Cohomology of Stacks

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Abstract

We construct the de Rham cohomology of differentiable stacks via a double complex associated to any Lie groupoid presenting the stack. This is a straightforward generalization of the Čech-de Rham complex of a differentiable manifold. We explain the relationship to the Cartan model of equivariant cohomology.

To get a theory of Poincaré duality, we construct the companion theory of cohomology with compact supports. We explain how cohomology acts on cohomology with compact supports, and how to integrate compact support cohomology classes.

We specialize to differentiable stacks of Deligne-Mumford type (these include orbifolds), where we prove that one can calculate cohomology, as well as compact support cohomology, via the complex of global differential forms. Finally, for proper differentiable stacks of Deligne-Mumford type we prove Poincaré duality between de Rham cohomology and itself.

We go on to define the cohomology class of a closed substack and intersection numbers. We do a few sample calculations from the theory of moduli stacks.

In the second part of these notes we construct the singular homology and cohomology of a topological stack. Among the results we prove is that singular homology equals equivariant homology in the case of a quotient stack. We also prove that the $\mathbb{Q}$-valued singular cohomology of a Deligne-Mumford stack is equal to that of its coarse moduli space. We conclude with a discussion of Chern classes and a few examples.

Contents

Introduction

In these lectures we give a short introduction to the cohomology of stacks. We first focus on the de Rham theory for differentiable stacks. Then we go on to singular homology of topological stacks. All the technical tools we use are explained in [2]. Our constructions are often straightforward generalizations of constructions in [ibid.]

When defining the cohomology groups $H^k(X,\mathbb{R})$ for a manifold X with values in the real numbers $\mathbb{R}$, there are two approaches:

(i) resolve the coefficient sheaf $\mathbb{R}$ using, for example, the de Rham complex

$$\mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \dots$$

and define $H^k(X,\mathbb{R})$ as the cohomology groups of the complex of global sections $\Gamma(X,\Omega^\bullet)$.

(ii) resolve the manifold X, using a good cover $\{U_i\}$ (a cover where all intersections are diffeomorphic to $\mathbb{R}^n$), and define $H^k(X,\mathbb{R})$ as the the Čech cohomology groups of the constant sheaf $\mathbb{R}$ with respect to the covering.

Approach (i) is justified because for every q, the higher cohomology groups of Ω^q on any manifold vanish, i.e., because the sheaves Ω^q are acyclic over manifolds.

Approach (ii) is justified because for every p, the manifold

$$U_p = \coprod_{i_0,\dots,i_p} U_{i_0} \cap \dots \cap U_{i_p}$$

has no higher cohomology groups (with values in $\mathbb{R}$). In other words U_p is acyclic, for every p.

It can be useful to combine the two approaches into one by considering the Čech-de Rham complex, a double complex made up from all $\Omega^q(U_p)$. (See [2], Chapter II.)

For stacks things are less simple. Approach (i) breaks down, because the Ω^q are not acyclic over stacks(see Exercise 6). Approach (ii) breaks down, because of the lack of good covers. But we can always resolve a stack using a simplicial manifold and all Ω^q are acyclic for manifolds. Therefore the combination of the two approaches using the Čech-de Rham complex works well for stacks.

We will explain this approach in detail.

Remark It is possible to define the cohomology of stacks by resolving the coefficient sheaf alone. For this we would have to use injective (or other) resolutions in the big site of the stack. For details, see [1].

To define the cohomology of a stack by resolving the stack alone, we would have to use hypercovers.

Neither of these two approaches will be discussed in these notes, except for the brief Remark 11.

De Rham cohomology

Differentiable stacks are stacks over the category of differentiable manifolds. They are the stacks associated to Lie groupoids. A groupoid $X_1 \rightrightarrows X_0$, is a *Lie groupoid* if both X_0 and X_1 are differentiable (i.e., C^∞) manifolds, all structure maps are differentiable and source and target map are (differentiable) submersions.

Two Lie groupoids $X_1 \rightrightarrows X_0$ and $Y_1 \rightrightarrows Y_0$ give rise to essentially the same stack, if and only if they are *Morita equivalent*, which means that there is a third Lie groupoid $Z_1 \rightrightarrows Z_0$, together with Morita morphisms $Z_\bullet \to X_\bullet$ and $Z_\bullet \to Y_\bullet$. A morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$ is a *Morita morphism* if $f_0 : X_0 \to Y_0$ is a surjective submersion and the diagram

$$\begin{tikzcd} X_1 \arrow[r, "(s,t)"] \arrow[d, "f_1"'] & X_0 \times X_0 \arrow[d, "f_0 \times f_0"] \\ Y_1 \arrow[r, "(s,t)"] & Y_0 \times Y_0 \end{tikzcd} \tag{1}$$

is cartesian, i.e., a pullback diagram of differentiable manifolds. We say that a Morita morphism $f : X_\bullet \to Y_\bullet$ *admits a section* if $X_0 \to Y_0$ admits a section.

Recall that a functor is an equivalence of categories if it is fully faithful and essentially surjective. Diagram (1) being cartesian translates into f being fully faithful. The requirement on f_0 is much stronger than essential surjectivity (and there are certain weakenings of the notion of Morita morphism taking this into account). For abstract categories, any equivalence has an inverse. For a Morita morphism of Lie groupoids, this is not the case, unless it admits a section.

Any section $s : X_0 \to Y_0$ of a Morita morphism $f : X_\bullet \to Y_\bullet$ induces uniquely a groupoid morphism $s : Y_\bullet \to X_\bullet$ with the properties

- $f \circ s = \mathrm{id}_{Y_\bullet}$,
- $s \circ f \cong \mathrm{id}_{X_\bullet}$, which means that there exits a 2-isomorphism $\theta : s \circ f \Rightarrow \mathrm{id}_{X_\bullet}$.

(Recall that a 2-isomorphism between two morphisms of Lie groupoids, $\theta : f \Rightarrow g$, where $f, g : X_\bullet \to Y_\bullet$ are morphisms,

$$\begin{tikzcd} X_\bullet \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\theta"] & Y_\bullet \end{tikzcd}$$

is a differentiable map $\theta : X_0 \to Y_1$ satisfying the formal properties of a natural transformation between functors.)

Another construction from the theory of groupoids we will use is *restriction.* Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Let $Y_0 \to X_0$ be a submersion. Define Y_1 by the cartesian diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d] & Y_0 \times Y_0 \arrow[d] \\ X_1 \arrow[r] & X_0 \times X_0 \end{tikzcd}$$

(Note that we need the assumption that $Y_0 \to X_0$ is a submersion, in order that Y_1 will be a manifold and the two maps $Y_1 \rightrightarrows Y_0$ will be submersions.) One checks that $Y_\bullet$ is again a Lie groupoid. It is called the *restriction* of $X_\bullet$ via $Y_0 \to X_0$. Note also that if $Y_0 \to X_0$ is surjective, then the natural morphism $Y_\bullet \to X_\bullet$ is a Morita morphism.

The simplicial nerve of a Lie groupoid

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid. Then we can produce a simplicial manifold $X_\bullet$ as follows. For every $p \geq 0$ we let X_p be the manifold of composable sequences of elements of X_1 of length p. In other words,

$$X_p = \underbrace{X_1 \times_{X_0} X_1 \times_{X_0} \ldots \times_{X_0} X_1}_{p} .$$

Then we have $p+1$ differentiable maps $\partial_i : X_p \to X_{p-1}$, for $i = 0, \ldots, p$, where ∂_i is given by 'leaving out the i-th object'. Thus ∂_0 leaves out the first arrow, ∂_p leaves out the last arrow, and $\partial_1, \ldots, \partial_{p-1}$ are given by composing two successive arrows. More precisely, ∂_i maps the element

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{i-1}} x_{i-1} \xrightarrow{\phi_i} x_i \xrightarrow{\phi_{i+1}} x_{i+1} \xrightarrow{\phi_{i+2}} \ldots \xrightarrow{\phi_p} x_p$$

of X_p to the element

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_{i-1}} x_{i-1} \xrightarrow{\phi_i * \phi_{i+1}} x_{i+1} \xrightarrow{\phi_{i+2}} \ldots \xrightarrow{\phi_p} x_p$$

of X_{p-1}. (There are also maps $X_{p-1} \to X_p$, given by inserting identity arrows, but they are less important for us.) Note that for the composition of maps $X_p \to X_{p-2}$ we have the relations

$$\partial_i \partial_j = \partial_{j-1} \partial_i \,, \quad \text{for all } 0 \le i < j \le p. \tag{2}$$

We summarize this data by the diagram of manifolds

$$\ldots \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} X_2 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} X_1 \rightrightarrows X_0 \,. \tag{3}$$

Čech cohomology

Let $X_1 \rightrightarrows X_0$ be a Lie groupoid and $X_\bullet$ the associated simplicial manifold. Letting Ω^q be the sheaf of q-forms, we get an induced cosimplicial set

$$\Omega^q(X_0) \rightrightarrows \Omega^q(X_1) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \Omega^q(X_2) \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \ldots \tag{4}$$

simply by pulling back q-forms. Since this is, in fact, a cosimplicial abelian group, we can associate a complex

$$\Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \Omega^q(X_2) \xrightarrow{\partial} \ldots$$

Here $\partial : \Omega^q(X_{p-1}) \to \Omega^q(X_p)$ is given by

$$\partial = \sum_{i=0}^{p} (-1)^i \partial_i^* \,.$$

We call this complex the *Čech complex* associated to the sheaf Ω^q and the Lie groupoid $X_\bullet$. Its cohomology groups $H^k(X, \Omega^q)$ are called the *Čech cohomology groups* of the groupoid $X = [X_1 \rightrightarrows X_0]$ with values in the sheaf of q-forms Ω^q.

Remark 1 (naturality) Given a morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$, we get an induced homomorphism of Čech complexes

$$f^* : \check{C}^\bullet(Y, \Omega^q) \to \check{C}^\bullet(X, \Omega^q) .$$

It is given by the formula

$$f^*(\omega)(\phi_1 \ldots \phi_p) = \omega\big(f(\phi_1) \ldots ..f(\phi_p)\big) ,$$

for $\omega \in \Omega^q(Y_p)$. Here $\phi_1 \ldots \phi_p$ abbreviates the element

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_p} x_p$$

in X_p. This follows directly from the presheaf property of Ω^q and the functoriality of f.

More interestingly, if we have a 2-isomorphism $\theta : f \Rightarrow g$ between the two morphisms $f, g : X_\bullet \to Y_\bullet$, then we get an induced homotopy $\theta^* : f^* \Rightarrow g^*$, between the two induced homomorphisms $f^*, g^* : \check{C}^\bullet(Y, \Omega^q) \to \check{C}^\bullet(X, \Omega^q)$. In fact, $\theta^* : \Omega^q(Y_{p+1}) \to \Omega^q(X_p)$ is defined by the formula

$$\theta^*(\omega)(\phi_1 \ldots \phi_p) = \sum_{i=0}^{p} (-1)^i \omega\big(f(\phi_1) \ldots f(\phi_i)\theta(x_i)g(\phi_{i+1}) \ldots g(\phi_p)\big) .$$

One checks (this is straightforward but tedious) that

$$\partial\theta^* + \theta^*\partial = g^* - f^* .$$

As consequences of these naturality properties we deduce that

- groupoid morphisms induce homomorphisms on Čech cohomology groups,
- 2-isomorphic groupoid morphisms induce identical homomorphisms on Čech cohomology groups,
- a Morita morphism admitting a section induces *isomorphisms* on Čech cohomology groups.

Note that these naturality properties follow formally from the presheaf properties of Ω^q, and thus hold for *any* contravariant functor F : (manifolds) $\to$ (abelian groups).

Proposition 2 *If $X_1 \rightrightarrows X_0$ is the banal groupoid associated to a surjective submersion of manifolds $X_0 \to Y$, then the Čech cohomology groups $H^k(X, \Omega^q)$ vanish, for all $k > 0$ and all $q \geq 0$. Moreover, $H^0(X, \Omega^q) = \Gamma(Y, \Omega^q)$.*

Proof. Recall that the banal groupoid associated to a submersion $X_0 \to Y$ is defined by setting $X_1 = X_0 \times_Y X_0$. Since $X_1 \to X_0 \times X_0$ is then an equivalence relation on X_0, we get a groupoid structure on $X_1 \rightrightarrows X_0$. Note that such a banal groupoid comes with a canonical Morita morphism $X_\bullet \to Y$, where Y is considered as a groupoid $Y \rightrightarrows Y$ in the trivial way.

