E_∞ ALGEBRAS AND p-ADIC HOMOTOPY THEORY

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ABSTRACT. Let $\mathbf{F}_p$ denote the field with p elements and $\bar{\mathbf{F}}_p$ its algebraic closure. We show that the singular cochain functor with coefficients in $\bar{\mathbf{F}}_p$ induces a contravariant equivalence between the homotopy category of connected p-complete nilpotent spaces of finite p-type and a full subcategory of the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras.

INTRODUCTION

Since the invention of localization and completion of topological spaces, it has proved extremely useful in homotopy theory to view the homotopy category from the perspective of a single prime at a time. The work of Serre, Quillen, Sullivan, and others showed that, viewed rationally, homotopy theory becomes completely algebraic. In particular, Sullivan showed that an important subcategory of the homotopy category of rational spaces is contravariantly equivalent to a subcategory of the homotopy category of commutative differential graded $\mathbf{Q}$-algebras, and that the functor underlying this equivalence is closely related to the singular cochain functor. In this paper, we offer a similar theorem for p-adic homotopy theory.

Since the non-commutativity of the multiplication of the $\mathbf{F}_p$ singular cochains is visible already on the homology level in the Steenrod operations, one would not expect that any useful subcategory of the p-adic homotopy category to be equivalent to a category of commutative differential graded algebras. We must instead look to a more sophisticated class of algebras, the E_∞ algebras [18]. E_∞ algebras, roughly, are differential graded modules with an infinitely coherent homotopy associative and commutative multiplication. They provide a generalization of commutative differential graded algebras that admits homology operations as commutativity obstructions generalizing the Steenrod operations. To capture p-adic homotopy theory, even the category of E_∞ $\mathbf{F}_p$-algebras is not quite sufficient; rather we consider E_∞ algebras over the algebraic closure $\bar{\mathbf{F}}_p$ of $\mathbf{F}_p$. We prove the following theorem.

Main Theorem. *The singular cochain functor with coefficients in $\bar{\mathbf{F}}_p$ induces a contravariant equivalence from the homotopy category of connected p-complete nilpotent spaces of finite p-type to a full subcategory of the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras.*

The homotopy category of connected p-complete nilpotent spaces of finite p-type is a full subcategory of the p-adic homotopy category, the category obtained from the category of spaces by formally inverting those maps that induce isomorphisms on singular homology with coefficients in $\mathbf{F}_p$. The p-adic homotopy category itself can be regarded as a full subcategory of the homotopy category, the category

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obtained from the category of spaces by formally inverting the weak equivalences. We remind the reader that a connected space is p-complete, nilpotent, and of finite p-type if and only if its Postnikov tower has a principal refinement in which each fiber is of type $K(\mathbf{Z}/p\mathbf{Z}, n)$ or $K(\mathbf{Z}_p^\wedge, n)$, where $\mathbf{Z}_p^\wedge$ denotes the p-adic integers.

By the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras, we mean the category obtained from the category of algebras over a particular but unspecified E_∞ $\bar{\mathbf{F}}_p$ operad by formally inverting the maps in that category that are quasi-isomorphisms of the underlying differential graded $\bar{\mathbf{F}}_p$-modules, the maps that induce an isomorphism of homology groups. It is well-known that up to equivalence, this category does not depend on the operad chosen. We refer the reader to [18, I] for a good introduction to operads, E_∞ operads, and E_∞ algebras.

To complete the picture, we need to identify intrinsically the subcategory of the homotopy category of E_∞ $\bar{\mathbf{F}}_p$-algebras that the Main Theorem asserts an equivalence with. Although we can write a necessary and sufficient condition for an E_∞ $\bar{\mathbf{F}}_p$-algebra to be quasi-isomorphic to the singular cochain complex of a connected p-complete nilpotent space of finite p-type, it is relatively unenlightening and difficult to verify in practice. This condition is stated precisely in Section 7 and is essentially the E_∞ $\bar{\mathbf{F}}_p$-algebra analogue of the existence of a finite p-type principal Postnikov tower. Unsurprisingly, restricting consideration to simply connected spaces makes the identification significantly easier. In fact, we can write necessary and sufficient conditions for an E_∞ $\bar{\mathbf{F}}_p$-algebra to be quasi-isomorphic to the singular cochain complex of a 1-connected space of finite p-type in terms of its homology and the generalized Steenrod operation P^0.

Characterization Theorem. *An E_∞ differential graded $\bar{\mathbf{F}}_p$-algebra A is quasi-isomorphic in the category of E_∞ $\bar{\mathbf{F}}_p$-algebras to the singular cochain complex of a 1-connected (p-complete) space of finite p-type if and only if $H^i A$ is zero for $i < 0$, $H^0 A = \bar{\mathbf{F}}_p$, $H^1 A = 0$, and for $i > 1$, $H^i A$ is finite dimensional over $\bar{\mathbf{F}}_p$ and generated as an $\bar{\mathbf{F}}_p$-module by the fixed points of the operation P^0.*

Succinctly, the Characterization Theorem states that an E_∞ $\bar{\mathbf{F}}_p$-algebra A is quasi-isomorphic to the singular cochain complex of a 1-connected space of finite p-type if and only if the homology of A looks like the cohomology of such a space as a module over the generalized Steenrod algebra.

Comparison with Other Approaches. The papers [13, 17, 26] and the unpublished ideas of [8] all compare p-adic homotopy theory to various homotopy categories of algebras (or coalgebras). We give a short comparison of these results to the results proved here.

The first announced results along the lines of our Main Theorem appeared in [26]. The arguments there are not well justified, however, and some of the results appear to be wrong.

More recently, [13, 17] have compared the p-adic homotopy category with the homotopy categories of simplicial cocommutative coalgebras and cosimplicial commutative algebras. In particular, [13] proves that the p-adic homotopy category embeds as a full subcategory of the homotopy category of cocommutative simplicial $\bar{\mathbf{F}}_p$-coalgebras. The analogue of the Characterization Theorem is not known in this context. It is straightforward to describe the relationship between the results of [13] and our Main Theorem. There is a functor from the homotopy category of simplicial cocommutative coalgebras to the homotopy category of E_∞ algebras

given by normalization of the dual cosimplicial commutative algebra [15] (see also §1 below). Applied to the singular simplicial chains of a space, we obtain the singular cochain complex of that space. Our Main Theorem implies that on the subcategory of nilpotent spaces of finite p-type, this refined functor remains a full embedding. This gives an affirmative answer to the question asked in [17, 6.3].

The unpublished ideas of [8] for comparing the p-adic homotopy category to the homotopy category of E_∞ ring spectra under the Eilenberg-Mac Lane spectrum $H\bar{\mathbf{F}}_p$, would give a "brave new algebra" version of our Main Theorem. A proof of such a comparison can be given along similar lines to the proof of our Main Theorem. We sketch the argument in Appendix C. The analogue of the Characterization Theorem in this context was not considered in [8], but can be proved by essentially the same arguments as the proof of our Characterization Theorem. A direct comparison between our approach and this approach to p-adic homotopy theory would require a comparison of the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra and the category of E_∞ $\bar{\mathbf{F}}_p$-algebras, and also an identification of the composite functor from spaces to E_∞ differential graded $\bar{\mathbf{F}}_p$-algebras as the singular cochain functor. We will provide this comparison and this identification in [19] and [20].

1. Outline of the Paper

Since the main objects we work with in this paper are the cochain complexes, it is convenient to grade differential graded modules "cohomologically" with the differential raising degrees. This makes the cochain complexes concentrated in non-negative degrees, but forces E_∞ operads to be concentrated in non-positive degrees. Along with this convention, we write the homology of a differential graded module M as H^*M. We work almost exclusively with ground ring $\bar{\mathbf{F}}_p$; throughout this paper, C^*X and H^*X always denote the cochain complex and the cohomology of X taken with coefficients in $\bar{\mathbf{F}}_p$. We write $C^*(X; \mathbf{F}_p)$ and $H^*(X; \mathbf{F}_p)$ for the cochain complex and the cohomology of X with coefficients in $\mathbf{F}_p$ or $C^*(X; k)$ and $H^*(X; k)$ for these with coefficients in a commutative ring k.

The first prerequisite to the Main Theorem is recognizing that the singular cochain functor can be regarded as a functor into the category of $\mathcal{E}$-algebras for some E_∞ $\bar{\mathbf{F}}_p$-operad $\mathcal{E}$. In fact, for the purpose of this paper, the exact construction of this structure does not matter so long as the (normalized) cochain complex of a simplicial set is naturally an $\mathcal{E}$-algebra. However, we do need to know that such a structure exists. This can be shown as follows.

The work of Hinich and Schechtman in [15] gives the singular cochain complex of a space or the cochain complex of a simplicial set the structure of a "May algebra", an algebra over an acyclic operad $\mathcal{Z}$, the "Eilenberg-Zilber" operad. The operad $\mathcal{Z}$ is not an E_∞ operad however since it is not Σ-free and since it is non-zero in both positive and negative degrees. To fix this, let $\bar{\mathcal{Z}}$ be the "(co)-connective cover" of $\mathcal{Z}$: $\bar{\mathcal{Z}}(n)$ is the differential graded $\bar{\mathbf{F}}_p$-module that is equal to $\mathcal{Z}(n)$ in degrees less than zero, equal to the kernel of the differential in degree zero, and zero in positive degrees. The operadic multiplication of $\mathcal{Z}$ lifts to $\bar{\mathcal{Z}}$, making it an acyclic operad. Tensoring $\bar{\mathcal{Z}}$ with an E_∞ operad $\mathcal{C}$ gives an E_∞ operad $\mathcal{E}$ and a map of operads $\mathcal{E} \to \mathcal{Z}$. The cochain complex of a simplicial set then obtains the natural structure of an algebra over the E_∞ operad $\mathcal{E}$.

We write $\mathfrak{E}$ for the category of $\mathcal{E}$-algebras. Since we are assuming that the functor C^* from spaces to $\mathcal{E}$-algebras factors through the category of simplicial sets, we

can work simplicially. As is fairly standard, we refer to the category obtained from the category of simplicial sets by formally inverting the weak equivalences as *the homotopy category*; this category is equivalent to the category of Kan complexes and homotopy classes of maps and to the category of CW spaces and homotopy classes of maps. Since the cochain functor converts $\mathbf{F}_p$-homology isomorphisms and in particular weak equivalences of simplicial sets to quasi-isomorphisms of $\mathcal{E}$-algebras, the (total) derived functor exists as a contravariant functor from the homotopy category to the homotopy category of $\mathcal{E}$-algebras. We prove the Main Theorem by constructing a right adjoint $\mathbf{U}$ from the homotopy category of $\mathcal{E}$-algebras to the homotopy category and showing that it provides an inverse equivalence on the subcategories in question.

In order to construct the functor $\mathbf{U}$ and to analyze the composite $\mathbf{U}C^*$, we need some tools to help us understand the homotopy category of $\mathcal{E}$-algebras. The tools we need are precisely those provided by Quillen's theory of closed model categories [25] (see also [9]). Unfortunately, we have not been able to verify that the category of $\mathcal{E}$-algebras is a model category. Nevertheless, the category of $\mathcal{E}$-algebras is close enough that most of the standard model category arguments apply, and we obtain the results we need. These theorems are summarized in Section 2.

Various steps in the proofs of the Main Theorem and the Characterization Theorem require understanding of the derived coproduct and the homotopy pushout of $\mathcal{E}$-algebras. We summarize the results we need in Section 3; the proofs of these results are in Section 14.

We construct in Section 4 a contravariant functor U from the category of $\mathcal{E}$-algebras to the category of simplicial sets that is the right adjoint to C^*. Our model theoretic results allow us to show that the right derived functor of U exists and is right adjoint to the derived functor of C^*; this derived functor is our functor $\mathbf{U}$ mentioned above. Precisely, $\mathbf{U}$ is a contravariant functor from the homotopy category of $\mathcal{E}$-algebras to the homotopy category, and we have a canonical isomorphism

$$\mathfrak{Ho}(X, \mathbf{U}A) \cong \bar{h}\mathfrak{E}(A, C^*X)$$

for a simplicial set X and an $\mathcal{E}$-algebra A. Here and elsewhere $\mathfrak{Ho}$ denotes the homotopy category and $\bar{h}\mathfrak{E}$ denotes the homotopy category of $\mathcal{E}$-algebras.

We write u_X for the "unit" of the derived adjunction $X \to \mathbf{U}C^*X$. For the purposes of this paper, we say that a simplicial set X is *resolvable by E_∞ $\bar{\mathbf{F}}_p$-algebras* or just *resolvable* if the map u_X is an isomorphism in the homotopy category. In Section 5, we prove the following two theorems.

Theorem 1.1. *Let X be the limit of a tower of Kan fibrations $\cdots \to X_n \to \cdots X_0$. Assume that the canonical map from H^*X to $\operatorname{Colim} H^*X_n$ is an isomorphism. If each X_n is resolvable, then X is resolvable.*

Theorem 1.2. *Let X, Y, and Z be connected simplicial sets of finite p-type, and assume that Z is simply connected. Let $X \to Z$ be a map of simplicial sets, and let $Y \to Z$ be a Kan fibration. If X, Y, and Z are resolvable, then so is the fiber product $X \times_Z Y$.*

These theorems allow us to argue inductively up towers of principal Kan fibrations. The following theorem proved in Section 6 provides a base case.

Theorem 1.3. *$K(\mathbf{Z}/p\mathbf{Z}, n)$ and $K(\mathbf{Z}_p^\wedge, n)$ are resolvable for $n \geq 1$.*

We conclude that every connected p-complete nilpotent simplicial set of finite p-type is resolvable. The Main Theorem is now an elementary categorical consequence:

$$\mathfrak{Ho}(X, Y) \cong \mathfrak{Ho}(X, \mathbf{U}C^*Y) \cong \bar{h}\mathfrak{E}(C^*Y, C^*X)$$

for X, Y connected p-complete nilpotent simplicial sets of finite p-type.

The proof of the Characterization Theorem is presented in Sections 7–10.

We mention here one more result in this paper. This result is needed in the proof of Theorem 1.3 but appears to be of independent interest. The work of [22] provides the homology of E_∞ algebras in characteristic p with operations P^s and βP^s (when $p > 2$) for $s \in \mathbf{Z}$. It follows from a check of the axioms and the identification of βP^0 as the Bockstein that when these operations are applied to the $\mathbf{F}_p$-cochain complex of a simplicial set they perform the Steenrod operation of the same names, where we understand P^s to be the zero operation for $s < 0$ and the identity for $s = 0$. The "algebra of all operations" $\mathfrak{B}$ therefore surjects onto the Steenrod algebra $\mathfrak{A}$ with kernel containing the two-sided ideal generated by $1 - P^0$. The following theorem describes the precise relationship between $\mathfrak{B}$ and $\mathfrak{A}$.

Theorem 1.4. *The left ideal of $\mathfrak{B}$ generated by $(1 - P^0)$ is a two-sided ideal whose quotient $\mathfrak{B}/(1 - P^0)$ is canonically isomorphic to $\mathfrak{A}$.*

The analogue of the Main Theorem for fields other than $\bar{\mathbf{F}}_p$ is discussed in Appendix A. In particular, we show that the analogue of the Main Theorem does not hold when $\bar{\mathbf{F}}_p$ is replaced by any finite field.

A discussion of the composite $\mathbf{U}C^*$ when the Main Theorem does not apply and a comparison with p-pro-finite completion is given in Appendix B (see also Remark 5.1).

2. The Homotopy Theory of E_∞ Algebras

In this section we develop the homotopy theoretic results we need for the category of $\mathcal{E}$-algebras. In fact, the results of this section hold for the categories of algebras over the more general class of operads described in Section 13. This class of operads includes all E_∞ operads of differential graded modules over a commutative ground ring. For convenience of notation for later reference, we state everything in terms of the particular operad $\mathcal{E}$ of differential graded $\bar{\mathbf{F}}_p$-modules associated to the given natural E_∞ algebra structure on the cochain functor.

Although we do not prove that the category of $\mathcal{E}$-algebras is a closed model category, the model category framework provides a convenient language in which to present the results we need. We assume familiarity with this language; we refer the unfamiliar reader to [9] for a good introduction to model categories. In order to be able to use much of this language and in order to facilitate constructions, we begin with the following well-known fact about categories of algebras over operads of differential graded modules.

Proposition 2.1. *The category of $\mathcal{E}$-algebras is complete and cocomplete. Limits and filtered colimits commute with the forgetful functor to differential graded modules.*

The following definition specifies the cofibrations, fibrations, and weak equivalences for our model category results.

Definition 2.2. We say that a map of $\mathcal{E}$-algebras $f \colon A \to B$ is a:

(i) *weak equivalence* if it is a quasi-isomorphism.
(ii) *fibration* if it is a surjection.
(iii) *cofibration* if it has the left lifting property with respect to the acyclic fibrations.

It is convenient to have a shorthand for indicating weak equivalences, fibrations, and cofibrations in diagrams. The following usage has become relatively standard.

Notation 2.3. The symbol "$\sim$" decorating an arrow indicates a map that is known to be or is assumed to be a quasi-isomorphism. The arrow "$\twoheadrightarrow$" indicates a map that is known to be or is assumed to be a fibration. The arrow "$\rightarrowtail$" indicates a map that is known to be or is assumed to be a cofibration.

We can identify the cofibrations more intrinsically. In the following definition, for a differential graded module M, we denote by CM the *cone* on M; this is the differential graded module whose underlying graded module is the sum of M and a copy of M shifted down, with differential defined so that CM is contractible and the inclusion $M \to CM$ is a map of differential graded modules.

Definition 2.4. A map of $\mathcal{E}$-algebras $f \colon A \to B$ is *relative cell inclusion* if there exists a sequence of $\mathcal{E}$-algebra maps $A = A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} \cdots$ such that

(i) $B \cong \operatorname{Colim} i_n$ under A.
(ii) Each map i_n is formed as a pushout of $\mathcal{E}$-algebras

$$
\begin{array}{ccc}
\mathbb{E}M_{n+1} & \longrightarrow & \mathbb{E}CM_{n+1} \\
\downarrow & & \downarrow \\
A_n & \xrightarrow{\quad i_n \quad} & A_{n+1}
\end{array}
$$

where $\mathbb{E}$ denotes the free $\mathcal{E}$-algebra functor, M_{n+1} is a degreewise free differential graded module with zero differential, CM_{n+1} is the cone on M_{n+1}, and the map $M_{n+1} \to CM_{n+1}$ is the canonical inclusion.

We say that an $\mathcal{E}$-algebra A is a cell $\mathcal{E}$-algebra if the initial map $\bar{\mathbf{F}}_p = \mathcal{E}(0) \to A$ is a relative cell inclusion. A cell $\mathcal{E}$-algebra A is *finite* if each M_n is finitely generated and there is some N such that $M_n = 0$ for $n > N$.

Clearly the relative cell inclusions are cofibrations. The following proposition provides a near converse.

Proposition 2.5. *A map is a cofibration if and only if it is a the retract of a relative cell inclusion.*

The previous proposition is a formal consequence of a standard lift argument and the following proposition that follows from an elementary application of the small objects argument.

Proposition 2.6. *Any map of $\mathcal{E}$-algebras $f \colon A \to B$ can be factored functorially as $f = p \circ i$, where i is a relative cell inclusion and p is an acyclic fibration.*

We also mention the following lifting property. It follows by considering the left lifting property for the relative cell inclusions $\mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}C\bar{\mathbf{F}}_p[n]$, where $\bar{\mathbf{F}}_p[n]$ denotes the degreewise free differential graded module with zero differential with one generator, in degree n.

Proposition 2.7. *A map of $\mathcal{E}$-algebras $A \to B$ is an acyclic fibration if and only if it has the right lifting property with respect to the cofibrations if and only if it has the right lifting property with respect to cofibrations between cell $\mathcal{E}$-algebras.*

The previous two propositions give us one factorization property and one lifting property. We cannot prove the other factorization and lifting properties in general. However, we can prove them for cofibrant $\mathcal{E}$-algebras. We prove the following theorem in Section 13.

Theorem 2.8. *Any map of $\mathcal{E}$-algebras $f\colon A \to B$ can be factored functorially as $f = q \circ j$, where j is a relative cell inclusion that has the left lifting property with respect to the fibrations, and q is a fibration. If A is cofibrant then j is in addition a quasi-isomorphism.*

Corollary 2.9. *Let A be cofibrant. Then a map of $\mathcal{E}$-algebras $A \to B$ is an acyclic cofibration if and only if it has the left lifting property with respect to the fibrations.*

Corollary 2.10. *A map of $\mathcal{E}$-algebras $A \to B$ is a fibration if and only if it has the right lifting property with respect to the acyclic cofibrations between cell $\mathcal{E}$-algebras.*

Using the fact that all $\mathcal{E}$-algebras are fibrant, the factorization and lifting properties above provide sufficient tools to make the homotopy theory formalized by Quillen in [25] useful for studying the category $\bar{h}\mathfrak{E}$, the localization of the category $\mathfrak{E}$ obtained by formally inverting the quasi-isomorphisms. Anticipating Theorem 2.13 below, we have already started referring to $\bar{h}\mathfrak{E}$ as the homotopy category of $\mathcal{E}$-algebras; we now state the definition of homotopy.

Definition 2.11. Let A be an $\mathcal{E}$-algebra. A (Quillen) *cylinder object* for A is an $\mathcal{E}$-algebra IA equipped with maps $\partial_0, \partial_1\colon A \to IA$ and $\sigma\colon IA \to A$ such that $\partial_0 + \partial_1\colon A \amalg A \to IA$ is a cofibration, σ is a quasi-isomorphism, and the composite $\sigma \circ (\partial_0 + \partial_1)$ is the folding map $A \amalg A \to A$. We say that maps of $\mathcal{E}$-algebras $f_0, f_1\colon A \to B$ are (Quillen left) *homotopic* if there is a map $f\colon IA \to B$ such that $f_0 = f \circ \partial_0$ and $f_1 = f \circ \partial_1$; we call f a (Quillen left) *homotopy* from f_0 to f_1. We denote by $\pi\mathfrak{E}(A, B)$ the quotient of the mapping set $\mathfrak{E}(A, B)$ by the equivalence relation generated by "homotopic".

In the case when A is a cofibrant $\mathcal{E}$-algebra, we can glue cylinder objects as in [25, Lemma 1-3] and see that "homotopic" is already an equivalence relation on $\pi\mathfrak{E}(A, B)$.

Since our fibrations are the surjections, the map σ is always an acyclic fibration, and so it is easy to see that for arbitrary $\mathcal{E}$-algebras A, B, C, composition in $\mathfrak{E}$ induces an associative composition

$$\pi\mathfrak{E}(B, C) \times \pi\mathfrak{E}(A, B) \to \pi\mathfrak{E}(A, C),$$

making $\pi\mathfrak{E}$ a category. The following proposition, the $\mathcal{E}$-algebra analogue of the Whitehead Theorem, is straightforward to deduce from the factorization and lifting properties above.

