A RELATIVE LUBIN–TATE THEOREM VIA HIGHER FORMAL GEOMETRY

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ABSTRACT. We formulate a theory of punctured affine formal schemes, suitable for describing certain phenomena within algebraic topology. As a proof-of-concept we show that the Morava K-theoretic localizations of Morava E-theory, which arise in transchromatic homotopy theory, corepresent a Lubin–Tate-type moduli problem in this framework.

1. INTRODUCTION

Associated to a fixed formal group law Γ of finite height h over the field $\mathbb{F}_{p^h}$, there is a cohomology theory E_h called Morava E-theory. Its coefficient ring can be noncanonically identified as

$$E_h^0 \cong \mathbb{W}_{\mathbb{F}_{p^h}}[\![u_1, \ldots, u_{h-1}]\!].$$

Both E-theory and this ring are intimately connected to the infinitesimal deformation theory of formal groups: the associated formal scheme $LT_h = \operatorname{Spf} E_h^0$ is known as Lubin–Tate space, which classifies deformations of Γ [14]. More precisely, when R is a complete local ring with residue field $\mathbb{F}_{p^h}$, maps $\operatorname{Spf} R \to LT_h$ are in bijective correspondence with $\star$-isomorphism classes of formal group laws over R which reduce over the special fiber to the specified formal group law Γ. Given a complex orientation $\operatorname{Spf} E_h^0 \mathbb{C}\mathrm{P}^\infty \cong \operatorname{Spf} E_h^0[\![x]\!]$ of E-theory, the resulting formal group law associated to the formal group

$$\operatorname{Spf} E_h^0 \mathbb{C}\mathrm{P}^\infty \to \operatorname{Spf} E_h^0 = LT_h$$

is an example of such a versal formal group law; that is, the indicated bijection is induced by pulling back this formal group law along maps $\operatorname{Spf} R \to LT_h$.

Transchromatic homotopy theory studies phenomena related to the self-interaction of the height filtration of the moduli of formal groups as appearing in chromatic homotopy theory. One object of interest in this subfield is $L_{K(h')}E_h$, the Bousfield localization at $K(h')$ of Morava E-theory, where the Morava K-theory $K(h')$ is the reduction of $E_{h'}$ to the special fiber of $LT_{h'}$. For $h' > h$ the spectrum $L_{K(h')}E_h$ is zero, and for $0 \leq h' \leq h$ it follows from a formula of Mark Hovey [8, Lemma 2.3] that the coefficient ring of $L_{K(h')}E_h$ can be expressed as

$$(L_{K(h')}E_h)^0 \cong \mathbb{W}_{\mathbb{F}_{p^h}}[\![u_1, \ldots, u_{h-1}]\!][u_{h'}^{-1}]^\wedge_{I_{h'}},$$

where $I_{h'}$ denotes the ideal $I_{h'} = (p, u_1, \ldots, u_{h'-1})$. Haynes Miller has suggested that this ring should be similarly studied through the moduli problem it represents — but from the point of view of formal geometry, this ring is extremely unfriendly. Here is an inexhaustive list of complaints:

- It is not a complete local ring, so cannot give rise to a formal scheme in the most naive, 1970s sense of the phrase that has served an enormous amount of algebraic topology perfectly well.
- Inverting topologically nilpotent elements destroys adic topologies. The adic topology on $k[\![x]\!]$ comes from declaring linear translates of powers of the ideal (x) to be a basis of open sets. In turn, this has the effect of requiring that a continuous map $k[\![x]\!] \to R$ to a discrete ring R send x to a nilpotent element, as continuous maps respect the ideals of definition. There are therefore no maps $k(\!(x)\!) \to R$ restricting to continuous maps $k[\![x]\!] \to k(\!(x)\!) \to R$, which is highly unsatisfactory when identifying schemes with their functors of points.
- It is generally only complete against *some* of the power series generators. So, even making use of the available completeness, when $h' < h - 1$ there remain power series generators for which the ring is *not* complete. Power series rings without their completion taken into account are quite intimidating.

Thus, the classical framework of formal geometry fails to account for the subtleties of this ring.

Nevertheless, rings of this sort have been gaining importance in chromatic homotopy theory. Here's a small sampling of applications:

- Takeshi Torii has worked extensively with $L_{K(h-1)}E_h$ (i.e., in the case $h' = h-1$), producing many interesting results about transchromatic phenomena and interrelationships among the stabilizer groups [23, 24, 25, 26, 27].
- Tyler Lawson and Niko Naumann have encountered these kinds of rings while studying the interplay of the $K(1)$- and $K(2)$-local obstructions to realizing $BP\langle 2\rangle$ as an E_∞-ring spectrum [13].
- Such rings appear in and around the Tate construction, especially in connection to results about the pro-spectrum $\mathbb{C}\mathrm{P}^{\infty}_{-\infty}$ arising from stunted projective space. For example, this powers work of Matthew Ando, Jack Morava, and Hal Sadofsky on a phenomenon known as "blueshift" [2, 3].
- Meromorphic formal geometry and formal Laurent series rings appear in the "sharp construction" of Matthew Ando, Christopher French, and Nora Ganter, which connects $\mathbb{C}\mathrm{P}^{\infty}_{-\infty}$ to the study of $MU\langle 2k\rangle$-orientations [1].
- John Greenlees and Neil Strickland have studied a form of transchromatic character theory involving the ring $E_h[u_{h'}^{-1}]/I_{h'}$, which corresponds to the functions on the punctured neighborhood $X_{h'} \setminus X_{h'-1}$ for a filtration X_* on Lubin–Tate space [7].
- The third author has used $L_{K(h')}E_h$ and the p-divisible group $(\operatorname{Spf} E_h^0 \mathbb{C}\mathrm{P}^\infty)[p^\infty] \otimes L_{K(h')}E_h^0$ to develop a variety of results concerning transchromatic character theory and group-cohomological data [20, 21, 22].

It is therefore important to sort out what variation on formal geometry houses these objects, given the past successes of formal geometry at organizing and interpreting results in algebraic topology.

Our goal is to propose a framework in which the ring $L_{K(h')}E_h^0$ and various natural maps concerning it can be studied. Kazuya Kato has already ventured in this direction [12] as part of a program to understand "higher local fields", the relevant higher-dimensional analogues of local function fields of smooth points on algebraic curves. Other work in this same area — for instance, that of Shuji Saito [18, 19], of Denis Osipov ([15, Section 3], [16, Sections 2.2 and 3.2]), and generally of those trying to understand the program initiated by Beilinson and Parshin ([5], [9, Section 3.2]) — grapple with many of the phenomena we encounter. What we contribute lies atop their substantial groundwork. Geometrically, the broad idea is to study analytic functions on deleted neighborhoods within formal schemes. This necessitates not just working with a localization of a complete ring, but also remembering the complete ring from which it stems, together with both of their defining direct and inverse systems. Along with more minor examples, we prove a Lubin–Tate result for $L_{K(h')}E_h^0$ in this extended setting:

Main Theorem (Proven below as Theorem 30). *Fix a finite ground field k of positive characteristic p and a formal group law Γ of finite height n over k. Select an infinitesimal deformation X of $\operatorname{Spec} k$ and a "punctured affine formal scheme" $X' = \operatorname{Spp} R$ over X. Then, for each $h' \leq h$, there is an equivalence between commuting diagrams of the shape*

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d, equal] & X \arrow[d] & X' \arrow[l] \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spf} E_h^0 & \operatorname{Spp} L_{K(h')}E_h^0 \arrow[l] \end{tikzcd}$$

and formal group laws $\mathbb{G}$ over X which deform Γ and whose pullback to X' is of height h'.

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2. Continuity and higher formal geometry following Kato

The goal of this section is to construct and study a category of "pipe rings" satisfying the following desiderata:

(1) The usual category of profinite rings and continuous maps contributes a full subcategory of pipe rings. That is to say: pipe rings will carry a " generalized topology", which reduces to the adic topology in that full subcategory.
(2) The localization morphism $\pi_0 E_h \to \pi_0 L_{K(h')} E_h$ belongs to this category, as do the more general iterated localization morphisms
$$\pi_0 E_h \to \pi_0 L_{K(h')} E_h \to \pi_0 L_{K(h'')} L_{K(h')} E_h \to \cdots .$$

Kato has constructed such a category of rings with these properties in an effort to understand the continuity properties of higher local fields [12]. Using our terminology, we recall his construction and point out various properties and constructions which we now need but which he has not already recorded.

