STUFF ABOUT QUASICATEGORIES

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Date: May 4, 2020.

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(Note: this is a draft, which can change daily. It includes some unorganized material which might later be incorporated into the narrative, or removed entirely. In particular, everything involving Cartesian and coCartesian fibrations needs to be completely reworked.)

1. Introduction to ∞-categories

I'll give a brief discussion to motivate the notion of ∞-categories.

1.1. Groupoids. Modern mathematics is based on sets. The most familiar way of constructing new sets is as sets solutions to equations. For instance, given a commutative ring R, we can consider the set $X(R)$ of tuples $(x,y,z) \in R^3$ which satisfy the equation $x^5 + y^5 = z^5$. We can express such sets as limits; for instance, $X(R)$ is the pullback of the diagram of sets

$$R \times R \xrightarrow{(x,y)\mapsto x^5+y^5} R \xleftarrow{z^5 \mapsfrom z} R.$$

Another way to construct new sets is by taking "quotients"; e.g., as sets of equivalence classes of an equivalence relation. This is in some sense much more subtle than sets of solutions to equations: mathematicians did not routinely construct sets this way until they were comfortable with the set theoretic formalism introduced by the end of the 19th century.

Some sets of equivalence classes are nothing more than that; but some have "higher" structure standing behind them, which is often encoded in the form of a *groupoid*[1]. Here are some examples.

- Given a topological space X, we can define an equivalence relation on the set of points, so $x \sim x'$ if and only if there is a continuous path connecting them. The set of equivalence classes is the set $\pi_0 X$ of path components. Standing behind this equivalence relation is the *fundamental groupoid* $\Pi_1 X$, whose objects are points of X, and whose morphisms are path-homotopy classes of paths between two points.
- Given any category C, there is an equivalence relation on the collection of objects, so that $X \sim Y$ if there exists an isomorphism between them. Equivalence classes are the isomorphism classes of objects. Standing behind this equivalence relation is the *core* of C (also called the *maximal subgroupoid*), which is a groupoid having the same objects as C, but having as morphisms only the isomorphisms in C.
- As a special case of the above, let $C = \mathrm{Vect}_F$ be the category of finite dimensional vector spaces and linear maps over some field F. Then isomoprhism classes of objects correspond to non-negative integers, via the notion of dimension. The core $\mathrm{Vect}_F^{\mathrm{core}}$ is a groupoid whose objects are finite dimensional vectors spaces, and whose morphisms are *invertible* linear maps.

Note that many interesting problems are about describing isomorphism classes; e.g., classifying finite groups of a given order, or principal G-bundles on a space. In practice, one learns that when you try to classify some type of objects up to isomorphism, you will need to have a good handle on the isomorphisms between such objects, including the groups of automorphisms of such objects. So you will likely need to know about the groupoid, even if it is not the primary object of interest.

For instance, a problem such as: "describe the groupoid $\mathrm{Bun}_G(M)$ of principal G-bundles on a space M" is a more sophisticated analogue of: "find the set $X(R)$ of solutions to $x^5 + y^5 = z^5$ in the ring R". (In fact, the theory of "moduli stacks" exactly develops this analogy between the two problems.) To do this, you can imagine having a "groupoid-based mathematics", generalizing the usual set-based one. Here are some observations about this.

- We regard two sets as "essentially the same" if they are *isomorphic*, i.e., if there is a bijection $f\colon X \to X'$ between them. Any such bijection has a unique inverse bijection $f^{-1}\colon X' \to X$.

  On the other hand, we regard two categories as "essentially the same" if they are merely *equivalent*, i.e., if there is a functor $f\colon C \to C'$ which admits an inverse *up to natural*

[1] I assume familiarity with basic categorical concepts, such as in Chapter 1 of [Rie16].

isomorphism. It is not the case that such an inverse up to natural isomorphism is itself unique. These same remarks apply in particular to equivalences of groupoids.

Although any equivalence of categories admits some kind of inverse, the failure to be unique leads to complications. For example, one goal of every course in abstract linear algebra is to demonstrate and exploit an equivalence of categories

$$f\colon \mathrm{Mat}_F \to \mathrm{Vect}_F.$$

Here Mat_F is the *matrix* category, whose objects are non-negative integers, and whose morphisms $n \to m$ are $m \times n$-matrices with entries in F. The functor f is defined by an explicit construction; e.g., it sends the object n to the vector space F^n. However, there is no completely "natural" way to construct an inverse functor $f^{-1}\colon \mathrm{Vect}_F \to \mathrm{Mat}_F$: producing such an inverse functor requires making an arbitrary choice, for each abstract vector space V, of a basis for V.

- We can consider "solutions to equations" in groupoids (e.g., limits). However, the naive construction of limits of groupoids may not preserve equivalences of groupoids; thus, we need to consider "weak" or "homotopy" limits.

  For example, suppose M is a space which is a union of two open subsets U and V. The *weak pullback* of

  $$\mathrm{Bun}_G(U) \to \mathrm{Bun}_G(U \cap V) \leftarrow \mathrm{Bun}_G(V)$$

  is a groupoid, whose objects are triples (P, Q, α), where $P \to U$ and $Q \to U$ are G-bundles, and $\alpha\colon P|_{U\cap V} \xrightarrow{\sim} Q|_{U\cap V}$ is an isomorphism of G-bundles over $U \cap V$; the morphisms $(P, Q, \alpha) \to (P', Q', \alpha')$ are pairs $(f\colon P \to P', g\colon Q \to Q')$ are pairs of bundle maps which are compatible over $U \cap V$ with the isomorphisms α, α'. Compare this with the *strict pullback*, which consists of (P, Q) such that $P|_{U\cap V} = Q|_{U\cap V}$ as bundles; in particular, $P|_{U\cap V}$ and $Q|_{U\cap V}$ must be the *identical sets*.

  A basic result about bundles is that $\mathrm{Bun}_G(M)$ is equivalent to this weak pullback. The strict limit may fail to be equivalent to this; in fact, it is impossible to describe the strict pullback without knowing precisely what definition of G-bundle we are using: in this case we need to be able to say when two bundles are *equal*, rather than *isomorphic*. The weak pullback is however relatively insensitive to the precise definition of G-bundle. (The point being, there can exist many non-identical "precise definitions of G-bundle", because what we really care about in the end is understanding $\mathrm{Bun}_G(M)$ up to *equivalence*, rather than up to isomorphism.)

These kinds of issues persist when dealing with higher groupoids and categories.

1.2. **Higher groupoids.** There is a category Gpd of groupoids, whose objects are groupoids and whose morphisms are functors. However, there is even more structure here; there are *natural transformations* between functors $f, f'\colon G \to G'$ of groupoids. That is, $\mathrm{Fun}(G, G')$ forms not merely a set, but a category. We can consider the collection consisting of (0) groupoids, (1) *equivalences* between groupoids, and (2) natural isomorphisms between equivalences; this is an example of a *2-groupoid*[2]. There is no reason to stop at 2-groupoids: there are n-groupoids, the totality of which are an example of an $(n+1)$-groupoid. (In this hierarchy, 0-groupoids are sets, and 1-groupoids are groupoids.) We might as well take the limit, and consider ∞-groupoids.

It turns out to be difficult (though not impossible) to construct an "algebraic" definition of n-groupoid. The approach which in seems to work best in practice is to use homotopy theory. We start with the observation that every groupoid G has a *classifying space* BG. This is defined

[2]More precisely, a "quasistrict 2-groupoid".

explicitly as a quotient space

$$G \mapsto BG := \left(\coprod_{x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} x_n} \Delta^n \right) / \sim,$$

where we glue in a topological n-simplex Δ^n for each n-fold sequence of composable arrows in G, modulo certain identifications. It turns out (i) the fundamental groupoid of BG is equivalent to G, and (ii) the higher homotopy groups π_k of BG are trivial, for $k \geq 2$. A space like this is called a *1-type*. Furthermore, (iii) there is a bijection between *equivalence classes of groupoids up to equivalence* and *CW-complexes which are 1-types, up to homotopy equivalence*. (More is true, but I'll stop there for now.)

The conclusion is that groupoids and equivalences between them are modelled by 1-types and homotopy equivalences between them. This suggests that we should *define* n-groupoids as n-types (CW complexes with trivial homotopy groups in dimensions $> n$), with equivalences being homotopy equivalences. Removing the restriction on homotopy groups leads to modelling ∞-groupoids by CW-complexes up to homotopy equivalence.

There is a different approach, which we will follow. It uses the fact that the classiyfing space construction factors through a "combinatorial" construction, called the "nerve". That is, we have

$$(G \in \mathrm{Gpd}) \mapsto (NG \in s\mathrm{Set}) \mapsto (\|NG\| = BG \in \mathrm{Top}),$$

where NG is the *nerve* of the groupoid, and is an example of a *simplicial set*; $\|X\|$ denotes the *geometric realization* of a *simplicial set* X. In fact, the nerve is a particular kind of simplicial set called a *Kan complex*. It is a classical fact of homotopy theory that Kan complexes model all homotopy types. Thus, we will choose our definitions so that ∞-groupoids are precisely the Kan complexes.

1.3. **∞-categories.** An ∞-category is a generalization of ∞-groupoid in which morphisms are no longer required to be invertible in any sense.

There are a number of approaches to defining ∞-categories. Here are two which build on top of the identification of ∞-groupoids with Kan complexes.

- A category C consists of a set $\mathrm{ob}\, C$ of objects, and for each pair of objects a set $\hom_C(x, y)$ of maps from x to y. If we replace the set $\hom_C(x, y)$ with a Kan complex (or more generally a simplicial set) $\mathrm{map}_C(x, y)$, we obtain a category *enriched* over Kan complexes (or simplicial sets). This leads to one model for ∞-categories: categories enriched over simplicial sets.
- The nerve construction makes sense for categories: given a category C, we have a simplicial set NC. In general, NC is not a Kan complex; however, it does land in a special class of simplicial sets, which are called quasicategories. This leads to another model for ∞-categories: *quasicategories*.

In this paper we focus on the second case: the quasicategory model for ∞-categories.

1.4. **Historical remarks.** Quasicategories were invented by Boardman and Vogt [BV73, §IV.2], under the name *restricted Kan complex*. They did not use them to develop a theory of ∞-categories. This development began with the work of Joyal, first published in [Joy02]. Much of the material in this course was developed first by Joyal, in published papers and unpublished manuscripts [Joy08a], [Joy08b], [JT08]. Lurie [Lur09] gives a thorough treatment of quasicategories (which he simply calls "∞-categories"), recasting and extending Joyal's work significantly.

There are significant differences between the ways that Joyal and Lurie develop the theory. In particular, they give different definitions of the notion of a "categorical equivalence" between simplicial sets, though they do in fact turn out to be equivalent [Lur09, §2.2.5]. The approach I follow here is essentially that of Joyal. However, I have tried to follow Lurie's terminology and notation in most places.

1.5. **Goal of this book.** The goal of this book is to give a reasonably approachable introduction to the subject of higher category theory. In particular, I am writing with the following ideas in mind.

- The prerequisites are merely some basic notions of category theory, as seen in a first year algebraic topology or algebraic geometry course. No advanced training in homotopy theory is assumed: in particular, no knowledge of simplicial sets or model categories is assumed.
- The book is written in "lecture notes" style rather that "textbook" style. That is, I will try to avoid introducing a lot of theory in section 3 which is only to be used in section 42, even if that is the "natural" place for it. The goal is to introduce new ideas near where they are first used, so that motivations are clear.
- The structure of the exposition is organized around the following type of question: Here is a [definition we can make/theorem we can prove] for ordinary categories; how do we generalize it to quasicategories? In some cases the answer is easy. In others, it can require a significant detour.
- The exposition is largely from the bottom up, rather than from the top down. Thus, I attempt to give complete details about everything I prove, so that nothing is relegated to references. (The current document does not achieve this yet, but that is the plan; in some cases, such details will be put into appendices.)
- The idea is that, after you have read this book, you will be well-prepared to dip into the main references on quasicategories (e.g., Lurie's books) without too much difficulty. Note that this book is not meant to (and does not) supplant any such reference.

1.6. **Prerequisites.** I assume only familiarity with basic concepts of category theory, such as those discussed in the first few chapters of [Rie16]. It is helpful, but not essential, to know a little algebraic topology (such as fundamental groups and groupoids, and the definition of singular homology, as described in Chs. 1–3 of Hatcher's textbook).

Some categorical prerequisites: you should be at least aware of the following notions (or know where to turn to in order to learn them):

- categories, functors, and natural transformations;
- full subcategories;
- groupoids;
- products and coproducts;
- pushouts and pullbacks;
- general colimits and limits.
- adjoint functors.

1.7. **References and other sources.** As noted, the material depends mainly on the work of Joyal and Lurie.

- Joyal's first paper [Joy02] on the subject explicitly introduces quasicategories as a model for ∞-categories. It is worth looking at.
- There are several versions of unpublished lecture notes by Joyal [Joy08a], [Joy08b], which develop the theory of quasicategories from scratch. Also note the paper by Joyal and Tierney [JT08], which gives a summary of some of this unpublished work.
- Lurie's "Higher topos theory" [Lur09] gives a complete development of ∞-categories, including a number of topics not even touched in this book. The main general material on ∞-categories is in Chapters 1–4, together with quite of bit of material from the appendices. It is also worth looking at Chapter 5, which develops the very important notions of accessible and presentable ∞-categories. The final two chapters apply these ideas to the theory of ∞-topoi.

- Lurie's "Higher algebra" [Lur12] treats a number of "advanced topics", including *stable ∞-categories* (the ∞-categorical foundations for derived categories in homological algebra and stable homotopy), various notions of monoidal structures on ∞-categories (via the theory of ∞-*operads*), and other topics.
- After I came up with the first version of these notes, Cisinski published the book "Higher Categories and Homotopical Algebra". It covers much of the material in these notes (and much more), on roughly similar lines: in his book model categories play a more prominent role from the start than they do here.
- Bergner's "The homotopy theory of $(\infty, 1)$-categories" is a survey of various approaches to higher categories and their interrelationships.
- Groth's note "A short course on ∞-categories" provides a brief survey to some of the basic ideas about quasicategories and their applications. It is not a complete treatment, but it does get very quickly to some of the more advanced topics.
- Riehl and Verity ...

1.8. **Things to add.** This is a place for me to remind myself of things I might add.

- A discussion of n-truncation and n-groupoids, including the equivalence of ordinary groupoids to 1-groupoids (so connecting with the introduction).
- Pointwise criterion for limits/colimits: Show that $S^{\rhd} \to \mathrm{Fun}(D, C)$ is a colimit cone if each projection to $S^{\rhd} \to \mathrm{Fun}(\{d\}, C) \approx C$ is one.

1.9. **Acknowledgements.** Thanks to all those who have submitted corrections, including most notably: Lang (Robbie) Yin, Darij Grinberg, and Vigleik Angeltveit. I'd also like to thank the participants of courses I have given based on a version of these notes: (Math 595 at the University of Illinois in Fall 2016, and again in Spring 2019).

Part 1. **Basic notions**

2. Simplicial sets

In the subsequent sections, we will define quasicategories as a generalization of the notion of a category. To accomplish this, we will recharacterize categories as a particular kind of *simplicial set*; relaxing this characterization will lead us to the definition of quasicategories.

Simplicial sets were introduced as a combinatorial framework for the homotopy theory of spaces. There are a number of treatments of simplicial sets from this point of view. We recommend Greg Friedman's survey [Fri12] as a starting place for learning about this viewpoint, and we will discuss this point of view later on in §**??**. Here we will focus on what we need in order to develop quasicategories.

2.1. **The simplicial indexing category Δ.** We write Δ for the category whose **M 14 Jan 2019**

- **objects** are the non-empty totally ordered sets $[n] := \{0 < 1 < \cdots < n\}$ for $n \geq 0$, and
- **morphisms** $f\colon [n] \to [m]$ are weakly monotone functions, i.e., such that $x \leq y$ implies $f(x) \leq f(y)$.

Note that we exclude the empty set from Δ. Morphisms in Δ are often called **simplicial operators**.

Because $[n]$ is an ordered set, you can also think of it as a category: the objects are the elements of $[n]$, and there is a morphism (necessarily unique) $i \to j$ if and only if $i \leq j$. Thus, morphisms in the category Δ are precisely the functors between the categories $[n]$. We can, and will, also think of $[n]$ as the category "freely generated" by the picture

$$\boxed{0} \to \boxed{1} \to \cdots \to \boxed{n-1} \to \boxed{n}.$$

Arbitrary non-identity morphisms $i \to j$ in $[n]$ can be expressed uniquely as iterated composites of the arrows $i \to i + 1$ which are displayed in the picture.

We will often use the following notation for morphisms in Δ:

$$f = \langle f_0 \cdots f_n \rangle \colon [n] \to [m] \quad \text{with } f_0 \leq \cdots \leq f_n \quad \text{represents the function} \quad k \mapsto f_k.$$

2.2. *Remark.* There are distinguished simplicial operators called **face** and **degeneracy** operators:

$$d^i := \langle 0, \dots, \widehat{i}, \dots, n \rangle \colon [n-1] \to [n], \quad 0 \leq i \leq n,$$
$$s^i := \langle 0 \dots i, i, \dots n \rangle \colon [n+1] \to [n], \quad 0 \leq i \leq n.$$

All maps in Δ can be obtained as a composition of face and degeneracy operators, and in fact Δ can be described as the category *generated* by the above symbols, subject to a set of *relations* called the "simplicial identities", which can be found in various places, e.g., [Fri12, Def. 3.2].

2.3. **Simplicial sets.** A **simplicial set** is a functor $X \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$, i.e., a contravariant functor (or "presheaf") from Δ to sets.

It is typical to write X_n for $X([n])$, and call it the set of n*-simplices* in X. I generally prefer to call it the set of n**-dimensional elements** of X instead (because the word "simplices" also applies to the so called "standard n-simplices" defined below (2.8), and I would like to avoid to confusion between them). I will also speak of the set of all elements (or all simplices) of X, i.e., of the disjoint union $\coprod_{n\geq 0} X_n$ the sets X_n.

The 0-dimensional elements of a simplicial set are also called **vertices**, while the 1-dimensional elements are also called **edges**.

Given an element $a \in X_n$ and a simplicial operator $f \colon [m] \to [n]$, I will write $af \in X_m$ as shorthand for $X(f)(a)$. That is, I'll think of simplicial operators as acting on elements from the right; this is a convenient choice given that X is a *contravariant* functor. In this language, a simplicial set consists of

- a sequence of sets $X_0, X_1, X_2, \dots$,
- functions $a \mapsto af \colon X_n \to X_m$ for each simplicial operator $f \colon [m] \to [n]$, such that
- $a\,\mathrm{id} = a$, and $(af)g = a(fg)$ for any element a and simplicial operators f and g whenever this makes sense.

If I need to have the simplicial operator act from the left, I'll write $f^*(a) = af$.

Sometimes I'll use a subscript notation when speaking of the action of particular simplicial operators. So, given a simplicial operator of the form $f = \langle f_0 \cdots f_m \rangle \colon [m] \to [n]$, we can indicate the action of f on elements using subscripts:

$$a_{f_0 \cdots f_m} := af = a\langle f_0 \dots f_m \rangle.$$

In particular, applying simplicial operators of the form $\langle i \rangle \colon [0] \to [n]$ to an n-dimensional element $a \in X_n$ gives vertices $a_0, \dots, a_n \in X_0$, which we call the "*vertices* of a", while applying simplicial operators of the form $\langle ij \rangle \colon [1] \to [n]$ for $0 \leq i \leq j \leq n$ gives edges $a_{ij} \in X_1$, which we call the "*edges* of a".

2.4. **The category of simplicial sets.** A simplicial set is a functor; therefore a **map of simplicial sets** is a natural transformation of functors. Explicitly, a map $\phi \colon X \to Y$ between simplicial sets is a collection of functions $\phi \colon X_n \to Y_n$, $n \geq 0$, which commute with simplicial operators:

$$(\phi a)f = \phi(af) \qquad \text{for all simplicial operators } f \text{ and elements } a \text{ in } X \text{, when this makes sense.}$$

I'll write $s\mathrm{Set}$ for the category of simplicial sets and maps between them[3].

[3]Lurie [Lur09] uses Set_Δ to denote the category of simplicial sets. **Perhaps I should try to be consistent with this?**

2.5. **Discrete simplicial sets.** A simplicial set X is **discrete** if every simplicial operator f induces a bijection $f^*\colon X_n \to X_m$.

Every set S gives us a discrete simplicial set S^{disc}, defined so that $(S^{\text{disc}})_n = S$, and so that each simplicial operator acts according to the identity map of S. This construction defines a functor $S \mapsto S^{\text{disc}}\colon \text{Set} \to s\text{Set}$.

2.6. *Exercise* (Discrete simplicial sets come from sets). Show that (i) every discrete simplicial set X is isomorphic to S^{disc} for some set S, and (ii) for every pair of sets S and T, the evident function $\text{Hom}_{\text{Set}}(S,T) \to \text{Hom}_{s\text{Set}}(S^{\text{disc}}, T^{\text{disc}})$ is a bijection.

Let $s\text{Set}^{\text{disc}}$ denote the full subcategory of $s\text{Set}$ spanned by discrete simplicial sets. That is, objects of $s\text{Set}^{\text{disc}}$ are discrete simplicial sets, and morphisms of $s\text{Set}^{\text{disc}}$ are all simplicial maps between them. Then the above exercise amounts to saying that *the full subcategory of discrete simplicial sets is equivalent to the category of sets.*

For this reason, it is often convenient to (at least informally) "identify" sets with their corresponding discrete simplicial sets (i.e., for a set S we also write S for the discrete simplicial set S^{disc} defined above).

2.7. *Exercise.* Show that for any set S and simplical set X there is a bijection $\text{Hom}_{s\text{Set}}(S^{\text{disc}}, X) \to \text{Hom}_{\text{Set}}(S, X_0)$.

2.8. **Standard n-simplex.** The **standard n-simplex** Δ^n is the simplicial set defined by

$$\Delta^n := \text{Hom}_\Delta(-, [n]).$$

That is, the standard n-simplex is exactly the functor *represented* by the object $[n]$. Explicitly, this means that

$$(\Delta^n)_m = \text{Hom}_\Delta([m],[n]) = \{\text{simplicial operators } a\colon [m] \to [n]\},$$

while the action of simplicial operators on elements of Δ^n is given by composition: $f\colon [m'] \to [m]$ sends $a \in (\Delta^n)_m$ to $(af\colon [m'] \to [n]) \in (\Delta^n)_{m'}$.

The **generator** of Δ^n is the element

$$\iota_n := \langle 01 \dots n\rangle = \text{id}_{[n]} \in (\Delta^n)_n$$

corresponding to the identity map of $[n]$.

The **Yoneda lemma** (applied to the category Δ) asserts that the function

$$\begin{aligned} \text{Hom}_{s\text{Set}}(\Delta^n, X) &\to X_n, \\ g &\mapsto g(\iota_n), \end{aligned}$$

is a bijection for every simplicial set X. (*Exercise:* if this fact is not familiar to you, prove it.)

The Yoneda lemma can be stated this way: for each n-dimensional element $a \in X_n$ there exists a *unique* map $f_a\colon \Delta^n \to X$ of simplicial sets which sends the generator to it, i.e., such that $f_a(\iota_n) = a$. We call the map f_a the **representing map** of the element a.

We will often use the bijection provided by the Yoneda lemma implicitly. In particular, instead of using notation such as f_a, we will typicially abuse notation and write $a\colon \Delta^n \to X$ for the representing map of the element $a \in X_n$. We reiterate that the map $a\colon \Delta^n \to X$ is characterized as the *unique* map sending the generator ι_n of Δ^n to a. Thus with our notation we have $a = a(\iota_n)$, where the two appearances of "a" denote respectively the element of X_n and the representing morphism $\Delta^n \to X$.

2.9. *Exercise.* Show that the representing map $f\colon \Delta^n \to X$ of $a \in X_n$ sends $\langle f_0 \cdots f_k\rangle \in (\Delta^n)_k$ to $a\langle f_0 \cdots f_k\rangle \in X_k$.

Note that if $X = \Delta^m$ is also a standard simplex, then the Yoneda lemma gives a bijection

$$\text{Hom}_{s\text{Set}}(\Delta^n, \Delta^m) \xrightarrow{\sim} (\Delta^m)_n = \text{Hom}_\Delta([n],[m]).$$

The inverse of this bijection sends a simplicial operator $f\colon [n] \to [m]$ to the map $\Delta^f\colon \Delta^n \to \Delta^m$ of simplicial sets defined on elements $g \in (\Delta^n)_k = \mathrm{Hom}_\Delta([k],[n])$ by $g \mapsto fg$. (*Exercise:* prove this.)

I will commonly abuse notation, and write $f\colon \Delta^n \to \Delta^m$ instead of Δ^f for the map induced by the simplicial operator f, as it is also the representing map of the corresponding n-dimensional element $f \in (\Delta^m)_n$.

2.10. **The standard 0-simplex and the empty simplicial set.** The standard 0-simplex Δ^0 is the terminal object in sSet; i.e., for every simplicial set X there is a unique map $X \to \Delta^0$. Sometimes I write $*$ instead of Δ^0 for this object. Note that Δ^0 is the only standard n-simplex which is discrete.

The **empty simplicial set** $\varnothing$ is the functor $\Delta^{\mathrm{op}} \to \mathrm{Set}$ sending each $[n]$ to the empty set. It is the initial object in sSet; i.e., for every simplicial set X there is a unique map $\varnothing \to X$.

2.11. *Exercise.* Show that a simplicial set X is isomorphic to the empty simplicial set if and only if X_0 is isomorphic to the empty set.

2.12. **Standard simplices on totally ordered sets.** The definition of the standard simplices Δ^n can be extended to simplicial sets "generated" by arbitrary totally ordered sets.

For instance, for *any* non-empty finite totally ordered set $S = \{s_0 < s_1 < \cdots < s_n\}$, there is a *unique* order preserving bijection $[n] \xrightarrow{\sim} S$ for a unique $n \geq 0$. We write Δ^S for the simplicial set with $(\Delta^S)_k = \{\text{order preserving } [k] \to S\}$. There is a unique isomorphism $\Delta^S \approx \Delta^n$ of simplicial sets (*Exercise:* prove this). We can also apply this idea to the empty ordered set $S = \varnothing$, in which case $(\Delta^\varnothing)_k = \varnothing$ for all k, i.e., $\Delta^\varnothing$ is the empty simplicial set.

This notation is especially convenient for subsets $S \subseteq [n]$ with induced ordering, as the simplicial set Δ^S is in a natural way a *subcomplex* of Δ^n (i.e., a collection of subsets of the $(\Delta^n)_k$ closed under action of simplicial operators; we will return to the notion of subcomplex below §4.9).

Furthermore, any simplicial operator $f\colon [m] \to [n]$ factors through its image $S = f([m]) \subseteq [n]$, giving a factorization

$$[m] \xrightarrow{f_{\mathrm{surj}}} S \xrightarrow{f_{\mathrm{inj}}} [n]$$

of maps between ordered sets, and thus a factorization $\Delta^m \xrightarrow{\Delta^{f_{\mathrm{surj}}}} \Delta^S \xrightarrow{\Delta^{f_{\mathrm{inj}}}} \Delta^n$ of the induced map Δ^f of simplicial sets.

2.13. *Exercise.* Show that $\Delta^{f_{\mathrm{inj}}}$ and $\Delta^{f_{\mathrm{surj}}}$ respectively induce maps between simplicial sets which are (respectively) injective and surjective on sets of k-dimensional elements for all k. (The case of $\Delta^{f_{\mathrm{inj}}}$ is formal, but the case of $\Delta^{f_{\mathrm{surj}}}$ is not completely formal.)

2.14. **Pictures of standard simplices.** When we draw a "picture" of Δ^n, we draw a geometric n-simplex: the convex hull of $n+1$ points in general position, with vertices labelled by $0, \dots, n$. The faces of the geometric simplex correspond exactly to *injective* simplicial operators into $[n]$: these elements are called *non-degenerate*. For each non-degenerate simplex f in Δ^n, there is an infinite collection of *degenerate* elements with the same "image" as f (when viewed as a simplicial operator with target $[n]$).

Here are some "pictures" of standard simplices, which show their non-degenerate elements. Note that we draw the 1-dimensional elements of Δ^n as arrows; this lets us easily see the total ordering on the vertices of Δ^n.

Δ^0: $\langle 0\rangle$

Δ^1: $\langle 0\rangle \longrightarrow \langle 1\rangle$

Δ^2:
$$\begin{tikzcd} & \langle 1\rangle \arrow[d] \\ \langle 0\rangle \arrow[ur] \arrow[dr] & \\ & \langle 2\rangle \end{tikzcd}$$

Δ^3:
$$\begin{tikzcd} & \langle 1\rangle \arrow[dd] \arrow[dr] & \\ \langle 0\rangle \arrow[ur] \arrow[dr] \arrow[rr] & & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd}$$

We'll extend the terminology of "degenerate" and "non-degenerate" elements to arbitrary simplicial sets in §15.5.

3. The nerve of a category

The nerve of a category is a simplicial set which retains all the information of the original category. In fact, the nerve construction provides a full embedding of Cat, the category of (small) categories, into sSet, which means that we are able to think of categories as just a special kind of simplicial set.

3.1. **Construction of the nerve.** Given a category C, the **nerve** of C is the simplicial set NC defined so that

$$(NC)_n := \operatorname{Hom}_{\mathrm{Cat}}([n], C),$$

the set of functors from $[n]$ to C, and so that simplicial operators $f\colon [m] \to [n]$ act by precomposition: $a \mapsto af$ for an element $a\colon [n] \to C$ in $(NC)_n$.

3.2. *Example.* There is an evident isomorphism $N[n] \approx \Delta^n$, which is in fact the unique isomorphism between these two categories.

Given a functor $F\colon C \to D$ between categories, we obtain a map $NF\colon NC \to ND$ of simplicial sets, sending $(a\colon [n] \to C) \in (NC)_n$ to $(Fa\colon [n] \to D) \in (ND)_n$. Thus the nerve construction defines a functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$.

3.3. **Structure of the nerve.** We observe the following, whose verification we leave to the reader.

- $(NC)_0$ is canonically identified with the set of objects of C.
- $(NC)_1$ is canonically identified with the set of morphisms of C.
- The operators $\langle 0\rangle^*, \langle 1\rangle^*\colon (NC)_1 \to (NC)_0$ assign to a morphism its source and target respectively.
- The operator $\langle 00\rangle^*\colon (NC)_0 \to (NC)_1$ assigns to an object its identity map.
- $(NC)_2$ is in bijective correspondence with the set of pairs (f, g) of morphisms such that gf is defined, i.e., such that the target of f is the source of g. This bijection is given by sending $a \in (NC)_2$ to $(a_{01}, a_{12}) \in (NC)_1 \times (NC)_1$.
- The operator $\langle 02\rangle^*\colon (NC)_2 \to (NC)_1$ assigns, to an element corresponding to a pair (f, g) of morphisms, the composite morphism gf.

In particular, you can recover the category C from its nerve NC, up to isomorphism, since the nerve contains all information about objects, morphisms, and composition of morphisms in C.

We have the following general description of n-dimensional elements in the nerve.

3.4. **Proposition.** *Let C be a category.*

(1) *There is a bijective correspondence*

$$(NC)_n \xrightarrow{\sim} \left\{\, (g_1, \dots, g_n) \in (\operatorname{mor} C)^{\times n} \;\middle|\; \operatorname{target}(g_{i-1}) = \operatorname{source}(g_i) \,\right\},$$

which sends $(a\colon [n] \to C) \in (NC)_n$ to the sequence $(a\langle 0,1\rangle, \dots, a\langle n-1,n\rangle)$

(2) *With respect to the correspondence of (1), the map $f^*\colon (NC)_n \to (NC)_m$ induced by a simplicial operator $f\colon [m] \to [n]$ coincides with the function*

$$(g_1, \dots, g_n) \mapsto (h_1, \dots, h_m), \qquad h_k = \begin{cases} \mathrm{id} & \text{if } f(k-1) = f(k) \\ g_j g_{j-1} \cdots g_{i+1} & \text{if } f(k-1) = i < j = f(k). \end{cases}$$

Proof. For (1), one verifies that an inverse is given by the function which sends a sequence $(g_1, \dots, g_n)$ to $(a\colon [n] \to C) \in (NC)_n$ defined on objects by $a(k) = \operatorname{target}(g_{k-1}) = \operatorname{source}(g_k)$, and on morphisms by $a(\langle ij\rangle) = g_j g_{j-1}\cdots g_{i+1}$ for $i<j$. For (2), note that for $a\in (NC)_n$ corresponding to the tuple $(g_1,\dots,g_n)$ we can compute

$$(af)\langle k-1,k\rangle = a\langle f(k-1),f(k)\rangle = \begin{cases} \mathrm{id} & \text{if } f(k-1)=f(k), \\ g_j g_{j-1}\cdots g_{i+1} & \text{if } f(k-1)=i<j=f(k). \end{cases}$$

□

3.5. *Remark.* It is clear from the above remarks that most of the information in the nerve of C is redundant: we only needed $(NC)_k$ for $k=0,1,2$ and certain simplicial operators between them to recover complete information about the category C.

3.6. *Exercise.* Show that for any discrete simplicial set X there exists a category C and an isomorphism $NC \approx X$.

3.7. **Characterization of nerves.** This leads to the question: given a simplicial set X, how can we detect that it is isomorphic to the nerve of some category?

3.8. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if for all $n\geq 2$ the function*

$$\phi_n\colon X_n \to \left\{ (g_1,\dots,g_n)\in (X_1)^{\times n} \;\middle|\; g_{i-1}\langle 1\rangle = g_i\langle 0\rangle,\ 1\leq i\leq n \right\}$$

which sends $a\in X_n$ to $(a_{0,1},\dots,a_{n-1,n})$ is a bijection.

Proof. First, suppose $X=NC$ for some category C. Then the function ϕ_n is precisely the bijection of (3.4)(1). Thus, if X is isomorphic to the nerve of some category then its ϕ_n are bijections.

Now suppose X is a simplical set such that the ϕ_n are bijections. We *define* a category C, with

$$(\text{objects of } C) = X_0, \qquad (\text{morphisms of } C) = X_1,$$

following the discussion in (3.3). Thus, the source and target of $g\in X_1$ are g_0 and g_1 in X_0 respectively, the identity map of $x\in X_0$ is $x_{00}\in X_1$, while the composite of (g,h) such that $g_1=h_0$ is a_{02}, where $a\in X_2$ is the unique 2-dimensional element with $a_{01}=g$ and $a_{12}=h$. We leave the remaining details (e.g., unit and associativity properties) to the reader, though we note that proving associativity requires consideration of elements of X_3. (Or look ahead to (5.10), where we carry out the argument explicitly in a slighlty different context.)

Next, we claim that for $a\in X_n$, and for $0\leq i\leq j\leq k\leq n$, we have that

$$a_{i,k} = a_{j,k}a_{i,j},$$

where $a_{i,k},a_{i,j},a_{j,k}\in X_1$ are images of a under face operators $[1]\to[n]$, and right-hand side represents composition of two morphisms in C. To see this, note first that for $b\in X_2$, we have $b_{0,2}=b_{1,2}b_{0,1}$ by construction of C. The general case follows from this by setting $b=a_{i,j,k}$.

Now we can define maps $\psi_n\colon X_n\to (NC)_n$ by sending $a\in X_n$ to $\psi(a)\colon [n]\to C$ defined by $\psi(a)(i\to j)=a_{i,j}$, which is a functor by the above remarks. These maps ψ_n are seen to be bijections using the bijections ϕ_n and (3.4), since $\psi_n(a)(i-1\to i)=a_{i-1,i}$. If $f\colon [m]\to[n]$ is a simplicial operator, then we compute

$$\psi_m(af)(i\to j) = (af)_{i,j} = a_{f(i),f(j)} = (\psi_n(a))(f(i)\to f(j)) = (\psi_n(a)f)(i\to j),$$

whence ψ is a map of simplicial sets. We have thus constructed an isomorphism $\phi\colon X\to NC$ of simplicial sets, as desired.

□

3.9. **A characterization of maps between nerves.** Maps between nerves are the same as functors between categories.

3.10. **Proposition.** *The nerve functor* $N\colon \mathrm{Cat} \to s\mathrm{Set}$ *is fully faithful. That is, every simplicial set map* $g\colon NC \to ND$ *between nerves is of the form* $g = N(f)$ *for a unique functor* $f\colon C \to D$.

Proof. We need to show that $\mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{s\mathrm{Set}}(NC, ND)$ induced by the functor N is a bijection for all categories C and D. *Injectivity* is clear, as a functor f is determined by its effect on objects and morphisms, which is exactly the effect of $N(f)$ on 0- and 1-dimensional elements of the nerves.

For *surjectivity*, observe that for any map $g\colon NC \to ND$ of simplicial sets, we can define a candidate functor $f\colon C \to D$, defined on objects and morphisms by the action of g on 0-dimensional and 1-dimensional elements. That F has the correct action on identity maps follows from the fact that g commutes with the simplicial operator $\langle 00\rangle\colon [1] \to [0]$. That f preserves composition uses (3.4) and the fact that g commutes with the simplicial operator $\langle 02\rangle\colon [1] \to [2]$.

Note that given $g\colon NC \to ND$ and $f\colon C \to D$ as constructed above, the maps $g, N(f)\colon NC \to ND$ coincide on 0-dimensional and 1-dimensional elements by construction. It follows that $g = N(f)$ by (3.11) below. Thus, we have shown that $N\colon \mathrm{Hom}_{\mathrm{Cat}}(C, D) \to \mathrm{Hom}_{s\mathrm{Set}}(NC, ND)$ is surjective as desired. □

3.11. *Exercise.* Show that if C is a category and X is any simplicial set (not necessarily a nerve), then two maps $g, g'\colon X \to NC$ are equal if and only if $g_0 = g'_0$ and $g_1 = g'_1$, i.e., g and g' are equal if ond only if they coincide on 0-dimensional and 1-dimensional elements. (Hint: use (3.4).)

4. Spines

In this section we will restate our characterization of simplicial sets which are isomorphic to nerves, in terms of a certain "extension" condition. To state this condition we need the notion of a "spine" of a standard n-simplex.

4.1. **The spine of an n-simplex.** The **spine** of the n-simplex Δ^n is the simplicial set I^n defined by

$$(I^n)_k = \{\, \langle a_0 \cdots a_k\rangle \in (\Delta^n)_k \;\mid\; a_k \leq a_0 + 1 \,\}.$$

That is, a k-dimensional element of I^n is a simplicial operator $a\colon [k] \to [n]$ whose image is of the form either $\{j\}$ or $\{j, j+1\}$. The action of simplicial operators on elements of I^n is induced by their action on Δ^n. (To see that this action is well defined, observe that for $a\colon [k] \to [n]$ in $(I^n)_k$ and $f\colon [p] \to [k]$, the image of the simplicial operator af is contained in the image of a.)

There is an evident injective map $I^n \to \Delta^n$ of simplicial sets. (In fact, I^n is another example of a *subcomplex* of Δ^n, see below §4.9.) Here is a picture of I^3 in Δ^3:

$$\begin{tikzcd} & \langle 1\rangle \arrow[d] & \\ \langle 0\rangle \arrow[ur] & {} & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd} \qquad \text{is the spine inside} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[d] \arrow[dr] & \\ \langle 0\rangle \arrow[ur] \arrow[rr] \arrow[dr] & {} & \langle 3\rangle \\ & \langle 2\rangle \arrow[ur] & \end{tikzcd}$$

Note that $I^0 = \Delta^0$ and $I^1 = \Delta^1$.

The key property of the spine is the following.

4.2. **Proposition.** *Given a simplicial set* X*, for every* $n \geq 0$ *there is a bijection*

$$\mathrm{Hom}(I^n, X) \xrightarrow{\sim} \{\, (a_1, \ldots, a_n) \in (X_1)^{\times n} \;\mid\; a_i\langle 1\rangle = a_{i+1}\langle 0\rangle \,\},$$

defined by sending $f\colon I^n \to X$ *to* $(f(\langle 01\rangle), f(\langle 12\rangle), \cdots, f(\langle n-1, n\rangle))$. *(In the case* $n = 0$*, the target of the bijection is taken to be the set* X_0 *of vertices of* X*, and the bijection in this case sends* $f \mapsto f\langle 0\rangle$.*)*

We will give the proof at the end of this section, after we describe I^n as a colimit of a diagram of standard simplices; specifically, as a collection of 1-simplices "glued" together at their ends.

4.3. **Nerves are characterized by unique spine extensions.** We can now state our new characterization of nerves: they are simplicial sets such that every map $I^n \to X$ from a spine extends *uniquely* along $I^n \subseteq \Delta^n$ to a map from the standard n-simplex. That is, nerves are precisely the simplicial sets with "unique spine extensions".

4.4. **Proposition.** *A simplicial set X is isomorphic to the nerve of some category if and only if the restriction map* $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(I^n, X)$ *along $I^n \subseteq \Delta^n$ is a bijection for all $n \geq 2$.*

Proof. Immediate from (4.2) and (3.8). $\square$

4.5. **Colimits of sets and simplicial sets.** Given any functor $F\colon C \to \mathrm{Set}$ from a small category to sets, there is a "simple formula" for its colimit. First consider the coproduct (i.e., disjoint union) $\coprod_{c\in \operatorname{ob} C} F(c)$ of the values of the functor; I'll write (c, x) for a typical element of this coproduct, with $c \in \operatorname{ob} C$ and $x \in F(c)$. Consider the relation $\sim$ on this defined by

$$(c, x) \sim (c', x') \quad \text{if } \exists\ \alpha\colon c \to c' \text{ in } C \text{ such that } F(\alpha)(x) = x'.$$

Define

$$X := \Big(\coprod_{c\in \operatorname{ob} C} F(c)\Big)/\approx,$$

the set obtained as the quotient by the equivalence relation "$\approx$" which is *generated* by the relation "$\sim$". For each object c of C we have a function $i_c\colon F(c) \to X$ defined by $i_c(x) := [(c, x)]$, sending x to the equivalence class of (c, x). Then the data $(X, \{i_c\})$ is a colimit of the functor F: i.e., for any set S and collection of functions

$$f_c\colon F(c) \to S \quad \text{such that} \quad f_{c'} \circ F(\alpha) = f_c \quad \text{for all } \alpha\colon c \to c'$$

there exists a *unique* function $f\colon X \to S$ such that $f \circ i_c = f_c$.

4.6. *Example.* Verify that $(X, \{i_c\})$ is in fact a colimit of F.

We write $\operatorname{colim}_C F$ for a chosen colimit of F.

Note that the relation "$\sim$" is often not itself an equivalence relation, so it can be difficult to figure out what "$\approx$" actually is: the simple formula may not be so simple in practice.

4.7. *Exercise.* If C is a groupoid, then the relation $\sim$ is already an equivalence relation.

There are cases when things are more tractable.

4.8. **Proposition.** *Let $\mathcal{A}$ be a collection of subsets of a set S. Rgeard $\mathcal{A}$ as a partially ordered set under "$\subseteq$" and thus can be regarded as a category. Suppose $\mathcal{A}$ has the following property: for all $s \in S$, and $T, U \in \mathcal{A}$ such that $s \in T \cap U$, there exists $V \in \mathcal{A}$ such that $s \in V \subseteq T \cap U$. Then the tautological map*

$$\operatorname{colim}_{T\in\mathcal{A}} T \to \bigcup_{T\in\mathcal{A}} T$$

(sending $[(T, t)] \mapsto t$) is a bijection.

Proof. Show that $(T, t) \approx (T', t')$ if and only if $t = t'$. The remaining details are left for the reader. $\square$

Note: an easy way to satisfy the hypothesis of (4.8) is to show that $\mathcal{A}$ is closed under finite interseciton, i.e., that $T, U \in \mathcal{A}$ implies $T \cap U \in \mathcal{A}$.

