EQUIVARIANT K-THEORY

GRAEME SEGAL, Oxford

§ 1. INTRODUCTION AND PRELIMINARIES

The purpose of this paper is to set down the basic facts about equivariant K-theory. The theory was invented by Professor Atiyah, and most of the results are due to him. Various applications can be found in [2], [6], [11]. I shall assume the reader has some acquaintance with ordinary K-theory ([5], [3], [4], [2]), and shall only sketch the development ot the equivariant theory when it is parallel to the ordinary case.

The theory is defined on spaces with group action: Let us choose a fixed topological group G; then a G-space is a topological space X together with a continuous action $G \times X \to X$, written $(g, x) \mapsto g.x$, satisfying the usual conditions $g.(g'.x) = (gg')x$ and $1.x = x$.

We are going to construct a cohomology theory by considering the equivariant vector bundles on G-spaces: I shall have to begin with a collection of definitions and simple facts concerning equivariant vector bundles, modelled precisely on the discussion in [3].

If X is a G-space, a G-*vector bundle* on X is a G-space E together with a G-map $p : E \to X$ (i.e. $p(g.\xi) = g.p(\xi)$) such that

(i) $p : E \to X$ is a complex vector bundle on X, i.e. the fibres $E_x = p^{-1}(x)$ for $x \in X$ are finite-dimensional complex vector spaces, and the situation is locally trivial in a familiar sense [3], and

(ii) for any $g \in G$ and $x \in X$ the group action $g : E_x \to E_{gx}$ is a homomorphism of vector spaces.

If, for example, X is a trivial G-space (i.e. $g.x = x$ for all $g \in G$ and $x \in X$) a G-vector bundle is a family of representations E_x of G parametrized by the points x of X and varying continuously with x in a certain sense.

G-vector bundles are fairly common in nature. I shall mention three kinds of example:

a) If X is a differentiable manifold and G is a Lie group which acts smoothly on X then the complexified tangent bundle $T_X \otimes \mathbf{C}$ of X is a G-vector bundle, and so are all the associated tensor bundles.

b) If E is any vector bundle on a space X then the k-fold tensor product $E \otimes \ldots \otimes E$

is in a natural way an S_k-vector bundle on X, where S_k is the symmetric group which permutes the factors of the product, and X is regarded as a trivial S_k-space.

c) Homogeneous vector bundles. Let us determine the G-vector bundles on the space of cosets G/H, when H is a closed subgroup of G. If $\pi : E \to G/H$ is such a G-vector-bundle then the fibre E_0 over the neutral coset is an H-module which, as we shall see, determines E completely. The action of G on E induces a map $G \times E_0 \to E$, which can be regarded as a map $\alpha : G \times_H E_0 \to E$, where $G \times_H E_0$ means the space of orbits of $G \times E_0$ under H, when H acts on it by $(h, g, \xi) \mapsto (gh^{-1}, h\xi)$. If G acts on $G \times_H E_0$ by $(g, g', \xi) \mapsto (gg', \xi)$ then α is a G-map, and is a homeomorphism: for one can construct its inverse as follows. Consider the homeomorphism $\beta : G \times E \to G \times E$ defined by $\beta(g, \xi) = (g, g^{-1}\xi)$. The inverse image under β of $G \times E_0$ is $G \times_{G/H} E = \{(g, \xi) \in G \times E : gH = \pi\xi\}$. The natural map $G \times_{G/H} E \to G \times_H E_0$ factorizes through the projection $(g, \xi) \mapsto \xi$ of $G \times_{G/H} E$ on to E, which is an open map. The resulting map $\widetilde{\beta} : E \to G \times_H E_0$ is the inverse of α. Thus any G-vector bundle on G/H it is of the form $G \times_H E_0$ for some H-module E_0.

Conversely, if H is locally compact, and E_0 is any H-module, $G \times_H E_0$ is a G-vector-bundle on G/H. The only thing in question is local triviality. Now if $G \to G/H$ is locally trivial then $G \times_H E_0$ looks locally like $(U \times H) \times_H E_0$, i.e. $U \times E_0$, where U is an open set of G/H, and so it is also locally trivial. This deals with the case when H is a Lie group ([13], p. 315). But in general one can write $G \times_H E_0 = (G/N) \times_{(H/N)} E_0$, where N is the kernel of the action of H on E_0. H/N is a Lie group, so we are reduced to the earlier case.

Of course if M is any G-module (finite-dimensional complex representation space of G) and X is any G-space one can form the G-vector bundle $X \times M$ on X, which I shall call *trivial*, and denote by **M** when there is no risk of confusion.

The sections of a G-vector bundle $E \xrightarrow{p} X$ are the maps $s : X \to E$ such that $ps = \mathrm{id}$. They form a vector space ΓE. If a section is a G-map it is called equivariant: the equivariant sections form a vector subspace $\Gamma^G E$ of ΓE which is the space of fixed points of the natural action of G on ΓE.

If E and F are two G-vector bundles on X one can form their sum $E \oplus F$, a G-vector bundle on X with $(E \oplus F)_x = E_x \oplus F_x$; and similarly the tensor product $E \otimes F$, and a bundle $Hom(E; F)$ with $(Hom(E; F))_x = \mathrm{Hom}(E_x; F_x)$.

A homomorphism $f : E \to F$ of G-vector bundles on X is a continuous G-map which induces a homomorphism of vector spaces $f_x : E_x \to F_x$ for each $x \in X$. The homomorphisms form a vector space isomorphic to $\Gamma^G Hom(E; F)$.

If $\varphi : Y \to X$ is a G-map of G-spaces, and E is a vector bundle on X, then one can form a G-vector bundle $\varphi^* E$ on Y with $(\varphi^* E)_y = E_{\varphi(y)}$, just as in the ordinary case. More generally, if Y is an H-space, X a G-space, $\alpha : H \to G$ a homomorphism, and $\varphi : Y \to X$ such that $\varphi(h.y) = \alpha(h).\varphi(y)$, then $\varphi^* E$ is an H-vector bundle on Y. If $i : Y \to X$ is the inclusion of a subspace, $i^* E$ is often written $E|Y$.

For the rest of this paper I shall assume that G is a *compact* group, and I shall continually perform integrations over G with respect to the Haar measure. (One can integrate any continuous function $G \to \Gamma$ with values in a hausdorff, locally convex, and complete topological vector space ([8], Chap. 3, § 4).) Also, for the most part I shall confine myself to compact G-spaces X.

Let E be a G-vector bundle on a compact G-space X. If the vector space ΓE is given the compact-open topology then G acts continuously on it in the sense that $G \times \Gamma E \to \Gamma E$ is continuous. For the continuity of the G-action $G \times \mathrm{Map}(X; E) \to \mathrm{Map}(X; E)$ follows from that of the map $G \times X \times \mathrm{Map}(X; E) \to E$ defined by $(g, x, s) \mapsto g . s(g^{-1}x)$. It is obvious that ΓE is hausdorff, locally convex, and complete. (It becomes a Banach space if one chooses a hermitian metric in E.) So one can " average " a section of E over the group to obtain an equivariant section.

We need a string of lemmas generalizing those of [3].

Proposition **(1.1)**. — *If* E *is a* G-*vector bundle on a compact* G-*space* X, *and* A *is a closed* G-*subspace of* X, *then an equivariant section of* $E | A$ *can be extended to an equivariant section of* E.

One simply extends the section arbitrarily, as in [3], and then averages it over G.

By applying Proposition (1.1) to the G-vector bundle *Hom*(E; F) we obtain, just as in [3]:

Proposition **(1.2)**. — *In the situation of* (1.1), *if* F *is another* G-*vector bundle on* X *and* $f : E | A \to F | A$ *is an isomorphism then there is a* G-*neighbourhood* U *of* A *in* X *and an isomorphism* $f : E | U \to F | U$ *extending* f.

And Proposition (1.2) implies in turn [3]:

Proposition **(1.3)**. — *If* $\varphi_0, \varphi_1 : Y \to X$ *are* G-*homotopic* G-*maps, and* Y *is compact, and* E *is a* G-*vector bundle on* X, *then* $\varphi_0^* E \cong \varphi_1^* E$.

Example. — (1.3) implies that the representations of a compact group are " discrete ". For if X is a path-connected trivial G-space then E is just a continuous family of G-modules $\{E_x\}_{x \in X}$, and (1.3) implies that $E_x \cong E_y$ for any $x, y \in X$.

We need to know also that G-vector-bundles can be constructed by clutching: if X is the union of compact G-subspaces X_1, X_2 with intersection A, and E_1, E_2 are G-vector bundles on X_1, X_2, and $\alpha : E_1 | A \to E_2 | A$ is an isomorphism, then there is a unique G-vector bundle E on X with isomorphisms $E | X_1 \cong E_1$, $E | X_2 \cong E_2$ compatible with α. The group G is irrelevant in the proof of this proposition, so I shall not repeat it.

Finally, if $f : E \to F$ is a morphism of G-vector bundles on X such that $f_x : E_x \to F_x$ is an isomorphism for each $x \in X$, then f is an isomorphism, i.e. it has an inverse. Again G is irrelevant.

§ 2. EQUIVARIANT K-THEORY

Let X be a compact G-space.

The set of isomorphism classes of G-vector bundles on X forms an abelian semi-group under $\oplus$. The associated abelian group is called $K_G(X)$: its elements are formal differences $E_0 - E_1$ of G-vector bundles on X, modulo the equivalence relation $E_0 - E_1 = E_0' - E_1' \Leftrightarrow E_0 \oplus E_1' \oplus F \cong E_0' \oplus E_1 \oplus F$ for some G-vector bundle F on X.

The tensor product of G-vector bundles induces a structure of commutative ring in $K_G(X)$.

If $\varphi : Y \to X$ is a G-map of compact G-spaces the functor $E \mapsto \varphi^* E$ induces a morphism of rings $\varphi^* : K_G(X) \to K_G(Y)$, so that K_G is a contravariant functor from compact G-spaces to commutative rings. A homomorphism $\alpha : H \to G$ induces a morphism of " restriction " $K_G(X) \to K_H(X)$; and, more generally, if $\varphi : Y \to X$ is a map from an H-space to a G-space compatible with α, one has $\varphi^* : K_G(X) \to K_H(Y)$.

If $G = 1$ one writes, of course, $K(X)$ for $K_G(X)$.

Examples. — (i) If X is a point then $K_G(X) \cong R(G)$, the *representation ring*, or *character ring*, of G (cf. [5], [16]) — for a G-vector bundle is then just a G-module. As a group $R(G)$ is the free abelian group generated by the set $\hat{G}$ of simple G-modules. In general $K_G(X)$ is an algebra over $R(G)$, because any G-space X has a natural map on to a point. (The morphism $R(G) \to K_G(X)$ is just $M \mapsto \mathbf{M}$.)

