ETALE EXTENSIONS OF λ-RINGS

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ABSTRACT. Given a λ-ring A and a formally etale morphism $f\colon A \to B$ of commutative rings, one may ask: What are the possible λ-ring strutures on B such that f is a map of λ-rings? We give the answer: Such a lifted λ-ring structure on B is determined uniquely by a compatible choice of lifts of the Adams operations ψ^p from A to B for all primes p which satisfy Frobenius congruences. In other words, to extend a λ-ring structure along a formally etale morphism, we need not be concerned about the "non-linear" part of the λ-ring structures in question.

Contents

1. Introduction

We assume the reader is familiar with λ rings. A ψ-ring B is a commutative ring equipped with "Adams operations", i.e., ring endomorphisms $\psi^p \colon B \to B$ for each prime p which pairwise commute. A ψ-ring is said to satisfy the Frobenius condition if each ψ^p is a lift of Frobenius on B/pB. Every λ-ring has an underlying ψ-ring, which necessarily satisfies the Frobenius condition.

Fix a λ-ring A and a map $f\colon A \to B$ of commutative rings which is formally etale. We will prove the following.

- If B is a ψ-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of ψ-rings, then there exists a unique λ-ring structure on B compatible with the given ψ-ring structure, making f a map of λ-rings (5.4).
- If, in addition, $h\colon B \to C$ is a map of ψ-rings such that $hf\colon A \to C$ is a map of λ-rings, then h is a map of λ-rings (5.5).

We will prove these by first dealing with the analogous result in the case of θ^p-rings, which are a p-local analogue of λ-rings. We will then prove the result for λ-rings using a characterization of λ-rings in terms of their underlying θ^p-ring structures (4.10). This characterization is of independent interest; I am unaware that it has appeared in the literature up to now.[1]

2. θ^p-rings

2.1. θ^p **rings.** Fix a prime $p \in \mathbb{Z}$. A θ^p-**ring** is a commutative ring R equipped with a function $\theta^p \colon R \to R$ satisfying the following identities:

- $\theta^p(a + b) = \theta^p(a) + \theta^p(b) - C_p(a, b)$, where $C_p(x, y) = \sum_{i=1}^{p-1} p^{-1}\binom{p}{i}x^i y^{p-i} \in \mathbb{Z}[x, y]$;

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[1]I have a vague sense that I have run across it before somewhere, but I cannot find it.

- $\theta^p(ab) = \theta^p(a)b^p + a^p\theta^p(b) + p\theta^p(a)\theta^p(b)$;
- $\theta^p(1) = 0$.

A morphism of θ^p-rings is a ring homomorphism which commutes with θ^p. We write $\theta^p\mathrm{Ring}$ for the category of θ^p-rings.[2]

Note that for $n \in \mathbb{Z}$, the above identities for θ^p imply

- $\theta^p(na) = n\,\theta^p(a) - F_p(n)a^p$ where $F_p(n) = (n^p - n)/p \in \mathbb{Z}$.

(This has the amusing consequence that for p *odd* we have $\theta^p(-a) = -\theta^p(a)$, while on the other hand when $p = 2$ we have $\theta^2(-a) = -\theta^2(a) - a^2$.)

2.2. θ^p-subrings. Given any collection $\{a_s\}_{s \in S}$ of elements of a θ^p-ring R, and any polynomial $f(\underline{x}) \in \mathbb{Z}[x_s \,|\, s \in S]$ with integer coefficients with variables indexed by S, it is straightforward to see using the above identities that there exists a polynomial $g(\underline{x}, \underline{y}) \in \mathbb{Z}[x_s, y_s \,|\, s \in S]$ such that

$$\theta^p(f(a_s)) = g(a_s, \theta^p(a_s)).$$

This implies the following.

2.3. Proposition. *Let R be a θ^p-ring, and let $X \subseteq R$ be a subset closed under θ^p (i.e., such that $x \in X$ implies $\theta^p(x) \in X$). Then the subring $S \subseteq R$ generated as a ring by the set X is also a subset closed under θ^p, and thus is itself a θ^p-ring with θ^p-operation obtained by restriction from R.*

2.4. θ^p-ideals. An ideal $I \subseteq R$ in a θ^p-ring is a θ^p-**ideal** if for all $x \in I$ and $a \in R$, we have that $\theta^p(a + x) - \theta^p(a) \in I$. It is straightforward to show that an ideal $I \subseteq R$ is θ^p-ideal *if and only if* the quotient ring R/I inherits a (necessarily unique) θ^p-ring structure so that the quotient map $R \to R/I$ is a homomorphism of θ^p-rings.

2.5. Proposition. *Let R be a θ^p-ring, let $X \subseteq R$ be a subset, and let $I = (X)$ be the ideal in R generated by X. Then I is a θ^p-ideal if and only if $\theta^p(x) \in I$ for all $x \in X$.*

Proof. The "only if" part is clear, since $\theta^p(x) = \theta^p(0 + x) - \theta^p(0)$. Suppose then that $X \subseteq R$ satisfies the given condition; we want to show that $I = (X)$ is a θ^p-ideal.