4.9. **Subcomplexes.** Given a simplicial set X, a **subcomplex** is just a subfunctor of X; i.e., a collection of subsets $A_n \subseteq X_n$ which are closed under the action of simplicial operators, and thus form a simplicial set so that the inclusion $A \to X$ is a morphism of simplicial sets. We typically write $A \subseteq X$ when A is a subcomplex of X.

4.10. *Example.* Examples we have already seen include the spines $I^n \subseteq \Delta^n$ and the $\Delta^S \subseteq \Delta^n$ associated to subsets $S \subseteq [n]$.

4.11. *Exercise.* For any map $f\colon X \to Y$ of simplicial sets, the image $f(X) \subseteq Y$ of f is a subcomplex of Y.

For every set S of elements of a simplicial set, there is a smallest subcomplex which contains the set, namely the intersection of all subcomplexes containing S.

4.12. *Example.* For a vertex $x \in X_0$, we write $\{x\} \subseteq X$ for the smallest subcomplex which contains x. This subcomplex has exactly one n-dimensional element for each $n \geq 0$, namely $x\langle 0 \cdots 0\rangle$, and thus is isomorphic to Δ^0.

More generally, for a collection of vertices $a, b, c, \dots \in X_0$, we write $\{a, b, c, \dots\} \subseteq X$ for the smallest subcomplex which contains $a, b, c, \dots$. This subcomplex is a discrete simplicial set. This choice of notation is supported by our informal identification of discrete sets with sets (2.5).

The result (4.8) carries over to simplicial sets, where the role of subsets is replaced by subcomplexes.

4.13. **Proposition.** *Let $\mathcal{A}$ be a collection of subcomplexes of a simplicial set X. Regard $\mathcal{A}$ partially ordered set under "$\subseteq$", and hence as a category. Suppose $\mathcal{A}$ has the following property: for all $n \geq 0$, all $x \in X_n$, and all $K, L \in \mathcal{A}$ such that $x \in K_n \cap L_n$, there exists $M \in \mathcal{A}$ such that $x \in M_n$ and $M \subseteq K \cap L$. Then the tautological map*

$$\operatorname{colim}_{K \in \mathcal{A}} K \to \bigcup_{K \in \mathcal{A}} K$$

is a bijection.

Proof. Because simplicial sets are actually functors $\Delta^{\mathrm{op}} \to \mathrm{Set}$, colimits in simplicial sets are "computed degreewise". That is, if $F\colon C \to s\mathrm{Set}$ is a functor with colimit $Y = \operatorname{colim}_{c \in C} F(c) \in s\mathrm{Set}$, then for each $n \geq 0$ there is a canonical bijection

$$Y_n \approx \operatorname{colim}_{c \in C} F(c)_n.$$

The proposition follows using this observation and (4.8). □

4.14. *Remark* (Pushouts of subcomplexes). A special case of (4.13) applied to simplicial sets which we will use constantly is the following. If K and L are subcomplexes of a simplicial set X, then so are both $K \cap L$ and $K \cup L$, and furthermore the evident commutative square

$$\begin{tikzcd} K \cap L \arrow[r, tail] \arrow[d, tail] & L \arrow[d, tail] \\ K \arrow[r, tail] & K \cup L \end{tikzcd}$$

is a pushout square in simplicial sets, i.e., $K \cup L \approx \operatorname{colim}[K \leftarrow K \cap L \to L]$. (Proof: (4.13) with $\mathcal{A} = \{K, L, K \cap L\}$.)

4.15. **Subcomplexes of Δ^n.** For each $S \subseteq [n]$ we have a subcomplex $\Delta^S \subseteq \Delta^n$. The following says that every subcomplex of Δ^n is a union of Δ^Ss.

4.16. **Lemma.** *Let $K \subseteq \Delta^n$ be a subcomplex. If $(f\colon [m] \to [n]) \in K_m$ with $f([m]) = S$, then $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$.*

This proof uses the following elementary fact.

4.17. **Lemma.** *Any order preserving surjection $f\colon S \to T$ between finite totally ordered sets admits an order preserving section, i.e., $s\colon T \to S$ such that $fs = \mathrm{id}_T$.*

Proof. Let $s(t) = \min\{\, s \in S \mid f(s) = t \,\}$. $\square$

Proof of (4.16). Choose a section $t\colon S \to [m]$ of $f_{\mathrm{surj}}\colon [m] \to S$ using (4.17). Consider an element $\overline{g} \in (\Delta^S)_k \subseteq (\Delta^n)_k$, represented by a map $\overline{g}\colon [k] \to [n]$ whose image is contained in S. We get a commutative diagram

$$\begin{tikzcd} & {[k]} \arrow[dl, "s:=tg"'] \arrow[d, "g"] \arrow[dr, "\overline{g}"] & \\ {[m]} \arrow[r, two heads, "f_{\mathrm{surj}}"'] & S \arrow[r, tail, "f_{\mathrm{inj}}"'] & {[n]} \end{tikzcd}$$

so $\overline{g} = fs$ and hence is an element of the subcomplex K since f is. Thus $\Delta^S \subseteq K$, and it is immediate that $f \in (\Delta^S)_m$. $\square$

4.18. *Remark.* Thus, a subcomplex $K \subseteq \Delta^n$ determines and is determined by a collection $\mathcal{K}$ of subsets of $[n]$ with the property that $T \subseteq S$ and $S \in \mathcal{K}$ implies $T \in \mathcal{K}$: namely,

$$\mathcal{K} = \{\, S \subseteq [n] \mid \Delta^S \subseteq K \,\} \qquad \text{and} \qquad K = \bigcup_{S \in \mathcal{K}} \Delta^S.$$

In other words, a subcomplex of Δ^n is the "same thing" as an *abstract simplicial complex* whose vertex set is a subset of $[n]$.

We can sharpen (4.16): every subcomplex of Δ^n is a *colimit* of subcomplexes of the form Δ^S.

4.19. **Proposition.** *Let $K \subseteq \Delta^n$ be a subcomplex. Let $\mathcal{A}$ be the poset of all non-empty subsets $S \subseteq [n]$ such that the inclusion map $f\colon S \to [n]$ represents a $(|S|-1)$-dimensional element of K. Then the tautological map*

$$\operatorname{colim}_{S \in \mathcal{A}} \Delta^S \to K$$

is an isomorphism.

Proof. We must show that for each $m \geq 0$, the map $\operatorname{colim}_{S\in\mathcal{A}}(\Delta^S)_m \to K_m$ is a bijection. Each $(\Delta^S)_m = \{\, f\colon [m] \to [n] \mid f([m]) \subseteq S \,\}$ is a distinct subset of $K_m \subseteq (\Delta^n)_m$; i.e., $S \neq S'$ implies $(\Delta^S)_m \neq (\Delta^{S'})_m$. In view of (4.13), it suffices to show that for each $f \in K_m$ there is a minimal S in $\mathcal{A}$ such that $f \in (\Delta^S)_m$. This is immediate from (4.16), which says that $f \in (\Delta^S)_m$ and $\Delta^S \subseteq K$ where $S = f([m])$, and it is obvious that this S is minimal with this property. $\square$

4.20. **Proof of** (4.2)**.** Now we can prove our claim about maps out of a spine, using an explicit description of a spine as a colimit.

Proof of (4.2). Let $\mathcal{A}$ be the poset of all non-empty $S \subseteq [n]$ which correspond to elements of I^n; i.e., subsets of $[n]$ of the form $\{j\}$ or $\{j, j+1\}$. Explicitly the poset $\mathcal{A}$ has the form

$$\{0\} \to \{0,1\} \leftarrow \{1\} \to \{1,2\} \leftarrow \{2\} \to \cdots \leftarrow \{n-1\} \to \{n-1,n\} \leftarrow \{n\}.$$

By (4.19), $\operatorname{colim}_{S\in\mathcal{A}} \Delta^S \to I^n$ is an isomorphism. Thus $\operatorname{Hom}(I^n, X) \approx \operatorname{Hom}(\operatorname{colim}_{S\in\mathcal{A}} \Delta^S, X) \approx \lim_{S\in\mathcal{A}} \operatorname{Hom}(\Delta^S, X)$, and an elementary argument gives the result. $\square$

5. Horns and inner horns

We now are going to give another (less obvious!) characterization of nerves, in terms of "extending inner horns", rather than "extending spines". It will be this characterization that we "weaken" to obtain the definition of a quasicategory. **W 16 Jan 2019**

5.1. **Definition of horns.** We define a collection of subobjects of the standard simplices, called "horns". For each $n \geq 1$, these are subsimplicial sets $\Lambda^n_j \subset \Delta^n$ for each $0 \leq j \leq n$. The **horn** Λ^n_j is the subcomplex of Δ^n defined by

$$(\Lambda^n_j)_k = \{\, f\colon [k] \to [n] \;\mid\; ([n] \smallsetminus \{j\}) \not\subseteq f([k]) \,\}.$$

Using the fact (4.19) that subcomplexes of Δ^n are always unions of Δ^Ss, we see that Λ^n_j is the union of "faces" $\Delta^{[n]\smallsetminus i}$ of Δ^n other than the jth face:

$$\Lambda^n_j = \bigcup_{i \neq j} \Delta^{[n]\smallsetminus i} \subset \Delta^n.$$

When $0 < j < n$ we say that $\Lambda^n_j \subset \Delta^n$ is an **inner horn**. We also say it is a **left horn** if $j < n$ and a **right horn** if $0 < j$. Sometimes I'll speak of an **outer horn**, meaning a horn Λ^n_j with $j \in \{0, n\}$, i.e., a non-inner horn.

5.2. *Example.* The horns inside Δ^1 are just the vertices viewed as subobjects: $\Lambda^1_0 = \Delta^{\{0\}} = \{0\} \subset \Delta^1$ and $\Lambda^1_1 = \Delta^{\{1\}} = \{1\} \subset \Delta^1$. Neither is an inner horn, the first is a left horn, and the second is a right horn.

5.3. *Example.* These are the three horns inside the 2-simplex.

$$\begin{tikzcd} & \langle 1\rangle & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[ur, "\langle 01\rangle"] & & \langle 2\rangle \end{tikzcd} \qquad \begin{tikzcd} & \langle 1\rangle \arrow[dr, "\langle 12\rangle"] & \\ \langle 0\rangle \arrow[rr, "\langle 02\rangle"'] & & \langle 2\rangle \end{tikzcd}$$

$$\Lambda^2_0 \qquad\qquad\qquad \Lambda^2_1 \qquad\qquad\qquad \Lambda^2_2$$

Only Λ^2_1 is an inner horn, while Λ^2_0 and Λ^2_1 are left horns, and Λ^2_1 and Λ^2_2 are right horns. Note that Λ^2_1 is the same as the spine I^2.

5.4. *Exercise.* Visualize the four horns inside the 3-simplex. The simplicial set Λ^3_j actually kind of looks like a horn: you blow into the vertex $\langle j\rangle$, and sound comes out of the opposite missing face $\Delta^{[3]\smallsetminus j}$.

5.5. *Exercise.* Show that Λ^n_j is the largest subcomplex of Δ^n which does not contain the element $\langle 0 \cdots \widehat{j} \cdots n\rangle \in (\Delta^n)_{n-1}$, the "face opposite the vertex j".

We note that inner horns always contain spines: $I^n \subseteq \Lambda^n_j$ if $0 < j < n$. This is also true for outer horns if $n \geq 3$, but not for outer horns with $n = 1$ or $n = 2$.

5.6. **The inner horn extension criterion for nerves.** We can now characterize nerves as those simplicial sets which admit "unique inner horn extensions"; this is different than, but analogous to, the characterization in terms of unique spine extensions (4.4).

5.7. **Proposition.** *A simplicial set X is isomorphic to the nerve of a category, if and only if $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_j, X)$ is a bijection for all $n \geq 2$, $0 < j < n$.*

The proof will take up the rest of the section.

5.8. **Nerves have unique inner horn extensions.** First we show that nerves have unique inner horn extensions.

5.9. **Proposition.** *If C is a category, then for every* inner *horn $\Lambda^n_j \subset \Delta^n$ the evident restriction map*

$$\mathrm{Hom}(\Delta^n, NC) \to \mathrm{Hom}(\Lambda^n_j, NC)$$

is a bijection.

Proof. Since inner horns contain spines, we can consider restriction along $I^n \subseteq \Lambda^n_j \subseteq \Delta^n$. The composite

$$\operatorname{Hom}(\Delta^n, NC) \to \operatorname{Hom}(\Lambda^n_j, NC) \xrightarrow{r} \operatorname{Hom}(I^n, NC)$$

of restriction maps is a bijection (4.4), so r is a surjection. Thus, it suffices to show that r is injective. This is immediate when $n = 2$, since $\Lambda^2_1 = I^2$, so we can assume $n \geq 3$.

We claim that for any inner horn Λ^n_j with $n \geq 3$ there exists a finite chain

$$I^n = F_0 \subset F_1 \subset \cdots \subset F_d = \Lambda^n_j$$

of subcomplexes, together with a list of subsets $S_1, \dots, S_d \subset [n]$, such that (i) $F_i = F_{i-1} \cup \Delta^{S_i}$ and (ii) $I^{S_i} \subseteq F_{i-1} \cap \Delta^{S_i}$; here I^{S_i} denotes the spine of Δ^{S_i}. Given this, we see by (4.14) that F_i is isomorphic to a pushout:

$$F_i \approx \operatorname{colim}\bigl(F_{i-1} \leftarrow F_{i-1} \cap \Delta^{S_i} \to \Delta^{S_i}\bigr).$$

We then obtain a commutative diagram of sets

$$\begin{tikzcd}
\operatorname{Hom}(F_i, NC) \arrow[r, "b"] \arrow[d] & \operatorname{Hom}(F_{i-1}, NC) \arrow[d] & \\
\operatorname{Hom}(\Delta^{S_i}, NC) \arrow[r, "a"'] & \operatorname{Hom}(F_{i-1} \cap \Delta^{S_i}, NC) \arrow[r] & \operatorname{Hom}(I^{S_i}, NC)
\end{tikzcd}$$

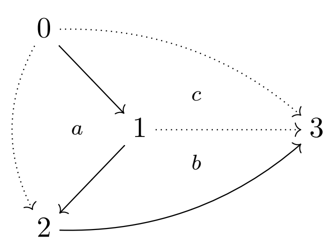
where all maps are induced by restriction, in which the square is a pullback (because F_i is a pushout), and such that the horizontal composition on the bottom is a bijection. It immediately follows that a, and hence b, are injective. We can thus conclude that $\operatorname{Hom}(\Lambda^n_j, NC) \to \operatorname{Hom}(I^n, NC)$ is injective as desired, since it is a composite of injective functions such as b.

Now we prove the existence of a filtration of Λ^n_j by suitable subcomplexes F_i as promised above.

When $n = 3$, we can "attach" simplices in order explicitly:

$$\Lambda^3_1 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,1,3\}}, \qquad \Lambda^3_2 = ((I^3 \cup \Delta^{\{0,1,2\}}) \cup \Delta^{\{1,2,3\}}) \cup \Delta^{\{0,2,3\}}.$$

Note that, for instance, in building Λ^3_1, we must add $\Delta^{\{0,1,3\}}$ after adding $\Delta^{\{1,2,3\}}$, so that the spine $I^{\{0,1,3\}}$ of $\Delta^{\{0,1,3\}}$ is already present.



When $n \geq 4$, we have that $(\Lambda^n_j)_1 = (\Delta^n)_1$ and $(\Lambda^n_j)_2 = (\Delta^n)_2$. The procedure to "build" Λ^n_j from I^n by adding subsimplices is: (1) first attach 2-simplices one at a time, in an allowable order; then (2) attach all needed higher dimensional subsimplices. In step (2) the order doesn't matter since all 1-simplices (and hence all spines) are present in what has already been built. We leave the details of step (1) to the reader. ☐

5.10. **Nerves are characterized by unique inner horn extension.** Let X be an arbitrary simplicial set, and suppose it has unique inner horn extensions, i.e., each restriction map $\operatorname{Hom}(\Delta^n, X) \to \operatorname{Hom}(\Lambda^n_j, X)$ is a bijection for all $0 < j < n$ with $n \geq 2$.

Observe that unique extension along $\Lambda^2_1 \subset \Delta^2$, defines a "composition law" on the set X_1. That is, given $f, g \in X_1$ such that $f_1 = g_0$ in X_0,[4] there is a unique map

$$u\colon \Lambda^2_1 = \Delta^{\{0,1\}} \cup \Delta^{\{1,2\}} \xrightarrow{(f,g)} X \qquad \text{such that} \qquad \langle 01\rangle \mapsto f \in X_1,\ \langle 12\rangle \mapsto g \in X_1.$$

Let $\widetilde{u}\colon \Delta^2 \to X$ be the *unique* extension of u along $\Lambda^2_1 \subset \Delta^2$, and define the "composite"

$$g \circ f := \widetilde{u}_{02}.$$

The 2-dimensional element $\widetilde{u}$ is uniquely characterized by: $\widetilde{u}_{01} = f$, $\widetilde{u}_{12} = g$, $\widetilde{u}_{02} = g \circ f$.

This composition law is automatically unital. Given $x \in X_0$, write $1_x := x\langle 00\rangle \in X_1$, so that $(1_x)_0 = x = (1_x)_1$. Then applying the composition law gives $1_x \circ f = f$ and $g \circ 1_x = g$. (Proof: consider the 2-dimensional elements $f\langle 011\rangle, g\langle 001\rangle \in X_2$, and use the fact that their representing maps $\Delta^2 \to X$ are the unique extensions of their restrictions to $\Lambda^2_1 \subset \Delta^2$.)

Now consider $\Lambda^3_1 \subset \Delta^3$. Recall (4.19) that Λ^3_1 is a union (and colimit) of $\Delta^S \subseteq \Delta^3$ such that $S \not\supseteq \{0,2,3\}$. A map $v\colon \Lambda^3_1 \to X$ can be pictured as

$$\begin{tikzcd}
0 \arrow[dr, "f"] \arrow[drr, bend left, "(h\circ g)\circ f"] \arrow[dd, bend right, "g\circ f"'] & & \\
& 1 \arrow[r, "h\circ g"] \arrow[dl, "g"] & 3 \\
2 \arrow[urr, bend right, "h"'] & &
\end{tikzcd}$$

so that the planar 2-cells in the picture correspond to non-degenerate 2-dimensional elements of Δ^3 which are contained in Λ^3_1, while the edges are labelled according to their images in X, using the composition law defined above. Let $\widetilde{v}\colon \Delta^3 \to X$ be any extension of v along $\Lambda^3_1 \subset \Delta^3$, and consider the restriction $w := \widetilde{v}\langle 023\rangle\colon \Delta^2 \to X$ to the face $\Delta^2 \approx \Delta^{\{0,2,3\}} \subset \Delta^3$. Then $w_{01} = g \circ f$, $w_{12} = h$, and $w_{02} = (h \circ g) \circ f$, and thus the existence of w demonstrates that

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

In other words, the *existence* of extensions along $\Lambda^3_1 \subset \Delta^3$ implies that the composition law we defined above is associative. (We could carry out this argument using $\Lambda^3_2 \subset \Delta^3$ instead.)

Thus, given an X with unique inner horn extensions, we can construct a category C, so that objects of C are elements of X_0, morphisms of C are elements of X_1, and composition is given as above.

Next we construct a map $X \to NC$ of simplicial sets. There are obvious maps $\alpha_n\colon X_n \to (NC)_n$, corresponding to restriction along spines $I^n \subseteq \Delta^n$; i.e., $\alpha(x) = (x_{01}, \dots, x_{n-1,n})$. These maps are compatible with simplicial operators, so that they define a map $\alpha\colon X \to NC$ of simplicial sets. *Proof:* For any n-dimensional element $x \in X_n$, *all* of its edges are determined by edges on its spine via the (associative) composition law: $x_{ij} = x_{j-1,j} \circ x_{j-2,j-1} \circ \cdots \circ x_{i,i+1}$, for all $0 \leq i \leq j \leq n$. Thus for $f\colon [m] \to [n]$ we have $\alpha(xf) = ((xf)_{01}, \dots, (xf)_{n-1,n}) = (x_{f_0 f_1}, \dots, x_{f_{n-1} f_n}) = (x_{01}, \dots, x_{n-1,n})_{f_0 \cdots f_n} = (\alpha x) f$.

Now we can prove that nerves are characterized by unique extension along inner horns.

Proof of (5.7). We have already shown (5.9) that nerves have unique extensions for inner horns. Consider a simplicial set X which has unique inner horn extension. By the discussion above, we obtain a category C and a map $\alpha\colon X \to NC$ of simplicial sets, which is clearly a bijection in degrees ≤ 1. We will show $\alpha_n\colon X_n \to (NC)_n$ is bijective by induction on n.

[4] Recall that $f_1 = f\langle 1\rangle$ and $g_0 = g\langle 0\rangle$, regarded as maps $\Delta^0 \to X$ and thus as elements of X_0, using the notation discussed in §2.3.

Fix $n \geq 2$, and consider the commutative square

$$\begin{tikzcd} \mathrm{Hom}(\Delta^n, X) \arrow[r, "\sim"] \arrow[d, "\alpha_{\Delta^n}"'] & \mathrm{Hom}(\Lambda^n_1, X) \arrow[d, "\alpha_{\Lambda^n_1}"] \\ \mathrm{Hom}(\Delta^n, NC) \arrow[r, "\sim"'] & \mathrm{Hom}(\Lambda^n_1, NC) \end{tikzcd}$$

The vertical maps are induced by post-composition with $\alpha\colon X \to NC$. The horizontal maps are induced by restriction along $\Lambda^n_1 \subset \Delta^n$, and are bijections (top by hypothesis, bottom by (5.9)). Because Λ^n_1 is a colimit of standard simplices of dimension $< n$ (4.19), the map $\alpha_{\Lambda^n_1}$ is a bijection by the induction hypothesis. Therefore so is α_{Δ^n}. ☐

6. Quasicategories

We can now define the notion of a quasicategory, by removing the uniqueness part of the inner horn extension criterion for nerves.

6.1. **Identifying categories with their nerves.** From this point on, I will (at least informally) often not distinguish a category C from its nerve. In particular, I may assert something like "let C be a simplicial set which is a category", which should be read as "C is a simplicial set which is isomorphic to the nerve of some category". This should not lead to much confusion, due to the fact that the nerve functor is a fully faithful embedding of Cat into sSet (3.10).

6.2. **Definition of quasicategory.** A **quasicategory** is a simplicial set C such that for every map $f\colon \Lambda^n_j \to C$ from an inner horn, there *exists* an extension of it to $g\colon \Delta^n \to C$. That is, C is a quasicategory if the function $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\Lambda^n_j, C)$ induced by restriction along $\Lambda^n_j \subset \Delta^n$ is *surjective* for all $0 < j < n$, $n \geq 2$, so there always exists a dotted arrow in any commutative diagram of the form

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & C \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Any category (more precisely, the nerve of any category) is a quasicategory. In fact, by what we have shown (5.7) a category is precisely a quasicategory for which there exist *unique* extensions of inner horns.

Let C be a quasicategory. We refer to elements of C_0 as the **objects** of C, and elements of C_1 as the **morphisms** of C. Every morphism $f \in C_1$ has a **source** and **target**, namely its vertices $f_0 = f\langle 0\rangle, f_1 = f\langle 1\rangle \in C_0$. For $f \in C_1$ we write $f\colon f_0 \to f_1$, just as we would for morphisms in a category. Likewise, for every object $x \in C_0$, there is a distinguished morphism $1_x\colon x \to x$, called the **identity morphism**, defined by $1_x = x_{00} = x\langle 00\rangle$. When C is (the nerve of) a category, all the above notions coincide with the usual ones. Note, however, that we cannot generally define *composition* of morphisms in a quasicategory in the same way we do for a category.

We now describe some basic categorical notions which admit immediate generalizations to quasicategories. Many of these generalizations apply to arbitrary simplicial sets.

6.3. **Products of quasicategories.** Simplicial sets are functors, so the product of simplicial sets X and Y is just the product of the functors. Thus, $(X \times Y)_n = X_n \times Y_n$, with the evident action of simplicial operators: $(x, y)f = (xf, yf)$.

6.4. **Proposition.** *The product of two quasicategories (as simplicial sets) is a quasicategory.*

Proof. Exercise, using the bijective correspondence between the sets of (i) maps $K \to X \times Y$ and (ii) pairs of maps $(K \to X, K \to Y)$. ☐

6.5. *Exercise.* If C and D are categories, then $N(C \times D) \approx NC \times ND$. Thus, the notion of product of quasicategories generalizes that of categories.

6.6. **Coproducts of quasicategories.** The coproduct of simplicial sets X and Y is just the coproduct of functors, whence $(X \amalg Y)_n = X_n \amalg Y_n$, i.e., the set n-dimensional elements of the coproduct is the disjoint union of the sets of n-dimensional elements of X and Y. More generally, $\left(\coprod_s X_s\right)_n = \coprod_s (X_s)_n$ for an indexed collection $\{X_s\}$ of simplicial sets.

6.7. **Proposition.** *The coproduct of any indexed collection of quasicategories is a quasicategory.*

To prove this, we introduce the set of **connected components** of a simplicial set. Given a simplicial set X, define an equivalence relation $\approx$ on the set $\coprod_{n \geq 0} X_n$ of elements of X, generated by the relation

$$a \sim af \qquad \text{for all } n \geq 0,\ a \in X_n,\ f\colon [m] \to [n].$$

An equivalence class for $\approx$ is called a **connected component** of X, and we write $\pi_0 X$ for the set of connected components. This construction defines a functor $\pi_0\colon s\mathrm{Set} \to \mathrm{Set}$.

6.8. *Exercise* (Connected components are path components)*.* Show that there is a canonical bijection

$$(X_0/\approx_1) \xrightarrow{\sim} \pi_0 X,$$

where the left-hand side denotes the set of equivalence classes in the vertex set X_0 with respect to the equivalence relation $\approx_1$ which is *generated* by the relation $\sim_1$ on X_0, defined by

$$a \sim_1 b \qquad \text{iff there exists } e \in X_1 \text{ such that } a = e_0, b = e_1.$$

6.9. *Exercise.* Show that there is a bijection $\operatorname{colim}_{\Delta^{\mathrm{op}}} X \xrightarrow{\sim} \pi_0 X$, between the set of connected components of X and the colimit of the functor $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$.

6.10. *Exercise* (Connected components respect colimits)*.* Show that if X is the colimit of a functor $F\colon D \to s\mathrm{Set}$ from some small category D, then $\pi_0 X \approx \operatorname{colim}_D \pi_0 F$. In particular, $\pi_0\left(\coprod_s X_s\right) \approx \coprod_s \pi_0(X_s)$ for any collection $\{X_s\}$ of simplicial sets.

We say that a simplicial set X is **connected** if $\pi_0 X$ is a singleton.

6.11. *Exercise.* Show that every standard simplex Δ^n is connected, and that every horn Λ^n_j is connected.

6.12. *Exercise* (Every simplicial set is a coproduct of its connected components)*.* Let X be a simplicial set. Given $a \in \pi_0 X$, let C_a denote its equivalence class (regarded as a subset of the set $\coprod_{n \geq 0} X_n$ of elements).

(1) Show that C_a is closed under the action of simplicial operators, and thus describes a subcomplex of X.
(2) Show that the evident map
$$\coprod_{a \in \pi_0 X} C_a \to X$$
is an isomorphism of simplicial sets.

Proof of (6.7)*.* If $X = \coprod_s X_s$ is a coproduct of simplicial sets, then any connected component of X must be contained in exactly one of the X_s summands, by (6.10). The proof is now straightforward, using (6.12) and the fact that horns and standard simplices are connected (6.11). ☐

6.13. *Exercise* (Important)*.* Show that the evident map $\pi_0(X \times Y) \to \pi_0 X \times \pi_0 Y$ induced by projections is a bijection.

6.14. **Subcategories of quasicategories.** We say that a subcomplex $C' \subseteq C$ of a quasicategory C is a **subcategory** if for all $n \geq 2$ and $0 < k < n$, every $f\colon \Delta^n \to C$ such that $f(\Lambda^n_k) \subseteq C'$ satisfies $f(\Delta^n) \subseteq C'$. That is, *all* inner horn extensions in C along "horns in C'" are themselves contained in C'. It is clear that a subcategory is in fact a quasicategory.

6.15. *Exercise.* Let C be a quasicategory, and consider $S \subseteq C_1$ a collection of morphisms in C. Define $C'_n := \{\, a \in C_n \mid a_{ij} \in S \text{ for all } 0 \leq i \leq j \leq n \,\}$. Show that the C'_ns describe a subcomplex C' of C if and only if $f_{00}, f_{11} \in S$ for all $f \in S$. Show that furthermore C' is a subcategory if and only if, in addition, for all $u \in C_2$ we have that $u_{01}, u_{12} \in S$ implies $u_{02} \in S$.

When C is an ordinary category, a subcategory of C in the above sense is the same as a subcategory in the usual sense, which correspond exactly to subsets $S \subseteq C_1$ of morphisms for which (i) if $(x \xrightarrow{a} y) \in S$ then $\mathrm{id}_x, \mathrm{id}_y \in S$, and (ii) S is closed under composition.

6.16. *Remark.* In general, if $C' \subseteq C$ is a subcomplex and C and C' are quasicategories, it need not be the case that C' is a subcategory of C. See (8.4) below.

6.17. **Full subcategories of quasicategories.** We say that a subcomplex $C' \subseteq C$ of a quasicategory C is a **full subcategory** if for all n and all $a \in C_n$, we have that $a \in C'_n$ if and only if $a_i \in C'_0$ for all $i = 0, \dots, n$. **F 18 Jan 2019**

6.18. *Exercise.* Show that a full subcategory $C' \subseteq C$ is in fact a subcategory as defined in (6.14), and thus in particular a full subcategory C' is itself a quasicategory.

Given a quasicategory C and a set $S \subseteq C_0$ of vertices, let

$$C'_n = \{\, a \in C_n \mid a_j \in S \text{ for all } j = 0, \dots, n \,\},$$

the set of n-dimensional elements whose vertices are in S. This is evidently a full subcategory of C, called the **full subcategory spanned by** S.

When C is an ordinary category, a full subcategory of C in the above sense is the same as a full subcategory in the usual sense.

6.19. **Opposite of a quasicategory.** Given a category C, the *opposite* category C^{op} has $\mathrm{ob}\, C^{\mathrm{op}} = \mathrm{ob}\, C$, and $\mathrm{Hom}_{C^{\mathrm{op}}}(x, y) = \mathrm{Hom}_C(y, x)$, and the sense of composition is reversed: $g \circ_{C^{\mathrm{op}}} f = f \circ_C g$.

This concept also admits a generalization to quasicategories, which we define using a non-trivial involution $\mathrm{op}\colon \Delta \to \Delta$ of the category Δ. This is the functor which on objects sends $[n] \mapsto [n]$, and on morphisms sends $\langle f_0, \dots, f_n\rangle \colon [n] \to [m]$ to $\langle m - f_n, \dots, m - f_0\rangle$.

6.20. *Remark.* You can visualize this involution as the functor which "reverses the ordering" of the totally-ordered sets $[n]$. Note that the totally ordered set "$[n]$ with the order of its elements reversed" isn't actually an object of Δ, but rather is uniquely isomorphic to $[n]$, via the function $x \mapsto n - x$.

The **opposite** of a simplicial set $X\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is the composite functor $X^{\mathrm{op}} := X \circ \mathrm{op}$. We have that $(\Delta^n)^{\mathrm{op}} = \Delta^n$, while $(\Lambda^n_j)^{\mathrm{op}} = \Lambda^n_{n-j}$, so that the opposite of an inner horn is another inner horn. As a consequence, the opposite of a quasicategory is a quasicategory. It is straightforward to verify that $(NC)^{\mathrm{op}} = N(C^{\mathrm{op}})$, so the notion of opposite quasicategory generalizes the notion of opposite category. The functor $\mathrm{op}\colon \Delta \to \Delta$ satisfies $\mathrm{op} \circ \mathrm{op} = \mathrm{id}_\Delta$, so $(X^{\mathrm{op}})^{\mathrm{op}} = X$.

7. Functors and natural transformations

7.1. **Functors.** A **functor** between quasicategories is merely a map $f\colon C \to D$ between the simplicial sets.

We write qCat for the category of quasicategories and functors between them.[5] Clearly $\mathrm{qCat} \subset s\mathrm{Set}$ is a full subcategory. Because the nerve functor is a full embedding of Cat into qCat, any functor between ordinary categories is also a functor between quasicategories.

[5] Lurie [Lur09] denotes this category by Cat_Δ.

7.2. *Exercise* (Mapping property of a full subcategory). Let C be a quasicategory, and $C' \subseteq C$ the full subcategory spanned by some subset $S \subseteq C_0$. Show that a functor $f\colon D \to C$ factors through a functor $f'\colon D \to C' \subseteq C$ if and only if $f(D_0) \subseteq S$.

7.3. **Natural transformations.** Given functors $F, G\colon C \to D$ between categories, a *natural transformation* $\phi\colon F \Rightarrow G$ is a choice, for each object c of C, of a map $\phi(c)\colon F(c) \to G(c)$ in D, such that for every morphism $\alpha\colon c \to c'$ in C the square

$$\begin{tikzcd} F(c) \arrow[r, "\phi(c)"] \arrow[d, "f(\alpha)"'] & G(c) \arrow[d, "g(\alpha)"] \\ F(c') \arrow[r, "\phi(c')"'] & G(c') \end{tikzcd}$$

commutes in D.

There is a standard convenient reformulation of this: a natural transformation $\phi\colon F \Rightarrow G$ is the same thing as a functor

$$H\colon C \times [1] \to D,$$

so that $H|C \times \{0\} = F$, $H|C \times \{1\} = G$, and $H|\{c\} \times [1] = \alpha(c)$ for each $c \in \operatorname{ob} C$. (Here we make implicit use of the evident isomorphisms $C \times \{0\} \approx C \approx C \times \{1\}$.)

This reformulation admits a straightforward generalization to quasicategories. A **natural transformation** $h\colon f_0 \Rightarrow f_1$ of functors $f_0, f_1\colon C \to D$ between quasicategories is defined to be a map

$$h\colon C \times N[1] = C \times \Delta^1 \to D$$

of simplicial sets such that $h|C \times \{i\} = f_i$ for $i = 0, 1$. For ordinary categories this coincides with the classical notion.

8. Examples of quasicategories

There are many ways to produce quasicategories, as we will see. Unfortunately, "hands-on" constructions of quasicategories which are not ordinary categories are relatively rare. Here I give a few reasonably explicit examples to play with.

8.1. **Large vs. small.** I have been implicitly assuming that certain categories are **small**; i.e., they have *sets* of objects and morphisms. For instance, for the nerve of a category C to be a simplicial set, we need $C_0 = \operatorname{ob} C$ to be a set.

However, in practice many categories of interest are only **locally small**; i.e., the collection of objects is not a set but is a "proper class", although for any pair of objects $\operatorname{Hom}_C(X, Y)$ is a set. For instance, the category Set of sets is of this type: there is no set of all sets. Other examples include the categories of abelian groups, topological spaces, (small) categories, simplicial sets, etc. It is also possible to have categories which are not even locally small, e.g., the category of locally small categories. These are called **large** categories.

We would like to be able to talk about large categories in exactly the same way we talk about small categories. This is often done by positing a hierarchy of (Grothendieck) "universes". A universe U is (informally) a collection of sets which is closed under the operations of set theory. We additionally assume that for any universe U, there is a larger universe U' such that $U \in U'$. Thus, if by "set" we mean "U-set", then the category *Set* is a "U'-category". This idea can be implemented in the usual set theoretic foundations by postulating the existence of suitable *strongly inaccessible cardinals*.

The same distinctions occur for simplicial sets. For instance, the nerve of a small category is a small simplicial set (i.e., the elements form a set), while the nerve of a large category is a large simplicial set.

I'm not going to be pedantic about this. I'll usually assume categories like Set, Cat, sSet, etc., are categories whose objects are "small" sets/categories/simplicial sets/whatever, i.e., are built from sets in a fixed universe U of "small sets". However, I sometimes need to consider examples of sets/categories/simplicial sets/whatever which are not small. I leave it to the reader to determine when this is the case.

In practice, a main point of concern involves constructions such as limits and colimits. Many typical examples of categories $C = \mathrm{Set}, \mathrm{Cat}, s\mathrm{Set}$, etc., in which objects are built out of small sets are **small complete** and **small cocomplete**: any functor $F\colon D \to C$ from a *small* category D has a limit and a colimit in C. This is *not* true if D is not assumed to be small. In this case care about the small/large distinction is necessary.

8.2. **The quasicategory of categories.** This is an example of a (large) quasicategory in which *objects* are (small) categories, *morphisms* are functors between categories, and *2-dimensional elements* are certain kinds of natural isomorphisms of functors.

Define a simplicial set Cat_1 so that $(\mathrm{Cat}_1)_n$ is a set whose elements are data $x := (C_i, F_{ij}, \phi_{ijk})$ where

(0) for each $i \in [n]$, C_i is a (small) category,
(1) for each $i \leq j$ in $[n]$, $F_{ij}\colon C_i \to C_j$ is a functor, and
(2) for each $i \leq j \leq k$ in $[n]$, $\zeta_{ijk}\colon F_{ik} \Rightarrow F_{jk}F_{ij}$ is a natural *isomorphism* of functors $C_i \to C_k$,

such that

- for each i in $[n]$, $F_{ii}\colon C_i \to C_i$ is the identity functor Id_{C_i} of C_i,
- for each $i \leq j$ in $[n]$, $\zeta_{iij}\colon F_{ij} \Rightarrow F_{ij}\,\mathrm{Id}_{C_i}$ and $\zeta_{ijj}\colon F_{ij} \Rightarrow \mathrm{Id}_{C_j} F_{ij}$ are the identity natural isomorphism of F_{ij}, and
- for each $i \leq j \leq k \leq \ell$, the diagram

$$\begin{tikzcd} F_{i\ell} \arrow[r, Rightarrow, "\zeta_{ij\ell}"] \arrow[d, Rightarrow, "\zeta_{ik\ell}"'] & F_{j\ell}F_{ij} \arrow[d, Rightarrow, "\zeta_{jk\ell}F_{ij}"] \\ F_{k\ell}F_{ik} \arrow[r, Rightarrow, "F_{k\ell}\zeta_{ijk}"'] & F_{k\ell}F_{jk}F_{ij} \end{tikzcd} \tag{8.3}$$

  of natural isomorphisms commutes.

For a simplicial operator $\delta\colon [m] \to [n]$ define

$$(C_i, F_{ij}, \zeta_{ijk})\delta = (C_{\delta(i)}, F_{\delta(i)\delta(j)}, \zeta_{\delta(i)\delta(j)\delta(k)}).$$

I claim that Cat_1 is a quasicategory. Fillers for $\Lambda^2_1 \subset \Delta^2$ always exist: a map $\Lambda^2_1 \to \mathrm{Cat}_1$ is a choice of data $(C_0 \xrightarrow{F_{01}} C_1 \xrightarrow{F_{12}} C_2)$, and an extension to Δ^2 can be given by setting $F_{02} = F_{12}F_{01}$ and $\zeta_{012} = \mathrm{id}_{F_{02}}$. Note that this is not the only possibly extension: even keeping F_{02} the same, there may be many choices of the isomorphism ζ_{012}.

Fillers for $\Lambda^3_1 \subset \Delta^3$ and $\Lambda^3_2 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $\zeta_{023} = f_{ik\ell}$ (for Λ^3_1) or $f_{013} = f_{ij\ell}$ (for Λ^3_2) making (8.3) commute. All fillers for inner horns $\Lambda^n_j \subset \Delta^n$ in higher dimensions $n \geq 4$ exist and are unique: there is no additional data to supply in these cases, and all properies of the data are automatically satisfied.

8.4. *Exercise.* Note that the (nerve of) the category Cat of small categories is isomorphic to the subcomplex of Cat_1 whose elements are $(C_i, F_{ij}, \zeta_{ijk})$ such that $F_{ik} = F_{jk}F_{ij}$ and $\zeta_{ijk} = \mathrm{id}_{F_{ik}}$. Show that this subcomplex is a quasicategory which not a subcategory in the sense of (6.14).

8.5. **Singular complex of a space.** The **topological n-simplex** is

$$\Delta^n_{\text{top}} := \Big\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \;\Big|\; \sum x_i = 1,\ x_i \geq 0 \Big\},$$

the convex hull of the standard basis vectors in $(n+1)$-dimensional Euclidean space. These fit together to give a functor $\Delta_{\text{top}} \colon \Delta \to \text{Top}$ to the category of topological spaces and continuous maps, with $\Delta_{\text{top}}([n]) = \Delta^n_{\text{top}}$. A simplicial operator $f \colon [m] \to [n]$ sends $(x_0, \dots, x_m) \in \Delta^m_{\text{top}}$ to $(y_0, \dots, y_n) \in \Delta^n_{\text{top}}$ with $y_j = \sum_{f(i)=j} x_i$.

For a topological space T, we define its **singular complex** $\operatorname{Sing} T$ to be the simplicial set with elements $[n] \mapsto \operatorname{Hom}_{\text{Top}}(\Delta^n_{\text{top}}, T)$, with the evident action of simplicial operators.

Define **topological horns**

$$(\Lambda^n_j)_{\text{top}} := \big\{ x \in \Delta^n_{\text{top}} \;\big|\; \exists\, i \in [n] \smallsetminus \{j\} \text{ such that } x_i = 0 \big\} \subset \Delta^n_{\text{top}},$$

and observe that continuous maps $(\Lambda^n_j)_{\text{top}} \to T$ correspond in a natural way with maps $\Lambda^n_j \to \operatorname{Sing} T$. (*Exercise.* This is a consequence of the fact that Λ^n_j is a colimit of all the $\Delta^S \subseteq \Lambda^n_j$, and that $(\Lambda^n_j)_{\text{top}}$ is similarly a colimit of all the $\Delta^S_{\text{top}} \subseteq (\Lambda^n_j)_{\text{top}}$.) There exists a continuous retraction $\Delta^n_{\text{top}} \to (\Lambda^n_j)_{\text{top}}$ (*Exercise.* describe such a retraction), and thus we see that

$$\operatorname{Hom}(\Delta^n, \operatorname{Sing} T) \to \operatorname{Hom}(\Lambda^n_j, \operatorname{Sing} T)$$

is surjective for every horn (not just inner ones).

8.6. *Remark* (Kan complexes). A simplicial set X which has extensions for all horns is called a **Kan complex**. Thus, $\operatorname{Sing} T$ is a Kan complex, and so in particular is a quasicategory (and as we will see below, a "quasigroupoid" (10.12)).

8.7. **Eilenberg-MacLane object.** Fix an abelian group A and an integer $d \geq 0$. We define a simplicial set $K = K(A, d)$, so that K_n is a set whose elements are data $a = (a_{i_0 \dots i_d})$ consisting of

- for each $0 \leq i_0 \leq \dots \leq i_d \leq n$, an element $a_{i_0 \dots i_d} \in A$, such that
- $a_{i_0 \dots i_d} = 0$ if $i_{u-1} = i_u$ for any u, and
- for each $0 \leq j_0 \leq \dots \leq j_{d+1} \leq n$ we have $\sum_u (-1)^u a_{j_0 \dots \widehat{j_u} \dots j_{d+1}} = 0$.

  (Here "$j_0 \dots \widehat{j_u} \dots j_{d+1}$" is shorthand for the subsequence $j_0, j_1, \dots, j_{u-1}, j_{u+1}, \dots, j_d, j_{d+1}$ with j_u omitted.)

For a map $\delta \colon [m] \to [n]$ we define

$$(a\delta)_{i_0 \dots i_d} = a_{\delta(i_0) \dots \delta(i_d)}.$$

The object $K(A, d)$ is a Kan complex, and hence a quasicategory (and in fact a quasigroupoid). When $d = 0$, this is just a discrete simplicial set, equal to A in each dimension.

8.8. *Exercise.* Show that $K(A, 1)$ is isomorphic to the nerve of a category, namely the nerve of the group A regarded as a category with one object.