(ii) $K_G(G/H) \cong R(H)$ when H is a closed subgroup of G. For we have seen that the category of G-vector bundles on G/H is equivalent to the category of H-modules.

(iii) More generally, if X is a compact H-space one can form a compact G-space $(G \times X)/H = G \times_H X$. There is an embedding $\varphi : X \to G \times_H X$ which identifies X with the H-subspace $H \times_H X$ of $G \times_H X$. The restriction φ^* is an equivalence between G-vector bundles on $G \times_H H$ and H-vector bundles on X, inverse to the extension $E \mapsto G \times_H E$: the argument of § 1, ex. *c)* applies without change.

For any compact G-space X the projection of X onto its orbit space X/G induces a morphism $\mathrm{pr}^* : K(X/G) \to K_G(X)$. Now if G acts *freely* on X (i.e. $g.x = x \Leftrightarrow g = 1$) and E is a G-vector bundle on X, then E/G is a vector bundle on X/G. The only non-trivial point is to show that E/G is locally trivial, which is always the case if G is a compact *Lie* group (see [7], Chap. 7). But we shall see presently that a G-vector bundle on X is always pulled back from a G/N-vector bundle on X/N, where N is some normal subgroup of G such that G/N is a Lie group, so E/G is locally trivial in any case. The functor $E \mapsto E/G$ is inverse to the functor pr^*, in fact the natural G-maps $E \to X$, $E \to E/G$ induce an isomorphism $E \to X \times_{X/G} E/G = \mathrm{pr}^*(E/G)$; while if F is a vector bundle on X/G, the projection on to the second factor induces an isomorphism $(X \times_{X/G} F)/G = (\mathrm{pr}^* F)/G \to F$.

Thus we have proved

Proposition (2.1). — *If* G *acts freely on* X *then* $\mathrm{pr}^*: \mathrm{K}(\mathrm{X}/\mathrm{G}) \xrightarrow{\cong} \mathrm{K}_{\mathrm{G}}(\mathrm{X})$. *More generally, if* N *is a normal subgroup of* G *which acts freely on* X *then* $\mathrm{pr}^*: \mathrm{K}_{\mathrm{G}/\mathrm{N}}(\mathrm{X}/\mathrm{N}) \xrightarrow{\cong} \mathrm{K}_{\mathrm{G}}(\mathrm{X})$.

(Observing that $\mathrm{X}/\mathrm{N} \cong (\mathrm{G}/\mathrm{N}) \times_{\mathrm{G}} \mathrm{X}$, one can combine everything said so far into the statement that a homomorphism $\alpha: \mathrm{G} \to \mathrm{G}'$ induces an isomorphism $\mathrm{K}_{\mathrm{G}'}(\mathrm{G}' \times_{\mathrm{G}} \mathrm{X}) \to \mathrm{K}_{\mathrm{G}}(\mathrm{X})$ if $\ker(\alpha)$ acts freely on X.)

Now let us consider the other extreme case, when G acts trivially on X. Then we have a homomorphism $\mathrm{K}(\mathrm{X}) \to \mathrm{K}_{\mathrm{G}}(\mathrm{X})$ which gives a vector bundle the trivial G-action. Combining this with the natural map $\mathrm{R}(\mathrm{G}) \to \mathrm{K}_{\mathrm{G}}(\mathrm{X})$ we have a morphism of rings $\mathrm{R}(\mathrm{G}) \otimes \mathrm{K}(\mathrm{X}) \to \mathrm{K}_{\mathrm{G}}(\mathrm{X})$. In fact

Proposition (2.2). — *If* X *is a trivial* G*-space the natural map*

$$\mu: \mathrm{R}(\mathrm{G}) \otimes \mathrm{K}(\mathrm{X}) \to \mathrm{K}_{\mathrm{G}}(\mathrm{X})$$

is an isomorphism of rings.

Proof. — I shall prove this by constructing an inverse to μ. The point is to show that a G-vector bundle can be decomposed into isotypical pieces which are locally trivial vector bundles. Because G acts trivially on X, it acts in each fibre of a G-vector bundle E on X, and there is an operation of averaging over G in each fibre, varying continuously. That is to say, there is a projection operator (cf. [3]) in E whose image is the subset E^{G} of E pointwise invariant under G. So ([3], Lemma (1.4)) E^{G} is a vector bundle on X, and the functor $\mathrm{E} \mapsto \mathrm{E}^{\mathrm{G}}$ induces a homomorphism of abelian groups $\varepsilon: \mathrm{K}_{\mathrm{G}}(\mathrm{X}) \to \mathrm{K}(\mathrm{X})$. And similarly for any G-module M the functor $\mathrm{E} \mapsto Hom^{\mathrm{G}}(\mathbf{M}; \mathrm{E}) = (Hom(\mathbf{M}; \mathrm{E}))^{\mathrm{G}}$ induces a homomorphism $\varepsilon_{\mathrm{M}}: \mathrm{K}_{\mathrm{G}}(\mathrm{X}) \to \mathrm{K}(\mathrm{X})$. I assert that the map $\nu: \mathrm{K}_{\mathrm{G}}(\mathrm{X}) \to \mathrm{R}(\mathrm{G}) \otimes \mathrm{K}(\mathrm{X})$ defined by $\nu(\xi) = \sum_{[\mathrm{M}] \in \widehat{\mathrm{G}}} [\mathrm{M}] \otimes \varepsilon_{\mathrm{M}}(\xi)$ is the inverse of μ. If E is a G-vector bundle on X we have a canonical isomorphism $\bigoplus_{[\mathrm{M}] \in \widehat{\mathrm{G}}} (\mathbf{M} \otimes Hom^{\mathrm{G}}(\mathbf{M}; \mathrm{E})) \to \mathrm{E}$ (it is an isomorphism because $\bigoplus (\mathrm{M} \otimes \mathrm{Hom}(\mathrm{M}; \mathrm{E}_x)) \xrightarrow{\cong} \mathrm{E}_x$ for each fibre E_x), so $\mu \circ \nu = \mathrm{id}$. On the other hand $Hom^{\mathrm{G}}(\mathbf{M}_1; \mathbf{M}_2 \otimes \mathrm{E}) \cong Hom^{\mathrm{G}}(\mathbf{M}_1; \mathbf{M}_2) \otimes \mathrm{E}$ if G acts trivially on E, and the last bundle is E or 0 according as the simple G-modules M_1, M_2 are isomorphic or not. So $\nu \circ \mu = \mathrm{id}$, also.

Example. — If E is a vector bundle on a space X, I have mentioned that $\mathrm{E}^{\otimes k} = \mathrm{E} \otimes \ldots \otimes \mathrm{E}$ is an S_k-vector bundle, where S_k is the symmetric group. The functor $\mathrm{E} \mapsto \mathrm{E}^{\otimes k}$ induces a natural transformation $\mathrm{K}(\mathrm{X}) \to \mathrm{K}_{\mathrm{S}_k}(\mathrm{X})$. We know now that $\mathrm{K}_{\mathrm{S}_k}(\mathrm{X}) \cong \mathrm{R}(\mathrm{S}_k) \otimes \mathrm{K}(\mathrm{X})$, so for each element of $\mathrm{R}'(\mathrm{S}_k) = \mathrm{Hom}(\mathrm{R}(\mathrm{S}_k); \mathbf{Z})$ we obtain a natural transformation $\mathrm{K}(\mathrm{X}) \to \mathrm{K}(\mathrm{X})$. It turns out that the operations of this type generate in a certain sense all the operations in K-theory [1].

Remark. — Proposition (2.2) is one of the few statements in this paper which does not generalize directly to *real* equivariant K-theory. If E is a real G-vector bundle on a trivial G-space X then it must be decomposed $\coprod_{\mathrm{M}}(\mathbf{M}\otimes_{\mathrm{D_M}} Hom^{\mathrm{G}}(\mathbf{M};\mathrm{E})) \xrightarrow{\cong} \mathrm{E}$, where M runs through the simple real G-modules, $\mathrm{D_M}$ is the field of endomorphisms of M (i.e. $\mathrm{D_M}=\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$), and $Hom^{\mathrm{G}}(\mathbf{M};\mathrm{E})$ is a G-vector bundle over the field $\mathrm{D_M}$. Thus $\mathrm{K}\mathbf{R}_{\mathrm{G}}(\mathrm{X})\cong(\mathrm{K}\mathbf{R}(\mathrm{X})\otimes\mathrm{R}(\mathrm{G};\mathbf{R}))\oplus(\mathrm{K}(\mathrm{X})\otimes\mathrm{R}(\mathrm{G};\mathbf{C}))\oplus(\mathrm{K}\mathbf{H}(\mathrm{X})\otimes\mathrm{R}(\mathrm{G};\mathbf{H}))$, where $\mathrm{R}(\mathrm{G};\mathrm{D})$ is the free abelian group generated by the simple real G-modules M with $\mathrm{D_M}=\mathrm{D}$.

I should record also the following consequence of (1.3).

Proposition **(2.3)**. — *If* $\varphi_0, \varphi_1 : \mathrm{Y}\to\mathrm{X}$ *are* G-*homotopic* G-*maps then*

$$\varphi_0^*=\varphi_1^* : \mathrm{K_G}(\mathrm{X}) \to \mathrm{K_G}(\mathrm{Y}).$$

Despite the simple results we have obtained we still know very little about the elements of $\mathrm{K_G}(\mathrm{X})$. The following proposition is fundamental for the development of the theory.

Proposition **(2.4)**. — *If* E *is a* G-*vector bundle on* X *there is a* G-*module* M *and a* G-*vector bundle* $\mathrm{E}^{\perp}$ *such that* $\mathrm{E}\oplus\mathrm{E}^{\perp}\cong\mathbf{M}$.

Proof. — Observe that it suffices to embed E in some **M**. For one can choose a G-invariant hermitian metric in M and can define $\mathrm{E}^{\perp}$ as the orthogonal complement of E in **M**. Similarly it suffices to find a surjection $\mathbf{M}\to\mathrm{E}$: one defines $\mathrm{E}^{\perp}$ as its kernel.

The proof depends on the following formulation of the Peter-Weyl theorem.

Theorem **(2.5)** ([12], p. 31). — *Let* Γ *be a topological vector space which is locally convex, hausdorff, and complete. If* G *acts continuously on* Γ (i.e. $\mathrm{G}\times\Gamma\to\Gamma$ *is continuous*), *and* Γ_a *is the union of the finite-dimensional invariant subspaces of* Γ, *then* Γ_a *is dense in* Γ. (Γ_a *is the image of the canonical injection* $\bigoplus_{[\mathrm{M}]\in\widehat{\mathrm{G}}}(\mathrm{M}\otimes\mathrm{Hom}^{\mathrm{G}}(\mathrm{M};\Gamma))\to\Gamma$.)

