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DE RHAM–WITT COHOMOLOGY FOR A PROPER AND SMOOTH MORPHISM

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Abstract We construct a relative de Rham–Witt complex $W\Omega^{\cdot}_{X/S}$ for a scheme X over a base scheme S. It coincides with the complex defined by Illusie (*Annls Sci. Ec. Norm. Super.* **12** (1979), 501–661) if S is a perfect scheme of characteristic $p > 0$. The hypercohomology of $W\Omega^{\cdot}_{X/S}$ is compared to the crystalline cohomology if X is smooth over S and p is nilpotent on S. We obtain the structure of a $3n$-display on the first crystalline cohomology group if X is proper and smooth over S.

Keywords: crystalline cohomology; de Rham cohomology

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Contents

Introduction

Let X be a smooth and proper scheme over a perfect field k. Assume X lifts to a smooth scheme $\tilde{X}$ over $W(k)$. It was discovered by Grothendieck that the hypercohomology of the de Rham complex $\Omega^{\cdot}_{\tilde{X}/W(k)}$ does not depend on the lifting but only on X. The crystalline cohomology defines this hypercohomology intrinsically in terms of X. It makes sense without the existence of any lifting $\tilde{X}$. Berthelot proved that this cohomology enjoys all good properties, i.e. it is a Weil cohomology on the category of proper and smooth schemes over k.

The de Rham–Witt complex $W\Omega^{\cdot}_{X/k}$ was defined by Illusie [**I**] relying on ideas of Lubkin, Bloch and Deligne. It is a complex of sheaves of $W(k)$-modules on X, whose hypercohomology is the crystalline cohomology.

The main goal of this paper is to extend Illusie's definition of the de Rham–Witt complex to a relative situation, where X is an arbitrary scheme over a $\mathbb{Z}_{(p)}$-algebra R. The de Rham–Witt complex is a projective system indexed by $\mathbb{N}$ of complexes $W_n\Omega^{\cdot}_{X/R}$ of $W_n(R)$-algebras on X. If p is nilpotent in R and X is smooth over $\operatorname{Spec} R$, the hypercohomology of $W_n\Omega^{\cdot}_{X/R}$ is isomorphic to the crystalline cohomology

$$H^*_{\mathrm{crys}}(X/W_n(R)) = H^*(X/W_n(R), \mathcal{O}^{\mathrm{crys}}_{X/W_n(R)})$$

of the crystalline structure sheaf.

We define a de Rham–Witt complex with coefficients in a crystal E on the crystalline site of $X/W_n(R)$. Its hypercohomology computes the crystalline cohomology of E.

As an application, we show that the first crystalline cohomology of an abelian scheme over a ring R where p is nilpotent has naturally the structure of a $3n$-display in the sense of Zink [**Z**]. This was known in the case where the geometric fibres of this abelian scheme have no p-torsion points, and trivially in the case where the ring R is reduced.

In the following we will give a more detailed description of the results of this paper. Let R be a $\mathbb{Z}_{(p)}$-algebra. In the first section we define the de Rham–Witt complex for any R-algebra S. It is projective system of complexes of $W_n(R)$-modules $\{W_n\Omega^{\cdot}_{S/R}\}_{n\in\mathbb{N}}$. We identify $\{W_n\Omega^{\cdot}_{S/R}\}_{n\in\mathbb{N}}$ as an initial object in the category of F-V-procomplexes over the R-algebra S. These procomplexes are defined as follows.

By a differential graded $W_n(S)/W_n(R)$-algebra P_n we mean the following: P_n is a graded $W_n(S)$-algebra with unit element,

$$P_n = \bigoplus_{i\in\mathbb{Z}_{\geqslant 0}} P_n^i,$$

and equipped with a $W_n(R)$-linear differential $\mathrm{d} : P_n \to P_n$, which is homogeneous of degree one such that

$$\begin{aligned} \omega\cdot\eta &= (-1)^{ij}\eta\omega, \quad \omega\in P_n^i, \quad \eta\in P_n^j,\\ \mathrm{d}(\omega\cdot\eta) &= (\mathrm{d}\omega)\eta + (-1)^i\omega\,\mathrm{d}\eta,\\ \mathrm{d}^2 &= 0. \end{aligned}$$

Let γ_k, $k \geqslant 0$, be the canonical divided powers on the ideal $VW_{n-1}(S) \subset W_n(S)$. We also denote by d the map $W_n(S) \to P_n^0 \xrightarrow{\mathrm{d}} P_n^1$. If this map d is a pd-differential, i.e. if

$$\mathrm{d}\gamma_k(x) = \gamma_{k-1}(x)\mathrm{d}x \quad \text{for } x \in VW_{n-1}(S),$$

we call P_n a pd-differential graded $W_n(S)/W_n(R)$-algebra.

Definition 1. An F-V-procomplex over an R-algebra S is a projective system of differential graded $W_n(S)/W_n(R)$-algebras P_n for $n \geqslant 1$,

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1.$$

This system is equipped with two sets of homomorphisms of graded abelian groups

$$F : P_{n+1} \to P_n, \quad V : P_n \to P_{n+1}, \quad n \geqslant 1,$$

such that the following properties hold.

(i) Let $P_{n,[F]}$ be the graded $W_{n+1}(S)$-algebra obtained by restriction of scalars by $F : W_{n+1}(S) \to W_n(S)$. Then F induces a homomorphism of graded algebras,

$$F : P_{n+1} \to P_{n,[F]}.$$

(ii) The structure morphism $W_n(S) \to P_n^0$ is compatible with F and V.

(iii) The following relations hold:

$$\begin{aligned} {}^{FV}\omega &= p\omega \quad \text{for } \omega \in P_n, \quad n \geqslant 1, \\ {}^{F}\mathrm{d}^{V}\omega &= \mathrm{d}\omega, \\ {}^{F}\mathrm{d}[x] &= [x^{p-1}]\mathrm{d}[x], \quad x \in S, \\ {}^{V}(\omega\, {}^{F}\eta) &= ({}^{V}\omega) \cdot \eta, \quad \eta \in P_{n+1}. \end{aligned}$$

This definition implies that P_n is even a pd-differential graded $W_n(S)/W_n(R)$-algebra for each n. Let $\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$ be the pd-differential de Rham complex (which is the universal pd-differential graded $W_n(S)/W_n(R)$-algebra). We obtain a natural epimorphism,

$$\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \to W_n\Omega^{\cdot}_{S/R}. \tag{1}$$

While Illusie works with V-procomplexes and identifies $\{W_n\Omega^{\cdot}_{S/R}\}_{n\in\mathbb{N}}$ (for $R = k$, a perfect field of characteristic p) as a universal V-procomplex and afterwards shows that the Frobenius on $\{W_n\Omega^{\cdot}_{S/R}\}_{n\in\mathbb{N}}$ is well defined by a rather long computation, the starting point of our construction is that we can already define the Frobenius on $\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$. The crucial observation here is the following.

If $\nu : W_n(S) \to M$ is a pd-derivation in some $W_n(S)$-module M, then

$$\begin{aligned} &{}^{F}\nu : W_{n+1}(S) \to M_{[F]}, \\ &\xi = [x] + {}^{V}\varrho \mapsto [x]^{p-1}\nu([x]) + \nu(\varrho) \end{aligned}$$

is also a pd-derivation. The verification of the additivity of ${}^F\nu$ requires that ν is a pd-derivation.

Then $W_n\Omega^{\cdot}_{S/R}$ is defined as a quotient of $\check{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$. If R is a perfect ring of characteristic p, our complex agrees with Illusie's complex.

By gluing arguments, the definition is then extended to schemes X over $\operatorname{Spec} R$ to obtain the de Rham–Witt complex $W_n\Omega^{\cdot}_{X/R}$. We set

$$W\Omega^{\cdot}_{X/R} = \varprojlim_n W_n\Omega^{\cdot}_{X/R}.$$

We remark that Hesselholt and Madsen [**HM**] defined independently an absolute de Rham–Witt complex $W_n\Omega^{\cdot}_S$ for a $\mathbb{Z}_{(p)}$-algebra S, which is closely related to ours. There exists a homomorphism $W_n\Omega_S \to W_n\Omega^{\cdot}_{S/\mathbb{Z}_{(p)}}$, which commutes with F and V, but this is in general not an isomorphism, e.g. $S = \mathbb{Z}_{(p)}$.

In the second section we give an explicit description of the de Rham–Witt complex $W\Omega^{\cdot}_{S/R}$ if $S = R[T_1, \dots, T_d]$ is a polynomial ring. In this description, we ignore the $W_n(S)$-module structure on $W\Omega^{\cdot}_{S/R}$ but consider it only as a $W_n(R)$-module.

Let us first consider the case of one variable, $S = R[T]$. We denote the Teichmüller representative of T by $X = [T] \in W(R[T])$. Let $k \in \mathbb{Z}_{\geqslant 0}[1/p]$ be an arbitrary element, which will be called a weight. We denote its denominator by $p^{u(k)}$. Any Witt vector $\omega \in W_n(S)$ has a unique expression

$$\omega = \sum_{k \text{ integral}} \xi_k X^k + \sum_{k \text{ not integral}} {}^{V^{u(k)}}(\eta_k X^{p^{u(k)}k}),$$

where $\xi_k \in W_n(R)$ and $\eta_k \in W_{n-u(k)}(R)$, and these elements are zero for almost all k. Actually, only weights such that $p^{n-1}k$ is integral appear in this expression.

An element in $\omega \in W_n\Omega^1_{S/R}$ has a unique expression

$$\omega = \sum_{k\geqslant 1,\ k \text{ integral}} \xi_k X^{k-1}\mathrm{d}X + \sum_{k \text{ not integral}} \mathrm{d}\,{}^{V^{u(k)}}(\eta_k X^{p^{u(k)}k}),$$

where $\xi_k \in W_n(R)$ and $\eta_k \in W_{n-u(k)}(R)$, and these elements are zero for almost all k.

This means that there are direct decompositions as $W(R)$-modules,

$$\left.\begin{aligned} W_n(S) &= \bigoplus_{k \text{ integral}} W_n(R)X^k \oplus \bigoplus_{k \text{ not integral}} {}^{V^u}W_{n-u}(R)X^k, \\ W_n\Omega^1_{S/R} &= \bigoplus_{k\geqslant 1, \text{ integral}} W_n(R)X^k\mathrm{d}\log X \oplus \bigoplus_{k \text{ not integral}} {}^{V^u}W_{n-u}(R)\mathrm{d}X^k. \end{aligned}\right\} \tag{2}$$

In these formulae, X^k, X^k, $X^k\mathrm{d}\log X$ and $\mathrm{d}X^k$ are viewed as symbols. For $l \neq 0, 1$, we have $W_n\Omega^l_{S/R} = 0$. The action of F, V, d on (2) can easily be made explicit.

We turn now to the case of several variables $S = R[T_1, \dots, T_d]$. For the description of the de Rham–Witt complex, we introduce the Cartier–Raynaud algebra $\mathbb{D}_R$ of the ring R. This algebra is a variant of the algebra introduced by Illusie and Raynaud in [**IR**]. The elements of $\mathbb{D}_R$ are formal sums,

$$\sum_{n\geqslant 0} V^n\xi_n + \sum_{n>0} \eta_n F^n + \sum_{n\geqslant 0} \mathrm{d}V^n\xi'_n + \sum_{n>0} \eta'_n F^n\mathrm{d}. \tag{3}$$

Here, n runs over integers as indicated. We consider F, V, d as indeterminates. By ξ_n, ξ'_n, η_n, η'_n we denote arbitrary elements in $W(R)$ that satisfy the following condition: for any given number $u > 0$, we have $\eta_n, \eta'_n \in {}^{V^u}W(R)$ for almost all $n > 0$.

On $\mathbb{D}_R$, we have the obvious structure of an abelian group. Let $c \geqslant 0$ be an integer. We denote by $\mathbb{D}_R(c)$ the subgroup that consists of all elements satisfying the conditions

$$\left.\begin{aligned} \xi_n, \xi'_n &\in {}^{V^{c-n}}W(R) \quad \text{for } c > n, \\ \eta_n, \eta'_n &\in {}^{V^c}W(R) \qquad \text{for } n > 0. \end{aligned}\right\} \tag{4}$$

There is a unique ring structure on $\mathbb{D}_R$ that is continuous with respect to the topology defined by the $\mathbb{D}_R(c)$, and such that the following relations hold:

$$\left.\begin{gathered} FV = p = V^0 p, \quad V\xi F = {}^V\xi \quad \text{for } \xi \in W(R), \\ F\xi = {}^F\xi F, \quad \xi V = V\, {}^F\xi, \\ \mathrm{d}\xi = \xi\mathrm{d}, \quad \mathrm{d}^2 = 0, \\ F\mathrm{d}V = \mathrm{d}, \quad V\mathrm{d} = \mathrm{d}Vp, \quad \mathrm{d}F = pF\mathrm{d}. \end{gathered}\right\} \tag{5}$$

The elements of (3) with $\xi'_n = \eta'_n = 0$ form a subring $\mathbb{E}_R \subset \mathbb{D}_R$, which is complementary to the two-sided ideal generated by d. The ring $\mathbb{E}_R$ is called the Cartier ring. The subgroup $\mathbb{D}_R(c)$ is a right ideal $\mathbb{D}_R(c) = V^c\mathbb{D}_R + \mathrm{d}V^c\mathbb{D}_R$, which is invariant under left multiplication by d.

For an arbitrary R-algebra S, we extend the $W(R)$-module structure on $W\Omega^{\cdot}_{S/R}$ to a $\mathbb{D}_R$-module structure by setting $F\omega = {}^F\omega$, $V\omega = {}^V\omega$, $\mathrm{d}\omega = \mathrm{d}\omega$ for $\omega \in W\Omega^{\cdot}_{S/R}$.

Let us denote by $[1, d]$ the interval in $\mathbb{N}$. We call a function $k : [1, d] \to \mathbb{Z}_{\geqslant 0}$ a primitive weight if not all of its values are divisible by p.

We fix for each primitive k an order of the set $\operatorname{Supp} k = \{i_1, \dots, i_r\}$ such that

$$\operatorname{ord}_p k_{i_1} \leqslant \cdots \leqslant \operatorname{ord}_p k_{i_r}.$$

Moreover, we consider partitions $\mathcal{P} : I_0 \sqcup I_1 \sqcup \cdots \sqcup I_l = \operatorname{Supp} k$ which are increasing, and such that the intervals I_j are not empty. For each primitive k and each partition $\mathcal{P}$ of $\operatorname{Supp} k$, we define a basic Witt differential $e(1, k, \mathcal{P}) \in W\Omega^l_{S/R}$ as follows. Let k_{I_j} be the vector with components k_i for $i \in I_j$. Let p^{τ_j} be the highest power of p which divides all these components. We set

$$X^{k_{I_j}} = \prod_{i \in I_j} [T_i]^{k_i}.$$

Then we define $e(1, k, \mathcal{P})$ to be the image of the following differential by the map (1)

$$X^{k_{I_0}}(p^{-\tau_1}\mathrm{d}X^{k_{I_1}}) \cdot \cdots \cdot (p^{-\tau_l}\mathrm{d}X^{k_{I_l}})$$

Theorem 2. *Each element of* $W\Omega^{\cdot}_{S/R}$ *has a unique expression*

$$\xi + \sum_{k, \mathcal{P}} \theta_{k, \mathcal{P}} e(1, k, \mathcal{P}).$$

Here, $\xi \in W(R)$ is regarded as an element of $W\Omega^0_{S/R} = W(S)$. The sum runs over all primitive weights and partitions as above. The elements $\theta_{k,\mathcal{P}} \in \mathbb{D}_R$ satisfy the following condition. Let $c > 0$ be an arbitrary integer. Then, for almost all primitive weights k, we have $\theta_{k,\mathcal{P}} \in \mathbb{D}_R(c)$.

We have a canonical isomorphism,

$$W_c\Omega^{\cdot}_{S/R} \cong \mathbb{D}_R/\mathbb{D}_R(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R}.$$

In the remaining part of this introduction we assume, for simplicity, that p is nilpotent in R. Let S be an arbitrary R-algebra. The de Rham–Witt complex has the following base change properties. Let $S \to S'$ be an étale morphism of R-algebras. Then $W_n(S) \to W_n(S')$ is étale and we have an isomorphism of F-V-procomplexes,

$$W_n\Omega^{\cdot}_{S'/R} \cong W_n(S') \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R}.$$

For the next base change property we consider an arbitrary ring homomorphism $R \to R'$. Let S be a smooth R-algebra. We set $S' = R' \otimes_R S$. There is a canonical isomorphism

$$W_c\Omega^{\cdot}_{S'/R'} \cong \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R}.$$

In the third section we prove that for a smooth scheme over $\operatorname{Spec} R$ the de Rham–Witt complex computes the crystalline cohomology. This is first done in the case where X lifts to a smooth formal scheme over $W_n(R)$. The essential point is to show that the de Rham complex of a lifting over $W_n(R)$ is quasi-isomorphic to the de Rham–Witt complex. If R is a perfect ring of characteristic p, Illusie shows this comparison theorem by computing the graded quotients of the canonical filtration of the de Rham–Witt complex. In this paper we follow a different approach, which is applicable to general R. We show that for $S = R[T_1, \dots, T_d]$ the de Rham–Witt complex decomposes naturally into a direct sum of the subcomplexes such that one of them is isomorphic to the de Rham complex of the lifting over $W(R)$ and the other one has zero cohomology, i.e. it is exact. (The reader notices quickly which are the two subcomplexes in the case $S = R[T]$.) For general smooth S/R, we use the étale base change property of the de Rham–Witt complex.

Then we construct the de Rham–Witt complex for crystals. Let E be a crystal on $\mathrm{Crys}(X/W_n(R))$. We consider an affine open set $U = \operatorname{Spec} S \subset X$ and a pd-thickening $A \to S$ relative to $W_n(R)$. Then we have the pd-differential de Rham complex with coefficients in E,

$$(E_A \otimes_A \breve{\Omega}^{\cdot}_{A/W_n(R)}, \nabla),$$

where E_A is the value of the crystal at the pd-thickening $A \to S$.

We apply this to the situation where $A = W_n(S)$. We set $E_n = E_{W_n(S)}$. It is easy to see that ∇ is well defined on the quotient obtained from (1),

$$E_n \otimes_{W_n(S)} \breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \to E_n \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R}.$$

This defines the de Rham–Witt complex with coefficients in E,

$$(E_n \otimes_{W_n(\mathcal{O}_X)} W_n\Omega^{\cdot}_{X/R}, \nabla).$$

Again, the hypercohomology of this complex is the crystalline cohomology of E if E is flat and if X is smooth over R.

Over a perfect ring R the de Rham–Witt complex with coefficients in E was defined by Étesse [**E**]. It was shown by Bloch [**Bl2**] that the crystal E may be recovered from ∇ for a perfect ring R. For a general base this is proved in [**LZ**].

Let X be a proper and smooth scheme over $\operatorname{Spec} R$, where R is a complete local ring. We assume that the Frobenius $R/pR \to R/pR$ is a finite ring homomorphism. We generalize the slope spectral sequence to this case,

$$E_1^{j,i} = H^i(X, W\Omega^j_{X/R}) \;\Rightarrow\; H^{i+j}_{\mathrm{crys}}(X/W(R)).$$

If R is a perfect field of characteristic p, Bloch [**Bl**] shows that the spectral sequence degenerates up to p-torsion. In our general situation, we do not know at this time how to prove any analogous result, e.g. whether this spectral sequence degenerates up to V-torsion.

In the end, we give an application of the de Rham–Witt complex in the theory of displays [**Z**]. Let R be a ring such that p is nilpotent in R. Let A be an abelian scheme over R of dimension g. By [**BBM**], the crystalline cohomology $H^1_{\mathrm{crys}}(A/W(R))$ is a projective $W(R)$-module of rank $2g$. We show that the de Rham–Witt complex of A over R defines on $P = H^1_{\mathrm{crys}}(A/W(R))$ the structure of a $3n$-display (P, Q, F, V^{-1}). This structure is functorial and commutes with base change $R \to R'$.

The definition of the $3n$-display is as follows. The Frobenius on A modulo p defines a Frobenius operator $F : P \to P$ on the $W(R)$-module P. We need to define a $W(R)$-submodule Q of P that contains $VW(R)P$ such that P/Q is a projective R-module. Moreover, we need an F-linear epimorphism,

$$V^{-1} : Q \to P,$$

satisfying $V^{-1}({}^V\omega x) = \omega F x$ for $x \in P$, $\omega \in W(R)$.

We denote by $IW\Omega^{\cdot}_{A/R}$ the subcomplex of the de Rham–Witt complex $W\Omega^{\cdot}_{A/R}$, which is obtained by replacing the group $W(\mathcal{O}_A)$ in degree zero by ${}^VW(\mathcal{O}_A)$. We define $Q = \mathbb{H}^1(A, IW\Omega^{\cdot}_{A/R})$ as the hypercohomology. The natural inclusion $IW\Omega^{\cdot}_{A/R} \subset W\Omega^{\cdot}_{A/R}$ induces an exact sequence,

$$0 \to Q \to P \to H^1(A, \mathcal{O}_A) \to 0.$$

On the other hand, we have the following map of complexes:

$$\begin{tikzcd}
IW\Omega^{\cdot}_{A/R}: & VW(\mathcal{O}_A) \arrow[r, "\mathrm{d}"] \arrow[d, "V^{-1}"] & W\Omega^1_{A/R} \arrow[r, "\mathrm{d}"] \arrow[d, "F"] & W\Omega^2_{A/R} \arrow[r] \arrow[d, "pF"] & \cdots \\
W\Omega^{\cdot}_{A/R}: & W(\mathcal{O}_A) \arrow[r, "\mathrm{d}"] & W\Omega^1_{A/R} \arrow[r, "\mathrm{d}"] & W\Omega^2_{A/R} \arrow[r] & \cdots
\end{tikzcd}$$

The commutativity of the squares follows from the identities,

$$pF\mathrm{d} = \mathrm{d}F \quad \text{and} \quad F\mathrm{d}V = \mathrm{d}.$$

The diagram above induces a map

$$V^{-1} : Q := \mathbb{H}^1(X, IW\Omega^{\cdot}_{X/R}) \to P = \mathbb{H}^1(X, W\Omega^{\cdot}_{X/R}).$$

This defines the structure of a $3n$-display on P.

Assume, moreover, that the geometric fibres of A over Spec R have no p-division points. Then this construction gives the $3n$-display, which is dual to the $3n$-display associated to the p-divisible group of A in [**Z**].

We note that the knowledge of the $3n$-display is equivalent to that of the p-divisible group by [**Z**, Theorem 3.2] if R is excellent. The crystal associated to A in the sense of [**BBM**] may be recovered from the $3n$-display in this case, but not vice versa if the ring R contains nilpotent elements.

1. The de Rham–Witt complex

1.1. pd-Derivations

Let A be a unitary commutative ring, and let B be a unitary commutative A-algebra. Assume that $\mathfrak{b} \subset B$ is an ideal, which is equipped with divided powers $\gamma_n : \mathfrak{b} \to \mathfrak{b}$ for $n \geqslant 1$. We set $\gamma_0(b) = 1$ for $b \in \mathfrak{b}$.

Definition 1.1. Let M be a B-module. A pd-derivation $\nu : B \to M$ over A is an A-linear derivation ν that satisfies

$$\nu(\gamma_n(b)) = \gamma_{n-1}(b)\nu(b) \tag{1.1}$$

for $n \geqslant 1$ and each $b \in \mathfrak{b}$.

The pd-derivations form a B-module, which we denote by

$$\check{\mathrm{Der}}_{B/A}(B, M).$$

There is a universal pd-derivation

$$\mathrm{d} : B \to \check{\Omega}^1_{B/A}.$$

The B-module $\check{\Omega}^1_{B/A}$ is obtained as the factor module of $\Omega^1_{B/A}$ by the submodule generated by all elements $\mathrm{d}(\gamma_n(b)) - \gamma_{n-1}(b)\mathrm{d}(b)$.

On $\mathfrak{b}$ we introduce the function $\alpha_p = (p-1)!\gamma_p$. We will now assume that A is a $\mathbb{Z}_{(p)}$-algebra. Then the function α_p determines the functions γ_n for all n uniquely (see [**G**, p. 70]). A pd-derivation satisfies the relation

$$\nu(\alpha_p(b)) = b^{p-1}\nu(b) \quad \text{for } b \in \mathfrak{b}. \tag{1.2}$$

Lemma 1.2. *Let $\nu : B \to M$ be an A-linear derivation that satisfies (1.2). Then ν is a pd-derivation.*

Proof. This is a straightforward verification. Clearly, relation (1.2) is equivalent to

$$\nu(\gamma_p(b)) = \gamma_{p-1}(b)\nu(b).$$

We show by induction on n that this implies (1.1). This is clear for $n \leqslant p$.

For the induction, we represent n as a p-adic number $n = \sum_{i \geqslant 0} a_i p^i$, where $0 \leqslant a_i < p$. We have the well-known formula [**BO**, 3.3]

$$\operatorname{ord}_p n! = \frac{1}{p-1} \sum_{i \geqslant 0} a_i (p^i - 1).$$

Let a_k be the first non-zero digit. We set $m = n - p^k$. Then we find

$$\operatorname{ord}_p n! = \frac{1}{p-1} \sum_{i \geqslant k} a_i (p^i - 1) = \operatorname{ord}_p m! + \operatorname{ord}_p (p^k)!.$$

This shows that the binomial coefficient $\binom{n}{m}$ is a unit. Therefore, we obtain

$$\gamma_n(a) = \frac{m!(p^k)!}{n!} \gamma_m(a) \gamma_{p^k}(a). \tag{1.3}$$

First we assume $m > 0$. Then the formula (1.1) holds by the induction assumption for $\gamma_m(b)$ and $\gamma_{p^k}(b)$. Applying the derivation ν to (1.3), we obtain

$$\begin{aligned}
\nu(\gamma_n(b)) &= \frac{m!(p^k)!}{n!} (\gamma_{m-1}(b)\gamma_{p^k}(b) + \gamma_m(b)\gamma_{p^k-1}(b))\nu(b) \\
&= \frac{m!(p^k)!}{n!} \left(\frac{(n-1)!}{(m-1)!p^k!} + \frac{(n-1)!}{m!(p^k-1)!} \right) \gamma_{n-1}(a)\nu(a) \\
&= \gamma_{n-1}(a)\nu(a).
\end{aligned}$$

Hence it remains to consider the case $n = p^k$. It is easy to see that

$$\operatorname{ord}_p \frac{(p^k)!}{p!(p^{(k-1)}!)^p} = 0.$$

This implies, by [**BO**, 3.1], that

$$\gamma_{p^k}(a) = \frac{p!(p^{(k-1)}!)^p}{p^k!} \gamma_p(\gamma_{p^{k-1}}(a)).$$

If we apply ν to this identity we obtain, by the induction assumption,

$$\begin{aligned}
\nu(\gamma_{p^k}(b)) &= \frac{p!(p^{(k-1)}!)^p}{p^k!} \gamma_{p-1}(\gamma_{p^{(k-1)}}(b)) \cdot \gamma_{p^{(k-1)}-1}(b)\nu(b) \\
&= \frac{p!(p^{(k-1)}!)^p}{p^k!} \frac{((p-1)p^{(k-1)})!}{(p-1)!(p^{(k-1)}!)^{(p-1)}} \gamma_{(p-1)p^{(k-1)}}(b) \cdot \gamma_{p^{(k-1)}-1}(b)\nu(b) \\
&= \frac{p!(p^{(k-1)}!)^p}{p^k!} \frac{((p-1)p^{(k-1)})!}{(p-1)!(p^{(k-1)}!)^{(p-1)}} \frac{(p^k-1)!}{((p-1)p^{(k-1)})!(p^{(k-1)}-1)!} \gamma_{p^k-1}(b)\nu(b) \\
&= \gamma_{p^k-1}(b)\nu(b).
\end{aligned}$$

□

A differential graded B/A-algebra will be a unitary graded B-algebra,

$$P = \bigoplus_{i \in \mathbb{Z}_{i \geqslant 0}} P^i.$$

Moreover, P is equipped with an A-linear differential $\mathrm{d} : P \to P$ such that the following relations hold:

$$\left.\begin{aligned} \omega\eta &= (-1)^{ij}\eta\omega, \quad \omega \in P^i, \quad \eta \in P^j, \\ \mathrm{d}(\omega\eta) &= (\mathrm{d}\omega)\eta + (-1)^i \omega\, \mathrm{d}\eta, \\ \mathrm{d}^2 &= 0. \end{aligned}\right\} \tag{1.4}$$

A pd-differential graded algebra is a differential graded algebra such that the composite of the following maps is a pd-derivation,

$$B \to P^0 \to P^1.$$

We set $\breve{\Omega}^i_{B/A} = \wedge^i \breve{\Omega}^1_{B/A}$, and form the pd-de Rham complex. This is a pd-differential graded algebra. For any other algebra $P^{\cdot}$ of this sort we have a unique homomorphism

$$\breve{\Omega}^{\cdot}_{B/A} \to P^{\cdot}$$

of differential graded pd-algebras.

We will now consider a unitary commutative $\mathbb{Z}_{(p)}$-algebra R and a unitary commutative R-algebra S. The Witt vectors of any length $W_m(S)$ have a divided power structure on the ideal $I_S = {}^V W_{m-1}(S)$, which is defined by (see [**G**, p. 76])

$$\gamma_n({}^V\xi) = \frac{p^{n-1}}{n!} {}^V(\xi^n), \quad \xi \in W_{m-1}(S).$$

Then we have

$$\alpha_p({}^V\xi) = p^{p-2}{}^V(\xi^p).$$

A pd-derivation $\nu : W_m(S) \to M$ to a $W_m(S)$-module M is one with respect to these divided powers. In other words, the following relation is satisfied:

$$p^{p-2}\nu({}^V(\xi^p)) = p^{p-2}{}^V(\xi^{(p-1)})\nu({}^V\xi).$$

We will see that in the de Rham–Witt complex this relation remains true even if we divide it by $p^{(p-2)}$.

Our next aim is to define the action of the Frobenius on pd-derivations. It is convenient not to specify the length of the Witt vectors. We call a $W(S)$-module M discrete if it is obtained by restriction of scalars $W(S) \to W_m(S)$ for some natural number m. A map $W(S) \to M$ is called continuous if it factors through $W_l(S)$ for some number l.

Let us consider any continuous pd-derivation $\nu : W(S) \to M$ to a discrete $W(S)$-module M. The we define a map

$${}^F\nu : W(S) \to M \tag{1.5}$$

as follows. An arbitrary $\xi \in W(S)$ has a unique representation $\xi = [x] + {}^V\rho$ for $[x] \in S$ and $\rho \in W(S)$. We set

$$ {}^F\nu(\xi) = [x^{(p-1)}]\nu([x]) + \nu(\rho). \tag{1.6} $$

Clearly, ${}^F\nu$ is again a continuous map. We have the relation

$$ \nu({}^F\xi) = p\,{}^F\nu(\xi). \tag{1.7} $$

Indeed, this follows by applying ν to the equation ${}^F\nu = [x]^p + p\rho$.

We denote by $M_{[F]}$ the $W(S)$-module obtained via restriction of scalars by $F : W(S) \to W(S)$. This is again a discrete module.

Proposition 1.3. *Let $\nu : W(S) \to M$ be a continuous $W(R)$-linear pd-derivation. Then ${}^F\nu : W(S) \to M_{[F]}$ is a continuous $W(R)$-linear pd-derivation too.*

Proof. The problem is to show the additivity of ${}^F\nu$,

$$ {}^F\nu(\xi + \eta) = {}^F\nu(\xi) + {}^F\nu(\eta), \quad \xi, \eta \in W(S). $$

We set $\xi = [x] + {}^V\rho$, $\eta = [y] + {}^V\sigma$, and we define τ by the equation

$$ [x + y] = [x] + [y] + {}^V\tau. \tag{1.8} $$

We obtain $\xi + \eta = [x + y] - {}^V\tau + {}^V\rho + {}^V\sigma$ and hence, by definition,

$$ {}^F\nu(\xi + \eta) = [x + y]^{p-1}\nu([x + y]) - \nu(\tau) + \nu(\rho) + \nu(\sigma). $$

On the other hand, we have

$$ {}^F\nu(\xi) + {}^F\nu(\eta) = [x^{p-1}]\nu([x]) + [y^{p-1}]\nu([y]) + \nu(\rho) + \nu(\sigma). $$

Therefore, it suffices to show for arbitrary $x, y \in S$ the equation

$$ [x + y]^{p-1}\nu([x + y]) = [x^{p-1}]\nu([x]) + [y^{p-1}]\nu([y]) + \nu(\tau), \tag{1.9} $$

where τ is given by (1.8). To prove this, we first check the following identity in the Witt ring:

$$ \sum_{\substack{i+j+k=p \\ i\neq p,\, j\neq p,\, k\neq p \\ i\geqslant 0,\, j\geqslant 0,\, k\geqslant 0}} \frac{(p-1)!}{i!j!k!}[x]^i[y]^j({}^V\tau)^k + \alpha_p({}^V\tau) = \tau. \tag{1.10} $$

To prove this relation, we may restrict ourselves to the case where $S = \mathbb{Z}_{(p)}[x, y]$ is the polynomial ring in two variables. Since, in this case, the multiplication by p is injective in the Witt ring, it is enough to check the identity (1.10) after multiplication by p. But then, by the polynomial theorem, the identity becomes

$$ ([x] + [y] + {}^V\tau)^p - [x]^p - [y]^p = p\tau. $$

Using (1.8), it remains to verify that

$$[x+y]^p - [x]^p - [y]^p = p\tau.$$

But this is obtained by applying the Frobenius F to (1.8). Hence we have established (1.10).

Now we compute

$$\begin{aligned}
&[x+y]^{p-1}\nu[x+y] - [x^{p-1}]\nu[x] - [y^{p-1}]\nu[y] \\
&\quad = ([x]+[y]+^V\tau)^{p-1}\nu[x] - [x^{p-1}]\nu[x] \\
&\qquad + ([x]+[y]+^V\tau)^{p-1}\nu[y] - [y^{p-1}]\nu[y] + ([x]+[y]+^V\tau)^{p-1}\nu(^V\tau) \\
&\quad = \sum_{\substack{i+j+k=p-1\\ i\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]^i[y]^j(^V\tau)^k\nu[x] \\
&\qquad \times \sum_{\substack{i+j+k=p-1\\ j\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]^i[y]^j(^V\tau)^k\nu[y] \\
&\qquad\quad \times \sum_{\substack{i+j+k=p-1\\ k\neq p-1}} \frac{(p-1)!}{i!j!k!}[x]^i[y]^j(^V\tau)^k\nu(^V\tau) + (^V\tau)^{(p-1)}\nu(^V\tau).
\end{aligned} \tag{1.11}$$

The right-hand side of the last equality is just ν applied to the left-hand side of equality (1.10) because ν was assumed to be a pd-derivation, $\nu(\alpha_p(^V\tau)) = (^V\tau)^{(p-1)}\nu(^V\tau)$. Hence (1.11) is equal to $\nu(\tau)$. This proves that $^F\nu$ is additive.