Let $J = \{ x \in I \mid \theta^p(x) \in I \}$. Note that; (1) if $x, y \in J$, then $\theta^p(x+y) = \theta^p(x) + \theta^p(y) - C_p(x, y) \in I$ and thus $x + y \in J$; (2) if $x \in J$ and $a \in R$, then $\theta^p(ax) = \theta^p(a)x^p + a^p\theta^p(x) + p\theta^p(a)\theta^p(x) \in I$, and thus $ax \in J$. Thus J is a subideal of I containing X, whence $J = I$, and thus I is closed under the operation θ^p. Now note that if $x \in I$ and $a \in R$, we have that

$$\theta^p(a + x) - \theta^p(a) = \theta^p(x) - C_p(a, x) \in I,$$

as desired. $\square$

2.6. Corollary. *If R is a θ^p-ring and $I, J \subseteq R$ are θ^p-ideals, then $IJ \subseteq R$ is a θ^p-ideal.*

Proof. If $x \in I$ and $y \in J$, then $\theta^p(xy) = \theta^p(x)y^p + x^p\theta^p(y) + p\theta^p(x)\theta^p(y) \in IJ$, and thus IJ is a θ^p-ideal by (2.5). $\square$

2.7. ψ^p-rings. A ψ^p-**ring** is a pair (R, ψ^p) consisting of a commutative ring R equipped with a commutative ring map $\psi^p \colon R \to R$. There is an evident category $\psi^p\mathrm{Ring}$ of ψ^p-rings, whose morphisms are ring homomorphism which commute with ψ^p. (The "p" in the term "ψ^p-ring" and notation "ψ^p" and "$\psi^p\mathrm{Ring}$" is meant to stand for a prime p. In this paragraph it has served a purely decorative role, but it will matter in the following paragraph.)

We say that a ψ^p-ring (R, ψ^p) satisfies the **Frobenius condition** if ψ^p is a lift of Frobenius, i.e., if for all $a \in R$ we have that $\psi^p(a) \equiv a^p \mod pR$. We write $\psi^p\mathrm{Ring}_{\mathrm{Fr}}$ for the full subcategory of $\psi^p\mathrm{Ring}$ consisting of objects which satisfy the p-Frobenius condition.

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[2]These are also known as δ-ring relative to p (Joyal [Joy85]) or a ring with p-derivation (Buium). We are following Bousfield's [Bou96] terminology here.

2.8. **The Adams operation of a θ^p-ring.** Given a θ^p-ring R, we define the **Adams operation** $\psi^p = \psi^p \colon R \to R$ by $\psi^p(a) = a^p + p\theta^p(a)$. It is immediate that (R, ψ^p) is a ψ^p-ring, which furthermore satisfies the Frobenius condition. Thus, we have obtained a forgetful functor

$$\theta^p\text{Ring} \to \psi^p\text{Ring}_{\text{Fr}} \subset \psi^p\text{Ring}.$$

2.9. **The congruence criterion.** The following gives a complete criterion for constructing a θ^p-ring compatible with a given ψ^p-ring structure (R, ψ^p) the case that R has no p-torsion.

2.10. **Proposition.** *Let R be a ψ^p-ring satsifying the Frobenius condition. If R is also p-torsion free, then there exists a unique θ^p-ring structure on R, compatible with the given ψ^p.*

Proof. Given such (R, ψ^p), define a function $\theta^p \colon R \to R$ by $\theta^p(a) = (\psi^p(a) - a^p)/p$, and verify the identities for a θ^p-ring directly. $\square$

This implies that the forgetful functor $\mathcal{U} \colon \theta^p\text{Ring} \to \psi^p\text{Ring}_{\text{Fr}} \subseteq \psi^p\text{Ring}$ restricts to an equivalence of categories

$$\theta^p\text{Ring}_{\text{tf}} \xrightarrow{\sim} (\psi^p\text{Ring}_{\text{Fr}})_{\text{tf}},$$

where these denote full subcategories of p-torsion free objects in $\theta^p\text{Ring}$ and $\psi^p\text{Ring}_{\text{Fr}}$ respectively.

2.11. **θ^p-rings as coalgebras.** Given a ring R, we define a ring $V(R) = V_p(R)$ as follows. The underlying set of $V(R)$ is $R \times R$, and addition and multiplication are defined by

$$(x, y) + (x', y') := (x + x', y + y' - C_p(x, x')),$$
$$(x, y) \cdot (x', y') := (xx', yx'^p + x^p y' + p\, yy').$$

The map $\epsilon_V \colon V(R) \to R$ defined by $\epsilon_V(x, y) = x$ is a ring homomorphism.

A V-**coalgebra** is a pair (R, α) consisting of a ring R and a ring homomorphism $\alpha \colon R \to V(R)$ such that $\epsilon_V \alpha = \text{id}_R$. A map of V-coalgebras is a ring homomorphism which commutes with α.

2.12. **Proposition.** *There is an equivalence of categories $\theta^p\text{Ring} \xrightarrow{\sim} V\text{Coalg}$, which sends the θ^p-ring (R, θ^p) to the V-coalgebra (R, α), where $\alpha(x) = (x, \theta^p(x))$.*

Proof. Straightforward. $\square$

2.13. *Remark.* One may also describe θ^p-rings as the coalgebras for a certain comonad W on Ring. The comonad W is in fact the *cofree comonad* on the augmented endofunctor (V, ϵ_V). It is well-known that the underlying functor $W \colon \text{Ring} \to \text{Ring}$ of this comonad is in fact the p-typical Witt functor.[3]

2.14. **ψ^p-rings as coalgebras.** Similarly as above, let $G(R) = R \times R$ as a ring, and define $\epsilon_G \colon G(R) \to R$ by $\epsilon_G(x, y) = x$. A G-**coalgebra** is a pair (R, β) with $\beta \colon R \to G(R)$ a ring homomorphism such that $\epsilon_G \beta = \text{id}_R$. Let $\pi \colon V(R) \to G(R)$ denote the map $\pi(x, y) = (x, x^p + py)$. It is clear that π is a ring homomorphism natural in R, and that $\epsilon_G \pi = \epsilon_V$.

