The power operation structure on the $K(1)$-localization of E_2

Yifei Zhu

Dyer-Lashof theories organize power operations in cohomology. We give an overview of the structure of the Dyer-Lashof theories associated to Morava E-theories, and to their $K(1)$-localizations. When the E-theory is an elliptic cohomology theory, this structure enables us to compute power operations by doing calculations with elliptic curves.

1 Introduction

The study of cohomology operations has been central to algebraic topology since the 1950s, with applications to solving problems such as vector fields on spheres, and the non-existence of elements of Hopf invariant one. This latter problem has impact on the author most recently felt when he tried to answer a question raised by students in a calculus class (the interested reader might see [Mas83, theorem II]).

Among the cohomology operations involved in these applications, the Steenrod operations Sq^i in ordinary cohomology and the Adams operations ψ^k in K-theory are examples of *power operations*. In this paper we study power operations in Morava E-theories. Here is an outline.

In this section we introduce preliminary definitions, in particular, the Dyer-Lashof theory DL_{E_*} associated to a Morava E-theory E.

In section 2 we "translate" from DL_{E_*} and related categories to categories arising from the formal group and its finite flat subgroups associated to E. This "bridge" is the foundation of our discussion, so that later we can study the structure of one side by doing calculations on the other side.

In section 3 we describe power operations in the $K(1)$-local setting where the structure is relatively simple.

Section 4 contains calculations of power operations for a specific Morava E-theory spectrum, and its $K(1)$-localization, at the prime 3. Thanks to the connection in section 2, we work with elliptic curves as concrete objects, following a recipe in hope of generalizing our computation to larger primes.

1.1 Dyer-Lashof theories

One organizing principle for understanding the structure among cohomology operations is through *algebraic theories* (due to Lawvere, cf. [Law63] and [Bor94, chapter 3]). We begin with a collection of definitions and simple facts concerning algebraic theories, following precisely the discussion in [Reza, sections 5–9].

Definition 1 An *(algebraic) theory* is a category T with object set $\{T^0, T^1, T^2, \ldots\}$, together with a canonical map $T^0 \to T^1$, and *projection maps* $\pi_i\colon T^n \to T^1$ for all $n \geq 1$, $1 \leq i \leq n$ such that $T(T^k, T^n) \xrightarrow{\pi_i} \prod_{i=1}^n T(T^k, T^1)$ is a bijection for all k and n, i.e. T^n is the n-fold product of T^1.

A *morphism of theories* is a functor $\phi\colon R \to T$ which preserves the product structure of a theory, i.e. $\phi(R^k) = T^k$ and $\phi(R^k \xrightarrow{\pi_i} R^1) = T^k \xrightarrow{\pi_i} T^1$. □

Definition 2 A *model* of a theory T (or T*-model*) is a functor $A\colon T \to \mathrm{Set}$ which preserves finite products. □

We can think of a model of T as an underlying set $X = A(T^1)$ together with operations $\psi_f\colon X^k \to X$ for each $f \in T(T^k, T^1)$ (these determine all the other operations $X^k \to X^n$ with $n > 1$). In particular, a *free model* on n generators is the model $F_T(n)$ defined by $F_T(n)(T^m) = T(T^n, T^m)$. We write Model_T for the category of models of T.

For example, let R be a commutative ring, and let F be the full subcategory of the category of commutative R-algebras having as objects $\{F_0, F_1, F_2, \ldots\}$, where $F_0 = R$ and $F_n = R[x_1, \ldots, x_n]$ for $n \geq 1$. We then have the theory of commutative R-algebras $C_R = F^{\mathrm{op}}$. The projection maps $\pi_i\colon R[x_1, \ldots, x_n] \to R[x_1]$ send x_i to x_1 and send x_j, $j \neq i$, to 0.

Definition 3 A *commutative operation theory* (COT) is a triple (T, R, ϕ) consisting of a theory T, a commutative ring R, and a morphism $\phi\colon C_R \to T$ of theories, such that the induced functor $\phi^*\colon \mathrm{Model}_T \to \mathrm{Model}_{C_R}$ commutes with finite coproducts. □

In other words, if T is a COT, every T-model has an underlying structure of a commutative R-algebra, and coproducts in Model_T are computed as tensor products over R

(we will use $\otimes$ when writing coproducts). We denote by $R\{x_1, ..., x_n\}$ a free T-model on n generators, and we have $R\{x_1, ..., x_n\} \cong R\{x_1\} \otimes_R \cdots \otimes_R R\{x_n\}$.

We next introduce grading to a theory.

Definition 4 Let C be a fixed set C of *colors*, and let $\mathbb{N}[C]$ be the free commutative monoid on C. A *C-graded theory T* is a category with object set $\{T^n\}_{n\in\mathbb{N}[C]}$, together with, for each $n = \sum_{c\in C} n_c[c] \in \mathbb{N}[C]$, a specified identification of T^n with the product $\prod_{c\in C}(T^{[c]})^{n_c}$. □

In particular, given a $\mathbb{Z}$-graded theory T and a graded-commutative ring R, we can define a graded COT as a triple (T, R_*, ϕ) similarly to the above (the theory C_{R_*} of graded-commutative R_*-algebras is equipped with the graded tensor product). Given a T-model A, we write A_c for the piece in grading c of the model.

For a graded COT (T, R_*, ϕ), and free models $R_*\{x\}$ and $R_*\{x_1, x_2\}$ with $|x| = |x_1| = |x_2| = c$, let $\mathcal{A}(c,d)$ be the set of elements $f \in R_*\{x\}_d = T(T^{[c]}, T^{[d]})$ which are primitive under the comultiplication $R_*\{x\} \xrightarrow{x\mapsto x_1+x_2} R_*\{x_1, x_2\}$, i.e. $f \mapsto f\otimes 1 + 1\otimes f$. Such $f \in \mathcal{A}(c,d)$ give rise to additive maps $A_c \to A_d$ natural in A. In particular $x \in R_*\{x\}_c$ corresponds to the identity map on A_c. Thus we obtain a category $\mathcal{A}$ of additive operations whose object set is $\mathbb{Z}$, the set of colors of our graded COT.

For example, let $T = O_{H\mathbb{F}_p}$ be the graded COT given by

$$\begin{aligned} &T(O_{H\mathbb{F}_p}^{[c_1]+\cdots+[c_m]}, O_{H\mathbb{F}_p}^{[d_1]+\cdots+[d_n]}) \\ &= [K(\mathbb{F}_p, c_1) \times \cdots \times K(\mathbb{F}_p, c_m), K(\mathbb{F}_p, d_1) \times \cdots \times K(\mathbb{F}_p, d_n)], \end{aligned}$$

where we use homotopy classes of maps, and by convention $K(\mathbb{F}_p, c) = *$ for $c < 0$. $\mathrm{Model}_{O_{H\mathbb{F}_p}}$ is the category of unstable algebras over the mod-p Steenrod algebra (this is a restatement of results of Serre [Ser53] and Cartan [Car54]; see [Ste62, II.§5]). $\mathcal{A}(c,d)$ is the set of additive operations $H^c(-;\mathbb{F}_p) \to H^d(-;\mathbb{F}_p)$. If $p = 2$, the additive operations $H^c(-;\mathbb{F}_2) \to H^*(-;\mathbb{F}_2)$ are linear combinations of monomials which are admissible composites of Steenrod operations having excess less than c (cf. [Ser53, theorem 2 of §4] and [MT68, chapters 3 and 9]).

Having the COT describing cohomology operations on spaces, we next consider one describing operations on spectra.

Let S be the sphere spectrum, and R be a commutative S-algebra (cf. [EKMM97]). We denote by Alg_R the category of commutative R-algebras. Let $\mathbb{P}$ be the *free S-algebra* *functor* defined by

$$\mathbb{P}(X) = \bigvee_{m\geq 0} \mathbb{P}^m(X) = \bigvee_{m\geq 0} X^{\wedge m}/\Sigma_m,$$

and let $\mathbb{P}_R$ be the *free* R*-algebra functor* defined similarly using the smash product over R. These functors descend to the homotopy categories.

Definition 5 Given a commutative S-algebra R, the *Dyer-Lashof theory* DL_R is the $\mathbb{Z}$-graded theory T defined by

$$\begin{aligned} &T(T^{[c_1]+\cdots+[c_m]}, T^{[d_1]+\cdots+[d_n]}) \\ &= h\mathrm{Alg}_R\Big(\mathbb{P}_R\big(R\wedge(S^{d_1}\vee\cdots\vee S^{d_n})\big), \mathbb{P}_R\big(R\wedge(S^{c_1}\vee\cdots\vee S^{c_m})\big)\Big). \end{aligned} \qquad \square$$

The significance of DL_R is that it describes all homotopy operations on commutative R-algebras (as natural transformations of the functors $\pi_i(-)$):

$$T(T^{[c]}, T^{[d]}) = h\mathrm{Alg}_R\big(\mathbb{P}_R(R\wedge S^d), \mathbb{P}_R(R\wedge S^c)\big) = \{\pi_c(-)\to\pi_d(-)\}.$$

Free models are given by

$$F_T([c_1]+\cdots+[c_m])_d = \pi_d\mathbb{P}_R\big(R\wedge(S^{c_1}\vee\cdots\vee S^{c_m})\big) = \pi_d\Big(R\wedge\big(\mathbb{P}(S^{c_1})\vee\cdots\vee\mathbb{P}(S^{c_m})\big)\Big).$$

If $\pi_*\big(R\wedge\mathbb{P}(S^c)\big)$ are flat as left π_*R-modules, DL_R turns out to be a COT (cf. [Reza, lemma 7.5]). Moreover, in the homotopy category, it is convenient to identify $\mathbb{P}^m(S^c)$ with $B\Sigma_m^{cV_m}$. The latter is the Thom spectrum of a virtual bundle, where $V_m = \mathbb{R}^m$ is equipped with the Σ_m-action given by permuting coordinates.