Next we show that $^F\nu$ satisfies the Leibniz rule,

$$^F\nu(\xi\eta) = {}^F\xi\,{}^F\nu(\eta) + {}^F\eta\,{}^F\nu(\xi).$$

With the same notation as before, we find

$$\xi\eta = [xy] + {}^V([x]^p\sigma) + {}^V([y]^p\rho) + {}^V(p\rho\sigma).$$

Therefore, we obtain

$$\begin{aligned}
{}^F\nu(\xi\eta) &= [xy]^{(p-1)}\nu([xy]) + \nu([x]^p\sigma) + \nu([y]^p\rho) + \nu(p\rho\sigma) \\
&= [y]^p[x]^{(p-1)}\nu[x] + [x]^p[y]^{(p-1)}\nu[y] + [x]^p\nu(\sigma) \\
&\qquad + p\sigma[x]^{(p-1)}\nu([x]) + [y]^p\nu(\rho) + p\rho[y]^{(p-1)}\nu([y]) + p\sigma\nu(\rho) + p\rho\nu(\sigma) \\
&= [y]^p([x]^{(p-1)}\nu[x] + \nu(\rho)) + [x]^p([y]^{(p-1)}\nu[y] + \nu(\sigma)) \\
&\qquad + p\sigma([x]^{(p-1)}\nu[x] + \nu(\rho)) + p\rho([y]^{(p-1)}\nu([y]) + \nu(\sigma)) \\
&= ([y]^p + p\sigma)\,{}^F\nu(\xi) + ([x]^p + p\rho)\,{}^F\nu(\eta) \\
&= {}^F\eta\,{}^F\nu(\xi) + {}^F\xi\,{}^F\nu(\eta).
\end{aligned} \tag{1.12}$$

This shows the Leibniz rule. If ν is $W(R)$-linear, we obtain ${}^F\nu(W(R)) = 0$ from the definition. By the Leibniz rule, this implies that ${}^F\nu$ is $W(R)$-linear.

Finally, we have to check that ${}^F\nu$ is a pd-derivation. The assertion is the following equation:

$$ {}^F\nu(\alpha_p({}^V\rho)) = {}^F({}^V\rho)^{(p-1)}\,{}^F\nu({}^V\rho). \tag{1.13} $$

The left-hand side of this equation is, by definition,

$$ {}^F\nu(p^{p-2\,V}(\rho^p)) = p^{(p-2)}\nu(\rho^p) = p^{(p-1)}\rho^{(p-1)}\nu(\rho). $$

For the right-hand side of (1.13), we readily find the same result. □

If we start with a pd-derivation $\nu : W_m(S) \to M$, then we obtain a pd-derivation ${}^F\nu : W_{m+1}(S) \to M_{[F]}$. If we take for ν the universal pd-differential $\mathrm{d} : W_m(S) \to \breve{\Omega}^1_{W_m(S)/W_m(R)}$, we obtain a homomorphism of $W_{m+1}(S)$-modules,

$$ F : \breve{\Omega}^1_{W_{m+1}(S)/W_{m+1}(R)} \to (\breve{\Omega}^1_{W_m(S)/W_m(R)})_{[F]}. \tag{1.14} $$

By definition, this map satisfies the following equations:

$$ \left.\begin{aligned} &{}^F(\mathrm{d}\xi) = ({}^F\mathrm{d})(\xi), \quad \xi \in W_{m+1}(S),\\ &{}^F\mathrm{d}({}^V\eta) = \mathrm{d}\eta, \quad \eta \in W_m(S),\\ &{}^F\mathrm{d}([x]) = [x]^{(p-1)}\mathrm{d}[x], \quad x \in S,\\ &\mathrm{d}({}^F\nu) = p\,{}^F\mathrm{d}\xi. \end{aligned}\right\} \tag{1.15} $$

1.2. *F-V*-procomplexes

We will start with a ring homomorphism $R \to S$, and consider pd-differential graded $W_n(S)/W_n(R)$-algebras, with respect to the canonical divided powers on ${}^VW_{n-1}(S) \subset W_n(S)$.

Definition 1.4. An F-V-procomplex over the R-algebra S is a projective system $\{P_n\}$ of differential graded $W_n(S)/W_n(R)$-algebras P_n for $n \geqslant 1$,

$$ \cdots \to P_{n+1} \to P_n \to \cdots \to P_1. $$

Moreover, $\{P_n\}$ is equipped with two sets of homomorphisms of graded abelian groups,

$$ F : P_{n+1} \to P_n, \quad V : P_n \to P_{n+1}, \quad n \geqslant 1 $$

The following properties hold.

(i) Let $P_{n,[F]}$ be the graded $W_{n+1}(S)$-algebra obtained via restriction of scalars $F : W_{n+1}(S) \to W(S)$. Then F induces a homomorphism of graded $W_{n+1}(S)$-algebras,

$$ F : P_{n+1} \to P_{n,[F]}. $$

(ii) The structure morphism $W_n(S) \to P_n^0$ is compatible with F and V.

(iii) We have

$$
\begin{aligned}
{}^{FV}\omega &= p\omega \quad \text{for } \omega \in P_n, \quad n \geqslant 1,\\
{}^{F}\mathrm{d}{}^{V}\omega &= \mathrm{d}\omega,\\
{}^{F}\mathrm{d}[x] &= [x^{p-1}]\mathrm{d}[x], \quad x \in S,\\
{}^{V}(\omega\,{}^{F}\eta) &= ({}^{V}\omega)\eta, \quad \eta \in P_{n+1}.
\end{aligned}
$$

We indicate a few consequences of these relations.

For arbitrary $\omega_i \in P_n$, we have

$$
{}^{V}(\omega_0\mathrm{d}\omega_1\cdots\mathrm{d}\omega_r) = {}^{V}\omega_0\mathrm{d}{}^{V}\omega_1\cdots\mathrm{d}{}^{V}\omega_n. \tag{1.16}
$$

Indeed, we replace on the left-hand side $\mathrm{d}\omega_i$ by ${}^{F}\mathrm{d}{}^{V}\omega_i$. Then we obtain by using the fourth relation of (iii) the relation (1.16),

$$
{}^{V}\mathrm{d}\omega = {}^{V}1\mathrm{d}{}^{V}\omega = p\,\mathrm{d}{}^{V}\omega. \tag{1.17}
$$

The first relation is (1.16). Since d is $W(R)$ linear, the second relation follows because

$$
\begin{aligned}
{}^{V}1{}^{V}\omega &= {}^{V}({}^{FV}1\omega) = p{}^{V}\omega,\\
{}^{VF}\omega &= ({}^{V}1)\cdot\omega, \quad \omega \in P_n.
\end{aligned} \tag{1.18}
$$

Indeed, we have ${}^{VF}\omega = {}^{V}(1{}^{F}\omega) = ({}^{V}1)\omega$,

$$
\mathrm{d}{}^{F}\omega = p{}^{F}\mathrm{d}\omega. \tag{1.19}
$$

If we replace ω by ${}^{F}\omega$ in the second equation of (iii), we obtain

$$
\begin{aligned}
\mathrm{d}{}^{F}\omega &= {}^{F}\mathrm{d}{}^{VF}\omega = {}^{F}\mathrm{d}{}^{V}1\omega = {}^{FV}1{}^{F}\mathrm{d}\omega,\\
{}^{V}(\xi[x^{p-1}])\mathrm{d}{}^{V}[x] &= {}^{V}\xi\mathrm{d}[x] \quad \text{for } x \in S, \quad \xi \in W_n(S).
\end{aligned} \tag{1.20}
$$

Indeed, using (1.16) and the relations (iii), we obtain

$$
{}^{V}(\xi[x^{p-1}])\mathrm{d}{}^{V}[x] = {}^{V}(\xi[x^{p-1}]\mathrm{d}[x]) = {}^{V}(\xi{}^{F}\mathrm{d}[x]) = {}^{V}\xi\mathrm{d}[x].
$$

We note that (1.20) appears in Illusie's definition of a V-procomplex, but it is automatic if F is present. By (1.16) and the last relation of (iii), we conclude that F and V take exact elements of P_n to closed elements,

$$
\mathrm{d}{}^{V}\mathrm{d}\omega = 0, \qquad \mathrm{d}{}^{F}\mathrm{d}\omega = 0. \tag{1.21}
$$

We note that, by the requirements of the definition, F and V are uniquely determined on the subalgebra of P_n generated over $W_n(S)$ by 1 and the elements $\mathrm{d}\xi \in P_n^1$ for $\xi \in W_n(S)$. Indeed, if we write $\xi = [x] + {}^{V}\eta$ for $x \in S$ and $\eta \in W_{n-1}(S)$, we obtain from (iii)

$$
{}^{F}\mathrm{d}\xi = [x^{p-1}]\mathrm{d}[x] + \mathrm{d}\eta. \tag{1.22}
$$

The uniqueness of ^{V} is a consequence of (1.16).

Lemma 1.5. *Let P_n be an F-V-procomplex over the R-algebra S. Then, for each $n \geqslant 1$, the differential* $\mathrm{d} : W(S) \to P_n^1$ *satisfies the relation*

$$\mathrm{d}^V(\xi^p) = {}^V(\xi^{(p-1)})\mathrm{d}^V\xi, \quad \xi \in W_n(S). \tag{1.23}$$

In particular, d *is a pd-differential.*

Proof. The proof of (1.23) consists of three steps. First, we show that (1.23) holds for $\xi = [x]$, $x \in S$. Second, we show that (1.23) holds for $\xi = \eta_1 + \eta_2$, if (1.23) holds for $\xi = \eta_1$ and $\xi = \eta_2$. Third, we show that (1.23) hold for $\xi =^V \eta$, if it holds for $\xi = \eta$. Then (1.23) follows clearly from these three steps.

If $\xi = [x]$, we obtain

$$\begin{aligned} \mathrm{d}^V([x]^p) &= \mathrm{d}^{VF}[x] \\ &= \mathrm{d}^V 1 \cdot [x] \\ &= {}^V 1\mathrm{d}[x] \\ &= {}^{VF}\mathrm{d}[x] \\ &= {}^V([x^{p-1}]\mathrm{d}[x]) \\ &= {}^V[x^{p-1}]\mathrm{d}^V[x]. \end{aligned}$$

Next we assume that the relation (1.23) holds for $\xi = \eta_1$ and $\xi = \eta_2$. We have to prove that

$$\mathrm{d}^V((\eta_1 + \eta_2)^p) = {}^V((\eta_1 + \eta_2)^{p-1})\mathrm{d}^V(\eta_1 + \eta_2).$$

Because of our assumption, this is equivalent to the relation

$$\sum_{\substack{i+j=p \\ i\neq 0,\, j\neq 0}} \frac{p!}{i!j!}\mathrm{d}^V(\eta_1^i\eta_2^j) = {}^V(\eta_1^{p-1})\mathrm{d}^V\eta_2 + {}^V(\eta_2^{p-1})\mathrm{d}^V\eta_1 + \sum_{\substack{l+k=p-1 \\ l\neq 0,\, k\neq 0}} \frac{(p-1)!}{l!k!}{}^V(\eta_1^k\eta_2^k)\mathrm{d}({}^V\eta_1 + {}^V\eta_2). \tag{1.24}$$

One term of the sum on the left-hand side may be expressed as follows:

$$\begin{aligned} \frac{p!}{i!j!}\mathrm{d}^V(\eta_1^i\eta_2^j) &= \frac{(p-1)!}{i!j!}{}^V\mathrm{d}(\eta_1^i\eta_2^j) \\ &= \frac{(p-1)!}{(i-1)!j!}{}^V(\eta_1^{(i-1)}\eta_2^j\mathrm{d}\eta_1) + \frac{(p-1)!}{i!(j-1)!}{}^V(\eta_1^i\eta_2^{(j-1)}\mathrm{d}\eta_2) \\ &= \frac{(p-1)!}{(i-1)!j!}{}^V(\eta_1^{(i-1)}\eta_2^j)\mathrm{d}^V\eta_1 + \frac{(p-1)!}{i!(j-1)!}{}^V(\eta_1^i\eta_2^{(j-1)})\mathrm{d}^V\eta_2. \end{aligned}$$

This immediately gives the relation (1.24).

Next we assume that the relation (1.23) holds for a particular $\xi \in W_n(S)$. Then we want to show that

$$\mathrm{d}^V(({}^V\xi)^p) = {}^V(({}^V\xi)^{p-1})\mathrm{d}^{V^2}\xi.$$

This equation is clearly equivalent to

$$p^{p-1}\mathrm{d}^{V^2}(\xi^p) = p^{p-2\ V^2}(\xi^{(p-1)})\mathrm{d}^{V^2}\xi.$$

Hence it is enough to show that

$$p\,\mathrm{d}^{V^2}(\xi^p) = {}^{V^2}(\xi^{(p-1)})\mathrm{d}^{V^2}\xi.$$

This follows if we apply V to the Equation (1.23).

Finally, we have to check that d is a pd-differential. By definition, we have

$$\alpha_p({}^V\xi) = p^{p-2}\,{}^V(\xi^p).$$

Hence we have to verify that

$$p^{(p-2)}\mathrm{d}^V(\xi^p) = ({}^V\xi)^{p-1}\mathrm{d}^V\xi,$$

or, equivalently,

$$p^{(p-2)}\mathrm{d}^V(\xi^p) = p^{(p-2)}\,{}^V(\xi^{(p-1)})\mathrm{d}^V\xi.$$

This follows from Lemma 1.5 and is, by the way, trivial if $p \neq 2$, because then the left-hand side is $p^{(p-3)}\,{}^V\mathrm{d}\xi^p$. □

Since $\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$ is a universal *pd*-differential graded $W_n(S)/W_n(R)$-algebra, there is a canonical morphism of procomplexes,

$$\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \to P^{\cdot}_n. \tag{1.25}$$

Since the Frobenius on $\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$ satisfies (1.15), we conclude that (1.25) commutes with F,

$$\begin{tikzcd} \breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \arrow[r] \arrow[d, "F"] & P^{\cdot}_n \arrow[d, "F"] \\ \breve{\Omega}^{\cdot}_{W_{n-1}(S)/W_{n-1}(R)} \arrow[r] & P^{\cdot}_{n-1} \end{tikzcd}$$

1.3. Construction of the de Rham–Witt complex

We come now to the construction of the universal F-V-procomplex $W_n\Omega^{\cdot}_{S/R}$. We do this by induction. We set

$$W_1\Omega^{\cdot}_{S/R} = \Omega^{\cdot}_{S/R} = \breve{\Omega}^{\cdot}_{W_1(S)/W_1(R)}$$

and assume that we have already constructed a system $\{W_m\Omega^{\cdot}_{S/R}\}_{m\leqslant n}$ of pd-differential graded $W_m(S)/W_m(R)$-algebras,

$$W_n\Omega^{\cdot}_{S/R} \to W_{n-1}\Omega^{\cdot}_{S/R} \to \cdots \to \Omega^{\cdot}_{S/R}, \tag{1.26}$$

and surjective homomorphisms of differential graded algebras,

$$\breve{\Omega}^{\cdot}_{W_m(S)/W_m(R)} \to W_m\Omega^{\cdot}_{S/R}, \quad m \leqslant n,$$

which are compatible with the restriction maps and with F. This implies, in particular, that the system (1.26) meets the requirements (i), (ii) and the third equation of (iii) in Definition 1.4. Moreover, we assume that there are additive maps

$$V : W_m\Omega^{\cdot}_{S/R} \to W_{m+1}\Omega^{\cdot}_{S/R}, \quad 1 \leqslant m < n.$$

We require that $W_m\Omega^0_{S/R} = W_m(S)$, and that the following relations hold:

$$\left.\begin{aligned} {}^{FV}\omega &= p\omega \qquad \text{for } \omega \in W_m\Omega^{\cdot}_{S/R}, \quad m < n,\\ {}^{F}\mathrm{d}^{V}\omega &= \omega,\\ {}^{V}(\omega^{F}\eta) &= {}^{V}\omega \cdot \eta \quad \text{for } \eta \in W_{m+1}\Omega^{\cdot}_{S/R}. \end{aligned}\right\} \tag{1.27}$$

We define an ideal $I \subset \breve{\Omega}^{\cdot}_{W_{n+1}(S)/W_{n+1}(R)}$ as follows. We start with an arbitrary relation in $W_n\Omega^i_{S/R}$,

$$\sum_{l=1}^{M} \xi^{(l)}\mathrm{d}\eta_1^{(l)} \cdot \cdots \cdot \mathrm{d}\eta_i^{(l)} = 0. \tag{1.28}$$

Here, i and M are natural numbers greater than or equal to 1 and $\xi^{(l)}$, $\eta_k^{(l)} \in W_n(S)$ for $l = 1, \dots, M$ and $k = 1, \dots, i$.

Then we consider the following elements of $\breve{\Omega}^{\cdot}_{W_{n+1}(S)/W_{n+1}(R)}$:

$$\sum_{l} {}^{V}\xi^{(l)}\mathrm{d}^{V}\eta_1^{(l)} \cdot \cdots \cdot \mathrm{d}^{V}\eta_i^{(l)}, \tag{1.29}$$

$$\sum_{l} \mathrm{d}^{V}\xi^{(l)}\mathrm{d}^{V}\eta_1^{(l)} \cdot \cdots \cdot \mathrm{d}^{V}\eta_i^{(l)}. \tag{1.30}$$

These homogeneous elements for all possible relations (1.28) generate a homogeneous ideal $I \subset \breve{\Omega}^{\cdot}_{W_{n+1}(S)/W_{n+1}(R)}$. We see that $\mathrm{d}I \subset I$.

Moreover, it is clear that I is mapped to 0 by the map

$$F : \breve{\Omega}^{\cdot}_{W_{n+1}(S)/W_{n+1}(R)} \to \breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \to W_n\Omega^{\cdot}_{S/R}, \tag{1.31}$$

since we have

$${}^{FV}\xi^{(l)} = p\xi^{(l)} \quad \text{in } \breve{\Omega}^0_{W_n(S)/W_n(R)} = W_n(S)$$

and

$${}^{F}\mathrm{d}^{V}\eta^{(l)} = \mathrm{d}\eta^{(l)} \quad \text{in } \breve{\Omega}^1_{W_n(S)/W_n(R)}.$$

We set

$$\bar{\Omega}^{\cdot}_{n+1} = (\breve{\Omega}^{\cdot}_{W_{n+1}(S)/W_{n+1}(R)})/I.$$

This is a differential graded algebra.

The Frobenius (1.31) factors through a map of algebras

$$F : \bar{\Omega}^{\cdot}_{n+1} \to W_n\Omega^{\cdot}_{S/R}.$$

On the other hand, we have, by the definition of I, an additive map

$$\begin{aligned} V : W_n\Omega^{\cdot}_{S/R} &\to \bar{\Omega}^{\cdot}_{n+1}, \\ \xi \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i &\mapsto {}^V\xi \mathrm{d}^V\eta_1 \cdots \mathrm{d}^V\eta_i. \end{aligned}$$

We see that this definition of V implies that ${}^F\mathrm{d}^V\omega = \omega$ for all $\omega \in W_n\Omega^{\cdot}_{S/R}$.

Then we consider the ideal $\mathcal{I} \subset \bar{\Omega}^{\cdot}_{n+1}$, which is generated by the following elements,

$$^V(\omega^F\eta) - {}^V\omega\eta, \qquad \mathrm{d}({}^V(\omega^F\eta) - {}^V\omega\eta),$$

where $\omega \in W_n\Omega^{\cdot}_{S/R}$ and $\eta \in \bar{\Omega}^{\cdot}_{n+1}$ runs through all possible elements. This is a homogeneous d-invariant ideal. We set

$$W_{n+1}\Omega^{\cdot}_{S/R} = \bar{\Omega}^{\cdot}_{n+1}/\mathcal{I}.$$

It is immediately verified that $F : \bar{\Omega}^{\cdot}_{n+1} \to W_n\Omega^{\cdot}_{S/R}$ maps $\mathcal{I}$ to zero. Hence we have constructed operators,

$$F : W_{n+1}\Omega^{\cdot}_{S/R} \to W_n\Omega^{\cdot}_{S/R}, \qquad V : W_n\Omega_{S/R} \to W_{n+1}\Omega_{S/R},$$

meeting all requirements (1.27) of our induction assumption.

We note that the other two equations of Definition 1.4 are satisfied, since they are already satisfied in $\breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)}$.

Proposition 1.6. *Let $\{P_n\}$ be an F-V-procomplex such that P_n is a differential graded $W_n(S)/W_n(R)$-algebra. Then there is a unique morphism*

$$W_n\Omega^{\cdot}_{S/R} \to P_n$$

of F-V-procomplexes.

Proof. It is clear from our construction that the natural morphism (1.25) factors through $W_n\Omega^{\cdot}_{S/R}$. □

1.4. Base change for étale morphisms

We will now establish the base change property of the de Rham–Witt complex with respect to étale morphisms $S \to S'$.

Let (P, d) be a differential graded B/A-algebra. Let B' be a B-algebra. Let us assume that the differential $\mathrm{d} : B \to P^1$ extends to a differential $\mathrm{d} : B' \to B' \otimes_B P^1$. Then $B' \otimes_B P$ becomes a differential graded algebra if we define the differential as follows:

$$\mathrm{d}(b' \otimes p) = (\mathrm{d}b')(1 \otimes p) + b' \otimes \mathrm{d}p.$$

If B' is étale over B, we know that an extension of $\mathrm{d} : B \to P^1$ to $\mathrm{d} : B' \to B' \otimes_B P^1$ always exists and is unique (see [**EGA**, 17.2.4]). Hence the base change $(B' \otimes_B P, d)$ is defined.

Let R be a ring and S be an R-algebra. We assume that S is F-finite or that p is nilpotent in S. If $S \to S'$ is étale (respectively, unramified), so is $W_n(S) \to W_n(S')$ (see the appendix).

Assume we are given an F-V-procomplex $\{P_n\}$ of differential graded $W_n(S)/W_n(R)$-algebras P_n. Let S' be an étale S-algebra.

Since $W_n(S')$ is an étale $W_n(S)$-algebra, we obtain a projective system of differential graded $W_n(S')/W_n(R)$-algebras

$$\to \cdots \to W_n(S') \otimes_{W_n(S)} P_n \to \cdots \to W_1(S') \otimes_{W_1(S)} P_1.$$

We equip this system with the structure of an F-V-procomplex.

For this we have to define the operators F and V. The operator F is simply given by the formula

$$\begin{aligned} F : W_{n+1}(S') \otimes_{W_{n+1}(S)} P_{n+1} &\to W_n(S') \otimes_{W_n(S)} P_n, \\ \xi \otimes x &\mapsto {}^F\xi \otimes {}^F x. \end{aligned}$$

For the definition of V, we use the canonical isomorphism

$$W_{n+1}(S') \otimes_{W_{n+1}(S),F} W_n(S) \to W_n(S'),$$

which maps $\xi \otimes \eta$ to ${}^F\xi\eta$. To define

$$V : W_n(S') \otimes_{W_n(S)} P_n \to W_{n+1}(S') \otimes_{W_{n+1}(S)} P_{n+1},$$

we rewrite the left-hand side as

$$W_n(S') \otimes_{W_n(S)} P_n = W_{n+1}(S') \otimes_{W_{n+1}(S)} P_{n,[F]}.$$

Hence we may define V as

$$\begin{aligned} V : W_{n+1}(S') \otimes_{W_{n+1}(S)} P_{n,[F]} &\to W_{n+1}(S') \otimes_{W_{n+1}(S)} P_{n+1}, \\ \xi \otimes x &\mapsto \xi \otimes {}^V x. \end{aligned}$$

We omit the obvious verification that $\{W_n(S') \otimes_{W_n(S)} P_n\}$ becomes, with these operators, an F-V-procomplex.

By the universal property of the de Rham–Witt complex as an F-V-procomplex, we obtain for S' étale over S a canonical map of F-V-procomplexes,

$$W_n\Omega^{\cdot}_{S'/R} \to W_n(S') \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R}. \tag{1.32}$$

Proposition 1.7. *Assume that S is F-finite or that p is nilpotent in S. Let S' be étale over S. Then the morphism (1.32) is an isomorphism.*

Proof. If we view $W_n\Omega^{\cdot}_{S'/R}$ as an F-V-procomplex relative to S/R, we obtain a morphism of F-V-procomplexes

$$\beta_0 : W_n\Omega^{\cdot}_{S/R} \to W_n\Omega^{\cdot}_{S'/R}.$$

This extends to a homomorphism of $W_n(S')$-modules,

$$\beta : W_n(S') \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R} \to W_n\Omega^{\cdot}_{S'/R}. \tag{1.33}$$

Because both sides of (1.33) are quotients of

$$W_n(S') \otimes_{W_n(S)} \Omega^{\cdot}_{W_n(S)/W_n(R)} = \Omega^{\cdot}_{W_n(S')/W_n(R)},$$

the map β is an epimorphism. Let us denote the map (1.32) by α. The map $\alpha \circ \beta_0$ is a morphism of F-V-procomplexes, which must, by Proposition 1.6, coincide with the obvious map,

$$\begin{aligned} W_n\Omega^{\cdot}_{S/R} &\to W_n(S') \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R}, \\ x &\mapsto 1 \otimes x. \end{aligned}$$

This proves that $\alpha\beta = \mathrm{id}$. Since β is an epimorphism, we obtain that α is an isomorphism. $\square$

Remark 1.8. The differential on the right-hand side of (1.33) does not induce $1 \otimes \mathrm{d}$ on the left-hand side. To remedy this, we may proceed as follows. We fix the number n. Then we choose a number m such that $p^m W_n(R) = 0$. Then p^m annihilates all groups of (1.33). If we consider the groups $W_n\Omega^{\cdot}_{S/R}$ as $W_{m+n}(S)$-modules via restriction of scalars $F^m : W_{m+n}(S) \to W_n(S)$, the differential of $W_n\Omega^{\cdot}_{S/R}$ becomes $W_{m+n}(S)$-linear. By Proposition A.8 in the appendix, we have the following tensor product diagram:

$$\begin{tikzcd} W_{m+n}(S') \arrow[r, "F^m"] & W_n(S') \\ W_{m+n}(S) \arrow[u] \arrow[r, "F^m"] & W_n(S) \arrow[u] \end{tikzcd}$$

Inserting this in the isomorphism (1.33), we obtain an isomorphism

$$W_{m+n}(S') \otimes_{W_{m+n}(S),F^m} W_n\Omega^i_{S/R} \cong W_n\Omega^i_{S'/R}.$$

Here, the map $1 \otimes \mathrm{d}$ on the left-hand side induces the differential d on the right-hand side. Since $W_{m+n}(S) \to W_{m+n}(S')$ is flat, we obtain an isomorphism of cohomology groups,

$$W_{m+n}(S') \otimes_{W_{m+n}(S),F^m} H^i(W_n\Omega_{S/R}) \cong H^i(W_n\Omega_{S'/R}).$$

Proposition 1.9. *Assume we are given ring homomorphisms $R \to R' \to S$. Let p be nilpotent in R or let R be F-finite. If $R \to R'$ is an unramified ring homomorphism, we have an isomorphism of F-V-procomplexes*

$$W_n\Omega^{\cdot}_{S/R} \to W_n\Omega^{\cdot}_{S/R'}. \tag{1.34}$$

Proof. Clearly, $W_n\Omega^{\cdot}_{S/R'}$ is an F-V-procomplex relative to S/R. Hence we obtain the morphism (1.34). On the other hand, the differential $W_n(R') \to W_n(S) \xrightarrow{\mathrm{d}} W_n\Omega^1_{S/R}$ is zero, because the restriction to $W_n(R)$ is and because $W_n(R')/W_n(R)$ is unramified. This shows that $W_n\Omega^{\cdot}_{S/R}$ is an F-V-procomplex relative to S/R'. Hence we obtain an arrow inverse to (1.34). $\square$

Remark 1.10. Let R be an arbitrary $\mathbb{Z}_p$-algebra, and let $R \to S$ be a ring homomorphism. The proof of Proposition 1.7 shows that, for an arbitrary $f \in S$, there is an isomorphism

$$W_n(S_f) \otimes_{W_n(S)} W_n\Omega^{\cdot}_{S/R} \cong W_n\Omega^{\cdot}_{S_f/R}. \tag{1.35}$$

Moreover, if $g \in R$ is an element whose image in S is a unit, we have the isomorphism

$$W_n\Omega^{\cdot}_{S/R} \cong W_n\Omega^{\cdot}_{S/R_g}. \tag{1.36}$$

The above remark allows us to define the de Rham–Witt complex on a scheme. Let $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$. We set $W_n(X) = \operatorname{Spec} W_n(S)$ and $W_n(Y) = \operatorname{Spec} W_n(R)$. We denote by $W_n\Omega_{X/Y}$ the quasi-coherent sheaf on $W_n(X)$ associated to $W_n\Omega^{\cdot}_{S/R}$.

More generally, let $X \to Y$ be a morphism of schemes over $\mathbb{Z}_{(p)}$. Then there is a quasi-coherent sheaf $W_n\Omega^{\cdot}_{X/Y}$ on $W_n(X)$, which has the following property.

Let $U' = \operatorname{Spec} S'$ an affine open subscheme of X and $V' = \operatorname{Spec} R'$ an affine open subscheme of Y, such that U' is mapped to V' by $X \to Y$. Then we have a canonical isomorphism,

$$\Gamma(W_n(U'), W_n\Omega^{\cdot}_{X/Y}) = W_n\Omega^{\cdot}_{S'/R'}.$$

If the schemes X and Y are F-finite and $X \to Y$ is a morphism of finite type, the sheaves $W_n\Omega^{\cdot}_{X/Y}$ are coherent because they are quotients of the coherent sheaves $\Omega^{\cdot}_{W_n(X)/W_n(Y)}$. Moreover, if X is proper over $Y = \operatorname{Spec} R$ and R is noetherian, the cohomology groups $H^i(W_n(X), W_n\Omega^{\cdot}_{X/Y})$ are modules of finite type over $W_n(R)$. This follows because $W_n(X) \to W_n(Y)$ is a proper morphism of noetherian schemes.

If p is locally nilpotent on X, the schemes $W_n(X)$ and X have the same topological space. Therefore, in this case, the cohomology groups may be identified with $H^i(X, W_n\Omega^{\cdot}_{X/Y})$.

We may summarize our base change results as follows.

Proposition 1.11. *Let $X \to Y$ be any morphism of schemes. We assume either that p is locally nilpotent on Y or that X and Y are F-finite. Assume we are given the following commutative diagram:*

$$\begin{tikzcd} X' \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "\beta"] & Y \end{tikzcd}$$

We assume that α is étale and that β is unramified. Then there is a canonical isomorphism,

$$W_n(\alpha)^* W_n\Omega^i_{X/Y} \cong W_n\Omega^i_{X'/Y'}.$$

Under the assumption of the proposition, this allows us to consider $W_n\Omega^i_{X/Y}$ as a sheaf on the étale site X_{et}.

1.5. The completed de Rham–Witt complex

In this section we fix a scheme Y such that p is locally nilpotent on Y. Let $X \to Y$ be a morphism of schemes. Since the topological spaces of $W_n(X)$ and X are the same, we can regard $W_n\Omega^{\cdot}_{X/Y}$ as a sheaf on X. We define for an open set U of X,

$$W\Omega^{\cdot}_{X/Y}(U) = \varprojlim_n W_n\Omega^{\cdot}_{X/Y}(U). \tag{1.37}$$

This is a sheaf on X.

We gather a few facts about the projective limit, which we apply to this situation. We consider projective systems of abelian groups indexed by the natural numbers,

$$\cdots \xrightarrow{\pi} A_n \xrightarrow{\pi} \cdots \xrightarrow{\pi} A_1.$$

We associate the Eilenberg complex concentrated in degree 0 and 1,

$$\prod_n A_n \to \prod_n A_n. \tag{1.38}$$

An element (a_n) from the left-hand side is mapped to $(a_n - \pi(a_{n+1}))$. The kernel of the map (1.38) is, by definition, $\varprojlim A_n$ and the co-kernel is $\varprojlim^1 A_n$. This co-kernel is easily seen to be zero if all transition morphisms $\pi : A_{n+1} \to A_n$ are surjective.

For a projective system of exact sequences,

$$0 \to A_n \to B_n \to C_n \to 0.$$

We have the exact cohomology sequence,

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \varprojlim{}^1 A_n \to \varprojlim{}^1 B_n \to \varprojlim{}^1 C_n \to 0.$$

Each system A_n may be embedded in a system with surjective transition morphisms, namely, the system

$$A_1 \oplus \cdots \oplus A_{n+1} \to A_1 \oplus \cdots \oplus A_n,$$

where the transition morphism maps an element $(a_1, \ldots, a_{n+1})$ of the left-hand side to $(a_1, \ldots, a_{n-1}, a_n + \pi(a_{n+1}))$. One deduces that, for a Mittag–Leffler system A_n, we have $\varprojlim^1 A_n = 0$.

We consider a projective system of noetherian complete local rings,

$$\cdots \to R_n \to \cdots \to R_1,$$

such that the transition homomorphisms are local surjective ring homomorphisms. A projective system of modules is a projective system M_n such that M_n is an R_n module and such that the transition homomorphism $\pi : M_{n+1} \to M_n$ is an R_{n+1}-module homomorphism for each n.

Proposition 1.12. *Let M_n a projective system of noetherian modules. Then we have*

$$\varprojlim{}^1 M_n = 0. \tag{1.39}$$

Proof. Suppose we are given an exact sequence of projective systems of modules

$$0 \to M_n \to N_n \to L_n \to 0. \tag{1.40}$$

Since M_n may be embedded in a projective system of modules with surjective transition morphisms, it suffices to show that (1.40) remains exact if we apply the projective limit.

Let us denote by $\mathfrak{m}_n$ the maximal ideal of R_n. For each pair of natural numbers n, i, we consider the exact sequence

$$0 \to M_n/\mathfrak{m}^i M_n \to N_n/\mathfrak{m}^i M_n \to L_n \to 0. \tag{1.41}$$

If we fix n and pass to the projective limit over i, we obtain the exact sequence (1.40).