8.9. *Exercise.* Show that $K(A, d)$ is a Kan complex, i.e., that $\operatorname{Hom}(\Delta^n, K(A, d)) \to \operatorname{Hom}(\Lambda^n_j, K(A, d))$ is surjective for all horns $\Lambda^n_j \subset \Delta^n$. In fact, this map is *bijective* unless $n = d$. (Hint: there are four distinct cases to check, namely $n < d$, $n = d$, $n = d + 1$, and $n > d + 1$.)

8.10. *Exercise.* Given a simplicial set X, a **normalized d-cocycle** with values in an abelian group A is a function $f \colon X_d \to A$ such that

1. $f(x_{0,\dots i,i,\dots d-1}) = 0$ for all $x \in X_{d-1}$ and $0 \leq i \leq d - 1$, and
2. $\sum (-1)^i f(x_{0,\dots,\widehat{i},\dots,d+1}) = 0$ for all $x \in X_{d+1}$ and $0 \leq i \leq d + 1$.

Show that the set $Z^d_{\text{norm}}(X; A)$ of normalized d-cocycles on X is in bijective correspondence with $\operatorname{Hom}_{s\text{Set}}(X, K(A, d))$. (Hint: an element $a \in K_n$ is uniquely determined by the collection of elements $a\delta \in K_d = A$, as δ ranges over injective maps $[d] \to [n]$.)

8.11. *Remark.* Eilenberg-MacLane objects are an example of a *simplicial abelian group*: the map $+\colon K \times K \to K$ defined in each dimension by $(a+b)_{i_0\dots i_d} = a_{i_0\dots i_d} + b_{i_0\dots i_d}$ is a map of simplicial sets which satisfies the axioms of an abelian group, reflecting the fact that $Z^d_{\text{norm}}(X;A)$ is an abelian group.

9. Homotopy category of a quasicategory

Our next goal is to define the notion of an *isomorphism* in a quasicategory. This notion behaves much like that of *homotopy equivalence* in topology. We will define isomorphism by means of the *homotopy category* of a quasicategory. If we think of a quasicategory as "an ordinary category with higher structure", then its homotopy category is the ordinary category obtained by "flatting out the higher stucture". **W 23 Jan 2019**

9.1. **The fundamental category of a simplicial set.** The homotopy category of a quasicategory is itself a special case of the notion of the *fundamental category* of a simplicial set, which we turn to first.

A **fundamental category** for a simplicial set X consists of (i) a category hX, and (ii) a map $\alpha\colon X \to N(hX)$ of simplicial sets, such that for every category C, the map

$$\alpha^*\colon \operatorname{Hom}(N(hX), NC) \to \operatorname{Hom}(X, NC)$$

induced by restriction along α is a bijection. This is a universal property which characterizes the fundamental category up to unique isomorphism, if it exists.

9.2. **Proposition.** *Every simplicial set has a fundamental category.*

Proof sketch. Given X, we construct hX by generators and relations. First, consider the **free category** F, whose objects are the set X_0, and whose morphisms are finite "composable" sequences $[a_n, \dots, a_1]$ of edges of X_1. Thus, morphisms in F are "words", whose "letters" are edges a_i with $(a_{i+1})_0 = (a_i)_1$, and composition is concatenation of words; the element $[a_n, \dots, a_1]$ is then a morphism $(a_1)_0 \to (a_n)_1$. (Note: we also suppose that there is an empty sequence $[]_x$ in F for each vertex $x \in X_0$; these correspond to identity maps in F.)

Then hX is defined to be the largest quotient category of F subject to the following relations on the set of morphisms:

- $[a] \sim []_x$ for each $x \in X_0$ where $a = x_{00} \in X_1$, and
- $[g, f] \sim [h]$ whenever there exists $a \in X_2$ such that $a_{01} = f$, $a_{12} = g$, and $a_{02} = h$.

The map $\alpha\colon X \to N(hX)$ sends $x \in X_n$ to the equivalence class of $[x_{n-1,n}, \dots, x_{0,1}]$. Given this, verifying the desired universal property of α is formal.

(We will give another construction of the fundamental category in (13.27).) □

9.3. *Exercise.* Complete the proof of (9.2) by showing that $\alpha^*\colon \operatorname{Hom}(N(hX), NC) \to \operatorname{Hom}(X, NC)$ is a bijection for any category C.

As a consquence: the fundamental category construction describes a functor $h\colon s\mathrm{Set} \to \mathrm{Cat}$, which is left adjoint to the nerve functor $N\colon \mathrm{Cat} \to s\mathrm{Set}$.

In general, the fundamental category of a simplicial set is not an easy thing to get a hold of explicitly, because it is difficult to give an explicit description of a "quotient category" induced by a relation on its morphisms. We will not be making much use of it. When C is a quasicategory, there is a more concrete construction of hC, which in this context is called the *homotopy category* of C. *Warning:* Sometimes people will not distinguish "fundamental category" from "homotopy category" as I have here, and just call either the homotopy category.

9.4. **The homotopy relation on morphisms.** Fix a quasicategory C. For $x, y \in C_0$, let $\hom_C(x, y) := \{\, f \in C_1 \mid f_0 = x,\ f_1 = y \,\}$ denote the set of "morphisms" in C from x to y. We write 1_x for the element $x_{00} \in \hom_C(x, x)$.

Define relations $\sim_\ell$, $\sim_r$ on $\hom_C(x, y)$ (called **left homotopy** and **right homotopy**) by

- $f \sim_\ell g$ iff there exists $a \in C_2$ with $a_{01} = 1_x$, $a_{02} = f$, $a_{12} = g$,
- $f \sim_r g$ iff there exists $b \in C_2$ with $b_{12} = 1_y$, $b_{01} = f$, $b_{02} = g$.

Pictorally:

$$f \sim_\ell g\colon \quad \begin{tikzcd} x \arrow[dd, "1_x"'] \arrow[dr, "f"] & \\ {} \arrow[r, phantom, "a"] & y \\ x \arrow[ur, "g"'] & \end{tikzcd} \qquad\qquad f \sim_r g\colon \quad \begin{tikzcd} & y \arrow[dd, "1_y"] \\ x \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "b"] & {} \\ & y \end{tikzcd}$$

Note that $f \sim_\ell g$ in $\hom_C(x, y)$ if and only if $g \sim_r f$ in $\hom_{C^{\mathrm{op}}}(y, x)$.

9.5. *Remark.* If C is an ordinary category, then the left homotopy and right homotopy relations reduce to the equality relation on morphisms $x \to y$.

9.6. **Proposition.** *The relations $\sim_\ell$ and $\sim_r$ are equal to each other, and are an equivalence relation on* $\hom_C(x, y)$.

Proof. Given $f, g, h\colon x \to y$ in a quasicategory C, we will prove

1. (1) $f \sim_\ell f$,
2. (2) $f \sim_\ell g$ and $g \sim_\ell h$ imply $f \sim_\ell h$,
3. (3) $f \sim_\ell g$ implies $f \sim_r g$,
4. (4) $f \sim_r g$ implies $g \sim_\ell f$.

Statements (3) and (4) combine to show that $\sim_\ell$ is symmetric, and thus with (1) and (2) that $\sim_\ell$ is an equivalence relation. Statements (3) and (4) and symmetry imply that $\sim_r$ and $\sim_\ell$ coincide. The idea is to use the inner-horn extension condition for C to produce the appropriate relations.

Staetement (1) is exhibited by $f_{001} \in C_2$.

$$\begin{tikzcd} x \arrow[dd, bend right, "1_x"'] \arrow[dr, "f"] & \\ {} \arrow[r, phantom, "f_{001}"] & y \\ x \arrow[ur, "f"'] & \end{tikzcd}$$

Statements (2), (3), and (4) are demonstrated by the following diagrams, which present a map from an inner horn of Δ^3 (respectively Λ^3_1, Λ^3_1, and Λ^3_2) to C constructed from the given data. The restriction of any extension to Δ^3 along the remaining face (respectively $\Delta^{\{023\}}$, $\Delta^{\{023\}}$, and $\Delta^{\{013\}}$) gives the conclusion.

$$\begin{tikzcd} x \arrow[dd, bend right, "1_x"'] \arrow[dr, "1_x"] \arrow[drr, bend left, "f"] & & \\ {} \arrow[r, phantom, "x_{000}"] & x \arrow[r, "g"] \arrow[dl, "1_x"] \arrow[ur, phantom, "a"] \arrow[dr, phantom, "b"] & y \\ x \arrow[urr, bend right, "h"'] & & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, bend right, "f"'] \arrow[dr, "1_x"] \arrow[drr, bend left, "g"] & & \\ {} \arrow[r, phantom, "a"] & x \arrow[r, "g"] \arrow[dl, "g"] \arrow[ur, phantom, "g_{001}"] \arrow[dr, phantom, "g_{011}"] & y \\ y \arrow[urr, bend right, "1_y"'] & & \end{tikzcd} \qquad \begin{tikzcd} x \arrow[dd, bend right, "1_x"'] \arrow[dr, "f"] \arrow[drr, bend left, "g"] & & \\ {} \arrow[r, phantom, "f_{001}"] & y \arrow[r, "1_y"] \arrow[ur, phantom, "b"] \arrow[dr, phantom, "f_{011}"] & y \\ x \arrow[ur, "f"] \arrow[urr, bend right, "f"'] & & \end{tikzcd}$$

□

9.7. **Composition of homotopy classes of morphisms.** We now define $f \approx g$ to mean $f \sim_\ell g$ (equivalently $f \sim_r g$). We speak of **homotopy classes** $[f]$ of morphisms $f \in \hom_C(x, y)$, meaning equivalence classes under $\approx$. Next we observe that we can compose homotopy classes.

Given $f \in \hom_C(x, y)$, $g \in \hom_C(y, z)$, $h \in \hom_C(x, z)$, we say that h is a **composite** of (g, f) if there exists a 2-dimensional element $a \in C_2$ with $a\langle 01\rangle = f$, $a\langle 12\rangle = g$, $a\langle 02\rangle = h$; thus composition is a three-fold relation on $\hom(x, y) \times \hom(y, z) \times \hom(x, z)$. The composition relation is compatible with the homotopy relation, as shown by the following.

9.8. **Lemma.** *If $f \approx f'$, $g \approx g'$, h a composite of (g, f), and h' a composite of (g', f'), then $h \approx h'$.*

Proof. Since $\approx$ is an equivalence relation, it suffices prove the special cases (a) $f = f'$, and (b) $g = g'$. We prove case (b), as case (a) is analogous.

Let $a \in C_2$ exhibit $f \sim_\ell f'$, and let $b, b' \in C_2$ exhibit h as a composite of (g, f) and h' as a composite of (g, f') respectively. The inner horn $\Lambda^3_2 \to C$ defined by

$$\begin{tikzcd}
0 \arrow[dd, bend right, "1_x"'] \arrow[dr, "f"] \arrow[drr, bend left, "h"] \arrow[dr, phantom, shift left=3, "b"] & & \\
a & 2 \arrow[r, "g"] \arrow[r, phantom, shift right=3, "b'"'] & 3 \\
1 \arrow[ur, "f'"] \arrow[urr, bend right, "h'"'] & &
\end{tikzcd}$$

extends to $u\colon \Delta^3 \to C$, and $u|\Delta^{\{0,1,3\}}$ exhibits $h \sim_\ell h'$. $\square$

Thus, composites of (g, f) live in a unique homotopy class of morphisms in C, which only depends on the homotopy classes of g and f. I will write $[g] \circ [f]$ for the homotopy class containing composites of (g, f).

I'll leave the following as exercises; the proofs are much like what we have already seen.

9.9. **Lemma.** *Given $f\colon x \to y$, we have $[f] \circ [1_x] = [f] = [1_y] \circ [f]$.*

9.10. **Lemma.** *If $[g] \circ [f] = [u]$, $[h] \circ [g] = [v]$, then $[h] \circ [u] = [v] \circ [f]$.*

9.11. **The homotopy category of a quasicategory.** For any quasicategory, we define its **homotopy category** hC, with object set $\mathrm{ob}(hC) := C_0$, and with morphism sets $\hom_{hC}(x, y) := \hom_C(x, y)/\approx$, with composition defined by $[g] \circ [f]$. The above lemmas (9.9) and (9.10) exactly imply that hC is a category.

We define a map $\pi\colon C \to N(hC)$ of simplicial sets as follows. On vertices, π is the identity map $C_0 = N(hC)_0 = \mathrm{ob}\, hC$. On edges, the map is defined by the tautological quotient maps $\hom_C(x, y) \to \hom_C(x, y)/\approx$ sending $f \mapsto [f]$. The map π sends an n-dimensional element $a \in C_n$ to the unique element $\pi(a) \in N(hC)_n$ such that $\pi(a)_{i-1,i} = \pi(a_{i-1,i})$. These functions are seen to be compatible with simplicial operators using the following exercise.

9.12. *Exercise.* Let C be a quasicategory and $a \in C_n$ an n-dimensional element, and define $f_i := a_{i-1,i} \in C_1$ for $i = 1, \dots, n$ and $g := a_{0,n} \in C_1$. Show that $[f_n] \circ \cdots \circ [f_1] = [g]$ in the homotopy category hC.

Note that if C is an ordinary category, then $f \approx g$ if and only if $f = g$. Thus, $\pi\colon C \to N(hC)$ is an isomorphism of simplicial sets if and only if C is isomorphic to the nerve of a category.

The following says that the homotopy category of a quasicategory is its fundamental category, justifying the notation "hC".

9.13. **Proposition.** *Let C be a quasicategory and D a small category, and let $\phi\colon C \to N(D)$ be a map of simplicial sets. Then there exists a unique map $\psi\colon N(hC) \to N(D)$ such that $\psi\pi = \phi$.*

Proof. We first show existence, by constructing a suitable map ψ, which being a map between nerves can be described as a functor $hC \to D$. On objects, let ψ send $x \in \mathrm{ob}(hC) = C_0$ to $\phi(x) \in \mathrm{ob}(D) = (ND)_0$. On morphisms, let ψ send $[f] \in \hom_{hC}(x, y)$ to $\phi(f) \in \hom_D(\phi(x), \phi(y)) \subseteq (ND)_1$. Observe that the function on morphisms is well-defined since if $f \sim_\ell f'$, exhibited by some $a \in C_2$, then $\phi(a) \in (ND)_2$ exhibits the identity $\phi(f) = \phi(f')\phi(1_x) = \phi(f')$ in D. It is straightforward to show that ψ so defined is actually a functor, and that $\psi\pi = \phi$ as maps $C \to N(D)$.

The functor ψ defined above is the unique solution: the value of ψ on objects and morphisms is uniquely determined, and $\pi\colon C_k \to (hC)_k$ is bijective for $k = 0$ and surjective for $k = 1$. $\square$

In particular, the homotopy category construction gives a pair of adjoint functors

$$h\colon \mathrm{qCat} \rightleftarrows \mathrm{Cat} : N.$$

9.14. *Exercise.* Understand the homotopy categories of the various examples of quasicategories described in §8.

9.15. *Exercise* (Easy but important). Show that for quasicategories C and D there is an isomorphism of categories $hC \times hD \approx h(C \times D)$.

9.16. **A criterion for composition.** We have observed that for morphisms $f\colon x \to y$ and $g\colon y \to z$ **F 25 Jan 2019**
in a quasicategory that we can define a composite "$g \circ f$" using extension along $\Lambda^2_1 \subset \Delta^2$, and that though such compositions are not unique, they are unique up to homotopy, so we get a well-defined homotopy class $[g] \circ [f]$. The following proposition says that every element in this homotopy class is obtained from this construction.

9.17. **Proposition.** *If $f\colon x \to y$, $g\colon y \to z$, and $h\colon x \to z$ are morphisms in a quasicategory C, then $[h] = [g] \circ [f]$ if and only if there exists $u\colon \Delta^2 \to C$ such that*

$$u|\Delta^{\{0,1\}} = f, \qquad u|\Delta^{\{1,2\}} = g, \qquad u|\Delta^{\{0,2\}} = h.$$

Thus, every *morphism in the homotopy class of h can be interpreted as a composite of g with f.*

Proof. Clearly if u exists then $[h] = [g] \circ [f]$. Conversely, suppose given f, g, h with $h \in [g] \circ [f]$, and choose some $a\colon \Delta^2 \to C$ with $a_{01} = f$ and $a_{12} = g$, whence $[g] \circ [f] = [h']$ for $h' = a_{02}$. Since $h \in [h']$ there is a $b \in C_2$ witnessing the relation $h' \sim_r h$, and using this we can construct a map $\Lambda^3_2 \to C$ according to the diagram

$$\begin{tikzcd}
0 \arrow[dd, bend right, "f"'] \arrow[dr, "h'"] \arrow[drr, bend left, "h"] \arrow[drr, phantom, "b"] & & \\
a & 2 \arrow[r, "1_z"] & 3 \\
1 \arrow[ur, "g"] \arrow[urr, bend right, "g"'] \arrow[urr, phantom, "g_{011}"] & &
\end{tikzcd}$$

Extend to a map $v\colon \Delta^3 \to C$; then $u = v|\Delta^{\{0,1,3\}}$ exhibits h as a composite of (g, f) as desired. $\square$

9.18. *Exercise.* Let $C' \subseteq C$ be a subcategory (6.14) of a quasicategory C. Show that if $f, g\colon x \to y$ are morphisms of C which are homotopic, then $f \in C'_1$ if and only if $g \in C'_1$. Use this to show that there is a bijective correspondence

$$(\text{subcategories of } C) \leftrightarrow (\text{subcategories } hC),$$

and also a bijective correspondence

$$(\text{full subcategories of } C) \leftrightarrow (\text{full subcategories } hC).$$

10. Isomorphisms in a quasicategory

Let C be a quasicategory. We say that an edge $f \in C_1$ is an **isomorphism**[6] if its image in the homotopy category hC is an isomorphism in the usual sense of category theory.

Explicitly, $f\colon x \to y$ is an isomorphism if and only if there exists an edge $g\colon y \to x$ such that $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, where equality is in the homotopy category hC.

10.1. *Example.* Consider $f \in C_1$. If we can produce $g \in C_1$ and $a, b \in C_2$ such that

$$a_{01} = f = b_{12}, \qquad a_{12} = g = b_{01}, \qquad a_{02} = x_{00}, \qquad b_{02} = y_{00} :$$

$$\begin{tikzcd}
& x \arrow[rr, "1_x"] \arrow[dr, "f"] \arrow[d, phantom, "b"] & & x \\
y \arrow[ur, "g"] \arrow[rr, "1_y"'] & & y \arrow[ur, "g"'] \arrow[u, phantom, "a"] &
\end{tikzcd}$$

then $[g] \circ [f] = [1_x]$ and $[f] \circ [g] = [1_y]$, so f isomorphism. The converse also holds: if f is an isomorphism, then there exist $g \in C_1$ and $a, b \in C_2$ as above, which can be proved using (9.17).

10.2. *Example* (Identity maps are isomorphims)*.* For every $x \in C_0$ the identity map $1_x\colon x \to x$ is an isomorphism: for instance, use $a = b = x_{000}$ in the above diagram.

10.3. *Exercise.* Show that any functor $f\colon C \to D$ between quasicategories sends isomorphisms to isomorphisms.

10.4. **Preinverses and postinverses.** Let C be a quasicategory. Given $f\colon x \to y \in C_1$, a **postinverse**[7] of f is a $g\colon y \to x \in C_1$ such that $[g] \circ [f] = [1_x]$, and a **preinverse**[8] of f is an $e\colon y \to x \in C_1$ such that $[f] \circ [e] = [1_y]$. An **inverse** is an $f' \in C_1$ which is both a postinverse and a preinverse. The following is trivial, but very handy.

10.5. **Proposition.** *In a quasicategory C consider $f \in C_1$. The following are equivalent.*

- *f is an isomorphism.*
- *f admits an inverse f'.*
- *f admits a postinverse g and a preinverse e.*
- *f admits a postinverse g and g admits a postinverse h.*
- *f admits a preinverse e and e admits a preinverse d.*

If these equivalent conditions apply, then $f \approx d \approx h$ and $f' \approx e \approx g$, and all of them are isomorphisms.

Proof. All of these are equivalent to the corresponding statements about morphisms in the homotopy category hC, where they are seen to be equivalent to each other by elementary arguments. □

Note that inverses to a morphism in a quasicategory are generally *not unique*, though necessarily they are *unique up to homotopy*.

10.6. **Quasigroupoids.** A **quasigroupoid** is a quasicategory C such that hC is a groupoid, i.e., a quasicategory in which every morphism is an isomorphism.

10.7. *Exercise.* If every morphism in a quasicategory admits a preinverse, then it is a quasigroupoid. Likewise if every morphism admits a postinverse.

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[6]Lurie [Lur09, §1.2.4] uses the term “equivalence” for this. I prefer to go with “isomorphism” here, because it is in fact a generalization of the classical notion of isomorphism in a category, and also because so many other things get to be called some kind of equivalence. Other authors also use “isomorphism” in this context.

[7]or *left inverse*, or *retraction*,

[8]or *right inverse*, or *section*,

10.8. **The core of a quasicategory.** For an ordinary category A, the **core** (or **maximal subgroupoid**) of A is the subcategory $A^{\mathrm{core}} \subseteq A$ consisting of all the objects, and all the *isomorphisms* between the objects.

For a quasicategory C, we define the core[9] $C^{\mathrm{core}} \subseteq C$ to be the subcomplex consisting of elements all of whose edges are all isomorphisms. That is, C^{core} is defined so that the diagram

$$\begin{tikzcd} C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "\pi"] \\ (hC)^{\mathrm{core}} \arrow[r, tail] & hC \end{tikzcd}$$

is a pullback of simplicial sets. Observe that $N(A^{\mathrm{core}}) = (NA)^{\mathrm{core}}$ for a category A.

10.9. **Proposition.** *Given a quasicategory C, its core C^{core} is a subcategory and a quasigroupoid, and every subcomplex of C which is a quasigroupoid is contained in C^{core}.*

Proof. First we show that C^{core} is a subcategory (6.14). Suppose $f\colon \Delta^n \to C$ such that $f(\Lambda^n_k) \subseteq C^{\mathrm{core}}$ for $n \geq 2$ and $0 < k < n$. When $n \geq 3$ then $(\Lambda^n_k)_1 = (\Delta^n)_1$, so clearly $f(\Delta^n) \subseteq C^{\mathrm{core}}$. When $n = 2$ we have that $f(\Delta^n) \subseteq C^{\mathrm{core}}$ because the composite of two isomorphisms is an isomorphism.

It follows that C^{core} is a quasicategory, and clearly it is a quasigroupoid. The final statement is clear: if $G \subseteq C$ is a subcomplex which is a quasigroupoid, then every edge in G has in inverse in G, and hence an inverse in C. □

10.10. **Kan complexes.** Recall that a Kan complex (8.6) is a simplicial set which has the extension property with respect to all horns, not just inner horns. That is, K is a Kan complex iff

$$\mathrm{Hom}(\Delta^n, K) \to \mathrm{Hom}(\Lambda^n_j, K)$$

is surjective for all $0 \leq j \leq n$, $n \geq 1$.

10.11. *Exercise.* Show that every simplicial set X has extensions for 1-dimensional horns; i.e., every $\Lambda^1_j \to X$ extends over $\Lambda^1_j \subset \Delta^1$, where $j \in \{0, 1\}$. Thus, X is a Kan complex if and only if it has extensions just for the horns inside simplices of dimension ≥ 2.

10.12. **Proposition.** *Every Kan complex is a quasigroupoid.*

Proof. It is immediate that a Kan complex K is a quasicategory. To show K is a quasigroupoid, note that the extension condition for $\Lambda^2_0 \subset \Delta^2$ implies that every morphism in hK admits a postinverse. Explicitly, if $f\colon x \to y$ is an edge in K, let $u\colon \Lambda^2_0 \to K$ with $u_{01} = f$ and $u_{02} = f_{00} = 1_x$, so there is an extension $v\colon \Delta^2 \to K$ and $g := v_{12}$ satisfies $gf \approx 1_x$. Use (10.7). □

This proposition has a converse.

A. **Deferred Proposition.** *Quasigroupoids are precisely the Kan complexes.*

This is a very important technical result, and it is not trivial; it is the main result of [Joy02]. We will give the proof in (30.2).

Recall (§8.5) that we observed that the singular complex $\mathrm{Sing}\, T$ of a topological space is a Kan complex, and therefore a quasigroupoid. It is reasonable to think of $\mathrm{Sing}\, T$ as the **fundamental quasigroupoid** of the space T.

10.13. *Exercise* (for topologists). Show that if T is a topological space, then $h\,\mathrm{Sing}\, T$, the homotopy category of the singular complex of T, is precisely the usual fundamental groupoid of T.

[9] Lurie (along with many others) uses the notation $C^{\simeq}$ for what we are calling C^{core}.

10.14. **Quasigroupoids, components, and isomorphism classes.** We say that two objects in a quasicategory are **isomorphic** if there exists an isomorphism between them. This is an equivalence relation on C_0, and thus we speak of **isomorphism classes** of objects.

Recall (6.8) that the set of connected components of a simplicial set is given by

$$\pi_0 X \approx \left((\coprod_{n\geq 0} X_n)/\sim \right) \approx (X_0/\sim_1),$$

the equivalence classes of elements of X under the equivalence relation generated by "related by a simplicial operator", or equivalently the equivalence classes of vertices of X under the equivalence relation generated by "connected by an edge" . Note that if T is a topological space, then elements of $\pi_0 \operatorname{Sing} T$ correspond exactly to path components of T.

For *quasigroupoids*, π_0 recovers the set of isomorphism classes of objects.

10.15. **Proposition.** *If C is a quasicategory, then*

$$\pi_0\big(C^{\mathrm{core}}\big) \approx \textit{isomorphism classes of objects of } C.$$

Proof. Straightforward: edges in C^{core} are precisely the isomorphisms in C. □

10.16. *Exercise.* Show that for a quasicategory C, $\pi_0(C^{\mathrm{core}}) \approx \pi_0(h(C^{\mathrm{core}})) \approx \pi_0((hC)^{\mathrm{core}})$.

11. Function complexes and the functor quasicategory

Given ordinary categories C and D, the *functor category* $\mathrm{Fun}(C, D)$ has

- as objects, the functors $C \to D$, and
- as morphisms $f \to f'$, natural transformations of functors.

Furthermore, for any category A there is a bijective correspondence between sets of functors

$$\{A \times C \to D\} \quad \longleftrightarrow \quad \{A \to \mathrm{Fun}(C, D)\}.$$

Explicitly, a functor $\phi\colon A \to \mathrm{Fun}(C, D)$ corresponds to $\widetilde{\phi}\colon A \times C \to D$, given on objects by $\widetilde{\phi}(a, c) = \phi(a)(c)$ for $a \in \operatorname{ob} A$ and $c \in \operatorname{ob} C$, and on morphisms by $\widetilde{\phi}(\alpha, \gamma) = \phi(a')(\gamma) \circ \phi(\alpha)(c) = \phi(\alpha)(c') \circ \phi(a)(\gamma)\colon \phi(a)(c) \to \phi(a')(c')$ for $\alpha\colon a \to a' \in \operatorname{mor} A$ and $\gamma\colon c \to c' \in \operatorname{mor} C$.

The generalization of the functor category to quasicategories admits a similar adjunction, and in fact can be defined for arbitrary simplicial sets.

11.1. **Function complexes.** Given simplicial sets X and Y, we may form the **function complex** $\mathrm{Map}(X, Y)$. This is a simplicial set with

$$\mathrm{Map}(X, Y)_n = \mathrm{Hom}(\Delta^n \times X, Y),$$

so that the action of a simplicial operator $\delta\colon [m] \to [n]$ on $\mathrm{Map}(X, Y)$ is induced by

$$\mathrm{Hom}(\delta \times \mathrm{id}_X, Y)\colon \mathrm{Hom}(\Delta^n \times X, Y) \to \mathrm{Hom}(\Delta^m \times X, Y).$$

In particular, the set $\mathrm{Map}(X, Y)_0$ of vertices of the function complex is precisely the set of maps $X \to Y$ of simplicial sets.

11.2. **Proposition.** *The function complex construction defines a functor*

$$\mathrm{Map}\colon s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \to s\mathrm{Set}.$$

Proof. Left as an exercise. □

Note: I think I'm going to replace the "Map" notation with "Fun" throughout the text. It is awkward to have two different notations for the same thing, and anyway we can interpret "Fun" as meaining either "function complex" or "functor category".

By construction, for each n, there is a bijective correspondence

$$\{\Delta^n \times X \to Y\} \longleftrightarrow \{\Delta^n \to \mathrm{Map}(X, Y)\}.$$

In fact, we can replace Δ^n with an arbitrary simplicial set.

11.3. **Proposition.** *For simplicial sets X, Y, Z, there is a bijection*

$$\mathrm{Hom}(X \times Y, Z) \xrightarrow{\sim} \mathrm{Hom}(X, \mathrm{Map}(Y, Z))$$

natural in all three variables.

Proof. The bijection sends $f\colon X \times Y \to Z$ to $\widetilde{f}\colon X \to \mathrm{Map}(Y, Z)$ defined so that for $x \in X_n$, the element $\widetilde{f}(x) \in \mathrm{Map}(Y, Z)_n$ is represented by the composite

$$\Delta^n \times Y \xrightarrow{x \times \mathrm{id}} X \times Y \xrightarrow{f} Z.$$

The inverse of this bijection sends $g\colon X \to \mathrm{Map}(Y, Z)$ to $\widetilde{g}\colon X \times Y \to Z$, defined so that for $(x, y) \in X_n \times Y_n$, the element $\widetilde{g}(x, y) \in Z_n$ is represented by

$$\Delta^n \xrightarrow{(\mathrm{id}, y)} \Delta^n \times Y \xrightarrow{g(x)} Z.$$

The proof amounts to showing that both $\widetilde{f}$ and $\widetilde{g}$ are in fact maps of simplicial sets, and that the above constructions are in fact inverse to each other. This is left as an exercise, as is the proof of naturality. □

11.4. *Exercise.* Show, using the previous proposition, that there are natural isomorphisms

$$\mathrm{Map}(X \times Y, Z) \approx \mathrm{Map}(X, \mathrm{Map}(Y, Z)).$$

of simplicial sets. This implies that the function complex construction makes $s\mathrm{Set}$ into a *cartesian closed category.* (Hint: show that both objects represent *isomorphic* functors $s\mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$, and apply the Yoneda lemma.)

11.5. *Remark.* The construction of the function complex is not special to simplicial sets. The construction of $\mathrm{Map}(X, Y)$ (and its properties as described above) works the same way in any category of functors $C^{\mathrm{op}} \to \mathrm{Set}$, where C is a small category (e.g., $C = \Delta$). In this general setting, the role of the standard n-simplices is played by the representable functors $\mathrm{Hom}_C(-, c)\colon C^{\mathrm{op}} \to \mathrm{Set}$.

11.6. **Functor quasicategories.** Thus, we may expect the generalization of functor category to quasicategories to be defined by the function complex. In fact, if C and D are quasicategories, then the vertices of $\mathrm{Map}(C, D)$ are precisely the functors $C \to D$, and the edges of $\mathrm{Map}(C, D)$ are precisely the natural transformations. Furthermore, for ordinary categories, the function complex recovers the functor category.

11.7. *Exercise.* Show that for ordinary categories C and D that $N\,\mathrm{Fun}(C, D) \approx \mathrm{Map}(NC, ND)$. (Hint: use that $N([n]) = \Delta^n$, and the fact that the nerve preserves finite products (6.5).)

It turns out that a function complex between quasicategories is again a quasicategory. In fact, we have the following.

B. **Deferred Proposition.** *Let K be any simplicial set and C a quasicategory. Then* $\mathrm{Map}(K, C)$ *is a quasicategory.*

For this reason, we will sometimes write $\mathrm{Fun}(K, C)$ for $\mathrm{Map}(K, C)$ when C is a quasicategory.

To prove (B), we need a to take a detour to develop some technology about "weakly saturated" classes of maps and "lifting properties". After this, we will complete the proof in §16.

Part 2. Lifting properties

12. Weakly saturated classes and inner-anodyne maps

Quasicategories are defined by an "extension property": they are the simplicial sets C such that any map $K \to C$ extends over L, whenever $K \subset L$ is an inner horn inclusion $\Lambda^n_j \subset \Delta^n$. The set of inner horns "generates" a larger class of maps (which will be called the class of *inner anodyne* maps), which "automatically" shares the extension property of the inner horns. This class of inner anodyne maps is called the *weak saturation* of the set of inner horns. **M 28 Jan 2019**

For instance, we will observe that the spine inclusions $I^n \subset \Delta^n$ are inner anodyne, so that quasicategories admit "spine extensions", i.e., any $I^n \to C$ extends over $I^n \subset \Delta^n$ to a map $\Delta^n \to C$.

12.1. **Weakly saturated classes.** Consider a category (such as sSet) which has all small colimits. A **weakly saturated class** is a class $\mathcal{A}$ of morphisms in the category, which

(1) contains all isomorphisms,
(2) is closed under cobase change,
(3) is closed under composition,
(4) is closed under transfinite composition,
(5) is closed under coproducts, and
(6) is closed under retracts.

Given a class of maps S, its **weak saturation** $\overline{S}$ is the smallest weakly saturated class containing S.

We need to explain some of the elements of this definition.

- **Closed under cobase change** is also called **closed under pushout**: it means that if f' is the pushout of $f\colon X \to Y$ along some map $g\colon X \to Z$, then $f \in \mathcal{A}$ implies $f' \in \mathcal{A}$.
- **Closed under composition** means that if $g, f \in \mathcal{A}$ and gf is defined, then $gf \in \mathcal{A}$.
- We say that $\mathcal{A}$ is **closed under countable composition** if given a countable sequence of composable morphisms, i.e., maps
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \cdots$$
such that each $f_k \in \mathcal{A}$ for all $k \in \mathbb{Z}_{>0}$, the induced map $X_0 \to \operatorname{colim}_k X_k$ to the colimit is in $\mathcal{A}$.

  The notion **closed under transfinite composition** is a generalization of this, in which $\mathbb{N}$ is replaced by an arbitrary ordinal λ (i.e., a well-ordered set). This means that for any ordinal λ and any functor $X\colon \lambda \to s\mathrm{Set}$, if for every $i \in \lambda$ with $i \neq 0$ the evident map
$$(\operatorname{colim}_{j<i} X(j)) \to X(i)$$
is in $\mathcal{A}$, then the induced map $X(0) \to \operatorname{colim}_{j\in\lambda} X(j)$ is in $\mathcal{A}$.
- **Closed under coproducts** means that if $\{f_i\colon X_i \to Y_i\}$ is a set of maps in $\mathcal{A}$, then $\coprod_i f_i\colon \coprod_i X_i \to \coprod_i Y_i$ is in $\mathcal{A}$.
- We say that f is a **retract** of g if there exists a commutative diagram in C of the form
$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r] \arrow[d, "g"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
This is really a special case of the notion of a retract of an object in the functor category $\mathrm{Fun}([1], s\mathrm{Set})$. We say that $\mathcal{A}$ is **closed under retracts** if for every diagram as above, $g \in \mathcal{A}$ implies $f \in \mathcal{A}$.

12.2. *Remark.* This list of properties is not minimal: (3) is the special case of (4) when $\lambda = [2]$, and (5) can be deduced from (2) and (4). *Exercise:* Show this.

12.3. *Example.* Consider the category of sets. The class of all surjective maps is weakly saturated, and in fact is the weak saturation of $\{\, \{0,1\} \to \{1\} \,\}$. Likewise, the class of injective maps is weakly saturated, and in fact is the weak saturation of $\{\, \varnothing \to \{1\} \,\}$.

12.4. *Example.* The classes of monomorphisms and epimorphisms of simplicial sets are weakly saturated classes. Later we will identify the class of monorphisms of simplicial sets as the weak saturation of the set of "cell inclusions" (15.26).

There is a dual notion of a **weakly cosaturated class**: a weakly cosaturated class is the same thing as a weakly saturated class in the opposite category, and is characterized by being closed under properties formally dual to (1)–(6).

12.5. **Classes of "anodyne" morphisms.** We use the following notation for sets of types of horns:

$$\begin{aligned}
\mathrm{InnHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k < n,\ n \geq 2 \,\}, && \text{(inner horns)},\\
\mathrm{LHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 \leq k < n,\ n \geq 1 \,\}, && \text{(left horns)},\\
\mathrm{RHorn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \leq n,\ n \geq 1 \,\}, && \text{(right horns)},\\
\mathrm{Horn} &:= \{\, \Lambda^n_k \subset \Delta^n \mid 0 \leq k \leq n,\ n \geq 1 \,\}, && \text{(horns)}.
\end{aligned}$$

The weak saturation of each of these sets will play an important role in what follows. Right now, we focus on the weak saturation $\overline{\mathrm{InnHorn}}$ of the set of inner horns, which is called the class of **inner anodyne**[10] morphisms. (There are also classes of "left anodyne", "right anodyne", and plain old "anodyne" morphisms, about which we have more to say later.) Note that inner anodyne morphisms are always monomorphisms, since monomorphisms of simplicial sets themselves form a weakly saturated class.

12.6. **Proposition.** *If C is a quasicategory and $A \subseteq B$ is an inner anodyne inclusion, then any $f\colon A \to C$ admits an extension to $g\colon B \to C$ so that $g|A = f$.*

Proof. It suffices to show that the collection $\mathcal{A}$ of monomorphisms $i\colon A \to B$ such that every map from A to a quasicategory extends along i is weakly saturated. Since $\mathrm{InnHorn} \subseteq \mathcal{A}$ it then follows that $\overline{\mathrm{InnHorn}} \subseteq \mathcal{A}$. To prove this claim is a relatively straightforward exercise, which we leave for the reader: check that the class $\mathcal{A}$ satisfies each of the conditions (1)–(6) of a weakly saturated class. It is highly recommended that you work through this argument this if you haven't seen it before. □

12.7. *Exercise* (Easy but important). Show that every inner anodyne map induces a bijection on vertices. (Hint: show that the class of maps of simplicial sets which are a bijection on vertices is weakly saturated.)

12.8. **Examples of inner anodyne morphisms.** It is crucial to be able to prove that certain explicit maps are inner anodyne.

Let $S \subseteq [n]$. The associated **generalized horn** is the subcomplex $\Lambda^n_S \subset \Delta^n$ defined by

$$\Lambda^n_S := \bigcup_{i \in S} \Delta^{[n] \smallsetminus i},$$

[10]The "anodyne" terminology for the weak saturation of a set of horns was introduced by Gabriel and Zisman [GZ67]. "Anodyne" derives from ancient Greek, meaning "without pain"; we leave it to the reader to decide whether this choice of terminology is appropiate.

i.e., the union of codimension one faces of the n-simplex indexed by elements of S. In particular, $\Lambda^n_{[n]\smallsetminus\{j\}}$ is the usual horn Λ^n_j. I'll generalize this notation to arbitary totally ordered sets, so $\Lambda^T_S = \bigcup_{i\in S} \Delta^{T\smallsetminus i}$ when $S \subseteq T$.

We call $\Lambda^n_S \subset \Delta^n$ a **generalized inner horn** if S is not an "interval" in $[n]$, i.e., if there exist $s < t < s'$ with $s, s' \in S$ and $t \notin S$.

12.9. **Lemma.** *All generalized inner horn inclusions $\Lambda^n_S \subset \Delta^n$ are inner anodyne.*

There is a slick proof of this given by Joyal [Joy08a, Prop. 2.12], which we present in the appendix (60.1). **Why not just put it here?**

12.10. *Example.* Consider $\Lambda^3_{\{0,3\}}$, which can be pictured as the solid diagram in

$$\begin{tikzcd}
0 \arrow[dr] \arrow[dd, bend right] \arrow[drr, dotted, bend left] & & \\
& 2 \arrow[r] & 3 \\
1 \arrow[ur] \arrow[urr, bend right] & &
\end{tikzcd}$$

We can get from this to Δ^3 in two steps:

$$\begin{tikzcd}
\Lambda^{\{0,2,3\}}_{\{0,3\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{0,2,3\}} \arrow[d, tail] & \\
\Lambda^3_{\{0,3\}} \arrow[r, tail] & \Lambda^3_2 \arrow[r, tail] & \Delta^3
\end{tikzcd}$$

The square is a pushout of subcomplexes since $\Lambda^3_{\{0,3\}} \cap \Delta^{\{0,2,3\}} = \Lambda^{\{0,2,3\}}_{\{0,3\}}$, and the map along the top is isomorphic to $\Lambda^2_1 \subset \Delta^2$, an inner horn inclusion. This proves that $\Lambda^3_{\{0,3\}} \subset \Delta^3$ is inner anodyne.

Recall that every standard n-simplex contains a spine $I^n \subseteq \Delta^n$.

12.11. **Lemma.** *The spine inclusions $I^n \subset \Delta^n$ are inner anodyne for all n. Thus, for a quasicategory C, any $I^n \to C$ extends to $\Delta^n \to C$.*

This is proved in [Joy08a, Prop. 2.13]; we give the proof in the appendix (60.2).

12.12. *Example.* To show that $I^3 \subset \Delta^3$ is inner anodyne, observe that we can get from I^3 to a generalized inner horn two steps by gluing 2-simplices along inner horns inclusions:

$$\begin{tikzcd}
\Lambda^{\{0,1,2\}}_{\{0,2\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{0,1,2\}} \arrow[d, tail] & \Lambda^{\{1,2,3\}}_{\{1,3\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{1,2,3\}} \arrow[d, tail] \\
I^3 \arrow[r, tail] & I^3 \cup \Delta^{\{0,1,2\}} \arrow[r, equal] & I^3 \cup \Delta^{\{0,1,2\}} \arrow[r, tail] & \Lambda^3_{\{0,3\}}
\end{tikzcd}$$

since $I^3 \cap \Delta^{\{0,1,2\}} = \Lambda^{\{0,1,2\}}_{\{0,2\}}$ and $(I^3 \cup \Delta^{\{0,1,2\}}) \cap \Delta^{\{1,2,3\}} = \Lambda^{\{1,2,3\}}_{\{1,3\}}$.

12.13. *Exercise.* Use (12.11) to show that the tautological map $\pi\colon C \to N(hC)$ from a quasicategory to (the nerve of) its homotopy category is surjective in every degree.

13. Lifting calculus and inner fibrations

We have defined quasicategories by an "extension property": in general, we say that X satisfies the extension property for $f\colon A \to B$ if for any diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & X \\
B \arrow[ur, dotted, "s"'] &
\end{tikzcd}$$

there exists a morphism s making the diagram commute. In this section, we discuss a "relative" version of this, called a "lifting property".

13.1. **The lifting relation.** Given morphisms $f\colon A \to B$ and $g\colon X \to Y$ in a category, a **lifting problem** for (f, g) is a pair of morphisms (u, v) such that $vf = gu$. That is, a lifting problem is any commutative square of solid arrows of the form

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

A **lift** for the lifting problem is a morphism s such that $sf = u$ and $gs = v$, i.e., a dotted arrow making the diagram commute.

We may thus define the **lifting relation** on morphisms in our category: we write "$f \boxslash g$" if *every* lifting problem for (f, g) admits a lift[11]. Equivalently, $f \boxslash g$ exactly if

$$\mathrm{Hom}(B, X) \xrightarrow{s \mapsto (sf, gs)} \mathrm{Hom}(A, X) \times_{\mathrm{Hom}(A,Y)} \mathrm{Hom}(B, Y)$$

is a surjection, where the target is the set of pairs $(u\colon A \to X, v\colon B \to Y)$ such that $gu = vf$ (i.e., the target is exactly the set of lifting problems for (f, g)).

When $f \boxslash g$ holds, one sometimes says f has the **left lifting property** relative to g, or that g has the **right lifting property** relative to f. Or we just say that f **lifts against** g.

We extend the notation to classes of maps, so "$\mathcal{A} \boxslash \mathcal{B}$" means: $a \boxslash b$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

13.2. *Exercise.* Show that $f \boxslash f$ if and only if f is an isomorphism.

