

Drinfeld Modular Polynomials in Higher Rank

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Abstract

We study modular polynomials classifying cyclic isogenies between Drinfeld modules of arbitrary rank over the ring $\mathbb{F}_q[T]$. We derive bounds for the coefficients of these polynomials, and compute some explicit examples in the case where $q = 2$, the rank is 3 and the isogenies have degree T .

Keywords: Drinfeld modules, modular polynomials, sublattices.

1 Introduction and Results

Let F be an algebraically closed field which contains the ring $A := \mathbb{F}_q[T]$. Let $K = \mathbb{F}_q(T)$ be the quotient ring of A . Denote by $\text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,F})$ the ring of \mathbb{F}_q -linear endomorphisms of the additive group over F , it is isomorphic to the non-commutative ring of \mathbb{F}_q -linear polynomials in X with coefficients in F and multiplication defined by composition of polynomials.

A Drinfeld module $\rho : A \rightarrow \text{End}_{\mathbb{F}_q}(\mathbb{G}_{a,F})$ of rank r in generic characteristic is uniquely determined by

$$\rho(T)(X) = TX + g_1(\rho)X^q + \cdots + g_{r-1}(\rho)X^{q^{r-1}} + \Delta(\rho)X^{q^r}$$

with coefficients $g_1(\rho), \dots, g_{r-1}(\rho), \Delta(\rho) \in F$ and $\Delta(\rho) \neq 0$. We refer the reader to [4, chapter 4] for an overview of Drinfeld modules.

The coefficients $g_1(\rho), \dots, g_{r-1}(\rho), \Delta(\rho)$ describe the isomorphism class of ρ in the following way (see [7]). Let $g_1, \dots, g_{r-1}, \Delta$ be indeterminates and define

$$j_k = \frac{g_k^{(q^r-1)/(q^{\gcd(k,r)}-1)}}{\Delta^{(q^k-1)/(q^{\gcd(k,r)}-1)}} \text{ for } k = 1, \dots, r-1.$$

Let in addition u_1, \dots, u_{r-1} satisfy the Kummer equations

$$u_k^{(q^r-1)/(q^{\gcd(k,r)}-1)} = j_k \text{ for } k = 1, \dots, r-1.$$

Consider the ring $A[u_1, \dots, u_{r-1}]$. The group $G = \mathbb{F}_{q^r}^*/\mathbb{F}_q^*$ acts on it by

$$u_k^{\bar{\beta}} = \beta^{q^k-1} \cdot u_k \text{ for } \beta \in \mathbb{F}_{q^r}^*.$$

The subring of invariant elements, which is denoted by $B_r = A[u_1, \dots, u_{r-1}]^G$, plays the crucial part in the following isomorphism problem:

Two Drinfeld modules ρ and $\tilde{\rho}$ are isomorphic over F if and only if $I(\rho) = I(\tilde{\rho})$ for each invariant $I \in B_r$. In other words the affine space

$$M^r = \text{Spec}(B_r)$$

is the coarse moduli space for Drinfeld modules of rank r and no level structure.

Let \mathbb{C}_∞ denote the completion of an algebraic closure of $\mathbb{F}_q((\frac{1}{T}))$, and denote by

$$\Omega^r = \mathbb{P}^{r-1}(\mathbb{C}_\infty) \setminus \{\text{Linear subvarieties defined over } \mathbb{F}_q((\frac{1}{T}))\}$$

the Drinfeld upper half-space. Then the moduli space M^r is given analytically by (see [2])

$$M^r(\mathbb{C}_\infty) \cong \text{GL}_r(A) \backslash \Omega^r.$$

Points in Ω^r correspond to Drinfeld A -modules of rank r . Two points $\tau, \tau' \in \Omega^r$ correspond to isomorphic (respectively isogenous) Drinfeld modules ρ^τ and $\rho^{\tau'}$ if and only if $\tau' = \sigma(\tau)$ with $\sigma \in \text{GL}_r(A)$ (respectively $\text{GL}_r(K)$).

Let $n \in A$ be monic, and let $\mathcal{H}_n \subset \mathrm{GL}_r(K)$ be a set of representatives for

$$\mathrm{GL}_r(A) \backslash \mathrm{GL}_r(A) \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \mathrm{GL}_r(A).$$

Then \mathcal{H}_n represents isomorphism classes of cyclic n -isogenies in the following sense. If $h \in \mathcal{H}_n$ and $\tau \in \Omega^r$, then there exists an isogeny $f : \rho^{h(\tau)} \rightarrow \rho^\tau$ with kernel $\ker f \cong A/nA$ as A -modules. (Dually, there is an isogeny $\hat{f} : \rho^\tau \rightarrow \rho^{h(\tau)}$ with kernel $\ker \hat{f} \cong (A/nA)^{r-1}$). We write $\psi_r(n) := \#\mathcal{H}_n$.

Now, for every invariant $I \in B_r$ we define the *modular polynomial* of the invariant I and level n by

$$P_{I,n}(X) := \prod_{h \in \mathcal{H}_n} (X - I \circ h).$$

Its coefficients can be viewed as functions on Ω^r . Evaluating these coefficients at a point of Ω^r corresponding to a Drinfeld module ρ gives a polynomial whose roots are precisely those $I(\tilde{\rho})$ for which there exists a cyclic n -isogeny $\tilde{\rho} \rightarrow \rho$.

Let $f \in F[u_1, \dots, u_{r-1}]$, we denote by $w(f)$ the weighted degree of f , where each monomial is assigned the weight

$$w(u_1^{\alpha_1} \cdots u_{r-1}^{\alpha_{r-1}}) := \sum_{k=1}^{r-1} \alpha_k \frac{q^k - 1}{q^r - 1}.$$

For a polynomial $n \in A$ we denote $|n| = q^{\deg n}$.

The aim of this paper is to prove the following result:

Theorem 1.1 *Let $I \in B_r = A[u_1, \dots, u_{r-1}]^G$ be an invariant of weighted degree $w(I)$, and $n \in A$ monic. Then*

1. *We have $P_{I,n}(X) \in B_r[X]$, which is monic of degree*

$$\psi_r(n) = |n|^{r-1} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}}$$

in X , and is irreducible in $\mathbb{C}_\infty[u_1, \dots, u_{r-1}][X]$.

2. *The weighted degree of the coefficient $a_i \in B_r$ of X^i in $P_{I,n}(X)$ is bounded by:*

$$w(a_i) \leq \left(|n|^{2(r-1)} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}} - i |n|^{r-1} \right) w(I).$$

Example. When $r = 2$ we need only the usual j -invariant,

$$j = u_1^{q+1} = \frac{g_1^{q+1}}{\Delta}.$$

We have

$$P_{j,n}(X) = \Phi_n(X, j) = \Phi_n(X, u_1^{q+1}),$$

where $\Phi_n(X, Y) \in A[X, Y]$ is the modular polynomial of level n constructed by Bae [1]. An example with $r = 3$ is computed at the end of the paper.

Outline of the paper. In §2 we give a purely algebraic proof that the coefficients of $P_{I,n}(X)$ are integral over B_r , before starting with analytic considerations in §3.

We define in §3.2 a parameter $q_{\Lambda_{r-1}}(z_r)$, which may be viewed as a local parameter at the cusp of the compactification of M^r , although we make no use of this interpretation. Section 3 is devoted to computing the expansions of various lattice invariants in terms of this parameter.

In Section 4 we study sublattices with cyclic quotients, as these correspond to cyclic isogenies. After expanding invariants of sublattices in terms of the parameter, we count the number of such sublattices in §4.2, prove a non-cancellation result for pole orders in §4.3 and finally prove Theorem 1.1 in §4.4.

§5 is devoted to computing modular polynomials in the case $r = 3$, $q = 2$ and $n = T$.

2 Integrality of invariants

We first prove a result which implies that the coefficients of $P_{I,n}(X)$ are in fact integral over B_r . This is a first step towards showing that $P_{I,n}(X) \in B_r[X]$.

Proposition 2.1 *Let ρ and $\tilde{\rho}$ be isogenous Drinfeld modules, and denote by B_r and \tilde{B}_r their respective rings of invariants. Then B_r is integral over \tilde{B}_r , and \tilde{B}_r is integral over B_r .*

Proof. Let $f : \rho \rightarrow \tilde{\rho}$ be an isogeny, defined over a field F , so

$$f \circ \rho_T = \tilde{\rho}_T \circ f. \tag{1}$$

Since for any $c \in F^*$ we have $(cf) \circ \rho_T = c\tilde{\rho}_T c^{-1} \circ (cf)$, and we're only interested in $\tilde{\rho}$ up to isomorphism, we may assume that f , viewed as a linear polynomial in $F[X]$, is monic.

We write

$$\begin{aligned} f &= f_0 X + f_1 X^q + \cdots + X^{q^N}, \\ \rho_T &= TX + g_1 X^q + \cdots + \Delta X^{q^r}, \\ \tilde{\rho}_T &= TX + \tilde{g}_1 X^q + \cdots + \tilde{\Delta} X^{q^r}. \end{aligned}$$

We may further assume that $\Delta = 1$.

We may choose a monic $n \in A$ such that $\ker(f) \subset \rho[n] = \ker(\rho_n)$. Since ρ_n is monic, we see that all the points of $\ker(f)$, and hence also the coefficients of f , are integral over $A[g_1, g_2, \dots, g_{r-1}]$.

Next, we use (1) to compute the coefficients of $\tilde{\rho}$, to obtain successively

$$\begin{aligned}\tilde{\Delta}_r &= 1, \\ \tilde{g}_{r-1} &= g_{r-1}^{q^N} + f_{N-1} - f_{N-1}^{q^r}, \\ \tilde{g}_{r-2} &= g_{r-2}^{q^N} + f_{N-1}g_{r-1}^{q^{N-1}} + f_{N-2} - f_{N-2}^{q^r} - \tilde{g}_{r-1}f_{N-1}^{q^{r-1}},\end{aligned}$$

and, in general, $\tilde{g}_k \in A[f_1, \dots, f_{N-1}, g_1, \dots, g_{r-1}]$, which is integral over $A[g_1, \dots, g_{r-1}]$.

As $\Delta = \tilde{\Delta} = 1$, it follows that the \tilde{j}_k 's, and thus also the \tilde{u}_k 's are integral over $A[g_1, \dots, g_{r-1}]$, and thus also over $A[u_1, \dots, u_{r-1}]$. Hence \tilde{B}_r is integral over B_r .

Lastly, since isogeny is a symmetric relation, we also have B_r integral over \tilde{B}_r . \square

3 Lattice invariants

In this section we compute various lattice invariants and express them as Laurent series in a parameter “at the cusp”.

3.1 Eisenstein Series

Let $F = \mathbb{C}_\infty$. There is a one-to-one correspondence between Drinfeld modules ρ of rank s and A -lattices $\Lambda \subset \mathbb{C}_\infty$ of rank s given by the following relation (see e.g. [4, Chapter 4]): Let

$$e_\Lambda(z) = z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) = \sum_{i=0}^{\infty} e_{q^i}(\Lambda) z^{q^i} \quad (2)$$

be the exponential function for Λ . Then the corresponding Drinfeld module ρ_Λ determined by

$$\rho_\Lambda(T)(X) = TX + g_1(\Lambda)X^q + \dots + g_{s-1}(\Lambda)X^{q^{s-1}} + \Delta(\Lambda)X^{q^s} \quad (3)$$

satisfies the equation

$$e_\Lambda(Tz) = \rho_\Lambda(T)(e_\Lambda(z)). \quad (4)$$

Equation (4) and the definitions (2) and (3) yield for each $k \geq 1$ the equation

$$(T^{q^k} - T)e_{q^k}(\Lambda) = g_k(\Lambda) + \sum_{j=1}^{k-1} g_j(\Lambda)e_{q^{k-j}}(\Lambda)^{q^j}. \quad (5)$$

Hence there is a polynomial $F_k \in A[X_1, \dots, X_k]$, which is independent of the lattice Λ and which can be computed recursively, such that

$$g_k(\Lambda) = F_k(e_q(\Lambda), \dots, e_{q^k}(\Lambda)). \quad (6)$$

Since $e'_\Lambda(z) = 1$, we get for the logarithmic derivative of $e_\Lambda(z)$

$$\frac{1}{e_\Lambda(z)} = \sum_{\lambda \in \Lambda} \frac{1}{z - \lambda} = \frac{1}{z} - \sum_{i=0}^{\infty} \left(\sum_{0 \neq \lambda \in \Lambda} \lambda^{-(i+1)} \right) z^i. \quad (7)$$

Let

$$E_k(\Lambda) = \sum_{0 \neq \lambda \in \Lambda} \frac{1}{\lambda^k}$$

be the k -th *Eisenstein series* of Λ . $E_k(\Lambda) = 0$ if k is not divisible by $q - 1$. Moreover, we have $E_{kq}(\Lambda) = E_k(\Lambda)^q$. Using the expansion (2) of $e_\Lambda(z)$ and equation (7) we get

$$\left(\sum_{i=0}^{\infty} e_{q^i}(\Lambda) z^{q^i-1} \right) \left(\sum_{j=0}^{\infty} E_j(\Lambda) z^j \right) = -1. \quad (8)$$

Here we define $E_0(\Lambda) = -1$. Comparing coefficients yields in particular for each $k \geq 1$

$$e_{q^k}(\Lambda) = E_{q^{k-1}}(\Lambda) + \sum_{i=1}^{k-1} e_{q^i}(\Lambda) E_{q^{k-i-1}}(\Lambda)^{q^i}. \quad (9)$$

Here we see that there is a polynomial $G_k \in \mathbb{F}_q[X_1, \dots, X_k]$, which is independent of the lattice Λ and which can be computed recursively, such that

$$e_{q^k}(\Lambda) = G_k(E_{q-1}(\Lambda), \dots, E_{q^{k-1}}(\Lambda)). \quad (10)$$

Equations (6) and (10) show that for each $k \geq 1$ there is a polynomial $H_k \in A[X_1, \dots, X_k]$ which is independent of the lattice Λ such that

$$g_k(\Lambda) = H_k(E_{q-1}(\Lambda), \dots, E_{q^{k-1}}(\Lambda)). \quad (11)$$

Remark. The expression (11) can also be obtained using recursion formulas involving g_i and E_{q^i-1} directly (see e.g. [2, §II.2]).

3.2 Parameter at the cusp

The following calculations are deeply influenced by Goss [5].

Let Λ_{r-1} be an A -lattice in \mathbb{C}_∞ of rank $r - 1$ with $\Delta(\Lambda_{r-1}) = 1$. In addition let $z_r \in \mathbb{C}_\infty$ such that $\Lambda_r = \Lambda_{r-1} \oplus Az_r$ is a lattice of rank r . We consider

$$q_{\Lambda_{r-1}}(z_r) := \frac{1}{e_{\Lambda_{r-1}}(z_r)}$$

as a local parameter “at the cusp”. Our first aim is to expand various lattice functions of Λ_r in terms of this parameter.

To begin with the calculations, we take the logarithmic derivative of $e_{\Lambda_{r-1}}(z_r)$ and get

$$q_{\Lambda_{r-1}}(z_r) = \frac{e'_{\Lambda_{r-1}}(z_r)}{e_{\Lambda_{r-1}}(z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{\lambda + z_r}.$$

In this section we want to express for each i the sums $\sum_{\lambda \in \Lambda_{r-1}} (\lambda + z_r)^{-i}$ in terms of $q_{\Lambda_{r-1}}(z_r)$. Let x be arbitrary, we get

$$\frac{1}{e_{\Lambda_{r-1}}(x) - e_{\Lambda_{r-1}}(z_r)} = \frac{1}{e_{\Lambda_{r-1}}(x - z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{x - z_r - \lambda}. \quad (12)$$

Using the definition of the local parameter this yields

$$-q_{\Lambda_{r-1}}(z_r) \frac{1}{1 - e_{\Lambda_{r-1}}(x) q_{\Lambda_{r-1}}(z_r)} = \sum_{\lambda \in \Lambda_{r-1}} \frac{1}{x - (z_r + \lambda)}. \quad (13)$$

The Taylor expansion in terms of x of the right side of (13) equals

$$\sum_{i=0}^{\infty} \left(- \sum_{\lambda \in \Lambda_{r-1}} \left(\frac{1}{z_r + \lambda} \right)^{i+1} \right) x^i.$$

Let the left side of (13) be given with

$$\frac{1}{1 - e_{\Lambda_{r-1}}(x) q_{\Lambda_{r-1}}(z_r)} = \sum_{i=0}^{\infty} A_i x^i, \quad (14)$$

then comparing both sides of (13) yields

$$\sum_{\lambda \in \Lambda_{r-1}} \left(\frac{1}{z_r + \lambda} \right)^{i+1} = q_{\Lambda_{r-1}}(z_r) A_i. \quad (15)$$

Therefore we have to expand A_i in terms of the local parameter. Let as before

$$e_{\Lambda_{r-1}}(x) = \sum_{j=0}^{\infty} e_{q^j}(\Lambda_{r-1}) x^{q^j}.$$

Then (14) says

$$\left(\sum_{i=0}^{\infty} A_i x^i \right) \left(1 - q_{\Lambda_{r-1}}(z_r) \sum_{j=0}^{\infty} e_{q^j}(\Lambda_{r-1}) x^{q^j} \right) = 1.$$

This yields $A_0 = 1$ and the following recurrences for $i \geq 1$

$$A_i = q_{\Lambda_{r-1}}(z_r) A_{i-1} + q_{\Lambda_{r-1}}(z_r) \sum_{j \geq 1, q^j \leq i} A_{i-q^j} e_{q^j}(\Lambda_{r-1}). \quad (16)$$

We see that $A_i = P_i(q_{\Lambda_{r-1}}(z_r))$ where P_i is a monic polynomial of degree i , which is divisible by $q_{\Lambda_{r-1}}(z_r)$ if $i \geq 1$ and whose coefficients are elements of $\mathbb{F}_q[e_{q^j}(\Lambda_{r-1}) \mid q^j \leq i]$. Hence (15) can be written as

$$\sum_{\lambda \in \Lambda_{r-1}} \left(\frac{1}{z_r + \lambda} \right)^{i+1} = q_{\Lambda_{r-1}}(z_r) P_i(q_{\Lambda_{r-1}}(z_r)). \quad (17)$$

3.3 $q_{\Lambda_{r-1}}(z_r)$ -expansions of Eisenstein Series

In this section we want to expand the Eisenstein series $E_k(\Lambda_r)$ and the Drinfeld coefficients $g_k(\Lambda_r)$ in terms of the local parameter $q_{\Lambda_{r-1}}(z_r)$. We calculate

$$\begin{aligned} E_k(\Lambda_r) &= E_k(\Lambda_{r-1}) + \sum_{0 \neq a \in A} \sum_{\lambda \in \Lambda_{r-1}} \left(\frac{1}{\lambda + az_r} \right)^k \\ &= E_k(\Lambda_{r-1}) + \sum_{0 \neq a \in A} q_{\Lambda_{r-1}}(az_r) P_{k-1}(q_{\Lambda_{r-1}}(az_r)) \end{aligned} \quad (18)$$

using (17) with the polynomials P_{k-1} .