Definition 1. Let Pipes_{-1} denote the category of finite sets and Pipes_0 the category of profinite sets, which we refer to as (-1)*-pipes* and 0*-pipes* respectively. For $n \geq 1$, we inductively define the category Pipes_n of n*-pipes* by the formula
$$\mathsf{Pipes}_n = \mathrm{Pro}(\mathrm{Ind}\,\mathsf{Pipes}_{n-1}),$$
and we refer to n as the *length*. (Note that there is an exception in the bottom case: Pipes_0 is *not* equivalent to $\mathrm{Pro}(\mathrm{Ind}\,\mathsf{Pipes}_{-1})$.) There are fully faithful product-preserving inclusions $\mathsf{Pipes}_{n-1} \to \mathsf{Pipes}_n$, given by sending an object of Pipes_{n-1} to its associated constant system in $\mathrm{Pro}(\mathrm{Ind}\,\mathsf{Pipes}_{n-1}) = \mathsf{Pipes}_n$. We denote the sequential colimit along these inclusions by Pipes_∞, and its objects are referred to simply as *pipes*.[1]

Definition 2. The category Pipes_n admits finite products, so we may speak of its associated category of ring objects $\mathsf{PipeRings}_n = \mathrm{Rings}(\mathsf{Pipes}_n)$. As shorthand, we refer to an object of $\mathsf{PipeRings}_n$ as an n*-pipe ring*. The inclusions $\mathsf{Pipes}_{n-1} \to \mathsf{Pipes}_n$ are product-preserving, so induce inclusions of categories of ring objects. In the aggregate these form the category $\mathsf{PipeRings}_\infty$ of *pipe rings*.

2.1. An embedding condition.

To simplify notation, in this subsection we will write Hom-sets as $[-,-]$, regardless of the ambient category.

Definition 3. The constant system at a singleton set gives a terminal object $1 \in \mathsf{Pipes}_\infty$, and we define a functor $\mathsf{Pipes}_\infty \to \mathsf{Sets}$ by
$$S \mapsto [1, S] =: \underline{S},$$
called (*set-theoretic*) *realization*.

This definition can also be given inductively. On the one hand, if $S \in \mathsf{Pipes}_n$ is given as a pro-system $\{S_\alpha\}_\alpha$, then (considering each $S_\alpha \in \mathsf{Pipes}_n$) we have that
$$\underline{S} = [1, S] = [1, \lim_\alpha S_\alpha] = \lim_\alpha [1, S_\alpha] = \lim_\alpha \underline{S_\alpha}.$$
On the other hand, if each S_α is given as an ind-system $\{(S_\alpha)_\beta\}_\beta$ of $(n-1)$-pipes, then we analogously have
$$\underline{S_\alpha} = [1, S_\alpha] = [1, \operatorname*{colim}_\beta (S_\alpha)_\beta] = \operatorname*{colim}_\beta [1, (S_\alpha)_\beta] = \operatorname*{colim}_\beta \underline{(S_\alpha)_\beta}$$

[1] This is a superficial variation on Kato's $\mathscr{F}_n$ categories [12, p. 166], modified only to simplify notation in the present work. See also the recent preprint of Braunlich–Groechenig–Wolfson [6, Section 7.1]. The word "pipe" here is an acronym for Pro-Ind-Pro Ensemble.

since 1, as a finite set, is a small object. Thus, taking the realization is exactly the iterative process of taking limits of pro-systems and colimits of ind-systems. This implies that realization commutes with finite limits and in particular with finite products, so it induces a functor $\mathsf{PipeRings}_\infty \to \mathsf{Rings}$.

This functor should be thought of as "forgetful", in the sense of sending a topologized ring to its underlying ring. As one expects, this functor is not injective on objects; just as a given set can support many topologies, so do sets support many pipe structures. However, as we have set up the category Pipes_∞, there is no reason to even expect realization to be *faithful*. This is a more serious problem.

Definition 4 (cf. [12, Definition 1])**.** To control this aspect of realization, we make two further inductive definitions:

(a) Every (-1)-pipe and 0-pipe is called *fine*. An n-pipe Y is called *fine* if it can be expressed as a pro-system $\{Y_\alpha\}_\alpha$ of ind-systems $\{(Y_\alpha)_\beta\}_\beta$ such that:
- Each $(Y_\alpha)_\beta$ is a fine $(n-1)$-pipe.
- The induced map $\underline{(Y_\alpha)_\beta} \to \underline{Y_\alpha}$ is injective for every choice of α and β.

(b) Every (-1)-pipe is called *cofine*. A 0-pipe X is called *cofine* if it can be expressed as a pro-system $\{X_\lambda\}_\lambda$ of finite sets such that the induced map $\underline{X} \to \underline{X_\lambda}$ is surjective for each choice of λ. Generally, an n-pipe X is called *cofine* if it can be expressed as an ind-system $\{X_\lambda\}_\lambda$ of pro-systems $\{(X_\lambda)_\mu\}_\mu$ such that:
- Each $(X_\lambda)_\mu$ is a cofine $(n-1)$-pipe.
- The induced map $\underline{X} \to \underline{X_\lambda}$ is surjective for every choice of λ.

Lemma 5. *The properties of being fine and cofine are both preserved by the inclusions* $\mathsf{Pipes}_{n-1} \to \mathsf{Pipes}_n$. □

Altogether, the point is the following lemma:

Lemma 6 ([12, Lemma 1.2])**.** *The realization functor is faithful whenever the source is cofine and the target is fine.*

Proof. We consider the map $\varphi_{X,Y} : [X, Y] \to [\underline{X}, \underline{Y}]$ induced by the realization functor, where X is cofine and Y is fine. There exists an integer n for which both X and Y are realized as n-pipes and all maps $X \to Y$ are exhibited as maps of n-pipes. It thus suffices to check the situation where the source and target have the same length.

The statement is clear for (-1)- and 0-pipes. So, suppose that the statement holds for $(n-1)$-pipes. We then show that $\varphi_{X,Y}$ is injective for X cofine and Y fine as follows:

$$\begin{aligned}
[X, Y] &= \lim_\alpha \operatorname*{colim}_\lambda [X_\lambda, Y_\alpha] \\
&= \lim_\alpha \operatorname*{colim}_\lambda \lim_\nu \operatorname*{colim}_\beta [(X_\lambda)_\nu, (Y_\alpha)_\beta] \\
&\subset \lim_\alpha \operatorname*{colim}_\lambda \lim_\nu \operatorname*{colim}_\beta [\underline{(X_\lambda)_\nu}, \underline{(Y_\alpha)_\beta}] && \text{(inductive assumption)} \\
&\subset \lim_\alpha \operatorname*{colim}_\lambda \lim_\nu [\underline{(X_\lambda)_\nu}, \underline{Y_\alpha}] && \text{(fineness of } Y\text{)} \\
&= \lim_\alpha \operatorname*{colim}_\lambda [\operatorname*{colim}_\nu \underline{(X_\lambda)_\nu}, \underline{Y_\alpha}] \\
&= \lim_\alpha \operatorname*{colim}_\lambda [\underline{X_\lambda}, \underline{Y_\alpha}] \\
&\subset \lim_\alpha [\underline{X}, \underline{Y_\alpha}] && \text{(cofineness of } X\text{)} \\
&= [\underline{X}, \lim_\alpha \underline{Y_\alpha}] \\
&= [\underline{X}, \underline{Y}].
\end{aligned}$$
□

This lemma demonstrates that $\mathsf{PipeRings}_\infty$ does indeed satisfy our first desideratum. More to the point, it illustrates the analogy with the classical situation of topologized sets, and indicates that we can indeed consider an n-pipe as determining a "generalized topology" on its set-theoretic realization.