Given a class of morphisms $\mathcal{A}$, define the **right complement** $\mathcal{A}^{\boxslash}$ and **left complement** ${}^{\boxslash}\mathcal{A}$ by

$$\mathcal{A}^{\boxslash} = \{\, g \mid a \boxslash g \text{ for all } a \in \mathcal{A} \,\}, \qquad {}^{\boxslash}\mathcal{A} = \{\, f \mid f \boxslash a \text{ for all } a \in \mathcal{A} \,\}.$$

13.3. **Proposition.** *For any class $\mathcal{B}$, the left complement ${}^{\boxslash}\mathcal{B}$ is a weakly saturated class.*

13.4. *Exercise* (Important). Prove (13.3). (This is a "relative" version of the proof of (12.6).)

The above proposition (13.3) has a dual statement: any right complement $\mathcal{B}^{\boxslash}$ is weakly cosaturated, i.e., satisfies dual versions of the closure properties of a weakly saturated class, or equivalently, corresponds to a weakly saturated class in the opposite category.

13.5. *Exercise* (Easy). Prove that if $\mathcal{A} \subseteq \mathcal{B}$, then $\mathcal{A}^{\boxslash} \supseteq \mathcal{B}^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} \supseteq {}^{\boxslash}\mathcal{B}$. Use this to show $\mathcal{A}^{\boxslash} = ({}^{\boxslash}(\mathcal{A}^{\boxslash}))^{\boxslash}$ and ${}^{\boxslash}\mathcal{A} = {}^{\boxslash}(({}^{\boxslash}\mathcal{A})^{\boxslash})$.

13.6. *Exercise* (for those who know a little homological algebra). Fix an abelian category $\mathcal{C}$ (e.g., the category of modules over some ring R). Let $\mathcal{P}$ be the class of morphisms in $\mathcal{C}$ of the form $0 \to P$ where P is projective, and let $\mathcal{B}$ be the class of epimorphisms in $\mathcal{C}$. Show that $\mathcal{P} \boxslash \mathcal{B}$; also, show that $\mathcal{B} = \mathcal{P}^{\boxslash}$ if $\mathcal{C}$ has enough projectives.

13.7. *Exercise.* In the setting of the previous exercise, identify the class ${}^{\boxslash}\mathcal{B}$.

[11]Sometimes one sees the notation "$f \perp g$" or "$f \pitchfork g$" used instead. Our notation is taken from [Rie14, §11].

13.8. **Inner fibrations.** A map p of simplicial sets is an **inner fibration** if $\mathrm{InnHorn} \boxslash p$. The class of inner fibrations $\mathrm{InnFib} = \mathrm{InnHorn}^{\boxslash}$ is thus the right complement of the set of inner horns. Note that C is a quasicategory if and only if $C \to *$ is an inner fibration.

Because InnFib is a right complement, it is weakly cosaturated. In particular, it is closed under composition. This implies that if $p\colon C \to D$ is an inner fibration *and* D is a quasicategory, then C is also a quasicategory.

13.9. *Exercise.* Show that if $f\colon C \to D$ is any functor from a quasicategory C to a *category* D, then f is an inner fibration. In particular, all functors between categories are automatically inner fibrations. (Hint: use the fact that all inner horns mapping to a category have *unique* extensions to simplices.)

13.10. *Exercise.* Show that any inclusion $C' \subseteq C$ of a subcomplex of a quasicategory is an inner fibration if and only if C' is a subcategory (6.14) of C.

13.11. *Exercise.* Let $p\colon C \to D$ be a functor between quasicategories, and let $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ be the restriction of p to cores (10.8). Show that if p is an inner fibration then p^{core} is also an inner fibration. (Hint. There are two distinct cases of lifting problems $(\Lambda^n_k \subset \Delta^n) \boxslash p^{\mathrm{core}}$, namely $n = 2$ and $n \geq 3$.)

13.12. *Exercise.* Consider a pullback square of simplicial sets

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "\pi"'] & Y \end{tikzcd}$$

such that π is a *surjective* map. Show that if p' is an inner fibration then so is p.

13.13. *Example* (Campbell's example). Here is an example of an inner fibration whose target is not a quasicategory. Let $H^\ell := \Delta^2/\Delta^{\{0,1\}}$, i.e., the pushout of the diagram $\Delta^2 \hookleftarrow \Delta^{\{0,1\}} \to \Delta^0$ of simplicial sets, and let $f\colon \Delta^1 \to H^\ell$ be the composite $\Delta^1 \xrightarrow{\langle 02 \rangle} \Delta^2 \xrightarrow{\pi} H^\ell$, where π is the evident projection. Then f is an inner fibration. To see this, note that the base-change of f along the projection map π is the inclusion $\Lambda^2_0 \subseteq \Delta^2$, which is an inner fibration since both source and target are categories (13.10). Thus f is an inner fibration by (13.12).

The map f has been observed by Alexander Campbell [Cam19] to be a counterexample to a number of plausible-sounding statements, some of which we will discuss later.

13.14. *Exercise.* Show that the map $f\colon \Delta^1 \to H^\ell$ of (13.13) is not inner anodyne. (Hint: (13.2).)

13.15. *Example* (Campbell's counterexample). **This needs to be moved later, at least after categorical equivalence and 2-out-of-3 are introduced.**

Every inner anodyne map is a monomorphism that induces a bijection on vertices, and is also a categorical equivalence (20.14). Joyal asked whether the converse holds. Campbell gave the following countexample to show it does not [Cam19].

Recall the simplicial set $H^\ell = \Delta^2/\Delta^{\{0,1\}}$ of (13.13), and write $\pi\colon \Delta^2 \to \Delta^{\{0,1\}}$ for the evident quotient map. We have a commutative diagram

$$\begin{tikzcd} \Delta^1 \arrow[r, "f"] \arrow[dr, "\mathrm{id}"'] & H^\ell \arrow[d, "h"] & \Delta^1 \arrow[l, "g"'] \arrow[dl, "\mathrm{id}"] \\ & \Delta^1 & \end{tikzcd}$$

where f and g represent the edges $\pi(\langle 02 \rangle)$ and $\pi(\langle 12 \rangle)$ in H^ℓ, and h is the unique map such that $h\pi = \langle 001 \rangle$. Note that the maps f, g, h induce bijections on vertices.

The map g is actually inner anodyne: it is isomorphic to the cobase-change of the horn inclusion $\Lambda^2_1 \to \Delta^2$ along the unique map $\Lambda^2_1 \to \Delta^1$ which on vertices sends $0, 1 \mapsto 0$ and $2 \mapsto 1$.

Thus g is a categorical equivalence, whence so are f and h by the 2-out-of-3 property (22.10). On the other hand, we have observed that f is not an inner anodyne map (13.14). Therefore f is a categorical equivalence which is a monomorphism and a bijection on vertices, but is not an inner anodyne map.

13.16. **Factorizations.** It turns out that we can always factor any map of simplicial sets into an inner anodyne map followed by an inner fibration. This is a consequence of the following general observation.

13.17. **Proposition** ("Small object argument")**.** *Let S be a* set *of morphisms in sSet. Every map f between simplicial sets admits a factorization $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$.*

The proof of this proposition is by means of what is known as the "small object argument". I'll give the proof in the next section. For now we record a consequence.

13.18. **Corollary.** *For any set S of morphisms in sSet, we have that $\overline{S} = {}^{\boxslash}(S^{\boxslash})$.*

Proof. That $\overline{S} \subseteq {}^{\boxslash}(S^{\boxslash})$ is immediate from (13.3). Given f such that $f \boxslash S^{\boxslash}$, use the small object argument (13.17) to choose $f = pj$ with $j \in \overline{S}$ and $p \in S^{\boxslash}$. We have a commutative diagram of solid arrows

$$\begin{tikzcd} \bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}"] & \bullet \arrow[r, "\mathrm{id}"'] \arrow[d, "j"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, dotted, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & \bullet \arrow[r, "p"] & \bullet \end{tikzcd}$$

A map s exists making the diagram commute, because $f \boxslash p$, so there is a lift in

$$\begin{tikzcd} \bullet \arrow[r, "j"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, "s"] & \bullet \end{tikzcd}$$

The diagram exhibits f as a retract of j, whence $f \in \overline{S}$ since weak saturations are closed under retracts. $\square$

13.19. *Remark* (Retract trick)*.* The proof of the corollary is called the "retract trick": given $f = pj$, $f \boxslash p$ implies that f is a retract of j, while $j \boxslash f$ implies that f is a retract of p.

In the case we are currently interested in, we have that $\overline{\mathrm{InnHorn}} = {}^{\boxslash}\mathrm{InnFib}$ and $\overline{\mathrm{InnHorn}}^{\boxslash} = \mathrm{InnFib}$, and thus any map can be factored into an inner anodyne map followed by an inner fibration.

13.20. **Weak factorization systems.** A **weak factorization system** in a category is a pair $(\mathcal{L}, \mathcal{R})$ of classes of maps such that **F 1 Feb 2019**

- every map f admits a factorization $f = r\ell$ with $r \in \mathcal{R}$ and $\ell \in \mathcal{L}$, and
- $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ and $\mathcal{R} = \mathcal{L}^{\boxslash}$.

Thus, in any weak factorization the "left" class $\mathcal{L}$ is weakly saturated and the "right" class $\mathcal{R}$ is weakly cosaturated. The small object argument implies that $(\overline{S}, S^{\boxslash})$ is a weak factorization in sSet for every set of maps S. In particular, $(\overline{\mathrm{InnHorn}}, \mathrm{InnFib})$ is a weak factorization system.

13.21. *Exercise* (for those who know some homological algebra). In an abelian category, let $\mathcal{A}$ be the class of monomorphisms with projective cokernel, and let $\mathcal{B}$ be the class of epimorphisms. Show that the pair $(\mathcal{A}, \mathcal{B})$ is a weak factorization system if and only if the category has enough projectives. (This exercise is related to (13.6).)

13.22. *Exercise* (Goodwillie). Classify all weak factorization systems on the category of sets. (There are exactly six.)

13.23. **Uniqueness of liftings.** The relation $f \boxslash g$ says that lifting problems admit solutions, but not that the solutions are unique. However, we can incorporate uniqueness into the lifting calculus if our category has pushouts.

Given a map $f\colon A \to B$, let $f^\vee := (f, f)\colon B \amalg_A B \to B$ be the "fold" map, i.e., the unique map such that the composition with either of the canonical maps $B \to B \amalg_A B$ is f. It is straightforward to show that for a map $g\colon X \to Y$ we have that $\{f, f^\vee\} \boxslash g$ if and only if in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

there exists a *unique* lift s.

13.24. *Example.* Consider the category of topological spaces. Let $\mathcal{A}$ be the class of morphisms of the form $A \times \{0\} \to A \times [0,1]$, where A is an arbitrary space. Then $(\mathcal{A} \cup \mathcal{A}^\vee)^\boxslash$ contains all covering maps (by the "Covering Homotopy Theorem").

A weak factorization system $(\mathcal{L}, \mathcal{R})$ in which liftings of type $\mathcal{L} \boxslash \mathcal{R}$ are always unique is called an **orthogonal factorization system**.

13.25. *Exercise.* Show that in an orthogonal factorization system, the factorizations $f = r\ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$ are unique up to *unique isomorphism*.

13.26. *Exercise.* Show that $(\{\text{surjections}\}, \{\text{injections}\})$ is an orthogonal factorization system for Set.

The small object argument implies that $(\overline{S \cup S^\vee}, (S \cup S^\vee)^\boxslash)$ is an orthogonal factorization system for every set S of morphisms.

13.27. *Example* (The fundamental category via an orthogonal factorization system). In simplicial sets, the projection map $C \to *$ is in the right complement to $S := \text{InnHorn} \cup \text{InnHorn}^\vee$ if and only if C is *isomorphic* to a nerve of a category (5.7). The small object argument using S, applied to a projection $X \to *$, thus produces a morphism $\pi\colon X \to Y$ in $\overline{S}$ with Y the nerve of a category.

Uniqueness of liftings in this case implies that $\pi\colon X \to Y$ has precisely the universal property of the fundamental category of X defined in §9.1: given $f\colon X \to C$ with C a category, a unique extension of f over $X \to Y$ exists. Thus, the small object argument applied to S gives another construction of the fundamental category (9.1) of an arbitrary simplicial set S.

13.28. *Exercise.* Prove that if $f\colon X \to Y$ is any inner anodyne map, then the induced functor $h(f)\colon hX \to hY$ between fundamental categories is an isomorphism. (Hint: use the universal property of fundamental categories to construct an inverse to $h(f)$.)

14. The small object argument

In this section we give the proof of (13.17), i.e., that given a fixed set $S = \{s_i\colon A_i \to B_i\}$ of maps of simplical sets, we can factor any map $f\colon X \to Y$ as $f = pj$ with $j \in \overline{S}$ and $p \in S^\boxslash$. For the reader: it may be helpful to first work through the special case where $Y = \Delta^0$ (the terminal object in simplicial sets).

14.1. **A factorization construction.** Given any map $f\colon X \to Y$, we first produce a factorization

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y, \qquad (Rf)(Lf) = f$$

as follows. Consider the set

$$[S,f] := \{\,(s_i,u,v) \;\mid\; s_i \in S,\ fu = vs_i\,\} = \left\{ \begin{array}{ccc} A_i & \xrightarrow{u} & X \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle f} \\ B_i & \xrightarrow[v]{} & Y \end{array} \right\}$$

of all commutative squares which have an arrow from S on the left-hand side, and f on the right-hand side. We define Ef, Lf, and Rf using the diagram

$$\begin{tikzcd}
\coprod\limits_{(s_i,u,v)} A_i \arrow[r, "(u)"] \arrow[d, "\coprod s_i"'] & X \arrow[d, "Lf"'] \arrow[dr, "f"] & \\
\coprod\limits_{(s_i,u,v)} B_i \arrow[r] \arrow[rr, bend right, "(v)"'] & Ef \arrow[r, "Rf"] & Y
\end{tikzcd}$$

where the the coproducts are indexed by the set $[S,f]$, and the square is a pushout. Note that $Lf \in \overline{S}$ by construction; however, we do not expect that Rf in $S^{\boxslash}$.

We can iterate the construction:

$$\begin{tikzcd}
X \arrow[d, equal] \arrow[dr, "Lf"] \arrow[drr, "L^2f"] \arrow[drrr, "L^3f"] \arrow[drrrrr, bend left, "L^\omega f"] & & & & & \\
X \arrow[r, "Lf"] \arrow[d, "f"'] & Ef \arrow[r, "LRf"] \arrow[dl, "Rf"] & E^2f \arrow[r, "LR^2f"] \arrow[dll, "R^2f"] & E^3f \arrow[r] \arrow[dlll, "R^3f"] & \cdots \arrow[r] & E^\omega f \arrow[dlllll, bend left, "R^\omega f"] \\
Y & & & & &
\end{tikzcd}$$

Here each triple $(E^\alpha f, L^\alpha f, R^\alpha f)$ is obtained by factoring the "R" map of the previous one, so that

$$E^{\alpha+1}f := E(R^\alpha f), \quad L^{\alpha+1}f := L(R^\alpha f)\circ(L^\alpha f), \quad R^{\alpha+1}f := R(R^\alpha f). \tag{14.2}$$

Taking direct limits gives a factorization $X \xrightarrow{L^\omega f} E^\omega f \xrightarrow{R^\omega f} Y$ of f, with $E^\omega f = \operatorname{colim}_{n\to\infty} E^n f$.

We can go even further, using the magic of transfinite induction, and define compatible factorizations $(E^\lambda f, L^\lambda f, R^\lambda f)$ for each ordinal[12] λ. For successor ordinals $\alpha+1$ use the prescription of (14.2), while for limit ordinals β take a direct limit $E^\beta f := \operatorname{colim}_{\alpha<\beta} E^\alpha f$ as in the construction of $E^\omega f$ above.

It is immediate that every $L^\alpha f \in \overline{S}$, because weak saturations are closed under transfinite composition. The maps $R^\alpha f$ are not generally contained in $S^{\boxslash}$, though they do satisfy a "partial lifting property": whenever $\alpha < \beta$ there exists *by construction* a dotted arrow making

$$\begin{tikzcd}
A_i \arrow[r, "u'"] \arrow[d, "s_i"'] & E^\alpha f \arrow[r] & E^{\alpha+1}f \arrow[r] & E^\beta f \arrow[d, "R^\beta f"] \\
B_i \arrow[rrr, "v"'] \arrow[urr, dotted] & & & Y
\end{tikzcd}$$

commute, for any u' and v making the square commute. Thus, we get a solution to a lifting problem (u,v) of s_i against $R^\beta f$ whenever the map $u\colon A_i \to E^\beta f$ on the top of a commutative square that we want a lift for can be factored through one of the maps $E^\alpha f \to E^\beta f$ with $\alpha < \beta$. This is so

[12]For a treatment of ordinals, see for instance the chapter on sets in [TS14].

exactly because $E^{\alpha+1}f$ was obtained from $E^\alpha f$ by "formally adjoining" a solution to *every* such lifting problem.

The "small object argument" amounts to the following.

Claim. There exists an ordinal κ such that for every domain A_i of a map in S, every map $A_i \to E^\kappa f$ factors through some $E^\alpha f \to E^\kappa f$ with $\alpha < \kappa$.

Given this, it follows from the "partial lifting property" that $S \boxslash R^\kappa f$, and so we obtain the desired factorization: $f = (R^\kappa f) \circ (L^\kappa f)$ with $L^\kappa f \in \overline{S}$ and $R^\kappa f \in S^{\boxslash}$.

It remains to prove the claim, which we will do by choosing κ to be a *regular cardinal* which is "bigger" than all the simplicial sets A_i.

14.3. **Regular cardinals.** The **cardinality** of a set X is the smallest ordinal λ such that there exists a bijection between X and λ; we write $|X|$ for this. Ordinals which can appear this way are called **cardinals**. For instance, the first infinite ordinal ω is the countable cardinal.

Note: the class of infinite cardinals is an unbounded subclass of the ordinals, so is well-ordered and can be put into bijective correspondence with ordinals. The symbol $\aleph_\alpha$ denotes the αth infinite cardinal, e.g., $\aleph_0 = \omega$.

Say that λ is a **regular cardinal**[13] if it is an infinite cardinal, and if for every set A of ordinals such that (i) $\alpha < \lambda$ for all $\alpha \in A$, and (ii) $|A| < \lambda$, we have that $\sup A < \lambda$. For instance, ω is a regular cardinal, since any finite collection of finite ordinals has a finite upper bound. Not every infinite cardinal is regular[14]; however, there exist arbitrarily large regular cardinals[15].

Every ordinal α defines a category, which is the poset of ordinals strictly less than α. Colimits of functors $Y\colon \kappa \to \mathrm{Set}$ with κ a regular cardinal have the following property: the map

$$\operatorname{colim}_{\alpha<\kappa} \operatorname{Hom}(X, Y_\alpha) \to \operatorname{Hom}(X, \operatorname{colim}_{\alpha<\kappa} Y_\alpha) \tag{14.4}$$

is a bijection whenever $|X| < \kappa$. This generalizes the familiar case of $\kappa = \omega$: any map of a finite set into the colimit of a countable sequence factors through a finite stage.

14.5. *Exercise.* Prove that (14.4) is a bijection when $|X| < \kappa$.

14.6. **Small simplicial sets.** Given a regular cardinal κ, we say that a simplicial set is **κ-small** if it is isomorphic to the colimit of some functor $F\colon C \to s\mathrm{Set}$, such that (i) $|\operatorname{ob} C|, |\operatorname{mor} C| < \kappa$, and (ii) each $F(c)$ is isomorphic to a standard simplex Δ^n. Morally, we are saying that a simplicial set is κ-small if it can be "presented" with fewer than κ generators and fewer than κ relations.

Given a functor $Y\colon \kappa \to s\mathrm{Set}$ and a κ-small simplicial set X, we have a bijection as in (14.4). (This is sometimes phrased as: *κ-small simplicial sets are κ-compact.*) Thus, to prove the claim about the small object argument, we simply choose a regular cardinal κ greater than $\sup\{|A_i|\}$.

14.7. *Example.* The standard simplices Δ^n, as well as any subcomplex such as the horns Λ^n_j, are ω-small: this is a consequence of (4.19). Thus, when we carry out the small object argument for $S = \mathrm{InnHorn}$, we can take $(E^\omega f, L^\omega f, R^\omega f)$ to be the desired factorization.

14.8. **Functoriality.** The construction $f \mapsto (X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y)$ is a functor $\mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([2], s\mathrm{Set})$, and it follows that so is $f \mapsto (X \xrightarrow{L^\alpha f} E^\alpha f \xrightarrow{R^\alpha f} Y)$ for any α. Because the choice of regular cardinal κ depends only on S, not on the map f, we see that the small object argument actually produces a *functorial factorization* of a map into a composite of an element of $\overline{S}$ with an element $S^{\boxslash}$. We will have use of this later. **Do we?**

[13] In the terminology of [TS14, §3.7], a regular cardinal is one which is equal to its own cofinality.

[14] For instance, $\aleph_\omega = \sup\{\, \aleph_k \mid k < \omega \,\}$ is not regular.

[15] For instance, every successor cardinal $\aleph_{\alpha+1}$ is regular.

15. Non-degenerate elements and the skeletal filtration

We have noted that monomorphisms of simplicial sets form a weakly saturated class. Here we identify an important set of maps called Cell, so that the weak saturation of Cell is precisely the class of monomorphisms. We do so by getting a very explicit handle on monomorphisms of simplicial sets. This will involve the notion of *degenerate* and *non-degenerate* elements of a simplicial set.

15.1. **Boundary of a standard simplex.** For each $n \geq 0$, we define

$$\partial\Delta^n := \bigcup_{k\in[n]} \Delta^{[n]\smallsetminus\{k\}} \subset \Delta^n,$$

the union of all codimension-one faces of the n-simplex. Equivalently,

$$(\partial\Delta^n)_k = \{\, f\colon [k] \to [n] \;\mid\; f([k]) \neq [n] \,\}.$$

We call $\partial\Delta^n$ the **boundary** of Δ^n. Note that $\partial\Delta^0 = \varnothing$ and $\partial\Delta^1 = \Delta^{\{0\}} \amalg \Delta^{\{1\}}$.

15.2. *Exercise.* Show that $\partial\Delta^n$ is the largest subcomplex of Δ^n which does not contain the "generator" $\langle 0 \dots n\rangle \in (\Delta^n)_n$. In other words, $\partial\Delta^n$ is the maximal *proper* subcomplex of Δ^n.

15.3. *Exercise.* Show that if C is a category, then the evident maps $\mathrm{Hom}(\Delta^n, C) \to \mathrm{Hom}(\partial\Delta^n, C)$ defined by restriction are isomorphisms when $n \geq 3$, but not necessarily when $n \leq 2$.

15.4. **Trivial fibrations and monomorphisms.** Let Cell be the set consisting of the inclusions $\partial\Delta^n \subset \Delta^n$ for $n \geq 0$. The resulting right complement is $\mathrm{TrivFib} := \mathrm{Cell}^{\boxslash}$, the class of **trivial fibrations** (also sometimes called **acyclic fibrations**). By the small object argument (13.17), we obtain a weak factorization system $(\overline{\mathrm{Cell}}, \mathrm{TrivFib})$.

Since the elements of Cell are monomorphisms, and the class of all monomorphisms is weakly saturated, we see that all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. We are going to prove the converse, i.e., we will show that $\overline{\mathrm{Cell}}$ is precisely equal to the class of monomorphisms.

15.5. **Degenerate and non-degenerate elements.** Recall $\Delta^{\mathrm{surj}}, \Delta^{\mathrm{inj}} \subset \Delta$, the subcategories of the category Δ of simplicial operators, consisting of all the objects and the *surjective* and *injective* order-preserving maps respectively, and that every operator factors uniquely as $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$, a surjection followed by an injection.

An element $a \in X_n$ is said to be **degenerate** if there exists a *non-injective* simplicial operator $f \in \Delta$ and an element b in X such that $a = bf$. In view of the factorization $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$, we see that a is degenerate if and only if there exists a *non-identity surjective* simplicial operator $f \in \Delta^{\mathrm{surj}}$ and an element b in X such that $a = b\sigma$.

Likewise, an element $a \in X_n$ is said to be **non-degenerate** if it is not degenerate, i.e., if $a = bf$ for some f in Δ and b in X we must have $f \in \Delta^{\mathrm{inj}}$. Equivalently, a is non-degenerate if $a = bf$ for some f in Δ^{surj} and b in X implies $f = \mathrm{id}$.

We write $X_n = X_n^{\mathrm{deg}} \amalg X_n^{\mathrm{nd}}$ for the decomposition of X_n into complementary subsets of degenerate and non-degenerate elements. Note that if $f\colon A \to X$ is a map of simplcial sets, then $f(A_n^{\mathrm{deg}}) \subseteq X_n^{\mathrm{deg}}$, while $f^{-1}(X_n^{\mathrm{nd}}) \subseteq A_n^{\mathrm{nd}}$. Note that neither X_n^{deg} nor X_n^{nd} assemble to give a subcomplex of X (unless X is empty).

15.6. **Proposition.** *If X is a simplicial set and $A \subseteq X$ is a subcomplex, then $A_n^{\mathrm{nd}} = X_n^{\mathrm{nd}} \cap A_n$ and $A_n^{\mathrm{deg}} = X_n^{\mathrm{deg}} \cap A_n$.*

Proof. The first statement is a consequence of the second, since subsets of degenerate and non-degenerate element are complementary. It is clear that $A_n^{\mathrm{deg}} \subseteq X_n^{\mathrm{deg}} \cap A_n$. Conversely, suppose $a \in X_n^{\mathrm{deg}} \cap A_n$, so $a \in A_n$ and $a = bg$ for some non-identity $g\colon [n] \to [k] \in \Delta^{\mathrm{surj}}$ and $b \in X_k$. Any surjection in Δ has a section (4.17), so there exists $s\colon [k] \to [n]$ such that $gs = 1_{[k]}$. Then $b = bgs = as \in A_k$, whence $a \in A_n^{\mathrm{deg}}$ as desired. $\square$

15.7. *Exercise* (easy). For any simplicial set X, we have $X_0^{\mathrm{deg}} = \varnothing$ and $X_0^{\mathrm{nd}} = X_0$, while X_1^{deg} is the image of $\langle 00\rangle^*\colon X_0 \to X_1$ (which is an injective function) and X_1^{nd} is its complement.

15.8. *Example.* Here are all elements in the standard 2-simplex up to dimension 3, with the non-degenerate ones indicated by a box.

$(\Delta^2)_0$	$(\Delta^2)_1$	$(\Delta^2)_2$	$(\Delta^2)_3$
$\boxed{\langle 0\rangle}$	$\langle 00\rangle$	$\langle 000\rangle$	$\langle 0000\rangle$
$\boxed{\langle 1\rangle}$	$\langle 11\rangle$	$\langle 111\rangle$	$\langle 1111\rangle$
$\boxed{\langle 2\rangle}$	$\langle 22\rangle$	$\langle 222\rangle$	$\langle 2222\rangle$
	$\boxed{\langle 01\rangle}$	$\langle 001\rangle$ $\langle 011\rangle$	$\langle 0001\rangle$ $\langle 0011\rangle$ $\langle 0111\rangle$
	$\boxed{\langle 02\rangle}$	$\langle 002\rangle$ $\langle 022\rangle$	$\langle 0002\rangle$ $\langle 0022\rangle$ $\langle 0222\rangle$
	$\boxed{\langle 12\rangle}$	$\langle 112\rangle$ $\langle 122\rangle$	$\langle 1112\rangle$ $\langle 1122\rangle$ $\langle 1222\rangle$
		$\boxed{\langle 012\rangle}$	$\langle 0012\rangle$ $\langle 0112\rangle$ $\langle 0122\rangle$

15.9. *Exercise.* Describe the degenerate and non-degenerate elements of all the standard n-simplices Δ^n.

15.10. *Exercise.* For every $n \geq 0$, let S^n be the pushout of the diagram $\Delta^n \leftarrowtail \partial\Delta^n \to \Delta^0$, where $\partial\Delta^n \rightarrowtail \Delta^n$ is the usual inclusion and $\partial\Delta^n \to \Delta^0$ is the unique map to the terminal object. Describe all degenerate and non-degenerate elements of S^n.

15.11. *Exercise.* Show that if C is an ordinary category, then an element $a \in N(C)_k$ of the nerve is non-degenerate if and only if it is represented by a composable sequence of *non-identity* maps $c_0 \to \cdots \to c_k$ in the category C.

15.12. *Exercise.* Let X be a simplicial set. Show that

$$X_n^{\mathrm{deg}} = \{\, af \;\mid\; a \in X_k,\ f\colon [n] \to [k],\ k < n \,\}.$$

The following exercises show that the subcomplexes of a simplicial set X can be completely characterized by the sets of non-degenerate elements of X that they contain.

15.13. *Exercise.* Let $X^{\mathrm{nd}} = \coprod_{n\geq 0} X_n^{\mathrm{nd}}$ be the set of non-degenerate elements of X. For $x, y \in X^{\mathrm{nd}}$ write $y \leq x$ if there exists $f \in \Delta^{\mathrm{inj}}$ such that $y = xf$. Show that "$\leq$" is a partial order on the set X^{nd}; it is called the *face relation*.

15.14. *Exercise.* Show that if $xf = yg$ for some $x, y \in X^{\mathrm{nd}}$, $f \in \Delta$ and $g \in \Delta^{\mathrm{surj}}$, then $y \leq x$.

15.15. *Exercise.* Let $S \subseteq X^{\mathrm{nd}}$ be a subset of non-degenerate elements which is closed downward under "$\leq$", i.e., $y \leq x$ and $x \in S$ implies $y \in S$. Show that there exists a unique subcomplex $A \subseteq X$ such that $A^{\mathrm{nd}} = S$. (Hint: the elements of A are of the form xg where $x \in S$ and $g \in \Delta^{\mathrm{surj}}$.)

15.16. **Simplicial sets are canonically free with respect to surjective operators.** The key observation is that degenerate elements in a simplicial set are precisely determined by knowledge of the non-degenerate elements.

15.17. **Proposition** (Eilenberg-Zilber lemma)**.** *Let a be an element of X. Then there exists a* unique *pair (b, σ) consisting of a* non-degenerate *element b and a map σ in Δ^{surj} such that $a = b\sigma$.*

Proof. [GZ67, §II.3]. First note that for degenerate a such a pair (b, σ) exists by definition, while for nondegenerate a we can take the pair (a, id).

Given $\sigma\colon [n] \to [m]$, let $\Gamma(\sigma) = \left\{\, \delta\colon [m] \to [n] \;\middle|\; \sigma\delta = \mathrm{id}_{[m]} \,\right\}$ denote the set of sections of σ. The sets $\Gamma(\sigma)$ is non-empty when $\sigma \in \Delta^{\mathrm{surj}}$ (4.17). We note the following elementary observation, whose proof is left for the reader:

$$\text{If } \sigma, \sigma' \in \Delta^{\mathrm{surj}} \text{ are such that } \Gamma(\sigma) = \Gamma(\sigma'), \text{ then } \sigma = \sigma'.$$

Let $a \in X_n$ be such that $a = b_i\sigma_i$ for $b_i \in X^{\mathrm{nd}}_{m_i}$, $\sigma_i \in \Delta^{\mathrm{surj}}([n],[m_i])$, for $i=1,2$. We want to show that $m_1 = m_2$, $b_1 = b_2$, and $\sigma_1 = \sigma_2$.

Pick any $\delta_1 \in \Gamma(\sigma_1)$ and $\delta_2 \in \Gamma(\sigma_2)$. Then we have

$$b_1 = b_1\sigma_1\delta_1 = a\delta_1 = b_2\sigma_2\delta_1, \qquad b_2 = b_2\sigma_2\delta_2 = a\delta_2 = b_1\sigma_1\delta_2,$$

so b_1 and b_2 are related by the simplicial operators $\sigma_2\delta_1$ and $\sigma_1\delta_2$. Since b_1 and b_2 are both non-degenerate, $\sigma_2\delta_1 \colon [m_1] \to [m_2]$ and $\sigma_1\delta_2 \colon [m_2] \to [m_1]$ must be injective. This implies $m_1 = m_2$, and since the only order-preserving injective map $[m] \to [m]$ is the identity map, we must have $\sigma_2\delta_1 = \mathrm{id} = \sigma_1\delta_2$, from which it follows that $b_1 = b_2$. This also shows that $\delta_1 \in \Gamma(\sigma_2)$ and $\delta_2 \in \Gamma(\sigma_1)$. Since δ_1 and δ_2 were arbitrarily chosen sections, we have shown $\Gamma(\sigma_1) = \Gamma(\sigma_2)$, and therefore $\sigma_1 = \sigma_2$. ☐

We can reinterpret the Eilenberg-Zilber lemma as follows.

15.18. **Corollary.** *For any simplicial set X, the evident maps*

$$\coprod_{j\geq 0} X^{\mathrm{nd}}_j \times \mathrm{Hom}_{\Delta^{\mathrm{surj}}}([n],[j]) \to X_n$$

defined by $(j,x,\sigma) \mapsto x\sigma$ are bijections. Furthermore, these bijections are natural with respect to surjective simplicial operators $[n'] \to [n]$.

Proof. The bijection is a restatement of (15.17). For the second statement, note that if $\tau\colon [n'] \to [n]$ is a surjective simplicial operator, then $(k,x,\sigma\tau) \mapsto (x\sigma)\tau$. ☐

Another way to say this: the restricted functor $X|(\Delta^{\mathrm{surj}})^{\mathrm{op}} \colon (\Delta^{\mathrm{surj}})^{\mathrm{op}} \to \mathrm{Set}$ is *canonically isomorphic* to a coproduct of representable functors $\mathrm{Hom}_{\Delta^{\mathrm{surj}}}(-,[k])$ indexed by the nondegenerate simplicies of X. Or more simply: simplicial sets are *canonically free* with respect to surjective simplicial operators.

15.19. *Remark.* A simplicial set can be recovered up to isomorphism if you only know (i) its sets of non-degenerate elements, and (ii) the faces of the non-degenerate elements. The proposition we proved above tells how to reconstruct the degenerate elements; simplicial operators on degenerate elements are computed using the fact that any simplicial operator factors into a surjection followed by an injection. **M 4 Feb 2019**

Warning. The faces of a non-degenerate element can be degenerate; this happens for instance for S^n in (15.10) when $n \geq 2$. If X is such that all faces of non-degenerate elements are also non-degenerate, then we get a functor $X^{\mathrm{nd}} \colon (\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$, and the full simplicial set X can be recovered from X^{nd}. For instance, this is so for the standard simplices Δ^n, as well as any subcomplexes of such. Functors $(\Delta^{\mathrm{inj}})^{\mathrm{op}} \to \mathrm{Set}$ are the combinatorial data behind the notion of a Δ-*complex*, as seen in Hatcher's textbook on algebraic topology [Hat02, Ch. 2.1].

The following exercises give a different point of view of this principle.

15.20. *Exercise.* Fix an object $[n]$ in Δ, and consider the category $\Delta^{\mathrm{surj}}_{[n]/}$, which has

- **objects** the surjective morphisms $\sigma\colon [n] \to [k]$ in Δ, and
- **morphisms** commutative triangles in Δ of the form

$$\begin{tikzcd} & {[k]} \arrow[d, "\tau"] \\ {[n]} \arrow[ur, "\sigma"] \arrow[r, "\sigma'"'] & {[k']} \end{tikzcd}$$

Show that the category $\Delta^{\mathrm{surj}}_{[n]/}$ is *isomorphic* to the poset $\mathcal{P}(\underline{n})$ of subsets of the set $\underline{n} = \{1,\ldots,n\}$. In particular, $\Delta^{\mathrm{surj}}_{[n]/}$ is a lattice (i.e., has finite products and coproducts, called *meets* and *joins* in this context).

15.21. *Exercise.* Let X be a simplicial set. Given $n \geq 0$ and $\sigma\colon [n] \to [k]$ in Δ^{surj}, let $X_n^\sigma := \sigma^*(X_k)$, the image of the operator σ^* in X_n. Show that $X_n^{\sigma\vee\sigma'} = X_n^\sigma \cap X_n^{\sigma'}$, where $\sigma \vee \sigma'$ is join in the lattice $\Delta^{\mathrm{surj}}_{[n]/}$. Conclude that for each $x \in X_n$ there exists a maximal σ such that $x \in X_n^\sigma$.

15.22. **Skeleta.** Given a simplicial set X, the **k-skeleton** $\operatorname{Sk}_k X \subseteq X$ is the subcomplex with n-dimensional elements.

$$(\operatorname{Sk}_k X)_n = \bigcup_{0\leq j\leq k} \{\, yf \;\mid\; y \in X_j,\ f\colon [n] \to [j] \in \Delta \,\}.$$

It is immediate that this defines a subcomplex of X, which is in fact the smallest subcomplex containing all elements of dimensions $\leq k$. Note that $\operatorname{Sk}_{k-1} X \subseteq \operatorname{Sk}_k X$ and $X = \bigcup_k \operatorname{Sk}_k X$, and that a map $X \to Y$ of simplicial sets restricts to a map $\operatorname{Sk}_k X \to \operatorname{Sk}_k Y$. The skeleta constructions define functors $\operatorname{Sk}_k\colon s\mathrm{Set} \to s\mathrm{Set}$.

In view of (15.17) and (15.18), we see that

$$(\operatorname{Sk}_k X)_n \approx \coprod_{0\leq j\leq k} X_j^{\mathrm{nd}} \times \operatorname{Hom}_{\Delta^{\mathrm{surj}}}([n],[j]).$$

The complement of the set of elements of $\operatorname{Sk}_{k-1} X$ in $\operatorname{Sk}_k X$ consists precisely of the *nondegenerate* k-dimensional elements of X together with their associated degenerate elements (in dimensions $> k$).

15.23. *Example.* The $(n-1)$-skeleton of the stardand n-simplex is precisely what we have called its boundary: $\operatorname{Sk}_{n-1} \Delta^n = \partial\Delta^n$. The only elements of Δ^n not contained in its boundary are the generator $\iota = \langle 0 \dots n\rangle \in (\Delta_n)_n$ together with the degenate elements associated to it.

15.24. **Proposition.** *The evident square*

$$\begin{tikzcd}
\coprod_{a\in X_k^{\mathrm{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & \operatorname{Sk}_{k-1} X \arrow[d, tail] \\
\coprod_{a\in X_k^{\mathrm{nd}}} \Delta^n \arrow[r] & \operatorname{Sk}_k X
\end{tikzcd}$$

is a pushout of simplicial sets. More generally, for any subcomplex $A \subseteq X$, the evident square

$$\begin{tikzcd}
\coprod_{a\in X_k^{\mathrm{nd}}\smallsetminus A_k^{\mathrm{nd}}} \partial\Delta^k \arrow[r] \arrow[d, tail] & A \cup \operatorname{Sk}_{k-1} X \arrow[d, tail] \\
\coprod_{a\in X_k^{\mathrm{nd}}\smallsetminus A_k^{\mathrm{nd}}} \Delta^k \arrow[r] & A \cup \operatorname{Sk}_k X
\end{tikzcd}$$

is a pushout.

Proof. In each of the above squares, the complements of the vertical inclusions coincide precisely. In particular, the complement of the inclusion $(A \cup \operatorname{Sk}_{k-1} X)_n \subseteq (A \cup \operatorname{Sk}_k X)_n$ is in bijective correspondence with $(X_k^{\mathrm{nd}} \smallsetminus A_k^{\mathrm{nd}}) \times \operatorname{Hom}_{\Delta^{\mathrm{surj}}}([n],[k])$, and thus the square is a pushout (15.25). $\square$

In the proof, we used the following fact which generalizes (4.14), which is worth recording.

15.25. **Lemma.** *If*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d, "i"'] & X \arrow[d, tail, "j"] \\
Y' \arrow[r, "f"'] & Y
\end{tikzcd}$$

is a pullback of simplicial sets such that (i) j is a monomorphism, and (ii) f induces in each degree n a bijection $Y'_n \smallsetminus i(X'_n) \xrightarrow{\sim} Y_n \smallsetminus j(X_n)$, then the square is a pushout square.

Proof. Verify the analogous statement for a pullback square of sets. □

15.26. **Corollary.** $\overline{\mathrm{Cell}}$ *is precisely the class of monomorphisms.*

Proof. We know all elements of $\overline{\mathrm{Cell}}$ are monomorphisms. Any monomorphism is isomorphic to an inclusion $A \subseteq X$ of a subcomplex, so we only need show that such inclusions are contained in $\overline{\mathrm{Cell}}$. Since $X \approx \mathrm{colim}_k\, A \cup \mathrm{Sk}_k\, X$, (15.24) exhibits the inclusion as a countable composite of pushouts along coproducts of elements of Cell. □

15.27. **Geometric realization.** Recall the singular complex functor Sing: Top → sSet (8.5). This functor has a left adjoint $\|-\| : s\mathrm{Set} \to \mathrm{Top}$, called **geometric realization**, constructed explicitly by

$$\|X\| := \mathrm{Cok}\left[\coprod_{f\colon [m]\to[n]} X_n \times \Delta^m_{\mathrm{top}} \rightrightarrows \coprod_{[p]} X_p \times \Delta^p_{\mathrm{top}}\right]; \tag{15.28}$$

that is, take a collection of topological simplices indexed by elements of X, and make identifactions according to the simplicial operators in X. (Here the symbol "Cok" represeents taking a "coequalizer", i.e., the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$.)

15.29. *Exercise.* Describe the two unlabelled maps in (15.28). Then show that $\|-\|$ is in fact left adjoint to Sing.

Because geometric realization is a left adjoint, it commutes with colimits. It is straightforward to check that $\|\Delta^n\| \approx \Delta^n_{\mathrm{top}}$, and that $\|\partial\Delta^n\| \approx \partial\Delta^n_{\mathrm{top}}$. Applying this to the skeletal filtration, we discover that there are pushouts

$$\begin{tikzcd} \coprod_{a\in X^{\mathrm{nd}}_k} \partial\Delta^k_{\mathrm{top}} \arrow[r] \arrow[d, tail] & \|\mathrm{Sk}_{k-1}\, X\| \arrow[d, tail] \\ \coprod_{a\in X^{\mathrm{nd}}_k} \Delta^k_{\mathrm{top}} \arrow[r] & \|\mathrm{Sk}_k\, X\| \end{tikzcd}$$

of spaces, and that $\|X\| = \bigcup \|\mathrm{Sk}_k\, X\|$ with the direct limit topology. Thus, $\|X\|$ is presented to us as a *CW-complex*, whose cells are in an evident bijective correspondence with the set of non-degenerate elements of X.

16. Pushout-product and pullback-power

We are going to prove several "enriched" versions of lifting properties associated to inner anodyne maps and inner fibrations. As a consequence we'll be able to prove that function complexes of quasicategories are themselves quasicategories.

16.1. **Definition of pushout-product and pullback-hom.** Given maps $f\colon A \to B$, $g\colon K \to L$ and $h\colon X \to Y$ of simplicial sets, we define new maps $f\Box g$ and $g^{\Box h}$ called the **pushout-product**[16] and the **pullback-hom**[17][18]. The pushout-product $f\Box g\colon (A \times L) \amalg_{A\times K} (B \times K) \to B \times L$ is the

[16] This is sometimes called the **box-product**. Some also call it the **Leibniz-product**, as its form is that of the Leibniz rule for boundary of a product space: $\partial(X \times Y) = (\partial X \times Y) \cup_{\partial X\times\partial Y} (X \times \partial Y)$ (which is itself reminiscent of the original Leibniz rule $D(fg) = (Df)g + f(Dg)$ of calculus).

[17] Sometimes called the **box-power** or **pullback-power**. A common alternate notation is $g \pitchfork h$. This may also be called the **Leibniz-hom**, though I don't know what rule of calculus it is related to.

