MORAVA K-THEORIES AND INFINITE LOOP SPACES

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§1 Introduction and main results

For a fixed prime p, the Morava K-theories $K(0)_*$, $K(1)_*$, $K(2)_*$, ... are a sequence of p-local periodic homology theories generalizing complex K-theory: $K(0)_*$ is ordinary rational homology and $K(1)_*$ is one of the (p-1) isomorphic summands of K-theory with mod p coefficients. These theories have various nice properties - for example, the associated spectra are ring spectra, the coefficients form a graded field, and thus there is a Kunneth formula (see [R2, Chapter 4, §2]). Recently, their central role in stable homotopy has been demonstrated by the work of M. Hopkins and J. Smith on maps between finite complexes [HS], using the remarkable nilpotence theorem of [DHS].

In [B3], A.K. Bousfield proved a beautiful theorem - K(1)-localization factors through the 0th space functor - and used this to reprove and strengthen the various delooping results of Adams-Priddy [AP] and Madsen-Snaith-Tornehave [MST]. In this paper, I show how the Hopkins-Smith work allows one to generalize Bousfield's argument to all n.

To state our main theorem, we need some notation. Let "Spaces" and "Spectra" respectively denote the homotopy categories of p-local spaces and spectra (as in, say, [A2]), and let Ω^∞: Spectra $\to$ Spaces be the 0th space functor, right adjoint to the suspension Σ^∞. Let $L_{K(n)}$: Spectra $\to$ Spectra be K(n)-localization [B1]. We will often write $E_{K(n)}$ for $L_{K(n)}(E)$. Recall the characterizing properties: $E \to E_{K(n)}$ is a $K(n)_*$-equivalence, and $[X,E_{K(n)}] = 0$ if $K(n)_*(X) = 0$.

<u>Theorem 1.1</u> For each $n \geq 1$, there exists a functor ϕ_n: Spaces $\to$ Spectra such that $\phi_n \circ \Omega^\infty = L_{K(n)}$. Furthermore, ϕ_n preserves fibration

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sequences.

We note two pleasant corollaries.

Corollary 1.2 If $\Omega^\infty X \simeq \Omega^\infty Y$ then $K(n)_*(X) \simeq K(n)_*(Y)$ for all $n \geq 1$.

Proof Applying Φ_n to the homotopy equivalence $\Omega^\infty X \simeq \Omega^\infty Y$ shows that $X_{K(n)} \simeq Y_{K(n)}$.

Corollary 1.3 Let $f: X \to Y$ be a map between spectra. If $\Omega^\infty f$ has a section (i.e. a right inverse) then so does $K(n)_*(f)$ for $n \geq 1$. In particular, the $K(n)$-homology suspension $K(n)_*(\Omega^\infty E) \to K(n)_*(E)$ is onto for all E.

Proof Applying Φ_n to the section of $\Omega^\infty f$ shows that $f_{K(n)}$ has a section, and the first statement follows. For the second, note that the homology suspension is induced by the evaluation map $\varepsilon: \Sigma^\infty\Omega^\infty E \to E$, and $\Omega^\infty\varepsilon$ has a section.

Note that these corollaries would be false with $K(n)_*$ replaced by ordinary homology, while they are essentially tautologically true (when restricted to (-1)-connected spectra) with $K(n)_*$ replaced by π_*^S. The failure of 1.3 for HZ/p is the source of the "unstable" condition for A-modules, thus 1.3 implies that there is no analogous condition for $K(n)_*$.

The key to all of these results is the existence of interesting self maps of finite complexes inducing isomorphisms in $K(n)_*$. An introduction to the use of such maps in our context occurs in §2. This contains an elementary proof of corollary 1.3 that is independent of both the theory of localizations (and thus Theorem 1.1) and the nilpotence conjecture. The only nontrivial input needed here is

(1.4) There exists a finite complex Z with $K(n)_*(Z) \neq 0$ and a $K(n)_*$-equivalence $v: \Sigma^d Z \to Z$ with $d > 0$.