Let $\mathcal{I}$ be the set of pairs of natural numbers with the order $(n', i') \geqslant (n, i)$ if and only if $n' \geqslant n$ and $i' \geqslant i$. Then we have obvious projective systems indexed by $\mathcal{I}$ if we set

$$M_{n,i} = M_n/\mathfrak{m}^i M_n, \quad N_{n,i} = N_n/\mathfrak{m}^i M_n, \quad L^{n,i} = L_n.$$

This makes (1.41) into a projective system of short exact sequences indexed by $\mathcal{I}$. By assumption, $M_{n,i}$ consists of artinian modules, and is therefore a Mittag–Leffler system. Hence we obtain an exact sequence if we pass to the projective limit. On the other hand, this projective limit coincides with

$$0 \to \varprojlim M_n \to \varprojlim N_n \to \varprojlim L_n \to 0.$$

□

Let $L^{\cdot}_n$ be a projective system of complexes of abelian groups. We set $L^{\cdot} = \varprojlim_n L^{\cdot}_n$, and we assume that $\varprojlim_n{}^1 L^{\cdot}_n = 0$. Then we have, for each $q \in \mathbb{Z}$, a short exact sequence,

$$0 \to \varprojlim_n{}^1 H^{q-1}(L^{\cdot}_n) \to H^q(L^{\cdot}) \to \varprojlim_n H^q(L^{\cdot}_n) \to 0. \tag{1.42}$$

Indeed, we consider the Eilenberg complex $X^{\cdot} \to X^{\cdot}$ associated to $L^{\cdot}_n$ (see (1.38). By assumption, we obtain an exact sequence of complexes,

$$0 \to L^{\cdot} \to X^{\cdot} \to X^{\cdot} \to 0.$$

We obtain (1.42) from the spectral sequence of the double complex $X^{\cdot} \to X^{\cdot}$.

Proposition 1.13. *Let X be a separated scheme. Let $X \to Y$ be a morphism such that p is locally nilpotent on Y. Then we have the exact sequences*

$$0 \to \varprojlim_n{}^1 \mathbb{H}^{q-1}(X, W_n\Omega^{\cdot}_{X/Y}) \to \mathbb{H}^q(X, W\Omega^{\cdot}_{X/Y}) \to \varprojlim_n \mathbb{H}^q(X, W_n\Omega^{\cdot}_{X/Y}) \to 0,$$
$$0 \to \varprojlim_n{}^1 H^{q-1}(X, W_n\Omega^l_{X/Y}) \to H^q(X, W\Omega^l_{X/Y}) \to \varprojlim_n H^q(X, W_n\Omega^l_{X/Y}) \to 0.$$

Proof. We consider an affine covering $\mathcal{U}$ of X and consider the Cech complexes

$$\mathcal{C}^{\cdot}(\mathcal{U}, W\Omega^{\cdot}_{X/Y}) = \varprojlim_{n} \mathcal{C}^{\cdot}(\mathcal{U}, W_n\Omega^{\cdot}_{X/Y}). \tag{1.43}$$

By [**EGA**, 0_{III}, 13.3.1], the cohomology of $W\Omega^{l}_{X/Y}$ vanishes for each open set U of the nerve of $\mathcal{U}$. Therefore, the left-hand side of (1.43) computes the hypercohomology of $W\Omega^{\cdot}_{X/Y}$. We denote by $L^{\cdot}_n$ the simple complex associated to the Cech complex of $W_n\Omega^{\cdot}_{X/Y}$. Since the transition homomorphisms on the right-hand side of (1.43) are surjective, we obtain the proposition from (1.42). □

Corollary 1.14. *Let $Y = \operatorname{Spec} R$ be the spectrum of a noetherian complete local ring whose residue class field is a field of characteristic p with a finite p-basis. We assume that p is nilpotent in R. Let X be a proper scheme over Y. Then we have canonical isomorphisms*

$$\left.\begin{aligned} \mathbb{H}^q(X, W\Omega^{\cdot}_{X/Y}) &\cong \varprojlim_{n} \mathbb{H}^q(X, W_n\Omega^{\cdot}_{X/Y}),\\ H^q(X, W\Omega^{l}_{X/Y}) &\cong \varprojlim_{n} H^q(X, W_n\Omega^{l}_{X/Y}). \end{aligned}\right\} \tag{1.44}$$

Proof. By the appendix, the scheme $W_n(X)$ is proper over the noetherian ring $W_n(R)$. Therefore, the cohomology groups $H^q(X, W_n\Omega^{l}_{X/Y})$ are finite $W_n(R)$-modules. If we knew that $W_n(R)$ is a complete local ring, the corollary would follow from Propositions 1.13 and 1.12. Therefore, we conclude the proof with the following lemma.

Lemma 1.15. *Let R be a noetherian complete local ring whose residue class field is a field of characteristic p with a finite p-basis. We denote by $\mathfrak{m}$ the maximal ideal of R.*

Then $W_n(R)$ is, for each number n, a noetherian complete local ring, whose maximal ideal $\mathfrak{n}$ is the kernel of the homomorphism $W_n(R) \xrightarrow{w_0} R \to R/\mathfrak{m}$. The $\mathfrak{n}$-adic topology of $W_n(R)$ coincides with the topology defined by the filtration by the ideals $W_n(\mathfrak{m}^s)$.

Proof. The ring $W_n(R)$ is complete and separated in the filtration above

$$W_n(R) = \varprojlim_{s} W_n(R/\mathfrak{m}^s).$$

Since $\mathrm{Frob} : R/pR \to R/pR$ is finite, it is easy to see that the rings $W_n(R/\mathfrak{m}^s)$ are local artinian. It follows that $W_n(R)$ is a local ring with maximal ideal $\mathfrak{n}$.

Therefore, it suffices to show the last sentence of the lemma. It is clear that $\mathfrak{n}$ is nilpotent in each of the rings $W_n(R/\mathfrak{m}^s)$.

We have to show that, for each number u, there is a number s such that

$$W_n(\mathfrak{m}^s) \subset (W_n(\mathfrak{m}))^u.$$

We assume this for n and show it for $n+1$.

Let $\mathfrak{a} \subset R$ be the ideal generated by all products of the form

$$c_1 c_2^{p^n} \cdots c_u^{p^n}, \quad c_i \in \mathfrak{m}.$$

This is an $\mathfrak{m}$-primary ideal. We find a number s such that $\mathfrak{m}^s \subset \mathfrak{a}$. In $W_{n+1}(R)$ we have the following equation:

$$^{V^n}[c_1][c_2]\ldots[c_u] = {}^{V^n}([c_1 c_2^{p^n} \cdots c_u^{p^n}]).$$

Since the right-hand side is in $(W_{n+1}(\mathfrak{m}))^u$, it follows that

$$^{V^n}[x] \in (W_{n+1}(\mathfrak{m}))^u \quad \text{for } x \in \mathfrak{m}^s. \tag{1.45}$$

We choose a number $u_1 > u$ such that $(W_{n+1}(\mathfrak{m}))^{u_1} \subset W_{n+1}(\mathfrak{m}^s)$. By induction hypothesis, we find a number $s_1 > s$ such that $W_n(\mathfrak{m}^{s_1}) \subset (W_n(\mathfrak{m}))^{u_1}$. Let us consider an arbitrary $\xi \in W_{n+1}(\mathfrak{m}^{s_1})$. Then we find $\eta \in (W_{n+1}(\mathfrak{m}))^{u_1}$ such that

$$\xi = \eta + {}^{V^n}[c] \quad \text{for } c \in R.$$

Since $\eta \in W_{n+1}(\mathfrak{m}^s)$, we obtain $c \in \mathfrak{m}^s$. But then we obtain, from (1.45), that $\xi \in (W_{n+1}(\mathfrak{m}))^u$. □

2. The de Rham–Witt complex of a polynomial algebra

2.1. A basis of the de Rham complex

Let R be a $\mathbb{Z}_p$-algebra. We consider the polynomial ring $R[X_1, \ldots, X_n] = R[\boldsymbol{X}]$.

A weight is a function $k : [1, n] \to \mathbb{Z}_{\geqslant 0}$ to the non-negative integers. We denote the value at the natural number i by k_i. Let $\operatorname{Supp} k \subset [1, n]$ the subset, where k_i is not zero. We fix, for any weight k, a total order of $\operatorname{Supp} k$,

$$\operatorname{Supp} k = \{i_1, \ldots, i_r\}, \tag{2.1}$$

in such a way that

$$\operatorname{ord}_p k_{i_1} \leqslant \operatorname{ord}_p k_{i_2} \leqslant \cdots \leqslant \operatorname{ord}_p k_{i_r}.$$

We denote by I an interval of $\operatorname{Supp} k$,

$$I = \{i_{s+1}, i_{s+2}, \ldots, i_{s+t}\}.$$

We consider partitions of $\operatorname{Supp} k$ into disjoint intervals,

$$\operatorname{Supp} k = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_\ell. \tag{2.2}$$

The intervals are numbered in such a way that the elements of I_j are smaller than the elements of I_{j+1}. The intervals $I_1, \ldots, I_\ell$ are assumed to be not empty, but I_0 may be empty.

Let $I \subset \operatorname{Supp} k$ be an interval. Then we set

$$X^{k_I} = \prod_{j \in I} X_j^{k_j}.$$

Let $\operatorname{ord}_p k_I$ be the order $\operatorname{ord}_p k_j$, where j is the smallest element in the interval I. Then $p^{\operatorname{ord}_p k_I}$ is the biggest p-power, which divides all numbers k_j for $j \in I$. We set $Z = X^{(p^{-\operatorname{ord}_p k_I})\cdot k_I}$ and define

$$(p^{-\operatorname{ord}_p k_I} \mathrm{d}X^{k_I}) = Z^{(p^{\operatorname{ord}_p k_I}-1)}\mathrm{d}Z.$$

This is an honest equality if the ring R has no p-torsion.

To any weight k and any partition (2.2) of $\operatorname{Supp} k$ we associate a differential form,

$$X^{k_{I_0}}(p^{-\operatorname{ord}_p k_{I_1}}\mathrm{d}X^{k_{I_1}})\cdots(p^{-\operatorname{ord}_p k_{I_\ell}}\mathrm{d}X^{k_{I_\ell}}) \in \Omega^\ell_{R[\boldsymbol{X}]/R}. \tag{2.3}$$

These elements are called the p-basic elements of the de Rham complex. They depend on the total order (2.1), which we have chosen for each weight k.

Proposition 2.1. *The p-basic elements (2.3) for all weights and partitions form a base of the de Rham complex $\Omega^{\cdot}_{R[\boldsymbol{X}]/R}$ as an R-module.*

Proof. We use the notation

$$\mathrm{d}\log X_j = \frac{\mathrm{d}X_j}{X_j}.$$

The R-module $\Omega^\ell_{R[\boldsymbol{X}]/R}$ has the following elements as a basis:

$$X_1^{k_1}\cdots X_n^{k_n}\,\mathrm{d}\log X_{i_1}\cdot\cdots\cdot\mathrm{d}\log X_{i_\ell}. \tag{2.4}$$

Here, k runs through all weights and $i_1 < i_2 < \cdots < i_\ell$ through all subsets of $\operatorname{Supp} k$. The R-module spanned by all elements (2.4) for fixed k is called the module of forms of weight k,

$$\Omega^\ell_{R[\boldsymbol{X}]/R}(k) \subset \Omega^\ell_{R[\boldsymbol{X}]/R}.$$

It is free of rank $\binom{m}{\ell}$ if m is the cardinality of $\operatorname{Supp} k$. The number of p-basic elements (2.3) for fixed k and ℓ is exactly $\binom{m}{\ell}$. These p-basic elements lie in $\Omega^\ell_{R[\boldsymbol{X}]/R}(k)$. If we show that these p-basic elements generate $\Omega^\ell_{R[\boldsymbol{X}]/R}(k)$, our proposition follows. Hence it is enough to show the weaker assertion that the p-basic elements generate the de Rham complex as an R-module.

We fix a weight k and set $I = \operatorname{Supp} k$. By giving the variables new names, we may assume that the chosen order on I is the order of natural numbers. We then have

$$\operatorname{ord}_p k_i < \operatorname{ord}_p k_j \quad \text{for } i < j, \quad i, j \in I.$$

For $\ell = 1$, our proposition is a consequence of the following result.

Lemma 2.2. *Let $a, b : I \to \mathbb{Z}_{\geqslant 0}$ be two functions such that $a_j + b_j = k_j$ for $j \in I$. Let $i_1 < i_2 < \cdots < i_r$ be the support of the function a and $j_1 < \cdots < j_s$ be the support of the function b. Then the element*

$$X_{i_1}^{a_{i_1}}\cdots X_{i_r}^{a_{i_r}} p^{-\delta}\mathrm{d}(X_{j_1}^{b_{j_1}}\cdots X_{j_s}^{b_{j_s}}),$$

where p^δ is a p-power dividing $b_{j_1}, \ldots, b_{j_r}$, is a linear combination of p-basic elements of weight k.

Proof. Let a be a natural number and $h \in R[\boldsymbol{X}]$. We will use the notation

$$\frac{1}{a}\mathrm{d}h^a = h^{a-1}\mathrm{d}h.$$

By the Leibniz rule, it is enough to show the assertion for $s = 1$. We have the formula

$$X_{j_1}^{a_{j_1}} p^{-\delta}\mathrm{d}X_{j_1}^{b_{j_1}} = (p^{-\delta} b_{j_1})\left(\frac{1}{a_{j_1}+b_{j_1}}\mathrm{d}X_{j_1}^{a_{j_1}+b_{j_1}}\right). \tag{2.5}$$

By assumption, we have $a_{j_1} + b_{j_1} = k_{j_1}$. Element (2.5) is a multiple of $p^{-\operatorname{ord}_p k_{j_1}}\mathrm{d}X_{j_1}^{k_{j_1}}$. Therefore, we are reduced to proving the lemma for the case where the sets $\{i_1, \dots, i_r\}$ and $\{j_1, \dots, j_s\}$ are disjoint. That means that we consider elements of the form

$$X_{i_1}^{k_{i_1}} \cdots X_{i_r}^{k_{i_r}}\left(\frac{\mathrm{d}X_{j_1}^{k_{j_1}} \cdots X_{j_s}^{k_{j_s}}}{k_{j_1}}\right). \tag{2.6}$$

This makes sense because $p^{\operatorname{ord}_p k_{j_1}}/k_{j_1} \in \mathbb{Z}_{(p)}^*$ is a unit. Let $r' \leqslant r$ be the smallest number such that $i_{r'} > j_1$.

We prove our assertion by induction on $r - r'$. The induction starts with the case where no r' exists. Then (2.6) is already a p-basic element, and we are done.

If r' exists, we have $i_r > j_1$. We find a number $t \leqslant s$ with

$$j_t < i_r < j_{t+1}.$$

In the case $t = s$, the last inequality is absent. If $t < s$, our expression (2.6) is, according to the Leibniz rule,

$$\begin{aligned} &X_{i_1}^{k_{i_1}} \cdots X_{i_r}^{k_{i_r}} X_{j_1}^{k_{j_1}} \cdots X_{j_t}^{k_{j_t}}\left(\frac{\mathrm{d}X_{j_{t+1}}^{k_{j_{t+1}}} \cdots X_{j_s}^{k_{j_s}}}{k_{j_1}}\right) \\ &\qquad + X_{i_1}^{k_{i_1}} \cdots X_{i_r}^{k_{i_r}} X_{j_{t+1}}^{k_{j_{t+1}}} \cdots X_{j_s}^{k_{j_s}}\left(\frac{\mathrm{d}X_{j_1}^{k_{j_1}} \cdots X_{j_t}^{k_{j_t}}}{k_{j_1}}\right). \end{aligned} \tag{2.7}$$

The first summand here is already a multiple of a p-basic element. Hence we have to show that the second summand is a linear combination of p-basic elements. Note that, in the case $t = s$, the element (2.6) is already the second summand.

Applying the Leibniz rule to the second summand of (2.7), we obtain

$$\begin{aligned} &X_{i_1}^{k_{i_1}} \cdots X_{i_r}^{k_{i_r}} X_{j_{t+1}}^{k_{j_{t+1}}} \cdots X_{j_s}^{k_{j_s}}\left(\frac{\mathrm{d}X_{j_1}^{k_{j_1}} \cdots X_{j_t}^{k_{j_t}}}{k_{j_1}}\right) \\ &\quad = X_{i_1}^{k_{i_1}} \cdots X_{i_{r-1}}^{k_{i_{r-1}}}\left(\frac{\mathrm{d}X_{j_1}^{k_{j_1}} \cdots X_{j_t}^{k_{j_t}} X_{i_r}^{k_{i_r}} X_{j_{t+1}}^{k_{j_{t+1}}} \cdots X_{j_s}^{k_{j_s}}}{k_{j_1}}\right) \\ &\qquad - X_{i_1}^{k_{i_1}} \cdots X_{i_{r-1}}^{k_{i_{r-1}}} X_{j_1}^{k_{j_1}} \cdots X_{j_t}^{k_{j_t}}\left(\frac{\mathrm{d}X_{i_r}^{k_{i_r}} X_{j_{t+1}}^{k_{j_{t+1}}} \cdots X_{j_s}^{k_{j_s}}}{k_{j_1}}\right). \end{aligned}$$

The first summand is, by induction, a linear combination of p-basic elements, while the second is already a multiple of a p-basic element. This proves the lemma. $\square$

Since, by the lemma, $\Omega^1_{R[\boldsymbol{X}]/R}$ is generated by p-basic elements, it is clearly enough to show that a product of p-basic elements is again a linear combination of p-basic elements. We show that any p-basic element

$$X^{k_{I_0}} p^{-\delta_1} \mathrm{d}X^{k_{I_1}} p^{-\delta_2} \mathrm{d}X^{k_{I_2}} \cdots p^{-\delta_\ell} \mathrm{d}X^{k_{I_\ell}}, \tag{2.8}$$

with $\delta_i = \operatorname{ord}_p k_{I_i}$, multiplied with any monom X^{h_J} is again a linear combination of p-basic elements.

Indeed, we may assume that $J = \{j\}$ such that $X^{h_J} = X_j^h$. If j is smaller than any index of I_ℓ, we conclude by induction on ℓ. If not, we write, by the lemma, $X_j^h p^{-\delta_\ell} dX^{k_{I_\ell}}$ as a linear combination of p-basic elements and apply again induction on ℓ.

It remains to be shown that (2.8) multiplied with $p^{-\delta}\mathrm{d}X^{h_J}$, where $\delta = \operatorname{ord}_p h_J$, is a linear combination of p-basic elements. By the last argument, it is enough to do this in the case $I_0 = \emptyset$.

To see this, we define an R-algebra homomorphism

$$\alpha : \Omega^{\cdot}_{R[\boldsymbol{X}]/R} \to \Omega^{\cdot}_{R[\boldsymbol{X}]/R},$$

which satisfies the relation $\mathrm{d}\alpha = p\alpha\mathrm{d}$.

On $R[\boldsymbol{X}] = \Omega^0_{R[\boldsymbol{X}]/R}$, the R-algebra homomorphism α is defined by

$$\alpha(X_i) = X_i^p.$$

On $\Omega^1_{R[\boldsymbol{X}]/R}$, we define

$$\begin{aligned} \alpha : \quad & \Omega^1_{R[\boldsymbol{X}]/R} \to \Omega^1_{R[\boldsymbol{X}]/R}, \\ & \sum_{i=1}^n f_i \mathrm{d}X_i \mapsto \sum_{i=1}^n \alpha(f_i) X_i^{p-1} \mathrm{d}X_i. \end{aligned}$$

We extend this to the higher degrees,

$$\begin{aligned} \alpha : \quad & \Omega^i_{R[\boldsymbol{X}]/R} \to \Omega^i_{R[\boldsymbol{X}]/R}, \\ & \omega_1 \wedge \cdots \wedge \omega_i \mapsto \alpha(\omega_1) \wedge \cdots \wedge \alpha(\omega_i). \end{aligned}$$

The relation

$$\mathrm{d}\alpha = p\alpha\mathrm{d}$$

is easily verified.

p-basic elements may be written by using the identity:

$$p^{-\operatorname{ord}_p k_I} \mathrm{d}X^{k_I} = \alpha^{\operatorname{ord}_p k_I} \mathrm{d}X^{p^{-\operatorname{ord} k_I} \cdot k_I}.$$

It is clear that α maps p-basic elements to p-basic elements. The same is true for d.

Let us consider the element

$$p^{-\delta} \mathrm{d}X^{h_J} p^{-\delta_1} \mathrm{d}X^{k_{I_1}} \cdots p^{-\delta_\ell} X^{k_{I_\ell}}. \tag{2.9}$$

Let μ be the minimum of the numbers $\delta, \delta_1, \dots, \delta_\ell$. Then the element (2.9) may be rewritten using α,

$$\alpha^\mu(p^{-(\delta-\mu)}\mathrm{d}X^{p^{-\mu}hy}p^{-(\delta_1-\mu)}\mathrm{d}X^{p^{-\mu}k_{I_1}}\cdots p^{-(\delta_\ell-\mu_\ell)}\mathrm{d}X^{p^{-\mu}k_{I_\ell}}).$$

Since one of the p-powers in the bracket is 1, the element in the brackets is an exact differential of an element, which is, by induction on ℓ, a linear combination of p-basic elements. This proves that (2.9) is a linear combination of p-basic elements, too. Hence we obtain the proposition. □

2.2. The basic Witt differentials

Let R be a $\mathbb{Z}_{(p)}$-algebra and $S = R[T_1, \dots, T_d] = R[\boldsymbol{T}]$. We will give an explicit description of the de Rham–Witt complex $W\Omega_{S/R}$. The part of degree zero is the Witt ring $W\Omega^0_{S/R} = W(S)$. It has the following description.

We consider functions $k : [1, d] \to \mathbb{Z}_{\geqslant 0}[1/p]$, which we call weights. The value of k at i will be denoted by k_i. We call k integral if all k_i are integral.

We write $X_i = [T_i] \in W(S)$ for the Teichmüller representative of T_i. If k is integral, we set

$$X^k = X_1^{k_1} \cdots X_d^{k_d}.$$

We denote by $p^{u(k)}$ the denominator of k, i.e. $u(k)$ is the smallest non-negative integer such that $p^{u(k)}k$ is integral.

Proposition 2.3. *Any element of $W(R[\boldsymbol{T}])$ may be uniquely written as a convergent sum*

$$\sum_k {}^{V^{u(k)}}(\eta_k X^{p^{u(k)}k}). \tag{2.10}$$

The sum is over all weights k. The convergence means that for a given number m, we have ${}^{V^{u(k)}}\eta_k \in {}^{V^m}W(R)$ for almost all k. The last inclusion holds for all k if and only if (2.10) is an element of ${}^{V^m}W(R[\boldsymbol{T}])$.

Proof. Take an element $\xi \in W(R[\boldsymbol{T}])$ and consider the polynomial $\boldsymbol{w}_0(\xi) = \Sigma a_k T^k$, $a_k \in R$, where k runs over integral weights. Then we obtain

$$\xi - \Sigma[a_k]X^k \in {}^VW(R[\boldsymbol{T}]).$$

By induction, we obtain a unique expression for ξ,

$$\xi = \sum_{\substack{m \geqslant 0 \\ k \text{ integral}}} {}^{V^m}([a_{k,m}]X^k).$$

We note that each summand may be rewritten as follows. For given m, k, let the number ϱ be maximal such that $p^{-\varrho}k$ is integral and $\varrho \leqslant m$. Then we have

$${}^{V^m}([a_{k,m}]X^k) = {}^{V^{m-\varrho}}({}^{V^\varrho}[a_{k,m}] \cdot X^{p^{-\varrho}\cdot k}).$$

This gives the result. □

Corollary 2.4. *Each element of* $W_m(R[T_1 \cdots T_d])$ *may be uniquely written in the form*

$$\sum_k {}^{V^{u(k)}}(\eta_k X^{p^{u(k)}k}), \quad \eta_k \in W_{m-u(k)}(R),$$

where k runs through all weights such that $u(k) < m$, except for finitely many weights $\eta_k = 0$.

We will now introduce the basic Witt differentials of the de Rham–Witt complex. For each weight k, we fix, once for all, a total order on the arguments where k does not vanish,

$$\operatorname{Supp} k = \{i_1, \ldots, i_r\},$$

in such a way that

$$\operatorname{ord}_p k_{i_1} \leqslant \operatorname{ord}_p k_{i_2} \leqslant \cdots \leqslant \operatorname{ord}_p k_{i_r}.$$

For later purposes, we choose the total orders in such a way that, for each integer a and for each weight k, the orders on $\operatorname{Supp} k = \operatorname{Supp} p^a k$ agree. We will call a weight k primitive if it is integral and not all k_i are divisible by p. We choose the orders for primitive weights in an arbitrary way.

We set $t(k_{i_\ell}) = -\operatorname{ord}_p k_{i_\ell}$ and $u(k_{i_\ell}) = \max(0, t(k_{i_\ell}))$.

We will denote by I an interval of $\operatorname{Supp} k$ in the given order,

$$I = \{i_\ell, i_{\ell+1}, \ldots, i_{\ell+m}\}.$$

The restriction of k to I will be denoted by k_I. The extension by zero to $[1, d]$ will be denoted by the same letter k_I. Then we set

$$t(k_I) = t(k_{i_\ell}) = \max\{t(k_i) \mid i \in I\},$$
$$u(k_I) = u(k_{i_\ell}) = \max(0, t(k_I)).$$

If k is fixed in our discussion, we set $t(I) = t(k_I)$ and $u(I) = u(k_I)$ to avoid too many indices. We have

$$t(i_1) \geqslant t(i_2) \geqslant \cdots \geqslant t(i_r).$$

The common denominator of the values of k_I is $p^{u(I)}$. A basic Witt differential of degree zero, i.e. in $W\Omega^0_{S/R}$, is any element of the form

$${}^{V^{u(I)}}(\eta X^{p^{u(I)} \cdot k_I}), \quad \eta \in W(R). \tag{2.11}$$

For $I = \emptyset$, this is equal to η by definition.

In degree one, we have two further types of basic Witt differentials. If the weight k_I is not integral, we consider, for $I \neq \emptyset$,

$$\mathrm{d}^{V^{u(I)}}(\eta X^{p^{u(I)} \cdot k_I}). \tag{2.12}$$

If the weight k_I is integral, we have the basic Witt differential

$$F^{-t(I)}(\mathrm{d}X^{p^{t(I)} k_I}) = X^{(k_I - p^{t(I)} k_I)} \mathrm{d}X^{p^{t(I)} k_I}. \tag{2.13}$$

In the last case, $p^{-t(I)}$ is the greatest p-power that divides k_I, i.e. $p^{t(I)}k_I$ is integral but not divisible by p.

The following expressions for (2.11), (2.12) and (2.13) are suggestive, but they have only a symbolic meaning:

$$^{V^{u(I)}}\eta X^{k_I}, \qquad ^{V^{u(I)}}\eta \mathrm{d}X^{k_I}, \qquad \left(\frac{\mathrm{d}X^{k_I}}{p^{-t(I)}}\right).$$

In general, a basic Witt differential is obtained by taking products of these elements in a certain way:

We let k be fixed and consider a partition of $\operatorname{Supp} k$ in disjoint intervals

$$\operatorname{Supp} k = I_0 \sqcup I_1 \cdots \sqcup I_\ell = I. \tag{2.14}$$

The elements in I_k are smaller than the elements in I_{k+1}. The interval I_0 may be empty, but the intervals $I_1, \ldots, I_\ell$ are assumed to be non-empty.

For $\xi \in {}^{V^{u(I)}}W(R)$, we define a basic Witt differential

$$e = e(\xi, k, I_0, \ldots, I_\ell) \in W\Omega^\ell_{R[T_1,\ldots,T_d]/R}$$

of degree ℓ as follows.

We set $\xi = {}^{V^{u(I)}}\eta$. Let us denote by $r \in [0, \ell-1]$ the first index such that $k_{I_{r+1}}$ is integral. We set $r = \ell$ if k_{I_ℓ} is not integral.

We distinguish between three cases in the definition of e.

Case 1 ($I_0 \neq \emptyset$).

$$\begin{aligned} e = {}&{}^{V^{u(I_0)}}(\eta X^{p^{u(I_0)}k_{I_0}})(\mathrm{d}^{V^{u(I_1)}}X^{p^{u(I_1)}k_{I_1}}) \\ &\cdots(\mathrm{d}^{V^{u(I_r)}}X^{p^{u(I_r)}k_{I_r}})({}^{F^{-t(I_{r+1})}}\mathrm{d}X^{p^{t(I_{r+1})}k_{I_{r+1}}})\cdots({}^{F^{-t(I_\ell)}}\mathrm{d}X^{p^{t(I_\ell)}k_{I_\ell}}). \end{aligned} \tag{2.15}$$

Case 2 ($I_0 = \emptyset$ and k not integral, i.e. $r > 0$).

$$\begin{aligned} e = {}&(\mathrm{d}^{V^{u(I_1)}}(\eta X^{p^{u(I_1)}k_{I_1}}))(\mathrm{d}^{V^{u(I_2)}}X^{p^{u(I_2)}k_{I_2}}) \\ &\cdots(\mathrm{d}^{V^{u(I_r)}}X^{p^{u(I_r)}k_{I_r}})({}^{F^{-t(I_{r+1})}}\mathrm{d}X^{p^{t(I_{r+1})}k_{I_{r+1}}})\cdots({}^{F^{-t(I_\ell)}}\mathrm{d}X^{p^{t(I_\ell)}k_{I_\ell}}). \end{aligned} \tag{2.16}$$

Case 3 ($I_0 = \emptyset$ and k integral).

$$e = \eta({}^{F^{-t(I_1)}}\mathrm{d}X^{p^{t(I_1)}k_{I_1}})\cdots({}^{F^{-t(I_r)}}\mathrm{d}X^{p^{t(I_\ell)}k_{I_\ell}}). \tag{2.17}$$

In the first case we have $\xi = {}^{V^{u(I_0)}}\eta$, in the second case $\xi = {}^{V^{u(I_1)}}\eta$ and in the third case $\xi = \eta$.

If $\xi \in {}^{V^m}W(R)$, the image of the basic Witt differential in $W_m\Omega^{\cdot}_{S/R}$ is zero. The action of $\alpha \in W(R)$ on a basic Witt differential is given by

$$\alpha e(\xi, k, I_0, \ldots, I_\ell) = e(\alpha\xi, k, I_0, \ldots, I_\ell).$$

Proposition 2.5. *The action of F and V on the basic Witt differentials is as follows.*

(1) *If $I_0 \neq \emptyset$ or if k is integral, the following equality holds:*

$$^{F}e(\xi, k, I_0, \ldots, I_\ell) = e(^{F}\xi, pk, I_0, \ldots, I_\ell).$$

(2) *If $I_0 = \emptyset$ and k is not integral, then*

$$^{F}e(\xi, k, I_0, \ldots, I_\ell) = e(^{V^{-1}}\xi, pk, I_0, \ldots, I_\ell).$$

(3) *If $I_0 \neq \emptyset$ or k is integral and divisible by p, then*

$$^{V}e(\xi, k, I_0, \ldots, I_\ell) = e\left(^{V}\xi, \frac{1}{p}k, I_0, \ldots, I_\ell\right).$$

(4) *If $I_0 = \emptyset$ and $(1/p)k$ is not integral, then*

$$^{V}e(\xi, k, I_0, \ldots, I_\ell) = e\left(p^{V}\xi, \frac{1}{p}k, I_0, \ldots, I_\ell\right).$$

Proof. The first two equalities follow readily from the definition of the basic Witt differentials.

Let us consider the third equation in the case $I_0 \neq \emptyset$. Let $r \in [0, \ell-1]$ be the first index such that $k_{I_{r+1}}$ is integral and divisible by p. With this new r, we still have s the equality (2.15). Since $-t(I_j) > 0$ for $\ell \geqslant j \geqslant r+1$, we obtain, by the F-V-formula,

$$\begin{aligned} ^{V}e = {}^{V}\big(^{V^{u(I_0)}}(\eta X^{p^{u(I_0)}k_{I_0}}) \cdots \mathrm{d}^{V^{u(I_r)}}X^{p^{u(I_r)}k_{I_r}}\big) \\ \cdot \big(^{F^{-t(I_{r+1})-1}}\mathrm{d}X^{p^{t(I_{r+1})}k_{I_{r+1}}}\big) \cdots \big(^{F^{-t(I_\ell)-1}}\mathrm{d}X^{p^{t(I_\ell)}k_{I_\ell}}\big). \end{aligned}$$

Using the general identity in the de Rham–Witt complex,

$$^{V}(\omega_0 \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_r) = {}^{V}\omega_0 \mathrm{d}^{V}\omega_0 \mathrm{d}^{V}\omega_1 \cdots \mathrm{d}^{V}\omega_r,$$

we obtain the third equation of the proposition. In the case where k is integral and divisible by p, the same result follows if we apply the F-V-formula to (2.17).

Finally, we consider the fourth equation. In this case, we may take e of the form (2.16), with r defined as above, and possibly $u(I_1) = t(I_1) = 0$. Then we obtain

$$^{V}e = {}^{V}1\mathrm{d}^{V^{u(I_1)+1}}(\eta X^{p^{u(I_1)}k_{I_1}}) \cdots (^{F^{-t(I_{r+1})-1}}\mathrm{d}X^{p^{t(I_{r+1})}k_{I_{r+1}}}) \cdots .$$

Since

$$^{V}1^{V^{u(I_1)+1}}\eta = {}^{V}(p^{V^{u(I_1)}}\eta) = p^{V}\xi,$$

the last case of the proposition follows. □

Proposition 2.6. *Let k be a weight with support I. We set $t = t(k_I)$ and $t = 0$ if I is empty. With this notation, the action of the differential* d *on basic Witt differentials is as follows:*

$$\mathrm{d}e(\xi, k, I_0, \ldots, I_\ell) = \begin{cases} 0 & \textit{if } I_0 = \emptyset, \\ e(\xi, k, \phi, I_0, \ldots, I_\ell) & \textit{if } I_0 \neq \emptyset,\ k \textit{ not integral}, \\ p^{-t} e(\xi, k, \phi, I_0, \ldots, I_\ell) & \textit{if } I_0 \neq \emptyset,\ k \textit{ integral.} \end{cases}$$

Proof. Let us consider the last equality. In this case, a basic Witt differential has the form

$$\xi X^{k_{I_0}} ({}^{F^{-t(I_1)}}\mathrm{d}X^{p^{t(I_1)} k_{I_1}}) \cdot \cdots \cdot ({}^{F^{-t(I_\ell)}}\mathrm{d}X^{p^{t(I_\ell)} k_{I_\ell}}). \tag{2.18}$$

We have

$$\begin{aligned} \mathrm{d}(\xi X^{k_{I_0}}) &= \xi \mathrm{d}^{F^{-t(I_0)}} X^{p^{t(I_0)} k_{I_0}} \\ &= p^{-t(I_0)} \xi^{F^{-t(I_0)}} \mathrm{d}X^{p^{t(I_0)} k_{I_0}}. \end{aligned}$$

From this our result follows if we apply d to (2.18). The case where k is not integral is even more obvious. The first equation of the proposition is trivial. □

If we introduce in the definition of a basic Witt differential (2.15), (2.16), for each factor of the form $\mathrm{d}^{V^{u(I_j)}} X^{p^{u(I_j)} k_{I_j}}$, a factor $\mathrm{d}^{V^{u(I_j)}}(\eta_j X^{p^{u(I_j)} k_{I_j}})$, we obtain again a basic Witt differential because of the following lemma.