I apply the theorem when $\Gamma=\Gamma\mathrm{E}$ is the Banach space of sections of E. For any $x\in\mathrm{X}$ one can choose a finite set σ_x of sections of E such that $\{s(x)\}_{s\in\sigma_x}$ spans E_x. Because Γ_a is dense in Γ, and the evaluation map $\Gamma\to\mathrm{E}_x$ is continuous, one can suppose $\sigma_x\subset\Gamma_a$. The set $\{s(y)\}_{s\in\sigma_x}$ spans E_y for all y in a neighbourhood U_x of x. Suppose $\mathrm{U}_{x_1},\ldots,\mathrm{U}_{x_n}$ cover X. Let $\sigma=\bigcup_i\sigma_{x_i}$, and let M be the finite-dimensional G-subspace of Γ generated by σ. Then the evaluation map $\mathrm{X}\times\mathrm{M}\to\mathrm{E}$ is the required surjection.

Of numerous consequences of Proposition (2.4) I shall mention two.

(i) Two G-vector bundles E, E′ on X are called *stably equivalent* if there exist G-modules M, M′ such that $\mathrm{E}\oplus\mathbf{M}\cong\mathrm{E}'\oplus\mathbf{M}'$. Proposition (2.4) implies that the stable equivalence classes of G-vector bundles on X form an abelian group under $\oplus$. This group is called $\widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{X})$; it can be identified naturally with a quotient group of $\mathrm{K_G}(\mathrm{X})$.

(ii) Let M be a G-module, and let $\mathrm{Gr}(n,\mathrm{M})$ be the G-space of n-dimensional subspaces of M, with the usual topology, and let $\mathrm{Gr}(\mathrm{M})$ be the topological sum of all $\mathrm{Gr}(n, \mathrm{M})$. There is a canonical G-vector bundle $\mathrm{E_M} = \{(\xi, \mathrm{A}) : \xi \in \mathrm{A}\} \subset \mathrm{M} \times \mathrm{Gr}(\mathrm{M})$ on $\mathrm{Gr}(\mathrm{M})$, and Proposition (2.4) can be interpreted as the statement that any E on X is of the form $\varphi^* \mathrm{E_M}$ for some M and some G-map $\varphi : \mathrm{X} \to \mathrm{Gr}(\mathrm{M})$. (This justifies the reduction principle which I used in proving (2.1) above, because any G-module M is really a (G/N)-module, where G/N is the image of G in Aut(M), which is a Lie group.)

Now we can begin the topological study of the functor $\mathrm{K_G}$. As the discussion to follow is unaffected by the presence of the group G, I shall be fairly brief.

To begin with I shall work in the category of compact G-spaces with base point. (I shall call all base points 0; of course $g.0 = 0$ for all $g \in \mathrm{G}$.) If X is such a space I write CX for the *reduced* cone on X, i.e. CX is obtained from $\mathrm{X} \times [0, 1]$ by shrinking to a point the subspace $(\mathrm{X} \times 0) \cup (0 \times [0, 1])$. ($[0, 1]$ is the unit interval in $\mathbf{R}$.) If $i_1 : \mathrm{X} \to \mathrm{Y}_1$, $i_2 : \mathrm{X} \to \mathrm{Y}_2$ are two inclusions of compact G-spaces with base point then $\mathrm{Y}_1 \amalg_{\mathrm{X}} \mathrm{Y}_2$ means the space obtained from the topological sum $\mathrm{Y}_1 \amalg \mathrm{Y}_2$ by identifying $i_1(x)$ with $i_2(x)$ for each $x \in \mathrm{X}$. There is an obvious embedding of X in CX, and $\mathrm{CX} \amalg_{\mathrm{X}} \mathrm{CX}$ is called the *reduced suspension* of X, and written SX.

Proposition **(2.6)**. — *If* X *is a compact* G*-space with base point, and* A *is a closed* G*-subspace (with the same base point), then the sequence*

$$\widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}) \to \widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{X}) \to \widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{A})$$

is exact.

Proof. — The composition is zero because $\mathrm{A} \to \mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}$ is null-homotopic. On the other hand if the bundle E on X represents an element of $\widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{X})$ which vanishes in $\widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{A})$ then $\mathrm{E}|\mathrm{A} \oplus \mathbf{M} \cong \mathbf{N}$ for some M, N. Form a bundle E on $\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}$ by clutching $\mathrm{E} \oplus \mathbf{M}$ to $\mathbf{N}$ on CA by this isomorphism. Then E represents the desired element of $\widetilde{\mathrm{K}}_{\mathrm{G}}(\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA})$.

Let us iterate this proposition: first attach a cone to $\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}$ on the subspace X to obtain $\mathrm{CX} \amalg_{\mathrm{A}} \mathrm{CA}$; then attach a cone to $\mathrm{CX} \amalg_{\mathrm{A}} \mathrm{CA}$ on the subspace $\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}$ to obtain $\mathrm{CX} \amalg_{\mathrm{X}} \mathrm{C}(\mathrm{X} \amalg_{\mathrm{A}} \mathrm{CA}) \cong \mathrm{CX} \amalg_{\mathrm{X}} \mathrm{CX} \amalg_{\mathrm{CA}} \mathrm{C}(\mathrm{CA})$. There is a natural map $\mathrm{SX} = \mathrm{CX} \amalg_{\mathrm{X}} \mathrm{CX} \to \mathrm{CX} \amalg_{\mathrm{X}} (\mathrm{CX} \amalg_{\mathrm{CA}} \mathrm{CCA})$, and the diagram

$$\begin{tikzcd}
\mathrm{SA} \arrow[r] \arrow[d] & \mathrm{CX} \amalg_{\mathrm{A}} \mathrm{CA} \arrow[d] \\
\mathrm{SX} \arrow[r] & \mathrm{CX} \amalg_{\mathrm{X}} (\mathrm{CX} \amalg_{\mathrm{CA}} \mathrm{CCA})
\end{tikzcd}$$

commutes up to G-homotopy. In fact on the left-hand CA in SA it commutes trivially; on the right-hand CA we have the two different natural maps $\mathrm{CA} \to \mathrm{C}(\mathrm{CA})$, which are homotopic relative to A. Moreover $\mathrm{SA} \to \mathrm{CX} \amalg_{\mathrm{A}} \mathrm{CA}$ is a G-homotopy-equivalence; and so is $\mathrm{SX} \to \mathrm{CX} \amalg_{\mathrm{X}} (\mathrm{CX} \amalg_{\mathrm{CA}} \mathrm{CCA})$, because $\mathrm{CX} \to \mathrm{CX} \amalg_{\mathrm{CA}} \mathrm{C}(\mathrm{CA})$ is a homotopy-equivalence relative to X, because C(CA) collapses to CA leaving CA fixed. So we have an exact sequence

$$(*) \qquad \widetilde{K}_G(SX) \to \widetilde{K}_G(SA) \to \widetilde{K}_G(X \amalg_A CA) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A),$$

where the first map is induced by $A \to X$.

Definition **(2.7)**. — *If* X *is a compact* G-*space with base point, and* A *is a closed* G-*subspace, define* (*for any* $q \in \mathbf{N}$)

$$\widetilde{K}_G^{-q}(X) = \widetilde{K}_G(S^q X), \qquad \textit{where } S^q X = S(\ldots S(SX)\ldots),$$

and

$$\widetilde{K}_G^{-q}(X, A) = \widetilde{K}_G(S^q(X \amalg_A CA)).$$

Thus

$$\widetilde{K}_G^{-q}(X, o) = \widetilde{K}_G^{-q}(X).$$

Because $S^q(X \amalg_A CA) = S^q X \amalg_{S^q A} CS^q A$ one has at once by iterating the sequence $(*)$ an exact sequence, infinite to the left

$$\ldots \to \widetilde{K}_G^{-q}(X, A) \to \widetilde{K}_G^{-q}(X) \to \widetilde{K}_G^{-q}(A) \to \widetilde{K}_G^{-q+1}(X, A) \to \ldots \to \widetilde{K}_G(X, A) \to \widetilde{K}_G(X) \to \widetilde{K}_G(A).$$

By the device of [10], Chap. 10 one can obtain from a cohomology theory defined on compact spaces with base point a theory defined on locally compact spaces without base point. If X is a locally compact G-space which is not compact, let X^+ denote its one-point compactification, a compact G-space with base point. If X is already compact, define $X^+ = X \amalg o$, the sum of X and a base point.

Definition **(2.8)**. — *If* X *is a locally compact* G-*space, and* A *is a closed subspace, define* $K_G^{-q}(X) = \widetilde{K}_G^{-q}(X^+)$ *and* $K_G^{-q}(X, A) = \widetilde{K}_G^{-q}(X^+, A^+)$. *Thus* $K_G^{-q}(X, \varnothing) = K_G^{-q}(X)$.

Example. — $K_G^{-q}(X) = K_G(X \times \mathbf{R}^q)$ and $K_G^{-q}(X, A) = K_G(X \times \mathbf{R}^q, A \times \mathbf{R}^q)$ for any locally compact G-space X and closed G-subspace A, for $(X \times \mathbf{R}^q)^+ \xrightarrow{\cong} S^q(X^+)$.

The groups so defined should be thought of as " K_G with compact supports ". (They form an " LC-theory " in the sense of [10].) They are functorial only for *proper* G-maps. However if X is compact the new $K_G^0(X)$ coincides with the original $K_G(X)$: there is a homomorphism $K_G(X) \to \widetilde{K}_G(X \amalg o)$ defined by extending G-vector bundles by giving them the fibre zero at the point o; and its inverse is defined by assigning to a G-vector bundle E on $X \amalg o$ the element $(E|X) - (E_0 \times X)$ of $K_G(X)$, where E_0 is the fibre of E at o. A G-vector bundle E on X does not define an element of $K_G(X)$ unless X is compact, but, as we shall see, it does define a multiplication $\xi \mapsto \xi.[E]$ in $K_G(X)$.

$(X, A) \mapsto K_G(X, A)$ is a contravariant functor for proper maps. It is also a *covariant* functor for *open embeddings*, for if U is an open G-subspace of a locally compact G-space X there is a natural G-map $X^+ \to U^+$. We have the following excision theorem.

Proposition (**2.9**). — *If* A *is a closed* G-*subspace of a locally compact* G-*space* X *then the natural map*

$$K_G^{-q}(X-A) \to K_G^{-q}(X, A)$$

is an isomorphism.