These conditions are reminiscent of an analogous situation in abstract homotopy theory, where a model structure on a category determines which are the "right" objects to map to and from. Unfortunately, despite substantial effort, we have been unable to precisely pin down this analogy. Instead, let us record the following conjecture:

Conjecture 7 (Pipe dream). *For every pipe X, there exists an initial cofine pipe X^c over X such that the map $X^c \to X$ induces an isomorphism $\underline{X^c} \to \underline{X}$. Dually, for every pipe Y, there exists a terminal fine pipe Y^f under Y such that the map $Y \to Y^f$ induces an isomorphism $\underline{Y} \to \underline{Y^f}$. Finally, there is a class*[2] *of weak equivalences W for which these compute the derived mapping space:*

$$\mathrm{Hom}_{\mathsf{Pipes}_\infty[W^{-1}]}(X,Y) \cong \mathrm{Hom}_{\mathsf{Pipes}_\infty}(X^c, Y^f).$$

Remark 8. A cofineification functor for Pipes_0 can be constructed as follows. Let X be a profinite set, presented as a pro-system $\{X_\alpha\}_\alpha$ of finite sets. Each finite set X_α in the pro-system has a subset X'_α of elements that persist in the inverse limit: these are precisely the elements which are in the image of $X_\beta \to X_\alpha$ for all $\beta < \alpha$. These assemble into a sub-pro-system $X' \hookrightarrow X$ (indexed on the same diagram), which is clearly the desired cofineification and is clearly unique up to unique isomorphism.

This suggests the following more general construction. If X is an n-pipe, presented as a pro-system $\{X_\alpha\}_\alpha \in \operatorname{Ind}\mathsf{Pipes}_{n-1}$, then by fixing any particular X_α in the system we can consider the objects $X_\beta \to X_\alpha$ over it, each of which has a corresponding image factorization

$$X_\beta \to \mathrm{im}(X_\beta \to X_\alpha) \hookrightarrow X_\alpha.$$

For $\gamma < \beta$, then there is a canonical monomorphism $\mathrm{im}(X_\gamma \to X_\alpha) \to \mathrm{im}(X_\beta \to X_\alpha)$, and altogether this gives a pro-system of images over X_α. However, unlike in the case where X_α was a finite set, we cannot guarantee that the pro-system of images can be replaced by a constant system, and it seems that the core feat of any construction of a cofineification functor on n-pipes would be to work around this fact. Some further discussion of this and related issues can be found in Appendix A.1.

2.2. **Closed ideals.** In algebraic geometry, one studies rings through their associated categories of modules, and in particular through their ideals. From a categorical perspective, ideals should be thought of as kernels of ring maps; then, just as the restriction of continuity on maps of topologized rings gives rise to the notion of a closed ideal, so should our "generalized topologies" determine the correct notion of an ideal in our setting. In this section, we make this notion precise and study some of its basic features, although we explore little beyond what we will need in the remainder of the paper. The interested reader is encouraged to refer to Appendix A.2 for a more thorough exploration of these generalized topologies in terms of the spectrum of a pipe ring. Throughout, we will make quiet use of basic facts about pro-categories, an excellent reference for which is Isaksen's paper [10].

Definition 9. Let R be a pipe ring. An *ideal* of R is a monomorphism $I \to R$ of R-modules exhibiting I as the kernel of some pipe ring map $R \to R/I$ inducing a surjection $\underline{R} \to \underline{R/I}$ of rings under set-theoretic realization. An ideal $\underline{I} \subset \underline{R}$ of the set-theoretic realization is said to be *closed* when it is the image under realization of some ideal I of R as a pipe ring.

Central to basic algebraic geometry are the notions of being complete and of being local. We now give analogous definitions in our setting.

Definition 10. The *completion* of an n-pipe ring R at an ideal I is the pro-system of n-pipe rings

$$R^\wedge_I = \{\cdots \to R/I^2 \to R/I\}$$

when such a system exists. (Note that because $\operatorname{Pro}\operatorname{Pro}\mathsf{C}$ is naturally equivalent to $\operatorname{Pro}\mathsf{C}$, this system can again be thought of as an n-pipe ring.) The pipe ring R is said to be *complete* (with respect to the ideal I) if the natural map $R \to R^\wedge_I$ is an isomorphism.

Definition 11. Let R be an n-pipe ring. Then for any $x \in \underline{R}$ we have an action map

$$R = 1 \times R \xrightarrow{x \times \mathrm{id}} R \times R \xrightarrow{\mu} R.$$

[2]See Remark 34 for a further thought about this.

From this, we can construct $x^{-1}R$, the *localization of R away from x*, as the ind-object

$$x^{-1}R = \left\{ R \xrightarrow{x\cdot -} R \xrightarrow{x\cdot -} R \xrightarrow{x\cdot -} \cdots \right\}.$$

(Crucially, the resulting object can potentially be an $(n+1)$-pipe ring and *not* an n-pipe ring.) More generally, for any multiplicatively closed subset $T \subset \underline{R}$ we can analogously form $T^{-1}R$. If the natural map $R \to T^{-1}R$ is an isomorphism, we say that R is *local away from T*.

Proposition 12. *Suppose $T \subset \underline{R}$ contains no zerodivisors. Then $T^{-1}R$ is fine if R is, and $T^{-1}R$ is cofine if R is.*

Proof. The maps in the defining ind-system for $T^{-1}R$ induce injections on set-theoretic realizations since we are assuming all the elements of T are non-zerodivisors. On the other hand, inducing up to a pro-system, the cofineness condition is vacuous. □

Remark 13. We are deliberately careful to use the notation $x^{-1}R$ to avoid confusing this operation with the construction $R[\alpha]/(x\alpha - 1)$. This latter operation is poorly behaved in pipe rings — for instance, the finiteness restriction of Pipes_{-1} can prevent $R[\alpha]$ from existing at all.

It's worth pointing out that it's obviously difficult to get a handle on the closed ideals of a pipe ring in general. In the profinite case of Pipes_0, there is a comparison with Stone spaces [11, VI.2.3] that yields an equivalence

$$\mathsf{PipeRings}_0 = \mathrm{Rings}\,\mathsf{Pipes}_0 \simeq \mathrm{Pro}(\mathrm{Rings}\,\mathsf{Pipes}_{-1})$$

and hence a simple theory of closed ideals, but in general we do not expect any equivalence along these lines.

2.3. **Some formal geometry of pipe rings.** In this section, we generalize basic definitions and facts in formal geometry by introducing the functor of points of a pipe ring. (For a more geometric and less sheaf-theoretic description of n-pipe rings, see Appendix A.2.)

Definition 14. The *pipe spectrum* (or simply *spectrum*) of an n-pipe ring R, denoted $\mathrm{Spp}(R)$, is defined to be the corepresentable functor $\mathrm{Spp}(R)(-) = \mathsf{PipeRings}_\infty(R, -)$. Note that this coincides with Spec when restricted to pipe rings of length -1 and with Spf when restricted to pipe rings of length at most 0.

Example 15. We define *formal affine m-space* over the $(n-1)$-pipe ring R to be

$$\hat{\mathbb{A}}^m_R = \mathrm{Spp}(R[\![x_1, \ldots, x_m]\!]).$$

Here $R[\![x]\!]$ denotes the object of $\mathsf{PipeRings}_n$ given by the pro-system

$$\ldots \to R\{x^2, x^1, x^0\} \to R\{x^1, x^0\} \to R\{x^0\},$$

where each node carries the expected truncated polynomial ring structure. (Again, the notation is meant to distinguish from the process of forming a polynomial ring and taking quotients, which is not necessarily available.) The multivariate power series ring can be described similarly.

Lemma 16. *Maps $\hat{\mathbb{A}}^m_R \to \hat{\mathbb{A}}^k_R$ over $\mathrm{Spp}\,R$ are in bijection with k-tuples of m-variate power series over $\underline{R}$ with no constant terms.* □

Definition 17. An *m-dimensional formal variety* over R is a functor $V : \mathsf{PipeRings}_\infty \to \mathsf{Sets}$ isomorphic to $\hat{\mathbb{A}}^m_R$. A (smooth, 1-dimensional, commutative) *formal group* over R is a 1-dimensional formal variety V equipped with an abelian group object structure. A *coordinatized formal group* over R is an abelian group object structure on $\hat{\mathbb{A}}^1_R$ itself. Finally, the *formal group law* associated to a coordinatized formal group is given by the representing power series in $\underline{R}[\![x, y]\!]$ for the multiplication map

$$\hat{\mathbb{A}}^2_R \cong \hat{\mathbb{A}}^1_R \times \hat{\mathbb{A}}^1_R \to \hat{\mathbb{A}}^1_R.$$

Corollary 18. *Over a fixed pipe ring R, a coordinatized formal group and a formal group law are equivalent data.* □

Classically, formal group laws in positive characteristic are classified according to an invariant called their "height". We define the pipe ring analogue as follows:

Definition 19. Suppose that R is a pipe ring, and select a formal group law F over R with p-series given by the formula

$$[p]_F(x) = px + \sum_{i=2}^{\infty} a_i x^i \in \underline{R}[\![x]\!].$$

Then, F is said to be of *p-height h* if R is complete with respect to an ideal I, $\underline{I}$ contains p, $\underline{I}$ contains a_i for every $i < p^h$, and the coefficient a_{p^h} is invertible in $\underline{R/I}$.

