

# The Hecke algebra action and the Rezk logarithm on Morava E-theory of height 2

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Given a one-dimensional formal group of height 2, let  $E$  be the Morava E-theory spectrum associated to the Lubin–Tate universal deformation of this formal group. By computing with moduli spaces of elliptic curves, we provide an explicit description for an algebra of Hecke operators acting on  $E$ -cohomology. As an application, we obtain a vanishing result for Rezk’s logarithmic cohomology operation on the units of  $E$ . It identifies a family of elements in the kernel with meromorphic modular forms whose Serre derivative is zero. Our calculation connects to logarithms of modular units. In particular, we define an action of Hecke operators on certain logarithmic  $q$ -series, in the sense of Knopp and Mason, which agrees with our vanishing result and extends the classical Hecke action on modular forms.

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## 1 Introduction

Interactions between algebraic topology and number theory have brought fascinating advances in mathematics over the past two decades. This paper represents an attempt to understand the topological construction of Rezk’s logarithmic cohomology operations by calculating with Hecke operators from number theory.

Our main theorem identifies a family of elements in the kernel of a logarithmic cohomology operation with modular forms satisfying a certain differential property. The theorem explains and generalizes sample calculations in [Rezk2008, 2.8–2.9]. It introduces this novel family comparable, but not entirely analogous to the one in [Ando–Hopkins–Rezk2010, Theorem 12.3]. The latter was critical in the proof for the existence of certain highly structured cobordism invariants for manifolds. The techniques we employed rely on a careful and explicit description of a Hecke algebra action on elliptic cohomology theories, which builds upon recent progress with power operations in elliptic cohomology.

Algebraic topology concerns the stable category of spectra as well as the unstable category of topological spaces. Chromatic homotopy theory is the study of both categories through the lens filtered by height- $n$  homology theories. In this framework, Rezk constructed logarithmic cohomology operations that act on the units of spectra representing these homology theories. More precisely, they act on the spectra  $\mathrm{gl}_1(E)$  of a Morava E-theory  $E$  at height  $n$  [Rezk2006]. In the cases  $n = 1$  and  $2$ , these operations are crucial in the proof by Ando, Hopkins, and Rezk for the rigidification of the string-bordism elliptic genus [Ando–Hopkins–Rezk2010]. For the space of such highly structured genera, its set of components can be detected by elements in the kernel of a logarithmic cohomology operation. See Remark 1.2 below for more details in comparison with our main results, as well as [Douglas et al. 2014, p124] for further discussion on the (conjectural) significance of the fiber of a “topological logarithm” in number theory and smooth structures on spheres.

The units spectra of Morava E-theories at height  $n$  are also closely related to the  $\infty$ -groupoid  $\mathrm{Pic} C_n$  of invertible objects in a symmetric monoidal  $(\infty, n)$ -category  $C_n$ . This Picard groupoid is the target category of an invertible  $n$ -dimensional fully extended field theory as studied by Freed and Hopkins. They classified such field theories from computations in stable homotopy theory, with applications to condensed matter physics [Freed–Hopkins2016]. The homotopy types of these units spectra and related objects have recently been calculated by Rezk for  $n = 1$  and  $2$  using power operations [Rezk2013, Example 2.14] and by Hopkins and Lurie for arbitrary  $n$ . It is important to understand cohomology operations acting on these spectra.

Rezk’s logarithmic cohomology operations on the units spectra are built through spaces via the Bousfield–Kuhn functors. These functors send spaces to spectra and behave well with respect to the chromatic filtration by heights  $n$ . As such, they have played a prominent role in recent work of Behrens and Rezk, among others, generalizing Quillen’s and Sullivan’s work on rational homotopy to higher chromatic levels [Behrens–Rezk2017] (see also [Rezk2013, Example 2.13] and [Zhu2018] for applications). A further purpose of the current paper is to understand the Bousfield–Kuhn functors through the logarithmic cohomology operations, by exploring the arithmetic information these operations carry.

This paper was thus generated from the circle of ideas and results discussed above. Our approach to the Rezk logarithms on Morava E-theories is based on recent progress with structural and computational understanding of power operations for E-theories, particularly at height 2 as gleaned from moduli spaces of elliptic curves [Rezk2013, Zhu2015].

To explain the main ideas, we first remark that Hecke operators have been studied as cohomology operations on various sorts of elliptic cohomology theories by Baker [Baker1989, Baker1990], Ando [Ando1992, Ando1995], Ganter [Ganter2006, Ganter2009, Ganter2013a, Ganter2013b], and others.

Here, Morava E-theory spectra are  $E_\infty$ -ring spectra and are thus equipped with power operations, which then give rise to Hecke operators as cohomology operations. The work of Ando, Hopkins, and Strickland provides a correspondence between these power operations and deformations of Frobenius isogenies of formal groups [Ando–Hopkins–Strickland2004] (see also [Rezk2009, Theorem B]).

At height 2, the Serre–Tate theorem gives a second correspondence between isogenies of formal groups and isogenies of elliptic curves, in terms of their respective deformation theories [Lubin–Serre–Tate1964] (cf. [Katz–Mazur1985, Theorem 2.9.1]).

Via these two bridges, the classical action of Hecke operators on modular forms then corresponds to an action of “topological” Hecke operators on an E-theory at height 2 [Rezk2006, Section 14]. Rezk wrote down a formula that relates these actions to the logarithmic cohomology operations [ibid., 1.12].

In this paper, we provide an explicit and precise comparison between the classical and topological Hecke actions. Using this connection as a computational device, we then obtain the following main results.

**Theorem 1.1** (Proposition 2.10 and Theorem 3.22) *Let  $E$  be a Morava E-theory spectrum of height 2 at the prime  $p$ , and let  $N > 3$  be any integer prime to  $p$ .*

- (i) *There is a ring homomorphism  $\beta: \mathrm{MF}[\Gamma_1(N)] \rightarrow E^0$  from the graded ring of weakly holomorphic modular forms for  $\Gamma_1(N)$  to the coefficient ring of  $E$  in degree zero.*
- (ii) *Given  $f \in (\mathrm{MF}[\Gamma_1(N)])^\times$  with trivial Nebentypus, if its Serre derivative equals zero, then the element  $\beta(f)$  is contained in the kernel of Rezk’s logarithmic cohomology operation  $\ell_{2,p}: (E^0)^\times \rightarrow E^0$ .*

The statement and proof of Theorem 1.1 involve a model for the E-theory from moduli of elliptic curves. We shall address the equivalence of these models in Remark 2.4 and formulate the exact choices needed in Definitions 2.9 and 3.8. Thus, up to modifying  $\beta$  by an isomorphism on the target, the conclusion in part (ii) of Theorem 1.1 does not depend on the choice of such a model.

**Remark 1.2** Our results have potential applications to cobordism invariants for families of manifolds parametrized by spaces. More specifically, it is useful to understand the (full) kernel of the Rezk logarithms in view of Ando, Hopkins, and Rezk’s work on the rigidification of the string-bordism elliptic genus. Let us explain the relationship between their results and ours.

In [Ando–Hopkins–Rezk2010], the authors showed the existence of  $E_\infty$  string orientations of the spectrum of topological modular forms. For the space of such orientations, its set of components is detected by elements in the kernel of a Rezk logarithm at height 2. Explicitly, these elements are identified with Eisenstein series, which are eigenforms of Hecke operators [ibid., Theorem 12.3]. Subsequent results in this direction include [Naumann–Sprang2018, Theorem 3.2.6] and [Wilson2015, Theorem 0.4].

Theorem 1.1 above gives a different account of elements in the kernel of a Rezk logarithm at height 2. They are meromorphic modular forms with vanishing Serre derivative, including modular forms whose zeros and poles are located only at the cusps, such as the modular discriminant.

The discrepancy between these two sets of elements, Eisenstein series as opposed to the discriminant, results from the different domains of the Rezk logarithms. The former is the group of units in the zeroth cohomology of an even-dimensional sphere, while the latter is with respect to the infinite-dimensional complex projective space  $\mathbb{C}P^\infty$ , i.e., the logarithms are defined on  $(E^0(S^{2k}))^\times$  and  $(E^0(\mathbb{C}P^\infty))^\times$  respectively.

The finiteness of  $E^0(S^{2k})$  as a module over  $E^0$  leads to a simple formula for the logarithm. Specifically, the logarithm can be written as a combination of Hecke operators acting on  $\log(1+f)$ , where  $f$  is a generator of the truncated polynomial ring  $E^0(S^{2k})$ . Note that the formal power series expansion of  $\log(1+f)$  simply equals  $f$ , because  $f^2 = 0$  (see [Ando–Hopkins–Rezk2010, Proposition 4.8 and Example 4.9]).

In our case of  $E^0(\mathbb{C}P^\infty)$ , we calculate instead with  $\log(g)$  for units  $g$  in  $E^0$ . Through a generator  $u$  of the formal power series ring  $E^0(\mathbb{C}P^\infty)$ , certain  $g$  can be represented by meromorphic modular forms (cf. part (i) of Theorem 1.1). Without nilpotence of  $g$  as in the previous case, we apply a different set of tools from number theory, including differential operators and  $q$ -expansions for modular forms.

The Rezk logarithms arise from a purely topological construction via Bousfield–Kuhn functors. In the process of reducing our computations to number theory, it is notable that the formula of Rezk for these logarithms resembles the logarithms of ratios of Siegel functions studied by Katz in the context of  $p$ -adic  $L$ -functions [Katz1976,

Section 10.1] (see Remark 3.17 for more details).

Our explicit description of the topological Hecke algebra enables this translation from algebraic topology to number theory, with computational consequences. It should have further applications bridging homotopy theory and arithmetic. Section 3.4 initiates such a direction from homotopy theory to arithmetic, by extending the Hecke action on modular forms to certain logarithmic  $q$ -series. It may be of interest beyond its original motivation from homotopy theory.

**Remark 1.3** Placed in a slightly broader context, our work in this paper is a first step towards understanding a generalized form of elliptic cohomology theory. As explained in the previous remark, functions of the form  $\log(g)$  as well as the quasimodular Eisenstein series  $\mathcal{E}_2$  appear in our calculation of the logarithms on  $E$ -cohomology (see also the proof of Lemma 3.23 and Section 3.4 below). Their occurrence indicates that Morava  $E$ -theories at height 2 witness a larger class of functions on elliptic curves than modular forms. Here it is essential to have the completion in  $E^0$  and the property of logarithms taking products to sums. After our results had been communicated, Rezk formulated an elliptic cohomology theory of *topological quasimodular forms* in his Felix Klein lectures in Bonn 2015 [Rezk2018, the second remark below Theorem 1.29] (see also [Movasati2012] for related algebraic geometry).

Rezk also observed the following chain of connections, not yet well understood [Rezk2015]. Serre’s differential operator, which is singled out in the condition for part (ii) of Theorem 1.1, appears in the matrix for the Gauss–Manin connection on the de Rham cohomology  $H_{\mathrm{dR}}^1(\mathcal{C}_S)$  of a (universal) elliptic curve  $\mathcal{C}$  over a (moduli) scheme  $S$  [Katz1973, Section A1.4] (cf. Section 2.1 below for the notation). This de Rham cohomology group sits in a Hodge extension which generates an Ext-group of certain rank-1 modules over the Dyer–Lashof algebra of  $E$  [Rezk2013, 11.5] (quasimodular forms classify splittings of this Hodge extension). As mentioned earlier in this section, Rezk computed homotopy groups of the spectrum of functions from  $H\mathbb{Z}$  to  $\mathrm{gl}_1(E)$ , by applying spectral sequences with this Ext-group as an  $E_2$ -term [ibid., Example 2.14].

**Remark 1.4** We obtained the results in Sections 2 and 4 concerning power operations on a height-2 Morava  $E$ -theory at the prime 5, prior to obtaining our formulas for all primes as presented in [Zhu2015]. The explicit models of elliptic curves involved in the 5-primary case were crucial in guiding us towards the general formulas. As higher chromatic analogues for the Adams operations in  $K$ -theory, these formulas have potential applications in the computation of stable homotopy groups of spheres and have already been applied in the context of unstable chromatic homotopy theory (see [Zhu2018]).

Although largely subsumed by our later paper [Zhu2015], the 5-primary calculations serve in the present paper as concrete examples for both classical and topological Hecke operators. It is for the accessibility and convenience of the reader that we shall present them as organized below.

Section 4 also contains explicit formulas for the topological Hecke operators valid for all primes. This material is independent of the main theorem on the Rezk logarithms. As a first application, Theorem 4.8 clarifies a result attributed to Ando in 1995 from Rezk’s 2014 ICM lecture (p18 of Rezk’s talk slides). Unlike claimed previously, the Hecke algebra and the center of the Dyer–Lashof algebra only agree in part.

**Convention 1.5** Let  $p$  be a prime and  $N > 3$  be an integer such that  $p \nmid N$ . In this paper, we denote by  $\mathrm{MF}[\Gamma_1(N)]$  the graded ring over  $\mathbb{Z}[1/N]$  of modular forms with respect to  $\Gamma_1(N)$ . As  $N$  is invertible in the ground ring, we may equivalently view these modular forms as defined over  $\mathbb{C}$  (i.e., *weakly holomorphic modular forms for  $\Gamma_1(N)$* ) or as defined over  $\overline{\mathbb{F}}_p$ . Cf. [Katz1973, the first paragraph of Introduction and Section 1.2].

For a concise and useful reference on  $\mathrm{MF}[\Gamma_1(N)]$ , we recommend [Ono2004]. In particular, we shall follow the standard Definitions 1.8, 1.12, 1.14, and 1.15 therein of modular forms for congruence subgroups and their Fourier expansions. See also [Baker1994] from the viewpoint of elliptic cohomology.

We shall refer to the elements in  $\mathrm{MF}[\Gamma_1(N)]$  as *modular forms of level  $\Gamma_1(N)$* , or simply as *modular forms* if the congruence subgroup  $\Gamma_1(N)$  is clear from the context.

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## 1.2 Outline of the paper

In Section 2, we study in detail the action of Hecke operators on Morava E-theories of height 2, with a series of examples, and prepare for its applications in later sections.

We first introduce in Section 2.1 models for such an E-theory from moduli of elliptic curves. The formal definition of these models is postponed to Section 2.2 as Definition 2.9. There we explain how each modular form of level  $\Gamma_1(N)$  corresponds to an element in the coefficient ring of a Morava E-theory at height 2 and at prime  $p \nmid N$ .

Section 2.3 then concerns how this correspondence is functorial under Frobenius isogenies of elliptic curves on the one side, and under power operations on the other. In Section 2.4, using modular descriptions of the classical Hecke operators, we construct topological Hecke operators from power operations. We further compare the classical and topological Hecke operators in Remark 2.27 and Example 2.28.

In Section 3, based on our explicit description of the Hecke algebra action, we study the Rezk logarithms on Morava E-theories of height 2. We begin with Section 3.1 and strengthen the definition of models to Definition 3.8. There the notion of a preferred model serves as a technical preparation for our applications (unnecessary for the Hecke action discussed in Section 2).

In Section 3.2, we recall a formula of Rezk for computing the logarithmic cohomology operations, which leads to a connection with Hecke operators. We formulate this connection carefully in Proposition 3.14. Section 3.3 contains the statement and proof of the main theorem.

In Section 3.4, we provide an alternative and more conceptual proof for the theorem, in special cases, by defining an appropriate Hecke action on a class of “log-cuspidal” modular functions (Definition 3.36).

We end the paper with Section 4 giving formulas for topological Hecke operators in terms of generators in the ring of power operations. We apply those formulas in Theorem 4.8 concerning the relationship between the Hecke algebra and the center of the ring of power operations. An appendix with formulas for power operations at the prime 5 concludes the series of examples throughout the paper.

The reader may perhaps like to read Section 2.4 first, and then proceed straight to Sections 3 or 4, referring to Section 2 when forced by the applications.