Lemma 2.7. *Let S be any R-algebra. Let $u_0 \geqslant u_1 \geqslant 0$ be an integer. Let $\eta_0, \eta_1 \in W(R)$ and $s_0, s_1 \in S$. Then the following formula holds in $W\Omega_{S/R}$:*

$${}^{V^{u_0}}(\eta_0[s_0])\mathrm{d}^{V^{u_1}}(\eta_1[s_1]) = {}^{V^{u_0}}(\eta_0{}^{F^{u_0-u_1}}\eta_1[s_0])\mathrm{d}^{V^{u_1}}[s_1].$$

Proof. We set $w = u_0 - u_1$. Then $F\mathrm{d}V = \mathrm{d}$ and the F-V-relation shows that

$$\begin{aligned} {}^{V^{u_0}}(\eta_0[s_0])\mathrm{d}^{V^{u_1}}(\eta_1[s_1]) &= {}^{V^{u_0}}(\eta_0[s_0]^{F^w}\mathrm{d}(\eta_1[s_1])) \\ &= {}^{V^{u_0}}(\eta_0{}^{F^w}\eta_1[s_0]^{F^w}\mathrm{d}[s_1]). \end{aligned}$$

If we repeat this equality with $\eta_0{}^{F^w}\eta_1$ for η_0 and 1 for η_1, we obtain the assertion of the lemma. □

2.3. The main theorem

Let k be a weight and $I = \operatorname{Supp} k$. We will denote by

$$\mathcal{P} = \{I_0, I_1, \ldots, I_\ell\}$$

an arbitrary partition of I of the form (2.2)

$$I = I_0 \sqcup \cdots \sqcup I_\ell.$$

Theorem 2.8. *Each element $\omega \in W\Omega^{\cdot}_{R[T_1 \cdots T]/R}$ has a unique expression as a convergent sum*

$$\sum_{k,\mathcal{P}} e(\xi_{k,\mathcal{P}}, k, \mathcal{P}),$$

where k runs over all possible weights and $\mathcal{P}$ over all partitions of $\operatorname{Supp} k$, *and where, for any given number m, we have $\xi_{k,\mathcal{P}} \in {}^{V^m}W(R)$ for all but finitely many weights k.*

This theorem was proved by Illusie in the case where R is a perfect ring. We remark that all elements of the type $e(\xi, k, I_0, \ldots, I_\ell)$ for a fixed weight k and a fixed partition $\mathcal{P} = (I_0, \ldots, I_\ell)$ form a $W(R)$-submodule of $W\Omega^{\ell}_{R[\boldsymbol{T}]/R}$, which is, due to the theorem, isomorphic to ${}^{V^{u(I_0)}}W(R)$ and ${}^{V^{u(I_1)}}W(R)$ for the cases $I_0 \neq \emptyset$ and $I_0 = \emptyset$, respectively.

In this section we will prove the theorem without the uniqueness assertion. For the following, we use an obvious notation.

Let $f \in W(S)$ and let $a, b > 0$ be integers such that $\operatorname{ord}_p(a/b) \geqslant 0$. Then we define

$$\left(\frac{\mathrm{d}f^a}{b}\right) = \frac{a}{b} f^{a-1}\mathrm{d}f.$$

The left-hand side is a symbol, which depends on f, a and b and not only on f^a.

Lemma 2.9. *Let $\{i_1, \ldots, i_r\} \subset [1, d]$ be a subset and $a_{i_1}, \ldots, a_{i_r}$ arbitrary positive integers. Let $1 < k < r$ be a number and let c be the greatest common divisor of $a_{i_k}, \ldots, a_{i_r}$. Then the following element in $W\Omega_{R[T_1 \cdots T_d]/R}$ is a sum of basic Witt differentials:*

$$X_{i_1}^{a_{i_1}} \cdots X_{i_{k-1}}^{a_{i_{k-1}}} \left(\frac{\mathrm{d}X_{i_k}^{a_{i_k}} \cdots X_{i_r}^{a_{i_r}}}{c}\right).$$

Proof. We may assume that $\{i_1 \cdots i_r\} = [1, r]$ by renumeration of the variables. The function $i \mapsto a_i$ extended by zero to $[1, n]$ is a weight. Again, by renumeration, we may assume that the order on $[1, r]$ assigned to this weight function is the order on natural numbers. Then we have

$$\operatorname{ord}_p a_1 \leqslant \operatorname{ord}_p a_2 \leqslant \cdots \leqslant \operatorname{ord}_p a_r.$$

Then we may reformulate the assertion in a new notation. Assume we are given a partition

$$[1, r] = \{i_1, \ldots, i_h\} \sqcup \{j_1, \ldots, j_\ell\}, \tag{2.19}$$

where we assume $i_1 < \cdots < i_h$ and $j_1 < \cdots < j_\ell$. Then the element

$$X_{i_1}^{a_{i_1}} \cdot \cdots \cdot X_{i_h}^{a_{i_h}} \left(\frac{\mathrm{d}X_{j_1}^{a_{j_1}} \cdots X_{j_\ell}^{a_{j_\ell}}}{a_{j_1}}\right) \tag{2.20}$$

is a sum of basic Witt differentials for the weight function $i \mapsto a_i$. We show this by induction on h. The beginning of the induction is the case where $h = 0$ (i.e. the first set of the partition (2.19) is empty). In this case, Equation (2.20) is clearly a basic Witt differential.

Next we consider an element (2.20) for $h \geqslant 1$. If $i_h < j_1$, the element (2.20) is basic. If not, let g be the greatest number such that

$$j_g < i_h.$$

If $g < \ell$, we may apply the Leibniz rule to obtain

$$X_{i_1}^{a_{i_1}} \cdot \cdots \cdot X_{i_h}^{a_{i_h}} X_{j_1}^{a_{j_1}} \cdot \cdots \cdot X_{j_g}^{a_{j_g}} \left(\frac{\mathrm{d} X_{j_{g+1}}^{a_{j_{g+1}}} \cdot \cdots \cdot X_{j_\ell}^{a_{j_\ell}}}{a_{j_1}} \right) \\ + X_{i_1}^{a_{i_1}} \cdot \cdots \cdot X_{i_{h-1}}^{a_{i_{h-1}}} \left(\frac{\mathrm{d} X_{j_1}^{a_{j_1}} \cdot \cdots \cdot X_{j_g}^{a_{j_g}}}{a_{j_1}} \right) X_{i_h}^{a_{i_h}} X_{j_{g+1}}^{a_{j_{g+1}}} \cdot \cdots \cdot X_{j_\ell}^{a_{j_\ell}}.$$

The first summand is already a basic Witt differential. We have to consider the second summand. For $g = \ell$, our original element already has this form. By the Leibniz rule, we obtain for the second summand,

$$X_{i_1}^{a_{i_1}} \cdot \cdots \cdot X_{i_{h-1}}^{a_{i_{h-1}}} \left(\frac{\mathrm{d} X_{j_1}^{a_{j_1}} \cdot \cdots \cdot X_{j_g}^{a_{j_g}} \cdot X_{i_h}^{a_{i_h}} \cdot X_{j_{g+1}}^{a_{j_{g+1}}} \cdot \cdots \cdot X_{j_\ell}^{a_{j_\ell}}}{a_{j_1}} \right) \\ - X_{i_1}^{a_{i_1}} \cdot \cdots \cdot X_{i_{h-1}}^{a_{i_{h-1}}} X_{j_1}^{a_{j_1}} \cdot \cdots \cdot X_{j_g}^{a_{j_g}} \left(\frac{\mathrm{d} X_{i_h}^{a_{i_h}} X_{j_{g+1}}^{a_{j_{g+1}}} \cdot \cdots \cdot X_{j_\ell}^{a_{j_\ell}}}{a_{j_1}} \right).$$

Here, the first summand is a sum of basic Witt differentials by induction, while the second summand is already a basic Witt differential. $\square$

Lemma 2.10. *Let $I \subset [1, d]$ be a subset. Let $\tilde{\mathcal{I}}$ and $\mathcal{I}$ be subsets of I such that $I = \tilde{\mathcal{I}} \cup \mathcal{I}$. Let $\tilde{a} : \tilde{\mathcal{I}} \to \mathbb{N}$, $a : \mathcal{I} \to \boldsymbol{N}$ be functions, which we extend by zero to $[1, n]$. We define a weight function k with support I as follows:*

$$k_i = \begin{cases} \tilde{a}_i & \text{for } i \in \tilde{\mathcal{I}} \setminus \mathcal{I}, \\ a_i & \text{for } i \in \mathcal{I} \setminus \tilde{\mathcal{I}}, \\ a_i + \tilde{a}_i & \text{for } i \in \mathcal{I} \cap \tilde{\mathcal{I}}. \end{cases}$$

Then the element

$$\prod_{i \in \mathcal{I}} X_i^{\tilde{a}_i} \left(\frac{\mathrm{d} \prod_{i \in \mathcal{I}} X_i^{a_i}}{c} \right), \qquad c = \gcd(a_i \mid i \in \mathcal{I})$$

is a sum of basic Witt differentials of weight k.

Proof. If $\mathcal{I} \cap \tilde{\mathcal{I}} = \emptyset$, this is Lemma 2.9. We fix an element $j \in \tilde{\mathcal{I}} \cap \mathcal{I}$ and argue by induction on the number of elements in $\tilde{\mathcal{I}} \cap \mathcal{I}$. It is enough to prove our assertion for the element

$$X_j^{\tilde{a}_j} \left(\frac{\mathrm{d} \prod_{i \in \mathcal{I}} X_i^{a_j}}{c} \right).$$

Indeed, if this is represented as a sum of basic Witt differentials as in the proposition, we may multiply this sum by $\prod_{i\in\tilde{\mathcal{I}}\setminus\{j\}} X_i^{\tilde{a}_i}$ and apply the induction assumption.

Therefore, we may assume that $\{j\}=\tilde{\mathcal{I}}\subset\mathcal{I}=I$. After renumeration of the variables, we may assume that $I=[1,r]$, and that

$$\operatorname{ord}_p a_1 \leqslant \operatorname{ord}_p a_2 \leqslant \cdots \leqslant \operatorname{ord}_p a_r.$$

Then we have to consider an element of the form

$$X_j^b\left(\frac{\mathrm{d}X_1^{a_1}\cdot\dots\cdot X_r^{a_r}}{a_1}\right),\quad \text{where } b=\tilde{a}_j. \tag{2.21}$$

First, we represent this as a sum of basic Witt differentials in the case, where $\operatorname{ord}_p b < \operatorname{ord}_p a_1$. Using the Leibniz rule, we may write (2.21) as follows:

$$X_j^b\left(\frac{\mathrm{d}X_j^{a_j}}{a_1}\right)X_1^{a_1}\cdot\dots\cdot\hat{X}_j^{a_j}\cdot\dots\cdot X_r^{a_r}+X_j^{b+a_j}\left(\frac{\mathrm{d}X_1^{a_1}\cdot\dots\cdot\hat{X}_j^{a_j}\cdot\dots\cdot X_r^{a_r}}{a_1}\right).$$

The second summand is already basic of the right weight.

To see the same thing for the first summand, we apply the formula

$$X_j^b\left(\frac{\mathrm{d}X_j^{a_j}}{a_1}\right)=\left(\frac{a_j}{a_1}\right)\left(\frac{\mathrm{d}X_j^{a_j+b}}{a_j+b}\right).$$

This is immediate from the definition.

Finally, we consider the case $\operatorname{ord}_p b\geqslant\operatorname{ord}_p a_1$. Then we may apply the Leibniz rule as follows:

$$X_j^b\left(\frac{\mathrm{d}X_1^{a_1}\cdot\dots\cdot X_r^{a_r}}{a_1}\right)=\left(\frac{\mathrm{d}X_1^{a_1}\cdots X_j^{a_j+b}\cdots X_r^{a_r}}{a_1}\right)-X_j^{a_j}\left(\frac{\mathrm{d}X_j^b}{a_1}\right)X_1^{a_1}\cdot\dots\cdot\tilde{X}_j^{a_j}\cdots X_r^{a_r}.$$

To see that the second summand is a sum of basic Witt differentials, we apply the formula

$$X_j^{a_j}\left(\frac{\mathrm{d}X_j^b}{a_1}\right)=\left(\frac{b}{a_1}\right)\left(\frac{\mathrm{d}X_j^{a_j+b}}{a_j+b}\right)$$

and apply Lemma 2.9. □

We may now generalize Lemma 2.10 to differential forms of arbitrary degree.

Proposition 2.11. *Let $I\subset[1,d]$ be a subset. Let $\mathcal{I}_0,\mathcal{I}_1,\dots,\mathcal{I}_\ell$ be subsets of I such that $\mathcal{I}_1,\dots,\mathcal{I}_\ell$ are non-empty. Let $a_{\mathcal{I}_k}:\mathcal{I}_k\to\mathbb{N}$ be functions, which we extend by zero to $[1,d]$. We denote by $c_{\mathcal{I}_j}$ the greatest common divisor of the values $a_{\mathcal{I}_j,i}$ for $\mathcal{I}_j$. Then the element of $W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$,*

$$\prod_{i\in\mathcal{I}_0}X_i^{a_{\mathcal{I}_0},i}\left(\frac{\mathrm{d}\prod_{i\in\mathcal{I}_1}X_i^{a_{\mathcal{I}_1},i}}{c_{\mathcal{I}_1}}\right)\cdot\dots\cdot\left(\frac{\mathrm{d}\prod_{i\in\mathcal{I}_\ell}X_i^{a_{\mathcal{I}_\ell},i}}{c_{\mathcal{I}_\ell}}\right), \tag{2.22}$$

is a sum of basic Witt differentials for the weight function

$$k_i = \sum_{j=0}^{\ell} a_{\mathcal{I}_\ell, i}. \tag{2.23}$$

Proof. By Lemma 2.10, this holds for $\ell = 1$. We use induction on ℓ, and assume that the proposition holds for numbers smaller than ℓ. The induction assumption implies that (2.22) is a sum of basic Witt differentials if $\mathcal{I}_0$ is empty. Indeed, without loss of generality, we have

$$\operatorname{ord}_p c_{\mathcal{I}_1} \leqslant \operatorname{ord}_p c_{\mathcal{I}_2} \leqslant \cdots \leqslant \operatorname{ord}_p c_{\mathcal{I}_\ell}.$$

We set $e_j = \operatorname{ord}_p c_{\mathcal{I}_j}$ and $b_{j,i} = p^{-e_j} a_{\mathcal{I}_j, i}$ for $i \in \mathcal{I}_j$.

The $b_{j,i}$ are natural numbers, which do not have p as a common divisor for j fixed. Then the expression (2.22) may be written up to a unit in $\mathbb{Z}_{(p)}$,

$$\begin{aligned} &{}^{F^{e_1}}\Big(\mathrm{d} \prod_{i \in \mathcal{I}_1} X_i^{b_{1,i}}\Big) \cdot \cdots \cdot {}^{F^{e_\ell}}\Big(\mathrm{d} \prod_{i \in \mathcal{I}_\ell} X_i^{b_{\ell,i}}\Big) \\ &\qquad = {}^{F^{e_1}}\mathrm{d}\bigg(\prod_{i \in \mathcal{I}_1} X_1^{b_{1,i}} \cdot \Big({}^{F^{e_2 - e_1}}\mathrm{d} \prod_{i \in \mathcal{I}_2} X^{b_{2,i}}\Big) \cdot \cdots \cdot \Big({}^{F^{e_\ell - e_1}}\mathrm{d} \prod_{i \in \mathcal{I}_\ell} X^{b_{\ell,i}}\Big)\bigg). \end{aligned}$$

Applying the induction assumption to the element in the outer parentheses, we obtain the assertion for $\mathcal{I}_0 = \emptyset$.

Next we consider the case where the subsets $\mathcal{I}_0, \mathcal{I}_1, \ldots, \mathcal{I}_\ell$ are disjoint. The weight k defined by (2.23) puts an order on I. Let us denote by κ the smallest index in this order. We argue by induction on $\operatorname{card} I$. If $\kappa \in \mathcal{I}_0$, we divide the element (2.22) by $X_\kappa^{a_{\mathcal{I}_0, \kappa}}$. Then we apply the induction to the remaining expression. If we multiply the remaining sum with $X_\kappa^{a_{\mathcal{I}_0, \kappa}}$, we obtain again a sum of basic Witt differentials. If $\kappa \notin \mathcal{I}_0$, we may assume that $\kappa \in \mathcal{I}_1$. Then the element

$$\prod_{i \in \mathcal{I}_0} X_i^{a_{\mathcal{I}_0, i}} \left(\frac{\mathrm{d} \prod_{i \in \mathcal{I}_1} X_i^{a_{\mathcal{I}_1, i}}}{c_1}\right)$$

may be expressed as a sum of basic Witt differentials by Lemma 2.9. If we substitute this sum in the expression (2.22), we are, for each summand, either in the case where $\mathcal{I}_0 \neq \emptyset$ or where κ appears in $\mathcal{I}_0$. These cases were already treated.

Finally, we consider the general case. By induction on $\operatorname{card} \mathcal{I}$, we may reduce the proposition to the following assertion. Assume that element (2.22) is a basic Witt differential, i.e. $I = \mathcal{I}_0 \sqcup \cdots \sqcup \mathcal{I}_\ell$ respects the order on $\operatorname{Supp} k = I$. Then, for any $m \in [1, d]$ and $b \in \mathbb{N}$, the product of (2.22) with X_m^b is a sum of basic Witt differentials.

If m belongs to $\mathcal{I}_0$, ord does not belong to I and we are in the case of a disjoint union, which was already treated. If $m \in \mathcal{I}_1$, we first express

$$X_m^b \left(\frac{\mathrm{d} \prod_{i \in \mathcal{I}_1} X_i^{a_{\mathcal{I}_1, i}}}{c_{\mathcal{I}_1}}\right)$$

as a sum of basic Witt differential. If we multiply this with the remaining terms in (2.22), we are again in the case of a disjoint union. $\square$

We accomplish now the first step in proving Theorem (2.8).

Lemma 2.12. *Any element in $W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$ is a convergent sum of basic Witt differentials.*

Proof. By Proposition 2.3, any element in $W\Omega^{\cdot}_{S/R}$ is a convergent sum of elements of the form

$$^{V^{u_0}}(\eta_0 X^{p^{u_0}k^{(0)}})\mathrm{d}^{V^{u_1}}(\eta_1 X^{p^{u_1}k^{(1)}}) \cdot \dots \cdot \mathrm{d}^{V^{u_\ell}}(\eta_\ell X^{p^{u_\ell}k^{\ell}}). \tag{2.24}$$

Here, $k^{(0)}, \dots, k^{(\ell)}$ are arbitrary weights and p^{u_i} is the denominator of $k^{(i)}$. We have to show that an element (2.24) is a sum of basic Witt differentials. We proved this in the case where all weights $k^{(j)}$ are integral, i.e. $u_0 = \dots = u_\ell = 0$ (Proposition 2.11).

For the general case, we use an induction on the degree ℓ by the differential form (2.24). Let us first assume that $u_0 \geqslant u_j$ for $j = 1, \dots, k$. Then we may rewrite the expression (2.24) as

$$^{V^{u_0}}(\eta_0 X^{p^{u_0}k^{(0)}}\, {}^{F^{u_0-u_1}}\mathrm{d}\eta_1 X^{p^{u_1}k^{(1)}} \cdot \dots \cdot {}^{F^{u_0-u_\ell}}\mathrm{d}\eta_\ell X^{p^{u_\ell}k^{(\ell)}}).$$

Then Proposition 2.11 shows that the expression in brackets is a sum of basic Witt differentials. Hence we finish this case by Proposition 2.5.

Secondly, we assume $u_1 \geqslant u_j$ for $j = 0, 1, \dots, \ell$. Then we apply the Leibniz rule

$$\begin{aligned} &^{V^{u_0}}(\eta_0 X^{p^{u_0}k^{(0)}})\mathrm{d}^{V^{u_1}}(\eta_1 X^{p^{u_1}k^{(1)}}) \\ &\quad = \mathrm{d}(^{V^{u_0}}(\eta_0 X^{p^{u_0}k^{(0)}})^{V^{u_1}}(\eta_1 X^{p^{u_1}k^{(1)}})) - {}^{V^{u_1}}(\eta_1 X^{p^{u_1}k^{(1)}})\mathrm{d}^{V^{u_0}}(\eta_0 X^{p^{u_0}k^{(1)}}). \end{aligned} \tag{2.25}$$

Inserting this in expression (2.24), the first summand of the right-hand side of (2.25) gives a differential of a form of degree $(\ell - 1)$, while the second summand gives an element considered in the case $u_0 \geqslant u_j$. Using the induction assumption and the fact that d takes basic Witt differentials to basic Witt differentials, we are done. □

Corollary 2.13. *The kernel of $W\Omega^{\cdot}_{R[\boldsymbol{T}]/R} \to W_m\Omega^{\cdot}_{R[\boldsymbol{T}]/R}$ consists of convergent sums of basic Witt differentials $e(\xi, k, I_0, \dots, I_\ell)$ with $\xi \in {}^{V^m}W(R)$.*

Proof. By definition, the kernel is topologically generated by elements of the form (2.24), where, for some index j, we have $^{V^{u_j}}\eta_j \in {}^{V^m}W(R)$ (by Proposition 2.3). The proof of the lemma shows that these elements may be written as a sum of basic Witt differentials of the indicated form. □

2.4. The phantom components

To prove the 'linear independence' of basic Witt differentials, i.e. the uniqueness assertion in Theorem 2.8, we will now introduce the phantom components for the de Rham–Witt complex.

Let R be a $\mathbb{Z}_{(p)}$-algebra and S be an R-algebra. If M is an S-module, we will denote by $M_{\boldsymbol{w}_n}$ the $W(S)$-module induced by restriction of scalars $\boldsymbol{w}_n : W(S) \to S$ via the Witt polynomial $\boldsymbol{w}_n$. We consider the map for $n \geqslant 0$,

$$\begin{aligned}\delta_n : W(S) &\to \Omega^1_{S/R,\boldsymbol{w}_n},\\ (x_0, x_1, x_2 \dots) &\mapsto \sum_{i=0}^{n} X_i^{p^{n-i}-1} \mathrm{d}x_i.\end{aligned}$$

The map δ_0 is the usual differential $\mathrm{d}x_0$.

Lemma 2.14. *δ_n is a continuous $W(R)$-linear pd-derivation.*

Proof. Since δ_n factors through $W_{n+1}(S)$, it is continuous. In the case where $\Omega^1_{S/R,\boldsymbol{w}_n}$ has no p-torsion, the assertion is obvious, since $\delta_n = (1/p^n)\mathrm{d}\boldsymbol{w}_n$ and the torsion-freeness guarantees that any derivation is a pd-derivation. But we may restrict to this case by considering homomorphisms $R' \to R$, $S' \to S$, where R' has no p-torsion and $S' = R'[x_0, x_1, \dots]$ is the polynomial algebra in infinitely many variables. □

The maps δ_n define $W_{n+1}(S)$-linear maps

$$\omega_n : \breve{\Omega}^1_{W_{n+1}(S)/W_{n+1}(R)} \to \Omega^1_{S/R,\boldsymbol{w}_n} \quad \text{for } n \geqslant 0,$$

which we extend to the exterior powers

$$\omega_n : \breve{\Omega}^i_{W_{n+1}(S)/W_{n+1}(R)} \to \Omega^i_{S/R,\boldsymbol{w}_n} \tag{2.26}$$

by the following formula,

$$\omega_n(\xi \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i) = \boldsymbol{w}_n(\xi)\delta_n\eta_1 \cdots \delta_n\eta_i,$$

where $\xi \in W_{n+1}(S)$, $\eta_1, \dots, \eta_i \in W_{n+1}(S)$.

Consider the complex of $W_n(S)$-modules

$$P_n^{\cdot} = \bigoplus_{i=0}^{n-1} \Omega^{\cdot}_{S/R,\boldsymbol{w}_i}.$$

With respect to the natural projection $P_n \to P_{n-1}$, we obtain a procomplex. We consider P_n as an algebra with respect to component-wise addition and multiplication. Hence we have a procomplex of differential graded algebras.

We define operators F and V on P_n, but they will not satisfy the relations required for an F-V-procomplex. Let us denote an element of P_n as follows:

$$\varrho = [\varrho_0, \dots, \varrho_{n-1}], \quad \text{where } \varrho_i \in \Omega^{\cdot}_{S/R,\boldsymbol{w}_i}.$$

We set

$$\begin{aligned}{}^F[\varrho_0, \dots, \varrho_{n-1}] &= [\varrho_1, \varrho_2, \dots, \varrho_{n-1}] \in P_{n-1},\\ {}^V[\varrho_0, \dots, \varrho_{n-1}] &= [0, p\varrho_0, \dots, p\varrho_{n-1}] \in P_{n+1}.\end{aligned}$$

Then $F : P_n \to P_{n-1,F}$ is an algebra homomorphism. The F-V-formula holds,

$$^V(\varrho^F\tau) = {}^V\varrho \cdot \tau, \quad \varrho \in P_{n-1}, \quad \tau \in P_n.$$

The sum of the maps $(\omega_0, \dots, \omega_{n-1})$ define a homomorphism of $W_n(S)$-modules,

$$\underset{\sim}{\omega}^n : \breve{\Omega}^{\cdot}_{W_n(S)/W_n(R)} \to P_n, \quad n \geqslant 1, \tag{2.27}$$

which is, by (2.26), a homomorphism of projective systems of algebras.

Proposition 2.15. *The $\underset{\sim}{\omega}^n$ factor through a homomorphism of projective systems of algebras*

$$\underset{\sim}{\omega}^n : W_n\Omega^{\cdot}_{S/R} \to P_n.$$

This homomorphism commutes with F and V but not with d,

$$\mathrm{d}\underset{\sim}{\omega}^n = [1, p, p^2, \dots]\underset{\sim}{\omega}^n \mathrm{d},$$

where $[1, p, p^2, \dots] \in \Pi S = P^0_n$.

Proof. Since P_n is not an F-V-procomplex, the universality of $W^{\cdot}\Omega$ is not applicable. We must give a direct argument.

Let $\xi = (x_0, x_1, \dots, x_{n-1}) \in W_n(S)$. Then we have the relations

$$\delta_n(^V\xi) = \begin{cases} \delta_{n-1}(\xi), & n > 0,\ \delta_{n-2}(^F\xi) = p\delta_{n-1}(\xi), \\ 0, & n = 0,\ \text{for } n \geqslant 2. \end{cases} \tag{2.28}$$

This is an obvious calculation.

We consider an element

$$u = \xi \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i \in \breve{\Omega}^i_{W_n(S)/W_n(R)}, \quad \text{where } \eta_j \in W_n(S).$$

Then we have the formulae

$$\left.\begin{aligned} \underset{\sim}{\omega}^{n+1}(^V\xi \mathrm{d}^V\eta_1 \cdots \mathrm{d}^V\eta_i) &= {}^V(\underset{\sim}{\omega}^n(\xi \mathrm{d}\eta_1 \cdot \cdots \cdot \mathrm{d}\eta_i)), \quad n \geqslant 1, \\ \underset{\sim}{\omega}^{n-1}(^F u) &= {}^F(\underset{\sim}{\omega}^n(u)), \qquad\qquad\quad n \geqslant 2. \end{aligned}\right\} \tag{2.29}$$

Indeed, the first relation says that, for $0 < m \leqslant n$, we have

$$\omega_m(^V\xi \mathrm{d}^V\eta_1 \cdots \mathrm{d}^V\eta_i) = p\omega_{m-1}(\xi \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i). \tag{2.30}$$

The left-hand side is, by definition,

$$\boldsymbol{w}_m(^V\xi)\delta_m(^V\eta_1) \cdot \cdots \cdot \delta_m(^V\eta_i) = p\boldsymbol{w}_{m-1}(\xi)\delta_{m-1}(\eta_1) \cdot \cdots \cdot \delta_{m-1}(\eta_i).$$

Hence we obtain (2.30). For $m = 0$, the left-hand side of (2.30) is obviously zero.

The second equation of (2.29) asserts that, for $0 \leqslant m \leqslant n-2$,

$$\omega_m(^F\xi^F\mathrm{d}\eta_1 \cdots {}^F\mathrm{d}\eta_i) = \omega_{m+1}(\xi \mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i). \tag{2.31}$$

Clearly, it is enough to show that

$$\omega_m({}^F\mathrm{d}\eta) = \omega_{m+1}(\mathrm{d}\eta) \quad \text{for } \eta \in W_n(S). \tag{2.32}$$

Let $\eta = (y_0, \ldots, y_{n-1})$ and $\varrho = (y_1, \ldots, y_{n-1})$. Then we may write $\eta = [y_0] + {}^V\varrho$. By definition, we have ${}^F\mathrm{d}\eta = [y_0^{p-1}]\mathrm{d}[y_0] + \mathrm{d}\varrho$,

$$\begin{aligned}
\omega_m({}^F\mathrm{d}\eta) &= \boldsymbol{w}_m([y_0^{p-1}])\delta_m[y_0] + \delta_m\varrho \\
&= y_0^{(p-1)p^m} y_0^{p^m-1}\mathrm{d}y_0 + \delta_{m+1}({}^V\varrho) \\
&= y_0^{p^{m+1}-1}\mathrm{d}y_0 + \delta_{m+1}({}^V\varrho) \\
&= \delta_{m+1}([y_0] + {}^V\varrho) \\
&= \omega_{m+1}(\mathrm{d}\eta).
\end{aligned}$$

This proves the relation (2.31) and (2.32).

Next we prove the relation

$$p^m\omega_m(\mathrm{d}u) = \mathrm{d}\omega_m(u). \tag{2.33}$$

We may assume that $u = \xi\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i$. Then we obtain

$$\begin{aligned}
p^m\omega_m(\mathrm{d}\xi\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i) &= p^m\delta_m(\xi) \cdots \delta_m(\eta_i) \\
&= \mathrm{d}\boldsymbol{w}_m(\xi)\delta_m(\eta_1) \cdot \cdots \cdot \delta_m(\eta_i) \\
&= \mathrm{d}(\boldsymbol{w}_m(\xi)\delta_m(\eta_1) \cdots \delta_m(\eta_i)) \\
&= \mathrm{d}\omega_m(\xi\mathrm{d}\eta_1 \cdots \mathrm{d}\eta_i).
\end{aligned}$$

Here, the third equation holds because the form $\delta_m(\eta_j)$ are obviously closed.

Finally, we have to show that the map (2.27) factors through $\underset{\sim}{\omega}^n$. For this, it suffices to show that the map

$$\omega_n : \breve{\Omega}^i_{W_{n+1}(S)/W_{n+1}(R)} \to \Omega^i_{S/R,\boldsymbol{w}_n}$$

factors through $W_{n+1}\Omega^i_{S/R}$.

Let $\xi^{(\ell)}$ and $\eta_j^{(\ell)} \in W_n(S)$ be some elements such that

$$\sum_\ell \xi^{(\ell)}\mathrm{d}\eta_1^{(\ell)} \cdot \cdots \cdot \mathrm{d}\eta_i^{(\ell)} = 0$$

in $W_n\Omega_{S/R}$. We will show that the following elements are annihilated by ω_n:

$$\sum_\ell {}^V\xi^{(\ell)}\mathrm{d}^V\eta_1^{(\ell)} \cdots \mathrm{d}^V\eta_i^{(\ell)}, \qquad \sum_\ell \mathrm{d}^V\xi^{(\ell)}\mathrm{d}^V\eta_1^{(\ell)} \cdots \mathrm{d}^V\eta_i^{(\ell)}. \tag{2.34}$$

We compute, by (2.30), for $n > 0$,

$$\omega_n\Big(\sum_\ell {}^V\xi^{(\ell)}\mathrm{d}^V\eta_1^{(\ell)} \cdot \cdots \cdot \mathrm{d}^V\eta_i^{(\ell)}\Big) = p\omega_{n-1}\Big(\sum_\ell \xi^{(\ell)}\mathrm{d}\eta_1^{(\ell)} \cdots \mathrm{d}\eta_i^{(\ell)}\Big).$$

This expression is zero because ω_{n-1} factors by induction through $W_n\Omega_{S/R}$. The second element of (2.34) is annihilated by ω_n because, for $n > 0$,

$$\omega_n(\mathrm{d}^V\xi^{(\ell)}\mathrm{d}^V\eta_1^{(\ell)}\cdots\mathrm{d}\,^V\eta_i^{(\ell)}) = \omega_{n-1}(\mathrm{d}\xi^{(\ell)}\mathrm{d}\eta_1^{(\ell)}\cdots\mathrm{d}\eta_i^{(\ell)}). \tag{2.35}$$

This follows readily from (2.28).

