ANNALS OF MATHEMATICS
Vol. 65, No. 2, March, 1957
Printed in U.S.A.

THE GEOMETRIC REALIZATION OF A SEMI-SIMPLICIAL COMPLEX

BY JOHN MILNOR

(Received February 9, 1956)

Corresponding to each (complete) semi-simplicial complex K, a topological space $| K |$ will be defined. This construction will be different from that used by Giever [4] and Hu [5] in that the degeneracy operations of K are used. This difference is important when dealing with product complexes.

If K and K' are countable it is shown that $| K \times K' |$ is canonically homeomorphic to $| K | \times | K' |$. It follows that if K is a countable group complex then $| K |$ is a topological group. In particular $| K(\pi, n)|$ is an abelian topological group.

In the last section it is shown that the space $| K |$ has the correct singular homology and homotopy groups.

The terminology for semi-simplicial complexes will follow John Moore [7]. In particular the face and degeneracy maps of K will be denoted by $\partial_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ respectively.

1. The definition

As standard n-simplex Δ_n take the set of all $(n + 2)$-tuples $(t_0, \cdots, t_{n+1})$ satisfying $0 = t_0 \leqq t_1 \leqq \cdots \leqq t_{n+1} = 1$. The face and degeneracy maps

$$\partial_i : \Delta_{n-1} \to \Delta_n$$

and $s_i : \Delta_{n+1} \to \Delta_n$ are defined by

$$\partial_i(t_0, \cdots, t_n) = (t_0, \cdots, t_i, t_i, \cdots, t_n)$$

$$s_i(t_0, \cdots, t_{n+2}) = (t_0, \cdots, t_i, t_{i+2}, \cdots, t_{n+2}).$$

Let $K = \bigcup_{i \geqq 0} K_i$ be a semi-simplicial complex. Giving K the discrete topology, form the topological sum

$$\bar{K} = (K_0 \times \Delta_0) + (K_1 \times \Delta_1) + \cdots + (K_n \times \Delta_n) + \cdots .$$

Thus $\bar{K}$ is a disjoint union of open sets $k_i \times \Delta_i$. An equivalence relation in $\bar{K}$ is generated by the relations

$$(\partial_i k_n, \delta_{n-1}) \sim (k_n, \partial_i \delta_{n-1})$$

$$(s_i k_n, \delta_{n+1}) \sim (k_n, s_i \delta_{n+1}),$$

for each $k_n \in K_n$, $\delta_{n\pm1} \in \Delta_{n\pm1}$ and for $i = 0, 1, \cdots, n$. The identification space $| K | = \bar{K}/(\sim)$ will be called the *geometric realization* of K. The equivalence class of (k_n, δ_n) will be denoted by $| k_n, \delta_n |$. (The equivalence class $| k_0, \delta_0 |$ may be abbreviated by $| k_0 |$.)

THEOREM 1. *$|K|$ is a CW-complex having one n-cell corresponding to each non-degenerate n-simplex of K.*

For the definition of CW-complex see Whitehead [8].

LEMMA 1. *Every simplex $k_n \in K_n$ can be expressed in one and only one way as $k_n = s_{j_p} \cdots s_{j_1} k_{n-p}$ where k_{n-p} is non-degenerate and $0 \leqq j_1 < \cdots < j_p < n$. The indices j_α which occur are precisely those j for which $k_n \in s_j K_{n-1}$.*

The proof is not difficult. (See [3] 8.3). Similarly we have:

LEMMA 2. *Every $\delta_n \in \Delta_n$ can be written in exactly one way as $\delta_n = \partial_{i_q} \cdots \partial_{i_1} \delta_{n-q}$ where δ_{n-q} is an interior point (that is the coordinates t_i of δ_{n-q} satisfy $t_0 < t_1 < \cdots < t_{n-q+1}$) and $0 \leqq i_1 < \cdots < i_q \leqq n$.*

By a non-degenerate point of $\bar{K}$ will be meant a point (k_n, δ_n) with k_n non-degenerate and δ_n interior.

LEMMA 3. *Each $(k_n, \delta_n) \in \bar{K}$ is equivalent to a unique non-degenerate point.*

Define the map $\lambda : \bar{K} \to \bar{K}$ as follows. Given k_n choose $j_1, \cdots, j_p, k_{n-p}$ as in Lemma 1 and set

$$\lambda(k_n, \delta_n) = (k_{n-p}, s_{j_1} \cdots s_{j_p} \delta_n).$$

Define the discontinuous function $\rho : \bar{K} \to \bar{K}$ by choosing $i_1 \cdots i_q, \delta_{n-q}$ as in Lemma 2 and setting

$$\rho(k_n, \delta_n) = (\partial_{i_1} \cdots \partial_{i_q} k_n, \delta_{n-q}).$$

Now the composition $\lambda\rho : \bar{K} \to \bar{K}$ carries each point into an equivalent, non-degenerate point. It can be verified that if $x \sim x'$ then $\lambda\rho(x) = \lambda\rho(x')$; which proves Lemma 3.