For small n, this has been known for quite awhile. Adams constructed a $K(1)_*$-self equivalence of the Moore space in [A1], while, for $n = 2$ or 3, Toda's spaces $V(n)$ do the job [T]. For arbitrary n, a Z satisfying (1.4) is constructed in [HS], independent of the main theorems of [DHS]. (To paraphrase Mike Hopkins, this Z is the *first* kid on the block with an ice cream cone.)

Section 3 has the proof of the main theorem - a streamlined version of Bousfield's argument in [B3]. The proof is basically formal, except that (1.4) must be strengthened to

(1.5) There exists a commutative diagram of finite complexes

$$\begin{tikzcd} Z_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[dl] & Z_3 \arrow[r] \arrow[dll] & \cdots \\ S^0 & & & \end{tikzcd}$$

such that

(i) $\lim\limits_{\rightarrow} K(n)_*(Z_i) \approx K(n)_*(S^0)$.

(ii) each Z_i has a $K(n)_*$-equivalence v_i: $\Sigma^{d_i} Z_i \to Z_i$ with $d_i > 0$, and these self maps are compatible, for different i, after suitable finite iteration.

This is proved in §4 using the whole strength of Hopkins and Smith's work. It should perhaps be pointed out that the length of [B3] is partly due to the fact that Bousfield had to prove a version of (1.5) pre Devanitz-Hopkins-Smith.

Section 5 contains some questions, conjectures, and examples, e.g., a computation of $BP^*(g/p\ell)$ at the prime 2.

Finally, I wish to thank Pete Bousfield and Mike Hopkins for their help in this project. I came across [B2] after having already discovered Corollary 1.3 and the argument of §2. An exchange of letters (and preprint [B3]) with Pete, and subsequent conversations with Mike, led to Theorem 1.1. It is only excessive modesty that caused each of them to decline joint authorship.

§2 A proof that (1.4) => (1.3)

We begin with the following elementary observation.

Lemma 2.1 Let $f:X \to Y$ be a map between spectra. The following are equivalent:

(1) $\Omega^\infty f$ has a section.

(2) Any map $g: \Sigma^\infty W \to Y$ lifts

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \Sigma^\infty W \arrow[ur, dashed] \arrow[r, "g"] & Y \end{tikzcd} .$$

Now we assume that $f: X \to Y$ is a map satisfying the conditions of this last lemma. We wish to show that $K(n)_*(f)$ is onto.

Since Y is a direct limit of its finite subspectra, it suffices to show that given a map $g: F \to Y$, where F is a finite complex, there exists an algebraic lifting

(2.2)
$$\begin{tikzcd} & K(n)_*(X) \arrow[d, "f_*"] \\ K(n)_*(F) \arrow[ur, dashed] \arrow[r, "g_*"] & K(n)_*(Y) \end{tikzcd} .$$

Now let Z be as in (1.4). Because $K(n)_*$ is a field, $K(n)_*(Z) \neq 0$, and $K(n)_*$ satisfies a Kunneth formula, to show that the lifting exists in (2.2), it suffices to show that it exists in the diagram

(2.3)
$$\begin{tikzcd} & K(n)_*(Z \wedge X) \arrow[d, "(1 \wedge f)_*"] \\ K(n)_*(Z \wedge F) \arrow[ur, dashed] \arrow[r, "(1 \wedge g)_*"] & K(n)_*(Z \wedge Y) \end{tikzcd} .$$

Now choose N so large that $\Sigma^{dN} Z \wedge F \wedge DZ$ is a suspension spectrum, where DZ denotes the Spanier-Whitehead dual of Z. By our assumption on f, there is a lifting

(2.4)
$$\begin{tikzcd} & X \arrow[d, "f"] \\ \Sigma^{dN} Z \wedge F \wedge DZ \arrow[ur, dashed] \arrow[r, "g_N"] & Y \end{tikzcd} ,$$

where g_N is dual to $v^N \wedge g: \Sigma^{dN} Z \wedge F \to Z \wedge Y$. ($v^N$ is the Nth iterate of v.)

Adjointing yields a diagram

(2.5)
$$\begin{tikzcd} & & Z \wedge X \arrow[d, "1 \wedge f"] \\ \Sigma^{dN} Z \wedge F \arrow[rru, dashed] \arrow[rr, "v^N \wedge g"] & & Z \wedge Y \end{tikzcd} .$$

Since $K(n)_*(v)$ is an isomorphism, applying $K(n)_*(\quad)$ to (2.5) yields the lifting in (2.3).