For example, if R is a ring (no longer a spectrum) containing $\mathbb{F}_2$ and HR is the corresponding Eilenberg-Mac Lane spectrum, there is a complete description of the COT DL_{HR}. A DL_{HR}-model is a graded commutative R-algebra A_*, equipped with functions $Q^s\colon A_c\to A_{c+s}$ for all $s, c\in\mathbb{Z}$, satisfying a set of properties, e.g. a Cartan formula and Adem relations (cf. [BMMS86, VIII.3.3] and [Reza, section 10]).

1.2 Morava E-theories and associated Dyer-Lashof theories

One organizing principle for understanding large-scale phenomena in homotopy theory is through the *chromatic filtration* (cf. [Law09], [Hopb, section 17], [Rav92, section 2.5] and [Rav86]). It corresponds to a stratification of the moduli stack of formal groups into layers according to height (in this paper all formal groups are commutative of dimension 1; cf. [Frö68, III.§ 1]). For complex oriented cohomology theories, the formal groups come about in terms of formal group laws which express the first Chern class of the tensor product of two line bundles in terms of the first Chern classes of the individual line bundles (cf. [Hopb, section 1]).

For each formal group law F of height $n < \infty$ over a perfect field k of characteristic $p > 0$, there is a complete local ring $\mathrm{LT}(k, F)$, called the Lubin-Tate ring. It is universal among complete local rings with residue field k carrying a formal group law whose reduction to k is F. As k is perfect, $\mathrm{LT}(k, F)$ is isomorphic to $\mathbb{W}k[\![u_1, \ldots, u_{n-1}]\!]$ (cf. [Rez98, sections 4.3 and 4.5]). There is an E_∞ ring spectrum $E_n(k, F)$ whose homotopy groups are $\mathrm{LT}(k, F)[u^{\pm 1}]$ with $|u| = 2$ (cf. [GH04, corollary 7.6] and [Rez98]). This is the *Morava E-theory* spectrum (associated to k and F).

Closely related to Morava E-theories are $E(n)$ (*Johnson-Wilson theories*) and $K(n)$ (*Morava K-theories*) for $n \geq 0$, with $\pi_* E(n) = \mathbb{Z}_{(p)}[v_1, \ldots, v_n, v_n^{-1}]$, and $\pi_* K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$, where $|v_i| = 2(p^i - 1)$, and by convention $v_0 = p$. They are particularly useful when we study specific layers in the chromatic filtration through *Bousfield localization* (cf. [Rav92, chapter 7] and [Lur, lectures 20–23]). In [Bou79], for each generalized homology theory E, Bousfield defines an idempotent functor L_E on the stable homotopy category whose image is equivalent to the category of fractions defined by Adams in [Ada74, section III.14]. Via the connection to the moduli stack of formal groups, geometrically we can think of the stable homotopy category as approximated by a category of quasicoherent sheaves on a moduli stack $\mathcal{M}$ which has a sequence of open substacks $\mathcal{M}(n)$. The Bousfield localization $L_{E(n)}$ can be thought of as restricting to the open substack $\mathcal{M}(n)$. The difference $\mathcal{M}(n) \setminus \mathcal{M}(n-1)$ between two adjacent layers is a closed substack of $\mathcal{M}(n)$, and $L_{K(n)}$ acts as completing along this closed substack. Roughly speaking, $L_{K(n)}$ has the effect of isolating height n phenomena, and $L_{E(n)}$ sees all phenomena of height n and lower. Thus to understand the stable homotopy category, we can first examine one chromatic layer at a time and do specific calculations for $K(n)$-localizations. Then we need to understand how to patch these together into the $E(n)$-localizations, and we need to understand the "chromatic convergence", i.e. how to take the limit as n goes to infinity.

In the chromatic filtration, ordinary rational cohomology lives over the open substack $\mathcal{M}(0)$ and K-theory lives over $\mathcal{M}(1)$. The open substack $\mathcal{M}(2)$ is where *elliptic cohomology theories* (cf. [Lur09]) are concentrated. An elliptic cohomology theory has its associated formal group equipped with a chosen isomorphism to the formal completion of an elliptic curve along the identity; see [Lur09, definition 1.2] for precise definition. In this paper we study the $K(1)$-localization of E_2, i.e. the localization to height 1 of a certain Morava E-theory of height 2 which is an elliptic cohomology theory. Specifically we study the power operations for E_2 and its $K(1)$-localization. (Within context, E_2 is not to be confused with the homotopy group in degree 2 of a theory E.)

Given a Morava E-theory E, there is an associated Dyer-Lashof theory DL_{E_*} describing all cohomology operations. It is defined similarly as in definition 5, except that we need to apply a certain localization to have good values of $E_*B\Sigma_m$ (cf. [Str98, section 3] and [HS99, section 8]). The free model on one generator is $E_*\{x_c\} = \bigoplus_{m\geq 0} E_*^\wedge(B\Sigma_m^{cV_m})$, where $E_*^\wedge(-)$ reflects the localization. (We need to be slightly careful about this localization, as it does not preserve homotopy colimits; cf. [Reza, section 15] and [Hov08, section 1] for details.)

As is explained at the beginning of [And95], we hope to learn about the conjectural geometry of the theories E_n by examining cohomology operations – in particular, power operations – along the lines of ordinary rational cohomology or K-theory (which are the initial cases E_0 and E_1).

2 The structure of power operations

Let E be the Morava E-theory associated to a formal group Γ of height $n < \infty$ over a perfect field k of characteristic $p > 0$. Based on knowledge of the spectrum E, we study the structure of DL_{E_*}, the $\mathbb{Z}$-graded Dyer-Lashof theory describing all homotopy operations on commutative E-algebras. We restrict our attention to the degree 0 part DL_{E_0}. We can translate operations in higher degree E-cohomology to degree 0 by working with higher sphere spectra S^n; even better, Morava E-theories are even-periodic (recall that $E_* = \mathrm{LT}(k,\Gamma)[u^{\pm 1}]$ with $|u| = 2$) so that a lot of operations are already determined by those in degree 0.

To understand the structure of DL_{E_0}, the main input comes from deformations of Frobenius which we discuss below. In particular, when the E-theory is an elliptic cohomology theory, deformations of Frobenius are parametrized by finite flat subgroups of the formal group of the associated elliptic curve, and thus we may study the operations by calculations with elliptic curves. Also the Serre-Tate theorem (cf. [Kat81, theorem 1.2.1]) states that p-adically the deformation theory of an elliptic curve is equivalent to the deformation theory of its p-divisible group. As Morava E-theories are associated to the latter, this important result enables our approach to power operations in elliptic cohomology, and it underlies the later discussion.

This section is largely a summary of some of the results in [Reza, section 16] and [Rezb, sections 3 and 4]. See [Rezc] for an exposition of related topics.

2.1 Identifications of categories

First we consider additive operations.

Let $\mathcal{A}$ be the set of additive elements in the free DL_{E_0}-model on one generator $E_0\{x\} = \bigoplus_{m\geq 0} E_0^{\wedge} B\Sigma_m$. Write $\mathcal{A}_{[m]} \subset E_0^{\wedge} B\Sigma_m$ for the summand, and write $\mathcal{A}_r = \mathcal{A}_{[p^r]}$. It turns out that $\mathcal{A}_{[m]} = 0$ unless $m = p^r$ for some r (cf. [Str98, lemma 8.10]). Thus $\mathcal{A} = \bigoplus_{r\geq 0} \mathcal{A}_r$ is an associative (not necessarily commutative) graded ring with respect to the product given by "composition of operations", with the unit element given by the generator $x \in E_0\{x\}$ representing the identity operation. Moreover the category $\mathrm{Mod}_{\mathcal{A}}$ of left $\mathcal{A}$-modules naturally admits a tensor product which makes it into a symmetric monoidal category (cf. [Reza, proposition 7.6]).

We formulate a category equivalent to $\mathrm{Mod}_{\mathcal{A}}$, which is specific to Morava E-theories, using deformations of Frobenius.