Take as n-cells of $|K|$ the images of the non-degenerate simplexes of $\bar{K}$. By Lemma 3 the interiors of these cells partition $|K|$. Since the remaining conditions for a CW-complex are easily verified, this proves Theorem 1.

LEMMA 4. *A semi-simplicial map $f : K \to K'$ induces a continuous map $|K| \to |K'|$.*

In fact the map $|f|$ defined by $|k_n, \delta_n| \to |f(k_n), \delta_n|$ is clearly well defined and continuous.

As an example of the geometric realization, let C be an ordered simplicial complex with space $|C|$. (See [2] pp. 56 and 67). From C we can define a semi-simplicial complex K, where K_n is the set of all $(n + 1)$-tuples $(a_0, \cdots, a_n)$ of vertices of C which (1) all lie in a common simplex, and (2) satisfy $a_0 \leqq a_1 \leqq \cdots \leqq a_n$. The operations ∂_i, s_i are defined in the usual way.

ASSERTION. The space $|C|$ is homeomorphic to the geometric realization $|K|$. In fact the point $|(a_0, \cdots, a_n); (t_0, \cdots, t_{n+1})|$ of $|K|$ corresponds to the point of $|C|$ whose a^{th} barycentric coordinate, a being a vertex of C, is the sum, over all i for which $a_i = a$, of $t_{i+1} - t_i$. The proof is easily given.

2. Product complexes

Let $K \times K'$ be the cartesian product of two semi-simplicial complexes (that is $(K \times K')_n = K_n \times K'_n$). The projection maps $\rho : K \times K' \to K$ and $\rho' : K \times K' \to K'$ induce maps $|\rho|$ and $|\rho'|$ of the geometric realizations. A map

$$\eta : | K \times K' | \to | K | \times | K' |$$

is defined by $\eta = | \rho | \times | \rho' |$.

THEOREM 2. *η is a one–one map of $| K \times K' |$ onto $| K | \times | K' |$. If either* (a) *K and K' are countable, or* (b) *one of the two CW-complexes $| K |$, $| K' |$ is locally finite; then η is a homeomorphism.*

The restrictions (a) or (b) are necessary in order to prove that $| K | \times | K' |$ is a CW-complex. (For the proof in case (b) see [8] p. 227 and for case (a) see [6] 2.1.)

PROOF (Compare [2] p. 68). If x'' is a point of $| K \times K' |$ with non-degenerate representative $(k_n \times k'_n, \delta_n)$ we will first determine the non-degenerate representative of $| \rho | (x'') = | k_n, \delta_n |$. Since δ_n is an interior point of Δ_n, this representative has the form

$$(k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n) \quad \text{where} \quad k_n = s_{i_p} \cdots s_{i_1} k_{n-p}$$

(see proof of Lemma 3). Similarly $| \rho' |(x'')$ is represented by

$$(k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n)$$

where $k'_n = s_{j_q} \cdots s_{j_1} k'_{n-q}$. The indices i_α and j_β must be distinct; for if $i_\alpha = j_\beta$ for some α, β then $k_n \times k'_n$ would be an element of $s_{i_\alpha}(K_{n-1} \times K'_{n-1})$.

However the point x'' can be completely determined by its image.

$$| k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n | \times | k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n | .$$

In fact given any pair $(x, x') \in | K | \times | K' |$ define $\bar{\eta}(x, x') \in | K \times K' |$ as follows. Let (k_a, δ_a) and (k'_b, δ'_b) be the non-degenerate representatives: where $\delta_a = (t_0, \cdots, t_{a+1})$, $\delta'_b = (u_0, \cdots, u_{b+1})$. Let $0 = w_0 < \cdots < w_{n+1} = 1$ be the distinct numbers t_i and u_j arranged in order. Set $\delta''_n = (w_0, \cdots, w_{n+1})$. Then if $\mu_1 < \cdots < \mu_{n-a}$ are those integers $\mu = 0, 1, \cdots, n-1$ such that $w_{\mu+1}$ is not one of the t_i, we have $\delta_a = s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n$. Similarly $\delta'_b = s_{\nu_1} \cdots s_{\nu_{n-b}} \delta''_n$ where the sets $\{\mu_i\}$ and $\{\nu_j\}$ are disjoint. Now define

$$\bar{\eta}(x, x') = |(s_{\mu_{n-a}} \cdots s_{\mu_1} k_a) \times (s_{\nu_{n-b}} \cdots s_{\nu_1} k'_b), \delta''_n | .$$

