcd}$$

where the top line are maps of R-algebras. The map labeled f is the map of R-modules obtained by smashing

$$g \colon S^c \wedge S^c \approx \Sigma_2^+ \wedge_{\Sigma_2} (S^c \wedge S^c) \to E\Sigma_2^+ \wedge_{\Sigma_2} (S^c \wedge S^c) \approx \mathbb{P}^2(S^c) \approx B\Sigma_2^{cV_2}$$

with R, and $\tilde{f}$ is the extension of $i \circ f$ to a map of R-algebras. It is clear that aif represents the cup square of a, and so $\tilde{f}$ corresponds to the morphism in DL_R representing cup-square of an element of degree c. The map g is just the inclusion of the bottom cell in $B\Sigma_2^{cV_2}$, and its R-Hurewicz image (represented by f) is e_c.

Axiom (4) holds for degree reasons.

10.3. **Transfer and additivity.** Given a map $f \colon X \to Y$ between finite sets, we can construct a map $f^! \colon \Sigma^\infty_+ Y \to \Sigma^\infty_+ X$ of spectra going the other way, by "averaging". Thus, $f^!$ is the map $\bigvee_{y \in Y} S^0_y \to \bigvee_{x \in X} S^0_x$ which sends the summand S^0_y via a "pinch" map to the summands S^0_x with $x \in f^{-1}(y)$.

Here is a slick formal way to do this: given $f \colon X \to Y$, consider the map

$$\gamma_f \colon (X \times Y)_+ \to S^0, \qquad (x, y) \mapsto \begin{cases} 1 & \text{if } f(x) = y, \\ * & \text{if } f(x) \neq y, \end{cases}$$

which is the characteristic function of the graph of f. This gives a stable map $\Sigma^\infty \gamma_f \colon \Sigma^\infty_+ X \wedge \Sigma^\infty_+ Y \to S^0$, which is adjoint to a map $\Sigma^\infty_+ Y \to \hom(\Sigma^\infty_+ X, S^0)$. Note that if $f \colon X \to X$ is the identity map, the map $\Sigma^\infty_+ X \to \hom(\Sigma^\infty_+ X, S^0) \approx \prod_X S^0$ is a weak equivalence. Thus $f^!$ is the "composite"

$$\Sigma^\infty_+ Y \to \hom(\Sigma^\infty_+ X, S^0) \xleftarrow{\sim} \Sigma^\infty_+ X.$$

If G is a topological group acting on X and Y, so that f is G-equivariant, then γ_f is also G-equivariant (using $g \cdot (x, y) = (g^{-1}x, gy)$), and therefore so is $f^!$. Thus we can form

$$(f_!)_{hG} \colon \Sigma^\infty_+ Y_{hG} \to \Sigma^\infty_+ X_{hG}$$

.

Example 10.4. Let $g \colon E \to B$ be a covering map with fibers isomorphic to a finite set $\underline{m}$. There is an associated principle bundle $P \to B$, with finite structure group $G = \Sigma_m$ (so that

$P = \{p\colon E_b \xrightarrow{\sim} \underline{m}\}_{b\in B})$. Consider the pullback square

$$\begin{tikzcd} P \times_B E \arrow[r] \arrow[d, "h"'] & E \arrow[d, "g"] \\ P \arrow[r] & B \end{tikzcd}$$

The horizontal maps identify the right-hand spaces as quotients by a *free* G-action: $B \approx G\backslash P$ and $E \approx G\backslash P \times_B E$. Furthermore, the covering map trivializes over P, so that $P \times_B E \approx P \times \underline{m}$ as a G-space by $(p, e) \mapsto (p, p(e))$, and h is identified with the projection $\mathrm{id}_P \times \pi\colon P \times \underline{m} \to \underline{m}$ as a G-equivariant map. Thus we can a G-equivariant map of spectra

$$\mathrm{id} \wedge \pi^!\colon \Sigma^\infty_+ P \to \Sigma^\infty_+ (P \times \underline{m}),$$

and taking G-homotopy orbits gives

$$g^!\colon \Sigma^\infty_+ B \to \Sigma^\infty_+ E.$$

This is the usual covering space transfer.

From this we can derive the usual properties of the transfer, for instance the push-pull formula: if

$$\begin{tikzcd} E' \arrow[r, "h"] \arrow[d, "f'"'] & E \arrow[d, "f"] \\ B' \arrow[r, "g"'] & B \end{tikzcd}$$

is a pullback square of covering spaces, then $f^! \circ \Sigma^\infty_+ g = \Sigma^\infty_+ h \circ (f')^!$. To prove this, pullback along the principal fibration over B.

There is also a product formula: $(f \times g)^! \approx f^! \wedge g^!$.

There's a variant: if $V \to B$ is a vector bundle, and $f^*V \to E$ is the pullback to E, then there is a transfer map on Thom spectra: $f^!\colon B^V \to E^{f^*V}$. This comes from the commutative diagram

$$\begin{tikzcd} (f^*V)' \arrow[r] \arrow[d] & f^*V \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

where $V' = V - \{\text{zero section}\}$. Another way to construct this is to pull back along the frame bundle $P \to B$ of V, using the fact that the Thom spectrum can be recovered as $B^V \approx \Sigma^\infty_+ P \wedge_{hO(n)} \mathbb{R}^n$.

Let's record a fact about transfers which will become useful.

Proposition 10.5. *Let $f\colon X \to Y$ be a map of finite G-sets, where G is some topological group. If the fibers of f have order prime to p, then for any G-spectrum Z, the composite*

$$\Sigma^\infty_+ Y \wedge_{hG} Z \xrightarrow{f^! \wedge_{hG} Z} \Sigma^\infty_+ X \wedge_{hG} Z \xrightarrow{\Sigma^\infty_+ f \wedge_{hG} Z} \Sigma^\infty_+ Y \wedge_{hG} Z$$

is a p-local equivalence.

Proof. It is enough to check that the G-equivariant map

$$h\colon \Sigma^\infty_+ Y \xrightarrow{f^!} \Sigma^\infty_+ X \xrightarrow{\Sigma^\infty_+ f} \Sigma^\infty_+ Y$$

is a p-local equivalence as a map of spectra, since $- \wedge_{hG} Z$ takes G-maps which are p-local equivalences to p-local equivalences. It is easy to calculate that h is given by a diagonal matrix, with (y,y)-entry multiplication by $|f^{-1}(y)|$. $\square$

Now we can analyze the map $\mathbb{P}^m(X) \xrightarrow{\mathbb{P}^m(\nabla)} \mathbb{P}^m(X \vee X)$. First note that the Σ_m-equivariant map $X^{\wedge m} \to (X \vee X)^{\wedge m}$ is weakly equivalent to

$$((S^0)^{\wedge m} \xrightarrow{i} (S^0 \vee S^0)^{\wedge m}) \wedge X^{\wedge m}.$$

The map labelled i is the same as $f^!\colon \Sigma^\infty_+ \underline{1} \to \Sigma^\infty_+ \underline{2}$, where $f\colon \underline{2} \to \underline{1}$ is the usual projection. The product formula for transfers identifies $i^{\wedge m}$ with $(f^m)^!$, where $f^m\colon \underline{2}^m \to \underline{1}^m$. As a Σ_m-space, $\underline{2}^m \approx \coprod_{k=0}^m X_k$, with $X_k \approx \Sigma_m/\Sigma_k \times \Sigma_{m-k}$. Thus $i^{\wedge m} = (f^m)^! = (\pi_0^!, \dots, \pi_m^!)$, where $\pi_k\colon \Sigma_m/\Sigma_k \times \Sigma_{m-k} \to *$ is the projection.

Thus, if $X \approx S^0$, $X^{\wedge m} \approx S^0$ as a Σ_m-spectrum, and so

$$\mathbb{P}^m(\nabla) \approx \Sigma^\infty_+ B\Sigma_m \xrightarrow{(\pi_k^!)} \bigvee_{i+j=m} \Sigma^\infty_+ B(\Sigma_i \times \Sigma_j).$$

If we more generally set $X = S^c$, then we obtain Thom spectrum transfers $B\Sigma_m^{cV_m} \to B\Sigma_i^{cV_i} \wedge B\Sigma_j^{cV_j}$, using the fact that $V_m|_{\Sigma_i \times \Sigma_j} \approx V_i \times V_j$.

To show that the Q^s operations are additive, we need to consider the diagram

$$\begin{tikzcd}
S^c \arrow[r, "\nabla"] \arrow[d] & S^c \vee S^c \arrow[d] & \\
R \wedge \mathbb{P}^2(S^c) \arrow[r, "R\wedge\mathbb{P}^2(\nabla)"'] \arrow[d] & R \wedge \mathbb{P}^2(S^c \vee S^c) \arrow[d] & \\
R \wedge \mathbb{P}(S^c) \arrow[r, "R\wedge\mathbb{P}(\nabla)"'] & R \wedge \mathbb{P}(S^c \vee S^c) \arrow[r, "(a,b)"'] & A
\end{tikzcd}$$

and show that in homotopy, the composite

$$R \wedge \mathbb{P}^2(S^c) \to R \wedge \mathbb{P}^2(S^c \vee S^c) \to X_1 \wedge_{\Sigma_2} (S^c)^{\wedge 2}$$

is trivial. This map is obtained by smashing R with the "twisted" transfer $B\Sigma_2^{cV_2} \to (*)^{\mathbb{R}^{2c}} \approx S^{2c}$. For degree reasons it is clear that the map is zero, except for degree $2c$. For this case, it suffices to show that the composite $(*)^{\mathbb{R}^{2c}} \to B\Sigma_2^{cV_2} \to (*)^{\mathbb{R}^{2c}}$ has degree 2, which can be proved using the push-pull formula, for instance.

10.6. **Diagonal map and the Cartan formula.** Since $\mathbb{P}$ is a monad on $h\mathrm{Mod}_S$, there is a natural map $\iota_X\colon X \to \mathbb{P}(X)$, which is the inclusion of $X \approx \mathbb{P}^1(X) \subset \mathbb{P}(X)$. Thus, given spectra X and Y, there is a natural map $\iota_X \wedge \iota_Y\colon X \wedge Y \to \mathbb{P}(X) \wedge \mathbb{P}(Y)$ of spectra, which extends to a map

$$\delta\colon \mathbb{P}(X \wedge Y) \to \mathbb{P}(X) \wedge \mathbb{P}(Y)$$

of spectra. The following describes this map

Proposition 10.7. *There are commutative diagrams*

$$\begin{tikzcd} \mathbb{P}(X \wedge Y) \arrow[r, "\delta"] & \mathbb{P}(X) \wedge \mathbb{P}(Y) \\ \mathbb{P}^m(X \wedge Y) \arrow[u] \arrow[r, "\delta_m"'] & \mathbb{P}^m(X) \wedge \mathbb{P}^m(Y) \arrow[u] \end{tikzcd}$$

where the vertical maps are the evident inclusions, and δ_m is defined as follows. Consider the map

$$E\Sigma_m^+ \wedge (X \wedge Y)^{\wedge m} \xrightarrow{\Delta^+ \wedge \mathrm{id}} E\Sigma_m^+ \wedge E\Sigma_m^+ \wedge (X \wedge Y)^{\wedge m} \xrightarrow{\sim} E\Sigma_m^+ \wedge X^{\wedge m} \wedge E\Sigma_m^+ \wedge Y^{\wedge m}.$$

This is Σ_m equivariant, with the evident Σ_m-action on the domain, and action on the codomain obtained by restricting the evident $\Sigma_m \times \Sigma_m$ action to the diagonal copy of Σ_m. The map δ_m is obtained by taking homotopy orbits.

For instance, we see that for $X = Y = S^0$, the map δ^m is Σ_+^∞ applied to the diagonal inclusion $B\Sigma_m \to B\Sigma_m \times B\Sigma_m$. More generally, if $X = S^{c_1}$ and $Y = S^{c_2}$, we get that δ_m is the map

$$B\Sigma_m^{(c_1+c_2)V_m} \to B\Sigma_m^{c_1 V_m} \wedge B\Sigma_m^{c_2 V_m}$$

associated to the pullback diagram of virtual bundles.

$$\begin{tikzcd} (c_1+c_2)V_m \arrow[r] \arrow[d] & c_1 V_m \times c_2 V_m \arrow[d] \\ B\Sigma_m \arrow[r] & B\Sigma_m \times B\Sigma_m \end{tikzcd}$$

The universal example of the product operation on the homotopy of a commutative S is given by δ; if $a_i \in \pi_{c_i} A$ for some commutative S-algebra A, then

$$S^{c_1+c_2} \approx S^{c_1} \wedge S^{c_2} \to \mathbb{P}(S^{c_1} \wedge S^{c_2}) \xrightarrow{\delta} \mathbb{P}(S^{c_1}) \wedge \mathbb{P}(S^{c_2}) \approx \mathbb{P}(S^{c_1} \vee S^{c_2}) \xrightarrow{(a_1,a_2)} A$$

represents $a_1 a_2 \in \pi_{c_1+c_2} A$. To calculate the effect a Q^s on a product, we need to calculate

$$R \wedge \mathbb{P}^2(S^{c_1+c_2}) \to R \wedge \mathbb{P}^2(S^{c_1}) \wedge \mathbb{P}(S^{c_2})$$

in homotopy. By the Thom spectrum interpretation given above, we see that this sends

$$e_s \mapsto \sum_{i+j=s} e_i \otimes e_j;$$

when $c_1 = c_2 = 0$, this is determined by the ring struture of $H^*(B\Sigma_2)$, and the general case is derived from this using the Thom isomorphism.

10.8. **Adem relations.** Since $\mathbb{P}$ is a monad on $h\mathrm{Mod}_S$, there is a natural map $\mu_X \colon \mathbb{P}\mathbb{P}(X) \to \mathbb{P}(X)$. Given $k, \ell \geq 0$ there is a map

$$\mathbb{P}^k\mathbb{P}^\ell(X) \xrightarrow{\mathbb{P}^k \mathrm{incl.}} \mathbb{P}^k\mathbb{P}(X) \xrightarrow{\mathrm{incl.}} \mathbb{P}\mathbb{P}(X).$$

Let $\Sigma_k \wr \Sigma_\ell$ denote the semi-direct product $\Sigma_\ell^k \rtimes \Sigma_k$. We have the following.

Proposition 10.9. *There are commutative diagrams*

$$\begin{tikzcd} \mathbb{P}\mathbb{P}(X) \arrow[r, "\mu"] & \mathbb{P}(X) \\ \mathbb{P}^k\mathbb{P}^\ell(X) \arrow[u] \arrow[r, "\mu_{k\ell}"'] & \mathbb{P}^{k\ell}(X) \arrow[u] \end{tikzcd}$$

where the map $\mu_{k\ell}$ is obtained as follows. Consider any $\Sigma_k\wr\Sigma_\ell$-equivariant map $E\Sigma_k\times E\Sigma_\ell^k \to E\Sigma_{k\ell}$, where the action on the target is through the inclusion $\Sigma_k\wr\Sigma_\ell \subset \Sigma_{k\ell}$. (There is only one such map up to equivariant homotopy.) Let $\mu_{k\ell}$ be the map obtained from

$$E\Sigma_k^+ \wedge (E\Sigma_\ell \wedge X^{\wedge\ell})^{\wedge k} \xrightarrow{\sim} (E\Sigma_k \times E\Sigma_\ell^k)_+ \wedge X^{\wedge k\ell} \to E\Sigma_{k\ell} \wedge X^{\wedge k\ell}$$

by taking homotopy orbits.

For instance, if $X = S^0$, this is the map $B\Sigma_k\wr\Sigma_\ell \to B\Sigma_{k\ell}$ induced by inclusion. More generally, for $X = S^c$, this is

$$(B\Sigma_k\wr\Sigma_\ell)^{cV_{k\wr\ell}} \to B\Sigma_{k\ell}^{cV_{k\ell}}$$

obtained from the pullback square of virtual bundles

$$\begin{tikzcd} V_{k\wr\ell} \arrow[r] \arrow[d] & V_{k\ell} \arrow[d] \\ B\Sigma_k\wr\Sigma_\ell \arrow[r] & B\Sigma_{k\ell} \end{tikzcd}$$

Suppose $\alpha\colon R\wedge S^d \to R\wedge\mathbb{P}(S^c)$ and $\beta\colon R\wedge S^e \to R\wedge\mathbb{P}(S^d)$ are maps of R-modules, which each give rise to homotopy operations $\alpha\propto\colon \pi_c A\to\pi_d A$ and $\beta\propto\colon \pi_d A\to\pi_e A$ on an R-algebra A. Consider the commutative diagram

$$\begin{tikzcd} & & R\wedge S^c \arrow[rr, "a"] \arrow[d] & & A \\ & R\wedge S^d \arrow[r, "\alpha"] \arrow[d] & \mathbb{P}_R(R\wedge S^c) \arrow[rr, "\mathbb{P}_R(a)"'] & & \mathbb{P}_R(A) \arrow[u, "\mu"'] \\ R\wedge S^e \arrow[r, "\beta"'] & \mathbb{P}_R(R\wedge S^d) \arrow[r, "\mathbb{P}_R(\alpha)"'] & \mathbb{P}_R\mathbb{P}_R(R\wedge S^c) \arrow[u, "\mu"'] & & \end{tikzcd}$$

The diagram commutes, and the composite $\mu\circ\mathbb{P}_R(\alpha)\circ\mu\circ\mathbb{P}_R(\alpha)\circ\beta$ represents $\beta\propto\alpha\propto a\in\pi_e A$. Thus, the map $\mu\circ\mathbb{P}_R(\alpha)\circ\beta\colon R\wedge S^e\to\mathbb{P}_R(R\wedge S^c)$ correspond to the composite operation $\beta\propto\alpha$.

If $\alpha = Q^s$ and $\beta = Q^r$, then the relevant map is

$$R\wedge S^{r+s+c} \xrightarrow{e_r} \mathbb{P}_R^2(R\wedge S^{s+c}) \xrightarrow{\mathbb{P}_R^2(e_s)} \mathbb{P}_R^2\mathbb{P}_R^2(R\wedge S^c) \xrightarrow{\mu_{22}} \mathbb{P}_R^4(R\wedge S^c),$$

which corresponds to an element (which we might denote $e_r\wr e_s$) in $R_{r+s+c}B\Sigma_p^{cV_4}$. The Adem relation is thus a relation among certain elements of the homology of $B\Sigma_2$.