For example, if $\{U_i\}$ is an open cover of Y, and $X_0 = \coprod U_i$, then $X_1 = \coprod U_{ij}$, where $U_{ij} = U_i \cap U_j$. In this case the proposition is a standard fact, which follows essentially from the existence of partitions of unity. See for example Proposition 8.5 of [2], where this result is called the generalized Mayer-Vietoris sequence. (Note that in [loc. cit.] alternating Čech cochains are used, whereas we do not make this restriction. The result is the same.)

Another case where the proof is easy, is the case of a surjective submersion with a section. This is because a section $s : Y \to X_0$ induces a section of the Morita morphism $X_\bullet \to Y$. Thus by naturality we have $H^k(X, \Omega^q) = H^k(Y \rightrightarrows Y, \Omega^q)$, which vanishes for $k > 0$, and equals $\Gamma(Y, \Omega^q)$, for $k = 0$.

The general case now follows from these two special cases by a double fibration argument. Let $\{U_i\}$ be an open cover of Y over which $X_0 \to Y$ admits local sections and let $V_0 = \coprod U_i$. We consider the banal groupoid $V_\bullet$ given by $V_0 \to Y$.

The key is to introduce $W_{00} = X_0 \times_Y V_0$. Thus $W_{00} \to V_0$ is now a surjective submersion which admits a section. We define $W_{mn} = X_m \times_Y V_n$, for all $m, n \geq 0$.

$$\begin{array}{ccc} W_{mn} & \longrightarrow & V_n \\ \downarrow & & \downarrow \\ X_m & \longrightarrow & Y \end{array}$$

Then $W_{\bullet\bullet}$ is a bisimplicial manifold. This means that we have an array as in Figure 1. It is important to notice that $W_{\bullet n}$ is the simplicial nerve of the banal groupoid associated to $W_{0n} \to V_n$, and $W_{m\bullet}$ is the simplicial nerve of the banal groupoid associated to $W_{m0} \to X_m$. All $W_{0n} \to V_n$ are submersions admitting sections and all $W_{m0} \to X_m$ are submersions coming from open covers. Thus we already know the proposition for all of these submersions.

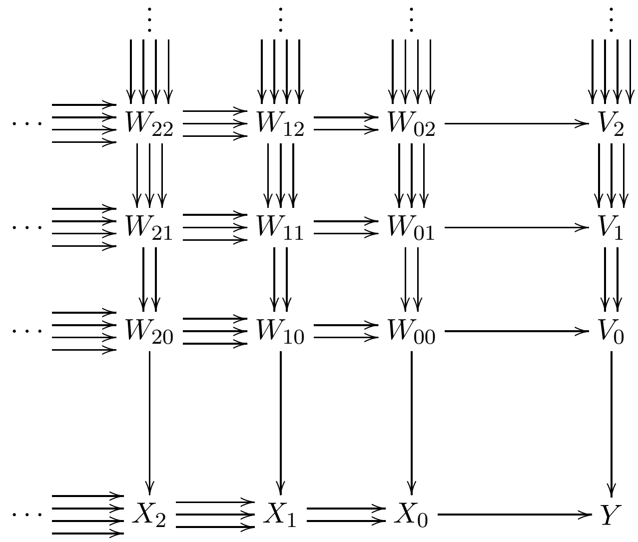


Figure 1: The bisimplicial manifold

We apply Ω^q to this array to obtain a double complex $\Omega^p(W_{\bullet\bullet})$ mapping to the two complexes $\Omega^q(X_\bullet)$ and $\Omega^q(V_\bullet)$, see Figure 2. Passing to cohomology we get a commutative diagram

$$\begin{tikzcd} H^*(W,\Omega^q) \arrow[r] \arrow[d] & H^*(V,\Omega^q) \arrow[d, "\cong"] \\ H^*(X,\Omega^q) \arrow[r] & H^*(Y \rightrightarrows Y,\Omega^q) \end{tikzcd}$$

and noticing that the two arrows originating at $H^*(W,\Omega^q)$ are isomorphisms, which follows by calculating cohomology of the double complex in two different ways, we get the required result. $\square$

Corollary 3 *Any Morita morphism of Lie groupoids $f : X_\bullet \to Y_\bullet$ induces isomorphisms on Čech cohomology groups $f^* : H^k(Y,\Omega^q) \xrightarrow{\sim} H^k(X,\Omega^q)$. Morita equivalent groupoids have canonically isomorphic Čech cohomology groups with values in Ω^q.*

Proof. Let $\mathfrak{X}$ be the differentiable stack given by the groupoid $Y_\bullet$. The composed morphism $X_\bullet \to \mathfrak{X}$ identifies $\mathfrak{X}$ as the stack given by $X_\bullet$. Form

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & \Omega^q(W_{22}) \arrow[r] \arrow[d] & \Omega^q(W_{12}) \arrow[r] \arrow[d] & \Omega^q(W_{02}) \arrow[r] \arrow[d] & \Omega^q(V_2) \arrow[d] \\
\cdots \arrow[r] & \Omega^q(W_{21}) \arrow[r] \arrow[d] & \Omega^q(W_{11}) \arrow[r] \arrow[d] & \Omega^q(W_{01}) \arrow[r] \arrow[d] & \Omega^q(V_1) \arrow[d] \\
\cdots \arrow[r] & \Omega^q(W_{20}) \arrow[r] \arrow[d] & \Omega^q(W_{10}) \arrow[r] \arrow[d] & \Omega^q(W_{00}) \arrow[r] \arrow[d] & \Omega^q(V_0) \\
\cdots \arrow[r] & \Omega^q(X_2) \arrow[r] & \Omega^q(X_1) \arrow[r] & \Omega^q(X_0) &
\end{tikzcd}$$

Figure 2: The double complex

the fibered product $Z_{00} = X_0 \times_{\mathfrak{X}} Y_0$. Then define a bisimplicial manifold $Z_{\bullet\bullet}$ as in the previous proof and apply the same kind of double fibration argument to produce isomorphisms $H^*(Z, \Omega^q) \to H^*(X, \Omega^q)$ and $H^*(Z, \Omega^q) \to H^*(Y, \Omega^q)$. $\square$

Thus we can make the following definition.

Definition 4 Let $\mathfrak{X}$ be a differentiable stack. Then

$$H^k(\mathfrak{X}, \Omega^q) = H^k(X_1 \rightrightarrows X_0, \Omega^q)\,,$$

for any Lie groupoid $X_1 \rightrightarrows X_0$ giving an atlas for $\mathfrak{X}$. In particular, this defines

$$\Gamma(\mathfrak{X}, \Omega^q) = H^0(\mathfrak{X}, \Omega^q)\,.$$

Example 5 If G is a Lie group then $H^k(BG, \Omega^0)$ is the group cohomology of G calculated with differentiable cochains. (Recall that BG is the differentiable stack given by the Lie group G itself, considered as a Lie groupoid $G \rightrightarrows *$.) Thus there are stacks for which these cohomology groups are non-trivial (see the following exercise).

Exercise 6 Calculate $H^1(B\mathbb{R}^+, \Omega^0)$.

Remark 7 We can formalize the above constructions. Let

$$F : (\text{manifolds}) \to (\text{abelian groups})$$

be a contravariant functor such that

(i) if we restrict F to any given manifold and its open subsets we get a sheaf on this manifold,

(ii) for every manifold this sheaf has vanishing Čech cohomology groups.

Then we can define $H^k(\mathfrak{X}, F)$ for any differentiable stack as the Čech cohomology of any groupoid presenting $\mathfrak{X}$. The above proof of well-definedness carries through.

A contravariant functor F satisfying (i), is called a *big sheaf*, on the category of manifolds. Thus Ω^q is an example of a big sheaf.

The de Rham complex

The exterior derivative $d : \Omega^q \to \Omega^{q+1}$ connects the various Čech complexes of a Lie groupoid with each other. We thus get a double complex

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^2(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^2(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^2(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots \\
\Omega^1(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^1(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots \\
\Omega^0(X_0) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(X_1) \arrow[r, "\partial"] \arrow[u, "d"] & \Omega^0(X_2) \arrow[r, "\partial"] \arrow[u, "d"] & \cdots
\end{tikzcd} \tag{5}$$

We make a total complex out of this by setting

$$C^n_{DR}(X) = \bigoplus_{p+q=n} \Omega^q(X_p)$$

and defining the total differential $\delta : C^n_{DR}(X) \to C^{n+1}_{DR}(X)$ by

$$\delta(\omega) = \partial(\omega) + (-1)^p d(\omega), \quad \text{for } \omega \in \Omega^q(X_p)\,.$$

The sign change is introduced in order that $\delta^2 = 0$.

Definition 8 The complex $C^\bullet_{DR}(X)$ is called the *de Rham complex* of the Lie groupoid $X_1 \rightrightarrows X_0$. Its cohomology groups

$$H^n_{DR}(X) = h^n\big(C^\bullet_{DR}(X)\big)$$

are called the *de Rham* cohomology groups of $X = [X_1 \rightrightarrows X_0]$.

If $X_1 \rightrightarrows X_0$ is the banal groupoid associated to an open cover $X_0 \to Y$ of a manifold Y, then the de Rham complex of $X_1 \rightrightarrows X_0$ is just the usual Čech-de Rham complex as treated, for example, in Chapter II of [2].

Definition 9 One can use Proposition 2 and a double fibration argument to prove that de Rham cohomology is invariant under Morita equivalence and hence well-defined for differentiable stacks:

$$H^n_{DR}(\mathfrak{X}) = H^n_{DR}(X_1 \rightrightarrows X_0)\,,$$

for any groupoid atlas $X_1 \rightrightarrows X_0$ of the stack $\mathfrak{X}$.

Remark 10 Because the de Rham complex $\Omega^\bullet$ of big sheaves resolves the big constant sheaf $\mathbb{R}$, we may consider $H_{DR}(\mathfrak{X})$ as the cohomology of $\mathfrak{X}$ with values in $\mathbb{R}$.

Remark 11 Recall that any sheaf F on a topological space has a canonical flabby resolution. (This resolution starts out by embedding F into its associated sheaf of discontinuous sections and continuing in like manner.) For example, denote by $Z^\bullet$ the canonical flabby resolution of $\mathbb{Z}$, the constant sheaf. Because of the canonical nature of $Z^\bullet$, we actually obtain for every q a big sheaf Z^q, satisfying the two conditions of Remark 7.

Now we can imitate the construction of de Rham cohomology: define the cohomology of the stack $\mathfrak{X}$ with values in $\mathbb{Z}$ as the cohomology of the total complex of the double complex $Z^\bullet(X_\bullet)$, where $X_\bullet$ is the nerve of any groupoid presenting $\mathfrak{X}$. The proof of well-definedness is identical to the case of de Rham cohomology.

For this construction, we don't even need the stack $\mathfrak{X}$ or the groupoid $X_1 \rightrightarrows X_0$ to be differentiable. It works for arbitrary topological stacks. In this way we can define the cohomology of an arbitrary topological stack with values in an arbitrary (abelian) big sheaf.

Equivariant Cohomology

We will explain why the de Rham cohomology of a quotient stack is equal to equivariant cohomology. (Another proof follows from the results of the section on singular homology.)

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and X a G-manifold. To fix ideas, assume that G always acts from the left. There is a well-known generalization of the de Rham complex to the equivariant case, namely the *Cartan complex* $\Omega^\bullet_G(X)$, defined by

$$\Omega^n_G(X) = \bigoplus_{2k+i=n} \big(S^k\mathfrak{g}^\vee \otimes \Omega^i(X)\big)^G .$$

Here $S^\bullet\mathfrak{g}^\vee$ is the symmetric algebra on the dual of $\mathfrak{g}$, which can also be thought of as the space of polynomial functions on $\mathfrak{g}$. The group G acts by the adjoint representation on $\mathfrak{g}^\vee$ and by pullback of differential forms on $\Omega^\bullet(X)$. The Cartan complex consists of the G-invariants for the induced action on $S^\bullet\mathfrak{g}^\vee \otimes \Omega^\bullet(X)$. The Cartan differential $d : \Omega^n_G(x) \to \Omega^{n+1}_G(X)$ is given as the sum $d = d_{DR} - \iota$, where $d_{DR} : S^k\mathfrak{g}^\vee \otimes \Omega^i(X) \to S^k\mathfrak{g}^\vee \otimes \Omega^{i+1}(X)$ is the de Rham differential and $\iota : S^k\mathfrak{g}^\vee \otimes \Omega^i(X) \to S^{k+1}\mathfrak{g}^\vee \otimes \Omega^{i-1}(X)$ is the tensor induced by the vector bundle homomorphism $\mathfrak{g}_X \to T_X$ coming from differentiating the action. Note that we have to pass to G-invariants in the Cartan complex before the Cartan differential satisfies $d^2 = 0$.