2.15. **Proposition.** *There is an equivalence of categories $\psi^p\text{Ring} \xrightarrow{\sim} G\text{Coalg}$ which sends (R, ψ^p) to the G-coalgebra $(R, (\text{id}, \psi^p))$. With respect to this equivalence and that of (2.12), the forgetful functor $\theta^p\text{Ring} \to \psi^p\text{Ring}$ corresponds to the functor $V\text{Coalg} \to G\text{Coalg}$ which sends (R, α) to $(R, \pi\alpha)$.*

Proof. Immediate. $\square$

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[3]I believe this is an observation of Joyal [Joy85].

2.16. Limits and colimits of θ^p-rings.

2.17. Proposition. *The category $\theta^p\mathrm{Ring}$ of θ^p-rings has all small limits and colimits, and the forgetful functor $\theta^p\mathrm{Ring} \to \mathrm{Ring}$ which sends a θ^p-ring to its underlying commutative ring preserves limits and colimits.*

Proof. To prove the statement about limits, note that if $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$ is a functor from a small category, and $U\colon \theta^p\mathrm{Ring} \to \mathrm{Ring}$ denotes the underlying ring functor, we can define a operator θ^p on the set $\lim_{\mathcal{C}} UA$ componentwise, and check that it satisfies the axioms for a θ^p-ring. It is then straightforward to check that this realizes the limit of $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$.

To prove statement about colimits, let $A\colon \mathcal{C} \to \theta^p\mathrm{Ring}$ be a functor from a small category, and $U\colon \theta^p\mathrm{Ring} \to \mathrm{Ring}$ the underlying ring functor. Let $S = \mathrm{colim}_{\mathcal{C}} UA$, the colimit of the diagram in Ring, with $i(c)\colon UA(c) \to S$ denoting the tautological maps. Each θ^p-ring $A(c)$ corresponds to a ring homomorphism $\alpha(c)\colon UA(c) \to V(UA(c))$ which is a section $\pi\colon V(UA(c)) \to UA(c)$, and these maps fit together to give a ring homomorphism

$$\alpha\colon S = \mathrm{colim}_{\mathcal{C}} UA \xrightarrow{\mathrm{colim}_{\mathcal{C}} \alpha} \mathrm{colim}_{\mathcal{C}} V(UA) \xrightarrow{(V(i(c)))} V(S),$$

and that $\pi\alpha = \mathrm{id}$. Thus (S, α) defines a θ^p-ring structure on S, and it is straightforward to verify that S is the colimit of A. $\qquad\qquad\square$

3. Lifting problems for θ^p-rings

3.1. Lifting an ψ^p-ring structure to a θ^p-ring structure. Given a ψ^p-ring (R, ψ^p), a θ^p-**ring structure** on it is a θ^p-ring structure on R such that $\psi^p(x) = x^p + p\theta^p(x)$.

We now consider the following problem: given a ψ^p-ring (R, ψ^p), what are the possible θ^p-ring structures on it? In view of (2.12) and (2.15), we see that a θ^p-ring structure on (R, ψ^p) corresponds exactly to a homomorphism α fitting in the diagram

$$
\begin{array}{ccc}
 & & V(R) \\
 & \alpha \nearrow & \downarrow \pi \\
R & \xrightarrow{(\mathrm{id}, \psi^p)} & R \times R
\end{array}
$$

Thus, to understand this lifting problem we must examine the homomorphism π.

Let $\overline{V}(R) = \pi(V(R)) \subseteq R \times R$ denote the image of π. It is a subring of $R \times R$, described as the subset $\{\, (x, y) \mid x^p \equiv y \mod p \,\}$. Let $I(R) = \mathrm{Ker}(\pi) = \mathrm{Ker}(V(R) \twoheadrightarrow \overline{V}(R)) \subseteq V(R)$.

3.2. Proposition. *Let R be a commutative ring, and let $\overline{V}(R)$ and $I(R)$ be defined as above.*

(1) *As an abelian group, the quotient group $(R \times R)/\overline{V}(R)$ is isomorphic to R/pR.*
(2) *As an abelian group, $I(R)$ is isomorphic to $R[p] = \mathrm{Ker}[p\colon R \to R]$, the additive group of p-torsion elements in R.*
(3) *The ideal $I(R) \subseteq V(R)$ is square-zero, and thus*

$$0 \to I(R) \to V(R) \twoheadrightarrow \overline{V}(R) \subseteq R \times R$$

presents $V(R)$ as a square-zero extension of $\overline{V}(R)$ by $I(R)$.
(4) *Let $\phi\colon \overline{V}(R) \to R/pR$ be the ring homomorphism defined by $\phi(x, y) = x^p = y$. As a $\overline{V}(R)$-module, $I(R)$ is isomorphic to $\phi^*(R[p])$, the module obtained by restricting scalars along ϕ from the evident R/pR-module structure on $R[p]$.*

Proof. The ring $\overline{V}(R)$ is isomorphic to the limit of the diagram

$$R \xrightarrow{x \mapsto x^p} R/pR \xleftarrow{y \leftarrow y} R,$$

and (1) follows immediately.

Note that as a set, $I(R) = \{\, (0, z) \mid pz = 0 \,\} \subseteq V(R)$. Statements (2) and (3) are immediate from the description of the ring structure on $V(R)$.