§3 Proof of Theorem 1.1

We start with our basic construction.

Construction 3.1 Suppose that Z is a *space*, and $v\colon \Sigma^d Z \to Z$ is a self map with $d > 0$. We construct a functor

$$\Phi'_Z : \text{Spaces} \to \text{Spectra}$$

as follows: $\Phi'_Z(X)$ is the spectrum with (md)th-space

$$\Phi'_Z(X)_{md} = \mathrm{Map}(Z,X)$$

with structure maps

$$\Phi'_Z(X)_{md} = \mathrm{Map}(Z,X) \xrightarrow{v^*} \Omega^d\, \mathrm{Map}(Z,X) = \Omega^d \Phi'_Z(X)_{(m+1)d} .$$

Note that a commutative diagram

$$\begin{tikzcd} \Sigma^{dN} Z \arrow[r, "\Sigma^{dN}\alpha"] \arrow[d, "v^N"] & \Sigma^{dN} Z' \arrow[d, "v'^N"] \\ Z \arrow[r, "\alpha"] & Z' \end{tikzcd} ,$$

with $N \in \mathbb{N}$, induces a natural transformation.

$$\alpha^* : \Phi'_{Z'} \longrightarrow \Phi'_Z .$$

We list some basic properties of Φ'.

Proposition 3.2

(1) $v^*: \Phi'_Z(X) \simeq \Phi'_{\Sigma^d Z}(X)$, naturally in both Z and X.

(2) $\Phi'_Z(X)$ preserves fibrations in the variable X, and cofibrations in the variable Z.

(3) $\Phi'_Z(X)$ is periodic with period d.

(4) $\Phi'_Z(\Omega^\infty E) \simeq v^{-1}F(Z,E)$, naturally in both Z and X.

In (4), $F(Z,E)$ denotes the function spectrum defined, by Brown Representability, so that $[Y \wedge Z,E] \simeq [Y, F(Z,E)]$, and $v^{-1}F(Z,E)$ is the direct limit $\lim\limits_{\to} \{F(Z,E) \xrightarrow{v} F(\Sigma^d Z,E) \xrightarrow{v} F(\Sigma^{2d} Z,E) \longrightarrow \ldots\}$.

Proof of Proposition 3.2 Properties (1), (2) and (3) are clear by inspection. For (4), note that $\mathrm{Map}(Z, \Omega^\infty E) \simeq \Omega^\infty F(Z,E)$. Then (4) follows from the next lemma by letting $E(m) = F(\Sigma^{md} Z,E)$ and $d_m = md$.

Lemma 3.3 Suppose given a sequence of spectra $E(0) \xrightarrow{f(0)} E(1) \xrightarrow{f(1)} E(2) \longrightarrow \ldots$, and an increasing sequence of natural numbers, $d_0 < d_1 < d_2 < \ldots$. Define a new spectrum E by letting $E_{d_m} = \Omega^\infty \Sigma^{d_m} E(m)$, with structure maps $\Omega^\infty \Sigma^{d_m} f_m : E_{d_m} \to \Omega^{d_{m+1} - d_m} E_{d_{m+1}}$. Then $E \simeq \lim\limits_{\to} E(m)$.

Proof We can assume $E(m)_i = \Omega^\infty \Sigma^i E(m)$. Thus, for $n \geq m$, the f_m induce maps $E(m)_{d_n} \to E_{d_n}$. These fit together to give a map $\lim\limits_{\to} E(m) \to E$, which is easily checked to be an isomorphism on homotopy groups.

Note that property (1) of Proposition 3.2 allows us to extend the construction Φ' to any *finite spectrum* Z with self map v: one simply replaces the pair (Z,v) by $(\Sigma^{dN} Z, \Sigma^{dN} v)$ with N large.

Now let $\Phi_Z = L_{K(n)} \circ \Phi'_Z$: Spaces $\to$ Spectra.