Proposition 2.12. *Let A be a cofibrant $\mathcal{E}$-algebra. A quasi-isomorphism of $\mathcal{E}$-algebras $B \to C$ induces a bijection $\pi(A, B) \to \pi(A, C)$.*

Since homotopic maps in $\mathfrak{E}(A, B)$ represent the same map in $\bar{h}\mathfrak{E}$, the localization functor $\mathfrak{E} \to \bar{h}\mathfrak{E}$ factors through the category $\pi\mathfrak{E}$. Let $\pi\mathfrak{E}_c$ denote the full subcategory of $\pi\mathfrak{E}$ consisting of the cofibrant $\mathcal{E}$-algebras. We therefore obtain a functor

$\pi\mathfrak{E}_c \to \bar{h}\mathfrak{E}$ by restriction. The following theorem, the analogue of [25, Theorem 1-1'], is now an immediate consequence of the previous proposition.

Theorem 2.13. *The functor $\pi\mathfrak{E}_c \to \bar{h}\mathfrak{E}$ is an equivalence of categories. In particular $\bar{h}\mathfrak{E}$ has small Hom sets.*

Another fundamental theorem that we can prove in this context is the analogue of [25, Theorem 4-3], needed for the construction of **U** in Section 4. The proof follows the standard one for model categories with only minor modifications and is left to the reader. Although we apply it to a contravariant functor in the construction of **U**, we write it here in the familiar covariant form because this gives a much clearer and unambiguous statement.

Theorem 2.14. *Let $L\colon \mathfrak{E} \to \mathfrak{M}$ and $R\colon \mathfrak{M} \to \mathfrak{E}$ be left and right adjoints between the category of $\mathcal{E}$-algebras $\mathfrak{E}$ and a closed model category $\mathfrak{M}$.*

  (i) *If L preserves cofibrations between cofibrant objects and R preserves fibrations, then the left derived functor of L and the right derived functor of R exist and are adjoint. Moreover, L converts quasi-isomorphisms between cofibrant $\mathcal{E}$-algebras to weak equivalences, and the restriction of the left derived functor of L to the cofibrant $\mathcal{E}$-algebras is naturally isomorphic to the derived functor of the restriction of L.*
 (ii) *Suppose that (i) holds and in addition for any cofibrant $\mathcal{E}$-algebra A and any fibrant object Y in $\mathfrak{M}$, a map $A \to RY$ is a quasi-isomorphism if and only if the adjoint $LA \to Y$ is a weak equivalence. Then the left derived functor of L and the right derived functor of R are inverse equivalences.*

The lifting properties provide the following useful alternative hypotheses.

Theorem 2.15. *The hypothesis of 2.14.(i) is equivalent to each of the following.*

  (i) *L preserves cofibrations between cofibrant objects and acyclic cofibrations between cofibrant objects.*
 (ii) *R preserves fibrations and acyclic fibrations.*

We note here for future reference that the analogues of the previous two theorems also hold when $\mathfrak{E}$ or $\mathfrak{M}$ (or both) is replaced by an undercategory $A/\mathfrak{E}$ for a cofibrant $\mathcal{E}$-algebra A.

3. The E_∞ Torsion Product

We use the results on the homotopy theory of $\mathcal{E}$-algebras of the last section to study coproducts and homotopy pushouts of $\mathcal{E}$-algebras in this section. We show that the homology of these is closely related to the differential torsion product. We state the results in this section for the category of algebras over the operad $\mathcal{E}$ of differential graded $\bar{\mathbf{F}}_p$-modules, but except as noted they apply more generally to the category of algebras over any E_∞ operad of differential graded modules over a commutative ring.

Definition 3.1. Let $A \to B$ and $A \to C$ be maps of $\mathcal{E}$-algebras. We define the E_∞ torsion product of B and C under A by

$$E_\infty \mathrm{Tor}_A^*(B, C) = H^*(B' \amalg_{A'} C')$$

where A' is a cofibrant approximation of A and B' and C' are cofibrant approximations of B and C in the category of $\mathcal{E}$-algebras under A', i.e. A' is cofibrant

and we have the following commutative diagrams with arrows quasi-isomorphisms, fibrations, and cofibrations as indicated.

$$
\begin{array}{ccc}
B' \longleftarrow\!\!\!\prec A' \succ\!\!\!\longrightarrow C' \\
\sim\downarrow \qquad \sim\downarrow \qquad \downarrow\sim \\
B \longleftarrow A \longrightarrow C
\end{array}
$$

We omit A from the notation, writing $E_\infty\mathrm{Tor}^*(B,C)$, when A is the initial object.

The standard lift and homotopy arguments combined with the following theorem imply that the E_∞ torsion product is well-defined and that it only depends on the diagram $B \leftarrow A \rightarrow C$ in the homotopy category of diagrams of this form.

Theorem 3.2. *Given the following diagram of $\mathcal{E}$-algebras, with the vertical maps quasi-isomorphisms and the right-hand horizontal maps cofibrations*

$$
\begin{array}{ccc}
B' \longleftarrow A' \succ\!\!\!\longrightarrow C' \\
\sim\downarrow \qquad \sim\downarrow \qquad \downarrow\sim \\
B \longleftarrow A \succ\!\!\!\longrightarrow C,
\end{array}
$$

if A, A', B, and B' are cofibrant, then the induced map of pushouts

$$B' \amalg_{A'} C' \rightarrow B \amalg_A C$$

is a quasi-isomorphism.

Proof. Factor the map $A' \rightarrow B'$ as a cofibration $A' \rightarrow B''$ followed by a quasi-isomorphism $B'' \rightarrow B'$; it suffices to show that the induced maps $B'' \amalg_{A'} C' \rightarrow B' \amalg_{A'} C'$ and $B'' \amalg_{A'} C' \rightarrow B \amalg_A C$ are both quasi-isomorphisms. As noted at the close of the last section, Theorems 2.14 and 2.15 hold with $\mathfrak{E}$ and $\mathfrak{M}$ replaced with the undercategories $X/\mathfrak{E}$ and $Y/\mathfrak{E}$ when X and Y are cofibrant $\mathcal{E}$-algebras. Applying this for the adjoint pair of functors induced by a map $X \rightarrow Y$ and applying the argument for K. Brown's lemma [9, 9.9], it follows that for cofibrant $\mathcal{E}$-algebras, the pushout of a weak equivalence along a cofibration is a weak equivalence. The theorem then follows by noting that the map $B'' \amalg_{A'} C' \rightarrow B' \amalg_{A'} C'$ is the pushout of a weak equivalence along a cofibration and the map $B'' \amalg_{A'} C' \rightarrow B \amalg_A C$ can be factored as a sequence of pushouts of weak equivalences along cofibrations. $\square$

Corollary 3.3. *Let A, B be cofibrant $\mathcal{E}$-algebras, $A \rightarrow B$ a map of $\mathcal{E}$-algebras and $A \rightarrow C$ a cofibration of $\mathcal{E}$-algebras. Then the canonical map $E_\infty\mathrm{Tor}^*_A(B,C) \rightarrow H^*(B \amalg_A C)$ is an isomorphism.*

The next theorem compares the E_∞ torsion product to the ordinary differential torsion product over the ground ring. Since our differential graded modules are integer graded, we should say a few words about what we mean by the differential torsion product. For differential graded modules M, N, let $\mathrm{Tor}^*(M,N)$ be the homology of the left derived functor of the bifunctor $(-) \otimes (-)$ on M, N. This derived functor is proved to exist for example in [18, §III.4], and it coincides with the left derived functor obtained by fixing one of the variables M or N. We have a canonical map $\mathrm{Tor}^*(M,N) \rightarrow H^*M \otimes H^*N$, which is an isomorphism in the case of main interest since $\bar{\mathbf{F}}_p$ is a field.

Returning to the context above, for A the initial object, the map

$$\mathcal{E}(2) \otimes B' \otimes C' \rightarrow \mathcal{E}(2) \otimes (B' \oplus C') \otimes (B' \oplus C') \rightarrow B' \amalg C'$$

induces a map from the differential torsion product $\operatorname{Tor}^*(B,C)$ to the E_∞ torsion product $E_\infty \operatorname{Tor}^*(B,C)$. In Section 14, we prove the following theorem.

Theorem 3.4. *The map* $\operatorname{Tor}^*(B,C) \to E_\infty \operatorname{Tor}^*(B,C)$ *is an isomorphism.*

We use the previous theorem to construct a spectral sequence for the calculation of $E_\infty \operatorname{Tor}^*_A(B,C)$ for general A. For this, we need the *bar construction* in the category of $\mathcal{E}$-algebras. Given $\mathcal{E}$-algebra maps $A \to B$ and $A \to C$, the bar construction $\beta_\bullet(B,A,C)$ is the simplicial $\mathcal{E}$-algebra that is given in simplicial degree n by

$$\beta_n(B,A,C) = B \amalg \underbrace{A \amalg \cdots \amalg A}_{n \text{ factors}} \amalg C.$$

Regarding $B \amalg_A C$ as a constant simplicial $\mathcal{E}$-algebra, the map $B \amalg C \to B \amalg_A C$ induces a map of simplicial $\mathcal{E}$-algebras $\beta_\bullet(B,A,C) \to B \amalg_A C$ and therefore a map of differential graded modules on their normalizations, $N(\beta_\bullet(B,A,C)) \to B \amalg_A C$. In fact the normalization of a simplicial $\mathcal{E}$-algebra is naturally an $\mathcal{E}$-algebra via the shuffle map [18, p. 51], and this is actually a map of $\mathcal{E}$-algebras, but we do not need this fact here. The fact we do need is given in the following theorem, proved in Section 14.

Theorem 3.5. *Let* A, B *be cofibrant* $\mathcal{E}$*-algebras,* $A \to B$ *a map of* $\mathcal{E}$*-algebras and* $A \to C$ *a cofibration of* $\mathcal{E}$*-algebras. Then the canonical map*

$$N(\beta_\bullet(B,A,C)) \to B \amalg_A C$$

is a quasi-isomorphism.

Since $\bar{\mathbf{F}}_p$ is a field, the following is an immediate consequence of the previous theorem and Theorem 3.4. Although less immediate it still holds for E_∞ operads over an arbitrary commutative ground ring.

Corollary 3.6. *There is a left half-plane cohomological spectral sequence with*

$$E_2^{p,q} = \operatorname{Tor}^{p,q}_{H^*A}(H^*B, H^*C),$$

converging strongly to $E_\infty \operatorname{Tor}^{p+q}_A(B,C)$.

4. Construction of the Functor $\mathbf{U}$

We construct the functor $\mathbf{U}$ whose restriction provides the inverse equivalence of the Main Theorem. As mentioned in the introduction, we construct $\mathbf{U}$ as the derived functor of a functor U from the category of $\mathcal{E}$-algebras to the category of simplicial sets, adjoint to the cochain functor. We begin by reinterpreting the cochain functor as a limit.

Consider the cosimplicial simplicial set $\Delta = \Delta[\cdot]$ given by the standard simplexes. Then $C^*\Delta[\cdot]$ is a simplicial $\mathcal{E}$-algebra. For an arbitrary set S, write $P(S, C^*\Delta[n])$ for the product of copies of $C^*\Delta[n]$ indexed on S. Then for a simplicial set X, $P(X, C^*\Delta[\cdot])$ is a cosimplicial simplicial $\mathcal{E}$-algebra. Write $M(X, C^*\Delta)$ for the *end*, the equalizer in the category of $\mathcal{E}$-algebras of the diagram

$$\prod_n P(X_n, C^*\Delta[n]) \Longrightarrow \prod_{\substack{f \colon \mathbf{m} \to \mathbf{n} \\ f \text{ in } \boldsymbol{\Delta}^{\mathrm{op}}}} P(X_m, C^*\Delta[n]).$$

By construction $M(X, C^*\Delta)$ is an $\mathcal{E}$-algebra, contravariantly functorial in the simplicial set X.

Proposition 4.1. *The cochain functor C^* is canonically naturally isomorphic to $M(-, C^*\Delta)$ as a functor from simplicial sets to $\mathcal{E}$-algebras.*

Proof. For each element of X_n, there is a canonical map $\Delta[n] \to X_n$, and the collection of all such maps induces a map of $\mathcal{E}$-algebras

$$C^*X \to \prod_n P(X_n, C^*\Delta[n]).$$

By naturality, this map factors through the equalizer to induce a map of $\mathcal{E}$-algebras $C^*X \to M(X, C^*\Delta)$. The underlying differential graded module of an equalizer of $\mathcal{E}$-algebras is the equalizer of the underlying differential graded modules. It follows that the induced map $C^*X \to M(X, C^*\Delta)$ is an isomorphism of the underlying differential graded modules and therefore an isomorphism of $\mathcal{E}$-algebras. $\square$

The description of C^* given by Proposition 4.1 makes it easy to recognize C^* as an adjoint. For an $\mathcal{E}$-algebra A, let UA be the simplicial set whose set of n-simplices U_nA is the mapping set $\mathfrak{E}(A, C^*\Delta[n])$. Clearly UA is a contravariant functor of A. For a simplicial set X, the set of simplicial maps from X to UA, $\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set}(X, UA)$ is by definition the end of the cosimplicial simplicial set $\mathbf{Set}_n^m(X, UA) = \mathbf{Set}(X_m, U_nA)$ that in cosimplicial degree m and simplicial degree n consists of the set of maps of sets from X_m to U_nA. Consider the cosimplicial simplicial bijection

$$\begin{aligned}\mathbf{Set}(X_m, U_nA) &= \mathbf{Set}(X_m, \mathfrak{E}(A, C^*\Delta[n])) \\ &\cong \prod_{X_m} \mathfrak{E}(A, C^*\Delta[n]) \cong \mathfrak{E}(A, \prod_{X_m} C^*\Delta[n]) = \mathfrak{E}(A, P(X_m, C^*\Delta[n])).\end{aligned}$$

Passing to ends gives a bijection

$$\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set}(X, UA) \cong \mathfrak{E}(A, C^*X),$$

natural in A and X. Thus, we have proved the following proposition.

Proposition 4.2. *The functors U and C^* are contravariant right adjoints between the category of simplicial sets and the category of $\mathcal{E}$-algebras.*

We now use the results of Section 2 on adjoint functors. Since we stated Theorems 2.14 and 2.15 in terms of covariant functors, we apply them to U, C^* viewed as an adjoint pair between the category of $\mathcal{E}$-algebras and the opposite to the category of simplicial sets. As such, U is the left adjoint. Taking the closed model category structure on the opposite category of simplicial sets as the one opposite to the standard one [25] on the category of simplicial sets, the "fibrations" are the maps opposite to monomorphisms and the "weak equivalences" are the maps opposite to weak equivalences. It follows that the functor C^* converts "fibrations" to surjections and "weak equivalences" to quasi-isomorphisms. It then follows from Theorems 2.14 and 2.15 that the left derived functor of $U\colon \mathfrak{E} \to (\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set})^{\mathrm{op}}$ exists and is adjoint to the right derived functor of $C^*\colon (\mathbf{\Delta}^{\mathrm{op}}\mathbf{Set})^{\mathrm{op}} \to \mathfrak{E}$. When we regard U as a contravariant functor, this derived functor is the right derived functor, and we obtain the following proposition.

Proposition 4.3. *The (right) derived functor $\mathbf{U}$ of U exists and gives an adjunction $\bar{h}\mathfrak{E}(A, C^*X) \cong \mathfrak{Ho}(X, \mathbf{U}A)$.*

Applying Theorem 2.15 again, we obtain the following proposition, which is needed in the proofs of Theorems 1.1 and 1.2 in the next section.

Proposition 4.4. *The functor U converts cofibrations of $\mathcal{E}$-algebras to Kan fibrations of simplicial sets.*

According to Theorem 2.14, the derived functor $\mathbf{U}$ is constructed by first approximating an arbitrary $\mathcal{E}$-algebra with a cofibrant $\mathcal{E}$-algebra and then applying U. This gives us the following standard observation.

Proposition 4.5. *Let X be a simplicial set and $A \to C^*X$ a quasi-isomorphism, where A is a cofibrant $\mathcal{E}$-algebra. The unit of the derived adjunction $X \to \mathbf{U}C^*X$ is represented by the map $X \to UA$.*

Instead of using the standard model structure on the category of simplicial sets, we can use the "$H_*(-;\mathbf{F}_p)$-local" model structure constructed in [1]. In this structure, the cofibrations remain the monomorphisms but the weak equivalences are the $\mathbf{F}_p$-homology equivalences. Since the functor C^* has the stronger property of converting $\mathbf{F}_p$-homology isomorphisms to quasi-isomorphisms, the derived adjunction factors as an adjunction between the homotopy category of $\mathcal{E}$-algebras and the p-adic homotopy category. Although we do not need it in the remainder of our work, we see that the functor U has the following strong $H_*(-;\mathbf{F}_p)$-local homotopy properties.

Proposition 4.6. *The functor U converts $\mathcal{E}$-algebra cofibrations to $H_*(-;\mathbf{F}_p)$-local fibrations. For a cofibrant $\mathcal{E}$-algebra A, UA is an $H_*(-;\mathbf{F}_p)$-local simplicial set.*

5. The Fibration Theorems

In this section, we prove Theorems 1.1 and 1.2 that allow us to construct resolvable simplicial sets out of other resolvable simplicial sets. The proofs proceed by choosing cofibrant $\mathcal{E}$-algebra approximations and applying Propositions 4.4 and 4.5 of the previous section.

Proof of Theorem 1.1. By Proposition 2.6, a map of $\mathcal{E}$-algebras can be factored as a relative cell inclusion followed by an acyclic fibration. Applying this to the $\mathcal{E}$-algebras C^*X_n, we can construct the following commutative diagram of $\mathcal{E}$-algebras.

$$
\begin{array}{ccccccccc}
\bar{\mathbf{F}}_p & \rightarrowtail & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots \rightarrowtail & A_n & \rightarrowtail & \cdots \\
& & \downarrow \sim & & \downarrow \sim & & & \downarrow \sim & & \\
& & C^*X_0 & \longrightarrow & C^*X_1 & \longrightarrow & \cdots \longrightarrow & C^*X_n & \longrightarrow & \cdots
\end{array}
$$

Let $A = \operatorname{Colim} A_n$. From the universal property, we obtain a map $A \to C^*X$. The assumption that $H^*X = \operatorname{Colim} H^*X_n$ then implies that the map $A \to C^*X$ is a quasi-isomorphism.

Applying the functor U, we see that UA is the limit of UA_n. We have the following commutative diagram.

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & X_n & \twoheadrightarrow & \cdots \longrightarrow & X_1 & \longrightarrow & X_0 \\
& \downarrow \sim & & & \downarrow \sim & & \downarrow \sim \\
\cdots \longrightarrow & UA_n & \twoheadrightarrow & \cdots \longrightarrow & UA_1 & \longrightarrow & UA_0
\end{array}
$$

The bottom row is a tower of Kan fibrations by Proposition 4.4 and the vertical maps are weak equivalences by Proposition 4.5 and the assumption that the X_n are resolvable. It follows that the map of the limits $X \to UA$ is a weak equivalence, and we conclude that X is resolvable. $\qquad\square$

Remark 5.1. The argument above actually proves a more general result than stated in Theorem 1.1. Let $X_\infty = \mathrm{Lim}\, X_n$ for a tower of Kan fibrations of Kan complexes. Assume that each X_n is resolvable and that X_∞ is non-empty. If $X \to X_\infty$ induces an isomorphism $H^*X \to \mathrm{Colim}\, H^*X_n$, then the argument above factors the unit of the derived adjunction $X \to \mathbf{U}C^*X$ through a natural isomorphism in the homotopy category $X_\infty \to \mathbf{U}C^*X$. If we assume the Main Theorem for a moment, then when X is connected and of finite p-type, we can apply this observation to the Bousfield–Kan p-completion tower R_nX for $R = \mathbf{F}_p$. We conclude that for any connected X of finite p-type, the unit of the derived adjunction $X \to \mathbf{U}C^*X$ is naturally isomorphic in the homotopy category to the Bousfield–Kan p-completion map $X \to R_\infty X$.
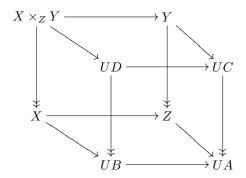
The proof of Theorem 1.2 is similar, but needs in addition the following result that relates the E_∞ torsion product to the usual differential torsion product and is proved at the end of this section.

Lemma 5.2. *Let X, Y, and Z be as in Theorem 1.2. The E_∞ torsion product $E_\infty \mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y)$ is canonically isomorphic to the usual differential torsion product $\mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y)$. Under this isomorphism, the composite*

$$\mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y) \to H^*(C^*X \amalg_{C^*Z} C^*Y) \to H^*(C^*(X \times_Z Y)) = H^*(X \times_Z Y)$$

is the Eilenberg–Moore map.

Proof of Theorem 1.2. Using Proposition 2.6, choose cell $\mathcal{E}$-algebras A, B, C, quasi-isomorphisms $A \to C^*Z$, $B \to C^*X$, $C \to C^*Y$, and relative cell inclusions $A \to B$, $A \to C$ such that the following diagram commutes.

$$\begin{array}{ccccc}
B & \longleftarrow\!\!\!\longleftarrow & A & \longrightarrow & C \\
{\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow \\
C^*X & \longleftarrow & C^*Z & \longrightarrow & C^*Y
\end{array}$$

Let $D = B \amalg_A C$ and consider the map $D \to C^*(X \times_Z Y)$. By Lemma 5.2 and well-known results on the Eilenberg–Moore map (e.g. [27, 3.2]), the map $D \to C^*(X \times_Z Y)$ is a quasi-isomorphism. It follows that the unit of the derived adjunction is represented for $X \times_Z Y$ as the map $X \times_Z Y \to UD$. We have the following commutative diagram.

The assumption that X, Y, and Z are resolvable implies that all four maps between the top and bottom squares are weak equivalences, and we conclude that $X \times_Z Y$ is resolvable. $\qquad\square$

The proof of Lemma 5.2 consists of a comparison of the bar construction in the category of $\mathcal{E}$-algebras with the cochain complex of the cobar construction of simplicial sets. Recall that for maps of simplicial sets $X \to Z$ and $Y \to Z$, the cobar construction $\mathrm{Cobar}^{\bullet}(X, Z, Y)$ is the cosimplicial simplicial set that is given in cosimplicial degree n by

$$\mathrm{Cobar}^n(X, Z, Y) = X \times \underbrace{Z \times \cdots \times Z}_{n \text{ factors}} \times Y$$

with face maps induced by diagonal maps and degeneracies by projections. The cochain complex $C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y))$ is then a simplicial $\mathcal{E}$-algebra. The normalization $N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y)))$ is a differential graded $\bar{\mathbf{F}}_p$-module; there is a canonical map from the usual differential torsion product to the homology

$$\mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y) \to H^*(N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y)))),$$

which is an isomorphism when X, Y, and Z are of finite p-type. On the other hand, considering $X \times_Z Y$ as a cosimplicial simplicial set constant in the cosimplicial direction, the inclusion $X \times_Z Y \to X \times Y$ induces a map of cosimplicial simplicial sets $X \times_Z Y \to \mathrm{Cobar}^{\bullet}(X, Z, Y)$ and therefore a map of differential graded $\bar{\mathbf{F}}_p$-modules

$$N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y))) \to C^*(X \times_Z Y).$$

The composite map

$$\mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y) \to H^*(N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y)))) \to H^*(X \times_Z Y)$$

is by definition the Eilenberg–Moore map; see for example [27].

