ON THE $K(2)$-LOCAL UNSTABLE HOMOTOPY GROUPS OF S^3 AT $p \geq 5$

GUOZHEN

In this paper, we will compute the homotopy groups of $\Phi_{K(2)}S^3$, where $\Phi_{K(2)}$ is the Bousfield-Kuhn functor. As a corollary, we find that the $K(2)$-local unstable homotopy groups of spheres have finite v_1-exponent.

In this paper, we will assume that p is a prime at least 5, and E_2 will denote the Morava E-theory, with $E_{2*} = \mathbb{Z}_p[[v_1]][v_2^{\pm}]$. (Note that we do not introduce the inessential (p^2-1)-st root of v_2.) Everything will be assumed $K(2)$-local, and we will apply $K(2)$-localization implicitly whenever necessary.

We will use the Goodwillie tower (of the identity functor) to do the computations. Recall that for any sphere S^k, $K(2)$-locally we have the finite Goodwillie tower $L(0)_k \to L(1)_k \to L(2)_k$. Since $K(2)$-locally S^1 is trivial, we can quotient out the Goodwillie tower of S^3 by that of S^1, and conclude that $\Phi_{K(2)}S^3$ is the fiber of $L(1)_1^3 \to L(2)_1^3$, for $L(n)_i^j$ the fiber of the suspension map $L(n)_i \to L(n)_j$.

1. Comodules of Hopf algebroids

Let X be a scheme. A groupoid over X is a scheme G, together with flat maps $s, t : G \to X$, and a multiplication $G_t \times_{X\,s} G \to X$, satisfying the usual axioms of a groupoid. When X and G are affine this is the dual notion of a Hopf algebroid.

A G-sheaf on X is a quasi-coherent sheaf M on X together with a morphism $s^*M \to t^*M$ of quasi-coherent sheaves on G, satisfying transitivity axioms. This is the same as a comodule on Hopf algebroids in the affine case.

Let $f : Y \to X$ be an étale map. Then a groupoid G on X pulls back to a groupoid $G_Y = Y \times_X G \times_X Y$ on Y. In this case, a G-sheaf on X is the same as a G_Y-sheaf on Y. Note that the descent data is automatically contained in G_Y.

Now suppose we have an algebraic group H acting on a scheme Y. Then we can construct a groupoid $H \times Y$ over Y, with the source and target map being the projection and the action respectively. We would say the groupoid splits in this case. In the split case, the notion of an $H \times Y$ sheaf is the same as an H-equivariant sheaf on Y in the usual sense.

Moreover, if K is a finite subgroup of H acting freely on Y so that $Y \to X = Y/K$ is étale, then we can construct a groupoid $K\backslash H \times Y/K$ over X, which pulls back to $H \times Y$ on Y.

Now let $\widehat{E(n)}$ denote the completed Johnson-Wilson theory, with $\widehat{E(n)}_* = \mathbb{Z}_p[[v_1, \dots, v_{n-1}]][v_n^{\pm}]$. Then

$$\widehat{E(n)}_*\widehat{E(n)} = \widehat{E(n)}_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^{p^i} t_i + \dots)$$

Let E_n be the Morave E-theory, so that $E_{n*} = W(\mathbb{F}_{p^n})[[u_1, \dots, u_{n-1}]][u^{\pm}]$. Then the Morava stabilizer group G_n acts on E_n. Moreover, E_{n*} is an étale extension of $\widehat{E(n)}_*$ with Galois group $\mathbb{F}_{p^n}^\times \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p) \subset G_n$. The split groupoid constructed using this action is dual to the Hopf algebroid $E_{n*}E_n$, which is also the pullback of $\widehat{E(n)}_*\widehat{E(n)}$.

The correspondence goes as follows. Over E_{n*}, the t_i's can be solved out as a power series $t_i = a_i u^{p^i - 1} + \dots$ such that $a_i \in \mathbb{F}_{p^n} \subset W(\mathbb{F}_{p^n})$ and $a_0 \in \mathbb{F}_{p^n}^\times \subset W(\mathbb{F}_{p^n})$. By choosing different choices of the a_i together with an element in the Galois group of $\mathbb{F}_{p^n}$ we get all the points of G_n. Then the formula for the coaction of an $E_{n*}E_n$ comodule corresponds to the action of G_n by substituting the values of t_i in the formula for coaction.

In summary, the data for an E_n-module with compatible G_n-action is the same as that for an $\widehat{E(n)}_*\widehat{E(n)}$ comodule.