Clearly

$$\begin{aligned} | \rho | \bar{\eta}(x, x') &= | s_{\mu_{n-a}} \cdots s_{\mu_1} k_a, \delta''_n | = | k_a, s_{\mu_1} \cdots s_{\mu_{n-a}} \delta''_n | \\ &= | k_a, \delta_a | = x \end{aligned}$$

and $| \rho' | \bar{\eta}(x, x') = x'$, which proves that $\eta\bar{\eta}$ is the identity map of $| K | \times | K' |$. On the other hand, taking x'' as above we have

$$\begin{aligned} \bar{\eta}\eta(x'') &= \bar{\eta}(| k_{n-p}, s_{i_1} \cdots s_{i_p} \delta_n |, \quad | k'_{n-q}, s_{j_1} \cdots s_{j_q} \delta_n |) \\ &= |(s_{i_p} \cdots s_{i_1} k_{n-p}) \times (s_{j_q} \cdots s_{j_1} k'_{n-q}), \delta_n | = x''. \end{aligned}$$

To complete the proof it is only necessary to show that $\bar{\eta}$ is continuous. However it is easily verified that $\bar{\eta}$ is continous on each product cell of $| K | \times | K' |$. Since we know that this product is a CW-complex, this completes the proof.

An important special case is the following. Let I denote the semi-simplicial complex consisting of a 1-simplex and its faces and degeneracies.

COROLLARY. *A semi-simplicial homotopy* $h:K \times I \to K'$ *induces an ordinary homotopy* $|K| \times [0, 1] \to |K'|$.

In fact the interval [0, 1] may be identified with $|I|$. The homotopy is now given by the composition

$$|K| \times |I| \xrightarrow{\bar{\eta}} |K \times I| \xrightarrow{|h|} |K'|.$$

3. Product operations

Now let K be a countable complex. Any semi-simplicial map $p:K \times K \to K$ induces by Lemma 4 and Theorem 2 a continuous product

$$|p|\bar{\eta}:|K| \times |K| \to |K|.$$

If there is an element e_0 in K_0 such that $s_0^n e_0$ is a two-sided identity in K_n for each n, then it follows that $|e_0|$ is a two-sided identity in $|K|$; so that $|K|$ is an H-space. If the product operation p is associative or commutative then it is easily verified that $|p|\bar{\eta}$ is associative or commutative. Hence we have the following.

THEOREM 3. *If K is a countable group complex (countable abelian group complex), then $|K|$ is a topological group (abelian topological group).*

Let $K(\pi, n)$ denote the Eilenberg MacLane semi-simplicial complex (see [1]). Since $K(\pi, n)$ is an abelian group complex we have:

COROLLARY. *If π is a countable abelian group, then for $n \geqq 0$ the geometric realization $|K(\pi, n)|$ is an abelian topological group.*

It will be shown in the next section that $|K(\pi, n)|$ actually is a space with one non-vanishing homotopy group.

The above construction can also be applied to other algebraic operations. For example a pairing $K \times K' \to K''$ between countable group complexes induces a pairing between their realizations. If K is a countable semi-simplicial complex of Λ-modules, where Λ is a discrete ring, then $|K|$ is a topological Λ-module.

4. The topology of $|K|$

For any space X let $S(X)$ be the total singular complex. For any semi-simplicial complex K a one–one semi-simplicial map $i:K \to S(|K|)$ is defined by

$$i(k_n)(\delta_n) = |k_n, \delta_n|.$$

Let $H_*(K)$ denote homology with integer coefficients.

LEMMA 5. *The inclusion $K \to S(|K|)$ induces an isomorphism $H_*(K) \approx H_*(S|K|)$ of homology groups.*

By the n-skeleton $K^{(n)}$ of K is meant the subcomplex consisting of all K_i, $i \leqq n$ and their degeneracies. Thus $|K^{(n)}|$ is just the n-skeleton of $|K|$ considered as a CW-complex. The sequence of subcomplexes

$$K^{(0)} \subset K^{(1)} \subset \cdots$$

gives rise to a spectral sequence $\{E^r_{pq}\}$; where E^∞ is the graded group corresponding to $H_*(K)$ under the induced filtration; and

$$E^1_{pq} = H_{p+q}(K^{(p)} \bmod K^{(p-1)}).$$

It is easily verified that $E^1_{pq} = 0$ for $q \neq 0$, and that E^1_{p0} is the free abelian group generated by the non-degenerate p-simplexes of K. From the first assertion it follows that $E^2_{p0} = E^\infty_{p0} = H_p(K)$.