10.10. **Suspension map.** Consider the category Alg_R/R of commutative R-algebra spectra equipped with an augmentation to R. There is an associated homotopy category $h(\mathrm{Alg}_R/R)$. There are loop and suspension functors

$$\Sigma_{\mathrm{Alg}_R}\colon \mathrm{Alg}_R/R \rightleftarrows \mathrm{Alg}_R/R : \Omega_{\mathrm{Alg}_R}.$$

If $A \in \mathrm{Alg}_R/R$, then as an R-module we have $A \approx R \times \bar{A}$, where $\bar{A} = \mathrm{fib}(A \to R)$. As an R-module, we have a weak equivalence $\Omega_{\mathrm{Alg}_R} A \approx R \times \Sigma^{-1}\bar{A}$. The ring structure on $\Omega_{\mathrm{Alg}_R} A$ can be non-trivial to describe, however, we can say that as a ring $\pi_*\Omega_{\mathrm{Alg}_R}$ has a square zero augmentation ideal.

Suspension is computed by a derived smash product: $\Sigma_{\mathrm{Alg}_R} A \approx R \wedge_A R$.

Example 10.11. If X is a spectrum, then $\Sigma^\infty_+\Omega^\infty X$ is a commutative S-algebra, augmented over S by $\Sigma^\infty_+\Omega^\infty X \to \Sigma^\infty_+ * \approx S^0$. There is a natural map

$$\Sigma_{\mathrm{Alg}_S}(\Sigma^\infty_+\Omega^\infty X) \to \Sigma^\infty_+\Omega^\infty(\Sigma X)$$

of commutative S-algebras, constructed by considering the commutative square

$$\begin{tikzcd}
\Sigma^\infty_+\Omega^\infty X \arrow[r] \arrow[d] & \Sigma^\infty_+\Omega^\infty(C_1 \wedge X) \arrow[d] \\
\Sigma^\infty_+\Omega^\infty(C_2 \wedge X) \arrow[r] & \Sigma^\infty_+\Omega^\infty S^1 \wedge X
\end{tikzcd}$$

where $S^1 \approx C_1 \cup_{S^0} C_2$, with C_1 and C_2 contractible. It turns out that this is a weak equivalence if X is (-1)-connected.

Example 10.12. Let T be a pointed topological space, and R a commutative S-algebra, so that R^T is a commutative R-algebra, augmented over R by $R^T \to R^* \approx R$. There is a natural map

$$\Sigma_{\mathrm{Alg}_R} R^T \to R^{\Omega T}$$

of commutative R-algebras, constructed using the commutative square

$$\begin{tikzcd}
R^{\Omega T} \arrow[r] \arrow[d] & R^{P_1 T} \arrow[d] \\
R^{P_2 T} \arrow[r] & R^T
\end{tikzcd}$$

where P_1T and P_2T are path fibrations over T. The map can sometimes be shown to be a weak equivalence, for instance if T is finite type and simply connected.

A free algebra $\mathbb{P}_R(M)$ has a standard augmentation, induced by the zero map $M \to R$. We see that $\Sigma_{\mathrm{Alg}_R}\mathbb{P}_R(M) \approx \mathbb{P}_R(\Sigma M)$.

Now specialize to the case of $R = HR$, where $R \supset \mathbb{F}_2$. Let's write $\bar{R}\{x_c\} \overset{\mathrm{def}}{=} \ker R\{x_c\} \to R$, where $R\{x_c\}$ denotes the free graded DL_R-model on some generator x_c in degree c. Thus,

$$\bar{R}\{x_c\}_d \approx h\mathrm{Alg}_R/R(\mathbb{P}_R(R \wedge S^d), \mathbb{P}_R(R \wedge S^c)),$$

and therefore the suspension functor Σ_{Alg_R} induces a function

$$\omega\colon \bar{R}\{x_c\}_d \to \bar{R}\{x_{c+1}\}_{d+1}.$$

I claim (and it is not hard to check) that

(1) ω kills products, i.e., it factors through $\bar{R}\{x_c\}_* \to \bar{R}\{x_c\}_*/(\bar{R}\{x_c\}_*)^2 \to \bar{R}\{x_{c+1}\}_{*+1}$;
(2) ω is compatible with the "composition" operator $\propto\colon \bar{R}\{x_d\}_e \times \bar{R}\{x_c\}_d \to \bar{R}\{x_c\}_e$.

Furthermore, suppose $A \in \mathrm{Alg}_R$, and write $\bar{A} \overset{\text{def}}{=} \mathrm{fib}(A \to R)$; recall that $\overline{\Omega_{\mathrm{Alg}_R} A} \approx \Sigma^{-1}\bar{A}$. Then for any $f \in \bar{R}\{x_c\}_d$ the two operations

$$f\propto\colon \pi_c\overline{\Omega_{\mathrm{Alg}_R} A} \to \pi_d\overline{\Omega_{\mathrm{Alg}_R} A} \quad \text{and} \quad \omega(f)\propto\colon \pi_{c+1}\bar{A} \to \pi_{d+1}\bar{A}$$

coincide; this is just an application of adjointness.

Finally, we can compute the suspension map using the evident natural transformation

$$\sigma\colon \Sigma\mathbb{P}_R(M) \to \Sigma_{\mathrm{Alg}_R}\mathbb{P}_R(M) \approx \mathbb{P}_R(\Sigma M)$$

of R-modules. The commutative diagram

$$\begin{tikzcd}
\Sigma R \wedge S^d \arrow[r] \arrow[dr, "f"'] & \Sigma\mathbb{P}_R(R \wedge S^d) \arrow[r, "\sigma"] \arrow[d, "\Sigma\tilde{f}"'] & \mathbb{P}_R(R \wedge S^{d+1}) \arrow[d, "\Sigma_{\mathrm{Alg}_R}(\tilde{f})"] \\
& \Sigma\mathbb{P}_R(R \wedge S^c) \arrow[r, "\sigma"'] & \mathbb{P}_R(R \wedge S^{c+1})
\end{tikzcd}$$

shows that if $\alpha \in \pi_d\mathbb{P}_R(R \wedge S^c)$ is represented by $f\colon R \wedge S^d \to \mathbb{P}_R(R \wedge S^c)$, the suspension $\omega(\alpha)$ is represented by

$$R \wedge S^{d+1} \approx \Sigma(R \wedge S^d) \xrightarrow{f} \Sigma\mathbb{P}_R(R \wedge S^c) \xrightarrow{\sigma} \mathbb{P}_R(\Sigma(R \wedge S^d)) \approx \mathbb{P}_R(M \wedge S^{c+1}).$$

So we need to calculate the map σ. Over S, this is the map obtained as the wedge of maps

$$\sigma_m\colon \Sigma\mathbb{P}^m(X) \to \mathbb{P}^m(\Sigma X)$$

obtained by taking homotopy orbits of the "diagonal" map

$$S^1 \wedge X^{\wedge m} \to (S^1)^{\wedge m} \wedge X^{\wedge m} \approx (S^1 \wedge X)^{\wedge m}.$$

Proposition 10.13. *In* DL_R, $\omega(Q^s) = Q^s$.

To prove this, consider $\sigma_m\colon \Sigma\mathbb{P}^m(S^c) \to \mathbb{P}^m(S^{c+1})$. By the above description, this can be identified with the map

$$B\Sigma_m^{\mathbb{R}\oplus cV_m} \to B\Sigma_m^{V_m\oplus cV_m},$$

induced by the inclusion $\mathbb{R} \to V_m$ of the trivial representation inside V_m. In other words, this is a kind of "zero section" map; in cohomology, using the Thom isomorphism, it corresponds to multiplication by the euler class of $\bar{V}_m$, where $V_m \approx \mathbb{R} \oplus \bar{V}_m$. When $m = 2$, $\bar{V}_2$ is the sign representation, whose Euler class is the generator of $H^1(B\Sigma_2; R)$. This implies that

$$\omega\colon \approx H_{s+c}(\mathbb{P}^2 S^c) \approx H_{s+c+1}(\Sigma\mathbb{P}^2 S^c) \to H_{s+c+1}(\mathbb{P}^2 S^{c+1})$$

sends $e_s \mapsto e_s$ (and in particular $e_c \mapsto 0$).

Note that this implies that $\omega(Q^{i_1} \cdots Q^{i_r}) = Q^{i_1} \cdots Q^{i_r}$.

11. Identification of DL_{HR} for $\mathbb{F}_2$-algebras

Let T be the graded theory defined by the axioms (1)–(6), and let $\mathrm{DL} = \mathrm{DL}_R$. I will sketch the proof that the map $\phi\colon T \to \mathrm{DL}$ is an isomorphism, following an argument due to McClure.

First, we might as well assume that $R = \mathbb{F}_2$, since the general case can be obtained by base change along $\mathbb{F}_2 \to R$.

I'll use the notation $\mathbb{F}_2\{S\}$ for the free T-model on the graded set S. We need to show that, if $S = \{x_1, \dots, x_r\}$ with $|x_i| = c_i$, that

$$\mathbb{F}_2\{S\} \to \pi_* R \wedge \mathbb{P}(S^{c_1} \vee \cdots \vee S^{c_r}) \approx H_*(\mathbb{P}(S^{c_1} \vee \cdots \vee S^{c_r})).$$

Define a new grading on $\mathbb{F}_2\{S\}$, which I'll call the **weight grading**, as follows: $Q^I(x_j)$ has weight 2^r, where $I = (i_1, \dots, i_r)$ is an admissible sequence, and the weight of a product is a sum of weights. If we write $\mathbb{F}_2\{S\}_{[m]}$ for the weight m piece, then we see that the map $\phi\colon T \to \mathrm{DL}$ preserves weight, so that the map restricts to

$$\phi\colon \mathbb{F}_2\{S\}_{[m]} \to H_*(\mathbb{P}^m(S^{c_1} \vee \cdots \vee S^{c_r})).$$

We show that ϕ is an isomorphism (for all finite graded sets S) by induction on m.

First note that weight is compatible with tensor product, so that

$$\mathbb{F}_2\{x_1, \dots, x_r\}_{[m]} \approx \bigoplus_{m = \sum m_i} \mathbb{F}_2\{x_1\}_{[m_1]} \otimes \cdots \otimes \mathbb{F}_2\{x_r\}_{[m_r]},$$

and there is a compatible decomposition

$$H_*(\mathbb{P}^m(S^{c_1} \vee \cdots \vee S^{c_r})) \approx \bigoplus_{m = \sum m_i} H_*\mathbb{P}^{m_1} S^{c_1} \otimes \cdots \otimes H_*\mathbb{P}^{m_r} S^{c_r}.$$

Thus, to prove case m, it will suffice to show that

$$\phi_m\colon \mathbb{F}_2\{x\}_{[m]} \to H_*\mathbb{P}^m(S^c)$$

for every grading $c \in \mathbb{Z}$. There are two separate cases: odd and even m.

11.1. **Case of odd m.**

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{F}\{x\}_{[m]} \arrow[d, "\phi_m"'] \arrow[rr, bend left, "\theta_m"] & \mathbb{F}\{x_1\}_{[1]} \otimes \mathbb{F}\{x_2\}_{[m]} \arrow[r, "\mu"'] \arrow[d, "\phi_1 \otimes \phi_{m-1}"', "\sim"] & \mathbb{F}\{x\}_{[m]} \arrow[d, "\phi_m"] \\
H_*\mathbb{P}^m S^c \arrow[r, "\tau"] \arrow[rr, bend right, "\sim"'] & H_*\mathbb{P}^1 S^c \otimes H_*\mathbb{P}^{m-1} S^c \arrow[r, "\mu"] & H_*\mathbb{P}^m S^c
\end{tikzcd}$$

where the maps labeled μ are induced by multiplication ($\mathbb{P}S^c \wedge \mathbb{P}S^c \to \mathbb{P}S^c$ or $\mathbb{F}\{x_1\} \otimes \mathbb{F}\{x_2\} \to \mathbb{F}\{x\}$), the map labeled τ is induced by $\mathbb{P}(\nabla)\colon \mathbb{P}S^c \to \mathbb{P}(S^c \vee S^c)$, and $\theta = \mu \circ (\phi_1 \otimes \phi_{m-1})^{-1} \circ \tau \circ \phi_m$, using the fact that ϕ_1 and ϕ_{m-1} are isomorphisms by the inductive hypothesis.

Since τ is calculated as a transfer map $B\Sigma_m^{cV_m} \to (B\Sigma_1 \times B\Sigma_{m-1})^{cV_1 \times cV_{m-1}}$ associated to an m-fold cover, we see that $\mu \circ \tau$ is an isomorphism in mod 2 homology, since m is odd. Thus we conclude that ϕ_m is epi, since $\phi_m \circ \mu = \mu \circ \phi_1 \otimes \phi_{m-1}$ is epi.

To show that ϕ_m is mono, it suffices to show that θ_m is mono, since $\phi_m = (\mu\circ\tau)^{-1}\circ\phi_m\circ\theta_m$. Since $\mathbb{F}\{x\}_{[m]}$ is a finite type $\mathbb{F}_2$-vector space, it suffices to show that θ_m is epi.

To show that θ_m is epi, we consider the following diagram.

$$\begin{tikzcd}
\bigvee_{0<i<m} \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c \arrow[rr, "\alpha"] \arrow[d, "\iota"'] & & \bigvee_{0<i<m} \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c \arrow[d, "\iota"'] \\
\mathbb{P}^m(S^c \vee S^c) \arrow[r, "\tau"'] \arrow[d] & \mathbb{P}^1(S^c \vee S^c) \wedge \mathbb{P}^{m-1}(S^c \vee S^c) \arrow[r] \arrow[d] & \mathbb{P}^m(S^c \vee S^c) \arrow[d] \\
\mathbb{P}^m S^c \arrow[r, "\tau"'] & \mathbb{P}^1 S^c \wedge \mathbb{P}^{m-1} S^c \arrow[r] & \mathbb{P}^m S^c
\end{tikzcd}$$

The maps marked ι are inclusions of summands. The maps marked τ are "transfer" maps, associated to the "projection" maps which follow them. We can summarize this by observing that they are induced by the following diagram of Σ_m sets.

$$\begin{tikzcd}
\underline{m} \times \Sigma_m/\Sigma_i \times \Sigma_{m-i} \arrow[r] \arrow[d, "\mu"'] & \Sigma_m/\Sigma_i \times \Sigma_{m-i} \arrow[d, "\mu"'] \\
\underline{m} \times \underline{2}^m \arrow[r] \arrow[d] & \underline{2}^m \arrow[d] \\
\underline{m} \arrow[r] & \underline{1}
\end{tikzcd}$$

The unmarked vertical maps are induced from the "fold" map $S^c \vee S^c \to S^c$, i.e., by the "multiplication" $\mathbb{P}S^c \wedge \mathbb{P}S^c \to \mathbb{P}S^c$. The map marked α is the restriction of the map in the middle row to the "middle" wedge summands in the decomposition $\mathbb{P}^m(X \vee Y) \approx \bigvee_{0\leq i\leq m} \mathbb{P}^i X \wedge \mathbb{P}^{m-i} Y$. To see that this works, notice that

$$\bigvee_{0<i<m} \mathbb{P}^m(X \wedge Y) \approx \operatorname{fib}\left[\mathbb{P}^m(X \vee Y) \to \mathbb{P}^m X \vee \mathbb{P}^m Y\right].$$

The horizontal composites $\mu\circ\tau$ are all p-local equivalences, since m is odd; thus α is also an equivalence. The vertical composites $\mu\circ\iota$ in the left-hand and right-hand columns are surjective in homology, by the transfer argument already given.

Consider the diagram

$$\begin{tikzcd}
\bigoplus\limits_{0<i<m} \mathbb{F}\{y\}_{[i]} \otimes \mathbb{F}\{z\}_{[m-i]} \arrow[rr, dotted, "\theta'_m"] \arrow[d, "\iota"'] & & \bigoplus\limits_{0<i<m} \mathbb{F}\{y\}_{[i]} \otimes \mathbb{F}\{z\}_{[m-i]} \arrow[d, "\iota"'] \\
\mathbb{F}\{y,z\}_{[m]} \arrow[r, dotted, "\tau"'] \arrow[d, "\mu"'] & \mathbb{F}\{y_1,z_1\}_{[1]} \otimes \mathbb{F}\{y_2,z_2\}_{[m-1]} \arrow[r] \arrow[d] & \mathbb{F}\{y,z\}_{[m]} \arrow[d, "\mu"'] \\
\mathbb{F}\{x\}_{[m]} \arrow[r, dotted, "\tau"'] \arrow[rr, bend right, "\theta_m"'] & \mathbb{F}\{x_1\}_{[1]} \otimes \mathbb{F}\{x_2\}_{[2]} \arrow[r] & \mathbb{F}\{x\}_{[m]}
\end{tikzcd}$$

This diagram maps to the homology of the previous diagram, by the maps ϕ. In the middle column, the ϕ maps are isomorphisms by the inductive hypothesis, and so we can define the dotted arrows marked τ using this. The arrow θ'_m is restriction of the middle row to the "middle" summands. The image of the vertical composites $\mu \circ \iota$ is equal to $\mathbb{F}\{x\}_{[m]}$, since m is odd and the odd weight part of $\mathbb{F}\{x\}$ consists entirely of decomposables. Thus, to show θ_m surjective it suffices to show that θ'_m are surjective. We apply the induction hypothesis, which shows that $\phi_i \otimes \phi_{m-i}$ is an isomorphism for $0 < i < m$, and therefore identifies θ'_m with the isomorphisms on homology induced by $\mu \circ \tau \colon \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c \to \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c$.

11.2. **Case of even m.** Write $m = 2k$. I need a new piece of notation: write $\mathbb{C}(V)$ for the free T-model on an underlying graded vector space, and write $\mathbb{C}_m(V)$ for the weight m piece of this, where elements of V are assumed to have weight 1. Thus $\mathbb{F}\{x\}_{[m]} \approx \mathbb{C}_m(\mathbb{F} \cdot x)$.

Consider the commutative diagram

$$\begin{tikzcd}
\mathbb{C}_m(\mathbb{F}\cdot x) \arrow[d, "\phi_m"'] \arrow[rr, bend left, "\theta_m"] & \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x) \arrow[r, "\mu"'] \arrow[d, "\phi"', "\sim"] & \mathbb{C}_m(\mathbb{F}\cdot x) \arrow[d, "\phi_m"] \\
H_*\mathbb{P}^m(S^c) \arrow[r, "\tau"] \arrow[rr, bend right, "\sim"'] & H_*\mathbb{P}^k\mathbb{P}^2(S^c) \arrow[r, "\mu"] & H_*\mathbb{P}^m(S^c)
\end{tikzcd}$$

where the maps labeled μ are induced by monad structure ($\mathbb{P}\mathbb{P}S^c \to \mathbb{P}S^c$ or $\mathbb{C}\mathbb{C}V \to \mathbb{C}V$). The bottom μ is given by $B(\Sigma_k \wr \Sigma_2)^{cV_{k\wr 2}} \to B\Sigma_m^{cV_m}$, and we define τ to be the transfer associated to the cover $B(\Sigma_k \wr \Sigma_2) \to B\Sigma_m$. The group $\Sigma_k \wr \Sigma_2 \approx \Sigma_k \rtimes \Sigma_2^k$ has odd index in Σ_m, since $\nu_2(m!) = m - \alpha(m)$ where $\alpha(m)$ is the number of 1s in the dyadic expansion, so that $\nu_2|\Sigma_k \wr \Sigma_2| = \nu_2(|\Sigma_k| \cdot |\Sigma_2|^k) = \nu_2(k!) + k\nu_2(2) = k - \alpha(m) + k = m - \alpha(m) = \nu_2(m!)$.