[18] This notation for pullback-hom is kinda awkward, and I'd like to change it. However, a new notation ought to admit compatible variants to describe the "pullback-slice" and "alternate pullback-slice" constructions which appear later on. I don't see a good way to do this.

unique map fitting in the diagram

$$\begin{tikzcd}
A \times K \arrow[r, "\mathrm{id} \times g"] \arrow[d, "f \times \mathrm{id}"'] & A \times L \arrow[d] \arrow[ddr, bend left, "f \times \mathrm{id}"] & \\
B \times K \arrow[r] \arrow[drr, bend right, "\mathrm{id} \times g"'] & (A \times L) \amalg_{A \times K} (B \times K) \arrow[dr, "f \square g"] & \\
 & & B \times L
\end{tikzcd}$$

while the pullback-hom $h^{\square g} \colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$ is the unique map fitting in the diagram

$$\begin{tikzcd}
\operatorname{Map}(L, X) \arrow[dr, "h^{\square g}"] \arrow[drr, bend left, "\operatorname{Map}(g,X)"] \arrow[ddr, bend right, "\operatorname{Map}(L,h)"'] & & \\
 & \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y) \arrow[r] \arrow[d] & \operatorname{Map}(K, X) \arrow[d, "\operatorname{Map}(\mathrm{id},h)"] \\
 & \operatorname{Map}(L, Y) \arrow[r, "\operatorname{Map}(f,\mathrm{id})"'] & \operatorname{Map}(K, Y)
\end{tikzcd}$$

16.2. *Remark.* Typically we form $f \square g$ when f and g are monomorphisms, in which case $f \square g$ is also a monomorphism. In this case, the elements $(b, \ell) \in B \times L$ which are *not* in the image of $f \square g$ are exactly those such that $b \in B \smallsetminus A$ *and* $\ell \in L \smallsetminus K$.

16.3. *Remark* (Important!). On vertices, the pullback-hom $h^{\square g}$ is just the "usual" map $\operatorname{Hom}(L, X) \to \operatorname{Hom}(K, X) \times_{\operatorname{Hom}(K,Y)} \operatorname{Hom}(L, Y)$ sending $s \mapsto (sg, hs)$. Thus, $h^{\square g}$ is surjective on vertices if and only if $g \boxslash h$.

We think of the pullback-hom as encoding an "enriched" version of the lifting problem for (g, h). Thus, the target of $h^{\square g}$ is an object which "parameterizes familes" of commutative squares involving g and h. Similarly, the source of $h^{\square g}$ "parameterizes families" of such commutative squares together with lifts.

16.4. *Remark.* The pushout-product construction is symmetric: $f \square g$ is isomorphic to $g \square f$ in the arrow category $\operatorname{Fun}([1], s\mathrm{Set})$. Ultimately, this is because product is symmetric. The pullback-hom construction however is not symmetric.

The product/function complex adjunction gives rise to the following relationship between lifting problems.

16.5. **Proposition.** *We have that* $(f \square g) \boxslash h$ *if and only if* $f \boxslash (h^{\square g})$.

Proof. Compare the two lifting problems using the product/map adjunction.

$$\begin{tikzcd}
(A \times L) \amalg_{A \times K} (B \times K) \arrow[r, "{(u,v)}"] \arrow[d, "f \square g"'] & X \arrow[d, "h"] \\
B \times L \arrow[r, "w"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}
\quad \Longleftrightarrow \quad
\begin{tikzcd}
A \arrow[r, "\widetilde{u}"] \arrow[d, "f"'] & \operatorname{Map}(L, X) \arrow[d, "h^{\square g}"] \\
B \arrow[r, "{(\widetilde{v},\widetilde{w})}"'] \arrow[ur, dotted, "\widetilde{s}"] & \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)
\end{tikzcd}$$

On the left-hand side are maps

$$u \colon A \times L \to X, \qquad v \colon B \times K \to X, \qquad w \colon B \times L \to Y, \qquad s \colon B \times L \to X,$$

while on the right-hand side are maps

$$\widetilde{u}\colon A \to \mathrm{Map}(L,X), \qquad \widetilde{v}\colon B \to \mathrm{Map}(K,X), \qquad \widetilde{w}\colon B \to \mathrm{Map}(L,Y), \qquad \widetilde{s}\colon B \to \mathrm{Map}(L,X).$$

The data of (u,v,w) giving a commutative square as on the left corresponds bijectively to data $(\widetilde{u},\widetilde{v},\widetilde{w})$ giving a commutative square as on the right. Similarly, lifts s correspond bijectively to lifts $\widetilde{s}$. $\square$

It is important to note the special cases where one or more of $A=\varnothing$, $K=\varnothing$, or $Y=*$ hold. For instance, if $K=\varnothing$ and $Y=*$, the proposition implies

$$(A\times L \xrightarrow{f\times L} B\times L) \boxslash (X\to *) \quad \text{iff} \quad (A\xrightarrow{f} B)\boxslash (\mathrm{Map}(L,X)\to *).$$

This is the kind of case we are interested in for proving that $\mathrm{Map}(K,C)$ is a quasicategory whenever C is. The more general statement of the proposition is a kind of "relative" version of the thing we want; it is especially handy for carrying out inductive arguments.

16.6. *Exercise* (if you like monoidal categories). Let $\mathcal{C} := \mathrm{Fun}([1], s\mathrm{Set})$, the "arrow category" of simplicial sets. Show that $\square\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ defines a symmetric monoidal structure on $\mathcal{C}$, with unit object $(\varnothing\subset\Delta^0)$. Furthermore, show that this is a *closed* monoidal structure, with $-\square g$ left adjoint to $(-)^{\square g}\colon \mathcal{C}\to\mathcal{C}$.

16.7. **Inner anodyne maps and pushout-products.** The key fact we want to prove is the following.

16.8. **Proposition.** *We have that* $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$*, i.e., that* $i\square j$ *is inner anodyne whenever* i *is inner anodyne and* j *is a monomorphism.*

To set up the proof we need the following. **W 6 Feb 2019**

16.9. **Proposition.** *For any sets of maps* S *and* T*, we have* $\overline{S}\square\overline{T}\subseteq\overline{S\square T}$.

Proof. Let $\mathcal{F} = (S\square T)^{\boxslash}$. From the small object argument we have that $\overline{S\square T} = {}^{\boxslash}\mathcal{F}$ (13.18), so we will show $(\overline{S}\square\overline{T})\boxslash\mathcal{F}$. First we show that $(\overline{S}\square T)\boxslash\mathcal{F}$. Consider

$$\begin{aligned}\mathcal{A} &:= \{\, a \mid (a\square T)\boxslash\mathcal{F}\,\}\\ &\approx \big\{\, a \;\big|\; a\boxslash(\mathcal{F}^{\square T})\,\big\}\end{aligned}$$

by correspondence between lifting problems for pushout-products and pullback-homs (16.5). Thus $\mathcal{A}$ is a left complement, and so is weakly saturated. Since $S\subseteq\mathcal{A}$ then $\overline{S}\subseteq\mathcal{A}$, i.e., $(\overline{S}\square T)\boxslash\mathcal{F}$. The same idea applied to

$$\mathcal{B} := \big\{\, b \;\big|\; (\overline{S}\square b)\boxslash\mathcal{F}\,\big\} \approx \Big\{\, b \;\Big|\; b\boxslash(\mathcal{F}^{\square\overline{S}})\,\Big\},$$

gives $\overline{T}\subseteq\mathcal{B}$, whence $(\overline{S}\square\overline{T})\boxslash\mathcal{F}$. $\square$

16.10. **Lemma.** *We have* $\mathrm{InnHorn}\square\mathrm{Cell}\subseteq\overline{\mathrm{InnHorn}}$.

Proof. This is a calculation, given in [Joy08a, App. H], and presented in the appendix (62.3). $\square$

Proof of (16.8)*.* We have that

$$\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}\square\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}}.$$

The first inclusion is (16.9), while the second is an immediate conseqence of $\mathrm{InnHorn}\square\mathrm{Cell}\subseteq\overline{\mathrm{InnHorn}}$ (16.10). $\square$

Let's carry out a proof of (16.10) explicitly in one case, by showing that $(\Lambda^2_1 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ is inner anodyne. This map is the inclusion

$$(\Lambda^2_1 \times \Delta^1) \cup_{\Lambda^2_1 \times \partial\Delta^1} (\Delta^2 \times \partial\Delta^1) \subset \Delta^2 \times \Delta^1,$$

whose target is a "prism", and whose source is a "trough". To show this is in $\overline{\text{InnHorn}}$, we'll give an explicit procedure for constructing the prism from the trough by succesively attaching simplices along inner horns.

Note that $\Delta^2 \times \Delta^1 = N([2] \times [1])$, so we are working inside the nerve of a poset, whose elements (objects) are "ij" with $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$. Here is a picture of the trough, showing all the non-degenerate simplicies.

$$\begin{tikzcd}
01 \arrow[r] \arrow[rr, bend left] & 11 \arrow[r] & 21 \\
00 \arrow[u] \arrow[r] \arrow[ur] \arrow[rr, bend right] & 10 \arrow[u] \arrow[r] \arrow[ur] & 20 \arrow[u]
\end{tikzcd}$$

The complement of this in the prism consists of three non-degenerate 3-dimensional elements, five non-degenerate 2-dimensional elements (two of which form the "lid" of the trough, while the other three are in the interior of the prism), and one non-degenerate edge element (separating the two 2-dimensional elements which form the lid).

The following chart lists all non-degenerate elements in the complement of the trough, along with their codimension one faces (in order). The "$\surd$" marks elements which are contained in the trough.

$\langle 00, 21\rangle$	$\langle 00, 20, 21\rangle$	$\langle 00, 01, 21\rangle$	$\langle 00, 10, 21\rangle$	$\langle 00, 11, 21\rangle$	$\langle 00, 10, 20, 21\rangle$	$\langle 00, 10, 11, 21\rangle$	$\langle 00, 01, 11, 21\rangle$
$\surd\langle 21\rangle$	$\surd\langle 20, 21\rangle$	$\surd\langle 01, 21\rangle$	$\surd\langle 10, 21\rangle$	$\surd\langle 11, 21\rangle$	$\surd\langle 10, 20, 21\rangle$	$\surd\langle 10, 11, 21\rangle$	$\surd\langle 01, 11, 21\rangle$
$\surd\langle 10\rangle$	$\langle 00, 21\rangle$	$\langle 00, 21\rangle$	$\langle 00, 21\rangle$	$\langle 00, 21\rangle$	$\langle 00, 20, 21\rangle$	$\langle 00, 11, 21\rangle$	$\langle 00, 11, 21\rangle$
	$\surd\langle 00, 20\rangle$	$\surd\langle 00, 01\rangle$	$\surd\langle 00, 10\rangle$	$\surd\langle 00, 11\rangle$	$\langle 00, 10, 21\rangle$	$\langle 00, 10, 21\rangle$	$\langle 00, 01, 21\rangle$
					$\surd\langle 00, 10, 20\rangle$	$\surd\langle 00, 10, 11\rangle$	$\surd\langle 00, 01, 11\rangle$

Note that the elements $\langle 00, 21\rangle$, $\langle 00, 10, 21\rangle$, and $\langle 00, 11, 21\rangle$ of the complement appear multiple times as faces. We can attach simplices to the domain in the following order:

$$①\langle 00, 10, 21\rangle, \quad ②\langle 00, 10, 20, 21\rangle, \quad ③\langle 00, 10, 11, 21\rangle, \quad ④\langle 00, 01, 11, 21\rangle.$$

In each case, the intersection of the simplex with (domain+previously attached simplices) is an inner horn. This directly exhibits $(\Lambda^2_1 \subset \Delta^2)\square(\partial\Delta^1 \subset \Delta^1)$ as an inner anodyne map.

17. Function complexes of quasicategories are quasicategories

17.1. **Enriched lifting properties.** We record the immediate consequences of $\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.8).

17.2. **Proposition.**

(1) *If $i\colon A \to B$ is inner anodyne and $j\colon K \to L$ a monomorphism, then*

$$i\square j\colon (A \times L) \cup_{A\times K} (B \times K) \to B \times L$$

is inner anodyne.

(2) *If $j\colon K \to L$ is a monomorphism and $p\colon X \to Y$ is an inner fibration, then*

$$p^{\square j}\colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$$

is an inner fibration.

(3) *If $i\colon A \to B$ is inner anodyne and $p\colon X \to Y$ is an inner fibration, then*

$$p^{\square i}\colon \operatorname{Map}(B, X) \to \operatorname{Map}(A, X) \times_{\operatorname{Map}(A,Y)} \operatorname{Map}(B, Y)$$

is a trivial fibration.

These can be summarized as

$$\overline{\text{InnHorn}}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}, \qquad \text{InnFib}^{\Box\overline{\text{Cell}}} \subseteq \text{InnFib}, \qquad \text{InnFib}^{\Box\overline{\text{InnHorn}}} \subseteq \text{TrivFib}.$$

Statement (1) is just restating (16.8). The other two statements follow from (1) using the correspondence between lifting problems for pushout-products and pullback-homs (16.5), together with the facts that $\text{InnFib} = \text{InnHorn}^{\boxslash}$ and $\text{TrivFib} = \text{Cell}^{\boxslash}$. For instance, (2) follows from the observation that $i \boxslash p^{\Box j}$ iff $(i\Box j) \boxslash p$, and that $i \in \overline{\text{InnHorn}}$ and $j \in \overline{\text{Cell}}$ imply $i\Box j \in \overline{\text{InnHorn}}$. Likewise (3) follows a similar argument using that $j \boxslash p^{\Box i}$ iff $(i\Box j) \boxslash p$.

We are going to use these consequences all the time. To announce that I am using any of these, I will simply assert "$\overline{\text{InnHorn}}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" without other explanation; sometimes, to indicate an application of statements (2) and (3), I will call it "enriched lifting". The following gives the most general statement, of which (16.8) amounts to the special case of $S = U = \text{InnHorn}$ and $T = \text{Cell}$.

17.3. **Proposition.** *Let S, T, and U be sets of morphisms in $s\text{Set}$. Write $\overline{S}$, $\overline{T}$, and $\overline{U}$ for the weak saturations of these sets, and let $S\text{Fib} := S^{\boxslash}$, $T\text{Fib} := T^{\boxslash}$, and $U\text{Fib} := U^{\boxslash}$ denote the respective right complements. If $S\Box T \subseteq \overline{U}$, then*

$$\overline{S}\Box\overline{T} \subseteq \overline{U}, \qquad U\text{Fib}^{\Box\overline{T}} \subseteq S\text{Fib}, \qquad U\text{Fib}^{\Box\overline{S}} \subseteq T\text{Fib}.$$

Proof. Exercise using (16.5). □

There are many useful special cases of (17.2), obtained by taking the domain of a monomorphism to be empty, or the target of an inner fibration to be terminal.

- If $i\colon A \to B$ is inner anodyne, so is $i \times \text{id}_L\colon A \times L \to B \times L$.
- If $p\colon X \to Y$ is an inner fibration, then so is $\text{Map}(L,p)\colon \text{Map}(L,X) \to \text{Map}(L,Y)$.
- If $j\colon K \to L$ is a monomorphism and C a quasicategory, then $\text{Map}(j,C)\colon \text{Map}(L,C) \to \text{Map}(K,C)$ is an inner fibration.
- If $i\colon A \to B$ is inner anodyne and C a quasicategory, then $\text{Map}(i,C)\colon \text{Map}(B,C) \to \text{Map}(A,C)$ is a trivial fibration.
- If C is a quasicategory, so is $\text{Map}(L,C)$. Thus we have proved (B).

Let's spell out the proof of (B) in a little more detail. Because $\overline{\text{InnHorn}}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$, we have (16.8) that

$$(\Lambda^n_j \subset \Delta^n)\Box(\varnothing \subseteq K) = (\Lambda^n_j \times K \to \Delta^n \times K)$$

is inner anodyne for any K and $0 < j < n$. Thus, for any diagram

$$\begin{tikzcd} \Lambda^n_j \times K \arrow[r] \arrow[d] & C \\ \Delta^n \times K \arrow[ur, dotted] & \end{tikzcd}$$

with C a quasicategory, a dotted arrow exists. By adjunction, this is the same as saying we can extend $\Lambda^n_j \to \text{Map}(K,C)$ along $\Lambda^n_j \subset \Delta^n$. That is, we have proved that $\text{Map}(K,C)$ is a quasicategory.

17.4. *Remark.* Most weakly saturated classes $\overline{S}$ that we will explictly discuss in these notes will have the property that $S\Box\text{Cell} \subseteq \overline{S}$, and thus analogues of the above remarks will hold for such classes.

17.5. *Exercise* (Important). Show that $\overline{\text{Cell}}\Box\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. (Hint: (15.26).) State the analogue of (17.2) associated to this inclusion.

17.6. *Exercise.* As a consequence of the previous exercise, show that every trivial fibration admits a section.

17.7. **Composition functors.** We can use the above theory to construct "composition functors". If C is an ordinary category, the operation of composing a sequence of n maps can be upgraded to a functor

$$\mathrm{Fun}([1],C)\times_C \mathrm{Fun}([1],C)\to \mathrm{Fun}([1],C)$$

which on *objects* describes composition of a sequence of maps. The source of this functor is the evident inverse limit in Cat of

$$\mathrm{Fun}([1],C)\xrightarrow{\langle 1\rangle^*}\mathrm{Fun}([0],C)\xleftarrow{\langle 0\rangle^*}\mathrm{Fun}([1],C),$$

which is isomorphic to $\mathrm{Fun}(I^2,C)$.

We can generalize this to quasicategories, with the proviso that the composition functor we produce is not uniquely determined. We use the following observation: any trivial fibration admits a section (17.6).

Let C be a quasicategory. Then map $r\colon \mathrm{Fun}(\Delta^2,C)\to \mathrm{Fun}(I^2,C)$ induced by restriction along $I^2\subseteq\Delta^2$ is a trivial fibration by (17.2), since $I^2\subset\Delta^2$ is an inner-horn inclusion. Therefore r admits a section s, so we get a diagram

$$\begin{tikzcd} \mathrm{Fun}(I^2,C) \arrow[r, bend left, "s"] & \mathrm{Fun}(\Delta^2,C) \arrow[l, "r"] \arrow[r, "r'"] & \mathrm{Fun}(\Delta^{\{0,2\}},C) \end{tikzcd}$$

where r' is restriction along $\Delta^{\{0,2\}}\subset\Delta^2$. The composite $r's$ can be thought of as a kind of "composition" functor. It is not unique, since s isn't, but we'll see (20.12) that this is ok: all functors constructed this way are "naturally isomorphic" to each other.

The same argument gives rise to a (non-unique) "n-fold composition functor"

$$\mathrm{Fun}([1],C)\times_C\cdots\times_C \mathrm{Fun}([1],C)\to \mathrm{Fun}([1],C),$$

whose source is isomorphic to $\mathrm{Fun}(I^n,C)$, using that spine inclusions are inner anodyne (12.11).

17.8. **A useful variant.** The proof of (16.8) actually proves something a little stronger.

17.9. **Proposition** ([Joy08a, §2.3.1], [Lur09, §2.3.2])**.** *We have that* $\overline{\{\Lambda^2_1\subset\Delta^2\}\square\mathrm{Cell}}=\overline{\mathrm{InnHorn}}$.

Proof. We give a proof in the appendix (62.3). □

A consequence of this is another characterization of quasicategories.

17.10. **Corollary.** *A simplicial set C is a quasicategory if and only if $f\colon \mathrm{Map}(\Delta^2,C)\to\mathrm{Map}(\Lambda^2_1,C)$ is a trivial fibration.*

Proof. First notice that $(\partial\Delta^k\subset\Delta^k)\boxslash f$ for all $k\geq 0$ iff $(\partial\Delta^k\subset\Delta^k)\square(\Lambda^2_1\subset\Delta^2)\boxslash(C\to *)$ for all $k\geq 0$, since $f=(C\to *)^{\square\{\Lambda^2_1\subset\Delta^2\}}$. Therefore $f\in\mathrm{TrivFib}=\mathrm{InnHorn}^{\boxslash}$ if and only if $(C\to *)\in(\mathrm{Cell}\square\{\Lambda^2_1\subset\Delta^2\})^{\boxslash}$. The conclusion immediately follows using (17.9). □

18. Natural isomorphisms

18.1. **Natural isomorphisms of functors.** Let C and D be quasicategories. Recall that a natural transformation between functors $f_0,f_1\colon C\to D$ is defined to be a morphism $\alpha\colon f_0\to f_1$ in the functor quasicategory $\mathrm{Fun}(C,D)$, or equivalently a map $\widetilde{\alpha}\colon C\times\Delta^1\to D$ such that $\widetilde{\alpha}|C\times\Delta^{\{i\}}=f_i$, $i=0,1$.

Say that $\alpha\colon f_0\to f_1$ is a **natural isomorphism** if a is an isomorphism in the quasicategory of functors $\mathrm{Fun}(C,D)$. Thus, α is a natural isomorphism iff there exists a natural transformation $\beta\colon f_1\to f_0$ such that $\beta\alpha\approx 1_{f_0}$ and $\alpha\beta\approx 1_{f_1}$, where "$\approx$" is homotopy between morphisms in the quasicategory $\mathrm{Fun}(C,D)$.

This notion of natural isomorphism corresponds with the usual one for ordinary categories, since in that case homotopy of morphisms is the same as equality of morphisms.

Observe that “there exists a natural isomorphism $f_0 \to f_1$” is an equivalence relation on the set of all functors $C \to D$, as this relation precisely coincides with “there exists an isomorphism $f_0 \to f_1$” in the category $h\,\mathrm{Fun}(C, D)$. We say that f_0 and f_1 are **naturally isomorphic** fuctors.

Furthermore, the “naturally isomorphic” relation is compatible with composition: if f, f' are naturally isomorphic and g, g' are naturally isomorphic, then so are gf and $g'f'$. You can read this off from the fact the operation of composition of functors extends to a functor $\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D) \to \mathrm{Fun}(C, E)$ between quasicategories, and so induces a functor

$$h\,\mathrm{Fun}(D, E) \times h\,\mathrm{Fun}(C, D) \approx h\big(\mathrm{Fun}(D, E) \times \mathrm{Fun}(C, D)\big) \to h\,\mathrm{Fun}(C, E).$$

(This uses (9.15) to identify the homotopy category of the product with the product of homotopy categories.)

18.2. **Objectwise criterion for natural isomorphisms.** Recall that if C and D are ordinary categories, a natural transformation $\alpha\colon f_0 \to f_1$ between functors $f_0, f_1 \colon C \to D$ is a natural isomorphism iff and only if α is “an isomorphism objectwise”; i.e., if for each object c of C the evident map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D. That natural isomorphisms are “objectwise isomorphisms” is immediate. The opposite implication follows from the fact that a natural transformation between functors of ordinary values can be completely recovered from its “values on objects”. Thus, given $\alpha\colon f_0 \to f_1$ such that each $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism, we may explicitly construct an inverse transformation $\beta\colon f_1 \to f_0$ by setting $\beta(c) := \alpha(c)^{-1}\colon f_1(c) \to f_0(c)$. Note that this β is in fact the *unique* inverse to α (since inverses to morphisms are unique when they exist).

One of these directions is straightforward for quasicategories.

18.3. **Proposition.** *Let C and D be quasicategories. If $\alpha\colon C \times \Delta^1 \to D$ is a natural isomorphism between functors $f_0, f_1 \colon C \to D$, then for each object c of C the induced map $\alpha(c)\colon f_0(c) \to f_1(c)$ is an isomorphism in D.*

Proof. The map $\mathrm{Fun}(C, D) \to \mathrm{Fun}(\{c\}, D) = D$ induced by restriction along $\{c\} \subseteq C$ is a functor between quasicategories, so it takes isomorphisms to isomorphisms (10.3). It sends α to $\alpha(c)$. ☐

The converse to this proposition is also true.

C. **Deferred Proposition.** *A natural transformation $\alpha\colon C \times \Delta^1 \to D$ of functors between quasicategories is a natural isomorphism if and only if each of the maps $\alpha(c)$ are isomorphisms in D.*

Unfortunately, this is much more subtle to prove, as it requires using the existence of inverses to the $\alpha(c)$s to produce an inverse to α, which though it exists is not at all unique. We will prove this converse later in §31.

18.4. *Remark.* An immediate consequence of (C) is that if D is a *quasigroupoid*, then so is $\mathrm{Fun}(C, D)$.

18.5. *Remark.* The objectwise criterion (C) can be reformulated in terms of homotopy categories. The homotopy category construction takes quasicategories to categories, and takes functors to functors. Furthermore, given a natural transformation $\alpha\colon f_0 \to f_1$ of functors $f_0, f_1 \colon C \to D$ between quasicategories (i.e., a functor $\alpha\colon C \times \Delta^1 \to D$ such that $(\alpha|C \times \{j\}) = f_j$), we obtain an induced transformation $h\alpha\colon hf_0 \to hf_1$ of functors $hf_0, hf_1 \colon hC \to hD$ between their homotopy categories (so that the value of $h\alpha$ at an object $c \in \mathrm{ob}\, hC = C_0$ is the homotopy class of the edge $\alpha(\{c\} \times \Delta^1) \subseteq D$). Then (C) asserts that α is a natural isomorphism of functors between quasicategories if and only if $h\alpha$ is a natural isomorphism of functors between ordinary categories.

19. Categorical equivalence

We are now in position to define the correct generalization of the notion of "equivalence" of categories. This will be called *categorical equivalence* of quasicategories, and will be a direct generalization of the classical notion. **F 8 Feb 2019**

Given this, we use it to define a notion of categorical equivalence which applies to arbitrary maps of simplicial sets. Finally, we will show that the two definitions agree for maps between quasicategories.

19.1. **Categorical equivalences between quasicategories.** A **categorical inverse** to a functor $f: C \to D$ between quasicategories is a functor $g: D \to C$ such that gf is naturally isomorphic to 1_C and fg is naturally isomorphic to 1_D. We provisionally say that a functor f between quasicategories is a **categorical equivalence** if it admits a categorical inverse.

19.2. *Remark.* Categorical equivalence between quasicategories is a kind of "homotopy equivalence", where homotopies are natural isomorphisms between functors.

If C and D are nerves of ordinary categories, then natural isomorphisms between functors in our sense are precisely natural isomorphisms between functors in the classical sense, and that categorical equivalence between nerves of categories coincides precisely with the usual notion of equivalence of categories.

If quasicategories are equivalent, then their homotopy categories are equivalent.

19.3. **Proposition.** *If $f: C \to D$ is a categorical equivalence between quasicategories, then $h(f): hC \to hD$ is an equivalence of categories.*

Proof. Immediate, given that natural isomorphisms $f \Rightarrow g: C \to D$ induce natural isomorphisms $h(f) \Rightarrow h(g): hC \to hD$. □

Note: the converse is not at all true. For instance, there are many examples of quasicategories which are not equivalent to Δ^0, but whose homotopy categories are: e.g., $\operatorname{Sing} T$ for any non-contractible simply connected space T, or $K(A, d)$ for any non-trivial abelian group A and $d \geq 2$.

19.4. *Exercise* (Categorical inverses are unique up to natural isomorphism)*.* Let $f: C \to D$ be a functor between quasicategories, and suppose $g, g': D \to C$ are both categorical inverses to f. Show that g and g' are naturally isomorphic.

19.5. **General categorical equivalence.** We can extend the notion of categorical equivalence to maps between arbitrary simplicial sets. Say that a map $f: X \to Y$ between arbitrary simplicial sets is a **categorical equivalence** if for every quasicategory C, the induced functor $\operatorname{Fun}(f, C): \operatorname{Fun}(Y, C) \to \operatorname{Fun}(X, C)$ of quasicategories admits a categorical inverse.

We claim that on maps between quasicategories this general definition of categorical equivalence coincides with the provisional notion described earlier.

19.6. **Lemma.** *For a map $f: C \to D$ between quasicategories, the two notions of categorical equivalence described above coincide. That is, the following are equivalent:*

1. *f admits a categorical inverse.*
2. *For every quasicategory E, the functor $\operatorname{Fun}(f, E): \operatorname{Fun}(D, E) \to \operatorname{Fun}(C, E)$ admits a categorical inverse.*

To prove this, we will need the following observation. The construction $X \mapsto \operatorname{Map}(X, E)$ is a functor $s\mathrm{Set}^{\mathrm{op}} \to s\mathrm{Set}$, and so in particular induces a natural map

$$\gamma_0: \operatorname{Hom}(X, Y) \to \operatorname{Hom}(\operatorname{Map}(Y, E), \operatorname{Map}(X, E))$$

of sets, which sends $f\colon X \to Y$ to $\mathrm{Map}(f, E)\colon \mathrm{Map}(Y, E) \to \mathrm{Map}(X, E)$. The observation we need is that this construction admits an "enrichment", to a map

$$\gamma\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(\mathrm{Map}(Y, E), \mathrm{Map}(X, E)),$$

which coincides with γ_0 on vertices. The map γ is defined to be adjoint to the "composition" map $\mathrm{Map}(X, Y) \times \mathrm{Map}(Y, E) \to \mathrm{Map}(X, E)$. (*Exercise:* Describe explicitly what γ does to n-dimensional elements.) We say that the functor $\mathrm{Map}(-, E)$ is an *enriched* functor, as it gives not merely a map between hom-sets (i.e., acts on vertices in function complexes), but in fact gives a map between function complexes.

Proof. (1) $\Longrightarrow$ (2). When C, D, and E are quasicategories so are the function complexes between them (B). In this case, the above map γ takes functors $C \to D$ to functors $\mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$ between quasicategories, natural transformations of such functors to natural transformations, and natural isomorphisms of such functors to natural isomorphisms. Using this observation, it is straightforward to show that a categorical inverse $g\colon D \to C$ to $f\colon C \to D$ gives rise to a categorical inverse $\mathrm{Map}(g, E)\colon \mathrm{Map}(C, E) \to \mathrm{Map}(C, D)$ to the induced functor $\mathrm{Map}(f, E)\colon \mathrm{Map}(D, E) \to \mathrm{Map}(C, E)$.

(2) $\Longrightarrow$ (1). Conversely, suppose $f\colon C \to D$ is a categorical equivalence in the general sense, so that $f^* = \mathrm{Map}(f, E)$ admits a categorical inverse for every quasicategory E, which implies that each functor

$$h(f^*)\colon h\,\mathrm{Fun}(D, E) \to h\,\mathrm{Fun}(C, E)$$

is an equivalence of ordinary categories (19.3). In particular, it follows that f^* induces a bijection of sets

$$f^*\colon \pi_0(\mathrm{Fun}(D, E)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(C, E)^{\mathrm{core}});$$

recall that $\pi_0(\mathrm{Fun}(D, E)^{\mathrm{core}}) \approx \pi_0((h\,\mathrm{Fun}(D, E))^{\mathrm{core}})$ is precisely the set of *natural isomorphism classes* of functors $D \to E$.

Taking $E = C$, this implies that there must exist $g \in \mathrm{Fun}(D, C)_0$ such that there exists a natural isomorphism $gf \to \mathrm{id}_C$ in $\mathrm{Fun}(C, C)_1$. Taking $E = D$, we note that since

$$f^*(\mathrm{id}_D) = \mathrm{id}_D f = f\,\mathrm{id}_C \approx fgf = f^*(fg),$$

we must have that $\mathrm{id}_D \approx fg$, i.e., there exists a natural isomorphism $\mathrm{id}_D \to fg$ in $\mathrm{Fun}(D, D)_1$. Thus, we have shown that g is a categorical inverse of f, as desired. □

19.7. *Remark.* The definition of categorical equivalence we are using here is *very* different to the definition adopted by Lurie [Lur09, §2.2.5]. It is also slightly different from the notion of "weak categorical equivalence" used by Joyal [Joy08a, 1.20]. As we will show soon (22.12), Joyal's weak categorical equivalence is equivalent to our definition of categorical equivalence. The discussion around [Lur09, 2.2.5.8] show's that Lurie's and Joyal's definitions are equivalent, and so they are both equivalent to the one we have used.

19.8. *Exercise.* Let $f\colon C \to D$ be a functor between quasicategories. Show that f is a categorical equivalence if and only if for all simplicial sets X, the induced functor $f_*\colon \mathrm{Map}(X, C) \to \mathrm{Map}(X, D)$ is a categorical equivalence.

20. Trivial fibrations and inner anodyne maps

Inner anodyne maps and trivial fibrations are particular kinds of categorical equivalences.

20.1. **Trivial fibrations to the terminal object.** Recall that a trivial fibration $p\colon X \to Y$ of simplicial sets is a map such that $(\partial\Delta^k \subset \Delta^k) \boxslash p$ for all $k \geq 0$. That is, $\mathrm{TrivFib} = \mathrm{Cell}^{\boxslash}$, so p is a trivial fibration if and only if $\mathrm{Cell} \boxslash p$.

20.2. *Exercise.* Consider an indexed collection of trivial fibrations $p_i\colon X_i \to Y_i$. Show that $p := \coprod p_i\colon \coprod X_i \to \coprod Y_i$ is a trivial fibration. (Hint: similar to proof of (6.7).)

20.3. **Proposition.** *Let X be a simplicial set and $p\colon X \to *$ be a trivial fibration whose target is the terminal simplicial set. Then X is a Kan complex (and thus a quasigroupoid) and p is a categorical equivalence.*

Proof. Enriched lifting (17.3) applied to $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ (17.5) means that for any monomorphism $i\colon A \to B$ of subcomplexes the pullback-hom map

$$p^{\square i} = \mathrm{Map}(i, X)\colon\ \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,*)} \mathrm{Map}(B, *) = \mathrm{Map}(A, X)$$

is a trivial fibration. In particular, it implies that $\mathrm{Map}(i, X)$ is surjective on vertices, so $\mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X)$ is surjective.

It follows immediately that X is a Kan complex, by taking i to be any horn inclusion.

To show that p is a categorical equivalence, first note that X is non-empty, since $\mathrm{Hom}(\Delta^0, X) \to \mathrm{Hom}(\varnothing, X) = *$ is surjective. Choose any $s \in \mathrm{Hom}(\Delta^0, X)$. Clearly $ps = \mathrm{id}_{\Delta^0}$. We will show that $sp\colon X \to X$ is naturally isomorphic to id_X. Consider the commutative diagram

$$\begin{tikzcd}
X \amalg X \arrow[r, equal] & X \times \partial\Delta^1 \arrow[r] \arrow[d, tail] \arrow[r, bend left, "{(\mathrm{id}_X, sp)}"] & X \arrow[d, "p"] \\
& X \times \Delta^1 \arrow[r] \arrow[ur, dotted, "h"] & *
\end{tikzcd}$$

Since p is a trivial fibration, a lift h exists, which exhibits a natural transformation $\mathrm{id}_X \to sp$; note that h represents a morphism in $\mathrm{Fun}(X, X)$. To show that h represents an isomorphism, it's enough to know that $\mathrm{Fun}(X, X)$ is actually a quasigroupoid. In fact, restriction along $\varnothing \to X$ a trivial fibration

$$\mathrm{Fun}(X, X) \to \mathrm{Fun}(\varnothing, X) = *,$$

whence $\mathrm{Fun}(X, X)$ is a Kan complex by the argument given above. ☐

We will prove a partial converse to this later (37.11): if C is a quasicategory which is categorically equivalent to $*$, then $C \to *$ is a trivial fibration.

20.4. **Preisomorphisms.** We need a way to produce categorical equivalences between simplicial sets which are not necessarily quasicategories.

Let X be a simplicial set. Say that an edge $a \in X_1$ is a **preisomorphism** if it projects to an isomorphism under $\alpha\colon X \to hX$, the tautological map to the (nerve of the) fundamental category (9.1). If X is actually a quasicategory, the preisomorphisms are just the isomorphisms (since in that case the fundamental category is the same as the homotopy category). Note that degenerate edges are always preisomorphisms, since they go to identity maps in the fundamental category.

20.5. **Proposition.** *An edge $a \in X_1$ is a preisomorphism if and only if for every map $g\colon X \to C$ to a quasicategory C, the image $g(a)$ is an isomorphism in C.*

Proof. Isomorphisms in C are exactly the edges which are sent to isomorphisms under $\gamma\colon C \to hC$. Given this the proof is straightforward, using the fact that the formation of fundamental categories is functorial, and that hX is itself a category and hence a quasicategory. ☐

As a consequence, any map $X \to Y$ of simplicial sets takes preisomorphisms to preisomorphisms. In particular, any map *from* a quasicategory takes isomorphisms to preisomorphisms. We will use this observation below.

20.6. *Example.* Consider the subcomplex $\Lambda^3_{\{0,3\}} = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3. Define Z to be the pushout of the diagram

$$\Lambda^3_{\{0,3\}} \xleftarrow{j} \Delta^{\{0,2\}} \amalg \Delta^{\{1,3\}} \xrightarrow{p} \Delta^{\{x\}} \amalg \Delta^{\{y\}}$$

where j is the evident inclusion, $\Delta^{\{x\}}$ and $\Delta^{\{y\}}$ are simplicial sets isomorphic to Δ^0, but with vertices labelled "x" and "y" respectively, and p is induced by the evident projections $\Delta^{\{0,2\}} \to \Delta^{\{x\}}$ and $\Delta^{\{1,3\}} \to \Delta^{\{y\}}$. The resulting complex Z looks like

$$\begin{tikzcd} y \arrow[r, "y_{00}"] \arrow[d, "g"'] \arrow[dr, phantom, "b", pos=0.3] & y \arrow[d, "h"] \\ x \arrow[ur, "f"] \arrow[r, "x_{00}"'] \arrow[ur, phantom, "a", pos=0.7] & x \end{tikzcd}$$

with seven non-degenerate elements: $x, y \in Z_0$, $f, g, h \in Z_1$, $a, b \in Z_2$. The simplicial set Z is not a quasicategory (*Exercise:* why not?). However, any map $\phi\colon Z \to C$ to a quasicategory sends f, g, h to morphisms $\phi(f), \phi(g), \phi(h)$ of C so that $\phi(g)$ is a preinverse of $\phi(f)$ and $\phi(h)$ is a postinverse of $\phi(f)$. Therefore these (and thus all) edges of Z are preisomorphisms.

20.7. *Example.* Here is a variant of the previous example. Consider the subcomplex $\Lambda^3_{\{0,3\}} = \Delta^{\{0,1,2\}} \cup \Delta^{\{1,2,3\}}$ of Δ^3. Define Z' to be the pushout of the diagram

$$\Lambda^3_{\{0,3\}} \xleftarrow{j} \Delta^{\{0,1\}} \cup \Delta^{\{0,2\}} \cup \Delta^{\{1,3\}} \cup \Delta^{\{2,3\}} \xrightarrow{p} \Delta^{\{y<x\}},$$

where j is the evident inclusion, $\Delta^{\{y<x\}}$ is a simplical set isomorphic to Δ^1 but with vertices labelled "y" and "x" instead of "0" and "1", and p is the unique map which on vertices sends $0, 2 \mapsto y$, $1, 3 \mapsto x$. The resulting complex Z' looks like

$$\begin{tikzcd} y \arrow[r, "y_{00}"] \arrow[d, "g"'] \arrow[dr, phantom, "b", pos=0.3] & y \arrow[d, "g"] \\ x \arrow[ur, "f"] \arrow[r, "x_{00}"'] \arrow[ur, phantom, "a", pos=0.7] & x \end{tikzcd}$$

with six non-degenerate elements: $x, y \in Z'_0$, $f, g \in Z'_1$, $a, b \in Z'_2$. Again, Z' is not a quasicategory, but all edges of Z' are preisomorphisms, since any map $\phi\colon Z' \to C$ to a quasicategory sends f, g to morphisms which are inverse to each other.

Say that vertices in a simplicial set X are **preisomorphic** if they can be connected by a chain of preisomorphisms (which can point in either direction). Clearly, any map $g\colon X \to C$ to a quasicategory takes preisomorphic vertices of X to isomorphic objects of C. **M 11 Feb 2019**

We can apply this to function complexes. If two maps $f_0, f_1\colon X \to Y$ are preisomorphic (viewed as vertices in $\mathrm{Map}(X, Y)$), then for any quasicategory C, the induced functors $\mathrm{Map}(f_0, C), \mathrm{Map}(f_1, C)\colon \mathrm{Map}(Y, C) \to \mathrm{Map}(X, C)$ are naturally isomorphic. To see this, consider

$$\Delta^1 \xrightarrow{a} \mathrm{Map}(X, Y) \xrightarrow{b} \mathrm{Map}(\mathrm{Map}(Y, C), \mathrm{Map}(X, C))$$

where b is adjoint to the composition map $\mathrm{Map}(Y, C) \times \mathrm{Map}(X, Y) \to \mathrm{Map}(X, C)$. If a represents a preisomorphism $f_0 \to f_1$ in $\mathrm{Map}(X, Y)$, then ba represents an isomorphism $\mathrm{Map}(f_0, C) \to \mathrm{Map}(f_1, C)$, since the target of b is a quasicategory. As a consequence we get the following.

20.8. **Lemma.** *If $f\colon X \to Y$ and $g\colon Y \to X$ are maps of simplicial sets such that gf is preisomorphic to id_X in $\mathrm{Map}(X, X)$ and fg is preisomorphic to id_Y in $\mathrm{Map}(Y, Y)$, then f and g are categorical equivalences.*

It is important to note that this is a sufficient condition for a map to be a categorical equivalence, but not a necessary one: there are many categorical equivalences of simplicial sets to which the lemma cannot be applied (see (21.3) below).

20.9. **Trivial fibrations are always categorical equivalences.**

20.10. **Proposition.** *Every trivial fibration between simplicial sets is a categorical equivalence.*

Here is some notation. Given maps $f\colon A \to Y$ and $g\colon B \to Y$, we write $\mathrm{Map}_{/Y}(f,g)$ or $\mathrm{Map}_{/Y}(A,B)$ for the simplicial set defined by the pullback square

$$\begin{tikzcd} \mathrm{Map}_{/Y}(A,B) \arrow[r] \arrow[d] & \mathrm{Map}(A,B) \arrow[d, "g_*=\mathrm{Map}(A,g)"] \\ \{f\} \arrow[r] & \mathrm{Map}(A,Y) \end{tikzcd}$$

Note that vertices of $\mathrm{Map}_{/Y}(A,B)$ correspond exactly to "sections of g over f", i.e., to $s\colon A \to B$ such that $gs = f$. You can think of $\mathrm{Map}_{/Y}(A,B)$ as a simplicial set which "parameterizes" sections of g over f. I'll call this the **relative function complex over** Y.

20.11. *Exercise.* Show that n-dimensional elements of $\mathrm{map}_{/Y}(A,B)$ correspond to maps $a\colon \Delta^n \times A \to B$ such that $ga = \pi(\mathrm{id} \times f)$, where $\pi\colon \Delta^n \times Y \to Y$ is the projection.

Proof of (20.10). Fix a trivial fibration $p\colon X \to S$. We regard both X and S as objects *over* S, via p and id_S, and consider various relative function complexes over S.

Note that since p is a trivial fibration, so are $\mathrm{Map}(X,p) = p^{\Box(\varnothing\subset X)}$ and $\mathrm{Map}(S,p) = p^{\Box(\varnothing\subset X)}$ by enriched lifting $\overline{\mathrm{Cell}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$. The maps

$$\mathrm{Map}_{/S}(S,X) \to \mathrm{Map}_{/S}(S,S) = * \qquad \text{and} \qquad \mathrm{Map}_{/S}(X,X) \to \mathrm{Map}_{/S}(X,S) = *$$

are (by construction) base changes of $\mathrm{Map}(S,p)$ and $\mathrm{Map}(X,p)$ respectively, and so are also trivial fibrations since TrivFib is closed under base change. It follows from (20.3) that both $\mathrm{Map}_{/S}(S,X)$ and $\mathrm{Map}_{/S}(X,X)$ are quasigroupoids which are categorically equivalent to the terminal object (and so are non-empty and such that all objects are isomorphic). Note that these are subcomplexes of simplicial sets $\mathrm{Map}(S,X)$ and $\mathrm{Map}(X,X)$ respectively, which however need not be quasicategories. The edges of $\mathrm{Map}_{/S}(S,X)$ and $\mathrm{Map}_{/S}(X,X)$ are preisomorphisms in $\mathrm{Map}(S,X)$ and $\mathrm{Map}(X,X)$.