2. Some $\widehat{E(2)} * \widehat{E(2)}$ comodules

In this section we will construct several $\widehat{E(2)} * \widehat{E(2)}$ comodules.

First we have the E_{2*} algebra $E_2^*\mathbb{CP}^\infty$ with compatible action of G_2. As an E_{2*} module it is $E_{2*}[t]$.

Next we have the algebra $E_{2*}[t]/[p](t)$ as $E_2^* B\mathbb{Z}/p$. By taking the fixed points of the $\mathbb{F}_p^\times$ action, we have the algebra $E_{2*}[x]/xq(x)$ as $E_2^* B\Sigma_p$, where $q(x)$ satisfies $[p](t) = tq(t^{p-1})$. Now the Morava stabilizer group G_n acts on these. In particular, $(xq(x))$ is an invariant ideal of G_2. We know (x) is also an invariant ideal of G_2. Since $E_{2*}[x]$ is a UFD, we conclude $(q(x))$ is also an invariant ideal, so that we can form the algebra $E_{2*}[x]/q(x)$ with compatible G_2 action.

Now we mod out by p. Then $[p](t) \equiv v_1 t^p +_F v_2 t^{p^2} \mod p$. So there is some $\bar{q}(x)$ such that $[p](t) \equiv t^p \bar{q}(t^{p-1}) \mod p$. So the same argument shows that we can construct an algebra $E_{2*}/p[x]/\bar{q}(x)$ with G_2 action.

When we assume p to be odd, we have $[-x] = -x$, so the ideal $(\bar{q}(x))$ is the same as $(v_1 + v_2 x^p)$. So we conclude the algebra $E_{2*}/p[x]/\bar{q}(x)$ is the same as $E_{2*}/p[(\frac{v_2}{v_1})^{\frac{1}{p}}]$.

The last algebra is over $\mathbb{F}_p$, so adding a pth root is purely inseparable, so the action of G_2 extends uniquely. To get the formula for the action, suppose $(t_1, t_2, \dots)$ is a certain set of solutions of the equation for $E_{2*}E_2$ representing an element of $S_2 \subset G_2$. Then it acts trivially on v_1 mod

p, and sends v_2 to $\eta_R(v_2) = v_2 + v_1 t_1^p - v_1^p t_1$. Hence this element sends $v_2^{\frac{1}{p}}$ to $v_2^{\frac{1}{p}} + v_1^{\frac{1}{p}} t_1 - v_1 t_1^{\frac{1}{p}}$. Here $t_1^{\frac{1}{p}}$ is literally the pth root of its value. By the equation $v_2 t_1^{p^2} - v_2^p t_1 + \dots$ we can transform $t_1^{\frac{1}{p}}$ into an expression with only integral powers of t_1. This in turn gives the formula for the coaction of $\widehat{E(2)}_* \widehat{E(2)}$.

3. Homological computations

Let $R = E_{2*}[y]/q(y)$ with $q(x^{p-1}) = \frac{[p](x)}{x}$. Then $q(y)$ is an irreducible polynomial, and R is a E_{2*}-module, free of rank $p+1$. We have the trace map $tr : R \to E_{2*}$ for the extension $E_{2*} \to R$. We find that $tr(a)$ is divisible by p if $a \in yR$.

Recall that $E_2^* L(1)_{2k-1} = y^k R$. By [4], the Goodwillie differential on cohomology $E_2^* L(0) \leftarrow E_2^* L(1)$ is the trace map $\frac{tr(-)}{p}$. From [1], we know that $L(n+1)_{2k-1}^{2k+1}$ is the fiber of a certain map $L(n)_{2pk+1} \to L(n)_{2pk-1}$, which is a lift of the multiplication by p map. Moreover, this is compatible with Goodwillie differentials. Hence we conclude that, cohomologically, the map $L(1)_1^3 \to L(2)_1^3$ is essentially the mod p reduction of the trace map $\frac{tr(-)}{p}$.

To be more precise, let $\bar{q}(x^{p-1}) = \frac{[p](x)}{x^p}$ as a power series in E_{2*}/p. Since $[p](x) = px +_F v_1 x^p +_F v_2 x^{p^2}$, we conclude that, up to units, $\bar{q}(y)$ is essentially $v_1 + v_2 y^p$. Let $\bar{R} = E_{2*}/p[y]/\bar{q}(y)$. $y^{p+1}\bar{R}$ equals $y^{p+1}R/p$, and we have the mod p reduction of the trace map $\frac{tr(-)}{p} : y^{p+1}\bar{R} \to E_2^*/p$. This is the map on cohomology of the map $L(1)_1^3 \to L(2)_1^3$. Hence the cohomology of $\Phi_{K(2)} S^3$ is the kernel of this trace map.

