Topological modular forms with level structure

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Abstract

The cohomology theory known as Tmf, for "topological modular forms," is a universal object mapping out to elliptic cohomology theories whose coefficient ring is closely connected to the classical ring of modular forms. We extend this to a functorial family of objects corresponding to elliptic curves with level structure and modular forms on them. These come equipped with highly structured multiplication. Along the way, we produce a natural way to restrict to the cusps, providing multiplicative maps from Tmf with level structure to forms of K-theory.

This is accomplished using the machinery of logarithmic structures. We construct a sheaf of locally even-periodic elliptic cohomology theories, equipped with highly structured multiplication, on the log-étale site of the moduli of elliptic curves. Evaluating this sheaf on modular curves produces Tmf with level structure. In particular, this allows us to construct a connective spectrum $\mathrm{tmf}_0(3)$ consistent with calculations of Mahowald and Rezk.

1 Introduction

The subject of topological modular forms traces its origin back to the Witten genus. The Witten genus is a function taking String manifolds and producing elements of the power series ring $\mathbb{C}[\![q]\!]$, in a manner preserving multiplication and bordism classes (making it a genus of String manifolds). It can therefore be described in terms of a ring homomorphism from the bordism ring $MO\langle 8\rangle_*$ to this ring of power series. Moreover, Witten explained that this should factor through a map $MO\langle 8\rangle_* \to MF_*$ taking values in a particular subring: the ring of modular forms.

An algebraic perspective on modular forms is that they are universal functions on elliptic curves. Given a ring R, an elliptic curve $\mathcal{E}$ over R, and an invariant 1-form ω on $\mathcal{E}$, a modular form g assigns an invariant $g(\mathcal{E}, \omega) \in R$.

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This is required to respect base change for maps $R \to R'$ and isomorphisms of elliptic curves $\mathcal{E} \to \mathcal{E}'$. The form is of weight k if it satisfies $g(\mathcal{E}, \lambda\omega) = \lambda^k g(\mathcal{E}, \omega)$, and this determines the grading.

Thus, given a graded ring R_*, we might have two pieces of data.

First, this ring may arise as the coefficient ring of a complex orientable cohomology theory E. This gives E a theory of Chern classes, and from this we derive a formal group $\mathbb{G}_E$ on R_* by work of Quillen [Qui69].

Second, the ring R_* may carry an elliptic curve $\mathcal{E}$ with an invariant 1-form ω, in a manner appropriately compatible with the grading. The elliptic curve also produces a formal group $\widehat{\mathcal{E}}$ on R_*.

The cohomology theory becomes an *elliptic cohomology theory* if we also specify an isomorphism $\mathbb{G}_E \to \widehat{\mathcal{E}}$. The universal property of modular forms automatically gives a ring homomorphism $MF_* \to R_*$. Witten's work then suggests a topological lift: elliptic cohomology theories should possess a map $MO\langle 8\rangle_* \to R_*$, and we can ask if this comes from a natural map $MO\langle 8\rangle \to E$. The subjects of elliptic cohomology and topological modular forms were spurred by the these developments [Lan88, LRS95, HM, AHS01].

In short, our goal is to produce cohomology theories solving a certain realization problem. One would like to take rings R equipped with elliptic curves $\mathcal{E}$, and produce elliptic cohomology theories derived from R and $\mathcal{E}$ [Goe09]. These have representing objects in the stable homotopy category called elliptic spectra. The object tmf, for "topological modular forms," arises as an object that maps in a universal way to these elliptic spectra.

However, within this general framework there are several related objects that are interchangeably described as topological modular forms.

The original definitions of elliptic cohomology ($\mathcal{E}ll$) and of topological modular forms (tmf) produced ring spectra, generating multiplicative cohomology theories, that enjoy a universality property for maps to certain elliptic spectra. More powerful than that is a functorial description. There is an assignment of elliptic spectra to certain rings equipped with an elliptic curve, functorially in certain maps, and the homotopy limit of this diagram is a ring spectrum of topological modular forms. Even further, this functor satisfies patching in the sense that it preserves certain limit diagrams of elliptic curves. This grants one the power to extend it to a larger functor, and the result is a sheaf of spectra on a moduli stack of elliptic curves.

However, there are several versions of this story. There is a division depending on whether one allows only smooth elliptic curves (represented the Deligne-Mumford stack $\mathcal{M}_{ell}$), elliptic curves with nodal singularities (represented by the compactification $\overline{\mathcal{M}}_{ell}$), or curves based on general cubic curves (represented by an algebraic stack $\mathcal{M}_{cub}$). These are sometimes informally given the names TMF, Tmf, and tmf respectively, and they represent a progressive decrease in our ability to obtain conceptual interpretations or construct objects. The stack $\mathcal{M}_{ell}$ can be extended by Lurie's work to a derived stack representing derived elliptic curves [Lur09]; the so-called "old" construction of topological modular forms due to Goerss-Hopkins-Miller, by obstruction theory, gives (the étale site of) $\overline{\mathcal{M}}_{ell}$ a sheaf of elliptic spectra [Beh]; the less-conceptual process of taking a connective cover produces a spectrum tmf, and the generalizations of tmf that exist are constructed in a somewhat ad-hoc manner (but see [Mat]), though these objects have exceptionally interesting properties.

There are many situations where extra functoriality for elliptic spectra can be a great advantage [Beh07, Sto12, Mei12]. In particular, considering elliptic curves equipped with extra structure, such as choices of subgroups or torsion points, leads to a family of generalizations. The corresponding moduli objects $\mathcal{M}(n)$, $\mathcal{M}_0(n)$, $\mathcal{M}_1(n)$, and more have been well-studied from the points of view of number theory and arithmetic geometry. Away from primes dividing n, these automatically inherit sheaves of elliptic spectra from TMF. For example, the maps $\mathcal{M}(n) \to \mathcal{M}_{ell}$ are étale covers with Galois group $GL_2(\mathbb{Z}/n)$, and so the étale sheaf defined on $\mathcal{M}_{ell}$ can be evaluated on $\mathcal{M}(n)$ to produce a $\mathrm{GL}_2(\mathbb{Z}/n)$-equivariant spectrum $\mathrm{TMF}(n)$.

However, the compactifications $\overline{\mathcal{M}}(n)$ are not étale over $\overline{\mathcal{M}}_{ell}$. In complex-analytic terms, if $\overline{\mathcal{M}}_{ell}$ is described in terms of a coordinate τ on the upper half-plane, then the compactification point or "cusp" has a coordinate $q = e^{2\pi i \tau}$; the cusps in $\overline{\mathcal{M}}(n)$ have coordinates expressed in terms of $q' = e^{2\pi i \tau/n}$, satisfying $q = (q')^n$. Since the leading coefficient of $dq = n(q')^{n-1}dq'$ is not a unit, the map is not an isomorphism on cotangent spaces and is ramified over the cusp of $\overline{\mathcal{M}}_{ell}$. This ramification complicates the key input to the Goerss-Hopkins-Miller obstruction theory.

The obstruction theory constructing Tmf does generalize readily to each individual moduli stack, and has been carried out in some instances to construct objects $\mathrm{Tmf}(n)$ [Sto12]. However, needing to re-do the entire construction of Tmf once per level structure is less than satisfying as a mathematical technique, and does it not provide any functoriality across different forms of level structure. Moreover, it does not give an immediate reason why one might expect a relationship between Tmf and the homotopy fixed-point object for the action of $\mathrm{GL}_2(\mathbb{Z}/n)$ on $\mathrm{Tmf}(n)$.

However, these ramified maps $\overline{\mathcal{M}}(n) \to \overline{\mathcal{M}}_{ell}$ do possess a slightly less restrictive form of regularity. The cusp determines a "logarithmic structure" on $\overline{\mathcal{M}}_{ell}$ [Kat89], and the various maps between moduli of elliptic curves are log-étale—this roughly expresses the fact that in the expression $\mathrm{dlog}(q^n) = n \cdot \mathrm{dlog}(q)$, the coefficient is a unit away from n. These ramified covers form part of a log-étale site enlarging the étale site of $\overline{\mathcal{M}}_{ell}$. Moreover, for the log schemes we will be considering, the fiber product in an appropriate category of objects with logarithmic structure is geared so that the Čech nerve of the cover $\overline{\mathcal{M}}_{ell}(n) \to \overline{\mathcal{M}}_{ell}$ is the simplicial bar construction for the action of $\mathrm{GL}_2(\mathbb{Z}/n)$ on $\overline{\mathcal{M}}_{ell}(n)$ (see 2.19).

These are precisely the conditions that might be useful from the point of view of extending functoriality for topological modular forms.

The goal of this paper is to extend Tmf to a sheaf of E_∞ ring spectra on the log-étale site of $\overline{\mathcal{M}}_{ell}$ (Theorem 7.6). In rough:

- there is an assignment of E_∞ ring spectra to certain generalized elliptic curves $\mathcal{E} \to X$ when the scheme X is equipped with a compatible logarithmic structure;

- this is functorial in certain diagrams

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

  such that the map $\mathcal{E}' \to \mathcal{E} \times_X X'$ is an isomorphism of elliptic curves over X';

- this functor satisfies descent with respect to log-étale covers $\{U_\alpha \to X\}$ (or hypercovers), in the sense that the value on X is the homotopy limit of the values on the Čech nerve; and

- in the special case where $\mathcal{E} \to X$ comes from a Weierstrass curve over $\mathrm{Spec}(R)$, the associated spectrum realizes it by an even-periodic elliptic cohomology theory E. (For more general elliptic curves on $\mathrm{Spec}(R)$, the functor produces a weakly even-periodic object, and the formal group scheme $\mathrm{Spf} E^*(\mathbb{CP}^\infty)$ will come equipped with a natural isomorphism to the formal group of $\mathcal{E}$.)

The specific property of $\mathcal{E}/X$ needed is that the resulting map $X \to \overline{\mathcal{M}}_{ell}$ classifying it must be log-étale.

The payoff is the establishment of the existence of topological modular forms spectra for all modular curves (Theorem 8.1). For a fixed integer N and a subgroup $\Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$ we have an E_∞ ring spectrum $\mathrm{Tmf}(\Gamma)$ (with N a unit in π_0). Functoriality then allows this to extend to a contravariant functor on a category of orbits. For $\Gamma \subset \Gamma'$ we have maps

$$\mathrm{Hom}_{GL_2(\mathbb{Z}/N)}(\mathrm{GL}_2(\mathbb{Z}/N)/\Gamma', \mathrm{GL}_2(\mathbb{Z}/N)/\Gamma) \to \mathrm{Hom}_{E_\infty}(\mathrm{Tmf}(\Gamma), \mathrm{Tmf}(\Gamma'))$$

respecting composition. We can then carry out an analysis of log-étale descent, and for $K \lhd \Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$ we find that the natural map

$$\mathrm{Tmf}(\Gamma) \to \mathrm{Tmf}(K)^{h\Gamma/K}$$

is an equivalence to the homotopy fixed-point spectrum. Finally, if we define $p\colon \mathrm{GL}_2(\mathbb{Z}/NM) \to \mathrm{GL}_2(\mathbb{Z}/N)$ to be the natural projection, there is a natural transformation $\mathrm{Tmf}(\Gamma) \to \mathrm{Tmf}(p^{-1}\Gamma)$ which is a localization, formed by inverting M.

(The reader who is surprised by the appearance of GL_2 rather than PSL_2 when discussing modular curves should be aware that this is an aspect of the difference between the complex-analytic theory and the theory over $\mathbb{Z}$. These modular curves may have several components when considered over $\mathbb{C}$, and this may show up in the form of a larger ring of constants. Two modular curves may have the same set of points over $\mathbb{C}$ because the group $\{\pm 1\}$ acts trivially on isomorphism classes, but the action is nontrivial on the associated spectra.)

The forms of K-theory from [LN13, Appendix A] play an important role in our construction, and allow us to generalize the main result of that paper. In particular, in the case above we have a generalized elliptic curve $\mathcal{E} \to X$, which has a restriction to the "cusps" $X^c \subset X$. The restriction of $\mathcal{E}$ to X^c is essentially a form of the multiplicative group $\mathbb{G}_m$, and there is a corresponding form of K-theory (after Morava [Mor89]). Our proof produces a natural map of E_∞ ring spectra from our functorial elliptic cohomology theory built from X to this form of K-theory built from X^c (Theorem 8.2).

We can then apply this to construct spectra $\mathrm{tmf}_1(3)$ and $\mathrm{tmf}_0(3)$, connective E_∞ ring spectra that realize calculations carried out by Mahowald and Rezk [MR09], together with a commutative diagram of E_∞ ring spectra

$$\begin{tikzcd} \mathrm{tmf}_0(3) \arrow[r] \arrow[d] & ko[1/3] \arrow[d] \\ \mathrm{tmf}_1(3) \arrow[r] & ku[1/3] \end{tikzcd}$$

(Theorem 8.3). In particular, this E_∞ connective version seems likely to coincide with one of their conjectural models. Carrying out calculations with further level structures remains a very interesting problem.