Proof of Lemma 5.2. Let A, B, and C be as in the proof of Theorem 1.2 above. The various projection maps of $X \times (Z \times \cdots \times Z) \times Y$ induce a map

$$B \amalg (A \amalg \cdots \amalg A) \amalg C \to C^*X \amalg (C^*Z \amalg \cdots \amalg C^*Z) \amalg C^*Y$$
$$\to C^*(X \times (Z \times \cdots \times Z) \times Y).$$

Theorem 3.4 and the Künneth theorem imply that the composite above is a quasi-isomorphism. We obtain a degreewise quasi-isomorphism of simplicial $\mathcal{E}$-algebras

$$\beta_{\bullet}(B, A, C)) \to C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y))$$

and therefore a quasi-isomorphism of differential graded $\bar{\mathbf{F}}_p$-modules

$$N(\beta_{\bullet}(B, A, C)) \to N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y)))$$

that makes the following diagram commute.

$$\begin{array}{ccc}
N(\beta_{\bullet}(B, A, C)) & \xrightarrow{\ \sim\ } & N(C^*(\mathrm{Cobar}^{\bullet}(X, Z, Y))) \\
{\scriptstyle \sim}\downarrow & & \downarrow \\
B \amalg_A C & \xrightarrow{\hspace{3cm}} & C^*(X \times_Z Y)
\end{array}$$

By Theorem 3.5, the left vertical arrow is a quasi-isomorphism. By definition, $H^*(B \amalg_A C)$ is the E_{∞} torsion product $E_{\infty}\mathrm{Tor}^*_{C^*Z}(C^*X, C^*Y)$ and the bottom horizontal map induces on homology the canonical map $E_{\infty}\mathrm{Tor}^{C^*Z}(C^*X, C^*Y) \to C^*X \amalg_{C^*Z} C^*Y$. The lemma now follows. $\qquad\square$

6. A Model for $C^* K(\mathbf{Z}/p\mathbf{Z}, n)$

In this section, we prove Theorem 1.3 that $K(\mathbf{Z}/p\mathbf{Z}, n)$ and $K(\mathbf{Z}_p^\wedge, n)$ are resolvable for $n \geq 1$. We prove the resolvability of $K(\mathbf{Z}/p\mathbf{Z}, n)$ by constructing an explicit cell $\mathcal{E}$-algebra model of $C^* K(\mathbf{Z}/p\mathbf{Z}, n)$ that lets us analyze the unit of the derived adjunction. The case of $\mathbf{Z}_p^\wedge$ follows easily from the case of $\mathbf{Z}/p\mathbf{Z}$ and the work of the previous section.

The construction of our cell model requires the use of the generalized Steenrod operations for E_∞ algebras [18, §I.7], [22]. The theory of [22] gives $\mathbf{F}_p$-linear (but not $\bar{\mathbf{F}}_p$-linear) operations on the homology of an $\mathcal{E}$-algebra. In this section, we only need the operation P^0. This operation preserves degree and performs the p-th power operation on elements in degree zero. Using this fact, naturality, and the fact that the operations commute with "suspension" [22, 3.3], the following observation can be proved by the argument of [22, 8.1].

Proposition 6.1. *For any simplicial set X, the operation P^0 on $H^* X$ induced by the $\mathcal{E}$-algebra structure is the identity on elements of $H^* X$ in the image of $H^*(X; \mathbf{F}_p)$.*

In Section 11, we describe all of the E_∞ algebra Steenrod operations on $H^* X$ in terms of the usual Steenrod operations on $H^*(X; \mathbf{F}_p)$.

For $n \geq 1$, let K_n be a model for $K(\mathbf{Z}/p\mathbf{Z}, n)$ such that the set of n-simplices of K_n is $\mathbf{Z}/p\mathbf{Z}$, e.g. the "minimal" model [21, §23]. Then we have a fundamental cycle k_n of $C^n K_n$ which represents the cohomology class in $H^n K_n$ that is the image of the fundamental cohomology class of $H^n(K(\mathbf{Z}/p\mathbf{Z}, n); \mathbf{F}_p)$. Write $\bar{\mathbf{F}}_p[n]$ for the differential graded $\bar{\mathbf{F}}_p$-module consisting of $\bar{\mathbf{F}}_p$ in degree n and zero in all other degrees, and let $\bar{\mathbf{F}}_p[n] \to C^* K_n$ be the map of differential graded $\bar{\mathbf{F}}_p$-modules that sends $1 \in \bar{\mathbf{F}}_p$ to k_n. Let $\mathbb{E}$ denote the free functor from differential graded $\bar{\mathbf{F}}_p$-modules to $\mathcal{E}$-algebras. We obtain an induced map of $\mathcal{E}$-algebras $a \colon \mathbb{E}\bar{\mathbf{F}}_p[n] \to C^* K_n$ that sends the fundamental class i_n of $\mathbb{E}\bar{\mathbf{F}}_p[n]$ to the fundamental class k_n of $C^* K_n$.

The operation P^0 is not the identity on the fundamental homology class of $\mathbb{E}\bar{\mathbf{F}}_p[n]$. We obtain our cell $\mathcal{E}$-algebra model of $C^* K_n$, by forcing $(1 - P^0)[i_n]$ to be zero as follows. Let p_n be an element of $\mathbb{E}\bar{\mathbf{F}}_p[n]$ that represents $(1 - P^0)[i_n]$. Since $(1 - P^0)[k_n]$ is zero in $H^* K_n$, $a(p_n)$ is a boundary in $C^n K_n$. Choose an element q_n of $C^{n-1} K_n$ such that $dq_n = a(p_n)$. Write $C\bar{\mathbf{F}}_p[n]$ for the cone on $\bar{\mathbf{F}}_p[n]$, the differential graded $\bar{\mathbf{F}}_p$-module that is $\bar{\mathbf{F}}_p$ in dimensions $n - 1$ and n and zero in all other dimensions, with the differential $\bar{\mathbf{F}}_p \to \bar{\mathbf{F}}_p$ the identity. We have a canonical map $q_n \colon C\bar{\mathbf{F}}_p[n] \to C^* K_n$ sending the generators to q_n and $a(p_n)$. We have a canonical map $\bar{\mathbf{F}}_p[n] \to C\bar{\mathbf{F}}_p[n]$, and a map $p_n \colon \bar{\mathbf{F}}_p[n] \to \mathbb{E}\bar{\mathbf{F}}_p[n]$ that sends the generator 1 to the element p_n. The diagram of differential graded k-modules on the left below then commutes.

$$
\begin{array}{ccc}
\bar{\mathbf{F}}_p[n] & \longrightarrow & C\bar{\mathbf{F}}_p[n] \\
{\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle q_n} \\
\mathbb{E}\bar{\mathbf{F}}_p[n] & \xrightarrow{\ a\ } & C^* K_n
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbb{E}\bar{\mathbf{F}}_p[n] & \longrightarrow & \mathbb{E}C\bar{\mathbf{F}}_p[n] \\
{\scriptstyle p_n}\downarrow & & \downarrow{\scriptstyle q_n} \\
\mathbb{E}\bar{\mathbf{F}}_p[n] & \xrightarrow{\ a\ } & C^* K_n
\end{array}
$$

It follows that the diagram of $\mathcal{E}$-algebras on the right above commutes. Let B_n be the $\mathcal{E}$-algebra obtained from the following pushout diagram in the category of

$\mathcal{E}$-algebras.

$$
\begin{array}{ccc}
\mathbb{E}\bar{\mathbf{F}}_p[n] & \longrightarrow & \mathbb{E}C\bar{\mathbf{F}}_p[n] \\
\downarrow{\scriptstyle p_n} & & \downarrow \\
\mathbb{E}\bar{\mathbf{F}}_p[n] & \longrightarrow & B_n
\end{array}
$$

We therefore obtain a map $\alpha\colon B_n \to C^*K_n$. We prove the following theorem in Section 12.

Theorem 6.2. *The map $\alpha\colon B_n \to C^*K_n$ is a quasi-isomorphism.*

Corollary 6.3. K_n *is resolvable.*

Proof. Applying U to the pushout diagram that defines B_n, we obtain the following pullback diagram of simplicial sets.

$$
\begin{array}{ccc}
UB_n & \longrightarrow & U\mathbb{E}C\bar{\mathbf{F}}_p[n] \\
\downarrow & & \downarrow \\
U\mathbb{E}\bar{\mathbf{F}}_p[n] & \xrightarrow[Up_n]{} & U\mathbb{E}\bar{\mathbf{F}}_p[n]
\end{array}
$$

The vertical maps are Kan fibrations since the inclusion $\mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a cofibration. The following two propositions, 6.4 and 6.5, then imply that UB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$.

By Theorem 6.2, the unit of the derived adjunction $K_n \to \mathbf{U}C^*K_n$ is represented by the map $K_n \to UB_n$. Since UB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$, to see that the map is a weak equivalence, we just need to check that the induced map on π_n is an isomorphism. The p distinct homotopy classes of maps from S^n to K_n induce maps $C^*K_n \to C^*S^n$ that differ on homology. It follows that the composite maps $B_n \to C^*S^n$ differ on homology and are therefore different maps in $\bar{h}\mathfrak{E}$. We conclude from the adjunction isomorphism $\bar{h}\mathfrak{E}(B_n, C^*S^n) \cong \mathfrak{Ho}(S^n, UB_n)$ that the map $K_n \to UB_n$ is injective on π_n, and is therefore an isomorphism on π_n. $\qquad\square$

Proposition 6.4. $U\mathbb{E}C\bar{\mathbf{F}}_p[n]$ *is contractible.*

Proof. $\mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a cell $\mathcal{E}$-algebra and the map $\bar{\mathbf{F}}_p \to \mathbb{E}C\bar{\mathbf{F}}_p[n]$ is a quasi-isomorphism, so the map $U\mathbb{E}C\bar{\mathbf{F}}_p[n] \to U\bar{\mathbf{F}}_p = *$ is a weak equivalence of Kan complexes. $\quad\square$

Proposition 6.5. $U\mathbb{E}\bar{\mathbf{F}}_p[n]$ *is a $K(\bar{\mathbf{F}}_p, n)$ and the map Up_n induces on π_n the map $1 - \Phi$, where Φ denotes the Frobenius automorphism of $\bar{\mathbf{F}}_p$.*

Proof. We have canonical isomorphisms

$$U\mathbb{E}\bar{\mathbf{F}}_p[n] = \mathfrak{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*\Delta) \cong \mathfrak{M}(\bar{\mathbf{F}}_p[n], C^*\Delta),$$

where $\mathfrak{M}$ denotes the category of differential graded $\bar{\mathbf{F}}_p$-modules. Thus, $U\mathbb{E}\bar{\mathbf{F}}_p[n]$ is the simplicial set which in dimension m is the set of cocycles in $C^n\Delta[m]$. This is the minimal $K(\bar{\mathbf{F}}_p, n)$ [21, 23.7ff].

The map of simplicial sets $\Delta[n] \to \Delta[n]/\partial\Delta[n]$ induces a bijection

$$\mathfrak{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*\Delta[n]) \cong \mathfrak{M}(\bar{\mathbf{F}}_p[n], C^*\Delta[n]) \cong \mathfrak{M}(\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])).$$

On the other hand, since $C^{n-1}(\Delta[n]/\partial\Delta[n]) = 0$, we have a canonical identification

$$\mathfrak{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])) \cong \mathfrak{M}(\bar{\mathbf{F}}_p[n], C^*(\Delta[n]/\partial\Delta[n])) \cong H^n(\Delta[n]/\partial\Delta[n]).$$

By naturality, the map $H^n(\Delta[n]/\partial\Delta[n]) \to H^n(\Delta[n]/\partial\Delta[n])$ induced by p_n must be $1 - P^0$. Under the isomorphism

$$H^n(\Delta[n]/\partial\Delta[n]) \cong H^n(\Delta[n]/\partial\Delta[n]; \mathbf{F}_p) \otimes \bar{\mathbf{F}}_p \cong \bar{\mathbf{F}}_p,$$

we can identify the operation $1 - P^0$ as $1 - \Phi$ by Proposition 6.1 and the Cartan formula [22, 2.7ff]. $\qquad\square$

We complete the proof of Theorem 1.3 by deducing that $K(\mathbf{Z}_p^\wedge, n)$ is resolvable for $n \geq 1$.

Proof of Theorem 1.3. We see by induction and Theorem 1.2 that $K(\mathbf{Z}/p^m\mathbf{Z}, n)$ is resolvable for $n \geq 1$ by considering the following fiber square

$$\begin{array}{ccc}
K(\mathbf{Z}/p^m\mathbf{Z}, n) & \longrightarrow & PK(\mathbf{Z}/p\mathbf{Z}, n+1) \\
\downarrow & & \downarrow \\
K(\mathbf{Z}/p^{m-1}\mathbf{Z}, n) & \longrightarrow & K(\mathbf{Z}/p\mathbf{Z}, n+1),
\end{array}$$

where $PK(\mathbf{Z}/p\mathbf{Z}, n+1)$ is some contractible simplicial set with a Kan fibration to $K(\mathbf{Z}/p\mathbf{Z}, n+1)$. Since $K(\mathbf{Z}_p^\wedge, n)$ can be constructed as the limit of a tower of Kan fibrations

$$\cdots \to K(\mathbf{Z}/p^m\mathbf{Z}, n) \to \cdots \to K(\mathbf{Z}/p\mathbf{Z}, n),$$

and the natural map $H^*K(\mathbf{Z}_p^\wedge, n) \to \operatorname{Colim} H^*K(\mathbf{Z}/p^m\mathbf{Z}, n)$ is an isomorphism, we conclude from Theorem 1.1 that $K(\mathbf{Z}_p^\wedge, n)$ is resolvable. $\qquad\square$

7. The Image Subcategory

The purpose of this section is to identify the $\mathcal{E}$-algebras that are quasi-isomorphic to the cochain complexes of connected (p-complete) nilpotent simplicial sets of finite p-type. As mentioned in the introduction, the condition characterizing these $\mathcal{E}$-algebras is essentially the $\mathcal{E}$-algebra analogue of the existence of a finite p-type principal Postnikov tower. Since the functors relating $\mathcal{E}$-algebras and simplicial sets are contravariant, towers of principal fibrations of simplicial sets correspond to "complexes" of $\mathcal{E}$-algebras, formed by attaching "cells". We make this precise in the following definitions.

Definition 7.1. An *augmented* $\mathcal{E}$-*algebra* is an $\mathcal{E}$-algebra B together with a map of $\mathcal{E}$-algebras $B \to \bar{\mathbf{F}}_p$ (the *augmentation*). A B-*cell* (CB, B) is an augmented $\mathcal{E}$-algebra CB together with a cofibration of augmented $\mathcal{E}$-algebras $B \to CB$ such that the augmentation $CB \to \bar{\mathbf{F}}_p$ is a quasi-isomorphism. For a map of $\mathcal{E}$-algebras $f\colon B \to A$, we say that $A \amalg_B CB$ is *formed by attaching a B-cell along f*.

The cells we use are built out of the cell $\mathcal{E}$-algebras B_n of Section 6. Since $\bar{\mathbf{F}}_p = \mathbb{E}0$, the maps of differential graded $\bar{\mathbf{F}}_p$-modules $\bar{\mathbf{F}}_p[n] \to 0$ and $C\bar{\mathbf{F}}_p[n] \to 0$ induce a map of $\mathcal{E}$-algebras $B_n \to \bar{\mathbf{F}}_p$ that we take as an augmentation. Let (CB_n, B_n) be a B_n-cell.

Let $B_{1,n} = B_n$. By the Main Theorem, we can choose a map $\beta\colon B_{n+1} \to B_{1,n}$ that sends the fundamental class of $H^{n+1}B_{n+1}$ to the Bockstein class of $H^{n+1}B_{1,n}$. Let $B_{2,n} = B_{1,n} \amalg_{B_{n+1}} CB_{n+1}$. Then $UB_{2,n}$ is a $K(\mathbf{Z}/p^2\mathbf{Z}, n)$ and it follows from Lemma 5.2 that the map $B_{2,n} \to C^*K(\mathbf{Z}/p^2\mathbf{Z}, n)$ is a quasi-isomorphism. By the Main Theorem, we can find a map $\beta_2\colon B_{n+1} \to B_{2,n}$ that sends the fundamental class of $H^{n+1}B_{n+1}$ to the class in $H^{n+1}B_{2,n}$ corresponding to the second Bockstein

in $H^{n+1}K(\mathbf{Z}/p^2Z, n)$. Inductively, we can form $B_{m,n}$ together with a cofibration $B_{m-1,n} \to B_{m,n}$ and a quasi-isomorphism $B_{m,n} \to C^*K(\mathbf{Z}/p^m\mathbf{Z}, n)$ by attaching the B_{n+1}-cell (CB_{n+1}, B_{n+1}) along the map β_{m-1}, and we can choose a map $\beta_m \colon B_{n+1} \to B_{m,n}$ that sends the fundamental class of $H^{n+1}B_{n+1}$ to the class in $H^{n+1}B_{m,n}$ corresponding to the m-th Bockstein in $H^{n+1}K(\mathbf{Z}/p^m\mathbf{Z}, n)$.

Let $B_{\infty,n} = \mathrm{Colim}\, B_{m,n}$. The quasi-isomorphisms $B_{m,n} \to C^*K(\mathbf{Z}/p^m\mathbf{Z}, n)$ induce a quasi-isomorphism $B_{\infty,n} \to C^*K(\mathbf{Z}_p^\wedge, n)$. Let $B_n^\wedge = B_{\infty,n}$, and choose a $B_n^\wedge$-cell $(CB_n^\wedge, B_n^\wedge)$. We can now define the "complexes" that we work with.

Definition 7.2. A B_*-*complex* is an $\mathcal{E}$-algebra $A = \mathrm{Colim}\, A_j$ such that $A_0 = \bar{\mathbf{F}}_p$ and for each $j > 0$ either $A_{j+1} = A_j$ or A_{j+1} is formed from A_j by attaching a B_{m_j, n_j+1}-cell for some $m_j \geq 1$ or $m_j = \infty$, where $\{n_j\}$ is some unbounded non-decreasing sequence of positive numbers. A *special B_*-complex* is a B_*-complex in which for each j, either $m_j = 1$ and the B_{n_j}-cell is (CB_{n_j}, B_{n_j}) or $m_j = \infty$ and the $B_{n_j}^\wedge$-cell is $(CB_{nj}^\wedge, B_{n_j}^\wedge)$.

We allow the case $A_{j+1} = A_j$ in order to permit the possibility that $A = A_j$ for some j. The assumption that the non-decreasing sequence of positive integers $\{n_j\}$ be unbounded is equivalent to the requirement that it repeat a given number at most finitely many times. Thus, a B_*-complex is an $\mathcal{E}$-algebra formed from $\bar{\mathbf{F}}_p$ by inductively attaching $B_{m,n+1}$-cells, for non-decreasing n, finitely many for each $n \geq 1$. The analogy between B_*-complexes and Postnikov towers is made clear by the following theorem and its proof.

Theorem 7.3. *The following conditions on an $\mathcal{E}$-algebra A are equivalent.*

   (i) *A is quasi-isomorphic to C^*X for some connected (p-complete) nilpotent simplicial set of finite p-type.*
  (ii) *A is quasi-isomorphic to a B_*-complex.*
 (iii) *A is quasi-isomorphic to a special B_*-complex.*

Thus, the homotopy category of connected p-complete nilpotent spaces of finite p-type is equivalent to the full subcategory of the homotopy category of $\mathcal{E}$-algebras consisting of the special B_*-complexes.

Proof. We prove (i) $\implies$ (iii) $\implies$ (ii) $\implies$ (i).

Suppose A is quasi-isomorphic to a connected nilpotent simplicial set of finite p-type X; replacing X by its p-completion if necessary, we can assume that X is p-complete. Then X has a principally refined Postnikov tower X_j whose fibers are all $K(\mathbf{Z}/p\mathbf{Z}, n)$'s and $K(\mathbf{Z}_p^\wedge, n)$'s with at most finitely many of each type for each n. Lemma 5.2 allows us to approximate inductively C^*X_j by the j-th stage of a special B_*-complex. In the colimit, we obtain a special B_*-complex and a quasi-isomorphism to C^*X. This proves (i) $\implies$ (iii).

The implication (iii) $\implies$ (ii) is trivial.

For the implication (ii) $\implies$ (i), start with a B_*-complex $A = \mathrm{Colim}\, A_j$. By Proposition 4.4, $UA = \mathrm{Lim}\, UA_j$ is the limit of principal Kan fibrations of Kan complexes. In fact, by the construction of the $B_{m,n}$'s, $UA = \mathrm{Lim}\, UA_j$ is a principally refined Postnikov tower whose fibers are $K(\mathbf{Z}/p^m\mathbf{Z}, n)$'s and $K(\mathbf{Z}_p^\wedge, n)$'s with only finitely many for each n. In particular, UA and each UA_j are connected p-complete nilpotent simplicial sets of finite p-type. Clearly, $\bar{\mathbf{F}}_p = A_0 \to C^*UA_0$ is a quasi-isomorphism. Inductive application of Lemma 5.2 shows that the maps $A_j \to C^*UA_j$ are quasi-isomorphisms, and we conclude that the map $A \to C^*X$ is a quasi-isomorphism. $\qquad\square$

Remark 7.4. We can refine the argument above to see that an $\mathcal{E}$-algebra is quasi-isomorphic to a finite stage special B_*-complex if and only if it is quasi-isomorphic to C^*X for some space X that has a finite stage finite type principally refined Postnikov tower. Likewise, an $\mathcal{E}$-algebra is quasi-isomorphic to a finite stage special B_*-complex with no $B_{n+1}^\wedge$-cells (for all n) if and only if it is equivalent to C^*X for some space X with only finitely many nontrivial homotopy groups, all of which are finite p-groups. If we choose CB_n to be a finite cell $\mathcal{E}$-algebra, such a B_*-complex is then also a finite cell $\mathcal{E}$-algebra.

8. The Characterization Theorem

We defined B_*-complexes in the last section having in mind an analogy with the definition of a principally refined Postnikov tower. We prove the Characterization Theorem in this section having in mind an analogy with the construction of the principal Postnikov tower of a simply connected space. Usually the main tool in the construction of a principal Postnikov tower is the Hurewicz theorem, of which we have no analogue in the category of $\mathcal{E}$-algebras. Instead, we are forced to work with the Eilenberg–Moore spectral sequence of Corollary 3.6 and implicitly a Bockstein spectral sequence. To avoid repeating lengthy hypotheses, we use the following terminology in this section and the next two.

Definition 8.1. We say that an $\mathcal{E}$-algebra A is *1-connected* if $H^n A = 0$ for $n < 0$, $H^0 A = \bar{\mathbf{F}}_p$, and $H^1 A = 0$. We say that A is *finite type* if for each n, $H^n A$ is a finite dimensional $\bar{\mathbf{F}}_p$-module. When A is finite type, we say that A is *spacelike* if for each n, $H^n A$ is generated as an $\bar{\mathbf{F}}_p$-module by fixed points of P^0.

Definition 8.2. Let $f\colon A \to B$ be a map of $\mathcal{E}$-algebras. We say that f is an *n-equivalence* if the induced map $H^i A \to H^i B$ is an isomorphism for $i < n$ and an injection for $i = n$. We say that f is an *n-approximation* if the induced map $H^i A \to H^i B$ is an isomorphism for $i \leq n$.

Most of the work needed for the proof of the Characterization Theorem goes into the following two lemmas. We prove these in the next two sections.