Let $\bar{\Omega}^i_{n+1}$ be the quotient of $\breve{\Omega}^i_{W_{n+1}(S)/W_{n+1}(R)}$ by the ideal generated by all possible elements (2.34). This is stable by d and we have that $\omega_n : \bar{\Omega}^i_{n+1} \to \Omega^i_{S/R,\boldsymbol{w}_n}$ is defined. By definition, we have a well-defined map

$$\begin{aligned} V: \qquad W_n\Omega^i_{S/R} &\to \bar{\Omega}^i_{n+1}, \\ \xi\mathrm{d}\eta_1\cdots\mathrm{d}\eta_i &\mapsto {}^V\xi\mathrm{d}^V\eta_1\cdots\mathrm{d}^V\eta_i. \end{aligned}$$

From (2.35), we obtain

$$\omega_n(\mathrm{d}^Vt) = \omega_{n-1}(\mathrm{d}t), \quad t \in W_n\Omega^i_{S/R}. \tag{2.36}$$

By construction, $W_{n+1}\Omega^{\cdot}$ is the quotient of $\bar{\Omega}^i_{n+1}$ by the d-stable ideal generated by the elements

$${}^V(t^Fu) - {}^Vtu, \quad t \in W_n\Omega^i_{S/R}, \quad u \in \bar{\Omega}^i_{n+1}. \tag{2.37}$$

The formulae (2.30) and (2.31) show that this element is annihilated by ω_n. We have to verify that $\mathrm{d}({}^V(t^Fu) - {}^Vtu)$ is annihilated by ω_n. Using (2.36), we obtain

$$\begin{aligned} &\omega_n(\mathrm{d}({}^V(t^Fu) - {}^Vtu)) \\ &\quad = \omega_{n-1}(\mathrm{d}(t^Fu)) - \omega_n(\mathrm{d}({}^Vtu)) \\ &\quad = \omega_{n-1}(\mathrm{d}t)\omega_{n-1}({}^Fu) + \omega_{n-1}(t)\omega_{n-1}(\mathrm{d}^Fu) - \omega_n(\mathrm{d}^Vt)\omega_n(u) - \omega_n({}^Vt)\omega_n(\mathrm{d}u). \end{aligned}$$

This vanishes because the following relations hold:

$$\begin{aligned} \omega_n(\mathrm{d}^Vt) &= \omega_{n-1}(\mathrm{d}t), & \omega_{n-1}({}^Fu) &= \omega_n(u), \\ \omega_{n-1}(\mathrm{d}^Fu) &= p\omega_n(\mathrm{d}u), & \omega_n({}^Vt) &= p\omega_{n-1}(t), \end{aligned}$$

by (2.31), (2.29), (2.28), respectively. This proves Proposition 2.15. □

We note that (2.31) may be written as

$$\underline{\omega}^{n+1}(\mathrm{d}^Vu) = [0, \omega_0(\mathrm{d}u), \ldots, \omega_{n-1}(\mathrm{d}u)] \quad \text{for } u \in W_n\Omega_{S/R}.$$

By Proposition 2.15, the map ω_n defined by (2.26) factors through

$$\omega_n : W_{n+1}\Omega^{\cdot}_{S/R} \to \Omega^{\cdot}_{S/R,\boldsymbol{w}_n}.$$

This is an algebra homomorphism which satisfies

$$\mathrm{d}\omega_n = p^n\omega_n\mathrm{d}.$$

Proposition 2.16. *Let $e = e(\xi, k, I_0, \dots, I_\ell) \in W\Omega^\ell_{S/R}$ be a basic Witt differential. Then $\omega_n(e) = 0$ unless $p^n \cdot k$ is integral. If $p^n k$ is integral, we have*

$$\omega_n(e) = \boldsymbol{w}_n(\xi) T^{p^n k_{I_0}} (p^{-\operatorname{ord} p^n k_{I_1}} \mathrm{d}T^{p^n k_{I_1}}) \cdots (p^{-\operatorname{ord} p^n k_{I_\ell}} \mathrm{d}T^{p^n k_{I_\ell}})$$

if $I_0 \neq \emptyset$ or if k is integral, and

$$\omega_n(e) = \boldsymbol{w}_{n-u}(\eta) (p^{-\operatorname{ord} p^n k_{I_1}} \mathrm{d}T^{p^n k_{I_1}}) \cdots (p^{-\operatorname{ord} p^n k_{I_\ell}} \mathrm{d}T^{p^n k_{I_\ell}}),$$

with ${}^{V^u}\eta = \xi$, if $I_0 = \emptyset$.

Proof. If k is a weight with support I and $u = u(k_I)$, we find, by (2.30),

$$\begin{aligned}\omega_n({}^{V^u}(\eta X^{p^u k})) &= p^u \omega_{n-u}(\eta X^{p^u k}) \\ &= p^u \boldsymbol{w}_{n-u}(\eta)(T^{p^u k})^{p^{n-u}} \\ &= \boldsymbol{w}_n({}^{V^u}\eta) T^{p^n \cdot k}.\end{aligned}$$

We note that the last expression is 0 for $u > n$.

Next we find

$$\omega_n(\mathrm{d}{}^{V^u}(\eta X^{p^u \cdot k})) = \delta_n({}^{V^u}(\eta X^{p^u k})).$$

We note that this is zero for $u > n$. For $u \leqslant n$, we obtain, for the last expression,

$$\begin{aligned}\delta_{n-u}(\eta X^{p^u \cdot k_I}) &= \boldsymbol{w}_{n-u}(\eta) T^{(p^u \cdot k_I)(p^{n-u}-1)} \mathrm{d}T^{p^u k_I} \\ &= \boldsymbol{w}_{n-u}(\eta)(p^{-n+u} \mathrm{d}T^{p^n \cdot k_I}) \\ &= \boldsymbol{w}_{n-u}(\eta)(p^{-\operatorname{ord} p^n k_I} \mathrm{d}T^{p^n k_I}).\end{aligned}$$

Finally, we consider an element ${}^{F^{-t}}\mathrm{d}X^{p^t k}$, where $t \leqslant 0$ and $p^t k$ is integral but not divisible by p,

$$\begin{aligned}\omega_n({}^{F^{-t}}\mathrm{d}X^{p^t k}) &= \omega_{n-t}(\mathrm{d}X^{p^t k}) \\ &= T^{p^t \cdot k(p^{n-t}-1)} \mathrm{d}T^{p^t k} \\ &= p^{-n+t} \mathrm{d}T^{p^n k} \\ &= p^{-\operatorname{ord} p^n k} \mathrm{d}T^{p^n k}.\end{aligned}$$

We obtain the proposition by multiplying these results together. □

2.5. The independence of basic Witt differentials

Let us denote by $e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ the image of $e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ in $W_n\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$. This element depends only on the residue class of $\xi_{k,\mathcal{P}}$ in $W_n(R)$. Let $p^{u(k)}$ be the common denominator of the values of k as before. If $k \equiv 0$, we set $u(k) = 0$. By definition of the basic Witt differentials, we have

$$\xi_{k,\mathcal{P}} \in {}^{V^{u(k)}}W_{n-u(k)}(R). \tag{2.38}$$

For $n \leqslant u(k)$, this should be read $\xi_{k,\mathcal{P}} = 0$, i.e. the elements $e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ are non-zero only if $p^{(n-1)} \cdot k$ is integral.

Proposition 2.17. *We assume that R is a $\mathbb{Z}_{(p)}$-algebra. Then any element ω of $W_n\Omega_{R[T_1,\dots,T_d]}$ may be written as a finite sum*

$$\omega = \sum_{k,\mathcal{P}} e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P}), \quad \xi_{k,\mathcal{P}} \in {}^{V^{u(k)}}W_{n-u(k)}(R). \tag{2.39}$$

Here, k runs over all weights such that $p^{n-1}\cdot k$ is integral. The coefficients $\xi_{k,\mathcal{P}}$ are uniquely determined by ω.

The kernel of the map

$$W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R} \to W_n\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$$

consists of a convergent series of basic Witt differentials $e(\xi,k,\mathcal{P})$ with $\xi \in {}^{V^n}W(R)$. In particular, this kernel is

$${}^{V^n}W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R} + \mathrm{d}^{V^n}W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}.$$

Proof. Clearly, an element $e(\xi,k,\mathcal{P})$ with $\xi \in {}^{V^n}W(R)$ maps to zero in $W_n\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$. Therefore, by Lemma 2.12, any ω may be written in the form (2.39). Let us first do the case where R has no p-torsion. Assume that we are given an expression (2.39) with $\omega = 0$. We want to show that $\xi_{k,\mathcal{P}} = 0$ for all $k,\mathcal{P}$. For this, we consider the map

$$\omega_m : W_n\Omega^{\cdot}_{R[T_1,\dots,T_d]/R} \to \Omega^{\cdot}_{R[T_1,\dots,T_d]/R,\boldsymbol{w}_m}$$

for $m = 0,\dots,n-1$. Proposition 2.16 shows that

$$\boldsymbol{w}_m(\xi_{k,\mathcal{P}}) = 0 \quad \text{for } m = 0,\dots,n-1,$$

since the basic Witt differentials in $\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$ are linearly independent by Proposition 2.1. Since R is p-torsion free, this implies $\xi_{k,\mathcal{P}} = 0$. This implies Theorem 2.8 in the case where R has no p-torsion. Then the assertion that the kernel in the corollary is generated by $e(\xi,k,\mathcal{P})$ with $\xi \in {}^{V^n}W(R)$ is clear. We set $\xi = {}^{V^n}\eta$. If the partition $\mathcal{P} = \{I_0,\dots,I_\ell\}$ considered has $I_0 \neq \emptyset$, we conclude that $e(\xi,k,\mathcal{P}) \in {}^{V^n}W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}$ by Proposition 2.5 (3). If $I_0 = \emptyset$ and k is integral, we find, by the same proposition,

$${}^{V^n}e(\eta, p^n k, \emptyset, I_1,\dots,I_\ell) = e(\xi,k,\emptyset,I_1,\dots,I_\ell).$$

If $I_0 = \emptyset$ and k is not integral, we apply Proposition 2.6,

$$\mathrm{d}e({}^{V^n}\eta, k, I_1,\dots,I_\ell) = e(\xi,k,\emptyset,I_1,\dots,I_\ell).$$

We have already seen that

$$e({}^{V^n}\eta, k, I_1,\dots,I_\ell) \in {}^{V^n}W\Omega^{\cdot}_{R[T_1,\dots,T_d]/R}.$$

This proves the proposition if R has no p-torsion.

Corollary 2.18. *Assume that R has no p-torsion. Then the natural map*

$$\underline{\omega} : W_n\Omega^{\cdot}_{R[T_1,\ldots,T_d]/R} \to \bigoplus_{m=0}^{n-1} \Omega^{\cdot}_{R[T_1,\ldots,T_d]/R,\boldsymbol{w}_m}$$

is injective.

We return now to the proof of the proposition, where R is arbitrary. We write $S = V_R[T_1,\ldots,T_d]$. The ring R may be represented in the form $R = \tilde{R}/\mathfrak{a}$, where $\tilde{R}$ is a ring without p-torsion. We set $\tilde{S} = \tilde{R}[T_1,\ldots,T_d]$. We consider the subgroup

$$W\Omega^{\cdot}_{\mathfrak{a}\tilde{S}/\tilde{R}} \subset W\Omega^{\cdot}_{\tilde{S}/\tilde{R}},$$

comprising convergent sums of basic Witt differentials $e(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ with $\xi_{k,\mathcal{P}} \in W(\mathfrak{a})$. From the proof of Lemma 2.12 and from Proposition 2.11, it follows that $W\Omega_{\mathfrak{a}\tilde{S}/R}$ is an ideal of the algebra $W\Omega_{S/R}$, which is invariant by F, V, d. We define a complex E as the quotient,

$$0 \to W\Omega^{\cdot}_{\mathfrak{a}\tilde{S}/\tilde{R}} \to W\Omega^{\cdot}_{\tilde{S}/\tilde{R}} \to E \to 0. \tag{2.40}$$

Then we have $E^0 = W(S)$. If we consider the exact sequence (2.40) for the truncated Witt vectors, we see that E is an F-V-procomplex over the R-algebra S. Therefore, we obtain a homomorphism

$$W\Omega^{\cdot}_{S/R} \to E$$

of F-V-procomplexes such that the following diagram is commutative:

$$\begin{tikzcd} W\Omega^{\cdot}_{S/R} \arrow[rr] & & E \\ & W\Omega^{\cdot}_{\tilde{S}/\tilde{R}} \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{2.41}$$

By the torsion-free case, any element $\omega \in E_n$ has a unique expression (2.39). By Lemma 2.12 and diagram (2.41), we conclude that the same holds for $W_n\Omega^{\cdot}_{S/R}$. The other assertions of the proposition follow formally as in the torsion-free case. This also completes the proof of Theorem 2.8. □

2.6. The filtration

In this section we extend the last statement of Proposition 2.17 to an arbitrary smooth R-algebra.

Let R be a ring such that p is nilpotent in R, or assume that R is F-finite. Let S be a smooth R-algebra.

Proposition 2.19. *Let n be a number. The kernel* Fil^n *of the map*

$$W\Omega^{\cdot}_{S/R} \to W_n\Omega^{\cdot}_{S/R} \tag{2.42}$$

is the subcomplex

$$V^n W\Omega^{\cdot}_{S/R} + \mathrm{d}V^n W\Omega^{\cdot}_{S/R}. \tag{2.43}$$

Proof. We begin with the case where S is étale over a polynomial algebra $S_0 = R[T_1, \ldots, T_d]$. Then we have the base change isomorphism,

$$W_m\Omega^{\cdot}_{S/R} \to W_m(S) \otimes_{W_m(S_0)} W_m\Omega^{\cdot}_{S_0/R}.$$

We denote by $\overline{\mathrm{Fil}}^m$ the kernel of the obvious map,

$$W(S) \otimes_{W(S_0)} W\Omega^{\cdot}_{S_0/R} \to W_m(S) \otimes_{W_m(S_0)} W_m\Omega^{\cdot}_{S_0/R}. \tag{2.44}$$

The completion of the left-hand side in the linear topology defined by the ideals $\overline{\mathrm{Fil}}^m$ will be denoted by $W(S)\hat{\otimes}_{W(S_0)}W\Omega^{\cdot}_{S_0/R}$. This is identified with $W\Omega^{\cdot}_{S/R}$ by base change. Then Fil^m is the completion of $\overline{\mathrm{Fil}}^m$. (We do not claim that this topology is separated.)

We claim that any element $\theta \in \mathrm{Fil}^n$ is of the form

$$^{V^n}\theta_1 + \mathrm{d}^{V^n}\theta_2, \quad \theta_1, \theta_2 \in W(S)\hat{\otimes}_{W(S_0)}W\Omega^{\cdot}_{S_0/R}.$$

Let us consider the case where θ is in the image of the canonical map $\overline{\mathrm{Fil}}^n \to \mathrm{Fil}^n$. We can compute the kernel of (2.44) by Proposition 2.17. This shows that θ is a sum of elements of the form

$$^{V^n}\xi \otimes \omega, \qquad \xi \otimes {}^{V^n}\omega, \qquad \xi \otimes \mathrm{d}^{V^n}\omega, \tag{2.45}$$

where $\xi \in W(S)$ and $\omega \in W\Omega^{\cdot}_{S_0/R}$. By the F-V formula, the elements of (2.45) may be rewritten as

$$^{V^n}(\xi \otimes {}^{F^n}\omega), \qquad ^{V^n}({}^{F^n}\xi \otimes \omega), \qquad \mathrm{d}^{V^n}({}^{F^n}\xi \otimes \omega).$$

This settles the case where θ is in the image of $\overline{\mathrm{Fil}}^n$.

Now we consider an arbitrary $\theta \in \mathrm{Fil}^n$. Then we find an element $\theta^{(n+1)}$ in the image of the map

$$W(S) \otimes_{W(S_0)} W\Omega^{\cdot}_{S_0/R} \to W(S)\hat{\otimes}_{W(S_0)}W\Omega^{\cdot}_{S_0/R} \tag{2.46}$$

such that $\theta - \theta^{(n+1)} \in \mathrm{Fil}^{n+1}$. Then we have that $\theta^{(n+1)}$ is in the image of $\overline{\mathrm{Fil}}^n$, since $\overline{\mathrm{Fil}}^n$ is the preimage of Fil^n by the map (2.46). Hence there exists a representation in $W(S)\hat{\otimes}_{W(S_0)}W\Omega^{\cdot}_{S_0/R}$,

$$\theta^{(n+1)} = {}^{V^n}\theta_1^{(n+1)} + \mathrm{d}^{V^n}\theta_2^{(n+1)}.$$

Inductively, we obtain elements $\theta^{(m)}$ in the image of (2.46) such that

$$\theta - \theta^{(n+1)} - \cdots - \theta^{(m+1)} \in \mathrm{Fil}^{(m+1)}.$$

This implies that $\theta^{(m+1)}$ is in the image of $\overline{\mathrm{Fil}}^m$, and therefore has the representation

$$\theta^{(m+1)} = {}^{V^m}\theta_1^{(m+1)} + \mathrm{d}^{V^m}\theta_2^{(m+1)}.$$

This yields the desired representation of θ,

$$\theta = {}^{V^n}\Biggl(\sum_{m>n} {}^{V^{m-1-n}}\theta_1^{(m)}\Biggr) + \mathrm{d}^{V^n}\Biggl(\sum_{m>n} {}^{V^{m-1-n}}\theta_2^{(m)}\Biggr).$$

This proves the result if S is étale over a polynomial algebra.

Finally, let S be an arbitrary smooth algebra. Then we consider the assertion at a finite level, i.e. we want to show that the following map is surjective:

$$V^n W_m \Omega^{\cdot}_{S/R} \oplus \mathrm{d}V^n W_m \Omega^{\cdot}_{S/R} \to \mathrm{Ker}(W_{m+n}\Omega^{\cdot}_{S/R} \to W_n \Omega^{\cdot}_{S/R}). \tag{2.47}$$

We remark that, by base change, all $W(S)$-modules involved in this map are compatible with localizations, e.g. $(W_n \Omega^{\cdot}_{S/R})_{[f]} \cong W_n \Omega^{\cdot}_{S_f/R}$. Therefore, it suffices to find elements $f_1, \ldots, f_s \in S$ that generate the unit ideal such that (2.47) becomes an isomorphism after localization with each Teichmüller representative $[f_i]$. But this is true if S_{f_i} is étale over a polynomial algebra. □

2.7. The Cartier–Raynaud ring

Let us consider the set $\mathbb{D}^0_R$, which consists of the following finite sums:

$$\sum_{n \geqslant 0} V^n \xi_n + \sum_{n>0} \eta_n F^n + \sum_{n \geqslant 0} \mathrm{d}V^n \xi'_n + \sum_{n>0} \eta'_n F^n \mathrm{d}. \tag{2.48}$$

Here, $\xi_n, \xi'_n, \eta_n, \eta'_n \in W(R)$ are arbitrary elements, which are almost all zero. The letters F, V, d denote indeterminates. We consider $\mathbb{D}^0_R$ as an abelian group which is isomorphic to a direct sum of copies of $W(R)$ with components ξ_n, ξ'_n, η_n, η'_n. Obviously, there is a unique ring structure on $\mathbb{D}^0_R$ which obeys the following rules:

$$\left.\begin{gathered} FV = p = V^0 p, \quad V\xi F = {}^V\xi \quad \text{for } \xi \in W(R), \\ F\xi = {}^F\nu F, \quad \xi V = V^F\nu, \\ \mathrm{d}\xi = \xi\mathrm{d}, \quad \mathrm{d}^2 = 0, \\ F\mathrm{d}V = \mathrm{d}, \quad V\mathrm{d} = \mathrm{d}Vp, \quad \mathrm{d}F = pF\mathrm{d}. \end{gathered}\right\} \tag{2.49}$$

For each number c, let us consider the right ideal $\mathbb{D}^0_R(c) = V^c \mathbb{D}^0_R + \mathrm{d}V^c \mathbb{D}^0_R$.

Lemma 2.20. *The right ideal $\mathbb{D}^0_R(c)$ consists of the elements (2.48) which satisfy the following conditions:*

$$\left.\begin{aligned} &\xi_n, \xi'_n \in {}^{V^{c-n}}W(R) && \text{for } c > n, \\ &\eta_n, \eta'_n \in {}^{V^c}W(R) && \text{for } n > 0. \end{aligned}\right\} \tag{2.50}$$

Proof. Let us denote the abelian group defined by (2.50) by $B(c)$. Consider an element (2.48) which belongs to $B(c)$. For $n < c$, we obtain

$$V^n \xi_n = V^{n\,V^{c-n}}\rho = V^n V^{c-n} \rho F^{n-c} \in V^c \mathbb{D}^0_R.$$

Here, ρ exists by the definition (2.50) of $B(c)$. The same consideration shows that all summands of (2.48) are in $\mathbb{D}^0_R(c)$. For the inverse inclusion $\mathbb{D}^0_R(c) \subset B(c)$, we apply consecutively V^c and then $\mathrm{d}V^c$ to an arbitrary element of the form (2.48). We have to show that the result is in $B(c)$. Since $\mathrm{d}B(c) \subset B(c)$, it is enough to look for the effect of V^c. If we apply V^c to the summand $\eta_m F^m$, we obtain, for $c \geqslant m$,

$$V^c \eta_m F^m = V^{c-m\,V^m}\eta_m \in B(c).$$

For $c < m$, we obtain

$$V^c \eta_m F^m = {}^{V^c}\eta_m F^{m-c} \in B(c).$$

The rest of the proof is done using the same argument. $\square$

The filtration by the right ideals $\mathbb{D}_R^0(c)$ defines a topology on $\mathbb{D}_R^0$. We call this the canonical topology. The next lemma implies that the ring multiplication is continuous for the canonical topology.

Lemma 2.21. *Let c be a number and $\alpha \in \mathbb{D}_R^0$ be an element. Then there is a number c' such that*

$$\alpha \mathbb{D}_R^0(c') \subset \mathbb{D}_R^0(c).$$

Proof. We may restrict to the case where α is just one summand of (2.48). We omit the straightforward verification. $\square$

Definition 2.22. The Cartier–Raynaud ring $\mathbb{D}_R$ is the completion of $\mathbb{D}_R^0$ with respect to the canonical topology

$$\mathbb{D}_R = \varprojlim_c \mathbb{D}_R^0/\mathbb{D}_R^0(c).$$

Indeed, $\mathbb{D}_R$ inherits a ring structure from $\mathbb{D}_R^0$ by the last lemma.

Any element of $\mathbb{D}_R$ may be written uniquely as a convergent sum,

$$\sum_{n \geqslant 0} V^n \xi_n + \sum_{n>0} \eta_n F^n + \sum_{n \geqslant 0} \mathrm{d} V^n \xi'_n + \sum_{n>0} \eta'_n F^n \mathrm{d}. \tag{2.51}$$

Here, ξ_n, ξ'_n for $n \geqslant 0$ and η_n, η'_n for $n > 0$ are any elements which satisfy the following condition.

For any given number $u > 0$, we have $\eta_n, \eta'_n \in {}^{V^u}W(R)$ for almost all $n > 0$.

The subring of $\mathbb{D}_R$ which consists of all sums

$$\sum_{n \geqslant 0} V^n \xi_n + \sum_{n>0} \eta_n F^n$$

is the Cartier ring $\mathbb{E}_R$. We denote by $\vartheta_R \subset \mathbb{D}_R$ the two-sided ideal generated by d. One checks easily that $\vartheta_R^2 = 0$. We have a direct decomposition,

$$\mathbb{D}_R = \mathbb{E}_R \oplus \vartheta_R.$$

We consider the Witt ring $W(R)$ as a $\mathbb{D}_R$-left module by the following rules:

$$V\rho = {}^V\rho, \quad F\rho = {}^F\rho, \quad \mathrm{d}\rho = 0 \quad \text{for } \rho \in W(R).$$

The subring $W(R) \subset \mathbb{D}_R$ acts on $W(R)$ by the natural multiplication.

Lemma 2.23. *The $\mathbb{D}_R$-module homomorphism*

$$\mathbb{D}_R/\mathbb{D}_R(F-1) + \mathbb{D}_R \mathrm{d} \to W(R)$$

which maps 1 to 1 is an isomorphism. If $R \to R'$ is a ring homomorphism, we have the natural isomorphism of $\mathbb{D}_{R'}$-modules,

$$\begin{aligned} \mathbb{D}_{R'} \otimes_{\mathbb{D}_R} W(R) &\cong W(R'), \\ \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W(R) &\cong W_c(R'). \end{aligned} \tag{2.52}$$

Proof. It is clear that the first isomorphism implies the other second. Then we also obtain the third, since, obviously,

$$\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_{R'}} W(R') \cong W_c(R').$$

If we consider $W(R)$ as an $\mathbb{E}_R$-left module, we have, by Cartier theory and isomorphism,

$$\mathbb{E}_R/\mathbb{E}_R(F-1) \cong W(R). \tag{2.53}$$

Therefore, it suffices to show that the two-sided ideal ϑ_R is contained in the left ideal $\mathbb{D}_R(F-1) + \mathbb{D}_R\mathrm{d}$. By (2.53), we have the congruence

$$\sum_n V^n \xi'_n = \sum {}^{V^n}\xi'_n \mod \mathbb{D}_R(F-1).$$

If we multiply the congruence with d, we obtain the result, since the element on the right-hand side commutes with d. $\square$

Let S be an R-algebra and consider the completed de Rham–Witt complex $W\Omega^{\cdot}_{S/R}$. We extend the action of $W(R)$ on this complex to an action of $\mathbb{D}^0_R$ by setting

$$V\omega = {}^V\omega, \quad F\omega = {}^F\omega \, \mathrm{d}\omega = \mathrm{d}\omega \quad \text{for } \omega \in W\Omega_{S/R}.$$

If $\xi \in W(R)$, then the projection of the elements ${}^{V^c}\xi$, $\mathrm{d}^{V^c}\xi \in W\Omega^{\cdot}_{S/R}$ to the complex $W_c\Omega^{\cdot}_{S/R}$ are zero. It follows that, for any $\alpha \in \mathbb{D}_R(c)$, the projection of $\alpha\omega$ to $W_c\Omega^{\cdot}_{S/R}$ is zero.

Let us fix a number c and an arbitrary element $\alpha \in \mathbb{D}^0_R$. It is clear that there is a number c' such that the action of α on $W\Omega^{\cdot}_{S/R}$ factors through $\alpha : W_{c'}\Omega^{\cdot}_{S/R} \to W_c\Omega^{\cdot}_{S/R}$. Moreover, we have just shown that any element in $\alpha + \mathbb{D}^0_R(c)$ has the same factorization with the same c'.

This shows that the action of $\mathbb{D}^0_R$ extends to an action of the Cartier–Raynaud algebra on the completed de Rham–Witt complex $W\Omega^{\cdot}_{S/R}$.

We consider now the case of the polynomial algebra $S = R[T_1, \ldots, T_d]$. The structure theorem for the de Rham–Witt complex as formulated in Proposition 2.17 and the formulae for the action of V, F, and d on the basic Witt differentials given in Propositions 2.5 and 2.6 show the following:

$$W_c\Omega^{\cdot}_{S/R} = W_c(R) \oplus \bigoplus_{\substack{k \text{ primitive} \\ \mathcal{P}}} \mathbb{D}_R/\mathbb{D}_R(c)e(1,k,\mathcal{P}). \tag{2.54}$$

The sum runs for each primitive weight k over all partitions of $\mathcal{P} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_l$ of $\operatorname{Supp} k$ such that I_0 is not empty. Moreover, we have already shown that

$$\mathbb{D}_R/\mathbb{D}_R(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R} \cong W\Omega^{\cdot}_{S/R}/V^c W\Omega^{\cdot}_{S/R} + \mathrm{d}V^c W\Omega^{\cdot}_{S/R} \cong W_c\Omega^{\cdot}_{S/R}. \tag{2.55}$$

In the completed form these results say the following.

Theorem 2.24. *Let $S = R[T_1, \ldots, T_d]$ be the polynomial ring. Each element of $W\Omega^{\cdot}_{S/R}$ has a unique expression*

$$\xi + \sum_{k,\mathcal{P}} \theta_{k,\mathcal{P}} e(1, k, \mathcal{P}).$$

Here, $\xi \in W(R)$ is regarded as an element of $W\Omega^0_{S/R} = W(S)$. The sum runs over all primitive weights and partitions as above. The elements $\theta_{k,\mathcal{P}} \in \mathbb{D}_R$ satisfy the following condition.

Let $c > 0$ be an arbitrary integer. Then, for almost all primitive weights k, we have $\theta_{k,\mathcal{P}} \in \mathbb{D}_R(c)$.

From this theorem, we obtain a base change property, which is similar to base change in Cartier theory.

Theorem 2.25. *Let R be a ring such that p is nilpotent in R, or assume that R is F-finite. Let S be a smooth algebra over R. Let R' be an arbitrary R-algebra. We set $S' = R' \otimes_R S$. Then we have a canonical isomorphism,*

$$\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R} \cong W_c\Omega^{\cdot}_{S'/R'}.$$

Proof. By the universal property of the de Rham–Witt complex, we have a canonical map

$$W\Omega^{\cdot}_{S/R} \to W\Omega^{\cdot}_{S'/R'}.$$

From this, we obtain a map $\mathbb{D}_{R'} \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R} \to W\Omega^{\cdot}_{S'/R'}$. The last map factors through

$$\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R} \to W_c\Omega^{\cdot}_{S'/R'}. \tag{2.56}$$

More precisely, we claim that this map is an isomorphism.

We begin with the case where S is a polynomial algebra over R. By (2.54), any element in $W_c\Omega_{S'/R'}$ has a unique expression as a finite sum,

$$\xi' + \sum \theta'_{k,\mathcal{P}} e(1, k, \mathcal{P}). \tag{2.57}$$

We denote by $\tau(\xi')$ the image of ξ' by the canonical map induced by (2.52),

$$\xi' \in W_c(R') \to \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W_c(R) \to \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R}.$$

Then we can define a map inverse to (2.56). It maps (2.57) to the element

$$\tau(\xi') + \sum \theta'_{k,\mathcal{P}} \otimes e(1, k, \mathcal{P}).$$

This proves the result for a polynomial algebra.

We now consider the case where S is étale over a polynomial algebra $S_0 = R[T_1, \dots, T_d]$. We set $S'_0 = R'[T_1, \dots, T_d]$. Then S' is étale over S'_0 and we have $S' = S'_0 \otimes_{S_0} S$. For the Witt rings, we obtain, by the appendix, the isomorphism $W(S') = W(S'_0) \otimes_{W(S_0)} W(S)$. We set

$$W(S)\hat{\otimes}_{W(S_0)} W\Omega^{\cdot}_{S_0/R} = \varprojlim_n W(S) \otimes_{W(S_0)} W_n\Omega^{\cdot}_{S_0/R}.$$

By base change, this group identifies with $W\Omega^{\cdot}_{S/R}$ and is therefore a $\mathbb{D}_R$-module. Hence we may rewrite the left-hand side of (2.56) as

$$\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W(S)\hat{\otimes}_{W(S_0)} W\Omega^{\cdot}_{S_0/R}.$$

We now rewrite the right-hand side of (2.56). By étale base change, we have an isomorphism,

$$W_c\Omega^{\cdot}_{S'/R'} = W(S') \otimes_{W(S'_0)} W_c\Omega_{S'_0/R'} = W(S) \otimes_{W(S_0)} W_c\Omega^{\cdot}_{S'_0/R'}.$$

If we apply to the last complex the base change for a polynomial algebra, we obtain that the right-hand side of (2.56) identifies with $W(S) \otimes_{W(S_0)} (\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_0/R})$. The $W(S_0)$-module structure on $\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_0/R}$ can be made explicit. For $\rho_0 \in W(S_0)$, $\xi \in W(R')$ and $\omega \in W\Omega^{\cdot}_{S_0/R}$, we have the following formulae:

$$\left.\begin{aligned} \rho_0(V^n\xi \otimes \omega) &= V^n\xi \otimes {}^{F^n}\rho_0\omega, \\ \rho_0(\mathrm{d}V^n\xi \otimes \omega) &= \mathrm{d}V^n\xi \otimes {}^{F^n}\rho_0\omega, \\ \rho_0(\xi F^n \otimes \omega) &= \xi \otimes \rho_0{}^{F^n}\omega, \\ \rho_0(\xi F^n\mathrm{d} \otimes \omega) &= \xi \otimes \rho_0{}^{F^n}\mathrm{d}\omega. \end{aligned}\right\} \tag{2.58}$$

Now we can rewrite the base change homomorphism (2.56) as follows:

$$\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} (W(S)\hat{\otimes}_{W(S_0)} W\Omega^{\cdot}_{S_0/R}) \to W(S) \otimes_{W(S_0)} (\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_0/R}). \tag{2.59}$$

An inverse map to (2.59) is given by the formulae. Let $\rho \in W(S)$, $\xi \in W(R')$ and $\omega \in W\Omega^{\cdot}_{S_0/R}$. Then we define

$$\left.\begin{aligned} \rho \otimes V^n\xi\omega &\mapsto V^n\xi \otimes {}^{F^n}\rho \otimes \omega, \\ \rho \otimes \mathrm{d}V^n\xi\omega &\mapsto \mathrm{d}V^n\xi \otimes {}^{F^n}\rho \otimes \omega, \\ \rho \otimes \xi F^n \otimes \omega &\mapsto \xi \otimes \rho \otimes {}^{F^n}\omega, \\ \rho \otimes \xi F^n\mathrm{d} \otimes \omega &\mapsto \xi \otimes \rho \otimes {}^{F^n}\mathrm{d}\omega. \end{aligned}\right\} \tag{2.60}$$

To see that this map is inverse to (2.59), we make (2.59) more explicit. We begin with the following remark. Fix an element $\theta \in \mathbb{D}_{R'}/\mathbb{D}_{R'}(c)$. Then there is a number c' such that, for any element α in the kernel of the map

$$W(S)\hat{\otimes}_{W(S_0)} W\Omega^{\cdot}_{S_0/R} \to W(S) \otimes_{W(S_0)} W_{c'}\Omega^{\cdot}_{S_0/R},$$

we have $\theta \otimes \alpha = 0$. Indeed, this follows, since, by Proposition 2.19, any element in this kernel is of the form $\alpha = V^{c'}\alpha_1 + \mathrm{d}V^{c'}\alpha_2$. Therefore, it is enough to see the effect of (2.59) on elements, which may be written in the form $\theta \otimes \rho \otimes \omega$ with $\theta \in \mathbb{D}_{R'}/\mathbb{D}_{R'}(c)$, $\rho \in W(S)$ and $\omega \in W\Omega^{\cdot}_{S_0/R}$. Moreover, we may assume that θ is an element of the following form, $V^n\xi$, $\mathrm{d}V^n\xi$, ξF^n or $\xi F^n\mathrm{d}$, where n is an arbitrary number and $\xi \in W(R')$. By Corollaries A.11 and A.18, we have the isomorphism

$$W_{c+n}(S) \otimes_{W_{c+n}(S_0),F^n} W_c(S_0) \cong W_c(S).$$

Therefore, the element ρ may be expressed as follows:

$$\rho \sum_i {}^{F^n}\rho_i\sigma_i + {}^{V^c}\rho', \quad \rho_i, \rho' \in W(S), \quad \sigma_i \in W(S_0).$$

Then the effect of (2.59) is

$$\left.\begin{aligned} V^n\xi \otimes \rho \otimes \omega &\mapsto \sum_i \rho_i \otimes V^n\xi \otimes \sigma_i\omega, \\ \mathrm{d}V^n\xi \otimes \rho \otimes \omega &\mapsto \sum_i \rho_i \otimes \mathrm{d}V^n\xi \otimes \sigma_i\omega. \end{aligned}\right\} \tag{2.61}$$

For the remaining cases, the effect is defined as follows:

$$\left.\begin{aligned} \xi F^n \otimes \rho \otimes \omega &\mapsto {}^{F^n}\rho \otimes \xi F^n \otimes \omega, \\ \xi F^n\mathrm{d} \otimes \rho \otimes \omega &\mapsto {}^{F^n}\rho \otimes \xi F^n\mathrm{d} \otimes \omega. \end{aligned}\right\} \tag{2.62}$$

That these formulae coincide with the definition of (2.59) is obvious if we identify the right-hand side of (2.59) with $W_c\Omega_{S'/R'}$. Finally, these formulae show that (2.60) is an inverse map. This proves the base change in the case where S is étale over the polynomial algebra S_0.