For statement (4), let $(x, y) \in \overline{V}(R)$, and choose any lift $(x, u) \in V(R)$, so that $pu = y - x^p$. Then for $(0, z) \in I(R)$, we have

$$(x, u) \cdot (0, z) = (0, x^p z + puz) = (0, x^p z) = (0, yz).$$

This verifies the claim about the module structure on $I(R)$. $\qquad\square$

Thus, to construct a θ^p-ring structure on (R, ψ^p), we must (1) show that the image of $(\mathrm{id}, \psi^p)\colon R \to R \times R$ lies in $\overline{V}(R)$, and (2) lift the resulting map $R \to \overline{V}(R)$ to a homomorphism $\alpha\colon R \to V(R)$. Step (1) exactly says that ψ^p must satisfy the Frobenious condition.

3.3. *Remark.* If R has no p-torsion, then $I(R) = 0$, and thus $V(R) \to \overline{V}(R)$ is an isomorphism. Thus we recover the congruence criterion (2.10) for torsion free θ^p-rings.

3.4. θ^p**-ring structures and p-localization.** Fix a ψ^p-ring (R, ψ^p). Tensoring with $\mathbb{Z}_{(p)}$ gives rise to a ψ^p-ring $(R_{(p)}, \psi^p)$.

3.5. **Proposition.** *There is a one-to-one correspondence*

$$\left\{ \theta^p\text{-ring structures on } (R, \psi^p) \right\} \longleftrightarrow \left\{ \theta^p\text{-ring structures on } (R_{(p)}, \psi^p) \right\}$$

Proof. From (3.2), we have natural exact sequences of abelian groups

$$0 \to R[p] \to V(R) \xrightarrow{\pi} R \times R \to R/pR \to 0.$$

When we plug in the homomorphism $j\colon R \to R_{(p)}$, and observe that $R[p] \xrightarrow{\sim} R_{(p)}[p]$ and $R/pR \xrightarrow{\sim} R_{(p)}/pR_{(p)}$, we obtain a pullback square of rings of the form

$$
\begin{array}{ccc}
V(R) & \xrightarrow{V(j)} & V(R_{(p)}) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
R \times R & \xrightarrow[j \times j]{} & R_{(p)} \times R_{(p)}
\end{array}
$$

Also, $V(R_{(p)})$ is a $\mathbb{Z}_{(p)}$-module, as it fits in a finite exact sequence whose other terms are $\mathbb{Z}_{(p)}$-modules.

To define the correspondence asserted by the proposition, we observe that there is a bijection

$$\{\, \alpha\colon R \to V(R) \mid \pi\alpha = (\mathrm{id}, \psi^p) \,\} \longleftrightarrow \{\, \alpha'\colon R_{(p)} \to V(R_{(p)}) \mid \pi\alpha' = (\mathrm{id}, \psi^p) \,\},$$

sending α to the unique homomorphism α' such that $\alpha'j = V(j)\alpha$. $\qquad\square$

Thus, the problem of lifting a ψ^p-structure to a θ^p-structure is (unsurprisingly) a purely p-local problem.

3.6. **The relative lifting problem for θ^p-rings.** Now we consider the following problem. Suppose we are given a θ^p-ring (A, θ^p) and a ψ^p-ring (B, ψ^p) which satisfies the Frobenius condition, together with homomophism $f\colon A \to B$ of ψ^p-rings (using the underlying ψ^p-structure of the (A, θ^p)). What are the possible θ^p-ring structures on (B, ψ^p) making f a map of θ^p-rings? In view of the previous sections, we see that providing such a structure amounts to producing a dotted arrow α_B in

$$
\begin{array}{ccccc}
A & \xrightarrow{V(f)\alpha_A} & V(B) & & \\
{\scriptstyle f}\downarrow & {\scriptstyle \alpha_B}\nearrow\!\!\cdots & \downarrow{\scriptstyle \pi} & & \\
B & \xrightarrow[(\mathrm{id}, \psi^p)]{} & \overline{V}(B) & \subseteq & B \times B
\end{array}
$$

making the diagram commute.

Recall that $f\colon A \to B$ is **formally etale** if for every ring R and square-zero ideal $I \subseteq R$, and every commutative diagram of ring homomorphisms of the form

$$
\begin{array}{ccc}
A & \longrightarrow & R \\
f\downarrow & \nearrow & \downarrow \\
B & \longrightarrow & R/I
\end{array}
$$

there exists a unique dotted arrow making the diagram commute.

3.7. Proposition. *If (A, θ^p) is a θ^p-ring, (B, ψ^p) a ψ^p-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of ψ^p-rings which is formally etale as a map of commutative rings, then there exists a unique θ^p-ring structure on (B, ψ^p) making f a map of θ^p-rings.*

Proof. Immediate in view of the above remarks and (3.2)(3). $\qquad\square$

The lift of the previous proposition is natural.

3.8. Proposition. *Consider a commutative diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\;g\;} & C \\
f\downarrow & \nearrow & \\
B & \raisebox{1ex}{h} &
\end{array}
$$

of ring maps, such that (i) A, B, and C are θ^p-rings, (ii) f and g are maps of θ^p-rings, (iii) h is a map of ψ^p-rings, and (iv) f is formally etale. Then h is a map of θ^p-rings.