Thus, the composite $\mu\tau$ is an equivalence. Thus ϕ_m is epi, since $\phi_m \circ \mu = \mu \circ \phi$, and ϕ is an iso by the inductive hypothesis. It remains to show that ϕ_m is mono, and it will suffice to prove this by showing θ_m epi, as we did in the case of odd m.

Consider the following diagram

$$\begin{tikzcd}
\bigvee\limits_{0<i<m} \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c \arrow[rr, "\alpha"'] \arrow[d, "\iota"'] & & \bigvee\limits_{0<i<m} \mathbb{P}^i S^c \wedge \mathbb{P}^{m-i} S^c \arrow[d, "\iota"'] \\
\mathbb{P}^m(S^c \vee S^c) \arrow[r, "\tau"'] \arrow[d] & \mathbb{P}^k\mathbb{P}^2(S^c \vee S^c) \arrow[r] \arrow[d] & \mathbb{P}^m(S^c \vee S^c) \arrow[d] \\
\mathbb{P}^m S^c \arrow[r, "\tau"'] & \mathbb{P}^k\mathbb{P}^2 S^c \arrow[r] & \mathbb{P}^m S^c
\end{tikzcd}$$

the τ are transfer maps, and ι are the evident inclusions. There is a corresponding algebraic diagram

$$\begin{tikzcd}
\bigoplus\limits_{0<i<m} \mathbb{C}_i(\mathbb{F}\cdot y) \otimes \mathbb{C}_{m-i}(\mathbb{F}\cdot z) \arrow[rr, dotted, "\theta'_m"] \arrow[d, "\iota"'] & & \bigoplus\limits_{0<i<m} \mathbb{C}_i(\mathbb{F}\cdot y) \otimes \mathbb{C}_{m-i}(\mathbb{F}\cdot z) \arrow[d, "\iota"'] \\
\mathbb{C}_m(\mathbb{F}\cdot y \oplus \mathbb{F}\cdot z) \arrow[r, dotted, "\tau"'] \arrow[d, "\mu"'] & \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot y \oplus \mathbb{F}\cdot z) \arrow[r] \arrow[d] & \mathbb{C}_m(\mathbb{F}\cdot y \oplus \mathbb{F}\cdot z) \arrow[d, "\mu"'] \\
\mathbb{C}_m(\mathbb{F}\cdot x) \arrow[r, dotted, "\tau"'] \arrow[rr, bend right, "\theta_m"'] & \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x) \arrow[r] & \mathbb{C}_m(\mathbb{F}\cdot x)
\end{tikzcd}$$

This maps to the homology of the previous diagram by the maps ϕ; these are isomorphisms in the middle column, and on the the objects in the top row, by induction. The maps τ are are induced by the transfer maps in the topological diagram; the map θ'_m is the restriction of the middle row to the "middle" summands.

Recall that we want to show that θ_m is an isomorphism. Let $\mathcal{D}' = \bigoplus_{0<i<m} \mathbb{C}_i(\mathbb{F}\cdot y) \otimes \mathbb{C}_{m-i}(\mathbb{F}\cdot z)$. Let $\mathcal{D} \subset \mathbb{C}_m(\mathbb{F}\cdot x)$ denote the image of the composites $\mu \circ \iota$ in the left and right-hand columns; these are precisely the sums of decomposable elements in $\mathbb{C}_m(\mathbb{F}\cdot x)$. Since the diagram commutes, this implies that $\theta_m(\mathcal{D}) \subset \mathcal{D}$. Since $\mathcal{D}$ and $\mathcal{D}'$ have finite type, to show $\theta_m|_{\mathcal{D}}$ is iso it is enough to show θ'_m is mono. By the inductive hypothesis, the maps $\phi_i \otimes \phi_{m-i}$ are isomorphisms for $0 < i < m$, and thus θ'_m can be identified with $H_*\alpha$, which is an isomorphism.

Let $\mathcal{I}_c = \mathbb{C}_m(\mathbb{F}\cdot x)/\mathcal{D}$, where $|x| = c$. The map θ_m descends to $\bar{\theta}\colon \mathcal{I} \to \mathcal{I}$, and we'll be done once we show $\bar{\theta}$ is iso. We have an explicit description of $\mathcal{I}$; it has a basis given by the set $\{Q^I(x)\}$, where I spans the admissible sequences $(i_1, \cdots, i_r)$ with $e(I) > c$ and with $2^r = m$. (In particular, $\mathcal{I} = 0$ except for $m = 2^r$.)

We define a map $\gamma\colon \mathcal{I}_c \to \mathbb{C}_m(\mathbb{F}\cdot x')$, where $|x'| = c+1$, by $\gamma(Q^I(x)) = Q^I(x')$. (This map raises degree by 1.) The calculation of the "suspension" operator ω that we did shows that

$$\begin{tikzcd} \mathbb{C}_m(\mathbb{F}\cdot x) \arrow[r, "\pi"'] \arrow[d, "\phi"'] & \mathcal{I}_c \arrow[r, "\gamma"] & \mathbb{C}_m(\mathbb{F}\cdot x') \arrow[d, "\phi"] \\ H_*\mathbb{P}^m S^c \arrow[rr, "\omega"'] & & H_{*+1}\mathbb{P}^m S^{c+1} \end{tikzcd}$$

commutes.

Lemma 11.3. *The diagram*

$$\begin{tikzcd} \mathcal{I}_c \arrow[r, "\bar\theta"] \arrow[d, "\gamma"'] & \mathcal{I}_c \arrow[d, "\gamma"] \\ \mathbb{C}_m(\mathbb{F}\cdot x') \arrow[r, "\theta_m"'] & \mathbb{C}_m(\mathbb{F}\cdot x') \end{tikzcd}$$

commutes, and also that γ is an injection.

Proof. Consider

$$\begin{tikzcd} \mathbb{C}_m(\mathbb{F}\cdot x) \arrow[r, "\phi_m"] \arrow[d, "\gamma\pi"'] & H_*\mathbb{P}^m S^c \arrow[r, "\tau"] \arrow[d, "\omega"'] & H_*\mathbb{P}^k\mathbb{P}^2 S^c \arrow[d, "\omega"'] & \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x) \arrow[l, "\phi", "\sim"'] \arrow[r, "\mu"] \arrow[d, "\gamma'\pi"'] & \mathbb{C}_m(\mathbb{F}\cdot x) \arrow[d, "\gamma\pi"'] \\ \mathbb{C}_m(\mathbb{F}\cdot x') \arrow[r, "\phi_m"] & H_{*+1}\mathbb{P}^m S^{c+1} \arrow[r, "\tau"] & H_{*+1}\mathbb{P}^k\mathbb{P}^2 S^{c+1} & \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x') \arrow[l, "\phi", "\sim"'] \arrow[r, "\mu"] & \mathbb{C}_m(\mathbb{F}\cdot x') \end{tikzcd}$$

where $\gamma'\colon \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x)/(\text{decomposables}) \to \mathbb{C}_k\mathbb{C}_2(\mathbb{F}\cdot x')$ sends $Q^I[Q^J(x)] \mapsto Q^I[Q^J(x')]$. The commutativity of these squares is clear except perhaps for the right hand one, we we check commutativity by checking the formula. $\square$

Now let $\mathcal{I}_c(k) \subset \mathcal{I}_c$ be the subspace spanned by monomials $Q^I(x)$ with $e(I) \leq c+k$; then $\mathcal{I}_c = \bigcup \mathcal{I}_c(k)$. I'll show that $\bar\theta\colon \mathcal{I}_c \to \mathcal{I}_c$ is monic, by showing that $\bar\theta|_{\mathcal{I}_c(k)}$ is monic for all $k \geq 1$.

First we show $\bar\theta|_{\mathcal{I}_c(1)}$ is monic. Note that by explicit calculation,

$$\begin{tikzcd} \mathcal{I}_c(1) \arrow[r] \arrow[d, tail] & \mathcal{D}_c \arrow[d, tail] \\ \mathcal{I}_c \arrow[r, tail, "\gamma"'] & \mathbb{C}_m(\mathbb{F}\cdot x') \end{tikzcd}$$

is a pullback square, where $\mathcal{D}_{c+1}$ denotes the set of decomposables. By what we have already proved, $\theta_m|_{\mathcal{D}_{c+1}}$ is iso, and therefore using the lemma we conclude that $\bar\theta|_{\mathcal{I}_c(1)}$ is mono.

We now show that $\bar{\theta}|_{\mathcal{I}_c(k)}$ is mono by induction on k. Another explicit calculation shows that

$$\begin{tikzcd} \mathcal{I}_c(k) \arrow[rr, "\beta"] \arrow[d, tail] & & \mathcal{I}_{c+1}(k-1) \arrow[d, tail] \\ \mathcal{I}_c \arrow[r, tail, "\gamma"'] & \mathbb{C}_m(\mathbb{F}\cdot x') \arrow[r, two heads, "\pi"'] & \mathcal{I}_{c+1} \end{tikzcd}$$

is a pullback square (suspension decreases excess by 1); the kernel of β is $\mathcal{I}_c(1)$. If $w \in \mathcal{I}_c(k)$ such that $\bar{\theta}(w) = 0$, then $0 = \pi\gamma\bar{\theta}(w) = \pi\theta_m\gamma(w) = \bar{\theta}\pi\gamma(w) = \bar{\theta}\beta(w)$. By induction, $\bar{\theta}|_{\mathcal{I}_{c+1}(k-1)}$ and so $\beta(w) = 0$, and thus $w \in \mathcal{I}_c(1)$. But we have already shown $\bar{\theta}|_{\mathcal{I}_c(1)}$ is mono, $w = 0$, and we are done.

12. Odd primes

We can relate the homology of the symmetric group Σ_p to the homology of the cyclic group C_p. Consider the pullback square

$$\begin{tikzcd} (\Sigma_p/C_p) \times (\Sigma_p/C_p) \arrow[r] \arrow[d] & \Sigma_p/C_p \arrow[d] \\ \Sigma_p/C_p \arrow[r] & * \end{tikzcd}$$

of Σ_p sets. If we twist this with $(S^c)^{\wedge p}$ and take orbits with respect to Σ_p, we get a commutative square

$$\begin{tikzcd} (\Sigma_p/C_p)^+ \wedge_{hC_p} (S^c)^{\wedge p} \arrow[d, "\pi'"'] & BC_p^{cV_p} \arrow[l, "\tau'"'] \arrow[d, "\pi"] \\ BC_p^{cV_p} & B\Sigma_p^{cV_p} \arrow[l, "\tau"] \end{tikzcd}$$

where the maps marked τ and τ' are transfers. The standard argument shows that $\pi \circ \tau$ is a p-local equivance, so that p-locally $B\Sigma^{cV_p}$ is a retract of $BC_p^{cV_p}$.

To calculate this splitting, we need to calculate the map $\tau\pi = \pi'\tau'$. We decompose Σ_p/C_p as a C_p set: the C_p fixed points in Σ_p/C_p is the set $NC_p/C_p \subset \Sigma_p/C_p$, which has order $p-1$ and can be identified with $\mathrm{Aut}(C_p) \approx (\mathbb{F}_p^\times)$. The remaining C_p orbits are free. Thus

$$\pi'\tau' \colon BC_p^{cV_p} \xrightarrow{\tau'} \coprod_{\lambda \in \mathrm{Aut}(C_p)} BC_p^{cV_p} \amalg \coprod_{\text{free orbits}} (*)^{cV_p} \xrightarrow{\pi'} BC_p^{cV_p}.$$

The composite

$$BC_p^{cV_p} \xrightarrow{t} (*)^{cV_p} \approx S^{cp} \to BC_p^{cV_p}$$

is 0 in mod p homology, since the transfer map t has order p on the bottom cell. The other composites are self-equivalences of $BC_p^{cV_p}$. Explicitly, these are equal to maps

$$f_\lambda \colon BC_p^{cV_p} \xrightarrow{\alpha} BC_p^{c\lambda^* V_p} \xrightarrow{B\lambda} BC_p^{cV_p}$$

associated to the diagram

$$\begin{tikzcd} V_p \arrow[r, "\alpha"] \arrow[d] & \lambda^* V_p \arrow[r] \arrow[d] & V_p \arrow[d] \\ BC_p \arrow[r, equal] & BC_p \arrow[r, "B\lambda"] & BC_p \end{tikzcd}$$

where $\lambda \in \operatorname{Aut}(C_p)$ and $\alpha\colon V_p \to \lambda^* V_p$ is the map of C_p representations given as follows. If we identify $V_p \approx \mathbb{R}[C_p]$ with action $g \cdot [h] = [gh]$, then $\lambda^* V_p$ has action $g \cdot [h] = [g^\lambda h]$. The isomorphism $V_p \to \lambda^* V_p$ sends $[h] \mapsto [h^\lambda]$.

Thus, in mod p-homology, the map $\pi'\tau' = \sum_{\lambda\in\operatorname{Aut}(C_p)} f_\lambda^*$. The upshot is that

$$H_*(B\Sigma_p^{cV_p}) \approx H_*(BC_p^{cV_p})^{\operatorname{Aut}(C_p)},$$

where $\lambda \in \operatorname{Aut}(C_p)$ acts in the evident way, using the natural isomorphism $\lambda^* V_p \approx V_p$ of C_p-representations.

To calculate the cohomology of $BC_p^{cV_p}$, write $V_p \approx \mathbb{R} \oplus \bar{V}_p$, so that $BC_p^{cV_p} \approx \Sigma^c BC_p^{c\bar{V}_p}$. If $c = 2d$, then we can identify $c\bar{V}_p \approx d\bar{V}_p \otimes \mathbb{C}$; in particular, this bundle admits a canonical orientation, preserved by the $\operatorname{Aut}(C_p)$-action. Therefore the Thom class $u \in H_{c(p-1)}(BC_p^{c\bar{V}_p})$ is fixed by $\operatorname{Aut}(C_p)$, and therefore

$$H_* B\Sigma_p^{cV_p} \approx H_*(BC_p^{cV_p})^{\operatorname{Aut}(C_p)} \approx H_{*-c}(BC_p^{c\bar{V}_p})^{\operatorname{Aut}(C_p)} \approx H_{*-pc}(BC_p)^{\operatorname{Aut}(C_p)} \approx H_{*-pc} B\Sigma_p.$$

Now suppose $c = 2d+1$, so that $c\bar{V}_p \approx \bar{V}_p \oplus (d\bar{V}_p \otimes \mathbb{C})$. The real bundle $\bar{V}_p \subset V_p$ admits an orientation, but it is not canonical. Fix a generator $g \in C_p$, and for $k \not\equiv 0 \mod p$, let $L_k \approx \mathbb{R}\cdot u_k \oplus \mathbb{R}\cdot v_k \subset \bar{V}_p \subset V_p$, where

$$u_k = [1] + \Re(\omega^k)[g] + \cdots + \Re(\omega^{k(p-1)})[g^{p-1}] \quad \text{and} \quad v_k = [1] + \Im(\omega^k)[g] + \cdots + \Im(\omega^{k(p-1)})[g^{p-1}]$$

where $\omega = e^{2\pi i/p}$. Fix an orientation on L_k using the ordering u_k, v_k. Since $u_{-k} = u_k$ and $v_{-k} = -v_k$, we see that L_{-k} is the same subspace as L_k, but with the opposite orientation. The natural isomorphism $V_p \to \lambda^* V_p$ associated to $\lambda \in \operatorname{Aut}(C_p)$ sends L_k to $L_{k/\lambda}$ in a way which preserves orientations.

Note that $\bar{V}_p \approx L_1 \oplus \cdots \oplus L_q$, where $q = (p-1)/2$. If we use this to fix an orientation on $\bar{V}_p$, then we see that since $\bar{V}_p \to \lambda^* \bar{V}_p$ sends L_k to $L_{k/\lambda}$ by an orientation preserving map, to determine whether it preserves the orientation of $\bar{V}_p$ it suffices to check which of the $\{k/\lambda\}_{k\in\{1,\dots,q\}}$ are in $\{1,\dots,q\}$ or $\{-1,\dots,-q\}$. Thus, orientation changes by

$$\epsilon = \frac{\prod_{k=1}^q k}{\prod_{k=1}^q k/\lambda} = \lambda^q \in \{\pm 1\}.$$

Given that $H^*(BC_p;\mathbb{F}_p) \approx \mathbb{F}_p[y] \otimes E(x)$, with $|x| = 1$ and $|y| = 2$, write $H^*(BC_p^{cV_p};\mathbb{F}_p) \approx u(\mathbb{F}_p[y] \otimes E(x))$, where $u = u_c$ represents the Thom class with $|u| = cp$. An element $\lambda \in \operatorname{Aut}(C_p)$ acts by

$$x \mapsto \lambda x, \quad y \mapsto \lambda y, \quad u \mapsto (\lambda^q)^c u.$$

If c is even, this gives

$$H^*(B\Sigma_p^{V_p};\mathbb{F}_p) \approx u(\mathbb{F}_p[b] \otimes E(a))$$

with $a = xy^{p-2}$ and $b = y^{p-1}$. If c is odd, this gives

$$H^*(B\Sigma_p^{V_p}; \mathbb{F}_p) \approx u(y^q \mathbb{F}_p[b] \oplus xy^{q-1}\mathbb{F}_p[b]).$$

We can describe the suspension map $B\Sigma_p^{\mathbb{R}\oplus cV_p} \to B\Sigma_p^{(c+1)V_p}$ as well. In cohomology, the map $BC_p^{\mathbb{R}\oplus cV_p} \to BC_p^{(c+1)V_p}$ sends the Thom class u_{c+1} to $e(V_p)u_c$. We can calculate that $e(V_p) = (q!)y^{p-1}$.

13. $K(n)$-local rings

For general ring spectra E, $E_*B\Sigma_m$ tends to be poorly behaved; in particular, it is mostly torsion. We need to apply a localization to get good values.

Some facts about Bousfield localization functors. Let

$$L_n = L_{K(0)\vee\cdots\vee K(n)}.$$

Let F_n be any type n finite complex.