Let $\rho_{\Lambda_{r-1}}$ be the Drinfeld module corresponding to the lattice Λ_{r-1} . Since $\Delta(\Lambda_{r-1}) = 1$ by assumption, we get for $a \in A$ with leading coefficient $l(a)$

$$\rho_{\Lambda_{r-1}}(a)(X) = aX + \dots + l(a)X^{q^{(r-1)\deg a}},$$

where all the coefficients are elements in $A[g_1(\Lambda_{r-1}), \dots, g_{r-2}(\Lambda_{r-1})]$. The fundamental relation

$$q_{\Lambda_{r-1}}(az_r)^{-1} = e_{\Lambda_{r-1}}(az_r) = \rho_{\Lambda_{r-1}}(a)(e_{\Lambda_{r-1}}(z_r)) = \rho_{\Lambda_{r-1}}(a)(q_{\Lambda_{r-1}}(z_r)^{-1})$$

yields a power series expansion

$$q_{\Lambda_{r-1}}(az_r) = l(a)^{-1} q_{\Lambda_{r-1}}(z_r)^{q^{(r-1)\deg a}} + \sum_{i > q^{(r-1)\deg a}} a_i(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i, \quad (19)$$

where the coefficients $a_i(\Lambda_{r-1})$ are in $A[g_1(\Lambda_{r-1}), \dots, g_{r-2}(\Lambda_{r-1})]$. This expansion (19), equation (18) and the properties of the polynomials P_k yield the following expansion of the Eisenstein series

$$E_k(\Lambda_r) = E_k(\Lambda_{r-1}) + \sum_{i=1}^{\infty} b_i^{(k)}(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i \quad (20)$$

where the coefficients $b_i^{(k)}(\Lambda_{r-1})$ are elements of $A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})]$. Here we used in addition the fact that the g_i are polynomials in e_{q^i} (cf. (6)).

In §3.1 (cf. (11)) we saw that the elements $g_k(\Lambda)$ are polynomials $H_k(E_{q^{-1}}(\Lambda), \dots, E_{q^{k-1}}(\Lambda))$ in the Eisenstein series, where the H_k 's are independent of the lattice Λ . Therefore (20) yields the expansion

$$g_k(\Lambda_r) = g_k(\Lambda_{r-1}) + \sum_{i=1}^{\infty} c_i^{(k)}(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i \quad (21)$$

with coefficients $c_i^{(k)}(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})]$.

3.4 Expansions of $\Delta(\Lambda_r)$

We want to expand the discriminant $\Delta(\Lambda_r)$. This can be done using equation (21) and the facts $\Delta(\Lambda_r) = g_r(\Lambda_r)$ and $g_r(\Lambda_{r-1}) = 0$. Then one sees immediately that the $q_{\Lambda_{r-1}}(z_r)$ -order of $\Delta(\Lambda_r)$ is positive. To get the exact value of this order one has to evaluate the coefficients $c_i^{(r)}(\Lambda_{r-1})$ in detail. We proceed in a different way, using a product expansion of $\Delta(\Lambda_r)$ due to Gekeler [3] and Hamahata [6].

The key ingredient is the fundamental relation for each lattice Λ :

$$\rho_\Lambda(T)(X) = \Delta(\Lambda) \cdot \prod_{z \in T^{-1}\Lambda/\Lambda} (X - e_\Lambda(z)). \quad (22)$$

We get immediately for $\Lambda = \Lambda_r$

$$\Delta(\Lambda_r) = T \cdot \prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} \frac{1}{e_{\Lambda_r}(z)}. \quad (23)$$

So it is enough to expand $\prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} e_{\Lambda_r}(z)$ in terms of the local parameter. It is not difficult to show (see [6, Lemma 1]) that

$$e_{\Lambda_r}(z) = e_{\Lambda_{r-1}}(z) \prod_{0 \neq a \in A} \frac{e_{\Lambda_{r-1}}(z + az_r)}{e_{\Lambda_{r-1}}(az_r)}. \quad (24)$$

We decompose $z \in T^{-1}\Lambda_r/\Lambda_r$ as $z = z' + \frac{\varepsilon}{T}z_r$ with $z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}$ and $\varepsilon \in \mathbb{F}_q$. In view of (23) and (24) we have to evaluate

$$\begin{aligned} \prod_{0 \neq z \in T^{-1}\Lambda_r/\Lambda_r} e_{\Lambda_{r-1}}(z + y) &= \\ \left(\prod_{0 \neq z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + y) \right) &\cdot \prod_{\varepsilon \neq 0} \left(\prod_{z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + \frac{\varepsilon}{T}z_r + y) \right) \end{aligned} \quad (25)$$

for $y = 0$ and $y = az_r$. We use the fundamental relation (22) now for $\Lambda = \Lambda_{r-1}$, where we always assume $\Delta(\Lambda_{r-1}) = 1$, and get

$$\prod_{0 \neq z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + y) = \frac{\rho_{\Lambda_{r-1}}(T)(e_{\Lambda_{r-1}}(y))}{e_{\Lambda_{r-1}}(y)} = \frac{e_{\Lambda_{r-1}}(Ty)}{e_{\Lambda_{r-1}}(y)},$$

this equals T if $y = 0$, and

$$\prod_{z' \in T^{-1}\Lambda_{r-1}/\Lambda_{r-1}} e_{\Lambda_{r-1}}(z' + \frac{\varepsilon}{T}z_r + y) = \rho_{\Lambda_{r-1}}(T)(e_{\Lambda_{r-1}}(\frac{\varepsilon}{T}z_r + y)) = e_{\Lambda_{r-1}}(\varepsilon z_r + Ty).$$

If we apply these two formulas to (25) we get with (23)

$$\begin{aligned} \frac{1}{\Delta(\Lambda_r)} &= \left(\prod_{\varepsilon \neq 0} e_{\Lambda_{r-1}}(\varepsilon z_r) \right) \cdot \prod_{0 \neq a \in A} \prod_{\varepsilon} \frac{e_{\Lambda_{r-1}}((aT + \varepsilon)z_r)}{e_{\Lambda_{r-1}}(az_r)^{q^{r-1}}} \\ &= - \frac{1}{q_{\Lambda_{r-1}}(z_r)^{q-1}} \cdot \prod_{0 \neq a \in A} \prod_{\varepsilon} \frac{q_{\Lambda_{r-1}}(az_r)^{q^{r-1}}}{q_{\Lambda_{r-1}}((aT + \varepsilon)z_r)}. \end{aligned} \quad (26)$$

We have used the fact that $\prod_{\varepsilon \neq 0} \varepsilon = -1$. In equation (19) we got the expansion of $q_{\Lambda_{r-1}}(bz_r)$ in terms of the parameter $q_{\Lambda_{r-1}}(z_r)$ for each $b \in A$. If we use this, we can evaluate

$$\frac{q_{\Lambda_{r-1}}(az_r)^{q^{r-1}}}{q_{\Lambda_{r-1}}((aT + \varepsilon)z_r)} = 1 + \sum_{i=1}^{\infty} d_i(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i \quad (27)$$

with $d_i(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})]$. Now (26) and (27) give the final result

$$\Delta(\Lambda_r) = -q_{\Lambda_{r-1}}(z_r)^{q-1} + \sum_{i=q}^{\infty} f_i(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i \quad (28)$$

with coefficients $f_i(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})]$.

3.5 Expansions of $u_k(\Lambda_r)$

Now we want to expand the invariants of Drinfeld modules in terms of the local parameter. Let as above

$$u_k^{(q^r-1)/(q^{\gcd(k,r)}-1)} = j_k = \frac{g_k^{(q^r-1)/(q^{\gcd(k,r)}-1)}}{\Delta^{(q^k-1)/(q^{\gcd(k,r)}-1)}}.$$

For a lattice Λ of rank r we set

$$u_k(\Lambda) = \frac{g_k(\Lambda)}{\Delta^{(q^k-1)/(q^r-1)}}.$$

This definition is not canonical, it could be changed by an $(q^r-1)/(q^k-1)$ -th root of unity. But since we are interested in the invariants rather than the u_k 's themselves, our setting is ultimately independent of the various choices.

If we combine (21) and (28) we get for $k = 1, \dots, r-1$

$$\begin{aligned} u_k(\Lambda_r) &= (-1)^{(q^k-1)/(q^r-1)} g_k(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^{-(q-1)(q^k-1)/(q^r-1)} \\ &\quad + \sum_{i > -(q-1)(q^k-1)/(q^r-1)} h_i^{(k)}(\Lambda_{r-1}) q_{\Lambda_{r-1}}(z_r)^i \end{aligned} \quad (29)$$

with coefficients $h_i^{(k)}(\Lambda_{r-1}) \in A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})]$.

4 Sublattices

In view of the main theorem we want to expand invariants of sublattices of Λ_r in terms of the local parameter.

4.1 Invariants of sublattices

Let $n \in A$ be a monic polynomial and let $\tilde{\Lambda}_r \subset \Lambda_r$ be a sublattice with cyclic quotient $\Lambda_r/\tilde{\Lambda}_r \simeq A/nA$. Then $\tilde{\Lambda}_{r-1} := \tilde{\Lambda}_r \cap \Lambda_{r-1}$ is a sublattice of Λ_{r-1} with $\Lambda_{r-1}/\tilde{\Lambda}_{r-1} \simeq A/n_2A$ where $n = n_1 \cdot n_2$ is a decomposition into monic factors. In addition we find a basis element $w_r = n_1 z_r + \lambda$ with $\lambda \in \Lambda_{r-1}$ such that $\tilde{\Lambda}_r = \tilde{\Lambda}_{r-1} \oplus Aw_r$.

If we want to expand the invariants of $\tilde{\Lambda}_r$ as in §3.5 we have to normalize $\tilde{\Lambda}_{r-1}$ such that its discriminant equals 1. Choose $\alpha \in \mathbb{C}_\infty$ such that $\Delta(\alpha\tilde{\Lambda}_{r-1}) = 1$ and consider the lattice $\alpha\tilde{\Lambda}_r = \alpha\tilde{\Lambda}_{r-1} \oplus \alpha Aw_r$. Then we get with (29) for $k = 1, \dots, r-1$

$$u_k(\tilde{\Lambda}_r) = u_k(\alpha\tilde{\Lambda}_r) = (-1)^{(q^k-1)/(q^r-1)} g_k(\alpha\tilde{\Lambda}_{r-1}) q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r)^{-(q-1)(q^k-1)/(q^r-1)} \\ + \sum_{i > -(q-1)(q^k-1)/(q^r-1)} h_i^{(k)}(\alpha\tilde{\Lambda}_{r-1}) q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r)^i. \quad (30)$$

We want to find α , express the parameter $q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r)$ in terms of $q_{\Lambda_{r-1}}(z_r)$ and compute the coefficients $g_k(\alpha\tilde{\Lambda}_{r-1})$, $h_i^{(k)}(\alpha\tilde{\Lambda}_{r-1})$ with formulas involving Λ_{r-1} .

Since $\Lambda_{r-1}/\tilde{\Lambda}_{r-1} \simeq A/n_2A$, we have $n_2\Lambda_{r-1} \subset \tilde{\Lambda}_{r-1}$. We consider the polynomial

$$P(X) = n_2 X \prod_{0 \neq \lambda \in n_2^{-1}\tilde{\Lambda}_{r-1}/\Lambda_{r-1}} \left(1 - \frac{X}{e_{\Lambda_{r-1}}(\lambda)}\right).$$

This is a polynomial of degree $q^{(r-2)\deg n_2} = |n_2^{r-2}|$, and its coefficients are elements of $A[q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n_2^{-1}\Lambda_{r-1}]$. These values $q_{\Lambda_{r-1}}(\lambda)$ are independent of $q_{\Lambda_{r-1}}(z_r)$.

$P(X)$ describes the isogeny corresponding to $n_2\Lambda_{r-1} \subset \tilde{\Lambda}_{r-1}$ and as usual (cf. [4, §4.7]) we get

$$P(e_{\Lambda_{r-1}}(X)) = e_{\tilde{\Lambda}_{r-1}}(n_2 X) \quad (31)$$

and

$$P(\rho_{\Lambda_{r-1}}(T)(X)) = \rho_{\tilde{\Lambda}_{r-1}}(T)(P(X)). \quad (32)$$

Let c_P be the leading coefficient of $P(X)$, then comparing leading coefficients in (32) yields

$$\Delta(\tilde{\Lambda}_{r-1}) = c_P^{-q^{r-1}+1}.$$

Hence we take $\alpha = c_P^{-1}$ and calculate

$$\Delta(\alpha\tilde{\Lambda}_{r-1}) = \alpha^{-q^{r-1}+1} \Delta(\tilde{\Lambda}_{r-1}) = 1.$$

Since $q_\Lambda(z) = e_\Lambda(z)^{-1}$, equation (31) can be written as

$$P\left(\frac{1}{q_{\Lambda_{r-1}}(X)}\right) = \frac{1}{q_{\tilde{\Lambda}_{r-1}}(n_2 X)},$$

which yields an expansion

$$q_{\tilde{\Lambda}_{r-1}}(n_2 X) = \alpha q_{\Lambda_{r-1}}(X)^{q^{(r-2) \deg n_2}} + \sum_{i > q^{(r-2) \deg n_2}} k_i q_{\Lambda_{r-1}}(X)^i$$

where $k_i \in A[q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n_2^{-1}\Lambda_{r-1}]$. We apply this formula to the parameter $q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r)$ and get

$$\begin{aligned} q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r) &= \alpha^{-1} q_{\tilde{\Lambda}_{r-1}}(w_r) = \alpha^{-1} q_{\tilde{\Lambda}_{r-1}}\left(n_2 \frac{w_r}{n_2}\right) \\ &= q_{\Lambda_{r-1}}\left(\frac{w_r}{n_2}\right)^{q^{(r-2) \deg n_2}} + \sum_{i > q^{(r-2) \deg n_2}} \alpha^{-1} k_i q_{\Lambda_{r-1}}\left(\frac{w_r}{n_2}\right)^i. \end{aligned} \quad (33)$$

Since $w_r = n_1 z_r + \lambda$ with $\lambda \in \Lambda_{r-1}$, we get

$$\begin{aligned} q_{\Lambda_{r-1}}\left(\frac{w_r}{n_2}\right) &= \left(q_{\Lambda_{r-1}}\left(\frac{n_1 z_r}{n_2}\right)^{-1} + q_{\Lambda_{r-1}}\left(\frac{\lambda}{n_2}\right)^{-1}\right)^{-1} \\ &= q_{\Lambda_{r-1}}\left(\frac{n_1 z_r}{n_2}\right) + \sum_{i > 1} l_i q_{\Lambda_{r-1}}\left(\frac{n_1 z_r}{n_2}\right)^i \end{aligned} \quad (34)$$

with $l_i \in A[q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n_2^{-1}\Lambda_{r-1}]$.

On the other hand we calculate with formula (19)

$$\begin{aligned} q_{\Lambda_{r-1}}\left(\frac{n_1 z_r}{n_2}\right) &= q_{\Lambda_{r-1}}\left(n_1^2 \frac{z_r}{n}\right) \\ &= q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^{q^{(r-1) \deg n_1^2}} + \sum_{i > q^{(r-1) \deg n_1^2}} a_i(\Lambda_{r-1}) q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^i. \end{aligned} \quad (35)$$

Now applying (34) and (35) to (33) we get

$$q_{\alpha\tilde{\Lambda}_{r-1}}(\alpha w_r) = q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^{|n_1^{2r-2} n_2^{r-2}|} + \sum_{i > |n_1^{2r-2} n_2^{r-2}|} m_i q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^i \quad (36)$$

with coefficients $m_i \in A[q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n_2^{-1}\Lambda_{r-1}]$.

The coefficients $g_k(\alpha\tilde{\Lambda}_{r-1})$ and $h_i^{(k)}(\alpha\tilde{\Lambda}_{r-1})$ in (30) are elements of the ring $A[e_q(\alpha\tilde{\Lambda}_{r-1}), \dots, e_{q^{r-2}}(\alpha\tilde{\Lambda}_{r-1})]$. Equation (31) shows that this is a subring of $A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})][q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n^{-1}\Lambda_{r-1}]$.

This remark and equations (30) and (36) show that $u_k(\tilde{\Lambda}_r)$ can be expanded in a series

$$u_k(\tilde{\Lambda}_r) = (-1)^{(q^k-1)/(q^r-1)} g_k(\alpha \tilde{\Lambda}_{r-1}) q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^{-|n_1^{2r-2} n_2^{r-2}|(q-1)(q^k-1)/(q^r-1)} \\ + \sum_{i > -|n_1^{2r-2} n_2^{r-2}|(q-1)(q^k-1)/(q^r-1)} r_i^{(k)}(\alpha \tilde{\Lambda}_{r-1}) q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^i \quad (37)$$

with coefficients $g_k(\alpha \tilde{\Lambda}_{r-1})$ and $r_i^{(k)}(\alpha \tilde{\Lambda}_{r-1})$ in the ring $A[e_q(\Lambda_{r-1}), \dots, e_{q^{r-2}}(\Lambda_{r-1})][q_{\Lambda_{r-1}}(\lambda) \mid \lambda \in n^{-1}\Lambda_{r-1}]$.