Proof. Consider the diagrams

The solid arrow diagrams commute, using that f and g are maps of θ^p-rings. The two outer rectangles are actually identical, since

$$
V(g)\alpha_A = V(h)V(f)\alpha_A \qquad \text{and} \qquad (\mathrm{id}, \psi_C^p)h = (h, \psi_C^p h) = (h, h\psi_B^p) = (h \times h)(\mathrm{id}, \psi_B^p).
$$

Thus the homomorphisms $\alpha_C h, V(h)\alpha_B \colon B \to V(C)$ must coincide, since f is formally etale and they are solutions to the same lifting problem, whence h is a map of θ^p-rings as desired. $\qquad\square$

Given a θ^p-ring A, let $\theta^p\mathrm{Ring}(A)$ denote the category of θ^p-rings under A, and $\psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)$ the category of ψ^p-rings under A which satisfy the Frobenius congruence. Let $\theta^p\mathrm{Ring}(A)_{\mathrm{f.etale}}$ and $\psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ denote the respective full subcategories consisting of objects $f\colon A \to B$ such that f is formally etale.

3.9. Proposition. *The evident forgetful functor $\theta^p\mathrm{Ring}(A)_{\mathrm{f.etale}} \to \psi^p\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ is an equivalence of categories.*

Proof. Clear from (3.7) and (3.8). $\qquad\square$

The statement of the previous proposition remains true if we replace "formally etale" with any subclass of maps, such as "etale" or "weakly etale".

3.10. Remark. In fact, we can generalize the above a little bit. Say that $f\colon A \to B$ is p-**formally etale** if $f \otimes \mathbb{Z}_{(p)}\colon A_{(p)} \to B_{(p)}$. In view of (3.5), the propositions (3.7), (3.8), and (3.9) apply with "formally etale" replaced by "p-formally etale".

4. λ-RINGS

A λ**-ring** is a commutative ring R equipped with functions $\lambda^n \colon R \to R$ for $n \geq 0$, satisfying identities of the form

- $\lambda^0(x) = 1$ and $\lambda^1(x) = x$,
- $\lambda^n(x + y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y)$,
- $\lambda^n(xy) = P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y))$,
- $\lambda^m \lambda^n(x) = P_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x))$,

where P_n and $P_{m,n}$ are certain polynomials with integer coefficients. We refer the reader elsewhere for a complete definition, for instance [Yau10]. We write λRing for the category of λ-rings.

The purpose of this section is to give a characterization of λ-rings in terms of θ^p-rings. That is, we will show that a λ-ring is nothing more than a commutative ring R equipped with θ^p-ring structures for each prime p, which are compatible in the sense that

$$\psi^p \theta^q = \theta^q \psi^p \qquad \text{for all distinct primes } p, q.$$

4.1. *Remark.* It is known to those who know these things that data of a λ-ring structure on R is equivalent to: a θ^p-structure on R for each prime, together with for each pair of distinct primes p, q a somewhat non-trivial relation relating $\theta^p \theta^q(x)$ and $\theta^q \theta^p(x)$ up to terms which do not involve compositions of θ-operations. (Probably this is in [Joy85]; see also [Bor11, §1.19].) Our characterization allows us to avoid explicit mention of this relation.

4.2. **Facts about lambda rings.** We note the following facts about λ-rings.

(1) λ-rings are a variety of universal algebra, and thus λRing is a locally presentable category. In particular, any functor $U \colon \lambda\text{Ring} \to \mathcal{C}$ which preserves small colimits admits a right adjoint.
(2) Limits and colimits in λRing exist, and the evident forgetful functor $\lambda\text{Ring} \to \text{Ring}$ preserves limits and colimits.
(3) The free λ-ring on one generator F has the form $F \approx \mathbb{Z}[\lambda^n(x) \,|\, n \geq 1]$, where $x = \lambda^1(x)$ is the generator.
(4) Any λ-ring R has natural Adams operations $\psi^n \colon R \to R$ for $n \geq 1$, which are ring homomorphisms; furthermore, $\psi^m \psi^n = \psi^{mn}$ and $\psi^1 = \text{id}$.
(5) For p prime, $\psi^p(x) \equiv x^p \mod pR$ in any λ-ring R.
(6) For every prime p there exist natural functions $\theta^p \colon R \to R$ on any λ-ring such that $\psi^p(x) = x^p + p\theta^p(x)$, and (R, θ^p) is in fact a θ^p-ring, as can be shown by checking the appropriate formulas in the free λ-ring on one generator F, which is torsion free.
(7) For distinct primes p and q, we have that $\psi^p \theta^q = \theta^q \psi^p$ as functions on any λ-ring, as can be shown by verifying that $\psi^p \theta^q(x) = \theta^q \psi^p(x)$ in F.
(8) Let $\delta \colon F \to \mathbb{Z}$ be the λ-ring homomorphism from the free λ-ring on one generator sending the generator x to 0, and let $J = \text{Ker}\,\delta$. Then J/J^2 is a free abelian group on $\{\lambda^n(x)\}_{n \geq 1}$.
(9) We have that $\psi^n(x) \equiv (-1)^{n-1} n \lambda^n(x) \mod J^2$. Thus, any sequence $\{u_n\}_{n \geq 0}$ of elements in J such that $nu_n \equiv \psi^n(x) \mod J^2$ is a basis for J/J^2.

4.3. Θ**-rings.** A Θ**-ring** is the data $(R, \{\theta^p\})$ consisting of a commutative ring R and a choice for each prime $p \in \mathbb{Z}$ of a θ^p-structure on R, such that for all distinct primes p and q, we have that

$$\psi^p \theta^q = \theta^q \psi^p,$$

where $\psi^p(x) = x^p + p\theta^p(x)$ is the Adams operation associated to θ^p. We note that it is also the case that $\psi^p \theta^p = \theta^p \psi^p$, as this is true in any θ^p-ring.