There are two basic facts about these localizations: L_n commutes with homotopy colimits, and L_{F_n} commutes with homotopy limits. The first fact is called the "smashing localization" theorem, and can be restated as $L_nX \approx X \wedge L_nS$. The second fact is a formal consequence of the fact that F_n is a finite complex, and in fact one can show that $L_{F_n}X \approx \hom(C_n, X)$, where C_n is a homotopy colimit of type n finite spectra.

Here is a quick construction of C_n, using only the thick subcategory theorem. Inductively define spectra Y_k and maps $f_k\colon Y_k \to S^0$ as follows. Set

$$Y_0 = *,$$

and if Y_k and f_k have been defined, let Z_k be the cofiber of f_k. Define Y_{k+1} by the fiber sequence

$$Y_{k+1} \to S^0 \to Z_{k+1}$$

where Z_{k+1} is defined by the cofiber sequence

$$\bigvee F \to Z_k \to Z_{k+1},$$

where the wedge is taken over a set of representatives of finite complexes of type at least n, and over all homotopy classes of maps to Z_k .

Let $C_n = \operatorname{hocolim} Y_k$, and write $f\colon C_n \to S^0$. You should think of C_n as the closest approximation to S^0 built from type n-finite complexes. I claim that

(1) $f \wedge F$ is an equivalence for all finite F of type at least n.

To see this, note that $f \wedge F$ is the same as $h\colon \hom(DF, C_n) \to \hom(DF, S^0)$, and this we need to show that $[DF, C_n] \to [DF, S^0]$ is an isomorphism. Equivalently, we must show that $[DF, Z] = 0$, where $Z = \operatorname{hocolim} Z_k$, which is clear.

Now note that $\hom(f, X)\colon X \approx \hom(S^0, X) \to \hom(C_n, X)$ must be an F-equivalence, since $\hom(f, X) \wedge F = \hom(f \wedge DF, X)$, and that $\hom(C_n, X)$ is F-local, since $[W, \hom(C_n, X)] = [W \wedge C_n, X]$, and $W \wedge F = 0$ implies $W \wedge C_n = 0$ by the construction of C_n.

Remark 13.1. The construction implies that $C_n \wedge Z = 0$, so that $Z \wedge Z \approx Z$. This means that $L_Z = Z \wedge -$ is itself a smashing localization, sometimes denoted L^f_{n-1}. The telescope conjecture asserts that $L^f_n \to L_n$ is always an equivalence.

Example 13.2. Let $M^0(p^e) = S^{-1} \cup_{p^e} e^0$ denote the mod p Moore spectrum with top cell in dimension 0. Let $C_1 = \operatorname{hocolim} M^0(p^e)$. Then

$$\hom(C_1, X) \approx \operatorname{holim} \hom(M^0(p^e), X) \approx \operatorname{holim} X \wedge M_0(p^e) \wedge X.$$

I claim that $\hom(C_1, X)$ is $M(p)$-localization, with $\iota\colon X \to \hom(C_1, X)$ induced by $f\colon C_1 \to S^0$ produced using $M^0(p^e) \to S^0$. To see this, note that if $W \wedge M(p) = 0$, then $W \wedge C_1 = 0$, and that $f \wedge M(p)$ is an equivalence.

The "universal coefficient theorem" gives a way to calculate $\pi_* L_{F_1} X$. Since $C_1 \approx S^{-1} \otimes \mathbb{Q}/\mathbb{Z}_{(p)}$, we have

$$0 \to \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, \pi_{k-1}X) \to \pi_k L_{F_1} X \to \operatorname{Ext}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, \pi_k X) \to 0.$$

Since $\mathbb{Q}/\mathbb{Z}_{(p)} \approx \operatorname{colim} \mathbb{Z}/p^e$, we have

$$0 \to \lim{}^1 \operatorname{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^e, \pi_* X) \to \operatorname{Ext}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, \pi_* X) \to \lim \operatorname{Ext}_{\mathbb{Z}_{(p)}}(\mathbb{Z}/p^e, \pi_* X).$$

Thus, if the tower $\cdots (\pi_* X)[p^e] \xrightarrow{p} (\pi_* X)[p^{e-1}] \to \cdots \to (\pi_* X)[p]$ has vanishing lim and $\lim^1$, (for instance, if there is a bound on the order of p-torsion in each $\pi_k X$), then

$$\pi_k L_{F_1} X \approx (\pi_k X)^\wedge_p.$$

The functor $\operatorname{Ext}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, -)$ is called **Ext-p-completion**, and it has a non-vanishing first right derived functor, which is $\operatorname{Hom}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, -)$. We can also describe this as "local homology" groups:

$$H_s^{(p)}(M) \stackrel{\text{def}}{=} \operatorname{Ext}^{1-s}_{\mathbb{Z}_{(p)}}(\mathbb{Q}/\mathbb{Z}_{(p)}, M), \qquad s = 0, 1,$$

and a spectral sequence

$$H_s^{(p)}(\pi_t X) \Longrightarrow \pi_{t+s} L_{F(1)} X.$$

There is a more general construction of the p-completion construction, using "generalized Moore complexes". If E is a complex oriented spectrum, and F is a type n-finite complex, and $f\colon \Sigma^{d|v_n|} F \to F$ is a "v_n-self map" (i.e., a $K(n)_*$-equivalence with $d > 0$), then the induced map $E_* f\colon E_{*-d|v_n|} F \to E_* F$ is v_n-multiplication up to nilpotence; i.e., there is some $q > 0$ such that $E_*(f^q)$ is multiplication by v_n^{dq}.

Proposition 13.3. *If E is a Landweber exact complex orientable theory, and M is an E-module spectrum, then there is a spectral sequence*

$$H_s^{(p, v_1, \dots, v_{n-1})}(\pi_t M) \Longrightarrow \pi_{s+t} L_{F(n)}(M),$$

where

$$H_s^{(p, v_1, \dots, v_{n-1})}(N) \stackrel{\text{def}}{=} \operatorname{Ext}^{n-s}_{E_*}(E_*/(p^\infty, \dots, v_{n-1}^\infty), N).$$

If $\pi_* M$ is a finitely generated E-module, then M is already $F(n)$-local.

Proof. See [HS99], and Hovey's paper on spectral sequences for E-theory. □

Proposition 13.4. $L_{K(n)} = L_{F_n} L_n$.

Proof. First I claim that $L_{F_n}L_nX$ is $K(n)$-local, i.e., that if $K(n)_*W = 0$ then $[W, L_{F_n}L_nX] = 0$. We have

$$[W, L_{F_n}L_nX] \approx [W, \mathrm{hom}(C_n, L_nX)] \approx [W \wedge C_n, L_nX].$$

We have that $K(i) \wedge C_n = 0$ for $i < n$ since C_n is a hocolim of type n finites, so that $K(i) \wedge C_n \wedge W = 0$ for $i \leq n$, using the Künneth theorem for $K(n)$. Thus $[W \wedge C_n, L_nX] = 0$.

Next I claim that $X \to L_nX \to L_{F_n}L_nX$ is a $K(n)_*$-equivalence. It is clear that $X \to L_nX$ is a $K(n)_*$-equivalence. A map $Y \to L_{F_n}Y$ is by hypothesis an $(F_n)_*$-equivalence. That is, $F_n \wedge Y \to F_n \wedge L_{F_n}Y$ is an equivalence, and therefore $K(n)_*(F_n \wedge Y) \to K(n)_*(F_n \wedge L_{F_n}Y)$ is an isomorphism. The $K(n)$ Künneth theorem and the fact that $K(n)_*F_n \neq 0$ imply that $K(n)_*Y \to K(n)_*L_{F_n}Y$ is an isomorphism as desired. □

Let E be an L_n-local spectrum. Then $E \wedge X$ is automatically L_n-local for any X, since $L_n(E \wedge X) \approx L_nS \wedge E \wedge X \approx (L_nE) \wedge X \approx E \wedge X$, and thus $L_{K(n)}(E \wedge X) \approx L_{F_n}(E \wedge X)$.

14. Morava E-theory

Let Γ be a height n formal group over a perfect field k of characteristic p.

A **deformation** of Γ to R is a triple (G, i, α), consisting of a formal group G over R, an inclusion $i\colon k \to R/\mathfrak{m}_R$, and an isomorphism $\alpha\colon G_0 \to i^*\Gamma$.

A **$\star$-isomorphism** $(G, i, \alpha) \to (G', i', \alpha')$ of deformations to R consists of an isomorphism $f\colon G \to G'$ of formal groups over R, such that $i = i'$ and $\alpha = \alpha' \circ f_0$.

Theorem 14.1 (Lubin-Tate)**.** *There is at most one $\star$-isomorphism between any two deformations of Γ to R. The set of $\star$-isomorphism classes of deformations of Γ to R are classified by a ring $\mathcal{O}$, isomorphic to $\mathbb{W}k[\![u_1, \dots, u_{n-1}]\!]$; we write $\chi_G\colon \mathcal{O} \to R$ for the map classifying the isomorphism class of G.*

There is an associated complex oriented cohomology theory E, with $\mathcal{O} = E^0(\mathrm{pt})$ and $\mathcal{O}_{G_{\mathrm{univ}}} = E^0(BS^1)$. We write $\eta\colon \mathcal{O} \to E^0(X)$ for the natural map.

Morava E-theory has Bousfield class of L_n, and is $K(n)$-local.

Hopkins-Miller-Goerss.

Completed Morava E-homology.

Proposition 14.2. *$E^\wedge_*(X) \approx 0$ if and only if $K(n)_*X \approx 0$.*

Proof. There is an E-module spectrum E obtained by killing $p, u_1, \dots, u_{n-1}$, so that $\pi_*K \approx k[u^\pm]$. The spectrum K is a wedge of copies of $K(n)$, and has the same Bousfield class. I want to show that $K \wedge_E M \approx 0$ if and only if $L_{F(n)}M \approx 0$, where M is an E-module spectrum. The idea is that if F is a type n finite complex, then $E \wedge F$ is built from finitely many copies of suspensions of K as an E-module. Therefore $K \wedge_E M = 0$ implies $M \wedge F = 0$, and thus $L_{F(n)}M = 0$. Conversely, if $L_{F(n)}M = 0$, then since $M \to L_{F(n)}M$ is a $K(n)$-equivalence, we have that $K(n)_*M = 0$, so $K \wedge_E M = 0$. □

A **completed E-module** is an E-module M which is $K(n)$-local, or equivalently which is $F(n)$-local. Note that

(a) If M and N are E-modules, and N is completed, then $\mathrm{hom}_E(M, N)$ is a completed E-module.

(b) If M and N are completed E-modules, then $L_{F(n)}(M\wedge_E N)$ is a completed E-module, called the completed smash product.

Proposition 14.3. *Let M be an E-module spectrum. The following are equivalent.*

(1) *$\pi_* M$ is a finitely generated (resp. finitely generated free) E_*-module.*
(2) *$[M, \Sigma^* E]_{\mathrm{Mod}_E}$ is a finitely generated (resp. finitely generated free) E_*-module.*

If either of these hold, then $M \approx \hom_E(\hom_E(M,E),E)$. Furthermore, such an M satisfies $L_{K(n)}M \approx M$.

In particular, if X is a space, E^*X is finitely generated (or free) if and only if $E^\wedge_* X$ is so.

Proof. See [HS99], section 8. □

Say that a E-module M is **finitely generated and free**, or **finite free** for short, if $\pi_* M$ is a finitely generated $\pi_* E$-module. It is easy to see that such $M \approx \bigvee_{i=1}^k \Sigma^{d_i} E$. Note that if M and N are finite free, then are $M \wedge_E N$ and $\hom_E(M,N)$.

15. Completed E-algebra spectra

If M is a E-module, we write $\hat{\mathbb{P}}^m_E(M) \overset{\mathrm{def}}{=} L_{F(n)}\mathbb{P}^m_E(M)$ for the completion. The goal of this section is to prove

Proposition 15.1. *If M is an E-module which is finite free, then $\hat{\mathbb{P}}^m_E(M)$ is also finite free.*

Lemma 15.2. *If p is a prime, then $E^\wedge_* BC_p^{cV_p}$ is a finitely generated free $\pi_* E$-module.*

Proof. First suppose $c = 0$. Then the cofiber sequence

$$BC_p \approx S(L^{\otimes p}) \to BS^1 \to (BS^1)^{L^{\otimes p}}$$

and the Thom isomorphism give

$$0 \leftarrow E^* BC_p \leftarrow E[\![x]\!] \xleftarrow{[p](x)} E[\![x]\!] \leftarrow 0,$$

and in particular since $[p](x) \equiv x^{p^n} \mod \mathfrak{m}$, $E^* BC_p$ is free over E_* on $1, \dots, x^{p^n - 1}$. Thus $\hom(\Sigma^\infty_+ BC_p, E)$ is a finitely generated free E-module, and therefore so is $L(E \wedge \Sigma^\infty_+ BC_p)$.

The Thom isomorphism immediately gives the result for c even, by identifying $2dV_p$ with the complex bundle $dV_p \otimes \mathbb{C}$. It remains to check the case of odd c, and the Thom isomorphism allows us to reduce to the case $c = 1$.

If p is odd, we can write $V_p \approx \mathbb{R} \oplus \bar{V}_p$, and $\bar{V}_p$ can be given a complex structure. Use the Thom isomorphism.

If $p = 2$, then $V_p \approx \mathbb{R} \oplus \bar{V}_2$, and $\bar{V}_2$ is the sign representation, so that

$$BC_2^{V_2} \approx \Sigma BC_2^{\bar{V}_2} \approx \Sigma\Sigma^\infty BC_2,$$

which is a retract of $\Sigma\Sigma^\infty_+ BC_2$. □

Proof of (15.1). Write $\mathbb{P}^G_E(M) \overset{\mathrm{def}}{=} M^{\wedge_E m}_{hG}$, where $G \subseteq \Sigma_m$. Then we have shown $\hat{\mathbb{P}}^{C_p}_E(\Sigma^c E)$ is finitely generated free, taking $C_p \subset \Sigma_p$.

The usual transfer argument shows that $\hat{\mathbb{P}}^p_E(M)$ is a retract of $\hat{\mathbb{P}}^{C_p}_E(M)$, so that $\hat{\mathbb{P}}^p(\Sigma^c E)$ is finitely generated free.

The "addition" formula shows that

$$\mathbb{P}^p_E(M \vee N) \approx \bigvee_{i+j=p} \mathbb{P}^i_E(M) \wedge_E \mathbb{P}^j_E(n),$$

and since clearly $\mathbb{P}^i_E(M)$ is a retract of $M^{\wedge_E i}$ (again using transfers), we conclude that $\hat{\mathbb{P}}^p_E(M)$ takes finite frees to finite frees.

We have that

$$\mathbb{P}^G_E \mathbb{P}^H_E(M) \approx \mathbb{P}^{H \wr G}_E(M).$$

If $\Sigma_p^{\wr r}$ denotes the r-fold wreath power, we have shown that $\hat{\mathbb{P}}^{\Sigma_p^{\wr r}}_E = \hat{\mathbb{P}}^p_E \cdots \hat{\mathbb{P}}^p_E$ preserves finite frees.

Finally, for $m \geq 0$ with $m = \sum a_i p^i$, with $a_i \in \{0, \dots, p-1\}$, the group Σ_m contains a subgroup $G = \prod_i (\Sigma_p^{\wr i})^{\times a_i}$, which acts on $\underline{m}$ in the evident way, and which contains a p-Sylow subgroup of G. Thus using transfers, $\mathbb{P}^m_E(M)$ is a retract of the smash product of finitely many $\mathbb{P}^{\Sigma_p^{\wr i}}_E(M)$, and therefore we are done. $\square$

Remark 15.3. The above proof shows a little bit more. Namely, if M is a finite free module with $\pi_* M$ concentrated in even degrees, then $\pi_* \hat{\mathbb{P}}^m_E(M)$ is also concentrated in even degrees.

Note that $\hat{\mathbb{P}}_E(M) \approx L_{F(n)}(\bigvee \mathbb{P}^m_E(M))$ is not a wedge of $\hat{\mathbb{P}}^m_E(M)$, but rather the $K(n)$-localization of such a wedge. Also, $\hat{\mathbb{P}}_E(M) \wedge_E \hat{\mathbb{P}}_E(N)$ is not equivalent to $\hat{\mathbb{P}}_E(M \vee N)$, but rather its localization is. I'd like to not have to worry about completions all the time, so I will define the theory DL_{E_*} is a funny way.

Thus, let $\mathrm{DL}_{E_*} = T$ be the $\mathbb{Z}$-graded theory defined by

$$T(T^{[c_1]+\cdots+[c_k]}, T^{[d_1]+\cdots+[d_l]}) \approx h\mathrm{Alg}'_E(\hat{\mathbb{P}}_E(\Sigma^{d_1} E \vee \cdots \vee \Sigma^{d_l} E, \hat{\mathbb{P}}_E(\Sigma^{c_1} E \vee \cdots \vee \Sigma^{c_k} E)),$$

where

$$h\mathrm{Alg}'_E(\hat{\mathbb{P}}_E(M), \hat{\mathbb{P}}_E(N)) \subset h\mathrm{Alg}_E(\hat{\mathbb{P}}_E(M), \hat{\mathbb{P}}_E(N)) \approx h\mathrm{Mod}_E(M, \hat{\mathbb{P}}_E(N))$$

is the subset of maps corresponding to $M \to \hat{\mathbb{P}}_E(N)$ which factor through $\hat{\mathbb{P}}^{\leq r}_E(N)$ for some r, where $\hat{\mathbb{P}}^{\leq r}_E(N) = \bigvee_{i=0}^r \hat{\mathbb{P}}^i_E(N)$. One checks that this really gives a category, since if $M \to \hat{\mathbb{P}}^{\leq r}_E(N) \to \hat{\mathbb{P}}_E(N)$, then $\hat{\mathbb{P}}^k_E(M) \to \hat{\mathbb{P}}^{\leq kr}_E(N)$.

With this definition, the free model on graded generators $[c_1], \dots, [c_k]$ is given by

$$\bigoplus_{m \geq 0} \pi_* \hat{\mathbb{P}}^m_E(\Sigma^{c_1} E \vee \cdots \vee \Sigma^{c_k} E) \approx \bigotimes_{i=1}^k \left(\bigoplus_{m \geq 0} \pi_* \hat{\mathbb{P}}^m_E(\Sigma^{c_i} E) \right),$$

which means that DL_{E_*} is a graded COT. In particular the free model on one generator is

$$E_*\{x\} \approx \bigoplus_{m \geq 0} E^\wedge_*(B\Sigma_m^{cV_m}).$$

16. "Main results"

Although there is a graded COT, I'm going to restrict attention to the degree 0 part. Thus, let DL_E denote the ordinary COT with free model on one generator given by $\bigoplus_{m\geq 0} E_0^\wedge B\Sigma_m$.