4.2 Counting sublattices

Let Λ_r be an A -lattice of rank r . For $n \in A$ we denote by $\psi_r(n)$ the number of sublattices $\tilde{\Lambda}_r \subset \Lambda_r$ with $\Lambda_r/\tilde{\Lambda}_r \simeq A/nA$. These sublattices correspond to the matrices in \mathcal{H}_n . Then we get

Proposition 4.1 *We have*

$$\psi_r(n) = |n|^{r-1} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}}$$

where the product is taken over all monic irreducible divisors p of n , and where $|m| = q^{\deg m}$ for each $m \in A$.

Proof. We could count the matrices in \mathcal{H}_n . Instead, we count sublattices. Since $\psi_r(n)$ is multiplicative, we may assume that $n = p^s$ for a monic irreducible $p \in A$. There is a bijection between sublattices $\tilde{\Lambda}_r \subset \Lambda_r$ with $\Lambda_r/\tilde{\Lambda}_r \cong A/p^s A$, and free rank one $A/p^s A$ -submodules of $\Lambda_r/p^s \Lambda_r \cong (A/p^s A)^r$. An element $(x_1, \dots, x_r) \in (A/p^s A)^r$ generates such a submodule if and only if $x_i \notin pA/p^s A$ for some i , hence there are $|p|^{sr} - |p|^{(s-1)r}$ generators. On the other hand, each such submodule has $|p|^s - |p|^{s-1}$ generators, and it follows that

$$\psi_r(p^s) = \frac{|p|^{sr} - |p|^{(s-1)r}}{|p|^s - |p|^{s-1}} = |p^s|^{r-1} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}},$$

from which the result follows. □

4.3 Non-cancelation

The coefficients of the modular polynomial $P_{I,n}(X)$ are polynomials in the basic invariants. We want to study the weighted degree of these polynomials by comparing it with its order as a series expansion in the local parameter. Therefore we need to show that the leading terms in these series do not cancel. For this we will need the following technical lemma.

We consider the homothety classes of lattices Λ_s of rank s as points in the moduli space $M^s(\mathbb{C}_\infty)$ equipped with the analytic topology, i.e. where closed sets are the zero-loci of sets of analytic functions.

For $k = 1, \dots, r-1$ let v_k be functions on $M^r(\mathbb{C}_\infty)$; suppose that their $q_{\Lambda_{r-1}}(\frac{z_r}{n})$ -expansions are of the form

$$v_k(\Lambda_{r-1} + Az_r) = a_k(\Lambda_{r-1}) q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^{-c(q^k-1)/(q^r-1)} + \text{higher terms},$$

where the a_k 's are algebraically independent analytic functions on $M^{r-1}(\mathbb{C}_\infty)$ and where c does not depend on Λ_{r-1} or on k .

Let $f \in \mathbb{C}_\infty[X_1, \dots, X_{r-1}]$ be a polynomial of weighted degree $w(f)$, where the weighted degree of a monomial is given by $w(X_1^{\alpha_1} \dots X_{r-1}^{\alpha_{r-1}}) = \sum_{k=1}^{r-1} \alpha_k \frac{q^k-1}{q^r-1}$.

Lemma 4.2 *There exists a non-empty open subset $S \subset M^{r-1}(\mathbb{C}_\infty)$ such that for any $\Lambda_{r-1} \in S$ the weighted degree and the order of the $q_{\Lambda_{r-1}}(\frac{z_r}{n})$ -expansion satisfy*

$$\text{ord}_{q_{\Lambda_{r-1}}(\frac{z_r}{n})} (f(v_1(\Lambda_{r-1} + Az_r), \dots, v_{r-1}(\Lambda_{r-1} + Az_r))) = -c w(f).$$

Proof. Let

$$f(X_1, \dots, X_{r-1}) = \sum_{(\alpha_1, \dots, \alpha_{r-1})} b_{(\alpha_1, \dots, \alpha_{r-1})} X_1^{\alpha_1} \dots X_{r-1}^{\alpha_{r-1}}.$$

Then the leading term of the $q_{\Lambda_{r-1}}(\frac{z_r}{n})$ -expansion of $f(v_1(\Lambda_{r-1} + Az_r), \dots, v_{r-1}(\Lambda_{r-1} + Az_r))$ is given by

$$\left(\sum_{(\alpha_1, \dots, \alpha_{r-1})} b_{(\alpha_1, \dots, \alpha_{r-1})} a_1(\Lambda_{r-1})^{\alpha_1} \dots a_{r-1}(\Lambda_{r-1})^{\alpha_{r-1}} \right) q_{\Lambda_{r-1}}\left(\frac{z_r}{n}\right)^{-cw(f)},$$

where the sum is taken over all indices $(\alpha_1, \dots, \alpha_{r-1})$ satisfying

$$\sum_{k=1}^{r-1} \alpha_k \frac{q^k-1}{q^r-1} = w(f).$$

The coefficient is zero only if $a_1(\Lambda_{r-1}), \dots, a_{r-1}(\Lambda_{r-1})$ satisfy a polynomial relation. The locus of $\Lambda_{r-1} \in M^{r-1}(\mathbb{C}_\infty)$ for which this relation holds is a proper closed set, since the a_k 's are algebraically independent. The result follows. \square

Remark. We can apply Lemma 4.2 to $v_k = u_k$ resp. $v_k = u_k(\tilde{\Lambda}_{r-1})$ in view of (29) resp. (37) and due to the fact that g_1, \dots, g_{r-1} are algebraically independent on $M^{r-1}(\mathbb{C}_\infty)$.

4.4 Proof of the Main Result

Proof of Theorem 1.1. (1) The value for the degree follows from Proposition 4.1. Let ρ be a Drinfeld module with invariant I . The $I \circ h$'s correspond to cyclic submodules of $\rho[n] \cong (A/nA)^r$ of order n , which are permuted transitively by $\mathrm{GL}_r(A)$. It follows that the coefficients of $P_{I,n}(X)$ are functions on $\mathrm{GL}_r(A) \backslash \Omega^r \cong M^r(\mathbb{C}_\infty)$, and so $P_{I,n}(X)$ is an irreducible polynomial with coefficients in $\mathbb{C}_\infty[u_1, \dots, u_{r-1}]^G$.

We next show how to replace \mathbb{C}_∞ by K . Let $u'_1, \dots, u'_{r-1} \in K$ be arbitrary. These values correspond to a Drinfeld module ρ defined over K with $u_k = u'_k$ and invariant $I \in K$. The absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ permutes the set of cyclic submodules of $\rho[n]$ of order n , hence permutes the $I \circ h$'s. Thus the coefficients of $P_{I,n}(X)$, when specialized to $(u'_1, \dots, u'_{r-1}) \in K^{r-1}$, lie in K . Since $(u'_1, \dots, u'_{r-1}) \in K^{r-1}$ is arbitrary, it follows that the coefficients of $P_{I,n}(X)$ lie in $K[u_1, \dots, u_{r-1}]^G$.

Lastly, by Proposition 2.1, the coefficients of $P_{I,n}(X)$ are integral over B_r , and hence lie in B_r .

(2) Let $\tilde{\Lambda}_r \subset \Lambda_r$ be the sublattice corresponding to $h \in \mathcal{H}_n$. Equation (37) and Lemma 4.2 show, for suitably chosen Λ_{r-1} , that

$$\mathrm{ord}_{q_{\Lambda_r}(\frac{z_r}{n})}(u_k(\tilde{\Lambda}_r)) = -|n_1^{2r-2}n_2^{r-2}|(q-1)\frac{q^k-1}{q^r-1}$$

(for $k = 1, \dots, r-1$) implies

$$\mathrm{ord}_{q_{\Lambda_{r-1}}(\frac{z_r}{n})}(I \circ h) = -(q-1)|n_1^{2r-2}n_2^{r-2}|w(I) \geq -(q-1)|n|^{2(r-1)}w(I).$$

Now let $a_i \in B_r$ be the coefficient of X^i in $P_{I,n}(X)$. Then

$$a_i = (-1)^d \sum_{(h_1, \dots, h_d)} (I \circ h_1) \cdots (I \circ h_d), \quad \text{where } d = \#\mathcal{H}(n) - i = \psi_r(n) - i.$$

Hence we get

$$\mathrm{ord}_{q_{\Lambda_{r-1}}(\frac{z_r}{n})}(a_i) \geq -(q-1)(\psi_r(n) - i)|n|^{2(r-1)}w(I). \quad (38)$$

On the other hand, since $a_i \in B_r$, using Lemma 4.2, (19) and (29), we get

$$\mathrm{ord}_{q_{\Lambda_{r-1}}(\frac{z_r}{n})}(a_i) = |n|^{r-1} \mathrm{ord}_{q_{\Lambda_{r-1}}(z_r)}(a_i) = -(q-1)|n|^{r-1}w(a_i). \quad (39)$$

Now (38), (39) and Proposition 4.1 yield

$$w(a_i) \leq \left(|n|^{2(r-1)} \prod_{p|n} \frac{|p|^r - 1}{|p|^r - |p|^{r-1}} - i|n|^{r-1} \right) w(I).$$

□

5 The Case $r = 3, q = 2, n = T$.

In this section we explicitly calculate some modular polynomials in the case where $r = 3$, $q = 2$ and $n = T$.

5.1 The moduli space

According to [7], the ring B_3 is generated by the following seven invariants:

$$J_{70} := \frac{g_1^7}{\Delta}, \quad J_{07} := \frac{g_2^7}{\Delta^3}, \quad J_{12} := \frac{g_1 g_2^2}{\Delta}, \quad J_{41} := \frac{g_1^4 g_2}{\Delta},$$

and

$$J_{53} := \frac{g_1^5 g_2^3}{\Delta^2} = J_{12} J_{41}, \quad J_{65} := \frac{g_1^6 g_2^5}{\Delta^3} = J_{12}^2 J_{41}, \quad J_{77} := \frac{g_1^7 g_2^7}{\Delta^4} = J_{07} J_{70}.$$

It follows that

$$B_3 = \mathbb{F}_2[T][J_{07}, J_{12}, J_{41}, J_{70}],$$

and these four generators satisfy the relations

$$J_{07} J_{41} = J_{12}^4, \quad J_{12} J_{70} = J_{41}^2.$$

It is easy to check that no further algebraic relations are needed, and so, for $A = \mathbb{F}_2[T]$,

$$M^3 \cong \text{Spec}(B_3) \cong \text{Spec} \left(\frac{\mathbb{F}_2[T][X_1, X_2, X_3, X_4]}{\langle X_1 X_3 - X_2^2, X_2 X_4 - X_3^2 \rangle} \right).$$

Consider the Hecke correspondence \mathcal{T}_T on M^3 , which sends a Drinfeld module ρ to the set of Drinfeld modules ρ' for which there exist isogenies $\rho \rightarrow \rho'$ with kernels isomorphic to $\mathbb{F}_2[T]/\langle T \rangle$. Then the graph of \mathcal{T}_T in

$$M^3 \times M^3 \cong \text{Spec} \left(\frac{\mathbb{F}_2[T][X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4]}{\langle X_1 X_3 - X_2^2, X_2 X_4 - X_3^2, Y_1 Y_3 - Y_2^2, Y_2 Y_4 - Y_3^2 \rangle} \right)$$

is cut out by the ideal generated by

$$P_{J_{07}, T}(X_1; Y_1, Y_2, Y_3, Y_4), \quad P_{J_{12}, T}(X_2; Y_1, Y_2, Y_3, Y_4), \quad P_{J_{41}, T}(X_3; Y_1, Y_2, Y_3, Y_4), \quad \text{and}$$

$$P_{J_{70}, T}(X_4; Y_1, Y_2, Y_3, Y_4),$$

where we have written the modular polynomials in the form $P_{J, T}(X; J_{07}, J_{12}, J_{41}, J_{70}) \in \mathbb{F}_2[T][J_{07}, J_{12}, J_{41}, J_{70}][X]$.

5.2 Computing the modular polynomials

It is possible to compute the modular polynomial $P_{J_{ab},T}(X)$ for $J_{ab} := \frac{g_1^a g_2^b}{\Delta^d}$ (where $a + 3b = 7d$), using series expansions in $q_{\Lambda_2}(z_3)$. We did this for J_{70} , where the expansion up to 35 summands was enough to recover the polynomial. However, this method is very time and space consuming, so we seek inspiration from [8] instead.

Suppose given a generic Drinfeld module ρ , but normalised such that $\Delta(\rho) = 1$. Then we can choose $u_1(\rho) = g_1(\rho)$ and $u_2 = g_2(\rho)$, and in fact we write

$$\rho_T = TX + u_1X^2 + u_2X^4 + X^8.$$

Suppose $\tilde{\rho}$ is another Drinfeld module, and $P : \tilde{\rho} \rightarrow \rho$ is a cyclic T -isogeny. We write

$$\tilde{\rho}_T = TX + g_1X^2 + g_2X^4 + \Delta X^8, \quad \text{and}$$

$$P = X + v^{-1}X^2.$$

Then

$$P \circ \tilde{\rho}_T = \rho_T \circ P \tag{40}$$

and, since $\ker P \subset \ker \tilde{\rho}_T$,

$$TX + g_1X^2 + g_2X^4 + \Delta X^8 = (TX + u_1X^2 + u_2X^4 + X^8) \circ (X + v^{-1}X^2). \tag{41}$$

From (41) we obtain

$$\Delta = v^{-4}g_2 + v^{-6}g_1 + v^{-7}T, \tag{42}$$

and from (40) we obtain

$$g_1 = v^{-1}(T + T^2) + u_1 \tag{43}$$

$$g_2 = v^{-3}(T^2 + T^4) + v^{-2}u_1 + v^{-1}u_1^2 + u_2 \tag{44}$$

$$\Delta = v^{-7}(T^4 + T^8) + v^{-5}u_1^2 + v^{-3}u_1^4 + v^{-1}u_2^2 + 1 \tag{45}$$

$$\Delta^2 = v^{-7}. \tag{46}$$

Combining (42), (45) and (46), we obtain

$$\Delta^{-1} = T^4 + v^2u_1^2 + v^3u_2, \tag{47}$$

and v must satisfy the equation

$$T^8 + v^4u_1^4 + v^6u_2^2 + v^7 = 0. \tag{48}$$

Since $\psi_3(T) = 7$, each of the seven roots of (48) gives rise to a root of $P_{J_{ab},T}(X)$, and we get

$$P_{J_{ab},T}(X) = \prod_{\substack{v \\ T^8 + v^4u_1^4 + v^6u_2^2 + v^7 = 0}} (X - g_1^a g_2^b \Delta^{-d}), \tag{49}$$

where g_1 , g_2 and Δ^{-1} are given by (43), (44) and (47), respectively.

In practice, we construct a 7×7 matrix V with characteristic equation (48), and then $P_{J_{ab},T}(X)$ is computed as the characteristic polynomial of

$$(V^{-1}(T + T^2) + u_1)^a (V^{-3}(T^2 + T^4) + V^{-2}u_1 + V^{-1}u_1^2 + u_2)^b (T^4 + V^2u_1^2 + V^3u_2)^d.$$

The results of this calculation for $J_{ab} \in \{J_{07}, J_{12}, J_{41}, J_{70}\}$ are listed in the Appendix. As in the classical case, the coefficients, even in this simple case, are very involved. In the following table, we list the T -degree and weighted degree of the coefficients. We see that the weighted degrees are bounded by Theorem 1.1, and in each case the bound on $w(a_6)$ is sharp.

$P_{J_{ab},T}(X) = X^7 + a_6X^6 + a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0$								
	$P_{J_{70},T}(X)$		$P_{J_{12},T}(X)$		$P_{J_{41},T}(X)$		$P_{J_{07},T}(X)$	
coeff.	deg $_T$	weight	deg $_T$	weight	deg $_T$	weight	deg $_T$	weight
a_0	70	7	42	7	56	7	112	21
a_1	60	6	36	6	48	6	96	19
a_2	50	6	30	6	40	6	80	18
a_3	40	5	24	5	32	5	64	15
a_4	30	5	18	5	24	5	48	15
a_5	20	4	12	4	16	4	32	13
a_6	10	4	6	4	8	4	16	12

Appendix

Here we list the coefficients of the modular polynomials

$$P_{J,T}(X) = X^7 + a_6X^6 + a_5X^5 + a_4X^4 + a_3X^3 + a_2X^2 + a_1X + a_0 \in \mathbb{F}_2[T, J_{07}, J_{12}, J_{41}, J_{70}][X],$$

for $J \in \{J_{12}, J_{41}, J_{70}, J_{07}\}$. They were computed using Maple 11 on an Intel Centrino 2GHz with 2GB RAM. The first three polynomials took 3 minutes or less each, whereas the fourth took 18 hours and 1GB working memory.