## 2 Hecke operators on Morava E-theories of height 2

### 2.1 Moduli of elliptic curves and models for an E-theory

Morava E-theory spectra can be viewed as topological realizations of Lubin–Tate rings, which classify deformations of formal groups over complete Noetherian local rings. Specifically, given a formal group  $\mathbb{G}_0$  of height  $n < \infty$  over a perfect field  $k \subset \mathbb{F}_p$ , the associated Morava E-theory (of height  $n$  at the prime  $p$ ) is a complex-oriented cohomology theory  $E$  whose formal group  $\mathrm{Spf}(E^0(\mathbb{C}P^\infty))$  is a universal deformation of  $\mathbb{G}_0$  over the Lubin–Tate ring

$$W(k)[[u_1, \dots, u_{n-1}]] \cong E^0$$

(see [Lubin–Tate1966, Section 3] and [Goerss–Hopkins2004, Section 7]). Such a universal deformation is unique up to unique isomorphism. For height  $n = 2$ , via the Serre–Tate theorem [Lubin–Serre–Tate1964] (cf. [Katz–Mazur1985, 2.9.1]), this universal deformation of a formal group can be obtained from a universal deformation of a supersingular elliptic curve. We construct the latter as the universal family of elliptic curves equipped with a level- $\Gamma_1(N)$  structure.

Let  $N \geq 3$  with  $(N, p) = 1$ . Write  $\mathcal{P}_N$  for the representable moduli problem of smooth elliptic curves over  $\mathbb{Z}[1/N]$  with a choice of a point  $P_0$  of exact order  $N$  and a nonvanishing 1-form  $\omega$ . The following two universal families represent  $\mathcal{P}_N$  for  $N = 4$  and  $N = 5$  respectively, and they suffice to model E-theories of height 2 at all primes  $p$ .

**Example 2.1** ([Zhu2014, Proposition 2.1]) The moduli problem  $\mathcal{P}_4$  is represented by the curve

$$\mathcal{C}_4: y^2 + Axy + ABx = x^3 + Bx^2$$

over the graded ring

$$S_4 := \mathbb{Z}[1/4][A, B, \Delta^{-1}] = \mathbb{Z}[1/2][A, B, \Delta^{-1}]$$

where  $|A| = 1$ ,  $|B| = 2$ , and  $\Delta = A^2B^4(A^2 - 16B)$ . The chosen point is  $P_0 = (0, 0)$  and the chosen 1-form is  $\omega = du$  with  $u = x/y$ . There are 3 cusps of  $\Gamma_1(4)$ , corresponding to the 3 factors of  $\Delta$  (counting without multiplicity).

**Example 2.2** ([Behrens–Ormsby2016, Corollary 1.1.10]) The moduli problem  $\mathcal{P}_5$  is represented by

$$\mathcal{C}_5: y^2 + Axy + B^2(A - B)y = x^3 + B(A - B)x^2$$

over the graded ring

$$S_5 := \mathbb{Z}[1/5][A, B, \Delta^{-1}]$$

where  $|A| = |B| = 1$  and  $\Delta = B^5(B - A)^5(A^2 + 9AB - 11B^2)$ . Again the chosen point is  $P_0 = (0, 0)$  and the chosen 1-form is  $\omega = du$  with  $u = x/y$ . Moreover, writing  $\zeta := e^{2\pi i/5}$ , we have  $\Delta = B^5(B - A)^5(A - (5\zeta^4 + 5\zeta - 2)B)(A + (5\zeta^4 + 5\zeta + 7)B)$ . This factorization is useful for explicit calculations with the 4 cusps of  $\Gamma_1(5)$ .

**Remark 2.3** The moduli problem  $\mathcal{P}_3$  is also representable [Mahowald–Rezk2009, Proposition 3.2], but  $[\Gamma_1(3)]$  is not (cf. [Katz–Mazur1985, 2.7.4]). The beginning of [Behrens–Ormsby2016, Section 1] explains the relationship between these two moduli problems (their moduli stacks are denoted by  $\mathcal{M}_1^1(N)$  and  $\mathcal{M}_1(N)$  respectively). For  $p = 2$ , compare [ibid., Corollary 1.1.11] and [Lawson–Naumann2014, Section 3.1] (see also Proposition 2.5 below).

From each of the above examples, restricting  $\mathcal{C}_N$  over a closed point in the mod- $p$  supersingular locus, we obtain a supersingular elliptic curve  $C_0$  over  $\overline{\mathbb{F}}_p$ . By the Serre–Tate theorem, the formal completion  $\widehat{\mathcal{C}}_N$  of  $\mathcal{C}_N$  at the identity then gives a universal deformation of the formal group  $\widehat{C}_0$ . Here  $\widehat{C}_0/\overline{\mathbb{F}}_p$  is of height 2, and it models  $\mathbb{G}_0/k$  for the E-theory we begin with.

**Remark 2.4** For a fixed E-theory, the various models each involve a choice of  $N$  for the  $\mathcal{P}_N$ -structures, and a choice of a supersingular elliptic curve over  $\overline{\mathbb{F}}_p$  equipped with a  $\mathcal{P}_N$ -structure. Over a separably closed field of characteristic  $p$ , any two formal group laws of the same height are isomorphic [Lazard1955, Théorème IV]. In view of this and the universal property in the Lubin–Tate theorem [Lubin–Tate1966, Theorem 3.1], we see that up to isomorphism and extension of scalars, these models for the E-theory are equivalent.

Topologically, a  $K(2)$ -localization corresponds to the above completion along the mod- $p$  supersingular locus, as we now describe.

The graded rings  $S_N$  in Examples 2.1 and 2.2 can be identified with  $\mathrm{MF}[\Gamma_1(N)]$ . Their topological realizations are the periodic spectra  $\mathrm{TMF}[\Gamma_1(N)]$  of topological modular forms of level  $\Gamma_1(N)$  (cf. [Mahowald–Rezk2009, Section 2] and [Hill–Lawson2016]). Following the convention that elements in algebraic degree  $k$  lie in topological degree  $2k$ , we have

$$\pi_*(\mathrm{TMF}[\Gamma_1(4)]) \cong \mathbb{Z}[1/2][A, B, \Delta^{-1}]$$

with  $|A| = 2$ ,  $|B| = 4$ ,  $\Delta = A^2 B^4 (A^2 - 16B)$  and

$$\pi_* (\mathrm{TMF} [\Gamma_1(5)]) \cong \mathbb{Z}[1/5][A, B, \Delta^{-1}]$$

with  $|A| = |B| = 2$ ,  $\Delta = B^5 (B - A)^5 (A^2 + 9AB - 11B^2)$ .

Topological modular forms of level  $\Gamma_1(N)$  and Morava E-theories of height 2 are related as follows. Let  $K(2)$  be the Morava K-theory spectrum at height 2 and prime  $p$  with  $\pi_* (K(2)) \cong \overline{\mathbb{F}}_p[v_2^{\pm 1}]$ , where  $|v_2| = 2(p^2 - 1)$ . Let  $N > 3$  be an integer prime to  $p$ . Denote by  $L_{K(2)}(\mathrm{TMF} [\Gamma_1(N)])$  the Bousfield localization of  $\mathrm{TMF} [\Gamma_1(N)]$  with respect to  $K(2)$ .

**Proposition 2.5** *Given a Morava E-theory spectrum  $E$  of height 2 at the prime  $p$ , there is a non-canonical isomorphism*

$$L_{K(2)}(\mathrm{TMF} [\Gamma_1(N)]) \cong \underbrace{\overline{E} \times \cdots \times \overline{E}}_{m \text{ copies}}$$

of  $E_\infty$ -ring spectra, where  $\overline{E}$  is the extension of  $E$  over  $W(\overline{\mathbb{F}}_p)$  and  $m$  is the number of isomorphism classes of supersingular elliptic curves over  $\overline{\mathbb{F}}_p$  equipped with a level- $\Gamma_1(N)$  structure.

**Proof** Let  $C_0$  be a supersingular elliptic curve over  $\overline{\mathbb{F}}_p$  equipped with a level  $\Gamma_1(N)$ -structure. Its formal group  $\widehat{C}_0/\overline{\mathbb{F}}_p$  gives a model for the E-theory. By the Goerss–Hopkins–Miller theorem [Goerss–Hopkins2004, Corollary 7.6], the spectrum  $E$  admits an action of the automorphism group  $\mathrm{Aut}(C_0/\overline{\mathbb{F}}_p)$ .

Let  $G$  be the subgroup of  $\mathrm{Aut}(C_0/\overline{\mathbb{F}}_p)$  consisting of automorphisms that preserve the  $\Gamma_1(N)$ -structure on  $C_0$ . In view of Remark 2.4, we then obtain  $L_{K(2)}\mathrm{TMF}(\Gamma_1(N))$  by taking homotopy fixed points for the action of  $G$  on  $E$ , one such copy for each closed point in the mod- $p$  supersingular locus. By [Katz–Mazur1985, 2.7.4], the moduli problem  $[\Gamma_1(N)]$  is rigid when  $N > 3$ , and thus  $G$  is trivial. Again by the Goerss–Hopkins–Miller theorem, we obtain the stated isomorphism as one between  $E_\infty$ -ring spectra.  $\square$

## 2.2 Modular forms and homotopy groups; definition of $\mathcal{P}_N$ -models

Thanks to the explicit models for E-theories from elliptic curves, the global-to-local relationship between  $\mathrm{TMF} [\Gamma_1(N)]$  and  $E$  in Proposition 2.5 can be spelled out on

homotopy groups. The next example illustrates a general process of passing from  $\pi_*(\mathrm{TMF}[\Gamma_1(N)])$  to

$$\pi_*(E) \cong W(\overline{\mathbb{F}}_p)[[u_1]][u^{\pm 1}]$$

where we extend the scalars to  $W(\overline{\mathbb{F}}_p)$ . In particular, the *deformation parameter*  $u_1$  in degree 0 arises from a mod- $p$  Hasse invariant, and the 2-periodic unit  $u$  in degree  $-2$  corresponds to a local uniformizer at the identity of the universal elliptic curve  $\mathcal{C}_N$ .

**Example 2.6** Let  $p = 5$  and  $N = 4$  as in Example 2.1. By [Silverman2009, V.4.1a], the mod-5 Hasse invariant of  $\mathcal{C}_4$  equals

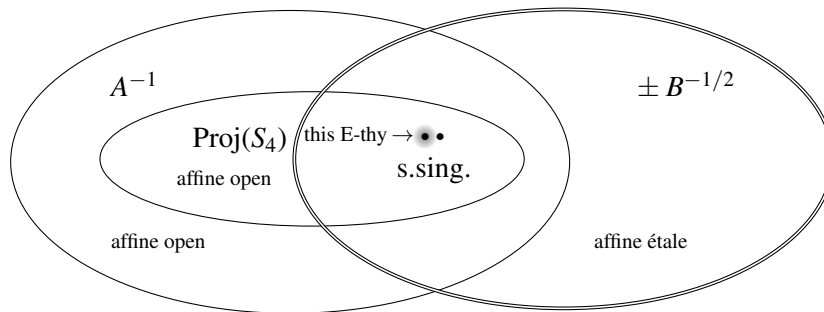
$$A^4 - A^2B + B^2 = (A^2 + 2(1 + \eta)B)(A^2 + 2(1 - \eta)B) \in \overline{\mathbb{F}}_5[A, B]$$

with  $\eta^2 = 2$ . Fix this  $\eta$ , and fix one of its lifts to  $W(\overline{\mathbb{F}}_5)$ , also denoted by  $\eta$  as an abuse of notation. For a reason that will become clear in (2.15), we choose an integral lift of this Hasse invariant given by

$$H := A^4 - 16A^2B + 26B^2 = (A^2 + 2(-4 + \eta)B)(A^2 + 2(-4 - \eta)B) \in S_4[\eta]$$

with  $\eta^2 = 19/2$ . The mod-5 supersingular locus of  $\mathcal{C}_4$  is then the closed subscheme of  $\mathrm{Proj}(S_4)$  cut out by the ideal  $(5, H)$ . Since  $S_4$  is a graded ring with generators  $A$  and  $B$  in different degrees, we have taken its *weighted projective scheme* (see, e.g., [Reid2002]). Over  $\overline{\mathbb{F}}_5$ , the supersingular locus consists of two closed points given the factorization above. Setting  $A = \eta$  and  $B = 1 - \eta$ , we obtain a supersingular curve  $C_0$  over  $\mathbb{F}_{25}$  corresponding to one of the closed points.

Since  $\Delta = A^2B^4(A^2 - 16B)$  gets inverted in  $S_4$ , the scheme  $\mathrm{Proj}(S_4)$  is affine and is contained in the affine open chart  $\mathrm{Proj}(\mathbb{Z}[1/2][A, B][A^{-1}])$  of the weighted projective space  $\mathrm{Proj}(\mathbb{Z}[1/2][A, B])$ . There is another affine chart  $\mathrm{Proj}(\mathbb{Z}[1/2][A, B][u])$  with  $u^2 = B^{-1}$  which is étale over  $\mathrm{Proj}(\mathbb{Z}[1/2][A, B])$ . The supersingular points (abbreviated “s.sing.”) are contained in both charts as illustrated below. The gray disk depicts a formal neighborhood of the point, where sits the formal group of a Morava E-theory of height 2 (abbreviated “this E-thy”).



Let us now pass to the homotopy groups of the E-theory spectrum  $E$  by a procedure of *dehomogenization* as follows. Define elements

$$\begin{aligned}
 (2.7) \quad & a := uA \\
 & u_1 := u^2(A^2 + 2(-4 + \eta)B) = a^2 - 8 + 2\eta \\
 & h := u^4H = a^4 - 16a^2 + 26 = (a^2 - 8 + 2\eta)(a^2 - 8 - 2\eta) \\
 & \delta := u^{12}\Delta = h - 26
 \end{aligned}$$

Following the convention that elements in algebraic degree  $k$  lie in topological degree  $2k$ , we then have

$$E_* \cong W(\overline{\mathbb{F}}_5)[[u_1]][u^{\pm 1}]$$

with  $|u_1| = 0$  and  $|u| = -2$ . In particular, by Hensel's lemma, both  $a$  and  $\delta$  are contained in  $(E_0)^\times$ . Moreover,  $u$  corresponds to a coordinate on  $\widehat{\mathcal{C}}_4$ , i.e., a local uniformizer at the identity of  $\mathcal{C}_4$ . This correspondence is via a chosen isomorphism  $\mathrm{Spf}(E^0(\mathbb{C}P^\infty)) \cong \widehat{\mathcal{C}}_4$  of formal groups over

$$\mathrm{Spf}(E^0) \cong \mathrm{Spf}(S_4[\eta]_{(5, A^2+2(-4+\eta)B)}^\wedge \otimes_{W(\mathbb{F}_5)} W(\overline{\mathbb{F}}_5))$$

(cf. [Ando–Hopkins–Strickland2001, Definition 1.2] and [Ando2000, Definition 1.4 and Remark 1.7]).

**Remark 2.8** As an abuse of notation, in Examples 2.1 and 2.2 we have written the local uniformizer

$$u = \frac{x}{y}$$

where  $x$  and  $y$  are the affine coordinates in the Weierstrass equation for  $\mathcal{C}_N$ . The resulting algebraic degree  $-1$  of  $u$  matches the topological degree  $-2$  of  $u$  in  $E_*$ .

To summarize, we arrive at the following.

**Definition 2.9** Let  $E$  be a Morava E-theory of height 2 at the prime  $p$ . With notation from above, a  $\mathcal{P}_N$ -model for  $E$  consists of the data as follows:

- a supersingular elliptic curve  $C_0/\overline{\mathbb{F}}_p$ ;
- a universal deformation  $\mathcal{C}_N/S_N$  of  $C_0$ ;
- a coordinate  $u$  on  $\widehat{\mathcal{C}}_N$ ;
- upon extension of scalars to  $W(\overline{\mathbb{F}}_p)$ , an isomorphism  $\mathrm{Spf}((S_N)_{\mathfrak{m}}^\wedge) \cong \mathrm{Spf}(E^0)$ , where  $\mathfrak{m}$  corresponds to the special fiber  $C_0/\overline{\mathbb{F}}_p$ ; and
- an isomorphism  $\widehat{\mathcal{C}}_N \cong \mathrm{Spf}(E^0(\mathbb{C}P^\infty))$  of formal groups, under which  $u$  corresponds to a unit  $u \in E_{-2}$ .