Lemma 8.3. *Let A be a B_*-complex, let B be a 1-connected finite type spacelike $\mathcal{E}$-algebra, and let $f\colon A \to B$ be an n-equivalence of $\mathcal{E}$-algebras for some $n \geq 1$. Then f factors through an n-approximation $f'\colon A' \to B$ such that A' is formed from A by attaching a finite number of $B_{1,n+1}$-cells.*

Lemma 8.4. *Let A be a B_*-complex, let B be a 1-connected finite type spacelike $\mathcal{E}$-algebra, and let $f\colon A \to B$ be an n-approximation of $\mathcal{E}$-algebras for some $n \geq 1$. If f is not an $(n+1)$-equivalence, then f factors through a map $f'\colon A' \to B$ such that $\dim(\ker H^{n+1} f') < \dim(\ker H^{n+1} f)$, and A' is formed from A by attaching a single $B_{m,n+1}$-cell for some $m \geq 1$ or $m = \infty$.*

Proof of the Characterization Theorem. Let B be a 1-connected finite type spacelike $\mathcal{E}$-algebra. Then the map $\bar{\mathbf{F}}_p \to B$ is a 2-equivalence. Alternately applying Lemma 8.3 and applying Lemma 8.4 (multiple times) inductively constructs a B_*-complex A and a quasi-isomorphism $A \to B$. Since in the construction of A, each $B_{m,n+1}$-cell attached has $n \geq 2$, UA is 1-connected. The Characterization Theorem then follows from the Main Theorem and Theorem 7.3. $\qquad\square$

9. Cofiber Sequences and the Proof of Lemma 8.3

This section is devoted to the proof of Lemma 8.3. Thinking in terms of the analogous lemma for spaces, we should be able to attach the $B_{1,n+1}$-cells in the statement along the trivial map. The proof of Lemma 8.3 then reduces to finding maps from $\bar{\mathbf{F}}_p \amalg_{B_{1,n+1}} CB_{1,n+1}$ to B for $B_{1,n+1}$-cells $(CB_{1,n+1}, B_{n+1})$. We find these by working with cofiber sequences; the following definitions are standard.

Definition 9.1. Let A be a cofibrant augmented $\mathcal{E}$-algebra, and let IA be a cylinder object for A. We define the *cone* of A to be the augmented $\mathcal{E}$-algebra $CA = IA \amalg_A \bar{\mathbf{F}}_p$ (via ∂_1). We define the suspension of A to be the $\mathcal{E}$-algebra $SA = \bar{\mathbf{F}}_p \amalg_A CA$ (via ∂_0). For any cofibrant $\mathcal{E}$-algebra B and any map of $\mathcal{E}$-algebras $f\colon A \to B$, the *cofiber* of f is the $\mathcal{E}$-algebra $Cf = B \amalg_A CA$.

Choosing a diagonal lift in the diagram

$$
\begin{array}{ccc}
A \amalg A & \xrightarrow{\ \partial_0\amalg\partial_1\ } & IA \amalg A \amalg IA \\
{\scriptstyle \partial_0+\partial_1}\Big\downarrow & \nearrow^{\partial_1} \quad \Big\downarrow{\scriptstyle \partial_0} \\
IA & \xrightarrow[\ \ \sigma\ \ ]{} & A
\end{array}
$$

induces a map $SA \to SA \amalg SA$ and a map $Cf \to Cf \amalg SA$. Just as in a closed model category, these maps make SA is a co-group object in $\bar{h}\mathfrak{E}$ and give Cf an SA co-action in $\bar{h}\mathfrak{E}$. In particular, for any $\mathcal{E}$-algebra D, $\bar{h}\mathfrak{E}(SA, D)$ is naturally a group and $\bar{h}\mathfrak{E}(Cf, D)$ is naturally a $\bar{h}\mathfrak{E}(SA, D)$-set. It is not hard to see that these structures are independent of the choice of lift used, and that S extends to a functor from $\bar{h}\mathfrak{E}$ to co-group objects in $\bar{h}\mathfrak{E}$. We have a canonical inclusion map $b\colon B \to Cf$, and when in addition B is an augmented $\mathcal{E}$-algebra and f is a map of augmented $\mathcal{E}$-algebras, we obtain a canonical collapse map $c\colon Cf \to SA$. Although usually stated in the "pointed" context, the usual arguments (e.g. [25, §3]) apply in this context to show that the maps $S^q f$, $S^q b$, and $S^q c$ induce a long "exact" sequence of mapping set functors for $q \geq 0$. We state this only in the case we need, avoiding the complications of the $q = 0$ case.

Proposition 9.2. *Let A and B be cofibrant augmented $\mathcal{E}$-algebras, and let $f\colon A \to B$ be a map of augmented $\mathcal{E}$-algebras. For an arbitrary $\mathcal{E}$-algebra D, the sequence*

$$
\bar{h}\mathfrak{E}(S^2 B, D) \xrightarrow{\ S^2 f^*\ } \bar{h}\mathfrak{E}(S^2 A, D) \xrightarrow{\ Sc^*\ } \bar{h}\mathfrak{E}(SCf, D) \xrightarrow{\ Sb^*\ } \bar{h}\mathfrak{E}(SB, D) \xrightarrow{\ Sf^*\ } \bar{h}\mathfrak{E}(SA, D)
$$

is an exact sequence of groups.

For a free $\mathcal{E}$-algebra $\mathbb{E}X$, the diagonal map $X \to X \oplus X$ induces a co-multiplication $\mathbb{E}X \to \mathbb{E}(X \oplus X) \cong \mathbb{E}X \amalg \mathbb{E}X$, making $\mathbb{E}X$ into an abelian co-group object in $\mathfrak{E}$. It is not hard to see that suspension commutes with the free functor. We need this observation only in the simplest case, which we state as the first part of the following proposition.

Proposition 9.3. *For each n, there is a canonical isomorphism of $\mathbb{E}\bar{\mathbf{F}}_p[n]$ and $S\mathbb{E}\bar{\mathbf{F}}_p[n+1]$ as co-group objects in $\bar{h}\mathfrak{E}$. The induced natural transformation*

$$
\sigma\colon H^{n+1}\bar{\mathbf{F}}_p[m+1] = \bar{h}\mathfrak{E}(\bar{\mathbf{F}}_p[n+1], \bar{\mathbf{F}}_p[m+1]) \to
$$
$$
\bar{h}\mathfrak{E}(S\bar{\mathbf{F}}_p[n+1], S\bar{\mathbf{F}}_p[m+1]) \cong \bar{h}\mathfrak{E}(\bar{\mathbf{F}}_p[n], \bar{\mathbf{F}}_p[m]) = H^n\bar{\mathbf{F}}_p[m]
$$

commutes with the homology operations P^s for all s.

Proof. Let $\Delta_*[1] = C_*\Delta[1]$ denote the standard 1-simplex differential graded $\bar{\mathbf{F}}_p$-module. Then $\mathbb{E}(\bar{\mathbf{F}}_p[n+1] \otimes \Delta_*[1])$ is a cylinder object for $\mathbb{E}\bar{\mathbf{F}}_p[n+1]$ and there are canonical isomorphisms

$$\mathbb{E}\bar{\mathbf{F}}_p[n] \cong \mathbb{E}(\bar{\mathbf{F}}_p[n+1] \otimes \Delta_*[1]/(\bar{\mathbf{F}}_p \oplus \bar{\mathbf{F}}_p))$$
$$\cong \bar{\mathbf{F}}_p \amalg_{\mathbb{E}\bar{\mathbf{F}}_p[n+1]} \mathbb{E}(\bar{\mathbf{F}}_p[n+1] \otimes \Delta_*[1]) \amalg_{\mathbb{E}\bar{\mathbf{F}}_p[n+1]} \bar{\mathbf{F}}_p = S\mathbb{E}\bar{\mathbf{F}}_p[n+1].$$

It is easy to check that this is an isomorphism of abelian co-groups in $\mathfrak{E}$. The fact that the operations P^s commute with σ for $A = \mathbb{E}\bar{\mathbf{F}}_p[n]$ follows from looking at the sequence

$$\mathbb{E}\bar{\mathbf{F}}_p[m] \to \mathbb{E}C\bar{\mathbf{F}}_p[m+1] \to \mathbb{E}\bar{\mathbf{F}}_p[m+1],$$

and applying [22, 3.3]. $\qquad\square$

The following proposition is an easy consequence of the previous proposition or of the Main Theorem and Theorem 7.3.

Proposition 9.4. *For $n > 0$, there is an isomorphism (in $\bar{h}\mathfrak{E}$), $B_n \simeq SB_{n+1}$.*

The augmented $\mathcal{E}$-algebra B_{n+1} is the cofiber of the map $p_{n+1}\colon \mathbb{E}\bar{\mathbf{F}}_p[n+1] \to \mathbb{E}\bar{\mathbf{F}}_p[n+1]$. Proposition 9.3 and the naturality of the operation $1 - P^0$ identify the exact sequence of Proposition 9.2 for B_{n+1} as the sequence

$$H^{n-1}D \xrightarrow{1-P^0} H^{n-1}D \to \bar{h}\mathfrak{E}(SB_{n+1}, D) \to H^n D \xrightarrow{1-P^0} H^n D.$$

This allows us to identify the mapping group $\bar{h}\mathfrak{E}(SB_{n+1}, D)$ when we understand the operation P^0 on H^*D. In particular, when $1 - P^0$ is surjective on $H^{n-1}D$, we obtain an isomorphism (of groups) between $\bar{h}\mathfrak{E}(SB_{n+1}, D)$ and the kernel of $1 - P^0$ on $H^n D$. Although we use this principally for SB_{n+1}, the identification is most naturally stated via Proposition 9.4 in terms of B_n.

Proposition 9.5. *Let $n > 0$ and let D be an $\mathcal{E}$-algebra. If the operation $1 - P^0$ is surjective on $H^{n-1}D$, then the association of a map $B_n \to D$ to the element of $H^n D$ given by image of the fundamental class induces a bijection between $\bar{h}\mathfrak{E}(B_n, D)$ and the fixed points of P^0 in $H^n D$.*

Proof of Lemma 8.3. Choose a basis $\langle a_1, \ldots, a_q \rangle$ for the fixed points of P^0 in $H^n A$ and expand this to a basis $\langle a_1, \ldots, a_q, b_1, \ldots, b_r \rangle$ of the fixed points of P^0 in $H^n B$. By the previous two propositions, we can find maps $f_j\colon SB_{n+1} \to B$ such that the composite $B_n \to SB_{n+1} \to B$ sends the fundamental class of B_n to b_j. Then

$$A' = A \amalg SB_{n+1} \amalg \cdots \amalg SB_{n+1}$$

is formed from A by attaching a finite number of $B_{1,n+1}$-cells and the map

$$f' = f + f_1 + \cdots + f_r\colon A \amalg SB_{n+1} \amalg \cdots \amalg SB_{n+1} \to B$$

is easily seen by Theorem 3.4 to be an n-approximation. $\qquad\square$

10. Proof of Lemma 8.4

The definition of B_*-complex in Section 7 and the statement of Lemma 8.4 leave us complete freedom in choosing the cells $(CB_{m,n+1}, B_{m,n+1})$. We take advantage of this freedom here in the proof of Lemma 8.4. In particular, we choose specific cells $(C_{m,n+1}, B_{m,n+1})$ which admit cofibrations $C_{m,n+1} \to C_{m+1,n+1}$ under the maps $B_{m,n+1} \to B_{m+1,n+1}$ such that for $C_{\infty,n+1} = \operatorname{Colim}_m C_{m,n+1}$, $(C_{\infty,n+1}, B_{\infty,n+1})$ is a $B_{\infty,n+1}$-cell. Regarding these cells, we have the following lemma, the proof of which occupies most of this section.

Lemma 10.1. *Let $f\colon A \to B$ be as in Lemma 8.4, and let $x \in \ker(H^{n+1}f)$ be a non-zero fixed point of P^0. Let $g\colon B_{m,n+1} \to A$ be a map that sends the fundamental class of $H^{n+1}B_{m,n+1}$ to x for some $1 \le m < \infty$, and let*

$$f_g\colon A_g = A \amalg_{B_{m,n+1}} C_{m,n+1} \to B$$

be some map extending f. If $\dim(\ker H^{n+1}f_g) = \dim(\ker H^{n+1}f)$, then g extends to a map $h\colon B_{m+1,n+1} \to A$ and f_g extends to a map

$$f_h\colon A_h = A \amalg_{B_{m+1,n+1}} C_{m+1,n+1} \to B.$$

Proof of Lemma 8.4 from Lemma 10.1. Choose a map $g_1\colon B_{1,n+1} \to A$ sending the fundamental class to x. By Proposition 9.5, the composite $f \circ g_1\colon B_{1,n+1} \to B$ is homotopic to the trivial map (the augmentation of $B_{1,n+1}$ composed with the unit of B). It follows that $f \circ g_1$ extends to a map

$$A \amalg_{B_{1,n+1}} IB_{1,n+1} \amalg_{B_{1,n+1}} \bar{\mathbf{F}}_p \to B$$

for a cylinder object $IB_{1,n+1}$. By choosing a diagonal lift in the diagram

$$
\begin{array}{ccc}
B_{1,n+1} & \longrightarrow & IB_{1,n+1} \amalg_{B_{1,n+1}} \bar{\mathbf{F}}_p \\
\downarrow & \nearrow & \downarrow{\scriptstyle \sim} \\
C_{1,n+1} & \longrightarrow & \bar{\mathbf{F}}_p
\end{array}
$$

and composing with the map above, we obtain a map

$$f_{g_1}\colon A_{g_1} = A \amalg_{B_{1,n+1}} C_{1,n+1} \to B$$

extending f. Inductively, for as long as possible, choose maps $g_{m+1}\colon B_{m+1,n+1} \to A$ and $f_{g_{m+1}}\colon A_{g_{m+1}} \to B$ extending the maps g_m and f_{g_m}. If g_m, f_{g_m} cannot be extended to some g_{m+1}, f_{m+1}, then $A' = A_{g_m}$, $f' = f_{g_m}$ satisfies the conclusion of Lemma 8.3 by Lemma 10.1. Otherwise, let $g = \operatorname{Colim} g_m$, let $A' = A \amalg_{B_{\infty,n+1}} C_{\infty,n+1}$, and let $f' = \operatorname{Colim} f_{g_m}$,

$$f'\colon A' = A \amalg_{B_{\infty,n+1}} C_{\infty,n+1} = A \amalg_{B_{\infty,n+1}} \operatorname{Colim} C_{m,n+1} \to B.$$

We have assumed that the map $B_{\infty,n+1} \to C_{\infty,n+1}$ is a cofibration, and so we can use the Eilenberg–Moore spectral sequence of Corollary 3.6 to calculate the effect on homology of the inclusion of A in A': It is an isomorphism on H^i for $i < n+1$ and is the quotient by the submodule generated by x on H^{n+1}. It follows that the image of $H^{n+1}A'$ in $H^{n+1}B$ coincides with the image of $H^{n+1}A$, but the dimension of $H^{n+1}A'$ is one less than the dimension of $H^{n+1}A$, and so the dimension of $\ker H^{n+1}f'$ is one less than the dimension of $\ker H^{n+1}f$. $\qquad\square$

Recall that the augmented $\mathcal{E}$-algebras $B_{m,n+1}$ for $m > 1$ are constructed inductively by attaching a B_{n+2}-cell along a map $\beta_{m-1}\colon B_{n+2} \to B_{m-1,n+1}$ representing the $(m-1)$-st Bockstein. We did not specify how to choose the B_{n+2}-cell in Section 7; we now assume that CB_{n+2} is a cone $IB_{n+2} \amalg_{B_{n+2}} \bar{\mathbf{F}}_p$ for some cylinder object IB_{n+2}. Let $S = CB_{n+2} \amalg_{B_{n+2}} CB_{n+2}$. For later convenience, choose and fix a quasi-isomorphism $q\colon B_{n+1} \to SB_{n+2} \to S$. Let IS be a cylinder object for S, and let CS be the cone $IS \amalg_S \bar{\mathbf{F}}_p$.

We choose the cell $(C_{1,n+1}, B_{1,n+1})$ arbitrarily and for $m > 1$ we choose the cells $(C_{m,n+1}, B_{m,n+1})$ inductively as follows. Choose a map $CB_{n+2} \to C_{m,n+1}$ so that

the restriction $B_{n+2} \to C_{m,n+1}$ factors through the map $\beta_m\colon B_{n+2} \to B_{m,n+1}$, i.e. find a diagonal lift in the following diagram.

$$\begin{array}{ccc} B_{n+2} & \xrightarrow{\beta_m} B_{m,n+1} \rightarrowtail & C_{m,n+1} \\ \downarrow & & \downarrow{\scriptstyle\sim} \\ CB_{n+2} & \longrightarrow & \bar{\mathbf{F}}_p \end{array}$$

Consider the map

$$\gamma_n\colon S = CB_{n+2} \amalg_{B_{n+2}} CB_{n+2} \to CB_{n+2} \amalg_{B_{n+2}} C_{m,n+1}$$

and let $C_{m+1,n+1} = C\gamma_n$ be the cofiber. Since γ_n is a quasi-isomorphism by Theorem 3.2 and an augmented map, $C_{m+1,n+1}$ is augmented and its augmentation is a quasi-isomorphism. Note that $B_{m+1,n+1} \amalg_{B_{m,n+1}} C_{m,n+1} = CB_{n+2} \amalg_{B_{n+2}} C_{m,n+1}$, and so we obtain cofibrations $B_{m+1,n+1} \to C_{m+1,n+1}$ and $C_{m,n+1} \to C_{m+1,n+1}$ under $B_{m,n+1}$. We let $C_{\infty,n+1} = \operatorname{Colim} C_{m,n+1}$ as required. Since by construction the maps $B_{m+1,n+1} \amalg_{B_{m,n+1}} C_{m,n+1} \to C_{m+1,n+1}$ are cofibrations for all m, we have that the map $B_{\infty,n+1} = \operatorname{Colim} B_{m,n+1} \to C_{\infty,m+1}$ is a cofibration.

Proof of Lemma 10.1. Let g, A_g, and f_g be as in the statement and suppose that $\dim(\ker H^{n+1} f_g) = \dim(\ker H^{n+1} f)$. Looking at the Eilenberg–Moore spectral sequence of Corollary 3.6 that calculates the homology of A_g, we see that the composite map $g \circ \beta_m\colon B_{n+2} \to A$ must send the fundamental class to zero, and the image of $H^{n+1} f_g$ must be the same as the image of $H^{n+1} f$, since otherwise we would have $\dim(\ker H^{n+1} f_g) = \dim(\ker H^{n+1} f) - 1$. Since $g \circ \beta_m$ sends the fundamental class to zero, it follows from Proposition 9.5 that we can extend $g \circ \beta_m$ to a map $b\colon CB_{n+2} \to A$. Using the map $CB_{n+2} \to C_{m,n+1}$ in the construction of $C_{m,n+1}$, we obtain a map $a\colon S \to A_g$; let y be the image in $H^{n+1} A_g$ of the fundamental class of $H^{n+1} B_{n+1}$ under the map $a \circ q\colon B_{n+1} \to A_g$.

We can change the choice of map b by "adding" a map $c\colon SB_{n+2} \to A$ via the map $CB_{n+2} \to CB_{n+2} \amalg SB_{n+2}$ under B_{n+2}. Doing so changes y by adding the image of the fundamental class in $H^{n+1} A_g$ of c composed with $A \to A_g$. In particular, since the image of $H^{n+1} A$ in $H^{n+1} B$ coincides with the image of $H^{n+1} A_g$ in $H^{n+1} B$, we can choose b so that y is in the kernel of $H^{n+1} f_g$. Let $h\colon B_{m+1,n+1} \to A$ be the map induced by g together with such a choice of b. Then the composite map

$$S \to B_{m+1,n+1} \amalg_{B_{m,n+1}} C_{m,n+1} \to A_g \to B$$

sends the fundamental class of $H^{n+1} S$ to zero, and so this map extends to a map $CS \to B$. This specifies a map

$$f_h\colon A \amalg_{B_{m+1,n+1}} C_{m+1,n+1} \cong (A_g \amalg_{B_{m,n+1}} B_{m+1,n+1}) \amalg_S CS \to B$$

that extends f_g. $\qquad\square$

11. The Algebra of Generalized Steenrod Operations

The key to the proof of Theorem 6.2 is a study of the algebra of all generalized Steenrod operations of [22]. Precisely, let $\mathfrak{B}$ be the free associative $\mathbf{F}_p$-algebra generated by the P^s and (if $p > 2$) the βP^s [22, 2.2,§5] for all $s \in \mathbf{Z}$ modulo the two-sided ideal consisting of those operations that are zero on all "Adem objects" [22, 4.1] of "$\mathfrak{C}(p,\infty)$" of [22, 2.1]. The Adem objects of $\mathfrak{C}(p,\infty)$ include all E_∞ algebras over any E_∞ k-operad for any commutative $\mathbf{F}_p$-algebra k. In this section,

we prove Theorem 1.4 and provide the main results needed in the next section to prove Theorem 6.2. We use the standard arguments effective in studying the Steenrod and Dyer-Lashoff algebras to analyze the structure of $\mathfrak{B}$.

Definition 11.1. We define length, admissibility, and excess as follows

(i) $p = 2$: Consider sequences $I = (s_1, \ldots, s_k)$. The sequence I determines the operation $P^I = P^{s_1} \cdots P^{s_k}$. We define the *length* of I to be k. Say that I is *admissible* if $s_j \geq 2s_{j+1}$ for $1 \leq j < k$. We define the *excess* of I by

$$e(I) = s_k + \sum_{j=1}^{k-1}(s_j - 2s_{j+1}) = s_1 - \sum_{j=2}^{k} s_j$$

(ii) $p > 2$: Consider sequences $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ such that ϵ_i is 0 or 1. The sequence I determines the operation $P^I = \beta^{\epsilon_1} P^{s_1} \cdots \beta^{\epsilon_k} P^{s_k}$, where $\beta^0 P^s$ means P^s and $\beta^1 P^s$ means βP^s. We define the *length* of I to be k. Say that I is *admissible* if $s_j \geq ps_{j+1} + \epsilon_{j+1}$. We define the *excess* of I by

$$e(I) = 2s_k + \epsilon_1 + \sum_{j=1}^{k-1}(2s_j - 2ps_{j+1} - \epsilon_{j+1}) = 2s_1 + \epsilon_1 - \sum_{j=2}^{k}(2s_j(p-1) + \epsilon_j)$$

In either case, by convention, the empty sequence determines the identity operation, has length zero, is admissible, and has excess $-\infty$. If I and J are sequences, we denote by (I, J) their concatenation.