Coefficients of $P_{J_{12},T}(X)$:

$$\begin{aligned} a_0 = & J_{07}^2 J_{70} + (T^8 + T^6 + T^4 + T^2) J_{07}^2 J_{12} + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3) J_{07}^2 + (T^8 + T^6 + T^2 + 1) J_{07} J_{70} J_{41} + (T^{12} + T^{10} + T^4 + T^2) J_{07} J_{12} J_{41} + (T^{14} + T^{13} + T^{12} + T^{11} + T^6 + T^5 + T^4 + T^3) J_{07} J_{41} + J_{70}^3 J_{12}^2 + J_{70}^3 T^4 + (T^8 + T^6 + T^4 + T^2 + 1) J_{70}^2 J_{12}^3 + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3) J_{70}^2 J_{12}^2 + (T^{10} + T^8 + T^6 + T^4) J_{70}^2 J_{12} + (T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7) J_{70}^2 + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3) J_{70} J_{12}^3 + (T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + 1) J_{70} J_{12}^2 + (T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7) J_{70} J_{12} + (T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8 + T^4) J_{70} + (T^{24} + T^{22} + T^{20} + T^{18} + T^8 + T^6 + T^4 + T^2) J_{12}^3 + (T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19} + T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3) J_{12}^2 + (T^{40} + T^{38} + T^8 + T^6) J_{12} + (T^{42} + T^{41} + T^{40} + T^{39} + T^{10} + T^9 + T^8 + T^7) \end{aligned}$$

$$\begin{aligned} a_1 = & J_{07} J_{12}^2 + (T^8 + T^6) J_{07} J_{70} J_{12} + (T^2 + T) J_{07} J_{70} J_{41} + (T^{12} + T^{11} + T^{10} + T^9 + T^6 + T^5 + T^4 + T^3) J_{07} J_{70} + (T^4 + T^3 + T^2 + T) J_{07} J_{12}^3 + J_{07} J_{12}^2 J_{41} + (T^{12} + T^8 + T^6 + T^2) J_{07} J_{12}^2 + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^2 + T) J_{07} J_{12} J_{41} + (T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^6 + T^5) J_{07} J_{12} + (T^{16} + T^{14} + T^{12} + T^{10}) J_{07} J_{41} + (T^{20} + T^{18} + T^4 + T^2) J_{07} + (T^8 + T^6 + 1) J_{70}^2 J_{12}^2 + (T^4 + T^3) J_{70}^2 J_{12} J_{41} + (T^{12} + T^{11}) J_{70}^2 J_{12} + J_{70}^2 J_{41} T^6 + (T^{14} + T^{12} + T^{10} + T^4) J_{70}^2 + (T^{12} + T^8 + T^4 + T^2) J_{70} J_{12}^3 + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3 + T^2 + T) J_{70} J_{12}^2 J_{41} + (T^{18} + T^{17} + T^{12} + T^{11} + T^8 + T^7 + T^2 + T) J_{70} J_{12}^2 + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4) J_{70} J_{12} J_{41} + (T^{24} + T^{22} + T^8 + T^6) J_{70} J_{12} + (T^{20} + T^{19} + T^{18} + T^{17} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^6 + T^5) J_{70} J_{41} + (T^{28} + T^{27} + T^{26} + T^{25} + T^{22} + T^{21} + T^{12} + T^{11} + T^{10} + T^9 + T^6 + T^5) J_{70} + (T^{20} + T^{19} + T^{16} + T^{15} + T^{12} + T^{11} + T^8 + T^7) J_{12}^3 + (T^{10} + T^8 + T^2 + 1) J_{12}^2 J_{41} + (T^{28} + T^{22} + T^{20} + T^{18} + T^{12} + T^6 + T^4 + T^2) J_{12}^2 + (T^{26} + T^{25} + T^{24} + T^{23} + T^{20} + T^{19} + T^{10} + T^9 + T^8 + T^7 + T^4 + T^3) J_{12} J_{41} + (T^{34} + T^{33} + T^{26} + T^{25} + T^{18} + T^{17} + T^{10} + T^9) J_{12} + (T^{32} + T^{30} + T^{26} + T^{20} + T^{16} + T^{14} + T^{10} + T^4) J_{41} + (T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6) \end{aligned}$$

$$\begin{aligned}
\mathbf{a}_2 &= (T^{10} + T^2)J_{07}J_{70} + (T^2 + 1)J_{07}J_{12}^3 + (T^4 + T^3 + T^2 + T)J_{07}J_{12}^2 + J_{07}J_{12}J_{41} + (T^{16} + T^{12} + T^8 + T^4)J_{07}J_{12} + (T^8 + T^7 + T^2 + T)J_{07}J_{41} + (T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^6 + T^5)J_{07} + J_{70}^2J_{12}J_{41}T^2 + J_{70}^2J_{12}T^6 + (T^{10} + T^9)J_{70}^2 + (T^4 + T^3)J_{70}J_{12}^3 + (T^8 + T^6 + T^4 + T^2)J_{70}J_{12}^2J_{41} + (T^{12} + T^{10} + T^4 + 1)J_{70}J_{12}^2 + (T^8 + T^7 + T^6 + T^5 + T^4 + T^3)J_{70}J_{12}J_{41} + (T^{12} + T^{11} + T^8 + T^7)J_{70}J_{12} + (T^{18} + T^{16} + T^{12} + T^6)J_{70}J_{41} + (T^{22} + T^{20} + T^{18} + T^{14} + T^{10} + T^4)J_{70} + (T^{16} + T^{12} + T^8 + T^4)J_{12}^3 + (T^{14} + T^{13} + T^2 + T)J_{12}^2J_{41} + (T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + T^6 + T^5 + T^4 + T^3 + T^2 + T)J_{12}^2 + (T^{24} + T^{22} + T^{16} + T^{14} + T^{12} + T^{10} + T^4 + T^2)J_{12}J_{41} + (T^{28} + T^{24} + T^{12} + T^8)J_{12} + (T^{24} + T^{23} + T^{22} + T^{21} + T^8 + T^7 + T^6 + T^5)J_{41} + (T^{30} + T^{29} + T^{22} + T^{21} + T^{14} + T^{13} + T^6 + T^5) \\
\mathbf{a}_3 &= (T^4 + T^3)J_{07}J_{70} + (T^4 + T^2)J_{07}J_{12}^2 + (T^{10} + T^9 + T^8 + T^7)J_{07}J_{12} + J_{07}J_{41}T^8 + (T^{12} + T^4)J_{07} + J_{70}^2J_{12}T^4 + (T^{12} + T^8)J_{70}^2 + J_{70}J_{12}^3T^2 + (T^4 + T^3)J_{70}J_{12}^2 + (T^6 + T^4 + T^2)J_{70}J_{12}J_{41} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6)J_{70}J_{12} + (T^{10} + T^9 + T^8 + T^7)J_{70}J_{41} + (T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9)J_{70} + (T^{10} + T^9 + T^8 + T^7)J_{12}^3 + (T^{10} + T^6)J_{12}^2J_{41} + (T^{20} + T^{18} + T^{12} + T^8 + T^2 + 1)J_{12}^2 + (T^{18} + T^{17} + T^{14} + T^{13} + T^{12} + T^{11} + T^8 + T^7)J_{12}J_{41} + (T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4)J_{41} + (T^{24} + T^{20} + T^8 + T^4) \\
\mathbf{a}_4 &= J_{07}J_{12}J_{41} + (T^4 + T^2)J_{07}J_{12} + (T^6 + T^5 + T^4 + T^3)J_{07} + J_{70}^3 + (T^2 + 1)J_{70}^2J_{12} + (T^6 + T^5 + T^4 + T^3)J_{70}^2 + (T^8 + T^6 + T^2 + 1)J_{70}J_{12}^3 + (T^{10} + T^9 + T^8 + T^7 + T^4 + T^3)J_{70}J_{12} + J_{70}J_{41}T^4 + (T^{12} + T^{10} + T^4 + 1)J_{70} + (T^{10} + T^4 + T^2)J_{12}^3 + (T^6 + T^5)J_{12}^2J_{41} + (T^{12} + T^{11} + T^6 + T^5)J_{12}^2 + (T^{12} + T^8 + T^6 + T^2)J_{12}J_{41} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2)J_{12} + (T^{14} + T^{13} + T^6 + T^5)J_{41} + (T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3) \\
\mathbf{a}_5 &= (T^2 + T)J_{07}J_{12} + J_{07}J_{41} + (T^4 + T^2)J_{07} + J_{70}^2 + (T^2 + T)J_{70}J_{12}^2 + J_{70}J_{12}J_{41} + J_{70}J_{12}T^2 + (T^2 + T)J_{70}J_{41} + (T^6 + T^5 + T^2 + T)J_{70} + (T^2 + T)J_{12}^3 + J_{12}^2J_{41}T^4 + J_{12}^2T^4 + (T^6 + T^5)J_{12}J_{41} + (T^{10} + T^9 + T^6 + T^5)J_{12} + (T^8 + T^4 + T^2 + 1)J_{41} + (T^{12} + T^{10} + T^4 + T^2) \\
\mathbf{a}_6 &= J_{07}J_{12} + (T^2 + T)J_{07} + J_{70} + J_{12}^3 + (T^2 + T)J_{12}^2 + J_{12}T^4 + (T^2 + T)J_{41} + (T^6 + T^5 + T^2 + T)
\end{aligned}$$

Coefficients of $P_{J_{41},T}(X)$:

$$\begin{aligned}
\mathbf{a}_0 &= (T^{32} + T^{24} + T^{16} + T^8)J_{07}^2J_{41} + J_{07}J_{70}^4 + (T^{32} + T^{24} + T^{16} + T^8)J_{07}J_{70}J_{12}^2 + (T^{32} + T^{24} + T^{16} + T^8)J_{07}J_{70}J_{12}J_{41} + (T^{38} + T^{34} + T^{30} + T^{26} + T^{22} + T^{18} + T^{14} + T^{10})J_{07}J_{70}J_{12} + (T^{34} + T^{26} + T^{18} + T^{10})J_{07}J_{70}J_{41} + (T^{34} + T^{32} + T^{26} + T^{24} + T^{18} + T^{16} + T^{10} + T^8)J_{07}J_{12}^2J_{41} + (T^{40} + T^8)J_{07}J_{12}J_{41} + (T^{48} + T^{46} + T^{44} + T^{42} + T^{16} + T^{14} + T^{12} + T^{10})J_{07}J_{41} + (T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12})J_{07} + J_{70}^5J_{12} + J_{70}^5T^2 + (T^2 + 1)J_{70}^4J_{12}^2 + J_{70}^4J_{12}J_{41} + (T^8 + 1)J_{70}^4J_{12} + (T^6 + T^2)J_{70}^4J_{41} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2)J_{70}^4 + (T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12})J_{70}J_{12} + (T^{42} + T^{38} + T^{34} + T^{30} + T^{26} + T^{22} + T^{18} + T^{14})J_{70} + (T^{42} + T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12})J_{12}^2 + (T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12})J_{12}J_{41} + (T^{48} + T^{44} + T^{16} + T^{12})J_{12} + (T^{46} + T^{14})J_{41} + (T^{56} + T^{54} + T^{48} + T^{46} + T^{24} + T^{22} + T^{16} + T^{14}) \\
\mathbf{a}_1 &= (T^4 + 1)J_{07}J_{70}^3 + (T^{10} + T^2)J_{07}J_{70}^2J_{41} + (T^{26} + T^{22} + T^{10} + T^6)J_{07}J_{70}^2 + (T^{20} + T^4)J_{07}J_{70}J_{12}^2 + (T^{18} + T^{14} + T^{10} + T^6)J_{07}J_{70}J_{12}J_{41} + (T^{28} + T^{22} + T^{12} + T^6)J_{07}J_{70}J_{12} + (T^{26} + T^{10})J_{07}J_{70}J_{41} + (T^{26} + T^{22} + T^{10} + T^6)J_{07}J_{70} + (T^{26} + T^{22} + T^{10} + T^6)J_{07}J_{12}^2J_{41} + (T^{32} + T^{28} + T^{16} + T^{12})J_{07}J_{12}J_{41} + (T^{38} + T^{26} + T^{22} + T^{10})J_{07}J_{41} + (T^{32} + T^{24} + T^{16} + T^8)J_{07} + (T^4 + T^2)J_{70}^4J_{12} + J_{70}^4J_{41}T^2 + (T^{10} + T^6)J_{70}^4J_{12}^2 + (T^{10} + T^2)J_{70}^3J_{12}J_{41} + (T^{18} + T^{14} + T^{12} + T^8 + T^6 + T^4 + T^2 + 1)J_{70}^3J_{12} + (T^{10} + T^8 + T^4 + T^2)J_{70}^3J_{41} + (T^{20} + T^{18} + T^{16} + T^{14} + T^8 + T^6 + T^4 + T^2)J_{70}^3 + (T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^6)J_{70}^2J_{12}^2 + (T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^2)J_{70}^2J_{12}J_{41} + (T^{26} + T^{22} + T^{18} + T^{14})J_{70}^2J_{12}^2 + (T^{10} + T^{12})J_{70}^2J_{12}J_{41} + (T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4)J_{70}^2J_{41} + (T^{30} + T^{26} + T^{22} + T^{14} + T^{10} + T^6)J_{70}^2 + (T^{34} + T^{28} + T^{20} + T^{18} + T^{12} + T^4)J_{70}J_{12}^3 + (T^{26} + T^{24} + T^{22} + T^{16} + T^{12} + T^{10} + T^6 + T^4)J_{70}J_{12}^2J_{41} + (T^{36} + T^{34} + T^{32} + T^{30} + T^{26} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{10} + T^6)J_{70}J_{12}^2 + (T^{30} + T^{26} + T^{24} + T^{20} + T^{14} + T^{10} + T^8 + T^4)J_{70}J_{12}J_{41} + (T^{38} + T^{32} + T^{26} + T^{24} + T^{22} + T^{16} + T^{10} + T^8)J_{70}J_{12} + (T^{32} + T^{26} + T^{24} + T^{22} + T^{16} + T^{10} + T^8 + T^4)J_{70}J_{41} + (T^{40} + T^{38} + T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{18} + T^{16} + T^{14} + T^{10})J_{70} + (T^{32} + T^{24} + T^{16} + T^8)J_{12}^2J_{41} + (T^{32} + T^{24} + T^{16} + T^8)J_{12}^2 + (T^{40} + T^{38} + T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{18} + T^{16} + T^{14} + T^{10})J_{12}J_{41} + (T^{42} + T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12})J_{41} + (T^{48} + T^{44} + T^{16} + T^{12}) \\
\mathbf{a}_2 &= (T^8 + 1)J_{07}J_{70}J_{12}J_{41} + (T^{22} + T^{20} + T^{16} + T^{12} + T^8 + T^6)J_{07}J_{70}J_{12} + (T^{14} + T^{10} + T^6 + T^2)J_{07}J_{70}J_{41} + (T^{24} + T^8)J_{07}J_{12}J_{41} + (T^{32} + T^{26} + T^{24} + T^{16} + T^{10} + T^8)J_{07}J_{41} + (T^6 + T^4)J_{70}^4 + (T^{10} + T^8 + T^6 + T^4)J_{70}^3J_{12} + (T^8 + T^4)J_{70}^3J_{41} + (T^{14} + T^{12} + T^{10} + T^4)J_{70}^3 + (T^6 + T^2)J_{70}^2J_{12}^3 + (T^8 + T^6 + T^4)J_{70}^2J_{12}^2J_{41} + (T^{22} + T^{20} + T^{16} + T^{10})J_{70}^2J_{12}^2 + (T^{14} + T^{10})J_{70}^2J_{12}J_{41} + (T^{22} + T^8 + T^6 + 1)J_{70}^2J_{12} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2)J_{70}^2J_{41} + (T^{26} + T^{20} + T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^2)J_{70}^2 + (T^{20} + T^{16} + T^{12} + T^8)J_{70}J_{12}^3 + (T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + 1)J_{70}J_{12}^2J_{41} + (T^{26} + T^{20} + T^{16} + T^{12} + T^{10} + T^8)J_{70}J_{12}^2 + (T^{24} + T^{22} + T^8 + T^6)J_{70}J_{12}J_{41} + (T^{24} + T^{22} + T^{20} + T^8 + T^6 + T^4)J_{70}J_{12} + (T^{26} + T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + T^8 + T^4)J_{70}J_{41} + (T^{32} + T^{28} + T^{26} + T^{22} + T^{16} + T^{12} + T^{10} + T^6)J_{70} + (T^{26} + T^{22} + T^{10} + T^6)J_{12}^2 + (T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8 + T^4)J_{12}J_{41} + (T^{34} + T^{22} + T^{18} + T^6)J_{41} + (T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10}) \\
\mathbf{a}_3 &= (T^{10} + T^6)J_{07}J_{70}^2 + (T^{10} + T^2)J_{07}J_{70}J_{12} + (T^{16} + T^{12} + T^8 + T^4)J_{07}J_{12}J_{41} + (T^{18} + T^{14} + T^{10} + T^6)J_{07}J_{41} + J_{70}^4T^2 + (T^{10} + T^6 + T^2)J_{70}^3J_{12} + (T^4 + T^2)J_{70}^3J_{41} + (T^{12} + T^8 + T^6 + T^4 + T^2)J_{70}^3 + J_{70}^3J_{12}T^4 + J_{70}^3J_{12}^2J_{41}T^2 + J_{70}^3J_{12}^2T^{12} + J_{70}^3J_{12}J_{41}T^{10} + (T^{10} + T^6)J_{70}^2J_{12} + (T^{12} + T^4)J_{70}^2J_{41} + (T^{18} + T^{16} + T^{12} + T^{10} + T^8 + T^4)J_{70}^2 + (T^{16} + T^{12} + T^{10} + T^8 + T^6 + T^4)J_{70}J_{12}^2 + (T^{20} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^2)J_{70} + (T^{16} + T^{12} + T^8 + T^4)J_{12}^2J_{41} + (T^{32} + T^{24} + T^{16} + T^8) \\
\mathbf{a}_4 &= (T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^4)J_{07}J_{41} + J_{70}^4 + (T^2 + 1)J_{70}^3J_{12} + J_{70}^3J_{41} + (T^4 + T^2 + 1)J_{70}^3 + J_{70}^2J_{12}^2J_{41} + J_{70}^2J_{12}^2T^6 + J_{70}^2J_{12}J_{41}T^2 + J_{70}^2J_{12}T^6 + J_{70}^2J_{41}T^4 + (T^{10} + T^6 + T^4 + T^2)J_{70}^2 + (T^8 + T^6 + T^2)J_{70}J_{12}^2J_{41} + (T^{10} + T^6)J_{70}J_{12}^2 + J_{70}J_{12}J_{41}T^8 + (T^{10} + T^6)J_{70}J_{41} + (T^{12} + T^4)J_{70} + (T^{12} + T^4)J_{12}J_{41} + (T^{16} + T^{12} + T^8 + T^4)J_{12} + (T^{14} + T^{10} + T^6 + T^2)J_{41} + (T^{24} + T^{22} + T^8 + T^6) \\
\mathbf{a}_5 &= (T^6 + T^2)J_{07}J_{41} + (T^2 + 1)J_{70}^2 + J_{70}J_{12}^3T^2 + J_{70}J_{12}^2J_{41} + J_{70}J_{12}^2T^4 + J_{70}J_{12}T^6 + J_{70}J_{41}T^2 + (T^8 + T^6 + T^4 + T^2)J_{70} + (T^8 + T^6 + T^4 + T^2)J_{12}J_{41} + (T^{10} + T^8 + T^6 + T^4)J_{41} + (T^{16} + T^{12} + T^8 + T^4) \\
\mathbf{a}_6 &= J_{07}J_{41} + J_{70} + J_{12}J_{41} + J_{41}T^2 + (T^8 + T^6 + T^4 + T^2)
\end{aligned}$$