Recall that we define $\mathcal{A}$ to be the primitives in the algebra $\mathcal{O}_{\mathrm{univ}}\{x\} \approx \bigoplus E_0^\wedge B\Sigma_m$. Write $\mathcal{A}_{[m]} \subset E_0^\wedge B\Sigma_m$ for the summand. Write $\mathcal{A}_r = \mathcal{A}_{[p^r]}$.

Lemma 16.1 (Strickland)**.** *The module $\mathcal{A}_{[m]} = 0$ unless m is a pth power. Each $\mathcal{A}_r$ is a free $\mathcal{O}_{\mathrm{univ}}$-module, and in fact is a summand of $E_0^\wedge B\Sigma_m$ as an $\mathcal{O}_{\mathrm{univ}}$-module.*

Thus, $\mathcal{A} \approx \bigoplus \mathcal{A}_r$ is a graded ring, and the category $\mathrm{Mod}_{\mathcal{A}}$ of left $\mathcal{A}$-modules admits a tensor structure.

An **isogeny** of formal groups over R is a homomorphism $f\colon G \to G'$ such that the corresponding map $\mathcal{O}_f\colon \mathcal{O}_{G'} \to \mathcal{O}_G$ is finite and free. If R is a complete local ring, this amounts to saying that f is represented by a map $R[\![y]\!] \to R[\![x]\!]$ sending y to $f(x)$, such that $g(0) = 0$ and $f(x) \equiv cx^n$ modulo $(\mathfrak{m}_R, x^{n+1})$, with c invertible. This is a consequence of the "Weierstrass preparation theorem".

Proposition 16.2. *Let R be a complete local ring, and let $f(x) \in R[\![x]\!]$ be a power series such that $f(x) \equiv cx^m \mod (\mathfrak{m}, x^{m+1})$, where $c \neq 0$ in $R/\mathfrak{m}$. Then there exists a unique pair (g, h) consisting of a monic polynomial $g(x) \in R[x]$ of degree m and an invertible power series $h(x) \in R[\![x]\!]^\times$ such that $f = gh$.*

16.3. **Deformations of Frobenius.** Let R be a complete local ring with $p = 0$. Let $\sigma = \sigma_R\colon R \to R$ denote the ring homomorphisms defined by $\sigma(x) = x^p$. If A is an R algebra, we write $\mathrm{Frob}\colon \sigma^*A \to A$ for the map of R-algebras which fits into the diagram

$$\begin{tikzcd}
R \arrow[r, "\sigma"] \arrow[d] & R \arrow[r, equal] \arrow[d] & R \arrow[d] \\
A \arrow[r] \arrow[rr, bend right, "\sigma"'] & \sigma^*A \arrow[r, "\mathrm{Frob}^*"] & A
\end{tikzcd}$$

where the left-hand square is a pushout of rings.

In particular, if G is a formal group over such a ring R, then there is an isogeny $\mathrm{Frob}\colon \sigma^*G \to G$ defined by the map $\mathrm{Frob}^*\colon \mathcal{O}_{\sigma^*G} = \sigma^*\mathcal{O}_G \to \mathcal{O}_G$.

We define $\mathrm{DefFrob}_\Gamma(R)$ to be the category whose objects are deformations of Γ to R, and whose morphisms are isogenies which are "deformations of Frobenius". The morphisms $(G, i, \phi) \to (G', i', \phi')$ of this category are isogenies $f\colon G \to G'$ such that there is an $r \geq 0$ such that $i' = \sigma^r \circ i$, and the diagram

$$\begin{tikzcd}
G_0 \arrow[r, "f_0"] \arrow[d, "\alpha"'] & G_0' \arrow[d, "\alpha'"] \\
i^*\Gamma \arrow[r, "\mathrm{Frob}^r"'] & (i')^*\Gamma
\end{tikzcd}$$

commutes. Note that if $r = 0$, then we are just saying that f is a $\star$-isomorphism.

Fix a height n formal group Γ over a perfect field k. We will define a category $\mathrm{Mod}_{DF} = \mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ as follows. An object $\mathcal{F}$ of Mod_{DF} consists of:

(1) for each complete local ring R, a functor
$$\mathcal{F}_R\colon \mathrm{DefFrob}(R)^{\mathrm{op}} \to \mathrm{Mod}_R$$
to the category of R-modules, and

(2) for each local homomorphism $f\colon R \to R'$, a natural isomorphism
$$\mathcal{F}_f\colon f^*\mathcal{F}_R \to \mathcal{F}_{R'} f^*,$$
(in terms of the functors $f^*\colon \mathrm{DefFrob}(R) \to \mathrm{DefFrob}(R')$ and $f^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$),

such that

(a) there are coherent natural isomorphisms $\mathcal{F}_{\mathrm{id}} \approx \mathrm{id}$ and $\mathcal{F}_{gf} \approx \mathcal{F}_g f^* \circ g^*\mathcal{F}_f$.

Morphisms $g\colon \mathcal{F} \to \mathcal{G}$ are collections of natural transformations $g_R\colon \mathcal{F}_R \to \mathcal{G}_R$ such that the diagrams
$$\begin{tikzcd}
\mathcal{F}_R \arrow[r, "g_R"] \arrow[d, "f^*"'] & \mathcal{G}_R \arrow[d, "f^*"] \\
\mathcal{F}_{R'} \arrow[r, "g_{R'}"'] & \mathcal{G}_{R'}
\end{tikzcd}$$
commute, where $f\colon R \to R'$.

The category Mod_{DL} is a tensor category: define $\mathcal{F} \otimes \mathcal{F}'$ by $(\mathcal{F} \otimes \mathcal{F}')_R(G) \overset{\mathrm{def}}{=} \mathcal{F}_R(G) \otimes_R \mathcal{F}'_R(G)$.

Pre-Theorem 16.4. *There is an equivalence of tensor categories*
$$U\colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_{DF}.$$

We define a category $\mathrm{Alg}_{DF} = \mathrm{Alg}_{\mathrm{DefFrob}_\Gamma}$ as follows. An object $\mathcal{B}$ of Alg_{DF} consists of a ring object in Mod_{DF} such that

(b) if R is a ring with $p = 0$, and G is a deformation of Γ to R, then the composite
$$\sigma^*\mathcal{B}_R(G) \xrightarrow[\sim]{\mathcal{B}_\sigma} \mathcal{B}_R(\sigma^*G) \xrightarrow{\mathcal{B}_R(\mathrm{Frob})} \mathcal{B}_R(G)$$
is identical to the Frobenius map $\sigma^*\mathcal{B}_R(G) \to \mathcal{B}_R(G)$.

Morphisms are maps of ring objects.

An object $\mathcal{B}$ in Alg_{DF} is said to be **torsion free** if $\mathcal{B}_R(G)$ is p-torsion free for every p-torsion free ring R, and every deformation G to R. Equivalently, $\mathcal{B}$ is torsion free if $\mathcal{B}_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}})$ is p-torsion free.

Pre-Theorem 16.5. *There is a functor*
$$U\colon \mathrm{Model}_{\mathrm{DL}_{E_0}} \to \mathrm{Alg}_{DF}.$$
It restricts to an equivalence
$$U\colon \mathrm{Model}^{\mathrm{tf}}_{\mathrm{DL}_{E_0}} \to \mathrm{Alg}^{\mathrm{tf}}_{DF},$$
of full subcategories of torsion free objects.

16.6. **Divisors and Subgroups.** Let R be a complete local ring, and let G be a formal group over R, so that $\mathcal{O}_G \approx R[\![x]\!]$. An **effective divisor** D **of degree** d on G is an ideal $\mathcal{I}(D) \subset \mathcal{O}_G$ such that $\mathcal{O}_D \overset{\text{def}}{=} \mathcal{O}_G/\mathcal{I}(D)$ is finite and free of rank d as an R-module. We think of D is a closed formal subscheme of G. The Cayley-Hamilton theorem together with the Weierstrass preparation theorem show that the set of effective divisors D of degree d on G is in bijective correspondence with the set of monic polynomials

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in R[\![x]\!] \approx \mathcal{O}_G$$

of degree d such that $a_0, \dots, a_{n-1} \in \mathfrak{m}_R$. In that case, $\mathcal{I}(D) = (f(x))$.

If $a\colon \mathcal{O}_G \to R$ is a continuous ring homomorphism, then let $[a]$ denote the divisor associated to the ideal $(x +_G x(a)) \subset R[\![x]\!]$. In particular, $[e]$ is the divisor associated to the ideal (x).

A **subgroup** of G is an effective divisor K which is a subgroup; that is, the addition law of G restricts to K, and K contains the identity.

$$\begin{tikzcd} K \times K \arrow[r, dotted] \arrow[d] & K \arrow[d] & \\ G \times G \arrow[r] & G & \ast \arrow[l] \arrow[ul, dotted] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O}_K \otimes_R \mathcal{O}_K & \mathcal{O}_K \arrow[l, dotted] \arrow[dr, dotted] & \\ \mathcal{O}_G \otimes_R \mathcal{O}_G \arrow[u] & \mathcal{O}_G \arrow[l] \arrow[u] \arrow[r] & R \end{tikzcd}$$

In terms of a coordinate x, K is a subgroup if $\mathcal{O}_K \approx R[\![x]\!]/(f(x))$ such that $f(x_1 +_G x_2) \in (f(x_1), f(x_2)) \subset R[\![x_1, x_2]\!]$ and $f(x) \in xR$.

One can show that if K is a subgroup of degree d, then the homomorphism $[d]_G\colon G \to G$ restricts to zero on K. (See Oort-Tate.) More concretely, this means that $f(x)$ must divide $[d]_G(x)$. As a consequence, we see that subgroups of a formal group over a p-local ring must have degree p^r.

Example 16.7. Consider the multiplicative formal group $\mathbb{G}_m$ over a ring R. Since $[p^r](x) \in R[\![x]\!]$ is a monic polynomial of degree p^r, we see that the only subgroups of $\mathbb{G}_m$ are $\mathbb{G}_m[p^r]$, where $\mathcal{O}_{\mathbb{G}_m[p^r]}/([p^r](x))$.

Proposition 16.8. *If G is a formal group over a field k of characteristic p, there is exactly one subgroup of degree p^r. In terms of a coordinate x on G, the subgroup is given by the ideal (x^{p^r}).*

Proof. In fact, there is only one divisor of degree p^r, and it is easy to check that it is a subgroup. It is the kernel of Frob^r. □

The **kernel** of an isogeny $f\colon G \to G'$ is the group scheme K with $\mathcal{O}_K = \mathcal{O}_G \otimes_{\mathcal{O}_{G'}} R$. Since $\mathcal{O}_{G'} \to \mathcal{O}_G$ is finite and free, we see that $\mathcal{O}_K$ is finite and free over R, and so it really is a subgroup in the sense above. If R is a complete local ring, and the isogeny is represented by a map $R[\![y]\!] \to R[\![x]\!]$ sending $y \mapsto f(x)$, then $\mathcal{O}_K \approx R[\![x]\!]/(f(x))$.

Given a subgroup $K < G$, one can construct the **quotient** G/K, which is a formal group. The idea is to define the ring of functions on $\mathcal{O}_{G/K}$ as the equalizer

$$\begin{tikzcd} \mathcal{O}_{G/K} \arrow[r, "f^*"] & \mathcal{O}_G \arrow[r, shift left, "\mu^*"] \arrow[r, shift right, "\pi_1^*"'] & \mathcal{O}_G \otimes_R \mathcal{O}_K \end{tikzcd} .$$

It turns out that $\mathcal{O}_{G/K} \approx R[\![y]\!] \subset R[\![x]\!] \approx \mathcal{O}_G$, that $f^*\colon \mathcal{O}_{G/K} \to \mathcal{O}_G$ is finite and free, and the kernel of f is K. Furthermore, the map $f\colon G \to G/K$ has the requisite universal property: given a homomorphism $g\colon G \to G'$ of formal groups such that $g|_K$ is zero, there is a unique dotted arrow in

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[dr, "g"'] & G/K \arrow[d, dotted] \\ & G' \end{tikzcd}$$

Finally, if $g\colon G \to G'$ is an isogeny of formal groups, with kernel K, then $G' \approx G/K$.

Lubin-Tate says that there is only one $\star$-isomorphism between any two deformations G and G'. More generally, let $\mathrm{DefFrob}^r(R)$ denote the category of morphisms (of degree p^r) in $\mathrm{DefFrob}(R)$. That is, the objects of $\mathrm{DefFrob}^r(R)$ are isogenies $f\colon G \to G'$ which deform Frob^r, and the morphisms $g\colon f_1 \to f_2$ are commutative diagrams

$$\begin{tikzcd} G_1 \arrow[r, "f_1"] \arrow[d, "g"', "\sim"] & G_1' \arrow[d, "g'", "\sim"'] \\ G_2 \arrow[r, "f_2"'] & G_2' \end{tikzcd}$$

in $\mathrm{DefFrob}(R)$ such that g and g' are isomorphisms.

Proposition 16.9. *Between any two objects in* $\mathrm{DefFrob}^r(R)$ *there is at most one isomorphism. The set of isomorphism classes of in* $\mathrm{DefFrob}^r(R)$ *is in bijective correspondence with the set $X_r(R)$ of $\star$-isomorphism classes of pairs $(G > K)$, where $G = (G, i, \alpha)$ is a deformation of Γ to R, and $K < G$ is a subgroup of degree p^r. (Pairs $(G_i > K_i)$ are $\star$-isomorphic if there is a $\star$-isomorphism $G_1 \to G_2$ carrying K_1 isomorphically to K_2.)*

Proof. The isomorphisms in $\mathrm{DefFrob}(R)$ are $\star$-isomorphisms, and so it is clear that there is at most one isomorphism between any two objects in $\mathrm{DefFrob}^r(R)$.

If $f\colon G \to G'$ is an object in $\mathrm{DefFrob}^r(R)$, we associate it to its kernel K, which is a subgroup of G. It is clear that isomorphic maps f_1 and f_2 give rise to $\star$-isomorphic pairs $(G_1 > K_1)$ and $(G_2 > K_2)$. Conversely, if f_1 and f_2 are maps such that the associated pairs $(G_1 > K_1)$ and $(G_2 > K_2)$ are $\star$-isomorphic, consider the diagram

$$\begin{tikzcd} K_1 \arrow[r, tail] \arrow[d, "\sim"] & G_1 \arrow[r, "f_1"] \arrow[d, "g"', "\sim"] & G_1' \arrow[d, dotted] \\ K_2 \arrow[r, tail] & G_2 \arrow[r, "f_2"'] & G_2' \end{tikzcd}$$

where g is the $\star$-isomorphism. Since the G_i' are cokernels, we can construct the dotted arrow, which is necessarily an isomorphism. Passing to the special fibers, we check that the dotted arrow is a $\star$-isomorphism. Thus, the assignment $\pi_0\mathrm{DefFrob}^r(R) \to X_r(R)$ is injective; it remains to show surjectivity.

Given a pair $(G > K)$, we can form the quotient formal group G/K, thus obtaining an isogeny $f\colon G \to G/K$. If we restrict to the special fiber, we get an exact sequence

$0 \to K_0 \to G_0 \to (G/K)_0 \to 0$ of group schemes. Since G_0 has only one subgroup, we can identify K_0 with the kernel of $\mathrm{Frob}^r\colon G_0 \to (\sigma^r)^*G_0$. The above argument shows that there is a unique isomorphism $\alpha'\colon (G/K)_0 \to (\sigma^r)^*G_0$ such that $\alpha' \circ f_0 = \mathrm{Frob}^r$. Thus, we can view G/K as a deformation of Γ, and so f is an object in $\mathrm{DefFrob}_r(R)$. □

Remark 16.10. This means that $\mathrm{DefFrob}(R)$ is equivalent to the following category. The objects are $\star$-isomorphism classes of deformations $[G]$. The morphisms are $\star$-isomorphism classes of pairs $[G > K]$; the source of $[G > K]$ is $[G]$, while the target of $[G > K]$ is $[G/K]$, where G/K is viewed as a deformation of Γ as described in the proof of the previous proposition. If $G/K \approx G'$, then the composite $[G'/K'] \circ [G/K] = [G/K'']$, where K'' is the kernel of $G \to G/K \approx G' \to G'/K'$.

Example 16.11. Let $\Gamma/\mathbb{F}_p$ be the multiplicative formal group. Since $\mathcal{O}_{\mathrm{univ}} = \mathbb{Z}_p$, we can take the universal deformation G_{univ} to also be the multiplicative formal group. Thus, for any R, every object of $\mathrm{DefFrob}(R)$ is $\star$-isomorphic to $\mathbb{G}_m/R$. Furthermore, the category $\mathrm{DefFrob}^r(R)$ has only one object up to isomorphism, corresponding to the subgroup $\mathbb{G}_m[p^r]$, whose cokernel is the isogeny $[p^r]\colon \mathbb{G}_m \to \mathbb{G}_m$.

Thus, to describe an object $\mathcal{B} \in \mathrm{Alg}_{DF}$, it is enough to give

(1) a $\mathbb{Z}_p$-algebra $B = \mathcal{B}_{\mathbb{Z}_p}(\mathbb{G}_m)$, and
(2) ring homomorphisms $\psi^{p^r}\colon B \to B$ (corresponding to the isogeny $[p^r]\colon \mathbb{G}_m \to \mathbb{G}_m$),

such that

(a) $\psi^{p^r}\psi^{p^s} = \psi^{p^{r+s}}$, and
(b) $\psi^p(b) \equiv b^p \mod pB$.

The other data for $\mathcal{B}$ are determined by base change.

More generally, the functor X_r which associates to a ring R the set of $\star$-isomorphism classes of pairs $[G > K]$, where K is a subgroup of G of degree p^r, is represented by a certain complete local ring $\mathcal{O}_{X_r}$. There are two ring homomorphisms

$$s^*, t^*\colon \mathcal{O}_{\mathrm{univ}} \to \mathcal{O}_{X_r},$$

where s^* represents $(G > K) \mapsto G$, and t^* represents $(G > K) \mapsto G/K$.

To describe an object $\mathcal{B}$ of Alg_{DF}, it suffices to give

(1) an $\mathcal{O}_{\mathrm{univ}}$-algebra $B = \mathcal{B}_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}})$, and
(2) a map of $\mathcal{O}_{\mathrm{univ}}$-algebras $\phi_r\colon B \to B \otimes^{s^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r}$, where we use $t^*\colon \mathcal{O}_{\mathrm{univ}} \to \mathcal{O}_{X_r}$ to map the target an algebra over $\mathcal{O}_{X_r}$,

satisfying a bunch of properties. The map ϕ_r is the composite of

$$B \to B \otimes^{t^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r} \approx \mathcal{B}_{\mathcal{O}_{X_r}}(t^*G_{\mathrm{univ}}) \xrightarrow{\mathcal{B}(f)} \mathcal{B}_{\mathcal{O}_{X_r}}(s^*G_{\mathrm{univ}}) \approx B \otimes^{s^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r},$$

where $f\colon s^*G_{\mathrm{univ}} \to t^*G_{\mathrm{univ}}$ is the universal example of a deformation of Frob^r.