Proposition 3.4 If Z is finite and $K(n)_*(v)$ is an isomorphism, there is a natural equivalence $\Phi_Z(\Omega^\infty E) \simeq F(Z, E_{K(n)})$.

<u>Proof</u> By Proposition 3.2 (4), we need to show that there is an equivalence

$$F(Z,E_{K(n)}) \simeq (v^{-1}F(Z,E))_{K(n)} \quad .$$

First note that $F(Z, E_{K(n)})$ is $K(n)$-local, since $E_{K(n)}$ is. Thus, to finish the proof, it suffices to show that the natural maps

$$F(Z,E_{K(n)}) \longleftarrow F(Z,E) \longrightarrow v^{-1}\ F(Z,E)$$

induce isomorphisms in $K(n)_*$. But, since Z is finite, these maps are equivalent to the maps

$$DZ \wedge E_{K(n)} \longleftarrow DZ \wedge E \longrightarrow v^{-1}(DZ \wedge E) \quad .$$

Both of these maps are clearly $K(n)_*$-equivalences, the second because $K(n)_*$ (v) is an isomorphism.

The definition of Φ_n, and the proof of theorem 1.1 are now remarkably easy, assuming (1.5).

<u>Definition 3.5</u> With $Z_1 \to Z_2 \to Z_3 \to \ldots$ as in (1.5), let

$\Phi_n = \lim\limits_{\leftarrow} \Phi_{Z_i}$: Spaces $\to$ Spectra.

<u>Proof of Theorem 1.1</u> There are natural equivalence

$$\begin{aligned} \Phi_n(\Omega^\infty E) &\simeq \lim_{\leftarrow} \Phi_{Z_i}\ (\Omega^\infty E) \\ &\simeq \lim_{\leftarrow} F(Z_i\ , E_{K(n)}) \\ &\simeq F(\lim_{\rightarrow} Z_i\ , E_{K(n)}) \\ &\simeq F(S^0\ , E_{K(n)}) \\ &\simeq E_{K(n)} \ . \end{aligned}$$

Here the second equivalence follows from Proposition 3.4. The fourth equivalence holds because $\lim\limits_{\rightarrow} Z_i \to S^0$ is a $K(n)_*$-equivalence ((1.5) (i)), and $E_{K(n)}$ is $K(n)$-local.

Finally, that Φ_n preserves fibrations is a direct consequence of Proposition 3.2 (2).

Exercise Prove Corollary 1.2 (without using Theorem 1.1!), by just using the Φ_Z' construction applied to the pair (Z,v) of (1.4).

§4 C_n-resolutions

In this section we develop the theory of what we dub "C_n-resolutions": approximations to a fixed finite complex by complexes admitting $K(n)_*$-equivalences. A special case will be (1.5).

We need some notation and definitions from [HS]. Let C be the p-local, stable homotopy category of finite complexes, and let C_n be the full subcategory consisting of the $K(n-1)_*$-acyclic complexes.

Definition 4.1 For $X \in C$, a map $v\colon \Sigma^d X \to X$ is a *v_n-self map* if $K(n)_*(v)$ is an isomorphism, and $K(m)_*(v)$ is nilpotent for $m \neq n$.

For the rest of this section we will repress suspensions "Σ^d" (i.e. view morphisms as having possibly nonzero degrees).

The main theorem of [HS] is

Theorem 4.2

(1) $X \in C_n$ if and only if X has a v_n-self map.

(2) Given $X,Y \in C_n$, with respective v_n-self maps v_X, v_Y, and $f\colon X \to Y$, there exist integers i, j such that

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle v_X^i} & & \downarrow{\scriptstyle v_Y^j} \\ X & \xrightarrow{f} & Y \end{array}$$

commutes.

Note that this has the following consequence.

Corollary 4.3 Given $X(1) \xrightarrow{f(1)} X(2) \xrightarrow{f(2)} X(3) \xrightarrow{f(3)} \cdots$ with $X(i) \in C_n$, there exist $k_i \in \mathbb{N}$ and v_n-self maps $v(i)\colon X(i) \to X(i)$

such that

$$\begin{tikzcd}
X(i) \arrow[r, "f(i)"] \arrow[d, "v(i)^{k_i}"'] & X(i+1) \arrow[d, "v(i+1)"] \\
X(i) \arrow[r, "f(i)"] & X(i+1)
\end{tikzcd}$$

commutes for all i.