Coefficients of $P_{J_{70},T}(X)$:

$$\begin{aligned} \mathbf{a}_0 = & (T^{56} + T^{54} + T^{48} + T^{46} + T^{24} + T^{22} + T^{16} + T^{14})J_{07}^2J_{70} + (T^{32} + T^{24} + T^{16} + T^8)J_{07}J_{70}^3J_{41} + (T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + \\ & T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10})J_{07}J_{70}^2J_{12}J_{41} + (T^{42} + T^{41} + T^{40} + T^{39} + T^{38} + T^{37} + T^{36} + T^{35} + T^{34} + T^{33} + T^{32} + T^{31} + \\ & T^{30} + T^{29} + T^{28} + T^{27} + T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11})J_{07}J_{70}^2J_{41} + (T^{48} + T^{44} + \\ & T^{16} + T^{12})J_{07}J_{70}J_{12}^2J_{41} + (T^{52} + T^{50} + T^{48} + T^{46} + T^{20} + T^{18} + T^{16} + T^{14})J_{07}J_{70}J_{41} + (T^{58} + T^{57} + T^{56} + T^{55} + T^{50} + T^{49} + T^{48} + T^{47} + T^{26} + \\ & T^{25} + T^{24} + T^{23} + T^{18} + T^{17} + T^{16} + T^{15})J_{07}J_{12}^2J_{41} + (T^{60} + T^{56} + T^{52} + T^{48} + T^{28} + T^{24} + T^{20} + T^{16})J_{07}J_{12}J_{41} + (T^{62} + T^{61} + T^{58} + T^{57} + T^{54} + \\ & T^{53} + T^{50} + T^{49} + T^{30} + T^{29} + T^{26} + T^{25} + T^{22} + T^{21} + T^{18} + T^{17})J_{07}J_{41} + J_{70}^7 + (T^8 + T^6 + T^4 + T^2)J_{70}^6J_{12} + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + \\ & T^3)J_{70}^6 + (T^{16} + T^{12} + T^8 + T^4)J_{70}^5J_{12}^2 + (T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6)J_{70}^5 + (T^{24} + T^{22} + T^8 + T^6)J_{70}^4J_{12}^3 + (T^{26} + T^{25} + T^{24} + T^{23} + \\ & T^{10} + T^9 + T^8 + T^7)J_{70}^4J_{12}^2 + (T^{28} + T^{24} + T^{12} + T^8)J_{70}^4J_{12} + (T^{30} + T^{29} + T^{26} + T^{25} + T^{14} + T^{13} + T^{10} + T^9)J_{70}^4 + (T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + \\ & T^{20} + T^{16} + T^{12})J_{70}^3 + (T^{48} + T^{46} + T^{16} + T^{14})J_{70}^2J_{12} + (T^{50} + T^{49} + T^{48} + T^{47} + T^{18} + T^{17} + T^{16} + T^{15})J_{70}^2 + (T^{56} + T^{48} + T^{24} + T^{16})J_{70}J_{12}^2 + \\ & (T^{60} + T^{58} + T^{52} + T^{50} + T^{28} + T^{26} + T^{20} + T^{18})J_{70} + (T^{64} + T^{62} + T^{60} + T^{58} + T^{56} + T^{54} + T^{52} + T^{50} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + \\ & T^{18})J_{12}^3 + (T^{66} + T^{65} + T^{64} + T^{63} + T^{62} + T^{61} + T^{60} + T^{59} + T^{58} + T^{57} + T^{56} + T^{55} + T^{54} + T^{53} + T^{52} + T^{51} + T^{34} + T^{33} + T^{32} + T^{31} + T^{30} + T^{29} + \\ & T^{28} + T^{27} + T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19})J_{12}^2 + (T^{68} + T^{52} + T^{36} + T^{20})J_{12} + (T^{70} + T^{69} + T^{54} + T^{53} + T^{38} + T^{37} + T^{22} + T^{21}) \end{aligned}$$

$$\begin{aligned} \mathbf{a}_1 = & (T^{26} + T^{25} + T^{22} + T^{21} + T^{10} + T^9 + T^6 + T^5)J_{07}J_{70}^3J_{12} + (T^{34} + T^{33} + T^{32} + T^{31} + T^{26} + T^{25} + T^{24} + T^{23} + T^{18} + T^{17} + T^{16} + T^{15} + T^{10} + \\ & T^9 + T^8 + T^7)J_{07}J_{70}^2J_{12} + (T^{36} + T^{30} + T^{28} + T^{22} + T^{20} + T^{14} + T^{12} + T^6)J_{07}J_{70}^2J_{41} + (T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8)J_{07}J_{70}^2J_{12} + \\ & (T^{44} + T^{42} + T^{12} + T^{10})J_{07}J_{70}J_{12}J_{41} + (T^{46} + T^{45} + T^{44} + T^{43} + T^{14} + T^{13} + T^{12} + T^{11})J_{07}J_{70}J_{41} + (T^{50} + T^{49} + T^{48} + T^{47} + T^{18} + T^{17} + \\ & T^{16} + T^{15})J_{07}J_{70} + (T^{56} + T^{54} + T^{48} + T^{46} + T^{24} + T^{22} + T^{16} + T^{14})J_{07} + J_{70}^6T^4 + (T^{12} + T^{10} + T^6 + 1)J_{70}^5J_{12} + (T^{14} + T^{13} + T^{12} + T^{11} + T^8 + \\ & T^7 + T^2 + T)J_{70}^5 + (T^{14} + T^{10} + T^6 + T^2)J_{70}^4J_{12}^2 + (T^{10} + T^9 + T^2 + T)J_{70}^4J_{12}J_{41} + (T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4)J_{70}^4 + (T^{22} + \\ & T^{20} + T^6 + T^4)J_{70}^3J_{12}^3 + (T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3)J_{70}^3J_{12}^2J_{41} + (T^{24} + T^{23} + \\ & T^{22} + T^{21} + T^8 + T^7 + T^6 + T^5)J_{70}^3J_{12}^2 + (T^{20} + T^4)J_{70}^3J_{12}J_{41} + (T^{30} + T^{28} + T^{24} + T^{22} + T^{14} + T^{12} + T^8 + T^6)J_{70}^3J_{12} + (T^{26} + T^{25} + T^{10} + \\ & T^9)J_{70}^3J_{41} + (T^{32} + T^{31} + T^{30} + T^{29} + T^{26} + T^{25} + T^{24} + T^{23} + T^{16} + T^{15} + T^{14} + T^{13} + T^{10} + T^9 + T^8 + T^7)J_{70}^3 + (T^{38} + T^{34} + T^{30} + T^{26} + \\ & T^{22} + T^{18} + T^{14} + T^{10})J_{70}^2J_{12}^2 + (T^{34} + T^{33} + T^{32} + T^{31} + T^{26} + T^{25} + T^{24} + T^{23} + T^{18} + T^{17} + T^{16} + T^{15} + T^{10} + T^9 + T^8 + T^7)J_{70}^2J_{12}J_{41} + \\ & (T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8)J_{70}^2J_{41} + (T^{44} + T^{42} + T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{10})J_{70}^2 + (T^{46} + T^{44} + T^{14} + \\ & T^{12})J_{70}J_{12}^3 + (T^{42} + T^{41} + T^{10} + T^9)J_{70}J_{12}^2J_{41} + (T^{48} + T^{47} + T^{46} + T^{45} + T^{16} + T^{15} + T^{14} + T^{13})J_{70}J_{12}^2 + (T^{52} + T^{48} + T^{46} + T^{44} + T^{20} + T^{16} + \\ & T^{14} + T^{12})J_{70}J_{12} + (T^{46} + T^{45} + T^{44} + T^{43} + T^{14} + T^{13} + T^{12} + T^{11})J_{70}J_{41} + (T^{54} + T^{53} + T^{50} + T^{49} + T^{48} + T^{47} + T^{46} + T^{45} + T^{22} + T^{21} + T^{18} + \\ & T^{17} + T^{16} + T^{15} + T^{14} + T^{13})J_{70} + (T^{52} + T^{44} + T^{20} + T^{12})J_{12}^2J_{41} + (T^{56} + T^{54} + T^{48} + T^{46} + T^{24} + T^{22} + T^{16} + T^{14})J_{12}^2 + (T^{54} + T^{53} + T^{46} + \\ & T^{45} + T^{22} + T^{21} + T^{14} + T^{13})J_{12}J_{41} + (T^{56} + T^{54} + T^{48} + T^{46} + T^{24} + T^{22} + T^{16} + T^{14})J_{41} + (T^{60} + T^{58} + T^{52} + T^{50} + T^{28} + T^{26} + T^{20} + T^{18}) \end{aligned}$$

$$\begin{aligned} \mathbf{a}_2 = & (T^{26} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^8 + 1)J_{07}J_{70}^3J_{12} + (T^{32} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{12} + T^{10} + T^8 + T^6 + T^4)J_{07}J_{70}^2J_{12} + \\ & (T^{34} + T^{33} + T^{30} + T^{29} + T^{26} + T^{25} + T^{22} + T^{21} + T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^6 + T^5)J_{07}J_{70}^2J_{41} + (T^{34} + T^{32} + T^{26} + T^{24} + T^{18} + T^{16} + \\ & T^{10} + T^8)J_{07}J_{70}J_{12}^2 + (T^{36} + T^{35} + T^{34} + T^{33} + T^{28} + T^{27} + T^{26} + T^{25} + T^{20} + T^{19} + T^{18} + T^{17} + T^{12} + T^{11} + T^{10} + T^9)J_{07}J_{70}J_{12} + (T^{36} + T^{32} + \\ & T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8)J_{07}J_{70}J_{41} + (T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + \\ & T^{10})J_{07}J_{70} + J_{70}^5J_{41}T^2 + J_{70}^5T^6 + (T^{10} + T^4)J_{70}^4J_{12}J_{41} + (T^{14} + T^{10} + T^8 + T^4)J_{70}^4J_{12} + (T^{12} + T^{10} + T^9 + T^8 + T^7 + T^6 + T^5)J_{70}^4J_{41} + \\ & (T^{16} + T^{15} + T^{12} + T^{11} + T^{10} + T^9 + T^6 + T^5)J_{70}^4 + (T^{18} + T^{16} + T^{14} + T^{10} + T^8 + T^6)J_{70}^3J_{12}^2J_{41} + (T^{22} + T^{20} + T^{12} + T^8 + T^6 + 1)J_{70}^3J_{12}^2 + \\ & (T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^6 + T^5)J_{70}^3J_{12}J_{41} + (T^{18} + T^{17} + T^{12} + T^2)J_{70}^3J_{12} + (T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^8 + 1)J_{70}^3J_{41} + \\ & (T^{22} + T^{20} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^2)J_{70}^2J_{12}^3 + (T^{30} + T^{26} + T^{22} + T^{20} + T^{14} + T^{10} + T^6 + T^4)J_{70}^2J_{12}^2 + (T^{28} + T^{27} + T^{26} + T^{25} + T^{22} + T^{21} + \\ & T^{20} + T^{19} + T^{16} + T^{15} + T^{14} + T^{13} + T^8 + T^7 + T^2 + T)J_{70}^2J_{12}J_{41} + (T^{32} + T^{31} + T^{30} + T^{29} + T^{26} + T^{25} + T^{22} + T^{21} + T^{16} + T^{15} + T^{14} + T^{13} + \\ & T^{10} + T^9 + T^6 + T^5)J_{70}^2J_{12}^2 + (T^{30} + T^{28} + T^{26} + T^{22} + T^{18} + T^{14} + T^{12} + T^{10} + T^6 + T^2)J_{70}^2J_{12}J_{41} + (T^{32} + T^{30} + T^{28} + T^{22} + T^{16} + T^{14} + T^{12} + \\ & T^6)J_{70}^2J_{12} + (T^{32} + T^{31} + T^{30} + T^{29} + T^{28} + T^{27} + T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + \\ & T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3)J_{70}^2J_{41} + (T^{34} + T^{33} + T^{32} + T^{31} + T^{24} + T^{23} + T^{22} + T^{21} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{10} + T^9)J_{70}J_{12}^3 + (T^{36} + T^{34} + T^{30} + T^{28} + T^{26} + T^{22} + \\ & T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^6)J_{70}J_{12}^2J_{41} + (T^{36} + T^{34} + T^{28} + T^{26} + T^{20} + T^{18} + T^{12} + T^{10})J_{70}J_{12}^2 + (T^{34} + T^{33} + T^{32} + T^{31} + T^{26} + T^{25} + \\ & T^{24} + T^{23} + T^{18} + T^{17} + T^{16} + T^{15} + T^{10} + T^9 + T^8 + T^7)J_{70}J_{12}J_{41} + (T^{38} + T^{37} + T^{36} + T^{35} + T^{34} + T^{33} + T^{32} + T^{31} + T^{30} + T^{29} + T^{28} + T^{27} + \\ & T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7)J_{70}J_{12} + (T^{40} + T^{38} + T^{36} + \\ & T^{34} + T^{28} + T^{26} + T^{20} + T^{18} + T^{12} + T^{10} + T^8 + T^6)J_{70}J_{41} + (T^{44} + T^{42} + T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{10})J_{70} + (T^{42} + T^{41} + \\ & T^{40} + T^{39} + T^{38} + T^{37} + T^{36} + T^{35} + T^{34} + T^{33} + T^{32} + T^{31} + T^{30} + T^{29} + T^{28} + T^{27} + T^{26} + T^{25} + T^{24} + T^{23} + T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + \\ & T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11})J_{12}^2J_{41} + (T^{42} + T^{40})J_{12}J_{41} + (T^{46} + T^{45} + T^{14} + T^{13})J_{41} + (T^{50} + T^{49} + T^{48} + T^{47} + T^{18} + T^{17} + T^{16} + T^{15}) \end{aligned}$$

$$\begin{aligned} \mathbf{a}_3 = & (T^{24} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^6 + T^4)J_{07}J_{70}^2J_{41} + (T^{28} + T^{24} + T^{12} + T^8)J_{07}J_{70}J_{12} + (T^{26} + T^{25} + T^{24} + T^{23} + T^{10} + T^9 + T^8 + \\ & T^7)J_{07}J_{70}J_{41} + (T^4 + T^3)J_{70}^5 + (T^{12} + T^{11} + T^8 + T^7 + T^6 + T^5)J_{70}^4J_{12} + (T^8 + T^6)J_{70}^4J_{41} + (T^{14} + T^{12} + T^6)J_{70}^4 + (T^{18} + T^{17} + T^{14} + T^{13} + T^8 + \\ & T^7 + T^4 + T^3)J_{70}^3J_{12}^2 + (T^{16} + T^{10} + T^8 + T^2)J_{70}^3J_{12}J_{41} + (T^{18} + T^{16} + T^{12} + T^6)J_{70}^3J_{12} + (T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^4 + T^3)J_{70}^3J_{41} + \\ & (T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7)J_{70}^3 + (T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9 + T^6 + T^5)J_{70}^2J_{12}^3 + (T^{20} + T^{14} + T^6 + T^4)J_{70}^2J_{12}^2J_{41} + \\ & (T^{24} + T^{20} + T^{18} + T^{14} + T^{10} + T^8 + T^6 + T^4)J_{70}^2J_{12}^2 + (T^{22} + T^{21} + T^{18} + T^{17} + T^{14} + T^{13} + T^{10} + T^9)J_{70}^2J_{12}J_{41} + (T^{22} + T^{21} + T^6 + T^5)J_{70}^2J_{12} + \\ & (T^{24} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^6 + T^4)J_{70}^2J_{41} + (T^{24} + T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^6 + 1)J_{70}^2 + (T^{28} + T^{26} + T^{12} + T^{10})J_{70}J_{12}^3 + (T^{26} + \\ & T^{25} + T^{20} + T^{19} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^8 + T^7 + T^6 + T^5)J_{70}J_{12}^2J_{41} + (T^{30} + T^{29} + T^{26} + T^{25} + T^{14} + T^{13} + T^{10} + \\ & T^9)J_{70}J_{12}^2 + (T^{22} + T^6)J_{07}J_{12}J_{41} + (T^{30} + T^{24} + T^{22} + T^{20} + T^{14} + T^8 + T^6 + T^4)J_{70}J_{12} + (T^{30} + T^{29} + T^{26} + T^{25} + T^{24} + T^{23} + T^{14} + T^{13} + \\ & T^{10} + T^9 + T^8 + T^7)J_{70}J_{41} + (T^{34} + T^{33} + T^{32} + T^{31} + T^{30} + T^{29} + T^{24} + T^{23} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^8 + T^7)J_{70} + (T^{34} + T^{32} + \\ & T^{26} + T^{24} + T^{18} + T^{16} + T^{10} + T^8)J_{12}^2 + (T^{36} + T^{34} + T^{28} + T^{26} + T^{20} + T^{18} + T^{12} + T^{10})J_{41} + (T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12}) \end{aligned}$$