A morphism $A \to B$ of Θ-rings is a map which commutes with all the structure, i.e., a ring homomorphism $f \colon A \to B$ such that $f\theta^p = \theta^p f$ for all p. We write ΘRing for the category of Θ-rings.

An ideal $I \subseteq R$ of a Θ-ring is a θ^p-**ideal** if it is a θ^p-ideal for all p. It is clear that if I is a Θ-ideal, then R/I admits a unique Θ-ring structure as a quotient of the structure on R.

4.4. **Facts about subrings of Θ-rings.** We collect some facts for use in the proof in the next section.

Let R be a Θ-ring, and consider an ordinary subring $S \subseteq R$. Write $\Theta S \subseteq R$ for the ordinary subring generated by the set $S \cup \bigcup_p \theta^p(S)$. It is straightforward to show (see §2.2) that if S is generated as a subring by a subset $X \subseteq R$, then ΘS is generated as a subring by the subset $X \cup \{ \theta^p(x) \mid x \in S, \ p \text{ prime} \}$.

It is clear from (2.3) that if $S \subseteq R$ is a subring, then $\bigcup_k \Theta^k S$ is the Θ-subring in R generated by S (i.e., the smallest subring of R containing S and closed under the θ^p operations).

Although a subring is not generally an ideal, it is a subgroup, and so it makes sense to talk about congruence modulo a subring: we say $x \equiv y \in S$ if $x - y \in S$, when $S \subseteq R$ is a subring and $x, y \in R$.

4.5. **Proposition.** *Let R be a Θ-ring, $S \subseteq R$ an ordinary subring, and $x \in S$. Then for all primes p and q, we have that*
$$\theta^p \theta^q(x) \equiv \theta^q \theta^p(x) \mod \Theta S.$$

Proof. If $p = q$ this is obvious. For distinct primes p and q we have
$$\psi^p \theta^q(x) = \theta^q(x)^p + p\, \theta^p \theta^q(x) \equiv p\, \theta^p \theta^q(x) \mod \Theta S,$$

and
$$\theta^q \psi^p(x) = \theta^q(x^p + p\theta^p(x)) = \theta^q(x^p) + \left(p\theta^q \theta^p(x) - F_q(p)\theta^p(x)^q\right) - C_p(x^p, p\theta^p(x))$$
$$\equiv p\theta^q \theta^p(x) \mod \Theta S.$$

Therefore $p(\theta^p \theta^q(x) - \theta^q \theta^p(x)) \in \Theta S$. By symmetry we also have $q(\theta^p \theta^q(x) - \theta^q \theta^p(x)) \in \Theta S$, and since p and q are relatively prime, a suitable integer combination of these congruences gives $\theta^p \theta^q(x) - \theta^q \theta^p(x) \in \Theta S$, as desired. $\qquad\square$

4.6. *Remark.* The argument of (4.5) actually shows that
$$\theta^p \theta^q(x) - \theta^q \theta^p(x) = f(x, \theta^p(x), \theta^q(x))$$

where f is some polynomial with integer coefficients. It is not hard to describe this polynomial explicitly, e.g., [BS09, Def. 2.2] or [Bor11, (1.19.4)].

Let $\mathcal{P}$ denote the set of finite sequences $P = (p_1, \dots, p_k)$ of primes $p_i \in \mathbb{Z}$, including the empty sequence, and write $|P| = k$ for the length of the sequence P. For $x \in R$ and $P \in \mathcal{P}$ write $\theta^P(x) = \theta^{p_1} \cdots \theta^{p_k}(x)$.

4.7. **Proposition.** *Let R be a Θ-ring and $S \subseteq R$ ordinary subring. If $x, y \in \Theta S$ are such that $x \equiv y \mod S$, then $\theta^P(x) \equiv \theta^P(y) \mod \Theta^{|P|} S$ for all $P \in \mathcal{P}$.*

Proof. If $|P| = 0$, there is nothing to prove. If $P = (p)$, then if $x = y + a$ with $a \in S$, we have that $\theta^p(x) - \theta^p(y) = \theta^p(a) - C_p(y, a) \in \Theta S$.

The case of $|P| > 1$ is handled by induction on the length: given $\theta^P(x) \equiv \theta^P(y) \mod \Theta^{|P|} S$ with $\theta^P(x), \theta^P(y) \in \Theta^{|P|+1} S$, it follows using the length-one case already proved that $\theta^q \theta^P(x) \equiv \theta^q \theta^P(y) \mod \Theta^{|P|+1} S$ and $\theta^q \theta^P(x), \theta^q \theta^P(x) \in \Theta^{|P|+2} S$. $\qquad\square$

4.8. **Proposition.** *Let R be a Θ-ring, $S \subseteq R$ an ordinary subring, and $x \in S$. Let $P, Q \in \mathcal{P}$ be two sequences of the same length, where Q is obtained from P by reordering its elements. Then*
$$\theta^P(x) \equiv \theta^Q(x) \mod \Theta^{|P|-1} S.$$

Proof. It suffices to consider the case of pairs of sequences obtained by reordering an adjacent pair of elements. Thus, let p, q be primes, $U, V \in \mathcal{P}$ and let $P = (U, p, q, V)$ and $Q = (U, q, p, V)$. Then $\theta^V(x) \in \Theta^{|V|}S$, whence

$$\theta^p \theta^q \theta^V(x) \equiv \theta^q \theta^p \theta^V(x) \mod \Theta^{|V|+1}S$$

by (4.5). As $\theta^p \theta^q \theta^V(x), \theta^q \theta^p \theta^V(x) \in \Theta^{|V|+2}S$, it follows that

$$\Theta^U \theta^p \theta^q \Theta^V(x) \equiv \theta^U \theta^q \theta^p \theta^V(x) \mod \Theta^{|U|+1+|V|}S = \Theta^{|P|-1}S$$

by (4.7). $\qquad \square$

4.9. Equivalence of Θ-rings and λ-rings. There is an evident "forgetful" functor $U \colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$, by §4.2(6), (7).