We now define our approximations.

<u>Definition 4.4</u> For $X \in \mathcal{C}$, a *$\mathcal{C}_n$-resolution of X* (written $X_* \to X$) is a commutative diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[dl] & X_3 \arrow[r] \arrow[dll] & \cdots \\
X & & &
\end{tikzcd}$$

such that

(1) $X_i \in \mathcal{C}_n$

(2) $\lim\limits_{\substack{\to \\ i}} K(m)_*(X_i) \to K(m)_*(X)$ is an isomorphism for $m \geq n$.

In light of Corollary 4.3, (1.5) is essentially the case $X = S^0$ of the following theorem.

<u>Theorem 4.5</u> Every $X \in \mathcal{C}$ has a $\mathcal{C}_n$-resolution.

It is handy to have the following notion: A *map* $f_*\colon X_* \to Y_*$ of $\mathcal{C}_n$-resolutions, over $f\colon X \to Y$, is a collection of maps $f_n\colon X_n \to Y_n$ making all the obvious diagrams commute. Similarly, given a commutative diagram D of spectra in $\mathcal{C}$, there is an obvious notion of a commutative diagram of $\mathcal{C}_n$-resolutions over D.

As a first step towards proving Theorem 4.5, we prove

<u>Proposition 4.6</u> Given $X(1) \xrightarrow{f(1)} X(2) \xrightarrow{f(2)} X(3) \xrightarrow{f(3)} \cdots$ with

$X(i) \in \mathcal{C}_n$, there exists a commutative diagram of $\mathcal{C}_{n+1}$-resolutions

$$\begin{array}{ccccccc}
X(1)_* & \xrightarrow{f(1)_*} & X(2)_* & \xrightarrow{f(2)_*} & X(3)_* & \xrightarrow{f(3)_*} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X(1) & \xrightarrow{f(1)} & X(2) & \xrightarrow{f(2)} & X(3) & \xrightarrow{f(3)} & \cdots
\end{array}$$

<u>Proof</u> Let $v(i)$ and k_i be as in Corollary 4.3. For each i, we have a diagram of cofibration sequences, defining $X(i)_j$:

$$\begin{array}{ccccccc}
X(i) & \xrightarrow{v(i)} & X(i) & \longrightarrow & X(i)_1 & \longrightarrow & X(i) \\
\| & & \downarrow{\scriptstyle v(i)} & & \downarrow & & \| \\
X(i) & \xrightarrow{v(i)^2} & X(i) & \longrightarrow & X(i)_2 & \longrightarrow & X(i) \\
\| & & \downarrow{\scriptstyle v(i)} & & \downarrow & & \| \\
X(i) & \xrightarrow{v(i)^3} & X(i) & \longrightarrow & X(i)_3 & \longrightarrow & X(i) \\
\| & & \downarrow & & \downarrow & & \| \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

Furthermore, we then get a commutative diagram

$$\begin{array}{ccccccc}
X(1)_* & \xrightarrow{f(1)_*} & X(2)_* & \xrightarrow{f(2)_*} & X(3)_* & \xrightarrow{f(3)_*} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
X(1) & \xrightarrow{f(1)} & X(2) & \xrightarrow{f(2)} & X(3) & \xrightarrow{f(3)} & \cdots
\end{array}$$

by letting $f(i)_j$ be the composite $X(i)_j \to X(i)_{k_i j} \to X(i+1)_j$.

We claim that $X(i)_* \to X(i)$ is a $\mathcal{C}_{n+1}$-resolution. By construction $X(i)_j \in \mathcal{C}_{n+1}$ for all j. For $m \geq n+1$, $K(m)_*(v(i))$ is nilpotent. Thus applying $K(m)_*$ to (4.7) yields an isomorphism

$$\lim_{\substack{\to \\ j}} K(m)_*(X(i)_j) \xrightarrow{\sim} K(m)_*(X(i)).$$

<u>Proof of Theorem 4.5</u> We prove this by induction on n. Assuming that

there is a C_n-resolution of X,

$$\begin{tikzcd} X(1) \arrow[r, "f(1)"] \arrow[d] & X(2) \arrow[r, "f(2)"] \arrow[dl] & X(3) \arrow[r, "f(3)"] \arrow[dll] & \cdots \\ X \end{tikzcd},$$

we will construct a C_{n+1}-resolution.