Proof. — $(X-A)^+ \cong (X^+-A^+)^+ \cong X^+/A^+$, so it suffices to show that $S^q(X^+ \amalg_{A^+} CA^+) = S^q X^+ \amalg_{S^q A^+} CS^q A^+ \to S^q(X^+/A^+) \cong$

$$S^q X^+/S^q A^+ \cong (S^q X^+ \amalg_{S^q A^+} CS^q A^+)/CS^q A^+$$

induces an isomorphism in $\widetilde{K}_G$. That follows from

Proposition (**2.10**). — *If* A *is a closed* G-*contractible subspace of a compact* G-*space* X *then* $K_G(X/A) \xrightarrow{\cong} K_G(X)$.

Proof. — Given a G-vector bundle E on X we construct a bundle $\check{E}$ on X/A as follows. Because A is contractible, $E|A \cong \mathbf{M}$ for some G-module M. Extend this isomorphism to an open G-neighbourhood U of A in X. Now $X-A \cong X/A-A/A$. Construct $\check{E}$ by clutching $E|X-A$ and $M \times (U/A)$ by the isomorphism between them on $(X/A-A/A) \cap (U/A) \cong U-A$. One must check that the isomorphism class of $\check{E}$ depends only on E; $E \mapsto \check{E}$ is then obviously additive, and defines a map $K_G(X) \to K_G(X/A)$ inverse to the natural map.

The following continuity property of K_G is often useful.

Proposition (**2.11**). — *If* $\mathcal{J}$ *is a filtering family of pairs of closed* G-*subspaces of a locally compact* G-*space* X *then*

$$\varinjlim_{(Y,B)\in\mathcal{J}} K_G^*(Y, B) \xrightarrow{\cong} K_G^*\Big(\bigcap_{(Y,B)\in\mathcal{J}} Y, \bigcap_{(Y,B)\in\mathcal{J}} B\Big).$$

In particular, if A *is a closed* G-*subspace of* X *then* $\varinjlim K_G^*(U) \xrightarrow{\cong} K_G^*(A)$ *when* U *runs through the closed* G-*neighbourhoods of* A.

(*Filtering* means that if $(Y, B), (Y', B') \in \mathcal{J}$ then there is $(Y'', B'') \in \mathcal{J}$ such that $(Y'', B'') \subset (Y \cap Y', B \cap B')$.)

Proof. — Because $\bigcap S^q(Y^+) = S^q(\bigcap Y^+)$, and

$$\bigcap(Y^+ \amalg_{B^+} CB^+) = (\bigcap Y^+) \amalg_{(\bigcap B^+)} C(\bigcap B^+)$$

it suffices to show that $\varinjlim \widetilde{K}_G(Y) \xrightarrow{\cong} \widetilde{K}_G(\bigcap Y)$ when all the Y are compact and have a common base point.

If any bundle on $A = \bigcap Y$ can be extended to a neighbourhood of A in X then the last map is surjective, because any neighbourhood contains some Y. On the other hand, if E and E′ in $\widetilde{K}_G(Y)$ define the same element of $\widetilde{K}_G(A)$, then $(E|A) \oplus \mathbf{N} \cong (E'|A) \oplus \mathbf{N}'$ for some G-modules N and N′, and this isomorphism can be extended to some Y′, so that E and E′ have the same image in $\widetilde{K}_G(Y')$.

To prove that a bundle E on a closed subspace A can be extended to a neighbourhood of A one can, for example, proceed as follows. Express E as the image of a projection operator in a trivial bundle **M**. The operator can be identified with a continuous G-map $\alpha : A \to \mathrm{Proj}(M)$, where Proj(M) is the set of projection operators in M, regarded as a closed G-subspace of the vector space End(M) of endomorphisms of M. It suffices to show that α can be extended to a neighbourhood of A. First extend α to a G-map $\beta : X \to \mathrm{End}(M)$. Let V be the open subset of the vector-space End(M) consisting of endomorphisms with no eigenvalues on the circle $\gamma = \left\{ z \in \mathbf{C} : |z-1| = \frac{1}{2} \right\}$. Then $T \mapsto \rho(T) = \frac{1}{2\pi i} \int_\gamma (z-T)^{-1}\, dz$ is an equivariant retraction of V onto Proj(M), and $\rho\beta : \beta^{-1}(V) \to \mathrm{Proj}(M)$ is the required extension of α.

Remark. — With a little more effort one can show that $\varinjlim K_G^*(X_\alpha) \xrightarrow{\cong} K_G^*(\varprojlim X_\alpha)$ for any directed inverse system of compact G-spaces.

Corollary **(2.12)**. — *If* X *is a locally compact* G-*space, then* $\varinjlim K_G^*(U) \xrightarrow{\cong} K_G^*(X)$, *where* U *runs through the relatively compact open* G-*subspaces of* X, *or, more generally, through any antifiltering open covering of* X.

(A covering $\mathscr{U}$ is *antifiltering* if for all $U, V \in \mathscr{U}$ there is $W \in \mathscr{U}$ such that $U \cap V \subset W$.)

§ 3. COMPLEXES, THE THOM HOMOMORPHISM, PERIODICITY

For many purposes it is convenient to know that K_G can be defined by complexes of G-vector bundles. Once again the group G is not relevant, so I shall simply state the result here. There is a proof in the compact case in [2]; and a proof for CW-complexes in [4]. The general case is no more difficult, but nevertheless I shall give in an appendix to this paper a proof in a slightly different spirit from that of [2] and [4].

A *complex* on a G-space is a sequence

$$E^\bullet : \ldots \xrightarrow{d} E^{i-1} \xrightarrow{d} E^i \xrightarrow{d} E^{i+1} \xrightarrow{d} \ldots \qquad (i \in \mathbf{Z})$$

of G-vector bundles on X such that $E^i = 0$ when $|i|$ is large, and of homomorphisms d such that $d^2 = 0$. A morphism of complexes $f : E^\bullet \to F^\bullet$ is a sequence of morphisms $f^i : E^i \to F^i$ such that $fd = df$. The complex $E^\bullet$ is called *acyclic* if the sequence $E_x^\bullet$ of vector spaces is exact for all x in X.

The *support* of a complex $E^\bullet$ is the closed G-subset of X consisting of the points x for which $E_x^\bullet$ is not exact. I shall write it $\mathrm{supp}(E^\bullet)$. It is closed because if f_x is a homomorphism of vector spaces depending continuously on x then $\mathrm{rank}(f_x)$ is a lower-semi-continuous function of x; and $\dim(\ker f_x)$ is upper-semi-continuous.

If A is a closed G-subspace of a locally compact G-space X, let $L_G(X, A)$ be the set of isomorphism classes of complexes $E^\bullet$ on X such that $\mathrm{supp}(E^\bullet)$ is a *compact* subset of $X-A$. The set $L_G(X, A)$ is a semi-group under direct sum. Two elements $E_0^\bullet$, $E_1^\bullet$ of

$L_G(X, A)$ are called *homotopic*, $E_0^\bullet \simeq E_1^\bullet$, if there is an object $E^\bullet$ of $L_G(X \times [0, 1], A \times [0, 1])$ such that $E_0^\bullet = E^\bullet | (X \times 0)$ and $E_1^\bullet = E^\bullet | (X \times 1)$. Introduce the equivalence relation $\sim$ in $L_G(X, A)$ defined by

$$E_0^\bullet \sim E_1^\bullet \Leftrightarrow E_0^\bullet \oplus F_0^\bullet \simeq E_1^\bullet \oplus F_1^\bullet$$

for some acyclic complexes $F_0^\bullet$ and $F_1^\bullet$ on X.

Proposition **(3.1)**. — $L_G(X, A)/\sim$ *is an abelian group naturally isomorphic to* $K_G(X, A)$.

This is easy when X is compact and $A = \emptyset$. The map $L_G(X, \emptyset) \to K_G(X)$ is simply $E^\bullet \mapsto \sum_k (-1)^k E^k$; it is trivially surjective, and is injective because a complex is homotopic to the complex obtained by replacing its differential by zero.

If $E^\bullet$ and $F^\bullet$ are complexes on X one can form their tensor product $E^\bullet \otimes F^\bullet$, with $(E^\bullet \otimes F^\bullet)^k = \bigoplus_{p+q=k} E^p \otimes F^q$. One has $\operatorname{supp}(E^\bullet \otimes F^\bullet) = \operatorname{supp}(E^\bullet) \cap \operatorname{supp}(F^\bullet)$. In view of Proposition (3.1) the tensor product of complexes induces a homomorphism

$$K_G(X, A) \otimes K_G(X, B) \to K_G(X, A \cup B),$$

which, when $A = B = \emptyset$, reduces to the product in the ring $K_G(X)$. This pairing is associative, and in particular it makes $K_G(X, A)$ into a commutative ring, which has a unit element if and only if $X - A$ is compact. (The product for relative groups can also be expressed as a product $K_G(U) \otimes K_G(V) \to K_G(U \cap V)$ for open G-subsets U, V of X.)

The product in $K_G(X)$ extends to make $K_G^*(X)$ into a graded ring. If $\xi_i \in K_G^{-p_i}(X)$ (for $i = 1, 2$) is represented by a complex $E_i^\bullet$ on $X \times \mathbf{R}^{p_i}$ with compact support, then the product $\xi_1 . \xi_2$ in $K_G^{-p_1-p_2}(X)$, is represented by the complex $\mathrm{pr}_1^* E_1^\bullet \otimes \mathrm{pr}_2^* E_2^\bullet$ on $X \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2}$, which also has compact support. ($\mathrm{pr}_i : X \times \mathbf{R}^{p_1} \times \mathbf{R}^{p_2} \to X \times \mathbf{R}^{p_i}$ is the projection.) To relativize this definition is automatic, and in any case unnecessary. The graded ring $K^*(X)$ is anticommutative: to see that one has to look at the effect of permuting the factors $\mathbf{R}$ in $K_G(X \times \mathbf{R}^p)$, and is immediately reduced to showing that if $\theta : X \times \mathbf{R} \to X \times \mathbf{R}$ is defined by $\theta(x, t) = (x, -t)$ then $\theta^* E^\bullet = -E^\bullet$ in $K_G(X \times \mathbf{R})$, when $E^\bullet$ is a complex on $X \times \mathbf{R}$ with compact support. But it is easy to see that $E^\bullet \oplus \theta^* E^\bullet$ is homotopic to an acyclic complex.

The most important application of Proposition (3.1) is the definition of the Thom homomorphism for K_G. First observe that if E is a G-vector bundle on X and s is an equivariant section of E one can form the *Koszul complex*

$$\dots \to 0 \to \mathbf{C} \xrightarrow{d} \Lambda^1 E \xrightarrow{d} \Lambda^2 E \xrightarrow{d} \dots$$

where d is defined by $d(\xi) = \xi \wedge s(x)$ if $\xi \in \Lambda^i E_x$. This complex is acyclic at all points x at which $s(x) \neq 0$, so its support is the set of zeros of s.