To understand the homology, we will consider the duals. First observe that $\frac{tr(y^s)}{p}$ lies in (p, v_1) for $1 \leq s \leq p$, and $\frac{tr(y^{p+1})}{p}$ is a unit. Thus the pairing $< a, b >= \frac{tr(ab)}{p}$ defines a perfect pairing between $y^{-k}R$ and $y^{k+1}R$. Modulo p, we find that we have a perfect pairing between $y^{-k+1}\bar{R}$ and $y^{k+1}\bar{R}$. Thus the homology of $L(2)_1^3$ can be identified with $y^{-p+1}\bar{R}$, and we find that the dual of the trace map $\frac{tr(-)}{p}$ is the inclusion $E_{2*}/p \to \bar{R} \to y^{-p+1}\bar{R}$, and the homology of $\Phi_{K(2)} S^3$ is the cokernel.

Because modulo p, $v_2 y^p = -v_1$ is a permanent cycle, we can also identify $E_{2*}\Phi_{K(2)} S^3$ with the cokernel of the map $E_{2*}/p \xrightarrow{v_1} y\bar{R}$.

As a final remark, since $\bar{R}$ is an inseparable extension of E_{2*}, there is a unique extension of the action of the Morava stabilizer group, and all the map above are compatible with the action.

4. computations of the AHSS differentials

In this section, we will implicitly mod out by p everywhere.

There is a natural filtration on the homology of $\Phi_{K(2)} S^3$ defined by powers of y. To simplify the notations, we will alter the sign of y in this section, so we set $y=(\frac{v_1}{v_2})^{\frac{1}{p}}$. So there is an AHSS to compute its homotopy groups. As a $\mathbb{F}_2$-vector space, $E_{2*}\Phi_{K(2)}S^3$ has generators $(y^k)_s=v_2^s y^k$ for $k\geq 1$ and relations $y^{kp}=0$.

Since everything is killed by p, the element ζ in the cohomology of the Morava stabilizer group is a permanent cycle, and we will ignore this factor. So we are to compute $H^*(\mathbb{G}_2^1, E_{2*}\Phi_{K(2)}S^3)$.

Recall that $H^*(\mathbb{G}_2^1,\mathbb{F}_2)$ has a basis $1,h_0,h_1,g_0,g_1,h_0g_1=h_1g_0$. The representatives are $h_0=[t_1]$, $h_1=[t_1^p]$, $g_0=<h_0,h_0,h_1>=\frac{1}{2}[t_1^2|t_1^p]+[t_1|t_2]$, $g_1=<h_0,h_1,h_1>=[t_2|t_1^p]+\frac{1}{2}[t_1|t_1^{2p}]$.

So the E_1 term of the AHSS are the $(y^k)_s$ multiples of these generators. We will compute the differentials.

Lemma 1. $\eta_R(v_2^{\frac{1}{p}})=v_2^{\frac{1}{p}}+v_1^{\frac{1}{p}}t_1-v_1t_1^{\frac{1}{p}}$.

Proof. This follows from the formula $\eta_R(v_2)=v_2+v_1t_1^p-v_1^pt_1$. □

To make $t_1^{\frac{1}{p}}$ into an integral expression, note that modulo v_1, $t_i=t_i^{p^2}$, hence we can inductively transform the expression into one without fraction exponent on t_i's.

To trace the effects, we have the following formula:

Lemma 2. $\eta_R(v_3)=v_3-v_2^pt_1+v_2t_1^{p^2}+v_1t_2^p+v_1w_1(v_2,-v_1^pt_1,v_1t_1^p)-v_1^pt_1^{p^2+1}-v_1^{p^2}t_2+v_1^{p^2}t_1^{1+p}$, *where* $w_1(a,b,c)=-\frac{1}{p}((a+b+c)^p-(a^p+b^p+c^p))$.

Proof. See [2]. □

Lemma 3. *For* $1\leq k\leq p-1$, $\eta_R((y^k)_{1+s})=v_1^{\frac{k}{p}}(v_2^{\frac{1}{p}}+v_1^{\frac{1}{p}}t_1-v_1t_1^{\frac{1}{p}})^{p-k}(v_2+v_1t_1^p-v_1^pt_1)^s$.