We should mention some directions for further investigation and some desirable generalizations of the results of this paper.

One obvious missing component is the connection to elliptic genera. The work of Ando, Hopkins, Rezk, and Strickland produced highly multiplicative lifts of the sigma orientation and the Atiyah-Bott-Shapiro orientation [AHS04, AHR], ultimately in the form of E_∞ maps $MO\langle 8\rangle \to \mathrm{Tmf}$ and $MO\langle 4\rangle \to KO$. It would be very useful to know which objects $\mathrm{Tmf}(\Gamma)$ (such as $\mathrm{Tmf}_1(3)$, which is complex orientable) accept refinements of the sigma orientation to extensions of $MO\langle 8\rangle$.

Rognes has recently developed a closely related concept of topological logarithmic structures for applications in algebraic K-theory [Rog09]. The core construction in this paper is built on a map of E_∞ spaces $\mathbb{N} \to \Omega^\infty KO[\![q]\!]$ modeling the logarithmic structure of $\overline{\mathcal{M}}_{ell}$ at the cusp. This ultimately imparts a topological logarithmic structure to $KO[\![q]\!]$ that should lift to Tmf, and it is natural to suspect that logarithmic obstruction theory constructs our sheaf. There does not seem to be an obstruction theory for maps between ring spectra with logarithmic structures in the literature that is yet developed enough carry out this program, but this is almost entirely due to how recently topological logarithmic structures have appeared.

The functor Tmf should extend to a functor on separable isogenies of elliptic curves, which produce isomorphisms of formal group laws. After inverting ℓ, this would allow the construction of two maps $\mathrm{Tmf} \rightrightarrows \mathrm{Tmf}_0(\ell)$ classified by the two canonical isogenous curves over $\mathrm{Tmf}_0(\ell)$, extend to the construction of global versions of Behrens' $Q(\ell)$ spectra [Beh07], and allow an "adelic" formulation of the functoriality of Tmf. However, $K(1)$-local obstruction theory seems to be an inadequate tool for this job. For example, this would require constructing an action of $\mathbb{N}^k$ on $N^{-1}\mathrm{Tmf}$, in the form of a commuting family of Adams operations $[\ell]$ for primes ℓ dividing N. Since this obstruction theory is not in the category of Tmf-algebras, it does not degenerate as easily. We hope that a consequence of Lurie's constructive methods for associating spectra to p-divisible groups (as employed in [BL10]) will be that the smooth object TMF becomes functorial in separable isogenies, and that the patching construction in this paper will inherit these isogeny operations from compatibility with the Adams operations on complex K-theory. We have chosen to write this paper without appealing to Lurie's forthcoming work.

Finally, the construction of the object tmf by connective cover remains wholly unsatisfactory, and this is even more true when considering level structure. In an ideal world, tmf should be a functor on a category of Weierstrass curves equipped with some form of extra structure. We await the enlightenment following discovery of what exact form this structure should take.

1.1 Outline of the method

In order to minimize the amount of repeated effort, our proof is based heavily on the tools used to construct topological modular forms in [Beh]. Our work follows the general Hopkins-Miller paradigm: we already have TMF defined away from the cusp, so in this paper we construct it explicitly in a formal neighborhood of the cusp and patch the two constructions together.

Section § 2 is the longest of the paper, but has only two main goals. The first is to discuss the moduli objects in question: coarsely, they parametrize log schemes equipped with an elliptic curve and some data expressing compatibility of the logarithmic structure with the j-invariant. The second is to give enough discussion to show that producing our desired derived structure sheaf $\mathcal{O}^{der}$ is equivalent to defining functorial elliptic cohomology theories on a much smaller category, one of log rings equipped with suitable Weierstrass curves. This requires delving into some details about Grothendieck topologies on log schemes. Our approach to this might be described as utilitarian.

In § 3, we describe elliptic spectra and point out that elliptic structures automatically lift under pullback.

In § 4 we recall how the Goerss-Hopkins-Miller theorem gives rise to a $K(2)$-local construction of $\mathcal{O}^{der}$ at any fixed prime p.

In § 5 we use real K-theory to produce a $K(1)$-local structure sheaf for log-étale maps to a formal neighborhood $\mathcal{M}_{Tate}$ of the cusp, classifying forms of the Tate curve. We start with Tate K-theory, a $\mathbb{Z}/2$-equivariant elliptic cohomology theory called $K[\![q]\!]$ studied in [AHS01], and extend it to a sheaf for log-étale maps to $\mathcal{M}_{Tate}$ by direct construction. In this section we also construct a natural map to forms of K-theory corresponding to evaluating at the cusps.

In § 6 we recall enough obstruction theory to produce the $K(1)$-local structure sheaf on the locus of smooth, ordinary elliptic curves. This obstruction theory is then applied to glue this structure sheaf to the sheaf on the supersingular locus and the one at the cusps. The basic construction at the cusps starts with a map $\mathrm{tmf} \to KO[\![q]\!]$ (an E_∞ map factoring the Witten genus), which is constructed in Appendix A. This is localized to produce a map $\mathrm{TMF} \to q^{-1}KO[\![q]\!]$, and degeneration of the Goerss-Hopkins obstruction theory for Tmf-algebras allows us to extend to a map of sheaves.

Finally, § 7 uses an arithmetic square to glue the p-complete constructions together with a rational sheaf to produce a global, integral, version. This relies very specifically on *formality* of the rationalization of the constructions in previous sections: they are equivalent to functors factoring through the Eilenberg-Mac Lane functor.

This assembles all the pieces necessary for the main results of the paper in § 8.

1.2 Background assumptions

We take as given that there exists an E_∞ ring spectrum tmf with the properties described in [Beh]. We additionally assume that at the prime $p = 2$, $L_{K(1)}\mathrm{tmf}$ is described by two pushout diagrams in the category of $K(1)$-local E_∞ ring spectra as in [Hop, Lau04].

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The ideas in this paper would not have existed without the Loen conference "p-Adic Geometry and Homotopy Theory" introducing us to logarithmic structures in 2009; the authors would like to thank the participants there, as well as Clark Barwick and John Rognes for organizing it.

This paper is written in dedication to Mark Mahowald.

2 Geometry

2.1 Logarithmic structures

Our primary reference for logarithmic structures is [Kat89], supplemented by [Rog09, Ogu]. For us, a prelog scheme is a scheme X equipped with an étale sheaf of commutative monoids M_X, together with a map $\alpha\colon M_X \to \mathcal{O}_X$ sending the multiplication in M_X to the multiplication in $\mathcal{O}_X$. This is a log scheme if the map α restricts to an isomorphism of sheaves $\alpha^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$, and there is a "logification" functor that is left adjoint to the forgetful functor from prelog schemes to log schemes. We will denote such a log scheme by (X, M_X), with α implicit.

If π is a global section of $\mathcal{O}_X$, there is an induced map from the free commutative monoid on a generator $\tilde{\pi}$ to $\mathcal{O}_X(X)$. This generates a prelog structure on X via the map from the constant sheaf $\mathbb{N} \to \mathcal{O}_X$, and we write $(X, \langle\pi\rangle)$ for the associated log scheme.

In this paper, we will only be working with fine and saturated log schemes. We write **LogSch** for the category of such objects, with associated forgetful functor **LogSch** $\to$ **Sch** to the category of schemes. This functor has a left adjoint **Sch** $\to$ **LogSch**, which takes a scheme X and gives it the trivial log structure $(X, \mathcal{O}_X^\times)$. We will implicitly view **Sch** as a full subcategory of **LogSch** via this map. Similarly, we have a full subcategory **AffLog** of affine log schemes, consisting of objects whose underlying scheme is affine.

Following Niziol [Niz08, §2.1.2], there are certain maps in the category **LogSch** which are referred to as Kummer log-étale maps, and Niziol shows that these give rise to a *Kummer log-étale topology.* Specifically, it is a Grothendieck topology generated by covers which are collections $\{(U_\beta, M_{U_\beta}) \to (X, M_X)\}$ of Kummer log-étale maps such that the underlying maps of schemes $\{U_\beta \to X\}$ are jointly surjective. Niziol shows that representable functors are sheaves for this topology, and that quasicoherent sheaves on X automatically extend to sheaves on the Kummer log-étale site of X.

We first note the following property about log-étale morphisms. (For the notion of a log-étale morphism we refer to [Kat89, 3.3].)

Proposition 2.1. *Suppose $\mathbb{N}^r \to Y$ generates a log structure M_Y on Y, and let $\mathbb{N}^r \to (\frac{1}{n}\mathbb{N})^r$ be isomorphic to the multiplication-by-n map on $\mathbb{N}^r$. Then for any étale map of schemes*

$$U \to Y \times_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[(\tfrac{1}{n}\mathbb{N})^r, n^{-1}],$$

the monoid $(\frac{1}{n}\mathbb{N})^r$ generates a log structure M_U on U making the map $(U, M_U) \to (Y, M_Y)$ flat and Kummer log-étale.

Such covers are cofinal in log-étale covers of (Y, M_Y).

Proof. The given map is evidently flat over Y, and the property of being Kummer is essentially by definition [Niz08, 2.11]. Therefore, it suffices to prove the cofinality property.

Our proof is very similar to [Niz08, Proposition 2.15]. By [Kat89, Theorem 3.5], any log-étale map from a fine and saturated log scheme to (Y, M_Y) has an étale cover by certain log schemes (V, M_V) determined by the following data:

- a finitely generated commutative monoid P which is integral and saturated;
- a map of monoids $\mathbb{N}^r \to P$ such that, on group completion, the map $\mathbb{Z}^r \to P^{gp}$ is an injection with cokernel of order $n < \infty$; and
- an étale map of schemes $V \to Y \times_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[P, n^{-1}]$.

The monoid P generates the log structure M_V on V.

The hypotheses on P imply that there is an injective monoid map $P \to (\frac{1}{n}\mathbb{N})^r \times F$, where F is a finite abelian group of order dividing n, factoring the inclusion $\mathbb{N}^r \to (\frac{1}{n}\mathbb{N})^r \times F$.

We define

$$U = V \times_{\mathbb{Z}[P]} \mathbb{Z}[(\tfrac{1}{n}\mathbb{N})^r \times F].$$

which has a log structure M_U generated by $(\frac{1}{n}\mathbb{N})^r \times F$. This makes (U, M_U) a log-étale cover of (V, M_V) and hence log-étale over (Y, M_Y) since n is invertible on V.

Without loss of generality we can then replace V with U. Because F consists of units, the log structure on U is equivalent to that generated by $(\frac{1}{n}\mathbb{N})^r$. As $\mathbb{Z}[F, n^{-1}]$ is étale over $\mathbb{Z}$, this gives us the desired cover

$$U \to Y \times_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[(\tfrac{1}{n}\mathbb{N})^r \times F, n^{-1}] \to Y \times_{\mathbb{Z}[\mathbb{N}^r]} \mathbb{Z}[(\tfrac{1}{n}\mathbb{N})^r, n^{-1}]. \qquad \square$$

If D is a normal crossing divisor on X, then the trivial log structure on $X \setminus D$ has a direct image log structure on X. Specifically, there is an associated submonoid $M_D \subset \mathcal{O}_X$ consisting of functions invertible on $X \setminus D$, and this makes (X, M_D) into a log scheme. The normal crossing condition implies that X is fine and saturated, and that locally this log structure is generated by a map $\mathbb{N}^r \to \mathcal{O}_X$. The above proposition then shows that the log-étale covers of (X, M_D) have a cofinal family of flat, Kummer log-étale covers.

In the particular case $r = 1$, this shows that there is a cofinal collection of log-étale covers of $(Y, \langle\pi\rangle)$ consisting of étale covers of

$$(Y \times_{\mathbb{Z}[\pi]} \mathbb{Z}[\pi^{1/n}, n^{-1}], \langle\pi^{1/n}\rangle).$$

In Section 5 we will make use of the special case of a power series ring.

Corollary 2.2. *Any log-étale map $(\mathbb{Z}[\![q]\!], \langle q\rangle) \to (R, M)$, where R is a formal $\mathbb{Z}[\![q]\!]$-algebra, has an étale cover by maps*

$$(\mathbb{Z}[\![q]\!], \langle q\rangle) \to (\mathbb{Z}[\![q^{1/m}]\!], \langle q^{1/m}\rangle) \to (S, \langle q^{1/m}\rangle),$$

where S is an étale formal $\mathbb{Z}[\![q^{1/m}]\!]$ algebra with m invertible.

Remark 2.3. We note that the integer m is recovered as a ramification index of the underlying map of schemes, and that the étale extension S of $\mathbb{Z}[\![q^{1/m}]\!]$ is uniquely of the form $\bar{S}[\![q^{1/m}]\!]$, where $\bar{S}$ is the étale extension $S/(q^{1/m})$ of $\mathbb{Z}[1/m]$. In other words, such an extension is determined up to isomorphism by the ramification index and the residue extension, but isomorphisms may differ by a map $q^{1/m} \mapsto \zeta q^{1/m}$.