Now, if $p : E \to X$ is the projection, the bundle $p^* E$ on E has a natural section which is the diagonal map $\delta : E \to E \times_X E = p^* E$. This section δ vanishes precisely on the zero-section of E. I shall denote by $\Lambda_E^\bullet$ the Koszul complex on E formed from p^*E and δ.

If $F^\bullet$ is a complex with compact support on X then $p^*F^\bullet$ is a complex on E with support $p^{-1}(\operatorname{supp}(F^\bullet))$, and $\Lambda_E^\bullet \otimes p^*F^\bullet$ is a complex with compact support on E. The assignment $F^\bullet \mapsto \Lambda_E^\bullet \otimes p^* F^\bullet$ induces an additive homomorphism $\varphi_* : K_G(X) \to K_G(E)$ which is called the *Thom homomorphism*. (It is a homomorphism of $K_G(X)$-modules in the obvious sense.)

If $\varphi : X \to E$ is the zero-section, then $\varphi^* \varphi_*(F^\bullet)$ is just the alternating sum of the complexes $\Lambda^i E \otimes F^\bullet$, i.e. $\varphi^* \varphi_*(\xi) = \xi . \lambda_{-1} E$ for any $\xi \in K_G(X)$.

If X is compact, $\Lambda_E^\bullet$ has compact support and defines the *Thom class* $\varphi_*(1) = \lambda_E$ in $K_G(E)$.

By replacing X and E by $X \times \mathbf{R}^q$ and $E \times \mathbf{R}^q$ one obtains a Thom homomorphism $\varphi_* : K_G^{-q}(X) \to K_G^{-q}(E)$ for each $q \in \mathbf{N}$.

The most important theorem in equivariant K-theory is

Proposition **(3.2)**. — *The Thom homomorphism* $\varphi_* : K_G^*(X) \to K_G^*(E)$ *is an isomorphism for any* G-*vector bundle* E *on a locally compact* G-*space* X.

I shall not prove this theorem here, as the proof for a general group G uses families of elliptic differential operators. But I shall perform some reductions, and in particular prove it when G is abelian.

First observe that it suffices to prove

Proposition **(3.3)**. — *The Thom homomorphism* $\varphi_* : K_G(X) \to K_G(E)$ *is an isomorphism for any* G-*vector bundle on a compact* G-*space* X.

Proof that (3.3)$\Rightarrow$(3.2). — By the continuity (2.12) of K_G^* it suffices to show $\varphi_* : K_G^*(U) \xrightarrow{\cong} K_G^*(E|U)$ when U is a relatively compact open G-subspace of X. Then by the exact sequence for the pair $(\overline{U}, \overline{U} - U)$ one is reduced to the case of a compact base space. Finally because of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_G^{-q}(E) & \longrightarrow & K_G(E \times S^q) & \longrightarrow & K_G(E) & \longrightarrow & 0 \\
 & & \uparrow \varphi_* & & \uparrow \varphi_* & & \uparrow \varphi_* & & \\
0 & \longrightarrow & K_G^{-q}(X) & \longrightarrow & K_G(X \times S^q) & \longrightarrow & K_G(X) & \longrightarrow & 0,
\end{array}$$

where the rows are split exact sequences, one is reduced to the case of (3.3).

When E is a G-line bundle, Proposition (3.3) is a generalization of the Bott periodicity theorem, and is proved in [3]. (The equivariant case is not considered in [3], but the argument applies without change.) To be precise, in [3] it is proved that $K_G(P(E\oplus \mathbf{C}))$ is a free $K_G(X)$-module generated by the unit element 1 and the class of the Hopf bundle H. (If E is a G-vector bundle on X, P(E) denotes the associated bundle of projective spaces, whose points are lines in E. There is a canonical G-line bundle $H^* = \{(\xi, x) \in P(E)\times E : X\in\xi\}$ on P(E). The Hopf bundle H is the dual of H^*.) Now E can be identified with $P(E\oplus \mathbf{C}) - P(E)$, so $K_G(E)$ is the kernel of the restriction $K_G(P(E\oplus\mathbf{C})) \to K_G(P(E)) = K_G(X)$, and is generated by $H^* - \pi^*E$ or $\mathbf{C} - \pi^*E\otimes H$, where $\pi : P(E\oplus\mathbf{C}) \to X$ is the projection. Because H^* is a sub-bundle of $\pi^*(E\oplus\mathbf{C})$, there is a canonical morphism $H^* \to \pi^*E$ or $\mathbf{C} \to \pi^*E\otimes H$. Restricted to $E \subset P(E\oplus\mathbf{C})$ the last thing becomes the complex $\Lambda_E^\bullet$: observe that $H|E$ is canonically trivial. So $K_G(E)$ is the free $K_G(X)$-module generated by λ_E, as desired.

If (3.3) is known for line bundles then so is (3.2). And hence (3.2) is true whenever E is a sum of line bundles, because the Thom homomorphism is transitive:

Proposition **(3.4)**. — *If* E *and* F *are bundles on* X, *and* $p : E\oplus F \to E$, $q : E\oplus F \to F$ *are the projections, then* $\Lambda^\bullet_{E\oplus F} \cong p^*\Lambda^\bullet_E \otimes q^*\Lambda^\bullet_F$, *and the diagram*

$$\begin{tikzcd} K_G(X) \arrow[rr, "\varphi_*"] \arrow[dr, "\varphi_*"'] & & K_G(E\oplus F) \\ & K_G(E) \arrow[ur, "\varphi_*"'] & \end{tikzcd}$$

commutes.

Proof. — The first statement is trivial; the second follows from $\Lambda^\bullet_{E\oplus F} \cong p^*\Lambda^\bullet_E \otimes \Lambda^\bullet_{\pi^*F}$ (where $\pi : E \to X$), which is true because $\Lambda^\bullet_{\pi^*F} \cong q^*\Lambda^\bullet_F$.

Applying (3.3) to the trivial bundle $\mathbf{C}$ one finds

Proposition **(3.5)**. — $K_G^{-q}(X)$ *is naturally isomorphic to* $K_G^{-q-2}(X)$, *the map being multiplication by a certain element of* $K_G^{-2}(\text{point})$.

Proposition (3.5) suggests that one should define $K_G^q(X)$ for positive q as $K_G^{q-2n}(X)$, where $n \geqslant q/2$. Then one has cohomological exact sequences extending infinitely in both directions, which are very much more powerful tools than the semi-infinite ones which exist not only for K_G but for any " half-exact functor " (1) For example they permit one to prove the following:

Proposition **(3.6)**. — (3.2) *is true when* E *is locally a sum of* G-*line bundles.* (*Locally* means " in a neighbourhood of each orbit ".)

(1) But nevertheless it is usually convenient to regard $K_G^*(X)$ as graded modulo two. In the sequel $K_G^*(X)$ will mean $K_G^0(X) \oplus K_G^{-1}(X)$.

Proof. — One reduces oneself very simply to showing that if Y is a closed G-subspace of X such that $E|(X-Y)$ is a sum of line bundles, and $\varphi_* : K_G^*(Y) \to K_G^*(E|Y)$ is an isomorphism, then $\varphi_* : K_G^*(X) \to K_G^*(E)$ is an isomorphism. Because

$$\varphi_* : K_G^*(X-Y) \xrightarrow{\cong} K_G^*(E|(X-Y)),$$

that follows on applying the 5-lemma to the exact sequences for the pairs (X, Y) and $(E, E|Y)$.

In particular (3.2) is true when G is abelian, because

Proposition **(3.7)**. — *If* G *is abelian then any* G*-vector bundle* E *on* X *is locally a sum of* G*-line bundles.*

Given $x \in X$, let G_x be its stabilizer or isotropy group. The fibre E_x is a G_x-module, but one can extend the G_x-action to make it a G-module, because an inclusion $G_x \to G$ of abelian groups induces a surjection $\hat{G} \to \hat{G}_x$. Then $E|Gx \cong G \times_{G_x} E_x \cong (G/G_x) \times E_x$. So E and $X \times E_x$ are isomorphic on the orbit Gx, and hence in a neighbourhood of it. But $X \times E_x$ is a sum of G-line bundles, because E_x is a sum of one-dimensional modules.

As to the proof of (3.2) or (3.3) in the general case, it depends on the following proposition, whose proof involves families of elliptic differential operators.

Proposition **(3.8)**. — *If* G *is a compact connected Lie group, and* $i : T \to G$ *is the inclusion of a maximal torus, then for each locally compact* G*-space* X *there is a natural homomorphism of* $K_G^*(X)$*-modules* $i_* : K_T^*(X) \to K_G^*(X)$ *such that* $i_*(1) = 1$, *and hence* $i_* i^* =$ identity.

Observe that by considering $U \times_G X$ instead of X, and using Example (iii) of § 2, one need prove (3.8) only when G is a unitary group U.

If one allows (3.8) the proof of (3.3) is very simple. For (3.3) is stable under the operation of extending the group, and any compact Lie group can be embedded in a unitary group, so if G is a Lie group one reduces oneself first to a unitary group, and then to a torus, and then applies (3.7). In fact one can even avoid (3.7), for the G-bundle E on X can be lifted to a trivial $(G \times U(n))$-bundle **M** on its principal bundle P, so one can reduce oneself to the case $K_G(X) \to K_G(X \times M)$ when G is abelian; and the M is a sum of one-dimensional modules. The case when G is not a Lie group follows by a simple continuity argument.

To conclude this section I should point out that by standard arguments [2] using (3.2) and (3.8) one can calculate $K_G^*(P)$ in terms of $K_G^*(X)$ when P is a bundle of projective spaces, Grassmannians, Stiefel manifolds, flag manifolds, or lens spaces associated to a G-vector bundle on X. For example

Proposition **(3.9)**. — *If* E *is a* G*-vector bundle on* X *then* $K_G^*(P(E))$ *is generated as* $K_G^*(X)$*-algebra by the Hopf bundle* H, *modulo the relation* $\sum_k (-1)^k \Lambda^k E . H^k = 0$.