Let R be a complete local ring containing $\mathbb{F}_p$. Given an R-algebra A, let $\mathrm{Frob}^*\colon \sigma^*A \to A$ be the unique map of R-algebras that fits into the diagram

$$\begin{tikzcd}
R \arrow[r, "\sigma"] \arrow[d] & R \arrow[r, equal] \arrow[d] & R \arrow[d] \\
A \arrow[r] \arrow[rr, bend right, "\sigma"'] & \sigma^*A \arrow[r, "\mathrm{Frob}^*"] \arrow[ul, phantom, "\ulcorner", very near start] & A,
\end{tikzcd}$$

where σ sends an element to its p'th power, and the left-hand square is a pushout of rings. In particular, if G is a formal group over R, there is an isogeny $\mathrm{Frob}\colon G \to \sigma^*G$ of formal groups over R defined by $\mathrm{Frob}^*\colon \mathcal{O}_{\sigma^*G} = \sigma^*\mathcal{O}_G \to \mathcal{O}_G$, $R[\![y]\!] \to R[\![x]\!]$ sending y to x^p.

The Frobenius can be defined more generally for any complete local ring R with maximal ideal $\mathfrak{m}$ (and R-algebras), by imposing the above commutative diagram on their mod-p reductions. This is indeed the generality we will be working with henceforth. Our restricted first definition is simply for the clarity of exposition.

A *deformation of* Γ *to* R is a triple (G, i, α) consisting of a formal group G over R, an inclusion $i\colon\ k \to R/\mathfrak{m}$ and an isomorphism $\alpha\colon\ \pi^*G \to i^*\Gamma$ of formal groups over $R/\mathfrak{m}$, where $\pi\colon\ R \to R/\mathfrak{m}$ is the natural quotient map. (We will simply write π^*G as G_0, i.e. the special fiber of G as a formal scheme over R. Similarly, given an isogeny ϕ of formal groups, we write ϕ_0 for the induced isogeny on the special fibers.) A $\star$*-isomorphism* $(G, i, \alpha) \to (G', i', \alpha')$ is an isomorphism $\phi\colon\ G \to G'$ of formal groups over R such that $i' = i$ and $\alpha' \circ \phi_0 = \alpha$.

We define the *category of deformations of Frobenius over* R as follows.

Definition 6 Let $\mathrm{DefFrob}_\Gamma(R)$ be the category whose objects are deformations of Γ to R, and whose morphisms are isogenies which are deformations of Frobenius, i.e. a morphism $(G, i, \alpha) \to (G', i', \alpha')$ is an isogeny $\phi\colon\ G \to G'$ such that $i' = \sigma^r \circ i$ and $\alpha' \circ \phi_0 = \mathrm{Frob}^r \circ \alpha$ for some $r \geq 0$. □

Remark 7 In the above definition, when $r = 0$, a morphism $(G, i, \alpha) \to (G', i', \alpha')$ is precisely a $\star$-isomorphism. □

We then consider the category of sheaves of modules on $\mathrm{DefFrob}_\Gamma = \{\mathrm{DefFrob}_\Gamma(R)\}$.

Definition 8 Define a category $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ as follows. An object $\mathcal{F}$ of this category consists of

(1) for each complete local ring R, a functor

$$\mathcal{F}_R\colon\ \mathrm{DefFrob}_\Gamma(R)^{\mathrm{op}} \to \mathrm{Mod}_R,$$

(2) for each local homomorphism $f\colon\ R \to S$, a natural isomorphism

$$\mathcal{F}_f\colon\ f^*\mathcal{F}_R \to \mathcal{F}_S f^*,$$

where the first f^* is the functor $\mathrm{Mod}_R \to \mathrm{Mod}_S$ of extending scalars along f, and the second $f^*\colon\ \mathrm{DefFrob}_\Gamma(R)^{\mathrm{op}} \to \mathrm{DefFrob}_\Gamma(S)^{\mathrm{op}}$ is induced by f ($\mathrm{DefFrob}_\Gamma(-)$ is a functor),

together with natural isomorphisms

$$\mathcal{F}_{\mathrm{id}} \cong \mathrm{id} \ \text{ and } \ \mathcal{F}_{gf} \cong \mathcal{F}_g(f^*) \circ g^*(\mathcal{F}_f)$$

for all local homomorphisms $\mathrm{id}\colon\ R \to R$, $f\colon\ R \to S$ and $g\colon\ S \to T$.

A morphism $\eta\colon\ \mathcal{F} \to \mathcal{G}$ in this category is a collection of natural transformations $\eta_R\colon\ \mathcal{F}_R \to \mathcal{G}_R$ together with natural isomorphisms $\mathcal{G}_f \circ f^*(\eta_R) \cong \eta_S(f^*) \circ \mathcal{F}_f$. □

For example, there is an object $\mathcal{O}$ of $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ described as follows. $\mathcal{O}_R$ is the functor sending deformations to their base ring R and sending morphisms between deformations to the identity map of R. $\mathcal{O}_f$ is determined by the isomorphism $R \otimes_R^f S \to S$, $r \otimes s \mapsto f(r)s$. The natural isomorphisms $\mathcal{O}_{\mathrm{id}} \cong \mathrm{id}$ and $\mathcal{O}_{gf} \cong \mathcal{O}_g(f^*) \circ g^*(\mathcal{O}_f)$ are determined by the isomorphisms $R \otimes_R^{\mathrm{id}} R \to R$, $r_1 \otimes r_2 \mapsto r_1 r_2$, and $R \otimes_R^{gf} T \to (R \otimes_R^f S) \otimes_S^g T$, $r \otimes t \mapsto (r \otimes 1) \otimes t$, respectively. See [Rez09, 11.13 and 11.17].

Remark 9 $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ is a symmetric monoidal category with the tensor product $\mathcal{F} \otimes \mathcal{G}$ given by $(\mathcal{F} \otimes \mathcal{G})_R(G) = \mathcal{F}_R(G) \otimes_R \mathcal{G}_R(G)$. □

Theorem 10 ([Reza, pre-theorem 16.4]) *The symmetric monoidal categories* $\mathrm{Mod}_{\mathcal{A}}$ *and* $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ *are equivalent.* □

Next we consider $\mathrm{Model}_{\mathrm{DL}_{E_0}}$, the category of models for the theory DL_{E_0}, on which $\mathcal{A}$ acts. By [Reza, proposition 7.6], there is a forgetful functor $\mathrm{Model}_{\mathrm{DL}_{E_0}} \to \mathrm{Mod}_{\mathcal{A}}$ along which the coproduct of DL_{E_0}-models and the tensor product of $\mathrm{Mod}_{\mathcal{A}}$ agree.

Definition 11 Define a category $\mathrm{Alg}_{\mathrm{DefFrob}_\Gamma}$ as follows. An object $\mathcal{B}$ of this category is a ring object in $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ which satisfies the *Frobenius congruence*, i.e. the diagram

$$\begin{tikzcd}
\sigma^*\mathcal{B}_R(G) \arrow[r, "\mathcal{B}_\sigma(G)", "\sim"'] \arrow[dr, "\mathrm{Frob}^*"'] & \mathcal{B}_R\sigma^*(G) \arrow[d, "\mathcal{B}_R(\mathrm{Frob})"] \\
& \mathcal{B}_R(G)
\end{tikzcd}$$

commutes for all complete local rings R and deformations G of Γ to R.

Morphisms in this category are maps of ring objects.

An object $\mathcal{B}$ is said to be *torsion free* if $\mathcal{B}_R(G)$ is p-torsion free for every p-torsion free R and every deformation G to R. We denote by $\mathrm{Alg}^{\mathrm{tf}}_{\mathrm{DefFrob}_\Gamma}$ the full subcategory of $\mathrm{Alg}_{\mathrm{DefFrob}_\Gamma}$ consisting of torsion free objects. □

Remark 12 We note as in [Rez09, 11.18] that roughly speaking, the Frobenius congruence is the requirement that $\mathcal{B}$ carry the Frobenius on formal groups to the Frobenius on algebras. □

Theorem 13 ([Reza, pre-theorem 16.5]) *There is a forgetful functor* $\mathrm{Model}_{\mathrm{DL}_{E_0}} \to \mathrm{Alg}_{\mathrm{DefFrob}_\Gamma}$ *which restricts to an equivalence* $\mathrm{Model}^{\mathrm{tf}}_{\mathrm{DL}_{E_0}} \cong \mathrm{Alg}^{\mathrm{tf}}_{\mathrm{DefFrob}_\Gamma}$ *between the full subcategories of torsion free objects.* □

2.2 Deformations of Frobenius are parametrized by subgroups

Having identified the categories, we now analyze the essential data that are encoded in $\mathrm{Mod}_{\mathrm{DefFrob}_\Gamma}$ and $\mathrm{Alg}^{\mathrm{tf}}_{\mathrm{DefFrob}_\Gamma}$, by studying the structure of the category $\mathrm{DefFrob}_\Gamma(R)$ of deformations of Frobenius. This turns out to be parametrized by the finite flat subgroups of deformations of Γ to R, as we explain below.

Choosing a coordinate x on a formal group G over R, a *degree* (or *rank*) *d subgroup K of G* is an effective divisor on G with $\mathcal{O}_K = R[\![x]\!]/\big(f(x)\big)$ for some degree d monic polynomial $f(x)$ such that $f(x_1 +_G x_2) \in \big(f(x_1), f(x_2)\big)$ and $f(x) \in (x)$. In other words, the group law of G restricts to K, and K contains the identity. In particular K is finite and flat over R (we will assume finiteness and flatness of subgroups even when we do not mention their degrees). We can define the *quotient group G/K* which is again a formal group (cf. [Str97, section 5]).