Proposition 11.2. *The set $\{P^I \mid I$ is admissible$\}$ is a basis of the underlying $\mathbf{F}_p$-module of $\mathfrak{B}$.*

Proof. It follows from the Adem relations [22, 4.7] that the set generates $\mathfrak{B}$ as a $\mathbf{F}_p$-module. Linear independence follows by examination of the action on $H^*(\mathbb{G}\mathbf{F}_p[n])$ as n gets large, where $\mathbb{G}$ denotes the free $\mathcal{G}$-algebra functor for some E_∞ $\mathbf{F}_p$-operad $\mathcal{G}$. This follows for example from [23, 2.2 or 2.6]. $\qquad\square$

Proposition 11.3. *If $s > 0$ then $P^{-s}(P^0)^s = 0$ and (if $p > 2$) $\beta P^{-s}(P^0)^s = 0$.*

Proof. Here $P^{-s}(P^0)^s$ and $\beta P^{-s}(P^0)^s$ are meant to denote P^{-s} or βP^{-s} composed with s factors of P^0. The Adem relations [22, 4.7] for $\beta^\epsilon P^{-s} P^0$ when $s > 0$ are given by

$$\beta^\epsilon P^{-s} P^0 = \sum_{i=-\infty}^{\infty} (-1)^{-s-i}\binom{(p-1)i-1}{s-i-1}\beta^\epsilon P^{-(s-i)} P^{-i},$$

where we understand $\epsilon = 0$ when $p = 2$, and we understand the binomial coefficient $\binom{n}{k} = 0$ when $k < 0$ or $k > n$. The binomial coefficient in the expression above therefore can be non-zero only when $s/p \leq i \leq s - 1$. Then $P^{-1}P^0 = 0$ and $\beta P^{-1} P^0 = 0$ since for these the coefficients are zero for all values of i. Assume by induction that $P^{-t}(P^0)^t = 0$ for all t such that $1 \leq t < s$; we see that $P^{-s}P^0$ and $\beta P^{-s}P^0$ are both in the left ideal generated by $\{P^{-t} \mid 1 \leq t < s\}$ and hence by the inductive hypothesis are annihilated by $(P^0)^{s-1}$; therefore, $P^{-s}(P^0)^s = 0$ and $\beta P^{-s}(P^0)^s = 0$. $\qquad\square$

We can now prove the first half of Theorem 1.4.

Proposition 11.4. *The left ideal of $\mathfrak{B}$ generated by $(1 - P^0)$ is a two-sided ideal.*

Proof. By Proposition 11.2, it suffices to show that for every admissible sequence I, $(1 - P^0)P^I$ is an element of the left ideal generated by $(1 - P^0)$. Let $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ be an admissible sequence (where if $p = 2$ each $\epsilon_j = 0$ and we think of this sequence as $(s_1, \ldots, s_k)$). If $s_k < 0$ then by the previous proposition

$$P^I = P^I(1 - (P^0)^{-s_k}) = P^I(1 + P^0 + \cdots + (P^0)^{-s_k-1})(1 - P^0)$$

is in the ideal and hence $(1 - P^0)P^I$ is as well. We can therefore assume that $s_k \geq 0$, and it follows from admissibility that $s_j \geq 0$ for all j. We proceed by induction on k, the length of I.

The statement is trivial for $k = 0$ (the empty sequence); now assume by induction that the statement holds for all sequences J of length less than k. We can write I as the concatenation $((\epsilon, s), J)$ for some sequence J of length $k - 1$. If $s = 0$, the Adem relation for $P^0 \beta P^0$ is $P^0 \beta P^0 = \beta P^0 P^0$, and we see that

$$(1 - P^0)P^I = (1 - P^0)\beta^\epsilon P^0 P^J = \beta^\epsilon P^0 (1 - P^0)P^J$$

is in the ideal by induction. For $s > 0$, the Adem relation for $P^0 P^s$ takes the form

$$P^0 P^s = \sum (-1)^i \binom{(p-1)(s-i)-1}{(p-1)s+i-1} P^{s-i} P^i.$$

When $i > 0$ the binomial coefficient is zero, when $i = 0$ we get the term $P^s P^0$, and when $i < 0$ we get terms of the form binomial coefficient times $P^{s-i}P^i$ that we know from the work above are in the ideal; therefore, we can write $P^0 P^s = P^s P^0 + \alpha(1 - P^0)$ for some α. An entirely similar argument shows that $P^0 \beta P^s$ can also be written $P^0 \beta P^s = \beta P^s P^0 + \alpha(1 - P^0)$ for some α. It follows that

$$(1 - P^0)P^I = (1 - P^0)\beta^\epsilon P^s P^J = (\beta^\epsilon P^s + \alpha)(1 - P^0)P^J$$

is in the ideal by induction, and this completes the argument. $\qquad\square$

For the other half of Theorem 1.4, we need a canonical map from $\mathfrak{B}$ to the Steenrod algebra $\mathfrak{A}$. It can be shown [22, 10.5] that the Steenrod operations on the cohomology of a simplicial set arise from the action of $\mathfrak{B}$ from a $\mathfrak{C}(p, \infty)$ structure on the cochains with coefficients in $\mathbf{F}_p$. However, it is important for our purposes to relate the action of $\mathfrak{B}$ obtained from the $\mathcal{E}$-algebra structure to the Steenrod algebra. The previous proposition implies that if x is an element of a left $\mathfrak{B}$-module that is fixed by P^0, then the submodule $\mathfrak{B}x$ generated by x is fixed by P^0. It follows from this observation and Proposition 6.1 that for any simplicial set X, the $\mathbf{F}_p$-submodule $H^*(X; \mathbf{F}_p)$ of $H^* X$ is a $\mathfrak{B}$-submodule. It then follows from the axioms that uniquely identify the Steenrod operations that the action of P^s on $H^*(X; \mathbf{F}_p)$ coincides with the Steenrod operation of the same name. Furthermore, by looking at $C^* K_n$, it is possible to identify βP^s as the composite of the operation P^s and the Bockstein. Thus, we understand the canonical map $\mathfrak{B} \to \mathfrak{A}$ as follows.

Proposition 11.5. *Let k be a commutative $\mathbf{F}_p$-algebra and let $\mathcal{G}$ be an E_∞ operad of differential graded k-algebras. For any $\mathcal{G}$-algebra structure on $C^*(X; k)$ that is natural in the simplicial set X, the operations P^s and (for $p > 2$) βP^s act on an element of $H^*(X; \mathbf{F}_p) \subset H^*(X; k)$ by the Steenrod operations of the same name.*

Remark 11.6. The previous proposition and the Cartan formula [22, 2.2] allow the identification the operations on $H^* X$ in terms of the Steenrod operations. When X is of finite p-type, $H^* X \cong H^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p$, and so every element of $H^n X$ can

be written as a linear combination $a_1 x_1 + \cdots + a_m x_m$ for some elements $x_1, \ldots, x_m$ in $H^*(X; \mathbf{F}_p)$ and $a_1, \ldots, a_m$ in $\bar{\mathbf{F}}_p$. Then

$$\beta^\epsilon P^s (a_1 x_1 + \cdots + a_m x_m) = \Phi(a_1) \beta^\epsilon P^s x_1 + \cdots + \Phi(a_m) \beta^\epsilon P^s x_m,$$

where Φ denotes the Frobenius automorphism of $\bar{\mathbf{F}}_p$. In general $H^* X$ is the limit of $H^* X_\alpha$ where X_α ranges over the finite subcomplexes of X.

Proof of Theorem 1.4. The map $\mathfrak{B} \to \mathfrak{A}$ is clearly surjective. Since the relation $P^0 = 1$ holds in $\mathfrak{A}$, the map $\mathfrak{B} \to \mathfrak{A}$ factors through the ring $\mathfrak{B}/(1 - P^0)$ and certainly remains surjective. To see that it is injective, note that Propositions 11.2 and 11.3 imply that $\mathfrak{B}/(1 - P^0)$ is generated as an $\mathbf{F}_p$-module by those P^I for admissible sequences $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ such that $s_j > 0$ for each j; the image of these elements in $\mathfrak{A}$ form an $\mathbf{F}_p$-module basis, and in particular are linearly independent. $\qquad\square$

12. Unstable Modules over $\mathfrak{B}$

In this section, we prove Theorem 6.2. The proof is based on a comparison of free unstable modules over $\mathfrak{A}$ with free unstable modules over $\mathfrak{B}$.

Definition 12.1. A module M over $\mathfrak{B}$ is *unstable* if for every $x \in M$, $P^I x = 0$ for any I with excess $e(I)$ greater than the degree of x.

Observe that a module over the Steenrod algebra is unstable if and only if it is unstable as a module over $\mathfrak{B}$. Also observe that if $M = H^* A$ for an object of $\mathfrak{C}(p, \infty)$, e.g. an E_∞ k-algebra A for a commutative $\mathbf{F}_p$-algebra k, then M is unstable [22, 5.(3)–(4)].

We denote by $\mathcal{A}_n^{\mathrm{un}}$ and $\mathcal{B}_n^{\mathrm{un}}$ the free unstable $\mathfrak{A}$ and $\mathfrak{B}$-modules on one generator in degree n; we denote these generators as a_n and b_n respectively. The following proposition generalizes the standard basis theorem for $\mathcal{A}_n^{\mathrm{un}}$ and follows easily from Proposition 11.2.

Proposition 12.2. *The set $\{P^I b_n \mid I \text{ is admissible and } e(I) \leq n\}$ is an $\mathbf{F}_p$-module basis of $\mathcal{B}_n^{\mathrm{un}}$.*

We can identify the $\mathfrak{B}$-modules $H^* \mathbb{E} \bar{\mathbf{F}}_p[n]$ in terms of free unstable $\mathfrak{B}$-modules. For this, we need the following terminology.

Definition 12.3. A *restricted $\mathbf{F}_p$-module* is a graded $\mathbf{F}_p$-module M together with an additive endomorphism (the *restriction*) that multiplies degrees by p, i.e. takes elements of degree n to elements of degree np. The *enveloping algebra* of M is the free graded commutative $\mathbf{F}_p$-algebra on M modulo the relation that the restriction is the p-th power operation.

An unstable $\mathfrak{B}$-module is naturally a restricted $\mathbf{F}_p$-module by neglect of structure; its enveloping algebra inherits an unstable $\mathfrak{B}$-module structure via the Cartan formula.

Proposition 12.4. *If $\mathcal{G}$ is an E_∞ $\mathbf{F}_p$-operad, then $H^* \mathbb{G} \mathbf{F}_p[n]$ is the enveloping algebra of $\mathcal{B}_n^{\mathrm{un}}$. $H^* \mathbb{E} \bar{\mathbf{F}}_p[n]$ is the extended $\bar{\mathbf{F}}_p$-algebra on the enveloping algebra of $\mathcal{B}_n^{\mathrm{un}}$.*

Proof. The argument of [23, 2.6] applies to prove the first statement. The second statement follows from the first. $\qquad\square$

To use this in the proof of Theorem 6.2, we need to understand the map on homology induced by the map $\mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}\bar{\mathbf{F}}_p[n]$ in the construction of B_n. In the following proposition, $(1 - P^0)$ denotes the map of $\mathfrak{B}$-modules $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ that sends the generator to $1 - P^0$ times the generator.

Proposition 12.5. *For $n \geq 1$, the sequence*

$$0 \to \mathcal{B}_n^{\mathrm{un}} \xrightarrow{(1-P^0)} \mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}} \to 0$$

is exact and split in the category of restricted $\mathbf{F}_p$-modules.

Proof. The fact that $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$ is onto is clear since it is a map of $\mathfrak{B}$-modules and $\mathcal{A}_n^{\mathrm{un}}$ is generated as a $\mathfrak{B}$-module by the image of the generator of $\mathcal{B}_n^{\mathrm{un}}$. Similarly, exactness in the middle is clear from examination of the $\mathbf{F}_p$-module bases of $\mathcal{B}_n^{\mathrm{un}}$ and $\mathcal{A}_n^{\mathrm{un}}$. Thus, it remains to show that the map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ is injective and split in the category of restricted $\mathbf{F}_p$-modules.

We proceed by writing an explicit splitting $f \colon \mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$ in the category of restricted $\mathbf{F}_p$-modules as follows. It suffices to specify f on $P^I b_n$ for each admissible $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ with $e(I) \leq n$. If $s_k < 0$, choose

$$f(P^I b_n) = P^I(1 + P^0 + (P^0)^2 + \cdots)b_n.$$

This is well-defined by Proposition 11.3. If $\epsilon_k + s_k > 0$ or if I is empty, then choose $f(P^I b_n)$ to be zero. Let $n(I)$ denote the largest number n such that the subsequence $(\epsilon_{k-n+1}, s_{k-n+1}, \ldots, \epsilon_k, s_k)$ is all zeros; if $\epsilon_k \neq 0$ or $s_k \neq 0$ then $n(I)=0$. We have chosen $f(P^I)$ when $n(I)$ is zero; when $n(I) > 0$, writing I as the concatenation $(J, (0, 0))$ where $n(J) = n(I) - 1$, we inductively choose

$$f(P^I b_n) = -P^J b_n + f(P^J b_n).$$

It is immediate from the construction and the fact that p-th power operations do not raise the excess above n that f is a map of restricted $\mathbf{F}_p$-modules. We need to verify that the composite of f and the map $(1 - P^0)$ is the identity. Let us denote by M_- the $\mathbf{F}_p$-submodule of $\mathcal{B}_n^{\mathrm{un}}$ generated by $P^I b_n$ for those $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ with $s_k < 0$; let us denote by M_+ the $\mathbf{F}_p$-submodule generated by $P^I b_n$ for those $I = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ with $s_k \geq 0$ or $k = 0$; clearly $\mathcal{B}_n^{\mathrm{un}}$ is the internal direct sum $M_- \oplus M_+$. The map $(1 - P^0)$ sends $P^I b_n$ to $P^I(1 - P^0)b_n$; it clearly sends M_+ into M_+, and it follows from Propositions 11.3 and 12.2 that it sends M_- to M_-. Thus, it suffices to check that the composite is the identity on each of these submodules.

On M_-, f sends αb_n to $\alpha(1 + P^0 + (P^0)^2 + \cdots)b_n$. It follows that the composite sends αb_n to $\alpha(1 - P^0)(1 + P^0 + (P^0)^2 + \cdots)b_n = \alpha b_n$, and so the composite is the identity on M_-.

To see that the composite is the identity on M_+, it suffices to check it on a standard basis element, $P^J b_n$, where $J = (\epsilon_1, s_1, \ldots, \epsilon_k, s_k)$ is an admissible sequence with $e(J) \leq n$ and $s_k \geq 0$. Write I for the concatenation $(J, (0, 0))$. Observe that I is admissible and $e(I) = e(J) \leq n$, so

$$f(P^J(1 - P^0)b_n) = f(P^J b_n) - f(P^I b_n) = f(P^J b_n) - (-P^J b_n + f(P^J b_n)) = P^J b_n.$$

It follows that the composite is the identity. $\square$

Proof of Theorem 6.2. Let $\bar{\mathbb{V}}$ denote the composite of the enveloping algebra functor and the functor $(-) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p$. Since this is the free functor from restricted $\mathbf{F}_p$-modules to graded commutative $\bar{\mathbf{F}}_p$-algebras, it preserves colimits. To avoid confusion, let us denote the (isomorphic) image of $\mathcal{B}_n^{\mathrm{un}}$ in $\mathcal{B}_n^{\mathrm{un}}$ under the map

$(1 - P^0)$ discussed above by I_n; by Proposition 12.5, $\mathcal{B}_n^{\mathrm{un}}$ is isomorphic as a restricted $\mathbf{F}_p$-module to the direct sum $I_n \oplus \mathcal{A}_n^{\mathrm{un}}$, and it follows that the ring $\bar{\mathbb{V}}\mathcal{B}^{\mathrm{un}}$ is isomorphic to the ring $\bar{\mathbb{V}}I_n \otimes \bar{\mathbb{V}}\mathcal{A}_n^{\mathrm{un}}$. We therefore obtain isomorphisms

$$\mathrm{Tor}_{\bar{\mathbb{V}}I_n}^{*,*}(\bar{\mathbf{F}}_p, \bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}}) \cong \bar{\mathbf{F}}_p \otimes_{\bar{\mathbb{V}}I_n} \bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}} \cong \bar{\mathbb{V}}\mathcal{A}_n^{\mathrm{un}}$$

where the first map is the projection from the torsion product to the tensor product and the second map is induced by $\bar{\mathbb{V}}$ applied to the quotient map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$.

On the other hand, Proposition 12.4 identifies $H^*\mathbb{E}\bar{\mathbf{F}}_p[n]$ as $\bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}}$. It is well-known that $H^*K_n = H^*K(\mathbf{Z}/p\mathbf{Z}, n)$ can be identified with $\bar{\mathbb{V}}\mathcal{A}_n^{\mathrm{un}}$. We see from Proposition 11.5 that the map a from $\mathbb{E}\bar{\mathbf{F}}_p[n]$ to C^*K_n in the construction of B_n induces on homology groups the map $\bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}} \to \bar{\mathbb{V}}\mathcal{A}_n^{\mathrm{un}}$ obtained by applying $\bar{\mathbb{V}}$ to the quotient map $\mathcal{B}_n^{\mathrm{un}} \to \mathcal{A}_n^{\mathrm{un}}$. Likewise, the map $p_n \colon \mathbb{E}\bar{\mathbf{F}}_p[n] \to \mathbb{E}\bar{\mathbf{F}}_p[n]$ in the construction of B_n induces the map $\bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}} \to \bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}}$ obtained by applying $\bar{\mathbb{V}}$ to the map $(1 - P^0) \colon \mathcal{B}_n^{\mathrm{un}} \to \mathcal{B}_n^{\mathrm{un}}$; in other words, we can identify the map induced by p_n on homology as the inclusion $\bar{\mathbb{V}}I_n \to \bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}}$. Corollary 3.6 provides a spectral sequence that calculates the homology groups of the pushout B_n and that has as its E^2 term $\mathrm{Tor}_{\bar{\mathbb{V}}I_n}^{*,*}(\bar{\mathbf{F}}_p, \bar{\mathbb{V}}\mathcal{B}_n^{\mathrm{un}})$. From the discussion of the last paragraph, we see that this spectral sequence degenerates at E^2 with no extension problems and that the map from B_n to C^*K_n is a quasi-isomorphism. $\qquad\square$

13. Cell Algebras over Flat Operads

The purpose of this section is to present the proof of Theorem 2.8. The argument is no more complicated to describe in full generality, and so we present it here this way. This should cause no confusion since the discussion in this section relates to the main lines of argument in this paper only through Theorem 2.8. In this section, k denotes a fixed but arbitrary commutative ground ring, and we consider the following class of operads of differential graded k-modules.

Definition 13.1. For a ring R, we say that a differential graded right R-module M is *right R-flat* if the functor $M \otimes_R (-)$ preserves quasi-isomorphisms in the category of differential graded left R-modules, or equivalently, if the natural map from the differential torsion product $\mathrm{Tor}_R^*(M, N)$ to $H^*(M \otimes_R N)$ is an isomorphism for every differential graded left R-module N. We say that an operad of differential graded k-modules $\mathcal{F}$ is *flat* if for each n, $\mathcal{F}(n)$ is right $k[\Sigma_n]$-flat.

We have definitions of fibrations, cofibrations, and relative cell inclusions of $\mathcal{F}$-algebras completely analogous to those given in Section 2 for $\mathcal{E}$-algebras. We denote by $\mathbb{F}$ the free $\mathcal{F}$-algebra functor on differential graded k-modules. The following theorem is the main result we prove in this section.

Theorem 13.2. *Let $\mathcal{F}$ be a flat operad of differential graded k-modules, let A be a cofibrant $\mathcal{F}$-algebra, and let Z be a differential graded k-module. Then the canonical map $A \to A \amalg \mathbb{F}CZ$ is a quasi-isomorphism.*

Let $A \to B$ be a map of $\mathcal{F}$-algebras, and let FB be the free differential graded k-module with zero differential that has one generator z_b in dimension n for each element b of B in dimension $n - 1$. Let x_b denote the unique element of the differential graded k-module CFB whose differential is z_b. We then have a map of differential graded k-modules $FB \to B$ that sends x_b to b for all b. The induced map of $\mathcal{F}$-algebras $A \amalg \mathbb{F}CFB \to B$ is a surjection and the canonical map $A \to A \amalg \mathbb{F}CFB$ is a relative cell inclusion and clearly has the left lifting property with respect to

fibrations. Thus, we obtain the following corollary of the previous theorem, of which Theorem 2.8 is the special case $k = \bar{\mathbf{F}}_p$, $\mathcal{F} = \mathcal{E}$.

Corollary 13.3. *Let $\mathcal{F}$ be a flat operad of differential graded k-modules. Any map of $\mathcal{F}$-algebras $f \colon A \to B$ can be factored functorially as $f = q \circ j$, where j is a relative cell inclusion that has the left lifting property with respect to the fibrations, and q is a fibration. If A is cofibrant then j is in addition a quasi-isomorphism.*

We begin the proof of Theorem 13.2 by noticing that the underlying differential graded k-module of a coproduct of the form $A \amalg \mathbb{F}X$ decomposes as a direct sum of pieces homogeneous in X. We make this precise as follows.

Notation 13.4. For an $\mathcal{F}$-algebra A, define $U_i A$ to be the differential graded $k[\Sigma_i]$-module that makes the following diagram a coequalizer.

$$\bigoplus_{j \geq 0} \mathcal{F}(j+i) \otimes_{k[\Sigma_j]} (\mathbb{F}A)^{(j)} \rightrightarrows \bigoplus_{j \geq 0} \mathcal{F}(j+i) \otimes_{k[\Sigma_j]} A^{(j)} \longrightarrow U_i A$$

Here we understand the superscript (j) to denote the j-th tensor over k power with $(\mathbb{F}A)^{(0)} = A^{(0)} = k$. One map is induced by the $\mathcal{F}$-algebra structure map $\mathbb{F}A \to A$, and the other is induced by the operadic multiplication of $\mathcal{F}$.

Note that $U_0 A$ is canonically isomorphic to A, and $U_1 A$ is by definition the "universal enveloping algebra" of A [12, 1.6.4]. More generally, the collection $\mathcal{U}_A = \{U_n A\}$ assembles into an operad with the universal property that the set of $\mathcal{U}_A$-algebra structures on a differential graded k-module X is naturally in one-to-one correspondence with the set of pairs (ξ, η) where $\xi \colon \mathbb{F}X \to X$ is an $\mathcal{F}$-algebra structure on X and $\eta \colon A \to X$ is a map $\mathcal{F}$-algebras for this structure, c.f. [11, 1.18]. Our use for this construction is given by the following proposition.

Proposition 13.5. *For a differential graded k-module X, there is a natural isomorphism of differential graded k-modules*

$$A \amalg \mathbb{F}X \cong \bigoplus U_i A \otimes_{k[\Sigma_i]} X^{(i)}.$$

Proof. A check of universal properties reveals that the coproduct on the left is the coequalizer of a pair of maps from $\mathbb{F}((\mathbb{F}A) \oplus X)$ to $\mathbb{F}(A \oplus X)$. The proposition follows by a comparison of coequalizers. $\square$

The inclusion of A into $A \amalg \mathbb{F}X$ corresponds to the inclusion of the summand $U_0 A$ on the right hand side. Since $(CZ)^{(i)}$ is acyclic for all $i > 0$, Theorem 13.2 is an immediate consequence of the following lemma.

Lemma 13.6. *For a flat operad $\mathcal{F}$ and a cofibrant $\mathcal{F}$-algebra A, $U_i A$ is right $k[\Sigma_i]$-flat for each i.*

The remainder of this section is devoted to proving Lemma 13.6. We fix the flat operad $\mathcal{F}$ and the cofibrant $\mathcal{F}$-algebra A. We can assume without loss of generality that A is a cell $\mathcal{F}$-algebra, and so we can write $A = \operatorname{Colim} A_n$ for some cell $\mathcal{F}$-algebras $\mathcal{F}(0) = A_0 \rightarrowtail A_1 \rightarrowtail \cdots$, degreewise free differential graded k-modules $M_1, M_2, \ldots$ with zero differential, and maps $M_{n+1} \to A_n$ such that $A_{n+1} = A_n \amalg_{\mathbb{F}M_{n+1}} \mathbb{F}CM_{n+1}$. Let $N_n = M_1 \oplus \cdots \oplus M_n$, and let $N = \operatorname{Colim} N_n = \bigoplus M_n$. For convenience, we understand $N_0 = M_0 = 0$.