Pick any vertex s of $\mathrm{Map}_{/S}(S,X)$, so that s can be regarded as a map $s\colon S \to X$ such that $ps = \mathrm{id}_S$. Pick any isomorphism $a\colon \mathrm{id}_X \to sp$ in $\mathrm{Map}_{/S}(X,X)$, which is hence a preisomorphism in $\mathrm{Map}(X,X)$.

Thus, we have exhibited maps p and s whose composites are preisomorphic to identity functors, and therefore they are categorical equivalences by (20.8). □

20.12. *Remark* ("Uniqueness" of sections of trivial fibrations). Suppose that $p\colon C \to D$ is a trivial fibration between quasicategories. As we have noted, the relative function complex $\mathrm{Map}_{/D}(D,C)$ "parameterizes sections of p". Since this is a quasigroupoid equivalent to the terminal quasicategory (20.10), not only is p a categorical equivalence, but also

- p admits a section, which is a categorical inverse to p, and
- any two sections of p are naturally isomorphic.

We will often make use of this observation.

20.13. **Inner anodyne maps are always categorical equivalences.**

20.14. **Proposition.** *Every inner anodyne map between simplicial sets is a categorical equivalence.*

Proof. Let $j\colon X \to Y$ be a map in $\overline{\text{InnHorn}}$, and let C be any quasicategory. The induced map $\text{Map}(j, C)\colon \text{Map}(Y, C) \to \text{Map}(X, C)$ is a trivial fibration by enriched lifting and $\overline{\text{InnHorn}}\square\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (17.2), and therefore is a categorical equivalence. □

20.15. **Every simplicial set is categorically equivalent to a quasicategory.**

20.16. **Proposition.** *Fix a simplicial set X.*

(1) *There exists a quasicategory C and an inner anodyne map $f\colon X \to C$, which is therefore a categorical equivalence.*
(2) *For any two $f_i\colon X \to C_i$ as in (1), there exists a categorical equivalence $g\colon C_1 \to C_2$ such that $gf_1 = f_2$.*
(3) *Any two categorical equvialences $g_1, g_2\colon C_1 \to C_2$ such that $g_i f_1 = f_2$ are naturally isomorphic.*

Here is some more notation. Given maps $f\colon X \to A$ and $g\colon X \to B$, we write $\text{Map}_{X/}(f, g)$ or $\text{Map}_{X/}(A, B)$ for the simplicial set defined by the pullback square

$$\begin{tikzcd} \text{Map}_{X/}(A,B) \arrow[r] \arrow[d] & \text{Map}(A,B) \arrow[d, "f^*=\text{Map}(f,B)"] \\ \{g\} \arrow[r] & \text{Map}(X,B) \end{tikzcd}$$

This is the **relative function complex under** X.

20.17. *Exercise.* Show that n-dimensional elements of $\text{map}_{X/}(A, B)$ correspond to maps $a\colon \Delta^n \times A \to B$ such that $a(\text{id} \times f) = g\pi$, where $\pi\colon \Delta^n \times X \to X$ is the projection

Proof of (20.16). (1) By the small object argument (13.17), we can factor $X \to *$ into $X \xrightarrow{j} C \xrightarrow{p} *$ where $j \in \overline{\text{InnHorn}}$ and $p \in \text{InnFib}$. The inner anodyne map j is the desired categorical equivalence to a quasicategory.

(2) For $i, j \in \{1, 2\}$, we have a restriction map $f^*_{i,j}\colon \text{Map}(C_i, C_j) \to \text{Map}(X, C_j)$, which is necessarily a trivial fibration by enriched lifting since $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. Therefore the maps $\text{Map}_{X/}(C_i, C_j) \to *$ (obtained by base-change from the $f^*_{i,j}$) are all trivial fibrations, i.e., each $\text{Map}_{X/}(C_i, C_j)$ is a quasigroupoid with only one isomorphism class of objects (20.3). As in the proof of (20.10) we construct $g\colon C_1 \to C_2$ and $g'\colon C_2 \to C_1$ which are categorically inverse to each other; details are left to the reader.

(3) The maps g_1, g_2 correspond to vertices in $\text{Map}_{X/}(C_1, C_2)$, which as we have observed is a quasigroupoid with only one isomorphism class of objects. □

Thus, we can always "replace" a simplicial set X by a categorically equivalent quasicategory C. Although such C is not unique, it is unique up to categorical equivalence.

You can think of such a replacement $X \to C$ of X as a quasicategory "freely generated" by the simplicial set X, an idea which is validated by the fact that $\text{Fun}(j, D)\colon \text{Fun}(C, D) \to \text{Map}(X, D)$ is a categorical equivalence for every quasicategory D.

21. Some examples of categorical equivalences

21.1. **Free monoid on one generator.** Let F denote the free monoid on one generator g. This is a category with one object x, and morphism set $\{\, g^n \mid n \geq 0 \,\}$.

Associated to the generator g is a map

$$\gamma\colon S^1 := \Delta^1/\partial\Delta^1 \to N(F)$$

sending the image of the generator $\iota \in (\Delta^1)_1$ in S^1 to g. (We use "L/K" as a shorthand for "$L \amalg_K *$" whenever $K \subseteq L$. The object S^1 is called the "simplicial circle", which has exactly two nondegenerate simplicies, one in dimension 0 and one in dimension 1.)

It is not hard to see that F is "freely generated" as a category by S^1, in the sense that $h(S^1) = F$ (the fundamental category of S^1 is F). It turns out that $N(F)$ is actually freely generated as a *quasicategory* by S^1.

21.2. **Proposition.** *The map $\gamma\colon S^1 \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.* **W 13 Feb 2019**

Proof. This is an explicit calculation. Note that a general element in $N(F)_d$ corresponds to a sequence $(g^{m_1}, \dots, g^{m_d})$ of elements of the monoid F, where $m_1, \dots, m_d \geq 0$. Let $a_k \in N(F)_k$ denote the k-dimensional element corresponding to the sequence $(g, g, \dots, g)$, and let $Y_k \subseteq N(F)$ denote the subcomplex which is the image of the representing map $a_k\colon \Delta^k \to N(F)$. For $f\colon [d] \to [k]$ we compute that $a_k f = (g^{m_1}, \dots, g^{m_d})$ where $m_i = f(i) - f(i-1)$, so that

$$(Y_k)_d = \{\, a_k f \;\mid\; f\colon [d] \to [k] \,\} = \{\, (g^{m_1}, \dots, g^{m_d}) \;\mid\; m_1 + \cdots + m_d = f(d) - f(0) \leq k \,\},$$

Clearly $N(F) = \bigcup_{k \geq 1} Y_k$, with $Y_1 \approx S^1$ and $Y_2 \approx Y_1 \cup_{\Lambda^2_1} \Delta^2$. Furthermore we have the following:

- An element f of $(\Delta^k)_d$ is such that $a_k f$ is in the subcomplex Y_{k-1} of Y_k if and only if $f(d) - f(0) < k$, if and only if either $f(d) < k$ or $f(0) > 0$, i.e., if and only if f is in the subcomplex $\Lambda^k_{\{0,k\}} = \Delta^{\{0,\dots,k-1\}} \cup \Delta^{\{1,\dots,k\}}$.
- Every element y of Y_k not in Y_{k-1} is the image under a_k of a *unique* element in Δ^k. (I.e.,, if $f\colon [d] \to [k]$, then $m_1 + \cdots + m_d = f(d) - f(0)$, which is equal to k if and only if $f(0) = 0$ and $f(d) = k$, and if this is the case then $f(i) = m_1 + \cdots + m_i$.)

In other words, the square

$$\begin{tikzcd} \Lambda^k_{\{0,k\}} \arrow[r] \arrow[d, tail] & Y_{k-1} \arrow[d, tail] \\ \Delta^k \arrow[r, "a_k"'] & Y_k \end{tikzcd}$$

is a pullback, and a_k induces in each dimension d d a bijection $(\Delta^k)_d \smallsetminus (\Lambda^k_{\{0,k\}})_d \xrightarrow{\sim} (Y_k)_d \smallsetminus (Y_{k-1})_d$. It follows (15.25) that the square is a pushout.

The inclusion $\Lambda^k_{\{0,k\}} \subset \Delta^k$ is a *generalized inner horn*, and we have noted this is inner anodyne when $k \geq 2$ (12.9). It follows that each $Y_{k-1} \to Y_k$ is inner anodyne for $k \geq 2$, whence $S^1 \to N(F)$ is inner anodyne. □

21.3. *Remark.* This gives an explicit example of a categorical equivalence to which (20.8) does not apply: γ does not admit an "inverse up to preisomorphims". There is only one map $\delta\colon N(F) \to S^1$, namely the composite $N(F) \to * \to S^1$, and it is clear that neither $\delta\gamma\colon N(F) \to N(F)$ nor $\gamma\delta\colon S^1 \to S^1$ are preisomorphic to identity functors.

21.4. **Free categories.** We can generalize the above to free monoids with arbitrary sets of generators, and in fact to free categories. Let S be a **1-dimensional** simplicial set, i.e., one such that $S = \mathrm{Sk}_1 S$. These are effectively the same thing as *directed graphs* (allowed to have multiple parallel edges and loops): S_0 corresponds to the set of vertices of the directed graph, and S_1^{nd} corresponds to the set of edges of the directed graph.

Let $F := hS$. We call F the **free category** on the 1-dimensional simplicial set S. In this case, the morphisms of the fundamental category are precisely the words in the edges S_1^{nd} of the directed graph (including empty words for each vertex, corresponding to identity maps).

21.5. **Proposition.** *The evident map $\gamma\colon S \to N(F)$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is virtually the same as the proof of (21.2). In this case, $Y_k \subseteq N(F)$ is the subcomplex generated by all $a\colon \Delta^k \to N(F)$ such that each spine-edge $a_{i-1,i}$ is in S_1^{nd}, and Y_k is obtained by attaching a generalized horn to Y_{k-1} for each such a. $\square$

As a consequence, it is "easy" to construct functors $F \to C$ from a free category to a quasicategory: start with a map $S \to C$, which amounts to specifying vertices and edges in C corresponding to elements S_0 and S_1^{nd}, and extend over $S \subseteq F$. The evident restriction map $\mathrm{Fun}(F, C) \to \mathrm{Map}(S, C)$ is a categorical equivalence, and in fact a trivial fibration. In other words, free categories are also "free quasicategories".

21.6. *Exercise.* Describe the ordinary category $A := h\Lambda^3_0$ "freely generated" by Λ^3_0. Show that the tautological map $\Lambda^3_0 \to N(A)$ is inner anodyne.

21.7. **Free commutative monoids.** Let F be the free monoid on one generator again, with generator corresponding to simplicial circle $S^1 = \Delta^1/\partial\Delta^1 \subset F$ Recall that $F^{\times n}$ is the free commutative monoid on n generators. Recall that the nerve functor preserves products, so $N(F^{\times n}) \approx N(F)^{\times n}$. We obtain a map

$$\delta = \gamma^{\times n}\colon (S^1)^{\times n} \to N(F^{\times n})$$

from the "simplicial n-torus".

21.8. **Proposition.** *The map $\delta\colon (S^1)^{\times n} \to N(F^{\times n})$ is a categorical equivalence, and in fact is inner anodyne.*

Proof. This is a consequence of the fact that if $j\colon A \to B$ is inner anodyne and K an arbitrary simplicial set, then $j \times K \times \colon A \times K \to B \times K$ is inner anodyne (because $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$). It follows that $A^{\times n} \to B^{\times n}$ is a composite of inner anodyne maps, and so is inner anodyne and thus a categorical equivalence (20.14). Also use the fact that the nerve construction preserves products (6.5), so $N(F^{\times}n) = N(F)^{\times n}$. $\square$

21.9. *Exercise.* Let $S^1 \vee S^1 \subset (S^1)^{\times 2}$ be the subcomplex obtained as the evident "one-point union" of the two "coordinate circles"; i.e., $S^1 \vee S^1 = (S^1 \times \{*\}) \cup (\{*\} \times S^1)$. Suppose given a map $\phi\colon S^1 \vee S^1 \to C$ to a quasicategory C, corresponding to a choice of object $x \in C_0$ together with two morphisms $f, g\colon x \to x$ in C_1. Show that there exists an extension of ϕ along $S^1 \vee S^1 \subset N(F^{\times 2})$ if and only if $[f][g] = [g][f]$ in hC.

21.10. *Remark.* The analogue of the above exercise for $n = 3$ isn't true. That is, consider the subcomplex $S^1 \vee S^1 \vee S^1 \subset (S^1)^{\times 3}$ which is a one-point union of three circles, suppose we have $S^1 \vee S^1 \vee S^1 \to C$ corrposding to three morphisms $f, g, h\colon x \to x$ in C, and suppose we also know that $[f][g] = [g][f]$, $[g][h] = [h][g]$, and $[f][h] = [h][f]$ in hC. Then you can show that there exists an extension to a map $K \to C$ as in (21.9), where $K \subseteq (S^1)^{\times 3}$ is the subcomplex $(S^1 \times S^1 \times \{*\}) \cup (S^1 \times \{*\} \times S^1) \cup (\{*\} \times S^1 \times S^1)$. However, there need not exist an extension to a map $(S^1)^{\times 3} \to C$, and thus there may not exist an extension to a map $N(F^{\times 3}) \to C$. (For an explicit example where this fails, take $C = \mathrm{Sing}\, T$, where $T \subseteq (S^1_{\mathrm{top}})^{\times 3}$ is the subspace of the topological 3-torus consisting of tuples (x_1, x_2, x_3) such that at least one x_i is the basepoint of S^1_{top}.)

Thus, this is a situation where the "higher structure" of a quasicategory plays a role. When C is an ordinary category, it is easy to show that the desired extension does always exist. However, for a general quasicategory C, three pairwise-commuting endomorphisms of an object do not generally give rise to a functor $N(F^{\times 3}) \to C$ from the free commutative monoid on 3 generators.

21.11. **Finite groups are not finite.** If A is any ordinary category, then $\mathrm{Sk}_2\, N(A)$ "freely generates $N(A)$ as a category", in the sense that $h(\mathrm{Sk}_2\, N(A)) \approx A$, or equivalently that $\mathrm{Map}(N(A), N(B)) \to \mathrm{Map}(\mathrm{Sk}_2\, N(A), N(B))$ is an isomorphism for any category B. However, it is often the case that *no finite dimensional simplicial set* "freely generates $N(A)$ as a quasicategory". In fact, this is the case for every non-trivial finite group.

21.12. *Example.* Let G be a finite group, and let $C = N(G)$. The geometric realization $BG := \|N(G)\|$ is the *classifying space* of G. I want to show that if G is not the trivial group, then NG is not categorically equivalent to any *finite* dimensional simplicial set K (i.e., one with no non-degenerate elements above a certain dimension). We need to use some homotopy theory, along with a fact to be proved later[19]: if $f\colon X \to Y$ is any categorical equivalence of simplicial sets, then the induced map $\|f\| : \|X\| \to \|Y\|$ on geometric realizations is a homotopy equivalence of spaces.

First consider $G = \mathbb{Z}/n$ with $n > 1$. A standard calculation in topology says that $H^{2k}(\|N(G)\|, \mathbb{Z}) \approx \mathbb{Z}/n \not\approx 0$ for all $k > 0$. This implies that $\|N(G)\|$ cannot be homotopy equivalent to any finite dimensional complex.

Now consider a general non-trivial finite group G, and choose a non-trivial cyclic subgroup $H \leq G$. We know the fundamental group: $\pi_1 \|K\| \approx \pi_1 \|N(G)\| = G$. Covering space theory tells us we can construct a covering map $p\colon E \to \|N(G)\|$ so that $\pi_1 E \to \pi_1 \|N(G)\|$ is the inclusion $H \to G$. In fact, E is homotopy equivalent to the classifying space BH (because $\pi_k E \approx 0$ for $k \geq 2$). But if $\|N(G)\|$ is finite dimensional then so is E, but this would then contradicting the observation that $H^*(BH, \mathbb{Z}) \approx H^*(E, \mathbb{Z}) \approx 0$ for infinitely many $*$.

Thus, non-trivial *finite* groups are never "freely generated as a quasicategory" by finite dimensional complexes.

21.13. *Remark.* Let T be a *finite* CW-complex, and G a finite group. A theorem of Haynes Miller (the "Sullivan conjecture") implies that every functor $N(G) \to \operatorname{Sing} T$ is naturally isomorphic to a constant functor (i.e., one which factors through Δ^0). Thus, the singular complex of a finite CW-complex cannot "contain" any non-trivial finite group, even if its fundamental group contains a non-trivial finite subgroup.

22. The homotopy category of quasicategories

22.1. **The homotopy category of** qCat**.** The homotopy category hqCat of quasicategories is defined as follows. The objects of hqCat are the quasicategories. Morphisms $C \to D$ in hqCat are natural isomorphism classes of functors. That is,

$$\operatorname{Hom}_{h\mathrm{qCat}}(C, D) := \text{isomorphism classes of objects in } h\operatorname{Fun}(C, D) = \pi_0\big(\operatorname{Fun}(C, D)^{\mathrm{core}}\big).$$

That this defines a category results from the fact that composition of functors passes to a functor $h\operatorname{Fun}(D, E) \times h\operatorname{Fun}(C, D) \to h\operatorname{Fun}(C, E)$, and thus is compatible with natural isomorphism.

It comes with an obvious functor qCat $\to h$qCat. Note that a map $f\colon C \to D$ of quasicategories is a categorical equivalence if and only if its image in hqCat is an isomorphism.

22.2. *Remark.* We can similarly define a category hCat, whose objects are categories and whose morphisms are *isomorphism classes* of functors. The nerve functor evidently induces a full embedding $h\mathrm{Cat} \to h\mathrm{qCat}$.

22.3. *Warning.* Although we use the word "homotopy", the definition of hqCat given above is not an example of the notion of the homotopy category of a quasicategory defined in §9: qCat is a (large) ordinary category, so is isomorphic to its own homotopy category in that sense. Here we are using the equivalence relation on morphisms(=functors) defined by natural isomorphism.

We define $h\mathrm{Kan} \subset h\mathrm{qCat}$ to be the full subcategory of the homotopy category spanned by quasicategories which are Kan complexes. **Use** qGpd **instead?**

For future reference, we note that hqCat and hKan have finite products, which just amount to the usual products of simplicial sets.

22.4. **Proposition.** *The terminal simplicial set Δ^0 is a terminal object in* hqCat*. If C_1, C_2 are quasicategories, then the projection maps exhibit $C_1 \times C_2$ as a product in* hqCat*.*

[19] I don't know if this will actually get proved later. It is proved in [GJ09]

Proof. This is straightforward. The key observation for the second statement is the fact that isomorphism classes of objects in a product of quasicategories correspond to pairs of isomorphism classes in each (6.13), and the fact that $\mathrm{Map}(X, C_1 \times C_2) \xrightarrow{\sim} \mathrm{Map}(X, C_1) \times \mathrm{Map}(X, C_2)$. □

22.5. **The 2-out-of-6 and 2-out-of-3 properties.** A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-6 property** if (i) $\mathcal{W}$ contains all identity maps, and (ii) given sequence (h, g, f) of maps such that the composites gf and hg are defined, if $gf, hg \in \mathcal{W}$ then also $f, g, h, hgf \in W$.

A class of morphisms $\mathcal{W}$ in a category is said to satisfy the **2-out-of-3 property** if (i) $\mathcal{W}$ contains all identity maps, and (ii) given a sequence (g, f) of maps such that the composite gf is defined, if any two of (f, g, gf) are in $\mathcal{W}$, so is the third.

22.6. *Example.* In any category, the class of isomorphisms satisfies 2-out-of-6 property and the 2-out-of-3 property. The class of identity maps satisfies 2-out-of-3, but does not generally satisfy 2-out-of-6.

22.7. **Proposition.** *If $\mathcal{W}$ satisfies 2-out-of-6, then it satisfies 2-out-of-3.*

Proof. Given f, g such that gf is defined, apply 2-out-of-6 to the composable sequences (id, g, f), (g, id, f), (g, f, id). □

22.8. *Exercise.* Given a functor $f\colon C \to D$ between categories, let $\mathcal{W}$ be the class of maps in C that f takes to isomorphisms in D. Show that $\mathcal{W}$ satisfies 2-out-of-6, and thus 2-out-of-3.

22.9. *Example* (2-out-of-6 for equivalences of categories)*.* In Cat, the category of small categories and functors, the class of equivalences satisfies 2-out-of-6, and thus 2-out-of-3.

To see this, first suppose (h, g, f) is a triple of functors such that there are natural isomorphisms $gf \approx \mathrm{id}$ and $hg \approx \mathrm{id}$. Then, since (i) natural isomorphism is an equivalence relation on functors and (ii) is compatible with composition, we see that

$$h = h\,\mathrm{id} \approx h(gf) = (hg)f \approx \mathrm{id}\, f = f,$$

and thus that g is an equivalence since $hg \approx \mathrm{id}$ and $gh \approx gf \approx \mathrm{id}$.

Next, note that composites of equivalences are equivalences, by a straightforward argument: if g and f are equivalences and composable, and g' and f' are categorical inverses to them, then $f'g'$ is easily seen to be a categorical inverese to gf.

Now suppose that (h, g, f) are such that gf and hg are categorical equivalences. Choose categorical inverses u and v for these, so that

$$gfu \approx \mathrm{id}, \quad ugf \approx \mathrm{id}, \quad hgv \approx \mathrm{id}, \quad vhg \approx \mathrm{id}\,.$$

Apply the above remarks to the triples (ug, f, ug), (vh, g, fu), (gv, h, gv), and (ugv, hgf, vgu) to show that f, g, h are equivalences, where we use that

$$fug \approx (vhg)fug = vh(gfu)g \approx vhg \approx \mathrm{id}, \quad gvh \approx gvh(gfu) = g(vhg)fu \approx gfu \approx \mathrm{id}\,.$$

It follows that the composite hgf is also an equivalence.

Alternately, we can apply (22.8) to the tautological functor $\mathrm{Cat} \to h\mathrm{Cat}$, which sends a functor to an isomorphism in $h\mathrm{Cat}$ if and only if it is an equivalence.

22.10. **Proposition.** *The class* CatEq *of categorical equivalences in* sSet *satisfies 2-out-of-6, and thus 2-out-of-3.*

Proof. It is immediate that the identity map of a simplicial set is a categorical equivalence.

Next consider functors f, g, h between *quasicategories* such that gf and hg are are defined and are categorical equivalences. Then f, g, h and hgf are categorical equivalences by an argument which is word-for-word the same as in (22.9).

For the general case, we reduce to the quasicategory case by applying $\mathrm{Fun}(-, C)$, where C is an arbitrary quasicategory. □

22.11. **Weak categorical equivalence.** It turns out that we can replace the condition in the definition of categorical equivalence with some seemingly weaker conditions.

22.12. **Proposition.** *Let $f\colon X \to Y$ be a map of simplicial sets. The following are equivalent.*

(1) *The map f is a categorical equivalence: i.e., for every quasicategory C, the map $\mathrm{Fun}(Y, C) \to \mathrm{Fun}(X, C)$ induced by restriction along f admits a categorical inverse.*
(2) *For every quasicategory C, the map $h\,\mathrm{Fun}(Y, C) \to h\,\mathrm{Fun}(X, C)$ induced by restriction along f is an equivalence of ordinary categories.*
(3) *For every quasicategory C, the map $\pi_0(\mathrm{Fun}(Y, C)^{\mathrm{core}}) \to \pi_0(\mathrm{Fun}(X, C)^{\mathrm{core}})$ induced by restriction along f is a bijection of sets.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate. We prove that (3) implies (1).

In the case that f is a map between *quasicategories*, this is really what the second half of the proof of (19.6) actually shows. That is, we use the bijections

$$f^*\colon \pi_0(\mathrm{Fun}(Y, X)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(X, X)^{\mathrm{core}}), \qquad f^*\colon \pi_0(\mathrm{Fun}(Y, Y)^{\mathrm{core}}) \xrightarrow{\sim} \pi_0(\mathrm{Fun}(X, Y)^{\mathrm{core}}),$$

to (a) produce a $g\colon Y \to X$ such that $gf \approx \mathrm{id}_X$, and (b) show that $fgf \approx f\,\mathrm{id}_X = \mathrm{id}_Y\, f$ implies $fg \approx \mathrm{id}_Y$.

We reduce the case of a general map f to that of a map f' between quasicategories as follows. Use factorization to construct a commutative square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ X' \arrow[r, "f'"'] & Y' \end{tikzcd}$$

so that u and v are inner anodyne (and so categorical equivalences), and X' and Y' are quasicategories. If we apply $\mathrm{Fun}(-, C)$ to the square with C a quasicategory, the vertical maps become trivial fibrations, and hence induce bijections on isomorphism classes of objects. Therefore $\mathrm{Fun}(f, C)$ induces a bijection on isomorphism classes of objects if and only if $\mathrm{Fun}(f', C)$ does. $\square$

Joyal [Joy08a, 1.20] singles out statement (2) of (22.12) as his basic notion of equivalence, which he calls **weak categorical equivalence**[20]. We see that his notion is equivalent to our definition of categorical equivalence.

22.13. **The homotopy 2-category of** qCat**.** A **2-category** E is a category which is itself "enriched" over Cat. That is,

- for each pair of objects $x, y \in \mathrm{ob}\, E$, there is a category $\underline{\mathrm{Hom}}_E(x, y)$, so that
- the objects of $\underline{\mathrm{Hom}}_E(x, y)$ are precisely the set $\mathrm{Hom}_E(x, y)$ of morphisms of E, and
- there are "composition functors" $\mathrm{Hom}_E(y, z) \times \mathrm{Hom}_E(x, y) \to \mathrm{Hom}_E(x, z)$ for all $x, y, z \in \mathrm{ob}\, E$ which on objects is just ordinary composition of morphisms in E, which
- is unital and associative in the evident sense.

One refers to the objects of $\mathrm{Hom}_E(x, y)$ as **1-morphisms** $f\colon x \to y$ of E, and the morphisms of $\mathrm{Hom}_E(x, y)$ as **2-morphisms** $\alpha\colon f \Rightarrow g$ of E. The **underlying category** of E consists of the objects and 1-morphisms only.

The standard example of a 2-category is Cat, the category of categories, with objects=categories, 1-morphisms=functors, 2-morphisms=natural transformations.

We can enlarge the category qCat of quasicategories to a **homotopy 2-category** $h_2\mathrm{qCat}$, so that

$$\mathrm{Hom}_{h_2\mathrm{qCat}}(C, D) := h\,\mathrm{Fun}(C, D).$$

[20]This is not to be confused with "weak equivalence", which we will talk about later (36.1).

That is,

- **objects** of h_2qCat are quasicategories,
- **1-morphisms** of h_2qCat are functors between quasicategories,
- **2-morphisms** of h_2qCat are *isomorphism classes* of natural transformations of functors.

Note that qCat sits inside h_2qCat as its underlying category; thus, h_2qCat contains all the information of qCat. On the other hand hqCat is obtained from h_2qCat by first identifying 1-morphisms (functors) which are 2-isomorphic (i.e., naturally isomorphic), and then throwing away the 2-morphisms.

Part 3. Joins, slices, and Joyal's extension and lifting theorems

In this part we describe and apply two new methods to construct new quasicategories from old, called "joins" and "slices". They are both generalizations of constructions which can be carried out on categories: the most familiar of these classical constructions is *slice category* $C_{/x}$ associated to an object x of a category C, in which objects of slice $C_{/x}$ are morphisms $c \to x$ in C.

With these constructions in hand, we will be able to define notions of *limits* and *colimits* in quasicategories. We will also be able to prove some of the results we have deferred up until now, including the equivalence of quasigroupoids and Kan complexes (A) and the objectwise criterion for natural isomorphisms (C). Much of the material in this part comes from Joyal's seminal paper [Joy02].

23. Joins

23.1. **Join of categories.** If A and B are ordinary categories, we can define a category $A \star B$ called the join. This has

$$\operatorname{ob}(A \star B) = \operatorname{ob} A \amalg \operatorname{ob} B, \qquad \operatorname{mor}(A \star B) = \operatorname{mor} A \amalg (\operatorname{ob} A \times \operatorname{ob} B) \amalg \operatorname{mor} B,$$

so that we put in a *unique* map from each object of A to each object of B. Explicitly,

$$\operatorname{Hom}_{A\star B}(x, y) := \begin{cases} \operatorname{Hom}_A(x, y) & \text{if } x, y \in \operatorname{ob} A, \\ \operatorname{Hom}_B(x, y) & \text{if } x, y \in \operatorname{ob} B, \\ \{*\} & \text{if } x \in \operatorname{ob} A,\ y \in \operatorname{ob} B, \\ \varnothing & \text{if } x \in \operatorname{ob} B,\ y \in \operatorname{ob} A, \end{cases}$$

with composition defined so that the evident inclusions $A \to A \star B \leftarrow B$ are functors. (Check that this really defines a category, and that A and B are identified with full subcategories of $A \star B$.)

23.2. *Example.* We have that $[p] \star [q] \approx [p + 1 + q]$. **F 15 Feb 2019**

23.3. *Exercise* (Functors from a join of categories). Show that functors $f : A \star B \to C$ are in bijective correspondence with triples $(f_A : A \to C,\ f_B : B \to C,\ \gamma : f_A \circ \pi_A \Rightarrow f_B \circ \pi_B)$, where f_A and f_B are functors, and γ is a natural transformation of functors $A \times B \to C$, where $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ denote the evident projection functors.

23.4. *Exercise* (Functors to a join of categories). Show that functors $f : C \to A \star B$ are in bijective correspondence with triples of functors $(\pi : C \to [1],\ f_{\{0\}} : C^{\{0\}} \to A,\ f_{\{1\}} : C^{\{1\}} \to B)$, where $C^{\{j\}} := \pi^{-1}(\{j\}) \subseteq C$ is the fiber of π over $j \in \operatorname{ob}[1]$, i.e., the subcategory of C consisting of objects which π sends to j and morphisms which π sends to id_j.

23.5. *Exercise.* Describe an isomorphism $(A \star B)^{\mathrm{op}} \approx B^{\mathrm{op}} \star A^{\mathrm{op}}$.

An important special case are the **left cone** and **right cone** of a category, defined by $A^{\triangleleft} := [0] \star A$ and $A^{\triangleright} := A \star [0]$. For instance, the right cone $A^{\triangleright}$ is the category obtained by adjoining one additional object v to A, as well as a unique map $x \to v$ for each object x of $A^{\triangleright}$. In this case, v

becomes a terminal object for $A^{\triangleright}$, and we can say that $A \mapsto A^{\triangleright}$ *freely adjoins a terminal object* to A. (Note that a terminal object of A will not be terminal in $A^{\triangleright}$ anymore.) Likewise, $A \mapsto A^{\triangleleft}$ *freely adjoins an initial object* to A.

Limits and colimits of functors can be characterized using cones: if $p\colon A \to C$ is a functor, a *colimit* of p is a functor $\widehat{p}\colon A^{\triangleright} \to C$ which is initial among functors which extend p, and likewise, a *limit* of p is a functor $\widehat{p}'\colon A^{\triangleleft} \to C$ which is terminal among functors which extend p.

23.6. *Remark.* It is worthwhile to spell this out in detail. Given a functor $p\colon A \to C$, to describe a functor $q\colon A^{\triangleright} \to C$ which *extends* p, it suffices to give

(1) an object $q(v)$ in C,
(2) for each object $a \in \operatorname{ob} A$ a morphism $q(a \to v)\colon p(a) = q(a) \to q(v)$ in C, such that
(3) for each morphism $\alpha\colon a \to a'$ in A we have an equality $q(a' \to v) \circ p(\alpha) = q(a \to v)$ of morphisms $p(a) \to q(v)$ in C.

$$\begin{tikzcd} a \arrow[d, "\alpha"'] \\ a' \end{tikzcd} \qquad \Longrightarrow \qquad \begin{tikzcd} p(a) \arrow[d, "p(\alpha)"'] \arrow[dr, "q(a\to v)"] & \\ p(a') \arrow[r, "q(a'\to v)"'] & q(v) \end{tikzcd}$$

Given functors $q, q'\colon A^{\triangleright} \to C$, we may consider natural transformations $\phi\colon q \to q'$ which*extend the identity transformation* of p. Explicitly, such a transformation ϕ is exactly determined by

(1) a morphism $\phi(v)\colon q(v) \to q'(v)$ in C such that
(2) for each object $a \in \operatorname{ob} A$ we have an equality $q'(a \to v) = \phi(v) \circ q(a \to v)$ of morphisms $p(a) \to q'(v)$ in C.

$$a \qquad \Longrightarrow \qquad \begin{tikzcd} & q(v) \arrow[d, "\phi(v)"] \\ p(a) \arrow[ur, "q(a\to v)"] \arrow[r, "q'(a\to v)"'] & q'(v) \end{tikzcd}$$

An extension $\widehat{p}\colon A^{\triangleright} \to C$ of p is a *colimit* of p if for every q extending p there exists a unique map $\phi(v)\colon \widehat{p}(v) \to q(v)$ in C such that $q(a \to v) = \phi(v) \circ \widehat{p}(a \to v)$ for all $a \in \operatorname{ob} A$. The object $\widehat{p}(v)$ is what is colloquially known as "the colimit of p", although the full data of a colimit of p is actually the functor $\widehat{p}$. We will call the functor $\widehat{p}$ a *colimit cone* it what follows.

23.7. **Ordered disjoint union.** As noted above (23.2), the join operation on categories effectively descends to Δ. We will call this the **ordered disjoint union**. It is a functor $\sqcup\colon \Delta \times \Delta \to \Delta$, defined so that $[p] \sqcup [q] := [p+1+q]$, to be thought of as the disjoint union of underlying sets, ordered so that the subsets $[p]$ and $[q]$ retain their ordering, and elements of $[p]$ come before elements of $[q]$.

It is handy to extend this to the category Δ_+, the full subcategory of ordered sets obtained by adding the empty set $[-1] := \varnothing$ to Δ. The functor $\sqcup$ extends in an evident way to $\sqcup\colon \Delta_+ \times \Delta_+ \to \Delta_+$. This extended functor makes Δ_+ into a (strict, but nonsymmetric) monoidal category, with unit object $[-1]$.

Note that for any map $f\colon [p] \to [q_1] \sqcup [q_2]$ in Δ_+, there is a *unique* decomposition $[p] = [p_1] \sqcup [p_2]$ such that $f = f_1 \sqcup f_2$ for some (necessarily unique) $f_i\colon [p_i] \to [q_i]$ in Δ_+. (We need an object $[-1]$ to be able to say this, even if $p, q_1, q_2 \geq 0$; if $f([p]) \subseteq [q_1]$ then $p_2 = -1$.)

23.8. **Join of simplicial sets.** Let X and Y be simplicial sets. The **join** of X and Y is a simplicial set $X \star Y$ defined as follows. It has n-dimensional elements

$$(X \star Y)_n := \coprod_{[n]=[n_1]\sqcup[n_2]} X_{n_1} \times Y_{n_2},$$

where $[n_1], [n_2] \in \operatorname{ob}\Delta_+$, and we declare $X_{-1} = * = Y_{-1}$ to be a one-point set. The action of simplicial operators is defined in the evident way, using the observation of the previous paragraph: for $(x, y) \in X_{n_1} \times Y_{n_2} \subseteq (X \star Y)_n$ and $f\colon [m] \to [n]$, we have $(x, y)f = (xf_1, yf_2) \in X_{m_1} \times Y_{m_2} \subseteq (X \star Y)_m$, where $f = f_1 \sqcup f_2$, $f_j\colon [m_j] \to [n_j]$ is the unique decomposition of f over $[n] = [n_1] \sqcup [n_2]$.

23.9. *Exercise.* Check that the above defines a simplicial set.

In particular,

$$\begin{aligned}(X \star Y)_0 &= X_0 \amalg Y_0,\\ (X \star Y)_1 &= X_1 \amalg X_0 \times Y_0 \amalg Y_1,\\ (X \star Y)_2 &= X_2 \amalg X_1 \times Y_0 \amalg X_0 \times Y_1 \amalg Y_2,\end{aligned}$$

and so on.

Note that there are evident maps $X \to X \star Y \leftarrow Y$, which give isomorphisms from X and Y to subcomplexes of $X \star Y$, and these subcomplexes are disjoint from each other.

There are isomorphisms

$$(X \star Y) \star Z \xrightarrow{\sim} X \star (Y \star Z),$$

natural in X, Y, Z: on either side, the set of n-dimensional elements can described as $\coprod_{[n]=[n_1]\sqcup[n_2]\sqcup[n_3]} X_{n_1} \times Y_{n_2} \times Z_{n_3}$. Together with the evident isomorphisms $\varnothing \star X \approx X \star X \star \varnothing$, the join gives a monoidal structure on sSet with unit object $\Delta^{-1} := \varnothing$. Note that $\star$ is *not* symmetric monoidal, though it is true that $(Y \star X)^{\mathrm{op}} \approx X^{\mathrm{op}} \star Y^{\mathrm{op}}$. (*Exercise:* verify this.)

23.10. **Joins of simplices.** We have the (unique) isomorphism

$$\Delta^p \star \Delta^q \approx \Delta^{p+1+q}.$$

Furthermore, if $f\colon [p'] \to [p]$ and $g\colon [q'] \to [q]$ are simplicial operators, then the induced map $f \star g\colon \Delta^{p'} \star \Delta^{q'} \to \Delta^p \star \Delta^q$ between joins of simplices is uniquely isomorphic to $(f \sqcup g)\colon \Delta^{p'+1+q'} \to \Delta^{p+1+q}$.

In particular, if $S \subseteq [p]$ and $T \subseteq [q]$ are subsets, giving rise to subcomplexes $\Delta^S \subseteq \Delta^p$ and $\Delta^T \subseteq \Delta^q$, then the evident map $\Delta^S \star \Delta^T \to \Delta^p \star \Delta^q \approx \Delta^{p+1+q}$ realizes the inclusion of the subcomplex $\Delta^{S\sqcup T} \subseteq \Delta^{p+1+q}$ associated to the subset $S \sqcup T \subseteq [p] \sqcup [q] = [p+1+q]$. This makes it relatively straightforward to describe the join of subcomplexes of standard simplices.

23.11. **Left and right cones of simplicial sets.** An important example of joins of simplicial sets are the cones. The **left cone** and **right cone** of a simplicial set X are

$$X^{\lhd} := \Delta^0 \star X, \qquad X^{\rhd} := X \star \Delta^0.$$

Note that outer horns are examples of cones:

$$(\partial\Delta^n)^{\lhd} = \Delta^0 \star \partial\Delta^n \approx \Lambda^{n+1}_0, \qquad (\partial\Delta^n)^{\rhd} = \partial\Delta^n \star \Delta^0 \approx \Lambda^{n+1}_{n+1}.$$

23.12. *Exercise.* Let $f\colon [m] \to [n]$ be any simplicial operator. Show that the induced map $f\colon \Delta^m \to \Delta^n$ on standard simplices is uniquely isomorphic to a join of maps $f_0 \star f_1 \star \cdots \star f_n$, with $f_j\colon \Delta^{m_j} \to \Delta^0$, where each $m_j \geq -1$.

It is straightforward to show that the nerve takes joins of categories to joins of simplicial sets: $N(A \star B) \approx N(A) \star N(B)$, and thus $N(A^{\lhd}) \approx (NA)^{\lhd}$ and $N(A^{\rhd}) \approx (NA)^{\rhd}$. (*Exercise:* prove this.)

23.13. **The join of quasicategories is a quasicategory.** Here is a handy rule for constructing maps into a join (compare (23.4)). Note that every join admits a canonical map $\pi\colon X \star Y \to \Delta^0 \star \Delta^0 \approx \Delta^1$, namely the join applied to the projections $X \to \Delta^0$ and $Y \to \Delta^0$.

23.14. **Lemma** ([Joy08a, Prop. 3.5], compare (23.4))**.** *Maps $f\colon K \to X \star Y$ are in bijective correspondence with the set of triples*

$$\big(\pi\colon K \to \Delta^1, \quad f_{\{0\}}\colon K^{\{0\}} \to X, \quad f_{\{1\}}\colon K^{\{1\}} \to Y\big),$$

where $K^{\{j\}} := \pi^{-1}(\{j\}) \subseteq K$, the pullback of $\{j\} \to \Delta^1$ along π.

Proof. This is a straightforward exercise. In one direction, the correspondence sends f to $(\overline{\pi}f, f|K^{\{0\}}, f|K^{\{1\}})$, where $\overline{\pi}\colon X \star Y \to \Delta^0 \star \Delta^0 = \Delta^1$. ☐

23.15. **Proposition.** *If C and D are quasicategories, so is $C \star D$.*

Proof. Use the previous lemma (23.14), together with the observations (which we leave as an exercise) that for any map $\pi\colon \Lambda^n_j \to \Delta^1$ from an *inner* horn, the preimages $\pi^{-1}(\{0\})$ and $\pi^{-1}(\{1\})$ are either inner horns, standard simplices, or are empty, and for any map $\pi\colon \Delta^n \to \Delta^1$ from a standard simplex, the preimages are either a standard simplex or empty. ☐

24. Slices

24.1. **Slices of categories.** Given an ordinary category C, and an object $x \in \operatorname{ob} C$, we may form the **slice categories** $C_{x/}$ and $C_{/x}$, also called **undercategory** and **overcategory**, or **slice-over category** and **slice-under category**.

For instance, the slice-over category $C_{/x}$ is the category whose *objects* are maps $f\colon c \to x$ with target x, and whose *morphisms* $(f\colon c \to x) \to (f'\colon c' \to x)$ are maps $g\colon c \to c'$ such that $f'g = f$.

This can be reformulated in terms of joins. Let "T" denote the terminal category (isomorphic to $[0]$). Note that $\operatorname{ob} C_{/x}$ corresponds to the set of functors $f\colon [0] \star T \to C$ such that $f|T = x$, and $\operatorname{mor} C_{/x}$ corresponds to the set of functors $g\colon [1] \star T \to C$ such that $g|T = x$.

More generally, given a functor $p\colon A \to C$ of categories, we obtain slice categories $C_{p/}$ and $C_{/p}$ defined as follows. The category $C_{/p}$ has

- **objects:** functors $f\colon [0] \star A \to C$ such that $f|A = p$,
- **morphisms $f \to f'$:** functors $g\colon [1] \star A \to C$ such that $g|A = p$.

Likewise, the category $C_{p/}$ has

- **objects:** functors $f\colon A \star [0] \to C$ such that $f|A = p$,
- **morphisms $f \to f'$:** functors $g\colon A \star [1] \to C$ such that $g|A = p$.

24.2. *Exercise.* Describe composition of morphisms in $C_{/p}$ and $C_{p/}$.

24.3. *Exercise.* Show that $(C_{p/})^{\mathrm{op}} \approx (C^{\mathrm{op}})_{p^{\mathrm{op}}/}$ (isomorphism of categories).

24.4. *Exercise.* Fix a functor $p\colon A \to C$, and let B be a category. Construct bijections

$$\{\text{functors } f\colon B \to C_{/p}\} \leftrightarrow \{\text{functors } g\colon B \star A \to C \text{ s.t. } g|A = p\}$$

and

$$\{\text{functors } f\colon B \to C_{p/}\} \leftrightarrow \{\text{functors } g\colon A \star B \to C \text{ s.t. } g|A = p\}.$$

24.5. *Remark.* The notions of limits and colimits can be formulated very compactly in terms of the general notion of slices. Thus, given a functor $p\colon A \to C$, a *colimit* of p is the same data as an initial object of $C_{p/}$, while a *limit* of p is the same data as a terminal object of $C_{/p}$. (*Exercise:* prove this; we will directly generalize this forumlation to define limits and colimits for quasicategories. Compare (23.6).)

24.6. **Joins and colimits of simplicial sets.** The join functor $\star\colon s\mathrm{Set}\times s\mathrm{Set}\to s\mathrm{Set}$ is in some ways analogous to the product functor $\times$, e.g., it is a monoidal functor.