I don't want to write down all the properties these need to satisfy; most of them are just the axioms for comodules over a Hopf bialgebroid (i.e., a category object in formal schemes). It is worthwhile to say something about the congruence axiom, however.

Recall that given a deformation G to a ring of R characteristic p, we may consider the Frobenius isogeny $\mathrm{Frob}\colon G \to \sigma^*G$, which is an object in $\mathrm{DefFrob}^1(R)$. The universal example of such a deformation G lies over $\mathcal{O}_{\mathrm{univ}}/(p)$, and thus Frob is represented by a certain

map
$$u^*\colon \mathcal{O}_{X_1} \to \mathcal{O}_{\mathrm{univ}}/(p).$$
The congruence condition then amounts to the requirement that
$$B \xrightarrow{\phi_1} B \otimes^{s^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_1} \xrightarrow{\mathrm{id}\otimes u^*} B \otimes_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{\mathrm{univ}}/(p) \approx B/pB$$
factors through the pth power map $\sigma\colon B/pB \to B/pB$.

17. The isogenies associated to power operations

17.1. **Vector bundles and divisors.** Given a vector bundle $W \to X$, we get a cofiber sequence of spaces
$$W' \to X \to X^W.$$
Here I write W' for the unit sphere bundle in W (or, if you prefer, the the complement of the zero section in W).

Let $\mathbb{T}$ denote the circle group. Let L denote the usual one-dimensional $\mathbb{T}$ representation. Given a non-equivariant bundle $W \to X$, there is a $\mathbb{T}$-equivariant bundle $W \otimes L \to X$. If we take $\mathbb{T}$-homotopy orbits, we obtain a bundle over $X \times B\mathbb{T}$, and thus a cofiber sequence
$$(W \otimes L)' \to X \times B\mathbb{T} \to (X \times B\mathbb{T})^{W\otimes L}.$$

Given a bundle $W \to X$, we can form the projective bundle $\mathbb{P}(V) \to X$, whose fiber $\mathbb{P}(V)_x$ is the space of lines in V_x. There is a tautological line bundle $H \to \mathbb{P}(V)$, and hence a map $\mathbb{P}(V) \to B\mathbb{T}$ representing H.

Proposition 17.2. *The maps $(W \otimes L)' \to X \times B\mathbb{T}$ and $\mathbb{P}(V) \to X \times B\mathbb{T}$ are weakly equivalent as spaces over $X \times B\mathbb{T}$.*

Proof. The space $(W \otimes L)'/\mathbb{T}$ is homeomorphic to $\mathbb{P}(V)$, and H can be identified with the principal $\mathbb{T}$-bundle $(W\otimes L)' \to (W\otimes L)'/\mathbb{T}$. Since $\mathbb{T}$ acts freely, $(W\otimes L)' \times_{\mathbb{T}} E\mathbb{T} \to (W\otimes L)'/\mathbb{T}$ is a weak equivalence. □

The long exact sequence in cohomology becomes a short exact sequence
$$0 \leftarrow E^0\mathbb{P}(W) \leftarrow E^0(X \times B\mathbb{T}) \leftarrow \widetilde{E}^0((X \times B\mathbb{T})^{W\otimes L}) \leftarrow 0,$$
where the right-hand map sends the Thom class to the Euler class of $W \otimes L$. Thus, for finite complexes X we can identify this sequence as that associated to a divisor D_W on G over $E^0(X)$:
$$0 \leftarrow \mathcal{O}_{D_W} \leftarrow E^0(X) \otimes_{E^0} \mathcal{O}_G \leftarrow \mathcal{I}(D_W) \leftarrow 0.$$

17.3. **Vector bundles and power operations.** In the following, I write $E^0_G(X)$ for $E^0(X_{hG})$.

Recall the total power operation
$$E^0X \xrightarrow{P_m} E^0_{\Sigma_m}(X^m).$$
An element represented by $\alpha\colon \Sigma^\infty_+ X \to E$ is sent to the element represented by the composite $\mathbb{P}^m(\Sigma^\infty_+ X) \xrightarrow{\mathbb{P}^m(\alpha)} \mathbb{P}^m E \to E$.

There is a "relative version" of this construction. If Y is a pointed space, then we can define

$$\widetilde{E}^0(Y) \xrightarrow{P_m} \widetilde{E}^0_{\Sigma_m}(Y^{\wedge m})$$

as the map sending $\alpha\colon \Sigma^\infty Y \to E$ to

$$\mathbb{P}^m(\Sigma^\infty Y) \xrightarrow{\mathbb{P}^m(\alpha)} \mathbb{P}^m E \to E.$$

If $Y = X/A$, then the evident diagram

$$\begin{tikzcd} E^0 X \arrow[r, "P_m"] & E^0_{\Sigma_m} X^m \\ \widetilde{E}^0 Y \arrow[u] \arrow[r, "P_m"'] & \widetilde{E}^0_{\Sigma_m} Y^m \arrow[u] \end{tikzcd}$$

commutes.

The power operation

$$P_m\colon \widetilde{E}^0 X^W \to \widetilde{E}^0_{\Sigma_m}(X^W)^{\wedge m}$$

is a function relating the cohomology of two Thom spaces, since $(X^W)^{\wedge m} \approx (X^m)^{W^m}$, where $W^m \to X^m$ is the external Whitney sum, which obtains an evident Σ_m-action.

Since E is complex orientable, for any complex bundle $W \to X$ there is an element $u \in \widetilde{E}^0 X^W$ making $\widetilde{E}^* X^W$ a free rank one module over $E^* X$. We we call any such element u an **E-trivialization** of W; note that u need not be the class arising from the complex orientation of E.

Proposition 17.4. *If $u \in \widetilde{E}^0 X^W$ is an E-trivialization, then $P_m(u) \in \widetilde{E}^0_{\Sigma_m}(X^m)^{W^m}$ is an E-trivialization.*

Proof. We know that an element $u \in \widetilde{E}^0 X^W$ is a trivialization if and only if, for every point $x \in X$, the restriction $u_x \in \widetilde{E}^0\{x\}^{W_x} \approx \widetilde{E}^0 S^{2d}$ is a trivialization. In particular, this means that if $f\colon Y \to X$ is a map which is surjective on π_0, then u is an E-trivialization for $W \to X$ if and only if $f^* u$ is an E-trivialization for $f^{-1}W \to Y$.

Furthermore, if $u_i \in \widetilde{E}^0(X_i)^{W_i}$ is an E-trivialization of $W_i \to X_i$, then the external cup product $u_1 \times \cdots \times u_k$ is an E-trivialization of $W_1 \times \cdots \times W_k \to X_1 \times \cdots \times X_k$.

The result now follows from the observation that $f\colon X^m \to X^m_{h\Sigma_m}$ is surjective on π_0, and $f^* P_m(u) = u^{\times m}$. □

17.5. **Transfer ideals.** Let X be a space. Let $I \subset E^0(X \times B\Sigma_m)$ denote the sum of the images of transfers

$$\mathrm{tr}\colon E^0(X \times B(\Sigma_i \times \Sigma_{m-i})) \to E^0(X \times B\Sigma_m)$$

for $0 < i < m$. Then I is an ideal.

Suppose $W \to X \times B\Sigma_m$ is a bundle, and let $W_i \to X \times B(\Sigma_i \times \Sigma_{m-i})$ denote the pullback. Then we can define subgroups

$$I' \subset E^0(W') \qquad \text{and} \qquad I'' \subset \widetilde{E}^0(X \times B\Sigma_m)^W$$

as sums of images of "transfer maps"

$$\mathrm{tr}\colon E^0 W_i' \to E^0 W'$$

and
$$\mathrm{tr}\colon \widetilde{E}^0(X \times B(\Sigma_i \times \Sigma_{m-i}))^{W_i} \to \widetilde{E}^0(X \times B\Sigma_m)$$
respectively. The object I' is an ideal, and I'' is a sub $E^0(X \times B\Sigma_m)$-module.

Note that the "double-coset formula" implies that $\pi^* I \subseteq I'$ and $z^* I'' \subseteq I$, where
$$E^0 W' \xleftarrow{\pi^*} E^0(X \times B\Sigma_m) \xleftarrow{z^*} \widetilde{E}^0(X \times B\Sigma_m)^W.$$
Furthermore, if each of the restriction maps $E^0 X \times B(\Sigma_i \times \Sigma_{m-i}) \to E^0 W_i'$ is surjective, then $\pi^* I = I'$.

Finally, recall that $E^* B\Sigma_m$ is a finite free E^*-module, concentrated in even degrees. Then $I = E^0 X \otimes_{E^0} J$, where $J \subseteq E^0 B\Sigma_m$ is the images of transfers from $E^0 B(\Sigma_i \times \Sigma_{m-i})$. Let $S_m = E^0 B\Sigma_m / J$.

We are going to consider the composite
$$E^0 X \xrightarrow{P_m} E^0 X^m_{h\Sigma_m} \to E^0 X \times B\Sigma_m \approx E^0 X \otimes_{E^0} E^0 B\Sigma_m \to E^0 X \otimes_{E_0} S_m.$$
This is a ring homomorphism, natural in X.

17.6. **Construction of the isogeny.** Consider the following diagram.

$$\begin{tikzcd}
0 & & 0 & 0 \\
E^0 L' \arrow[u] \arrow[rrr, dotted, bend left] & E^0(L^m)' \arrow[r] & E^0(L \otimes V_m)' \arrow[u] \arrow[r, two heads] & E^0(L \otimes V_m)'/I' \arrow[u] \\
E^0 B\mathbb{T} \arrow[u, "\pi^*"] \arrow[r, "P_m"] & E^0 B(\mathbb{T} \wr \Sigma_m) \arrow[u] \arrow[r] & E^0 B(\mathbb{T} \times \Sigma_m) \arrow[u, "\pi^*"] \arrow[r, two heads] & E^0 B(\mathbb{T} \times \Sigma_m)/I \arrow[u] \\
\widetilde{E}^0 B\mathbb{T}^L \arrow[u, "z^*"] \arrow[r, "P_m"'] & \widetilde{E}^0 B(\mathbb{T} \wr \Sigma_m)^{L^m} \arrow[u] \arrow[r] & \widetilde{E}^0 B(\mathbb{T} \times \Sigma_m)^{L \otimes V_m} \arrow[u, "z^*"] \arrow[r, two heads] & \widetilde{E}^0 B(\mathbb{T} \times \Sigma_m)^{L \otimes V_m}/I'' \arrow[u] \\
0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd}$$

The first three columns are Gysin sequences associated to bundles $L \to B\mathbb{T}$, $L^m \to B(\mathbb{T} \wr \Sigma_m)$, and $B(\mathbb{T} \times \Sigma_m)^{L \otimes V_m}$. Here L^m denotes the bundle obtained from the evident $\mathbb{T} \wr \Sigma_m$-representation on L^m. The fourth column is obtained from the third by passing to quotients.

The sphere bundles of L and $L \otimes V_m$ are equivalent to projective bundles $\mathbb{P}(\mathbb{C}) \approx *$ and $\mathbb{P}(V_m)$, and so the first and third columns are exact. In particular, we note that $E^0(L')$ and $E^0(L \otimes V_m)'$ are obtained from $E^0 B\mathbb{T}$ and $E^0 B(\mathbb{T} \times \Sigma_m)$ by quotienting by the cyclic ideal generated by an Euler class, i.e., by any element of the from $z^*(u)$ where u is any E-trivialization of the Thom space. Note that in fact L' is contractible, and this identification gives $E^0 L' \approx E^0[\![x]\!]/(x)$, where x is the Euler class of the canonical line bundle.

Exactness of the fourth column follows from the fact that $\pi^* I = I'$, since the restriction of $L \otimes V_m$ over $B(\mathbb{T} \times \Sigma_i \times \Sigma_{m-i})$ can be identified with $L \otimes V_i \otimes V_{m-i}$, and the associated sphere bundle is again a projective bundle, so that $E^0 B(\mathbb{T} \times \Sigma_i \times \Sigma_{m-i}) \to E^0(L \otimes V_i \otimes V_{m-i})'$ is surjective.

The maps marked "P_m" are the total power operation. The horizontal maps between the second and third columns are induced by the "diagonal" map $B(\mathbb{T}\times\Sigma_m)\to B(\mathbb{T}\wr\Sigma_m)$ and the associated pullback of vector bundles.

Since $\pi^*I = I'$, the square in the upper right-hand corner is a pushout of rings. This means that $\mathcal{O}_D \overset{\text{def}}{=} E^0(L\otimes V_m)'/I'$ is a quotient ring of $E^0B(\mathbb{T}\times\Sigma_m)/I \approx \mathcal{O}_G\otimes_{\mathcal{O}_{\text{univ}}}S_m$ which is finite and free of rank m as an S_m-module. In other words, we obtain a divisor D of degree m on G pulled back along $s\colon \mathcal{O}_{\text{univ}}\to S_m$. The kernel $\mathcal{I}(D) = \ker(E^0B(\mathbb{T}\times\Sigma_m)/I\to E^0(L\otimes V_m)'/I')$ is generated by the image of $\widetilde{E}^0B(\mathbb{T}\times\Sigma_m)^{L\otimes V_m}$, and is in fact a cyclic ideal generated by the image of any E-trivialization of $L\otimes V_m\to B(\mathbb{T}\times\Sigma_m)$.

The maps P_m are not additive. However, the long horizontal composites involving P_m are ring homomorphisms, since the total power operation is "additive modulo transfers". Furthermore, we know that P_m carries E-trivializations to E-trivializations. Therefore, if $x\in E^0B\mathbb{T}$ is the generator of the augmentation ideal, then its image in $E^0B(\mathbb{T}\times\Sigma_m)/I$ generates the cyclic ideal $\mathcal{I}(D)\subset \mathcal{O}_G\otimes^{s^*}_{\mathcal{O}_{\text{univ}}}S_m$.

Therefore, the dotted arrow exists, and is a map of rings. In other words, we have obtained a pushout square in rings

$$\begin{tikzcd}
\mathcal{O}_{\text{univ}} \arrow[r] & \mathcal{O}_D \\
\mathcal{O}_G \arrow[u, "e^*"] \arrow[r] & \mathcal{O}_G\otimes^{s^*}_{\mathcal{O}_{\text{univ}}} S_m \arrow[u]
\end{tikzcd}$$

Now we consider the projection $B\mathbb{T}\to *$.

$$\begin{tikzcd}
E^0B\mathbb{T} \arrow[r, "P_m"] & E^0B(\mathbb{T}\times\Sigma_m) \arrow[r, two heads] & E^0B(\mathbb{T}\times\Sigma_m)/I \\
E^0 \arrow[u] \arrow[r, "P_m"'] & E^0B\Sigma_m \arrow[u] \arrow[r, two heads] & E^0B\Sigma_m/J \arrow[u]
\end{tikzcd}$$

This leads to a commutative square of rings

$$\begin{tikzcd}
\mathcal{O}_G \arrow[r] & \mathcal{O}_G\otimes^{s^*}_{\mathcal{O}_{\text{univ}}} S_m \\
\mathcal{O}_{\text{univ}} \arrow[u] \arrow[r, "t^*"'] & S_m \arrow[u]
\end{tikzcd}$$

Putting this together, we get a diagram

$$\begin{tikzcd}
\mathcal{O}_{\text{univ}} \arrow[r, "t^*"] & S_m \arrow[r] & \mathcal{O}_D \\
\mathcal{O}_G \arrow[u, "e^*"] \arrow[r] & \mathcal{O}_G\otimes^{t^*}_{\mathcal{O}_{\text{univ}}} S_m \arrow[u, "e^*"] \arrow[r, "\phi^*"'] & \mathcal{O}_G\otimes^{s^*}_{\mathcal{O}_{\text{univ}}} S_m \arrow[u] \\
\mathcal{O}_{\text{univ}} \arrow[u] \arrow[r, "t^*"'] & S_m \arrow[u] &
\end{tikzcd}$$

in which the squares are all pushouts.

In general, the composite

$$E^0X \xrightarrow{P_m} E^0X^m_{h\Sigma_m} \to E^0X \times B\Sigma_m \to E^0X \otimes_{E^0} E^0B\Sigma_m/J$$

is functorial in X. Inserting the map $B\mathbb{T}\times B\mathbb{T} \to B\mathbb{T}$ for X shows that ϕ is a homomorphism of formal groups. Thus we conclude that ϕ^* gives an isogeny

$$\phi\colon s^*G \to t^*G$$

of degree m with kernel D.

Now I want to show that ϕ is a deformation of Frobenius. To do that, I first need to explain how S_m is a local ring (or 0, if m is not a pth power). It suffices to show that $E^0B\Sigma_m$ is a local ring. For any pointed and connected finite type CW-complex X, the augmentation ideal $I = \ker(i^*\colon E^0X \to E^0)$ is such that $\bigcap I^n = 0$. In particular, since $E^0B\Sigma_m$ is finitely generated as an E^0-module, we must have that the image of I in $E^0B\Sigma_m/\mathfrak{m}(E^0B\Sigma_m)$ is nilpotent, where $\mathfrak{m} \subset E^0$ is the maximal ideal. Again, since $E^0B\Sigma_m$ is a finitely generated E^0-module, for any maximal ideal $P \subset E^0B\Sigma_m$ we must have $P \cap E^0 = \mathfrak{m}$. Therefore $I^n \subseteq P$ for some n, whence $I \subseteq P$. Therefore $I + \mathfrak{m}E^0B\Sigma_m$ is the unique maximal ideal of $E^0B\Sigma_m$.

Consider the homotopy pullback square

$$\begin{tikzcd}
\Sigma_m/(\Sigma_i \times \Sigma_{m-i}) \arrow[r, "j"] \arrow[d, "g"'] & B(\Sigma_i \times \Sigma_{m-i}) \arrow[d, "f"] \\
* \arrow[r, "i"'] & B\Sigma_m
\end{tikzcd}$$

We see that $i^*f_! = g_!j^* = \binom{m}{i}q^*$, where $q\colon * \to B(\Sigma_i \times B\Sigma_{m-i})$. Thus

$$i^*J = \sum_{0<i<m} \binom{m}{i} q^*(E^0B(\Sigma_i \times B\Sigma_{m-i})) = \sum_{0<i<m} \binom{m}{i} E^0.$$

Since

$$\gcd\{\, \tbinom{m}{i} \mid 0 < i < m \,\} = \begin{cases} \ell & \text{if } m = \ell^k \text{ for some prime } \ell, \\ 1 & \text{otherwise,} \end{cases}$$

we see that if m is not a pth power, i^*J is the unit ideal (and thus $S_m = 0$), while if $m = p^k$, then $i^*J = pE^0$.