Recall that elements in  $\mathrm{MF}[\Gamma_1(N)]$  are functions  $f$  on elliptic curves  $C/R$  equipped with a  $\mathcal{P}_N$ -structure  $(P_0, \omega)$ . Each value  $f(C/R, P_0, \omega) \in R$  depends only on the  $R$ -isomorphism class of the triple  $(C/R, P_0, \omega)$  and is subject to a modular transformation property that encodes the weight of  $f$ . Moreover, its formation commutes with arbitrary base change (see, e.g., [Katz1973, Section 1.2]).

**Proposition 2.10** *Let  $E$  be a Morava  $E$ -theory of height 2 at the prime  $p$ . There is a non-canonical ring homomorphism  $\beta: \mathrm{MF}[\Gamma_1(N)] \rightarrow E^0$ .*

**Proof** Choose a  $\mathcal{P}_N$ -model for the  $E$ -theory. As in Example 2.6, via dehomogenization, a modular form  $f$  of weight  $k$  maps to  $u^k \cdot f(\mathcal{C}_N, P_0, du) \in E^0$ .  $\square$

### 2.3 Frobenius isogenies and power operations

Let  $\mathcal{C}_N$  over  $S_N$  be the universal curve in Section 2.1 for the moduli problem  $\mathcal{P}_N$ . Denote by  $\mathcal{G}_N^p$  the universal example of a degree- $p$  subgroup scheme of  $\mathcal{C}_N$ . It is defined over an extension ring  $S_N^p$ , which is free of rank  $p + 1$  as an  $S_N$ -module [Katz–Mazur1985, Theorem 6.6.1]. More explicitly,

$$S_N^p \cong S_N[\kappa] / (V(\kappa))$$

where  $\kappa$  is a generator with  $V(\kappa) = 0$  for a monic polynomial  $V$  of degree  $p + 1$ . The roots  $\kappa_0, \kappa_1, \dots, \kappa_p$  of  $V$  each correspond to a degree- $p$  subgroup scheme of  $\mathcal{C}_N$ . In particular, let  $\kappa_0$  correspond to the subgroup whose formal completion over an ordinary point is the unique degree- $p$  subgroup of the formal group  $\widehat{\mathcal{C}}_N$ .

We write

$$\Psi_N^p: \mathcal{C}_N \rightarrow \mathcal{C}_N / \mathcal{G}_N^p$$

for the universal degree- $p$  isogeny over  $S_N^p$ , and write  $\mathcal{C}_N^p$  for the quotient curve  $\mathcal{C}_N / \mathcal{G}_N^p$ . We next construct  $\Psi_N^p$  as a *deformation of Frobenius*, i.e., over any closed point in the mod- $p$  supersingular locus,  $\Psi_N^p$  restricts as the  $p$ -power Frobenius isogeny on the corresponding supersingular curve.

**Construction 2.11** (cf. [Lubin1967, proof of Theorem 1.4] and [Katz–Mazur1985, Section 7.7]) Let  $P$  be any point on  $\mathcal{C}_N$ . Let  $u \in S_N$  be a local uniformizer at the identity  $O$  of  $\mathcal{C}_N$ , so that  $|u| = -1$  tautologically.

- (i) Define  $\Psi_N^p: \mathcal{C}_N \rightarrow \mathcal{C}_N^p$  by the formula

$$u(\Psi_N^p(P)) := \prod_{Q \in \mathcal{G}_N^p} u(P - Q)$$

where the  $u$  on the left-hand side, by an abuse of notation, denotes the local coordinate on  $\mathcal{C}_N^p$  induced by  $\Psi_N^p$  as given on the right-hand side (cf. [Ando2000, Section 4.3]).

- (ii) Define  $\kappa \in S_N^p$  in degree  $-p + 1$  as

$$\kappa := \prod_{Q \in \mathcal{G}_N^p \setminus \{O\}} u(Q)$$

We verify that the isogeny  $\Psi_N^p$  has kernel precisely the subgroup  $\mathcal{G}_N^p$ . Moreover, it is a deformation of Frobenius, since at a supersingular point the  $p$ -divisible group is formal so that  $Q = O$  for all  $Q \in \mathcal{G}_N^p$ .

**Remark 2.12** The element  $\kappa$  in Construction 2.11 gives a *norm parameter* for the moduli problem  $[\Gamma_0(p)]$  as an “open arithmetic surface,” the other parameter being a deformation parameter (see [Katz–Mazur1985, Section 7.7]). As  $u$  is a local coordinate near the identity  $O$ , we observe that by the above construction the cotangent map  $(\Psi_N^p)^*$  at  $O$  sends  $du$  to  $\kappa \cdot du$ . Note that both  $\Psi_N^p$  and  $\kappa$  as defined above depend on a choice of  $u$ .

Via completion at a mod- $p$  supersingular point, we see in Section 2.2 that the ring  $S_N \cong \mathrm{MF}[\Gamma_1(N)]$  representing  $\mathcal{P}_N$  is locally realized in homotopy theory as the coefficient ring  $E^0$  of a Morava E-theory  $E$ . Here, given the ring  $S_N^p$  that represents the simultaneous moduli problem  $\mathcal{P}_N \times [\Gamma_0(p)]$ , Strickland’s theorem identifies the completion of  $S_N^p$  at the supersingular point as  $E^0(B\Sigma_p)/I$ , where  $I$  is an ideal of images of transfer maps [Strickland1998, Theorem 1.1]. In particular,  $E^0(B\Sigma_p)/I$  is free over  $E^0$  of rank  $p + 1$ , isomorphic to  $E^0[\alpha]/(w(\alpha))$  for a monic polynomial  $w$  of degree  $p + 1$ .

The universal degree- $p$  isogeny  $\Psi_N^p: \mathcal{C}_N \rightarrow \mathcal{C}_N^p$  is constructed above as a deformation of Frobenius. By [Rezk2009, Theorem B], it then corresponds to an (*additive*) *total power operation*

$$(2.13) \quad \psi^p: A^0 \rightarrow A^0(B\Sigma_p)/J$$

natural in  $K(2)$ -local commutative  $E$ -algebras  $A$ , where  $J$  is the corresponding transfer ideal. Since  $E^0(B\Sigma_p)$  is free over  $E^0$  of finite rank [Strickland1998, Theorem 3.2],

we have  $J \cong A^0 \otimes_{E^0} I$  and

$$A^0(B\Sigma_p)/J \cong (A^0 \otimes_{E^0} E^0(B\Sigma_p))/J \cong A^0 \otimes_{E^0} (E^0(B\Sigma_p)/I) \cong A^0[\alpha] / (w(\alpha))$$

Note that up to isomorphism,  $\psi^p$  is independent of the choice of a  $\mathcal{P}_N$ -model (cf. Remark 2.4). In particular, its independence on  $N$  follows from the functoriality with respect to base change of the moduli problem  $\mathcal{P}_N \times [\Gamma_0(p)]$ , which is finite flat over  $\mathcal{P}_N$ . Moreover, taking quotient by the transfer ideal makes  $\psi^p$  additive and hence a homomorphism of local rings.

**Example 2.14** We continue Example 2.6 with  $p = 5$  and  $N = 4$ . Let  $u = x/y$  as in Example 2.1 and let  $v = 1/y$ . The universal isogeny  $\Psi_4^5: \mathcal{C}_4 \rightarrow \mathcal{C}_4^5$  in Construction 2.11 is defined over the graded ring  $S_4^5 \cong S_4[\kappa] / (V(\kappa))$ , where  $|\kappa| = -4$  and

$$(2.15) \quad V(\kappa) = \kappa^6 - \frac{10}{B^2}\kappa^5 + \frac{35}{B^4}\kappa^4 - \frac{60}{B^6}\kappa^3 + \frac{55}{B^8}\kappa^2 - \frac{H}{B^{12}}\kappa + \frac{5}{B^{12}}$$

This polynomial is computed from the division polynomial  $\psi_5$  in [Silverman2009, Exercise 3.7] for the curve  $\mathcal{C}_4$ . Indeed, we first deduce from  $\psi_5$  identities satisfied by the  $uv$ -coordinates of a universal example  $Q \in \mathcal{G}_4^5 \setminus \{O\}$  as in [Zhu2014, proof of Proposition 2.2]. We then compute an explicit formula for  $\kappa = u(Q) \cdot u(-Q) \cdot u(2Q) \cdot u(-2Q)$  using methods analogous to [Silverman2009, III.2.3]. Finally we solve for a monic degree-6 equation satisfied by  $\kappa$  and obtain the polynomial  $V$ . (We do not compute an equation for  $\mathcal{C}_4^5$  as in [Zhu2014, Proposition 2.3].)

Passing to the corresponding power operation, we write

$$(2.16) \quad \alpha := u^{-4}\kappa_0$$

(see Remark 2.8), where  $\kappa_0$  corresponds to the subgroup of  $\mathcal{C}_4$  whose formal completion over an ordinary point is the unique degree-5 subgroup of  $\widehat{\mathcal{C}}_4$ . The total power operation  $\psi^5$  then lands in  $E^0(B\Sigma_5)/I \cong W(\overline{\mathbb{F}}_5)[[u_1, \alpha]] / (w(\alpha))$ , where  $w(\alpha) = \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - h\alpha + 5$ . We can now compute the effect of  $\psi^5$  on  $h$  as follows.

Consider a second universal degree-5 isogeny  $\widetilde{\Psi}_4^5: \mathcal{C}_4^5 \rightarrow \mathcal{C}_4^5 / \widetilde{\mathcal{G}}_4^5$ , where  $\widetilde{\mathcal{G}}_4^5 = \mathcal{C}_4[5] / \mathcal{G}_4^5$ . It is defined similarly as in Construction 2.11, with a parameter  $\widetilde{\kappa} \in S_4^5$  in degree  $-20 = 5 \cdot |\kappa|$ . Over  $S_4^5$ , the assignment

$$(\mathcal{C}_4, P_0, du, \mathcal{G}_4^5) \mapsto (\mathcal{C}_4^5, \Psi_4^5(P_0), du, \widetilde{\mathcal{G}}_4^5)$$

is an involution on the moduli problem  $\mathcal{P}_4 \times [\Gamma_0(5)]$  (cf. [Katz–Mazur1985, 11.3.1]). By rigidity, we have an identity  $\widetilde{\Psi}_4^5 \circ \Psi_4^5 = \tau \circ [5]$  that lifts  $\text{Frob}^2 = [5]$  over the supersingular point, where  $\text{Frob}$  is the 5-power Frobenius isogeny and  $\tau: \mathcal{C}_4 / \mathcal{C}_4[5] \rightarrow$



$\mathcal{C}_4^5 / \tilde{\mathcal{G}}_4^5$  is the canonical isomorphism. Thus in view of Remark 2.12 and  $\kappa_0 \kappa_1 \cdots \kappa_5 = 5/B^{12}$  from (2.15) we obtain a relation  $\tilde{\kappa} \cdot \kappa = 5/B^{12}$  in  $S_4^5$  (cf. [Zhu2014, proof of Corollary 3.2]).

Correspondingly, there is an involution  $(h, \alpha) \mapsto (\tilde{h}, \tilde{\alpha})$  on  $E^0(B\Sigma_5)/I$  coming from the Atkin–Lehner involution of modular forms on  $\Gamma_0(5)$  (cf. [Atkin–Lehner1970, Lemmas 7–10]). In particular, the relation

$$(2.17) \quad \alpha^6 - 10\alpha^5 + 35\alpha^4 - 60\alpha^3 + 55\alpha^2 - h\alpha + 5 = 0$$

has an analogue

$$(2.18) \quad \tilde{\alpha}^6 - 10\tilde{\alpha}^5 + 35\tilde{\alpha}^4 - 60\tilde{\alpha}^3 + 55\tilde{\alpha}^2 - \tilde{h}\tilde{\alpha} + 5 = 0$$

with

$$(2.19) \quad \tilde{\alpha} \cdot \alpha = 5$$

Based on these identities, we compute that

$$\begin{aligned}
 \psi^5(h) &= \tilde{h} \\
 &= \tilde{\alpha}^5 - 10\tilde{\alpha}^4 + 35\tilde{\alpha}^3 - 60\tilde{\alpha}^2 + 55\tilde{\alpha} + \alpha && \text{by (2.18) and (2.19)} \\
 &= (-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^5 \\
 &\quad - 10(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^4 \\
 &\quad + 35(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^3 \\
 &\quad - 60(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h)^2 \\
 &\quad + 55(-\alpha^5 + 10\alpha^4 - 35\alpha^3 + 60\alpha^2 - 55\alpha + h) \\
 &\quad + \alpha && \text{by (2.19) and (2.17)} \\
 (2.20) \quad &= h^5 - 10h^4 - 1065h^3 + 12690h^2 + 168930h \\
 &\quad - 1462250 + (-55h^4 + 850h^3 + 39575h^2 \\
 &\quad - 608700h - 1113524)\alpha + (60h^4 - 775h^3 \\
 &\quad - 45400h^2 + 593900h + 2008800)\alpha^2 \\
 &\quad + (-35h^4 + 400h^3 + 27125h^2 - 320900h \\
 &\quad - 1418300)\alpha^3 + (10h^4 - 105h^3 - 7850h^2 \\
 &\quad + 86975h + 445850)\alpha^4 + (-h^4 + 10h^3 \\
 &\quad + 790h^2 - 8440h - 46680)\alpha^5 && \text{by (2.17)}
 \end{aligned}$$

We also have  $\psi^5(c) = Fc$  for  $c \in W(\overline{\mathbb{F}}_5)$ , where  $F$  is the Frobenius automorphism. We would like to compute  $\psi^5(u_1)$  as well. As  $\psi^5$  is a homomorphism of local rings,

a formula for  $\psi^5(u_1)$  determines  $\psi^5(x)$  for all  $x \in E^0 \cong W(\overline{\mathbb{F}}_5)[[u_1]]$ . We will address this calculation using a modified presentation for  $E^0(B\Sigma_5)/I$  in a subsequent paper [Zhu2015].

The previous example illustrates a general recipe for computing power operations on a Morava E-theory at height 2 and prime  $p$ , with a model based on the moduli problem  $\mathcal{P}_N \times [\Gamma_0(p)]$ . Crucial in this computation is an explicit expression for

$$(2.21) \quad w(\alpha) = \alpha^{p+1} + w_p \alpha^p + \cdots + w_1 \alpha + w_0 \in E^0[\alpha]$$

Cf. [ibid., Theorem A].

## 2.4 Classical and topological Hecke operators: constructions, calculations, and comparison

We first describe in our setting the classical action of Hecke operators on modular forms in terms of isogenies between elliptic curves. The  $p$ 'th Hecke operator  $T_p$  which acts on  $\text{MF}[\Gamma_1(N)]$  can be built from universal isogenies as follows.

**Construction 2.22** (cf. [Katz1973, (1.11.0.2)]) Let the notation be as in Section 2.3, with the subscripts  $N$  suppressed. Given any  $f \in \text{MF}[\Gamma_1(N)]$  of weight  $k \geq 1$ ,  $T_p f \in \text{MF}[\Gamma_1(N)]$  is of weight  $pk$  such that

$$(2.23) \quad T_p f(\mathcal{C}_S, P_0, du) := \frac{1}{p} \sum_{i=0}^p \kappa_i^k \cdot f(\mathcal{C}_{S^p}/\mathcal{G}_i^p, \Psi_i^p(P_0), du)$$

where each  $\mathcal{G}_i^p$  denotes a degree- $p$  subgroup scheme of  $\mathcal{C}$  over  $S^p$ , and  $\Psi_i^p$  is the quotient map with kernel  $\mathcal{G}_i^p$  as in Construction 2.11.

**Remark 2.24** The terms  $\kappa_i^k$  appear in the above formula so that  $T_p$  is independent of the choice of a basis for the cotangent space (cf. Remark 2.12). Specifically, we have

$$(\Psi_i^p)^* du = \kappa_i \cdot du \quad \text{and} \quad (\Psi_i^p)^*(\check{\Psi}_i^p)^* du = p \cdot du$$

where each  $\check{\Psi}_i^p : \mathcal{C}/\mathcal{G}_i^p \rightarrow \mathcal{C}$  is a dual isogeny and  $(\check{\Psi}_i^p)^* du$  is the choice in [Katz1973] for a nonvanishing 1-form on a quotient curve (see [ibid., discussion above (1.11.0.0)]).