One can show that the homomorphism $[d]_G\colon\ G \to G$ restricts to zero on K (cf. [TO70, section 1]). More concretely, this means that $f(x)$ must divide $[d]_G(x)$. As a consequence, subgroups of a formal group over a p-local ring must have degree p^r. In particular, if G is a formal group over a field k of characteristic $p > 0$, there is exactly one subgroup of degree p^r, given by $f(x) = x^{p^r}$, which is the kernel of the r-fold Frobenius isogeny Frob^r.

We have seen in remark 7 that in $\mathrm{DefFrob}_\Gamma(R)$ the degree 1 morphisms (when $r = 0$) are precisely the $\star$-isomorphisms of deformations. In general, with morphisms corresponding to all $r \geq 0$, $\mathrm{DefFrob}_\Gamma(R)$ is equivalent to the following category (cf. [Reza, proposition 16.9]). The objects of this category are $\star$-isomorphism classes of deformations $[G]$. The morphisms are $\star$-isomorphism classes of pairs $[G > K]$: the source of $[G > K]$ is $[G]$, and the target of $[G > K]$ is $[G/K]$, where G/K is a deformation of Γ with $i_{G/K} = \sigma^r \circ i_G$ (p^r being the degree of K). Moreover, by the Lubin-Tate theorem (cf. [LT66, theorem 3.1] and [Rez98, section 4.3]), there is at most one $\star$-isomorphism between any two deformations. If $[G/K] = [G']$, then $[G' > K'] \circ [G > K] = [G > K'']$, where K'' is the kernel of the composite $G \to G/K \cong G' \to G'/K'$. Thus deformations of Frobenius with source (G, i, α) correspond *exactly* to subgroups of G.

Example 14 Let Γ be the multiplicative formal group over $\mathbb{F}_p$ of height 1. For the multiplicative formal group $\mathbb{G}_m$ over a p-local ring R, since the formal group law is defined by $1 + (x_1 +_{\mathbb{G}_m} x_2) = (1 + x_1)(1 + x_2)$, we have $[p^r](x) = (1 + x)^{p^r} - 1 = x^{p^r}$. Thus the only subgroups of $\mathbb{G}_m$ are $\mathbb{G}_m[p^r]$ with $\mathcal{O}_{\mathbb{G}_m[p^r]} = \mathcal{O}_{\mathbb{G}_m}/(x^{p^r})$. Moreover, by the Lubin-Tate theorem (cf. [LT66, theorem 3.1] and [Rez98, section 4.3]), every object of $\mathrm{DefFrob}_\Gamma(R)$ is $\star$-isomorphic to $\mathbb{G}_m$. In particular the set of $\star$-isomorphism classes of deformations of Γ to R is classified by the ring $\mathcal{O}_{\mathrm{univ}} = \mathbb{Z}_p$, and we can take the universal deformation G_{univ} to be the multiplicative formal group over $\mathbb{Z}_p$. Thus by functoriality, to describe an object $\mathcal{B}$ of $\mathrm{Alg}^{\mathrm{tf}}_{\mathrm{DefFrob}_\Gamma}$, it is enough to give

(1) a p-torsion free $\mathbb{Z}_p$-algebra $B = \mathcal{B}_{\mathbb{Z}_p}(\mathbb{G}_m)$,

(2) maps of $\mathbb{Z}_p$-algebras $\psi^{p^r}\colon B \to B$ (corresponding to the isogenies $[p^r]\colon \mathbb{G}_m \to \mathbb{G}_m$) such that
   - $\psi^1 = \mathrm{id}_B$ and $\psi^{p^r} \circ \psi^{p^s} = \psi^{p^{r+s}}$,
   - $\psi^p(b) \equiv b^p \mod pB$ (the Frobenius congruence).

We note as in [Rez09, example 1.3] that this is a "p-typicalization" of the original theorem of Wilkerson (cf. [Wil82, proposition 1.2]) which characterizes the torsion free λ-rings in terms of the Adams operations satisfying the Frobenius congruences at all primes. More concretely, let K be the complex K-theory spectrum. Then for $B = \pi_0 A$, where A is a p-complete K-algebra (commutative K-algebra such that $A \cong A^\wedge_p$), ψ^p recovers the p'th Adams operation studied by McClure (cf. [BMMS86, chapters VIII and IX]). $\square$

In general, consider the functor X_r which associates to a ring R the set of $\star$-isomorphism classes of pairs $[G > K]$ with K a degree p^r subgroup of G. It is represented by the complete local ring $\mathcal{O}_{X_r} = E^0 B\Sigma_{p^r}/I$, where $I = \sum_{0<i<p^r} \mathrm{Image}\big(E^0 B(\Sigma_i \times \Sigma_{p^r-i}) \xrightarrow{\text{transfer}} E^0 B\Sigma_{p^r}\big)$ is the *transfer ideal* (roughly speaking, the corresponding power operation should be additive, so modulo the "mixing terms" in the Cartan formula); cf. [Str98, theorem 9.2]. This can be viewed as a generalization of the Lubin-Tate theorem for $\mathcal{O}_{\mathrm{univ}} = \mathcal{O}_{X_0}$. Moreover there are two ring homomorphisms $s^*, t^*\colon \mathcal{O}_{\mathrm{univ}} \to \mathcal{O}_{X_r}$, where s^* represents the source map $[G > K] \mapsto [G]$, and t^* represents the target map $[G > K] \mapsto [G/K]$. (In example 14, $\mathcal{O}_{X_r} \cong \mathcal{O}_{\mathrm{univ}}$ for all r, and $s^* = t^* = \mathrm{id}$.) Thus to describe an object $\mathcal{B}$ of $\mathrm{Alg}^{\mathrm{tf}}_{\mathrm{DefFrob}_\Gamma}$, it is enough to give

(1) a p-torsion free $\mathcal{O}_{\mathrm{univ}}$-algebra $B = \mathcal{B}_{\mathcal{O}_{\mathrm{univ}}}(G_{\mathrm{univ}})$,

(2) maps of $\mathcal{O}_{\mathrm{univ}}$-algebras $\psi^{p^r}\colon B \to B \otimes^{s^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r}$ as the composite

$$B \xrightarrow{f^*} B \otimes^{t^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r} \overset{\mathcal{B}_f}{\cong} \mathcal{B}_{\mathcal{O}_{X_r}}(t^* G_{\mathrm{univ}}) \xrightarrow{\mathcal{B}_{\mathcal{O}_{X_r}}(\psi)} \mathcal{B}_{\mathcal{O}_{X_r}}(s^* G_{\mathrm{univ}}) \overset{\mathcal{B}_g}{\cong} B \otimes^{s^*}_{\mathcal{O}_{\mathrm{univ}}} \mathcal{O}_{X_r},$$

where $f = t^*$ and $g = s^*$ are local homomorphisms, and $\psi\colon s^* G_{\mathrm{univ}} \to t^* G_{\mathrm{univ}}$ is the universal deformation of Frob^r (cf. [Str97, section 13]), satisfying a set of formal properties. In particular, if we denote by u^* the map $\mathcal{O}_{X_1} \to \mathcal{O}_{\text{univ}}/(p)$ which represents the universal Frobenius isogeny, the Frobenius congruence amounts to requiring that

$$B \xrightarrow{\psi^p} B \otimes^{s^*}_{\mathcal{O}_{\text{univ}}} \mathcal{O}_{X_1} \xrightarrow{\text{id}\otimes u^*} B \otimes_{\mathcal{O}_{\text{univ}}} \mathcal{O}_{\text{univ}}/(p) = B/pB$$

be the p'th power map $B \to B/pB \xrightarrow{\sigma} B/pB$ which sends x to $\bar{x}^p$.

Example 15 Consider the elliptic curve $C_0 \subset \mathbb{P}^2_{\mathbb{F}_2}$ defined by

$$Y^2Z + YZ^2 = X^3,$$

which is supersingular so that its formal group $\widehat{C_0}$ is of height 2. It has a universal deformation C over the Lubin-Tate ring $\mathbb{W}\mathbb{F}_2[\![u_1]\!] \cong \mathbb{Z}_2[\![a]\!]$, with a choice of 3-torsion point, given by

$$Y^2Z + aXYZ + YZ^2 = X^3,$$

where a is the Hasse invariant (cf. [KM85, 2.2.10] and [MR09, proposition 3.2]). Setting $a = 0$ we recover the supersingular elliptic curve C_0. Let E be the Morava E-theory spectrum associated to C_0, so that $\pi_*E = \mathbb{Z}_2[\![a]\!][u^{\pm 1}]$ with $|u| = 2$. The power operations on E are constructed in [And95, section 3], with explicit formulas computed in [Rezb, sections 3 and 4]. What follows is directly from the latter reference.