The functorial properties of the Cartan complex are analogous to the functorial properties of the de Rham complex: if $f : X \to Y$ is an equivariant map of G-manifolds, we get a natural morphism of complexes $\Omega^\bullet_G(Y) \to \Omega^\bullet_G(X)$. More generally, if $G \to H$ is a morphism of Lie groups and $X \to Y$ is equivariant from a G-manifold X to an H-manifold Y, we have a natural map $\Omega^\bullet_H(Y) \to \Omega^\bullet_G(X)$. In particular, there is always a quotient map $\Omega^\bullet_G(X) \to \Omega^\bullet(X)$. If G acts trivially on X, there is a canonical section $\Omega^\bullet(X) \to \Omega^\bullet_G(X)$, making $\Omega^\bullet(X)$ a direct summand of $\Omega_G(X)$.

Recall that if G is compact, the cohomology groups $H^i_G(X) = h^i\big(\Omega^\bullet_G(X)\big)$ are the *equivariant cohomology groups* of X.

We will now formulate an equivariant analog of Proposition 2. Let $X \to Y$ be a surjective submersion of G-manifolds. Then the simplicial nerve of the associated banal groupoid $X_\bullet$ consists of G-manifolds and equivariant maps. Thus we get an associated double complex $\Omega^\bullet_G(X_\bullet)$ together with an augmentation map $\Omega^\bullet_G(Y) \to \Omega^\bullet_G(X_\bullet)$.

Lemma 12 *If G is compact, the augmentation is a quasi-isomorphism, i.e., induces isomorphisms*

$$H^i_G(Y) \longrightarrow h^i\big(\operatorname{tot}\Omega^\bullet_G(X_\bullet)\big)\,,$$

for all i.

Proof. We cannot imitate the proof of Proposition 2, because $X \to Y$ will, in general, not have sufficiently many local equivariant sections. This is also why we have to restrict to the case that G is compact. Recall that a compact Lie group admits a left invariant *gauge form*, i.e., a top degree form ω on G, which is left invariant and satisfies $\int_G \omega = 1$. Integrating against ω, we can define a natural projection operator $V \otimes \Omega^\bullet(X) \to \big(V \otimes \Omega^\bullet(X)\big)^G$, for any (finite-dimensional) representation V of G.

Now, to prove the lemma, we note that by Proposition 2, the augmentation $\Omega^\bullet(Y) \to \Omega^\bullet(X_\bullet)$ is a quasi-isomorphism. Tensoring with the representation $S^\bullet\mathfrak{g}^\vee$, this remains a quasi-isomorphism. Moreover, because of the existence of the natural projection onto G-invariants, after taking G-invariants, we still have a quasi-isomorphism. Of course, so far, we are using only the differential d_{DR} on $\Omega^\bullet_G(X_\bullet)$. But there is a spectral sequence, starting with the cohomology of $\Omega^\bullet_G(X_\bullet)$ with respect to d_{DR} and abutting to the cohomology of $\Omega^\bullet_G(X_\bullet)$ with respect to $d = d_{DR} - \iota$. There is a corresponding spectral sequence for $\Omega^\bullet_G(Y)$, but it degenerates. Convergence of the spectral sequences now implies the result. □

We can use this lemma to define a Morita-invariant notion of equivariant cohomology for groupoids. This is not our goal here. Rather we are interested in the following setup:

Let X be a G-manifold. Consider the transformation groupoid $G \times X \rightrightarrows X$, which has projection and operation as structure maps. (The associated stack is the quotient stack $[X/G]$.) We consider the pair (g,x) as a morphism with source gx and target x:

$$gx \xrightarrow{(g,x)} x$$

Proposition 13 *If G is compact, there is a natural isomorphism*

$$H^i_G(X) \longrightarrow H^i_{DR}(G \times X \rightrightarrows X) = H^i_{DR}([X/G])\,.$$

PROOF. Denote the simplicial nerve of the transformation groupoid $G \times G \to G$ (left multiplication of G on itself) by $EG_\bullet$. Then the simplicial manifold $EG_\bullet \times X$ is isomorphic to the simplicial nerve of the banal groupoid associated to the projection $G \times X \to X$. In particular, by Lemma 12, we have a quasi-isomorphism $\Omega^\bullet_G(X) \to \Omega^\bullet_G(EG_\bullet \times X)$.

On the other hand, we have a morphism of simplicial manifolds $\pi_\bullet : EG_\bullet \times X \to \Gamma_\bullet$, where $\Gamma_\bullet$ is the simplicial nerve of the transformation groupoid $G \times X \rightrightarrows X$. Thus $\Gamma_n = G^n \times X$. The morphism $EG_n \times X = G^{n+1} \times X \to \Gamma_n = G^n \times X$ is given by ∂_{n+1}, which maps $(g_0, \dots, g_n, x)$ to $(g_0, \dots, g_{n-1}, g_{n+1}x)$. Note that $\pi_\bullet$ is a (level-wise) principal G-bundle.

Now we invoke the theorem of Cartan: if $P \to M$ is a principal G-bundle, the canonical homomorphism $\Omega^\bullet(M) \to \Omega^\bullet_G(M) \to \Omega^\bullet_G(P)$ is a quasi-isomorphism. Here we need to use the fact that G is compact for the second time. We apply the theorem of Cartan levelwise to the principal G-bundle $\pi_\bullet$ and obtain a quasi-isomorphism $\Omega^\bullet(\Gamma_\bullet) \to \Omega^\bullet_G(EG_\bullet \times X)$. $\square$

Corollary 14 *For a compact Lie group G we have $H^*_{DR}(BG) = (S^{2*}\mathfrak{g}^\vee)^G$.*

Exercise 15 For example, $H^\bullet_{DR}(BS^1) = \mathbb{R}[c]$. Let dt be a gauge form on S^1, i.e., $\int_{S^1} dt = 1$. This form defines a basis element in $(\mathfrak{s}^1)^\vee$, which we denote by c. Prove that dt defines a 2-cocycle in the de Rham complex of the groupoid $S^1 \rightrightarrows *$. Find the sign ϵ, such that under the identification of $H^2_{DR}(S^1 \rightrightarrows *)$ with $H^2_G(*)$, the class of dt corresponds to ϵc.

Remark 16 If G is not compact, then $H_{DR}([X/G])$ is still equal to equivariant cohomology. This is not difficult to believe, as the main ingredient in our proof of Proposition 13 was the fact that $H^\bullet_G(P) = H^\bullet(M)$, for every principal G-bundle P over a manifold M. This fact holds for equivariant cohomology in general. Only, in general, the Cartan complex is insufficient to calculate equivariant cohomology.

Remark 17 Most differentiable stacks occurring 'in nature' are quotient stacks of a group action on a manifold (although not always quotients by a compact group). Thus, the cohomology of all such stacks is simply equivariant cohomology. The stack point of view adds one essential insight: if transformation groupoids $G \times X \rightrightarrows X$ and $H \times Y \rightrightarrows Y$ are Morita equivalent, then $H^\bullet_G(X) = H^\bullet_H(Y)$. In other words, if G and H act freely and compatibly on a manifold Z, then $H^\bullet_G(Z/H) = H^\bullet_H(Z/G)$.

Exercise 18 Let G be a compact group acting on a manifold X. Prove that there exists a manifold Y with a $U(n)$-action, for some n, such that $H^\bullet_G(X) = H^\bullet_{U(n)}(Y)$.

Multiplicative structure

We define a multiplication on the double complex (5) as follows. Let $\omega \in \Omega^q(X_p)$ and $\eta \in \Omega^{q'}(X_{p'})$. Then we set

$$\omega \cup \eta = (-1)^{qp'} \pi_1^*\omega \wedge \pi_2^*\eta \quad \in \Omega^{q+q'}(X_{p+p'})\,. \tag{6}$$

Here the map $\pi_1 : X_{p+p'} \to X_p$ projects the element

$$\circ \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_p} \circ \xrightarrow{\phi_{p+1}} \ldots \xrightarrow{\phi_{p+p'}} \circ \qquad \in X_{p+p'} \tag{7}$$

to

$$\circ \xrightarrow{\phi_1} \ldots \xrightarrow{\phi_p} \circ$$

and $\pi_2 : X_{p+p'} \to X_{p'}$ projects the same element (7) to

$$\circ \xrightarrow{\phi_{p+1}} \ldots \xrightarrow{\phi_{p+p'}} \circ\,.$$

One checks that

$$\delta(\omega \cup \eta) = \delta(\omega) \cup \eta + (-1)^{p+q}\omega \cup \delta(\eta)\,,$$

and so we get an induced cup product

$$H^n_{DR}(X_\bullet) \otimes H^m_{DR}(X_\bullet) \longrightarrow H^{n+m}_{DR}(X_\bullet)\,.$$

The cup product is associative on the level of cochains. But note that this is not true for (skew) commutativity. The cup product is commutative only on the level of cohomology.

Cohomology with compact supports

As with cohomology, cohomology with compact supports is defined via a double complex. As usual, let $X_1 \rightrightarrows X_0$ be a Lie groupoid. But now we have to also assume that $X_1 \rightrightarrows X_0$ is oriented. This means that both the manifolds X_1 and X_0 and the submersions s and t are oriented, in a compatible way. Moreover, assume that both X_0 and X_1 have constant dimension. Define two numbers r, n by the formulas

$$r = \dim X_1 - \dim X_0, \quad n = 2\dim X_0 - \dim X_1 \,.$$

Note that n is the dimension of the stack defined by $X_1 \rightrightarrows X_0$ and r is the relative dimension of X_1 over X_0.

Let $\Omega_c^q(X_p)$ denote the space of differential forms on X_p which have compact support. Note that Ω_c^q is not a sheaf. We consider the double complex

$$\begin{tikzcd}
\cdots \arrow[r, "-\partial_!"] & \Omega_c^{n+3r}(X_2) \arrow[r, "\partial_!"] & \Omega_c^{n+2r}(X_1) \arrow[r, "-\partial_!"] & \Omega_c^{n+r}(X_0) \\
\cdots \arrow[r, "-\partial_!"] & \Omega_c^{n+3r-1}(X_2) \arrow[r, "\partial_!"] \arrow[u, "d"] & \Omega_c^{n+2r-1}(X_1) \arrow[r, "-\partial_!"] \arrow[u, "d"] & \Omega_c^{n+r-1}(X_0) \arrow[u, "d"] \\
\cdots \arrow[r, "-\partial_!"] & \Omega_c^{n+3r-2}(X_2) \arrow[r, "\partial_!"] \arrow[u, "d"] & \Omega_c^{n+2r-2}(X_1) \arrow[r, "-\partial_!"] \arrow[u, "d"] & \Omega_c^{n+r-2}(X_0) \arrow[u, "d"] \\
 & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"]
\end{tikzcd} \tag{8}$$

For a form $\gamma \in \Omega_c^{n+(p+1)r-j}(X_p)$ its horizontal degree is $-p$ and its vertical degree is $n-j$. The vertical differential is the usual exterior derivative d. The horizontal differential is defined in terms of

$$\partial_! = \sum_i (-1)^i \partial_{i!} \,,$$

where $\partial_{i!} : \Omega_c^{q+r}(X_p) \to \Omega_c^q(X_{p-1})$ is the map obtained from $\partial_i : X_p \to X_{p-1}$ by integration over the fiber. In fact, for $\gamma \in \Omega_c^q(X_p)$, the horizontal differential is defined as

$$\gamma \longmapsto (-1)^p \partial_! \gamma \,.$$

To make a single complex out of (8), we define

$$C_c^\nu(X) = \bigoplus_{q-rp-p-r=\nu} \Omega_c^q(X_p) \,,$$

and set the total differential equal to

$$\delta(\gamma) = (-1)^p \big(\partial_! \gamma + d\gamma\big), \quad \text{for } \gamma \in \Omega_c^q(X_p) \,.$$

Note that the total degree of an element of $\Omega^q_c(X_p)$ is equal to $q-r(p+1)-p$, which is the sum of its vertical and horizontal degrees.