Find C_{n+1}-resolutions of the X(i) as in the last proposition. Then let $Y_i = X(i)_i$, and let $Y_i \to Y_{i+1}$ be the composite $X(i)_i \to X(i+1)_i \to X(i+1)_{i+1}$. We claim that $Y_* \to X$ is a C_{n+1}-resolution. This is easy to check: $\{Y_i\}$ are cofinal in $\{X(i)_j\}$, so that, for $m \geq n+1$,

$$\varinjlim_i K(m)_*(Y_i) \simeq \varinjlim_i \varinjlim_j K(m)_*(X(i)_j) \simeq \varinjlim_i K(m)_*(X(i))$$
$$\simeq K(m)_*(X) \qquad .$$

§5 <u>Examples and Conjectures</u>

Our first example uses a consequence of Corollary 1.2. Recall a definition from [R1]: *Harmonic* localization is localization with respect to $\bigvee_{n\geq 0} K(n)$. Ravenel shows that BP is harmonic, as is any finite complex.

<u>Proposition 5.1</u> Suppose that $\Omega^\infty X$ is homotopic to a weak product of spaces $\Omega^\infty Y_i$, where each Y_i has only finitely many non zero homotopy groups. Then

$$X \longrightarrow X_Q$$

is harmonic localization.

<u>Proof</u> Eilenberg-MacLane spectra are $K(n)_*$-acyclic for $n \geq 1$[R1], thus so are spectra with only a finite number of homotopy groups. By hypothesis, $\Omega^\infty X \simeq \Omega^\infty Y$ where Y is a wedge of such spectra. It follows that $K(n)_*(X) = 0$ for $n \geq 1$, so that $X \to X_Q$ is a $K(n)_*$-equivalence for all $n \geq 0$.

Example 5.2 Let g, $p\ell$, and top be the usual spectra with 0th spaces G, PL and Top, as in [MM]. (G is the group of stable homotopy equivalences of spheres, etc.) By [MM, Theorem 4.8 and remark 4.36], if X is either $g/p\ell$ or g/top, then X satisfies the hypothesis of the last proposition at the prime 2. It follows, e.g., that, at the prime 2, $BP^*(g/p\ell) \approx BP^*((g/p\ell)_Q)$.

Remark 5.3 We would like to thank Frank Adams for explaining to us how easy it is to calculate $[HQ,E]$. In particular, $BP^*(\Sigma^{-1} HQ) \approx Z_{\hat{p}} / Z_{(p)} \otimes BP^*$, from which one can calculate $BP^*(X_Q)$.

Example 5.4 $K(n)_*(g) \approx K(n)_*(S^0)$ for $n \geq 1$.

Proof Let Q_0S^0 be the basepoint component of $\Omega^\infty\Sigma^\infty S^0$, so that $Q_0S^0 = \Omega^\infty(S\langle 0\rangle)$ where $S\langle 0\rangle$ is the 0-connected cover of $\Sigma^\infty S^0$. Then $\Omega^\infty g = G \approx Q_0S^0 \times Z/2 = \Omega^\infty(S\langle 0\rangle \vee HZ/2)$. Thus 1.2 implies that $K(n)_*(g) \approx K(n)_*(S\langle 0\rangle \vee HZ/2)$. But $HZ/2$ and HZ are both $K(n)_*$-acyclic, so $K(n)_*(S\langle 0\rangle \vee HZ/2) \approx K(n)_*(S^0)$.