Thus we rewrite (2.23) as

$$\begin{aligned}
 T_p f(\mathcal{C}_S, P_0, du) &= \frac{1}{p} \sum_{i=0}^p \kappa_i^k \cdot f(\mathcal{C}_{S^p} / \mathcal{G}_i^p, \Psi_i^p(P_0), du) \\
 &= \frac{1}{p} \sum_{i=0}^p \kappa_i^k \cdot f(\mathcal{C}_{S^p} / \mathcal{G}_i^p, \Psi_i^p(P_0), \frac{\kappa_i}{p} \cdot (\check{\Psi}_i^p)^* du) \\
 &= \frac{1}{p} \sum_{i=0}^p p^k \cdot f(\mathcal{C}_{S^p} / \mathcal{G}_i^p, \Psi_i^p(P_0), (\check{\Psi}_i^p)^* du)
 \end{aligned}$$

and the last line agrees with [ibid., (1.11.0.2)].

**Construction 2.25** (cf. [Rezk2006, 1.12]) There is a *topological* Hecke operator  $t_p: E^0 \rightarrow p^{-1}E^0$  defined by

$$(2.26) \quad t_p(x) := \frac{1}{p} \sum_{i=0}^p \psi_i^p(x)$$

where  $\psi_i^p$  denotes the power operation  $\psi^p: E^0 \rightarrow E^0(B\Sigma_p)/I \cong E^0[\alpha] / (w(\alpha))$  with the parameter  $\alpha$  replaced by  $\alpha_i = u^{-p+1}\kappa_i$  (cf. (2.16)).

Since the parameters  $\alpha_i$  are the roots of  $w(\alpha) \in E^0[\alpha]$ ,  $t_p$  indeed lands in  $p^{-1}E^0$ .

**Remark 2.27** Along the ring homomorphism  $\beta$  in Proposition 2.10, the topological Hecke operator above is *not* compatible with the classical one on modular forms. More precisely, comparing (2.26) to (2.23), note that there are no terms  $\alpha_i^k$  corresponding to  $\kappa_i^k$ . This is related to the fact that  $\beta$  is not injective. If we include  $\alpha_i^k$  in the definition, for each  $x \in E^0$  we need to determine a unique value of its “weight”  $k$  so that  $t_p$  is well-defined.

Ando originally constructed Hecke operators in terms of power operations, which include such factors, for a version of  $\mathbb{Z}$ -graded elliptic cohomology theory due to Landweber, Ravenel, and Stong [Ando1992, Theorem 6.5.2]. In contrast, Morava E-theories are 2-periodic and hence essentially  $\mathbb{Z}/2$ -graded (cf. [Ando1995, Proposition 3.6.2]).

Note that the map  $\beta$  is not surjective either, so an element in  $E^0$  may not come from any modular form. On the other hand, the total power operation  $\psi^p$  (and hence  $t_p$ ) is defined on the entire  $E^0$ .

As a result of the definition of the total power operation (2.13), the operation  $t_p$  is independent of the choice of  $\alpha$ . Thus both the classical and topological Hecke operators are canonical (cf. Remark 2.24).

**Example 2.28** Again, let  $p = 5$  and  $N = 4$ . Recall from Example 2.6 that we have  $\beta(\Delta) = \delta = h - 26$ . In view of (2.17), we then compute from (2.20) that

$$\begin{aligned} t_5(\delta) &= \frac{1}{5} \sum_{i=0}^5 \psi_i^5(\delta) \\ &= \frac{1}{5} \sum_{i=0}^5 (\psi_i^5(h) - 26) \\ &= \frac{1}{5} (h^5 - 10h^4 - 1340h^3 + 18440h^2 + 267430h - 3178396) \\ &= \frac{1}{5} (h^4 + 16h^3 - 924h^2 - 5584h + 122246) \cdot \delta \end{aligned}$$

In contrast, since the modular form  $\Delta$  is of weight 12, we define and compute

$$\begin{aligned} \tilde{\tau}_5(\delta) &:= \frac{1}{5} \sum_{i=0}^5 \alpha_i^{12} \cdot \psi_i^5(\delta) \\ &= 4830h - 125580 \\ &= \tau(5) \cdot \delta \end{aligned}$$

where  $\tau: \mathbb{N} \rightarrow \mathbb{Z}$  is the Ramanujan tau-function, with  $\tau(5) = 4830$ . This calculation recovers the action of  $T_5$  on  $\Delta$ .

More generally, in the theory of automorphic forms on  $\Gamma_1(N) \subset \mathrm{SL}_n(\mathbb{Z})$  for  $n \geq 2$ , the above Hecke operator  $T_p$  can be renamed as  $T_{1,p}$ . It belongs to a family of operators  $T_{i,p}$ ,  $1 \leq i \leq n$  that generate the  $p$ -primary Hecke algebra (see, e.g., [Shimura1994, Theorems 3.20 and 3.35] and [Miyake1989, Sections 2.7–2.8]).

For  $n = 2$ , the other Hecke operator  $T_{2,p}$  arises from the isogeny of multiplication by  $p$ , whose kernel is the degree- $p^2$  subgroup of the  $p$ -torsion points. Explicitly (again with the subscripts  $N$  suppressed), if  $f \in \mathrm{MF}[\Gamma_1(N)]$  is of weight  $k \geq 2$ , then  $T_{2,p}f \in \mathrm{MF}[\Gamma_1(N)]$  has weight  $p^2k$  such that

$$\begin{aligned} (2.29) \quad T_{2,p}f(\mathcal{C}, P_0, du) &:= \frac{1}{p^2} ((p\lambda)^k \cdot f(\mathcal{C}/\mathcal{C}[p], [p]P_0, du)) \\ &= p^{k-2} \cdot f(\mathcal{C}/\mathcal{C}[p], [p]P_0, \lambda^{-1} du) \\ &= p^{k-2} \cdot f(\mathcal{C}, [p]P_0, du) \end{aligned}$$

for some  $\lambda \in S^\times$  of degree  $-p^2 + 1$ , where the last identity follows from the isomorphism  $\mathcal{C}/\mathcal{C}[p] \rightarrow \mathcal{C}$ . For example, when  $N = 4$  with  $p = 3$  or  $p = 5$ , we have  $\lambda = B^{(1-p^2)/2}$  (cf. Example 2.1). Via the canonical  $S^p$ -isomorphism

$$(2.30) \quad \mathcal{C}/\mathcal{C}[p] \cong \frac{\mathcal{C}/\mathcal{G}^p}{\mathcal{C}[p]/\mathcal{G}^p} =: \frac{\mathcal{C}^p}{\widetilde{\mathcal{G}}^p}$$

the quotient curve  $\mathcal{C}/\mathcal{C}[p]$  can be identified with the target in the composite  $\mathcal{C} \xrightarrow{\Psi^p} \mathcal{C}^p \xrightarrow{\widetilde{\Psi}^p} \mathcal{C}^p/\widetilde{\mathcal{G}}^p$  of deformations of Frobenius isogenies.

Correspondingly, given any  $K(2)$ -local commutative  $E$ -algebra  $A$ , there is a composite  $\phi$  of total power operations  $\psi^p \circ \psi^p: A^0 \rightarrow A^0$  (cf. [Zhu2014, (26)]). In view of (2.13), note that  $\phi$  lands in  $A^0$  because up to the isomorphism (2.30) the composite  $\widetilde{\Psi}^p \circ \Psi^p$  is an endomorphism on  $\mathcal{C}$  over  $S$ . In particular, on  $W(\mathbb{F}_{p^2})[[u_1]] \subset E^0$ , the operation  $\phi$  is the identity map (this is also true on  $W(\mathbb{F}_{p^2})[[u_1, \alpha]]/(w(\alpha)) \subset E^0(B\Sigma_p)/I$ ). The trivial action of  $\phi$  on  $u_1$  and  $\alpha$  is a manifest of the Atkin-Lehner involution (cf. Example 2.14).

Taking  $A = E$ , we define a *topological* Hecke operator  $t_{2,p}: E^0 \rightarrow p^{-1}E^0$  by

$$(2.31) \quad t_{2,p}(x) := p^{-2}\phi(x) = p^{-2}\psi^p(\psi^p(x))$$

Rewrite  $t_{1,p} := t_p$ . The ring  $\mathbb{Z}[t_{1,p}, t_{2,p}]$  then acts on  $p^{-1}E^0$ . As we see in Remark 2.27, along the map  $\beta: \text{MF}[\Gamma_1(N)] \rightarrow E^0$ , this action is not compatible with the action of the Hecke algebra  $\mathbb{Z}[T_{1,p}, T_{2,p}]$  on modular forms. Nevertheless, in certain instances, the two actions do interact well. We shall exploit such a connection in Section 3 (see (3.16) particularly).

### 3 Kernel of the Rezk logarithm on a Morava E-theory of height 2

#### 3.1 Preferred models for an E-theory

For our applications of the topological Hecke operators built from power operations, we need to further impose two technical conditions on a model for an E-theory as previously stated in Definition 2.9.

The first condition is for the supersingular elliptic curve  $C_0/\overline{\mathbb{F}}_p$  of the model. Let  $E$  be a Morava E-theory of height 2 at  $p$  and choose a  $\mathcal{P}_N$ -model for it. To ease notation,

henceforth we write

$$\pi := (-1)^{p-1}p$$

Given the supersingular elliptic curve  $C_0$  over  $\overline{\mathbb{F}}_p$ , there exists a supersingular curve  $C'_0$  over  $\mathbb{F}_{p^2}$ , isomorphic over  $\overline{\mathbb{F}}_p$  to  $C_0$ , such that its  $p^2$ -power Frobenius endomorphism  $\text{Frob}^2: C'_0 \rightarrow C'_0$  satisfies

$$(3.1) \quad \text{Frob}^2 = [\pi] = \begin{cases} [-p] & \text{if } p = 2 \\ [p] & \text{if } p \neq 2 \end{cases}$$

[Poonen2010] (cf. [Baker et al. 2005, Lemma 3.21], [Zhu2014, Remark 3.3], and [Rezk2012, 3.8]). For a reason that will become clear in (3.9), let us replace  $C_0$  in this model by  $C'_0$  over  $\overline{\mathbb{F}}_p$  (after base change from  $\mathbb{F}_{p^2}$ ).

Correspondingly, replace  $\mathcal{C}_N$  up to an isomorphism as needed so that its restriction over a mod- $p$  supersingular point satisfies (3.1). By rigidity [Katz–Mazur1985, 2.4.2], this identity of endomorphisms lifts to be  $\tilde{\Psi}_N^p \circ \Psi_N^p = \tau \circ [\pi]$  between isogenies, where  $\tau$  is the *canonical* isomorphism (2.30). In view of Remark 2.12, since  $\tau$  induces the identity map on relative cotangent spaces, we then obtain

$$(3.2) \quad \tilde{\alpha} \cdot \alpha = \pi$$

Cf. (2.19).

The second condition to impose on a  $\mathcal{P}_N$ -model is for the coordinate  $u$  on  $\widehat{\mathcal{C}}_N$ . Ando constructed power operations for a family of Landweber-exact cohomology theories  $E_n$ , using power operations in the complex cobordism  $MU$  [Ando1995]. For a fixed prime  $p$ , each  $E_n$  is a Morava E-theory whose formal group is a universal deformation of the Honda formal group  $\Phi$  of height  $n$  over  $\mathbb{F}_p$ . This universal deformation is defined over the Lubin–Tate ring  $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]$ . In particular, for each  $\star$ -isomorphism class  $F$  of such a universal deformation, Ando constructed a unique coordinate  $x$  on  $F$  that lifts a particular coordinate on  $\Phi$  [ibid., 2.5.5–2.5.6] and satisfies

$$(3.3) \quad f_p^x(t) = [p]_{F_x}(t)$$

The left-hand side is the formal power series of a degree- $p^n$  *Lubin isogeny* (cf. Construction 2.11), defined using the coordinate  $x$ , for the subgroup of  $p$ -torsions on  $F$ . The right-hand side is the formal power series of multiplication by  $p$  under the group law  $F_x$ . See [ibid., Theorems 2.5.7 and 2.6.4]. Ando showed that such coordinates give rise to precisely those orientations for  $E_n$  that intertwine the power operations on  $MU$  and on  $E_n$  [ibid., Theorem 4.1.1].

In our context of elliptic cohomology, with notation as in (2.23), Construction 2.11 extends to produce a degree- $p^2$  isogeny  $\Psi^{p^2}: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{C}[p]$  such that

$$u(\Psi^{p^2}(P)) = \prod_{Q \in \mathcal{C}[p]} u(P - Q) = \prod_{i=0}^p u(\Psi_i^p(P))$$

(again, with an abuse of the notation  $u$ ). This Lubin isogeny of elliptic curves (as opposed to that of *formal groups* studied by Ando) is defined over  $S$ , because the parameters  $\kappa_i$  for  $\Psi_i^p$  are the roots of the polynomial  $V(\kappa) \in S[\kappa]$ . Since  $\Psi^{p^2}$  has the same kernel as the endomorphism  $[p]$  on  $\mathcal{C}$  of multiplication by  $p$ , we have  $\Psi^{p^2} = \sigma \circ [p]$  for some *non-canonical* isomorphism  $\sigma$  which depends on the choice of  $u$ . In particular, comparing the induced cotangent maps at the identity, we obtain

$$(3.4) \quad \alpha_0 \alpha_1 \cdots \alpha_p = s \cdot p$$

for some degree-0 unit  $s \in S$ . We next show that there exists a coordinate  $u_A$  on  $\widehat{\mathcal{C}}$ , as a restriction of a coordinate on  $\mathcal{C}$ , such that its corresponding  $s = 1$  for all  $p$ .

Over a punctured formal neighborhood of each cusp of the compactified moduli scheme for  $\mathcal{P}_N$ , the universal curve  $\mathcal{C}_N$  is isomorphic to the Tate curve  $\text{Tate}(q^N)$  with the level- $\Gamma_1(N)$  structure corresponding to that cusp. Over  $\mathbb{Z}[1/N]\langle\langle q \rangle\rangle$ , the formal group of  $\text{Tate}(q^N)$  is canonically isomorphic to the multiplicative formal group  $\widehat{\mathbb{G}}_m$ . After base change to  $\mathbb{Z}_p$ , there is a unique coordinate  $u_A$  of Ando's on this  $\star$ -isomorphism class of universal deformations for  $\Phi$ , the Honda formal group of height 1 over  $\mathbb{F}_p$  (cf. [ibid., Example 2.7]). In particular, the Lubin isogeny  $\Psi_0^p$  on  $\text{Tate}(q^N)$  in this coordinate gives

$$(3.5) \quad \alpha_0 = p$$

if we compare the induced cotangent maps at the identity.

In view of Remark 2.24 and (3.1), we then have

$$(3.6) \quad \alpha_1 = \cdots = \alpha_p = (-1)^{p-1}$$

by construction of the degree- $p$  isogenies on  $\text{Tate}(q^N)$  in [Katz1973, Section 1.11] (see particularly the first new paragraph on page Ka-23 concerning the subgroup  $\#_{\ell}$ ). In fact, (3.1) lifts so that, when  $p$  is odd, the dual isogenies used by Katz in his construction coincide with the Lubin isogenies as deformations of Frobenius. When  $p = 2$ , they differ by a sign.

Thus with respect to this coordinate  $u_A$ , (3.4) becomes

$$(3.7) \quad \alpha_0 \alpha_1 \cdots \alpha_p = p$$

near the cusps. Since  $u_A$  is the restriction of a globally defined coordinate on  $\mathcal{C}$ , the modular unit  $s \in S$  in (3.4) must be 1. In particular, upon changing coordinates as needed, (3.7) holds over a formal neighborhood of the supersingular point of this model. (Cf. [Zhu2015, Lemma 2.15 and Figure 3.6]. This lemma provides an alternative approach to the desired coordinate without referring to Ando's results. See also our subsequent work [Zhu2017] for a more general treatment of Ando's coordinates which leads to a third approach.)

**Definition 3.8** Given a  $\mathcal{P}_N$ -model for an E-theory  $E$  in Definition 2.9, we call it a *preferred model for  $E$*  if in addition it satisfies the following two conditions.