Our argument for Lemma 13.6 is an inductive analysis of the following filtration on the differential graded modules $U_i A_n$ that generalizes the filtration given by the direct sum decomposition of Proposition 13.5.

Notation 13.7. Let B be an $\mathcal{F}$-algebra, let X be a differential graded k-module, and let $g\colon X \to B$ be a map of differential graded k-modules. Let $U_i^m g$ be the differential graded $k[\Sigma_i]$-submodule of $U_i(B \amalg_{\mathbb{F}X} \mathbb{F}CX)$ of elements of degree m or less in CX, i.e. the submodule generated by the image of the elements

$$f \otimes x_{a_1} \otimes \cdots \otimes x_{a_j} \in \mathcal{F}(j + i) \otimes (B \oplus CX)^{(j)} \qquad \text{for } j \geq 0$$

in which at most m of $x_{a_1}, \ldots, x_{a_j} \in B \oplus CX$ map to a non-zero element under the canonical projection $B \oplus CX \to CX$. We write $U_i^m A_n$ for $U_i^m g$ when g is the given map $M_n \to A_{n-1}$; we understand $U_i^m A_0 = U_i A_0 = \mathcal{F}(i)$.

In order to understand this filtration, it is convenient to do some work in the category of graded k-modules. Forgetting the differential, we can regard $\mathcal{F}$ as an operad in the category of graded k-modules. We denote by $\mathbb{F}^\sharp$ the free functor from graded k-modules to $\mathcal{F}$-algebras of graded k-modules. To avoid confusion, we refer to $\mathcal{F}$-algebras of graded k-modules as $\mathbb{F}^\sharp$-algebras, reserving the term $\mathcal{F}$-algebra for $\mathcal{F}$-algebras of differential graded k-modules. Note that when X is a differential graded k-module, then the underlying $\mathbb{F}^\sharp$-algebra of $\mathbb{F}X$ is canonically isomorphic to $\mathbb{F}^\sharp X$.

Recall that for a differential graded k-module X, the underlying graded k-module of CX is the direct sum of X and a copy of X shifted one degree down. We denote by σX the graded k-submodule of CX consisting of the shifted copy of X. Then since $A_{n+1} = A_n \amalg_{\mathbb{F}M_{n+1}} \mathbb{F}CM_{n+1}$, we have that as $\mathbb{F}^\sharp$-algebras, $A_{n+1} = A_n \amalg \mathbb{F}^\sharp \sigma M_{n+1}$. Passing to colimits, we obtain the following proposition.

Proposition 13.8. *The map of graded k-modules $\sigma N \to A$ induces an isomorphism of $\mathbb{F}^\sharp$-algebras $\mathbb{F}^\sharp \sigma N \to A$.*

The differential that makes $\mathbb{F}^\sharp \sigma N$ into A is the obvious one determined by the Leibniz rule and the operadic multiplication of $\mathcal{F}$, writing the differential of an element of σM_n (the image of the corresponding element of M_n in A_{n-1}) as an element of $\mathbb{F}^\sharp \sigma N$. We can give a description of the underlying graded $k[\Sigma_i]$-module of $U_i A$, generalizing the description of A above. For $i \geq 0$, let $U_i^\sharp A$ be the graded right $k[\Sigma_i]$-module

$$U_i^\sharp A = \bigoplus_{j \geq 0} \mathcal{F}(j + i) \otimes_{k[\Sigma_j]} (\sigma N)^{(j)}$$

A comparison of (now split) coequalizers gives the following result.

Proposition 13.9. *The underlying graded $k[\Sigma_i]$-module of $U_i A$ is canonically isomorphic to $U_i^\sharp A$.*

An entirely analogous description of the underlying graded $k[\Sigma_i]$-modules of $U_i A_n$ holds. More generally, we can describe the underlying graded $k[\Sigma_i]$-module of $U_i^m A_n$ as the graded submodule of $U_i A_n$ generated by elements of

$$f \otimes x_{a_1} \otimes \cdots \otimes x_{a_j} \in \mathcal{F}(j + i) \otimes_{k[\Sigma_j]} (\sigma N_n)^{(j)}$$

in which at most m of $x_{a_1}, \ldots, x_{a_j}$ map to a non-zero element under the canonical projection $\sigma N_n \to \sigma M_n$. Since M_n is a direct summand of N_n, for $n > 0$ we can

identify the inclusion of $U_i^m A_n$ in $U_i A_n$ as the map of graded $k[\Sigma_i]$-modules

$$\bigoplus_{j=0}^{\infty} \bigoplus_{l=0}^{\max(j,m)} \mathcal{F}(j+i) \otimes_{k[\Sigma_{j-l} \times \Sigma_l]} (\sigma N_{n-1})^{(j-l)} \otimes (\sigma M_n)^{(l)}$$

$$\to \bigoplus_{j=0}^{\infty} \bigoplus_{l=0}^{j} \mathcal{F}(j+i) \otimes_{k[\Sigma_{j-l} \times \Sigma_l]} (\sigma N_{n-1})^{(j-l)} \otimes (\sigma M_n)^{(l)}$$

$$\cong \bigoplus_{j=0}^{\infty} \mathcal{F}(j+i) \otimes_{k[\Sigma_j]} (\sigma N_n)^{(j)}.$$

This identification is vital to our argument; we use it through the following immediate consequence.

Proposition 13.10. *For all $i \geq 0$, $m, n > 0$, the inclusion $U_i^{m-1} A_n \to U_i^m A_n$ is a split monomorphism of the underlying graded $k[\Sigma_i]$-modules. We have an isomorphism of differential graded $k[\Sigma_i]$-modules*

$$U_i^m A_n / U_i^{m-1} A_n \cong U_{i+m} A_{n-1} \otimes_{k[\Sigma_m]} (CM_n/M_n)^{(m)}.$$

Monomorphisms of differential graded modules that are split on the underlying graded modules play an important role in the proof of Lemma 13.6, and so we introduce the following terminology.

Definition 13.11. Let R be a ring and let $f \colon L \to M$ be a map of differential graded right R-modules. We say that f is an *almost split* monomorphism if the map of underlying graded right R-modules is a split monomorphism.

If $L \to M$ is an almost split monomorphism of differential graded right R-modules, then for any left R-module P, the sequence

$$0 \to L \otimes_R P \to M \otimes_R P \to (M/L) \otimes_R P \to 0$$

is exact, and so induces a long exact sequence on homology groups. From this observation and the definition of right R-flat, we obtain the following proposition.

Proposition 13.12. *Let $L \to M$ be an almost split monomorphism of differential graded right R-modules. If any two of L, M, M/L are right R-flat then so is the third.*

We need one more observation on flat differential graded modules. The following proposition is an easy consequence of the definition. We apply it below with $R = k[\Sigma_{m+i}]$, $S = k[\Sigma_m]$, and $T = k[\Sigma_i]$ for $i, m \geq 0$.

Proposition 13.13. *Let R, S, and T be k-algebras, let $S \otimes_k T \to R$ be a map of k-algebras, let L be a differential graded left S-module, and let M be a differential graded right R-module. If M is right R-flat, R is right $S \otimes_k T$-flat, and L is right k-flat, then $M \otimes_S L$ is right T-flat.*

Finally, we complete our argument with the proof of Lemma 13.6.

Proof of Lemma 13.6. By passage to the sequential colimit, it suffices to prove that $U_i^m A_n$ is right $k[\Sigma_i]$-flat for each $i, m, n \geq 0$. In the case $n = 0$, this is equivalent to the assumption that $\mathcal{F}$ is a flat operad. Assume by induction that this holds for $U_i^m A_{n-1}$ for all i, m.

Since $U_i^0 A_n = U_i A_{n-1} \cong \operatorname{Colim} U_i^m A_{n-1}$, it is right $k[\Sigma_i]$-flat by the inductive hypothesis. In general, for $m > 0$, the inclusion of $U_i^{m-1} A_n$ in $U_i^m A_n$ is an almost split monomorphism of differential graded right $k[\Sigma_i]$-modules, and the quotient

$$U_i^m A_n / U_i^{m-1} A_n \cong U_{m+i} A_{n-1} \otimes_{k[\Sigma_m]} (CM_n/M_n)^{(m)}$$

is right $k[\Sigma_i]$-flat by Proposition 13.13. Then by Proposition 13.12 and induction on m, we conclude that $U_i^m A_n$ is right $k[\Sigma_i]$-flat. $\qquad\square$

14. Proof of Theorems 3.4 and 3.5

The proofs of Theorems 3.4 and 3.5 rely heavily on the work of the last section and we follow the conventions and notations introduced there. In particular k is a commutative ground ring and $\mathcal{F}$ is a flat operad of differential graded k-modules, and we prove the theorems in this context. Of course, we do not expect the homology of a coproduct to be the differential torsion product for an arbitrary flat operad, e.g. the operad for associative k-algebras, so for the generalization of Theorem 3.4, we must restrict to operads that are also "acyclic": We say that an operad $\mathcal{F}$ is *acyclic* if it comes equipped with an acyclic augmentation, a map of operads $\mathcal{F} \to \mathcal{N}$ that is a componentwise quasi-isomorphism, where $\mathcal{N}$ is the operad for commutative k-algebras, $\mathcal{N}(n) = k$. We prove the following generalizations of Theorems 3.4 and 3.5.

Theorem 14.1. *Let $\mathcal{F}$ be an acyclic flat operad, and let B and C be cofibrant $\mathcal{F}$-algebras. The canonical map*

$$\mathcal{F}(2) \otimes B \otimes C \to \mathcal{F}(2) \otimes (B \oplus C) \otimes (B \oplus C) \to B \amalg C$$

induces an isomorphism $\operatorname{Tor}_k^*(B, C) \to H^*(B \amalg C)$.

Theorem 14.2. *Let $\mathcal{F}$ be a flat operad, let A, B be cofibrant $\mathcal{F}$-algebras, $A \to B$ a map of $\mathcal{F}$-algebras and $A \to C$ a cofibration of $\mathcal{F}$-algebras. Then the canonical map $N(\beta(B, A, C)) \to B \amalg_A C$ is a quasi-isomorphism.*

The proofs of these theorems consist of very similar arguments that study the filtrations described in 13.7 on the underlying differential graded module of a pushout of $\mathcal{F}$-algebras. We use the following observation many times in these arguments; it is an immediate consequence of the Tor version of the definition of a right R-flat differential graded module.

Proposition 14.3. *If $X \to Y$ is a quasi-isomorphism of right R-flat differential graded modules, then the map $X \otimes_R Z \to Y \otimes_R Z$ is a quasi-isomorphism for any differential graded left R-module Z.*

We begin with the proof of Theorem 14.1. The following proposition gives the base case for the main part of the argument below.

Proposition 14.4. *Let $\mathcal{F}$ be an acyclic flat operad, and let B be a cofibrant $\mathcal{F}$-algebra. The natural map*

$$\mathcal{F}(2) \otimes (B \otimes \mathcal{F}(i)) \to \mathcal{F}(i+1) \otimes B \to U_i B$$

is a quasi-isomorphism.

Proof. The natural map we have in mind is the composite of the map induced by the operadic multiplication and the canonical map in Definition 13.4. The first map is a quasi-isomorphism since $\mathcal{F}$ is acyclic and B is k-flat by Lemma 13.6. Thus, to

see that the composite is a quasi-isomorphism, we only need to check that the map $\mathcal{F}(i+1) \otimes B \to U_i B$ is a quasi-isomorphism.

We assume without loss of generality that B is a cell $\mathcal{F}$-algebra; write $B = \operatorname{Colim} B_n$ where $B_0 = \mathcal{F}(0)$ and $B_n = B_{n-1} \amalg_{\mathbb{F}M_n} \mathbb{F}CM_n$ for some M_n as in Definition 2.4. Since $\mathcal{F}(i+1) \otimes B \cong \operatorname{Colim} \mathcal{F}(i+1) \otimes B_n$ and $U_i B \cong \operatorname{Colim} U_i B_n$, it suffices to show that the map $\mathcal{F}(i+1) \otimes B_n \to U_i B_n$ is a quasi-isomorphism for each n. The case $n = 0$ follows from the assumption that $\mathcal{F}$ is flat and acyclic. Assume by induction that this map is a quasi-isomorphism for B_{n-1} for all i.

Write $U_i^m B_n$ for $U_i^m g$ as in 13.7 for g the given map $M_n \to B_{n-1}$ above. The map $\mathcal{F}(i+1) \otimes B_n \to U_i B_n$ restricts to a map

$$\mathcal{F}(i+1) \otimes U_0^m B_n \to U_i^m B_n.$$

In other words, $\mathcal{F}(i+1) \otimes B_n \to U_i B_n$ is a filtered map. Consider the map of strongly convergent spectral sequences associated to this filtered map. The map on E^1-terms consists of the maps

$$\mathcal{F}(i+1) \otimes (U_s B_{n-1} \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)}) \to U_{s+i} B_{n-1} \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)}.$$

These map are quasi-isomorphisms by Proposition 14.3 since each map $\mathcal{F}(i+1) \otimes U_s B_{n-1} \to U_{s+i} B_{n-1}$ is a quasi-isomorphism by the inductive hypothesis. It follows that the map of spectral sequences is an isomorphism from E^2 onwards. The map

$$\mathcal{F}(i+1) \otimes B_n \cong \operatorname{Colim}_m \mathcal{F}(i+1) \otimes U_0^m B_n \to \operatorname{Colim}_m U_i^m B_n \cong U_i B_n$$

is therefore a quasi-isomorphism. $\qquad\square$

Proof of Theorem 14.1. We can assume without loss of generality that B and C are cell $\mathcal{F}$-algebras. Write $C = \operatorname{Colim} C_n$ where $C_0 = \mathcal{F}(0)$ and $C_n = C_{n-1} \amalg_{\mathbb{F}M_n} \mathbb{F}CM_n$ for some M_n as in Definition 2.4. It suffices to prove that the map $\mathcal{F}(2) \otimes (B \otimes C_n) \to B \amalg C_n$ is a quasi-isomorphism for each n. In fact, it is convenient for our inductive argument to prove that the map

$$\mathcal{F}(2) \otimes (B \otimes U_i C_n) \to U_i(B \amalg C_n)$$

is a quasi-isomorphism for all $i, n \geq 0$. In the case $n = 0$, this follows from the previous proposition; assume by induction that this holds for C_{n-1} for all i.

Let $U_i^m C_n$ denote $U_i^m g$ as in 13.7 for g the given map $M_n \to C_{n-1}$, and let $U_i^m(B \amalg C_n)$ denote $U_i^m h$ for h the composite $M_n \to C_{n-1} \to B \amalg C_{n-1}$; we understand $U_i^m C_0 = U_i C_0 = \mathcal{F}(i)$ and $U_i^m(B \amalg C_0) = U_i(B \amalg C_0) = U_i B$. The map displayed above restricts to a map

$$\mathcal{F}(2) \otimes (B \otimes U_i^m C_n) \to U_i^m(B \amalg C_n)$$

and induces a map of the strongly convergent spectral sequences associated to these filtrations. The map on E^1-terms consists of the maps

$$\mathcal{F}(2) \otimes B \otimes U_{s+i} C_{n-1} \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)} \to$$
$$U_{s+i}(B \amalg C_{n-1}) \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)},$$

which are quasi-isomorphisms by the inductive hypothesis and Proposition 14.3. It follows that the map of spectral sequences is an isomorphism from E^2 onwards. The map

$$\mathcal{F}(2) \otimes (B \otimes U_i C_n) \cong \operatorname{Colim}_m \mathcal{F}(2) \otimes (B \otimes U_i^m C_n) \to$$
$$\operatorname{Colim}_m U_i^m(B \amalg C_n) \cong U_i(B \amalg C_n)$$

is therefore a quasi-isomorphism. □

We now proceed with the proof of Theorem 14.2. As in the proof of Theorem 14.1 above, it is convenient to prove the more general result that (with the hypothesis of Theorem 14.2) the natural map

$$N(U_i\beta_\bullet(B, A, C)) \to U_i(B \amalg_A C)$$

is a quasi-isomorphism for all i. We need the following observation for our argument.

Proposition 14.5. *Let $\mathcal{F}$ be a flat operad, let $A \to B$ and $A \to C$ be maps of cofibrant $\mathcal{F}$-algebras. Then $N(U_i\beta_\bullet(B, A, C))$ is a right $k[\Sigma_i]$-flat differential graded module.*

Proof. For a differential graded left $k[\Sigma_i]$-module X,

$$N(U_i\beta_\bullet(B, A, C)) \otimes_{k[\Sigma_i]} X \cong N(U_i\beta_\bullet(B, A, C) \otimes_{k[\Sigma_i]} X).$$

The proposition now follows from Lemma 13.6. □

The proof of Theorem 14.2 begins with the following special case.

Proposition 14.6. *Let $\mathcal{F}$ be a flat operad, and let $A \to B$ be a map of cofibrant $\mathcal{F}$-algebras. The natural map $N(U_i\beta_\bullet(B, A, A)) \to U_iB$ is a quasi-isomorphism.*

Proof. By the usual argument, the map of simplicial $\mathcal{F}$-algebras $\beta_\bullet(B, A, A) \to B$ is a homotopy equivalence. Applying the functor U_i, we have that the map $U_i\beta_\bullet(B, A, A) \to U_iB$ is a homotopy equivalence of simplicial differential graded $k[\Sigma_i]$-modules and so its normalization is a chain homotopy equivalence of differential graded $k[\Sigma_i]$-modules. □

Proof of Theorem 14.2. We assume without loss of generality that B is a cell $\mathcal{F}$-algebra and the map $A \to C$ is a relative cell inclusion. Write $C = \operatorname{Colim} C_n$ where $C_0 = A$ and $C_n = C_{n-1} \amalg_{\mathbb{F}M_n} \mathbb{F}CM_n$ for some differential graded k-modules M_n as in Definition 2.4. It suffices to prove that the natural map

$$N(U_i\beta_\bullet(B, A, C_n)) \to U_i(B \amalg_A C_n)$$

is a quasi-isomorphism for all $i, n \geq 0$. The case for C_0 follows from the previous proposition; assume by induction that this holds for C_{n-1}.

Define $U_i^m(B\amalg_A C_n) = U_i^m g$ for g the composite map $M_n \to C_{n-1} \to B\amalg_A C_{n-1}$. Define $U_i^m\beta_j(B, A, C)$ analogously. The simplicial map $\beta_\bullet(B, A, C_n) \to B \amalg_A C_n$ restricts to a simplicial map

$$U_i^m\beta_\bullet(B, A, C_n) \to U_i^m(B \amalg_A C_n).$$

We take the normalization and consider the induced map on the strongly convergent spectral sequences associated to these U_i^m filtrations. We can identify the map on E^1-terms as the map

$$N(U_{s+i}\beta_\bullet(B, A, C_{n-1})) \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)} \to$$
$$U_{s+i}(B \amalg_A C_{n-1}) \otimes_{k[\Sigma_s]} (CM_n/M_n)^{(s)},$$

which is a quasi-isomorphism by the inductive hypothesis and Proposition 14.3. It follows that the map of spectral sequences is an isomorphism from E^2 onwards.

The map

$$N(U_i\beta_\bullet(B,A,C_n)) \cong \mathrm{Colim}_m\, N(U_i^m\beta_\bullet(B,A,C_n)) \to$$
$$\mathrm{Colim}_m\, U_i^m(B\amalg_A C_n) \cong U_i(B\amalg_A C_n)$$

is therefore a quasi-isomorphism. $\square$

Appendix A. Other Fields

We use the techniques developed in the body of the paper to discuss when the analogue of the Main Theorem holds for a field k. We prove the following theorem. In this theorem, Φ denotes the Frobenius endomorphism on a field of positive characteristic.

Theorem A.1. *Let k be a field. The singular cochain functor with coefficients in k induces an equivalence between the homotopy category of $H_*(-;k)$-local [1] nilpotent spaces of finite k-type and a full subcategory of the homotopy category of E_∞ k-algebras if and only if k satisfies one of the following two conditions*

   (i) $k = \mathbf{Q}$, *the field of rational numbers.*
   (ii) *k has positive characteristic and $1 - \Phi$ is surjective.*

It follows in particular that the analogue of the Main Theorem does not hold when k is a finite field. The smallest field of characteristic p for which $1 - \Phi$ is surjective is the fixed field in $\bar{\mathbf{F}}_p$ of $\ker(\mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p) \to \mathbf{Z}_p^\wedge)$.

For the finite fields $\mathbf{F}_q$, we can be more specific about the the relationship between the p-adic homotopy category and the homotopy category of E_∞ $\mathbf{F}_q$-algebras. The following theorem was suggested by W. G. Dwyer and M. J. Hopkins.

Theorem A.2. *Let $q = p^n$. For connected p-complete nilpotent spaces of finite p-type X and Y, there is a natural bijection*

$$\bar{h}\mathfrak{E}_{\mathbf{F}_q}(C^*(X;\mathbf{F}_q), C^*(Y;\mathbf{F}_q)) \cong \mathfrak{Ho}(Y, \Lambda X)$$

where $\bar{h}\mathfrak{E}_{\mathbf{F}_q}$ denotes the homotopy category of E_∞ $\mathbf{F}_q$-algebras and Λ denotes the free loop space functor.

Outline of the proof of Theorem A.1. For an arbitrary field k, there is no difficulty in providing a natural $\mathcal{E}_k$-algebra structure on the cochains of simplicial sets, for some E_∞ k-operad $\mathcal{E}_k$. For example the work of [15] and the construction described in Section 1 produce such a structure. Write $\mathfrak{E}_k$ for the category of $\mathcal{E}_k$-algebras. We can form the adjoint functor $U(-;k)$ from $\mathcal{E}_k$-algebras to simplicial sets by the simplicial mapping set

$$U_\bullet(A;k) = \mathfrak{E}(A, C^*(\Delta[\,\cdot\,];k)).$$

Arguing as in Section 4, we obtain the following proposition.

Proposition A.3. *The functors $C^*(-;k)$ and $U(-;k)$ are contravariant right adjoints between the category of $\mathcal{E}_k$-algebras and the category of simplicial sets. Their right derived functors exist and give an adjunction between the homotopy category of $\mathcal{E}_k$-algebras and the homotopy category.*

We say that a simplicial set is k-resolvable if the unit of the derived adjunction $X \to \mathbf{U}(C^*(X;k);k)$ is an isomorphism in the homotopy category. As an elementary consequence of the previous proposition, we see that $C^*(-;k)$ gives an equivalence as in the statement of the theorem if and only if every connected

$H_*(-;k)$-local nilpotent simplicial set of finite k-type is k-resolvable. The base field $\bar{\mathbf{F}}_p$ is irrelevant in Sections 2–5, and the arguments there apply to prove the following propositions that allow us to argue inductively up principally refined Postnikov towers.