$$\begin{aligned} \mathbf{a}_4 = & (T^{16} + T^{10} + T^8 + T^2)J_{07}J_{70}J_{41} + (T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6)J_{07}J_{70} + J_{70}^5 + J_{70}^4J_{12}T^2 + (T^6 + T^5 + T^4 + \\ & T^3)J_{70}^4 + (T^8 + T^6 + T^4)J_{70}^3J_{12}^2 + (T^{10} + T^9 + T^8 + T^7 + T^4 + T^3)J_{70}^3J_{12} + (T^{12} + T^8 + T^4)J_{70}^3 + (T^{10} + T^6)J_{70}^2J_{12}^3 + (T^{14} + T^{13} + T^6 + \\ & T^5)J_{70}^2J_{12}^2 + (T^8 + T^4)J_{70}^2J_{12}J_{41} + (T^{14} + T^{12} + T^8 + T^2)J_{70}^2J_{12} + (T^{10} + T^9 + T^8 + T^7)J_{70}^2J_{41} + (T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + \\ & T^7)J_{70}^2 + (T^{16} + T^{15} + T^{12} + T^{11} + T^8 + T^7 + T^4 + T^3)J_{70}J_{12}^3 + (T^{16} + T^{14} + T^{10} + T^8 + T^6 + T^2)J_{70}J_{12}^2J_{41} + (T^{16} + T^{12} + T^8 + T^4)J_{70}J_{12}^2 + \\ & (T^{16} + T^{15} + T^{12} + T^{11} + T^8 + T^7 + T^4 + T^3)J_{70}J_{12}J_{41} + (T^{22} + T^{21} + T^6 + T^5)J_{70}J_{12} + (T^{20} + T^{18} + T^{14} + T^{10} + T^6 + T^4)J_{70}J_{41} + (T^{22} + \\ & T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4)J_{70} + (T^{22} + T^{21} + T^{20} + T^{19} + T^{18} + T^{17} + T^{16} + T^{15} + T^{14} + T^{13} + T^{12} + T^{11} + T^{10} + T^9 + T^8 + \\ & T^7)J_{12}^2J_{41} + (T^{24} + T^8)J_{12}J_{41} + (T^{28} + T^{24} + T^{12} + T^8)J_{12} + (T^{24} + T^{23} + T^8 + T^7)J_{41} + (T^{30} + T^{29} + T^{26} + T^{25} + T^{14} + T^{13} + T^{10} + T^9) \end{aligned}$$

$$\begin{aligned} \mathbf{a}_5 = & (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3)J_{07}J_{70} + J_{70}^3J_{12} + (T^2 + T)J_{70}^3 + (T^4 + T^2)J_{70}^2J_{12}^2 + (T^2 + T)J_{70}^2J_{12}J_{41} + (T^6 + \\ & T^5)J_{70}^2J_{12} + J_{70}^2T^4 + (T^8 + T^4)J_{70}J_{12}^3 + (T^4 + T^3)J_{70}J_{12}^2J_{41} + (T^{10} + T^9 + T^8 + T^7 + T^6 + T^5 + T^4 + T^3)J_{70}J_{12}^2 + (T^8 + T^4)J_{70}J_{12}J_{41} + \\ & (T^{10} + T^8 + T^6 + T^4)J_{70}J_{12} + (T^{10} + T^9 + T^4 + T^3)J_{70}J_{41} + (T^{12} + T^{11} + T^{10} + T^9 + T^8 + T^7 + T^6 + T^5)J_{70} + (T^{12} + T^4)J_{12}^2J_{41} + (T^{14} + \\ & T^{13} + T^6 + T^5)J_{12}J_{41} + (T^{14} + T^{10} + T^6 + T^2)J_{41} + (T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6) \end{aligned}$$

$$\begin{aligned}
& T^{12} + T^{10}) J_{70}^4 J_{12} J_{41} + (T^{48} + T^{46} + T^{44} + T^{42} + T^{36} + T^{34} + T^{32} + T^{30} + T^{20} + T^{18} + T^{12} + T^{10}) J_{70}^4 J_{12} + (T^{42} + T^{40} + T^{38} + T^{34} + T^{30} + T^{28} + \\
& T^{26} + T^{24} + T^{22} + T^{18} + T^{14} + T^{12}) J_{70}^4 J_{41} + (T^{50} + T^{48} + T^{40} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{18} + T^{12}) J_{70}^4 + (T^{48} + T^{46} + T^{44} + T^{42} + T^{40} + T^{38} + \\
& T^{36} + T^{34} + T^{30} + T^{28} + T^{26} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12}) J_{70}^3 J_{12}^3 + (T^{46} + T^{14}) J_{70}^3 J_{12}^2 J_{41} + (T^{54} + T^{50} + T^{44} + T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + \\
& T^{28} + T^{26} + T^{24} + T^{20} + T^{16} + T^{14} + T^{10}) J_{70}^3 J_{12}^2 + (T^{52} + T^{48} + T^{20} + T^{16}) J_{70}^3 J_{12} J_{41} + (T^{66} + T^{60} + T^{52} + T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{34} + \\
& T^{28} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10}) J_{70}^3 J_{12} + (T^{56} + T^{54} + T^{24} + T^{22}) J_{70}^3 J_{41} + (T^{68} + T^{64} + T^{54} + T^{48} + T^{46} + T^{44} + T^{36} + T^{32} + T^{22} + T^{16} + \\
& T^{14} + T^{12}) J_{70}^3 + (T^{52} + T^{46} + T^{42} + T^{40} + T^{20} + T^{14} + T^{10} + T^{18}) J_{70}^2 J_{12}^3 + (T^{50} + T^{48} + T^{46} + T^{42} + T^{18} + T^{16} + T^{14} + T^{10}) J_{70}^2 J_{12}^2 J_{41} + (T^{70} + T^{68} + \\
& T^{64} + T^{62} + T^{58} + T^{54} + T^{52} + T^{48} + T^{46} + T^{42} + T^{38} + T^{36} + T^{32} + T^{30} + T^{26} + T^{22} + T^{20} + T^{16} + T^{14} + T^{10}) J_{70}^2 J_{12}^2 + (T^{66} + T^{62} + T^{52} + T^{46} + \\
& T^{44} + T^{42} + T^{34} + T^{30} + T^{20} + T^{14} + T^{12} + T^{10}) J_{70}^2 J_{12} J_{41} + (T^{74} + T^{72} + T^{70} + T^{68} + T^{66} + T^{56} + T^{54} + T^{52} + T^{50} + T^{40} + T^{38} + T^{36} + T^{34} + T^{24} + \\
& T^{22} + T^{20} + T^{18} + T^{10}) J_{70}^2 J_{12} + (T^{72} + T^{64} + T^{58} + T^{54} + T^{52} + T^{50} + T^{46} + T^{44} + T^{40} + T^{32} + T^{26} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12}) J_{70}^2 J_{41} + (T^{82} + \\
& T^{80} + T^{68} + T^{60} + T^{52} + T^{44} + T^{36} + T^{28} + T^{20} + T^{18} + T^{16} + T^{12}) J_{70}^2 + (T^{74} + T^{72} + T^{68} + T^{66} + T^{62} + T^{60} + T^{52} + T^{50} + T^{46} + T^{44} + T^{36} + T^{34} + \\
& T^{30} + T^{28} + T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^{18}) J_{70} J_{12}^3 + (T^{70} + T^{66} + T^{60} + T^{58} + T^{54} + T^{50} + T^{44} + T^{42} + T^{38} + T^{34} + T^{28} + T^{26} + T^{22} + T^{18} + T^{12} + \\
& T^{10}) J_{70} J_{12}^2 J_{41} + (T^{78} + T^{76} + T^{74} + T^{72} + T^{70} + T^{68} + T^{66} + T^{64} + T^{62} + T^{60} + T^{58} + T^{56} + T^{54} + T^{52} + T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{40} + T^{38} + \\
& T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16}) J_{70} J_{12}^2 + (T^{76} + T^{68} + T^{66} + T^{64} + T^{62} + T^{52} + T^{50} + T^{48} + T^{46} + T^{36} + T^{34} + T^{32} + \\
& T^{30} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12}) J_{70} J_{12} J_{41} + (T^{86} + T^{84} + T^{80} + T^{78} + T^{74} + T^{72} + T^{22} + T^{20} + T^{16} + T^{14} + T^{10}) J_{70} J_{12} + (T^{82} + T^{80} + T^{78} + T^{76} + T^{18} + \\
& T^{16} + T^{14} + T^{12}) J_{70} + (T^{82} + T^{78} + T^{74} + T^{72} + T^{18} + T^{14} + T^{10} + T^{18}) J_{12}^3 + (T^{80} + T^{78} + T^{70} + T^{68} + T^{66} + T^{64} + T^{58} + T^{56} + T^{54} + T^{52} + \\
& T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^{10}) J_{12}^2 J_{41} + (T^{90} + T^{88} + T^{82} + \\
& T^{80} + T^{26} + T^{24} + T^{18} + T^{16}) J_{12}^2 + (T^{80} + T^{78} + T^{76} + T^{74} + T^{16} + T^{14} + T^{12} + T^{10}) J_{12} J_{41} + (T^{88} + T^{86} + T^{84} + T^{82} + T^{80} + T^{78} + T^{76} + T^{74} + T^{24} + \\
& T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10}) J_{12} + (T^{86} + T^{84} + T^{78} + T^{76} + T^{22} + T^{20} + T^{14} + T^{12}) J_{41} + (T^{96} + T^{92} + T^{80} + T^{76} + T^{32} + T^{28} + T^{16} + T^{12})
\end{aligned}$$