By studying degree 2 subgroups, i.e. subgroups of 2-torsion points on C, we can identify $\mathcal{O}_{X_1} \cong \mathbb{Z}_2[\![a,d]\!]/(d^3 - ad - 2)$: in the affine coordinate chart $u = X/Y$, $v = Z/Y$, degree 2 subgroups are generated by points Q of the form $\big(u(Q), v(Q)\big) = (d, -d^3)$ such that $d^3 - ad - 2 = 0$. Thus we have a power operation

$$\psi^2\colon\ E^0X \to E^0X[\![d]\!]/(d^3 - ad - 2).$$

Moreover, by studying the canonical isogeny with source C and kernel the degree 2 subgroup generated by Q (cf. [Lub67, theorem 1.4]), Rezk computes that

$$t^*(a) = \psi^2(a) = a^2 + 3d - ad^2$$

(cf. [Rezb, p. 6]). There are also formulas for a set of functions $Q_0(x)$, $Q_1(x)$ and $Q_2(x)$ which express

$$\psi^2(x) = Q_0(x) + Q_1(x)d + Q_2(x)d^2.$$

In particular the Frobenius congruence takes the form $Q_0(x) \equiv x^2 \bmod 2$.

We will discuss in detail such calculations for Morava E-theories associated to supersingular elliptic curves in section 4. □

3 $K(1)$-local power operations

In this section we discuss how to pass to the $K(1)$-local setting from the power operations at arbitrary height described in the previous section. For general background of $K(1)$-local operations, see [Hopa].

Let F be an even-periodic E_∞ ring spectrum such that F^0 is a p-torsion free, complete, noetherian, local ring with maximal ideal $\mathfrak{m}$ containing p, and the mod-$\mathfrak{m}$ reduction of the formal group over F^0 is of height $n < \infty$, and let $E = L_{K(1)}F$ be its $K(1)$-localization. For example, the Morava E-theory spectrum associated to the supersingular elliptic curve in example 15 is such an F. We write this spectrum as E_2, specifying its height.

Henceforth when talking about a Morava E-theory and its $K(1)$-localization, we will use F to denote the E-theory spectrum itself, and use E to denote the $K(1)$-local spectrum which is primary to our study. However, to avoid conflict with the usual notation in the literature, we reserve E_2 for the above Morava E-theory of height 2 (and in general a height 2 Morava E-theory which is an elliptic cohomology theory of interest), and we will write $L_{K(1)}E_2$ for its $K(1)$-localization.

The general pattern of the relationship between $K(1)$-local power operations and the power operations in section 2.2 is as follows:

$$\begin{tikzcd}
F^0 \arrow[r, "\psi^p"] \arrow[d] & F^0B\Sigma_p/I \arrow[d, "L"] \\
E^0 \arrow[r, "\psi^p"] & E^0B\Sigma_p/I \cong E^0.
\end{tikzcd}$$

Recall that the top operation arises from the universal deformation of Frobenius which is represented by the ring $\mathcal{O}_{X_1} = F^0B\Sigma_p/I$. The vertical maps are induced by the $K(1)$-localization $F \to E$. In terms of homotopy groups, this is obtained by inverting the generator u_1 (so that the resulting formal group is of height at most 1) and completing at the ideal (p), i.e. $\pi_*F = \mathbb{W}k[\![u_1, ..., u_{n-1}]\!][u^{\pm 1}]$ and $\pi_*E = \mathbb{W}k[\![u_1, ..., u_{n-1}]\!][u_1^{-1}]^\wedge_p[u^{\pm 1}]$. For example, we have

$$\pi_0 L_{K(1)}E_2 = \varprojlim_i \mathbb{Z}_2(\!(a)\!)/(2^i) = \left\{ \sum_{n=-\infty}^{\infty} c_n a^n \;\middle|\; c_n \in \mathbb{Z}_2, \lim_{n\to-\infty} c_n = 0 \right\}$$

(the Hasse invariant a can be taken as the generator u_1). In particular the formal group over F^0 obtains a unique degree p subgroup after being pulled back to E^0, and the map L classifies it. We will explain this uniqueness of subgroup and the isomorphism at the bottom right corner shortly.

In order to do calculations we need to examine L more carefully. Let $\mathbb{G}$ be a p-divisible group over a base scheme X, and $x \in X$ be a point whose local ring is complete and noetherian. The restriction of $\mathbb{G}$ to x lives in the following short exact sequence which is natural in $\mathbb{G}$:

$$0 \to \mathbb{G}_x^{\text{for}} \to \mathbb{G}_x \to \mathbb{G}_x^{\text{ét}} \to 0,$$

where the subobject (the connected component of the identity) is the *formal* component, and the quotient is the *étale* component (cf. [Tat67, section 2.2]). The formal component $\mathbb{G}_x^{\text{for}}$ is a formal group on X. The localization $L\colon\ F^0B\Sigma_p/I \to E^0B\Sigma_p/I$ factors through $E^0 \otimes_{F^0} F^0B\Sigma_p/I$. Along the base change $F^0B\Sigma_p/I \to E^0 \otimes_{F^0} F^0B\Sigma_p/I$, the special fiber of a p-divisible group that consists solely of a formal component may split into formal and étale components. We want to take the formal component so as to keep track of the unique subgroup classified by L which lands in the formal group over the ring $E^0B\Sigma_p/I$.

Example 16 We continue example 15 in the $K(1)$-local setting. After base change to $L_{K(1)}E_2$, the universal elliptic curve C has a unique degree 2 subgroup in its formal component which is the canonical subgroup introduced in [Lub67, theorem 1.4]. The degree 2 subgroup generated by $(d, -d^3)$ is contained in the formal component if and only if $(d, -d^3)$ is in the formal neighborhood of the identity $(0,0)$. The equation $d^3 - ad - 2 = 0$ which parametrizes degree 2 subgroups has a unique root in $\mathbb{F}_2(\!(a)\!)$, and Hensel's lemma implies that this lifts to a root in $\pi_0 L_{K(1)}E_2 = \mathbb{Z}_2(\!(a)\!)^\wedge_2$. Plugging this specific value of d into $\psi^2\colon\ \pi_0E_2 \to \pi_0E_2[\![d]\!]/(d^3 - ad - 2)$, we get an endomorphism of the ring $\pi_0 L_{K(1)}E_2$, and this endomorphism is the $K(1)$-local power operation. For an application of this calculation, see [LN, section 6]. $\square$

Lastly we note that the commutative square on the previous page describes the pattern for $K(m)$-local operations with $m > 1$ as well, but the isomorphism at the bottom right corner is specific to the height 1 case.

Lemma 17 $E^0B\Sigma_p/I \cong E^0$.

This generalizes what we have seen in example 14 about the multiplicative formal group. A formal group $\mathbb{G}$ of height 1 has a unique degree p subgroup given by $\mathbb{G}[p]$, and the ring $E^0B\Sigma_p/I$ classifying subgroups of degree p is isomorphic to E^0. Thus the power operation takes the form $\psi^p\colon E^0 \to E^0$ which is a lift of the Frobenius map (cf. [Hopa, section 4]). In general a height n formal group has more than one degree p subgroup, as in the example of section 4 where $n = 2$ and $p = 3$ (cf. remark 22). See [Str97, theorem 42 and lemma 46] and [KM85, section 1.4] for details, and cf. [And95, section 3.5] for an approach of making power operations of higher height land in E^0.

Proof of the lemma First we identify $E^0B\Sigma_p$ as the E^0-submodule of E^0BC_p fixed by the action induced by $\mathrm{Aut}C_p \cong \mathbb{F}_p^\times$. This is a special case of the calculations in [Reza, section 12]. We have

$$B\Sigma_p \xrightarrow{\mathrm{tr}} BC_p \xrightarrow{\mathrm{res}} B\Sigma_p,$$

where tr is the transfer map, and res is the restriction map. Since $[\Sigma_p : C_p]$ is prime to p, the composite is a p-local equivalence, and thus p-locally $B\Sigma_p$ is a retract of BC_p. Moreover $\mathrm{Aut}C_p$ acts on $\mathrm{Hom}(C_p, \Sigma_p)$ as conjugation (it preserves the "cycle type" of the element generating the image of C_p in Σ_p). Thus two maps $C_p \to \Sigma_p$ that differ by an $\mathrm{Aut}C_p$-action induce homotopic maps $BC_p \to B\Sigma_p$, and hence the same map $E^0B\Sigma_p \to E^0BC_p$.

We calculate E^0BC_p by considering the cofiber sequence associated to the construction of the Thom space

$$S(L^{\otimes p}) \to BS^1 \to (BS^1)^{L^{\otimes p}},$$

where L is the tautological complex line bundle over BS^1. E^*BS^1 can be calculated as $E^*[\![x]\!]$ with $|x| = 2$, where x is the first Chern class of L (cf. [Hopb, section 1]). Recall that $\pi_*E = \mathbb{W}k[\![u_1, ..., u_{n-1}]\!][u_1^{-1}]_p^\wedge[u^{\pm 1}]$. By the Thom isomorphism and the even-periodicity of E^*, the map on cohomology induced by the right-hand map in the cofiber sequence can be identified as the E^*-algebra map $E^*BS^1 \to E^*BS^1$ sending x to $[p](x) = px + \cdots + u_1x^p + \cdots$. Also note that we can identify the sphere bundle $S(L^{\otimes p})$ with BC_p. Thus again as E^* is even-periodic, the long exact sequence induced by the cofiber sequence implies that $E^0BC_p \cong E^0[\![x]\!]/\big([p](x)\big)$.