4.10. Proposition. *The forgetful functor $U \colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$ is an equivalence of categories.*

Proof. Let R be a Θ-ring.

By (2.3), it is clear that for any $x \in R$, the subring $S \subseteq R$ generated as a ring by the set $\{\theta^P(x)\}_{P \in \mathcal{P}}$ is a sub-Θ-ring. We may thus construct the free Θ-ring on one generator E as the quotient of the polynomial ring $\mathbb{Z}[\theta^P(x) \mid P \in \mathcal{P}]$ by the ideal consisting of relations which hold in every Θ-ring.

Let $\mathcal{P}_{\leq} \subseteq \mathcal{P}$ denote the subset consisting of sequences $P = (p_1, \ldots, p_k)$ such that $p_1 \leq \cdots \leq p_k$. By inductive application of (4.8), we see that E is in fact generated as a commutative ring by the set $\{\theta^P(x)\}_{P \in \mathcal{P}_{\leq}}$. That is, if we write S_k for the subring of E generated by $\{\theta^P(x) \mid P \in \mathcal{P}_{\leq}, |P| \leq k\}$, we show by induction on k that $\Theta^k S_0 \subseteq S_k$, using (4.8). Thus $E = \bigcup_k \Theta^k S_0 = \bigcup_k S_k$.

Let $\delta \colon E \to \mathbb{Z}$ be the Θ-homomorphism sending $x \mapsto 0$, and let $I = \mathrm{Ker}\, \delta$. Clearly I is a Θ-ideal, and thus I^2 is a Θ-ideal by (2.6). Furthermore, as an ideal I is generated by the set $\{\theta^P(x)\}_{P \in \mathcal{P}_{\leq}}$, since these elements generate E as a ring and are contained in I. Therefore I/I^2 is generated as an abelian group by the $\theta^P(x)$ with $P \in \mathcal{P}_{\leq}$. Note also that $p\theta^p(x) \equiv \psi^p(x) \mod I^2$, and thus that $n\theta^P(x) \equiv \psi^n(x) \mod I^2$ where $P = (p_1, \ldots, p_k) \in \mathcal{P}$ is such that $n = \prod p_i$.

Now let F denote the free λ-ring on one generator, with augmentation ideal J, and let $\phi \colon E \to U(F)$ be the Θ-ring map sending generator to generator. By §4.2(3), (9), the induced map

$$I/I^2 \to J/J^2 \approx \mathbb{Z}\{\text{``}n^{-1}\psi^n(x)\text{''}\}_{n \geq 1}$$

the generators $\theta^P(x)$ of I/I^2 with $P \in \mathcal{P}_{\leq}$ bijectively to a basis of J/J^2, whence $I/I^2 \to J/J^2$ is a bijection, whence ϕ is an isomorphism.

The forgetful functor $U \colon \lambda\mathrm{Ring} \to \Theta\mathrm{Ring}$ preserves small colimits (since in both $\lambda\mathrm{Ring}$ and $\Theta\mathrm{Ring}$ colimits are computed as in commutative rings, by §4.2(2) and (2.17) respectively), and thus U admits a right adjoint $G \colon \Theta\mathrm{Ring} \to \lambda\mathrm{Ring}$ by §4.2(1).

Now let $\eta \colon \mathrm{Id} \to GU$ and $\epsilon \colon UG \to \mathrm{Id}$ be unit and counit of the $U \dashv G$ adjunction. For a λ-ring R, the composite

$$R \approx \lambda\mathrm{Ring}(F, R) \xrightarrow{U} \Theta\mathrm{Ring}(UF, UR) \xrightarrow{\phi^*} \Theta\mathrm{Ring}(E, UR) \approx R$$

is the identity function on the set R, and thus $U \colon \lambda\mathrm{Ring}(F, R) \to \Theta\mathrm{Ring}(UF, UR)$ is a bijection for every λ-ring R. The composite map

(4.11) $$R \approx \lambda\mathrm{Ring}(F, R) \xrightarrow{U} \Theta\mathrm{Ring}(UF, UR) \approx \lambda\mathrm{Ring}(F, GUR) \approx GUR$$

is therefore a bijection, whence $\eta \colon \mathrm{Id} \to GU$ is an isomorphism. For a Θ-ring S, the composite map

(4.12) $$S \approx \Theta\mathrm{Ring}(E, S) \xrightarrow{(\phi^*)^{-1}} \Theta\mathrm{Ring}(UF, S) \approx \lambda\mathrm{Ring}(F, GS) \xrightarrow[\sim]{\phi^* \circ U} \Theta\mathrm{Ring}(E, UGS) \approx UGS$$

is a bijection of sets (using that (4.11) is always an isomorphism), and it is straightforward to check that the composite of (4.12) is an inverse to $\epsilon \colon UGS \to S$. Thus we have shown that $U \dashv G$ is an equivalence of categories. $\qquad \square$

5. Lifting problems for λ-rings

In view of (4.10), we will treat λ-rings as synonymous with Θ-rings.