First proof. — I shall confine myself to the case when E is a sum of line bundles, $E = L_1 \oplus \ldots \oplus L_n$. Then one can proceed by induction on n. Let

$E_0 = L_1 \oplus \ldots \oplus L_{n-1}$. Then $E_0 \cong P(E) - P(E_0)$. The Hopf bundle on $P(E)$ satisfies the relation $\sum_k (-1)^k \Lambda^k E . H^k = 0$ over $K_G^*(X)$, because the bundle $\pi^* E \otimes H$ on $P(E)$ has a natural non-vanishing section, and so its Koszul complex is acyclic. Also, H restricts to the Hopf bundle on $P(E_0)$, and is trivial on E_0. Write $A = K_G^*(X)$, and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & A(H)/(f_n) & \xrightarrow{\beta} & A(H)/(f_{n-1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_{n-1}} & & \\ \ldots & \longrightarrow & K_G^*(E_0) & \longrightarrow & K_G^*(P(E)) & \xrightarrow{\rho} & K_G^*(P(E_0)) & \longrightarrow & \ldots \end{array}$$

Here $f_n = \sum_k (-1)^k H^k . \Lambda^k E$, $f_{n-1} = \sum_k (-1)^k H^k . \Lambda^k E_0$, so that $f_n = (1 - L_n . H) . f_{n-1}$; α is defined by $\alpha(1) = f_{n-1}$, making the top line exact; φ_* is the Thom isomorphism. The diagram commutes. We suppose θ_{n-1} is an isomorphism, and want to prove θ^n is one, too. Because β and θ_{n-1} are surjective, ρ is surjective, so one can add 0 at each end of the lower line. Then θ_n is an isomorphism by the 5-lemma.

Second proof. — $P(E)$ is the quotient by the group $\mathbf{T}$ of complex numbers of modulus 1 of $S(E)$, the sphere bundle of E. Because $\mathbf{T}$ acts freely on $S(E)$ one has $K_G(P(E)) = K_{G \times \mathbf{T}}(S(E))$. Let $D(E)$ be the disc bundle associated to E, so that $S(E)$ is the " boundary " of $D(E)$, and $D(E) - S(E) = E$. Then $D(E)$ is contractible to X, so $K_{G\times\mathbf{T}}^*(D(E)) \cong K_{G\times\mathbf{T}}^*(X) \cong K_G^*(X) \otimes R(\mathbf{T}) \cong A[H, H^{-1}]$, where $A = K_G^*(X)$. The group $K_{G\times\mathbf{T}}^*(E)$ is the same, by (3.2), and the map $K_{G\times\mathbf{T}}^*(X) \to K_{G\times\mathbf{T}}^*(X)$ corresponding to the inclusion $E \to D(E)$ is the multiplication by $\lambda_{-1}E$, where E is regarded as a $(G \times \mathbf{T})$-bundle. That means $\lambda_{-1}E = \sum_k (-1)^k \Lambda^k E . H^k = f_n \in A[H, H^{-1}]$.

One has an exact sequence

$$\ldots \to K_{G\times\mathbf{T}}^*(E) \to K_{G\times\mathbf{T}}^*(D(E)) \to K_{G\times\mathbf{T}}^*(S(E)) \to \ldots,$$

and can identify it with

$$\ldots \longrightarrow A[H, H^{-1}] \xrightarrow{x f_n} A[H, H^{-1}] \longrightarrow K_G^*(P(E)) \longrightarrow \ldots$$

The multiplication by f_n is injective, so one can add 0 at each end of the sequence, so $K_G^*(P(E))$ can be identified with $A[H, H^{-1}]/(f_n) = A[H]/(f_n)$, as desired. I leave it to the reader to check that the H in this proof is the same as the earlier one.

§ 4. LOCALIZATION

I have pointed out that $K_G^*(X)$ is an algebra over $R(G)$. It turns out that localizing it at a prime ideal of $R(G)$ corresponds to restricting one's attention to the set of fixed points of a conjugacy class of elements of G associated to the ideal.

An obvious prime of $R(G)$ consists of all the elements whose characters vanish at a certain element g of G. This prime depends only on the conjugacy class of the closed subgroup of G generated by g. In [16] I described all the prime ideals of $R(G)$, and showed that to each prime $\mathfrak{p}$ is associated a cyclic [1] subgroup S of G determined up to conjugation, which I called the *support* of $\mathfrak{p}$. The subgroup S is characterized as minimal among the subgroups of G such that $\mathfrak{p}$ is the inverse image of a prime of $R(S)$. If $\mathfrak{p}$ is the ideal of characters vanishing at g, then S is the cyclic subgroup generated by g. I proved in [16] that, if H is any subgroup of G, then $R(H)_{\mathfrak{p}}$, i.e. $R(H)$ regarded as an $R(G)$-module and localized at $\mathfrak{p}$, is non-zero if and only if S is conjugate to a subgroup of H.

I shall write G_x for the stabilizer or isotropy group of a point x of a G-space X, and, if S is a cyclic subgroup of G, I shall write $X^{(S)}$ for the closed G-subset of X consisting of points x for which S is conjugate to a subgroup of G_x. That means $X^{(S)}=G.X^S$, where X^S is the set of points left fixed by S.

The localization theorem is

Proposition **(4.1)**. — *If* X *is a locally compact* G-*space, and* $\mathfrak{p}$ *is a prime of* $R(G)$ *with support* S, *then the restriction*

$$K_G^*(X)_{\mathfrak{p}} \to K_G^*(X^{(S)})_{\mathfrak{p}}$$

is an isomorphism.

Remark. — In view of the preceding remarks, $X^{(S)}$ is precisely the union of the orbits T of X such that $K_G^*(T)_{\mathfrak{p}} \neq 0$, for $K_G^*(Gx) \cong R(G_x)$.

Proof of (4.1). — Because of the cohomological exact sequence it suffices to show that $K_G^*(X-X^{(S)})_{\mathfrak{p}}=0$, i.e. one can suppose $X^{(S)}=\emptyset$. So one has to show that $K_G^*(X)_{\mathfrak{p}}=0$ if $K_G^*(T)_{\mathfrak{p}}=0$ for each orbit T in X. By the continuity of K_G^* (2.11) it suffices to show that $K_G^*(U)_{\mathfrak{p}}=0$ for all relatively compact open G-subspaces of X. Then by the exact sequence for the pair $(\overline{U}, \overline{U}-U)$ one reduces oneself to the case of a compact G-space.

If X is a compact G-space then there exists a *slice* at each point x of X, i.e. a G_x-subspace S of X containing x such that the natural map $G\times_{G_x}S \to X$ is an open embedding ([7], Chap. 7). The projection $G\times_{G_x}S\to G/G_x$ is an equivariant retraction of a neighbourhood of the orbit Gx on to Gx. In view of this one can choose a finite number of points $x_1, \ldots, x_n$ of X with compact G-neighbourhoods $X_1, \ldots, X_n$ which cover X and are such that X_i admits a G-retraction onto the orbit T_i of x_i. Now, assuming that each $K_G^*(T_i)_{\mathfrak{p}}$ is zero, I want to prove $K_G^*(X)_{\mathfrak{p}}=0$. It suffices to show that if Y is a G-subspace of X such that $K_G^*(Y)_{\mathfrak{p}}=0$ then $K_G^*(Y\cup X_i)_{\mathfrak{p}}=0$. So it

---

[1] S is *cyclic* if it contains an element g whose powers are dense in S, i.e. if it is the product of a torus and a finite cyclic group.

suffices to show $K_G^*(Y \cup X_i, Y)_\mathfrak{p} = 0$. But this is $K_G^*(X_i, X_i \cap Y)_\mathfrak{p}$, which is a unitary module over the ring $K_G^*(X_i)_\mathfrak{p}$. The projection $X_i \to T_i$ induces a homomorphism of rings $K_G^*(T_i)_\mathfrak{p} \to K_G^*(X_i)_\mathfrak{p}$; as the first ring is zero so is the second, and so therefore is the module $K_G^*(X_i, X_i \cap Y)_\mathfrak{p}$.

Some interesting applications of the localization theorem can be found in [6] and [11].

§ 5. THE FILTRATION AND THE SPECTRAL SEQUENCE

If X is a CW-complex, which is filtered by its skeletons $\{X^p\}$, it is customary to define [5] a filtration of $K^*(X)$ by setting $K_p^*(X) = \text{kernel}\,(K^*(X) \to K^*(X^{p-1}))$. Then

$$K^*(X) = K_0^*(X) \supset K_1^*(X) \supset K_2^*(X) \supset \ldots,$$

and $K^*(X)$ is a filtered ring in the sense that $K_p^*(X) . K_q^*(X) \subset K_{p+q}^*(X)$.

In the equivariant theory there are several quite different filtrations of $K_G^*(X)$. The one I am going to discuss corresponds to filtering X by the G-subspaces $\pi^{-1}(Y^p)$ when the orbit space $Y = X/G$ is a CW-complex. ($\pi : X \to Y$ is the projection.) But to avoid making assumptions about the orbit space I shall define the filtration by a Čech method. For a fuller and more sophisticated discussion of the construction I refer the reader to [15].

To each finite covering $U = \{U_\alpha\}_{\alpha \in A}$ of a compact G-space X by G-stable closed sets I am going to associate a compact G-space W_U with a G-map $w : W_U \to X$ and a filtration by G-subspaces $W_U^0 \subset W_U^1 \subset \ldots \subset W_U$, so that the following conditions are satisfied:

(i) $w^* : K_G^*(X) \to K_G^*(W_U)$ is an isomorphism, and

(ii) when V is a refinement of U there is a G-map $W_V \to W_U$, defined up to G-homotopy, respecting the filtrations and the projections on to X.

Then I shall say that an element of $K_G^*(X)$ is in $K_{G,p}^*(X)$ if, for some finite covering U, it is in the kernel of $w^* : K_G^*(X) \to K_G^*(W_U^{p-1})$. Thus $K_{G,p}^*(X)$ is an ideal in $K_G^*(X)$. To see that $K_{G,p}^*(X) . K_{G,q}^*(X) \subset K_{G,p+q}^*(X)$ one needs a further property of W_U:

(iii) the diagonal map $W_U \to W_U \times W_U$ is G-homotopic to a filtration preserving map, when the filtration of $W_U \times W_U$ is defined by $(W_U \times W_U)^n = \bigcup_{p+q=n} W_U^p \times W_U^q$.

The definition of W_U is as follows. Let N_U be the *nerve* of U, a finite simplicial complex whose simplexes are the finite subsets σ of A such that $U_\sigma = \bigcap_{\alpha \in \sigma} U_\alpha$ is non-empty. Let $|N_U|$ be the geometrical realization of N_U, a compact space. Then W_U is the closed subspace $\bigcup_\sigma (U_\sigma \times |\sigma|)$ of the product $X \times |N_U|$, and $w : W_U \to X$ is the projection onto the first factor. Define $W_U^p = \bigcup_{\dim(\sigma) \leqslant p} (U_\sigma \times |\sigma|)$, i.e. it is the inverse image of the p-skeleton of $|N_U|$.