Proof. This follows from $(y^k)_{1+s}=(v_1^{\frac{k}{p}}v_2^{\frac{p-k}{p}})v_2^s$. □

Lemma 4. *For* $1\leq k\leq p-2$, $d(y^k)_{1+s}=(p-k)h_0(y^{k+1})_{1+s}$, $dg_1(y^k)_{1+s}=(p-k)g_1h_0(y^{k+1})_{1+s}$.

Proof. This follows from the previous lemma by collecting the leading terms. □

Lemma 5. *For* $1\leq k\leq p-3$, $dh_1(y^k)_{1+s}=-(p-k)(p-k-1)g_0(y^{k+2})_{1+s}$.

Proof. We have the leading terms: $d(v_2^sv_1^{\frac{k}{p}}v_2^{\frac{p-k}{p}}[t_1^p])=-v_2^sv_1^{\frac{k}{p}}((p-k)v_2^{\frac{p-k-1}{p}}v_1^{\frac{1}{p}}[t_1|t_1^p]+\binom{p-k}{2}v_2^{\frac{p-k-2}{p}}v_1^{\frac{2}{p}}[t_1^2|t_1^p])=-(p-k)v_2^sv_1^{\frac{k+1}{p}}v_2^{\frac{p-k-1}{p}}[t_1|t_1^p]-\binom{p-k}{2}v_2^sv_1^{\frac{k+2}{p}}v_2^{\frac{p-k-2}{p}}[t_1^2|t_1^p]$. We also have $d(v_2^sv_1^{\frac{k+1}{p}}v_2^{\frac{p-k-1}{p}}[t_2])=v_2^sv_1^{\frac{k+1}{p}}v_2^{\frac{p-k-1}{p}}[t_1|t_1^p]-(p-k-1)v_2^sv_1^{\frac{k+2}{p}}v_2^{\frac{p-k-2}{p}}[t_1|t_2]$. After killing the leading terms, we have the following differential: $d(v_2^sv_1^{\frac{k}{p}}v_2^{\frac{p-k}{p}}[t_1^p]+(p-k)v_2^sv_1^{\frac{k+1}{p}}v_2^{\frac{p-k-1}{p}}[t_2])=-(p-k)(p-k-1)v_2^sv_1^{\frac{k+2}{p}}v_2^{\frac{p-k-2}{p}}([t_1|t_2]+\frac{1}{2}[t_1^2|t_1^p])=-(p-k)(p-k-1)v_2^sv_1^{\frac{k+2}{p}}v_2^{\frac{p-k-2}{p}}g_0$. □

Now we will study the long differentials.

Lemma 6. *If $s-1$ is not divisible by p, then $d(y^{p-1})_{1+s} = (s-1)h_1(y^{2p-1})_{1+s}$. If $s-1$ is divisible by p, then $d(y^{p-1})_{1+s} = h_0(y^{2p+1})_{2+s}$.*

Proof. Up to order $v_1^{\frac{2p+2}{p}}$, we have $d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}}) = -v_2^s v_1^{\frac{2p-1}{p}} t_1^{\frac{1}{p}} + s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p = -v_2^{s-1} v_1^{\frac{2p-1}{p}} (v_2^{\frac{1}{p}} t_1^p - v_1^{\frac{2}{p}} v_2^{\frac{p-1}{p}} t_1 - \frac{p-1}{2} v_1^{\frac{3}{p}} v_2^{\frac{p-2}{p}} t_1^2) + s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p = (s-1) v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^p + v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_1 + \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} t_1^2)$. □

Lemma 7. *For s not divisible by p, we have $dh_0(y)_{1+s} = sg_0(y^{p+2})_{1+s}$.*

Proof. From the previous lemma, we know that $d(v_1^{\frac{p-1}{p}} v_2^{\frac{p+1}{p}}) = v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_1 - \frac{1}{2} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{2}} t_1^2 + \cdots =: v_1^2 \eta$ is a permanent cycle. Hence up to order $v_1^{\frac{p+2}{p}}$, we have $d(v_2^s \eta) = -s v_2^{s-1} (v_1^{\frac{p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^p | t_1] - \frac{1}{2} v_1^{\frac{p+2}{p}} v_2^{\frac{p-2}{p}} [t_1^p | t_1^2])$. And $d(v_2^s \eta + s v_2^{s-1} v_1^{\frac{p+1}{p}} v_2^{\frac{p-1}{p}} (-t_2 + t_1^{p+1})) = s v_2^{s-1} v_1^{\frac{p+2}{p}} v_2^{\frac{p-2}{p}} g_0$. □