For any $r \in \mathbb{F}_p^\times$, the induced action on E^0BC_p sends x to $[r](x) = rx + \cdots$. Hence as the E^0-submodule of E^0BC_p fixed by the $\mathrm{Aut}C_p$-action, $E^0B\Sigma_p$ can be identified with $E^0 \oplus \big(E^0 \cdot \prod_{r=1}^{p-1}[r](x)\big) \cong E^0 \oplus \big(E^0 \cdot (x^{p-1} + \cdots)\big)$.

Next we identify the transfer ideal $I = \sum_{0<i<p} \mathrm{Image}\big(E^0B(\Sigma_i \times \Sigma_{p-i}) \xrightarrow{\text{transfer}} E^0B\Sigma_p\big)$ as the second summand in $E^0B\Sigma_p$. Similarly to the above, for all $0 < i < p$, we can identify the composite

$$E^0B(\Sigma_i \times \Sigma_{p-i}) \xrightarrow{\mathrm{res}} E^0 \xrightarrow{\mathrm{tr}} E^0B(\Sigma_i \times \Sigma_{p-i})$$

as multiplication by an invertible scalar, and thus $E^0B(\Sigma_i \times \Sigma_{p-i}) \cong E^0$. Moreover by the "double-coset formula" the composite

$$E^0 \xrightarrow{\mathrm{tr}} E^0B\Sigma_p \xhookrightarrow{\mathrm{res}} E^0BC_p$$

has image the same as $E^0 \xrightarrow{\mathrm{tr}} E^0BC_p$. Thus I is generated by $\mathrm{tr}(1)$ as an E^0-module.

Define $\langle p\rangle(x) = x^{-1} \cdot [p](x)$ so that $\langle p\rangle(x) = p + \cdots + u_1x^{p-1} + \cdots$ and $E^0BC_p = E^0[\![x]\!]/\big(x \cdot \langle p\rangle(x)\big)$. Write $\mathrm{tr}(1) = f(x) \in E^0[\![x]\!]$ by abuse of notation. We can identify the composite

$$E^0 \xrightarrow{\mathrm{tr}} E^0BC_p \xrightarrow{\mathrm{res}} E^0$$

as multiplication by p. Since this composite sends 1 to $\mathrm{res}\big(f(x)\big) = f(0)$, we have $f(0) = p$. Moreover since $x \cdot \mathrm{tr}(1) = \mathrm{tr}\big(\mathrm{res}(x)\big)$ and $\mathrm{res}(x) = 0$, $x \cdot f(x)$ is divisible by $[p](x)$. We claim that these two conditions on $f(x)$ forces it to be $\langle p\rangle(x)$. Clearly $\langle p\rangle(0) = p$, and $\langle p\rangle(x)$ is annihilated by x in $E^0[\![x]\!]/\big(x \cdot \langle p\rangle(x)\big)$. Applying the snake lemma to two rows of copies of

$$0 \longrightarrow E^0[\![x]\!] \xrightarrow{\cdot x} E^0[\![x]\!] \longrightarrow E^0 \longrightarrow 0$$

with vertical maps being multiplication by $[p](x)$ on $E^0[\![x]\!]$ and zero on E^0, we see that any element annihilated by x is a multiple of $\langle p\rangle(x)$ by an element of E^0. As p is not a zero-divisor in E^0, the claim follows. Thus $I \cong E^0 \cdot \langle p\rangle(x)$.

Finally as u_1 is invertible in E^0, $E^0B\Sigma_p/I \cong E^0$. $\square$

We also note that at height 1, as in example 14, the power operation ψ^p determines the other ψ^{p^r} with $r > 1$ by iterated composition. Thus in the $K(1)$-local setting, among the operations ψ^{p^r} it suffices to study only ψ^p as above. Moreover if the $K(1)$-local homotopy groups are *p-torsion free*, it turns out that ψ^p determines *all* the power operations (cf. [Reza, section 3]). For this reason we will simply write ψ for *the* $K(1)$-local power operation. At higher height, the relationship among these operations is more complicated.

4 Power operations at the prime 3

In [Rezb], Rezk gives explicit calculations of the algebraic theory of power operations for a specific Morava E-theory of height 2 at the prime 2 (cf. example 15). We record some calculations analogous to some of the results there, at the prime 3, together with calculations of the corresponding $K(1)$-local power operation.

The Morava E-theory F of height 2 that we consider here is associated to a supersingular elliptic curve over a field of characteristic 3. Our calculations break into the following steps:

(1) Find the universal elliptic curve with a choice of 4-torsion point, so that a posteriori the supersingular elliptic curve that we are interested in is one with the Hasse invariant zero;

(2) Study the degree 3 subgroups of this universal elliptic curve. In particular we need to compute

- the coordinate ring parametrizing degree 3 subgroups,
- the equation of the quotient curve which is the image of the universal degree 3 isogeny;

(3) $K(1)$-localize.

4.1 The universal elliptic curve with a choice of 4-torsion point

Let k be a field of characteristic 3. First we note that a supersingular elliptic curve over k cannot have a 3-torsion point. Moreover, in order to have a *unique* lift of the elliptic curve together with a choice of N-torsion point along a deformation of k, N must be prime to 3. As the moduli stack $\mathcal{M}\big(\Gamma_1(2)\big)$ associated to a level $\Gamma_1(2)$-structure on the elliptic curve is not representable by a scheme due to the existence of nontrivial automorphism of order 2 (cf. [KM85, corollaries 4.7.2 and 2.7.2]), a natural choice for the universal elliptic curve is one equipped with a level $\Gamma_1(4)$-structure.

The computation for the equation of the universal elliptic curve with a choice of 4-torsion point P is analogous to [KM85, 2.2.10] and [MR09, proposition 3.2] where the universal elliptic curve for the prime 2 case has a choice of 3-torsion point (cf. example 15). In xy-coordinates, with the constraints that P be at the origin, $2P$ be on the x-axis, and $4P$ be the identity at the infinity, we have the Weierstrass equation

$$y^2 + axy + acy = x^3 + cx^2$$

over the graded ring $\mathbb{Z}_{(3)}[a,c]$, where $|a| = 1$ and $|c| = 2$. The grading comes from the action of $\mathbb{G}_m = \mathrm{Spec}\mathbb{Z}[\lambda^{\pm 1}]$ given by $a \mapsto \lambda a$ and $c \mapsto \lambda^2 c$. By [Sil09, V.4.1(a)], the Hasse invariant can be computed as $h = a^2 + 4c$, so that over k the curve is supersingular precisely when $h = 0$ (note that the minimal field of definition of this supersingular elliptic curve is $\mathbb{F}_9$).

To facilitate calculations, we work in the affine coordinate chart $c = 1$ of $\mathcal{M}\big(\Gamma_1(4)\big)$ so that the elliptic curve is given by

$$y^2 + axy + ay = x^3 + x^2,$$

with the discriminant of the elliptic curve $\Delta = a^2(a+4)(a-4)$ and the Hasse invariant $h = a^2 + 4$. This reduction is analogous to the one discussed in detail in [LN, section 4]. In uv-coodinates, with $u = x/y$ and $v = 1/y$, the equation becomes

$$v + auv + av^2 = u^3 + u^2 v$$

which is the form we will use most often later.

We denote this elliptic curve by $\mathcal{E}$.

Remark 18 We note that the curve $\mathcal{E}$, together with the universal elliptic curve with a choice of 3-torsion point for the prime 2 case, give models for studying power operations at *all* primes. □

Let F be the Morava E-theory associated to the restriction of $\mathcal{E}$ to the supersingular locus. This locus consists of a single closed point, as the ideal $\mathfrak{m} = (3, h)$ is the maximal ideal of the complete local ring $F^0 = \mathbb{Z}_9[\![h]\!]$, where the Hasse invariant h is taken as the generator u_1.

Remark 19 For later calculations, we note that both a and Δ^{-1} lie in F^0. In fact, let i be an element generating $\mathbb{Z}_9$ over $\mathbb{Z}_3$ with $i^2 = -1$, and let $f(x) = x^2 + 4 - h \in F^0[x]$. Then F^0 is complete with respect to $\mathfrak{m}$, and we have

$$f(i) = 3 - h \equiv 0 \mod f'(i)^2\mathfrak{m}.$$

Thus by Hensel's lemma, since $f'(i) = 2i$ is not a zero-divisor in F^0, there is a unique root b of f such that

$$b \equiv i \mod f'(i)\mathfrak{m}.$$

As $f(a) = 0$, we can identify a with b, and hence $a \in F^0$.

To see $\Delta = a^2(a+4)(a-4)$ is invertible in F^0, note that

$$\Delta = (h-4)(h-20) \equiv -1 \mod \mathfrak{m}.$$

Thus $\Delta = m - 1$ for some $m \in \mathfrak{m}$. Since F^0 is complete with respect to $\mathfrak{m}$, $-\sum_{i=0}^{\infty} m^i \in F^0$, and this is the inverse of Δ. □

4.2 Degree 3 subgroups

4.2.1 3-torsion points

In order to compute the coordinate ring parametrizing degree 3 subgroups, we need to find an equation characterizing the coodinates of a 3-torsion point.