We can also obtain control on cohomology in this category.

Proposition 2.4 ([Niz08, 3.27])**.** *Suppose (X, M_X) is an affine log scheme with a sheaf $\mathcal{F}$ on its Kummer log-étale site which is locally quasicoherent. Then the cohomology of (X, M_X) with coefficients in $\mathcal{F}$ vanishes in positive degrees.*

From a standard hypercover argument, we then obtain the following.

Corollary 2.5. *Suppose (X, M_X) is a log scheme (or Deligne-Mumford stack with a log structure) with a quasicoherent sheaf $\mathcal{F}$ on X. Then the natural map from Zariski cohomology of X to the Kummer log-étale cohomology of (X, M_X) with coefficients in $\mathcal{F}$ is an isomorphism.*

2.2 The moduli of elliptic curves

Recall that there is a stack $\overline{\mathcal{M}}_{ell}$ in the fppf topology on **Sch** such that maps $X \to \overline{\mathcal{M}}_{ell}$ classify generalized elliptic curves on X. In particular, $\overline{\mathcal{M}}_{ell}$ is a category whose the objects are pairs $(X, \mathcal{E})$ consisting of a scheme X and a generalized elliptic curve $\mathcal{E} \to X$. The morphisms in $\overline{\mathcal{M}}_{ell}$ are pullback diagrams

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ X' \arrow[r] & X \end{tikzcd}$$

respecting the chosen section of the elliptic curve. The functor which forgets $\mathcal{E}$ makes $\overline{\mathcal{M}}_{ell}$ into a category fibered in groupoids over **Sch**.

In particular, there is a ring

$$A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \tag{2.1}$$

parametrizing Weierstrass curves of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

This ring A contains modular quantities c_4, c_6, and Δ [Sil09, III.1], and the complement of the common vanishing locus of c_4 and Δ is an open subscheme

$$\mathrm{Spec}(A)^o = \mathrm{Spec}(A) - \mathrm{Spec}(A/(c_4, \Delta)) \tag{2.2}$$

parametrizing generalized elliptic curves in Weierstrass form. In particular, there is a universal generalized elliptic curve $\mathcal{E} \to \mathrm{Spec}(A)^o$. This determines a map from $\mathrm{Spec}(A)^o$ to the Deligne-Mumford stack $\overline{\mathcal{M}}_{ell}$ which is a smooth cover.

The j-invariant is a map $\overline{\mathcal{M}}_{ell} \to \mathbb{P}^1$, and this allows us to equip $\overline{\mathcal{M}}_{ell}$ with a log structure.

Definition 2.6. The log scheme $\mathbb{P}^1_{\mathrm{log}}$ is the log structure on $\mathbb{P}^1$ defined by the divisor (∞) in $\mathbb{P}^1$.

In particular, the monoid sheaf $M \subset \mathcal{O}_{\mathbb{P}^1}$ consists of functions whose restriction to $\mathbb{P}^1 \setminus \{\infty\} = \mathrm{Spec}(\mathbb{Z}[j])$ is invertible.

Remark 2.7. For any integer z, viewed as an integral point $z \in \mathbb{A}^1$, there is an isomorphism of log schemes

$$\mathbb{P}^1_{log} \setminus \{z\} \cong (\mathrm{Spec}(\mathbb{Z}[(j-z)^{-1}]), \langle (j-z)^{-1} \rangle).$$

Definition 2.8. The category $\overline{\mathcal{M}}_{log}$ is the category fibered in groupoids over **LogSch** defined as a pullback in the diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{log} \arrow[r] \arrow[d, "j_{log}"'] & \overline{\mathcal{M}}_{ell} \arrow[d, "j"] \\ \mathbb{P}^1_{log} \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

In particular, a log scheme over $\overline{\mathcal{M}}_{log}$ consists of a log scheme (X, M_X), a generalized elliptic curve $\mathcal{E}$ on X, and a lift of the j-invariant $j(\mathcal{E})\colon X \to \mathbb{P}^1$ to a map of log schemes $j_{log}(\mathcal{E})\colon (X, M_X) \to \mathbb{P}^1_{log}$.

Similarly, there is a universal scheme parametrizing pairs of isomorphic Weierstrass curves. Let

$$\Gamma = A[r, s, t, \lambda^{\pm 1}] \tag{2.3}$$

be the ring parametrizing the change-of-coordinates $y \mapsto \lambda^3 y + rx + s$, $x \mapsto \lambda^2 x + t$ on Weierstrass curves. The pair (A, Γ) form a Hopf algebroid with an invariant ideal (c_4, Δ). Defining

$$\mathrm{Spec}(\Gamma)^o = \mathrm{Spec}(\Gamma) - \mathrm{Spec}(\Gamma/(c_4, \Delta)), \tag{2.4}$$

we obtain a groupoid $(\mathrm{Spec}(A)^o, \mathrm{Spec}(\Gamma)^o)$ in schemes that maps naturally to $\overline{\mathcal{M}}_{ell}$.

As $\mathrm{Spec}(\Gamma)^o$ parametrizes pairs of isomorphic elliptic curves in Weierstrass form, we have a natural isomorphism

$$\mathrm{Spec}(\Gamma)^o \to \mathrm{Spec}(A)^o \times_{\overline{\mathcal{M}}_{ell}} \mathrm{Spec}(A)^o.$$

For Weierstrass curves, we note that the identity $j^{-1} = \Delta c_4^{-3}$ and the constraint that the units of M_X map isomorphically to $\mathcal{O}_X^\times$ imply that a lift of the j-invariant is equivalent to a lift of the element $\Delta \in \mathcal{O}_X(c_4^{-1}X)$ to a section of $M_X(c_4^{-1}X)$. We will casually refer to this as a lift of the elliptic discriminant to M_X.

The divisor defined by the vanishing of Δ determines a log structure on $\mathrm{Spec}(A)^o$. The fiber product $\mathrm{Spec}(A)^o \times_{\mathbb{P}^1} \mathbb{P}^1_{log}$ is, in fact, the log scheme

$$U_{log} = (\mathrm{Spec}(A)^o, \langle\Delta\rangle).$$

This classifies the universal Weierstrass curve in log schemes.

The fiber product of $(\mathrm{Spec}(A)^o, \langle\Delta\rangle)$ with itself over $\overline{\mathcal{M}}_{log}$ is then, by invariance of Δ up to unit,

$$R_{log} = (\mathrm{Spec}(\Gamma)^o, \langle\Delta\rangle).$$

The pair (U_{log}, R_{log}) form a smooth groupoid object in **LogSch** that parametrizes the groupoid of Weierstrass curves with lifts of the log structure.

Proposition 2.9. *The natural map $(U_{log}, R_{log}) \to \overline{\mathcal{M}}_{log}$ of groupoids induces an equivalence of stacks in the Kummer log-étale topology.*

Proof. The induced map of groupoids is fully faithful, and any elliptic curve is locally isomorphic to a Weierstrass curve. □

Proposition 2.10. *The map $U_{log} \to \overline{\mathcal{M}}_{log}$ is representable and a smooth cover.*

Proof. Given $(X, M_X) \to \overline{\mathcal{M}}_{log} = \overline{\mathcal{M}}_{ell} \times_{\mathbb{P}^1} \mathbb{P}^1_{log}$, the pullback is

$$(X, M_X) \times_{\overline{\mathcal{M}}_{ell} \times_{\mathbb{P}^1} \mathbb{P}^1_{log}} (\mathrm{Spec}(A)^o \times_{\mathbb{P}^1} \mathbb{P}^1_{log}) \cong (X, M_X) \times_{\overline{\mathcal{M}}_{ell}} \mathrm{Spec}(A)^o.$$

In particular, this follows from the fact that the map $\mathrm{Spec}(A)^o \to \overline{\mathcal{M}}_{ell}$ is representable and smooth. □

As a result, within the Grothendieck topology on **LogSch**, (U_{log}, R_{log}) gives a presentation of the same stack as $\overline{\mathcal{M}}_{log}$.

Remark 2.11. This smooth cover can be refined to a Kummer log-étale cover. Away from the cusps, for example, there are *schemes* $\mathcal{M}(4)[1/2]$ and $\mathcal{M}_1(3)[1/3]$, parametrizing generalized elliptic curves with full level 4 structures away from the prime 2 and 3-torsion points away from the prime 3 [KM85]. To cover the cusps, the curve

$$y^2 + xy = x^3 - \tfrac{36}{j-1728}x - \tfrac{1}{j-1728}$$

[Sil09, III.1.4.c] describes an étale map from $\mathrm{Spec}(\mathbb{Z}[j^{-1}, (1728j^{-1} - 1)^{-1}]$ to $\overline{\mathcal{M}}_{log}$ covering the complement of the j-invariants 0 and 1728; the induced log structure is $\langle j^{-1} \rangle$ makes this map log-étale.

We remark that $\overline{\mathcal{M}}_{log}$ is merely a prestack, rather than a stack, because elliptic curves do not obviously satisfy descent for log-étale covers.

2.3 Log-étale objects over $\overline{\mathcal{M}}_{log}$

Proposition 2.10 allows us to define *log-étale* objects over $\overline{\mathcal{M}}_{log}$. It suffices to check on the cover defined by the moduli of Weierstrass curves (2.2).

For any log scheme (X, M_X) over $\overline{\mathcal{M}}_{log}$ classifying an elliptic curve $\mathcal{E}/X$, we can form the pullback

$$\begin{tikzcd}
(X, M_X) \times_{\overline{\mathcal{M}}_{log}} U_{log} \arrow[r, "p"] \arrow[d] & U_{log} \arrow[d] \\
(X, M_X) \arrow[r] & \overline{\mathcal{M}}_{log}.
\end{tikzcd}$$

This fiber product is universal among log schemes (Y, M_Y) equipped with a map $f\colon (Y, M_Y) \to (X, M_X)$ and an isomorphism of $f^*(\mathcal{E})$ with a Weierstrass curve. The object (X, M_X) is (Kummer) log-étale over $\overline{\mathcal{M}}_{log}$ if and only if the map p of log schemes is (Kummer) log-étale.

In particular, if the elliptic curve on X is smooth, then Δ is invertible and the logarithmic structure on X must be trivial.

Definition 2.12. The *small log-étale site* of $\overline{\mathcal{M}}_{log}$ is the category of log schemes (X, M_X) equipped with a Kummer log-étale map $(X, M_X) \to \overline{\mathcal{M}}_{log}$, with maps being the Kummer log-étale maps.

The affine examples of log schemes are *log rings*, determined by a ring R and an appropriate étale sheaf of monoids M on $\mathrm{Spec}(R)$. We now give some details about the data needed on a Weierstrass curve over $\mathrm{Spec}(R)$, classified by a map $f\colon A \to R$, to get a map $\mathrm{Spec}(R, M) \to \overline{\mathcal{M}}_{log}$, and when this map is log-étale.

The map f determines a generalized elliptic curve when the ideal $(f(c_4), f(\Delta))$ is the unit ideal of R. As in the previous section, a lift of this to a map of log schemes is a lift of Δ/c_4^3 to a section of M over $\mathrm{Spec}(c_4^{-1}R)$; the fact that M is a logarithmic structure implies that this is equivalent to a lift of Δ to a section $\tilde{\Delta}$ of M over $\mathrm{Spec}(R)$.

We then have a composite of pullback diagrams

$$\begin{tikzcd} \mathrm{Spec}(R \otimes_A \Gamma, (\eta_L)^* M) \arrow[r] \arrow[d] & R_{log} \arrow[r] \arrow[d] & U_{log} \arrow[d] \\ \mathrm{Spec}(R, M) \arrow[r] & U_{log} \arrow[r] & \overline{\mathcal{M}}_{log}. \end{tikzcd}$$

We find that the lower map is log-étale if and only if the resulting map

$$(A, \langle \Delta \rangle) \to (R \otimes_A \Gamma, (\eta_L)^* M),$$

induced by the right unit of Γ, is log-étale. Here $(\eta_L)^* M$ is the pullback logarithmic structure.

In particular, an object (X, M_X) can be log-étale over $\overline{\mathcal{M}}_{log}$ only if it is locally Noetherian. In this case, the Artin-Rees lemma gives us the following.

Proposition 2.13. *For a log ring* $\mathrm{Spec}(R, M)$ *log-étale over* $\overline{\mathcal{M}}_{log}$*, the diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R^\wedge_\Delta \arrow[d] \\ \Delta^{-1} R \arrow[r] & \Delta^{-1} R^\wedge_\Delta \end{tikzcd}$$

is both cartesian and cocartesian.

This tells us that an object of the log-étale site is completely determined by an object $\Delta^{-1} X \to \mathcal{M}_{ell}$ étale over the moduli of smooth elliptic curves, an object $(X^\wedge_\Delta, M_X)$ over the completion at the cusp, and a patching map $\Delta^{-1} X^\wedge_\Delta \to \Delta^{-1} X$ over $\mathcal{M}_{ell}$.