To demonstrate that this notion is indeed an isomorphism invariant, we use the following lemma:

Lemma 20. *Let $A(x)$ and $A'(x)$ be formal power series over the (classical) ring S with expansions*

$$A(x) = \sum_{i=1}^{\infty} a_i x^i,$$
$$A'(x) = \sum_{i=1}^{\infty} a'_i x^i,$$

and let $\varphi(x)$ be a compositionally-invertible power series with $A'(x) = \varphi^{\circ(-1)}(A(\varphi(x)))$. Suppose that S is complete with respect to some ideal $\mathfrak{a}$. If $A(x)$ has the property that $a_i \in \mathfrak{a}$ for $i < n$ and $a_n \in S^\times$, called property $\mathbf{P}_n(\mathfrak{a})$, then the property $\mathbf{P}_n(\mathfrak{a})$ holds for $A'(x)$.

Proof. Write $\varphi(x) = \sum_{j=1}^{\infty} b_j x^j$ with $b_1 \in S^\times$. We would like to prove that property $P_n(\mathfrak{a})$ holds for the series

$$A'(x) = \varphi^{\circ(-1)}(A(\varphi(x))).$$

We begin by proving that property $\mathbf{P}_n(\mathfrak{a})$ holds for the argument $A(\varphi(x))$, which we expand as

$$\begin{aligned}
A(\varphi(x)) &= \sum_{i=1}^{\infty} a_i \cdot \varphi(x)^i \\
&= \sum_{i=1}^{\infty} a_i \cdot \left(\sum_{j=1}^{\infty} b_j x^j \right)^i \\
&= \sum_{i=1}^{\infty} \sum_{j_1,\ldots,j_i > 0} a_i b_{j_1} \cdots b_{j_i} x^{j_1+\cdots+j_i} \\
&=: \sum_{m=1}^{\infty} \beta_m x^m.
\end{aligned}$$

First, it is clear that $\beta_m \in \mathfrak{a}$ for $m < n$, since the coefficients in the penultimate line that sum to give β_m all contain some factor a_i with $i < n$, which by assumption lives in $\mathfrak{a}$. Furthermore, among the coefficients summing to give β_n, we have $a_n b_1^n \in S^\times$ when $i = n$ and $b_{j_1} = \ldots = b_{j_i} = 1$, and then all the rest contain some factor a_i with $i < n$ and hence live in $\mathfrak{a}$. Thus $\beta_n \in S^\times$, as the sum of a unit and an element of $\mathfrak{a}$ is again a unit since S is complete with respect to $\mathfrak{a}$. So, $A(\varphi(x))$ does indeed satisfy property $\mathbf{P}_n(\mathfrak{a})$.

Let us write $\varphi^{\circ(-1)} = \sum_{j=1}^{\infty} c_j x^j$, which has $c_1 = b_1^{-1} \in S^\times$. Consider the expansion

$$A'(x) = \varphi^{\circ(-1)}(A(\varphi(x))) = \sum_{j=1}^{\infty} c_j \left(\sum_{m=1}^{\infty} \beta_m x^m \right)^j = \sum_{j=1}^{\infty} \sum_{m_1,\ldots,m_j > 0} c_j \beta_{m_1} \cdots \beta_{m_j} x^{m_1+\cdots+m_j}.$$

It is again the case that the coefficients contributing to the terms of order less than n are formed from products of elements in $\mathfrak{a}$, hence are all in $\mathfrak{a}$ themselves. In degree n, all but one of the summands contributes a coefficient in $\mathfrak{a}$, where the exception is $c_1 \beta_n$ at $m_1 = n$. Since this coefficient is a unit, this series also satisfies property $\mathbf{P}_n(\mathfrak{a})$. □

Corollary 21. *The p-height is independent of the chosen coordinate on a given formal group.*

Proof. Let G and G' be two coordinatized formal groups, and let φ be an isomorphism between them. Apply Lemma 20 to the choices $A(x) = [p]_G(x)$ and $A'(x) = [p]_{G'}(x)$. □

2.4. **The Lubin–Tate pipe rings.** For the remainder of the paper, we fix the following data: a finite field k of positive characteristic p; a finite, positive integer h; a coordinatized formal group Γ of finite p-height h over k; an integer $N \geq 0$; and a weakly decreasing sequence of nonnegative integers

$$h = h_0 \geq h_1 \geq h_2 \geq \cdots \geq h_N \geq 0.$$

As indicated, we will refer to h_0 simply as h, as it plays a privileged role in the theory.

In this section of the paper, we turn to our second desideratum for pipe rings: the instantiation of $\pi_0 L_{K(h')} E_h$ and its associated localization morphism. Let us recall the basics of the Lubin–Tate theory of deformations of p-complete formal group laws of finite height [14]. The space of infinitesimal deformations of the formal group Γ of height h chosen above is represented by a formal affine variety LT_h of dimension $h-1$ over the ring $\mathbb{W}_k$ of Witt vectors over k. This space comes equipped with a nested sequence of relative divisors, identified by a flag of ideals

$$(0) = I_0 \leq (p) = I_1 \leq I_2 \leq \cdots \leq I_h.$$

This space is not canonically coordinatized; rather, a family of "good" choices of coordinates exist:

Definition 22. Let $\widetilde{\Gamma}$ be a versal [4, Definition 7.2] deformation of the coordinatized formal group Γ to LT_h. Then, a *Lubin–Tate coordinate system* on LT_h is a set of coordinate functions $p, u_1, \ldots, u_{h-1}$ satisfying the following property for each t:

$$[p]_{\widetilde{\Gamma}}(x) \equiv u_t x^{p^t} \pmod{(p, u_1, \ldots, u_{t-1}, x^{p^t+1})}.$$

Such coordinate systems always exist. Moreover, the specific choice of versal deformation $\widetilde{\Gamma}$ does not affect whether a given coordinate system is a Lubin–Tate coordinate system. (Indeed, any sequence of elements u_t in $I_{t+1} \setminus (I_t \cup I_{t+1}^2)$ forms such a system.)

Recall that $LT_h \cong \operatorname{Spf} E_h^0 = \operatorname{Spf} \pi_0 E_h$, where E_h is a Morava E-theory. Of course, in using this notation we implicitly endow E_h with a topology — in fact, it is endowed with a *profinite* topology, specified by the ideal $I_h = (p, u_1, \ldots, u_{h-1})$, where these generators are taken to form a Lubin–Tate coordinate system. Of course, this is the same as considering $\pi_0 E_h$ as a 0-pipe ring.