Thus we conclude that there is a canonically defined map $u^*\colon S_m \to \mathcal{O}_{\mathrm{univ}}/(p)$.

Proposition 17.7. *Let $m = p^k$. For any space X, the composite*

$$E^0X \xrightarrow{P_m} E^0(X \times B\Sigma_m) \approx E^0X \otimes_{E^0} E^0B\Sigma_m \to E^0X \otimes_{E^0} S_m \xrightarrow{\mathrm{id}\otimes u^*} E^0X/p(E^0X)$$

is the map sending $a \mapsto a^m$.

Proof. The diagram

$$\begin{tikzcd} E^0X \arrow[r, "P_m"] \arrow[dr, "a\mapsto a^m"'] & E^0X\otimes_{E^0}E^0B\Sigma_m \arrow[r] \arrow[d, "\mathrm{id}\otimes q^*"] & E^0X\otimes_{E^0}S_m \arrow[d, "\mathrm{id}\otimes u^*"] \\ & E^0X\otimes_{E^0}E^0 \arrow[r] & E^0X\otimes_{E^0}E^0/pE^0 \end{tikzcd}$$

commutes. ∎

18. Sketch of the proof of Pre-Theorem 1

Recall that $\mathcal{A}\approx\bigoplus_{m\geq 1}\mathcal{A}_m$, with $\mathcal{A}_m\subseteq E_0^\wedge B\Sigma_m$ being the module of primitives. We noted (16.1) that $\mathcal{A}_m$ is a module summand of $E_0^\wedge B\Sigma_m$, and that $\mathcal{A}_m=0$ if m is not a pth power.

The ring $\mathcal{A}$, together with a ring map $\eta\colon\mathcal{O}\to\mathcal{A}$ and maps $\epsilon\colon\mathcal{A}\to\mathcal{O}$ and $\Delta\colon\mathcal{A}\to\mathcal{A}^r\underset{\mathcal{O}}{\otimes}{}^l\mathcal{A}$, satisfy axioms (a)–(e). Note that $\eta\colon\mathcal{O}\to\mathcal{A}_1\subset\mathcal{A}$, while $\Delta\colon\mathcal{A}_m\to{\mathcal{A}_m}^r\underset{\mathcal{O}}{\otimes}{}^l\mathcal{A}_m$, and that Δ and ϵ make each $\mathcal{A}_m$ into a cocommutative coalgebra.

Let

$$\mathcal{A}_m^*\overset{\text{def}}{=}\hom_{\ell_{\mathcal{O}}\ell}(\mathcal{A}_m,\mathcal{O}).$$

Then $\mathcal{A}_m^*$ becomes a commutative ring, with unit element corresponding to $\epsilon\colon\mathcal{A}_m\to\mathcal{O}$, and multiplication defined by

$$f\cdot g\overset{\text{def}}{=}(a\mapsto\sum f(a')g(a'')),\qquad f,g\in\mathcal{A}_m^*,\quad a\in\mathcal{A}.$$

That is, we make $\mathcal{A}_m^*$ into a ring using the coalgebra struture on $\mathcal{A}_m$.

Let $s^*\colon\mathcal{O}\to\mathcal{A}_m^*$ be given by

$$r\mapsto(a\mapsto\epsilon(\eta(r)a)=r\epsilon(a)),\qquad r\in\mathcal{O},\quad a\in A_m.$$

One checks that s^* is a ring homomorphism; this amounts to the fact that ϵ is the counit of the coalgebra structure. In particular, s^* makes $\mathcal{A}_m^*$ into an $\mathcal{O}$-module according to the formula

$$s^*(r)f=(a\mapsto rf(a)=f(\eta(r)a)),\qquad r\in\mathcal{O},\quad a\in\mathcal{A}_m,\quad f\in\mathcal{A}_m^*.$$

Let $t^*\colon\mathcal{O}\to\mathcal{A}_m^*$ be given by

$$r\mapsto(a\mapsto\epsilon(a\eta(r))),\qquad r\in\mathcal{O},\quad a\in\mathcal{A}_m.$$

One checks that t^* is a ring homomorphism:

$$\begin{aligned}(a\mapsto\epsilon(a\eta(r_1)))\cdot(a\mapsto\epsilon(a\eta(r_2))) &= (a\mapsto\sum\epsilon(a'\eta(r_1))\epsilon(a''\eta(r_2)))\\ &= (a\mapsto(\epsilon\otimes\epsilon)\Delta(a\eta(r_1)\eta(r_2))\\ &= \epsilon(a\eta(r_1r_2))).\end{aligned}$$

In particular, t^* makes $\mathcal{A}_m$ into an $\mathcal{O}$-module according to the formula

$$t^*(r)f=(a\mapsto f(a\eta(r))),\qquad r\in\mathcal{O},\quad a\in\mathcal{A}_m,\quad f\in\mathcal{A}_m^*.$$

In general, if N is an $\mathcal{O}$-module, there is an isomorphism

$$N\underset{\mathcal{O}}{\otimes}{}^{s^*}\mathcal{A}_m^*\xrightarrow{\sim}\hom_{\ell_{\mathcal{O}}}(\mathcal{A}_m,N),$$

defined by

$$n \otimes f \mapsto (a \mapsto f(a)n).$$

This is an isomorphism because $\mathcal{A}_m$ is a finitely generated free left $\mathcal{O}$-module. It is an isomorphism of $\mathcal{O}$-modules, where the left-hand side becomes an $\mathcal{O}$-module through t^*, while the right-hand side becomes an $\mathcal{O}$-module using the right $\mathcal{O}$-module structure on $\mathcal{A}_m$. That is

$$r(n \otimes f) = n \otimes t^*(r)f \mapsto (a \mapsto f(a\eta(r))n).$$

Apply this observation to $\mathcal{A}_m^* {}^{t^*}\underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_n^*$, so that

$$\mathcal{A}_m^* {}^{t^*}\underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_n^* \xrightarrow{\sim} \hom_{{}^\ell\mathcal{O}^{t^*}}(\mathcal{A}_n, \mathcal{A}_m^*) \xrightarrow{\sim} \hom_{{}^\ell\mathcal{O}^r}(\mathcal{A}_n, \hom_{{}^\ell\mathcal{O}}(\mathcal{A}_m, \mathcal{O})) \approx \hom_{{}^\ell\mathcal{O}}(\mathcal{A}_m{}^r \underset{\mathcal{O}}{\otimes}{}^\ell \mathcal{A}_n, \mathcal{O}).$$

Tracing this through, we see that this isomorphism is given by the map sending

$$f \otimes g \mapsto (a \otimes b \mapsto f(a \cdot \eta g(b))).$$

Using the above isomorphism, we define $\mu^*\colon \mathcal{A}_{mn}^* \to \mathcal{A}_m^* {}^{t^*}\underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_n^*$ by

$$f \mapsto (a \otimes b \mapsto f(ab)).$$

One can check that μ^* is a ring homomorphism. To prove this, it is useful to be able to describe the ring structure on $\mathcal{A}_m^* {}^{t^*}\underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_n^*$ in terms of the isomorphism with $\hom_{{}^\ell\mathcal{O}}(\mathcal{A}^r \underset{\mathcal{O}}{\otimes}{}^\ell \mathcal{A}_n, \mathcal{O})$. Thus, consider the map

$$\psi\colon \mathcal{A}_m{}^r \underset{\mathcal{O}}{\otimes}{}^\ell \mathcal{A}_n \to (\mathcal{A}_m{}^r \underset{\mathcal{O}}{\otimes}{}^\ell \mathcal{A}_n)^\ell \underset{\mathcal{O}}{\otimes}{}^\ell (\mathcal{A}_m{}^r \underset{\mathcal{O}}{\otimes}{}^\ell \mathcal{A}_n)$$

defined by

$$a \otimes b \mapsto \sum (a_i' \otimes b_j') \otimes (a_i'' \otimes b_j''),$$

where $\Delta(a) = \sum a_i' \otimes a_i''$ and $\Delta(b) = \sum b_j' \otimes b_j''$. It is perhaps not immediately clear that the above expression is well-defined, since we have identified the left $\mathcal{O}$-module structures on the $\mathcal{A}_m$s, but not on the $\mathcal{A}_n$s. Thus, we must show that if we express $\Delta(b)$ using a different sum of tensors, say $\Delta(b) = (\eta(r)\bar{b}' \otimes \bar{b}'' - \bar{b}' \otimes \eta(r)\bar{b}'') + \sum b_j' \otimes b_j''$, the above formula for ψ still takes the same value. We have

$$\begin{aligned}\sum (a_i' \otimes \eta(r)\bar{b}') \otimes (a_i'' \otimes \bar{b}'') &= \sum (a_i'\eta(r) \otimes \bar{b}') \otimes (a_i'' \otimes \bar{b}'') \\ &= \sum (a_i' \otimes \bar{b}') \otimes (a_i''\eta(r) \otimes \bar{b}'') \\ &= \sum (a_i' \otimes \bar{b}') \otimes (a_i'' \otimes \eta(r)\bar{b}''),\end{aligned}$$

using property (e), which says that $\sum a_i'\eta(r) \otimes a_i'' = \sum a_i' \otimes a_i''\eta(r)$.

The map ψ is thus a coproduct, which can be shown to be coassociative and cocommutative. It induces a product on $\mathcal{A}_m^* {}^{t^*}\underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_n^*$, which can be shown to be the same as the evident one. It is now clear that μ^* is a ring homomorphism.

We have produced enough structure to allow us to state the following.

Proposition 18.1. *Given a complete local ring R, there is a category $D(R)$, with object set $\hom(\mathcal{O}, R)$, and morphism set $\coprod_{m\geq 0} \hom(\mathcal{A}_m^*, R)$. The source and target of a morphism are determined by s^* and t^*, and composition of morphisms is determined by μ^*.*

Proving the proposition is a "straightforward" exercise, and is left to the reader. This essentially amounts to showing $(\mathcal{O}, \prod_{m\geq 1} \mathcal{A}_m^*)$ is a Hopf algebroid (except that there is no involution map, since the categories $D(R)$-need not have inverses, and we should be careful to think of morphisms as a graded set, graded by the integer m).

Now let Mod_D be the category whose objects are collections of functors $\mathcal{F}_R\colon D(R)^{\mathrm{op}} \to \mathrm{Mod}_R$, together with natural isomorphism $\mathcal{F}_f\colon f^* \circ \mathcal{F}_R \to \mathcal{F}_{R'} \circ f^*$ for each local homomorphism $f\colon R \to R'$, satisfying the evident coherence properties. It is not hard to see that Mod_D is equivalent to the category of comodules over the Hopf algebroid.

Proposition 18.2. *There is a functor $\tilde{U}\colon \mathrm{Mod}_{\mathcal{A}} \to \mathrm{Mod}_D$, which is an equivalence of tensor categories.*

Proof. Given an $\mathcal{A}$-module N, the "action" maps ${\mathcal{A}_m}^r \underset{\mathcal{O}}{\otimes} N \to \mathcal{A}_m$ give rise to "coaction" maps

$$\psi_m^*\colon N \to N \underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_m^* \approx \hom_{\ell_{\mathcal{O}}}(\mathcal{A}_m, N)$$

by

$$n \mapsto (a \mapsto a \cdot n).$$

Note that the coaction map ψ_m^* is a map of $\mathcal{O}$-modules, where $\mathcal{O}$ acts on the target through $t^*\colon \mathcal{O} \to \mathcal{A}_m^*$. That is,

$$\psi_m^*(rn) = (a \mapsto a \cdot rn).$$

Consider an object G in $D(R)$, that is, a ring homomorphism $\chi_G\colon \mathcal{O} \to R$. Let $(\tilde{U}N)_R(G) \overset{\text{def}}{=} N \underset{\mathcal{O}}{\otimes}{}^{\chi_G} R$.

For a morphism $f\colon G \to G'$ in $D(R)$, that is, a ring homomorphism $\chi_f\colon \mathcal{A}_m^* \to R$, define $(\tilde{U}N)_R(f)$ by

$$N \underset{\mathcal{O}}{\otimes}{}^{\chi_{G'}} R \approx N \underset{\mathcal{O}}{\otimes}{}^{t^*} \mathcal{A}_m^* \underset{\mathcal{A}_m^*}{\otimes}{}^{\chi_f} R \xrightarrow{\psi_m^* \otimes \mathrm{id}} N \underset{\mathcal{O}}{\otimes}{}^{s^*} \mathcal{A}_m^* \underset{\mathcal{A}_m^*}{\otimes}{}^{\chi_f} R \approx N \underset{\mathcal{O}}{\otimes}{}^{\chi_G} R,$$

keeping in mind the commutative diagram

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "t^*"] \arrow[d, "\chi_{G'}"'] & \mathcal{A}_m^* \arrow[d, "\chi_f"'] & \mathcal{O} \arrow[l, "s^*"'] \arrow[d, "\chi_G"'] \\
R \arrow[r, equal] & R \arrow[r, equal] & R
\end{tikzcd}$$

It is straightforward to check that this really defines an object of Mod_D.

Conversely, it is straightforward to define an inverse functor $\mathrm{Mod}_D \to \mathrm{Mod}_{\mathcal{A}}$, by evaluating a D-module $\mathcal{F}$ at rings $R = \mathcal{A}_m^*$. $\square$

Recall that if F is a commutative E-algebra spectrum, we can make $\pi_0 F$ into an $\mathcal{A}$-module, using

$$(a, b) \mapsto \mu \circ \mathbb{P}_E^m(b) \circ a\colon \pi_0 \mathbb{P}_E^m E \times \pi_0 F \to \pi_0 F,$$

where $\mu\colon \mathbb{P}^m_E F \to F$ is the structure map of F. We obtain the $\mathcal{A}$-module structure by restricting to $\mathcal{A}_m \subset E_0^\wedge B\Sigma_m$.

On the other hand, there is a "coaction" map $\pi_0 F \to \pi_0 F \otimes_{\mathcal{O}} S_m$, where $S_m = E^0 B\Sigma_m/J$, defined using

$$P_m\colon \pi_0 F \to \pi_0 F^{B\Sigma_m} \approx \pi_0 F \otimes_{\mathcal{O}} E^0 B\Sigma_m.$$

Recall that $\mathcal{A}_m$ is defined by the exact sequence

$$0 \to \mathcal{A}_m \xrightarrow{j} E_0^\wedge B\Sigma_m \xrightarrow{\mathrm{tr}} \bigoplus_{0<i<m} E_0^\wedge B(\Sigma_i \times \Sigma_{m-i}),$$

and that furthermore the inclusion j admits an $\mathcal{O}$-module retraction. This means that if we take $\mathcal{O}$-linear duals, we obtain an exact sequence

$$0 \leftarrow \mathcal{A}_m^* \leftarrow E^0 B\Sigma_m \xleftarrow{\mathrm{tr}} \bigoplus_{0<i<m} E^0 B(\Sigma_i \times \Sigma_{m-i}),$$

and thus there is a canonical identification of $\mathcal{A}_m^*$ and S_m. Moreover, a straightforward diagram chase shows that the "action" and "coaction" described above are actually adjoint to each other in the expected way.

Now take $F = E^{B\mathbb{T}}$. The calculations of the previous section thus show that the ring $\mathcal{O}_{G_{\mathrm{univ}}} = E^0 B\mathbb{T}$ is an $\mathcal{A}$-module, and thus we may consider the object $\tilde{U}\mathcal{O}_{G_{\mathrm{univ}}} \in \mathrm{Mod}_D$. Note that is means

$$(\tilde{U}\mathcal{O}_{G_{\mathrm{univ}}})_R(G) = \mathcal{O}_{G_{\mathrm{univ}}} \underset{\mathcal{O}}{\otimes}{}^{\chi_G} R$$

is a formal group over R, and in fact is a deformation of Γ (by Lubin-Tate), and that $(\tilde{U}\mathcal{O}_{G_{\mathrm{univ}}})_R(f)$ is a deformation of Frobenius, by (17.7). Thus, we obtain a functor

$$H\colon D(R) \to \mathrm{DefFrob}(R).$$

Theorem 18.3 (Strickland [Str98])**.** *The ring $S_m = E^0 B\Sigma_m/J$ classifies $\star$-isomorphism classes of pairs $(G > K)$ where G is a deformation of Γ, and K is a subgroup scheme of G of degree m. The universal example of a subgroup scheme is the divisor on G over S_m defined by $\mathcal{O}_D = E^0(L \otimes V_m)'/I'$.*

This gives the theoremification of Pre-Theorem 1.

Corollary 18.4. *The functor H is an equivalence of categories, and thus* $\mathrm{Mod}_{\mathcal{A}} \approx \mathrm{Mod}_D \approx \mathrm{Mod}_{DF}$.

Proof. The functor H is essentially surjective; this is the main content of the Lubin-Tate theorem. Strickland's theorem and (16.10) show that it is fully faithful. ☐

19. Sketch of the proof of Pre-Theorem 2

Let $T = \mathrm{DL}_{E_0}$ denote the algebraic theory we are studying. Using Pre-theorem 1, we can define a functor

$$U\colon \mathrm{Model}_T \to \mathrm{Alg}_{DF}.$$

It is clear that we get a functor to ring objects in Mod_{DF}, and (17.7) (which works more generally with $E^0 X$ replaced by π_0 of an E-algebra) shows that the output of U satisfies the congruence condition. We want to show that U restricts to an equivalence of subcategories of torsion free objects.

First, consider the diagram of functors

$$\begin{tikzcd} \mathrm{Model}_T \arrow[rr, "U"] \arrow[dr, "F"'] & & \mathrm{Alg}_{DF} \arrow[dl, "G"] \\ & \mathrm{Alg}_{\mathcal{O}} & \end{tikzcd}$$

where the functor F sends a T-model to its underlying $\mathcal{O}$-algebra, and the functor G sends an object $B \in \mathrm{Alg}_{DF}$ to the $\mathcal{O}$-algebra $B_{\mathrm{univ}} \overset{\mathrm{def}}{=} B_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}})$ obtained by evaluating B at the universal deformation. The functors F and G are both faithful, and reflect isomorphisms. Furthermore, the diagram commutes up to natural isomorphism.

Next, note that the category Alg_{DF} is cocomplete. In fact, colimits in Alg_{DF} are computed by taking colimits "pointwise".

Next, observe that U must preserve colimits, since colimits in Model_T are computed in the same way.