2.4 The Tate curve

For references for the following material, we refer the reader to [DR73, VII] or [And00, §2.3].

The Tate curve T is a generalized elliptic curve over $\mathbb{Z}[\![q]\!]$ defined by the formula

$$y^2 + xy = x^3 + a_4(q)x + a_6(q),$$

where

$$a_4(q) = -5 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}, \qquad a_6(q) = -\frac{1}{12} \sum_{n=1}^{\infty} \frac{(5n^3 + 7n^5)q^n}{1-q^n}.$$

The j-invariant of this curve is

$$j(q) = q^{-1} + 744 + 196884q + \dots \tag{2.5}$$

At $q = 0$, the Tate curve is a curve of genus one with a nodal singularity, whose smooth locus is the group scheme $\mathbb{G}_m$.

The Tate curve possesses an isomorphism of formal groups $\widehat{T} \cong \widehat{\mathbb{G}}_m$ over $\mathbb{Z}[\![q]\!]$, and a canonical nowhere-vanishing invariant differential. In addition, there is a compatible sequence of diagrams of group schemes

$$\begin{tikzcd} \mu_n \arrow[r] \arrow[d] & T[n] \arrow[d] \\ \widehat{\mathbb{G}}_m \arrow[r] & T. \end{tikzcd}$$

For any $n \in \mathbb{N}$, define $\psi^n(q) = q^n$. This map has a lift $\psi^n_T \colon T \to T$ making the following diagram commute:

$$\begin{tikzcd} T \arrow[r, "\psi^n_T"] \arrow[d] & T \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}[\![q]\!]) \arrow[r, "\psi^n"'] & \mathrm{Spec}(\mathbb{Z}[\![q]\!]) \end{tikzcd}$$

The resulting map $T \to (\psi^n)^* T$ is an isogeny with μ_n mapping isomorphically to the kernel.

2.5 The Tate moduli

The Tate curve is classified by a map $\mathrm{Spec}(\mathbb{Z}[\![q]\!]) \to \overline{\mathcal{M}}_{ell}$. It also has a $\mathbb{Z}/2$-action by negation; both j and $(j - 1728)$ have inverses in $\mathbb{Z}[\![q]\!]$, and so [KM85, 8.4.3] implies that there are no automorphisms of curves parametrized by the Tate curve other than the identity and negation. As Equation (2.5) allows us to express j^{-1} as an invertible function of q, the coefficient q is uniquely determined for any such specialization. This implies that we have an identification of fiber products

$$\mathrm{Spf}(\mathbb{Z}[\![q]\!]) \times_{\overline{\mathcal{M}}_{ell}} \mathrm{Spf}(\mathbb{Z}[\![q]\!]) \cong \mathrm{Spf}(\mathbb{Z}[\![q]\!]) \times \mathbb{Z}/2.$$

In particular, the substack of $\overline{\mathcal{M}}_{ell}$ parametrizing generalized elliptic curves locally isomorphic to the Tate curve is the quotient stack

$$\mathcal{M}_{Tate} = \mathrm{Spf}(\mathbb{Z}[\![q]\!]) \times B\mathbb{Z}/2.$$

The following shows that this identification is compatible with the logarithmic structure.

Proposition 2.14. *Log maps to $\mathcal{M}_{Tate}$ are equivalent to log maps to $(\mathbb{Z}[\![q]\!], \langle q \rangle)$, together with a choice of principal $\mathbb{Z}/2$-torsor.*

Proof. Our identification of $\mathcal{M}_{Tate}$ has already shown that maps $X \to \mathcal{M}_{Tate}$ are equivalent to maps $j^{-1}\colon X \to \mathrm{Spec}(\mathbb{Z}[\![q]\!])$, together with a principal $\mathbb{Z}/2$-torsor $Y = X \times_{\mathcal{M}_{Tate}} \mathrm{Spec}(\mathbb{Z}[\![q]\!])$.

As the j-invariant of the Tate curve has the expression from Equation (2.5), the logarithmic structure $\langle j^{-1} \rangle$ on $\mathrm{Spf}(\mathbb{Z}[\![q]\!])$ is equivalent to the logarithmic structure $\langle q \rangle$. $\square$

Therefore, if (X, M_X) is a log scheme over $\mathcal{M}_{Tate}$, an extension to a log map is equivalent to a choice of lift of q to an element $\tilde{q} \in M_X(X)$.

Let $\mathcal{M}_{\mathbb{G}_m} = \mathrm{Spec}(\mathbb{Z}) \times B\mathbb{Z}/2$ be the substack of $\mathcal{M}_{Tate}$ where $q = 0$.

Definition 2.15. For a log scheme (X, M_X) over $\overline{\mathcal{M}}_{log}$ carrying the elliptic curve $\mathcal{E}$, the cusp subscheme $X^c \to \mathcal{M}_{\mathbb{G}_m}$ is the closed subscheme of X whose ideal of definition is generated by the image of M_X.

The associated form of the multiplicative group scheme is the smooth locus of the restriction $\mathcal{E}|_{X^c}$, classified by the resulting map $X^c \to \mathcal{M}_{\mathbb{G}_m}$.

We note that the cusp subscheme is functorial, and there is a natural diagram

$$\begin{tikzcd} X^c \arrow[r] \arrow[d] & X \arrow[d] \\ \mathcal{M}_{\mathbb{G}_m} \arrow[r] & \overline{\mathcal{M}}_{log}. \end{tikzcd}$$

The definition of log-étale implies the following.

Proposition 2.16. *If (X, M_X) is log-étale over $\overline{\mathcal{M}}_{log}$, then X^c is étale over over $\mathcal{M}_{\mathbb{G}_m}$.*

2.6 Modular curves

In this section we will describe why the modular curves, for various forms of level structure, give a natural tower of log-étale maps to $\overline{\mathcal{M}}_{log}$.

Definition 2.17. Let $G = \mathrm{GL}_2(\widehat{\mathbb{Z}})$. We define a category O_G of *orbits* whose objects are open subgroups $H < G$, with

$$\mathrm{Hom}_{O_G}(H, H') = \mathrm{Hom}_G(G/H, G/H').$$

Similarly, we define a category $\mathcal{L}$ of *levels* whose objects are pairs (N, Γ) of a positive integer N and a subgroup $\Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$, with

$$\mathrm{Hom}_{\mathcal{L}}((N,\Gamma),(N',\Gamma')) = \begin{cases} \mathrm{Hom}_{O_G}(p_N^{-1}\Gamma, p_{N'}^{-1}\Gamma') & \text{if } N'|N \\ \emptyset & \text{otherwise,} \end{cases}$$

where p_N is the surjection $G \twoheadrightarrow \mathrm{GL}_2(\mathbb{Z}/N)$.

In particular, morphisms in $\mathcal{L}$ are generated by morphisms of the following three types:

1. inclusions $\Gamma < \Gamma'$ for subgroups of $\mathrm{GL}_2(\mathbb{Z}/N)$;
2. conjugation maps $\Gamma \to g\Gamma g^{-1}$ for elements $g \in \mathrm{GL}_2(\mathbb{Z}/N)$; and
3. changes-of-level $(NM, p^{-1}\Gamma) \to (N, \Gamma)$, where $p\colon\ \mathrm{GL}_2(\mathbb{Z}/NM) \to \mathrm{GL}_2(\mathbb{Z}/N)$ is the projection.

The morphisms in O_G correspond only to the first two types. There is a forgetful map $\mathcal{L} \to O_G$.

Proposition 2.18. *For any pair (N, Γ) in $\mathcal{L}$, there is a Deligne-Mumford stack denoted $\overline{\mathcal{M}}(\Gamma)$ with a logarithmic structure over $\overline{\mathcal{M}}_{log}$, parametrizing elliptic curves with level Γ structure away from primes dividing N. Moreover, this is functorial in the sense that there is a (lax) functor from $\mathcal{L}$ to the 2-category of log Deligne-Mumford stacks.*

Proof. It is that there is a functorial family of Deligne-Mumford stacks $\overline{\mathcal{M}}(\Gamma)$ in **Sch** parametrizing elliptic curves with level Γ [DR73]. The submonoid of $\mathcal{O}_X$ of functions invertible away from the cusps defines a logarithmic structure on $\overline{\mathcal{M}}(\Gamma)$, natural in (N, Γ). (On any étale open $U \to \overline{\mathcal{M}}(\Gamma)$, this logarithmic structure is defined by the divisor of cusps, which is a normal crossing divisor.) □

Proposition 2.19. *For a fixed N, any map $(N, \Gamma) \to (N, \Gamma')$ induces a log-étale cover $\overline{\mathcal{M}}(\Gamma') \to \overline{\mathcal{M}}(\Gamma)$. If $K \lhd \Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$, then the Čech nerve of the associated cover represents, in* **LogSch**, *the simplicial bar construction for the action of Γ/K on $\overline{\mathcal{M}}(\Gamma)$.*

Proof. Consider a map in étale coordinate charts on these two stacks. Since we are working away from the primes dividing N, the map in coordinates is finite flat and tamely ramified at the cusps [DR73]. The source is smooth, and so the log purity theorem [Moc99, Theorem B] implies that the induced map is log-étale.

We now consider the case where the map comes from the inclusion of a normal subgroup. Away from the cusps, this map is an étale cover, and so it suffices to work over the étale coordinate chart W of $\overline{\mathcal{M}}_{ell}$ where $j \neq 0, 1728$ described in Remark 2.11. The fiber product $W \times_{\overline{\mathcal{M}}_{ell}} (-)$ with the map $\overline{\mathcal{M}}(K) \to \overline{\mathcal{M}}(\Gamma)$

is a log-étale map $U \to V$ whose underlying map of smooth schemes is tamely ramified over the cusps.

Consider the shearing map

$$\Gamma/K \times \overline{\mathcal{M}}(K) \to \overline{\mathcal{M}}(K) \times_{\overline{\mathcal{M}}(\Gamma)} \overline{\mathcal{M}}(K),$$

which in these coordinates is the map $\Gamma/K \times U \to U \times_V U$. Away from the cusps, the map $U \to V$ is Galois with Galois group Γ/K, and so this map is an isomorphism away from the cusps. In particular, this forces this map to be the normalization of the fiber product, which coincides with the fiber product in the category of fine and saturated log schemes. By induction, we find that the Čech nerve in fine and saturated log schemes, in these coordinates, is the bar construction $\{U \times (\Gamma/K)^p\}$. $\square$

As in the previous section, the logarithmic structure determines a natural cusp substack $\overline{\mathcal{M}}(\Gamma)^c$ which is étale over $\mathcal{M}_{\mathbb{G}_m}$.

2.7 Grothendieck sites

For convenience, we consider the following convenient category of elliptic curves. Recall that (U_{log}, R_{log}) forms a groupoid in schemes parametrizing generalized elliptic curves in Weierstrass form with compatible logarithmic structures.

Definition 2.20. The category $\overline{\mathcal{W}}_{log}$ of affine étale opens of $\overline{\mathcal{M}}_{log}$ is the intersection of the small log-étale site of $\overline{\mathcal{M}}_{log}$ with the category over **AffLog** represented by (U_{log}, R_{log}).

The objects of $\overline{\mathcal{W}}_{log}$ are fine and saturated log rings (R, M), equipped with a generalized elliptic curve $\mathcal{E}$ in Weierstrass form and a lift of the elliptic discriminant to $\tilde{\Delta} \in M(\mathrm{Spec}(R))$, such that the associated to $\overline{\mathcal{M}}_{log}$ is log-étale. (Maps do not preserve Weierstrass form.)

Equivalently, $\overline{\mathcal{W}}_{log}$ is the intersection of the small étale site of $\overline{\mathcal{M}}_{log}$ with essential image of the map $\mathbf{AffLog}/U_{log} \to \mathbf{LogSch}/\overline{\mathcal{M}}_{log}$.

The category $\overline{\mathcal{W}}_{log}$, while it is not closed under limits in the small étale site of $\overline{\mathcal{M}}_{log}$, still inherits a Grothendieck topology.

Proposition 2.21. *The Grothendieck site $\overline{\mathcal{W}}_{log}$ is equivalent to the small étale site of $\overline{\mathcal{M}}_{log}$.*

Proof. It suffices, by [Joh02, C.2.2.3] to show that any object Kummer log-étale over $\overline{\mathcal{M}}_{log}$ has an étale cover by objects isomorphic to those from $\overline{\mathcal{W}}_{log}$. However, this merely expresses the fact that a log scheme can be covered by affine charts, and that elliptic curves on affine schemes are locally isomorphic to elliptic curves in Weierstrass form. $\square$

As our goal is to construct a sheaf $\mathcal{O}^{der}$ on the small étale site of $\overline{\mathcal{M}}_{log}$, we can equivalently carry out a construction on $\overline{\mathcal{W}}_{log}$.

2.8 Sheafification

The following recapitulates [Beh, §2].