- The supersingular elliptic curve  $C_0$  is defined over  $\mathbb{F}_{p^2} \subset \overline{\mathbb{F}}_p$  such that  $\text{Frob}^2 = [\pi]$  as in (3.1).
- With the chosen coordinate  $u$  on  $\widehat{\mathcal{C}}_N$ , the identity (3.7) holds over a formal neighborhood of the supersingular point.

Given a preferred model, the constant term in (2.21) satisfies

$$(3.9) \quad w_0 = (-1)^{p-1} \alpha_0 \alpha_1 \cdots \alpha_p = \pi = \tilde{\alpha} \cdot \alpha$$

by (3.7) and (3.2). Note that the rightmost equality depends only on the curve  $C_0$  satisfying (3.1), not on the choice of a coordinate.

Our model with  $p = 5$  and  $N = 4$  in Examples 2.6 and 2.14 is preferred, so is Rezk's model with  $p = 2$  and  $N = 3$  in [Rezk2008, Section 3]. The one with  $p = 3$  and  $N = 4$  in [Zhu2014] is not. We will choose preferred models for E-theories at height 2 throughout the rest of this paper.

### 3.2 The Rezk logarithms and Hecke operators

Given any  $E_\infty$ -ring spectrum  $R$ , let  $\text{gl}_1(R)$  be the spectrum of units of  $R$  so that  $(\text{gl}_1(R))^0(X) \cong R^0(X)^\times$  for any space  $X$ . Rezk constructed a family of operations that naturally acts on  $\text{gl}_1(R)$  [Rezk2006, Definition 3.6]. Specifically, given a positive integer  $n$  and a prime  $p$ , let  $L_{K(n)}$  denote localization of spectra with respect to the  $n$ 'th Morava K-theory at  $p$ . Write  $\Phi_n$  for the corresponding  $(K(n)$ -local) Bousfield–Kuhn functor from the category of based topological spaces to the category of spectra. In particular, there is a natural weak equivalence of functors between  $\Phi_n \circ \Omega^\infty$  and  $L_{K(n)}$ . Consider the composite

$$(3.10) \quad \text{gl}_1(R) \rightarrow L_{K(n)}(\text{gl}_1(R)) \simeq \Phi_n \Omega^\infty(\text{gl}_1(R)) \xrightarrow{\sim} \Phi_n \Omega^\infty(R) \simeq L_{K(n)}(R)$$



Note that  $\Omega^\infty(\mathrm{gl}_1(R))$  and  $\Omega^\infty(R)$  have weakly equivalent basepoint components, but the standard inclusion  $\Omega^\infty(\mathrm{gl}_1(R)) \hookrightarrow \Omega^\infty(R)$  is not basepoint-preserving. The equivalence  $\Phi_n \Omega^\infty(\mathrm{gl}_1(R)) \xrightarrow{\sim} \Phi_n \Omega^\infty(R)$  thus involves a “basepoint shift” (see [ibid., 3.4]).

Let  $E$  be a Morava E-theory of height  $n$  at the prime  $p$ . Setting  $R = E$  and applying  $\pi_0(-)$  to (3.10), we then obtain the logarithmic operation

$$\ell_{n,p}: (E^0)^\times \rightarrow E^0$$

which is a homomorphism from a multiplicative group to an additive group. More generally, let  $X$  be a space and take  $R$  to be the spectrum of functions from  $\Sigma_+^\infty X$  to  $E$ . We then obtain the operation  $\ell_{n,p}$  that acts on  $E^0(X)^\times$ , naturally in both  $X$  and  $E$ .

Rezk proved a formula for this operation [ibid., Theorem 1.11]. In particular, for any  $x \in (E^0)^\times$ ,

$$(3.11) \quad \ell_{n,p}(x) = \frac{1}{p} \log(1 + p \cdot M(x))$$

Here  $M: (E^0)^\times \rightarrow E^0$  is a cohomology operation that can be expressed in terms of power operations  $\psi_A$  associated to finite subgroups  $A$  of  $(\mathbb{Q}_p/\mathbb{Z}_p)^n$ . Explicitly,

$$1 + p \cdot M(x) = \prod_{j=0}^n \prod_{\substack{A \subset (\mathbb{Q}_p/\mathbb{Z}_p)^n[p] \\ |A|=p^j}} \psi_A(x)^{(-1)^j p^{(j-1)(j-2)/2}}$$

Now let the E-theory be of height  $n = 2$ . In the presence of a model for  $E$ , the operations  $\psi_A$  above coincide with the power operations in Section 2. In particular,

$$(3.12) \quad \ell_{2,p}(x) = \frac{1}{p} \log \left( x^p \cdot \frac{1}{\psi_0^p(x) \cdots \psi_p^p(x)} \cdot \phi(x) \right)$$

Note that  $\ell_{2,p}(x) = 0$  for any  $x \in \mathbb{Z}_p \cap (E^0)^\times = \mathbb{Z}_p^\times$ .

In Section 2.4, we gave a comparison between classical and topological Hecke operators, particularly as illustrated in Example 2.28. In view of this comparison, we next write (3.12) in terms of Hecke operators.

Let  $E$  be a Morava E-theory of height 2 at the prime  $p$ . Choose a preferred model for  $E$  as in Definition 3.8. Write

$$(3.13) \quad F_X := 1 - T_{1,p} \cdot X + p T_{2,p} \cdot X^2 \in \mathbb{Z}[T_{1,p}, T_{2,p}][X]$$

where  $T_{1,p}$  and  $T_{2,p}$  are the Hecke operators. Let  $\beta$  be the ring homomorphism in Proposition 2.10.

**Proposition 3.14** (cf. [Rez2006, 1.12]) *Given  $x = \beta(f) \in (E^0)^\times$  for some unit  $f$  in  $\text{MF}[\Gamma_1(N)]$ , we have*

$$(3.15) \quad \ell_{2,p}(x) = \beta(F_1(\log f))$$

**Proof** Recall that  $\psi_i^p$  and  $\phi$  in (3.12) are ring homomorphisms. Multiplications by the terms  $\kappa_i^k$  in (2.23) and by  $(p\lambda)^k$  in (2.29) can also be made into homomorphisms of graded rings. In fact, we let the exponent  $k$  vary according to the weight  $k$  of the modular form that  $\kappa_i^k$  or  $(p\lambda)^k$  multiplies. We denote such ring homomorphisms by replacing the exponent  $k$  with  $\bullet$ . Defined as a formal power series, the usual logarithmic function  $\log$  commutes with these continuous ring homomorphisms (cf. [Koblitz1984, § IV.1–2]).

Recall from Definition 3.8 that the parameters  $\alpha_i$  in  $\psi_i^p$  satisfy  $\alpha_0 \cdots \alpha_p = p =: (-1)^{p-1} \pi$ . Thus

$$\log \frac{\pi^k}{\prod_{i=0}^p \alpha_i^k} = \log \frac{((-1)^{p-1} p)^k}{p^k} = \log ((-1)^{p-1})^k = 0$$

for all  $k$ , where for the last identity we interpret  $\log$  as the  $p$ -adic logarithm. Given the element  $x = \beta(f)$ , we then have

$$\begin{aligned} \ell_{2,p}(x) &= \frac{1}{p} \log \left( x^p \cdot \frac{1}{\psi_0^p(x) \cdots \psi_p^p(x)} \cdot \phi(x) \right) \\ &= \frac{1}{p} \log \frac{x^p \cdot \pi^\bullet \phi(x)}{\prod_{i=0}^p \alpha_i^\bullet \psi_i^p(x)} \\ (3.16) \quad &= \left( 1 - \frac{1}{p} \sum_{i=0}^p \alpha_i^\bullet \psi_i^p + p \cdot \frac{1}{p^2} \pi^\bullet \phi \right) \log x \\ &= \beta((1 - T_{1,p} + p \cdot T_{2,p}) \log f) \\ &= \beta(F_1(\log f)) \end{aligned}$$

□

**Remark 3.17** Let  $x = \beta(f) \in (E^0)^\times$  for some unit  $f$  in  $\text{MF}[\Gamma_1(N)]$  of weight  $k$ . The formula (3.12) expresses  $p \ell_{2,p}(x)$  as the logarithm of a ratio: the numerator, via  $\beta$ , corresponds to a modular form of weight  $pk + p^2k$ , and the denominator corresponds to one having the same weight  $(p+1)pk = pk + p^2k$ . Thus  $p \ell_{2,p}(x)$  corresponds to a  $p$ -adic modular form of weight 0 in view of (3.11) (cf. [Katz1976, Section 10.1]). We note the similarity between the logarithm in [ibid., 10.2.7] and the one below [Rez2006, Theorem 1.9]. The former is a formula for certain  $p$ -adic  $L$ -function in

terms of logarithms of ratios of Siegel functions. The latter is an analogue of (3.12) at height  $n = 1$ . This connection has not yet been well understood (see [ibid., 1.12]).

**Example 3.18** We revisit the case  $p = 5$  with the preferred model for the E-theory given by the moduli problem  $\mathcal{P}_4$ . Consider  $\delta = \beta(\Delta) \in (E^0)^\times$ . As in Example 2.28, since  $\delta = h - 26$ , we compute from (2.20) and use (2.17) to substitute values of symmetric functions in  $\alpha_i$ . We then obtain

$$\begin{aligned} \ell_{2,5}(\delta) &= \frac{1}{5} \log \left( \delta^5 \cdot \frac{1}{\psi_0^5(\delta) \cdots \psi_5^5(\delta)} \cdot \phi(\delta) \right) \\ &= \frac{1}{5} \log \left( \delta^5 \cdot \frac{1}{\delta^6} \cdot \delta \right) \\ &= \frac{1}{5} \log 1 \\ &= 0 \end{aligned}$$

Compare this calculation to the case  $p = 2$ , for which we choose a preferred  $\mathcal{P}_3$ -model from [Mahowald–Rezk2009, Proposition 3.2]. Mahowald and Rezk showed that  $\mathcal{P}_3$  is represented by  $y^2 + Axy + By = x^3$  over  $\mathbb{Z}[1/3][A, B, \Delta^{-1}]$  with  $|A| = 1$ ,  $|B| = 3$ , and  $\Delta = B^3(A^3 - 27B)$ . In [Rezk2008, 2.8], using this model, Rezk computed that

$$\ell_{2,2}(\beta(\Delta)) = \frac{1}{2} \log(-1) = 0$$

For the last identity, as in the proof of Proposition 3.14, we interpret  $\log$  as the 2-adic logarithm. In this case, the modular form  $\Delta$  again produces an element in the kernel of the logarithmic operation.

Moreover, since here  $\Delta = B^3(A^3 - 27B)$ , we have  $\beta(\Delta) = (a - 3)(a^2 + 3a + 9)$  with  $E^0 \cong W(\overline{\mathbb{F}}_2)[[a]]$  (cf. [ibid., Section 4]). Using Rezk’s formula for the total power operation on  $E^0$ , we compute that

$$\begin{aligned} \ell_{2,2}(a - 3) &= \frac{1}{2} \log(-1) = 0 \\ \ell_{2,2}(a^2 + 3a + 9) &= \frac{1}{2} \log 1 = 0 \end{aligned}$$

As  $\ell_{2,2}$  is a group homomorphism, the inverses of  $a - 3$  and of  $a^2 + 3a + 9$  in  $W(\overline{\mathbb{F}}_2)[[a]]$  are also contained in the kernel of  $\ell_{2,2}$ .

The above turn out to be instances of a general vanishing result for the logarithms  $\ell_{2,p}$  which we discuss next.

### 3.3 A vanishing theorem for the Rezk logarithms

In this section, via the formula (3.15) for a logarithmic operation in terms of Hecke operators, we detect a family of elements contained in the kernel of this operation. It includes those elements computed in Example 3.18.

We first collect some preliminaries and set the notation about differential structures on rings of modular forms (see, e.g., [Zagier2008, §5] and [Ono2004, Section 2.3]). In connection with Morava E-theories, we have been considering the  $p$ -local behavior of integral modular forms of level  $\Gamma_1(N)$ , with  $p$  not dividing  $N$ . These modular forms embed into the ring of  $p$ -adic modular forms. Thus for our purpose they can equivalently be viewed as defined over  $\mathbb{C}$  (cf. [Katz1973, the first paragraph of Introduction]). Henceforth we will freely move between the algebraic and analytic perspectives.

Recall that there is a differential operator  $D$  which acts on meromorphic modular forms over  $\mathbb{C}$ . Specifically, any meromorphic modular form  $f$  has a  $q$ -expansion at  $\infty$   $f(z) = \sum_{j > -\infty} a_j q^j$  with  $a_j \in \mathbb{C}$ , where  $q = e^{2\pi iz}$  as usual (see Convention 1.5). We then have  $Df := \frac{1}{2\pi i} \frac{df}{dz} = q \frac{df}{dq}$ . If the  $q$ -expansion of  $f$  has coefficients in  $\mathbb{Z}$ , so does the  $q$ -series for  $Df$ .

In general, the function  $Df$  is no longer modular. There is another derivation  $\vartheta$  which preserves modularity. If  $f$  has weight  $k$ , its *Serre derivative* is defined as

$$(3.19) \quad \vartheta f := Df - \frac{k}{12} \mathcal{E}_2 \cdot f$$

where  $\mathcal{E}_2(z) = 1 - 24 \sum_{j=1}^{\infty} \sigma_1(j) q^j$  is the quasimodular Eisenstein series of weight 2 (the divisor function  $\sigma_s(m) := \sum_{1 \leq d|m} d^s$ ). This  $\vartheta f$  is a meromorphic modular form of weight  $k+2$  [Ono2004, Proposition 2.11]. Cf. [Serre1973, Théorème 5(a)] for the action of  $D$  on  $p$ -adic modular forms.

**Example 3.20** Consider the modular discriminant  $\Delta$ . Its product expansion  $\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$  implies that

$$(3.21) \quad \begin{aligned} \log \Delta &= \log q + 24 \sum_{n=1}^{\infty} \log(1 - q^n) \\ &= \log q + 24 \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(-q^n)^k}{k} \\ &= \log q - 24 \sum_{m=1}^{\infty} \sigma_{-1}(m) q^m \end{aligned}$$

Thus  $D \log \Delta = \mathcal{E}_2$  and hence  $\vartheta \Delta = 0$ .

Let  $E$  be a Morava E-theory of height 2 at the prime  $p$ , and  $N > 3$  be any integer prime to  $p$ . Choose any preferred  $\mathcal{P}_N$ -model for  $E$  in the sense of Definition 3.8. Let  $\beta: \mathrm{MF}[\Gamma_1(N)] \rightarrow E^0$  be the ring homomorphism in Proposition 2.10 built from this model. Let  $\ell_{2,p}: (E^0)^\times \rightarrow E^0$  be Rezk's logarithmic cohomology operation.

**Theorem 3.22** *Suppose that  $f \in (\mathrm{MF}[\Gamma_1(N)])^\times$  has trivial Nebentypus character. If its Serre derivative  $\vartheta f = 0$ , then  $\beta(f)$  is contained in the kernel of  $\ell_{2,p}$ .*

Our proof consists of two parts. In the first part (Lemma 3.23 below), we show that  $\ell_{2,p}(\beta(f))$  is constant, i.e., it is the image of a constant modular form under  $\beta$ . This is based on an interplay between the differential structures and the action of Hecke operators on modular forms. Looking at  $q$ -expansions, we then show in the second part that this constant equals zero. It boils down to an analysis of the behavior of Tate curves under isogenies.