To prove (i) above, define X_k as the subset of points of X which are contained in at least $k+1$ of the sets U_α. Thus $X = X_0 \supset X_1 \supset X_2 \supset \dots$ Define also $W_k = w^{-1}(X_k) \subset W_U$. Consider the diagram

$$\begin{array}{ccc} \coprod_\sigma ((U_\sigma - U'_\sigma) \times |\sigma|) & \longrightarrow & W_k - W_{k+1} \\ \big\downarrow & & \big\downarrow w \\ \coprod_\sigma (U_\sigma - U'_\sigma) & \longrightarrow & X_k - X_{k+1}, \end{array}$$

where σ runs through the p-simplexes of N_U, and $U'_\sigma = U_\sigma \cap X_{k+1}$. The horizontal arrows are homeomorphisms, and the vertical arrow on the left is a proper homotopy-equivalence, so $w^* : K^*_G(X_k - X_{k+1}) \to K^*_G(W_k - W_{k+1})$ is an isomorphism. As this is true for all k, it follows from the cohomology exact sequence that $w^* : K^*(X) \xrightarrow{\cong} K^*_G(W_U)$.

I shall not give here the proofs of the statements (ii) and (iii). They are obtained in the same way as the analogous ones for $|N_U|$, and the details can be found in [15].

I should record the following simple facts about the filtration of $K^*_G(X)$.

Proposition (**5.1**). — *If* X *is a compact* G-*space, then*

(i) *an element of* $K^*_G(X)$ *is in* $K^*_{G,1}(X)$ *if and only if its restriction to each orbit is zero*, i.e. $K^*_{G,1}(X) = \text{kernel } (K^*_G(X) \to \prod_{x \in X} R(G_x))$;

(ii) *the elements of* $K^*_{G,1}(X)$ *are nilpotent.*

Proof. — (i) Because $W^0_U = \coprod_{\alpha \in A} U_\alpha$, an element ξ belongs to $K^*_{G,1}(X)$ if there is a finite G-stable covering U such that ξ restricts to zero in each $K^*_G(U_\alpha)$. This is equivalent to (i) because K^*_G is continuous.

(ii) If $\xi \in K^*_{G,1}(X)$, choose U so that ξ vanishes in $K^*_G(W^0_U)$. The covering U has some finite dimension n, so that $W^n_U = W_U$. Then ξ^{n+1} vanishes in $K^*_G(W_U) = K^*_G(X)$, so ξ is nilpotent.

This proposition implies the localization theorem of § 4, at least in the form that if $R(G_x)_\mathfrak{p} = 0$ for all $x \in X$ then $K^*_G(X)_\mathfrak{p} = 0$. For by localizing the exact sequence $0 \to K^*_{G,1}(X) \to K^*_G(X) \to \prod R(G_x)$ at $\mathfrak{p}$ one finds that every element of $K^*_G(X)_\mathfrak{p}$, including the unit element, is nilpotent.

To the filtration of the space W_U there corresponds a spectral sequence, defined by the method of [9], p. 333, terminating in $K^*_G(W_U) \cong K^*_G(X)$, and with $E_1^{pq} = K_G^{p+q}(W^p - W^{p-1})$. There is a homeomorphism $\coprod_\sigma (U_\sigma \times \mathring{\sigma}) \to W^p - W^{p-1}$, where σ runs through the p-simplexes of N_U, and $\mathring{\sigma}$ is the interior of $|\sigma|$. So $E_1^{pq} \cong \prod_\sigma K_G^{p+q}(U_\sigma \times \mathring{\sigma}) \cong \prod_\sigma K^q_G(U_\sigma)$. One can verify that the differential $d : E_1^{pq} \to E_1^{p+1,q}$ corresponds to the differential of the complex of cochains of N_U with coefficients in the system $\sigma \mapsto K^q_G(U_\sigma)$. That is :

Proposition **(5.2)**. — *If* U *is a finite closed* G*-stable covering of a compact* G*-space* X, *there is a spectral sequence* $H^p(N_U; K^q_G(U)) \Rightarrow K^*_G(X)$, *where* $K^q_G(U)$ *means the coefficient system* $\sigma \mapsto K^q_G(U_\sigma)$.

If one lets U run through the directed family of closures of the finite open G-stable coverings of X, and takes the direct limit of the family of spectral sequences corresponding, then, because G-stable coverings of X can be identified with coverings of X/G, one obtains

Proposition **(5.3)**. — *If* X *is a compact* G*-space there is a spectral sequence* $H^p(X/G; \mathscr{K}^q_G) \Rightarrow K^*_G(X)$, *where* $\mathscr{K}^q_G$ *is the sheaf on* X/G *associated to the presheaf* $V \mapsto K^q_G(\pi^{-1}\overline{V})$. ($\pi : X \to X/G$ *is the projection.*) *The stalk of* $\mathscr{K}^q_G$ *at an orbit* $Gx = G/G_x$ *is* $R(G_x)$ *if* q *is even, and* $\mathscr{K}^q_G = 0$ *if* q *is odd.*

For the details I refer again to [15]. The assertion about the stalk follows from the continuity of K^*_G. (Prop (2.11)).

Remark. — More generally, if $f : X \to Y$ is a map of compact G-spaces and G acts trivially on Y, the argument shows there is a spectral sequence $H^p(Y; \mathscr{K}^q_G f) \Rightarrow K^*_G(X)$, where $\mathscr{K}^q_G f$ is a sheaf on Y whose stalk at y is $K^q_G(f^{-1}y)$.

One application of the spectral sequence (5.3) is to prove the following useful finiteness theorem.

Let us call a G-space X *locally* G*-contractible* if each point $x \in X$ has arbitrarily small G_x-stable neighbourhoods which are G_x-contractible in themselves to x, or, what is the same thing, if each orbit has arbitrarily small G-neighbourhoods of which it is a G-deformation retract. For example, a differentiable manifold X on which a compact Lie group G acts smoothly is locally G-contractible. Then one has

Proposition **(5.4)**. — *If* X *is a locally* G*-contractible compact* G*-space such that* X/G *has finite covering dimension, then* $K^*_G(X)$ *is a finite* R(G)*-module.*

To prove this one observes first that because X/G has finite dimension the spectral sequence $H^*(X/G; \mathscr{K}^*_G) \Rightarrow K^*_G(X)$ is convergent, and so it suffices to show that $H^*(X/G; \mathscr{K}^*_G)$ is finite over R(G). (R(G) is noetherian [16].) Because X is locally G-contractible one can show that each point Gx of X/G has arbitrarily small neighbourhoods U such that $\mathscr{K}^*_G(U) \cong R(G_x)$, which is finite over R(G). This implies, after a little manipulation, that $H^*(X/G; \mathscr{K}^*_G)$ is finite over R(G); but I shall not give the details here.

The hypothesis that X/G has finite dimension is satisfied in the case of a smooth G-manifold, because X/G is then a finite union of open manifolds (cf. [13], (1.7.31)).

APPENDIX

Proof of Proposition (3.1)

I shall begin with some definitions.

A complex $E^\bullet$ is *elementary* if $E^i = 0$ except for two values $i = n$, $n+1$, and $d : E^n \to E^{n+1}$ is an isomorphism; it is *trivial* if it is elementary with trivial bundles. Because exact sequences of G-vector bundles split, any acyclic complex is a sum of elementary complexes.

Two morphisms $f_0, f_1 : E^\bullet \to F^\bullet$ are *equivalent* if there is a sequence of homomorphisms $h^i : E^i \to F^{i-1}$ such that $f_1 - f_0 = dh + hd$. Complexes on X and equivalence classes of morphisms form a category denoted by $C_G(X)$. To avoid confusion I shall use the word *equivalence* for an isomorphism in this category. An elementary complex is equivalent to 0, and hence so is any acyclic complex.

If A is a subspace of X, I shall write $C_G(X, A)$ for the full subcategory of $C_G(X)$ whose objects are the complexes $E^\bullet$ such that $E^\bullet | A$ is equivalent to zero, or, what is the same, acyclic.

Two objects $E_0^\bullet$, $E_1^\bullet$ of $C_G(X, A)$ are *homotopic* if there is an object $E^\bullet$ of $C_G(X \times [0, 1], A \times [0, 1])$ with equivalences $E_i^\bullet \to E^\bullet | (X \times i)$ for $i = 0, 1$.

Proposition (3.1) can be reformulated in the following way, which seems to me more appealing.

Proposition (*A*.1). — *If* X *is a compact* G*-space and* A *a closed* G*-subspace, then the set of homotopy classes of objects of* $C_G(X, A)$ *forms an abelian group* $Q_G(X, A)$ *under* $\oplus$. *This group is naturally isomorphic to* $K_G(X, A)$.

I shall prove the equivalence of (3.1) and (A.1) after proving (A.1); and before proving (A.1) I need a few more definitions, and a lemma.

If $E^\bullet$ is a complex, $TE^\bullet$ denotes the complex with the grading translated: $(TE)^k = E^{k-1}$.

A morphism $f : E^\bullet \to F^\bullet$ of complexes has a *mapping cone* $C_f^\bullet$, which is the complex obtained by regarding the double complex $\ldots \to 0 \to E^\bullet \to F^\bullet \to 0 \to \ldots$ as a single complex; $C_f^\bullet$ is acyclic if and only if f is an equivalence, for a family of splitting maps for $C_f^\bullet$ is the same thing as an inverse equivalence to f.

If $f, g : E^\bullet \to F^\bullet$ are two morphisms then the complexes $C_f^\bullet$, $C_g^\bullet$ are obviously homotopic. In particular, taking $E^\bullet = F^\bullet$, the complexes $C_{\mathrm{id}}^\bullet$ and $C_0^\bullet = E^\bullet \oplus TE^\bullet$ are homotopic. $C_{\mathrm{id}}^\bullet$ is equivalent to zero, so $E^\bullet = -TE^\bullet$ in $C_G(X, A)$. This proves that $Q_G(X, A)$ is an abelian group.

Lemma (*A*.2). — *If a* G*-space is the union of two compact* G*-subspaces* X *and* Y *with intersection* A, *and if* $E^\bullet$ *is a complex of* $C_G(X, A)$, *then there is a complex* $\widetilde{E}^\bullet$ *of*

$C_G(X \amalg_A Y, Y)$ *such that* $\widetilde{E}^\bullet | X$ *is equivalent to* $E^\bullet$. *The complex* $\widetilde{E}^\bullet$ *is unique up to equivalence. That is, the categories* $C_G(X, A)$ *and* $C_G(X \amalg_A Y, Y)$ *are equivalent.*

Proof. — To see $\widetilde{E}^\bullet$ exists it suffices to show that one can add an acyclic complex to $E^\bullet$ so that $E^\bullet | A$ becomes isomorphic to a sum of trivial complexes, for the latter can be extended over Y. First add elementary complexes to $E^\bullet$ so as to make all the bundles trivial except the first, say E^a. Then $\sum_k (-1)^k (E^k | A) = 0$ in $\widetilde{K}_G(A)$ because $E^\bullet | A$ is acyclic. $E^k | A$ represents zero in $K_G(A)$ for $k \neq a$, and hence also for $k = a$, i.e. $E^a | A$ is stably trivial. One sees inductively that when $E^\bullet | A$ is expressed as a sum of elementary complexes all the bundles occurring are stably trivial, and so by adding trivial complexes one can make $E^\bullet | A$ into a sum of trivial complexes.