The small étale site of $\overline{\mathcal{M}}_{log}$—an equivalent Grothendieck site to $\overline{\mathcal{W}}_{log}$—has enough points, and so the category of presheaves of symmetric spectra has a Jardine model structure [Jar00]. In this model structure, fibrant objects satisfy homotopy descent with respect to hypercovers.

In the following, we fix a presheaf $\mathcal{F}$ of spectra on $\overline{\mathcal{W}}_{log}$, take a Jardine fibrant replacement, and extend it along a Quillen equivalence to a Jardine fibrant presheaf $\mathcal{F}'$ of spectra on the small étale site of $\overline{\mathcal{M}}_{log}$.

By [Beh, 2.4, 2.5] and Proposition 2.4, if the homotopy groups of $\mathcal{F}$ are quasicoherent sheaves, the values of $\mathcal{F}'$ on objects of $\overline{\mathcal{W}}_{log}$ are weakly equivalent to the values of $\mathcal{F}$. There is a cohomology spectral sequence

$$H^s_{\text{log-ét}}(\mathcal{X}, \underline{\pi}_t(\mathcal{F})) \Rightarrow \pi_{t-s}\Gamma(\mathcal{X}, \mathcal{F}').$$

Here $\underline{\pi}_t$ is the sheaf on $\overline{\mathcal{M}}_{log}$ associated to the presheaf $\pi_t\mathcal{F}$.

As a consequence of Proposition 2.4, the log-étale cohomology of quasicoherent sheaves coincides with ordinary cohomology of the underlying stack, and we have the following result.

Proposition 2.22. *Suppose that the presheaves $\pi_t(\mathcal{F})$ form a quasicoherent sheaf on $\overline{\mathcal{W}}_{log}$. Then for a Deligne-Mumford stack $\mathcal{X}$ log-étale over $\overline{\mathcal{M}}_{log}$, there is a spectral sequence*

$$H^s(\mathcal{X}, \underline{\pi}_t(\mathcal{F})) \Rightarrow \pi_{t-s}\Gamma(\mathcal{X}, \mathcal{F}').$$

In particular, for $\mathcal{X}$ affine the map $\mathcal{F}(\mathcal{X}) \to \mathcal{F}'(\mathcal{X})$ is a weak equivalence.

By applying Proposition 2.19, we can then get a homotopy fixed-point description of the values of $\mathcal{F}'$ on modular curves.

Proposition 2.23. *If $K \lhd \Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$, then the natural map*

$$\mathcal{F}'(\overline{\mathcal{M}}(\Gamma)) \to \mathcal{F}'(\overline{\mathcal{M}}(K))^{h\Gamma/K}$$

is an equivalence. In particular, there is a group cohomology spectral sequence

$$H^s(\Gamma/K, \pi_t\mathcal{F}'(\overline{\mathcal{M}}(K)) \Rightarrow \pi_{t-s}\mathcal{F}'(\overline{\mathcal{M}}(\Gamma)).$$

3 Elliptic cohomology and lifting orientations

We being by recalling the following (see [AHS01, Lur09, Beh]).

Definition 3.1. An elliptic spectrum consists of a homotopy-commutative ring spectrum E which is weakly even-periodic, an elliptic curve $\mathcal{E}$ over π_0E, and an isomorphism $\alpha\colon \mathbb{G}_E \to \widehat{\mathcal{E}}$ between the formal group of the complex orientable theory and the formal group of $\mathcal{E}$ over π_0E. We say that this elliptic spectrum realizes the elliptic curve $\mathcal{E}$.

A map of elliptic spectra is a multiplicative map $E \to E'$ together with a compatible isomorphism $\mathcal{E}' \to \mathcal{E} \otimes_{\pi_0 E} \pi_0 E'$ of elliptic curves which respects the identification of formal groups. We will say that a diagram of elliptic spectra realizes the corresponding diagram of elliptic curves.

Such an elliptic spectrum has a natural identification of $\pi_{2t}E$ with the tensor power $\omega^{\otimes t}$ of the cotangent sheaf of $\mathcal{E}$.

Consider the case where $\mathcal{E}$ is given the structure of a Weierstrass curve. It carries a coordinate $-x/y$ near the unit which trivializes the relative cotangent sheaf ω, and hence identifies $\pi_* E \cong \pi_0 E[u^{\pm 1}]$. (This also gives its formal group a standard lift to a formal group law classified by a map from the Lazard ring to $\pi_* E$, and E has a corresponding standard orientation $MU \to R$.)

We will find the following result convenient in showing that many objects defined by pullback naturally remain elliptic spectra (compare [LN12, Lemma 3.9]).

Lemma 3.2. *Suppose that we have a homotopy pullback diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \arrow[d] \\ S' \arrow[r] & T \end{tikzcd}$$

of homotopy commutative ring spectra which are weakly even-periodic and complex orientable. Suppose $\pi_0 R$ carries an elliptic curve $\mathcal{E}$, and the subdiagram $S \to T \leftarrow S'$ is given the structure of a diagram of elliptic spectra realizing $\mathcal{E}$. Then there is a unique way to give R the structure of an elliptic spectrum realizing $\mathcal{E}$ so that the square commutes.

Proof. The diagram of elliptic spectra gives us a commutative diagram of formal groups

$$\begin{tikzcd} \mathbb{G}_R \otimes \pi_* S \arrow[r] \arrow[d, "\sim"] & \mathbb{G}_R \otimes \pi_* T \arrow[d, "\sim"] & \mathbb{G}_R \otimes \pi_* S' \arrow[l] \arrow[d, "\sim"] \\ \widehat{\mathcal{E}} \otimes \pi_* S \arrow[r] & \widehat{\mathcal{E}} \otimes \pi_* T & \widehat{\mathcal{E}} \otimes \pi_* S'. \arrow[l] \end{tikzcd}$$

The hypotheses imply that, for any n, applying π_n to the homotopy pullback gives a bicartesian square. Therefore, taking pullbacks along rows gives us the formal groups $\mathbb{G}_R$ and $\widehat{\mathcal{E}}$, and so there is a unique isomorphism $\mathbb{G}_R \otimes \widehat{\mathcal{E}}$ compatible with the given isomorphisms. $\square$

Remark 3.3. In this paper, we will mostly be working with ordinary even-periodic ring spectra; this is mostly to simplify discussions related to localization and completion with respect to elements in the ring of modular forms.

4 Construction on the supersingular locus

Consider the category of pairs $(\mathrm{Spec}(k), \mathcal{E})$ with k a finite product of perfect fields of characteristic p and $\mathcal{E}$ a supersingular elliptic curve over k, and morphisms induced by ring maps and isomorphism of elliptic curves. The associated formal group law $\widehat{\mathcal{E}}$ is of height 2 over k, and isomorphisms of curves induce isomorphisms of formal group laws.

We recall the following consequence of the Goerss-Hopkins-Miller theorem.

Theorem 4.1. *There exists a Lubin-Tate functor E to the category of E_∞ ring spectra, sending $(k, \mathcal{E})$ to the Lubin-Tate spectrum $E(k, \mathcal{E})$ associated to the universal deformation of $\mathcal{E}$.*

Proposition 4.2. *There exists a functor $\mathcal{O}_{K(2)}$ from $\overline{\mathcal{W}}_{log}$ to the category of E_∞ tmf-algebras realizing $\mathcal{E}$, such that $\mathcal{E}/\mathrm{Spec}(R)$ is sent to the Lubin-Tate spectrum associated to $\widehat{\mathcal{E}}/\mathrm{Spec}(R^\wedge_{(p,v_1)})$.*

Proof. Given a map $\mathrm{Spec}(R, M) \to \overline{\mathcal{W}}_{log}$, the preimage of the supersingular locus is the closed subscheme $\mathrm{Spec}(R/(p, v_1)) \subset \mathrm{Spec}(\Delta^{-1}R)$. As $\mathrm{Spec}(\Delta^{-1}R)$ is étale over $\mathcal{M}_{ell}$, $\mathrm{Spec}(R/(p, v_1))$ decomposes as a disjoint union of Spec of finite fields equipped with supersingular elliptic curves, and Serre-Tate theory implies that the completion $R^\wedge_{(p,v_1)}$ at the supersingular locus then decomposes into a product of the associated Lubin-Tate rings.

The spectrum tmf is constructed so that the $K(2)$-localization is precisely the homotopy limit over supersingular elliptic curves of these Lubin-Tate spectra [Beh, § 4]. □

5 Construction at the cusps

5.1 The construction of $\mathcal{O}$ at the cusp

Proposition 2.2 will allow us to define $\mathcal{O}$ in a formal neighborhood of the cusp by restricting our attention to a subcategory of log schemes over $\mathcal{M}_{Tate}$ (§ 2.5).

Let $\mathcal{C}$ be the full subcategory of schemes log-étale over $\mathcal{M}_{Tate}$, whose underlying log schemes come from connected étale $\mathbb{Z}[\![q^{1/m}]\!]$-algebras S with m invertible and logarithmic structure $\langle q^{1/m}\rangle$. By Proposition 2.2, the category $\mathcal{C}$ inherits a Grothendieck topology equivalent to the category of schemes log-étale over $\mathcal{M}_{Tate}$; thus, as in § 2.8, to define the sheaves we are interested in it suffices to define them on $\mathcal{C}$. We will abuse notation by simply writing S for the log ring $(S, \langle q^{1/m}\rangle)$ equipped with a form of the Tate curve.

Definition 5.1. Let the functor $A^\times$, from $\mathcal{C}$ to abelian groups, send S to the set of elements $\zeta \in S$ such that $\zeta^k = 1$ for k an integer invertible in S. The functor A from $\mathcal{C}$ to commutative monoids sends S to the set of formal products $\zeta q^{k/m}$, where $\zeta \in A^\times(S)$ and $k \in \mathbb{N}$.

The functor $A^\times$ factors through reduction by the defining ideal, and therefore through the map $\mathcal{C} \to \mathcal{M}_{\mathbb{G}_m}$. We have the following lift.

Proposition 5.2. *There exists a functor $\overline{\mathcal{O}}_{Tate}$ from $\mathcal{C}$ to E_∞ ring spectra, together with a natural transformation of E_∞ ring spectra*

$$\Sigma^\infty_+ A^\times \to \overline{\mathcal{O}}_{Tate}.$$

The spectrum $\overline{\mathcal{O}}_{Tate}(S)$ is a form of K-theory associated to the reduction of this curve to $\bar{S} = S/(q^{1/m})$, and the map $\Sigma^\infty_+ A^\times(S) \to \overline{\mathcal{O}}_{Tate}(S)$ is, on π_0, a realization of the map $\mathbb{Z}[A^\times(S)] \to \bar{S}$.

Proof. We recall from [LN13, Appendix A] that there exists a sheaf $\mathcal{O}$ of E_∞ ring spectra (in fact, KO-algebras) on the étale site of $\mathcal{M}_{\mathbb{G}_m}$, classifying forms of the formal multiplicative group; specifically, there is a functor taking an étale $\mathbb{Z}/2$-algebra $\bar{S}$ with a form of the formal multiplicative group to an E_∞ form of K-theory. We therefore define $\overline{\mathcal{O}}_{Tate}$ as the composite of the functor $\mathcal{C} \to \mathcal{M}_{\mathbb{G}_m}$ with $\mathcal{O}$.

We now must define the natural transformation $A^\times \to \overline{\mathcal{O}}_{Tate}$, which we have already defined on π_0. The unit map $\mathbb{S} \to \overline{\mathcal{O}}_{Tate}$ factors naturally through $\mathbb{S}[1/m]$, and so it suffices to define a natural extension

$$\mathbb{S}[1/m] \to \Sigma^\infty_+ A^\times[1/m] \to \overline{\mathcal{O}}_{Tate}.$$

On homotopy groups, the map $\mathbb{S}[1/m] \to \Sigma^\infty_+ A^\times[1/m]$ is always an étale map because $A^\times$ has order invertible in $\mathbb{Z}[1/m]$. As this is an étale map, the obstruction theory for E_∞ maps then collapses, showing that the space of derived E_∞ maps $\Sigma^\infty_+ A^\times(S) \to \overline{\mathcal{O}}_{Tate}(S)$ is always homotopically discrete and equivalent to the space of maps $A^\times(S) \to (\pi_0 \overline{\mathcal{O}}_{Tate}(S))^\times = S^\times$. Using [DKS89], we can therefore replace these functors $\Sigma^\infty_+ A^\times$ and $\overline{\mathcal{O}}_{Tate}$ with equivalent functors possessing a natural transformation as desired. $\square$

Remark 5.3. We note that a map $(S, \langle q^{1/m}\rangle) \to (S', \langle q^{1/dm}\rangle)$ over $\mathcal{M}_{Tate}$ is equivalent to a map $\bar{S} \to \bar{S}'$ over $\mathcal{M}_{\mathbb{G}_m}$, together with a lift to a map $A(S) \to A(S')$ fixing q.