On the other hand the sequence

$$S(|K^{(0)}|) \subset S(|K^{(1)}|) \subset \cdots$$

gives rise to a spectral sequence $\{\bar{E}^r_{pq}\}$ where $\bar{E}^\infty$ is the graded group corresponding to $H_*(S(|K|))$. Since it is easily verified that the induced map $E^1_{pq} \to \bar{E}^1_{pq}$ is an isomorphism, it follows that the rest of the spectral sequence is also mapped isomorphically; which completes the proof.

Now suppose that K satisfies the Kan extension condition, so that $\pi_1(K, k_0)$ can be defined.

LEMMA 6. *If K is a Kan complex then the inclusion i induces an isomorphism of $\pi_1(K, k_0)$ onto $\pi_1(S(|K|), i(k_0)) = \pi_1(|K|, |k_0|)$.*

Let K' be the Eilenberg subcomplex consisting of those simplices of K whose vertices are all at k_0. Then $\pi_1(K, k_0)$ can be considered as a group with one generator for each element of K'_1 and one relation for each element of K'_2.

The space $|K'|$ is a CW-complex with one vertex. For such a space the group π_1 is known to have one generator for each edge and one relation for each face. Comparing these two descriptions it follows easily that the homomorphism $\pi_1(K) = \pi_1(K') \to \pi_1(|K'|)$ is an isomorphism.

We may assume that K is connected. Then it is known (see [7] Chapter I, appendix C) that the inclusion map $K' \to K$ is a semi-simplicial homotopy equivalence. By the corollary to Theorem 2 this proves that the inclusion $|K'| \to |K|$ is a homotopy equivalence; which completes the proof of Lemma 6.

REMARK 1. From Lemmas 5 and 6 it can be proved, using a relative Hurewicz theorem, that the homomorphisms

$$\pi_n(K, k_0) \to \pi_n(|K|, |k_0|)$$

are isomorphisms for all n. (The proof of the relative Hurewicz theorem given in [9] §3 carries over to the semi-simplicial case without essential change, making use of [7] Chapter I, appendices A and C. This theorem is applied to the pair $(S(|\tilde{K}|), \tilde{K})$ where $\tilde{K}$ denotes the universal covering complex of K.)

REMARK 2. The space $|K(\pi, n)|$ has n^{th} homotopy group π, and other homotopy groups trivial. This clearly follows from the preceding remark. Alternatively the proof given by Hu [5] may be used without essential change.

Now let X be any topological space. There is a canonical map

$$j: | S(X)| \to X$$

defined by $j(| k_n , \delta_n |) = k_n(\delta_n)$.

THEOREM 4. *The map $j: | S(X)| \to X$ induces isomorphisms of the singular homology and homotopy groups.*

(This result is essentially due to Giever [4]).

The map j induces a semi-simplicial map $j_{\#}: S(| S(X)|) \to S(X)$. A map i in the opposite direction was defined at the beginning of this section. The composition $j_{\#}i: S(X) \to S(X)$ is the identity map. Together with Lemma 5 this implies that j induces isomorphisms of the singular homology groups of $| S(X)|$ onto those of X. Together with Remark 1 it implies that j induces isomorphisms of the homotopy groups of $| S(X) |$ onto those of X. This completes the proof.

PRINCETON UNIVERSITY

REFERENCES

1. S. EILENBERG and S. MACLANE, *Relations between homology and homotopy groups of spaces* II, Ann. of Math, 51 (1950), 514–533.
2. ——— and N. STEENROD, Foundations of Algebraic Topology, Princeton, 1952.
3. ——— and J. A. ZILBER, *Semi-simplicial complexes and singular homology*, Ann. of Math., 51 (1950), 499–513.
4. J. B. GIEVER, *On the equivalence of two singular homology theories*, Ann. of Math., 51 (1950), 178–191.
5. S. T. HU, *On the realizibility of homotopy groups and their operations*, Pacific J. Math., 1 (1951), 583–602.
6. J. MILNOR, *Construction of universal bundles* I, Ann. of Math., 63 (1956), 272–284.
7. J. MOORE, Algebraic homotopy theory (Lecture notes), Princeton, 1955–56.
8. J. H. C. WHITEHEAD, *Combinatorial homotopy* I, Bull. Amer. Math. Soc., 55 (1949), 213–245.
9. J. MOORE, *Some applications of homology theory to homotopy problems*, Ann. of Math., 58 (1953), 325–350.