Definition 23. By work of Hovey [8, Lemma 2.3], we have that $\pi_0 L_{K(h_1)} E_h \cong (u_{h_1}^{-1} \pi_0 E_h)^\wedge_{I_{h_1}}$. More generally, there is the inductive formula

$$\pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h \cong (u_{h_n}^{-1}(\pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h))^\wedge_{I_{h_n}}.$$

We inductively endow this with the structure of an n-pipe ring: assuming the existence of an $(n-1)$-pipe ring structure on $\pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h$, we construct the ind-system

$$u_{h_n}^{-1} \pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h = \left(\pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h \xrightarrow{- \cdot u_{h_n}} \pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h \xrightarrow{- \cdot u_{h_n}} \cdots \right),$$

and from this we construct the pro-system

$$(u_{h_n}^{-1}(\pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h))^\wedge_{I_{h_n}} = \left(\cdots \to (u_{h_n}^{-1} \pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h)/I_{h_n}^2 \to (u_{h_n}^{-1} \pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h)/I_{h_n} \right).$$

Lemma 24. *The n-pipe ring $\pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h$ is bifine.*

Proof. This is immediate from the inductive definition: the quotient maps in the completion systems are all surjective on set-theoretic realizations, and the multiplication maps in the localization systems are all likewise injective. □

Proposition 25. *The universal formal group law over $\pi_0 E_h$ pulled back to $\pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h$ is of p-height h_n.*

Proof. This follows from the definition of a Lubin–Tate coordinate system, along with the fact that completion and localization have the expected actions on realizations. □

As one would hope, the n-pipe ring $\pi_0 L_{K(h_n)} \dots L_{K(h_1)} E_h$ does not depend upon the choice of coordinates, and rather is determined by data stemming from this flag of relative divisors.[3]

Theorem 26. *The n-pipe ring structure on $\pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h$ is independent of choice of Lubin–Tate coordinate system $u_1, \dots, u_{h-1}$ used in its construction.*

Proof. Let X^u_{n-1} denote the pipe ring $\pi_0 L_{K(h_{n-1})} \cdots L_{K(h_1)} E_h$ as presented with u-coordinates. Suppose that X^v_{n-1} is a second such ring, presented in v-coordinates, and suppose that the change of coordinates isomorphism $\varphi_0 : X^u_0 \to X^v_0$ has been shown to extend uniquely to an isomorphism $\varphi_{n-1} : X^u_{n-1} \to X^v_{n-1}$. The remainder of the situation is then summarized in the following diagram:

$$\begin{tikzcd}
X^u_{n-1} \arrow[rr, "\varphi_{n-1}"] \arrow[d] \arrow[ddrr] & & X^v_{n-1} \arrow[d] \\
u_{h_n}^{-1} X^u_{n-1} \arrow[d] \arrow[drr, dashed] & & v_{h_n}^{-1} X^v_{n-1} \arrow[d] \\
X^u_n = (u_{h_n}^{-1} X^u_{n-1})^\wedge_{I^u_{h_n}} \arrow[rr, dashed, "\varphi_n"] & & (v_{h_n}^{-1} X^v_{n-1})^\wedge_{I^v_{h_n}} = X^v_n.
\end{tikzcd}$$

We seek to construct the dashed arrows. (As we will see, in general there is not a map $u_{h_n}^{-1} X^u_{n-1} \to v_{h_n}^{-1} X^v_{n-1}$.) All of the vertical maps are injective on set-theoretic points and all the pipe rings in question are bifine, so the existence of the diagonal dashed arrow is determined by whether u_{h_n} is sent to an invertible element in the target. Since

$$\varphi_{n-1}(u_{h_n}) = v_{h_n} + c$$

for some $c \in I^v_{h_n}$, then $\varphi_{n-1}(h_n)$ has an inverse in X^v_n given by

$$\varphi_{n-1}(u_{h_n})^{-1} = \frac{1}{v_{h_n} + c} = \frac{v_{h_n}^{-1}}{1 + v_{h_n}^{-1} c} = \sum_{j=0}^{\infty} (-1)^j c^j v_{h_n}^{-1-j}.$$

(Note that the existence of this infinite sum depends on the completeness of X^v_n against the ideal $I^v_{h_n}$.) This gives us the diagonal dashed arrow. Finally, we obtain φ_n from the fact that $\varphi_{n-1}(I^u_{h_n}) \subset I^v_{h_n}$. Switching the two pipe rings and applying this same method produces an inverse to φ_n. □

Remark 27. Beilinson and Parshin [5, 9] defined higher local fields to formulate a theory of adeles for higher dimensional schemes, notably studied in recent years by Osipov [15, 16]. A necessary input to the construction he considers is a flag of relative divisors, from which he produces a higher-dimensional local field. For instance, the flag of divisors $(0) \le (x) \le (x, y)$ in $k[x, y]$ determines the 2-dimensional local field

$$k((y))((x)) := x^{-1}((y^{-1}(k[x,y]^\wedge_{(y)}))^\wedge_{(x)}),$$

where y is an element $y \in (x, y) \setminus (x) + (y)^2$ and x is an element $x \in (x) \setminus (0) + (x)^2$. Hovey's calculation and our presentation above constitute a mild generalization of Osipov's construction, since we allow for resultant objects other than (higher) local fields.

[3]Of course, this must be true if we expect to give a formal-geometric interpretation of its functor of points and if the n-pipe ring in question indeed comes from a homotopy-theoretic construction which *a priori* has nothing to do with coordinates.

3. A Lubin–Tate theorem

We remind the reader that we have fixed the following data: a finite field k of positive characteristic p; a finite, positive integer h; a coordinatized formal group Γ of finite p-height h over k; an integer $N \geq 0$; and a weakly decreasing sequence of nonnegative integers

$$h = h_0 \geq h_1 \geq h_2 \geq \cdots \geq h_N \geq 0.$$

Again, we will often refer to h_0 simply as h, as it plays a privileged role in the theory.

We now describe the moduli problem represented by the pipe spectrum

$$\mathrm{Spp}\, \pi_0 L_{K(h_n)} \cdots L_{K(h_1)} E_h$$

described in the previous section. To do so, we need one further definition:

Definition 28. Write $[N]$ for the category freely generated by a chain of N composable morphisms. We define *N-staged pipe rings*, or $\mathsf{StagedRings}_N$, to be the full subcategory of $\mathrm{Fun}([N], \mathsf{PipeRings}_\infty)$ of objects R_* such that R_n is an n-pipe ring for each $n \leq N$.

Definition 29 (Staged Lubin–Tate moduli problem)**.** Consider the full subcategory of those N-staged pipe rings whose leading term is a complete local ring with residue field k, and let $\mathcal{M} = \mathcal{M}_{\Gamma/k}$ be the moduli problem which assigns to such an N-staged ring

$$R_0 \xrightarrow{i_1} R_1 \xrightarrow{i_2} \cdots \xrightarrow{i_N} R_N$$

the following groupoid:

- The objects are given by commuting diagrams of the form

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & F_0 \arrow[d] & F_1 \arrow[l] \arrow[d] & \cdots \arrow[l] & F_N \arrow[l] \arrow[d] \\ \mathrm{Spp}\, k \arrow[r] & \mathrm{Spp}\, R_0 & \mathrm{Spp}\, R_1 \arrow[l] & \cdots \arrow[l] & \mathrm{Spp}\, R_N, \arrow[l] \end{tikzcd}$$

  where each F_n is a coordinatized formal group over $\mathrm{Spp}\, R_n$ of p-height h_n, each square is a pullback square, and the arrows in the top row are morphisms of coordinatized formal groups (i.e., the formal group law F_n is specified by pushing forward F_{n-1} along i_n).
- Morphisms in the groupoid correspond to commuting diagrams of shape:

$$\begin{tikzcd} & \Gamma \arrow[rr] \arrow[dl] \arrow[dd, equal] & & F_0 \arrow[dl] \arrow[dd] & & F_1 \arrow[ll] \arrow[dl] \arrow[dd] & \cdots \arrow[l] \\ \mathrm{Spp}\, k \arrow[rr] & & \mathrm{Spp}\, R_0 & & \mathrm{Spp}\, R_1 \arrow[ll] & & \cdots \arrow[ll] \\ & \Gamma \arrow[rr] \arrow[ul] & & F_0' \arrow[ul] & & F_1' \arrow[ll] \arrow[ul] & \cdots \arrow[l] \end{tikzcd}$$

  where the first vertical arrow $\Gamma \to \Gamma$ is an equality and the other vertical arrows are formal group isomorphisms (not necessarily respecting the chosen coordinatizations). Note that the pullback condition for the object diagrams implies that the isomorphism $F_0 \to F_0'$ determines the isomorphisms $F_n \to F_n'$ for all $1 \leq n \leq N$.