Theorem 5.4. *There exists a functor $\mathcal{O}_{Tate}$ from $\mathcal{C}$ to E_∞ ring spectra, together with natural homotopy pushout diagrams of E_∞ rings*

$$\begin{tikzcd}
\Sigma^\infty_+ A \arrow[r] \arrow[d] & \mathcal{O}_{Tate} \arrow[d] \\
\Sigma^\infty_+ A^\times \arrow[r] & \overline{\mathcal{O}}_{Tate}.
\end{tikzcd}$$

These satisfy the following properties:

- *$\mathcal{O}_{Tate}(S)$ realizes the form of the Tate curve on S;*
- *the map $A(S) \times \mathbb{N} \to \pi_0 \mathcal{O}_{Tate}(S) = S$ is the map inducing the logarithmic structure on S; and*
- *the map $\Sigma^\infty_+ A \to \Sigma^\infty_+ A^\times$ is induced by the projection $A_+ \to A^\times_+$ sending all multiples of $q^{1/m}$ to the basepoint.*

Proof. We define our functor as a limit of derived smash products:

$$\mathcal{O}_{Tate}(S) = \operatorname*{holim}_k \left[\overline{\mathcal{O}}_{Tate}(S) \underset{\Sigma^\infty_+ A^\times(S)}{\wedge} \Sigma^\infty_+ (A(S)/q^k) \right]$$

Then the zero'th homotopy groups of $\mathcal{O}_{Tate}(S)$ are functorially isomorphic to $S \cong \bar{S}[\![q^{1/m}]\!]$, and the formal group of $\mathcal{O}_{Tate}(S)$ carries an isomorphism to the formal group associated to the form of the multiplicative formal group over $\bar{S}$. We then compose with the isomorphism of this with the formal group of the form of the Tate curve over $\bar{S}$.

The cofiber sequence

$$\Sigma^\infty_+ A \overset{q^{1/m}}{\to} \Sigma^\infty_+ A \to \Sigma^\infty_+ A^\times$$

remains a cofiber sequence after completing with respect to the powers of q, and so the homotopy pushout diagram follows by associativity of the derived smash product. $\square$

By construction, the functor $\mathcal{O}$ naturally takes values in the category of algebras over

$$KO[\![q]\!] = \operatorname*{holim}_k KO \wedge \{1, q, q^2, \cdots, q^{k-1}\}_+.$$

By the results in Appendix A, we therefore have the following.

Corollary 5.5. *The functor $\mathcal{O}_{Tate}$ naturally takes values in the category of E_∞* tmf*-algebras.*

We will abuse notation by referring to the resulting extension

$$(R, M) \mapsto \mathcal{O}_{Tate}(R^\wedge_\Delta, M)$$

as a functor $\mathcal{O}_{Tate}$ from $\overline{\mathcal{W}}_{log}$ (Definition 2.20) to E_∞ tmf-algebras.

5.2 Rationalization at the cusp

We have the following formality of the rationalization. We recall from [Beh, §9] that the Eilenberg-Mac Lane functor H extends to a functor from graded-commutative $\mathbb{Q}$-algebras to E_∞ $H\mathbb{Q}$-algebras (corresponding to the underlying differential graded algebra with trivial differential). A functor to the category of commutative $H\mathbb{Q}$-algebras is *formal* if it is weakly equivalent to a functor factoring through H.

Theorem 5.6. *The rationalized functors $(\mathcal{O}_{Tate})_\mathbb{Q}$ and $(\prod_p (\mathcal{O}_{Tate})^\wedge_p)_\mathbb{Q}$ are formal, and the natural transformation $(\mathcal{O}_{Tate})_\mathbb{Q} \to (\prod_p (\mathcal{O}_{Tate})^\wedge_p)_\mathbb{Q}$ preserves this structure.*

Proof. The proof consists of constructing a free algebra out of π_0 and π_2 which explicitly maps to the original functor and rationalizes to a formal object.

We must recall the definition of $\overline{\mathcal{O}}_{Tate}$ from [LN13]. This functor takes as input an étale $\mathbb{Z}$-algebra $\bar{S}$ equipped with a principal C_2-torsor, classified by a C_2-Galois extension $T/\bar{S}$. Associated to T there is a homotopically unique E_∞ ring spectrum $\mathbb{S}(T)$ with an action of C_2. We have that the value of $\overline{\mathcal{O}}_{Tate}$ on this object is

$$\overline{\mathcal{O}}_{Tate} = (K \wedge \mathbb{S}(T))^{hC_2},$$

where the group acts on K by complex conjugation and on $\mathbb{S}(T)$ by its given action. We note that the homotopy fixed point spectral has no higher cohomology, and so rationalization commutes with homotopy fixed points.

Second, we note that there is a C_2-equivariant map $\Sigma^\infty \mathbb{CP}^1 \to K$ induced by $\mathbb{CP}^1 \to BU(1) \to BU$ which is an isomorphism on π_2, where $\mathbb{CP}^1$ has the complex conjugation action.

Let $\bar{R} = \mathbb{S}(\bar{S})$, and let $\bar{M} = (\mathbb{S}(T) \wedge \mathbb{CP}^1)^{hC_2}$. We note that the homotopy groups of R are finite except in degree 0, and similarly for $\bar{M}$ except in degree 2. There is a natural map of E_∞ ring spectra $\bar{R} \to \overline{\mathcal{O}}_{Tate}$ which is an isomorphism on π_0, and one of $\bar{R}$-modules $\bar{M} \to \overline{\mathcal{O}}_{Tate}$ which is an isomorphism on π_2.

Finally, we define

$$R = \operatorname*{holim}_k \left[\bar{R} \underset{\Sigma^\infty_+ A^\times(S)}{\wedge} \Sigma^\infty_+ (A(S)/q^k) \right]$$

and $M = R \wedge_{\bar{R}} \bar{M}$. Then $\pi_0 R \cong \pi_0 \bar{R}[\![q]\!]$ and $\pi_2 M \cong \pi_2 \bar{M}[\![q]\!]$. There is a natural transformation $\bar{R} \to \mathcal{O}_{Tate}$ which is an isomorphism on π_0, and by extension of scalars a map of R-modules $M \to \mathcal{O}_{Tate}$ which is an isomorphism on π_2. Both R and M have torsion homotopy groups except in degrees 0 and 2 respectively; after rationalization, they become Eilenberg-Mac Lane spectra.

The natural map $M \to \mathcal{O}_{Tate}$ of R-modules produces a map of free R-algebras $\mathbb{P}_R(M) \to \mathcal{O}_{Tate}$. After rationalization, the map $\mathbb{P}_R(M)_\mathbb{Q} \to (\mathcal{O}_{Tate})_\mathbb{Q}$ is an isomorphism on homotopy groups in nonnegative degrees. The Eilenberg-Mac Lane functor preserves free algebras, and so $\mathbb{P}_R(M)_\mathbb{Q} \simeq H\mathbb{P}_{\pi_0 \otimes \mathbb{Q}}(\pi_2 \otimes \mathbb{Q})$ is formal. Finally, $(\mathcal{O}_{Tate})_\mathbb{Q}$ takes values in even-periodic objects, and is obtained as a localization of $\mathbb{P}_R(M)_\mathbb{Q}$. The functor H preserves this localization, and so $(\mathcal{O}_{Tate})_\mathbb{Q}$ is formal.

The same argument, after applying the functor $\prod (-)^\wedge_p$, gives a map $\mathbb{P}_{R^\wedge}(M^\wedge) \to (\mathcal{O}_{Tate})^\wedge$ which is a rational isomorphism in nonnegative degrees, and shows $(\mathcal{O}_{Tate})^\wedge$ is formal. Moreover, upon rationalization the natural map $\mathbb{P}_R(M)_\mathbb{Q} \to \mathbb{P}_{R^\wedge}(M^\wedge)_\mathbb{Q}$ becomes one in the image of the Eilenberg-Mac Lane functor, and similarly for the localization. □

6 Patching in the $K(1)$-local smooth locus

In this section we describe the construction of cohomology theories associated to smooth elliptic curves.

A proof is given in [Beh, § 7] of the following application of Goerss-Hopkins obstruction theory.

Proposition 6.1. *Let $\operatorname{Spf}(R) \to (\overline{\mathcal{M}}_{ell})^\wedge_p$ be an étale formal affine open classifying an elliptic curve $\mathcal{E}/R$ with no supersingular fibers. Then there is a lift of this data to a $K(1)$-local E_∞ tmf-algebra $\mathcal{O}_{K(1)}(R)$ realizing $\mathcal{E}$.*

Given a second formal map $\operatorname{Spf}(R') \to (\overline{\mathcal{M}}_{ell})^\wedge_p$ classifying an elliptic curve $\mathcal{E}'$ with no supersingular fibers, equipped with an E_∞ tmf-algebra E, the space

$\mathrm{Map}_{E_\infty-\mathrm{tmf}\text{-}alg}(\mathcal{O}_{K(1)}R, E)$ *of* $K(1)$*-local* E_∞ tmf*-algebra maps is homotopically discrete and equivalent to the set* $\mathrm{Hom}_{\overline{\mathcal{M}}_{ell}}(\mathrm{Spf}(R'), \mathrm{Spf}(R))$ *of pullback diagrams*

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \mathrm{Spf}(R') \arrow[r] & \mathrm{Spf}(R). \end{tikzcd}$$

Proposition 4.2 and Theorem 5.4 have given constructions of functors $\mathcal{O}_{K(2)}$ and $\mathcal{O}_{Tate}$ from $\overline{\mathcal{W}}_{log}$ to elliptic spectra; these both take values in the category of E_∞ tmf-algebras (the latter by Corollary 5.5).

Proposition 6.2. *For objects* $\mathrm{Spec}(R, M)$ *of* $\overline{\mathcal{W}}_{log}$*, the natural maps*

$$(v_1\Delta^{-1}R)^\wedge_p \to (v_1^{-1}R^\wedge_{(p,v_1)})^\wedge_p \text{ and}$$
$$(v_1\Delta^{-1}R)^\wedge_p \to (\Delta^{-1}R^\wedge_\Delta)^\wedge_p$$

over $\overline{\mathcal{M}}_{ell}$ *lift to a diagram of maps of* $K(1)$*-local elliptic* E_∞ tmf*-algebras:*

$$\mathcal{O}_{K(1)}((v_1\Delta^{-1}R)^\wedge_p) \to L_{K(1)}\mathcal{O}_{K(2)}(R)$$
$$\mathcal{O}_{K(1)}((v_1\Delta^{-1}R)^\wedge_p) \to L_{K(1)}\Delta^{-1}\mathcal{O}_{Tate}(R, M)$$

Proof. The rings $(\Delta^{-1}R)^\wedge_p$ satisfies the criteria for Proposition 6.1, so the spaces of algebra maps are contractible. The results of [DKS89] then allow us to replace this with a genuinely commutative diagram. □

Definition 6.3. The functor $\mathcal{O}^\wedge_p$, from $\overline{\mathcal{W}}_{log}$ to p-complete elliptic spectra equipped with E_∞ tmf-algebra structure, sends $\mathrm{Spec}(R, M) \to \overline{\mathcal{M}}_{ell}$ to the homotopy limit of the solid portion in the following diagram:

$$\begin{tikzcd} \mathcal{O}^\wedge_p(R,M) \arrow[r, dotted] \arrow[d, dotted] & \mathcal{O}^{log}_{K(1)}(R,M) \arrow[r, dotted] \arrow[d, dotted] & L_{K(1)}\mathcal{O}_{Tate}(R,M) \arrow[d] \\ \mathcal{O}^{sm}_p(R) \arrow[r, dotted] \arrow[d, dotted] & \mathcal{O}_{K(1)}((v_1\Delta^{-1}R)^\wedge_p) \arrow[r] \arrow[d] & L_{K(1)}\Delta^{-1}\mathcal{O}_{Tate}(R,M) \\ \mathcal{O}_{K(2)}(R) \arrow[r] & L_{K(1)}\mathcal{O}_{K(2)}(R) & \end{tikzcd}$$

We have the following.

Proposition 6.4. *For any affine log scheme* $(R, M) \in \overline{\mathcal{W}}_{log}$ *carrying the generalized elliptic curve* $\mathcal{E}$*, the spectrum* $\mathcal{O}^\wedge_p(R)$ *is an elliptic cohomology theory associated to the elliptic curve* $\mathcal{E}$ *over* $R^\wedge_p$*.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
(\omega^{\otimes n})^\wedge_p \arrow[r, dotted] \arrow[d, dotted] & (v_1^{-1}(\omega^{\otimes n}))^\wedge_p \arrow[r, dotted] \arrow[d, dotted] & (\omega^{\otimes n})^\wedge_{(p,\Delta)} \arrow[d] \\
(\Delta^{-1}(\omega^{\otimes n}))^\wedge_p \arrow[r, dotted] \arrow[d, dotted] & ((v_1\Delta)^{-1}(\omega^{\otimes n}))^\wedge_p \arrow[r] \arrow[d] & (\Delta^{-1}(\omega^{\otimes n})^\wedge_\Delta)^\wedge_p \\
(\omega^{\otimes n})^\wedge_{(p,v_1)} \arrow[r] & (v_1^{-1}(\omega^{\otimes n})^\wedge_{v_1})^\wedge_p &
\end{tikzcd} \tag{6.1}$$

We claim that that it consists entirely of bicartesian squares, and that it is the result of applying π_{2n} to Definition 6.3.

