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The Goodwillie tower and the EHP sequence. (English summary)
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This monograph studies the interaction between two machines for computing the 2-primary unstable homotopy groups of spheres: the EHP sequence and the Goodwillie tower of the identity functor.

Let all spaces be implicitly localized at 2. Let S^n be the n-dimensional sphere. The Goodwillie tower of the identity functor evaluated at S^n is a tower of fibrations whose homotopy inverse limit is equivalent to S^n. The "layers" (i.e., the homotopy fibers) in the Goodwillie tower are infinite loop spaces. The k-th layer is equivalent to $\Omega^\infty\Sigma^{n-k}L(k)_n$, where $\{L(k)_n\}_{k,n\geq 0}$ is a family of spectra that have been studied before. In particular, the homology of the spectra $L(k)_n$ is well known. The Goodwillie tower gives rise to a spectral sequence relating the stable and unstable homotopy groups of spheres. Following the author, we call it the Goodwillie spectral sequence, and denote it GSS.

For every $n>0$ there is a homotopy fibration sequence, constructed by I. M. James,

$$\Omega^2S^{2n+1}\xrightarrow{P}S^n\xrightarrow{E}\Omega S^{n+1}\xrightarrow{H}\Omega S^{2n+1}.$$

This is the EHP sequence. Combining EHP sequences for all values of n, one gets another spectral sequence relating the stable and unstable homotopy groups of spheres (EHPSS). Calculating the differentials in this spectral sequence is a difficult problem.

In the author's own words: "In this book we shall demonstrate that when the EHPSS and the GSS are computed in tandem, well understood aspects of each shed light on the more mysterious aspects of the other."

There are many interesting ideas in this book. We can mention only a few. For starters, it is observed that the EHP sequence gives rise to a fibration sequence between layers in the Goodwillie tower. Specifically, for each n and k there is a fibration sequence of spectra

$$\Sigma^nL(k-1)_{2n+1}\xrightarrow{P}L(k)_n\xrightarrow{E}L(k)_{n+1}. \tag{1}$$

The author shows that this sequence induces a short exact sequence in homology. To prove it, he introduces certain homology operations on the layers in the Goodwillie tower. The existence of these operations should be of independent interest. En route, he describes the action of the Steenrod algebra on the cohomology of $L(k)_n$. Here he corrects an imprecise statement in the paper of G. Z. Arone and M. E. Mahowald [Invent. Math. **135** (1999), no. 3, 743–788; MR1669268]. The description given in that paper is only valid up to lower terms in a certain filtration. Fortunately, none of the results of [op. cit.] are affected.

The author studies the Atiyah-Hirzebruch spectral sequence (AHSS) for the spectra $L(k)_n$. Letting k vary, these spectral sequences calculate the E_1 term of the Goodwillie spectral sequence. The author uses the sequence (1), for varying n and k, to extract from the AHSS information about differentials in the EHP spectral sequence. Conversely, he shows that information about the EHP sequence can be used to extract differentials in the Goodwillie spectral sequence. To illustrate his methods the author recalculates the unstable homotopy groups $\pi_{n+i}(S^i)$ for $i\leq 6$ and $n\leq 19$ (the "Toda range").

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.