5.1. **Ψ-rings.** A **Ψ-ring** is the data $(R, \{\psi^p\})$ consisting of a commutative ring R and a choice for each prime $p \in \mathbb{Z}$ of a ψ^p-ring structure on R, such that for all distinct primes p and q we have that $\psi^p \psi^q = \psi^q \psi^p$. We write ΨRing for the category of Ψ-rings.

We say that a Ψ-ring satisfies the **Frobenius condition** if each ψ^p does, i.e., if $\psi^p(x) \equiv x^p$ mod p for all p. We write ΨRing$_{\mathrm{Fr}} \subseteq \Psi$Ring for the full subcategory of Ψ-rings satisfying the Frobenius condition. There is an evident forgetful functor

$$\lambda\mathrm{Ring} \to \Psi\mathrm{Ring}_{\mathrm{Fr}} \subseteq \Psi\mathrm{Ring}.$$

We observe that if $F\colon \mathrm{Ring} \to \mathrm{Ring}$ is any functor and R is a Ψ-ring, then $F(R)$ inherits a natural Ψ-ring structure, with operations $\psi^p_{F(R)} = F(\psi^p_R)$. For instance, this applies to the functors $V_p\colon \mathrm{Ring} \to \mathrm{Ring}$ of §2.11.

5.2. **Lifting a Ψ-ring structure to a λ-ring structure.** We now consider the following problem: given a Ψ-ring R, what are the possible λ-ring structures on R compatible with the given Ψ-ring structure? In view of (4.10), (2.12), and (2.15), such a λ-ring structure on R corresponds exactly to a choice for each prime p of Ψ-ring homomorphism α_p fitting in the commutative diagram

$$
\begin{array}{ccc}
 & & V_p(R) \\
 & \overset{\alpha_p}{\nearrow} & \downarrow \pi \\
R & \xrightarrow[(\mathrm{id}, \psi^p)]{} & R \times R
\end{array}
$$

That is, a Ψ-ring map $\alpha_p\colon R \to V_p(R)$ such that $\pi\alpha_p = (\mathrm{id}, \psi^p)$ corresponds exactly to a θ^p-ring structure on (R, ψ^p) such that $\psi^q \theta^p = \theta^p \psi^q$ for all primes $q \neq p$, by the discussion in §3.1. (Recall that for any θ^p-ring structure on (R, ψ^p) we automatically have that $\psi^p \theta^p = \theta^p \psi^p$.) A choice of such α_p for all primes p thus amounts to a λ-ring structure on R by (4.10).

Clearly, a necessary condition to lift Ψ-ring structure on R to a λ-ring structure is that the Ψ-ring structure should satisfy the Frobenius condition. As a corollary of (3.2)(2) we obtain Wilkerson's criterion: if R is torsion free as an abelian group, then any Ψ-ring structure on R satisfying the Frobenius condition lifts uniquely to a λ-ring structure.

5.3. **The relative lifting problem for λ-rings.** Now consider the following problem. Suppose we are given a λ-ring A, a Ψ-ring B which satisfies the Frobenius condition, and a homomorphism $f\colon A \to B$ of Ψ-rings. What are the possible λ-ring structures on B making f a map of λ-rings?

5.4. **Proposition.** *If A is a λ-ring, B a Ψ-ring satisfying the Frobenius condition, and $f\colon A \to B$ is a map of Ψ-rings which is formally etale as a map of commutative rings, then there exists a unique λ-ring structure on B making f a map of λ-rings.*

Proof. In view of (3.7), there are unique θ^p-ring structures on B making f a map of θ^p-rings for each p. In view of (3.8), these θ^p-structures on B must commute with the given ψ^q-operations on B for $q \neq p$. That is, apply (3.8) for θ^p-rings to the commutative triangle of ring maps

$$
\begin{array}{ccc}
A & \xrightarrow{f\psi^q_A} & B \\
\downarrow{\scriptstyle f} & \nearrow{\scriptstyle \psi^q_B} & \\
B & &
\end{array}
$$

where f and thus $f\psi^q_A$ are θ^p-ring maps, to show that $\psi^q_B\colon B \to B$ is a map of θ^p-rings. $\qquad\square$

5.5. Proposition. *Consider a commutative diagram*

$$
\begin{array}{ccc}
A & \xrightarrow{\ g\ } & C \\
{\scriptstyle f}\downarrow & \nearrow{\scriptstyle h} & \\
B & &
\end{array}
$$

of commutative ring maps, such that (i) A, B, and C are λ-rings, (ii) f and g are maps of λ-rings, (iii) h is a map of Ψ-rings, and (iv) f is formally etale. Then h is a map of λ-rings.

Proof. Apply (3.8) to show that h is a map of θ^p-rings for all p. $\qquad\square$

Given a λ-ring A, let $\lambda\mathrm{Ring}(A)$ denote the category of λ-rings under A, and $\Psi\mathrm{Ring}_{\mathrm{Fr}}(A)$ the category of ψ^p-rings under A which satisfy the Frobenius congruence. Let $\lambda\mathrm{Ring}(A)_{\mathrm{f.etale}}$ and $\Psi\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ denote the respective full subcategories consisting of objects $f\colon A \to B$ such that f is formally etale.

5.6. Proposition. *The evident forgetful functor $\lambda\mathrm{Ring}(A)_{\mathrm{f.etale}} \to \Psi\mathrm{Ring}_{\mathrm{Fr}}(A)_{\mathrm{f.etale}}$ is an equivalence of categories.*

The statement of the previous proposition remains true if we replace "formally etale" with any subclass of maps, such as "etale" or "weakly etale".

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