We also introduce notation for the horizontal cohomology of (8). Namely, we denote the k-th homology of $\big(\Omega_c^{(\bullet+1)r-q}(X_\bullet), \pm\partial_!\big)$ by $H^k_c(X,\Omega^q)$. This defines $H^k_c(X,\Omega^q)$ for $k \leq 0$ and $q \leq n$. We also denote $H^0_c(X,\Omega^q)$ by $\Gamma_c(X,\Omega)$.

Module structure

Now we shall turn (8) into a module over (5). Thus, given $\omega \in \Omega^q(X_p)$ and $\gamma \in \Omega_c^{q'}(X_{p'})$, we set

$$\omega \cap \gamma = (-1)^{-qp'} \pi_{1!}\big(\pi_2^*\omega \wedge \gamma\big)\,,$$

where π_1 and π_2 have similar meanings as in (6). More precisely, they are defined according to the cartesian diagram

$$\begin{tikzcd} X_{p'} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & X_p \arrow[d, "\text{0-th projection}"] \\ X_{p'-p} \arrow[r, "(p'-p)\text{-th projection}"'] & X_0\,. \end{tikzcd}$$

Note that $\omega\cap\gamma \in \Omega_c^{q+q'-pr}(X_{p'-p})$ and hence we have

$$\deg(\omega\cap\gamma) = \deg\omega + \deg\gamma\,.$$

Of course, if $p' < p$, then it is understood that $\omega\cap\gamma = 0$.

Using the projection formula

$$f_!(f^*\eta\wedge\tau) = \eta\wedge f_!(\tau)$$

it is not difficult to check associativity

$$(\tau\cup\omega)\cap\gamma = \tau\cap(\omega\cap\gamma)\,.$$

Using that integration over the fiber commutes with exterior derivative, one can also check the derivation property

$$\delta(\omega\cap\gamma) = \delta\omega\cap\gamma + (-1)^{\deg\omega}\omega\cap\delta\gamma\,,$$

which implies that the cap product passes to cohomology, and we have that $H^*_c(X)$ is a graded module over the graded ring $H^*(X)$.

The integral, Poincaré duality

We can define and integral

$$\int_X : H_c^n(X) \longrightarrow \mathbb{R}$$

by noticing that the integral $\Omega_c^{n+r}(X_0) \to \mathbb{R}$ vanishes on coboundaries of the total complex of $C_c^\bullet(X_\bullet)$.

Finally, we define a pairing

$$\begin{aligned} H_{DR}^\bullet(X) \otimes H_c^\bullet(X) &\longrightarrow \mathbb{R} \\ \omega \otimes \gamma &\longmapsto \int_X \omega \cap \gamma \,, \end{aligned} \tag{9}$$

For Poincaré duality, let us assume that X_1 and X_0 are of finite type, i.e., that they both have a finite good cover (see [2, §5]) and that these covers are compatible via s and t.

Proposition 19 (Poincaré duality) *Under this assumption, the pairing (9) sets up a perfect pairing*

$$H^p(X) \otimes H_c^{n-p}(X) \longrightarrow \mathbb{R}\,,$$

for all $p \geq 0$.

Proof. Consider the homomorphism of complexes

$$\begin{aligned} C^\bullet(X) &\longrightarrow \big(C_c^\bullet(X)[n]\big)^\vee \\ \omega &\longmapsto \int_{X_0} \omega \cap (\,\cdot\,)\,. \end{aligned}$$

It suffices to prove that this is a quasi-isomorphism. But this we can check by considering the associated spectral sequences whose E_1-terms are given by $H^q(X_p)$ and $H_c^{n-q}(X_p)^\vee$, respectively. Thus we conclude using usual Poincaré duality for manifolds (see [ibid.]) □

Definition 20 A differentiable stack is of **finite type**, if there exists a Lie groupoid $X_1 \rightrightarrows X_0$ presenting it, where X_0 and X_1 are differentiable manifolds of finite type admitting compatible finite good covers.

Let $\mathfrak{X}$ be a finite type differentiable stack. Then Poincaré duality implies that cohomology with compact supports is independent of the groupoid chosen to present $\mathfrak{X}$. Thus we get well-defined $H^p_c(\mathfrak{X})$ and an integral

$$\int_{\mathfrak{X}} H^n_c(\mathfrak{X}) \longrightarrow \mathbb{R}\,,$$

where $n = \dim \mathfrak{X}$. Poincaré duality holds:

$$H^p(\mathfrak{X}) \otimes H^{n-p}_c(\mathfrak{X}) \longrightarrow \mathbb{R}$$

is a perfect pairing.

Example 21 Recall that $H^*(BS^1) = \mathbb{R}[c]$ is a polynomial ring in one variable. By Poincaré duality, we have

$$H^p_c(BS^1) \cong \begin{cases} \mathbb{R} & \text{if } p \text{ is odd and negative,} \\ 0 & \text{otherwise.} \end{cases}$$

To exhibit the module structure, let

$$\psi_i = (-1)^{\frac{1}{2}i(i+1)} 1 \in \Omega^0\big((S^1)^i\big)\,,$$

which represents an element $\psi_i \in H^{-2i-1}_c(BS^1)$. Note that $c \cap \psi_i = \psi_{i-1}$, for all i and hence

$$\int_{BS^1} c^i \cap \psi_i = 1\,,$$

for all i, so that $\{\psi_i\}$ is the dual basis of $\{c^i\}$. Note that the $\mathbb{R}[c]$-module $H^*_c(BS^1)$ is divisible.

Deligne-Mumford stacks

From now on, we will assume that our Lie groupoids are étale, which means that $s : X_1 \to X_0$ and $t : X_1 \to X_0$ are étale (i.e., induce isomorphisms on tangent spaces). We will also assume that $X_1 \to X_0 \times X_0$ is proper and unramified, with finite fibers (unramified means injective on tangent spaces). These conditions mean that the associated differentiable stack is of Deligne-Mumford type.

Definition 22 A *partition of unity* for the groupoid $X_1 \rightrightarrows X_0$ is an $\mathbb{R}$-valued C^∞-function ρ on X_0 with the property that $s^*\rho$ has proper support with respect to $t : X_1 \to X_0$ and

$$t_! s^* \rho \equiv 1\,.$$

Partitions of unity may not exist, unless we pass to a Morita equivalent groupoid. This process works as follows.

For a groupoid as above there always exists an open cover $\{U_i\}$ of X_0, with the property that the restricted groupoid $V_i \rightrightarrows U_i$ (which is the restriction of $X_1 \rightrightarrows X_0$ via $U_i \to X_0$) is a transformation groupoid associated to the action of a finite group G_i on U_i. Given such a cover, we let $U = \coprod U_i$ and $V \rightrightarrows U$ be the restriction of $X_1 \rightrightarrows X_0$ via $U \to X_0$. Thus we have a Morita morphism from $V \rightrightarrows U$ to $X_1 \rightrightarrows X_0$.

Now we consider the coarse moduli space $\overline{X}$ of $X_\bullet$, which is also the coarse moduli space of $V \rightrightarrows U$. One way to define $\overline{X}$ is as the quotient of X_0 by the equivalence relation given by the image of $X_1 \to X_0 \times X_0$. Note that the coarse moduli space is Morita invariant and thus depends only on the stack defined by $X_\bullet$.

The open cover $\{U_i\}$ of X_0 induces an open cover of $\overline{X}$. Choose a differentiable partition of unity for $\overline{X}$ subordinate to this cover. Pull back to U. This gives over each U_i a G_i-invariant differentiable function ρ_i. Define $\rho : U \to \mathbb{R}$ by setting $\rho\,|\,U_i = \frac{1}{\#G_i}\rho_i$. It is then straightforward to check that ρ is, indeed, a partition of unity for the groupoid $V \rightrightarrows U$.

Proposition 23 *Assume that the groupoid $X_1 \rightrightarrows X_0$ admits a partition of unity. Then for every q we have long exact sequences*

$$\ldots \xrightarrow{\partial_!} \Omega^q_c(X_1) \xrightarrow{\partial_!} \Omega^q_c(X_0) \longrightarrow \Gamma_c(X, \Omega^q) \longrightarrow 0$$

and

$$0 \longrightarrow \Gamma(X, \Omega^q) \longrightarrow \Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \ldots$$

If we can find a partition of unity with compact support, then there is a long exact sequence

$$\ldots \xrightarrow{-\partial_!} \Omega^q_c(X_1) \xrightarrow{\partial_!} \Omega^q_c(X_0) \longrightarrow \Omega^q(X_0) \xrightarrow{\partial} \Omega^q(X_1) \xrightarrow{\partial} \ldots \tag{10}$$

Here the central map $\Omega^q_c(X_0) \to \Omega^q(X_0)$ is given by $\omega \mapsto s_! t^* \omega = t_! s^* \omega$. So in this latter case, we have a canonical isomorphism

$$\Gamma_c(X_\bullet, \Omega^q) \xrightarrow{\sim} \Gamma(X_\bullet, \Omega^q)\,.$$

Proof. To prove (10), let $\rho : X_0 \to \mathbb{R}$ be a partition of unity for $X_\bullet$, such that ρ has compact support. We define a contraction operator

$$\begin{aligned} K : \Omega^q(X_p) &\longrightarrow \Omega^q(X_{p-1}) \\ \omega &\longmapsto \partial_{0!}\big((\pi_0^* \rho)\omega\big)\,. \end{aligned}$$

Here $\pi_0 : X_p \to X_0$ maps onto the zeroth object, $\partial_0 : X_p \to X_{p-1}$ leaves out the zeroth object. This definition is valid for $p > 0$. We also define

$$\begin{aligned} K : \Omega^q_c(X_p) &\longrightarrow \Omega^q_c(X_{p+1}) \\ \omega &\longmapsto (-1)^{p+1}\pi_0^*\rho\,\partial_0^*\omega\,. \end{aligned}$$

This definition is valid for $p \geq 0$. We finally define $K : \Omega^q(X_0) \to \Omega^q_c(X_0)$ as multiplication by ρ. This defines a contraction operator for the total complex (10), i.e. we have $K\delta + \delta K = \mathrm{id}$, where δ is the boundary operator of (10).

The only place where we used properness was when we used multiplication by ρ to define $\Omega^q(X_0) \to \Omega_c(X_0)$. For this, ρ needs to have compact support, which is only true if $\mathfrak{X}$ is proper. In this case, we may choose X_0 to come from a finite cover U_i.

The first two claims follow by just using part of K. $\square$

Definition 24 A differentiable Deligne-Mumford stack $\mathfrak{X}$ is **proper** if

(i) we can find a groupoid presentation $X_1 \rightrightarrows X_0$ such that $X_1 \to X_0 \times X_0$ is proper

(ii) the coarse moduli space of $\mathfrak{X}$ (which is locally the quotient of a differentiable manifold by a finite group action) is proper.

Corollary 25 *For a differentiable Deligne-Mumford stack $\mathfrak{X}$ we have:*

- *the de Rham cohomology groups $H^k(\mathfrak{X})$ can be calculated as the cohomology groups of the global de Rham complex $\big(\Gamma(\mathfrak{X}, \Omega^\bullet), d\big)$.*
- *the compact support cohomology groups $H^k_c(\mathfrak{X})$ can be calculated using the global complex $\big(\Gamma_c(\mathfrak{X}, \Omega^\bullet), d\big)$.*

If $\mathfrak{X}$ is proper, we also have:

- *these two complexes are equal, i.e., for every q we have*

$$\Gamma_c(\mathfrak{X}, \Omega^q) = \Gamma(\mathfrak{X}, \Omega^q)\,,$$

- *for every k we have*

$$H^k(\mathfrak{X}) = H^k_c(\mathfrak{X})\,,$$

in particular, there exists an integral

$$\int_{\mathfrak{X}} : H^n(\mathfrak{X}) \longrightarrow \mathbb{R}\,, \tag{11}$$

- *the induced pairing*

$$H^k(\mathfrak{X}) \otimes H^{n-k}(\mathfrak{X}) \longrightarrow \mathbb{R}$$

is perfect.

Let us denote the structure map of an atlas X_0 admitting a partition of unity by $\pi : X_0 \to \mathfrak{X}$. With this notation, we may write the integral (11) as follows:

$$\int_{\mathfrak{X}} \omega = \int_{X_0} \rho \, \pi^* \omega \, .$$

Example 26 Let us consider a finite type Deligne-Mumford stack $\mathfrak{X}$ of dimension zero. We can present $\mathfrak{X}$ by a groupoid $X_1 \rightrightarrows X_0$, where both X_1 and X_0 are zero-dimensional, i.e., just finite collections of points. Then it is obvious, that $X_1 \rightrightarrows X_0$ is Morita equivalent to a disjoint union of groups: $X_0 = \{*_1, \ldots, *_n\}$, $X_1 = \coprod_{i=1}^n G_i$, for finite groups G_i, and one-point manifolds $*_i$.