Let S be an arbitrary smooth algebra over R. First, we will see that the question of whether (2.56) is an isomorphism is local for the Zariski-topology on $\operatorname{Spec} S$. Let $f \in S$ be an element and $[f] \in W(S)$ be its Teichmüller representative. We will show that the localization of (2.56) by $[f]$ coincides with the base change map for S_f/R, if S_f is étale over a polynomial algebra over R.

We know that the right-hand side of (2.56) is compatible with localization,

$$(W_c\Omega^{\cdot}_{S'/R'})_{[f]} \cong W_c\Omega^{\cdot}_{S'_f/R'}.$$

We have to prove the same thing for the left-hand side of (2.56). This means that the following natural map is an isomorphism:

$$(\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R})_{[f]} \to \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_f/R}.$$

This map is defined because $[f]$ acts bijectively on the right-hand side. Indeed, by what we have shown, the right-hand side is canonically isomorphic to $W\Omega^{\cdot}_{S'_f/R'}$.

We define the inverse map. Consider an element $\theta \otimes \alpha \in \mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_f/R}$. We know that there is an index c' depending on θ such that $\theta \otimes \alpha = 0$ whenever α is in the kernel of the map

$$W\Omega^{\cdot}_{S_f/R} \to W_{c'}\Omega^{\cdot}_{S_f/R} \cong (W_{c'}\Omega^{\cdot}_{S/R})_{[f]}.$$

We choose $\beta \in (W\Omega^{\cdot}_{S/R})_{[f]}$ with the same image in $(W_{c'}\Omega^{\cdot}_{S/R})_{[f]}$ as α. Hence we may write $\theta\otimes\alpha$ in the form $\theta\otimes[f]^{-m}\omega$ for some number m and some $\omega \in W\Omega^{\cdot}_{S/R}$. We consider separately the cases where θ is $V^n\xi$, $\mathrm{d}V^n\xi$, ξF^n and $\xi F^n\mathrm{d}$. Then we find the following relation in $\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_f/R}$:

$$[f]^m(V^n\xi \otimes [f]^{-m}\omega) = V^n\xi \otimes [f]^{m(p^n-1)}\omega.$$

Hence we map $V^n\xi \otimes [f]^{-m}\omega$ to

$$[f]^{-m}(V^n\xi \otimes [f]^{m(p^n-1)}\omega) \in (\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S/R})_{[f]}.$$

If $\theta = \mathrm{d}V^n\xi$, we proceed in the same way. An element of the form $\xi F^n \otimes [f]^{-m}\omega$ is mapped to $[f]^{-mp^n}(\xi F^n \otimes \omega)$ and finally $\xi F^n\mathrm{d} \otimes [f]^{-m}\omega$ to $[f]^{-mp^n}(\xi F^n\mathrm{d} \otimes \omega)$. One checks that these definitions are bilinear and therefore give a well-defined map on the tensor product $\mathbb{D}_{R'}/\mathbb{D}_{R'}(c) \otimes_{\mathbb{D}_R} W\Omega^{\cdot}_{S_f/R}$. This completes the proof of the theorem. □

3. The comparison to crystalline cohomology

3.1. Liftings over the Witt vectors

Let R be a ring such that p is nilpotent in R. Let X be a smooth scheme over R. We consider, for a fixed number n, the crystalline topos $(X/W_n(R))_{\mathrm{crys}}$ with respect to the canonical divided powers on the kernel of $\boldsymbol{w}_0 : W_n(R) \to R$. Let $\mathcal{O}_{X/W_n(R)}$ be the structure sheaf on $(X/W_n(R))_{\mathrm{crys}}$ (see [**BO**] for the notation). In this section we prove the following result.

Theorem 3.1. *There is a canonical isomorphism*

$$H^i((X/W_n(R))_{\mathrm{crys}}, \mathcal{O}_{X/W_n(R)}) \cong \mathbb{H}^i(X, W_n\Omega^{\cdot}_{X/R}).$$

The right-hand side of this isomorphism is the hypercohomology of the de Rham–Witt complex with respect to the Zariski topology. To prove this, we use the fact that the crystalline cohomology on the left-hand side is the de Rham cohomology of a lifting of X to a smooth scheme Y over $W_n(R)$, provided a lifting exists. In this section we choose local liftings carefully.

Proposition 3.2. *Let p be nilpotent in R. Let A be a smooth R-algebra. Then, locally for the Zariski topology on* $\operatorname{Spec} A$, *the following set of data exists.*

(1) *For each number $n \geqslant 1$, a smooth lifting A_n over $W_n(R)$ of A, and isomorphisms $W_n(R) \otimes_{W_{n+1}(R)} A_{n+1} \cong A_n$, where $A_1 = A$.*

(2) *For each $n > 1$, a homomorphism $\phi_n : A_n \to A_{n-1}$, which is compatible with the Frobenius on the Witt ring $F : W_n(R) \to W_{n-1}(R)$, and with the absolute Frobenius* $\mathrm{Frob} : A/pA \to A/pA$.

(3) *For each $n \geqslant 1$, a homomorphism*

$$\delta_n : A_n \to W_n(A)$$

such that $\boldsymbol{w}_0\delta_n$ is the natural map $A_n \to A$ and such that the following diagrams commute:

$$\begin{tikzcd} A_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & W_{n+1}(A) \arrow[d] \\ A_n \arrow[r, "\delta_n"] & W_n(A) \end{tikzcd} \qquad \begin{tikzcd} A_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "\phi_{n+1}"'] & W_{n+1}(A) \arrow[d, "F"] \\ A_n \arrow[r, "\delta_n"] & W_n(A) \end{tikzcd}$$

We will call the system (A_n, ϕ_n, δ_n) a Frobenius lift of A to $W(R)$.

Proof. This is trivial if A is a polynomial algebra over R. Indeed, let $A = R[T_1, \dots, T_d]$. We set $A_n = W_n(R)[T_1, \dots, T_d]$. Then we extend the Frobenius $F : W_n(R) \to W_{n-1}(R)$ to a homomorphism,

$$\begin{aligned} F: \quad W_n(R)[T_1, \dots, T_d] &\to W_{n-1}(R)[T_1, \dots, T_d], \\ T_i &\mapsto T_i^p. \end{aligned}$$

Finally, δ_n is the $W_n(R)$-algebra homomorphism

$$\delta_n : W_n(R)[T_1, \dots, T_d] \to W_n(R[T_1, \dots, T_d]),$$

which maps T_i to its Teichmüller representative $[T_i]$. This meets all requirements of a Frobenius lift.

Since, locally, A is étale over a polynomial algebra, it suffices to prove the following. Let $A \to B$ be an étale homomorphism of R-algebras. Assume we are given a Frobenius lift (A_n, ϕ_n, δ_n) of A. Then there is a unique Frobenius lift $(B_n, \psi_n, \epsilon_n)$ of B such that $A \to B$ lifts to a homomorphism $(A_n, \phi_n, \delta_n) \to (B_n, \psi_n, \epsilon_n)$.

We obtain the Frobenius lift B_n as follows. Since the surjection $A_n \to A$ has nilpotent kernel, there is a unique étale A_n-algebra B_n which lifts B. Hence we obtain a projective system of liftings of B,

$$\begin{tikzcd} \cdots \arrow[r] & B_{n+1} \arrow[r] & B_n \arrow[r] & \cdots \arrow[r] & B_1 = B \\ \cdots \arrow[r] & A_{n+1} \arrow[r] \arrow[u] & A_n \arrow[r] \arrow[u] & \cdots \arrow[r] & A_1 = A \arrow[u] \end{tikzcd}$$

For the construction of ψ_n, we consider the étale A_{n-1}-algebra $B_n^* = B_n \otimes_{A_n, \phi_n} A_{n-1}$. Since ϕ_n lifts the absolute Frobenius on A/pA, we obtain the isomorphisms

$$\begin{aligned} B_n^* \otimes_{A_{n-1}} A/pA &\cong (B_n \otimes_{A_n} A/pA) \otimes_{A/pA, \mathrm{Frob}} A/pA \\ &\cong B/pB \otimes_{A/pA, \mathrm{Frob}} A/pA. \end{aligned} \tag{3.1}$$

Because B/pB is étale over A/pA, we have the isomorphism

$$\begin{aligned} B/pB \otimes_{A/pA, \mathrm{Frob}} A/pA &\cong B/pB, \\ b \otimes a &\mapsto b^p a. \end{aligned}$$

Therefore, B_n^* is a lifting of the étale A/pA-algebra B/pB with respect to the morphism $A_{n-1} \to A/pA$. Since B_{n-1} has the same property, there is a unique isomorphism of A_{n-1}-algebras $B_n^* \cong B_{n-1}$. This induces the desired morphism $\psi_n : B_n \to B_{n-1}$. It is the unique morphism compatible with ϕ_n.

The morphisms ϵ_n are obtained by the same kind of argument. The $W_n(A)$-algebra $B_n \otimes_{A_n,\delta_n} W_n(A)$ is étale and is a lifting of B with respect to the morphism $\boldsymbol{w}_0 : W_n(A) \to A$. By the appendix, the same is true for the étale $W_n(A)$-algebra $W_n(B)$. We obtain a canonical isomorphism,

$$B_n \otimes_{A_n,\delta_n} W_n(A) \cong W_n(B). \tag{3.2}$$

This provides the desired morphism $\epsilon_n : B_n \to W_n(B)$.

The isomorphism (3.2) above lifts the identity on B with respect to the morphism $W_n(A) \to A$. This shows that $\boldsymbol{w}_0\epsilon_n$ coincides with the restriction $B_n \to B$.

Finally,

$$\psi_n \otimes F : B_n \otimes_{A_n,\delta_n} W_n(A) \to B_{n-1} \otimes_{A_{n-1},\delta_{n-1}} W_{n-1}(A)$$

is the unique map that lifts the Frobenius on B/pB and is compatible with $F : W_n(A) \to W_{n-1}(A)$. Since the same is true for $F : W_n(B) \to W_{n-1}(B)$, the isomorphism (3.2) takes $\psi_n \otimes F$ to F. This shows the last property required in the lemma. □

Let A, A' be smooth R-algebras. Assume we are given the Frobenius lifts (A_n, ϕ_n, δ_n), $(A'_n, \phi'_n, \delta'_n)$. Then we may form the tensor product $(A_n \otimes_{W_n(R)} A'_n, \phi_n \otimes \phi'_n, \delta_n \otimes \delta'_n)$. Here, $\delta_n \otimes \delta'_n$ denotes the composition of the following obvious homomorphisms:

$$A_n \otimes_{W_n(R)} A'_n \to W_n(A) \otimes_{W_n(R)} W_n(A') \to W_n(A \otimes_R A').$$

In this way, we obtain a Frobenius lift of $A \otimes_R A'$.

For many purposes, a weaker type of lifting is sufficient, which we call a Witt lift.

Definition 3.3. Let p be nilpotent in R. Let A be a smooth R-algebra. A Witt lift of A consists of the following set of data.

(1) For each number $n \geqslant 1$, a smooth lifting A_n over $W_n(R)$ of A and isomorphisms $W_n(R) \otimes_{W_{n+1}(R)} A_{n+1} \cong A_n$, where $A_1 = A$.

(2) For each $n \geqslant 1$, a homomorphism,

$$\delta_n : A_n \to W_n(A),$$

such that $\boldsymbol{w}_0\delta_n$ is the natural map $A_n \to A$ and such that the following diagram commutes:

$$\begin{tikzcd} A_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d] & W_{n+1}(A) \arrow[d] \\ A_n \arrow[r, "\delta_n"] & W_n(A) \end{tikzcd}$$

It is easy to see that a Witt lift (A_n, δ_n) always exists.

Proposition 3.4. *Any morphism of smooth R-algebras $\phi : B \to A$ extends to a morphism of Witt lifts $(B_n, \epsilon_n) \to (A_n, \delta_n)$.*

Proof. We take arbitrary Witt liftings (B_n, ϵ_n) and (A_n, δ_n) but we forget the data ϵ_n. Then we construct, by induction, homomorphisms $\phi_n : B_n \to A_n$ and maps $\epsilon_n : B_n \to W_n(B)$ such that the ϕ_n become a morphism of Witt lifts. We consider the following diagram of W_{n+1}-algebras:

$$\begin{tikzcd} & A_{n+1} \times_{W_{n+1}(A)} W_{n+1}(B) \arrow[d] \\ B_{n+1} \arrow[r] & A_n \times_{W_n(A)} W_n(B) \end{tikzcd} \tag{3.3}$$

The lower horizontal arrow is the composition $B_{n+1} \to B_n \to A_n \times_{W_n(A)} W_n(B)$, where the last arrow is ϕ_n on the first factor and ϵ_n on the second factor. We note that the kernel of the vertical arrow is nilpotent. Since B_{n+1} is smooth over $W_{n+1}(R)$, the diagram (3.3) may be extended to a commutative diagram of $W_{n+1}(R)$-algebras by an arrow $B_{n+1} \to A_{n+1} \times_{W_{n+1}(A)} W_{n+1}(B)$. □

3.2. The comparison morphism

Let X_{zar} denote the topos of Zariski sheaves on X. Let us denote by u_n the natural map of topoi (see [**BO**, Proposition 5.18]),

$$u_n : (X/W_n(R))_{\mathrm{crys}} \to X_{\mathrm{zar}}.$$

The structure sheaf $\mathcal{O}_{X/W_n(R)}$ on the crystalline topos will be denoted by $\mathcal{O}_n$. It is a sheaf of $W_n(R)$-modules. We will define a morphism in the derived category $D^+(X, W_n(R))$ of sheaves of $W_n(R)$-modules on X_{zar},

$$Ru_{n*}\mathcal{O}_n \to W_n\Omega^{\cdot}_{X/R}. \tag{3.4}$$

For the definition we use the comparison between crystalline cohomology and de Rham cohomology (see [**BO**, Theorem 7.1]).

Let us first assume that X admits an embedding in a smooth scheme Y over R, which has a Witt lift (Y_n, Δ_n). Here, Y_n is a system of smooth liftings of Y over $W_n(R)$ and $\Delta_n : W_n(Y) \to Y_n$ are morphisms, which are global versions of the homomorphisms δ_n in Definition 3.3. Let us denote by $\bar{Y}_n$ the divided power envelope of X in Y_n relative to the canonical divided powers on ${}^VW(R)$. By the properties of a Witt lift we have the following commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\boldsymbol{w}_0"'] & Y_n \\ W_n(X) \arrow[r] & W_n(Y) \arrow[u, "\Delta_n"'] \end{tikzcd} \tag{3.5}$$

Since $X \to W_n(X)$ is a pd-thickening relative to $W_n(R)$, it follows that the morphism $W_n(X) \to Y_n$ given by the last diagram factors through a morphism,

$$W_n(X) \to \bar{Y}_n. \tag{3.6}$$

Now the left-hand side of Equation (3.4) is represented by the de Rham complex $\mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{Y_n/W_n(R)}$, which can be viewed as a complex of sheaves on X, since $X \to \bar{Y}_n$ is a nilimmersion. We define the comparison morphism (3.4) as the composition of the following morphisms:

$$\begin{tikzcd}
\mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{Y_n/W_n(R)} \arrow[d] & \\
\mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{W_n(Y)/W_n(R)} \arrow[r] & W_n(\mathcal{O}_X) \otimes_{W_n(\mathcal{O}_Y)} \Omega^{\cdot}_{W_n(Y)/W_n(R)} \arrow[d] \\
& W_n\Omega^{\cdot}_{X/R}
\end{tikzcd} \tag{3.7}$$

Two different embeddings, $X \to Y$ and $X \to Y'$, into smooth schemes, Y and Y', respectively, over R, which have a Witt lift, lead to the same morphism (3.4). This follows from a standard argument, since we may take fibre products (cf. [**I**]).

The case where X admits no embedding into a smooth scheme Y over R, which has a Witt lift one (see [**I**]), proceeds by simplicial methods. Let $X(i)$ for $i \in I$ be an open covering of X such that each $X(i)$ admits an embedding in a smooth scheme $Y(i)$ which has a Witt lift $Y_n(i)$. We set

$$X(i_1, \dots, i_r) = X(i_1) \cap \cdots \cap X(i_r) \quad \text{and} \quad Y_n(i_1, \dots, i_r) = Y_n(i_1) \times_{W_n(R)} \cdots \times_{W_n(R)} Y_n(i_r).$$

We denote by $\bar{Y}_n(i_1, \dots, i_r)$ the pd-envelope of the canonical morphism $X(i_1, \dots, i_r) \to Y_n(i_1, \dots, i_r)$. This gives us three simplicial schemes,

$$X^{\cdot} \to \bar{Y}^{\cdot}_n \to Y^{\cdot}_n.$$

Let $\epsilon : X^{\cdot} \to X$ be the natural augmentation. From [**BO**, § 7], we obtain an isomorphism

$$Ru_{n*}\mathcal{O}_n \cong R\epsilon_*(\mathcal{O}_{\bar{Y}^{\cdot}_n} \otimes_{Y^{\cdot}_n} \Omega^{\cdot}_{Y^{\cdot}_n/W_n(R)}).$$

By the liftable case, we have a natural morphism of simplicial sheaves,

$$\mathcal{O}_{\bar{Y}^{\cdot}_n} \otimes_{Y^{\cdot}_n} \Omega^{\cdot}_{Y^{\cdot}_n/W_n(R)} \to W_n\Omega^{\cdot}_{X^{\cdot}/R}.$$

If we apply $R\epsilon_*$ to this morphism, we obtain the desired comparison morphism (3.2). Indeed, by étale base change for the de Rham–Witt complex, we have a natural isomorphism,

$$R\epsilon_* W_n\Omega^{\cdot}_{X^{\cdot}/R} \cong W_n\Omega^{\cdot}_{X/R}.$$

3.3. The comparison theorem

Theorem 3.5. *Let R be a ring such that p is nilpotent in R. Let X be a smooth scheme over R. Then the canonical homomorphism (3.4),*

$$Ru_{n*}\mathcal{O}_n \to W_n\Omega^{\cdot}_{X/R},$$

is an isomorphism. This isomorphism is functorial in X.

Proof. The question is local for the Zariski topology on X. We may therefore assume that $X = \operatorname{Spec} B$ is affine and that B is étale over a polynomial algebra $A = R[T_1, \ldots, T_d]$. We set $A_n = W_n(R)[T_1, \ldots, T_d]$ and give it its natural structure of a Frobenius lift $\phi : A_{n+1} \to A_n$ (see proof of Proposition 3.2). Then the morphism of R-algebras $A \to B$ extends to a morphism of Frobenius lifts $A_n \to B_n$. Let us denote by $\psi : B_{n+1} \to B_n$ the Frobenius structure. We are then exactly in the situation of the isomorphism (3.2), and we use the notation there.

Since B is smooth over R, we may use the Frobenius lift B_n to compute the comparison morphism of the theorem. It becomes the map

$$\Omega^{\cdot}_{B_n/W_n(R)} \to W_n\Omega^{\cdot}_{B/R}, \tag{3.8}$$

which is induced by the map $\epsilon_n : B_n \to W_n(B)$ of the Frobenius lift B_n.

Let us assume that (3.8) is a quasi-isomorphism if we replace B by A. We fix n and choose m such that $p^m W_n(R) = 0$. Then the differential of $\Omega^{\cdot}_{A_n/W_n(R)}$ becomes linear if we consider this complex as a complex of A_{m+n}-modules via restriction of scalars by $\phi^m : A_{m+n} \to A_n$. By the tensor product diagram

$$\begin{tikzcd} B_{m+n} \arrow[r, "\psi^m"] & B_n \\ A_{m+n} \arrow[u] \arrow[r, "\psi^m"] & A_n \arrow[u] \end{tikzcd}$$

we find a quasi-isomorphism,

$$\Omega^{\cdot}_{B_n/W_n(R)} \cong B_n \otimes_{A_n} \Omega^{\cdot}_{A_n/W_n(R)} \cong B_{m+n} \otimes_{A_{m+n},\phi^m} \Omega^{\cdot}_{A_n/W_n(R)}.$$

The point is that the differential d on the first complex commutes with $1 \otimes \mathrm{d}$ on the last complex. Similarly, we find, by the remark to Proposition 1.7, the quasi-isomorphisms

$$W_n\Omega^{\cdot}_{B/R} \cong W_{m+n}(B) \otimes_{W_{m+n},F^m} W_n\Omega^{\cdot}_{A/R} \cong B_{m+n} \otimes_{A_{m+n},\phi^m} W_n\Omega^{\cdot}_{A/R}.$$

Since B_{m+n} is flat over A_{m+n}, the quasi-isomorphism (3.8) is obtained from the corresponding quasi-isomorphism for the polynomial algebra A by tensoring with $B_{m+n} \otimes_{A_{m+n},\phi^m}$. To show that (3.8) is a quasi-isomorphism we may therefore, without loss of generality, assume that $B = A$ is a polynomial algebra over R.

We will use the basic Witt differentials of the de Rham–Witt complex $W_n\Omega^{\cdot}_{A/R}$. We call $\omega \in W_n\Omega^{\cdot}_{A/R}$ integral if the unique expression of ω as a sum of basic Witt differentials contains only integral weights, i.e. if, in the expression (2.39), $\xi_{k,\mathcal{P}} = 0$ if k is not integral. The integral elements of $W_n\Omega^{\cdot}_{A/R}$ form a subcomplex which we denote by C_{int}.

If the unique expression of ω as a sum of basic Witt differentials contains only non-integral weights, we call Ω fractional. The subcomplex of fractional elements of $W_n\Omega^{\cdot}_{A/R}$ will be denoted by C_{frac}. We obtain a direct decomposition,

$$W_n\Omega^{\cdot}_{A/R} = C_{\mathrm{int}} \oplus C_{\mathrm{frac}}. \tag{3.9}$$

In the introduction, we wrote this decomposition explicitly (formula (2)) in the case $A = R[T]$ of one variable. One sees immediately that the integral part is just the de Rham–Witt complex of $\Omega_{W_n(R)[T]/W_n(R)}$, while the fractional part is acyclic. It is enough to verify that the same holds for several variables.

By Proposition 2.1, we know that the elements

$$T^{k_{I_0}}(p^{-\operatorname{ord}_p k_{I_1}}\mathrm{d}T^{k_{I_1}})\cdot\cdots\cdot(p^{-\operatorname{ord}_p k_{I_\ell}}\mathrm{d}T^{k_{I_\ell}}) \tag{3.10}$$

form a basis of $\Omega_{A_n/W_n(R)}$ if k runs through all integral weights and

$$\operatorname{Supp} k = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_\ell$$

runs through all partitions as in this proposition. The comparison morphism (3.8) maps the element (3.10) to the following basic Witt differential:

$$X^{k_{I_0}}({}^{F^{-t(I_1)}}\mathrm{d}X^{p^{t(I_1)}k_{I_1}})\cdot\cdots\cdot({}^{F^{-t(I_\ell)}}\mathrm{d}X^{p^{t(I_\ell)}k_{I_\ell}}). \tag{3.11}$$

The independence of basic Witt differentials shows that the comparison morphism maps $\Omega_{A_n/W_n(R)}$ isomorphically to the complex C_{int}.

It therefore remains to be shown that C_{frac} is acyclic. This is a consequence of Proposition 2.6. Indeed, an element $\omega \in C_{\mathrm{frac}}$ has the form

$$\omega = \sum e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P}),$$

where the sum runs over all k which are not integral and $\mathcal{P}$ runs over all partitions. The independence of basic Witt differentials and Proposition 2.6 shows that ω is a cycle if and only if $\xi_{k,\mathcal{P}} = 0$ for partitions $\mathcal{P}$ with $I_0 \neq \emptyset$. On the other hand, $e_n(\xi_{k,\mathcal{P}}, k, \mathcal{P})$ is a boundary if $I_0 = \emptyset$. This completes the proof of the theorem.

Finally, we must verify the functoriality. Let $X \to X'$ be a morphism of smooth schemes over R. Then we obtain a commutative diagram of topoi,

$$\begin{tikzcd}
(X/W_n(R))_{\mathrm{crys}} \arrow[r, "u_n"] \arrow[d] & X_{\mathrm{zar}} \arrow[d, "\alpha"] \\
(X'/W_n(R))_{\mathrm{crys}} \arrow[r, "u'_n"] & X'_{\mathrm{zar}}
\end{tikzcd} \tag{3.12}$$

Let $\mathcal{O}_n$, $\mathcal{O'}_n$ be the structure sheaves on $(X/W_n(R))_{\mathrm{crys}}$, $(X'/W_n(R))_{\mathrm{crys}}$, respectively. Our claim is the commutativity of the following diagram:

$$\begin{array}{ccc} \alpha^* Ru'_{n*}\mathcal{O'}_n & \longrightarrow & W_n\Omega^{\cdot}_{X'/R} \\ \downarrow & & \downarrow \\ Ru_{n*}\mathcal{O}_n & \longrightarrow & W_n\Omega^{\cdot}_{X/R} \end{array}$$

The horizontal arrows are defined simplicially by affine open coverings of X and X', respectively. By the construction at the and of § 1.2 we are therefore reduced to prove the following statement.

Assume we are given the embeddings $X \to Y$ and $X' \to Y'$ into smooth affine schemes Y and Y' over R, and the following commutative diagram:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\tilde{\alpha}} \\ X' & \longrightarrow & Y' \end{array} \tag{3.13}$$

By Proposition 3.4, there are Witt lifts (Y_n, Δ_n) and (Y'_n, Δ'_n) such that $\tilde{\alpha}$ extends to a map of these Frobenius lifts. Let $\bar{Y}_n$ be the pd-envelope of $X \to Y_n$ and $\bar{Y}'_n$ the pd-envelope of $X' \to Y'_n$. Then our assertion is the commutativity of the following diagram given by (3.7):

$$\begin{array}{ccc} \tilde{\alpha}^*(\mathcal{O}_{\bar{Y}'_n} \otimes_{\mathcal{O}_{Y'_n}} \Omega^{\cdot}_{Y'_n/W_n(R)}) & \longrightarrow & \alpha^* W\Omega^{\cdot}_{X'/R} \\ \downarrow & & \downarrow \\ \mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{Y_n/W_n(R)} & \longrightarrow & W\Omega^{\cdot}_{X/R} \end{array} \tag{3.14}$$

But this is obvious. □

We are going to explain the compatibility of the Frobenius with the comparison morphism. This is the point where we need Frobenius lifts. Let A be an R-algebra. The commutative diagram

$$\begin{array}{ccc} W_n(A) & \xrightarrow{F} & W_{n-1}(A) \\ \uparrow & & \uparrow \\ W_n(R) & \xrightarrow{F} & W_{n-1}(R) \end{array} \tag{3.15}$$

induces a map $\boldsymbol{F} : \Omega^{\cdot}_{W_n(A)/W_n(R)} \to \Omega^{\cdot}_{W_{n-1}(A)/W_{n-1}(R)}$ which factors through a map of the de Rham–Witt complexes,

$$\boldsymbol{F} : W_n\Omega^{\cdot}_{A/R} \to W_{n-1}\Omega^{\cdot}_{A/R}.$$

We call this map the absolute Frobenius. On the group $W_n\Omega^i_{A/R}$ we have $\boldsymbol{F} = p^iF$. This follows from the equation $\mathrm{d}^F\nu = p^F\mathrm{d}\xi$ for $\xi \in W_n(A)$. More generally, we obtain for a scheme X over $S = \operatorname{Spec} R$ an absolute Frobenius,

$$\boldsymbol{F} : W_n\Omega^{\cdot}_{X/S} \to W_{n-1}\Omega^{\cdot}_{X/S}. \tag{3.16}$$

On the other hand, let $Ru_{n*}\mathcal{O}_n$ be the direct image of the structure sheaf by $u_n : (X/W_n(S))_{\mathrm{crys}} \to X_{\mathrm{zar}}$. Then again, we have an absolute Frobenius

$$\boldsymbol{F} : Ru_{n*}\mathcal{O}_n \to Ru_{n-1*}\mathcal{O}_{n-1}. \tag{3.17}$$

This map is defined as follows. We set $X_0 = X \times \operatorname{Spec}\mathbb{F}_p$ and $S_0 = S \times \operatorname{Spec}\mathbb{F}_p$. Then the nilimmersion $S_0 \to W_n(S)$ has a natural pd-structure which is an extension of the pd-structure $S \to W_n(S)$ which we considered so far. For this pd-structure, the Frobenius is a pd-morphism $F : W_{n-1}(S) \to W_n(S)$. We consider the morphism

$$\bar{u}_n : (X_0/W_n(S)) \to X_{\mathrm{zar}} = X_{0\mathrm{zar}}.$$

By [**BO**, 5.17], we have a canonical isomorphism $R\bar{u}_{n*}\mathcal{O}_{X_0/W_n(S)} \cong Ru_{n*}\mathcal{O}_n$. Then we consider the commutative square

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{Frob}"] \arrow[d] & X_0 \arrow[d] \\ W_{n-1}(S) \arrow[r, "F"] & W_n(S) \end{tikzcd} \tag{3.18}$$

It induces a map $R\bar{u}_{n*}\mathcal{O}_{X_0/W_n(S)} \to R\bar{u}_{n-1*}\mathcal{O}_{X_0/W_{n-1}(S)}$. Hence we obtain the absolute Frobenius (3.17).

Proposition 3.6. *Let X be a smooth scheme over $S = \operatorname{Spec} R$. The comparison isomorphism of Theorem 3.5 respects the absolute Frobenius, i.e. we have the commutative diagram*

$$\begin{tikzcd} Ru_{n*}\mathcal{O}_n \arrow[r] \arrow[d, "\boldsymbol{F}"'] & W_n\Omega^{\cdot}_{X/R} \arrow[d, "\boldsymbol{F}"] \\ Ru_{n-1*}\mathcal{O}_n \arrow[r] & W_{n-1}\Omega^{\cdot}_{X/R} \end{tikzcd} \tag{3.19}$$

Proof. By the simplicial methods above we may reduce the assertion to the case where X is embedded in a smooth affine scheme Y which admits a Frobenius lift Y_n. Let $\Phi_n : Y_{n-1} \to Y_n$ be the given lift of the Frobenius. As before, we denote by $\bar{Y}_n$ the pd-envelope of $X \to Y_n$. Since Φ is a lift of the absolute Frobenius, we obtain from [**BO**, 7.1] that the map (3.17) is represented by the following map of complexes induced by Φ_n:

$$\mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{Y_n/S_n} \to \mathcal{O}_{\bar{Y}_{n-1}} \otimes_{\mathcal{O}_{Y_{n-1}}} \Omega^{\cdot}_{Y_n/S_{n-1}}.$$

Here we write $S_n = \operatorname{Spec} W_n(R)$. Therefore, our assertion is the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{O}_{\bar{Y}_n} \otimes_{\mathcal{O}_{Y_n}} \Omega^{\cdot}_{Y_n/S_n} & \longrightarrow & \mathcal{O}_{\bar{Y}_{n-1}} \otimes_{\mathcal{O}_{Y_{n-1}}} \Omega^{\cdot}_{Y_n/S_{n-1}} \\ \downarrow & & \downarrow \\ W_n\Omega^{\cdot}_{X/S} & \xrightarrow{\boldsymbol{F}} & W_{n-1}\Omega^{\cdot}_{X/S} \end{array} \tag{3.20}$$

This follows from the properties of the Frobenius lift Φ_n. □

3.4. Displays

Let R be a ring where p is nilpotent. We set $S = \operatorname{Spec} R$. Let A be an abelian scheme over S. Using the de Rham–Witt complex we will equip the Dieudonné crystal associated to A by [**BBM**] with the structure of a $3n$-display (see [**Z**, Introduction]).

We start with a more general situation. Let $f : X \to S$ be a smooth and proper morphism. Then we consider the $W(R)$-module

$$P = \varprojlim \mathbb{H}^1(X, W_n\Omega^{\cdot}_{X/S}).$$

We define $I_n\Omega^{\cdot}_{X/S}$ as the subcomplex of $W_n\Omega^{\cdot}_{X/S}$ obtained by replacing the group $W_n(\mathcal{O}_X)$ in degree zero by the subgroup $VW_{n-1}(\mathcal{O}_X)$ but leaving the other degrees untouched. Then we obtain an exact sequence of complexes of sheaves on X,

$$0 \to I_n\Omega^{\cdot}_{X/S} \to W_n\Omega^{\cdot}_{X/S} \to \mathcal{O}_X \to 0. \tag{3.21}$$

Here, $\mathcal{O}_X$ is viewed as a complex with $\mathcal{O}_X$ placed in degree zero and zero otherwise. We set $Q_n = \mathbb{H}^1(X, I_n\Omega^{\cdot}_{X/S})$ and $P_n = \mathbb{H}^1(X, W_n\Omega^{\cdot}_{X/S})$. Then the sequence of hypercohomology of (3.21) gives

$$\mathbb{H}^0(X, W_n\Omega^{\cdot}_{X/S}) \to H^0(X, \mathcal{O}_X) \to Q_n \to P_n \to H^1(X, \mathcal{O}_X). \tag{3.22}$$

We claim that the first arrow is surjective if S is noetherian. Indeed, $R' = H^0(X, \mathcal{O}_X)$ is an étale R-algebra by [**EGA**, III, 7.8.10]. By definition, the group $\mathbb{H}^0(X, W_n\Omega_{X/S})$ is the kernel of the differential

$$H^0(X, W_n(\mathcal{O}_X)) \to H^0(X, W_n\Omega^1_{X/S}).$$

The ring $W_n(R')$ is naturally a subring of $H^0(X, W_n(\mathcal{O}_X))$. Because $W_n(R')$ is étale over $W_n(R)$ and because the differential is zero on $W_n(R)$ by definition, it is also zero on $W_n(R')$. Hence $W_n(R')$ is contained in the first term of (3.22), and therefore the first arrow of (3.22) is surjective.