The product operation $(-)\times(-)$ on simplicial sets commutes with colimits in each input, and the functors $X\times -$ and $-\times X$ admit right adjoints (in both cases, the right adjoint is $\mathrm{Map}(X,-)$). The join functor does not commute with colimits in each variable, but *almost* does so; the only obstruction is the value on the initial object

More precisely, the functors $X\star -$ and $-\star X\colon s\mathrm{Set}\to s\mathrm{Set}$ do not preserve the initial object, since $X\star\varnothing\approx X\approx\varnothing\star X$. However, (the identity map of) X is tautologically the initial object of $s\mathrm{Set}_{X/}$, the slice category of simplicial sets under X.

24.7. **Proposition.** *For every simplicial set X, the induced functors*

$$X\star -,\quad -\star X\colon s\mathrm{Set}\to s\mathrm{Set}_{X/}$$

preserve colimits.

Proof. Ths follows from the degreewise formula for the join, which has the form:

$$(X\star Y)_n = X_n \amalg (X_{n-1}\times Y_0)\amalg\cdots\amalg(X_0\times Y_{n-1})\amalg Y_n = X_n\amalg(\text{terms which are “linear” in }Y).$$

That is, for each $n\geq 0$ the functor $Y\mapsto (X\star Y)_n\colon s\mathrm{Set}\to\mathrm{Set}_{X_n/}$ is seen to be colimit preserving, since each functor $X_k\times(-)\colon\mathrm{Set}\to\mathrm{Set}$ is colimit preserving. $\square$

24.8. *Exercise* (Trivial, but important). Show that the functors $X\star -$ and $-\star X\colon s\mathrm{Set}\to s\mathrm{Set}$ preserve *pushouts*.

24.9. **Slices of simplicial sets.** We have seen that the functors

$$S\star -\colon s\mathrm{Set}\to s\mathrm{Set}_{S/}\qquad\text{and}\qquad -\star T\colon s\mathrm{Set}\to s\mathrm{Set}_{T/}$$

preserve colimits, and therefore we predict that they admit right adjoints. These exist, and are called **slice** functors, denoted

$$(p\colon S\to X)\mapsto X_{p/}\colon s\mathrm{Set}_{S/}\to s\mathrm{Set}$$

and

$$(q\colon T\to X)\mapsto X_{/q}\colon s\mathrm{Set}_{T/}\to s\mathrm{Set}.$$

I will sometimes distinguish these as **slice-under** and **slice-over**, respectively. Explicitly, there are are bijective correspondences

$$\left\{\begin{tikzcd} S \arrow[r, "p"] \arrow[d] & X \\ S\star K \arrow[ur, dotted] \end{tikzcd}\right\}\iff\{K\dashrightarrow X_{p/}\},\qquad \left\{\begin{tikzcd} T \arrow[r, "q"] \arrow[d] & X \\ K\star T \arrow[ur, dotted] \end{tikzcd}\right\}\iff\{K\dashrightarrow X_{/q}\}. \tag{24.10}$$

Here we write “$S\to S\star K$” and “$T\to K\star T$” for the inclusions $S\star\varnothing\subseteq S\star K$ and $\varnothing\star T\subseteq K\star T$, using the canonical isomorphisms $S\star\varnothing = S$ and $\varnothing\star T=T$.

Taking $K=\Delta^n$ we obtain the formulas **M 18 Feb 2019**

$$(X_{p/})_n = \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S\star\Delta^n, X),\qquad (X_{/q})_n=\mathrm{Hom}_{s\mathrm{Set}_{T/}}(\Delta^n\star T, X),$$

which we regard as the definition of slices. (I.e., these formulas specify the n-dimensional elements of the slices, and naturality in “Δ^n” specifies the action of simplicial operators.)

24.11. *Exercise.* Given this explicit definition of slices in terms of their elements and the action of simplicial operators, verify the bijective corrrespondences (24.10).

In particular, we note the special cases associated to $x\colon\Delta^0\to X$:

$$\mathrm{Hom}_{s\mathrm{Set}}(K, X_{x/}) = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(\Delta^0\star K, X)\approx\mathrm{Hom}_{s\mathrm{Set}_*}((K^{\triangleleft}, v),(X,x)),$$

$$\mathrm{Hom}_{s\mathrm{Set}}(K, X_{/x}) = \mathrm{Hom}_{s\mathrm{Set}_{\Delta^0/}}(K\star\Delta^0, X)\approx\mathrm{Hom}_{s\mathrm{Set}_*}((K^{\triangleright}, v),(X,x)).$$

The notation (X,x) with $x \in X_0$ represents a **pointed simplical set**, the category of which is $s\mathrm{Set}_* := s\mathrm{Set}_{\Delta^0/}$. We write v for the cone point of $K^{\triangleleft}$ and $K^{\triangleright}$.

The slice construction for simplicial sets agrees with that for categories.

24.12. **Proposition.** *The nerve preserves slices; i.e., if $p\colon A \to C$ is a functor, then $N(C_{p/}) \approx (NC)_{Np/}$ and $N(C_{/p}) \approx (NC)_{/Np}$.*

Proof. Left as an exercise. ☐

24.13. **Slice as a functor.** The function complex construction $\mathrm{Map}(-,-)$ is a functor in two variables, contravariant in the first and covariant in the second. The slice constructions also behave something like a functor of two variables, though it is a little more complicated, because the slice constructions also depend on a map between the two objects. A precise statement is that every diagram on the left gives rise to commutative diagrams as on the right.

$$\begin{tikzcd} S \arrow[r, "p"] & X \arrow[d, "f"] \\ T \arrow[u, "j"] \arrow[r, "fpj"'] & Y \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} X_{/p} \arrow[r] \arrow[d] & Y_{/fp} \arrow[d] \\ X_{/pj} \arrow[r] & Y_{/fpj} \end{tikzcd} \qquad \begin{tikzcd} X_{p/} \arrow[r] \arrow[d] & Y_{fp/} \arrow[d] \\ X_{pj/} \arrow[r] & Y_{fpj/} \end{tikzcd}$$

There seems to be no decent notation for the maps in the right-hand squares. The whole business of joins and slices can get pretty confusing because of this.

24.14. *Remark.* A very precise formulation is that each kind of slice defines a functor $s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}$ from the *twisted arrow category* of simplicial sets, whose objects are maps p of simplicial sets, and whose morphisms are pairs $(j,f)\colon p \to fpj$, where j and f are themselves maps of simplicial sets.

Let's spell this out in terms of the correspondence between "maps into slices" and "maps from joins". Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider "restriction map" $X_{p/} \to Y_{fpj/}$. The composite of a map $u\colon K \to X_{p/}$ with this restriction map is described in terms of the bijection of (24.10) as follows. The map u corresponds to a dotted arrow in

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, tail] & S \arrow[r, "p"] \arrow[d, tail] & X \arrow[r, "f"] & Y \\ T \star K \arrow[r, "j\star K"'] & S \star K \arrow[ur, dotted, "\widetilde{u}"'] & & \end{tikzcd}$$

The composite $K \xrightarrow{u} X_{p/} \to Y_{fpj/}$ corresponds to $f\widetilde{u}(j \star K)$.

A particular special case which we will see a lot of are the "restriction" functors

$$X_{/p} \to X \qquad \text{and} \qquad X_{p/} \to X$$

induced by sequence $\varnothing \xrightarrow{j} S \xrightarrow{p} X$, using that $X_{/\varnothing} = X = X_{\varnothing/}$. For instance, $X_{/p} \to X$ sends an n-dimensional element $x \in (X_{/p})_n$ corresponding to $\widetilde{x}\colon \Delta^n \star S \to X$ extending p to the n-dimensional element of X represented by the map $\widetilde{x}|(\Delta^n \star \varnothing)$ defined as the composite

$$\Delta^n = \Delta^n \star \varnothing \to \Delta^n \star S \xrightarrow{x} X.$$

Another special case of interest are the "projection" functors

$$X_{/p} \to Y_{/fp} \qquad \text{and} \qquad X_{p/} \to Y_{fp/}$$

induced by the sequence $S \xrightarrow{p} X \xrightarrow{f} Y$. For instance, $X_{/p} \to Y_{/fp}$ sends an n-dimensional element $x \in (X_{/p})_n$ corresponding to $\widetilde{x}\colon \Delta^n \star S \to X$ extending p to the n-dimensional element of $Y_{fp/}$ represented by $f\widetilde{x}\colon \Delta^n \star S \to Y$.

24.15. *Exercise.* Let $p\colon S \to X$ and $q\colon T \to X$ be maps of simplicial sets. Describe and prove bijections between the following sets of solutions to lifting problems:

$$\left\{ \begin{tikzcd} & X_{p/} \arrow[d] \\ T \arrow[r, "q"'] \arrow[ur, dotted] & X \end{tikzcd} \right\} \iff \left\{ \begin{tikzcd} S \amalg T \arrow[r, "{(p,q)}"] \arrow[d, tail] & X \\ S \star T \arrow[ur, dotted] & \end{tikzcd} \right\} \iff \left\{ \begin{tikzcd} & X_{/q} \arrow[d] \\ S \arrow[r, "p"'] \arrow[ur, dotted] & X \end{tikzcd} \right\}$$

Here $X_{p/} \to X$ and $X_{/q} \to X$ are the evident restriction maps, and $S \amalg T \to S \star T$ the tautological inclusion.

25. Slices of quasicategories

In this section we show that, given a quasicategory C and an object $x \in C_0$, both $C_{/x}$ and $C_{x/}$ are also quasicategories.

We recall the sets **left horns**

$$\mathrm{LHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 \le k < n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_0 \subset \Delta^n \mid n \ge 1 \,\}$$

and the **right horns**

$$\mathrm{RHorn} := \{\, \Lambda^n_k \subset \Delta^n \mid 0 < k \le n,\ n \ge 1 \,\} = \mathrm{InnHorn} \cup \{\, \Lambda^n_n \subset \Delta^n \mid n \ge 1 \,\}.$$

The associated weak saturations $\overline{\mathrm{LHorn}}$ and $\overline{\mathrm{RHorn}}$ are the **left anodyne** and **right anodyne** maps. The associated right complements

$$\mathrm{LFib} := \mathrm{LHorn}^{\boxslash}, \qquad \mathrm{RFib} := \mathrm{RHorn}^{\boxslash}$$

are the **left fibrations** and **right fibrations**. Note that

$$\overline{\mathrm{InnHorn}} \subseteq \overline{\mathrm{LHorn}} \cap \overline{\mathrm{RHorn}} \qquad \text{and} \qquad \mathrm{LFib} \cup \mathrm{RFib} \subseteq \mathrm{InnFib}.$$

These classes correspond to each other under the opposite involution $(-)^{\mathrm{op}}\colon s\mathrm{Set} \to s\mathrm{Set}$; i.e., $\mathrm{LHorn}^{\mathrm{op}} = \mathrm{RHorn}$, $\mathrm{LFib}^{\mathrm{op}} = \mathrm{RFib}$.

25.1. **Proposition.** *Let C be a quasicategory and $x \in C_0$. The evident maps $C_{x/} \to C$ and $C_{/x} \to C$ which "forget x" (i.e., induced by the sequence $\varnothing \to \{x\} \to C$) are left fibration and right fibration respectively. In particular, $C_{x/}$ and $C_{/x}$ are also quasicategories.*

Proof. I claim that $\pi\colon C_{/x} \to C$ is a right fibration. Explicitly, this map sends the n-dimensional element $a\colon \Delta^n \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^n \star \Delta^0 \to C$ such that $\widetilde{a}|(\varnothing \star \Delta^0) = x$, to the n-dimensional element represented by $\widetilde{a}|(\Delta^n \star \varnothing) \to C$. Using the join/slice adjunction, there is a bijective correspondence between lifting problems

$$\begin{tikzcd} \Lambda^n_j \arrow[r, "f"] \arrow[d, tail] & C_{/x} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd} \iff \begin{tikzcd} \varnothing \star \Delta^0 \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Lambda^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing) \arrow[r, "{(\widetilde{f},g)}"] \arrow[d, tail] & C \\ & \Delta^n \star \Delta^0 \arrow[ur, dotted] & \end{tikzcd}$$

Note that there is a unique isomorphism $\Delta^n \star \Delta^0 \approx \Delta^{n+1}$. For any subset $S \subset [n]$, the above isomorphism identifies the subcomplex $\Delta^S \star \Delta^0 \subset \Delta^n \star \Delta^0$ with $\Delta^{S \cup \{n+1\}} \subset \Delta^{n+1}$, while $\Delta^S \star \varnothing \subset \Delta^n \star \Delta^0$ is identified with $\Delta^S \subseteq \Delta^{n+1}$. Since $\Lambda^n_j = \bigcup_{k \in [n] \smallsetminus j} \Delta^{[n] \smallsetminus k}$, we see that

(1) the subcomplex $(\Delta^n_j \star \Delta^0) \cup_{\Lambda^n_j \star \varnothing} (\Delta^n \star \varnothing)$ of $\Delta^n \star \Delta^0$ is the horn $\Lambda^{n+1}_j \subset \Delta^{n+1}$, and
(2) the subcomplex $\varnothing \star \Delta^0$ of $\Delta^n \star \Delta^0$ is the vertex $\{n+1\}$.

Thus, the right hand diagram above is isomorphic to

$$\begin{tikzcd} \{n+1\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_j \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

If C is a quasicategory, then an extension exists for $0 < j \leq n$.

Since right fibrations are inner fibrations, the composite $C_{/x} \to C \to *$ is an inner fibration, and thus $C_{/x}$ is a quasicategory.

The case of $C_{x/} \to C$ is similar, using the correspondence

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, tail] & C_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C \end{tikzcd} \iff \begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \Lambda^{n+1}_{j+1} \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

□

26. Initial and terminal objects

We give the definition of initial and terminal object in a quasicategory, and we reformuate it in terms of slices.

26.1. **Initial and terminal objects.** An **initial object**[21] of a quasicategory C is an $x \in C_0$ such that every $f\colon \partial\Delta^n \to C$ (for all $n \geq 1$) such that $f|\{0\} = x$, there exists an extension $f'\colon \Delta^n \to C$.

A **terminal object** of C is an initial object of C^{op}. That is, a $y \in C_0$ such that every $f\colon \partial\Delta^n \to C$ with $f|\{n\} = y$ extends to Δ^n.

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd} \qquad \begin{tikzcd} \{n\} \arrow[r, tail] \arrow[rr, bend left, "y"] & \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted, "\exists"'] & \end{tikzcd}$$

Let's spell out the first parts of the definition of initial object applied to $x \in C_0$:

- The condition for $n = 1$ says that for every object c in C there exists $f\colon x \to c$,
- The condition for $n = 2$ says that for every triple of maps $f\colon x \to c$, $g\colon c \to c'$, and $h\colon x \to c'$, we must have $[h] = [g][f]$. In particular (taking $f = 1_x$), we see there is at most one homotopy class of maps from x to any object.

If C is the nerve of an ordinary category, then $\mathrm{Hom}(\Delta^n, C) \xrightarrow{\sim} \mathrm{Hom}(\partial\Delta^n, C)$ for all $n \geq 3$. Thus, for ordinary categories, this definition coincides with the usual notion of initial object.

For general quasicategories, we see that an initial object $x \in C_0$ necessarily satisfies $\mathrm{Hom}_{hC}(x, y) \approx *$ for all $y \in C_0$, so that x represents an initial object in the homotopy category hC, but this is not sufficient to be initial in C: there are also an infinite sequence of "higher" conditions that an initial object of a quasicategory must satisfy.

We will now reformulate these notions using slice categories.

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[21] We use Joyal's definition of initial and terminal object [Joy02, §4] here. Lurie's definition [Lur09, 1.2.12.1] is different, but is equivalent to what we use, by [Lur09, 1.2.12.5] and (26.3).

26.2. **Reformulation of initial/terminal via slices.** We can restate the definition of initial/terminal object using the "forgetful" functor of the relevant slice.

26.3. **Proposition.** *If C is a quasicategory, then $x \in C_0$ is initial if and only if $C_{x/} \to C$ is a trivial fibration, and terminal if and only if $C_{/x} \to C$ is a trivial fibration.*

Proof. This is an application of the join/slice adjunction. Applied to $\partial\Delta^n \subset \Delta^n$ with $n \geq 0$ and $C_{x/} \to C$, this has the form

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "f"] \arrow[d, tail] & C_{x/} \arrow[d, "\pi"] \\ \Delta^n \arrow[r, "g"'] \arrow[ur, dotted] & C \end{tikzcd} \iff \begin{tikzcd} \Delta^0 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Delta^0 \star \partial\Delta^n) \cup_{\varnothing\star\partial\Delta^n} (\varnothing \star \Delta^n) \arrow[r, "(\widetilde{f},g)"] \arrow[d, tail] & C \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

The right-hand diagram is isomorphic to

$$\begin{tikzcd} \{0\} \arrow[r, tail] \arrow[rr, bend left, "x"] & \partial\Delta^{n+1} \arrow[r] \arrow[d, tail] & C \\ & \Delta^{n+1} \arrow[ur, dotted] & \end{tikzcd}$$

Thus $C_{x/}$ is in TrivFib $=$ Cell$^{\boxslash}$ if and only if x is an initial object of C, as desired. $\square$

26.4. *Remark.* This implies that if x is initial, then $C_{x/} \to C$ is a categorical equivalence. Later (41.1) we'll be able to show the converse: if $C_{x/} \to C$ is a categorical equivalence, then x is initial.

26.5. **Uniqueness of initial and terminal objects.** A crucial fact about initial and terminal objects in an ordinary category is that they are *unique up to unique isomoprhism.* One way to formulate this is as follows: given a category C, let $C^{\text{init}} \subseteq C$ be the full subcategory spanned by the initial objects. Then one of two cases applies: either there are no initial objects, so C^{init} is empty, or there is at least one initial object, and C^{init} is equivalent to the terminal category $[0]$. **W 20 Feb 2019**

This leads to an analogous formulation for quasicategories.

26.6. **Proposition.** *Let C be a quasicategory. Let C^{init} and C^{term} denote respectively the full subcategories spanned by initial objects and terminal objects. Then (i) either C^{init} is empty or is categorically equivalent to the terminal quasicategory Δ^0, and (ii) either C^{term} is empty or is categorically equivalent to the terminal quasicategory Δ^0.*

Proof. Since $C^{\text{term}} = ((C^{\text{op}})^{\text{init}})^{\text{op}}$, we just need to consider the case of initial objects. By definition of initial object, any $f\colon \partial\Delta^n \to C^{\text{init}}$ with $n \geq 1$ can be extended to $g\colon \Delta^n \to C$, and the image of g must lie in the full subcategory C^{init} since all of its vertices do. If $C^{\text{init}} \neq \varnothing$, then this extension condition also holds for $n = 0$, whence $C \to \Delta^0$ is a trivial fibration, and thus C is categorically equivalent to Δ^0 by (20.1). $\square$

There are some seemingly obvious facts about initial and terminal objects that we can't prove just yet.

D. **Deferred Proposition.**

(1a) *Let $f\colon x \to y$ be a morphism in a quasicategory C, and let $\widetilde{f} \in (C_{x/})_0$ be the object of the slice which corresponds to $f \in C_1$. Then $\widetilde{f}$ is initial in $C_{x/}$ if and only if f is an isomorphism.*

(1b) *Let $f\colon x \to y$ be a morphism in a quasicategory C, and let $\widetilde{f} \in (C_{/y})_0$ be the object of the slice which corresponds to $f \in C_1$. Then $\widetilde{f}$ is terminal in $C_{/y}$ if and only if f is an isomorphism.*

(2) *In a quasicategory, every object which is isomorphic to an initial object is initial, and any object isomorphic to a terminal object is terminal.*

Proofs will be given in (30.7).

26.7. **Initial and terminal objects in functor categories.** Here is a sample of a property of initial/terminal objects that we can now prove. A functor between ordinary categories whose values are all initial (or terminal) objects is itself initial (or terminal) as an object of the functor category. The same holds with categories replaced by quasicategories.

26.8. **Proposition.** *Consider a map $f\colon X \to C$ from a simplicial set to a quasicategory. Suppose that for every vertex $x \in X_0$ the object $f(x) \in C_0$ is initial (resp. terminal) in C. Then the functor f is initial (resp. terminal) viewed as an object of* $\mathrm{Fun}(X, C)$.

As a consquence, if C has an initial (or terminal) object c_0, then the "constant" map (defined as the composite $X \to \{c_0\} \to C$) is an initial (or terminal) object of $\mathrm{Fun}(X, C)$.

26.9. *Remark.* In other words, there is an inclusion $\mathrm{Fun}(X, C^{\mathrm{init}}) \subseteq \mathrm{Fun}(X, C)^{\mathrm{init}}$ of the full subcategories of "objectwise initial functors" and "initial functors" in $\mathrm{Fun}(X, C)$. Using (D)(2) you can also show that when C^{init} is non-empty then $\mathrm{Fun}(X, C^{\mathrm{init}}) = \mathrm{Fun}(X, C)^{\mathrm{init}}$. To see this, pick an initial object $c_0 \in C^{\mathrm{init}}$ and let $f_0\colon X \to C$ be the constant map with image $\{c_0\} \subseteq C$. Since any two initial objects are isomorphic, every $f \in \mathrm{Fun}(X, C)_0^{\mathrm{init}}$ is naturally isomorphic to f_0, and therefore $f(x)$ is isomorphic to $f_0(x) = c_0$ for every $x \in X_0$. By (D)(2), $f(x)$ must be initial in C, so $f \in \mathrm{Fun}(X, C^{\mathrm{init}})_0$.

On the other hand, it is possible for $\mathrm{Fun}(X, C)^{\mathrm{init}}$ to be non-empty when C^{init} is empty. (*Exercise:* give an example. *Hint:* think small.)

Proof. (26.8) Assume $f(x) \in C_0$ is initial in C for all $x \in X_0$. Suppose given $g\colon \partial\Delta^n \to \mathrm{Fun}(X, C)$ with $n \geq 1$ and $g|\{0\} = f$. We want to show that there exists an extension $g'\colon \Delta^n \to \mathrm{Fun}(X, C)$ of g. We convert this to the adjoint lifting problem:

$$\begin{tikzcd}
\{0\} \times X \arrow[r] \arrow[rr, bend left, "f"] & \partial\Delta^n \times X \arrow[r, "\widetilde{g}"] \arrow[d, tail] & C \\
& \Delta^n \times X \arrow[ur, dotted, "\widetilde{g}'"'] &
\end{tikzcd}$$

The strategy is to construct the extension by inductively constructing extensions $\widetilde{g}_k\colon F_k \to C$ where $F_k = (\partial\Delta^n \times X) \cup \mathrm{Sk}_k(\Delta^n \times X)$, $k \geq 0$ is the skeletal filtration (15.24) of the inclusion $\partial\Delta^n \times X \to \Delta^n \times X$. That is, we need to inductively construct lifts $\widetilde{g}_k$ in

$$\begin{tikzcd}
\coprod_{h \in F_k^{\mathrm{nd}} \smallsetminus F_{k-1}^{\mathrm{nd}}} \partial\Delta^k \arrow[r, "(h|\partial\Delta^k)"] \arrow[d, tail] & F_{k-1} \arrow[r, "\widetilde{g}_{k-1}"] \arrow[d, tail] & C \\
\coprod_{h \in F_k^{\mathrm{nd}} \smallsetminus F_{k-1}^{\mathrm{nd}}} \Delta^k \arrow[r, "(h)"'] & F_k \arrow[ur, dotted, "\widetilde{g}_k"'] &
\end{tikzcd}$$

for all $k \geq 0$. For $k = 0$ we have $F_{-1} = F_0$, since $n \geq 1$ so $(\partial\Delta^n \times X)_0 = (\Delta^n \times X)_0$.

For $k \geq 1$, note that a k-dimensional element $h = (a, b)\colon \Delta^k \to \Delta^n \times X$ is not contained in in the subcomplex $\partial\Delta^n \times X$ if and only if $a \in (\Delta^n)_k \smallsetminus (\partial\Delta^n)_k$, i.e., if the corresponding simplciial operator $a\colon [k] \to [n]$ is surjective. Therefore such $a\colon \Delta^k \to \Delta^n$ sends the vertex $0 \in (\Delta^k)_0$ to $0 \in (\Delta^n)_0$. Therefore, each composite

$$\partial\Delta^k \xrightarrow{h|\partial\Delta^k} F_{k-1} = (\partial\Delta^n \times X) \cup \mathrm{Sk}_{k-1}(\Delta^n \times X) \xrightarrow{\widetilde{g}_{k-1}} C$$

sends the vertex 0 to $\widetilde{g}_{k-1}(0, b(0)) = \widetilde{g}(0, b(0)) = f(0, b(0))$, which by hypothesis is an inital object of C. Therefore an extension of $(\widetilde{g}_{k-1}h)|\partial\Delta^k$ along $\partial\Delta^k \subset \Delta^k$ exists as desired. □

27. Joins and slices in lifting problems

Recall that for an object x in a quasicategory C, the slice objects $C_{x/}$ and $C_{/x}$ are also quasicategories. It turns out that the conclusion remains true for more general kinds of slices of quasicategories.

27.1. **Proposition.** *Let $p\colon S \to C$ be a map of simplicial sets, and suppose C is a quasicategory. Then both $C_{p/}$ and $C_{/p}$ are quasicategories.*

The proof is just like that of (25.1): we will show below (27.15) that $C_{p/} \to C$ is a left fibration and $C_{/p} \to C$ is a right fibration.

To set this up, we need a little technology about how joins interact with lifting problems.

27.2. **Pushout-joins.** We define an analogue of the pushout-product for the the join. Given maps $i\colon A \to B$ and $j\colon K \to L$ of simplicial sets, the **pushout-join** (or **box-join**) $i \circledast j$ is the map

$$i \circledast j\colon (A \star L) \amalg_{A\star K} (B \star K) \xrightarrow{(i\star L, B\star j)} B \star L.$$

27.3. *Warning.* Unlike the pushout-product, the pushout-join is not symmetric, since the join is not symmetric: $i \circledast j \not\approx j \circledast i$.

27.4. *Example.* We have already observed examples of pushout-joins in the proof of (25.1), namely

$$(\Lambda^n_j \subset \Delta^n) \circledast (\varnothing \subset \Delta^0) \approx (\Lambda^{n+1}_j \subset \Delta^{n+1}), \quad (\varnothing \subset \Delta^0) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{1+n}_{1+j} \subset \Delta^{1+n}),$$

and also

$$(\varnothing \subset \Delta^0) \circledast (\partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{1+n} \subset \Delta^{1+n}), \quad (\partial\Delta^n \subset \Delta^n) \circledast (\varnothing \subset \Delta^0) \approx (\partial\Delta^{n+1} \subset \Delta^{n+1})$$

in the proof of (26.3). These generalize to arbitrary horns and cells. The pushout-join of a horn with a cell is always a horn:

$$(\Lambda^n_j \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\Lambda^{n+1+k}_j \subset \Delta^{n+1+k}),$$
$$(\partial\Delta^k \subset \Delta^k) \circledast (\Lambda^n_j \subset \Delta^n) \approx (\Lambda^{k+1+n}_{k+1+j} \subset \Delta^{k+1+n}).$$

Also, the pushout-join of a cell with a cell is always a cell:

$$(\partial\Delta^n \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) \approx (\partial\Delta^{n+1+k} \subset \Delta^{n+1+k}).$$

We leave proofs as an exercise for the reader.

27.5. *Exercise.* Prove the isomorphisms asserted in (27.4). (*Hint:* use (23.10).)

27.6. *Remark.* Both pushout-product and pushout-join are special cases of a general construction: given any functor $F\colon s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$ of two variables, you get a corresponding "pushout-F" functor: $F_\square\colon \mathrm{Fun}([1], s\mathrm{Set}) \times \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$. We will meet more examples later.

27.7. **Pullback-slices.** Just as the pushout-product is associated to the pullback-hom, so the pushout-join is associated to two kinds of **pullback-slices** (or **box-slices**). Given a sequence of maps $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, we define the map

$$f^{\boxast_p j}\colon X_{/p} \to X_{/pj} \times_{Y_{/fpj}} Y_{/fp},$$

where the maps defining the pullback and the components of $f^{\boxast_p j}$ are the evident maps induced from the sequence, as described in (24.13). In a similar way, we define the map

$$f^{j \boxast_p}\colon X_{p/} \to X_{pj/} \times_{Y_{fpj/}} Y_{fp/}.$$

27.8. *Remark.* When $Y = *$, these pullback-slice maps are just the restriction maps $X_{/p} \to X_{/pj}$ and $X_{p/} \to X_{pj/}$. When $T = \varnothing$, these pullback-slice maps are just the projection maps $X_{/p} \to Y_{/fp}$ and $X_{p/} \to Y_{fp/}$. When both $Y = *$ and $T = \varnothing$, we get $X_{/p} \to X$ and $X_{p/} \to X$.

27.9. *Remark.* Both pullback-hom and pullback-slices are special cases of a general construction: given any functor $F\colon s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}$ from the twisted arrow category (24.14), you get a corresponding "pullback-F" functor $F^{\square}\colon s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}$. In the case of pullback-hom, the F in question is a composite functor $s\mathrm{Set}^{\mathrm{tw}} \to s\mathrm{Set}^{\mathrm{op}} \times s\mathrm{Set} \xrightarrow{\mathrm{Map}} s\mathrm{Set}$.

27.10. **Joins, slices, and lifting problems.** The pushout-join and pullback-slice interact with lifting problems in much the same way that pushout-product and pullback-hom do.

27.11. **Proposition.** *Given $i\colon A \to B$, $j\colon K \to L$, and $h\colon X \to Y$, the following are equivalent.*
  (1) $(i \boxast j) \boxslash h$.
  (2) $i \boxslash (h^{\boxast_q j})$ *for all* $q\colon L \to X$.
  (3) $j \boxslash (h^{i \boxast_p})$ *for all* $p\colon B \to X$.

Proof. A straightforward exercise. The equivalence of (1) and (2) looks like:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_{/q} \arrow[d, "h^{\boxast_q j}"] \\ B \arrow[r] \arrow[ur, dotted] & X_{/qj} \times_{Y_{/hqj}} Y_{/hq} \end{tikzcd} \qquad \iff \qquad \begin{tikzcd} \varnothing \star L \arrow[r] \arrow[rr, bend left, "q"] & (A \star L) \cup_{A \star K} (B \star K) \arrow[r] \arrow[d, "i \boxast j"'] & X \arrow[d, "h"] \\ & B \star L \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

□

Now we can set up "join/slice analogues" of the "enriched lifting theory" we have seen for products and function complexes. **F 22 Feb 2019**

27.12. **Proposition.** *Let S and T be sets of maps in sSet. Then $\overline{S} \boxast \overline{T} \subseteq \overline{S \boxast T}$.*

Proof. This is formal and nearly identical to the proof of the weak saturation result for box-products (16.9). □

27.13. **Proposition.** *We have*

$$\overline{\mathrm{Cell}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}, \qquad \overline{\mathrm{RHorn}} \boxast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}, \qquad \textit{and} \qquad \overline{\mathrm{Cell}} \boxast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}.$$

Proof. Immediate from (27.4) and (27.12). □

27.14. **Proposition.** *Given $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$, consider the pullback-slice maps*

$$\ell\colon X_{p/} \to X_{pj/} \times_{Y_{fpj/}} Y_{fp/}, \qquad r\colon X_{/p} \to X_{/pj} \times_{Y_{/fpj}} Y_{/fp}.$$

We have the following.
  (1) $j \in \overline{\mathrm{Cell}}$, $f \in \mathrm{TrivFib}$ *implies* $\ell, r \in \mathrm{TrivFib}$.
  (2) $j \in \overline{\mathrm{Cell}}$, $f \in \mathrm{InnFib}$ *implies* $\ell \in \mathrm{LFib}$, $r \in \mathrm{RFib}$.

(3) $j \in \overline{\text{RHorn}}$, $f \in \text{InnFib}$ *implies* $\ell \in \text{TrivFib}$.
(4) $j \in \overline{\text{LHorn}}$, $f \in \text{InnFib}$ *implies* $r \in \text{TrivFib}$.

Proof. Exercise, using (27.13). □

We are mostly interested in special cases when $X = C$ is a quasicategory, and $Y = *$.

27.15. **Corollary.** *Given $T \xrightarrow{j} S \xrightarrow{p} C$ with C a quasicategory, consider the pullback-slice maps*

$$\ell\colon C_{p/} \to C_{pj/}, \qquad r\colon C_{/p} \to C_{/pj}.$$

We have the following.
(1) $j \in \overline{\text{Cell}}$ *implies* $\ell \in \text{LFib}$, $r \in \text{RFib}$.
(2) $j \in \overline{\text{RHorn}}$ *implies* $\ell \in \text{TrivFib}$.
(3) $j \in \overline{\text{LHorn}}$ *implies* $r \in \text{TrivFib}$.

In particular, (1) when $T = \varnothing$ gives
(1') $\ell\colon C_{p/} \to C$ *is a left fibration and* $r\colon C_{/p} \to C$ *is a right fibration.*

Here is another useful special case when $T = \varnothing$: slices preserve trivial fibrations.

27.16. **Corollary.** *Given $S \xrightarrow{p} X \xrightarrow{f} Y$ where f is a trivial fibration, all of the maps in*

$$X_{p/} \to X \times_Y Y_{fp/} \to Y_{fp/} \qquad \text{and} \qquad X_{/p} \to X \times_Y Y_{/fp} \to Y_{/fp}$$

are trivial fibrations.

Proof. The two pullback-slice maps are trivial fibrations by (27.14). The projections are each base changes of the trivial fibration f, and so are trivial fibrations. □

We'll also meet the following consequence now and again: joins preserve monomorphisms.

27.17. **Proposition.** *If $i\colon A \to B$ is a monomorphism of simplical sets, then so are $S\star i\colon S\star A \to S\star B$ and $i \star S\colon A \star S \to B \star S$ for any S.*

Proof. The map $S \star i$ is the composite

$$S \star A \to (S \star A) \cup_{\varnothing\star A} (\varnothing \star B) \xrightarrow{(\varnothing\subseteq S)\boxtimes i} S \star B.$$

the second map is a monomorphism by $\overline{\text{Cell}} \boxtimes \overline{\text{Cell}} \subseteq \overline{\text{Cell}}$ (27.13), while the first map is a cobase change of the monomorphism i.

Note: that join preserves monomorphisms can also be proved directly from the definition of join. □

27.18. **Composition functors for slices.** Here is a nice consequence of the above. Let C be a quasicategory and $f\colon x \to y$ a morphism in it; we represent f by a map $\Delta^1 \to C$ of simplicial sets, which we also call f. We obtain two restriction functors

$$C_{/x} \xleftarrow{r_0} C_{/f} \xrightarrow{r_1} C_{/y}$$

associated to the inclusions $\{0\} \subset \Delta^1 \supset \{1\}$. The first inclusion $\{0\} \subset \Delta^1$ is a left-horn inclusion, and thus by (27.15) the restriction map r_0 is a trivial fibration, and hence we can choose a section $s\colon C_{/x} \to C_{/f}$ of p.

The resulting composite $r_1 s\colon C_{/x} \to C_{/y}$ can be thought of as a functor realizing the operation which sends an object $(c \xrightarrow{g} x)$ of $C_{/x}$ to an object "$(c \xrightarrow{f\circ g} y)$" of $C_{/y}$ defined by "composing f and g" (but remember that such composition is not uniquely defined in a quasicategory C; the choice of section s gives a collection of such choices for all g.)

27.19. *Exercise.* Show that if C is a category, then r_0 is an isomorphism, and that $r_1 s$ is precisely the functor $C_{/x} \to C_{/y}$ described above.

28. Limits and colimits in quasicategories

28.1. **Definition of limits and colimits.** Now we can define the notion of a limit and colimit of a functor between quasicategories (and in fact of a map from a simplicial set to a quasicategory). Given a map $p\colon K \to C$ where C is a quasicategory, a **colimit** of p is defined to be an initial object of the slice quasicategory $C_{p/}$. Explicitly, a colimit of $p\colon K \to C$ is a map $\widehat{p}\colon K \star \Delta^0 = K^{\triangleright} \to C$ extending p, such that for $n \geq 1$ a lift exists in every diagram of the form

$$\begin{tikzcd} K \star \{0\} \arrow[r, tail] \arrow[rr, bend left, "\widehat{p}"] & K \star \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\ & K \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Sometimes it is better to call $\widehat{p}$ a **colimit cone** for p, in which case the restriction $\widetilde{p}|\varnothing \star \Delta^0$ to the cone point is an object in C which can be called a "colimit of p".

Similarly, a **limit** of p is a terminal object of $C_{/p}$; explicitly, this is a map $\widehat{p}\colon \Delta^0 \star K = K^{\triangleleft} \to C$ extending p such that for $n \geq 1$ a lift exists in every diagram of the form

$$\begin{tikzcd} \{n\} \star K \arrow[r, tail] \arrow[rr, bend left, "\widehat{p}"] & \partial\Delta^n \star K \arrow[r] \arrow[d, tail] & C \\ & \Delta^n \star K \arrow[ur, dotted] & \end{tikzcd}$$

Again, we will also sometimes refer to $\widehat{p}\colon \Delta^0 \star K = K^{\triangleleft} \to C$ as a **limit cone** for p.

28.2. *Example.* Consider the empty simplicial set $K = \varnothing$ and the unique map $p\colon \varnothing \to C$. Then $C_{p/} = C$, so a colimit of p is precisely the same as an initial object of C. Likewise, a limit of p is precisely the same as a terminal object of C.

28.3. *Example.* Consider $K = \Lambda^2_0$, which is the nerve of a category which we can draw as the picture $\big(1 \leftarrow 0 \rightarrow 2\big)$. Then $(\Lambda^2_0)^{\triangleright} \approx \Delta^1 \times \Delta^1$ is also an ordinary category; explicitly it has the form of a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \\ 2 \arrow[r] & v \end{tikzcd}$$

where v is the "cone vertex". A colimit cone $(\Lambda^2_0)^{\triangleright} \to C$ is called a **pushout diagram** in C.

Similar considerations give $(\Lambda^2_2)^{\triangleleft} \approx \Delta^1 \times \Delta^1$; a limit cone $(\Lambda^2_2)^{\triangleleft} \to C$ is called a **pullback diagram** in C.

28.4. *Exercise.* Let $C' \subseteq C$ be an inclusion of a full subcategory. Show that if $p\colon K \to C'$ has a colimit $\widehat{p}$ in C, and if the image of $\widehat{p}$ is contained in C', then $\widehat{p}$ is in fact a colimit of p in C'.

28.5. **Uniqueness of limits and colimits.** Limits and colimits are unique if they exist.

28.6. **Proposition.** *Let $p\colon K \to C$ be a map to a quasicategory, and let $(C_{p/})^{\mathrm{colim}} \subseteq C_{p/}$ and $(C_{/p})^{\mathrm{lim}} \subseteq C_{/p}$ denote the full subcategories spanned by colimit cones and limit cones respectively. Then (i) either $(C_{p/})^{\mathrm{colim}}$ is empty or is categorically equivalent to Δ^0, and (ii) either $(C_{/p})^{\mathrm{lim}}$ is empty or is categorically equivalent to Δ^0.*

Proof. This is just the uniqueness of initial and terminal objects (26.6), since $(C_{p/})^{\mathrm{colim}} = (C_{p/})^{\mathrm{init}}$ and $(\mathbb{C}_{/p})^{\mathrm{lim}} = (C_{/p})^{\mathrm{term}}$. $\square$

We have noted above (26.3) that an object x in a quasicategory C is initial iff $C_{x/} \to C$ is a trivial fibration, and terminal iff $C_{/x} \to C$ is a trivial fibration. There is a similar characterization of limit and colimit cones.

28.7. **Proposition.** *Let C be a quasicategory. Let $\widetilde{p}\colon K^{\triangleright} \to C$ be a map, and write $p := \widetilde{p}|K$. Then $\widetilde{p}$ is a colimit diagram if and only if $C_{\widetilde{p}/} \to C_{p/}$ is a trivial fibration.*

Likewise, let $\widetilde{q}\colon K^{\triangleleft} \to C$ be a map, and write $q := \widetilde{q}|K$. Then $\widetilde{q}$ is a limit diagram if and only if $C_{/\widetilde{q}} \to C_{/q}$ is a trivial fibration.

Proof. I'll just do the case of colimits.

We make an elementary observation about iterated slices (see (28.8) below). There is an isomorphism $(C_{p/})_{\widetilde{p}/} \approx C_{\widetilde{p}/}$, where the symbol "$\widetilde{p}$" refers to both a morphism $\widetilde{p}\colon K^{\triangleright} \to C_{p/}$ (on the right-hand side of the isomorphism) and the corresponding object $\widetilde{p} \in (C_{p/})_0$ (on the left-hand side of the isomorphism). The point is that in either simplical set, a k-dimensional element corresponds to a map $K \star \Delta^0 \star \Delta^k \to C$ which restricts to $\widetilde{p}$ on $K \star \Delta^0 \star \varnothing$.

Using this, the statement amounts to the special case for initial and terminal objects (26.3). □

28.8. *Exercise* (Iterated slices). Let $f\colon A \star B \to C$ be a map of simplicial sets. Describe isomorphisms

$$C_{f/} \approx (C_{f_A/})_{\widetilde{f_B}/}, \qquad C_{/f} \approx (C_{/f_B})_{/\widetilde{f_A}},$$

where $f_A\colon A \to C$ and $f_B\colon B \to C$ are the evident restrictions of f to subcomplexes, and $\widetilde{f_A}\colon A \to C_{/f_B}$ and $\widetilde{f_B}\colon B \to C_{f_A/}$ are the adjoints to f.

28.9. **Limits and colimits in slices.** Given a map $p\colon S \to C$ to a quasicategory, we have "forgetful functors" $\pi\colon C_{/p} \to C$ and $\pi\colon C_{p/} \to C$ from the slices to C.

The following proposition says that an initial object of C implies a compatible initial object of $C_{/p}$, and a terminal object of C implies a compatible terminal object of $C_{p/}$. Note that when C is an ordinary category this is entirely straightforward: e.g., given an initial object c_0 of C, there is a a *unique* cone $\widetilde{p}\colon S^{\triangleright} \to C$ extending p which sends the cone vertex to c_0, and it's an easy exercise to show that $\widetilde{p}$ represents an initial object of the slice $C_{/p}$.

28.10. **Proposition.** *Let $p\colon S \to C$ be a map from a simplicial set to a quasicategory.*

(1a) *If $x \in (C_{/p})_0$ is an object such that $\pi(x) \in C_0$ is initial in C, then x is initial in $C_{/p}$.*
(1b) *If C has an initial object then so does $C_{/p}$.*
(2a) *If $x \in (C_{p/})_0$ is an object such that $\pi(x) \in C_0$ is terminal in C, then x is terminal in $C_{/p}$.*
(2b) *If C has a terminal object then so does $C_{p/}$.*

Proof. (See [Lur09, 1.2.13.8].) I'll only prove (1a) and (1b), as the other parts are analogous.

To prove (1a), let $x \in (C_{/p})_0$ and $y = \pi(x) \in C_0$; we need to show that if y is initial then so is x. To show that x is initial we must produce a lift in any diagram of the form

$$\begin{tikzcd}
\Delta^0 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "x"] & (\Delta^0 \star \partial\Delta^n) \cup_{\varnothing\star\partial\Delta^n} (\varnothing \star \Delta^n) \arrow[r] \arrow[d, tail] & C_{/p} \\
& \Delta^0 \star \Delta^n \arrow[ur, dotted] &
\end{tikzcd}$$

for $n \geq 0$, using the identification $(\varnothing \subset \Delta^0) \star (\partial\Delta^n \subset \Delta^n) \approx (\partial\Delta^{n+1} \subset \Delta^{n+1})$. This lifting problem is equivalent to one of the form

$$\begin{tikzcd}
\Delta^0 \star \varnothing \star S \arrow[r, tail] \arrow[rr, bend left, "x'"] & (\Delta^0 \star \partial\Delta^n \star S) \cup_{\varnothing\star\partial\Delta^n\star S} (\varnothing \star \Delta^n \star S) \arrow[r] \arrow[d, tail] & C \\
& \Delta^0 \star \Delta^n \star S \arrow[ur, dotted] &
\end{tikzcd}$$

(because $(-) \star S$ preserves *pushouts* (24.8)), which in turn is equivalent to one of the form

$$\begin{tikzcd}
S \arrow[r, tail] \arrow[rr, bend left, "x''"] & \partial\Delta^n \star S \arrow[r] \arrow[d, tail] & C_{y/} \arrow[d, "q"] \\
& \Delta^n \star S \arrow[r] \arrow[ur, dotted] & C
\end{tikzcd}$$

(In these diagrams the maps marked x, x', x'' are all adjoints of each other.) Since y is initial, q is a trivial fibration (26.3), and therefore a lift exists since $\partial\Delta^n \star S \to \Delta^n \star S$ is a monomorphism, because joins preserve monomorphisms (27.17). We conclude that x is initial when y is.