We now come to the main theorem:

Theorem 30 (Lubin–Tate theorem for pipe rings). *The moduli problem $\mathscr{M}$ is essentially discrete*[4] *and is corepresented by the levelwise-bifine N-staged pipe ring*

$$\pi_0 E_h \to \pi_0 L_{K(h_1)} E_h \to \cdots \to \pi_0(L_{K(h_N)} \cdots L_{K(h_1)} E_h).$$

Proof. Select a versal deformation $\mathbb{G}_0 := \tilde{\Gamma}$ of the coordinatized formal group Γ to $LT_h = \operatorname{Spf} \pi_0 E_h$. Taking u_n to be the coefficient of x^{p^n} in $[p]_{\mathbb{G}_0}(x)$, we obtain an isomorphism $\pi_0 E_h \cong \mathbb{W}_k[\![u_1, \ldots, u_{h-1}]\!]$. The elements $p, u_1, \ldots, u_{h-1}$ form a Lubin–Tate coordinate system on $\operatorname{Spf} \pi_0 E_h = LT_h$. (Note that these satisfy the required identities on the nose, not just modulo an ideal.) From here on, we refer to the N-staged pipe ring in the theorem statement as $X_* = (X_0 \to X_1 \to X_2 \to \cdots)$ and to the pullback of the coordinatized formal group $\mathbb{G}_0$ along $\operatorname{Spp} X_n \to \operatorname{Spp} X_0$ as $\mathbb{G}_n$. These coordinatized formal groups will collectively form our versal object.

At $N = 0$, this is the classical Lubin–Tate moduli problem, which is known to be essentially discrete [17, Theorem 4.4] and representable by $\operatorname{Spp} X_0$ [14, Proposition 1.1]. So, suppose that we are given some $R_* = (R_0 \xrightarrow{i_1} \cdots \xrightarrow{i_N} R_N) \in \mathsf{StagedRings}_N$ with R_0 complete and local with residue field k. As a point $(F_0, \ldots, F_N) \in \mathscr{M}(R_*)$ is completely determined by the coordinatized formal group F_0 over $\operatorname{Spp} R_0$, it follows that $\mathscr{M}$ is a subfunctor of the classical Lubin–Tate moduli problem (pulled back along the forgetful functor to 0-staged pipe rings, i.e., to profinite rings), and hence $\mathscr{M}$ is also essentially discrete. This means that the natural transformation

$$\mathsf{StagedRings}_N(X_*, -) \to \mathscr{M}$$

given on R_* by pulling back each $\mathbb{G}_n$ along the chosen maps $\operatorname{Spp} R_n \to \operatorname{Spp} X_n$ is fully faithful, and so it only remains to check that it is also essentially surjective.

To accomplish this, we will construct a section of the composite natural transformation

$$\mathsf{StagedRings}_N(X_*, -) \to \mathscr{M} \xrightarrow{\simeq} \pi_0 \mathscr{M}.$$

More precisely, given a point

$$(F_0, \ldots, F_N) \in \mathscr{M}(R_0 \xrightarrow{i_1} \cdots \xrightarrow{i_N} R_N)$$

representing an equivalence class in $\pi_0 \mathscr{M}(R_*)$, we must produce another point in the same path component of $\mathscr{M}(R_*)$ which is in the image of $\mathsf{StagedRings}_N(X_*, R_*)$ under the above natural transformation. Altogether, this amounts to the data of an element $\{f_* : X_* \to R_*\} \in \mathsf{StagedRings}_N(X_*, R_*)$ along with commuting isomorphisms of formal groups $\varphi_n : F_n \to f_n^* \mathbb{G}_n$ over $\operatorname{Spp} R_n$ for $0 \leq n \leq N$. This data fits into the diagram displayed in Figure 1. Noting again that the case $n = 0$ is handled by the classical Lubin–Tate moduli problem, we assume that the solid maps have been constructed and that our inductive task is to construct the morphisms f_n and φ_n.

Now, let us expand the p-series of F_n as

$$[p]_{F_n}(x) = px + \sum_{i=2}^{\infty} a_i x^i.$$

By assumption, F_n has p-height h_n, and hence $\underline{R_n}$ is complete against the ideal

$$\mathfrak{a}_n := (p, a_2, \ldots, a_{p^{h_n - 1}})$$

and $a_{p^{h_n}}$ is invertible in $\underline{R_n}$. In other words, $[p]_{F_n}(x)$ has property $\mathbf{P}_n(\mathfrak{a}_n)$. In light of the isomorphism

$$F_n = i_n^* F_{n-1} \xrightarrow{i_n^* \varphi_{n-1}} i_n^* f_{n-1}^* \mathbb{G}_{n-1},$$

Lemma 20 implies that the p-series of $i_n^* f_{n-1}^* \mathbb{G}_{n-1}$ has property $\mathbf{P}_n(\mathfrak{a}_n)$ as well. This is exactly what is needed to conclude that the map $i_n \circ f_{n-1}$ extends uniquely along the composite

$$X_{n-1} \to u_{h_n}^{-1} X_{n-1} \to (u_{h_n}^{-1} X_{n-1})^\wedge_{I_{h_n}} = X_n.$$

[4] That is, it is valued in groupoids which are naturally weakly equivalent to sets, or equivalently whose objects admit no nontrivial automorphisms.

$$
\begin{tikzcd}
& \Gamma \arrow[dl] \arrow[dd, equals] \arrow[rr] & & F_0 \arrow[dl] \arrow[dd, "\varphi_0"] & \cdots \arrow[l] & & F_{n-1} \arrow[ll] \arrow[dl] \arrow[dd, "\varphi_{n-1}"] & & F_n \arrow[ll] \arrow[dl] \arrow[dd, dotted, "\varphi_n"] \\
\operatorname{Spp} k \arrow[rr, dashed] \arrow[dd, equals] & & \operatorname{Spp} R_0 \arrow[dd] & & \cdots \arrow[ll] & \operatorname{Spp} R_{n-1} \arrow[l] \arrow[dd] & & \operatorname{Spp} R_n \arrow[ll] \arrow[dd, dotted] & \\
& \Gamma \arrow[ul] \arrow[dd, equals] \arrow[rr] & & f_0^*\mathbb{G}_0 \arrow[ul] \arrow[dd] & \cdots \arrow[l] & & f_{n-1}^*\mathbb{G}_{n-1} \arrow[ll] \arrow[ul] \arrow[dd] & & f_n^*\mathbb{G}_n \arrow[ll] \arrow[ul, dotted] \arrow[dd, dotted] \\
\operatorname{Spp} k \arrow[rr, dashed] & & \operatorname{Spp} X_0 & & \cdots \arrow[ll] & \operatorname{Spp} X_{n-1} \arrow[l] & & \operatorname{Spp} X_n \arrow[ll] & \\
& \Gamma \arrow[ul] \arrow[rr] & & \mathbb{G}_0 \arrow[ul] & \cdots \arrow[l] & & \mathbb{G}_{n-1} \arrow[ll] \arrow[ul] & & \mathbb{G}_n . \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

FIGURE 1. Diagram of the morphisms in play in Theorem 30.

We take this extension to be the desired map f_n and the accompanying isomorphism φ_n to be

$$F_n = i_n^* F_{n-1} \xrightarrow{i_n^*\varphi_{n-1}} i_n^* f_{n-1}^* \mathbb{G}_{n-1} = f_n^* \mathbb{G}_n. \qquad \square$$

Remark 31 (*Ceci n'est pas une pipe*). Our original formulation of this moduli problem used a different source category. Let $\mathsf{StagedRings}'_N$ be defined as the category of commutative squares

$$
\begin{tikzcd}
{[N]}^{\text{discrete}} \arrow[r] \arrow[d] & {[N]} \arrow[d] \\
\mathsf{CompleteTopologicalRings} \arrow[r] & \mathsf{Rings},
\end{tikzcd}
$$

with morphisms commuting natural transformations of the vertical functors. Writing R' for the functor $R' : [N] \to \mathsf{Rings}$, this is to say: the morphism of rings $R'_n \to R'_{n+1}$ *within* a sequence need not be continuous, but we require that a morphism $R'_n \to S'_n$ across sequences be continuous. Using this category, we also define a second moduli problem $\mathscr{M}'$, identical in definition to $\mathscr{M}$ except that it uses $\mathsf{StagedRings}'_N$ as a source. Now suppose that we have a point $(F'_0, F'_1, \ldots) \in \mathscr{M}'(R'_*)$. Then we can also given a Lubin–Tate type theorem for $\mathscr{M}'$: there is again a unique sequence of compatible maps $f'_n : \underline{X_n} \to R'_n$, continuous in the I'_{h_n}-adic topology, expressing F'_n as the pullback $(f'_n)^*\mathbb{G}_n$.