$$\begin{aligned}
& \mathbf{a}_2 = J_{07}^6 J_{70}^3 + J_{07}^5 J_{70} J_{70} T^2 + (T^8 + T^4 + 1) J_{07}^5 J_{12} + (T^6 + T^2) J_{07}^5 J_{41} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4 + T^2) J_{07}^5 J_{70}^3 + J_{07}^4 J_{70}^2 J_{12} T^2 + (T^6 + T^4) J_{07}^4 J_{70}^2 + \\
& (T^8 + T^6) J_{07}^4 J_{70} J_{12}^2 + J_{07}^4 J_{70} J_{12} J_{41} T^6 + (T^{16} + T^{14} + T^8 + T^6 + T^4 + 1) J_{07}^4 J_{70} J_{12} + (T^8 + T^6 + T^4) J_{07}^4 J_{70} J_{41} + (T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + \\
& T^4) J_{07}^4 J_{70} + (T^{10} + T^8 + T^6 + T^4 + 1) J_{07}^4 J_{12}^3 + J_{07}^4 J_{12}^2 J_{41} T^6 + (T^{18} + T^{16} + T^{12} + T^8 + T^6 + T^4 + T^2 + 1) J_{07}^4 J_{12}^2 + (T^{16} + T^{14} + T^8 + T^6 + 1) J_{07}^4 J_{12} J_{41} + \\
& (T^{22} + T^{20} + T^{18} + T^{16} + T^{12} + T^{10} + T^8 + T^6) J_{07}^4 J_{12} + (T^{18} + T^{12} + T^8 + T^6) J_{07}^4 J_{41} + (T^{32} + T^{28} + T^{22} + T^{18} + T^{14} + T^{10} + T^8 + T^4) J_{07}^4 + (T^6 + \\
& T^4) J_{07}^3 J_{70}^3 + J_{07}^3 J_{70}^2 J_{12}^2 T^2 + (T^8 + T^2) J_{07}^3 J_{70}^2 J_{41} + (T^{16} + T^{14} + T^{12} + T^2) J_{07}^3 J_{70}^2 + (T^8 + T^4 + T^2) J_{07}^3 J_{70}^2 J_{12}^2 + (T^6 + T^2) J_{07}^3 J_{70}^2 J_{12} J_{41} + (T^{16} + \\
& T^{14} + T^8 + T^4 + T^2) J_{07}^3 J_{70}^2 J_{12}^2 + (T^6 + T^4 + T^2) J_{07}^3 J_{70}^2 J_{12} J_{41} + (T^{18} + T^{16} + T^{10} + T^4) J_{07}^3 J_{70}^2 J_{12} + (T^{18} + T^{12} + T^8 + T^4 + T^2) J_{07}^3 J_{70}^2 J_{41} + \\
& (T^{22} + T^{16} + T^{10} + T^8 + T^6) J_{07}^3 J_{70}^2 + (T^{16} + T^{14} + T^{12} + T^8 + T^2) J_{07}^3 J_{70} J_{12}^2 + (T^{12} + T^6) J_{07}^3 J_{70} J_{12} J_{41} + (T^{22} + T^{16} + T^{10} + T^8 + T^6 + T^4 + T^2 + \\
& 1) J_{07}^3 J_{70} J_{12}^2 + (T^8 + T^6 + T^4 + T^2) J_{07}^3 J_{70} J_{12} J_{41} + (T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{14} + T^8 + T^2) J_{07}^3 J_{70} J_{12} + (T^{20} + T^{18} + T^{10} + 1) J_{07}^3 J_{70} J_{41} + \\
& (T^{30} + T^{28} + T^{26} + T^{24} + T^{20} + T^{14} + T^{12} + T^{10} + T^8 + T^4) J_{07}^3 J_{70} + (T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + 1) J_{07}^3 J_{12}^3 + (T^{20} + T^{18} + T^{10} + T^6 + T^2 + \\
& 1) J_{07}^3 J_{12}^2 J_{41} + (T^{32} + T^{28} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{12} + T^8 + T^6 + T^4 + T^2) J_{07}^3 J_{12}^2 + (T^{24} + T^{18} + T^{16} + T^{14} + T^{10} + T^8 + T^6 + 1) J_{07}^3 J_{12} J_{41} + \\
& (T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{18} + T^{16} + T^{14} + T^{10} + T^4) J_{07}^3 J_{12} + (T^{38} + T^{30} + T^{28} + T^{24} + T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + \\
& T^8 + T^2 + 1) J_{07}^3 J_{41} + (T^{38} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^8 + T^6) J_{07}^3 + (T^6 + T^4) J_{07}^2 J_{70}^5 J_{12} + J_{07}^2 J_{70}^5 T^6 + (T^4 + T^2) J_{07}^2 J_{70}^4 J_{12}^2 + \\
& (T^8 + T^6 + T^4) J_{07}^2 J_{70}^4 J_{12} J_{41} + (T^{10} + T^6) J_{07}^2 J_{70}^4 J_{12} + (T^{16} + T^{14} + T^{12} + T^{10} + T^6) J_{07}^2 J_{70}^4 J_{41} + (T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^2) J_{07}^2 J_{70}^4 + (T^{16} + T^{14} + \\
& T^{12} + T^2) J_{07}^2 J_{70}^3 J_{12}^2 + (T^{10} + T^8) J_{07}^2 J_{70}^3 J_{12} J_{41} + (T^{18} + T^{16} + T^{14} + T^6) J_{07}^2 J_{70}^3 J_{12} + (T^{14} + T^{12} + T^6 + T^4) J_{07}^2 J_{70}^3 J_{12} J_{41} + (T^{16} + T^{12} + T^8 + T^4 + \\
& T^2) J_{07}^2 J_{70}^3 J_{12} + (T^{20} + T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^2) J_{07}^2 J_{70}^3 J_{41} + (T^{20} + T^{18} + T^{16} + T^6 + T^2) J_{07}^2 J_{70}^3 + (T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{10} + \\
& T^6 + T^2) J_{07}^2 J_{70}^2 J_{12}^3 + (T^{20} + T^{18} + T^{10} + T^6 + T^4 + T^2) J_{07}^2 J_{70}^2 J_{12}^2 J_{41} + (T^{32} + T^{28} + T^{24} + T^{22} + T^{18} + T^{14} + T^{12} + T^6 + T^2 + 1) J_{07}^2 J_{70}^2 J_{12}^2 + (T^{26} + \\
& T^{22} + T^{20} + T^{16} + T^{14} + T^{12} + T^{10} + T^8) J_{07}^2 J_{70}^2 J_{12} J_{41} + (T^{28} + T^{24} + T^{20} + T^{14} + T^{12} + T^8 + T^6 + T^4) J_{07}^2 J_{70}^2 J_{12} + (T^{30} + T^{24} + T^{20} + T^{10} + T^6 + \\
& T^2) J_{07}^2 J_{70}^2 J_{41} + (T^{36} + T^{28} + T^{26} + T^{24} + T^{20} + T^{18} + T^{16} + T^{12} + T^{10} + T^8 + T^2 + 1) J_{07}^2 J_{70}^2 + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2) J_{07}^2 J_{70} J_{12}^3 + \\
& (T^{28} + T^{18} + T^{16} + T^8 + T^4 + T^2) J_{07}^2 J_{70} J_{12}^2 J_{41} + (T^{36} + T^{32} + T^{30} + T^{28} + T^{26} + T^{22} + T^{14} + T^{12} + T^{10} + T^6 + T^4 + 1) J_{07}^2 J_{70} J_{12}^2 + (T^{34} + T^{32} + T^{28} + \\
& T^{26} + T^{18} + T^{16} + T^{14} + T^{12} + T^6 + T^2) J_{07}^2 J_{70} J_{12} J_{41} + (T^{42} + T^{32} + T^{30} + T^{28} + T^{26} + T^{16} + T^{14} + T^{12}) J_{07}^2 J_{70} J_{12} + (T^{38} + T^{36} + T^{34} + T^{30} + T^{26} + \\
& T^{20} + T^{16} + T^{14} + T^{10} + T^6 + T^2 + 1) J_{07}^2 J_{70} J_{41} + (T^{44} + T^{36} + T^{32} + T^{12} + T^4 + 1) J_{07}^2 J_{70} + (T^{40} + T^{30} + T^{28} + T^{22} + T^{20} + T^{14} + T^{12} + T^8 + T^6 + \\
& T^4) J_{07}^2 J_{12}^3 + (T^{38} + T^{34} + T^{32} + T^{30} + T^{22} + T^{20} + T^{16} + T^{14} + T^4 + T^2) J_{07}^2 J_{12}^2 J_{41} + (T^{42} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + \\
& T^{18} + T^{16} + T^{14} + T^{12} + T^8 + T^6) J_{07}^2 J_{12}^2 + (T^{44} + T^{36} + T^{34} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{10} + T^6 + T^4 + 1) J_{07}^2 J_{12} J_{41} + \\
& (T^{56} + T^{52} + T^{46} + T^{44} + T^{42} + T^{32} + T^{24} + T^{20} + T^{14} + T^{12} + 1) J_{07}^2 J_{12} + (T^{54} + T^{52} + T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{36} + T^{34} + T^{32} + T^{20} + T^{18} + T^{12} + \\
& T^{10} + T^4 + T^2) J_{07}^2 J_{41} + (T^{54} + T^{50} + T^{48} + T^{46} + T^{40} + T^{34} + T^{22} + T^{18} + T^{16} + T^{14} + T^8 + T^2) J_{07}^2 + (T^8 + T^6 + T^4) J_{07} J_{70}^7 + (T^6 + T^4) J_{07} J_{70}^5 J_{12}^2 + \\
& (T^{10} + T^6 + T^4) J_{07} J_{70}^6 J_{12} + J_{07} J_{70}^6 J_{41} T^6 + (T^{16} + T^{12} + T^{10} + T^8 + T^6) J_{07} J_{70}^6 + (T^6 + T^4) J_{07} J_{70}^5 J_{12}^3 + J_{07} J_{70}^5 J_{12}^2 J_{41} T^8 + (T^{16} + T^{14}) J_{07} J_{70}^5 J_{12}^2 + \\
& (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^4) J_{07} J_{70}^5 J_{12} J_{41} + (T^{22} + T^{20} + T^{18} + T^{14}) J_{07} J_{70}^5 J_{12} + (T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^6 + T^4) J_{07} J_{70}^5 J_{41} + (T^{22} + \\
& T^{16} + T^{14} + T^{12} + T^{10} + T^6 + T^4) J_{07} J_{70}^5 + (T^{16} + T^{10} + T^8 + T^6 + T^4) J_{07} J_{70}^4 J_{12}^3 + (T^{18} + T^{16} + T^{14} + T^8 + T^6 + T^4) J_{07} J_{70}^4 J_{12}^2 J_{41} + (T^{14} + \\
& T^{10} + T^4 + T^2) J_{07} J_{70}^4 J_{12}^2 + (T^{24} + T^{22} + T^{18} + T^{16} + T^6 + T^2) J_{07} J_{70}^4 J_{12} J_{41} + (T^{26} + T^{24} + T^{20} + T^{18} + T^{14} + T^{10} + T^8 + T^4 + T^2) J_{07} J_{70}^4 J_{12} + \\
& (T^{32} + T^{26} + T^{22} + T^{20} + T^{16} + T^{12}) J_{07} J_{70}^4 J_{41} + (T^{38} + T^{36} + T^{32} + T^{26} + T^{20} + T^{18} + T^{16} + T^{14} + T^8) J_{07} J_{70}^4 + (T^{24} + T^{12} + T^6 + T^4) J_{07} J_{70}^3 J_{12}^3 + \\
& (T^{24} + T^{22} + T^{16} + T^{14} + T^{12} + T^6) J_{07} J_{70}^3 J_{12}^2 J_{41} + (T^{32} + T^{30} + T^{26} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^8 + T^6) J_{07} J_{70}^3 J_{12}^2 + (T^{26} + T^{14} + T^{10} + T^8 + \\
& T^6 + T^2) J_{07} J_{70}^3 J_{12} J_{41} + (T^{38} + T^{36} + T^{32} + T^{30} + T^{24} + T^{12}) J_{07} J_{70}^3 J_{12} + (T^{40} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{12} + T^4) J_{07} J_{70}^3 J_{41} + (T^{48} + T^{46} + \\
& T^{42} + T^{40} + T^{32} + T^{30} + T^{28} + T^{26} + T^{22} + T^{20} + T^{12} + T^8 + T^6 + T^4) J_{07} J_{70}^3 + (T^{40} + T^{38} + T^{30} + T^{28} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^8 + \\
& T^2) J_{07} J_{70}^2 J_{12}^3 + (T^{38} + T^{36} + T^{30} + T^{26} + T^{22} + T^{20} + T^{12} + T^6 + T^4 + T^2) J_{07} J_{70}^2 J_{12}^2 J_{41} + (T^{48} + T^{46} + T^{42} + T^{34} + T^{26} + T^{22} + T^{20} + T^{16} + T^{14} + \\
& T^6 + T^4 + T^2) J_{07} J_{70}^2 J_{12}^2 + (T^{40} + T^{38} + T^{26} + T^{24} + T^{22} + T^{14} + T^6 + T^2) J_{07} J_{70}^2 J_{12} J_{41} + (T^{44} + T^{38} + T^{32} + T^{30} + T^{24} + T^{22} + T^{16} + T^{14} + T^{12} + \\
& T^8) J_{07} J_{70}^2 J_{12} + (T^{50} + T^{48} + T^{46} + T^{42} + T^{36} + T^{34} + T^{26} + T^{16} + T^{14} + T^4) J_{07} J_{70}^2 J_{41} + (T^{50} + T^{46} + T^{38} + T^{18} + T^{14} + T^6) J_{07} J_{70}^2 + (T^{34} + T^{32} + \\
& T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4) J_{07} J_{70} J_{12}^3 + (T^{36} + T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{18} + T^{16} + \\
& T^{14} + T^{10} + T^8 + T^6 + T^4) J_{07} J_{70} J_{12}^2 J_{41} + (T^{56} + T^{54} + T^{52} + T^{40} + T^{36} + T^{24} + T^{22} + T^{20} + T^8 + T^4) J_{07} J_{70} J_{12}^2 + (T^{54} + T^{52} + T^{48} + T^{44} + T^{34} + T^{22} + \\
& T^{20} + T^{16} + T^{12} + T^2) J_{07} J_{70} J_{12} J_{41} + (T^{60} + T^{52} + T^{48} + T^{40} + T^{28} + T^{20} + T^{16} + T^8) J_{07} J_{70} J_{12} + (T^{56} + T^{54} + T^{52} + T^{50} + T^{44} + T^{40} + T^{38} + T^{24} + \\
& T^{22} + T^{20} + T^{18} + T^{12} + T^8 + T^6) J_{07} J_{70} J_{41} + (T^{58} + T^{54} + T^{48} + T^{44} + T^{42} + T^{40} + T^{38} + T^{36} + T^{26} + T^{22} + T^{16} + T^{12} + T^{10} + T^8 + T^6 + T^4) J_{07} J_{70} + \\
& (T^{58} + T^{48} + T^{40} + T^{34} + T^{26} + T^{16} + T^8 + T^2) J_{07} J_{12}^3 + (T^{54} + T^{52} + T^{50} + T^{40} + T^{36} + T^{34} + T^{22} + T^{20} + T^{18} + T^8 + T^4 + T^2) J_{07} J_{12}^2 J_{41} + (T^{64} + T^{60} + \\
& T^{58} + T^{56} + T^{54} + T^{52} + T^{48} + T^{46} + T^{44} + T^{40} + T^{36} + T^{34} + T^{32} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^4 + T^2) J_{07} J_{12}^2 + (T^{60} + T^{58} + \\
& T^{56} + T^{52} + T^{48} + T^{44} + T^{36} + T^{34} + T^{28} + T^{26} + T^{24} + T^{20} + T^{16} + T^{12} + T^4 + T^2) J_{07} J_{12} J_{41} + (T^{72} + T^{70} + T^{56} + T^{52} + T^{40} + T^{36} + T^{24} + T^{20} + T^6 + \\
& T^4) J_{07} J_{12} + (T^{70} + T^{66} + T^{60} + T^{54} + T^{52} + T^{48} + T^{46} + T^{42} + T^{40} + T^{34} + T^{28} + T^{22} + T^{20} + T^{16} + T^{14} + T^{10} + T^8 + T^6) J_{07} J_{41} + (T^{64} + T^{56} + T^{48} + \\
& T^{40} + T^{32} + T^{24} + T^{16} + T^8) J_{07} + (T^6 + T^4) J_{70}^7 J_{12}^3 + (T^8 + T^6) J_{70}^7 J_{12}^2 + (T^{10} + T^8) J_{70}^7 J_{12} + (T^{12} + T^{10}) J_{70}^7 + (T^{12} + T^{10} + T^8 + T^4) J_{70}^6 J_{12}^3 + (T^{12} + \\
& T^8 + T^6) J_{70}^6 J_{12}^2 J_{41} + (T^{22} + T^{18} + T^{16} + T^{12} + T^6) J_{70}^6 J_{12}^2 + (T^{12} + T^8) J_{70}^6 J_{12} J_{41} + (T^{20} + T^{18} + T^8) J_{70}^6 J_{12} + (T^{16} + T^{12} + T^4) J_{70}^6 J_{41} + (T^{48} + T^{18} + \\
& T^{10}) J_{70}^6 J_{12}^3 + T^4 + (T^{24} + T^{22} + T^{14} + T^{12} + T^8) J_{70}^5 J_{12}^2 J_{41} + (T^{22} + T^{16} + T^{12} + T^6) J_{70}^5 J_{12}^2 + (T^{24} + T^{18} + T^{16} + T^{14} + T^{12}) J_{70}^5 J_{12} J_{41} + (T^{38} + \\
& T^{36} + T^{32} + T^{28} + T^{26} + T^{22} + T^{12} + T^8) J_{70}^5 J_{12} + (T^{28} + T^{22} + T^{12}) J_{70}^5 J_{41} + (T^{40} + T^{38} + T^{36} + T^{34} + T^{24} + T^{16} + T^{14} + T^{12} + T^{10}) J_{70}^5 + (T^{30} + T^{28} + \\
& T^{24} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4) J_{70}^4 J_{12}^3 + (T^{26} + T^{24} + T^{20} + T^{18} + T^{16} + T^{12} + T^8 + T^6) J_{70}^4 J_{12}^2 J_{41} + (T^{38} + T^{34} + T^{32} + T^{28} + \\
& T^{26} + T^{20} + T^{18} + T^{16} + T^8 + T^6) J_{70}^4 J_{12}^2 + (T^{40} + T^{38} + T^{36} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{12} + T^8) J_{70}^4 J_{12} J_{41} + (T^{44} + T^{42} + T^{40} + T^{36} + T^{34} + \\
& T^{30} + T^{14} + T^8) J_{70}^4 J_{12} + (T^{44} + T^{38} + T^{36} + T^{34} + T^{28} + T^{16} + T^{14} + T^{10}) J_{70}^4 J_{41} + (T^{54} + T^{50} + T^{36} + T^{32} + T^{24} + T^{14} + T^{12} + T^{10}) J_{70}^4 + (T^{48} + T^{46} + \\
& T^{44} + T^{42} + T^{36} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{20} + T^{14} + T^{12} + T^8 + T^6 + T^4) J_{70}^3 J_{12}^3 + (T^{42} + T^{32} + T^{30} + T^{26} + T^{24} + T^{14}) J_{70}^3 J_{12}^2 J_{41} + (T^{50} +
\end{aligned}$$

$$\begin{aligned}
& T^{48} + T^{44} + T^{42} + T^{38} + T^{32} + T^{28} + T^{24} + T^{18} + T^{10} + T^8 + T^6 + J_{70}^3 J_{12}^2 + (T^{48} + T^{44} + T^{40} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{18} + T^{16}) J_{70}^3 J_{12} J_{41} + \\
& (T^{50} + T^{44} + T^{36} + T^{34} + T^{30} + T^{24} + T^{20} + T^{14} + T^{12} + T^8) J_{70}^3 J_{12}^2 + (T^{56} + T^{54} + T^{40} + T^{36} + T^{22} + T^{20}) J_{70}^3 J_{41} + (T^{52} + T^{48} + T^{38} + T^{34} + T^{30} + \\
& T^{26} + T^{22} + T^{20} + T^{18} + T^{16} + T^{14} + T^{10}) J_{70}^3 + (T^{56} + T^{54} + T^{50} + T^{46} + T^{42} + T^{38} + T^{36} + T^{30} + T^{24} + T^{18} + T^{10} + T^4) J_{70}^2 J_{12}^2 + (T^{54} + T^{52} + T^{50} + \\
& T^{46} + T^{44} + T^{42} + T^{38} + T^{36} + T^{30} + T^{26} + T^{24} + T^{22} + T^{18} + T^{12} + T^8 + T^6) J_{70}^2 J_{12}^2 J_{41} + (T^{64} + T^{58} + T^{54} + T^{52} + T^{48} + T^{46} + T^{42} + T^{40} + T^{36} + T^{34} + \\
& T^{30} + T^{28} + T^{24} + T^{18} + T^{12} + T^6) J_{70}^2 J_{12}^2 + (T^{50} + T^{46} + T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^8) J_{70}^2 J_{12} J_{41} + (T^{58} + T^{56} + T^{54} + \\
& T^{52} + T^{50} + T^{44} + T^{42} + T^{40} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{12} + T^{10} + T^8) J_{70}^2 J_{12} + (T^{56} + T^{48} + T^{42} + T^{24} + T^{16} + T^{10}) J_{70}^2 J_{41} + (T^{66} + T^{64} + \\
& T^{52} + T^{48} + T^{44} + T^{42} + T^{34} + T^{32} + T^{20} + T^{16} + T^{12} + T^{10}) J_{70}^2 + (T^{58} + T^{56} + T^{52} + T^{50} + T^{46} + T^{44} + T^{42} + T^{36} + T^{26} + T^{24} + T^{20} + T^{18} + T^{14} + T^{12} + \\
& T^{10} + T^4) J_{70} J_{12}^2 + (T^{54} + T^{50} + T^{44} + T^{40} + T^{22} + T^{18} + T^{12} + T^8) J_{70} J_{12}^2 J_{41} + (T^{62} + T^{60} + T^{58} + T^{56} + T^{54} + T^{52} + T^{48} + T^{46} + T^{42} + T^{38} + T^{30} + \\
& T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{14} + T^{10} + T^6) J_{70} J_{12}^2 + (T^{60} + T^{52} + T^{50} + T^{42} + T^{28} + T^{20} + T^{18} + T^{10}) J_{70} J_{12} J_{41} + (T^{70} + T^{68} + T^{64} + T^{60} + \\
& T^{58} + T^{56} + T^{54} + T^{52} + T^{48} + T^{44} + T^{42} + T^{40} + T^{38} + T^{36} + T^{32} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{12} + T^{10} + T^8) J_{70} J_{12} + (T^{66} + T^{58} + T^{50} + \\
& T^{42} + T^{34} + T^{26} + T^{18} + T^{10}) J_{70} + (T^{66} + T^{62} + T^{58} + T^{56} + T^{54} + T^{52} + T^{50} + T^{46} + T^{42} + T^{40} + T^{38} + T^{36} + T^{34} + T^{30} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + \\
& T^{14} + T^{10} + T^8 + T^6 + T^4) J_{12}^3 + (T^{64} + T^{56} + T^{54} + T^{48} + T^{40} + T^{38} + T^{32} + T^{24} + T^{22} + T^{16} + T^8 + T^6) J_{12}^2 J_{41} + (T^{74} + T^{72} + T^{66} + T^{62} + T^{58} + T^{54} + \\
& T^{50} + T^{46} + T^{42} + T^{38} + T^{34} + T^{30} + T^{26} + T^{22} + T^{18} + T^{14} + T^8 + T^6) J_{12}^2 + (T^{64} + T^{56} + T^{48} + T^{40} + T^{32} + T^{24} + T^{16} + T^8) J_{12} J_{41} + (T^{72} + T^8) J_{12} + \\
& (T^{70} + T^{66} + T^{62} + T^{58} + T^{54} + T^{50} + T^{46} + T^{42} + T^{38} + T^{34} + T^{30} + T^{26} + T^{22} + T^{18} + T^{14} + T^{10}) J_{41} + (T^{80} + T^{78} + T^{76} + T^{74} + T^{16} + T^{14} + T^{12} + T^{10})
\end{aligned}$$