Lemma 8. *If s is divisible by p, then $dh_1(y^{p-1})_{1+s} = g_0(y^{2p+2})_{2+s}$.*

Proof. We have $d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_1^p) = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p | t_1^p] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1 | t_1^p] - \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} [t_1^2 | t_1^p])$. So $d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_1^p - \frac{1}{2} v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} t_1^{2p} + v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} t_2) = v_2^{s-1} v_1^{\frac{2p+2}{p}} v_2^{\frac{p-2}{p}} g_0$. □

Lemma 9. *If $s+2$ is not divisible by p, we have $dh_1(y^{p-2})_{1+s} = 2(s+2)g_1(y^{2p-1})_{1+s}$. If $s+2$ is divisible by p, we have $dh_1(y^{p-2})_{1+s} = -2g_0(y^{2p+1})_{2+s}$.*

Proof. Up to order $v_1^{\frac{2p+1}{p}}$, we have $d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p) = -2 v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} [t_1 | t_1^p] + 2 v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}} | t_1^p] + 2 v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}} | t_1^p] - s v_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p | t_1^p] - 2 s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1} | t_1^p]$.

Following [3], we have $dt_2 = [t_1 | t_1^p] + v_1 T$, and $dt_3^p = v_2^{p^2-1} [t_1^p | t_2] + v_2^{p-1} [t_2^p | t_1^p] + v_2^{p^2} T \mod v_1$, where $T = w_1([1|t_1], [t_1|1])$.

So we have $d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_2) = v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} [t_1 | t_1^p] + v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}} | t_2] - s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p | t_2] + v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} T$. Then $d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p + 2 v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} t_2) = 2 v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}} | t_1^p] + 2 v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}} | t_1^p] + 2 v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}} | t_2] + 2 v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} T - s v_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p | t_1^p] - 2 s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1} | t_1^p] - 2 s v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p | t_2]$.

$v_2^s v_1^{\frac{2p-2}{p}} v_2^{\frac{1}{p}} [t_1^{\frac{1}{p}} | t_1^p] = v_2^{s-1} v_1^{\frac{2p-2}{p}} v_2^{\frac{2}{p}} [t_1^p | t_1^p] + v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_2 | t_1^p] - \frac{p-1}{2} v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^2 | t_1^p]$. Also $v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{p+1}{p}} | t_1^p] = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^{p+1} | t_1^p] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1^2 | t_1^p]$. Also $v_2^s v_1^{\frac{2p-1}{p}} [t_1^{\frac{1}{p}} | t_2] = v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} [t_1^p | t_2] - v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} [t_1 | t_2]$.

Collecting the terms, we find that, when $s+2$ is not divisible by p, we have $d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p + \dots) = 2(s+2) v_2^{s-1} v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}} g_1$. And when $s+2$ is divisible by p, we have $d(v_2^s v_1^{\frac{p-2}{p}} v_2^{\frac{2}{p}} t_1^p + \dots) = -2 v_2^{s-1} v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}} g_0$. □

Lemma 10. *If s is not divisible by p, then $dg_0(y)_{-2+s} = sg_0h_1(y^{p+1})_{-2+s}$.*

Proof. From the previous lemma, we know $g_0(y)_{-2}$ is a permanent cycle, so $dg_0(y)_{-2+s} = sg_0h_1v_1v_2^{-1}(y)_{-2+s}$. □

Lemma 11. *If $s+3$ is divisible by p, then $dg_1(y^{p-1})_{1+s} = h_0g_1(y^{2p+1})_{2+s}$.*

Proof. We have $d(v_2^s v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}} g_1) = -v_2^s v_1^{\frac{2p-1}{p}}[t_1^{\frac{1}{p}}|g_1] + sv_2^{s-1}v_1^{\frac{p-1}{p}} v_2^{\frac{1}{p}}[t_1^p|g_1] - v_2^s v_1^{\frac{2p-1}{p}} v_2^{\frac{1}{p}}[T|t_1^p]$.