However, this moduli problem "factors" through the moduli problem $\mathscr{M}$ considered here. Note that the set-theoretic realization functor has an obvious lift

$$\mathsf{StagedRings} \to \mathsf{StagedRings}',$$

where we topologize the realizations by, after length 0, taking our colimits of ind-systems and limits of pro-systems in topologized sets – i.e., in topological spaces – rather than simply in sets. (As the forgetful functor $\mathsf{Spaces} \to \mathsf{Sets}$ admits both adjoints, these will have the same underlying sets as before.) So, given some $R'_* \in \mathsf{StagedRings}'_N$ with R'_0 profinite, we can construct an N-staged pipe ring $R_* \in \mathsf{StagedRings}_N$ by taking $R_0 = R'_0$ and $R_n = R_0 \hat{\otimes}_{X_0} X_n$ for $n \geq 1$ (for an appropriate definition of $\hat{\otimes}$), and this will satisfy

$$\mathsf{StagedRings}'_N(\underline{R_*}, R'_*) \xrightarrow{\cong} \mathsf{StagedRings}'_N(\underline{X_*}, R'_*).$$

It is also worth pointing out that the moduli problem $\mathscr{M}'$ is often empty: since there is no topological restriction on the maps $R'_n \to R'_{n+1}$, there is no reason to think that an ideal of definition of R'_{n+1} will have as its preimage something contained in the ideal of definition of R'_n. In this situation, no formal group law can have the height property demanded by $\mathscr{M}'$. This is all to say that the aggregate category $\mathsf{PipeRings}_\infty$ directly gives "continuity"-like control over the internal morphisms in these staging sequences, which is not present in the category $\mathsf{StagedRings}'_N$ and whose absence allows for much more pathological behavior there.

Appendix A. The portrait of a pipe ring and basic examples

The point of algebraic geometry in general and schemes in particular is to provide very literally an interface between commutative algebra and geometry. In particular, a scheme comes with a (complicated and difficult) recipe for drawing a picture of it, which is extremely useful for building geometric intuition about the behavior of algebraic objects when it can even partially be carried out. In this appendix, we produce a construction that provides similar information, and we use it to describe a handful of instructive examples.

A.1. j-ideals.

We can further refine our definition of closed ideals. Notice that there are $(n+1)$ naturally occurring inclusions i_m of Pipes_n into Pipes_{n+1}. For instance, in the case of $n=0$, given a profinite set $\{X_\alpha\}_\alpha$ one can specify two new pro-ind-profinite sets i_0X and i_1X using the following pair of formulas:

$$\{\{(i_0X)_\beta\}_\gamma\}_\delta = X_\delta, \qquad\qquad \{\{(i_1X)_\beta\}_\gamma\}_\delta = X_\beta.$$

That is, to construct i_0X we consider the pro-system $\{X_\alpha\}_\alpha \in \mathsf{Pipes}_0$ as a constant system in $\operatorname{Ind}\mathsf{Pipes}_0$, then consider that as a constant system in $\operatorname{Pro}(\operatorname{Ind}\mathsf{Pipes}_0) = \mathsf{Pipes}_1$. On the other hand, to construct i_1X we consider each finite set $X_\alpha \in \mathsf{Pipes}_{-1}$ as a constant system in $\operatorname{Pro}\mathsf{Pipes}_{-1} = \mathsf{Pipes}_0$, then consider each of those as a constant system in $\operatorname{Ind}\mathsf{Pipes}_0$, and then finally piece these objects together using the original structure maps of X to get a system in $\operatorname{Pro}(\operatorname{Ind}\mathsf{Pipes}_0) = \mathsf{Pipes}_1$. The standard inclusion $\mathsf{Pipes}_0 \to \mathsf{Pipes}_1$ used in the sequential colimit defining Pipes_∞ is i_0. The general pattern is similar, and to capture a coarse part of this phenomenon we make the following definition:

Definition 32. We call a pipe ideal a *j-ideal* if it appears as the kernel of a map to an j-pipe ring.

Example 33. The inclusions $i_m : \mathsf{Pipes}_0 \to \mathsf{Pipes}_1$ can be used to produce examples of strange behavior in pipe rings. For instance, for a finite ring $R \in \mathsf{PipeRings}_{-1}$, let $R[\![x]\!]$ denote the 0-pipe ring

$$R[\![x]\!] := \{\cdots \to R[x]/x^3 \to R[x]/x^2 \to R\},$$

and consider the 1-pipe rings

$$\begin{aligned} Q_0 &= i_0(R[\![x]\!]) = \operatorname{const}(R[\![x]\!]), \\ Q_1 &= i_1(R[\![x]\!]) = \{\cdots \operatorname{const}(R[x]/x^3) \to \operatorname{const}(R[x]/x^2) \to \operatorname{const}(R)\}. \end{aligned}$$

There is a map $Q_0 \to Q_1$ given by a levelwise quotient $R[\![x]\!] \to R[x]/x^n$, and while this map realizes to the identity on set-theoretic realizations, no inverse map exists. Instead, its kernel is given by the nonzero pro-ideal $\{(x^n)\}_{n\geq 1}$, which (given the construction of Q_1) we see is a 1-ideal. Similar examples can be produced for n-pipe rings, $n > 1$.

Remark 34. Examples of this flavor also point out an interesting feature of the co/fine properties defined in Definition 4. The only cofine ideal which realizes to the zero ideal under set-theoretic realization is isomorphic to the zero ideal itself. Hence, no ideal stemming from this sort of construction can ever be cofine. From the perspective of Conjecture 7, this implies that it cannot suffice to simply define the class W of weak equivalences to be those maps which set-theoretically realize to isomorphisms.

A.2. Portraits.

In this section we describe a useful visualization for pipe rings.

Definition 35. We define $\mathsf{FilteredSpaces}$ to be the category of sequences

$$X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

of topological spaces with each map the inclusion of a closed subspace. A morphism in this category is a natural transformation of such diagrams (without further restriction).

Definition 36. We define a functor

$$\square : (\mathsf{PipeRings}_\infty)^{\mathrm{op}} \to \mathsf{FilteredSpaces},$$

called the *portrait* of a pipe ring. The filtration level $\square(R)_n$ is given by the set of those isomorphism classes of closed m-ideals of R which are of length $m \le n$. The topology on $\square(R)_n$ is generated by declaring that the closure of a point $I \in \square(R)_n$ is the collection of ideals which contain it. Hence, any closed ideal determines a sub-filtered space of the portrait $\square(R)$. This construction is functorial in the obvious way.

Remark 37. Our definition of the portrait functor lacks an accompanying locally ringed structure sheaf. We have a definition for the localization of a pipe ring, and so could make an ostensible guess as to a definition of the structure sheaf by mimicking the classical setting, but it doesn't appear to lead where we would like; see Remark 38. So, we omit it, with some disappointment.

A.2.1. *Example:* $k[\![x]\!] \to k(\!(x)\!)$. The basic recipe is to draw a point for each power of each closed prime n-ideal, to label the points by the index n, and to topologize the space by defining the closure of a point to be the collection of ideals which contain it.

As a first example, let's begin with $k[\![x]\!]$, where k is a finite field. This 0-pipe ring has a single closed point corresponding to the kernel of the map $k[\![x]\!] \to k$; we represent it by a large dot. Then, there is an ascending sequence of closed nilpotent thickenings of this closed point given by maps

$$k[\![x]\!] \to k[x]/x^n$$

onto nilpotent extensions of k. Each of these targets is a finite ring, and so the corresponding ideal is a (-1)-ideal. Together, we imagine these as an infinite sequence of smaller dots, topologized as indicated. Finally, there is a single 0-ideal, corresponding to the identity map and the ideal (0), which we draw as a fuzzy dot at the limit of the smaller ones. As the generic point, its closure is the entire space.

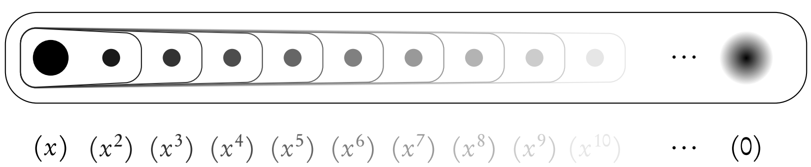

FIGURE 2. Portrait of $k[\![x]\!]$. The solid black dots correspond to (-1)-ideals, the fuzzy dot to a 0-ideal.