As to the uniqueness of $\widetilde{E}^\bullet$, if $\widetilde{E}^\bullet_1$ is another candidate, then one has an equivalence $f : \widetilde{E}^\bullet | X \to \widetilde{E}^\bullet_1 | X$ and would like to extend it over Y. But $f | A$ is equivalent to the zero-morphism (because $\widetilde{E}^\bullet_1 | A$ is equivalent to 0), so one can write $f | A = dh + hd$ for suitable h defined on A. Then any extension of h over Y defines an extension of f by the formula $dh + hd$.

Now I can prove (A.1). By excision $K_G(X, A) = K_G(X \amalg_A X, X)$, where the subspace is the second summand, and by (A.2) $Q_G(X, A) = Q_G(X \amalg_A X, X)$, so it suffices to show that $Q_G(X \amalg_A X, X)$ is naturally isomorphic to $K_G(X \amalg_A X, X)$. This is more convenient because $K_G(X \amalg_A X, X)$ can be identified with the kernel of the split restriction $i_2^* : K_G(X \amalg_A X) \to K_G(X)$ onto the second summand. In fact one is reduced to proving the following:

Proposition (A.3). — *If* A *is a closed* G-*subspace of a compact* G-*space* X, *and is a retract of* X *by a map* $p : X \to A$, *then the sequence*

$$0 \to Q_G(X, A) \xrightarrow{\alpha} K_G(X) \to K_G(A) \to 0,$$

where α *is* $E^\bullet \mapsto \sum_k (-1)^k E^k$, *is split exact.*

Proof. — If E is a bundle on X then $E | A$ and $(p^* i^* E) | A$, where $i : A \to X$ is the inclusion, are isomorphic. Form a complex $\ldots \to 0 \to E \to p^* i^* E \to 0 \to \ldots$ on X by extending arbitrarily this isomorphism. Because different extensions lead to homotopic complexes this construction defines a homomorphism $\beta : K_G(X) \to Q_G(X, A)$. It remains to see that $p^* i^* + \alpha\beta = 1$ and that $\alpha\beta = 1$. The first is trivial. As to the second: if $E^\bullet$ is a complex of $C_G(X, A)$, choose a morphism $f : E^\bullet \to p^* i^* E^\bullet$ which extends the canonical isomorphism on A. (That is possible because $p^* i^* E^\bullet$ is acyclic and hence a sum of elementary complexes.) The mapping cone $C^\bullet_f$ is equivalent to $E^\bullet$, and on the other hand is homotopic in $C_G(X, A)$ to the mapping cone of $f : E^\bullet_0 \to p^* i^* E^\bullet_0$, where $E^\bullet_0$ is obtained from $E^\bullet$ by replacing the differential by zero. This last mapping cone, however, represents $\beta\alpha(E^\bullet)$.

It remains to show that $Q_G(X, A)$ is the same as the semi-group $L_G(X, A)/\sim$ introduced in § 3. There is an obvious surjection $L_G(X, A)/\sim \to Q_G(X, A)$, and one has only to show that, if $f : E_0^\bullet \to E_1^\bullet$ is an equivalence, then $E_0^\bullet \sim E_1^\bullet$, i.e. $E_0^\bullet \oplus F_0^\bullet \simeq E_1^\bullet \oplus F_1^\bullet$ for some acyclic complexes $F_0^\bullet$, $F_1^\bullet$. But in fact $TE_0^\bullet \oplus C_f^\bullet \simeq TE_1^\bullet \oplus C_I^\bullet$, where I means the identity morphism of $E_0^\bullet$, for $TE_0^\bullet \oplus C_f^\bullet$ is the mapping cone of $o \oplus f : E_0^\bullet \to E_0^\bullet \oplus E_1^\bullet$, and $TE_1^\bullet \oplus C_I^\bullet$ is the mapping cone of $I \oplus o : E_0^\bullet \to E_0^\bullet \oplus E_1^\bullet$.

So far in this appendix I have confined myself to compact G-spaces. The generalization to locally compact spaces amounts to the proof of the following lemma.

Proposition (*A*.**4**). — $Q_G(X, A) \xrightarrow{\cong} Q_G(X-A)$ *when* X *and* A *are compact* G-*spaces.* ($Q_G(X-A)$ is formed, of course, from the category of complexes with compact support on $X-A$.)

Proof. — In fact the categories $C_G(X, A)$ and $C_G(X-A)$ are equivalent. I shall define a functor $C_G(X-A) \to C_G(X, A)$. Let $E^\bullet$ be a complex of $C_G(X-A)$, and let K be a compact G-neighbourhood of its support. Apply (A.2) to $E^\bullet|K$, which is acyclic on $K-\mathring{K}$, and extend it to $\widetilde{E}^\bullet$ defined on X. Then $\widetilde{E}^\bullet|(X-A)$ is canonically equivalent to $E^\bullet$, for if $\theta : E^\bullet|K \to \widetilde{E}^\bullet|K$ is the canonical equivalence and $\varphi : X-A \to \mathbf{R}_+$ is a function vanishing outside K and equal to 1 on $\mathrm{supp}(E^\bullet)$ then $\varphi\theta$ is an equivalence between $E^\bullet$ and $\widetilde{E}^\bullet|(X-A)$ and does not depend on the choice of φ. If one chooses such an $\widetilde{E}^\bullet$ for each $E^\bullet$, one can clearly define a functor $\varepsilon : C_G(X-A) \to C_G(X, A)$ with $\varepsilon(E^\bullet) = \widetilde{E}^\bullet$ which is inverse to the restriction, and is the required equivalence of categories. By (A.2) the composition $C_G(X, A) \to C_G(X-A) \to C_G(X, A)$ is the " identity ". As to the composition in the other order, let $\theta : E^\bullet|K \to \widetilde{E}^\bullet|K$ be the canonical equivalence, and let $\varphi : X-A \to \mathbf{R}_+$ be a function equal to 1 on $\mathrm{supp}(E^\bullet)$, and with $\mathrm{supp}(\varphi) \subset K$. Then $\varphi\theta$ is a canonical equivalence between $E^\bullet$ and $\widetilde{E}^\bullet|(X-A)$.

To conclude this appendix I would like to point out that the lemma (A.2) is just a special case of the following clutching property of complexes, which illustrates the naturalness of the categories $C_G(X)$.

Proposition (*A*.**5**). — *If a* G-*space* X *is the union of compact* G-*subspaces* X_1, X_2 *with intersection* A, *and if* $E_1^\bullet$, $E_2^\bullet$ *are complexes on* X_1, X_2, *and* $\alpha : E_1^\bullet|A \to E_2^\bullet|A$ *is an equivalence, then there is a complex* $E^\bullet$ *on* X *with equivalences* $\beta_i : E|X_i \to E_i$ *such that* $(\beta_2|A) = \alpha.(\beta_1|A)$. *The complex* $E^\bullet$ *is unique up to canonical equivalence.*

Proof. — First replace the situation by an equivalent one in which each $\alpha^k : E_1^k|A \to E_2^k|A$ is surjective — for example by adding trivial complexes to $E_1^\bullet$. Then, as in (A.2), add an acyclic complex to $E_1^\bullet$ so that $K^\bullet = \ker(\alpha)$ becomes a sum of trivial complexes. Now whenever one has a short exact sequence of complexes $o \to F_1^\bullet \to F_2^\bullet \to F_3^\bullet \to o$ one can identify F_2^q with $F_1^q \oplus F_3^q$ for each q, and this identifies $F_2^\bullet$

with the mapping-cone of a map $f: F_3^\bullet \to F_1^\bullet$. Extend $K^\bullet$ over X_2 and extend the corresponding f to $f: E_2^\bullet \to K^\bullet$. Then $C_f^\bullet$ is a complex on X_2 canonically equivalent to $E_2^\bullet$, and such that $C_f^\bullet | A$ can be identified with $E_1^\bullet | A$. So form $E^\bullet$ by joining together $E_1^\bullet$ and $C_f^\bullet$. This proves the existence of $E^\bullet$.

One can prove the uniqueness of $E^\bullet$ just as in (A.2).

REFERENCES

[1] M. F. Atiyah, Power operations in K-theory, *Quart. J. of Math.* (Oxford), **17** (1966), 165-193.
[2] M. F. Atiyah, *Lectures on K-theory*, mimeographed, Harvard, 1964.
[3] M. F. Atiyah and R. Bott, On the periodicity theorem for complex vector bundles, *Acta mathematica*, **112** (1964), 229-247.
[4] M. F. Atiyah, R. Bott and A. Shapiro, Clifford modules, *Topology*, **3** (Suppl. 1) (1964), 3-38.
[5] M. F. Atiyah and F. Hirzebruch, Vector bundles and homogeneous spaces, *Differential geometry, Proc. of Symp. in Pure Math.*, **3** (1961), Amer. Math. Soc., 7-38.
[6] M. F. Atiyah, I. M. Singer, etc., The index of elliptic operators I, II (To appear).
[7] A. Borel et al., Seminar on transformation groups, *Ann. of Math. Studies*, n° 46, Princeton, 1960.
[8] N. Bourbaki, *Intégration*, chap. 1-4, Paris, Hermann, 1952, A.S.I., 1175.
[9] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, 1956.
[10] S. Eilenberg and N. E. Steenrod, *Foundations of algebraic topology*, Princeton University Press, 1952.
[11] L. Illusie, Nombres de Chern et groupes finis (To appear).
[12] G. D. Mostow, Cohomology of topological groups and solvmanifolds, *Ann. of Math.*, **73** (1961), 20-48.
[13] R. S. Palais, The classification of G-spaces, *Mem. Amer. Math. Soc.*, n° 36, 1960.
[14] R. S. Palais, On the existence of slices for actions of non-compact Lie groups, *Ann. of Math.*, **73** (1961), 295-323.
[15] G. B. Segal, Classifying-spaces and spectral sequences, *Publ. Math. Inst. des Hautes Études Scient.* (Paris), **34** (1968).
[16] G. B. Segal, The representation-ring of a compact Lie group, *Publ. Math. Inst. des Hautes Études Scient.* (Paris), **34** (1968).

Reçu le 6 juin 1967.

1968. — Imprimerie des Presses Universitaires de France. — Vendôme (France)
ÉDIT. N° 30 135 IMPRIMÉ EN FRANCE IMP. N° 20 887