We have $H^0(\mathfrak{X}) = H^0_c(\mathfrak{X}) = \mathbb{R}^n$ and all other cohomology groups vanish. There is the canonical element $1 \in H^0(\mathfrak{X})$, which together with the integral $\int_{\mathfrak{X}} : H^0(\mathfrak{X}) \to \mathbb{R}$, defines a canonical number $\int_{\mathfrak{X}} 1 \in \mathbb{R}$.

To calculate $\int_{\mathfrak{X}} 1$, note that $\rho(*_i) = \frac{1}{\#G_i}$ defines a partition of unity for $X_1 \rightrightarrows X_0$. Thus we have

$$\int_{\mathfrak{X}} 1 = \sum_{i=1}^n \int_{*_i} \frac{1}{\#G_i} = \sum_{i=1}^n \frac{1}{\#G_i} = \#\mathfrak{X} \, .$$

The *number of points* of an abstract finite groupoid $\mathfrak{X}$ is defined as

$$\#\mathfrak{X} = \sum_{x \in \mathfrak{X}/\cong} \frac{1}{\# \operatorname{Aut}(x)} \, .$$

The class of a substack, intersection numbers

Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a proper representable morphism of differentiable oriented Deligne-Mumford stacks. In terms of presenting groupoids, this means that we can find $X_1 \rightrightarrows X_0$ for $\mathfrak{X}$, $Y_1 \rightrightarrows Y_0$ for $\mathfrak{Y}$ and a morphism of groupoids $f_\bullet : Y_\bullet \to X_\bullet$ presenting f, with the properties:

(i) the diagram

$$\begin{tikzcd} Y_1 \arrow[r, "s"] \arrow[d, "f_1"'] & Y_0 \arrow[d, "f_0"] \\ X_1 \arrow[r, "s"] & X_0 \end{tikzcd}$$

is a pullback diagram of manifolds (this is the representability of f)

(ii) f_0 is proper.

One checks that a proper representable morphism f admits a wrong way map $\Omega^q_c(X_p) \to \Omega^q_c(Y_p)$, by pulling back compactly supported forms. This map passes to cohomology with compact supports and we denote the induced map by $f^* : H^i_c(\mathfrak{X}) \to H^i_c(\mathfrak{Y})$. By duality, we get an induced map on de Rham cohomology, which goes in the opposite direction. We denote it by $f_! : H^i(\mathfrak{Y}) \to H^{i-c}(\mathfrak{X})$, where $c = \dim \mathfrak{Y} - \dim \mathfrak{X}$.

Let $\dim \mathfrak{Y} = k$. We get a linear form

$$\begin{aligned} H^k_c(\mathfrak{X}) &\longrightarrow \mathbb{R} \\ \gamma &\longmapsto \int_{\mathfrak{Y}} f^*\gamma\,, \end{aligned}$$

and hence by duality and element $\mathrm{cl}(\mathfrak{Y}) \in H^{n-k}(\mathfrak{X})$, the *class* of $\mathfrak{Y}$. Alternatively, $\mathrm{cl}(\mathfrak{Y}) = f_!(1)$.

Example 27 If $E \to \mathfrak{X}$ is a vector bundle of rank r, then the class of the zero section in $H^r(E)$, pulls back (via the zero section) to an element $e(E) \in H^r(\mathfrak{X})$, known as the *Euler class* of E.

Every differentiable Deligne-Mumford stack $\mathfrak{X}$ has a tangent bundle $T_{\mathfrak{X}}$. The *Euler number* of a compact $\mathfrak{X}$ is defined as

$$e(\mathfrak{X}) = \int_{\mathfrak{X}} e(T_{\mathfrak{X}})\,.$$

Proposition 28 *Consider a cartesian (i.e. pullback) diagram of differentiable stacks of Deligne-Mumford type*

$$\begin{tikzcd} \mathfrak{W} \arrow[r] \arrow[d, "u"'] & \mathfrak{Z} \arrow[d] \\ \mathfrak{Y} \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Assume that all maps are proper and representable. Moreover, assume that for all $w \in \mathfrak{W}$ we have $T_{\mathfrak{W},w} = T_{\mathfrak{Y},w} \cap T_{\mathfrak{Z},w} \subset T_{\mathfrak{X},w}$ (a condition which can be checked and defined by pulling back to an étale presentation $X_0 \to \mathfrak{X}$). Then we have

$$\mathrm{cl}(\mathfrak{Y}) \cup \mathrm{cl}(\mathfrak{Z}) = f_! e(E)\,,$$

where E is the excess bundle

$$E = u^* N_{\mathfrak{Y}/\mathfrak{X}} / N_{\mathfrak{W}/\mathfrak{Z}}\,,$$

and $f : \mathfrak{W} \to \mathfrak{X}$ is the structure morphism. (Strictly speaking, this formula should be a sum over the connected components of $\mathfrak{W}$, because the excess bundle will not have constant rank, in general.)

Proof. We have been very careful to state a proposition that does not contain any global compactness assumptions on $\mathfrak{X}$. Thus the proof can be reduced to the case of manifolds by using an étale presentation of $\mathfrak{X}$. □

If $\mathfrak{X}$ is proper, we define the *intersection number* of two proper representable stacks $\mathfrak{Y} \to \mathfrak{X}$ and $\mathfrak{Z} \to \mathfrak{X}$ to be

$$\int_{\mathfrak{X}} \mathrm{cl}(\mathfrak{Y}) \cup \mathrm{cl}(\mathfrak{Z}) \in \mathbb{R}\,.$$

As applications of Proposition 28, we get

(i) if $\mathfrak{Y}$ and $\mathfrak{Z}$ intersect transversally, (again a condition that can be checked and defined after pullback to any étale presentation of $\mathfrak{X}$) and $\dim \mathfrak{Y} + \dim \mathfrak{Z} = \dim \mathfrak{X}$, we have

$$\int_{\mathfrak{X}} \mathrm{cl}(\mathfrak{Y}) \cup \mathrm{cl}(\mathfrak{Z}) = \#\mathfrak{W}\,.$$

(ii) if $\mathfrak{Y} = \mathfrak{Z}$ we have the self-intersection formula

$$\int_{\mathfrak{X}} \mathrm{cl}(\mathfrak{Y})^2 = \int_{\mathfrak{W}} e(u^* N_{\mathfrak{Y}/\mathfrak{X}})\,.$$

Example: the stack of elliptic curves

Consider the stack $\overline{M}_{1,2}$ of stable genus one curves with two marked points. If we consider $\overline{M}_{1,2}$ as a stack over $\mathbb{C}$, ignoring its arithmetic structure, we obtain a proper differentiable Deligne-Mumford stack. Alternatively, we can think of $\overline{M}_{1,2}$ as the stack of degenerate elliptic curves with a marked point (an elliptic curve is a genus one curve with a marked point serving as origin for the group law). The complex dimension of $\overline{M}_{1,2}$ is two and $\overline{M}_{1,2}$ is generically a scheme. (Exercise: determine the stacky points of $\overline{M}_{1,2}$.)

Recall that $\overline{M}_{1,2}$ has a natural morphism π to $\overline{M}_{1,1}$, exhibiting it as the universal family of (degenerate) elliptic curves over the stack $\overline{M}_{1,1}$ of (degenerate) elliptic curves. Thus we can picture the surface $\overline{M}_{1,2}$ as elliptically fibered over the curve $\overline{M}_{1,1}$.

There are two natural 'boundary divisors' on $\overline{M}_{1,2}$. First there is the universal section of $\pi : \overline{M}_{1,2} \to \overline{M}_{1,1}$. It maps every elliptic curve E in $\overline{M}_{1,1}$ onto its base point (zero element) $P \in E$, where we identify E with the fiber of π over E. Another way to think of the universal section is as the image of the morphism

$$\overline{M}_{1,1} = \overline{M}_{0,3} \times \overline{M}_{1,1} \longrightarrow \overline{M}_{1,2}\,,$$

which maps a genus one curve with one mark (E, P) to the degenerate genus one curve with two marks obtained by gluing $\mathbb{P}^1$ with three marks to E, by identifying the third mark on $\mathbb{P}^1$ with the mark P on E. By abuse of notation, we will denote this divisor on $\overline{M}_{1,2}$ by $\overline{M}_{1,1}$. It is in fact a substack, and hence an 'honest' divisor.

From this description it follows that the normal bundle of $\overline{M}_{1,1}$ in $\overline{M}_{1,2}$ is the pullback of he relative tangent bundle of π to $\overline{M}_{1,1}$. This may also be thought of as the bundle of Lie algebras associated to the family of groups π. Let us call this complex line bundle N.

The other divisor we are interested in is the degenerate fiber of our elliptic fibration. Already, there is a lot of ambiguity in this statement. To be more precise, let us consider the morphism $\overline{M}_{0,4} \to \overline{M}_{1,2}$ which takes a genus zero curve with four marks and glues together the marks labeled '3' and '4' to obtain a degenerate elliptic curve (with a marked point). This morphism gives rise to the class $\mathrm{cl}(\overline{M}_{0,4}) \in H^2(\overline{M}_{1,2})$. The image of this morphism is a closed substack of $\overline{M}_{1,2}$. It is irreducible of codimension one, so a Weil divisor on $\overline{M}_{1,2}$, which we shall denote by W.

Note that W is generically a scheme, but it has three stacky points: two smooth and one singular. (Exercise: what are the three corresponding degenerate curves with two marks?)

Another way to describe W is as the fiber of π over the closed substack $B\mathbb{Z}_2 \subset \overline{M}_{1,1}$ representing the degenerate curve. (Reduced closed substacks are determined by underlying point sets.) This shows that the morphism $\mathbb{P}^1 = \overline{M}_{0,4} \to W$ factors through the fiber of π over $\{*\} \to \overline{M}_{1,1}$. This fiber is a nodal elliptic curve, let us call it $\widetilde{W}$. The morphism $\mathbb{P}^1 = \overline{M}_{0,4} \to \widetilde{W}$ is the normalization map, the morphism $\widetilde{W} \to W$ is finite étale of degree $\frac{1}{2}$. It is the quotient map by the action of the inverse map on the nodal elliptic curve $\widetilde{W}$.

Yet another way to describe W is as the zero locus of a section of a complex line bundle. Consider the coarse moduli map $j : \overline{M}_{1,1} \to \mathbb{P}^1$ (the j-invariant). Pulling back, via j, the line bundle $\mathcal{O}(\infty)$ with its global section $1 \in \Gamma\big(\mathbb{P}^1, \mathcal{O}(\infty)\big)$, gives a line bundle L on $\overline{M}_{1,1}$ with a section whose zero locus is $B\mathbb{Z}_2 \subset \overline{M}_{1,1}$. Pulling back further to $\overline{M}_{1,2}$, we get the line bundle π^*L, which has a global section, whose zero locus is W.

To compute the self-intersection of $\overline{M}_{0,4}$ in $\overline{M}_{1,2}$ note that $\mathrm{cl}(\overline{M}_{0,4}) = \mathrm{cl}(\widetilde{W}) = \frac{1}{2}e(\pi^*L)$. Thus, we have

$$\begin{aligned}\int_{\overline{M}_{1,2}} \mathrm{cl}(\overline{M}_{0,4})^2 &= \frac{1}{4}\int_{\overline{M}_{1,2}} \pi^* e(L)^2 \\ &= \frac{1}{4}\int_{\overline{M}_{1,1}} e(L)^2 \cup \pi_!(1) \\ &= 0\,,\end{aligned}$$

as expected.