If we pass in (3.22) to the projective limit, we obtain an exact sequence

$$0 \to Q \to P \to H^1(X, \mathcal{O}_X). \tag{3.23}$$

We set $I_R = VW(R)$. Then we obtain $I_R P \subset Q$, since this holds on the level of complexes.

We denote by
$$F : W_n\Omega^{\cdot}_{X/S} \to W_{n-1}\Omega^{\cdot}_{X/S} \tag{3.24}$$
the homomorphism which is p^iF in degree i. It induces a Frobenius linear endomorphism of the $W(R)$-module P,
$$F : P \to P,$$
which is called a Frobenius.

Next we define a morphism of complexes,
$$V^{-1} : I_n\Omega^{\cdot}_{X/S} \to W_{n-1}\Omega^{\cdot}_{X/S}, \tag{3.25}$$
by the commutative diagram
$$\begin{tikzcd} VW_{n-1}(\mathcal{O}_X) \arrow[r, "\mathrm{d}"] \arrow[d, "V_{-1}"'] & W_n\Omega^1_{X/S} \arrow[r, "\mathrm{d}"] \arrow[d, "F"'] & W_n\Omega^2_{X/S} \arrow[r] \arrow[d, "pF"] & \cdots \\ W_{n-1}(\mathcal{O}_X) \arrow[r, "\mathrm{d}"] & W_{n-1}\Omega^1_{X/S} \arrow[r, "\mathrm{d}"] & W_{n-1}\Omega^2_{X/S} \arrow[r] & \cdots \end{tikzcd} \tag{3.26}$$
The commutativity is obvious from the relations
$$F\mathrm{d}V = \mathrm{d}, \qquad pF\mathrm{d} = \mathrm{d}F.$$
We obtain a Frobenius linear homomorphism of $W(R)$-modules
$$V^{-1} : Q \to P.$$
Let $\omega \in W_n\Omega^i_{X/S}$ and $\xi \in W(R)$. Then ${}^V\xi\omega \in I_n\Omega^i_{X/S}$. One checks easily the relation
$$V^{-1}({}^V\xi\omega) = \xi F\omega.$$
Here, F and V^{-1} are the homomorphisms of complexes (3.24) and (3.25), respectively. This shows that, for $x \in P$ and $\xi \in W(R)$, we have the relation
$$V^{-1}({}^V\xi x) = \xi Fx. \tag{3.27}$$

Let us now consider the case where $X = A$ is an abelian scheme over $S = \operatorname{Spec} R$. Then we can drop the assumption that R is noetherian. We denote by $\mathbb{D}(A)$ the Dieudonné crystal associated to A (cf. [**BBM**]). By the comparison isomorphism, we have, for each n, a canonical isomorphism,
$$\mathbb{H}^1(A, W_n\Omega^{\cdot}_{X/S}) \cong \mathbb{D}(A)_{(S, W_n(S), \gamma)}. \tag{3.28}$$
Here, γ denotes the canonical divided powers on $W_n(S)$.

The right-hand side of (3.28) is a finitely generated projective $W_n(R)$-module of rank $2\dim A$. Using that $\mathbb{D}(A)$ is a crystal, we conclude that P is a finitely generated projective

$W(R)$-module of rank $2\dim A$. Using [**BBM**, 2.5.8], we conclude the exactness of the sequence,

$$0 \to Q \to P \to H^1(A, \mathcal{O}_A) \to 0. \tag{3.29}$$

We want to show that (P, Q, F, V^{-1}) is a $3n$-display. Since $H^1(A, \mathcal{O}_A)$ is known to be a finitely generated projective R-module (see [**BBM**]), we have a decomposition $P = L \oplus T$ as $W(R)$-module such that $Q = L \oplus I_R T$. This is called a normal decomposition in [**Z**].

We have to show that $V^{-1} : Q \to P$ is an F-linear epimorphism. All other requirements of a $3n$-display are trivially fulfilled. It is easy to see (cf. [**Z**]) that V^{-1} is an F-linear epimorphism if and only if the following map is an F-linear isomorphism:

$$V^{-1} \oplus F : L \oplus T \to P. \tag{3.30}$$

Since the question is local, we may assume that P is a free $W(R)$-module. We consider $\delta = \det(V^{-1} \oplus F)$ with respect to some basis of P. We have to show that δ is a unit in $W(R)$. If $R = k$ is a perfect field, we know that $\operatorname{ord}_p \det F = \dim A = \dim_{W(k)} L$. Since we have $F = pV^{-1}$ on L, we conclude that $\operatorname{ord}_p \delta = 0$. Hence δ is a unit.

In the general case it clearly suffices to check that $\boldsymbol{w}_0(\delta)$ is a unit in R, i.e. non-zero in $R/\mathfrak{m}$, for any maximal ideal $\mathfrak{m}$. Since $\mathbb{D}(A)$ is a crystal on the big crystalline situs, it commutes with arbitrary base change. This shows that it is enough to treat the case $R = R/\mathfrak{m}$. Finally, we see by a base change to the perfect closure of $R/\mathfrak{m}$ that (P, Q, F, V^{-1}) is a $3n$-display.

We will now give the comparison to the theory in [**Z**]. Let us assume that R/pR is essentially of finite type over a perfect field k. Then we may write R as a quotient,

$$W(k)[T_1, \ldots, T_r]_M \to R, \tag{3.31}$$

of a polynomial ring over $W(k)$ localized in a multiplicative closed system M. Let $\mathfrak{a}$ be the kernel of the map (3.31). Let S be the completion of $W(k)[T_1, \ldots, T_r]$ with respect to the $\mathfrak{a}$-adic topology. Then the ring S is without p-torsion. We set $S_n = S/\mathfrak{a}^n$.

Let A be an abelian scheme over R. We have defined the structure of a $3n$-display on the finitely generated projective $W(R)$-module

$$P = H^1_{\text{crys}}(A/W(R)) = \varprojlim \mathbb{H}^1_{\text{crys}}(X/W_n(R)) = \varprojlim \mathbb{H}^1(X, W_n\Omega^{\cdot}_{X/S}). \tag{3.32}$$

We set $P_n = H^1_{\text{crys}}(A_n/W(S_n))$ and $\hat{P} = \varprojlim P_n$. Then $\hat{P}$ is a finitely generated projective $W(S)$-module. We define Q_n to be the kernel of the canonical map,

$$H^1_{\text{crys}}(A_n/W(S_n)) \to H^1_{DR}(A_n/S_n) \to H^1(A_n, \mathcal{O}_{A_n}).$$

The maps $Q_{n+1} \to Q_n$ are surjective. We set $\hat{Q} = \varprojlim Q_n$. One checks that $FQ_n \subset pP_n$. Indeed, we can reduce the problem modulo the ideal $pW(S_n) + {}^V W(S_n)$. Let $\bar{A}_n$ be the abelian variety obtained by base change over $\bar{S}_n = S_n/pS_n$. We have to show that the Frobenius induces the zero map on $H^0(\bar{A}_n, \Omega_{\bar{A}_n/\bar{S}_n})$. This is clear. Since the ring S has no p-torsion, we obtain a unique map $V^{-1} : \hat{Q} \to \hat{P}$ such that $pV^{-1} = F$. Therefore, $(\hat{P}, \hat{Q}, F, V^{-1})$ coincides with the $3n$-display defined by the de Rham–Witt complex.

Let (P, Q, F, V^{-1}) be the $3n$-display we have associated to A. Then V^{-1} is uniquely determined by the commutative diagram

$$\begin{tikzcd} \hat{Q} \arrow[r, "V^{-1}"] \arrow[d] & \hat{P} \arrow[d] \\ Q \arrow[r, "V^{-1}"] & P \end{tikzcd} \tag{3.33}$$

We can summarize our considerations as follows. Assume we are given a functor which associates to an abelian scheme A over R a $3n$-display (P_A, Q_A, F, V^{-1}) such that $(P_A, F) = (H^1_{\mathrm{crys}}(A/W(R)), F)$ is the crystalline cohomology equipped with the Frobenius and such that Q_A is the kernel of the morphism

$$H^1_{\mathrm{crys}}(A/W(R)) \to H^1(A, \mathcal{O}_A).$$

Assume, moreover, that the functor commutes with base change. Then the functor is uniquely determined. This proves, in particular, the following result.

Proposition 3.7. *Let A be an abelian variety over R with no p-division points in the geometric fibres. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be the $3n$-display associated to the p-divisible group of A by [**Z**]. Then the dual $3n$-display $\hat{\mathcal{P}}$ is canonically isomorphic to the $3n$-display given on $H^1_{\mathrm{crys}}(A/W(R))$ by the de Rham–Witt complex.*

Proof. It is shown in [**Z**] that P is canonically isomorphic to the Lie algebra of the universal extension of A over $W(R)$. Therefore, the proposition follows from [**MM**, Theorem 1] and the duality theory of [**BBM**]. □

One might define the structure of a $3n$-display on $H^1_{\mathrm{crys}}(A/W(R))$ by a lifting as above, without using the de Rham–Witt complex. But then the point is that it seems difficult to show that this structure is independent of the lifting.

3.5. The de Rham–Witt complex for a crystal

We consider an arbitrary scheme X over a ring R, where p is nilpotent. Let us denote by $\mathrm{Crys}(X/W_n(R))$ the crystalline site. We recall that an object of this site is a triple (U, T, δ), where U is a Zariski open subset of X, $U \to T$ is a closed immersion of $W_n(R)$-schemes defined by an ideal $J \subset \mathcal{O}_T$ and δ is a pd-structure on J which is compatible with the pd-structure on ${}^V W_n(R)$. If there is no confusion possible, we will denote this object simply by T. As before, we denote by $\mathcal{O}_n = \mathcal{O}_{X/W_n(R)}$ the structure sheaf of this site, i.e. $\mathcal{O}_n(T) = \mathcal{O}_T$.

A sheaf E of $\mathcal{O}_n$-modules on $\mathrm{Crys}(X/W_n(R))$ induces a sheaf E_T of $\mathcal{O}_T$-modules on the scheme T. We call E quasi-coherent if, for all objects T in $\mathrm{Crys}(X/W_n(R))$, the $\mathcal{O}_T$-module E_T is quasi-coherent.

In this work a crystal is a quasi-coherent sheaf E of $\mathcal{O}_n$-modules such that, for any morphism $\alpha : T' \to T$ in $\mathrm{Crys}(X/W_n(R))$, the induced homomorphism of $\mathcal{O}_{T'}$-modules $\alpha^* E_T \to E_{T'}$ is an isomorphism. Let us denote the Zariski sheaf $E_{W_n(X)}$ given by the pd-thickening $X \to W_n(X)$ by E_n. Since X and $W_n(X)$ have the same topological space, we can view E_n as a sheaf on X. The aim of this section is to build a procomplex for varying $n \geqslant 1$,

$$(W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n, \nabla).$$

For $n = 1$, the $\mathcal{O}_X$-module E_1 is equipped with an integrable connection (also see below) and the complex above coincides with the de Rham complex defined by this integrable connection.

Let (U, T, γ) be an object of $\mathrm{Crys}(X/W_n(R))$ such that U is affine. We set $U = \operatorname{Spec} A$ and $T = \operatorname{Spec} S$. Then we have a surjective map $\alpha : S \to A$ whose kernel $\mathfrak{a}$ is equipped with divided powers γ_n which are compatible with the canonical divided powers on ${}^{V}W_{n-1}(R) \subset W_n(R)$.

Let $\nu : S \to \Omega$ be a $W_n(R)$-linear pd-derivation to an S-module Ω. By definition, we have, for each number $n \geqslant 1$ and each $a \in \mathfrak{a}$, the equation

$$\nu(\gamma_n(a)) = \gamma_{n-1}(a)\nu(a). \tag{3.34}$$

The direct sum $S \oplus \Omega$ has a natural ring structure such that Ω is an ideal whose square is zero. We define on the kernel $\mathfrak{a} \oplus \Omega$ of the ring homomorphism

$$\alpha \oplus 0 : S \oplus \Omega \to A \tag{3.35}$$

a pd-structure denoted by the same letter γ_n as follows:

$$\gamma_n(a + \omega) = \gamma_n(a) + \gamma_{n-1}(a)\omega \quad \text{for } a \in \mathfrak{a}, \quad \omega \in \Omega. \tag{3.36}$$

Clearly, this extends the canonical pd-structure on ${}^{V}W_{n-1}(R)$. Hence we may view (3.35) as an object in $\mathrm{Crys}(X/W_n(R))$. The homomorphism of $W_n(R)$-algebras,

$$\left.\begin{aligned} \tilde{\nu} : \quad & S \to S \oplus \Omega, \\ & s \mapsto s + \nu(s), \end{aligned}\right\} \tag{3.37}$$

is a morphism of pd-thickenings of A, i.e. induces a morphism in the category $\mathrm{Crys}(X/W_n(R))$. Indeed, this is equivalent to the requirement that ν is a pd-derivation. On one hand, we have

$$\gamma_n(\tilde{\nu}(a)) = \gamma_n(a + \nu(a)) = \gamma_n(a) + \gamma_{n-1}(a)\nu(a). \tag{3.38}$$

On the other hand, we have

$$\tilde{\nu}(\gamma_n(a)) = \gamma_n(a) + \nu(\gamma_n(a)). \tag{3.39}$$

The expressions (3.38) and (3.39) are equal if and only if (3.34) holds.

There is a second morphism of $W_n(R)$-algebras,

$$\left.\begin{aligned} \tilde{\nu}_0 : \quad & S \to S \oplus \Omega, \\ & s \mapsto s + 0, \end{aligned}\right\} \tag{3.40}$$

which is also a morphism of pd-thickenings of A.

Now let E be a quasi-coherent crystal on X. Then we obtain a quasi-coherent sheaf $E_{\mathrm{Spec}\, S}$ on $\mathrm{Spec}\, S$. We denote the associated S-module by E_S. In the same way, the pd-thickening (3.35) defines an $S \oplus \Omega$-module $E_{S\oplus\Omega}$. Since E is a crystal, we have isomorphisms of $S \oplus \Omega$-modules,

$$(S \oplus \Omega) \otimes_{\tilde{\nu},S} E_S \cong E_{S\oplus\Omega} \cong (S \oplus \Omega) \otimes_{\tilde{\nu}_0,S} E_S. \tag{3.41}$$

This induces the identity when tensored with the map $S \oplus \Omega \to S$ of pd-thickenings which sends Ω to 0. We identify the right-hand side of (3.41) with $E_S \oplus \Omega \otimes_S E_S$. Then an element $1 \otimes m$ from the left-hand side of (3.41) is mapped to an element of the form $m \oplus \nabla m \in E_S \oplus \Omega \otimes_S E_S$. One checks easily that

$$\nabla : E_S \to \Omega \otimes_S E_S \tag{3.42}$$

is a connection, i.e. an additive map which satisfies the equation

$$\nabla(sm) = \nu(s)m + s\nabla m. \tag{3.43}$$

We apply this to the canonical pd-thickening $W_n(A) \to A$. If we denote by $E_{n,A}$ the value of the crystal E at $W_n(A)$, we obtain a connection

$$\nabla : E_{n,A} \to \breve{\Omega}^1_{W_n(A)/W_n(R)} \otimes_{W_n(A)} E_{n,A}. \tag{3.44}$$

We have to check that this connection is integrable. Then we may extend the connection to a complex $(\breve{\Omega}^{\cdot}_{W_n(A)/W_n(R)} \otimes_{W_n(A)} E_{n,A}, \nabla)$ by the formula

$$\nabla(\omega \otimes m) = \mathrm{d}\omega \otimes m + (-1)^{\deg \omega}\omega \wedge \nabla m \quad \text{for } m \in E_{n,A}.$$

Let $\mathcal{I}^{\cdot} \subset \breve{\Omega}^{\cdot}_{W_n(A)/W_n(R)}$ be a graded d-invariant ideal. Then ∇ leaves

$$\mathcal{I}^{\cdot} \otimes_{W_n(A)} E_{n,A} \subset \breve{\Omega}^{\cdot}_{W_n(A)/W_n(R)} \otimes_{W_n(A)} E_{n,A}$$

stable. Indeed, for $a \in \mathcal{I}^i$ and $m \in E_{n,A}$, we write

$$\nabla(a \otimes m) = \mathrm{d}a \otimes m + a\nabla m.$$

Clearly, each summand of the right-hand side is in $\mathcal{I}^{i+1} \otimes_{W_n(A)} E_{n,A}$. If we apply this remark to the kernel $\mathcal{I}^{\cdot}$ of the canonical surjection $\breve{\Omega}^{\cdot}_{W_n(A)/W_n(R)} \to W_n\Omega^{\cdot}_{A/R}$, we obtain a complex $(W_n\Omega^{\cdot}_{A/R} \otimes_{W_n(A)} E_{n,A}, \nabla)$.

For varying $U = \mathrm{Spec}\, A$, these complexes glue to the desired complex

$$(W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n, \nabla).$$

The integrability is a consequence of the theory of HPD-stratifications. Indeed, since the question is local on X, we may assume that there is a smooth $W_n(R)$-algebra B and a surjection $B \to W_n(A)$. Let us denote by $\mathcal{D}_\gamma(B)$ the pd-envelope of the surjection $B \to W_n(A) \xrightarrow{\boldsymbol{w}_0} A$ of $W_n(R)$-algebras relative to the canonical divided powers γ on the ideal ${}^{V}W_{n-1}(R)$. Since $\boldsymbol{w}_0 : W_n(A) \to A$ is a pd-thickening, we obtain by the universal property of the pd-envelope a morphism of pd-thickenings of A,

$$\mathcal{D}_\gamma(B) \to W_n(A).$$

We apply our general construction of a connection above to the case where $S = \mathcal{D}_\gamma(B)$ and $d : \mathcal{D}_\gamma(B) \to \breve{\Omega}^{\cdot}_{\mathcal{D}_\gamma(B)/W_n(R)}$. Then we obtain a connection,

$$\nabla : E_{\mathcal{D}_\gamma(B)} \to \breve{\Omega}^{\cdot}_{\mathcal{D}_\gamma(B)/W_n(R)} \otimes_{\mathcal{D}_\gamma(B)} E_{\mathcal{D}_\gamma(B)} \tag{3.45}$$

Taking into account the canonical isomorphism (cf. [**I**, Chapter 0, Proposition 3.1.6]):

$$\breve{\Omega}^{\cdot}_{\mathcal{D}_\gamma(B)/W_n(R)} \cong \mathcal{D}_\gamma(B) \otimes_B \Omega_{B/W_n(R)},$$

we obtain from the proof of the theorem in [BO] 6.6 that the connection (3.45) is just the connection associated to the crystal E, and hence integrable. The connection (3.44) is by construction the push-forward of (3.45) by the morphism $\mathcal{D}_\gamma(B) \to W_n(A)$ and therefore is integrable too.

Let $U = \operatorname{Spec} A$, as before, and denote by $u_n : (U/W_n(R))_{\mathrm{crys}} \to U$ the canonical morphism of topoi. Then $Ru_{n*}E_U$ is in the derived category $D^+(U, W_n(R))$ represented by the de Rham complex $\breve{\Omega}^{\cdot}_{\mathcal{D}_\gamma(B)/W_n(R)} \otimes_{\mathcal{D}_\gamma(B)} E_{\mathcal{D}_\gamma(B)}$. Therefore, the morphism

$$(\breve{\Omega}^{\cdot}_{\mathcal{D}_\gamma(B)/W_n(R)} \otimes_{\mathcal{D}_\gamma(B)} E_{\mathcal{D}_\gamma(B)}, \nabla) \to (W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n, \nabla)$$

provides, by [**BO**, 7.1], a morphism in $D^+(U, W_n(R))$,

$$Ru_{n*}E_U \to (W_n\Omega^{\cdot}_{U/R} \otimes_{W_n(\mathcal{O}_U)} E_n, \nabla), \tag{3.46}$$

As in § 1.2, this morphism is independent of the embedding $B \to W_n(A)$ and globalizes by the method described in § 1.2 to a morphism,

$$Ru_{n*}E \to (W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n, \nabla). \tag{3.47}$$

Theorem 3.8. *Let $u_n : (X/W_n(R))_{\mathrm{crys}} \to X_{\mathrm{zar}}$ be the natural morphism of topoi. Then the morphism (3.47) above is a quasi-isomorphism for any crystal E of flat modules.*

Proof. We have proved this in the case where E is the structure sheaf $\mathcal{O}_n$ of $(X/W_n(R))_{\mathrm{crys}}$. The proof will be a reduction to this case using the ideas of [**BO**, Theorem 7.1].

Since the question is local on X, we may assume that $X = \operatorname{Spec} A$ where A is étale over a polynomial algebra $R[T_1, \dots, T_d]$. We lift A to an étale algebra A_n over $W_n(R)[T_1, \dots, T_d]$, as in the proof of Proposition 3.2. In particular, we obtain a map

$$\delta_n : A_n \to W_n(A). \tag{3.48}$$

We set $S_0 = \operatorname{Spec} R$, $S = \operatorname{Spec} W_n(R)$ and $Y = \operatorname{Spec} A_n$. Then we obtain the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d] & Y \arrow[d] \\ S_0 \arrow[r] & S \end{tikzcd} \tag{3.49}$$

We note that $S_0 \to S$ is a pd-thickening with respect to the natural pd-structure on the ideal ${}^V W_{n-1}(R) \subset W_n(R)$. This pd-structure extends to Y, and hence $i : X \to Y$ becomes a pd-thickening.

Let $\mathcal{D}_{Y/S}(1)$ be the pd-envelope of the diagonal $Y \to Y \otimes_S Y$ considered as a quasi-coherent sheaf on Y. If we set $\xi_i = 1 \otimes T_i - T_i \otimes 1$, we may identify $\mathcal{D}_{Y/S}(1)$ with the pd-polynomial algebra $\mathcal{O}_Y\langle\xi_1, \ldots, \xi_d\rangle$ (see [**BO**, Proposition 3.32]) in such a way that the canonical $\mathcal{O}_Y$-module structure on the pd-polynomial algebra corresponds to the $\mathcal{O}_Y$-module structure on $\mathcal{D}_{Y/S}(1)$ from the right (sic).

Let $\delta : \mathcal{D}_{Y/S}(1) \to \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} \mathcal{D}_{Y/S}(1)$ be the map defined by

$$\delta(\xi^{[k]}) = (\xi \otimes \mathbf{1} + \mathbf{1} \otimes \xi)^{[k]} = \sum_{i+j=k} \xi^i \otimes \xi^j.$$

This map is needed for Grothendieck's linearization L_Y of HPD-differential operators. L_Y is a functor from the category of quasi-coherent $\mathcal{O}_Y$-modules and HPD-differential operators to the category of $\mathcal{O}_Y$-modules with an HPD-stratification and horizontal maps. By [**BO**, (6.9)], the last category is equivalent to the category of crystals on $(Y/S)_{\text{crys}}$. We will denote the corresponding crystal by $\boldsymbol{L}_Y$ if it is necessary to distinguish it from the HPD-stratification.

If M is a $\mathcal{O}_Y$-module, then $L_Y(M) = \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M$ is equipped with an HPD-stratification. A HPD-differential operator

$$\mathrm{D} : \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M \to N$$

induces a horizontal map of HPD-stratified $\mathcal{O}_Y$-modules,

$$L_Y(\mathrm{D}) : \mathcal{D}_{Y/S}(1) \otimes \xrightarrow{\delta \otimes \mathrm{id}_M} \mathcal{D}_{Y/S}(1) \otimes \mathcal{D}_{Y/S}(1) \otimes M \xrightarrow{\mathrm{id} \otimes \mathrm{D}} \mathcal{D}_{Y/S}(1) \otimes N,$$

where all tensor products are taken over $\mathcal{O}_Y$.

We will apply this construction to the case where D is a differential operator of order less than or equal to 1. In this case, D is given by an $\mathcal{O}_S$-linear map

$$\mathrm{D} : M \to N.$$

For $f \in \mathcal{O}_Y$, we define $[\mathrm{D}, f] : M \to N$ by the formula

$$[\mathrm{D}, f](m) = \mathrm{D}(fm) - f\mathrm{D}(m).$$

This is an $\mathcal{O}_Y$-linear map since D is a differential operator of order less than or equal to 1. We linearize D to an $\mathcal{O}_Y$-linear map,

$$\mathrm{D}^\sharp : \mathcal{O}_Y \otimes_{\mathcal{O}_S} M \to N.$$

Let J be the kernel of the multiplication $\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_Y \to \mathcal{O}_Y$. Then $\mathrm{D}^\sharp$ factors through a quotient,

$$\mathrm{D}^\sharp : (\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_Y)/J^2 \otimes_{\mathcal{O}_Y} M \to N.$$

By [**BO**, 4.2], we have a natural surjection

$$\mathcal{D}_{Y/S}(1) \to (\mathcal{O}_Y \otimes_{\mathcal{O}_S} \mathcal{O}_Y)/J^2.$$

Hence we obtain an HPD-differential operator

$$\mathrm{D}^\sharp : \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M \to N.$$

We denote its linearization simply by $L_Y(\mathrm{D}) = L_Y(\mathrm{D}^\sharp)$. In local coordinates $T_1, \dots, T_d$ as above this linearization is given as follows. An element of $\mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M$ may be uniquely written as a finite sum,

$$\sum_k \xi^{[k]} \otimes m_k, \quad m_k \in M. \tag{3.50}$$

Here, $k = (k_1, \dots, k_d)$ runs through all vectors of non-negative integers. In this notation, we find

$$L_Y(\mathrm{D})(\xi^{[k]} \otimes m_k) = \xi^{[k]} \otimes \mathrm{D}m_k + \sum_{i=1}^{d} \xi_1^{[k_1]} \cdots \xi_i^{[k_i-1]} \cdots \xi_d^{[k_d]} [\mathrm{D}, T_i](m_k), \tag{3.51}$$

with the convention $\xi_i^{[-1]} = 0$.

In the following lemma, S can be an arbitrary scheme where p is locally nilpotent and Y can be an arbitrary smooth scheme over S.

Lemma 3.9. *Let $\mathrm{D}_1 : M_1 \to M_2$ and $\mathrm{D}_2 : M_2 \to M_3$ be differential operators of order less than or equal to 1 between quasi-coherent $\mathcal{O}_Y$-modules such that $\mathrm{D}_2\mathrm{D}_1 = 0$. If $p = 2$, we require, moreover, that $[\mathrm{D}_2, f][\mathrm{D}_1, f] = 0$ for any element $f \in \mathcal{O}_Y$. Assume that the sequence of $\mathcal{O}_S$-linear maps*

$$M^1 \xrightarrow{\mathrm{D}_1} M^2 \xrightarrow{\mathrm{D}_2} M^3$$

is exact in M^2.

Then $L_Y(\mathrm{D}_2)L_Y(\mathrm{D}_1) = 0$ and the following sequence is exact:

$$\mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^1 \xrightarrow{L_Y(\mathrm{D}_1)} \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^2 \xrightarrow{L_Y(\mathrm{D}_2)} \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^3.$$

We postpone the proof to the end of this section.

More generally, we can consider a complex of differential operators of order less than or equal to 1, i.e. a sequence of quasi-coherent $\mathcal{O}_Y$-modules M^i for $i \in \mathbb{Z}$ and differential operators $\mathrm{D}_i : M^i \to M^{i+1}$ of order less than or equal to 1 such that

$$\cdots \to M^i \xrightarrow{\mathrm{D}_i} M^{i+1} \to \cdots$$

is a complex of $\mathcal{O}_S$-modules. If $p = 2$, we add the condition that, for any $f \in \mathcal{O}_Y$ and $i \in \mathbb{Z}$,

$$[\mathrm{D}_{i+1}, f][\mathrm{D}_i, f] = 0. \tag{3.52}$$

By the last lemma, this ensures that $L_Y(M^\cdot)$ is a complex.

A morphism $\alpha : M^{\cdot} \to N^{\cdot}$ of complexes of differential operators of order less than or equal to 1 is a graded homomorphism of $\mathcal{O}_Y$-modules $\alpha : M^i \to N^i$ which is also a homomorphism of complexes. Since the compositions $\alpha \mathrm{D}_i$, $\mathrm{D}_i \alpha$ as $\mathcal{O}_S$-module homomorphisms correspond to the composition as HPD-differential operators, we obtain that $L_Y(\alpha) : L_Y(M^{\cdot}) \to L_Y(N^{\cdot})$ is a morphism of complexes.

Corollary 3.10. *Let $\alpha : M^{\cdot} \to N^{\cdot}$ be a quasi-isomorphism of complexes of differential operators of order less than or equal to 1. Then*

$$L_Y(\alpha) : \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^{\cdot} \to \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} N^{\cdot}$$

is a quasi-isomorphism of complexes of $\mathcal{O}_Y$-modules.

Proof. The mapping cone $C = M^{i+1} \oplus N^i$ of α is an acyclic complex of differential operators of order less than or equal to 1. Clearly, the functor L_Y respects mapping cones. Since, by the last lemma, $L_Y(C) = \operatorname{Cone} L_Y(\alpha)$ is acyclic, we are done. $\square$

We apply the functor L_Y to the de Rham–Witt complex $(W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n, \nabla)$ defined before. We view this as a complex consisting of $\mathcal{O}_Y$-modules by (3.48). If E_Y denotes the value of E at the pd-thickening $X \to Y$, we have the isomorphism

$$W_n\Omega^{\cdot}_{X/R} \otimes_{W_n(\mathcal{O}_X)} E_n \cong W_n\Omega^{\cdot}_{X/R} \otimes_{\mathcal{O}_Y} E_Y.$$

Using that ∇ is a connection, we find

$$[\nabla, f](\alpha) = \nabla(f\alpha) - f\nabla(\alpha) = \mathrm{d}f \wedge \alpha \quad \text{for } \alpha \in W_n\Omega^i_{X/R} \otimes_{\mathcal{O}_Y} E_Y f \in \mathcal{O}_Y. \tag{3.53}$$

Hence we find that $(W_n\Omega^{\cdot}_{X/R} \otimes_{\mathcal{O}_Y} E_Y, \nabla)$ is a complex of differential operators of order less than or equal to 1. Indeed, the extra condition (3.52) for $p = 2$ is fulfilled by (3.53). It therefore induces an HPD-differential operator which is explicitly given as follows:

$$\left.\begin{aligned} \nabla : \quad \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} W_n\Omega^i_{X/R} \otimes_{\mathcal{O}_Y} E_Y &\to W_n\Omega^{i+1}_{X/R} \otimes_{\mathcal{O}_Y} E_Y, \\ y \otimes \alpha &\mapsto y\nabla\alpha, \\ \xi_i \otimes \alpha &\mapsto \mathrm{d}[T_i] \wedge \alpha, \\ \xi_i^{[k]} \otimes \alpha &\mapsto 0. \end{aligned}\right\} \tag{3.54}$$

Here, $y \in \mathcal{O}_Y$, $k \geqslant 2$ and $\alpha \in W_n\Omega^i_{X/R} \otimes_{\mathcal{O}_Y} E_Y$. The differential $\mathrm{d}[T_i]$ appears because δ_n maps T_i to the Teichmüller representative $[T_i]$.

By Theorem 3.5, we have a quasi-isomorphism $\Omega^{\cdot}_{Y/S} \to W_n\Omega^{\cdot}_{X/R}$ which is transformed by $\boldsymbol{L}_Y$ into a quasi-isomorphism of complexes of crystals (Corollary 3.10). By [**BO**, 7.1], we obtain the quasi-isomorphisms

$$\mathcal{O}_{Y/S} \to \boldsymbol{L}_Y(\Omega^{\cdot}_{Y/S}) \to \boldsymbol{L}_Y(W_n\Omega^{\cdot}_{X/R}).$$

We note that the category of crystals over Y is equivalent to the category of crystals over X by [**BO**, 6.7]. If $\boldsymbol{K}$ is a crystal over Y, we will denote by $\iota^*\boldsymbol{K}$ the corresponding crystal over X.

If we apply the functor ι^*, we obtain a resolution in the category of crystals on X,

$$\mathcal{O}_{X/S} \to \iota^* \boldsymbol{L}_Y(W_n\Omega^{\cdot}_{X/R}).$$

Then we obtain a chain of isomorphisms in the derived category $D^+(X_{\mathrm{zar}})$,

$$\begin{aligned} Ru_{n*}E &\overset{(1)}{\cong} u_{n*}\boldsymbol{L}_Y(\Omega^{\cdot}_{Y/S} \otimes_{\mathcal{O}_Y} E_Y) \\ &\overset{(2)}{\cong} u_{n*}(\boldsymbol{L}_Y(\Omega^{\cdot}_{Y/S}) \otimes_{\mathcal{O}_Y} E_Y) \\ &\overset{(3)}{\cong} u_{n*}(\boldsymbol{L}_Y(W_n\Omega^{\cdot}_{X/R}) \otimes_{\mathcal{O}_Y} E_Y) \\ &\overset{(4)}{\cong} u_{n*}\boldsymbol{L}_Y(W_n\Omega^{\cdot}_{X/R} \otimes_{\mathcal{O}_Y} E_Y) \\ &\overset{(5)}{\cong} W_n\Omega^{\cdot}_{X/R} \otimes_{\mathcal{O}_Y} E_Y. \end{aligned} \tag{3.55}$$

Indeed, isomorphisms (1) and (5) follow from the proof of Theorem 7.1 in [**BO**]. Isomorphism (3) follows because E_Y is a flat $\mathcal{O}_Y$-module by assumption and because we have shown that $\boldsymbol{L}_Y(\Omega^{\cdot}_{Y/S}) \to \boldsymbol{L}_Y(W_n\Omega^{\cdot}_{X/R})$ is a quasi-isomorphism. Finally, we obtain isomorphisms (2) and (4) from [**BO**, Proposition 6.15]. Therefore, the proof of the Theorem 3.8 is finished modulo the missing proof of Lemma 3.9. □

Proof of Lemma 3.9. The question is local. We may assume that $S = \operatorname{Spec} R$ and that Y is étale over $\operatorname{Spec} R[T_1, \dots, T_d]$. We set $\xi_i = 1 \otimes T_i - T_i \otimes 1$ as before.