**Lemma 3.23** *Given any  $f \in (\mathrm{MF}[\Gamma_1(N)])^\times$  with  $\vartheta f = 0$ , the function  $F_1(\log f)$  is constant, where  $F_1 = 1 - T_{1,p} + p T_{2,p}$  is the operator defined in (3.13).*

**Proof** Suppose that  $f$  is of weight  $k$ . By (3.19), since  $\vartheta f = 0$ , we have

$$D \log f = \frac{Df}{f} = \frac{k\mathcal{E}_2}{12}$$

Note that  $\mathcal{E}_2$  is a weight-2 eigenform for each Hecke operator  $T_{1,m}$ ,  $m \geq 1$  with eigenvalue  $\sigma_1(m)$ . By comparing the effects on  $q$ -expansions, we have

$$(3.24) \quad D \circ T_{i,p} = \frac{1}{p^i} \cdot T_{i,p} \circ D$$

for  $i = 1$  and  $2$ . We then compute that

$$\begin{aligned} D(F_1(\log f)) &= D((1 - T_{1,p} + p T_{2,p}) \log f) \\ &= \left(1 - \frac{1}{p} \cdot T_{1,p} + \frac{1}{p^2} \cdot p T_{2,p}\right) D \log f \\ &= \left(1 - \frac{1}{p} \cdot T_{1,p} + \frac{1}{p^2} \cdot p T_{2,p}\right) \frac{k\mathcal{E}_2}{12} \\ &= \left(1 - \frac{1}{p} \cdot (1 + p) + \frac{1}{p^2} \cdot p \left(\frac{1}{p^2} \cdot p^2\right)\right) \frac{k\mathcal{E}_2}{12} \\ &= 0 \end{aligned}$$

Thus, as a function of the complex variable  $z$ ,  $F_1(\log f)$  is constant.  $\square$

**Proof of Theorem 3.22** Let  $x := \beta(f)$ . Then  $x \in (E^0)^\times$  since  $f \in (\text{MF}[\Gamma_1(N)])^\times$  and  $\beta$  is a ring homomorphism. Recall from (3.16) that

$$(3.25) \quad \ell_{2,p}(x) = \frac{1}{p} \log \frac{x^p \cdot \pi^\bullet \phi(x)}{\prod_{i=0}^p \alpha_i^\bullet \psi_i^p(x)} = \beta(F_1(\log f))$$

By Lemma 3.23, the above equals a constant, i.e., an element in  $W(\overline{\mathbb{F}}_p)$ .

Suppose that  $f$  has weight  $k$ . Via the correspondence between power operations and deformations of Frobenius in [Rezk2009, Theorem B], the ratio of values of power operations on  $x$  in (3.25) equals a ratio of values of  $f$  on the corresponding universal elliptic curves. Explicitly, with notation as in (2.23) and (2.29), we have

$$(3.26) \quad \frac{x^p \cdot \pi^\bullet \phi(x)}{\prod_{i=0}^p \alpha_i^\bullet \psi_i^p(x)} = \frac{f(\mathcal{C}_S, P_0, du)^p \cdot \pi^k f(\mathcal{C}_S, \tilde{\Psi}^p \circ \Psi^p(P_0), du)}{\prod_{i=0}^p \kappa_i^k f(\mathcal{C}_{S^p}/\mathcal{G}_i^p, \Psi_i^p(P_0), du)}$$

To determine the constant  $\ell_{2,p}(x)$ , we need only inspect the constant term in the  $q$ -expansion at  $\infty$  of the right-hand side.

Over a punctured formal neighborhood of each cusp, the universal curve  $\mathcal{C}$  is isomorphic to the Tate curve  $\text{Tate}(q^N)$  with the level- $\Gamma_1(N)$  structure corresponding to that cusp. The universal degree- $p$  isogeny on  $\text{Tate}(q^N)$  is defined over the ring  $\mathbb{Z}[1/pN, \zeta_p][[q^{1/p}]]$ , where  $\zeta_p$  is a primitive  $p$ 'th root of unity (see [Katz1973, Sections 1.2, 1.4, and 1.11]). In particular, the  $(p+1)$  subgroups of order  $p$  are

$$G_0^p, \text{ generated by } \zeta_p, \quad \text{and} \quad G_i^p, 1 \leq i \leq p, \text{ generated by } (\zeta_p^i q^{1/p})^N$$

Let  $\sum_{j=m}^\infty a_j q^j$  be the  $q$ -expansion of  $f$  at  $\infty$ , with leading coefficient  $a_m$ . We now compare as follows the lowest powers of  $q$  at the denominator and the numerator of (3.26).

By [ibid., (1.11.0.3) and (1.11.0.4)]<sup>1</sup> and Remark 2.24, we have at the denominator a leading term as the product of

$$p^k \cdot a_m (q^p)^m \quad \text{and} \quad p^k \cdot \pi^{-k} a_m (\zeta_p^i q^{1/p})^m$$

where  $i$  runs from 1 to  $p$ . Note that in the term on the right we have  $\pi^{-k}$  instead of  $p^{-k}$  due to our choice of a preferred model, where  $\text{Frob}^2 = [\pi]$  over the supersingular point (see (3.1)). Also note that since the Nebentypus character of  $f$  is trivial, the coefficient  $a_m$  is independent of where the level- $\Gamma_1(N)$  structure goes under each degree- $p$  isogeny.

<sup>1</sup>The last line of (1.11.0.4) should begin with  $\ell^{-k}$  instead of  $\ell^{-1}$ .

At the numerator, for the first factor, we have a leading term  $(a_m q^m)^p$ . For the second factor, we have a leading term  $\pi^k a_m q^m$ , again under the assumption that  $f$  has trivial Nebentypus character.

Combining these terms, we see that the ratio (3.26) has a leading constant term

$$\begin{aligned} \frac{(a_m q^m)^p \cdot \pi^k a_m q^m}{p^k a_m (q^p)^m \cdot \prod_{i=1}^p p^k \pi^{-k} a_m (\zeta_p^i q^{1/p})^m} &= (-1)^{k(p-1)} \zeta_p^{-mp(p+1)/2} \\ &= \begin{cases} (-1)^{k+m} & \text{if } p = 2 \\ 1 & \text{if } p \neq 2 \end{cases} \end{aligned}$$

Applying the  $p$ -adic logarithm, we obtain  $\ell_{2,p}(x) = 0$ .  $\square$

**Remark 3.27** Let  $f$  be any nonzero meromorphic modular form on  $\mathrm{SL}_2(\mathbb{Z})$ . Bruinier, Kohnen, and Ono gave an explicit formula for  $\vartheta f$  as a multiple of  $-f$  by a certain function  $f_\Theta$  [Bruinier–Kohnen–Ono2004, Theorem 1]. The function  $f_\Theta$  encodes a sequence of modular functions  $j_m$ ,  $m \geq 0$  defined by applying Hecke operators to the usual  $j$ -invariant.

In particular, the formula of Bruinier, Kohnen, and Ono immediately shows that a nonzero meromorphic modular form  $f$  has vanishing Serre derivative precisely when its zeros and poles are located only at the cusp (cf. [Dumas–Royer2014, Proposition 6]). The functions in Example 3.18, and in fact any unit in  $\mathrm{MF}[\Gamma_1(N)]$ , all have the latter property, though they are associated to  $\Gamma_1(N)$  instead of  $\mathrm{SL}_2(\mathbb{Z})$ .

Bruinier, Kohnen, and Ono’s theorem has been generalized by Ahlgren to  $\Gamma_0(p)$  with  $p \in \{2, 3, 5, 7, 13\}$  [Ahlgren2003, Theorem 2] and further by Choi to  $\Gamma_0(n)$  for any square-free  $n$  [Choi2006, Theorem 3.4]. In view of the assumption on Nebentypus character in Theorem 3.22, we note that modular forms of level  $\Gamma_1(N)$  with trivial Nebentypus character are precisely those of level  $\Gamma_0(N)$ .

### 3.4 An action of Hecke operators on logarithmic $q$ -series; an alternative conceptual proof of the vanishing theorem

The purpose of this subsection is to give an account for functions of the form  $\log f$ , with  $f$  a meromorphic modular form, which have appeared in the formula (3.16) for Rezk’s logarithmic operation  $\ell_{2,p}$ . Such functions have connections to mock theta functions, logarithmic  $q$ -series, and  $p$ -adic modular forms in the literature. We shall give an alternative and more conceptual proof of Theorem 3.22 by defining a suitable action of Hecke operators on these functions.

**Example 3.28** Recall the function  $\log \Delta$  in Example 3.20. In the final form of (3.21), the second summand is a convergent  $q$ -series. We may call it an *Eisenstein series of weight 0*, by analogy to  $q$ -expansions for the usual Eisenstein series of higher weight (also cf. the real analytic Eisenstein series of weight 0 discussed in [Funke2007, Sections 3.3 and 4.1]).

The first summand  $\log q$  never shows up in the  $q$ -expansion of a meromorphic modular form. It is this term that we shall address, given the prominence of  $\Delta$  in the context of logarithmic operations (see Theorem 3.22). Specifically, with motivations from homotopy theory, we propose to extend the classical action of Hecke operators on modular forms to incorporate series such as (3.21). We then apply the extended action in Example 3.38 at the end of this section.

**Remark 3.29** Given a cusp form  $f = \sum_{n=1}^{\infty} a_n q^n$  of weight  $k$ , its Eichler integral  $\tilde{f} = \sum_{n=1}^{\infty} n^{-k+1} a_n q^n$  is a mock modular form of weight  $2 - k$  (see [Zagier2009, the end of §6]). Recall from Example 3.20 that  $D \log \Delta = \mathcal{E}_2$ . Since  $D^{k-1} \tilde{f} = f$ , we may then view  $\log \Delta$  as a generalized Eichler integral, “generalized” in that  $\mathcal{E}_2$  is not a cusp form. We may even approach proving Lemma 3.23 from this viewpoint.

In [Knopp–Mason2011], given a representation  $\rho: \mathrm{SL}_2(\mathbb{Z}) \rightarrow \mathrm{GL}_n(\mathbb{C})$ , Knopp and Mason considered  $n$ -dimensional vector-valued modular forms associated to  $\rho$ . In particular, they showed in [ibid., Theorem 2.2] that the components of certain vector-valued modular forms are functions

$$(3.30) \quad f(z) = \sum_{j=0}^t (\log q)^j h_j(z)$$

where  $t \geq 0$  is an integer and each  $h_j$  is a convergent  $q$ -series with at worst real exponents (cf. [ibid., (7), (13), and Sections 3.2–3.3]). They remarked that  $q$ -expansions of this form occur in logarithmic conformal field theory (e.g., cf. [Zhu1996, (5.3.9)] and [Dong–Li–Mason2000, (6.12)]). The one in (3.21) gives another example. Following Knopp and Mason, we call the series in (3.30) a *logarithmic  $q$ -series*.

**Proposition 3.31** *Let  $N > 3$  be any integer prime to  $p$ . Consider the action of the Hecke operator  $T_p$  on  $q$ -expansions at  $\infty$  of modular forms of level  $\Gamma_0(N)$ . Then this action extends naturally onto  $\log q$  with*

$$T_p(\log q) = (p^{-1} + p^{-2}) \log q$$

**Proof** Consider modular forms in  $\mathrm{MF}[\Gamma_1(N)]$  with trivial Nebentypus character, i.e., those on  $\Gamma_0(N)$ . We follow the modular description in [Katz1973, Section 1.11] for



Hecke operators in the presence of the Tate curve  $\text{Tate}(q^N)$  over  $\mathbb{Z}[1/pN, \zeta_p](\!(q^{1/p})\!)$ , where  $\zeta_p$  is a primitive  $p$ 'th root of unity.

Write  $\mathcal{F} := \log q$ . Let  $\omega_{\text{can}}$  be the canonical differential on  $\text{Tate}(q^N)$ . Let  $\mathcal{M}_N := \text{Proj}(S_N)$  be the scheme over  $\mathbb{Z}[1/N]$  representing the moduli problem  $\mathcal{P}_N$  (see Examples 2.1, 2.2, and 2.6). Denote by  $\underline{\omega} := \text{pr}_* \Omega^1_{\mathcal{C}_N/\mathcal{M}_N}$  the pushforward along the structure morphism  $\text{pr}: \mathcal{C}_N \rightarrow \mathcal{M}_N$  of the relative cotangent sheaf  $\Omega^1_{\mathcal{C}_N/\mathcal{M}_N}$ .

By [Katz–Mazur1985, Theorem 10.13.11], the isomorphism  $\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\mathcal{M}_N/\mathbb{Z}[1/N]}$  on  $\mathcal{M}_N$  extends to an isomorphism  $\underline{\omega}^{\otimes 2} \xrightarrow{\sim} \Omega^1_{\overline{\mathcal{M}}_N/\mathbb{Z}[1/N]}(\log \text{cusps})$  on the compactification  $\overline{\mathcal{M}}_N$ , where the target is the invertible sheaf of 1-forms with at worst simple poles along the cusps. In particular, over the cusps,  $\omega_{\text{can}}^2$  corresponds to  $N \cdot d\mathcal{F}$  under this isomorphism (cf. [Katz1973, Section 1.5]). Therefore, as  $\mathcal{F} = \log q = 2\pi iz$  is linear in  $z$  (say, choose the principal branch of the logarithm), we have

$$(3.32) \quad \mathcal{F}(\text{Tate}(q^N), P_0, p \cdot \omega_{\text{can}}) = p^2 \cdot \mathcal{F}(\text{Tate}(q^N), P_0, \omega_{\text{can}})$$

By [ibid., (1.11.0.3) and (1.11.0.4)],<sup>2</sup> given the assumption of trivial Nebentypus character, we then calculate that

$$\begin{aligned} T_p(\log q) &= \frac{1}{p} \cdot p^k \left( \log(q^p) + \sum_{i=1}^p p^{-k} \log(\zeta_p^i q^{1/p}) \right) \\ (3.33) \quad &= p^{k-1} \left( p \log q + p^{-k} \sum_{i=1}^p (\log \zeta_p^i + p^{-1} \log q) \right) \\ &= p^{k-1} \left( p \log q + p^{-k} \sum_{i=1}^p p^{-1} \log q \right) \\ &= p^{k-1} (p + p^{-k}) \log q \end{aligned}$$

where we interpret  $\log$  as the  $p$ -adic logarithm so that  $\log \zeta_p^i = 0$  (cf. Example 3.18). Setting  $k = -2$  in view of (3.32), we obtain the stated identity.  $\square$

**Remark 3.34** More generally, if we view  $\log q$  as generalizing modular forms in  $\text{MF}[\Gamma_1(N)]$  with Nebentypus character  $\chi$ , we compute as in (3.33) and obtain  $T_p(\log q) = (p^{-1} + \chi(p)p^{-2}) \log q$ . For the rest of this section, we focus on the case when  $\chi$  is trivial (cf. Theorem 3.22).

<sup>2</sup>The last line of (1.11.0.4) should begin with  $\ell^{-k}$  instead of  $\ell^{-1}$ .

Let  $K$  be a number field. Bruinier and Ono studied meromorphic modular forms  $g$  for  $\mathrm{SL}_2(\mathbb{Z})$  with  $q$ -expansion

$$(3.35) \quad g(z) = q^m \left( 1 + \sum_{n=1}^{\infty} a_n q^n \right)$$

where  $m \in \mathbb{Z}$  and  $a_n \in \mathcal{O}_K$ . They showed that if  $g$  satisfies a certain condition with respect to a prime  $p$ , its logarithmic derivative  $D \log(g)$  is a  $p$ -adic modular form of weight 2 [Bruinier–Ono2003, Theorem 1].

This theorem has been generalized to meromorphic modular forms for  $\Gamma_0(p)$  with  $p \geq 5$  [Getz2004, Theorem 4]. Examples include  $\mathcal{E}_{p-1}$  at each  $p \geq 5$  and, for all  $p$ , meromorphic modular forms whose zeros and poles are located only at the cusps (cf. [Bruinier–Ono2003, Definition 3.1]). In particular, when  $g = \Delta$ , we have  $D \log \Delta = \mathcal{E}_2$  (cf. [Serre1973, discussion above Théorème 5]).

Given any  $g$  as in (3.35), note that  $\log(g)$  is a logarithmic  $q$ -series. We now arrive at the following definition based on Proposition 3.31 (esp. (3.32) and (3.33)) and the theorem of Bruinier and Ono above.

**Definition 3.36**

- (i) Given any integer  $j \geq 0$ , define the *weight* of  $(\log q)^j$  to be  $-2j$ .
- (ii) Let  $g$  be a meromorphic modular form such that  $D \log(g)$  is a  $p$ -adic modular form of weight 2 for all  $p$ . Define the *weight* of  $\log(g)$  to be 0.
- (iii) Given any prime  $p$  and any logarithmic  $q$ -series  $f(z) = \sum_{j=0}^t (\log q)^j h_j(z)$  of weight  $k$ , define  $T_p f$  as follows. For each  $j$ , suppose  $h_j(z) = \sum_{m > -\infty} a_m q^m$  (the index  $m$  and the coefficients  $a_m$  depend on  $j$ ). Define

$$T_p((\log q)^j h_j(z)) := (\log q)^j \sum_{m > -\infty} b_m q^m$$

where

$$b_m = p^{j+k-1} a_{m/p} + p^{-j} a_{pm}$$

with the convention that  $a_{m/p} = 0$  unless  $p|m$ . We then define  $T_p f(z) := \sum_{j=0}^t T_p((\log q)^j h_j(z))$ .