Next, let us calculate the intersection of $\overline{M}_{0,4}$ and $\overline{M}_{1,1}$ inside $\overline{M}_{1,2}$. We have a cartesian diagram

$$\begin{tikzcd} \{*\} \arrow[r, hook] \arrow[d] & \overline{M}_{0,4} \arrow[d] \\ \overline{M}_{1,1} \arrow[r, hook] & \overline{M}_{1,2} \end{tikzcd}$$

To see that this intersection is transversal, note that the intersection point correponds to a curve with two nodes and smoothing each node gives a tangent direction to $\{*\}$ in $\overline{M}_{1,2}$. Along $\overline{M}_{1,1}$, one of the nodes is smoothed, along $\overline{M}_{0,4}$, the other. We conclude that

$$\int_{\overline{M}_{1,2}} \mathrm{cl}(\overline{M}_{1,1}) \cup \mathrm{cl}(\overline{M}_{0,4}) = 1\,.$$

More interesting is the self-intersection of $\overline{M}_{1,1}$. Suppose given a family of degenerate elliptic curves E, parameterized by $\mathbb{P}^1$. We then get an induced cartesian diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \overline{M}_{1,2} \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f"] & \overline{M}_{1,1} \end{tikzcd}$$

by the universal mapping property of π. It follows that $f^*e(N) = e(N_{\mathbb{P}^1/E})$ and

$$\begin{aligned}\int_{\mathbb{P}^1} e(N_{\mathbb{P}^1/E}) &= \int_{\mathbb{P}^1} f^*e(N) \\ &= \int_{\overline{M}_{1,1}} f_! f^*e(N) \\ &= \deg(f) \int_{\overline{M}_{1,1}} e(N)\,,\end{aligned}$$

which we can solve for $\int_{\overline{M}_{1,1}} e(N)$.

The degree of f is in fact twice the number of rational fibres of $E \to \mathbb{P}^1$ (supposing that f is unramified over $j = \infty$). If we call $\int_{\mathbb{P}^1} e(N_{\mathbb{P}^1/E})$ the *degree* of the fibration $E \to \mathbb{P}^1$, denoted $\deg(E)$, we see that

$$\int_{\overline{M}_{1,2}} e(N) = \frac{\deg E}{2\#\{\text{ratl. fibers}\}}\,,$$

for any elliptic fibration $E/\mathbb{P}^1$.

For example, we may consider the pencil of plane cubics through 8 generic points. In this case E is the blow up of $\mathbb{P}^2$ in 9 points. Thus the number of rational fibres is equal to

$$\chi(E) = \chi(\mathbb{P}^2) + 9 = 12\,.$$

Any of the exceptional lines of the blow up can be chosen as section of $E \to \mathbb{P}^1$, proving that the degree of this elliptic fibration is -1. We conclude that

$$\int_{\overline{M}_{1,2}} \mathrm{cl}(\overline{M}_{1,1})^2 = \int_{\overline{M}_{1,1}} e(N) = -\frac{1}{24}\,.$$

The Lefschetz trace formula

Let $f : \mathfrak{X} \to \mathfrak{X}$ be an endomorphism of a proper oriented differentiable Deligne-Mumford stack $\mathfrak{X}$. Assume that f has non-degenerate fixed locus, which means that there exists a differentiable stack $\mathfrak{F}$ fitting into a cartesian diagram

$$\begin{tikzcd} \mathfrak{F} \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "\Gamma_f"] \\ \mathfrak{X} \arrow[r, "\Delta"] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

such that $T_{\mathfrak{F}} = T_{\mathfrak{X}} \cap T_{\mathfrak{X}}$. The usual proof applies and we get the Lefschetz trace formula

$$\operatorname{tr} f^*|H^*(\mathfrak{X}) = e(\mathfrak{F})\,.$$

If $\mathfrak{F}$ is zero-dimensional, then this says

$$\operatorname{tr} f^*|H^*(\mathfrak{X}) = \#\mathfrak{F}\,.$$

For f the identity of $\mathfrak{X}$, our assumption on $\mathfrak{F}$ is automatically satisfied, and then the fixed stack $\mathfrak{F}$ equals the inertia stack $I_{\mathfrak{X}}$ of $\mathfrak{X}$. We get

$$\chi_{DR}(\mathfrak{X}) = e(I_{\mathfrak{X}})\,.$$

In particular, the inertia stack has integer Euler number.

The Euler number of the inertia stack is hence a cohomological invariant. This is not true for the Euler number of the stack itself.

Example 29 Consider the trivial example of a finite group G acting on a finite set X, with zero-dimensional quotient stack $\mathfrak{X}$. The Euler number of $\mathfrak{X}$ is $\frac{\#X}{\#G}$, the Euler number of the inertia stack of $\mathfrak{X}$ is $\#(X/G)$, the number of orbits.

Example 30 We can use these results to compute the Euler number of $\overline{M}_{1,1}$. The inertia stack of $\overline{M}_{1,1}$ is the disjoint union of two identical copies of $\overline{M}_{1,1}$, two copies of $B\mathbb{Z}_4$ and four copies of $B\mathbb{Z}_6$. Thus the Euler number of the inertia stack equals $2e(\overline{M}_{1,1}) + \frac{2}{4} + \frac{4}{6}$. On the other hand, the cohomological Euler characteristic of $\overline{M}_{1,1}$ is the same as the cohomological Euler characteristic of the coarse moduli space $\mathbb{P}^1$ (see Proposition 36), which is 2. Hence $e(\overline{M}_{1,1}) = \frac{5}{12}$.

Singular homology

The de Rham theory has the drawback that it works only for differentiable stacks. Many algebraic stacks are singular, and hence do not have de Rham cohomology groups in the sense of the first section. That is why we need to develop a cohomology theory for topological stacks. In this part we will do this by generalizing singular homology and cohomology to stacks.

We do not try to develop the most general notion of topological stack here. A good notion is given by those stacks on the category of topological spaces which can be presented by topological groupoids $X_1 \rightrightarrows X_0$ satisfying the properties

(i) both X_1 and X_0 are topological spaces, all structure maps are continuous

(ii) the source and target maps $s, t : X_1 \to X_0$ are topological submersions.

We will tacitly assume all topological groupoids to satisfy these properties. This notion of topological stack includes all stacks appearing in algebraic geometry, where one assumes s and t always to be at least smooth.

The singular chain complex of a topological groupoid

To set up notation, recall the singular chain complex of a topological space X. We shall denote it by $C_\bullet(X)$. Thus $C_q(X)$ is the abelian group of formal integer linear combinations of continuous maps $\Delta_q \to X$. Let us denote the boundary maps by $d_i : \Delta_{q-1} \to \Delta_q$, for $i = 0, \ldots, q$. Then we have induced maps $d_i : \mathrm{Maps}(\Delta_q, X) \to \mathrm{Maps}(\Delta_{q-1}, X)$ and $d : C_q(X) \to C_{q-1}(X)$ defined by $d(\gamma) = \sum_{j=0}^{q}(-1)^j d_j(\gamma)$. It is a standard fact that $d^2 = 0$, so that $C_\bullet(X)$ is, indeed, a complex. The singular chain complex is covariant: if $f : X \to Y$ is continuous, we get an induced homomorphism of complexes $f_* : C_\bullet(X) \to C_\bullet(Y)$. Often we write f for f_*.

As in de Rham theory, we study stacks via presenting groupoids. Every topological groupoid defines a simplicial nerve

$$\ldots \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; X_2 \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; X_1 \rightrightarrows X_0\,, \tag{12}$$

because we can form fibered products liberally.

Now applying $C_\bullet$ to (12), we get the diagram

$$\ldots \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; C_\bullet(X_2) \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; C_\bullet(X_1) \rightrightarrows C_\bullet(X_0)\,. \tag{13}$$

By defining $\partial = \sum_{i=0}^{p}(-1)^i \partial_j$ we get a morphism of complexes $\partial : C_\bullet(X_p) \to$

$C_\bullet(X_{p-1})$. Thus we have defined a double complex

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_0(X_2) \arrow[r, "\partial"] & C_0(X_1) \arrow[r, "\partial"] & C_0(X_0) \\
\cdots \arrow[r, "\partial"] & C_1(X_2) \arrow[r, "\partial"] \arrow[u, "d"'] & C_1(X_1) \arrow[r, "\partial"] \arrow[u, "d"'] & C_1(X_0) \arrow[u, "d"'] \\
\cdots \arrow[r, "\partial"] & C_2(X_2) \arrow[r, "\partial"] \arrow[u, "d"'] & C_2(X_1) \arrow[r, "\partial"] \arrow[u, "d"'] & C_2(X_0) \arrow[u, "d"'] \\
 & \vdots \arrow[u, "d"'] & \vdots \arrow[u, "d"'] & \vdots \arrow[u, "d"']
\end{tikzcd}$$

We define the associated total complex $C_\bullet(X)$

$$C_n(X) = \bigoplus_{p+q=n} C_q(X_p)$$

with the differential $\delta : C_n(X) \to C_{n-1}(X)$ given by

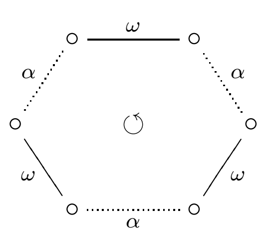
$$\delta(\gamma) = (-1)^{p+q}\partial(\gamma) + (-1)^q d(\gamma)\,, \quad \text{if } \gamma \in C_q(X_p)\,.$$

It is immediate that $\delta^2 = 0$.

Definition 31 The complex $\big(C_\bullet(X), \delta\big)$ is called the *singular chain complex* of the topological groupoid $X = [X_1 \rightrightarrows X_0]$. Its homology groups, denoted $H_n(X, \mathbb{Z})$, are called the *singular homology groups* of $X_1 \rightrightarrows X_0$.

What do cycles look like?

Typical examples of 1-cycles look like this:



(14)

Here the solid lines are paths in X_0, in other words paths of objects in the groupoid $X_1 \rightrightarrows X_0$. The dotted lines are elements in X_1, in other words morphisms in the groupoid $X_1 \rightrightarrows X_0$. The little circles represent elements of X_0, i.e., objects of $X_1 \rightrightarrows X_0$

Moreover, the cycle (14) has to be endowed with an orientation. This induces an orientation on each of the edges. Thus a path of objects (labeled ω) is then an oriented path in X_0. Each dotted line corresponds more precisely to an arrow and its inverse in the groupoid $X_1 \rightrightarrows X_0$. Among these two arrows we choose the one which points in the direction given by the orientation of (14).

Thus there are two ways to travel around a circle such as (14), connecting several objects of our groupoid: we can continuously deform one object to the next, or we can use an isomorphism to move us along.

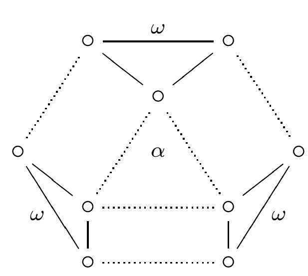
2-cycles are a little more difficult to describe. Picture an oriented closed surface S. Assume that S has been tiled with triangles and quadrilaterals. The edges of this tiling come it two types: solid ones called ω-edges and dotted ones called α-edges, just as in (14).

The triangles in our tiling also come in two types, type α and type ω. Triangles of type α are always bounded by three α-edges and triangles of type ω are always bounded by three ω-edges.

Finally, the quadrilaterals are all bounded by two α- and two ω-edges, in an alternating fashion.

Given such a tiled surface, every vertex will correspond to an object of $X_1 \rightrightarrows X_0$. Every ω-edge will represent a path of objects, every α-edge a morphism and its inverse in $X_1 \rightrightarrows X_0$. The ω-triangles correspond to continuous maps from Δ_2 to X_0, the quadrilaterals to paths in X_1 and the α-triangles to points in X_2. Thus the ω-triangles represent 2-simplices of objects, the quadrilaterals paths of morphisms and the α-triangles represent commutative triangles in $X_1 \rightrightarrows X_0$.

Here is an example of such a tiled surface, but this one has a boundary, so it does not give rise to a cycle. Instead, its boundary is the cycle given by (14).



(15)

This time we have labeled the four triangles according to their type.

Let us be more precise about sign questions. Our tiled surface is oriented, so it induces an orientation on each ω-triangle and so every ω-triangle does, in fact, give rise to a well-defined element of $C_2(X_0)$, at least an element which is well-defined up to a boundary in the complex $C_\bullet(X_0)$.

For the quadrilaterals, we make the convention that if we look at one in such a way that the α-edges are on the left and right, and the ω-edges on the top and bottom

$$\begin{tikzcd} \circ \arrow[r, dash] & \circ \\ \circ \arrow[u, dotted, "1"] \arrow[r, dash] \arrow[ur, phantom, "\circlearrowleft"] & \circ \arrow[u, dotted, "0"'] \end{tikzcd}$$

then we choose the morphisms in the groupoid to point 'up' and let the right arrow correspond to the value $t = 0$, and the left arrow to the value $t = 1$, where t is a coordinate on Δ_1. (Exercise: check that the appearant ambiguity in this definition leads to two choices which differ by a boundary in the total complex $C_\bullet(X_1 \rightrightarrows X_0)$.)