Given the elliptic curve

$$\mathcal{E}\colon\ y^2 + axy + ay = x^3 + x^2,$$

(x, y) is a 3-torsion point if and only if the division polynomial

$$\psi_3(x) = 3x^4 + (a^2 + 4)x^3 + 3a^2x^2 + 3a^2x + a^2$$

equals zero (cf. [Sil09, exercise 3.7(d)]; here ψ_3 is not to be confused with the power operation ψ^3). (This polynomial is exactly what one gets for the characterization, away from the prime 2, of a flex point in terms of the second derivative y'' calculated by implicit differentiation.) We want to translate this into uv-coordinates which are more convenient to work with in the formal neighborhood of the identity (in xy-coordinates, the identity is at the infinity). In this way we will get a "characteristic" equation comparable to $d^3 - ad - 2 = 0$ in example 15, where d was the u-coordinate of a 2-torsion point.

From the formula of multiplication by 3 in xy-coordinates, we have

$$[3](u, v) = \left(\frac{\phi_3(\frac{u}{v})\psi_3(\frac{u}{v})}{\omega_3(\frac{u}{v})}, \frac{\psi_3(\frac{u}{v})^3}{\omega_3(\frac{u}{v})} \right),$$

with notation following [Sil09, exercise 3.7(d)], and thus our preliminary equation is $\psi_3(\frac{u}{v}) = 0$. Clearing the denominators in $\psi_3(\frac{u}{v})$, we get

$$\widetilde{\psi_3}(u, v) = 3u^4 + (a^2 + 4)u^3v + 3a^2u^2v^2 + 3a^2uv^3 + a^2v^4.$$

To eliminate v, note that we can rewrite the equation of $\mathcal{E}$ as a quadratic equation in v:

$$av^2 + (1 + au - u^2)v - u^3 = 0.$$

We then multiply $\widetilde{\psi_3}(u, v)$ by its "conjugate" $\widetilde{\psi_3}(u, v')$, where v and v' are conjugate roots of the above equation. We get a degree 8 polynomial in u:

$$f(u) = -3 - 3au + 8u^2 - a^2u^2 + 9au^3 - 6u^4 + 6a^2u^4 + 7au^5 + a^3u^5 + 3a^2u^6 + 3au^7 + u^8.$$

Thus if we denote by (d, e) the coordinates of a 3-torsion point, d must satisfy $f(d) = 0$.

Remark 20 According to section 3, in particular example 16, we might expect that $f(u)$ have a unique root reduced to zero modulo 3 after $h = a^2 + 4$ gets inverted, corresponding to the unique subgroup in the formal neighborhood of the identity. However, as we obtain $f(u)$ out of a conjugation procedure, this is not exactly the case. We have

$$f(u) \equiv u^2(u+a)^6 \mod 3,$$

and in view of $a \not\equiv 0$ by the invertibility of $\Delta = a^2(a+4)(a-4)$ (cf. remark 19), such a root has multiplicity 2 (corresponding to the two nontrivial elements in the subgroup). $f(u)$ is the best possible polynomial solely in terms of u that we have at this point, and in later steps we will have to "rescale" it back to a degree 4 polynomial (cf. the polynomial $w(t)$ above proposition 23). ◻

Next, for the coordinates (d, e) of a 3-torsion point, we want to express e in terms of d. At the prime 2 (cf. example 15), the relation $e = -d^3$ can be obtained by manipulating the division polynomial ψ_2 and its "conjugate" as above, or simply by computing the inversion formula for a 2-torsion point on the elliptic curve as in [Rezb, section 3]. The prime 3 case is a little more involved.

Using the Euclidean algorithm, we compute the gcd of

$$A(v) = \widetilde{\psi_3}(u,v) = 3u^4 + (a^2+4)u^3v + 3a^2u^2v^2 + 3a^2uv^3 + a^2v^4$$

and

$$B(v) = av^2 + (1 + au - u^2)v - u^3 \quad \text{(rewriting the equation of } \mathcal{E}\text{)},$$

both of which vanish at (d, e). We have

$$A(v) = B(v)Q_1(v) + R_1(v)$$

$$B(v) = R_1(v)Q_2(v) + R_2(v),$$

and it turns out that $R_2(e) = 0$ as a result of $f(d) = 0$. Thus

$$R_1(d,e) = p(d) + q(d)e = 0$$

is a relation between d and e which is linear in e. As we have formulas for $p(d)$ and $q(d)$, we can compute the inverse of $q(d)$ by applying the Euclidean algorithm and find

$$1 = m(d)q(d) + n(d)f(d) = m(d)q(d).$$

Thus

$$e = -q(d)^{-1}p(d) = -m(d)p(d).$$

In the end we have a degree 7 polynomial $e = g(d)$, comparable to $e = -d^3$ in the prime 2 case:

$$g(d) = -\frac{1}{(-4+a)a(4+a)}(-18 - 12ad + 18d^2 - 15ad^3 + 4a^3d^3 + 2d^4 + a^2d^4 + a^4d^4 \\ -6ad^5 + 3a^3d^5 - 2d^6 + 3a^2d^6 + ad^7).$$

To summarize, in this subsection we find two polynomials f and g, so that any 3-torsion point with uv-coordinates (d, e) satisfies $f(d) = 0$ and $e = g(d)$.

4.2.2 The universal degree 3 isogeny

Now we are ready to compute the universal degree 3 isogeny, and thus the equation of the quotient curve which is its image. From there we can find a formula for the power operation $\psi^3\colon F^0 \to F^0[\![d]\!]/\big(f(d)\big)$, where $F^0 = \mathbb{Z}_9[\![h]\!]$ (to be precise, the target should actually be a certain improved coordinate ring as promised in remark 20).

To compute the coordinates (u', v') of the isogeny, we follow the "Lubin isogeny" construction (cf. [Lub67, proof of theorem 1.4]). Let $P = \big(u, v(u)\big)$ be a general point on $\mathcal{E}$, and $Q = \big(d, e(d)\big)$ be a 3-torsion point. For $v(u)$, we use the equation of $\mathcal{E}$ rewritten as a quadratic equation

$$av^2 + (1 + au - u^2)v - u^3 = 0.$$

We take the first few terms (up to at least u^9 for our purpose) in the power series expansion of the root satisfying $v(0) = 0$ (in the formal neighborhood of the identity). For $e(d)$, we use the polynomial $g(d)$ computed at the end of section 4.2.1. We set

$$u' = u(P)u(P - Q)u(P + Q),$$

and similarly for v'. By computing the inversion and addition formulas for the curve $\mathcal{E}$, we can write down formulas for $u' = \alpha u + \cdots$ and $v' = \beta u^3 + \cdots$ in terms of the uniformizer u (at the identity) and parameters a and d. For example,

$$\alpha = -\frac{1}{(-4+a)(4+a)}(-18 - 12ad + 2d^2 + a^2d^2 - 15ad^3 + 4a^3d^3 + 2d^4 + a^2d^4 + a^4d^4 \\ -6ad^5 + 3a^3d^5 - 2d^6 + 3a^2d^6 + ad^7).$$

Remark 21 In order to have the equation of the quotient curve in the Weierstrass form, we need to include an adjusting constant factor α^3/β into v' by multiplying it to the original formula of v'. This factor is comparable to the term -1 appearing in the formula for u' in [Rezb, p. 6]. $\square$

We then solve for the Weierstrass equation which u' and v' satisfy. The equation of the quotient curve turns out to be

$$v + ruv + rv^2 = u^3 + u^2 v,$$

where

$$\begin{aligned} r(a,d) = -\frac{1}{(-4+a)(4+a)}(&-126a+28a^3-a^5+120d-9a^2d+3a^4d+258ad^2-67a^3d^2\\ &+3a^5d^2-152d^3+208a^2d^3-40a^4d^3+a^6d^3+198ad^4-33a^3d^4-3a^5d^4+8d^5\\ &+63a^2d^5-15a^4d^5+70ad^6-17a^3d^6+24d^7-6a^2d^7). \end{aligned}$$

Comparing the above equation with that of $\mathcal{E}$

$$v + auv + av^2 = u^3 + u^2 v,$$

we see that $\psi^3(a) = r(a,d)$. As $d \equiv 0 \bmod 3$ (cf. remark 20), we check that $\psi^3(a) \equiv a^3 \bmod 3$ which is the Frobenius congruence at height 1.

Lastly we compute the coordinate ring which is the target of the power operation ψ^3.

Set $t = g(d)/d$, the reciprocal of the x-coordinate of a 3-torsion point. This is a quantity that is invariant under negation using the group law of $\mathcal{E}$, since the x-coordinate of any point on the elliptic curve is invariant under negation (cf. [Sil09, III.2.3(a)]). Moreover, as the reciprocal of the x-coordinate, t is "distinguishable" from the identity, since in xy-coodinates the identity is at the infinity. In view of $f(d) = 0$, we compute that t is the root of a quartic polynomial

$$w(t) = a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2+4)t + 3$$

which has a *unique* root reducing to zero modulo 3.