**Remark 3.37** The definitions of weight above are compatible with the action of the differential operator  $D$ . Specifically, we have  $D \log q = 1$ . Thus applying  $D$  to  $\log q$  increases the weight by 2, which agrees with [Serre1973, Théorème 5(a)]. More generally, for  $j \geq 0$ , since  $D(\log q)^j = j(\log q)^{j-1}$ , the same compatibility holds.

The definition for  $T_p((\log q)^j h_j(z))$  extends [Katz1973, Formula 1.11.1] by a computation analogous to (3.33) under the assumption of trivial Nebentypus character (see Remark 3.34). Moreover, the following identities for operators acting on modular forms extend to the series  $(\log q)^j h_j(z)$ .

$$\begin{aligned} D \circ T_p &= \frac{1}{p} \cdot T_p \circ D && \text{cf. (3.24)} \\ T_\ell \circ T_p &= T_p \circ T_\ell && \text{for primes } \ell \text{ and } p \end{aligned}$$

We can also define the Hecke operators  $T_m$  acting on  $(\log q)^j h_j(z)$  for any positive integer  $m$  as in [Serre1973, Remarque below Théorème 4].

**Example 3.38** Let us return to Example 3.28. By Definition 3.36(ii),  $\log \Delta$  is a logarithmic  $q$ -series of weight 0. In view of (3.21), we then compute by Definition 3.36(iii) that

$$T_p(\log \Delta) = \sigma_{-1}(p) \log \Delta$$

Thus in (3.15) we have  $F_1(\log \Delta) = (1 - \sigma_{-1}(p) + p^{-1}) \log \Delta = 0$ . This calculation gives a second proof of Theorem 3.22 in the case  $f = \Delta$ .

## 4 Hecke operators as elements in the Dyer–Lashof algebra of Morava E-theory

Constructed from total power operations, the topological Hecke operators in (2.26) and (2.31) can be defined more generally on  $E^0(X)$  for any space  $X$ . In this section, we examine their role in the Dyer–Lashof algebra of additive  $E$ -cohomology operations. Specifically, we first prove in Proposition 4.5 formulas for these Hecke operators in terms of *individual power operations*, which are generators of the Dyer–Lashof algebra. With the explicit formulas, we then compare in Theorem 4.8 the Hecke algebra and the center of the Dyer–Lashof algebra (see Remark 1.4).

Let  $E$  be a Morava E-theory of height  $n$  at the prime  $p$ . Its Dyer–Lashof algebra  $\Gamma$  is a ring of additive power operations that controls in a precise sense all homotopy operations on  $K(n)$ -local commutative  $E$ -algebras [Rezk2009, Theorem A]. This algebra  $\Gamma$  is a graded associative ring generated in degree 1 over the coefficient ring  $E^0$  by a set of individual power operations  $Q_i: A^0 \rightarrow A^0$ ,  $i = 0, 1, \dots, p^{n-1}$ , natural in the  $K(n)$ -local commutative  $E$ -algebra  $A$ .

Indeed,  $\Gamma$  has the structure of a *twisted bialgebra* over  $E^0$ . The twists, product, and coproduct of this structure are given respectively by commutation relations, Adem relations, and Cartan formulas for the operations  $Q_i$ .

**Example 4.1** Let the height  $n = 2$ . Recall from Section 2.3 that we have the additive total power operation

$$(4.2) \quad \psi^p: A^0 \rightarrow A^0(B\Sigma_p)/J \cong A^0[\alpha] / (w(\alpha))$$

where  $J$  is an ideal of images of transfers and  $w(\alpha) \in E^0[\alpha]$  is a monic polynomial of degree  $p + 1$ . In terms of individual power operations,

$$(4.3) \quad \psi^p(x) = \sum_{i=0}^p Q_i(x) \alpha^i$$

For explicit presentations of  $\Gamma$  in the cases  $p = 2$  and  $3$ , see [Rezk2008, Section 2] and [Zhu2014, Section 3.3]. For  $p = 5$ ,  $\Gamma$  is a graded twisted bialgebra over  $E_0 \cong W(\mathbb{F}_5)[[u_1]]$  with generators  $Q_i$ ,  $0 \leq i \leq 5$ . In the absence of a formula for  $\psi^5(u_1)$ , using those for  $\tilde{h}$  and  $\tilde{\alpha}$  in (2.20), we computed as in [ibid., Proposition 3.6] and derived *some* of the relations for  $\Gamma$ . In a subsequent paper independent of the results here, we have obtained a presentation for  $\Gamma$  uniform with all  $p$  which subsumes the earlier cases [Zhu2015, Theorem C]. Nevertheless, below we shall include the formulas for  $p = 5$  in the appendix for the reader's convenience to get a concrete sense of our discussion. A further purpose is to demonstrate the functoriality of the Dyer–Lashof algebra under base residue field extension, which will become clear in connection with the methods for proving the general result (see [ibid., Remark 1.8]).

We now reach the main result of this section. Let  $E$  be a Morava E-theory of height 2 at the prime  $p$ . Choose any preferred  $\mathcal{P}_N$ -model for  $E$  in Definition 3.8. Given any  $K(2)$ -local commutative  $E$ -algebra  $A$ , let  $\psi^p$  be the total power operation in (4.2) with the polynomial

$$(4.4) \quad w(\alpha) = w_{p+1}\alpha^{p+1} + \cdots + w_1\alpha + w_0 \quad w_{p+1} = 1$$

associated to this model, and let  $Q_i$ ,  $0 \leq i \leq p$  be the corresponding individual power operations in (4.3).

**Proposition 4.5** For  $\mu = 1$  and  $2$ , let  $t_{\mu,p}: A^0 \rightarrow p^{-1}A^0$  be the topological Hecke operators defined as in (2.26) and (2.31). Then the following identities hold.

$$t_{1,p} = \frac{1}{p} \sum_{i=0}^p c_i Q_i \quad \text{and} \quad t_{2,p} = \frac{1}{p^2} \sum_{j=0}^p \sum_{i=0}^j w_0^i d_{j-i} Q_i Q_j$$

where recursively

$$c_i = \begin{cases} p+1 & i=0 \\ -\sum_{k=0}^{i-1} w_{p+1+k-i} c_k + (p+1-i) w_{p+1-i} & 1 \leq i \leq p \end{cases}$$

$$d_\tau = \begin{cases} 1 & \tau=0 \\ -\sum_{k=0}^{\tau-1} w_0^{\tau-k-1} w_{\tau-k} d_k & 1 \leq \tau \leq p \end{cases}$$

Moreover, for  $i \geq 1$  and  $\tau \geq 1$ , we have closed formulas

$$c_i = i \sum_{\substack{m_1+2m_2+\dots+vm_v=i \\ m_s \geq 1}} (-1)^{m_1+\dots+m_v} \frac{(m_1+\dots+m_v-1)!}{m_1! \dots m_v!} w_{p+1-1}^{m_1} \dots w_{p+1-v}^{m_v}$$

$$d_\tau = \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1+\dots+m_{\tau-n}=\tau \\ 1 \leq m_s \leq p}} w_{m_1} \dots w_{m_{\tau-n}}$$

**Proof** By definitions (2.26) and (4.3),

$$t_{1,p}(x) = \frac{1}{p} \sum_{j=0}^p \psi_j^p(x) = \frac{1}{p} \sum_{j=0}^p \sum_{i=0}^p Q_i(x) \alpha_j^i = \frac{1}{p} \sum_{i=0}^p \left( \sum_{j=0}^p \alpha_j^i \right) Q_i(x)$$

Since the parameters  $\alpha_j$  are the roots of  $w$  in (4.4), the formulas for  $c_i = \sum_{j=0}^p \alpha_j^i$  then follow from Newton's and Girard's formulas relating power sums and elementary symmetric functions (see, e.g., [Milnor–Stasheff1974, Problem 16-A]).<sup>3</sup>

For  $t_{2,p}$ , we first write by (2.31) that

$$\begin{aligned} t_{2,p}(x) &= \frac{1}{p^2} \psi^p(\psi^p(x)) \\ &= \frac{1}{p^2} \psi^p \left( \sum_{j=0}^p Q_j(x) \alpha^j \right) \\ &= \frac{1}{p^2} \sum_{j=0}^p \psi^p(Q_j(x)) \psi^p(\alpha)^j \\ &= \frac{1}{p^2} \sum_{j=0}^p \left( \sum_{i=0}^p Q_i Q_j(x) \alpha^i \right) \tilde{\alpha}^j \end{aligned}$$

<sup>3</sup>In the cited reference, the left-hand side of Girard's formula should read  $(-1)^n s_n/n$ . The summation on the right-hand side is over  $i_1 + 2i_2 + \dots + ki_k = n$ .

Since the target of  $t_{2,p}$  is  $p^{-1}A^0$ , the above identity should simplify to contain neither  $\alpha$  nor  $\tilde{\alpha}$ . Thus in view of (3.9), we rewrite

$$t_{2,p}(x) = \frac{1}{p^2} \sum_{j=0}^p \sum_{i=0}^j w_0^i Q_i Q_j(x) \tilde{\alpha}^{j-i}$$

For  $0 \leq \tau \leq p$ , we next express each  $\tilde{\alpha}^\tau$  as a polynomial in  $\alpha$  of degree at most  $p$  with coefficients in  $E^0$ , and verify that the constant term of this polynomial is  $d_\tau$  as stated in the proposition.

The case  $\tau = 0$  is clear. For  $1 \leq \tau \leq p$ , we have

$$\begin{aligned} \tilde{\alpha}^\tau &= \left( \frac{w_0}{\alpha} \right)^\tau && \text{by (3.9)} \\ &= \frac{w_0^{\tau-1}(-w_{p+1}\alpha^{p+1} - \dots - w_1\alpha)}{\alpha^\tau} && \text{by (4.4)} \\ &= w_0^{\tau-1}(-w_{p+1}\alpha^{p+1-\tau} - \dots - w_{\tau+1}\alpha) \\ &\quad - w_0^{\tau-1}w_\tau - \frac{w_0^{\tau-1}w_{\tau-1}}{\alpha} - \dots - \frac{w_0^{\tau-1}w_1}{\alpha^{\tau-1}} \\ &= w_0^{\tau-1}(-w_{p+1}\alpha^{p+1-\tau} - \dots - w_{\tau+1}\alpha) \\ &\quad - w_0^{\tau-1}w_\tau - w_0^{\tau-2}w_{\tau-1}\tilde{\alpha} - \dots - w_1\tilde{\alpha}^{\tau-1} && \text{by (3.9)} \end{aligned}$$

and thus

$$\begin{aligned} d_\tau &= -w_0^{\tau-1}w_\tau - w_0^{\tau-2}w_{\tau-1}d_1 - \dots - w_1d_{\tau-1} \\ &= -\sum_{k=0}^{\tau-1} w_0^{\tau-k-1}w_{\tau-k}d_k \end{aligned}$$

We have obtained the first identity for  $d_\tau$  stated in the proposition. From this relation, we show the second stated identity for  $d_\tau$  by induction on  $\tau$ . The base case  $\tau = 1$ , with  $d_1 = -w_1$ , can be checked directly. For  $\tau \geq 2$ , by a change of indices  $\nu = \tau - k$ , we first rewrite

$$d_\tau = -w_0^{\tau-1}w_\tau - \sum_{\nu=1}^{\tau-1} w_0^{\nu-1}w_\nu d_{\tau-\nu}$$

Next we expand  $d_{\tau-\nu}$  for  $1 \leq \nu \leq \tau - 1$  by the induction hypothesis. Let  $n$  index the

power of  $w_0$  in the expansion of the term  $\sum_{\nu=1}^{\tau-1} w_0^{\nu-1} w_\nu d_{\tau-\nu}$  above. We then have

$$\begin{aligned}
d_\tau &= -w_0^{\tau-1} w_\tau - \sum_{n=0}^{\tau-2} \left( \sum_{\nu=1}^{n+1} w_0^{\nu-1} w_\nu (-1)^{\tau-\nu-(n-\nu+1)} w_0^{n-\nu+1} \right. \\
&\quad \left. \sum_{\substack{m_{1,\nu} + \dots + m_{\tau-\nu-(n-\nu+1),\nu} = \tau-\nu \\ 1 \leq m_{s,\nu} \leq p}} w_{m_{1,\nu}} \cdots w_{m_{\tau-\nu-(n-\nu+1),\nu}} \right) \\
&= -w_0^{\tau-1} w_\tau - \sum_{n=0}^{\tau-2} (-1)^{\tau-n-1} w_0^n \sum_{\nu=1}^{n+1} w_\nu \sum_{\substack{m_{1,\nu} + \dots + m_{\tau-n-1,\nu} = \tau-\nu \\ 1 \leq m_{s,\nu} \leq p}} w_{m_{1,\nu}} \cdots w_{m_{\tau-n-1,\nu}} \\
&= -w_0^{\tau-1} w_\tau + \sum_{n=0}^{\tau-2} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1 + \dots + m_{\tau-n} = \tau \\ 1 \leq m_s \leq p}} w_{m_1} \cdots w_{m_{\tau-n}} \\
&= \sum_{n=0}^{\tau-1} (-1)^{\tau-n} w_0^n \sum_{\substack{m_1 + \dots + m_{\tau-n} = \tau \\ 1 \leq m_s \leq p}} w_{m_1} \cdots w_{m_{\tau-n}}
\end{aligned}$$

□

**Remark 4.6** In the proof above, the method for computing  $t_{2,p}$  applies more generally. It enables us to find formulas for Adem relations, one for each  $Q_i Q_0$  with  $1 \leq i \leq p$ , in terms of the coefficients  $w_j$  of  $w(\alpha)$  (see Example 4.1, the appendix below, and cf. [Zhu2014, proof of Proposition 3.6 (iv)]). We can also express Cartan formulas using these coefficients. However, commutation relations are determined by both the terms  $w_j$  and  $\psi^p(w_j)$ .

**Example 4.7** For  $p = 5$ , by Proposition 4.5, we compute from (2.17) that

$$\begin{aligned}
t_{1,5} &= \frac{6}{5} Q_0 + 2Q_1 + 6Q_2 + 26Q_3 + 126Q_4 + (h + 600)Q_5 \\
t_{2,5} &= \frac{1}{25} (Q_0 Q_0 + h Q_0 Q_1 + (h^2 - 275) Q_0 Q_2 + (h^3 - 550h + 1500) Q_0 Q_3 \\
&\quad + (h^4 - 825h^2 + 3000h + 71250) Q_0 Q_4 + (h^5 - 1100h^3 + 4500h^2 \\
&\quad + 218125h - 818750) Q_0 Q_5 + 5Q_1 Q_1 + 5h Q_1 Q_2 + (5h^2 - 1375) Q_1 Q_3 \\
&\quad + (5h^3 - 2750h + 7500) Q_1 Q_4 + (5h^4 - 4125h^2 + 15000h \\
&\quad + 356250) Q_1 Q_5 + 25Q_2 Q_2 + 25h Q_2 Q_3 + (25h^2 - 6875) Q_2 Q_4 + (25h^3 \\
&\quad - 13750h + 37500) Q_2 Q_5 + 125Q_3 Q_3 + 125h Q_3 Q_4 + (125h^2 \\
&\quad - 34375) Q_3 Q_5 + 625Q_4 Q_4 + 625h Q_4 Q_5 + 3125Q_5 Q_5)
\end{aligned}$$

Observing the coefficients in  $t_{1,5}$ , we see that setting  $h = 26$  will make the last term fit the pattern. The importance of this value of  $h$ , as a Hasse invariant evaluated at the cusps, will become clear in [Zhu2015, Example 2.18].