The right and lower squares are bicartesian by the Artin-Rees lemma: for a Noetherian ring R with a finitely generated module M and an element x, the diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & M^\wedge_x \arrow[d] \\
x^{-1}M \arrow[r] & x^{-1}M^\wedge_x
\end{tikzcd}$$

is bicartesian. As the terms consist of torsion-free groups, this property is preserved by p-adic completion. Similarly, we have a square

$$\begin{tikzcd}
\omega^{\otimes n} \arrow[r] \arrow[d] & v_1^{-1}(\omega^{\otimes n}) \arrow[d] \\
\Delta^{-1}(\omega^{\otimes n}) \arrow[r] & (v_1\Delta)^{-1}(\omega^{\otimes n})
\end{tikzcd}$$

which becomes bicartesian on p-adic completion, as the supersingular locus does not intersect the cusps.

To show that this is the diagram of homotopy groups, it then remains to show that the solid portion of (6.1) is the result of applying π_{2n} to Definition 6.3. However, this is a consequence of the following isomorphisms:

$$\begin{aligned}
(v_1^{-1}(\Delta^{-1}(\omega^{\otimes n}))^\wedge_p)^\wedge_p &\cong ((v_1\Delta)^{-1}(\omega^{\otimes n}))^\wedge_p \\
((v_1\Delta)^{-1}(\omega^{\otimes n}))^\wedge_p &\cong (v_1^{-1}(\omega^{\otimes n})^\wedge_{(p,v_1)})^\wedge_p
\end{aligned}$$

□

7 Rational construction

In this section, we will use an arithmetic square to construct $\mathcal{O}_{\mathbb{Q}}$ and $\mathcal{O}$. We recall from §5.2 that $(\mathcal{O}_{Tate})_{\mathbb{Q}}$ is equivalent to a functor factoring through the Eilenberg-Mac Lane functor.

In [Beh, §9], a functorial E_∞ elliptic spectrum $\mathcal{O}_{\mathbb{Q}}$ is constructed on the étale site of $\overline{\mathcal{M}}_{ell}$, together with an arithmetic attaching map $\mathcal{O}_{\mathbb{Q}} \to (\prod_p \mathcal{O}^\wedge_p)_{\mathbb{Q}}$.

Moreover, this data is formal: these functors and the natural transformation between them are in the image of the Eilenberg-Mac Lane functor.

For an object $(R, M) \in \overline{\mathcal{W}}_{log}$, we have a well-defined value of $\mathcal{O}_{\mathbb{Q}}$ on the restriction $\Delta^{-1}R$ to the smooth locus.

Proposition 7.1. *There exists a natural transformation of E_∞ elliptic* tmf-*algebras*

$$\mathcal{O}_{\mathbb{Q}}(\Delta^{-1}R) \to \Delta^{-1}\mathcal{O}_{Tate}(R, M)_{\mathbb{Q}}$$

such that the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{Q}}(\Delta^{-1}R) \arrow[r] \arrow[d] & \Delta^{-1}\mathcal{O}_{Tate}(R, M)_{\mathbb{Q}} \arrow[d] \\ (\prod_p \mathcal{O}_p^{\wedge}(\Delta^{-1}R))_{\mathbb{Q}} \arrow[r] & \Delta^{-1}(\prod_p \mathcal{O}_{Tate}(R, M)_p^{\wedge})_{\mathbb{Q}} \end{tikzcd}$$

commutes.

Proof. Theorem 5.6 shows that the natural transformation on the right-hand side of the diagram is also in the image of the Eilenberg-Mac Lane functor. Therefore, to construct this commutative diagram it suffices to take the diagram of homotopy groups on the level of graded algebras and apply the Eilenberg-Mac Lane functor. This is automatically compatible with elliptic structure, as the diagram on the level of graded rings is. ☐

Definition 7.2. The functor $\mathcal{O}_{\mathbb{Q}}$, from $\overline{\mathcal{W}}_{log}$ to the category of rational E_∞ tmf-algebras, sends $\mathrm{Spec}(R, M)$ to the homotopy pullback in the diagram

$$\begin{tikzcd} \mathcal{O}_{\mathbb{Q}}(R, M) \arrow[r] \arrow[d] & \mathcal{O}_{Tate}(R, M)_{\mathbb{Q}} \arrow[d] \\ \mathcal{O}_{\mathbb{Q}}(\Delta^{-1}R) \arrow[r] & \Delta^{-1}\mathcal{O}_{Tate}(R, M)_{\mathbb{Q}}. \end{tikzcd}$$

Proposition 7.3. *For any affine log scheme $(R, M) \in \overline{\mathcal{W}}_{log}$ carrying the generalized elliptic curve $\mathcal{E}$, the spectrum $\mathcal{O}_{\mathbb{Q}}(R, M)$ is an elliptic cohomology theory associated to the elliptic curve $\mathcal{E}$ over $R_{\mathbb{Q}}$.*

There is a natural map

$$\mathcal{O}_{\mathbb{Q}}(R, M) \to \left(\prod_p \mathcal{O}_p^{\wedge}(R, M)\right)_{\mathbb{Q}}$$

of E_∞ elliptic tmf-*algebras.*

Proof. For any n, we have a bicartesian square

$$\begin{tikzcd} \omega_{\mathbb{Q}}^{\otimes n} \arrow[r] \arrow[d] & ((\omega^{\otimes n})^{\wedge}_{\Delta})_{\mathbb{Q}} \arrow[d] \\ \Delta^{-1}\omega_{\mathbb{Q}}^{\otimes n} \arrow[r] & \Delta^{-1}((\omega^{\otimes n})^{\wedge}_{\Delta})_{\mathbb{Q}} \end{tikzcd}$$

which is obtained by rationalizing the corresponding bicartesian square for the finitely generated module $\omega^{\otimes n}$. Therefore, the homotopy groups of $\mathcal{O}_{\mathbb{Q}}(R, M)$ are the modules $\omega_{\mathbb{Q}}^{\otimes n}$, and by Lemma 3.2 this is an elliptic spectrum realizing $\mathcal{E}$.

To construct the given map, we note that we now have a natural diagram of tmf-algebras:

$$\begin{array}{ccccc} \mathcal{O}_{Tate}(R,M)_{\mathbb{Q}} & \longrightarrow & \Delta^{-1}\mathcal{O}_{Tate}(R,M)_{\mathbb{Q}} & \longleftarrow & \mathcal{O}_{\mathbb{Q}}(\Delta^{-1}R) \\ \downarrow & & \downarrow & & \downarrow \\ (\prod_p \mathcal{O}_{Tate}(R,M)_p^{\wedge})_{\mathbb{Q}} & \longrightarrow & \Delta^{-1}(\prod_p \mathcal{O}_{Tate}(R,M)_p^{\wedge})_{\mathbb{Q}} & \longleftarrow & (\prod_p \mathcal{O}_p^{\wedge}(\Delta^{-1}R))_{\mathbb{Q}} \end{array}$$

Taking homotopy pullbacks in rows gives the desired natural map. $\square$

Definition 7.4. The functor $\mathcal{O}$, from $\overline{\mathcal{W}}_{log}$ to elliptic cohomology theories equipped with E_∞ tmf-algebra structures, sends $\mathrm{Spec}(R, M)$ to the homotopy pullback in the diagram

$$\begin{array}{ccc} \mathcal{O}(R,M) & \longrightarrow & \prod_p \mathcal{O}_p^{\wedge}(R,M) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathbb{Q}}(R,M) & \longrightarrow & (\prod_p \mathcal{O}_p^{\wedge}(R,M))_{\mathbb{Q}}. \end{array}$$

As a consequence of Lemma 3.2, we find that $\mathcal{O}(R, M)$ is a functorial elliptic spectrum realizing the elliptic curve $\mathcal{E}$ on R.

The reduced Tate functor of Proposition 5.2 now allows us to evaluate at the cusps.

Proposition 7.5. *There is a natural transformation of functors on $\overline{\mathcal{W}}_{log}$*

$$\mathcal{O}(R,M) \to \overline{\mathcal{O}}_{Tate}(R,M) = \mathcal{O}_{\mathcal{M}_{\mathbb{G}_m}}(R/(ImM)),$$

where the latter is the form of K-theory associated to the cusp subscheme of $\mathrm{Spec}(R, M)$.

As in Section 2.8, we may automatically extend this to a sheaf of spectra on the log-étale site of $\overline{\mathcal{M}}_{log}$.

Theorem 7.6. *There exists a realization of the universal elliptic curve on the small log-étale site of $\overline{\mathcal{M}}_{log}$ by a sheaf $\mathcal{O}$ of locally even-periodic E_∞ elliptic spectra.*

We note that the sheaf property automatically allows us to take sections on a Deligne-Mumford stack equipped with a logarithmic structure which is log-étale over $\overline{\mathcal{M}}_{log}$.

8 Tmf with level structure

Having constructed our derived structure sheaf $\mathcal{O}$, we can now evaluate it on the modular curves $\overline{\mathcal{M}}(\Gamma)$ from Section 2.6.

Theorem 8.1. *There exists a contravariant functor* Tmf *from the category* $\mathcal{L}$ *of Definition 2.17 to* E_∞ *ring spectra, taking a pair* (N, Γ) *to an object* $\mathrm{Tmf}(\Gamma)$*. When* $N = 1$ *(and hence* Γ *is trivial) this recovers the nonconnective spectrum* Tmf*.*

For any such Γ*, there is a spectral sequence*

$$H^s(\overline{\mathcal{M}}(\Gamma); \omega^{\otimes t/2}) \Rightarrow \pi_{t-s}\mathrm{Tmf}(\Gamma),$$

and for any $K \lhd \Gamma < \mathrm{GL}_2(\mathbb{Z}/N)$*, the natural map*

$$\mathrm{Tmf}(K) \to \mathrm{Tmf}(\Gamma)^{h\Gamma/K}$$

is an equivalence.

If $p\colon \mathrm{GL}_2(\mathbb{Z}/NM) \to \mathrm{GL}_2(\mathbb{Z}/N)$ *is the natural projection, the map* $\mathrm{Tmf}(\Gamma) \to \mathrm{Tmf}(p^{-1}\Gamma)$ *is a localization formed by inverting* M*.*

Proof. The functoriality of the modular curves $\overline{\mathcal{M}}(\Gamma)$ as log-étale objects over $\overline{\mathcal{M}}_{log}$ was established in Proposition 2.18. Therefore, we may apply $\mathcal{O}$ to obtain a functorial diagram of E_∞ ring spectra. By construction, the value on the terminal object is Tmf. The statement follows from about localizations is true due to the existence of vanishing lines in the cohomology spectral sequences for these stacks [MM].

The spectral sequence for the cohomology of $\mathrm{Tmf}(\Gamma)$ is Proposition 2.22, while the equivalence from $\overline{\mathcal{M}}(K)$ to the homotopy fixed-point object $\overline{\mathcal{M}}(\Gamma)^{h\Gamma/K}$ is Proposition 2.23. □

We can also evaluate at the cusps, using Proposition 7.5.

Theorem 8.2. *Let* $K(\Gamma)$ *be the natural form of* K*-theory associated to the cusp substack of* $\overline{\mathcal{M}}(\Gamma)$*. There is a natural transformation of* E_∞ *ring spectra*

$$\mathrm{Tmf}(\Gamma) \to K(\Gamma).$$

In particular, we can apply Theorem 8.1 to the specific cover $\overline{\mathcal{M}}_1(3) \to \overline{\mathcal{M}}_0(3)$, obtaining an integral lift of the work in [LN13].

Theorem 8.3. *There exists an* E_∞ *ring spectrum* $\mathrm{tmf}_0(3)$ *whose homotopy groups form the "positive" portion of the homotopy groups of* $\mathrm{TMF}_0(3)$ *described in [MR09, §7]. This fits into a commutative diagram of* E_∞ *ring spectra*

$$\begin{tikzcd}
\mathrm{tmf}_0(3) \arrow[r] \arrow[d] & ko[1/3] \arrow[d] \\
\mathrm{tmf}_1(3) \arrow[r] & ku[1/3].
\end{tikzcd}$$

Proof. The cohomology of the moduli stack $\overline{\mathcal{M}}_1(3)$ was essentially determined in [MR09]; see also [LN13]. The cohomology groups $H^0(\overline{\mathcal{M}}_1(3);\omega^{\otimes t})$ form the graded ring $\mathbb{Z}[1/3, a_1, a_3]$, which come from the universal cubic curve

$$y^2 + a_1 xy + a_3 y = x^3$$

carrying a triple intersection with the line $y = 0$. The other cohomology groups $H^s(\overline{\mathcal{M}}_1(3);\omega^{\otimes t})$ are concentrated in $s = 1$, $t \le -4$, and so the positive-degree homotopy groups of $\mathrm{Tmf}_1(3)$ form the graded ring $\mathbb{Z}[1/3, a_1, a_3]$. Moreover, the open subscheme $\mathcal{M}_1(3) \subset \overline{\mathcal{M}}_1(3)$ induces a map of E_∞ ring spectra $\mathrm{Tmf}_1(3) \to \mathrm{TMF}_1(3)$ which, on homotopy groups, inverts the elliptic discriminant $\Delta = a_3^3(a_1^3 - 27a_3)$.