Proposition A.4. *Let $X = \mathrm{Lim}\, X_n$ be the limit of a tower of Kan fibrations. Assume that the canonical map from $H^*(X;k)$ to $\mathrm{Colim}\, H^*(X_n;k)$ is an isomorphism. If each X_n is k-resolvable, then X is k-resolvable.*

Proposition A.5. *Let X, Y, and Z be connected simplicial sets of finite k-type, and assume that Z is simply connected. Let $X \to Z$ be a map of simplicial sets, and let $Y \to Z$ be a Kan fibration. If X, Y, and Z are k-resolvable, then so is the fiber product $X \times_Z Y$.*

A connected space is nilpotent, $H_*(-;k)$-local, and of finite k-type if and only if its Postnikov tower has a principal refinement with fibers:

(i) $K(\mathbf{Q}, n)$ when k is characteristic zero.
(ii) $K(\mathbf{Z}/p\mathbf{Z}, n)$ or $K(\mathbf{Z}_p^\wedge, n)$ when k is characteristic $p > 0$.

By the argument in Section 6, $K(\mathbf{Z}_p^\wedge, n)$ is easily seen to be k-resolvable when $K(\mathbf{Z}/p\mathbf{Z}, n)$ is. The theorem is therefore a consequence of the following two propositions.

Proposition A.6. *Let k be a field of characteristic zero. $K(\mathbf{Q}, n)$ is k-resolvable if and only if $k = \mathbf{Q}$.*

Proof. Write $\mathbb{E}$ for the free $\mathcal{E}_k$-algebra functor. Let $a\colon \mathbb{E}k[n] \to C^*(K(\mathbf{Q}, n); k)$ be any map of $\mathcal{E}_k$-algebras that sends the fundamental class of $k[n]$ to the fundamental class of $H^*(K(\mathbf{Q}, n); \mathbf{Q}) \subset H^*(K(\mathbf{Q}, n); k)$. Since k is characteristic zero, it is easy to see that a is a quasi-isomorphism, so the unit of the derived adjunction is represented by the map $K(\mathbf{Q}, n) \to U\mathbb{E}k[n]$. It is straightforward to check that $U\mathbb{E}k[n]$ is a $K(k, n)$ and the map $K(\mathbf{Q}, n) \to K(k, n)$ induces on π_n the inclusion $\mathbf{Q} \subset k$. $\qquad\square$

Proposition A.7. *Let k be a field of characteristic $p > 0$. $K(\mathbf{Z}/p\mathbf{Z}, n)$ is k-resolvable if and only if $1 - \Phi$ is surjective on k.*

Proof. We can construct a model $B_{n;k}$ for $C^*(K_n, k)$ exactly as in Section 6 and prove that the map $\alpha_k\colon B_{n;k} \to C^*(K_n; k)$ is a quasi-isomorphism just as in Section 12. We are therefore reduced to checking when the map $K_n \to UB_{n;k}$ is a weak equivalence. Again, we have $UB_{n;k}$ given by a Kan fibration square

$$
\begin{array}{ccc}
UB_{n;k} & \longrightarrow & U\mathbb{E}Ck[n] \\
\downarrow & & \downarrow \\
U\mathbb{E}k[n] & \xrightarrow{\ Up_n\ } & U\mathbb{E}k[n].
\end{array}
$$

The argument of Proposition 6.5 then applies to show that $U\mathbb{E}k[n]$ is a $K(k, n)$ and the map Up_n induces on π_n the map $1 - \Phi$. It follows that $UB_{n;k}$ is a $K(\mathbf{Z}/p\mathbf{Z}, n)$ if and only if $1 - \Phi$ is surjective. When $1 - \Phi$ is surjective, it is straightforward to verify that the map $K_n \to UB_{n;k}$ is a weak equivalence. $\qquad\square$

Outline of the proof of Theorem A.2. Let $q = p^n$ and consider the finite field $\mathbf{F}_q$. From the work above, it suffices to show that there is a natural isomorphism $\Lambda X \to \mathbf{U}(C^*(X; \mathbf{F}_q); \mathbf{F}_q)$ in the homotopy category for X connected, p-complete, nilpotent, and of finite p-type.

To make the argument, we need to assume that we have a map of operads of $\bar{\mathbf{F}}_p$-algebras $\mathcal{E}_{\mathbf{F}_q} \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p \to \mathcal{E}$; we have such a map in the case when $\mathcal{E}_{\mathbf{F}_q}$ and $\mathcal{E}$ are constructed from the Eilenberg–Zilber operad of [15] as outlined above and in Section 1. By changing $\mathcal{E}$ if necessary, we can assume without loss of generality that this map is an isomorphism. Then we have an extension of scalars functor $E: \mathcal{E}_{\mathbf{F}_q} \to \mathcal{E}$ defined by $E(-) = (-) \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p$. The functor E preserves cofibrations and quasi-isomorphisms and is left adjoint to the forgetful functor that regards an $\mathcal{E}$-algebra as an $\mathcal{E}_{\mathbf{F}_q}$-algebra. In particular, we have the following proposition.

Proposition A.8. *There is a canonical natural isomorphism of simplicial sets* $U(E(-)) \cong \mathfrak{E}_{\mathbf{F}_q}(-; C^*\Delta[\,\cdot\,])$.

Let $\Psi = \Phi^n$ denote the n-th iterate of the Frobenius automorphism on $\bar{\mathbf{F}}_p$. Since Ψ is a map of $\mathbf{F}_q$-algebras, we obtain a map of simplicial $\mathcal{E}_{\mathbf{F}_q}$-algebras

$$C^*\Delta[\,\cdot\,] \cong C^*(\Delta[\,\cdot\,]; \mathbf{F}_q) \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p \xrightarrow{\mathrm{id} \otimes \Psi} C^*(\Delta[\,\cdot\,]; \mathbf{F}_q) \otimes_{\mathbf{F}_q} \bar{\mathbf{F}}_p \cong C^*\Delta[\,\cdot\,].$$

We obtain a natural automorphism Ψ on $U(E-)$. Thus, we can regard $U(E-)$ as a functor from the category of $\mathcal{E}_{\mathbf{F}_q}$-algebras to the category of $\mathbf{Z}$-equivariant simplicial sets. We can regard $U(-; \mathbf{F}_q)$ as a functor to the category of $\mathbf{Z}$-equivariant simplicial sets by giving $U(-; \mathbf{F}_q)$ the trivial $\mathbf{Z}$-action. The natural map $U(-; \mathbf{F}_q) \to U(E-)$ induced by the inclusion $C^*(\Delta[\,\cdot\,]; \mathbf{F}_q) \to C^*\Delta[\,\cdot\,]$ is then $\mathbf{Z}$-equivariant.

For a $\mathbf{Z}$-equivariant simplicial set X, let $X^{h\Psi}$ be the homotopy equalizer of the maps id and Ψ (where as above Ψ generates the $\mathbf{Z}$-action): Let $X^{h\Psi}$ be the simplicial set that makes the following diagram a pullback.

$$
\begin{array}{ccc}
X^{h\Psi} & \longrightarrow & X^{\Delta[1]} \\
\downarrow & & \downarrow \\
X & \xrightarrow[\mathrm{id} \times \Psi]{} & X \times X
\end{array}
$$

Since the natural transformation $U(-; \mathbf{F}_q) \to U(E-)$ factors through the fixed points of Ψ, we obtain a natural map $U(-; \mathbf{F}_q) \to U(E-)^{h\Psi}$. We prove below the following theorem.

Theorem A.9. *The natural map* $U(A; \mathbf{F}_q) \to U(EA)^{h\Psi}$ *is a weak equivalence when A is cofibrant.*

Theorem A.9 is the main fact needed for Theorem A.2.

Proof of Theorem A.2. Let X be a simplicial set, let $A \to C^*(X; \mathbf{F}_q)$ be a cofibrant approximation in the category of $\mathcal{E}_{\mathbf{F}_q}$-algebras, and let $B \to C^*X$ be a cofibrant approximation in the category of $\mathcal{E}$-algebras. Since EA is cofibrant, we can choose a map of $\mathcal{E}$-algebras $A \to B$ so that the composite $EA \to C^*X$ coincides with the composite of $EA \to EC^*(X; \mathbf{F}_q)$ and the natural map of $\mathcal{E}$-algebras $EC^*(X; \mathbf{F}_q) \to C^*X$. Then we have a composite map

$$X \to UB \to UEA,$$

natural in X in the homotopy category, which is a weak equivalence when X is connected p-complete nilpotent of finite p-type by the Main Theorem. It is straightforward to check that the map $X \to UEA$ factors through $U(A; \mathbf{F}_q)$, and so is $\mathbf{Z}$-equivariant when we give X the trivial action. Consider the maps

$$U(A; \mathbf{F}_q) \to (UEA)^{h\Psi} \leftarrow X^{h\Psi}$$

By Theorem A.9, the first map is a weak equivalence. When X is a connected p-complete nilpotent Kan complex of finite p-type, the second map is a weak equivalence and $X^{h\Psi}$ is a model for the free loop space ΛX. $\square$

For the proof of Theorem A.9, we recall the definition of a cosimplicial resolution from [7]. For an object A of $\mathfrak{C}_{\mathbf{F}_q}$, a *cosimplicial resolution of A* is a cosimplicial $\mathcal{E}_{\mathbf{F}_q}$-algebra $A^\bullet$ together with a quasi-isomorphism $A^0 \to A$ such that A^0 is cofibrant, each coface map in $A^\bullet$ is an acyclic cofibration, and each map $(d^*, A^n) \to A^{n+1}$ is a cofibration, where (d^*, A^n) is the object described in [7, 4.3]: the colimit of the diagram in $\mathfrak{C}_{\mathbf{F}_q}$ with objects

- For each i, $0 \le i \le n+1$, a copy of A^n labelled (d^i, A^n)
- For each (i, j), $0 \le i < j \le n+1$, a copy of A^{n-1} labelled $(d^j d^i, A^{n-1})$ (we understand $A^{-1} = \mathbf{F}_q$).

and maps

- For each (i, j), $0 \le i < j \le n+1$, a map $(d^j d^i, A^{n-1}) \to (d^j, A^n)$ given by the map $d^i : A^{n-1} \to A^n$.
- For each (i, j), $0 \le i < j \le n+1$, a map $(d^j d^i, A^{n-1}) \to (d^i, A^n)$ given by the map $d^{j-1} : A^{n-1} \to A^n$.

Although $\mathfrak{C}_{\mathbf{F}_q}$ is not a model category, the following analogues of the results of [7, §6] still hold.

Proposition A.10. *Let $A^\bullet$ be a cosimplicial resolution. The functor $\mathfrak{C}_{\mathbf{F}_q}(A^\bullet, -)$ from $\mathcal{E}_{\mathbf{F}_q}$-algebras to simplicial sets preserves fibrations and weak equivalences.*

Proof. That $\mathfrak{C}_{\mathbf{F}_q}(A^\bullet, -)$ preserves fibrations and acyclic fibrations follows from the standard arguments (omitted) in [7, §6]. Since $\mathfrak{C}_{\mathbf{F}_q}(A^\bullet, -)$ preserves acyclic fibrations, to see that it preserves all weak equivalences, it suffices to show that it preserves weak equivalences between cell $\mathcal{E}_{\mathbf{F}_q}$-algebras. Since for cell $\mathcal{E}_{\mathbf{F}_q}$-algebras, we can factor a map as an acyclic cofibration followed by a fibration, we can apply the dual of the argument for K. Brown's lemma [9, 9.9]. $\square$

Proposition A.11. *Let $k = \mathbf{F}_q$ or $\bar{\mathbf{F}}_p$. For any cosimplicial resolution of $\mathcal{E}_{\mathbf{F}_q}$-algebras $A^\bullet$, the maps of simplicial sets*

$$\mathfrak{C}_{\mathbf{F}_q}(A^\bullet, k) \to \operatorname{diag} \mathfrak{C}_{\mathbf{F}_q}(A^\bullet, C^*(\Delta[\,\cdot\,]; k)) \leftarrow \mathfrak{C}_{\mathbf{F}_q}(A^0, C^*(\Delta[\,\cdot\,]; k))$$

are weak equivalences.

Proof. Since all the face maps of $C^*(\Delta[\,\cdot\,]; k)$ are acyclic fibrations, the first map is a weak equivalence by the previous lemma. The simplicial $\mathcal{E}_k$-algebra $C^*(\Delta[\,\cdot\,]; k)$ has the dual property that mapping into it converts acyclic cofibrations to acyclic Kan fibrations, and so the second map is a weak equivalence. $\square$

Proof of Theorem A.9. Since the weak equivalences in Proposition A.11 are $\mathbf{Z}$-equivariant maps of Kan complexes (where for $k = \mathbf{F}_q$ we understand the trivial

action), it suffices to show that the map

$$\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \mathbf{F}_q) \to \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)^{h\Psi}$$

is a weak equivalence. Factor the diagonal map $\bar{\mathbf{F}}_p \to \bar{\mathbf{F}}_p \times \bar{\mathbf{F}}_p$ as an acyclic cofibration $\bar{\mathbf{F}}_p \to P$ composed with a fibration $P \to \bar{\mathbf{F}}_p \times \bar{\mathbf{F}}_p$, and let Q be the $\mathcal{E}_{\mathbf{F}_q}$-algebra that makes the following diagram a pullback.

$$
\begin{array}{ccc}
Q & \longrightarrow & P \\
\downarrow & & \downarrow \\
\bar{\mathbf{F}}_p & \xrightarrow{\text{id} \times \Psi} & \bar{\mathbf{F}}_p \times \bar{\mathbf{F}}_p
\end{array}
$$

The unit map $\mathbf{F}_q \to Q$ is a weak equivalence, and so the map $\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \mathbf{F}_q) \to \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, Q)$ is a weak equivalence. Since $\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, -)$ preserves pullbacks and fibrations, we have that the following diagram is the pullback of a Kan fibration.

$$
\begin{array}{ccc}
\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, Q) & \longrightarrow & \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, P) \\
\downarrow & & \downarrow \\
\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p) & \xrightarrow{\text{id} \times \Psi} & \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p) \times \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)
\end{array}
$$

Choosing a diagonal lift in the following diagram

$$
\begin{array}{ccc}
\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p) & \xrightarrow{\sim} & \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, P) \\
{\scriptstyle \sim} \downarrow & \nearrow & \downarrow \\
\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)^{\Delta[1]} & \longrightarrow & \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p) \times \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)
\end{array}
$$

we obtain a weak equivalence $\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)^{h\Psi} \to \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, Q)$ factoring the weak equivalence $\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \mathbf{F}_q) \to \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, Q)$ above through the map $\mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \mathbf{F}_q) \to \mathfrak{E}_{\mathbf{F}_q}(A^\bullet, \bar{\mathbf{F}}_p)^{h\Psi}$. $\qquad\square$

APPENDIX B. PRO-CATEGORIES AND p-PRO-FINITE COMPLETION

In this section we describe the relationship between the unit of the derived adjunction $X \to \mathbf{U}C^*X$ and p-pro-finite completion in the sense of Sullivan [28, §3], [24, §2.1]. The idea that there should be some relationship was first suggested by W. G. Dwyer. We prove the following theorem.

Theorem B.1. *For any connected simplicial set X, the composite map*

$$X \to \mathbf{U}C^*X \to \mathbf{U}(C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p)$$

is p-pro-finite completion.

Here we are giving $C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p$ the structure of an $\mathcal{E}$-algebra via the natural isomorphism $C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p \cong C^*_{\text{cont}} \hat{X} = \text{Colim}\, C^* X_\alpha$, where $\hat{X} = \{X_\alpha\}$ denotes the "completion of X" [24, §2.1], [14, 1.2.2], the projective system of levelwise finite quotients of X. The system of maps $X \to X_\alpha$ induces a map of $\mathcal{E}$-algebras $C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p \to C^*X$ that induces the map $\mathbf{U}C^*X \to \mathbf{U}(C^*(X; \mathbf{F}_p) \otimes_{\mathbf{F}_p} \bar{\mathbf{F}}_p)$ above.

In other words, for the theorem above, we have used a version of the cochain functor that factors through the category of pro-finite simplicial sets. From this

perspective, it is clear that the theorem we should try to prove is that the functor C^*_{cont} from the category of pro-finite simplicial sets to the category of $\mathcal{E}$-algebras is a Quillen adjoint (to "$\hat{U}$") and that the natural transformation $X \to \hat{U}LC^*_{\mathrm{cont}}X$ is p-pro-finite completion in the sense of Morel [24, §2.1], where L is some cofibrant approximation functor. Unfortunately, this is not true; there is no adjoint functor $\hat{U}$ from the category of $\mathcal{E}$-algebras to the category of pro-finite simplicial sets. To see this, note that the set of maps of pro-finite simplicial sets from the (constant) standard simplex $\Delta[n]$ to any pro-finite simplicial set is naturally a compact space, and so the set of maps from an $\mathcal{E}$-algebras A to $C^*_{\mathrm{cont}}\Delta[n] \cong C^*\Delta[n]$ would have to be a compact space with an action of $\mathfrak{E}(A, A)$ through continuous maps. On the other hand, $\mathfrak{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], C^*\Delta[n]) \cong \bar{\mathbf{F}}_p$ is countable and $\bar{\mathbf{F}}_p \subset \mathfrak{E}(\mathbb{E}\bar{\mathbf{F}}_p[n], \mathbb{E}\bar{\mathbf{F}}_p[n])$ acts transitively.

If we look at a larger category, the pro-category of simplicial sets, then an adjoint functor does exist. Letting pro-$\mathfrak{S}$ denote the pro-category of simplicial sets, the cochain functor $C^*_{\mathrm{cont}} \colon \mathrm{pro}\text{-}\mathfrak{S} \to \mathfrak{E}$ most natural to consider is the functor that takes a pro simplicial set $X = \{X_\alpha\}$ to the $\mathcal{E}$-algebra $\mathrm{Colim}\, C^*X_\alpha$. We prove the following lemma below.

Lemma B.2. *The functor $C^*_{\mathrm{cont}} \colon \mathrm{pro}\text{-}\mathfrak{S} \to \mathfrak{E}$ has a right adjoint U_{c}, i.e. there is a bijection* $\mathrm{pro}\text{-}\mathfrak{S}(X, U_{\mathrm{c}}A) \cong \mathfrak{E}(A, C^*_{\mathrm{cont}}X)$, *natural in pro simplicial sets X and $\mathcal{E}$-algebras A.*

For the proof of Lemma B.2, we consider the functor C^* from pro-$\mathfrak{S}$ to ind-$\mathfrak{E}$, the *ind-category* of $\mathcal{E}$-algebras, the opposite category of the pro-category of $\mathfrak{E}^{\mathrm{op}}$. The functor $U \colon \mathrm{ind}\text{-}\mathfrak{E} \to \mathrm{pro}\text{-}\mathfrak{S}$ is a right adjoint to C^*. We have an obvious functor $\mathrm{Colim} \colon \mathrm{ind}\text{-}\mathfrak{E} \to \mathfrak{E}$, and $C^*_{\mathrm{cont}} = \mathrm{Colim}\, C^*$. Of course Colim is a left adjoint (to the constant functor), but in fact it is also a right adjoint. Lemma B.2 is an immediate consequence of the following proposition, setting $U_{\mathrm{c}}A = U\mathrm{c}A$.

Proposition B.3. *The functor* $\mathrm{Colim} \colon \mathrm{ind}\text{-}\mathfrak{E} \to \mathfrak{E}$ *has a left adjoint* $\mathrm{c} \colon \mathfrak{E} \to \mathrm{ind}\text{-}\mathfrak{E}$.

The proof of the previous proposition is easy, but requires the following terminology.

Definition B.4. We say that an $\mathcal{E}$-algebra A is *compact* if for any $B = \{B_\alpha\}$ in ind-$\mathfrak{E}$, the natural map $\mathrm{Colim}\, \mathfrak{E}(A, B_\alpha) \to \mathfrak{E}(A, \mathrm{Colim}\, B)$ is a bijection. We say that an $\mathcal{E}$-algebra A is *finitely presented* if A a coequalizer (in $\mathfrak{E}$)

$$\mathbb{E}N \rightrightarrows \mathbb{E}M \longrightarrow A$$

for finitely generated differential graded $\bar{\mathbf{F}}_p$-modules M and N.

Clearly $\mathbb{E}M$ is compact when M is finitely generated, and so finitely presented $\mathcal{E}$-algebras are compact. For an arbitrary $\mathcal{E}$-algebra consider the category $\mathfrak{R}_A$ whose objects consist of ordered pairs (M, N) where M is a finitely generated differential graded submodule of A and N is a finitely generated differential graded submodule of $\mathbb{E}M$ sent to zero under the induced map $\mathbb{E}M \to A$; the maps in $\mathfrak{R}_A$ are the inclusions. We have a functor $\mathfrak{D}_A$ from $\mathfrak{R}_A$ to finitely presented $\mathcal{E}$-algebras, sending (M, N) to the coequalizer (in $\mathfrak{E}$) of the maps $\mathbb{E}N \to \mathbb{E}M$ induced by the inclusion $N \to \mathbb{E}M$ and the zero map $N \to \mathbb{E}M$. The category $\mathfrak{R}_A$ is filtered, and $\mathrm{c}(-) = \mathfrak{D}_{(-)}$ specifies a well-defined functor from $\mathfrak{E}$ to ind-$\mathfrak{E}$. Since the canonical map

$\mathrm{Colim}_{\mathfrak{R}_A} \mathfrak{D}_A \to A$ is an isomorphism, we have that for any $B = \{B_\alpha\}$ in ind-$\mathfrak{E}$,

$$\mathfrak{E}(A, \mathrm{Colim}\, B) \cong \mathfrak{E}(\mathrm{Colim}\, cA, \mathrm{Colim}\, B) \cong \mathrm{Lim}_{\mathfrak{R}_A} \mathfrak{E}(\mathfrak{D}_A, \mathrm{Colim}\, B)$$
$$\cong \mathrm{Lim}_{\mathfrak{R}_A} \mathrm{Colim}(\mathfrak{D}_A, B_\alpha) = \text{ind-}\mathfrak{E}(cA, B)$$

This proves Proposition B.3. We find it useful to note here the following easy observations.

Proposition B.5. *A finite cell $\mathcal{E}$-algebra is finitely presented.*

Proposition B.6. *The functor* $\mathrm{c}(-)$ *is an equivalence between* $\mathfrak{E}$ *and the full subcategory of* ind-$\mathfrak{E}$ *consisting of the inductive systems of compact $\mathcal{E}$-algebras.*

To take advantage of the adjoint functor U_c, we need a homotopy theory for the category pro-$\mathfrak{S}$ of pro simplicial sets. This theory is provided in the recent work of Isaksen [16], where it is shown that the category pro-$\mathfrak{S}$ is a closed model category. Following the terminology there, say that a map $f \colon X \to Y$ is a *level map* if X and Y are indexed on the opposite of the same filtered category $\mathfrak{J}$ and f is represented by a map of diagrams on $\mathfrak{J}^\mathrm{op}$. A map $f \colon X \to Y$ in pro-$\mathfrak{S}$ is a *strong weak equivalence* if it is a level map where for all $n \geq 0$, $\beta \in \mathfrak{J}$, there exists $\alpha \to \beta$ in $\mathfrak{J}^\mathrm{op}$ such that for every choice of basepoint in X_α, there is a map $\pi_n Y_\alpha \to \pi_n X_\beta$ that makes the following diagram commute.

$$\begin{array}{ccc} \pi_n X_\alpha & \xrightarrow{\ f_\alpha\ } & \pi_n Y_\alpha \\ \downarrow & \swarrow & \downarrow \\ \pi_n X_\beta & \xrightarrow[\ f_\beta\ ]{} & \pi_n Y_\beta \end{array}$$

A *weak equivalence* in pro-$\mathfrak{S}$ is a map in pro-$\mathfrak{S}$ that is isomorphic to a strong weak equivalence. It is proved in [16] that a level map is a weak equivalence if and only if it is a strong weak equivalence. Thus, since every map in pro-$\mathfrak{S}$ is isomorphic to a level map, when X is a pro connected based simplicial set, a map $X \to Y$ is a weak equivalence if and only if it induces a pro-isomorphism of each homotopy pro-group $\{\pi_n X_\alpha\} \to \{\pi_n Y_\beta\}$.