Let $F\colon T^{\mathrm{op}} \to \mathrm{Model}_T$ denote the functor taking an object T^n in the theory to the free model $F(T^n) \overset{\mathrm{def}}{=} F_T(n)$. Note that F is fully faithful, and that F preserves coproducts. Since U preserves colimits, and any T-model can be built as a colimit of free models, we see that the functor U is completely determined by the restriction $U \circ F\colon T^{\mathrm{op}} \to \mathrm{Alg}_{DF}$; we can even say that U is the "left Kan extension" of $U \circ F$ along F.

Define a functor

$$V\colon \mathrm{Alg}_{DF} \to \mathrm{Model}_T$$

by

$$(VC)(T^n) \overset{\mathrm{def}}{=} \mathrm{Alg}_{DF}(UF(T^n), C).$$

Note that this is really a T-model, because U preserves coproducts and thus $VC\colon T \to \mathrm{Set}$ preserves products. It is straightforward to check that

$$U\colon \mathrm{Model}_T \rightleftarrows \mathrm{Alg}_{DF} :V$$

form an adjoint pair, with unit and counit $\eta\colon I \to VU$ and $\epsilon\colon UV \to I$.

We can give a fairly explicit description of the counit ϵ. If $C \in \mathrm{Alg}_{DF}$, the map $\epsilon\colon UVC \to C$ in Alg_{DF}, evaluated at the universal deformation, is a map

$$\gamma\colon \mathrm{Alg}_{DF}(UF(T^1), C) \to C_{\mathrm{univ}}.$$

Explicitly, γ is the map which sends $f\colon UF(T^1) \to C$ to the element $f(x) \in C_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}})$, where $x \in (UF(T^1))_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}}) \approx UF(T^1)$ is the generator.

Proposition 19.1. *If $C \in \mathrm{Alg}_{DF}$ is torsion free, then $\gamma\colon \mathrm{Alg}_{DF}(UF(T^1), C) \to C_{\mathrm{univ}}$ is a bijection.*

19.2. **Proof of Pre-theorem 2.** Assuming (19.1) for the moment, we can give a proof of Pre-Theorem 2.

First note that a map in Alg_{DF} is an isomorphism if and only if it is an isomorphism when evaluated at the universal deformation. Thus (19.1) immediately implies that $\epsilon\colon UV(C) \to C$ is iso if C is torsion free.

Next observe that $B \in \mathrm{Model}_T$ is torsion free if and only if $U(B) \in \mathrm{Alg}_{DF}$ is torsion free, since for either category being torsion free is a property of the underlying ring, and U preserves the underlying ring.

We can immediately conclude that $U^{\mathrm{tf}}\colon \mathrm{Model}_T^{\mathrm{tf}} \to \mathrm{Alg}_{DF}^{\mathrm{tf}}$ is essentially surjective, for if C is a torsion free object of Alg_{DF}, then $C \approx UV(C)$, and we see that $V(C)$ must be torsion free since $UV(C)$ is so.

It remains to show that U^{tf} is fully faithful. Let B and B' be T-models, and consider

$$\phi\colon \mathrm{Model}_T(B,B') \xrightarrow{U} \mathrm{Alg}_{DF}(UB,UB') \xrightarrow{\sim} \mathrm{Model}_T(B,VU(B'));$$

this is given by composing $f\colon B \to B'$ with $\eta\colon B' \to VU(B')$. I claim that ϕ is a bijection if B' is torsion free, for which it is enough to show that $\eta_{B'}\colon B' \to VU(B')$ is an isomorphism when B' is torsion free.

In general, if $f\colon B \to B'$ is a map of T-models, then f is an isomorphism if and only if $U(f)$ is a isomorphism in Alg_{DF}. Thus, it suffices to show that $U(\eta_{B'})$ is an isomorphism. The identity map of $U(B')$ factors as

$$U(B') \xrightarrow{U\eta} UVU(B') \xrightarrow{\epsilon U} U(B'),$$

and thus it suffices to check that $\epsilon_{U(B')}\colon UV(UB') \to UB'$ is iso, which follows from (19.1) since UB' is torsion free.

19.3. **A worked example.** Before trying to prove (19.1), let's consider an example where we know the answers, namely p-adic K-theory.

In this case, T is the theory of **θ-rings**. A θ-ring is a $\mathbb{Z}_p$ algebra B, together with an function $\theta\colon B \to B$ satisfying

(a) $\theta(c) = (c - c^p)/p$ for $c \in \mathbb{Z}_p$,
(b) $\theta(x+y) = \theta(x) + \theta(y) - \sum_{0<i<p} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$,
(c) $\theta(xy) = x^p\theta(y) + \theta(x)y^p + p\theta(x)\theta(y)$.

These conditions imply that the function $\psi\colon B \to B$ defined by $\psi(x) = x^p + p\theta(x)$ is $\mathbb{Z}_p$-algebra homomorphism, with the property that $\psi(x) \equiv x^p \mod pB$. That this is true is essentially due to McClure, although the first hint is already in work of Hodgkin, who described the free θ-ring on one generator:

$$\Lambda = \mathbb{Z}_p[x_k,\ k \geq 0],$$

where $x = x_0$ is the generator, and the x_k satisfy

$$(\psi)^n(x) = \sum_{0 \leq i \leq n} p^i x_i^{p^{n-i}}.$$

As we observed in (16.11), the category Alg_{DF} is the category of **ψ-rings**, the objects of which are commutative $\mathbb{Z}_p$-algebras C equipped with a $\mathbb{Z}_p$-algebra map $\psi\colon C \to C$ such that $\psi(x) \equiv x^p \mod pC$. The functor U is the evident one. The functor V is defined by

$$VC = \hom_{\psi\text{-ring}}(\Lambda, C),$$

with the operation θ defined using the θ-ring homomorphism $x \mapsto \theta(x)\colon \Lambda \to \Lambda$.

In this context, the Wilkerson criterion says that if C is a torsion free ψ-ring, then C admits a unique operation θ making it into a θ-ring. In particular, this means that if C is torsion free, then $\hom_{\psi\text{-ring}}(\Lambda, C) \approx C$.

Recall that the ring of Witt vectors is defined by

$$WC = \hom_{\mathrm{Alg}_{Z_p}}(\Lambda, C).$$

Thus VC is a certain subring of the Witt ring.

If C is an $\mathbb{F}_p$-algebra, then C admits a unique structure of ψ-ring, by $\psi(x) = x^p$. Furthermore, any $\mathbb{Z}_p$-algebra homomorphism $\Lambda \to C$ is a ψ-homomorphism. Thus, if C has characteristic p, then $VC \approx WC$; the operation ψ on VC corresponds to the operation on WC usually called "Frobenius".

19.4. **Sketch of the proof of** (19.1)**.** I'm going to write $\mathbb{C}(M) = \bigoplus_{m\geq 0} \mathbb{C}_m(M)$ for the free T-model on an underlying $\mathcal{O}$-module M. Note that $\mathbb{C}_m(\mathcal{O}^{\oplus d}) \approx \pi_0 \mathbb{P}_E^m(E^{\vee d})$.

Say an **$\mathcal{A}$-ring** is an $\mathcal{O}$-algebra equipped B with a left $\mathcal{A}$-module structure, such that the $\mathcal{A}$-action applied to a product in B is given according to the coproduct of $\mathcal{A}$. There is a forgetful functor from T-models to $\mathcal{A}$-rings. There is a square of functors

$$\begin{tikzcd} \mathrm{Model}_T \arrow[r, "U"] \arrow[d, "U'"'] & \mathrm{Alg}_{DF} \arrow[d, "i"] \\ \mathcal{A}\mathrm{-rings} \arrow[r, "\sim"'] & \mathrm{Mod}_{DF}^{\mathrm{rings}} \end{tikzcd}$$

where $\mathrm{Mod}_{DF}^{\mathrm{rings}}$ denotes the category of rings in Mod_{DF}. By definition, the functor i is the inclusion of a full subcategory, while the functor along the bottom is an equivalence by Pre-Theorem 1. Thus, $\mathrm{Alg}_{DF}(UF(T^1), C) \approx \mathcal{A}\mathrm{-rings}(U'F(T^1), C_{\mathrm{univ}})$, where C_{univ} is the underlying $\mathcal{A}$-ring of C.

In view of this, we have reduced (19.1) to showing

Proposition 19.5. *If B is an $\mathcal{A}$-ring which arises as the underlying $\mathcal{A}$-ring of an object $C \in \mathrm{Alg}_{DF}$, then*

$$\gamma\colon \hom_{\mathcal{A}\mathrm{-ring}}(\mathbb{C}\mathcal{O}, B) \to B$$

is a bijection if B is torsion free.

There is a monad $\mathbb{D}$ on $\mathcal{O}$-modules, so that a $\mathbb{D}$-algebra is precisely an $\mathcal{A}$-ring. We have

$$\mathbb{D}(M) \approx \mathrm{Sym}_{\mathcal{O}}(\mathcal{A}),$$

and there is an evident natural transformation $\mathbb{D} \to \mathbb{C}$. This transformation supplies a weight-grading on $\mathbb{D}$, compatible with that on $\mathbb{C}$. Note that $\mathbb{D}(M \oplus N) \approx \mathbb{D}(M) \otimes_{\mathcal{O}} \mathbb{D}(N)$; in fact, the category of $\mathbb{D}$-algebras is a category of models for a COT.

Proposition 19.6. *The map $\mathbb{D}(M) \otimes \mathbb{Q} \to \mathbb{C}(M) \otimes \mathbb{Q}$ is an isomorphism.*

Proof. This proof is sketched in [ST97]. First note that it suffices to show that $\mathbb{D}\mathcal{O} \otimes \mathbb{Q} \to \mathbb{C}\mathcal{O} \otimes \mathbb{Q}$ is iso. The object $\mathbb{C}\mathcal{O} \otimes \mathbb{Q}$ is a graded, connected, commutative and cocommutative Hopf algebra over a ring of characteristic 0; the coproduct $\Delta^+\colon \mathbb{C}\mathcal{O} \to \mathbb{C}\mathcal{O} \otimes \mathbb{C}\mathcal{O}$ is the one giving the addition formula for operations. The structure theory of Hopf algebras implies that $\mathbb{C}\mathcal{O}$ is isomorphic to the symmetric algebra on $\operatorname{Prim}(\mathbb{C}\mathcal{O}\otimes\mathbb{Q})$. Since $\operatorname{Prim}(\mathbb{C}\mathcal{O}\otimes\mathbb{Q}) \approx (\operatorname{Prim}\mathbb{D}\mathcal{O})\otimes\mathbb{Q} \approx \mathcal{A}\otimes\mathbb{Q}$, the result follows. $\square$

Now suppose that B in an $\mathcal{A}$-ring corresponding to a torsion free $C \in \mathrm{Alg}_{DF}$. To prove (19.5), we need to show that given $b \in B$, there exists a unique map $f_b\colon \mathbb{C}\mathcal{O} \to B$ of $\mathcal{A}$-rings sending the generator to b. noted above.

Since B is an $\mathcal{A}$-ring there is a unique map $g = g_b\colon \mathbb{D}(\mathcal{O}) \to B$ of $\mathcal{A}$-rings sending the generator to b. By (19.6), there is a commutative diagram of $\mathcal{A}$-rings

$$\begin{tikzcd} \mathbb{D}\mathcal{O} \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ \mathbb{C}\mathcal{O} \arrow[r, "f"'] & B\otimes\mathbb{Q} \end{tikzcd}$$

where f is uniquely determined by g. Thus, there is at most one map $f_b\colon \mathbb{C}\mathcal{O} \to B$ sending the generator of $\mathbb{C}\mathcal{O}$ to b, and it exists if and only if f factors through B.

We will inductively prove the following statement:

P(m): Let $b \in B$, and let $f\colon \mathbb{C}\mathcal{O} \to B\otimes Q$ be the corresponding map of $\mathcal{A}$-rings. Then $f(\sum_{k\le m}\mathbb{C}_k\mathcal{O}) \subseteq B$.

Note that if we can prove P(m), then we will actually know that given a map $b\colon M \to B$ from a free $\mathcal{O}$-module M, we get a diagram

$$\begin{tikzcd} \mathbb{D}(M) \arrow[r] \arrow[d] & B \arrow[d] \\ \mathbb{C}(M) \arrow[r, "f"'] & B\otimes\mathbb{Q} \end{tikzcd}$$

of $\mathcal{A}$-rings, such that $f(\sum_{k\le m}\mathbb{C}_k(M)) \subseteq B$.

We prove $P(m)$ is several cases:

Case of $m = 0, 1$. This is clear, since $\mathbb{D}_m(\mathcal{O}) \approx \mathbb{C}_m(\mathcal{O})$ if $m = 0, 1$.

Case of m prime to p. Since f is a map of $\mathcal{A}$-rings, induction leads to a commutative diagram

$$\begin{tikzcd} \mathbb{C}_1\mathcal{O}\otimes\mathbb{C}_{m-1}\mathcal{O} \arrow[r, "f\otimes f"] \arrow[d, "\mu"'] & B\otimes B \arrow[r, "\mu"] & B \arrow[d] \\ \mathbb{C}_m\mathcal{O} \arrow[rr, "f"'] & & B\otimes\mathbb{Q} \end{tikzcd}$$

where the arrows marked μ are multiplication. The result follows from the fact that the map $\mathbb{C}_1\mathcal{O}\otimes\mathbb{C}_{m-1}\mathcal{O} \to \mathbb{C}_m\mathcal{O}$ (i.e., $E_0^\wedge B(\Sigma_1\times\Sigma_{m-1}) \to E_0^\wedge B\Sigma_m$) is surjective by the usual transfer argument.

Case of $m = p$. The assignment $(b \in B) \mapsto (g\colon \mathbb{D}_p\mathcal{O} \to B)$ gives a "total power operation" map $P\colon B \to \hom_{\mathcal{O}}(\mathbb{D}_p\mathcal{O}, B)$. To prove the case, we need to show that there is a dotted arrow in

$$\begin{tikzcd} B \arrow[r] \arrow[dr, dotted, "P"] & \hom_{\mathcal{O}}(\mathbb{D}_p\mathcal{O}, B) \\ & \hom_{\mathcal{O}}(\mathbb{C}_p\mathcal{O}, B) \arrow[u, tail, "i"'] \end{tikzcd}$$

Note that $\mathbb{D}_p\mathcal{O} \approx \mathrm{Sym}^p_{\mathcal{O}} \mathcal{A}_1 \oplus \mathcal{A}_p \approx \mathcal{O} \oplus \mathcal{A}_p$. Thus i is a map $\hom_{\mathcal{O}}(\mathbb{C}_p\mathcal{O}, B) \to B \times \hom_{\mathcal{O}}(\mathcal{A}_p, B)$, and $P\colon B \to B \times \hom_{\mathcal{O}}(\mathcal{A}_p, B)$ is given by $P(b) = (b^p, \psi_p^*(b))$, where $\psi_p^*\colon B \to \hom_{\mathcal{O}}(\mathcal{A}_p, B) \approx B \otimes_{\mathcal{O}} \mathcal{A}_p^*$ is the coaction map associated to the $\mathcal{A}$-module structure on B.

Recall the diagram of (17.7)

$$\begin{tikzcd} I \cap J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, tail] & I\mathcal{A}_p^* \arrow[d, tail] \\ J \arrow[r, tail] \arrow[d, tail] & E^0 B\Sigma_p \arrow[r, tail] \arrow[d, tail] & \mathcal{A}_p^* \arrow[d, tail, "u^*"] \\ pE^0 \arrow[r, tail] & E^0 \arrow[r, tail] & E^0/pE^0 \end{tikzcd}$$

where I is the augmentation ideal, and J is the ideal generated by transfers. The rows and columns of this diagram are short exact sequences. In the sequence of covering maps $E\Sigma_p \xrightarrow{h} B(\Sigma_i \times \Sigma_{p-i}) \to B\Sigma_p$, the map labelled h has degree prime to p when $0 < i < p$, and therefore the image of $\mathrm{tr}^{\Sigma_p}_{\Sigma_i \times \Sigma_{p-i}}$ can be identified with the image of the transfer from the trivial subgroup of Σ_p. In particular, this implies that J is a rank 1 $\mathcal{O}$-module, and thus the map $J \to pE^0$ is the diagram is iso, and hence $I \cap J = 0$. Thus, the lower right-hand square is a pullback square.

Now consider the induced square

$$\begin{tikzcd} \hom_{\mathcal{O}}(\mathbb{C}_p\mathcal{O}, B) \arrow[r] \arrow[d] & \hom_{\mathcal{O}}(\mathcal{A}_p, B) \arrow[d] \\ B \arrow[r] & B/pB \end{tikzcd}$$

By what we have just shown, this square is a pullback square if B is torsion free.

Since B is the underlying $\mathcal{A}$-ring of an object $C \in \mathrm{Alg}_{DF}$, it satisfyies a congruence condition, which says precisely that $\psi_p^*(b) \equiv b^p \mod pB$. Therefore, the coaction map $B \to \hom_{\mathcal{O}}(\mathbb{D}_p\mathcal{O}, B)$ factors through $\hom_{\mathcal{O}}(\mathbb{C}_p\mathcal{O}, B)$, as desired.

Case of $m = kp$ for $k > 1$. By induction, $f_p\colon \mathbb{C}_p\mathcal{O} \to B \otimes \mathbb{Q}$ factors through B. The map of $\mathcal{O}$-modules f_p gives rise to a map $h\colon \mathbb{C}\mathbb{C}_p\mathcal{O} \to B \otimes \mathbb{Q}$. I claim that

$$\begin{tikzcd} \mathbb{C}_k\mathbb{C}_p\mathcal{O} \arrow[r, "h"] \arrow[d, "\psi"] & B \arrow[d] \\ \mathbb{C}_m\mathcal{O} \arrow[r, "f"'] & B \otimes \mathbb{Q} \end{tikzcd}$$

commutes, where the map marked ψ is given by the structure map of the monad $\mathbb{C}$. This can be deduced from the fact that the analogous diagram

$$\begin{array}{ccc} \mathbb{D}_k\mathbb{D}_p\mathcal{O} & \longrightarrow & B \\ \downarrow & & \downarrow \\ \mathbb{D}_m\mathcal{O} & \longrightarrow & B \end{array}$$

commutes, since B is an algebra of the monad $\mathbb{D}$, and the two diagrams become identical when tensored with $\mathbb{Q}$.

The result follows from the fact that $\mathbb{C}_k\mathbb{C}_p\mathcal{O} \to \mathbb{C}_m\mathcal{O}$ (i.e., $E_0^\wedge B(\Sigma_k \wr \Sigma_p) \to E_0^\wedge B\Sigma_m$) is surjective by the usual transfer argument.

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