**Theorem 4.8** *Let  $E$  be a Morava  $E$ -theory of height 2 at the prime  $p$ , and let  $\Gamma$  be its Dyer–Lashof algebra. Define  $\tilde{t}_{\mu,p} := p^\mu \cdot t_{\mu,p}$  for  $\mu = 1$  and  $\mu = 2$ , where  $t_{\mu,p}$  are the topological Hecke operators in Proposition 4.5. Then  $\tilde{t}_{2,p}$  lies in the center of  $\Gamma$ , i.e., it commutes with all elements under multiplication. On the contrary,  $\tilde{t}_{1,p}$  does not.*

**Proof** By Proposition 4.5, both  $\tilde{t}_{1,p}$  and  $\tilde{t}_{2,p}$  are contained in  $\Gamma$ . In fact,  $\tilde{t}_{2,p} = \psi^p \circ \psi^p$  and is thus a ring homomorphism. Note that  $\tilde{t}_{2,p}(\alpha) = \alpha$  as a result of the relation (3.9). Since  $\psi^p = Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p$ , we can then write the three-fold composite  $\psi^p \circ \psi^p \circ \psi^p$  in two ways as follows.

$$\begin{aligned} \tilde{t}_{2,p}(Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p) &= \tilde{t}_{2,p}Q_0 + (\tilde{t}_{2,p}Q_1)(\tilde{t}_{2,p}\alpha) + \cdots + (\tilde{t}_{2,p}Q_p)(\tilde{t}_{2,p}\alpha)^p \\ &= \tilde{t}_{2,p}Q_0 + (\tilde{t}_{2,p}Q_1)\alpha + \cdots + (\tilde{t}_{2,p}Q_p)\alpha^p \\ (Q_0 + Q_1\alpha + \cdots + Q_p\alpha^p)\tilde{t}_{2,p} &= Q_0\tilde{t}_{2,p} + (Q_1\tilde{t}_{2,p})\alpha + \cdots + (Q_p\tilde{t}_{2,p})\alpha^p \end{aligned}$$

For each  $0 \leq i \leq p$ , comparing the coefficients for  $\alpha^i$ , we see that  $\tilde{t}_{2,p}$  commutes with  $Q_i$ . Thus  $\tilde{t}_{2,p}$  lies in the center of  $\Gamma$ .

It remains to show that the other operation  $\tilde{t}_{1,p}$  is not central in  $\Gamma$ . Seeking a contradiction, suppose that

$$(4.9) \quad \tilde{t}_{1,p}Q_1 = Q_1\tilde{t}_{1,p}$$

We examine the coefficient of the term  $Q_0Q_1$  on each side, modulo  $p$ . Since

$$\tilde{t}_{1,p}Q_1 = ((p+1)Q_0 - w_pQ_1 + c_2Q_2 + \cdots + c_pQ_p)Q_1$$

this coefficient is congruent to 1 modulo  $p$ . We rewrite the right-hand side as

$$Q_1\tilde{t}_{1,p} = Q_1((p+1)Q_0 - w_pQ_1 + c_2Q_2 + \cdots + c_pQ_p)$$

By means analogous to the proof of Proposition 4.5, we find that the term  $Q_0Q_1$  has coefficient  $w_2$  in the Adem relation for  $Q_1Q_0$  (see Remark 4.6). Let  $a$  be the coefficient of  $Q_0$  in  $Q_1(-w_p)$  after we apply commutation relations. Thus, comparing the two sides of (4.9) for  $Q_0Q_1$ , we must have

$$1 \equiv w_2 + a \pmod{p}$$

Via the ring homomorphism  $\beta$  in Proposition 2.10, we view this congruence as one between modular functions. In particular, near the cusps, our choice of a preferred  $\mathcal{P}_N$ -model guarantees (3.5) and (3.6), which imply that  $p|w_i$  for  $2 \leq i \leq p$ . Thus



$a \equiv 1 \pmod{p}$  as a modular function near the cusps. By definition of  $a$ , this in turn means that  $Q_1(-w_p) \not\equiv 0 \pmod{p}$  as an element in  $\Gamma$ , and so  $-w_p \not\equiv 0 \pmod{p}$  in  $E^0$ . This last inequality contradicts the divisibility  $p|w_p$  exhibited above near the cusps.  $\square$

## Appendix: Dyer–Lashof algebra for an E-theory of height 2 at the prime 5

We continue Example 4.1 and give the following three sets of formulas for the Dyer–Lashof algebra  $\Gamma$  as a twisted bialgebra over  $E_0 \cong W(\mathbb{F}_5)[[u_1]]$ . Let  $h$  be the product of  $u_1$  and its Galois conjugate, defined as in (2.7).

- Twists in terms of *commutation relations*:

$$\begin{aligned}
Q_i c &= (Fc)Q_i \text{ for } c \in W(\mathbb{F}_5) \text{ and all } i, \text{ with } F \text{ the Frobenius automorphism} \\
Q_0 h &= (h^5 - 10h^4 - 1065h^3 + 12690h^2 + 168930h - 1462250)Q_0 + (5h^4 \\
&\quad - 50h^3 - 3950h^2 + 42200h + 233400)Q_1 + (25h^3 - 250h^2 - 12875h \\
&\quad + 104750)Q_2 + (125h^2 - 1250h - 30000)Q_3 + (625h - 6250)Q_4 \\
&\quad + 3120Q_5 \\
Q_1 h &= (-55h^4 + 850h^3 + 39575h^2 - 608700h - 1113524)Q_0 + (-275h^3 \\
&\quad + 4250h^2 + 122250h - 1462250)Q_1 + (-1375h^2 + 21250h \\
&\quad + 233400)Q_2 + (-6875h + 104750)Q_3 - 30000Q_4 + (h - 6250)Q_5 \\
Q_2 h &= (60h^4 - 775h^3 - 45400h^2 + 593900h + 2008800)Q_0 + (300h^3 \\
&\quad - 3875h^2 - 144500h + 1453876)Q_1 + (1500h^2 - 19375h \\
&\quad - 310000)Q_2 + (7500h - 96600)Q_3 + 36000Q_4 + 4320Q_5 \\
Q_3 h &= (-35h^4 + 400h^3 + 27125h^2 - 320900h - 1418300)Q_0 + (-175h^3 \\
&\quad + 2000h^2 + 87500h - 792000)Q_1 + (-875h^2 + 10000h + 196876)Q_2 \\
&\quad + (-4375h + 50000)Q_3 - 21600Q_4 - 1440Q_5 \\
Q_4 h &= (10h^4 - 105h^3 - 7850h^2 + 86975h + 445850)Q_0 + (50h^3 - 525h^2 \\
&\quad - 25500h + 215500)Q_1 + (250h^2 - 2625h - 58750)Q_2 + (1250h \\
&\quad - 13124)Q_3 + 6250Q_4 + 240Q_5 \\
Q_5 h &= (-h^4 + 10h^3 + 790h^2 - 8440h - 46680)Q_0 + (-5h^3 + 50h^2 + 2575h \\
&\quad - 20950)Q_1 + (-25h^2 + 250h + 6000)Q_2 + (-125h + 1250)Q_3 \\
&\quad - 624Q_4 + 10Q_5
\end{aligned}$$

- A product structure subject to *Adem relations*:

$$\begin{aligned}
Q_1 Q_0 &= 55Q_0 Q_1 + (55h - 300)Q_0 Q_2 + (55h^2 - 300h - 14250)Q_0 Q_3 + (55h^3 \\
&\quad - 300h^2 - 29375h + 163750)Q_0 Q_4 + (55h^4 - 300h^3 - 44500h^2 \\
&\quad + 328750h + 3228750)Q_0 Q_5 + 275Q_1 Q_2 + (275h - 1500)Q_1 Q_3 \\
&\quad + (275h^2 - 1500h - 71250)Q_1 Q_4 + (275h^3 - 1500h^2 - 146875h \\
&\quad + 818750)Q_1 Q_5 - 5Q_2 Q_1 + 1375Q_2 Q_3 + (1375h - 7500)Q_2 Q_4 \\
&\quad + (1375h^2 - 7500h - 356250)Q_2 Q_5 - 25Q_3 Q_2 + 6875Q_3 Q_4 \\
&\quad + (6875h - 37500)Q_3 Q_5 - 125Q_4 Q_3 + 34375Q_4 Q_5 - 625Q_5 Q_4 \\
Q_2 Q_0 &= -60Q_0 Q_1 + (-60h + 175)Q_0 Q_2 + (-60h^2 + 175h + 16250)Q_0 Q_3 \\
&\quad + (-60h^3 + 175h^2 + 32750h - 138000)Q_0 Q_4 + (-60h^4 + 175h^3 \\
&\quad + 49250h^2 - 276125h - 3943750)Q_0 Q_5 - 300Q_1 Q_2 + (-300h \\
&\quad + 875)Q_1 Q_3 + (-300h^2 + 875h + 81250)Q_1 Q_4 + (-300h^3 + 875h^2 \\
&\quad + 163750h - 690000)Q_1 Q_5 - 1500Q_2 Q_3 + (-1500h + 4375)Q_2 Q_4 \\
&\quad + (-1500h^2 + 4375h + 406250)Q_2 Q_5 - 5Q_3 Q_1 - 7500Q_3 Q_4 \\
&\quad + (-7500h + 21875)Q_3 Q_5 - 25Q_4 Q_2 - 37500Q_4 Q_5 - 125Q_5 Q_3 \\
Q_3 Q_0 &= 35Q_0 Q_1 + (35h - 50)Q_0 Q_2 + (35h^2 - 50h - 9600)Q_0 Q_3 + (35h^3 \\
&\quad - 50h^2 - 19225h + 66250)Q_0 Q_4 + (35h^4 - 50h^3 - 28850h^2 \\
&\quad + 132500h + 2411875)Q_0 Q_5 + 175Q_1 Q_2 + (175h - 250)Q_1 Q_3 \\
&\quad + (175h^2 - 250h - 48000)Q_1 Q_4 + (175h^3 - 250h^2 - 96125h \\
&\quad + 331250)Q_1 Q_5 + 875Q_2 Q_3 + (875h - 1250)Q_2 Q_4 + (875h^2 \\
&\quad - 1250h - 240000)Q_2 Q_5 + 4375Q_3 Q_4 + (4375h - 6250)Q_3 Q_5 \\
&\quad - 5Q_4 Q_1 + 21875Q_4 Q_5 - 25Q_5 Q_2 \\
Q_4 Q_0 &= -10Q_0 Q_1 + (-10h + 5)Q_0 Q_2 + (-10h^2 + 5h + 2750)Q_0 Q_3 \\
&\quad + (-10h^3 + 5h^2 + 5500h - 16375)Q_0 Q_4 + (-10h^4 + 5h^3 + 8250h^2 \\
&\quad - 32750h - 705000)Q_0 Q_5 - 50Q_1 Q_2 + (-50h + 25)Q_1 Q_3 + (-50h^2 \\
&\quad + 25h + 13750)Q_1 Q_4 + (-50h^3 + 25h^2 + 27500h - 81875)Q_1 Q_5 \\
&\quad - 250Q_2 Q_3 + (-250h + 125)Q_2 Q_4 + (-250h^2 + 125h \\
&\quad + 68750)Q_2 Q_5 - 1250Q_3 Q_4 + (-1250h + 625)Q_3 Q_5 - 6250Q_4 Q_5 \\
&\quad - 5Q_5 Q_1 \\
Q_5 Q_0 &= Q_0 Q_1 + hQ_0 Q_2 + (h^2 - 275)Q_0 Q_3 + (h^3 - 550h + 1500)Q_0 Q_4 + (h^4 \\
&\quad - 825h^2 + 3000h + 71250)Q_0 Q_5 + 5Q_1 Q_2 + 5hQ_1 Q_3 + (5h^2 \\
&\quad - 1375)Q_1 Q_4 + (5h^3 - 2750h + 7500)Q_1 Q_5 + 25Q_2 Q_3 + 25hQ_2 Q_4 \\
&\quad + (25h^2 - 6875)Q_2 Q_5 + 125Q_3 Q_4 + 125hQ_3 Q_5 + 625Q_4 Q_5
\end{aligned}$$

- A coproduct structure given by *Cartan formulas*:

$$\begin{aligned}
Q_0(xy) &= Q_0(x)Q_0(y) - 5[Q_1(x)Q_5(y) + Q_2(x)Q_4(y) + Q_3(x)Q_3(y) \\
&\quad + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)] - 50[Q_2(x)Q_5(y) + Q_3(x)Q_4(y) \\
&\quad + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)] - 325[Q_3(x)Q_5(y) + Q_4(x)Q_4(y) \\
&\quad + Q_5(x)Q_3(y)] - 1800[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] \\
&\quad - 9350Q_5(x)Q_5(y) \\
Q_1(xy) &= [Q_0(x)Q_1(y) + Q_1(x)Q_0(y)] + h[Q_1(x)Q_5(y) + Q_2(x)Q_4(y) \\
&\quad + Q_3(x)Q_3(y) + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)] + (10h - 5)[Q_2(x)Q_5(y) \\
&\quad + Q_3(x)Q_4(y) + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)] + (65h \\
&\quad - 50)[Q_3(x)Q_5(y) + Q_4(x)Q_4(y) + Q_5(x)Q_3(y)] + (360h \\
&\quad - 325)[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] + (1870h - 1800)Q_5(x)Q_5(y) \\
Q_2(xy) &= [Q_0(x)Q_2(y) + Q_1(x)Q_1(y) + Q_2(x)Q_0(y)] - 55[Q_1(x)Q_5(y) \\
&\quad + Q_2(x)Q_4(y) + Q_3(x)Q_3(y) + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)] + (h \\
&\quad - 550)[Q_2(x)Q_5(y) + Q_3(x)Q_4(y) + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)] \\
&\quad + (10h - 3580)[Q_3(x)Q_5(y) + Q_4(x)Q_4(y) + Q_5(x)Q_3(y)] + (65h \\
&\quad - 19850)[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] + (360h - 103175)Q_5(x)Q_5(y) \\
Q_3(xy) &= [Q_0(x)Q_3(y) + Q_1(x)Q_2(y) + Q_2(x)Q_1(y) + Q_3(x)Q_0(y)] \\
&\quad + 60[Q_1(x)Q_5(y) + Q_2(x)Q_4(y) + Q_3(x)Q_3(y) + Q_4(x)Q_2(y) \\
&\quad + Q_5(x)Q_1(y)] + 545[Q_2(x)Q_5(y) + Q_3(x)Q_4(y) + Q_4(x)Q_3(y) \\
&\quad + Q_5(x)Q_2(y)] + (h + 3350)[Q_3(x)Q_5(y) + Q_4(x)Q_4(y) \\
&\quad + Q_5(x)Q_3(y)] + (10h + 18020)[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] + (65h \\
&\quad + 92350)Q_5(x)Q_5(y) \\
Q_4(xy) &= [Q_0(x)Q_4(y) + Q_1(x)Q_3(y) + Q_2(x)Q_2(y) + Q_3(x)Q_1(y) \\
&\quad + Q_4(x)Q_0(y)] - 35[Q_1(x)Q_5(y) + Q_2(x)Q_4(y) + Q_3(x)Q_3(y) \\
&\quad + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)] - 290[Q_2(x)Q_5(y) + Q_3(x)Q_4(y) \\
&\quad + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)] - 1730[Q_3(x)Q_5(y) + Q_4(x)Q_4(y) \\
&\quad + Q_5(x)Q_3(y)] + (h - 9250)[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] + [10h \\
&\quad - 47430]Q_5(x)Q_5(y) \\
Q_5(xy) &= [Q_0(x)Q_5(y) + Q_1(x)Q_4(y) + Q_2(x)Q_3(y) + Q_3(x)Q_2(y) \\
&\quad + Q_4(x)Q_1(y) + Q_5(x)Q_0(y)] + 10[Q_1(x)Q_5(y) + Q_2(x)Q_4(y) \\
&\quad + Q_3(x)Q_3(y) + Q_4(x)Q_2(y) + Q_5(x)Q_1(y)] + 65[Q_2(x)Q_5(y) \\
&\quad + Q_3(x)Q_4(y) + Q_4(x)Q_3(y) + Q_5(x)Q_2(y)] + 360[Q_3(x)Q_5(y) \\
&\quad + Q_4(x)Q_4(y) + Q_5(x)Q_3(y)] + 1870[Q_4(x)Q_5(y) + Q_5(x)Q_4(y)] \\
&\quad + (h + 9450)Q_5(x)Q_5(y)
\end{aligned}$$

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