Finally, we also get induced orientations on the α-triangles. For every α-triangle we choose arrows or their inverses in such a way that we end up with a commutative triangle in the groupoid $X_1 \rightrightarrows X_0$, whose orientation is compatible with the given one. In this case, we get an element of $C_0(X_2)$, which is well-defined up to a boundary in the complex $\big(C_0(X_\bullet), \partial\big)$.

One can now check that our closed oriented tiled surface, together with the additional data of singular triangles in X_0, paths in X_1 and points in X_2 does, indeed, give rise to a 2-cycle in the singular chain complex $C_\bullet(X_1 \rightrightarrows X_0)$. It is also true, that every 2-cycle is a linear combination of such tiled surfaces.

Note that a 1-cycle such as (14) represents 0 in $H_1(X_1 \rightrightarrows X_0)$, if and only if there exists a disc as in (15) (or of a more complicated type) whose boundary is the given 1-cycle.

Examples

Let us consider a transformation groupoid $G \times X \rightrightarrows X$, where the group G is discrete. Then our double complex gives rise to a spectral sequence

$$E^2_{p,q} = H_q\big(G, H_p(X)\big) \Longrightarrow H_{p+q}(G \times X \rightrightarrows X)\,.$$

To see this, note that when taking vertical cohomology of the singular chain complex of $G \times X \rightrightarrows X$, we end up with a complex computing the group homology of G with values in the G-module $H_*(X)$. A simple case where this spectral sequence degenerates is the case of contractible X. In this case we get immediately that

$$H_p(G \times X \rightrightarrows X) = H_p(G, \mathbb{Z}),$$

so that the homology of the transformation groupoid is equal to the homology of the group G. For example, the stack of triangles up to similarity may be represented by the groupoid $S_3 \times \Delta_2 \rightrightarrows \Delta_2$. Thus the homology of the stack of triangles is equal to the homology of the symmetric group S_3.

Similarly, the stack of elliptic curves $M_{1,1}$ may be represented by the action of $SL_2(\mathbb{Z})$ by linear fractional transformations on the upper half plane in $\mathbb{C}$. Thus the homology of the stack of elliptic curves is equal to the homology of $SL_2(\mathbb{Z})$.

Invariance under Morita equivalence

We have already said that the homology of a stack is defined via the homology of a topological groupoid presenting the stack. For this to make sense, the homology of a groupoid has to be invariant under Morita equivalence.

It is helpful to examine the 2-functorial properties of the singular chain complex of topological groupoids. These are analogous to the properties of a contravariant functor of Remark 1. In fact, let F be any covariant functor from the category of topological spaces to the category of abelian groups.

Then a morphism of groupoids $f : X_\bullet \to Y_\bullet$ induces a homomorphism of homological complexes $f_* : F(X_\bullet) \to F(Y_\bullet)$. A 2-morphism $\theta : f \Rightarrow g$ between the two morphisms of groupoids $f, g : X_\bullet \to Y_\bullet$ induces a homotopy $\theta_* : f_* \Rightarrow g_*$, defined as follows: The map $\theta : X_0 \to Y_1$ extends to maps $\theta_0, \ldots, \theta_p : X_p \to Y_{p+1}$. Here θ_i maps the element

$$x_0 \xrightarrow{\phi_1} x_1 \xrightarrow{\phi_2} \ldots \xrightarrow{\phi_p} x_p$$

of X_p to the element

$$\begin{tikzcd} f(x_0) \arrow[r, "f(\phi_1)"] & f(x_1) \arrow[r, "f(\phi_2)"] & \cdots \arrow[r, "f(\phi_i)"] & f(x_i) \arrow[d, "\theta(x_i)"] & & \\ & & & g(x_i) \arrow[r, "g(\phi_{i+1})"] & \cdots \arrow[r, "g(\phi_p)"] & g(x_p) \end{tikzcd}$$

of Y_{p+1}. Then $\theta_* : F(X_p) \to F(Y_{p+1})$ is the alternating sum of the maps induced by $\theta_0, \dots, \theta_p$.

Applying this to the singular chain complex functor $F = C_\bullet$, we get that groupoid morphisms induce homomorphisms of singular chain complexes and 2-isomorphic groupoid morphisms induce homotopic chain maps. We also get that groupoid morphisms induce homomorphisms on singular homology groups, 2-isomorphic groupoid morphisms induce identical homomorphisms on homology and Morita morphisms with a section induce isomorphisms on homology. We need to prove that this is true also for Morita morphisms admitting only local sections. This will use double fibrations and a Mayer-Vietoris argument.

Let $\mathfrak{X}$ be a topological stack and $X_0 \to \mathfrak{X}$ and $Y_0 \to \mathfrak{X}$ two presentations. Let $X_\bullet$ and $Y_\bullet$ be the induced topological groupoids, or rather their induced simplicial topological spaces. We define $W_{mn} = X_m \times_{\mathfrak{X}} Y_n$, for all $m, n \geq 0$.

$$\begin{tikzcd} W_{mn} \arrow[r] \arrow[d] & Y_n \arrow[d] \\ X_m \arrow[r] & \mathfrak{X} \end{tikzcd}$$

Then $W_{\bullet\bullet}$ is a bisimplicial topological space. We apply $C_\bullet$ to $W_{\bullet\bullet}$ to obtain a triple complex $C_\bullet(W_{\bullet\bullet})$ mapping to the two double complexes $C_\bullet(X_\bullet)$ and $C_\bullet(Y_\bullet)$, see Figure 3. We claim that both induced maps on total complexes $\operatorname{tot}\big(C_\bullet(W_{\bullet\bullet})\big) \to \operatorname{tot}\big(C_\bullet(X_\bullet)\big)$ and $\operatorname{tot}\big(C_\bullet(W_{\bullet\bullet})\big) \to \operatorname{tot}\big(C_\bullet(Y_\bullet)\big)$ are quasi-isomorphisms. But this follows immediately from the following lemma.

Lemma 32 *Let $Y \to X$ be a surjective submersion of topological spaces. Let*

$$Y_p = \underbrace{Y \times_X Y \times_X \ldots \times_X Y}_{p+1}$$

be the simplicial nerve of the banal *groupoid $Y_1 \rightrightarrows Y_0$ associated to $Y \to X$. Then $C_\bullet(Y_1 \rightrightarrows Y_0) \to C_\bullet(X)$ is a quasi-isomorphism.*

PROOF. Let $\{U_i\}$ be an open cover of X over which $Y \to X$ admits sections. Let $U = \coprod_i U_i$. Thus $Y \to X$ admits a section over $U \to X$. As above, we consider the fibered product

$$\begin{tikzcd} W \arrow[r] \arrow[d] & U \arrow[d] \\ Y \arrow[r] & X\,. \end{tikzcd}$$

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\cdots \arrow[r] & C_\bullet(W_{22}) \arrow[r] \arrow[d] & C_\bullet(W_{12}) \arrow[r] \arrow[d] & C_\bullet(W_{02}) \arrow[r] \arrow[d] & C_\bullet(Y_2) \arrow[d] \\
\cdots \arrow[r] & C_\bullet(W_{21}) \arrow[r] \arrow[d] & C_\bullet(W_{11}) \arrow[r] \arrow[d] & C_\bullet(W_{01}) \arrow[r] \arrow[d] & C_\bullet(Y_1) \arrow[d] \\
\cdots \arrow[r] & C_\bullet(W_{20}) \arrow[r] \arrow[d] & C_\bullet(W_{10}) \arrow[r] \arrow[d] & C_\bullet(W_{00}) \arrow[r] \arrow[d] & C_\bullet(Y_0) \\
\cdots \arrow[r] & C_\bullet(X_2) \arrow[r] & C_\bullet(X_1) \arrow[r] & C_\bullet(X_0) &
\end{tikzcd}
$$

Figure 3: The triple complex

A similar argument as above, reduces to considering the two cases $W \to Y$ and $W \to U$. The first one is the case of an open cover, the second is the case of a map with a section.

The case of a map with section follows from 2-functoriality of the singular chain complex construction.

The case of an open cover is a classical fact. It is used in the proof of the Mayer-Vietoris sequence of singular homology, see for example Proposition 15.2 of [2]. □

Thus we have now proved that the singular homologies of two topological groupoids presenting the same stack are *canonically* isomorphic. We can thus make the following definition.

Definition 33 Let $\mathfrak{X}$ be a topological stack. Then we define the k-th *homology group* of $\mathfrak{X}$ with values in the integers to be

$$H_k(\mathfrak{X}, \mathbb{Z}) = H_k(X_1 \rightrightarrows X_0, \mathbb{Z}),$$

for any groupoid $X_1 \rightrightarrows X_0$ presenting $\mathfrak{X}$.

Equivariant homology

Consider a Lie group G acting continuously on a topological space X. We get an associated topological groupoid $\Gamma = [G \times X \rightrightarrows X]$ and the associated topological quotient stack $[X/G]$. We will prove that the homology $H_*([X/G])$ of this quotient stack is equal to the G-equivariant homology of X.

For this, consider BG, the topological classifying space[1] of G. Over BG there is a principal G-bundle EG, and EG is a contractible topological space. We consider the action of G on the product $EG \times X$ given by $g(e,x) = (eg^{-1}, gx)$. This is a free action whose quotient is denoted by X_G. (This is also called the homotopy quotient of X by G.) The homology of X_G is by definition the equivariant homology of X.

Since the quotient map $EG \times X \to X_G$ is a principal G-bundle, the transformation groupoid $\widetilde{\Gamma} = [G \times (EG \times X) \rightrightarrows (EG \times X)]$ is the banal groupoid associated to the topological submersion $(EG \times X) \to X_G$. Thus, by Lemma 32, we have a canonical isomorphism

$$H_*(\widetilde{\Gamma}) \longrightarrow H_*(X_G) = H_*^G(X)\,.$$

Now the projection onto the second factor $EG \times X \to X$ is equivariant, so induces a morphism of groupoids $\widetilde{\Gamma} \to \Gamma$. Because EG is contractible, the morphism $\widetilde{\Gamma}_\bullet \to \Gamma_\bullet$ of simplicial topological spaces is a level-wise homotopy equivalence. This implies immediately that the induced homomorphism

$$H_*(\widetilde{\Gamma}) \longrightarrow H_*(\Gamma) = H_*([X/G])$$

is an isomorphism.

Cohomology

For a topological groupoid $X = [X_1 \rightrightarrows X_0]$, we denote the dual of the complex $C_\bullet(X)$ by $C^\bullet(X)$. Thus we have

$$C^n(X) = \operatorname{Hom}\big(C_n(X), \mathbb{Z}\big)\,.$$

The cohomology groups of $C^\bullet(X)$ are called the *singular cohomology groups* of X. By the above results, it is immediate that the singular cohomology

[1] Only in this paragraph do we use the notation BG for the topological classifying space. Throughout the rest of these notes BG always denotes the stack $[*/G]$.

groups are the same, for different presentations of a given topological stack. Thus, for a topological stack $\mathfrak{X}$, we set

$$H^n(\mathfrak{X},\mathbb{Z}) = H^n(X,\mathbb{Z})\,,$$

where X is any groupoid presenting $\mathfrak{X}$.

If A is an arbitrary abelian group we define

$$H_k(\mathfrak{X},A) = h_k(C_\bullet(X)\otimes_{\mathbb{Z}} A)$$

and

$$H^k(\mathfrak{X},A) = h^k(C^\bullet(X)\otimes_{\mathbb{Z}} A)\,,$$

for any groupoid X, presenting the stack $\mathfrak{X}$.

Directly by construction we have pairings

$$H_k(\mathfrak{X},\mathbb{Z})\otimes H^k(\mathfrak{X},\mathbb{Z}) \longrightarrow \mathbb{Z}\,.$$

After tensoring with $\mathbb{Q}$, these give rise to natural identifications $H^k(\mathfrak{X},\mathbb{Q}) = H_k(\mathfrak{X},\mathbb{Q})^\vee$.

Relation to de Rham cohomology

Consider a Lie groupoid $X_1 \rightrightarrows X_0$ and consider singular cohomology defined using differentiable chains.