Our next examples are applications of Corollary 1.3. As in [K2], call a sequence of spectra $\ldots \to X_2 \to X_1 \to X_0 \to E_0$ *exact* if it is formed by splicing together cofibration sequences $E_{i+1} \to X_i \xrightarrow{f_i} E_i$ where each $\Omega^\infty f_i$ has a section. Corollary 1.3 implies

Proposition 5.4 If $\ldots \to X_2 \to X_1 \to X_0 \to E$ is an exact sequence of spectra, then, for all $n \geq 1$, there is a long exact sequence

$$\ldots \to K(n)_*(X_1) \to K(n)_*(X_0) \to K(n)_*(E) \to 0.$$

Example 5.5 The $K(n)$-homology suspension epimorphism can be extended, using the canonical resolution based on the adjoint pair $(\Sigma^\infty,\Omega^\infty)$:

$$\ldots \to K(n)_*(Q\Omega^\infty E) \to K(n)_*(\Omega^\infty E) \xrightarrow{\varepsilon_*} K(n)_*E \to 0.$$

(Here QX denotes $\Omega^\infty\Sigma^\infty X$.)

Example 5.6 For a "smaller" bound on the kernel of ε_*, we use the main theorem of [K4]: If E is 0-connected, at the prime 2 there is an exact sequence

$$\Sigma^\infty D_2\Omega^\infty E \xrightarrow{f_2} \Sigma^\infty\Omega^\infty E \xrightarrow{\varepsilon} E .$$

Here $D_2(X)$ is the quadratic construction on X, and f_2 is the composite $\Sigma^\infty D_2(\Omega^\infty E) \hookrightarrow \Sigma^\infty Q(\Omega^\infty E) \xrightarrow{\Sigma^\infty\Omega^\infty\varepsilon} \Sigma^\infty\Omega^\infty E$. (See [K4] for the odd primary analogue.) It follows that there is an exact sequence:

$$K(n)_*(D_2(\Omega^\infty E)) \xrightarrow{f_{2*}} K(n)_*(\Omega^\infty E) \xrightarrow{\varepsilon_*} K(n)_*(E) \to 0$$

<u>Example 5.7</u> In [K1, KP1], we constructed a "minimal spacelike resolution" of $HZ_{(p)}$ extending the Kahn-Priddy epimorphism:

$$\dots \to L(2) \to L(1) \to L(0) \to HZ_{(p)} .$$

Here $L(0) = \Sigma^\infty S^0$, $L(1) = \Sigma^\infty B\Sigma_p$, and, in general, $L(m)$ is an indecomposable stable wedge summand of $B(Z/p)^m_+$. $L(m)$ is $K(n)_*$-acyclic if $m > n$ [W, K5]. It follows that, for all $n \geq 1$, there is an exact sequence

(5.8) $\quad 0 \to K(n)_*(L(n)) \to \dots \to K(n)_*(L(1)) \to K(n)_*(L(0)) \to 0 .$

This generalizes the well known isomorphism $K(1)_*(B\Sigma_p) \xrightarrow{\sim} K(1)_*(S^0)$ (see e.g. [K3]).

Note that (5.8) implies that

$$\sum_{m=0}^{n} (-1)^m \dim_{K(n)_*} K(n)_*(L(m)) = 0 .$$

This was observed computationally in [K5], and first caused us to try to prove Corollary 1.3.

With end this section with some questions and conjectures, aimed at making stronger use of Theorem 1.1.

<u>Question 5.8</u> How faithful is the functor $\bigvee_{n\geq 0} L_{K(n)}$: Spectra $\to$ Spectra?

Clearly, one should begin by restricting to the harmonic subcategory. Bousfield [B4] has pointed out that the cofibers of maps $S_K^{-2} \to S_K^0$ provide infinitely many distinct K-local spectra all having identical $K(0)$ and $K(1)$ localizations.

With Mike Hopkins, we conjecture

<u>Conjecture 5.9</u> If X and Y are finite spectra, then

$$X_{K(n)} \simeq Y_{K(n)} \text{ for all } n \Rightarrow X \simeq Y.$$

A consequence of Theorem 1.1 and the validity of this conjecture would be:

$$\Omega^{\infty}X \simeq \Omega^{\infty}Y \Rightarrow X \simeq Y \text{ for all finite } X \text{ and } Y.$$

We repeat a question from [R1].

Question 5.10 Is every suspension spectrum harmonic?

Conjecture 5.11 $QX \simeq QY \Rightarrow \Sigma^{\infty}X \simeq \Sigma^{\infty}Y$.

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