The homotopy fixed-point spectral sequence for the homotopy groups of $\mathrm{Tmf}_0(3)$, in positive degrees, consists of the part of the computation carried out by Mahowald-Rezk which involves no negative powers of Δ. The portion of the spectral sequence with $s > t - s$ consists of spurious classes which are annihilated by differentials, and so the portion of the spectral sequence with $s \le t - s$ converges to the homotopy groups of the connective cover $\mathrm{tmf}_0(3)$, which is the "positive" portion of the computation described in [MR09, §7].

Applying to the natural diagram classifying the reduced cusp described in [LN13], we obtain maps of E_∞ ring spectra

$$\begin{tikzcd} \mathrm{Tmf}_0(3) \arrow[r] \arrow[d] & KO[1/3] \arrow[d] \\ \mathrm{Tmf}_1(3) \arrow[r] & KU[1/3]. \end{tikzcd}$$

Taking connective covers, we obtain a diagram of E_∞ ring spectra giving the desired connective lifts. ☐

A Appendix: The Witten genus

The goal of this section is to show that there is a map of E_∞ ring spectra

$$\mathrm{tmf} \to KO[\![q]\!]$$

which, on homotopy groups, represents a segment of the Witten genus $MSpin_* \to \mathbb{Z}[\![q]\!]$. Here the power series notation $KO[\![q]\!]$ is shorthand for the homotopy limit of the monoid algebras

$$\operatorname{holim} KO \wedge \{1, q, \cdots, q^n\}_+$$

where q^{n+1} is identified with the basepoint (as in §5).

The following result is well-known and featured prominently in earlier, unpublished, constructions of tmf, but to the knowledge of the authors it does not appear in the literature. The relation of the Tate curve to power operations has been extensively explored, especially in this context by Baker [Bak90], Ando [And00], Ando-Hopkins-Strickland [AHS01], and Ganter [Gan13].

We recall that the p-adic ring V is Katz's ring of generalized modular functions [Kat75], which is a universal object carrying an ordinary elliptic curve $\mathcal{E}$ together with an isomorphism $\widehat{\mathbb{G}}_m \to \widehat{\mathcal{E}}$.

Proposition A.1. *At the prime* 2*, there exists a map of $K(1)$-local E_∞ ring spectra*

$$L_{K(1)}\mathrm{tmf} \to L_{K(1)}KO[\![q]\!]$$

which, on 2*-adic K-homology $K_0^\vee(-)$, is the map*

$$V \to \left[\mathrm{Hom}_c(\mathbb{Z}_2^\times/\{\pm 1\}, K_*) \otimes_{K_*} K_*[\![q]\!]\right]_2^\wedge$$

classifying the Tate curve with its isomorphism $\widehat{\mathbb{G}}_m \to \widehat{T}$ and its action of ± 1.

Proof. We first recall from [Hop, Lau04] the construction of $L_{K(1)}\mathrm{tmf}$ at the prime 2 as a $K(1)$-local E_∞ ring spectrum. One first forms a spectrum T_ζ as an E_∞ pushout:

$$\begin{tikzcd} \mathbb{P}_{K(1)}S^{-1} \arrow[r, "0"] \arrow[d, "\zeta"] & \mathbb{S}_{K(1)} \arrow[d] \\ \mathbb{S}_{K(1)} \arrow[r] & T_\zeta \end{tikzcd}$$

There is a "modular function" $f \in \pi_0 T_\zeta$ and a 2-adically convergent power series $h(x)$ such that $L_{K(1)}\mathrm{tmf}$ is an E_∞ pushout:

$$\begin{tikzcd} \mathbb{P}_{K(1)}S^{0} \arrow[r, "0"] \arrow[d, "\theta(f)-h(f)"] & \mathbb{S}_{K(1)} \arrow[d] \\ T_\zeta \arrow[r] & \mathrm{tmf}_{K(1)} \end{tikzcd}$$

This is built to satisfy a universal identity of modular functions: for any $K(1)$-local, E_∞, even-periodic elliptic cohomology theory E, there is a resulting map $T_\zeta \to E$ which automatically sends $\theta(f)$ and $h(f)$ to the same element, and so that the map on K-theory is the map from V classifying the elliptic curve.

As $KO[\![q]\!]_2^\wedge$ is the $K(1)$-localization of $KO[\![q]\!]$ and has trivial π_{-1}, we have a map of E_∞ ring spectra $T_\zeta \to KO[\![q]\!]_2^\wedge$. The composite map $T_\zeta \to K[\![q]\!]_2^\wedge$ detects the effect on π_0, and is a map to an elliptic cohomology theory, where the latter carries the Tate curve over the power series ring $\mathbb{Z}[\![q]\!]_2^\wedge$. Therefore, the element $\theta(f) - h(f)$ automatically maps to zero, and we obtain an extension $\mathrm{tmf}_{K(1)} \to L_{K(1)}KO[\![q]\!]$. $\square$

Proposition A.2. *For any odd prime p, there exists a map of $K(1)$-local E_∞ ring spectra*

$$L_{K(1)}\mathrm{tmf} \to L_{K(1)}KO[\![q]\!]$$

which, on p-adic K-theory $K_0^\vee$, induces the map of θ-algebras

$$V \to \mathrm{Hom}_c(\mathbb{Z}_p^\times/\{\pm 1\}, K_*)\widehat{\otimes}_{K_*} K_*[\![q]\!]$$

classifying the Tate curve with its isomorphism $\widehat{\mathbb{G}}_m \to \widehat{T}$ and its action of ± 1.

Proof. As $KO[\![q]\!]$ is the homotopy fixed-point spectrum of the action of $\{\pm 1\}$ on $K[\![q]\!]$, we have an adjunction

$$\mathrm{Map}_{E_\infty}(L_{K(1)}\mathrm{tmf}, L_{K(1)}KO[\![q]\!]) \simeq \mathrm{Map}_{E_\infty}(L_{K(1)}\mathrm{tmf}, L_{K(1)}K[\![q]\!])^{h\{\pm 1\}}.$$

In addition, we have a standard adjunction

$$\mathrm{Map}_{E_\infty}(L_{K(1)}\mathrm{tmf}, L_{K(1)}K[\![q]\!]) \simeq \mathrm{Map}_{K\text{-}alg}(L_{K(1)}(K \wedge \mathrm{tmf}), L_{K(1)}K[\![q]\!]).$$

Free $K(1)$-local algebras over K are, on homotopy groups, free p-complete θ-algebras. As a consequence, the Goerss-Hopkins obstruction theory for maps of $K(1)$-local K-algebras produces obstructions in André-Quillen cohomology groups for this category of algebras. We have a fringed spectral sequence with E_2-term given by

$$E_2^{s,t} = \begin{cases} \mathrm{Hom}_{\theta\text{-}alg/K_*^\vee}(K_*^\vee\mathrm{tmf}, K_*[\![q]\!]_p^\wedge) & \text{if } (s,t) = (0,0), \\ H^s_{\theta\text{-}alg/K_*^\vee}(K_*^\vee\mathrm{tmf}, \Omega^t K_*[\![q]\!]_p^\wedge) & \text{otherwise.} \end{cases}$$

This spectral sequence converges to the homotopy groups of the mapping space $\mathrm{Map}_{K\text{-}alg}(L_{K(1)}(K \wedge \mathrm{tmf}), L_{K(1)}K[\![q]\!])$.

On homotopy groups, the map $L_{K(1)}K \to L_{K(1)}(K \wedge \mathrm{tmf})$ is a map $\mathbb{Z}_p \to V$, the p-adic completion of a directed sequence of étale extensions. In particular, the relative André-Quillen cohomology groups vanish, and so for $s > 0$ we get an isomorphism

$$H^s_{\theta\text{-}alg/K_*^\vee}(K_*^\vee\mathrm{tmf}, -) \to H^s_{\theta\text{-}alg/K_*^\vee}(K_*^\vee, -) \cong 0.$$

Therefore, the Goerss-Hopkins obstruction theory degenerates to showing that the mapping space is homotopically discrete, and specifically

$$\mathrm{Map}_{K\text{-}alg}(L_{K(1)}(K \wedge \mathrm{tmf}), L_{K(1)}K[\![q]\!]) \simeq \mathrm{Hom}_{\theta\text{-}alg/K_*^\vee}(K_*^\vee\mathrm{tmf}, K_*[\![q]\!]_p^\wedge)$$

In particular, there is a map $V \to K_*[\![q]\!]$ classifying the Tate curve with its choice of isomorphism $\widehat{T} \cong \widehat{\mathbb{G}}_m$, and this takes the -1-automorphism of T to the -1 automorphism of $\widehat{\mathbb{G}}_m$. This determines a contractible component in the space of maps of E_∞ ring spectra $\mathrm{tmf} \to L_{K(1)}K[\![q]\!]$ which is preserved by the action of $\{\pm 1\}$, or equivalently an equivariant map $E\mathbb{Z}/2 \to \mathrm{Map}_{E_\infty}(\mathrm{tmf}, K[\![q]\!])$. This determines a map from tmf to the homotopy fixed-point spectrum, which is $L_{K(1)}KO[\![q]\!]$. □

We note in the following that, as KO-modules are automatically $E(1)$-local, $K(1)$-localizations and p-completions are interchangeable on them.

Proposition A.3. *There exists a map of rational E_∞ ring spectra*

$$\mathrm{tmf}_{\mathbb{Q}} \to (KO[\![q]\!])_{\mathbb{Q}}$$

which, on homotopy groups, is given by a map

$$\mathbb{Q}[c_4, c_6] \to \mathbb{Q} \otimes \mathbb{Z}[\![q]\!][\beta^{\pm 2}]$$

sending c_4 and c_6 to their q-expansions. The two maps

$$\mathrm{tmf} \to (\prod_p KO[\![q]\!]_p^\wedge)_{\mathbb{Q}},$$

induced by this map and the maps constructed in Propositions A.1 and A.2, are homotopic as maps of E_∞ ring spectra.

Proof. The elements c_4 and c_6 can be realized as maps $S^8 \to \mathrm{tmf}$ and $S^{12} \to \mathrm{tmf}$ respectively. The induced map of E_∞ ring spectra $\mathbb{P}_{\mathbb{Q}}(S^8 \vee S^{12}) \to \mathrm{tmf}_{\mathbb{Q}}$ is a weak equivalence, and so homotopy classes of E_∞ maps $\mathrm{tmf}_{\mathbb{Q}} \to (KO[\![q]\!])_{\mathbb{Q}}$ are defined uniquely, up to homotopy, by specifying the images of c_4 and c_6.

Similarly, maps $\mathrm{tmf}_{\mathbb{Q}} \to (\prod_p KO[\![q]\!]_p^\wedge)_{\mathbb{Q}}$ are uniquely determined by the images of c_4 and c_6. Therefore, as the $K(1)$-local and rational constructions are both obtained by q-expansion in a neighborhood of the Tate curve, the resulting pair of maps are homotopic as maps of E_∞ ring spectra. $\square$

Theorem A.4. *There exists a map of E_∞ ring spectra*

$$\mathrm{tmf} \to KO[\![q]\!]$$

compatible with the $K(1)$-local and rational maps constructed in Propositions A.1, A.2, and A.3.

Proof. We can express the spectrum $KO[\![q]\!]$ as a homotopy pullback in following arithmetic square of E_∞ ring spectra:

$$\begin{tikzcd}
KO[\![q]\!] \arrow[r] \arrow[d] & \prod_p L_{K(1)} KO[\![q]\!] \arrow[d] \\
(KO[\![q]\!])_{\mathbb{Q}} \arrow[r] & (\prod_p KO[\![q]\!]_p^\wedge)_{\mathbb{Q}}
\end{tikzcd}$$

However, from Propositions A.1, A.2, and A.3 we obtain maps from tmf to the rational and p-completed entries which are compatible, and therefore a map from tmf to the homotopy pullback. $\square$

Remark A.5. As the spectrum $(\prod_p KO[\![q]\!]_p^\wedge)_{\mathbb{Q}}$ has trivial homotopy groups in degrees 9 and 13, the path components of the mapping space

$$\mathrm{Map}_{E_\infty}\left(\mathrm{tmf}, (\prod_p KO[\![q]\!]_p^\wedge)_{\mathbb{Q}}\right)$$

are all simply connected. The Mayer-Vietoris square of mapping spaces shows that the lift to a map of E_∞ ring spectra from tmf to the pullback is uniquely determined up to homotopy class.

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