First we must verify that $L_Y(\mathrm{D}_2)L_Y(\mathrm{D}_1) = 0$. Using the explicit formula for the linearization, this reduces to the following identities. Let $f, g \in \mathcal{O}_Y$ and $m \in M_1$. Then the following relations hold:

$$[\mathrm{D}_2, f](\mathrm{D}_1 m) + \mathrm{D}_2([D_1, f]m) = 0, \tag{3.56}$$

$$[\mathrm{D}_2, f]([\mathrm{D}_1, g]m) + [\mathrm{D}_2, g]([D_1, f]m) = 0, \tag{3.57}$$

$$[\mathrm{D}_2, f]([\mathrm{D}_1, f]m) = 0. \tag{3.58}$$

We note that the last equation holds by assumption if $p = 2$.

The assumption that D_1 is a differential operator of order less than or equal to 1 is equivalent to the relation

$$\mathrm{D}_1(fgm) = f\mathrm{D}_1(gm) + g\mathrm{D}_1(fm) - fg\mathrm{D}_1(m). \tag{3.59}$$

A similar relation holds for D_2. From this and from $\mathrm{D}_2\mathrm{D}_1 = 0$, it is straightforward to verify the relations (3.56), (3.57) and (3.58). We do it only for the last relation with the assumption $p \neq 2$. We compute the left-hand side of (3.58),

$$\begin{aligned} [\mathrm{D}_2, f](\mathrm{D}_1(fm) - f\mathrm{D}_1(m)) &= \mathrm{D}_2(f\mathrm{D}_1(fm) - f^2\mathrm{D}_1(m)) - f\mathrm{D}_2(\mathrm{D}_1(fm) - f\mathrm{D}_1(m)) \\ &= \mathrm{D}_2(f\mathrm{D}_1(fm)) - \mathrm{D}_2(f^2\mathrm{D}_1(m)) + f\mathrm{D}_2(f\mathrm{D}_1 m). \end{aligned} \tag{3.60}$$

By (3.59), we find

$$\mathrm{D}_1(f^2m) = 2f\mathrm{D}_1(fm) - f^2\mathrm{D}_1(m).$$

Applying D_2 gives

$$\mathrm{D}_2(f^2\mathrm{D}_1(m)) = 2\mathrm{D}_2(f\mathrm{D}_1(fm)).$$

Using the fact that D_2 is a differential operator of order less than or equal to 1, we obtain

$$\mathrm{D}_2(f^2\mathrm{D}_1(m)) = 2f\mathrm{D}_2(f\mathrm{D}_1(m)).$$

If we add the last two equations, we find that (3.60) becomes zero when multiplied by 2. Hence the relation $L_Y(\mathrm{D}_2)L_Y(\mathrm{D}_1) = 0$ is established.

In the decomposition (3.50), we give $\xi_k \otimes m_k$ the grade $|k| = k_1 + \cdots + k_d$. Then we obtain $\mathbb{Z}_{\geqslant 0}$-graded abelian groups $\mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^i$ for $i = 1, 2, 3$. We write the formula (3.51) for D_1, D_2 as follows:

$$L_Y(\mathrm{D}_i)(\xi^{[k]} \otimes m_k) = \tilde{\mathrm{D}}_i^0(\xi^{[k]} \otimes m_k) + \tilde{\mathrm{D}}_i^-(\xi^{[k]} \otimes m_k). \tag{3.61}$$

Here, $\tilde{\mathrm{D}}_i^0(\xi^{[k]} \otimes m_k) = \xi^{[k]} \otimes \mathrm{D}_i m_k$ is the first summand on the right-hand side of (3.51) and $\tilde{\mathrm{D}}_i^-(\xi^{[k]} \otimes m_k)$ is the second. Then $\tilde{\mathrm{D}}_i^0$ for $i = 1, 2$ are homogeneous maps of degree 0 of graded abelian groups. Clearly, the sequence

$$\mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^1 \xrightarrow{\tilde{\mathrm{D}}_1^0} \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^2 \xrightarrow{\tilde{\mathrm{D}}_2^0} \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^3$$

is exact. The operators $\tilde{\mathrm{D}}_i^-$ for $i = 1, 2$ are homogeneous maps of degree -1 of graded abelian groups. Consider an element ω of degree h in $\mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^2$,

$$\omega = \sum_{|k| \leqslant h} \xi^{[k]} \otimes m_k, \quad m_k \in M^2,$$

which is in the kernel of $L_Y(\mathrm{D}_2)$. We set

$$\omega^h = \sum_{|k| = h} \xi^{[k]} \otimes m_k.$$

Then $L_Y(\mathrm{D}_2)(\omega) = 0$ implies, by homogeneity, that $\tilde{\mathrm{D}}_2^0(\omega^h) = 0$. Hence $\omega^h = \tilde{\mathrm{D}}_1^0 \eta^h$, where $\eta^h \in \mathcal{D}_{Y/S}(1) \otimes_{\mathcal{O}_Y} M^1$ is homogeneous of degree h. Then $\omega - L_Y(\mathrm{D}_1)(\eta^h)$ is of degree less than h and is in the kernel of $L_Y(\mathrm{D}_2)$. We conclude by induction on h that $\omega - L_Y(\mathrm{D}_1)(\eta^h)$ is in the image of $L_Y(\mathrm{D}_1)$. □

Appendix A. The ring of Witt vectors

In this appendix we collect some general facts about Witt vectors.

Let X be a scheme. Then we define, for any natural number n, a scheme $W_n(X)$ by a gluing process as follows. If $X = \operatorname{Spec} R$ is affine, we set $W_n(X) = \operatorname{Spec} W_n(R)$. We note that, for any element $f \in R$, there is a natural isomorphism

$$W_n(R)_{[f]} \cong W_n(R_f).$$

If elements $f_1, \ldots, f_r$ generate the unit ideal in R, then their Teichmüller representatives $[f_1], \ldots, [f_r]$ in $W_n(R)$ generate the unit ideal. Indeed, by induction, it suffices to show that any element of the form ${}^{V^{n-1}}[a]$ for $a \in R$ is in the ideal generated by the $[f_i]$. It suffices to find elements $x_i \in R$ such that the following equality holds in $W_n(R)$:

$$\sum_{i=1}^{r} ({}^{V^{n-1}}[x_i])[f_i] = {}^{V^{n-1}}[a].$$

This is equivalent to the following equality in R, which is clearly solvable:

$$\sum_{i=1}^{r} x_i f_i^{p^{n-1}} = a.$$

This shows that $W_n(X)$ is the union of the open subschemes $W_n(\operatorname{Spec} R_{f_i})$ for $i = 1, \ldots, r$. Morally, this means that the construction of $W_n(X)$ for an affine scheme X is local. If $\operatorname{Spec} S \to \operatorname{Spec} R$ is an open immersion of affine schemes, we easily deduce that $W_n(\operatorname{Spec} S) \to W_n(\operatorname{Spec} R)$ is an open immersion.

If U is a quasi-affine scheme, we choose an open embedding $U \to \operatorname{Spec} R$ and define $W_n(U)$ as the union of all affine subschemes $W_n(\operatorname{Spec} R_f)$ of $W_n(\operatorname{Spec} R)$ with $\operatorname{Spec} R_f \subset U$. One can show that this is independent of the chosen embedding.

Finally, if X is any scheme and U_α, $\alpha \in I$, is an affine covering, we define $W_n(X)$ as the ringed space obtained by gluing the affine schemes $W_n(U_\alpha)$ along the open subspaces $W_n(U_\alpha \cap U_\beta)$.

Proposition A.1. *If the scheme X is separated, so is $W_n(X)$.*

Proof. We apply the criterion [**EGA**, I, 5.5.6]. Then we are reduced to proving the following statement. Let $R_1 \to R$ and $R_2 \to R$ be ring homomorphisms, which induce open immersions of the affine schemes. We assume that the images of R_1 and R_2 generate R as a ring, i.e. $R = R_1 R_2$. Then the images of $W_n(R_1)$ and $W_n(R_2)$ generate $W_n(R)$ as a ring.

We assume this assertion for n and show it for $n + 1$. We consider the situation modulo p. It follows from the isomorphism (A.5) below that $R = R^{p^n} R_1 + pR$ and hence, by our assumption, that $R = R_2^{p^n} R_1 + pR$. Iterating this equation we find $R = R_2^{p^n} R_1 + p^n R_1 R_2$. This means that any element $a \in R$ may be expressed in the following form,

$$a = \sum_{i=1}^{r} x_i y_i^{p^n} + \sum_{j=1}^{s} x'_j p^n y'_j,$$

where the x_i and x'_j are elements of R_1 and the y_i and y'_j are elements of R_2. But this implies the following identity in $W_{n+1}(R)$:

$$V^n[a] = \sum_{i=1}^{r} (V^n[x_i])[y_i] + \sum_{j=1}^{s} V^n[x'_j] V^n[y'_j].$$

Hence $V^n W_1(R)$ is in the subring of $W_{n+1}(R)$ generated by $W_{n+1}(R_1)$ and $W_{n+1}(R_2)$. We obtain the result from the induction assumption. □

We remark that this construction becomes trivial if p is nilpotent in R. In this case, the kernel of $\boldsymbol{w}_0 : W_n(R) \to R$ is nilpotent. Therefore, if p is locally nilpotent on X, the scheme $W_n(X)$ has the same topological space as X but the structure sheaf is $W_n(\mathcal{O}_X)$.

We want to formulate finiteness conditions for $W_n(X)$ in terms of X.

Proposition A.2. *Let R be a $\mathbb{Z}_{(p)}$-algebra. Then the following conditions are equivalent.*

(i) *For some number $n \geqslant 1$, the Frobenius $F : W_{n+1}(R) \to W_n(R)$ is a finite ring homomorphism.*

(i bis) *For each number $n \geqslant 1$, the Frobenius $F : W_{n+1}(R) \to W_n(R)$ is a finite ring homomorphism.*

(ii) *For some number $n \geqslant 1$, the Witt polynomial $\boldsymbol{w}_n : W_{n+1}(R) \to R$ is a finite ring homomorphism.*

(ii bis) *For each number $n \geqslant 1$, the Witt polynomial $\boldsymbol{w}_n : W_{n+1}(R) \to R$ is a finite ring homomorphism.*

(iii) *The absolute Frobenius* $\mathrm{Frob} : R/pR \to R/pR$ *is a finite ring homomorphism.*

Proof. If, for some number $n \geqslant 1$, the homomorphism $\boldsymbol{w}_n : W_{n+1}(R) \to R$ is finite, then $\boldsymbol{w}_n : W_{n+1}(R/p^nR) \to R/p^nR$ is obviously finite too. The converse statement is also true. Indeed, let $x_1, \dots, x_k \in R$ generate $(R/p^nR)_{[\boldsymbol{w}_n]}$, i.e. R/p^nR considered as a $W_{n+1}(R/p^nR)$-module via $\boldsymbol{w}_n : W_{n+1}(R/p^nR) \to R/p^nR$. Then any element of R has the representation

$$\sum_{i=1}^{k} \boldsymbol{w}_n(\xi_i)x_i + p^n r = \sum_{i=1}^{k} \boldsymbol{w}_n(\xi_i)x_i + \boldsymbol{w}_n({}^{V^n}[r])1.$$

This shows that $R_{[\boldsymbol{w}_n]}$ is finitely generated too.

Moreover, by the lemma of Nakayama, $(R/p^nR)_{[\boldsymbol{w}_n]}$ is finitely generated if and only if $(R/pR)_{[\boldsymbol{w}_n]}$ is finitely generated. The map $\boldsymbol{w}_n : W_{n+1}(R) \to R/pR$ factors as follows:

$$W_{n+1}(R) \xrightarrow{\boldsymbol{w}_0} R/pR \xrightarrow{\mathrm{Frob}^n} R/pR.$$

Since the first map here is surjective, we see that $\boldsymbol{w}_n : W_{n+1}(R) \to R/pR$ is finite if and only if $\mathrm{Frob} : R/pR \to R/pR$ is finite. Therefore, we have shown that conditions (ii), (ii bis) and (iii) are equivalent. These conditions are also equivalent with (i) for $n = 1$, since the map $F : W_2(R) \to R$ coincides with $\boldsymbol{w}_1$.

We now show that condition (i) implies (ii). Knowing this for $n = 1$, we apply induction. For $n > 1$, we consider the following commutative diagram:

$$\begin{tikzcd} W_{n+1}(R) \arrow[r, "F"] \arrow[d, "\mathrm{Res}"'] & W_n(R) \arrow[d, "\mathrm{Res}"] \\ W_n(R) \arrow[r, "F"] & W_{n-1}(R) \end{tikzcd}$$

This shows that condition (i) holds for $n - 1$. We conclude by induction.

Finally, we show that condition (ii bis) implies (i bis). We prove by induction on n that the homomorphism $F : W_{n+1}(R) \to W_n(R)$ is finite. For $n = 1$, this is $\boldsymbol{w}_1$, which starts our induction. Let us assume that $n > 1$ and that $F : W_n(R) \to W_{n-1}(R)$ is finite. We denote by $W_{n-1}(R)_{[F]}$ the $W_{n+1}(R)$-module obtained by the homomorphism

$$W_{n+1}(R) \xrightarrow{\text{Res}} W_n(R) \xrightarrow{F} W_{n-1}(R).$$

Then $W_{n-1}(R)_{[F]}$ is finitely generated by induction.

We obtain an exact sequence,

$$0 \to R_{[\boldsymbol{w}_n]} \xrightarrow{V^{n-1}} W_n(R)_{[F]} \to W_{n-1}(R)_{[F]} \to 0.$$

Then the module in the middle is finitely generated because the other modules in this sequence are. □

Definition A.3. We call a ring R, which satisfies the equivalent conditions of the last proposition, F-finite.

Let X be a scheme. The Frobenius on the Witt vectors induces a morphism $F : W_n(X) \to W_{n+1}(X)$. Let $U \subset X$ be an open subset. By reduction to the affine case, we show that the following diagram is Cartesian:

$$\begin{array}{ccc} W_n(U) & \xrightarrow{F} & W_{n+1}(U) \\ \downarrow & & \downarrow \\ W_n(X) & \xrightarrow{F} & W_{n+1}(X) \end{array} \tag{A.1}$$

This shows that the morphism $F : W_{n+1}(X) \to W_n(X)$ is finite if and only if X admits an open covering by affine schemes $\operatorname{Spec} R_i$ such that each ring R_i is finite. In this case, we say that the scheme X is F-finite.

Proposition A.4. *Let R be an F-finite noetherian ring. Then $W_{n+1}(R)$ is a noetherian ring for each number $n \geqslant 0$.*

Proof. Since $\boldsymbol{w}_n : W_{n+1}(R) \to R$ is a finite ring homomorphism and since R is noetherian, we easily see that $R_{[\boldsymbol{w}_n]}$ is a noetherian $W_{n+1}(R)$-module. We consider the exact sequence

$$0 \to R_{[\boldsymbol{w}_n]} \xrightarrow{V^n} W_{n+1}(R) \to W_n(R) \to 0.$$

This shows the proposition by induction on n. □

Proposition A.5. *Let R be an F-finite ring and S be a finitely generated R-algebra. Then, for each number n, the $W_n(R)$-algebra $W_n(S)$ is finitely generated.*

Proof. It is enough to prove our proposition in the case of a polynomial algebra in one variable $S = R[T]$. We consider the morphism $W_{n+1}(R)[X] \to W_{n+1}(R[T])$ which maps

X to the Teichmüller representative $[T]$. We have to prove that this last homomorphism is of finite type. We will see that this homomorphism is even finite.

Let us consider the exact sequence

$$0 \to R[T]_{[\boldsymbol{w}_n]} \xrightarrow{V^n} W_{n+1}(R[T]) \to W_n(R[T]) \to 0.$$

By induction it is enough to prove that $R[T]_{[\boldsymbol{w}_n]}$ is a finitely generated module over $W_{n+1}(R)[X]$. But this module is obtained from the homomorphism

$$W_{n+1}(R)[X] \to R[X] \to R[T],$$

where the first morphism is induced by $\boldsymbol{w}_n$ and the second is the R-algebra homomorphism which maps X to T^{p^n}. Since both morphisms are finite, we are done. $\square$

From the last proposition we deduce the global version.

Proposition A.6. *Let T be an F-finite scheme. If $X \to T$ is a morphism of finite type, then X is F-finite. The morphism $W_n(X) \to W_n(T)$ is of finite type. If T is noetherian, then $W_n(X)$ and $W_n(T)$ are noetherian.*

Corollary A.7. *Let T be an F-finite scheme. If $X \to T$ is a proper morphism, then the morphism $W_n(X) \to W_n(T)$ is proper.*

Proof. We assume the corollary for n and show it for $n+1$. Consider the commutative diagram

$$\begin{tikzcd} W_n(X) \sqcup W_n(X) \arrow[r] \arrow[d] & W_{n+1}(X) \arrow[d] \\ W_n(T) \sqcup W_n(T) \arrow[r] & W_{n+1}(T) \end{tikzcd} \tag{A.2}$$

The horizontal arrows are induced by F on the first summand and by the restriction on the second summand. These morphisms are finite by the proposition. It follows from the induction that the diagonal in the diagram above is proper. The arrow $W_{n+1}(X) \to W_{n+1}(T)$ is separated by Proposition A.1. Therefore, it suffices to show that the upper horizontal arrow in the diagram is surjective. For this we may restrict ourselves to the case where $X = \operatorname{Spec} R$ is affine. Then it suffices to show that the kernel of the following map is nilpotent for $n \geqslant 1$,

$$W_{n+1}(R) \xrightarrow{(F,\mathrm{Res})} W_n(R) \times W_n(R).$$

But this kernel consists of elements ${}^{V^n}[a]$ with $a \in R$ and $pa = 0$. It is clear that the product of two of these elements is zero. $\square$

Next we find conditions which ensure that the functor $W_n(X)$ takes étale morphisms to étale morphisms. We begin with the case where the prime p is nilpotent.

Proposition A.8. *Let R be a ring such that p is nilpotent in R. Let $R \to S$ be an étale morphism. Then, for each number n, the morphism of Witt rings $W_n(R) \to W_n(S)$ is étale. For $m < n$, the natural restriction map $W_n(S) \to W_m(S)$ induces an isomorphism,*

$$W_n(S) \otimes_{W_n(R)} W_m(R) \cong W_m(S). \tag{A.3}$$

In particular, we obtain the isomorphisms

$$W_n(S) \otimes_{W_n(R)} {}^{V^m}W_{n-m}(R) \cong {}^{V^m}W_{n-m}(S).$$

Proof. Take any elements $u_1, \ldots, u_r$, which generate S as an R-algebra. We denote by $t_i = [u_i] \in W(S)$ their Teichmüller representatives.

Lemma A.9. *We have*

$$W(S) = W(R)[t_1, \ldots, t_r]$$

More precisely, any element of ${}^{V^m}W(S)$ may be written as a polynomial in $t_1, \ldots, t_r$ with coefficients in ${}^{V^m}W(R)$.

Proof. We show that

$$S = R[u_1^p, \ldots, u_r^p]. \tag{A.4}$$

Since p is nilpotent, we may restrict by the lemma of Nakayama to the case where $pR = 0$. One considers the relative Frobenius over R,

$$R \otimes_{\mathrm{Frob},R} S \to S. \tag{A.5}$$

This is known to be an isomorphism. Indeed, the morphism of affine schemes induced by (A.5) is obviously radical and surjective. On the other hand, both sides of (A.5) are étale over R, and therefore the morphism is also étale. Hence we have an isomorphism by [**EGA**, IV, 17.9.1]. From the isomorphism (A.5), we conclude (A.4) in the case $pR = 0$.

Let us consider an element ${}^{V^m}\xi \in W(S)$ and denote by $\xi_0 = \boldsymbol{w}_0(\xi)$ its first Witt component. By (A.4), we may write

$$\xi_0 = \sum_I a_I u^{p^m I},$$

where the sum goes over multi-indices $I = (i_1, \ldots, i_r)$ and $a_I \in R$. Hence

$$\xi - \sum_I [a_I]^{F^m} t^I \in {}^{V}W(S).$$

If we apply V^m to the last equation we obtain

$${}^{V^m}\xi - \sum_I {}^{V^m}[a_I] t^I = {}^{V^{m+1}}\eta$$

for some $\eta \in W(S)$. From this, the lemma follows by an easy induction. $\square$

We continue with the proof of the proposition. The lemma shows that $W_n(R) \to W_n(S)$ is of finite type and that the isomorphism (A.3) holds.

We want to think in terms of schemes. We set $Y = \operatorname{Spec} S$, $X = \operatorname{Spec} R$, $W_n(Y) = \operatorname{Spec} W_n(S)$ and $W_n(X) = \operatorname{Spec} W_n(R)$. For $m < n$, the Witt polynomials $\boldsymbol{w}_m$ define morphisms

$$\omega_m : Y \to W_n(Y).$$

These maps are radical and surjective, and the homeomorphism induced on the underlying spaces is independent of m.

Next we verify that the morphism $W_n(Y) \to W_n(X)$ is unramified. Since $\omega_0 : X \to W_n(X)$ is a nilimmersion, we may lift the étale scheme Y over X to an étale scheme Z_n over $W_n(X)$. Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\omega_0"'] & Z_n \arrow[d] \\ W_n(Y) \arrow[r] & W_n(X) \end{tikzcd}$$

Applying the infinitesimal criterion for étale to the étale morphism $Z_n \to W_n(X)$, we obtain an arrow $W_n(Y) \to Z_n$. We set $Z_n = \operatorname{Spec} C_n$ and consider the co-morphism $C_n \to W_n(S)$ of $W_n(R)$-algebras. Since the composite with $\boldsymbol{w}_0$ is surjective, we find that

$$\bar{C}_n + I_S = W_n(S),$$

where $\bar{C}_n$ denotes the image of C_n in $W_n(S)$. By the lemma, we know that $I_S = I_R W_n(S)$. Hence the lemma of Nakayama shows that $\bar{C}_n = W_n(S)$. Hence $W_n(Y) \to Z_n$ is a closed immersion, which shows that $W_n(Y) \to W_n(X)$ is unramified.

Now we show that the following diagram is a fibre product:

$$\begin{tikzcd} R\ Y \arrow[r, "\omega_m"] \arrow[d] & W_n(Y) \arrow[d] \\ X \arrow[r, "\omega_m"] & W_n(X) \end{tikzcd}$$

Indeed, consider the fibre product T and the canonical morphism $Y \to T$. Since T is unramified over X, by what we have shown, and since Y is étale over X by assumption, the morphism $Y \to T$ is étale (cf. [**EGA**, IV, 17.7.10, 17.1.4]). On the other hand, this morphism is radical and surjective. Therefore, we conclude that $Y = T$, as desired.

Let $I_{B,n-1} = {}^{V^{n-1}}W_1(S) \subset W_n(S)$. This is an ideal, which is isomorphic to S considered as a $W_n(S)$ via $\boldsymbol{w}_{n-1}$. Therefore, the Cartesian diagram above just says that we have the isomorphism

$$I_{R,n-1} \otimes_{W_n(R)} W_n(S) = I_{S,n-1}.$$

On the other hand, we have already remarked that $I_{R,n-1}W_n(S) = I_{B,n-1}$. Therefore, by the local criterion for flatness, we deduce that $W_n(R) \to W_n(S)$ is flat if $W_{n-1}(R) \to W_{n-1}(S)$ is flat. The proposition follows by induction. $\square$

Corollary A.10. *Let p be nilpotent in R. If $R \to S$ is an unramified homomorphism, then, for each $n \geqslant 0$, the homomorphism $W_n(R) \to W_n(S)$ is also unramified.*

Proof. This is clear if $R \to S$ is surjective or étale. The general case follows from [**EGA**, IV, 18.4.7]. □

Corollary A.11. *With the assumptions of Proposition A.8, the homomorphism $F : W_{n+1}(S) \to W_n(S)$ induces an isomorphism of $W_n(R)$-algebras,*

$$W_{n+1}(S) \otimes_{W_{n+1}(R),F} W_n(R) \to W_n(S). \tag{A.6}$$

Proof. We see that the left-hand side of (A.6) is étale over $W_n(R)$ if we tensor the étale morphism $W_{n+1}(R) \to W_{n+1}(S)$ by $\otimes_{W_{n+1}(R),F} W_n(R)$. Hence the morphism (A.6) is étale since it is a morphism of étale $W_n(R)$-algebras.

On the other hand, we have the commutative diagram

$$\begin{tikzcd} W_{n+1}(S) \otimes_{W_{n+1}(R),F} W_n(R) \arrow[r] \arrow[d] & W_n(S) \arrow[d] \\ S/pS \otimes_{R/pR,\mathrm{Frob}} R/pR \arrow[r] & S/pS \end{tikzcd} \tag{A.7}$$

The vertical arrows are induced by $\boldsymbol{w}_0$. They are surjective with nilpotent kernel. Since the arrow below is an isomorphism by (A.5), we conclude that (A.6) induces a morphism of affine schemes which is radical and surjective. Since we know that this morphism is étale, it is an isomorphism. □

Corollary A.12. *With the assumption of Proposition A.8, let R' be an R-algebra. We set $S' = S \otimes_R R'$. Then we have a canonical isomorphism for each number n,*

$$W_n(S) \otimes_{W_n(R)} W_n(R') \to W_n(S'). \tag{A.8}$$

Proof. Again, the canonical map (A.8) is étale, since both sides are étale over $W_n(R')$. Using the commutative diagram

$$\begin{tikzcd} W_n(S) \otimes_{W_n(R)} W_n(R') \arrow[r] \arrow[d] & W_n(S') \arrow[d] \\ S \otimes_R R' \arrow[r] & S' \end{tikzcd} \tag{A.9}$$

we conclude as in the proof of the last corollary. □

Corollary A.13. *With the assumption of Proposition A.8, let $\mathfrak{a} \subset R$ be an ideal. Then we have, for each number n, the natural isomorphisms*

$$\left.\begin{aligned} W_n(S) \otimes_{W_n(R)} W_n(R/\mathfrak{a}) &\cong W_n(S/\mathfrak{a}S), \\ W_n(S) \otimes_{W_n(R)} W_n(\mathfrak{a}) &\cong W_n(\mathfrak{a}S). \end{aligned}\right\} \tag{A.10}$$

Proof. The first isomorphism is a special case of the second. Since $W_n(S)$ is flat over $W_n(R)$, the first isomorphism implies the second. □

We want to remove the condition that p is nilpotent in Proposition A.8. Instead, we introduce the condition F-finite.

Proposition A.14. *Let R be an F-finite ring and let S be an étale R-algebra. Then $W_n(S)$ is étale over $W_n(R)$ for each number n.*

For the proof, we need a form of the local criterion of flatness (see [**BAC1**, Chapter III, § 5, Théorème 1]).

Lemma A.15. *Let $A \to B$ be a homomorphism of noetherian rings. Let $a \in A$ be an element such that the homomorphism obtained by localization $A_a \to B_a$ is flat. Assume, moreover, that for each number n the homomorphism $A/a^nA \to B/a^nB$ is flat.*

Then the homomorphism $A \to B$ is flat.

Proof. Consider the multiplicatively closed system $U = 1 + aB$ of B. Then the image of $\operatorname{Spec} B_U \to \operatorname{Spec} B$ contains $V(a) \subset \operatorname{Spec} B$. We set $C = B_a \times B_U$. This is a faithfully flat B-algebra. Hence it is enough to show that $A \to C$ is flat, i.e. we must show that $A \to B_U$ and $A \to B_a$ are flat. This is clear for the last arrow.

It remains to be shown that $A \to B_U$ is flat. By [**BAC1**, Chapter III, § 5, Proposition 2], B_U is an ideally separated A-module with respect to a. Therefore, [**BAC1**, Théorème 1] says that it is enough to verify that $A/a^nA \to B_U/a^nB_U$ is flat for all numbers $n \geqslant 1$. But, because of the isomorphism $B_U/a^nB_U \cong B/a^nB$, this is true by assumption. □

Corollary A.16. *Let $A \to B$ be a homomorphism of finite type of noetherian rings. Let $a \in A$ be an element such that the homomorphism obtained by localization $A_a \to B_a$ is étale. Assume, moreover, that for each number n the homomorphism $A/a^nA \to B/a^nB$ is étale.*

Then the homomorphism $A \to B$ is étale.

Proof. We have to show that $A \to B$ is unramified, i.e. $\Omega^1_{B/A} = 0$. By assumption, $\Omega^1_{B/A}$ is a B-module of finite type. Since the module of Kähler differentials commutes with base change, we have $(\Omega^1_{B/A})_a = 0$ and $\Omega^1_{B/A} \otimes_A A/aA = 0$. We conclude that $\Omega^1_{B/A} = 0$. □

Proof of Proposition A.14. We apply the last corollary to the homomorphism $W_n(R) \to W_n(S)$. We take for a the Teichmüller representative $[p] \in W_n(R)$. We have to prove that the following ring homomorphisms are étale:

$$W_n(R)_{[p]} \to W_n(S)_{[p]}, \tag{A.11}$$

$$W_n(R)/[p]^m W_n(R) \to W_n(S)/[p]^m W_n(S). \tag{A.12}$$

We have the isomorphisms

$$W_n(R)_{[p]} \cong W_n(R_p) \cong R_p \times \cdots \times R_p. \tag{A.13}$$

The last isomorphism is provided by the Witt polynomials. Since the same holds for S, we see that (A.11) is étale.

It remains to be shown that (A.12) is étale for each number m. Obviously, we have the following inclusions:

$$W_n(p^{mp^{n-1}}R) \subset p^m W_n(R).$$

We set $c = p^{mp^{n-1}}$. Then we find the isomorphisms

$$\begin{aligned} W_n(R)/[p]^m W_n(R) &\cong (W_n(R)/W_n(p^c R))/[p]^m (W_n(R)/W_n(p^c R)) \\ &\cong W_n(R/p^c R)/[p]^m W_n(R/p^c R). \end{aligned} \tag{A.14}$$

Since the same holds for S, the arrow (A.12) may be identified with

$$W_n(R/p^c R)/[p]^m W_n(R/p^c R) \to W_n(S/p^c S)/[p]^m W_n(S/p^c S).$$

But this is étale because, by Proposition A.8, $W_n(R/p^c R) \to W_n(S/p^c S)$ is étale. $\square$

Corollary A.17. *Let R be an F-finite ring, and let S be an unramified R-algebra. Then $W_n(S)$ is unramified over $W_n(R)$ for each number n.*

This follows in the same way as Corollary A.10.

Corollary A.18. *Let $R \to S$ be an étale morphism of F-finite rings. Then we have the following natural isomorphisms for arbitrary numbers $n \geqslant m \geqslant 1$:*

$$\left.\begin{aligned} W_n(S) \otimes_{W_n(R)} W_m(R) \to W_m(S), \\ W_{n+1}(S) \otimes_{W_{n+1}(R),F} W_n(R) \to W_n(S). \end{aligned}\right\} \tag{A.15}$$

Moreover, let R' be an R-algebra. Then we have the natural isomorphism

$$W_n(S) \otimes_{W_n(R)} W_n(R') \to W_n(S \otimes_R R'). \tag{A.16}$$

If $\mathfrak{a}$ is an ideal in R, we have the isomorphism

$$W_n(S) \otimes_{W_n(R)} W_n(\mathfrak{a}) \to W_n(\mathfrak{a}S). \tag{A.17}$$

Proof. If, in the notation of Lemma A.15, $A_a \to B_a$ and $A/a^m A \to B/a^m B$ are injective, then $A \to B$ is injective. This is a consequence of Krull's intersection theorem (see [**BAC2**]). If we assume that $A \to B$ is finite, the same statement holds for injective replaced by surjective.

Let us begin with the first homomorphism of (A.15). We have a canonical surjection,

$$W_n(S) \otimes_{W_n(R)} W_m(R) \to W_m(S).$$

We apply our starting remark to $a = [p]$. If we localize the surjection by $[p]$, it becomes an isomorphism by Equation (A.13). If we consider the morphism modulo $[p]^m$, we obtain an isomorphism using (A.14) and Proposition A.8.

The next homomorphism of (A.15) is, by assumption, finite. Therefore, it suffices to show that it becomes an isomorphism if we localize by $[p]$, and if we consider it modulo $[p^m]$. We conclude as above, using Corollary A.11.

We show now that the canonical homomorphism (A.16) is an isomorphism. Let us first consider the case where p is nilpotent in R'. In this case, it follows from Corollary A.12 that the homomorphism (A.16) is surjective. Therefore, we need only to verify that the homomorphism is injective modulo p^m, which can be done as above.

We use induction and assume that our assertion is true for n. We set $S' = S \otimes_R R'$. By the case where p is nilpotent in R' it is enough to prove that, for some m, the group $W_{n+1}(p^m S')$ is in the image of the homomorphism

$$W_{n+1}(S) \otimes_{W_{n+1}(R)} W_{n+1}(R') \to W_{n+1}(S'). \tag{A.18}$$

We consider the commutative diagram (with tensor products taken over $W_n(R)$)

$$\begin{tikzcd} W_n(S) \otimes W_n(p^m R') \arrow[r] \arrow[d] & W_n(S) \otimes W_n(R') \arrow[r] \arrow[d] & W_n(S) \otimes W_n(R'/p^m R') \arrow[d] \\ W_n(p^m S') \arrow[r] & W_n(S') \arrow[r] & W_n(S'/p^m S') \end{tikzcd} \tag{A.19}$$

Note that the first row is a short exact sequence because $W_n(S)$ is étale over $W_n(R)$. We have shown that the right vertical arrow is an isomorphism, and assume it by induction for the middle one. Hence the left vertical arrow is an isomorphism.

Now we consider an arbitrary $\xi \in W_{n+1}(p^m S')$. We want to show that it is in the image of (A.18). Let $\bar{\xi} \in W_n(p^m)(S')$ be its residue class. By the last diagram, $\bar{\xi}$ is the image of an element $\bar{\eta} \in W_n(S) \otimes_{W_n(R)} W_n(p^m R')$. This element we lift to $\eta \in W_{n+1}(S) \otimes_{W_{n+1}(R)} W_{n+1}(R')$. Then the image ξ_1 of η by (A.18) is in $W_{n+1}(p^m S')$. On the other hand, $\rho = \xi - \xi_1$ maps to zero in $W_n(p^m S')$. Hence $\rho = {}^{V^n}[p^m s']$ for some $s' \in S'$. We have to show that, for some fixed m, expressions of the form ${}^{V^n}[p^m s']$ are in the image of (A.18). But this is an immediate consequence of the equation in $W_{n+1}(S')$,

$${}^{V^n}[s]^{V^n}[r'] = {}^{V^n}([p^n s r']).$$

Taking $m = n$, this proves the surjectivity of (A.18). The injectivity is done as usual, by considering the morphism modulo powers of $[p]$ and by considering the localization with respect to p. This proves the isomorphism (A.16). The isomorphism (A.17) is a formal consequence (cf. Corollary A.13). □

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