The cofibrations are the maps isomorphic to level maps that are level cofibrations; in particular all objects are cofibrant. It is shown that the constant pro simplicial set on a Kan simplicial set with only finitely many nontrivial homotopy groups is fibrant in pro-$\mathfrak{S}$ and a Kan fibration between such simplicial sets is a fibration in pro-$\mathfrak{S}$. It follows that we can identify the functor $H^*_\mathrm{cont} X = H^*(C^*_\mathrm{cont} X)$ as the set of maps from X to $K(\bar{\mathbf{F}}_p, n)$ in the homotopy category of pro-$\mathfrak{S}$. Thus, C^*_cont converts cofibrations to fibrations and preserves weak equivalences. As an immediate consequence of Theorems 2.14 and 2.15, we obtain the following proposition.

Proposition B.7. *The (right) derived functor* $\mathbf{U}_\mathrm{c}$ *of* U_c *exists and gives an adjunction* $\bar{h}\mathfrak{E}(A, C^*_\mathrm{cont} X) \cong \mathrm{Hopro\text{-}}\mathfrak{S}(X, \mathbf{U}_\mathrm{c} A)$.

The functor C^* from the homotopy category to the homotopy category of $\mathcal{E}$-algebras factors as the composite of the constant functor and C^*_cont, and so it follows that the functor $\mathbf{U}$ is the composite of $\mathbf{U}_\mathrm{c}$ and the right derived functor of Lim. The forgetful functor from Morel's model category of pro-finite simplicial sets to Isaksen's model category of pro simplicial sets is a right adjoint that preserves fibrations and acyclic fibrations, and so the right derived functor of Lim from the

homotopy category of pro-finite simplicial sets to the homotopy category is the composite of the right derived functor of the forgetful functor and the right derived functor of Lim from the homotopy category of pro simplicial sets to the homotopy category. Since pro-finite completion in the sense of Sullivan is the composite of the completion functor from simplicial sets to pro-finite simplicial sets and the right derived functor of Lim [24, §2.1], Theorem B.1 is an immediate consequence of the following lemma.

Lemma B.8. *Let X be a connected simplicial set. There is a fibrant pro-finite simplicial set Y, a weak equivalence of pro-finite simplicial sets $\hat{X} \to Y$, and a cofibrant approximation $A \to C_{\mathrm{cont}}^* Y$ such that the map $Y \to U_{\mathrm{c}} A$ is a weak equivalence of pro simplicial sets.*

The remainder of the section is devoted to the proof of Lemma B.8. According to [24, §2.1], we can take $Y = \{Y_\alpha\}$ to have the property that each Y_α is a connected "*p-espace finis*", i.e. has finitely many non-trivial homotopy groups, all of which are finite p-groups. Choose such a Y and write $\mathfrak{I}$ for the filtering category opposite to the category that indexes Y. It is not hard to see that we can make an $\mathfrak{I}$ diagram of cofibrant $\mathcal{E}$-algebras A_α with a natural acyclic fibration $A_\alpha \to C^* Y_\alpha$ and with the property that $A = \mathrm{Colim}\, A_\alpha$ is also cofibrant. For example, it is straightforward to check that $LC^* Y_\alpha$ has this property where L is the cofibrant approximation functor obtained by the small object argument in Proposition 2.6. Alternatively, after replacing Y with an isomorphic object if necessary, we can assume that $\mathfrak{I}$ is a cofinite strongly directed category, and then such a diagram A_α is easily constructed by induction. Note that however the A_α are constructed, the induced map $A \to C_{\mathrm{cont}}^* Y$ is an acyclic fibration. We choose Y and A in this way in order to make the following observation.

Proposition B.9. *For Y and A as above, for each α, the map from the constant pro simplicial set Y_α to $U_{\mathrm{c}} A_\alpha$ is a weak equivalence.*

Proof. According to Remark 7.4, since Y_α has only finitely many nontrivial homotopy groups, all of which are finite p-groups, there is a finite cell $\mathcal{E}$-algebra B and a quasi-isomorphism $B \to C^* Y_\alpha$. By Proposition B.5, $U_{\mathrm{c}} B$ is isomorphic to the constant pro simplicial set on UB, and so by the Main Theorem, the map $Y_\alpha \to U_{\mathrm{c}} B$ is a weak equivalence. But by the left lifting property, the map $B \to C^* Y_\alpha$ can be factored through a quasi-isomorphism $B \to A_\alpha$, and so the map $Y_\alpha \to U_{\mathrm{c}} A_\alpha$ is also a weak equivalence. $\square$

Let $\mathfrak{J}$ be the category whose set of objects is the disjoint union of the sets of objects of the $\mathfrak{R}_{A_\alpha}$ where α ranges over the objects of $\mathfrak{I}$. For $a \in \mathfrak{R}_{A_\alpha}$, $b \in \mathfrak{R}_{A_\beta}$, we have a map $a \to b$ in $\mathfrak{J}$ for each map $A_\alpha \to A_\beta$ in $\mathfrak{I}$ that maps the pair of differential graded submodules (M, N) corresponding to a into the pair of differential graded submodules corresponding to b. Clearly $\mathfrak{J}$ is a filtered category. The functors $\mathfrak{D}_{A_\alpha} \colon \mathfrak{R}_{A_\alpha} \to \mathfrak{E}$ assemble to a functor $D \colon \mathfrak{J} \to \mathfrak{E}$, which we regard as an element of ind-$\mathfrak{E}$. We have a canonical map $D \to \{A_\alpha\}$ covering the forgetful functor $\mathfrak{J} \to \mathfrak{I}$ and inducing an isomorphism $\mathrm{Colim}\, D \to \mathrm{Colim}\, A_\alpha = A$. Since D is a diagram of compact $\mathcal{E}$-algebras, the map $D \to \{A_\alpha\}$ factors through an isomorphism $D \to \mathrm{c}A$ by Proposition B.6.

Proof of Lemma B.8. If we choose a basepoint for X, we obtain compatible basepoints for the Y_α so that Y is a system of based connected simplicial sets. Then

it suffices to show that the map $Y \to U_\mathrm{c}A$ induces a pro-isomorphism of each homotopy pro-group $\pi_n Y \to \pi_n U_\mathrm{c} A$. By construction, the map $Y \to U_\mathrm{c} A$ factors through the map $Y \to \{U A_\alpha\}$; we base $U A_\alpha$ and the simplicial sets in $U_\mathrm{c} A_\alpha$ at the image of the basepoint of Y_α. Looking at D, we can identify $\pi_n U_\mathrm{c} A$ as the limit (over α in $\mathfrak{I}$) in pro-groups of the pro-groups $\{\pi_n U_\mathrm{c} A_\alpha\}$. Since $\pi_n Y$ is the limit (over α in $\mathfrak{I}$) in pro-groups of the constant pro-groups $\pi_n Y_\alpha$, the lemma now follows from Proposition B.9. $\qquad\square$

Appendix C. E_∞ Ring Spectra under $H\bar{\mathbf{F}}_p$

We sketch how the arguments in this paper can be modified to prove the following unpublished theorem of W. G. Dwyer and M. J. Hopkins [8] comparing the p-adic homotopy category with the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra.

Theorem C.1. (Dwyer–Hopkins) *The free mapping spectrum functor $F((-)_+, \bar{\mathbf{F}}_p)$ induces an equivalence between the homotopy category of connected p-complete nilpotent spaces of finite p-type and a full subcategory of the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra.*

By the homotopy category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra, we mean the category obtained from the category of E_∞ ring spectra under the (cofibrant) E_∞ ring spectrum $H\bar{\mathbf{F}}_p$ by formally inverting the weak equivalences. The free mapping spectrum $F(X_+, H\bar{\mathbf{F}}_p)$ is naturally an E_∞ ring spectrum with an E_∞ ring map

$$H\bar{\mathbf{F}}_p = F(*_+, H\bar{\mathbf{F}}_p) \to F(X_+, H\bar{\mathbf{F}}_p)$$

induced by the collapse map $X \to *$. The functor $F((-)_+, \bar{\mathbf{F}}_p)$ therefore takes values in the category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra. This functor is the spectrum analogue of the singular chain complex. Its right derived functor represents unreduced ordinary cohomology with coefficients in $\bar{\mathbf{F}}_p$ in the sense that there is a canonical map $\pi_{-*}(F(X_+, \bar{\mathbf{F}}_p)) \to H^*(X; \bar{\mathbf{F}}_p)$ that is an isomorphism if X is a CW complex.

It is convenient for us to use a modern variant of the category of E_∞ $H\bar{\mathbf{F}}_p$ ring spectra, the category of commutative $H\bar{\mathbf{F}}_p$-algebras, a certain subcategory introduced in [10]. The forgetful functor from commutative $H\bar{\mathbf{F}}_p$-algebras to E_∞ $H\bar{\mathbf{F}}_p$ ring spectra induces an equivalence of homotopy categories. We have a commutative $H\bar{\mathbf{F}}_p$-algebra variation of the free mapping spectrum functor, given by

$$FX = S \wedge_{\mathcal{L}} F(X_+, H\bar{\mathbf{F}}_p).$$

There is a natural map $FX \to F(X_+, H\bar{\mathbf{F}}_p)$ that is always a weak equivalence, and so it suffices to prove that the functor F induces an equivalence between the homotopy category of connected p-complete nilpotent spaces of finite p-type and a full subcategory of the homotopy category of commutative $H\bar{\mathbf{F}}_p$-algebras. We denote the category of commutative $H\bar{\mathbf{F}}_p$-algebras as $\mathfrak{C}$. By [10, VII.4.10], $\mathfrak{C}$ is a closed model category with weak equivalences the weak equivalences of the underlying spectra; we denote its homotopy category as $\bar{h}\mathfrak{C}$.

The commutative $H\bar{\mathbf{F}}_p$-algebra FX is the "cotensor" of $H\bar{\mathbf{F}}_p$ with X [10, VII.2.9]. In general, the cotensor A^X of a commutative $H\bar{\mathbf{F}}_p$-algebra A with the space X is the commutative $H\bar{\mathbf{F}}_p$-algebra that solves the universal mapping problem $\mathfrak{C}(-, A^X) \cong \mathfrak{U}(X, \mathfrak{C}(-, A))$, where $\mathfrak{U}$ denotes the category of (compactly generated and weakly Hausdorff) spaces. Similarly, the tensor $A \otimes X$ of A with the space X is the commutative $H\bar{\mathbf{F}}_p$-algebra that solves the universal mapping problem $\mathfrak{C}(A \otimes X, -) \cong \mathfrak{U}(X, \mathfrak{C}(A, -))$. Clearly, when they exist A^X and $A \otimes X$ are unique

up to canonical isomorphism, and [10, VII.2.9] guarantees that they exist for any A and any X. The significance of the identification of FX as the cotensor is in the following proposition.

Proposition C.2. *The functor $T\colon \mathfrak{C} \to \mathfrak{U}$ defined by $TA = \mathfrak{C}(A, H\bar{\mathbf{F}}_p)$ is a continuous contravariant right adjoint to F. In other words, there is a homeomorphism $\mathfrak{U}(X, TA) \cong \mathfrak{C}(A, FX)$, natural in the space X and the commutative $H\bar{\mathbf{F}}_p$-algebra A.*

We have introduced the notion of tensor here to take advantage of [10, VII.4.16] that identifies the tensor $A \otimes I$ as a Quillen cylinder object when A is cofibrant. This allows us to relate the homotopies in the sense of Quillen with topological homotopies defined in terms of $(-) \otimes I$ or in terms of paths in mapping spaces. In particular, since all objects in $\mathfrak{C}$ are fibrant, it follows that the natural transformation $\pi_0(\mathfrak{C}(A, -)) \to \bar{h}\mathfrak{C}(A, -)$ is an isomorphism when A is cofibrant. Since the adjunction isomorphism $\mathfrak{U}(X, TA) \cong \mathfrak{C}(A, FX)$ is a homeomorphism, letting X vary over the spheres, we obtain the following proposition.

Proposition C.3. *The functor T preserves weak equivalences between cofibrant objects.*

As a slight generalization of the proof of [10, VII.4.16], it is elementary to check that when A is a cofibrant object of $\mathfrak{C}$ and $A \to B$ is a cofibration, the map $(A \otimes I) \amalg_A B \to B \otimes I$ is an acyclic cofibration and therefore (since every object is fibrant) the inclusion of a retract. Since T also converts pushouts to pullbacks, applying T and using the tensor adjunction, we obtain the following proposition.

Proposition C.4. *The functor T converts cofibrations to fibrations.*

The functors F and T are therefore a model category adjunction. In particular, we obtain the following proposition.

Proposition C.5. *The (right) derived functors $\mathbf{F}$ and $\mathbf{T}$ of F and T exist and give a contravariant right adjunction $\mathfrak{Ho}(X, \mathbf{T}A) \cong \bar{h}\mathfrak{C}(A, \mathbf{F}X)$.*

For the purposes of this section, let us say that a space X is $H\bar{\mathbf{F}}_p$-resolvable if the unit of the derived adjunction $X \to \mathbf{T}\mathbf{F}X$ is a weak equivalence. Thus, we need to show that if X is a connected p-complete nilpotent space of finite p-type, then X is $H\bar{\mathbf{F}}_p$-resolvable. Again, we work by induction up principally refined Postnikov towers. The following analogue of Theorem 1.1 can be proved from Proposition C.4 by essentially the same argument used to prove Theorem 1.1 from Proposition 4.4.

Proposition C.6. *Let $X = \mathrm{Lim}\, X_n$ be the limit of a tower of Serre fibrations. Assume that the canonical map from H^*X to $\mathrm{Colim}\, H^*X_n$ is an isomorphism. If each X_n is $H\bar{\mathbf{F}}_p$-resolvable, then X is resolvable.*

We have in addition the following analogue of Theorem 1.2.

Theorem C.7. *Let X, Y, and Z be connected spaces of finite p-type, and assume that Z is simply connected. Let $X \to Z$ be a map, and let $Y \to Z$ be a Serre fibration. If X, Y, and Z are $H\bar{\mathbf{F}}_p$-resolvable, then so is the fiber product $X \times_Z Y$.*

The proof of this theorem is essentially the same in outline as the proof of Theorem 1.2. The analogue of Lemma 5.2 can be proved by observing that the bar construction of the cofibrant approximations in $\mathfrak{C}$ is equivalent to the (thickened) realization of F applied to the cobar construction of the singular simplicial sets on the

spaces $X_\bullet$, $Y_\bullet$, and $Z_\bullet$. Some fiddling with the filtration induced by the cosimplicial direction of the cobar construction and the filtration induced by the skeletal filtration of the singular simplicial sets allows the identification of $\operatorname{Tor}_{-*}^{F|Z_\bullet|}(F|X_\bullet|, F|Y_\bullet|)$ as $\operatorname{Tor}_{C^*Z}^*(C^*X, C^*Y)$ and the composite map

$$\operatorname{Tor}_{C^*Z}^*(C^*X, C^*Y) \cong \operatorname{Tor}_{-*}^{F|Z_\bullet|}(F|X_\bullet|, F|Y_\bullet|) \to \pi_{-*}F(|X_\bullet \times_{Z_\bullet} Y_\bullet|) \cong H^*(X \times_Z Y)$$

as the Eilenberg–Moore map.

To complete the proof of Theorem C.1, we need to see that $K(\mathbf{Z}/p\mathbf{Z}, n)$ is $H\bar{\mathbf{F}}_p$-resolvable. It then follows as in Section 1.3 that $K(\mathbf{Z}_p^\wedge, n)$ is $H\bar{\mathbf{F}}_p$-resolvable and by induction up principal Postnikov towers that every connected p-complete nilpotent space of finite p-type is $H\bar{\mathbf{F}}_p$-resolvable. The remainder of the appendix is devoted to sketching a proof of the following theorem.

Theorem C.8. *For $n \geq 1$, $K(\mathbf{Z}/p\mathbf{Z}, n)$ is $H\bar{\mathbf{F}}_p$-resolvable.*

The homotopy groups of a commutative $H\bar{\mathbf{F}}_p$-algebra have an action by the algebra $\mathfrak{B}$, and it is elementary to show that for the "free" commutative $H\bar{\mathbf{F}}_p$-algebra on the spectrum S^{-n}, denoted $\mathbb{P}S_{H\bar{\mathbf{F}}_p}^{-n}$ in [10], $\pi_{-*}\mathbb{P}S_{H\bar{\mathbf{F}}_p}^{-n}$ is $\bar{\mathbb{V}}\mathfrak{B}_n^{\mathrm{un}}$, the extended $\bar{\mathbf{F}}_p$-algebra on the enveloping algebra of the free unstable $\mathfrak{B}$-module on one generator in degree n. We construct a commutative $H\bar{\mathbf{F}}_p$-algebra B_n as the commutative $H\bar{\mathbf{F}}_p$-algebra that makes the following diagram a pushout in $\mathfrak{C}$.

$$
\begin{array}{ccc}
\mathbb{P}S_{\bar{\mathbf{F}}_p}^{-n} & \longrightarrow & \mathbb{P}CS_{\bar{\mathbf{F}}_p}^{-n} \\
{\scriptstyle p_n}\downarrow & & \downarrow \\
\mathbb{P}S_{\bar{\mathbf{F}}_p}^{-n} & \longrightarrow & B_n
\end{array}
$$

Here p_n is any map in the unique homotopy class that on homotopy groups sends the fundamental class of $\pi_{-n}S_{H\bar{\mathbf{F}}_p}^{-n}$ to $1 - P^0$ applied to the fundamental class. Choosing a map $a\colon \mathbb{P}S_{\bar{\mathbf{F}}_p}^{-n} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ that represents the fundamental class of $H^n(K(\mathbf{Z}/p\mathbf{Z}, n))$, and a null homotopy $\mathbb{P}CS_{\bar{\mathbf{F}}_p}^{-n} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ for the map $p_n \circ a\colon \mathbb{P}S_{\bar{\mathbf{F}}_p}^{-n} \to FK(\mathbf{Z}/p\mathbf{Z}, n)$, we obtain an induced map $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n)$.

Proposition C.9. *For $n \geq 1$, the map $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n)$ is a weak equivalence.*

The proof uses the Eilenberg–Moore spectral sequence of [10, IV.4.1] in place of the Eilenberg–Moore spectral sequence of Section 3, but otherwise is the same as the proof of Theorem 6.2.

Since B_n is a cofibrant commutative $H\bar{\mathbf{F}}_p$-algebra, the unit of the derived adjunction is represented by the map $K(\mathbf{Z}/p\mathbf{Z}, n) \to TB_n$ adjoint to the map constructed above. Since B_n is defined as a pushout of a cofibration, Proposition C.4 allows us to identify TB_n as the pullback of fibration. Looking at the mapping spaces and using the freeness adjunction, we see that TB_n is the homotopy fiber of an endomorphism on $K(\bar{\mathbf{F}}_p, n)$. Write α_n for the induced endomorphism on $\bar{\mathbf{F}}_p$. To see that TB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$, it suffices to show that α_n is $1 - \Phi$. Once we know that TB_n is a $K(\mathbf{Z}/p\mathbf{Z}, n)$, the argument of Corollary 6.3 shows that the map $K(\mathbf{Z}/p\mathbf{Z}, n) \to TB_n$ is a weak equivalence, completing the proof of Theorem C.8.

Unfortunately, the simple argument given in Proposition 6.5 to identify α_n as $1 - \Phi$ in the algebraic case does not have a topological analogue. Here we must

use homotopy theory to make this identification. The key observation is that the commutative $H\bar{\mathbf{F}}_p$-algebras B_n are related by "suspension". We make this precise in the following proposition. For this proposition, note that the definition of B_n makes sense for $n = 0$, although the map $B_0 \to FK(\mathbf{Z}/p\mathbf{Z}, 0)$ may not be a weak equivalence.

Proposition C.10. *For $n > 0$, B_{n-1} is homotopy equivalent as a commutative $H\bar{\mathbf{F}}_p$-algebra to the pushout of the following diagram*

$$\begin{array}{ccc} B_n & \longrightarrow & B_n \otimes S^1 \\ \downarrow & & \\ H\bar{\mathbf{F}}_p & & \end{array}$$

where the map $B_n \to H\bar{\mathbf{F}}_p$ is the augmentation $B_n \to FK(\mathbf{Z}/p\mathbf{Z}, n) \to F* = H\bar{\mathbf{F}}_p$ induced by the inclusion of the basepoint of $K(\mathbf{Z}/p\mathbf{Z}, n)$ and the map $B_n \to B_n \otimes S^1$ is induced by the inclusion $* \to S^1$.

For an augmented commutative $H\bar{\mathbf{F}}_p$-algebra A, denote the analogous pushout for A as $\Sigma_{\mathfrak{C}} A$. If we give $\mathbb{P}S^{-n}_{H\bar{\mathbf{F}}_p}$ the augmentation induced by applying $\mathbb{P}$ to the map $S^n_{H\bar{\mathbf{F}}_p} \to *$, then $\Sigma_{\mathfrak{C}} \mathbb{P}S^{-n}_{H\bar{\mathbf{F}}_p}$ is canonically isomorphic to $\mathbb{P}S^{-n+1}_{H\bar{\mathbf{F}}_p}$. This gives us a canonical suspension homomorphism $\sigma \colon \tilde{\pi}_{-n} A \to \tilde{\pi}_{-n+1} \Sigma_{\mathfrak{C}} A$, where $\tilde{\pi}_*$ is the kernel of the augmentation map $\pi_* A \to \pi_* H\bar{\mathbf{F}}_p$. The following proposition is closely related to and can be deduced from [22, 3.3].

Proposition C.11. *The suspension homomorphism σ commutes with the operation P^s for all s.*

We can choose the map p_n in the construction of B_n to be augmented for the augmentation described on $\mathbb{P}S^n_{H\bar{\mathbf{F}}_p}$ above. Then it follows from the previous proposition that $\Sigma_{\mathfrak{C}} p_n$ is homotopic to p_{n-1}. This observation can be used to prove Proposition C.10.

It follows from Proposition C.10 that TB_{n-1} is the loop space of TB_n. In fact, we see from the discussion above that the fiber sequence for TB_{n-1}

$$TB_{n-1} \to K(\bar{\mathbf{F}}_p, n-1) \to K(\bar{\mathbf{F}}_p, n-1)$$

is the loop of the corresponding fiber sequence for TB_n. In particular, α_n and α_{n-1} are the same endomorphisms of $\bar{\mathbf{F}}_p$. Since P^0 performs the p-th power map on classes in degree zero, α_0 is $1 - \Phi$. We conclude that α_n is $1 - \Phi$.

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