Given a k-algebra R, continuous k-algebra maps $k[\![x]\!] \to R$ pick out elements which can be either nilpotent or merely topologically nilpotent. The difference is detected by whether the corresponding ideal is a (-1)- or 0-ideal — an observation that will be useful as we now study the ind-profinite ring $k(\!(x)\!) = x^{-1}k[\![x]\!]$. This ind-object is defined by iterating the multiplication-by-x map, which has a visible action on the ideals: it sends (x^n) to (x^{n+1}) and (0) to (0). Taking the colimit, we see that $\square(k(\!(x)\!))$ is supported on the subset (0) of $\square(k[\![x]\!])$, and the portrait for $k(\!(x)\!)$ reflects this. We label the point corresponding to (0) with a 1, to reflect that it's the kernel of a morphism with target in $\mathsf{PipeRings}_1$. It is also immediately apparent from functoriality of the portrait construction that a morphism of pipe rings $k(\!(x)\!) \to R$ cannot send x to a nilpotent element, as the inclusion of the portrait for $k[\![x]\!]/x^n$ into the one for $k[\![x]\!]$ evidently does not factor through the one for $k(\!(x)\!)$.

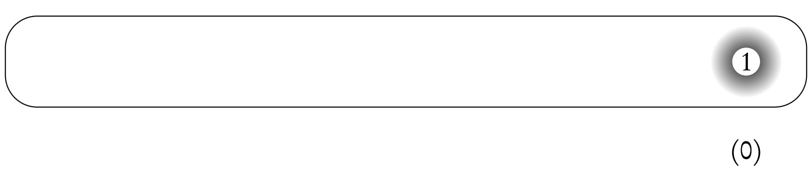

FIGURE 3. Portrait of $k(\!(x)\!) = x^{-1}k[\![x]\!]$.

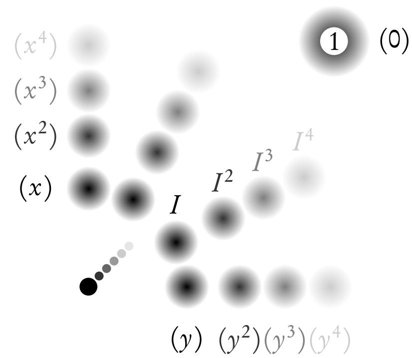

FIGURE 4. Portrait of $k[\![x,y]\!]$. Here, I is a closed prime 0-ideal. The horizontal and vertical axes are labeled by (y) and (x) respectively, as these are the subschemes selected by these ideals.

Remark 38. Having worked this example, one can explore the obvious definition of the locally ringed structure sheaf that could be attached to this space. The local ring assigned to the point (x) is the ring $k[\![x]\!]$ itself, but every other point is assigned the field $k(\!(x)\!)$, making it too coarse an invariant to effectively distinguish between the other points we've added. It's unclear how to correct that construction to something more sensitive.

Remark 39. This same method produces portraits for $\mathbb{Z}_p$ and $\mathbb{Q}_p$, isomorphic to the ones given above, though it's helpful when working to keep an "arithmetic direction" distinct from the rest.

Remark 40. Throughout, we have taken k to be a finite field, and this is an inescapable feature of our set-up. Similar to $k(\!(x)\!)$ in this example, more general local fields can be instantiated as n-pipe rings for $n > -1$; see work of Kato [12, Section 1.2].

A.2.2. *Example:* $k[\![x,y]\!] \to y^{-1}k[\![x,y]\!] \to (y^{-1}k[\![x,y]\!])^{\wedge}_{(x)}$. This is the main example. As before, we proceed in stages, beginning with the 0-pipe ring $k[\![x,y]\!]$. This has both closed (-1)-ideals and 0-ideals: examples of (-1)-ideals include (x,y), (x^i,y^j) for $i,j>0$, and (x^2,xy,y^2), whereas examples of 0-ideals include (x), $(x+y)$, (x^n) for $n>0$, and (xy). To analyze the topology, notice first that the closure of an n-ideal may only contain m-ideals for $m \le n$. For instance, the 0-ideal (x^2) is contained in both the 0-ideal (x) and the (-1)-ideal (x^2,y).

To give just an abbreviation of the full portrait, we draw only the points for powers of prime ideals, as in Figure 4. The closed point sits at the bottom left, together with its string of powers. The prime 0-ideals are also drawn in, with their powers marching away along lines of their own. The ideal (0) is the generic point: it is not contained in the closure of any other point, and its closure is the entire space. This abbreviation is reasonable because the portrait is topologized: we are working in a UFD, so an arbitrary closed ideal will be uniquely characterized by a finite collection of these points. We provide pictures of the closures of the non-prime ideals (y^3) and $(x^3(x+y)^2y)$ in Figure 5.

We now seek to understand the intermediate ring $y^{-1}k[\![x,y]\!]$. Just as before, multiplication by y sends a closed set corresponding to I to the closed set corresponding to yI. To produce a portrait of this pipe ring, we identify the closed sets associated to I and J when

$$\bigcup_k \overline{y^k I} = \bigcup_k \overline{y^k J}.$$

In terms of the abbreviated portrait, this action admits a simple description: a closed set in the original portrait is sent to its union with $\{(y),(y^2),\ldots\}$.

However, there are an awful lot of ideals (i.e., closed sets) in $y^{-1}k[\![x,y]\!]$. This uncomfortable fact is completely done away with by completing against the ideal (x), producing the ring $(y^{-1}k[\![x,y]\!])^{\wedge}_{(x)}$. The Weierstrass preparation theorem tells us that any element $f(x,y)$ of this ring takes the form

$$f(x,y) = x^n \cdot g(x,y),$$

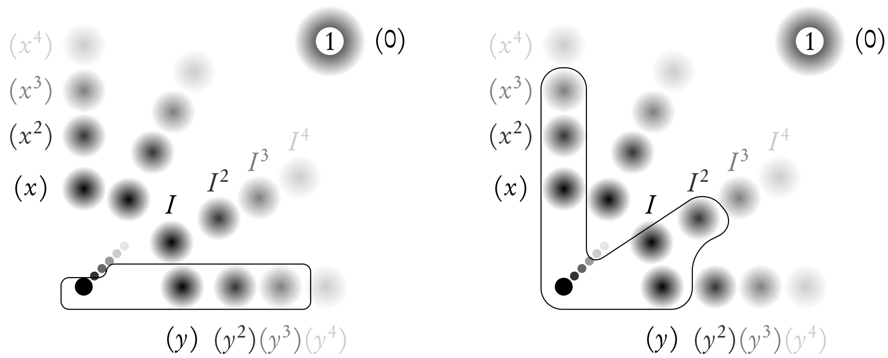

FIGURE 5. The closures of the ideals (y^3) and $(x^3(x+y)^2y)$ in $k[\![x,y]\!]$. Here, I now represents the specific closed prime 1-ideal $(x+y)$.

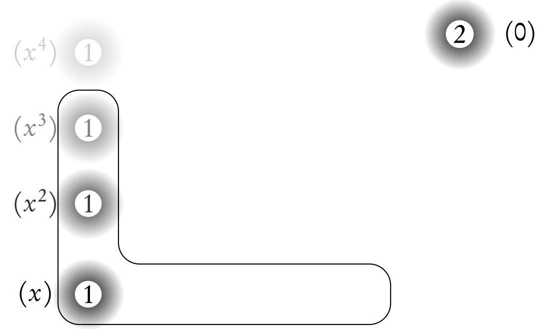

FIGURE 6. Portrait of $(y^{-1}k[\![x,y]\!])^{\wedge}_{(x)}$, with the closure of (x^3) indicated. This closed set has been drawn with the distention to suggest that it's been inherited from the previous stage $y^{-1}k[\![x,y]\!]$ of the construction.

where $g(x,y)$ is a unit (or equivalently, considered as a power series in x its constant coefficient is a unit). It is instructive to note that this completion allows for power series that extend infinitely in both directions in y, provided that the coefficients of the negative powers of y lie in increasing powers of the ideal (x). This reflects the fact that there is an isomorphism (of pipe rings)

$$(y^{-1}k[\![x,y]\!])^{\wedge}_{(x)} \cong k(\!(y)\!)[\![x]\!].$$

Pictorially, this completion also has a very simple description: two closed sets are identified if they have the same intersection with the set $\{(x),(x^2),\ldots\}$. For example, the closed sets defined by the ideals $(x^2(x+y)^3y^2)$ and (x^2) become identified because $(x+y)^3y^2$ is now a unit.

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