Remark 22 We note that via the relation $t = 1/x$ this polynomial $w(t)$ recovers the division polynomial $\psi_3(x)$. Thus the eight roots of $f(d)$ together with $d = 0$ correspond to the nine 3-torsion points on $\mathcal{E}$, and the four roots of $w(t)$ correspond to the four degree 3 subgroups consisting of 3-torsion points, one of which lies in the formal neighborhood of the identity. □

The equation of $\mathcal{E}$ implies that d satisfies a quadratic equation in terms of t:

$$(t+1)d^2 - at(t+1)d - t = 0.$$

From this equation and $w(t) = 0$, we can rewrite $\psi^3(a) = r(a,d)$ above in terms of t which lands in the correct target ring.

We summarize the above calculations as follows.

Proposition 23

(1) *The universal degree 3 isogeny with source $\mathcal{E}$ is defined over the ring*

$$F^0[\![t]\!]/\bigl(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2+4)t + 3\bigr),$$

and has target the elliptic curve

$$y^2 + rxy + ry = x^3 + x^2,$$

where

$$r(a,t) = a^3t^3 + 3a^3t^2 + 3a^3t - 4at + a^3 - 3a.$$

The kernel of this isogeny is generated by the 3-torsion point whose x-coordinate is $1/t$.

(2) *The power operation $\psi^3\colon\ F^0 \to F^0[\![t]\!]/\bigl(a^2t^4 + 3a^2t^3 + 3a^2t^2 + (a^2+4)t + 3\bigr)$ is given by*

$$\begin{aligned}\psi^3(h) = {} & (t+1)^3h^3 - (22t^3 + 69t^2 + 75t + 27)h^2 + (128t^3 + 424t^2 + 512t + 201)h \\ & -16(14t^3 + 49t^2 + 65t + 27),\\ \psi^3(a) = {} & (t+1)^3a^3 - (4t+3)a.\end{aligned}$$

□

4.3 $K(1)$-localize

As in example 16, with $h = a^2 + 4$ invertible, we can solve for t 3-adically from the equation $w(t) = 0$ by first writing

$$t = -\frac{1}{a^2+4}(a^2t^4 + 3a^2t^3 + 3a^2t^2 + 3)$$

and then substituting t recursively. Plugging this uniquely determined value of t into $\psi^3(h,t)$ and $\psi^3(a,t)$ computed in proposition 23, we get the $K(1)$-local power operation ψ as an endomorphism of the ring $E^0 = \mathbb{Z}_9(\!(h)\!)^\wedge_3$:

$$\psi(h) = \frac{1}{h^{15}}\,(34\,012\,224 - 127\,545\,840\,h + \cdots),$$

$$\psi(a) = \frac{a^3}{(a^2+4)^{15}}\,(4\,194\,304 - 207\,028\,224\,a^2 + \cdots).$$

References

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974, Chicago Lectures in Mathematics. MR0402720 (53 #6534)

[And95] Matthew Ando, *Isogenies of formal group laws and power operations in the cohomology theories E_n*, Duke Math. J. **79** (1995), no. 2, 423–485. MR1344767 (97a:55006)

[BMMS86] R. R. Bruner, J. P. May, J. E. McClure, and M. Steinberger, *H_∞ ring spectra and their applications*, Lecture Notes in Mathematics, vol. 1176, Springer-Verlag, Berlin, 1986. MR836132 (88e:55001)

[Bor94] Francis Borceux, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994, Categories and structures. MR1313497 (96g:18001b)

[Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281. MR551009 (80m:55006)

[Car54] Henri Cartan, *Sur les groupes d'Eilenberg-Mac Lane. II*, Proc. Nat. Acad. Sci. U. S. A. **40** (1954), 704–707. MR0065161 (16,390b)

[EKMM97] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997, With an appendix by M. Cole. MR1417719 (97h:55006)

[Frö68] A. Fröhlich, *Formal groups*, Lecture Notes in Mathematics, No. 74, Springer-Verlag, Berlin, 1968. MR0242837 (39 #4164)

[GH04] P. G. Goerss and M. J. Hopkins, *Moduli spaces of commutative ring spectra*, Structured ring spectra, London Math. Soc. Lecture Note Ser., vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200. MR2125040 (2006b:55010)

[Hopa] M. J. Hopkins, *K(1)-local E_∞ ring spectra*, available at `http://www.math.rochester.edu/u/faculty/doug/otherpapers/knlocal.pdf`.

[Hopb] Mike Hopkins, *Complex oriented cohomology theories and the language of stacks*, available at `http://www.math.rochester.edu/u/faculty/doug/otherpapers/coctalos.pdf`.

[Hov08] Mark Hovey, *Morava E-theory of filtered colimits*, Trans. Amer. Math. Soc. **360** (2008), no. 1, 369–382 (electronic). MR2342007 (2008g:55007)

[HS99] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100. MR1601906 (99b:55017)

[Kat81] N. Katz, *Serre-Tate local moduli*, Algebraic surfaces (Orsay, 1976–78), Lecture Notes in Math., vol. 868, Springer, Berlin, 1981, pp. 138–202. MR638600 (83k:14039b)

[KM85] Nicholas M. Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR772569 (86i:11024)

[Law63] F. William Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872. MR0158921 (28 #2143)

[Law09] Tyler Lawson, *An overview of abelian varieties in homotopy theory*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Geom. Topol. Monogr., vol. 16, Geom. Topol. Publ., Coventry, 2009, pp. 179–214. MR2544390 (2010i:55006)

[LN] Tyler Lawson and Niko Naumann, *Topological modular forms of level 3 and E_∞-structures on truncated Brown-Peterson spectra*, arXiv:1101.3897.

[LT66] Jonathan Lubin and John Tate, *Formal moduli for one-parameter formal Lie groups*, Bull. Soc. Math. France **94** (1966), 49–59. MR0238854 (39 #214)

[Lub67] Jonathan Lubin, *Finite subgroups and isogenies of one-parameter formal Lie groups*, Ann. of Math. (2) **85** (1967), 296–302. MR0209287 (35 #189)

[Lur] Jacob Lurie, *Chromatic homotopy theory*, available at `http://www.math.harvard.edu/~lurie/252x.html`.

[Lur09] J. Lurie, *A survey of elliptic cohomology*, Algebraic topology, Abel Symp., vol. 4, Springer, Berlin, 2009, pp. 219–277. MR2597740

[Mas83] W. S. Massey, *Cross products of vectors in higher-dimensional Euclidean spaces*, Amer. Math. Monthly **90** (1983), no. 10, 697–701. MR723943 (85b:15038)

[MR09] Mark Mahowald and Charles Rezk, *Topological modular forms of level 3*, Pure Appl. Math. Q. **5** (2009), no. 2, Special Issue: In honor of Friedrich Hirzebruch. Part 1, 853–872. MR2508904 (2010g:55010)

[MT68] Robert E. Mosher and Martin C. Tangora, *Cohomology operations and applications in homotopy theory*, Harper & Row Publishers, New York, 1968. MR0226634 (37 #2223)

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986. MR860042 (87j:55003)

[Rav92] ———, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992, Appendix C by Jeff Smith. MR1192553 (94b:55015)

[Reza] Charles Rezk, *Lectures on power operations*, available at `http://www.math.uiuc.edu/~rezk/power-operation-lectures.dvi`.

[Rezb] ———, *Power operations for Morava E-theory of height 2 at the prime 2*, arXiv:0812.1320.

[Rezc] ———, *Power operations in Morava E-theory: a survey*, available at `http://www.math.uiuc.edu/~rezk/midwest-2009-power-ops-handout.pdf`.

[Rez98] ———, *Notes on the Hopkins-Miller theorem*, Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997), Contemp. Math., vol. 220, Amer. Math. Soc., Providence, RI, 1998, pp. 313–366. MR1642902 (2000i:55023)

[Rez09] ______, *The congruence criterion for power operations in Morava E-theory*, Homology, Homotopy Appl. **11** (2009), no. 2, 327–379. MR2591924 (2011e:55021)

[Ser53] Jean-Pierre Serre, *Cohomologie modulo* 2 *des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198–232. MR0060234 (15,643c)

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, second ed., Graduate Texts in Mathematics, vol. 106, Springer, Dordrecht, 2009. MR2514094 (2010i:11005)

[Ste62] N. E. Steenrod, *Cohomology operations*, Lectures by N. E. Steenrod written and revised by D. B. A. Epstein. Annals of Mathematics Studies, No. 50, Princeton University Press, Princeton, N.J., 1962. MR0145525 (26 #3056)

[Str97] Neil P. Strickland, *Finite subgroups of formal groups*, J. Pure Appl. Algebra **121** (1997), no. 2, 161–208. MR1473889 (98k:14065)

[Str98] N. P. Strickland, *Morava E-theory of symmetric groups*, Topology **37** (1998), no. 4, 757–779. MR1607736 (99e:55008)

[Tat67] J. T. Tate, *p-Divisible groups*, Proc. Conf. Local Fields (Driebergen, 1966), Springer, Berlin, 1967, pp. 158–183. MR0231827 (38 #155)

[TO70] John Tate and Frans Oort, *Group schemes of prime order*, Ann. Sci. École Norm. Sup. (4) **3** (1970), 1–21. MR0265368 (42 #278)

[Wil82] Clarence Wilkerson, *Lambda-rings, binomial domains, and vector bundles over* $\mathbf{C}P(\infty)$, Comm. Algebra **10** (1982), no. 3, 311–328. MR651605 (83f:55003)