Define a pairing

$$\begin{aligned} C^\bullet_{DR}(X)\otimes C_\bullet(X) &\longrightarrow \mathbb{R} \\ \omega\otimes\gamma &\longmapsto \int_\gamma \omega\,. \end{aligned} \tag{16}$$

If $\omega\in\Omega^q(X_p)$ and $\gamma\in C_{q'}(X_{p'})$, then $\int_\gamma\omega$ is understood to vanish, unless $p=p'$ and $q=q'$. To make sure that we get a homomorphism of complexes $C^\bullet_{DR}(X)\otimes C_\bullet(X)\to\mathbb{R}$, we need to check that this pairing vanishes on coboundaries of total degree zero. (To turn $C_\bullet(X)$ into a cochain complex multiply all degrees by -1.) Thus we have to check that

$$\int_\gamma \delta\omega + (-1)^{p+q}\int_{\delta\gamma}\omega = 0\,.$$

But this fact follows directly from the chain rule and Stokes' theorem.

The pairing (16) induces a paring

$$H^k_{DR}(\mathfrak{X}) \otimes H_k(\mathfrak{X}) \longrightarrow \mathbb{R},,$$

for any differentiable stack $\mathfrak{X}$. This pairing can be used to define when a de Rham cohomology class $[\omega]$ is *integral*, namely by requiring $\int_\gamma \omega \in \mathbb{Z}$, for all $[\gamma] \in H_k(\mathfrak{X})$.

The pairing (16) also gives rise to a homomorphism of complexes

$$C^\bullet_{DR}(X) \longrightarrow C^\bullet(X) \otimes \mathbb{R}\,.$$

The fact that this homomorphism induces isomorphisms on cohomology, reduces via a spectral sequence argument immediately to the corresponding result for manifolds. We conclude that for every differentiable stack $\mathfrak{X}$ we have

$$H^*_{DR}(\mathfrak{X}) = H^*(\mathfrak{X}, \mathbb{R})\,.$$

Relation to the cohomology of the coarse moduli space

We define a topological Deligne-Mumford stack to be any topological stack which can be represented by a topological groupoid $X_1 \rightrightarrows X_0$, such that

(i) source and target maps $X_1 \to X_0$ are local homeomorphisms,

(ii) the diagonal $X_1 \to X_0 \times X_0$ is proper.

These conditions imply that the diagonal has finite fibers.

Exercise 34 If $X_1 \rightrightarrows X_0$ is a topological groupoid satisfying Conditions (i) and (ii), X_0 can be covered by open subsets U_i, such that the restriction of the groupoid $X_1 \rightrightarrows X_0$ to U_i is a transformation groupoid $G_i \times U_i \rightrightarrows U_i$ for a finite group G_i acting on U_i, for all i. We say that a topological Deligne-Mumford stack is locally a finite group quotient.

Recall that the image of the diagonal $X_1 \to X_0 \times X_0$ is an equivalence relation on X_0. In fact, by Assumption (ii), this equivalence relation is closed, and hence admits a Hausdorff quotient space $\overline{X}$.

Exercise 35 Prove that $\overline{X}$ depends only on the Morita equivalence class of the groupoid X, and is hence an invariant of the associated topological stack $\mathfrak{X}$. The topological space $\overline{X}$ is called the *coarse moduli space* of $\mathfrak{X}$, notation $\overline{\mathfrak{X}}$.

For example, the coarse moduli space of the stack BG, for a finite group G is the point $\{*\}$. This trivial example shows that the singular cohomology of a stack and its coarse moduli space can be quite different: the cohomology of BG is group cohomology, whereas the cohomology of $\{*\}$ is trivial. But note that all higher cohomology groups of finite groups are *torsion*.

This is a general fact: the difference between the cohomology of a Deligne-Mumford stack and its coarse moduli space is entirely due to torsion phenomena:

Proposition 36 *Let $\mathfrak{X}$ be a topological Deligne-Mumford stack with coarse moduli space $\overline{\mathfrak{X}}$. Then the canonical morphism $\mathfrak{X} \to \overline{\mathfrak{X}}$ induces isomorphisms on $\mathbb{Q}$-valued cohomology groups:*

$$H^k(\overline{\mathfrak{X}}, \mathbb{Q}) \xrightarrow{\sim} H^k(\mathfrak{X}, \mathbb{Q}) .$$

Proof. Using Exercise 34, we reduce to the case of a finite group quotient. This is because $\mathfrak{X}$ has an open cover by the quotient stacks $[U_i/G_i]$ and we can use the Čech spectral sequence.

So let us assume that $\mathfrak{X} = [X/G]$ for a finite group G acting on the topological space X. Then we have a spectral sequence

$$E_2^{p,q} = H^p\big(G, H^q(X, \mathbb{Q})\big) \Longrightarrow H^{p+q}(\mathfrak{X}, \mathbb{Q}) .$$

It degenerates, because all higher group cohomology over $\mathbb{Q}$ vanishes, and we deduce

$$H^p(\mathfrak{X}, \mathbb{Q}) = H^p(X, \mathbb{Q})^G ,$$

for all p. The coarse moduli space of $\mathfrak{X}$ is the quotient space X/G, and it is well-known that $H^p(X/G, \mathbb{Q}) = H^p(X, \mathbb{Q})^G$. $\square$

Chern classes

Consider the general linear group $GL_n = GL_n(\mathbb{C})$. We have

$$H^*(BGL_n, \mathbb{Z}) = \mathbb{Z}[t_1, \ldots, t_n] . \tag{17}$$

The element t_i is called the i-th universal Chern class, $t_i \in H^{2i}(BGL_n, \mathbb{Z})$. The equality (17) can be interpreted as meaning that every characteristic class of a complex vector bundle can be expressed in terms of Chern classes, and there are no universally true relations among Chern classes.

Given a rank n complex vector bundle $\mathfrak{E}$ over a stack $\mathfrak{X}$, we get an associated morphism of stacks $f : \mathfrak{X} \to BGL_n$. The diagram

$$\begin{tikzcd} \mathfrak{P} \arrow[r] \arrow[d] & * \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & BGL_n \end{tikzcd}$$

is a cartesian diagram of topological stacks. Here $\mathfrak{P}$ is the principal GL_n-bundle of frames of $\mathfrak{E}$. We also have, by passing back to the associated vector bundles, a cartesian diagram

$$\begin{tikzcd} \mathfrak{E} \arrow[r] \arrow[d] & {[\mathbb{A}^n/GL_n]} \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & BGL_n \end{tikzcd}$$

Giving $f : \mathfrak{X} \to BGL_n$ is entirely equivalent to giving a complex vector bundle over $\mathfrak{X}$. (A similar principal holds for algebraic stacks: a morphism of algebraic stacks $\mathfrak{X} \to BGL_n$ is the same thing as an algebraic vector bundle over $\mathfrak{X}$.)

We define the i-th Chern class $c_i(\mathfrak{E}) \in H^{2i}(\mathfrak{X}, \mathbb{Z})$ to be equal to the pullback of t_i via f:

$$c_i(\mathfrak{E}) = f^* t_i \,.$$

In everything that follows, we will be considering singular cohomology with integer coefficients.

Lemma 37 *Let $\pi : \mathfrak{E} \to \mathfrak{X}$ be a complex vector bundle with zero section $\iota : \mathfrak{X} \to \mathfrak{E}$. Then $\pi^* : H^*(\mathfrak{X}) \to H^*(\mathfrak{E})$ is an isomorphism with inverse ι^*.*

Proof. Given a groupoid $X_1 \rightrightarrows X_0$ presenting $\mathfrak{X}$, we obtain a groupoid $E_1 \rightrightarrows E_0$ presenting $\mathfrak{E}$ by forming the fibered products

$$\begin{tikzcd} E_i \arrow[r] \arrow[d] & \mathfrak{E} \arrow[d] \\ X_i \arrow[r] & \mathfrak{X} \end{tikzcd}$$

We also get a morphism of groupoids $E \to X$ and E_i is a vector bundle over X_i, for all i. (Conversely, any such groupoid $E \to X$, which consists of vector bundles, defines a vector bundle over $\mathfrak{X}$. For example, if X is a transformation groupoid, a vector bundle on $\mathfrak{X}$ is the same thing as an equivariant bundle.)

The induced homomorphism of cochain complexes $C^\bullet(X) \to C^\bullet(E)$ is easily seen to be a quasi-isomorphism by a spectral sequence argument. $\square$

We define a homomorphism $H^k(\mathfrak{X}) \to H^{k+2n}(\mathfrak{E})$ by sending $\omega \in H^k(\mathfrak{X})$ to $\pi^*(\omega \cup c_n\mathfrak{E})$. (In de Rham theory, this homomorphism is equal to $i_!$.)

Proposition 38 *There is a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r, "\cdot c_n"'] & H^{k+2n-1}(\mathfrak{E}) \arrow[r] & H^{k+2n-1}(\mathfrak{E}\setminus\mathfrak{X}) \arrow[dll] \\
H^k(\mathfrak{X}) \arrow[r, "\cdot c_n"'] & H^{k+2n}(\mathfrak{E}) \arrow[r] & H^{k+2n}(\mathfrak{E}\setminus\mathfrak{X}) \arrow[dll] \\
H^{k+1}(\mathfrak{X}) \arrow[r, "\cdot c_n"'] & H^{k+2n+1}(\mathfrak{E}) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. Let X be a groupoid presenting $\mathfrak{X}$, and E the induced groupoid presenting $\mathfrak{E}$. We wish to construct a short exact sequence of cochain complexes

$$0 \longrightarrow C^\bullet(X)[-2n] \longrightarrow C^\bullet(E) \longrightarrow C^\bullet(E\setminus X) \longrightarrow 0\ . \tag{18}$$

To do this, choose a cochain $\phi \in C^{2n}(X)$ representing $c_n(\mathfrak{E})$. Then the first map in (18) is given by cupping with ϕ and pulling back. The second map in (18) is simply restriction. Now we need to prove exactness of (18). This we do by filtering the complexes involved in such a way that the associated graded pieces look like

$$0 \longrightarrow C^\bullet(X_p)[-2n] \longrightarrow C^\bullet(E_p) \longrightarrow C^\bullet(E_p\setminus X_p) \longrightarrow 0\ ,$$

and we are reduced to the case of spaces. The key point is that even though ϕ itself might not be of pure bidegree, i.e., not entirely contained in $C^{2n}(X_0)$, after passing to the graded pieces, the other contributions drop out. $\square$

Corollary 39 *If all odd cohomology of $\mathfrak{X}$ vanishes, we have*

$$H^*(\mathfrak{E}\setminus\mathfrak{X}) = H^*(\mathfrak{X})/c_n(\mathfrak{E})\,.$$

Example 40 Consider $\mathfrak{X} = B\mathbb{G}_{\mathrm{m}} = B\mathbb{C}^*$, with $H^*(B\mathbb{G}_{\mathrm{m}}) = \mathbb{Z}[t]$. Any character (one-dimensional representation) of $\mathbb{G}_{\mathrm{m}} = \mathbb{C}^*$ defines a complex line bundle over $B\mathbb{G}_{\mathrm{m}}$. If the character is $\chi_j(\lambda) = \lambda^j$, then the first Chern class of the corresponding line bundle L_j is jt.

Note that $L_j \setminus B\mathbb{G}_{\mathrm{m}} = [\mathbb{G}_{\mathrm{m}}/\mathbb{G}_{\mathrm{m}}] = B\mu_j$, where μ_j is the group of j-th roots of unity. Thus we get

$$H^*(B\mu_j) = \mathbb{Z}[t]/jt = \mathbb{Z} \oplus \mathbb{Z}/j\mathbb{Z} \oplus \mathbb{Z}/j\mathbb{Z} \oplus \dots$$

Taking the direct sum of $L_{j_1}, \dots, L_{j_n}$, we get that $\mathfrak{E} \setminus \mathfrak{X} = \mathbb{A}^n - \{0\}/\mathbb{G}_{\mathrm{m}}$, where the action is through $\lambda(x_1, \dots, x_n) = (\lambda^{j_1} x_1, \dots, \lambda^{j_n} x_n)$. This is the *weighted projective space stack* $\mathbb{P}(j_1, \dots, j_n)$. The top Chern class of a direct sum of line bundles is the product of the first Chern classes of the line bundles. Hence we obtain

$$H^*\big(\mathbb{P}(j_1, \dots, j_n)\big) = \mathbb{Z}[t]/(j_1 \dots j_n t^n)\,.$$

Note how this reduces to the cohomology of projective space after moding out by torsion.

For example, $\overline{M}_{1,1} = \mathbb{P}(4,6)$. We conclude

$$H^*(\overline{M}_{1,1}) = \mathbb{Z}[t]/24t^2 = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z} \oplus \mathbb{Z}/24\mathbb{Z} \oplus \dots$$

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