$$\begin{aligned}
\alpha_3 = & (T^6 + T^4 + T^2) J_{70}^5 + J_{70}^4 J_{70}^2 T^2 + (T^6 + T^2) J_{70}^4 J_{70} J_{12} + (T^{10} + T^2) J_{70}^4 J_{70} + J_{70}^4 J_{70}^3 T^2 + J_{70}^4 J_{70}^2 J_{41} T^2 + (T^8 + T^2) J_{70}^4 J_{70}^2 J_{12}^2 + \\
& J_{70}^4 J_{12} J_{41} T^6 + (T^{14} + T^{12} + T^{10} + T^6 + T^2) J_{70}^4 J_{12} + (T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2) J_{70}^4 J_{41} + (T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^4) J_{70}^4 + \\
& J_{70}^3 J_{70}^2 J_{12}^2 + J_{70}^3 J_{70}^2 T^2 + (T^6 + T^4) J_{70}^3 J_{70}^2 J_{12}^2 + J_{70}^3 J_{70}^2 J_{12} J_{41} T^2 + (T^6 + T^2) J_{70}^3 J_{70}^2 J_{12} + (T^8 + T^2) J_{70}^3 J_{70}^2 J_{41} + (T^6 + T^2) J_{70}^3 J_{70}^2 + \\
& (T^8 + T^4) J_{70}^3 J_{70} J_{12}^2 + (T^4 + T^2) J_{70}^3 J_{70} J_{12}^2 J_{41} + (T^{10} + T^6 + T^4 + T^2) J_{70}^3 J_{70} J_{12}^2 + (T^{10} + T^8 + T^4) J_{70}^3 J_{70} J_{12} J_{41} + (T^{22} + T^{20} + T^{18} + T^{10} + T^8 + \\
& T^4) J_{70}^3 J_{70} J_{12} + (T^{18} + T^{14} + T^{12} + T^8 + T^6 + T^2) J_{70}^3 J_{70} J_{41} + (T^{26} + T^{24} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^8 + T^6) J_{70}^3 J_{70} + (T^{18} + T^{16} + T^{14} + \\
& T^{12} + T^{10} + T^8 + T^6) J_{70}^3 J_{12}^2 + (T^{18} + T^{16} + T^{14} + T^{10} + T^6 + T^4) J_{70}^3 J_{12}^2 J_{41} + (T^{26} + T^{20} + T^{18} + T^{12} + T^{10} + T^8 + T^6) J_{70}^3 J_{12}^2 + (T^{22} + T^{20} + T^{16} + \\
& T^2) J_{70}^3 J_{12} J_{41} + (T^{26} + T^{20} + T^{18} + T^{14} + T^{12} + T^6) J_{70}^3 J_{12} + (T^{28} + T^{18} + T^{12} + T^{10} + T^6 + T^2) J_{70}^3 J_{41} + (T^{38} + T^{36} + T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + \\
& T^{10} + T^8 + T^6) J_{70}^3 + J_{70}^2 J_{70}^5 T^2 + J_{70}^2 J_{70}^4 J_{12}^2 T^2 + J_{70}^2 J_{70}^4 J_{12} T^6 + (T^6 + T^4 + T^2) J_{70}^2 J_{70}^4 J_{41} + (T^{12} + T^8 + T^4 + T^2) J_{70}^2 J_{70}^4 + J_{70}^2 J_{70}^3 J_{12}^2 T^6 + (T^6 + \\
& T^4 + T^2) J_{70}^2 J_{70}^3 J_{12}^2 + (T^8 + T^6 + T^2) J_{70}^2 J_{70}^3 J_{12} J_{41} + J_{70}^2 J_{70}^3 J_{12} T^{14} + (T^{16} + T^{12} + T^{10} + T^6 + T^4) J_{70}^2 J_{70}^3 J_{41} + (T^{16} + T^{14} + T^{12} + T^6 + T^2) J_{70}^2 J_{70}^3 + \\
& (T^{16} + T^{14} + T^8 + T^6 + T^4) J_{70}^2 J_{70}^3 J_{12}^2 + (T^{10} + T^8 + T^6 + T^4 + T^2) J_{70}^2 J_{70}^3 J_{12} J_{41} + (T^{16} + T^{14} + T^{12} + T^8 + T^6) J_{70}^2 J_{70}^3 J_{41} + (T^{16} + T^{14} + T^{12} + T^8 + \\
& T^4 + T^2) J_{70}^2 J_{70}^3 J_{12} J_{41} + (T^{26} + T^{22} + T^{20} + T^{16} + T^{10} + T^4 + T^2) J_{70}^2 J_{70}^3 J_{12} + (T^{26} + T^{22} + T^{18} + T^4) J_{70}^2 J_{70}^3 J_{41} + (T^{34} + T^{32} + T^{26} + T^{24} + T^{16} + \\
& T^{14} + T^{10} + T^8 + T^2) J_{70}^2 J_{70}^3 + (T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{10} + T^6 + T^2) J_{70}^2 J_{70}^3 J_{12} + (T^{26} + T^{22} + T^{18} + T^4) J_{70}^2 J_{70}^3 J_{41} + (T^{30} + T^{28} + T^{24} + \\
& T^{24} + T^{20} + T^{18} + T^{16} + T^8 + T^2) J_{70}^2 J_{70} J_{12}^2 + (T^{28} + T^{22} + T^{18} + T^{14} + T^{10} + T^8 + T^6 + T^4 + T^2) J_{70}^2 J_{70} J_{12} J_{41} + (T^{38} + T^{34} + T^{28} + T^{24} + T^{22} + T^{20} + \\
& T^{18} + T^{16} + T^{12} + T^{10} + T^4 + T^2) J_{70}^2 J_{70} J_{12} + (T^{34} + T^{32} + T^{30} + T^{24} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2) J_{70}^2 J_{70} J_{41} + (T^{42} + T^{38} + T^{36} + \\
& T^{32} + T^{30} + T^{26} + T^{18} + T^{12} + T^{10} + T^8) J_{70}^2 J_{70} + (T^{30} + T^{20} + T^{18} + T^{12} + T^6 + T^2) J_{70}^2 J_{12}^2 + (T^{34} + T^{32} + T^{30} + T^{14} + T^{12} + T^6) J_{70}^2 J_{12} J_{41} + (T^{38} + \\
& T^{36} + T^{32} + T^{30} + T^{28} + T^{24} + T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^4) J_{70}^2 J_{12}^2 + (T^{32} + T^{28} + T^{26} + T^{24} + T^{20} + T^{10}) J_{70}^2 J_{12} J_{41} + (T^{48} + T^{44} + T^{42} + T^{40} + \\
& T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^6) J_{70}^2 J_{12} + (T^{44} + T^{40} + T^{38} + T^{34} + T^{30} + T^{28} + T^{26} + T^{22} + T^{14} + T^{10} + T^8 + \\
& T^2) J_{70}^2 J_{41} + (T^{54} + T^{50} + T^{36} + T^{28} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{12} + T^8 + T^4) J_{70}^2 + J_{70} J_{70}^6 J_{41} T^2 + (T^8 + T^6 + T^4 + T^2) J_{70} J_{70}^6 + J_{70} J_{70}^5 J_{12}^2 T^2 + \\
& J_{70} J_{70}^5 J_{12}^2 T^4 + (T^6 + T^2) J_{70} J_{70}^5 J_{12} J_{41} + (T^6 + T^2) J_{70} J_{70}^5 J_{12} + J_{70} J_{70}^5 T^4 + (T^{10} + T^8 + T^2) J_{70} J_{70}^4 J_{12}^2 + (T^8 + T^4) J_{70} J_{70}^4 J_{12} J_{41} + (T^{18} + T^{14} + \\
& T^8 + T^2) J_{70} J_{70}^4 J_{12} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4) J_{70} J_{70}^4 J_{12} J_{41} + (T^{22} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^4 + T^2) J_{70} J_{70}^4 J_{12} + (T^{22} + \\
& T^{16} + T^{10} + T^6 + T^4 + T^2) J_{70} J_{70}^4 J_{41} + (T^{28} + T^{20} + T^{18} + T^{16} + T^{12} + T^8 + T^2) J_{70} J_{70}^4 + (T^{22} + T^{20} + T^{16} + T^{14} + T^{12} + T^8 + T^6 + T^2) J_{70} J_{70}^3 J_{12}^2 + \\
& (T^{18} + T^{16} + T^{12} + T^{10} + T^6) J_{70} J_{70}^3 J_{12} J_{41} + (T^{22} + T^{14} + T^8 + T^6) J_{70} J_{70}^3 J_{12}^2 + (T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^6 + T^4) J_{70} J_{70}^3 J_{12} J_{41} + \\
& (T^{32} + T^{28} + T^{22} + T^{20} + T^{14} + T^{12} + T^6 + T^4 + T^2) J_{70} J_{70}^3 J_{12} + (T^{30} + T^{26} + T^{24} + T^{20} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8) J_{70} J_{70}^3 J_{41} + (T^{18} + \\
& T^{14} + T^2) J_{70} J_{70}^3 + (T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{14} + T^{12} + T^8 + T^6 + T^4) J_{70} J_{70}^2 J_{12}^2 + (T^{26} + T^{22} + T^{16} + T^{14} + T^{12} + T^{10} + T^8) J_{70} J_{70}^2 J_{12} J_{41} + \\
& (T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{16} + T^{10} + T^6 + T^2) J_{70} J_{70}^2 J_{12}^2 + (T^{34} + T^{26} + T^{18} + T^8 + T^4) J_{70} J_{70}^2 J_{12} J_{41} + (T^{42} + T^{40} + T^{38} + T^{36} + T^{32} + \\
& T^{30} + T^{24} + T^{22} + T^{18} + T^{14} + T^{12} + T^8) J_{70} J_{70}^2 J_{12} + (T^{42} + T^{40} + T^{34} + T^{30} + T^{26} + T^{22} + T^{20} + T^{18} + T^{12} + T^6 + T^4 + T^2) J_{70} J_{70}^2 J_{41} + (T^{34} + T^{32} + \\
& T^{24} + T^{22} + T^{20} + T^{14} + T^{12} + T^{10}) J_{70} J_{70}^2 + (T^{38} + T^{36} + T^{34} + T^{24} + T^{20} + T^{18} + T^{14} + T^{12} + T^8 + T^4) J_{70} J_{70} J_{12}^2 + (T^{38} + T^{36} + T^{32} + T^{22} + T^{18} + \\
& T^{14} + T^{10} + T^2) J_{70} J_{70} J_{12}^2 J_{41} + (T^{48} + T^{44} + T^{42} + T^{38} + T^{36} + T^{26} + T^{24} + T^{20} + T^{16} + T^{12} + T^8 + T^6) J_{70} J_{70} J_{12}^2 + (T^{44} + T^{42} + T^{38} + T^{34} + T^{30} + T^{24} + T^{12} + \\
& T^6) J_{70} J_{70} J_{12} J_{41} + (T^{50} + T^{48} + T^{42} + T^{30} + T^{28} + T^{24} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^2) J_{70} J_{70} J_{12} + (T^{48} + T^{46} + T^{38} + T^{36} + T^{34} + T^{30} + T^{28} + \\
& T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{12} + T^{10} + T^8 + T^2) J_{70} J_{70} J_{41} + (T^{42} + T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{28} + T^{24} + T^{20} + T^{16} + T^{12} + T^{10} + T^6 + \\
& T^2) J_{70} J_{70} + (T^{50} + T^{46} + T^{42} + T^{40} + T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + T^{22} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6) J_{70} J_{12}^2 + (T^{50} + T^{48} + T^{46} + T^{38} + T^{36} + T^{34} + T^{30} + T^{28} + \\
& T^8 + T^4 + T^2) J_{70} J_{12}^2 J_{41} + (T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^8) J_{70} J_{12}^2 + (T^{48} + T^{46} + T^{40} + \\
& T^{34} + T^{32} + T^{30} + T^{24} + T^{22} + T^6 + T^2) J_{70} J_{12} J_{41} + (T^{54} + T^{50} + T^{46} + T^{42} + T^{40} + T^{36} + T^{22} + T^{18} + T^{14} + T^{10} + T^8 + T^4) J_{70} J_{12} + (T^{52} + T^{48} + T^{44} + \\
& T^{38} + T^{36} + T^{34} + T^{20} + T^{16} + T^{12} + T^6 + T^4 + T^2) J_{70} J_{41} + (T^{54} + T^{52} + T^{48} + T^{46} + T^{40} + T^{36} + T^{22} + T^{20} + T^{16} + T^{14} + T^8 + T^4) J_{70} + J_{70}^7 J_{12}^2 T^2 + \\
& J_{70}^7 J_{12} J_{41} T^2 + (T^{10} + T^8 + T^4) J_{70}^7 J_{12} + J_{70}^7 J_{41} T^4 + (T^{12} + T^8) J_{70}^6 J_{12}^2 + (T^6 + T^2) J_{70}^6 J_{12}^2 + (T^6 + T^4) J_{70}^6 J_{12}^2 + (T^{12} + T^{10} + T^8 + T^4 + T^2) J_{70}^6 J_{12} J_{41} + (T^{18} + T^{10}) J_{70}^6 J_{12} J_{41} + \\
& (T^{12} + T^8 + T^6 + T^4 + T^2) J_{70}^6 J_{12}^2 J_{41} + (T^{22} + T^{20} + T^{14} + T^8 + T^6 + T^2) J_{70}^6 J_{12}^2 + (T^{22} + T^{20} + T^{14} + T^{12} + T^{10} + T^8 + T^4 + T^2) J_{70}^6 J_{12} J_{41} + (T^{28} + T^{26} + \\
& T^{22} + T^{16} + T^{14} + T^8 + T^4) J_{70}^5 J_{12} + (T^{26} + T^{24} + T^{22} + T^{20} + T^{14} + T^{10} + T^8 + T^4) J_{70}^5 J_{12} + (T^{32} + T^{30} + T^{24} + T^{20} + T^{16} + T^{14} + T^8) J_{70}^5 + (T^{10} + T^{26} + \\
& T^{20} + T^{16} + T^{14} + T^{12} + T^8 + T^4) J_{70}^5 J_{12}^2 + (T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{14} + T^8 + T^4 + T^2) J_{70}^5 J_{12}^2 J_{41} + (T^{34} + T^{32} + T^{22} + T^{20} + T^{18} + T^{16} + \\
& T^6) J_{70}^5 J_{12}^2 + (T^{32} + T^{28} + T^{26} + T^{24} + T^{16} + T^{12} + T^{10} + T^8 + T^6 + T^2) J_{70}^5 J_{12} J_{41} + (T^{28} + T^{26} + T^{20} + T^{18}) J_{70}^5 J_{12} + (T^{34} + T^{32} + T^{30} + T^{26} + T^{24} + \\
& T^{22} + T^{18} + T^{16} + T^{14} + T^{10} + T^8 + T^6 + T^4) J_{70}^4 J_{41} + (T^{38} + T^{36} + T^{18} + T^{16} + T^8) J_{70}^4 + (T^{38} + T^{34} + T^{16} + T^{12} + T^{10}) J_{70}^3 J_{12}^2 + (T^{34} + T^{32} + T^{26} + \\
& T^{24} + T^{18} + T^{14} + T^{10} + T^8 + T^4) J_{70}^3 J_{12} J_{41} + (T^{20} + T^{18} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^2) J_{70}^3 J_{12}^2 + (T^{32} + T^{30} + T^{28} + T^{26} + T^{22} + T^{20} + T^{18} + \\
& T^{16} + T^{14} + T^{12} + T^{10} + T^2) J_{70}^3 J_{12} J_{41} + (T^{20} + T^{18} + T^{10} + T^4) J_{70}^3 J_{12} + (T^{40} + T^{36} + T^{22} + T^{20} + T^{16} + T^{14} + T^{12} + T^4) J_{70}^3 J_{41} + (T^{22} + T^{20} + T^{18} + \\
& T^{16} + T^{14} + T^{12} + T^{10} + T^8) J_{70}^3 + (T^{32} + T^{26} + T^{24} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4 + T^2) J_{70}^2 J_{12}^2 + (T^{22} + T^{16} + T^{12} + T^{10}) J_{70}^2 J_{12}^2 J_{41} + \\
& (T^{34} + T^{30} + T^{28} + T^{24} + T^{22} + T^{20} + T^{18} + T^{14} + T^{12} + T^8 + T^6 + T^4) J_{70}^2 J_{12}^2 + (T^{32} + T^{26} + T^{24} + T^{22} + T^{20} + T^{16} + T^{14} + T^8 + T^4 + T^2) J_{70}^2 J_{12} J_{41} + \\
& (T^{38} + T^{28} + T^{26} + T^{24} + T^{22} + T^{12} + T^{10} + T^8) J_{70}^2 J_{12} + (T^{34} + T^{30} + T^{22} + T^{20} + T^{18} + T^{14} + T^6 + T^4) J_{70}^2 J_{41} + (T^{36} + T^{32} + T^{28} + T^{24} + T^{20} + \\
& T^{16} + T^{12} + T^8) J_{70}^2 + (T^{36} + T^{28} + T^{20} + T^{12}) J_{70} J_{12}^3 + (T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{28} + T^{22} + T^{16} + T^{14} + T^{12} + T^4 + T^2) J_{70} J_{12}^2 J_{41} + (T^{48} + \\
& T^{46} + T^{40} + T^{38} + T^{36} + T^{32} + T^{30} + T^{28} + T^{26} + T^{24} + T^{22} + T^{20} + T^{18} + T^{12} + T^{10} + T^4) J_{70} J_{12}^2 + (T^{40} + T^{38} + T^{36} + T^{34} + T^{32} + T^{30} + T^{24} + T^{22} + \\
& T^{16} + T^{14} + T^4 + T^2) J_{70} J_{12} J_{41} + (T^{48} + T^{46} + T^{42} + T^{36} + T^{16} + T^{14} + T^{10} + T^4) J_{70} J_{12} + (T^{48} + T^{46} + T^{40} + T^{36} + T^{16} + T^{14} + T^8 + T^4) J_{70} J_{41} + \\
& (T^{50} + T^{48} + T^{42} + T^{40} + T^{18} + T^{16} + T^{10} + T^8) J_{70} + (T^{42} + T^{40} + T^{38} + T^{36} + T^{10} + T^8 + T^6 + T^4) J_{12}^3 + (T^{50} + T^{48} + T^{38} + T^{34} + T^{18} + T^{16} + T^6 + \\
& T^2) J_{12}^2 J_{41} + (T^{50} + T^{46} + T^{42} + T^{38} + T^{18} + T^{14} + T^{10} + T^6) J_{12}^2 + (T^{48} + T^{46} + T^{42} + T^{40} + T^{38} + T^{34} + T^{16} + T^{14} + T^{10} + T^8 + T^6 + T^2) J_{12} J_{41} + \\
& (T^{50} + T^{48} + T^{46} + T^{44} + T^{42} + T^{40} + T^{38} + T^{36} + T^{18} + T^{16} + T^{14} + T^{12} + T^{10} + T^8 + T^6 + T^4) J_{41} + (T^{64} + T^{56} + T^{48} + T^{40} + T^{32} + T^{24} + T^{16} + T^8)
\end{aligned}$$

$$\begin{aligned}
\alpha_4 = & J_{70}^5 + J_{70}^4 J_{70} J_{12} + J_{70}^4 J_{70}^2 T^2 + (T^2 + 1) J_{70}^4 J_{12}^2 + J_{70}^4 J_{12} J_{41} + (T^6 + T^4 + T^2 + 1) J_{70}^4 J_{12} + (T^2 + 1) J_{70}^4 J_{41} + (T^{16} + T^{14} + T^{12} + T^{10} + T^8 + \\
& T^4 + T^2) J_{70}^4 + J_{70}^3 J_{70}^2 + J_{70}^3 J_{70}^2 J_{12}^2 + J_{70}^3 J_{70}^2 J_{12} + J_{70}^3 J_{70}^2 J_{12} J_{41} + J_{70}^3 J_{70}^2 J_{12} + (T^8 + T^4 + T^2 + 1) J_{70}^3 J_{70} J_{12}^2 + (T^6 + T^2) J_{70}^3 J_{70} J_{12} J_{41} + J_{70}^3 J_{70} J_{12} + \\
& (T^8 + T^6 + T^4 + T^2 + 1) J_{70}^3 J_{70} J_{41} + (T^{18} + T^{16} + T^{14} + T^{10} + T^8 + T^6 + T^4 + 1) J_{70}^3 J_{70} + (T^{10} + T^4) J_{70}^3 J_{12}^2 + (T^4 + T^2 + 1) J_{70}^3 J_{12} J_{41} + J_{70}^3 J_{12} + T^{12} + \\
& T^8 + T^6 + T^2 + 1) J_{70}^3 J_{12}^2 + (T^{10} + T^8 + T^4 + T^2 + 1) J_{70}^3 J_{12} J_{41} + (T^{24} + T^{22} + T^{20} + T^{14} + T^{10} + T^6 + T^4) J_{70}^3 J_{12} + (T^{22} + T^{16} + T^{10} + T^8 + T^4 +
\end{aligned}$$

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