We know that up to order v_1, T is homologous to $2v_2^{-1}g_1$ and h_1g_1 is homologous to 0. Also up to order $v_1^{\frac{2p+1}{p}}$, we have $v_2^s v_1^{\frac{2p-1}{p}}[t_1^{\frac{1}{p}}|g_1] = v_2^{s-1}v_1^{\frac{2p-1}{p}}[t_1^p|g_1] - v_2^{s-1}v_1^{\frac{2p+1}{p}} v_2^{\frac{p-1}{p}}[t_1|g_1]$. The lemma follows. □

Now we have computed all the differential, so we have:

Theorem 1. *$H^*(\mathbb{G}_2^1, E_{2*}\Phi_{K(2)}S^3)$ is a vector space over $\mathbb{F}_p$, with a set of basis $h_0(y)_{1+pt}$, $v_1h_0(y)_{1+pt}$, $h_1(y^{p-1})_{1+s}$, $g_0(y)_{-2+pt}$, $v_1g_0(y)_{-2+pt}$, $g_0(y^2)_s$, $g_0(y^2)_{pt}$, $v_1g_0(y^2)_{pt}$, $g_1(y^{p-1})_{-2+s}$, $g_0h_1(y)_{-3+s}$, $g_0h_1(y)_{-3+pt}$, $v_1g_0h_1(y)_{-3+pt}$. Here s runs over integers not divisible by p, and t runs over all integers.*

One can see that there are no differentials in ANSS, so the homotopy groups $\pi_*\Phi_{K(2)}S^3$ is a free module over $\mathbb{F}_p[\zeta]/\zeta^2$ with the above generators.

Remark 1. *We have the boundary map from $\Phi_{K(2)}S^3$ to $L_{K(2)}M(p)$. On E_2-term of ANSS, this is the boundary map coming from the exact sequence $E_{2*}/p \xrightarrow{v_1} y\bar{R} \to E_{2*}\Phi_{K(2)}S^3$. One can show this map, after projecting to the top cell in $M(p)$, is essentially the stabilization map $\Omega^3S^3 \to \Omega^\infty\Sigma^\infty S^0$.*

5. v_1-exponent of unstable spheres

We find that, from the computations in the last section, the homotopy groups of $\Phi_{K(2)}S^3$ is killed by v_1^2. This implies that all the $K(2)$-local unstable homotopy groups of spheres have finite v_1-exponent.

Definition 1. *We say that a spectrum X have finite v_n-exponent, if there exists a fixed type $n+1$ complex V_n, such that any map $S^k \to X$ can be lifted to a map $\Sigma^kV_n \to X$.*

The following lemma is straightforward:

Lemma 12. *A spectrum X has finite v_n-exponent, if and only if the following holds:*

1. *X has finite v_{n-1}-exponent. Let V_{n-1} be a choice of type n complex admitting liftings. Choose a v_n-self map v_n^k on V_{n-1}.*
2. *There exists a number N, such that for any map $f : V_{n-1} \to X$, the composition $f \circ (v_2^k)^N = 0$.*

Obviously, any complex of type $n+1$ has finite v_n-exponent. We also note that the class of spectra with finite v_n-exponent is closed under taking fibers.

We will show that for all $k \geq 1$, $\Phi_{K(2)}S^k$ has finite v_1-exponent. It is enough to treat the odd sphere case.

Theorem 2. *$\Phi_{K(2)}S^{2k+1}$ has finite v_1-exponent at prime $p \geq 5$.*

Proof. We will show this with induction. The case for S^1 is trivial, and the case for S^3 is already proved. Now let $W(k)$ be the fiber of $S^{2k+1} \to S^{2k+3}$. Then using the secondary suspension, one finds that the fiber of the secondary suspension $\Phi_{K(2)}W(k) \to \Phi_{K(2)}W(k+1)$ is $K(2)$-locally equivalent to a type 2 complex. So we prove inductively all the $W(k)$ has finite v_1-exponent, and the theorem follows with another induction. □

Remark 2. *We can actually show that the v_1-exponent in the E_2-term of ANSS is bounded by $\frac{k(k+3)}{2}$ on S^{2k+1}.*

Remark 3. *Using the tmf resolution, one can also prove the $p = 3$ case of the theorem.*

This theorem leads to the following conjecture, generalizing the theorem for the p-exponent of unstable spheres:

Conjecture 1. *The v_n-torsion of unstable groups of spheres have finite v_n-exponent for every n.*

References

[1] Mark Behrens, The Goodwillie tower and the EHP sequence, Memoirs of the AMS. 1026 (2012) i-xi, 1-90.

[2] Douglas C. Ravenel, Complex cobordism and stable homotopy groups of spheres, Academic Press, Orlando, 1986, xix+413pp.

[3] Katsumi Shimomura and Atsuko Yabe, The homotopy groups $\pi_*(L_2S^0)$, Topology Vol. 34, No.2, 261-289 (1995).

[4] Wang Guozhen, On the K(2)-local stable Hopf invariant, to appear.