Next we prove (1b). Suppose $y \in C_0$ is an initial object. This implies $q\colon C_{y/} \to C$ is a trivial fibration (26.3). In particular, a lift exists in

$$\begin{tikzcd}
& C_{y/} \arrow[d, "q"] \\
S \arrow[r, "p"'] \arrow[ur, dotted, "x''"] & C
\end{tikzcd}$$

By an adjunction argument (24.15), x'' corresponds to a map $x\colon \Delta^0 \to C_{/p}$ such that $\pi(x) = y$. By what we have already proved, x must be initial since $\pi(x) = y$ is initial. $\square$

28.11. *Remark.* In fact, the converses of (1a) and (2a) in (28.10) are also true, as long as we assume that C has an initial/terminal object. The proof of these converses requires (D), which we have not established yet.

We can now generalize the above to arbitrary limits in colimits.

The following proposition says that colimits in $C_{/p}$ or limits in $C_{p/}$ can be "computed in the underlying quasicategory" C (if the corresponding colimit or limit in C exists).

28.12. **Proposition.** *Let $p\colon S \to C$ be a map from a simplicial set to a quasicategory.*

(1) *Let $f\colon K \to C_{/p}$ be a map such that the composite map $f_0 = \pi f\colon K \xrightarrow{f} C_{/p} \xrightarrow{\pi} C$ has a colimit cone in C. Then*
   (a) *f admits a colimit cone, and*
   (b) *if $\widetilde{f}\colon K^{\triangleright} \to C_{/p}$ is such that the composite map $K^{\triangleright} \xrightarrow{\widetilde{f}} C_{/p} \to C$ is a colimit cone, then $\widetilde{f}$ is a colimit cone.*

(2) *Let $f\colon K \to C_{p/}$ be a map such that the composite map $f_0 = \pi f\colon K \xrightarrow{f} C_{p/} \xrightarrow{\pi} C$ has a limit cone in C. Then*
   (a) *f admits a limit cone, and*
   (b) *if $\widetilde{f}\colon K^{\triangleright} \to C_{/}p$ is such that the composite map $K^{\triangleright} \xrightarrow{\widetilde{f}} C_{p/} \to C$ is a limit cone, then $\widetilde{f}$ is a limit cone.*

The proof will make use an observation sketched in the following exercise: any composite of a slice-over followed by a slice-under can be reinterpreted as a slice-under followed by a slice-over.

28.13. *Exercise* (Two-sided slice). Fix a map $p\colon A \star B \to X$ of simplicial sets. Describe a simplicial set $X_{/p/}$ which admits bijective correspondences

$$\left\{ \begin{tikzcd} A \star B \arrow[r, "p"] \arrow[d] & X \\ A \star K \star B \arrow[ur, dotted] & \end{tikzcd} \right\} \iff \{K \dashrightarrow X_{/p/}\},$$

natural in K. Then construct natural isomorphisms

$$(X_{p_A/})_{/\widetilde{p}_B} \approx X_{/p/} \approx (X_{/p_B})_{\widetilde{p}_A/},$$

where $p_A\colon A \to X$ and $p_B\colon B \to X$ are the evident restrictions of p to subcomplexes, and $\widetilde{p}_A\colon A \to X_{/p_B}$ and $\widetilde{p}_B\colon B \to X_{p_A/}$ are adjoints to p.

Proof of (28.12). I prove (1), as (2) is analogous. Note that $f\colon K \to C_{/p}$ is adjoint to a map $g\colon K \star S \to C$, which in turn is adjoint to a map $q\colon S \to C_{f_0/}$. Colimit cones of f_0 correspond precisely to initial objects of $C_{f_0/}$; in particular, the hypothesis of (1) asserts that $C_{f_0/}$ has an initial object. Likwise, colimit cones of f correspond exactly to initial objects of $(C_{/p})_{f/}$. As in (28.13) we have isomorphisms

$$(C_{/p})_{f/} \approx C_{/g/} \approx (C_{f_0/})_{/q}.$$

To prove (1a) here it suffices to show that $(C_{f_0/})_{/q}$ has an initial object, which since $C_{f_0/}$ does using (28.10)(1a). To prove (1b) here it suffices to show that the projection $(C_{f_0/})_{/q} \to C_{f_0/}$ has the property that objects sent to initial objects of $C_{f_0/}$ are initial in $(C_{f_0/})_{/q}$, which is immediate from (28.10)(1b). $\square$

28.14. **Invariance of limits and colimits.** Here are some seemingly obvious facts about invariance of limits and colimits which we cannot prove yet. The first asserts that any two categorically equivalent quasicategories have thet "same" limits and colimits in them.

E. **Deferred Proposition.** *Let $f\colon C \to D$ be a categorical equivalence between quasicategories. Then a map $\widehat{p}\colon K^{\triangleright} \to C$ is a colimit cone in C if and only if $f\widehat{p}$ is a colimit cone in D, and a map $\widehat{q}\colon K^{\triangleleft} \to C$ is a limit cone in C if and only if $f\widehat{q}$ is a colimit cone in D.*

I will prove this in (**??**).

The second asserts that any two naturally isomorphic functors to a quasicategory have the "same" limits and colimits.

F. **Deferred Proposition.** *Let $f_0, f_1\colon K \to C$ be maps, and $\alpha\colon f_0 \to f_1$ an isomorphism of objects of* $\mathrm{Fun}(K, C)$.

(1) *The map f_0 admits a colimit if and only if f_1 does, and furthermore, if $\widehat{f_0}$ and $\widehat{f_1}$ are colimit cones for f_0 and f_1 respectively, there exists an isomorphism $\widehat{\alpha}\colon \widehat{f_0} \to \widehat{f_1}$ extending α.*
(2) *The map f_0 admits a limit if and only if f_1 does, and furthermore, if $\widehat{f_0}$ and $\widehat{f_1}$ are limit cones for f_0 and f_1 respectively, there exists an isomorphism $\widehat{\alpha}\colon \widehat{f_0} \to \widehat{f_1}$ extending α.*

I will prove this in (**??**).

29. The Joyal extension and lifting theorems

We are now at the point where we can state and prove Joyal's theorems about extending or lifting maps along outer horns. This will allow us to prove several of the results whose proofs we have deferred up to now.

29.1. **Joyal extension theorem.** Joyal's theorem gives a condition for extending maps from *outer horns* into a quasicategory.

29.2. **Theorem** (Joyal extension)**.** [Joy02, Thm. 1.3] *Let C be a quasicategory, and fix a map $f\colon \Delta^1 \to C$. The following are equivalent.*

1. *The edge represented by f is an isomorphism in C.*
2. *Every $a\colon \Lambda^n_0 \to C$ with $n \geq 2$ such that $f = a|\Delta^{\{0,1\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*
3. *Every $b\colon \Lambda^n_n \to C$ with $n \geq 2$ such that $f = b|\Delta^{\{n-1,n\}}\colon \Delta^1 \to C$ admits an extension to a map $\Delta^n \to C$.*

I'll call $\langle 01\rangle \in \Delta^n$ the **leading edge**, and $\langle n-1, n\rangle \in \Delta^n$ the **trailing edge**. Thus, the implications (1) ⇒ (2) and (1) ⇒ (3) say that we can always extend $\Lambda^n_0 \to C$ to an n-simplex if the *leading edge* goes to an isomorphism in C, and extend $\Lambda^n_n \to C$ to an n-simplex if the *trailing edge* goes to an isomorphism in C.

The implications (2) ⇒ (1) and (3) ⇒ (1) are easy, and are left as an exercise.

29.3. *Exercise* (Easy part of Joyal extension). Suppose C is a quasicategory with edge $f \in C_1$, and suppose that every map $a\colon \Lambda^n_0 \to C$ with $n \in \{2,3\}$ and $f = a|\Delta^{\{0,1\}}$ admits an extension along $\Lambda^n_0 \subset \Delta^n$. Prove that f is an isomorphism.

The non-trivial implications of Joyal extension will lead to proofs of the deferred propositions (A), (C), and (D).

The proof of the Joyal extension theorem will be an application of the fact that left fibrations and right fibrations are *conservative isofibrations.* **M 25 Feb 2019**

29.4. **Conservative functors.** A functor $p\colon C \to D$ between categories is **conservative** if whenever f is a morphism in C such that $p(f)$ is an isomorphism in D, then f is an isomorphism in C. The definition of a conservative functor between *quasicategories* is precisely the same.

29.5. **Proposition.** *All left fibrations and right fibrations between quasicategories are conservative.*

Proof. Consider a right fibration $p\colon C \to D$, and a morphism $f\colon x \to y$ in C such that $p(f)$ is an isomorphism. We first show that f admits a preinverse.

Let $a\colon \Lambda^2_2 \to C$ such that $a_{12} = f$ and $a_{02} = 1_y$. Let $b\colon \Delta^2 \to C$ be any 2-dimensional element exhibiting a preinverse of $p(f)$, i.e., such that $b_{12} = p(f)$ and $b_{02} = 1_{p(y)}$, so that b_{01} is a preinverse. Now we have a commutative diagram

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "s"] & D \end{tikzcd}$$

which admits a lift since p is a right fibration. The lift s exhibits a preinverse $g := s|\Delta^{\{0,1\}}$ for f.

Because $p(f)$ was assumed to be an isomorhism in D, its preinverse $p(g)$ is also an isomorphism, and therefore by the above argument g admits a preinverse as well. We conclude that f is an isomorphism by (10.5). □

29.6. **Isofibrations.** We say that a functor $p\colon C \to D$ of quasicategories is an **isofibration**[22] if

1. p is an inner fibration, and
2. we have "isomorphism lifting" along p. That is, for any $c \in C_0$ and *isomorphism* $g\colon p(c) \to d'$, there exists a $c' \in C_0$ and *isomorphism* $f\colon c \to c'$ such that $p(f) = g$.

[22]Joyal uses the term "quasifibration" in [Joy02]. Later in [Joy08a] this is called a "pseudofibration". Lurie uses this notion, but never names it. The term "isofibration" is used by Riehl and Verity [RV15].

Condition (2) is illustrated by the diagram

$$\begin{tikzcd} \{0\} \arrow[r, "c"] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "f"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

Recall that if C and D are nerves of ordinary categories, then any functor $C \to D$ is an inner fibration. Thus in the case of ordinary categories, being an isofibration amounts to condition (2) only. Also, it is clear that in the case of ordinary categories condition (2) is equivalent to

(2') for any $c \in C_0$ and *isomorphism* $g'\colon d' \to p(c)$, there exists a $c' \in C_0$ and isomorphism $f'\colon c' \to c$ such that $p(f) = g'$.

To derive (2) from (2') for ordinary categories, just apply (2') to the (unique) inverse of g.

The symmetry between (2) and (2') also holds for functors between quasicategories, by the following.

29.7. **Proposition.** *An inner fibration $p\colon C \to D$ between quasicategories is an isofibration if and only if $h(p)\colon h(C) \to h(D)$ is an isofibration of ordinary categories.*

Proof. ($\Longrightarrow$) Straightforward. ($\Longleftarrow$) Suppose given an isomophism $g\colon p(c) \to d'$ in D. If $h(p)\colon hC \to hD$ is an isofibration, there exists an isomorphism $f'\colon c \to c'$ in C such that $p(f') \sim_r g$. Now choose a lift in

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "a"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "s"] & D \end{tikzcd}$$

where b exhibits $p(f') \sim_r g$ and $a(\langle 01\rangle) = f'$ and $a(\langle 12\rangle) = 1_{c'}$. The edge $f = s_{02}$ is a lift of g, and is an isomorphism since $f' \sim_r f$. $\square$

29.8. *Exercise.* (i) Let Group denote the category of groups, whose objects are pairs $G = (S, \mu)$ consisting of a set S and a function $\mu\colon S \times S \to S$ satisfying a well-known list of axioms. Show that the functor $U\colon \mathrm{Group} \to \mathrm{Set}$ which on objects sends $(S, \mu) \mapsto S$ is an isofibration between ordinary categories.

(ii) Consider the functor $U'\colon \mathrm{Group} \to \mathrm{Set}$ defined on objects by $G \mapsto \mathrm{Hom}(\mathbb{Z}, G)$. Explain why, although U' is naturally isomorphic to U, you don't know how to show whether U' is an isofibration without explicit reference to the axioms of your set theory. The moral is that the property of being an isofibration is not "natural isomorphism invariant".

29.9. **Left and right fibrations are isofibrations.**

29.10. **Proposition.** *All left fibrations and right fibrations between quasicategories are isofibrations.*

Proof. Suppose $p\colon C \to D$ is a right fibration (and hence an inner fibration) between quasicategories, and consider

$$\begin{tikzcd} \{1\} \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "g"'] \arrow[ur, dotted, "f"] & D \end{tikzcd}$$

where g represents an isomorphism in D. Because p is a right fibration and $(\{1\} \subset \Delta^1) \in \mathrm{RHorn}$, there exists a lift f. Because right fibrations are conservative, f represents an isomorphism. $\square$

Note that the above proof explicitly checked isofibration condition (2') for right fibrations; thus, by symmetry we conclude that isofibration condition (2) holds for right fibrations. It seems difficult to give an elementary *direct* proof that right-fibrations satisfy (2).

29.11. **Proof of the Joyal extension theorem.**

Proof of (29.2). We prove (1) ⇒ (2). Suppose given $a\colon \Lambda^n_0 \to C$ such that $f = a|\Delta^{\{0,1\}}$ represents an isomorphism. Observe (27.4) that $(\Lambda^n_0 \subset \Delta^n)$ is the pushout-join of a 1-horn with an $(n-2)$-cell:

$$(\Lambda^n_0 \subset \Delta^n) \approx (\Delta^{\{0\}} \subset \Delta^{\{0,1\}}) \circledast (\partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}),$$

since $\Lambda^n_0 \approx (\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}}) \cup (\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$ inside $\Delta^n \approx \Delta^{\{0,1\}} \star \Delta^{\{2,\dots,n\}}$. Using this, we get a correspondence of lifting problems

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "a"] \arrow[d, tail] & C \\ & \Delta^n \arrow[ur, dotted] & \end{tikzcd} \iff \begin{tikzcd} \Delta^{\{0\}} \arrow[r, "h"] \arrow[d, tail] & C_{/(a|\Delta^{\{2,\dots,n\}})} \arrow[d, "q"] \\ \Delta^{\{0,1\}} \arrow[r, "g"] \arrow[ur, dotted] \arrow[dr, "f"'] & C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \arrow[d, "p"] \\ & C \end{tikzcd}$$

where g is adjoint to $a|(\Delta^{\{0,1\}} \star \partial\Delta^{\{2,\dots,n\}})$, and h is adjoint to $a|(\Delta^{\{0\}} \star \Delta^{\{2,\dots,n\}})$. Because C is a quasicategory, and because p and q are restrictions along monomorphisms $\varnothing \subset \partial\Delta^{\{2,\dots,n\}} \subset \Delta^{\{2,\dots,n\}}$, both p and q are right fibrations (27.15), and therefore are conservative isofibrations (29.5), (29.10). Thus since f represents an isomorphism, so does g since p is conservative, and therefore a lift exists since q is an isofibration.

The proof of (2) ⟹ (1) is left as an exercise (29.3). The proof of (1) ⟺ (3) is similar. □

29.12. **The Joyal lifting theorem.** There is a relative generalizaton.

29.13. **Theorem** (Joyal lifting)**.** *Let $p\colon C \to D$ be an inner fibration between quasicategories, and let $f \in C_1$ be an edge such that $p(f)$ is an isomorphism in D. The following are equivalent.*

(1) *The edge f is an isomorphism in C.*
(2) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

admits a lift.
(3) *For all $n \geq 2$, every diagram of the form*

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

admits a lift.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) are elementary, as in (29.3).

For (1) $\Rightarrow$ (2), the first step is to prove that

$$C_{/(a|\Delta^{\{2,\dots,n\}})} \xrightarrow{q} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p} C$$

are both right fibrations. For instance, the map q is the slice-power of the inner fibration p by a monomorphism, so is a right fibration by (27.14). The map p is the composite

$$C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \times_{D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}} D_{/(pa|\Delta^{\{2,\dots,n\}})} \xrightarrow{p'} C_{/(a|\partial\Delta^{\{2,\dots,n\}})} \xrightarrow{p''} C,$$

where p' is the base change of the right fibration $D_{/(pa|\Delta^{\{2,\dots,n\}})} \to D_{/(pa|\partial\Delta^{\{2,\dots,n\}})}$, and p'' is a right fibration (in both cases by (27.15)) Then the proof of (1) $\Longrightarrow$ (2) proceeds exactly as in (29.2). $\square$

30. Applications of the Joyal extension theorem

We can now prove all the statements whose proofs we have deferred until now, as well as some others. I'll prove (A) and (D) in this section, and (C) in the next section.

30.1. **Quasigroupoids are Kan complexes.** First we prove (A), the identification of quasigroupoids with Kan complexes.

30.2. **Proposition** (Deferred proposition (A))**.** *Every quasigroupoid is a Kan complex.*

Proof. In a quasigroupoid, the Joyal extension property (29.2) applies to all maps from Λ^n_0 and Λ^n_n with $n \geq 2$, since every edge is an isomorphism. (Recall that all simplicial sets automatically have extensions for 1-horns (10.11).) $\square$

From now on we will use terms "quasigroupoid" and "Kan complex" interchangeably.

30.3. **Invariance of slice categories.** Here is an equivalent reformulation of the Joyal extension theorem in terms of maps between slices.

30.4. **Proposition** (Reformulation of Joyal extension)**.** *If $f\colon x \to y$ is an edge in a quasicategory C, then the following are equivalent: (1) f is an isomorphism; (2) $C_{f/} \to C_{x/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y}$ is a trivial fibration.*

Proof. For all $n \geq 0$ we have a correspondence of lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, tail] & C_{f/} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & C_{x/} \end{tikzcd} \iff \begin{tikzcd} \Delta^1 \star \varnothing \arrow[r, tail] \arrow[rr, bend left, "f"] & (\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n) \arrow[r] \arrow[d] & C \\ & \Delta^1 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

and $\big((\Delta^1 \star \partial\Delta^n) \cup (\{0\} \star \Delta^n)\big) \subset \Delta^1 \star \Delta^n) \approx (\Lambda^{1+1+n}_0 \subseteq \Delta^{1+1+n})$. The lifting problems on the right-hand side are precisely those of statement (2) of the Joyal extension theorem (29.2). $\square$

30.5. *Exercise* (Reformulation of Joyal lifting). Let $p\colon C \to D$ be an inner fibration, and $f\colon x \to y$ an edge in C such that $p(f) \in D_1$ is an isomorphism. Show that the following are equivalent: (1) f is an isomorphism in C; (2) $C_{f/} \to C_{x/} \times_{D_{p(x)/}} D_{p(f)/}$ is a trivial fibration; (3) $C_{/f} \to C_{/y} \times_{D_{/p(y)}} D_{/p(f)}$ is a trivial fibration.

30.6. **Corollary.** *If $f\colon x \to y$ is an isomorphism in a quasicategory C, then $C_{x/}$ and $C_{y/}$ are categorically equivalent, and $C_{/x}$ and $C_{/y}$ are categorically equivalent.*

Proof. Consider $C_{/x} \xleftarrow{r_0} C_{/f} \xrightarrow{r_1} C_{/y}$. We have already observed (27.15) that $r_0 \in \mathrm{TrivFib}$, since $\{0\} \subset \Delta^0$ is left anodyne. The reformulation of Joyal extension (30.4) implies that $r_1 \in \mathrm{TrivFib}$ when f is an isomorphism. Therefore $C_{x/}$ and $C_{y/}$ are connected by a chain of categorical equivalences. The proof for slice-under categories is analogous. $\square$

30.7. **Invariance of initial objects.** Now we prove (D) about initial and terminal objects. We will explicitly prove the statements about initial objects, as the case of terminal objects is similar.

30.8. **Proposition** (Deferred proposition (D)(1))**.** *Let $f\colon x \to y$ be a morphism in a quasicategory C, and let $\widetilde{f} \in (C_{x/})_0$ be the object of the slice which corresponds to $f \in C_1$. Then $\widetilde{f}$ is initial in $C_{x/}$ if and only if f is an isomorphism.*

Proof. For all $n \geq 1$ we have a correspondence of lifting problems

$$\begin{tikzcd} \{0\} \arrow[r, rightarrowtail] \arrow[rr, bend left, "\widetilde{f}"] & \partial\Delta^n \arrow[r] \arrow[d, rightarrowtail] & C_{x/} \\ & \Delta^n \arrow[ur, dotted] & \end{tikzcd} \iff \begin{tikzcd} \Delta^0 \star \{0\} \arrow[r, rightarrowtail] \arrow[rr, bend left, "f"] & \Delta^0 \star \partial\Delta^n \arrow[r] \arrow[d] & C \\ & \Delta^0 \star \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

and $(\Delta^0 \star \partial\Delta^n \subset \Delta^0 \star \Delta^n) \approx (\Lambda^{1+n}_0 \subseteq \Delta^{1+n})$, so a lift exists in either if and only if f is an isomorphism, by the Joyal extension theorem applied to the right-hand liftig problem.

(Alternately, we can note that $\widetilde{f}$ is initial if and only if $\pi\colon (C_{x/})_{\widetilde{f}/} \to C_{x/}$ is a trivial fibration (26.3), and that π is isomorphic to $C_{f/} \to C_{x/}$ (28.8), so the claim follows from (30.4).) □

Note that (30.8) implies that the slice $C_{x/}$ necessarily has an initial object, namely the vertex corresponding to the edge $1_x \in C_1$.

30.9. **Proposition** (Deferred proposition (D)(2))**.** *Any object in a quasicategory isomorphic to an initial object is also initial.*

Proof. Let x be an initial object in C, and let c be an object isomorphic to x. It is easy to see that x is initial in the homotopy category hC, and therefore c is initial in hC also. This has a useful consequence: *any* map between x and c (in either direction) must be an isomorphism in C.

We next note another fact: if x is initial, any map $f\colon S \to C$ extends along $S \subset \Delta^0 \star S$ to a map $f'\colon \Delta^0 \star S \to C$ such that $f'|\Delta^0$ represents x. This is a consequence of the fact (26.3) that $p\colon C_{/x} \to C$ is a trivial fibration, whence (20.12) there exists a map $s\colon C \to C_{/x}$ such that $ps = \mathrm{id}_C$; set f' be the adjoint to $sf\colon S \to C_{/x}$.

To show c is initial in C, we need to extend any $a\colon \partial\Delta^n \to C$ with $a_0 = c$ to a map $\widetilde{a}\colon \Delta^n\colon C$. This follows from a succession of two extension problems:

$$\begin{tikzcd} \varnothing \star \partial\Delta^n \arrow[r, rightarrowtail] \arrow[dd, rightarrowtail] & (\Delta^0 \star \varnothing) \amalg (\varnothing \star \partial\Delta^n) \arrow[r, "{(x,a)}"] \arrow[d, rightarrowtail] & C \\ & \Delta^0 \star \partial\Delta^n \arrow[ur, dotted, "g"] \arrow[d, rightarrowtail] & \\ \varnothing \star \Delta^n \arrow[r, rightarrowtail] & \Delta^0 \star \Delta^n \arrow[uur, dotted, "h"'] & \end{tikzcd}$$

The extension g exists by the remarks of the previous paragraph since x is initial. The extension h exists because the leading edge of g is a map $x \to c$ in C, which is an isomorphism by the remarks of the first paragraph. The desired extension $\widetilde{a}$ is $h|(\varnothing \star \Delta^n)$. □

30.10. *Remark* (Slices of quasigroupoids are quasigroupoids). If C is a quasigroupoid, and $x \in C_0$ an object, then the slices $C_{/x}$ and $C_{x/}$ are quasigroupoids. This is immediate from the fact that the restriction maps $C_{/x} \to C$ and $C_{x/} \to C$ are conservative, being respectively right and left fibrations (27.15) (29.5).

30.11. *Remark* (Initial and terminal objects in quasigroupoids). If C is a quasigroupoid with object $x \in C_0$, then (30.8) and its analogue for final objects, together with the fact that slices of quasigroupoids are quasigroupoids (30.10), implies that *every* object of $C_{x/}$ is initial, and *every* object of $C_{/x}$ is terminal. That is, $C_{x/} = (C_{x/})^{\mathrm{init}}$ and $C_{/x} = (C_{/x})^{\mathrm{term}}$, and so both $C_{x/}$ and $C_{/x}$ are categorically equivalent to the terminal quasicategory (26.6).

Conversely, suppose C is a quasicategory such that $C_{x/}$ is categorically equivalent to the terminal category for every $x \in C_0$. Then $C_{x/} \to *$ is a trivial fibration by (37.11), and thus it is easy to see that every object of $C_{x/}$ is an initial object. Then (30.8) implies that every morphism of C is an isomorphism. A similar observation holds for slices $C_{/x}$.

In other words, a quasicategory C is a quasigroupoid if and only if $C_{x/}$ is contractible for every object x, if and only if $C_{/x}$ is contractible for every object x.

31. Proof of the objectwise criterion for natural isomorphisms

Recall that if C is a quasicategory then so is any function complex $\mathrm{Fun}(X, C) = \mathrm{Map}(X, C)$ **W 27 Feb 2019**
for an arbitrary simplicial set X. In this setting, say that an edge in $\mathrm{Fun}(X, C)_1$ is an **objectwise isomorphism** of maps $X \to C$ if for each for each vertex $x \in X_0$, the composite $\Delta^1 \xrightarrow{f} \mathrm{Fun}(X, C) \xrightarrow{\mathrm{res}} \mathrm{Fun}(\{x\}, C) \approx C$ represents an isomorphism in C, where f is the representing map of the edge.

Note that any isomorphism in $\mathrm{Fun}(X, C)$ is automatically an objectwise isomorphism. In this section we will prove the following.

31.1. **Proposition** (Deferred proposition (C))**.** *Let C be a quasicategory and X a simplicial set. Then an edge of* $\mathrm{Fun}(X, C)$ *is an isomorphism if and only if it is an objectwise isomorphism.*

As a consequence we obtain a proof of (C), which is the special case where X is also a quasicategory, in which case "isomorphisms" in $\mathrm{Fun}(X, C)$ are the same thing as "natural isomorphisms" of functors $X \to C$.

We first reformulate this a bit, using the following easy observation.

31.2. **Proposition.** *Let $\{C_\alpha\}_{\alpha \in A}$ be a collection of quasicategories indexed by a set A, and let $C = \prod_{\alpha \in A} C_\alpha$. Then $f \in C_1$ is an isomorphism if and only if each image $f_\alpha \in (C_\alpha)_1$ under projection is an isomorphism.*

Proof. The only if direction is clear. For the if direction, choose for each α a $g_\alpha \in (C_\alpha)_1$ together with $a_\alpha, b_\alpha \in (C_\alpha)_2$ which witness left-homotopies which demonstrate $[g_\alpha][f_\alpha] = \mathrm{id}$ and $[f_\alpha][g_\alpha] = \mathrm{id}$ in hC_α. Then the evident $g = (g_\alpha) \in C_1$ and $a = (a_\alpha), b = (b_\alpha) \in C_2$ witness g as an inverse of f.

(Alternate proof: use Joyal extension (29.2).) $\square$

Since $\mathrm{Map}(\mathrm{Sk}_0 X, C) \approx \prod_{x \in X_0} C$, this means that a map $f \in \mathrm{Map}(X, C)_1$ is an objectwise isomorphism if and only if its image in $\mathrm{Map}(\mathrm{Sk}_0 X, C)_1$ under restriction along $j\colon \mathrm{Sk}_0 X \to X$ is an isomorphism. So (31.1) is equivalent to the following slightly more general statement.

31.3. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets such that $j\colon K_0 \xrightarrow{\sim} L_0$ is a bijection. Then for every quasicategory C the restriction map* $\mathrm{Fun}(j, C)\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ *is conservative.*

We will give the proof of this below (31.8).

31.4. **A lifting property for objectwise isomorphisms.** We establish a "lifting property" for objectwise isomorphisms, somewhat analogous to the definition of isofibration.

31.5. **Lemma.** *Let C be a quasicategory, and let $i\colon S \to T$ be a monomorphism of simplicial sets such that $i\colon S_0 \to T_0$ is a bijection. Then for every diagram*

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(T, C) \arrow[d, "\mathrm{Fun}(i,C)"] \\
\Delta^1 \arrow[r, "v"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(S, C)
\end{tikzcd}$$

such that v represents an objectwise isomorphism of maps $S \to C$, a lift t exists, and any such lift t represents an objectwise isomorphism of maps $T \to C$.

Proof. First note that any lift t is necessarily an objectwise isomorphism, since $S_0 \xrightarrow{\sim} T_0$.

Let $\mathcal{C}$ denote the class of monomorphims $i\colon S \to T$ such that (i) $i\colon S_0 \to T_0$ is a bijection, and (ii) such that the conclusion of the lemma applies to i. I claim that $\mathcal{C}$ is a weakly saturated class. To see this, first note that the class of monomorphisms satisfying (i) is a weakly saturated class: either by directly verifying the definition, or using the fact that by the skeletal filtration (15.24), it is the weak saturation of the set $\mathrm{Cell}_{\geq 1} := \{\, \partial\Delta^n \subset \Delta^n \mid n \geq 1 \,\}$. Thus $\mathcal{C} \subseteq \overline{\mathrm{Cell}_{\geq 1}}$ by hypothesis.

The verification that $\mathcal{C}$ is itself weakly saturated is straightforward. This verification requires (i) to ensure that for any lifting problem we need to consider, the lift t of an objectwise isomorphism v is also an objectwise isomorphism. For instance, to show that $\mathcal{C}$ is closed under composition consider a sequence $S \xrightarrow{i} T \xrightarrow{j} U$ two maps in $\mathcal{C}$. Then in

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[dd] & \mathrm{Fun}(U,C) \arrow[d, "\mathrm{Fun}(j,C)"] \\
 & \mathrm{Fun}(T,C) \arrow[d, "\mathrm{Fun}(i,C)"] \\
\Delta^1 \arrow[r, "v"'] \arrow[uur, dotted, "t"] \arrow[ur, dotted, "u"] & \mathrm{Fun}(S,C)
\end{tikzcd}$$

where v is an objectwise isomorphism, we first have a lift u which is necessarily an objectwise isomorphism because $i \in \mathcal{C}$, and then a lift t because $j \in \mathcal{C}$. A similar argument shows that $\mathcal{C}$ is closed under transfinite composition.

Given this, the proof of the lemma amounts to showing that $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}$. By the evident correspondence of lifting problems, it suffices to show that for any $n \geq 1$ that a lift exists in

$$\begin{tikzcd}
(\{0\} \times \Delta^n) \cup_{\{0\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r, "\widetilde{u}"] \arrow[d, tail] & C \arrow[d] \\
\Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & \Delta^0
\end{tikzcd}$$

whenever $\widetilde{u}$ is such that $\widetilde{u}|\Delta^1 \times \{j\}$ represents an isomorphisms in C for all $j \in (\Delta^n)_0$. This amounts to the following proposition (31.6) applied to the case of $D = \Delta^0$ and $(i,j) = (0,0)$. ☐

The following is a kind of "pushout-product" version of Joyal lifting, where we replace the horn inclusion $\Lambda^n_0 \subset \Delta^n$ with the inclusion $(\{0\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$, with the role of the "leading edge" played by $\Delta^1 \times \{0\} \subset \Delta^1 \times \Delta^n$; or alternately, replace the horn inclusion $\Lambda^n_n \subset \Delta^n$ with the inclusion $(\{1\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$, with the role of the "trailing edge" played by $\Delta^1 \times \{n\} \subset \Delta^1 \times \Delta^n$.

31.6. **Proposition** (Pushout-product Joyal lifting)**.** *Suppose $p\colon C \to D$ is an inner fibration of quasicategories, and suppose $n \geq 1$, and either $(i,j) = (0,0)$ or $(i,j) = (1,n)$. For any diagram*

$$\begin{tikzcd}
\Delta^1 \times \{j\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{i\} \times \Delta^n) \cup_{\{i\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
 & \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphism in C, a lift exists.

Proof. This is a calculation, given in the appendix (62.5), which itself relies on Joyal lifting. ☐

31.7. *Example.* To give an idea of the proof (31.6), consider the case of $n = 1$ and $(i, j) = 0$, in which case $K = (\{0\} \times \Delta^1) \cup_{\{0\}\times\partial\Delta^1} (\Delta^1 \times \partial\Delta^1)$ can be pictured the solid-arrow part of the diagram

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[r, "\sim", "e"'] \arrow[ur, dotted] \arrow[u, phantom, "a", shift left=3ex] & (1,0) \arrow[u, dotted] \arrow[u, phantom, "b", shift right=3ex]
\end{tikzcd}$$

To lift to a map $\Delta^1 \times \Delta^1 \to C$, we first choose a lift on the 2-simplex a, which is attached along an inner horn $\Lambda^2_1 \subset \Delta^2$; then we choose a lift on the 2-simplex b, which is a non-inner horn $\Lambda^2_0 \subset \Delta^2$ such that $K \to C$ sends its leading edge (marked e) to an isomorphism in C, so Joyal-lifting applies.

31.8. **Proof of the objectwise criterion.** We now prove (31.3), using ideas from [Lur09, §3.1.1].

Proof of (31.3). Let $j \colon K \to L$ be a monomorphism which is a bijection on vertices, and C a quasicategory. Note that $p = \mathrm{Fun}(j, C)$ is always an inner fibration between quasicategories (by enriched lifting (17.2) using $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$). Let $f \colon \Delta^1 \to \mathrm{Fun}(L, C)$ be a map representing an edge in $\mathrm{Fun}(L, C)$, such that the edge in $\mathrm{Fun}(K, C)$ represented by pf is an isomorphism. We want to show that f also represents an isomorphism.

We do this by applying the Joyal lifting theorem (29.13): we will show that under these hypotheses, for every $n \geq 2$ and every diagram of the form

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r, "u"'] \arrow[d, tail] & \mathrm{Fun}(L, C) \arrow[d, "p=\mathrm{Fun}(j,C)"] \\
& \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "a"] & \mathrm{Fun}(K, C)
\end{tikzcd}$$

a lift a exists. The Joyal lifting theorem then tells us that f represents an isomorphism as desired. Note that since $p(f)$ is assumed to represent an isomorphism in $\mathrm{Fun}(K, C)$, it certainly represents an objectwise isomorphism in $\mathrm{Fun}(K, C)$; this is the only fact about $p(f)$ we will actually need below.

To do this, we use a strategy which will recur several more times in this book: we "deform" the given lifting problem to one which is easily seen to have a solution. More precisely, for any $n \geq 2$ we can define maps

$$\Delta^n \xrightarrow{s} \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n$$

uniquely characterized by their effect on vertices: $s(x) = (1, x)$, while $r(x, y) = y$ if $(x, y) \neq (0, 1)$ and $r(0, 1) = 0$. That is, r is the unique natural transformation

$$r \colon \langle 0023 \dots n\rangle \to \langle 0123 \dots n\rangle = \mathrm{id}_{\Delta^n} \quad \text{of functors} \quad \Delta^n \to \Delta^n.$$

We calculate that

- $rs = \mathrm{id}$,
- $s(\Lambda^n_0) = \{1\} \times \Lambda^n_0 \subseteq (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda^n_0)$,
- $r(\{0\} \times \Delta^n) = \Delta^{[n]\smallsetminus 1} \subseteq \Lambda^n_0$,
- $r(\Delta^1 \times \Delta^{[n]\smallsetminus j}) = \Delta^{[n]\smallsetminus j}$ if $j \neq 0$, whence $r(\Delta^1 \times \Lambda^n_0) = \Lambda^n_0$, and
- $r(\Delta^1 \times \{k\}) = \{k\}$ if $k \neq 1$, while $r(\Delta^1 \times \{1\}) = \Delta^{\{0,1\}}$.

Therefore we can form the solid arrow commutative diagram

$$\begin{tikzcd}
\Lambda^n_0 \arrow[r, "s"'] \arrow[d, tail] \arrow[rr, bend left, "\mathrm{id}"] & (\{0\} \times \Delta^n) \cup (\Delta^1 \times \Lambda^n_0) \arrow[r, "r"'] \arrow[d, tail] & \Lambda^n_0 \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}(L, C) \arrow[d, "p=\mathrm{Fun}(j,C)"] \\
\Delta^n \arrow[r, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & \Delta^1 \times \Delta^n \arrow[r, "r"] \arrow[urr, dotted, "b"] & \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "a"] & \mathrm{Fun}(K, C)
\end{tikzcd}$$

and observe that to produce a lift a, it suffices to produce a map b which is a lift in its rectangle: given b, we can take $a = bs$ as the desired lift.

Furthermore, producing a lift b in the above diagram amounts (by adjunction of lifting problems) to showing that a lift exists in a diagram of the form

$$\begin{tikzcd}
\{0\} \arrow[r] \arrow[d, tail] & \mathrm{Fun}(T, C) \arrow[d, "\mathrm{Fun}(i,C)"] \\
\Delta^1 \arrow[r, "\widetilde{v}"'] \arrow[ur, dotted, "\widetilde{b}"] & \mathrm{Fun}(S, C)
\end{tikzcd}$$

where $i\colon S \to T$ is the monomorphism $(\Lambda^n_0 \times L) \cup (\Delta^n \times K) \to \Delta^n \times L$. I claim that (i) i induces a bijection $S_0 \to T_0$ on vertices, and (ii) that $\widetilde{v}$ represents an objectwise isomorphism of maps $S \to C$. Therefore by (31.5) a lift $\widetilde{b}$ exists, whose adjoint b gives the solution we need.

Claim (i) is immediate, since $K_0 \xrightarrow{\sim} L_0$ implies that $\Delta^n \times K$ already contains all the vertices of $\Delta^n \times L$. For claim (ii), observe for a vertex $(k, x) \in S_0 = (\Delta^n)_0 \times K_0$, the restriction $\Delta^1 \xrightarrow{\widetilde{v}} \mathrm{Fun}(S, C) \to \mathrm{Fun}(\{(k, x)\}, C) = C$ is adjoint to the composite

$$\Delta^1 \times \{k\} \rightarrowtail \Delta^1 \times \Delta^n \xrightarrow{r} \Delta^n \xrightarrow{v} \mathrm{Fun}(K, C) \to \mathrm{Fun}(\{x\}, C) = C.$$

As noted earlier, when $k \neq 1$ we have $r(\Delta^1 \times \{k\}) = \{k\}$, so that the above composite represents an identity map in C. On the other hand when $k = 1$ we have $r(\Delta^1 \times \{k\}) = \Delta^{\{0,1\}}$, the leading edge of the n-simplex, and $v|\Delta^{\{0,1\}}\colon \Delta^{\{0,1\}} \to \mathrm{Fun}(K, C)$ is pf, which represents an isomorphism in $\mathrm{Fun}(K, C)$ by hypothesis, hence an objectwise isomorphism. □

31.9. *Remark.* Now that we have proved the objectwise criterion for objectwise isomorphisms (31.3), we can reinterpret the conclusion of (31.5): the restriction functor $\mathrm{Fun}(i, C)\colon \mathrm{Fun}(T, C) \to \mathrm{Fun}(S, C)$ is an isofibration when $S_0 \to T_0$ is a bijection. Later we will prove a far reaching generalization (40.6), in which we replace the restriction functor $\mathrm{Fun}(i, C)$ with a pullback power $p^{\square i}$ where $p\colon C \to D$ is an arbitrary isofibration and i an arbitrary monomorphism of simplicial sets.

31.10. *Exercise.* Show that if $f, g\colon C \to D$ are naturally isomorphic functors between quasicategories, then their restrictions $f^{\mathrm{core}}, g^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ to cores are also naturally isomorphic. Conclude that if $f\colon C \to D$ is a categorical equivalence between quasicategories, then the restriction $f^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ of f to cores is a categorical equivalence of quasigroupoids.

Part 4. The fundamental theorem

32. The fundamental theorem of category theory

Recall that a functor $f\colon C \to D$ between quasicategories is said to be an *equivalence* there exists a $g\colon D \to C$ such that gf and fg are naturally isomorphic to the respective identity functors. When C and D are ordinary categories, there is a well-known criterion for the existence of such a g, namely: *f is an equivalence if and only if f is fully faithful and essentially surjective.* Here

- *fully faithful* means that $\mathrm{Hom}_C(x,y) \to \mathrm{Hom}_D(f(x), f(y))$ is a bijection of sets for every pair of objects $x, y \in \mathrm{ob}\, C$, and
- *essentially surjective* means that for every object $d \in \mathrm{ob}\, D$ there exists an object $c \in \mathrm{ob}\, C$ such that $f(c)$ is isomorphic to d.

I like to call this fact the *Fundamental Theorem of Category Theory.* This is non-standard and frankly pretentious terminology[23], though I am unaware of any standard abbreviated name for this result[24]. I want to give this fact a fancy name in order to signpost it, as it is quite *nonconstructive*: to prove it requires making a *choice* for each object d in D of an object c of C and an isomorphism $f(c) \approx d$ (so it in fact relies on an appropriate form of the axiom of choice).

32.1. *Exercise.* Prove the "Fundamental Theorem" for ordinary categories as follows: given $f: C \to D$ which is fully faithful and essentially surjective, make a choice of object $g(d) \in \mathrm{ob}\, C$ and isomorphism $\alpha(d): f(g(d)) \to d$ for each object of d, and extend this to the data of a categorical inverse of f.

32.2. *Example.* Fix a field k. Let Mat be the category whose objects are non-negative integers $n \geq 0$, and whose morphims $A: n \to m$ are $(m \times n)$-matrices with entres in k, so that composition is matrix multiplication. Let Vect be the category of finite dimensional k-vector spaces and linear maps. Every basic class in linear algebra proves that the evident functor $F: \mathrm{Mat} \to \mathrm{Vect}$ is fully faithful and essentially surjective. Therefore F is an equivalence of categories. However, there is no *canonical* choice of an inverse functor, whose construction amounts to making an arbitrary choice of basis for each vector space.

We are going to state and then prove an analogue of this result for functors between quasicategories. This will first require an analogue of hom-sets, namely the *quasigroupoid* of maps between two objects, also called the *mapping space*.

33. Mapping spaces of a quasicategory

Given a quasicategory C and objects $x, y \in C_0$, the **mapping space** (or **mapping quasigroupoid**) from x to y is the simplicial set defined by the pullback square **F 1 Mar 2019**

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, C) \arrow[d] \\ \{(x,y)\} \arrow[r] & C \times C \end{tikzcd}$$

That is, $\mathrm{map}_C(x,y)$ is the fiber of the restriction map $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ over the point $(x,y) \in (C \times C)_0$, where we use the isomorphism $\mathrm{Fun}(\partial\Delta^1, C) \approx C \times C$ induced by the isomorphism $\partial\Delta^1 \approx \Delta^0 \amalg \Delta^0$.

If $C = N(A)$ is the nerve of a category, then $\mathrm{map}_C(x,y)$ is a discrete simplicial set (2.5) corresponding to the set $\mathrm{Hom}_C(x,y)$.

33.1. **Mapping spaces are Kan complexes.** The terminology "space" is justified by the following

33.2. **Proposition.** *The simplicial sets* $\mathrm{map}_C(x,y)$ *are quasigroupoids (and hence Kan complexes by* (30.2)*).*

We prove this as a special case of (33.4) below, applied to the restriction $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$ along $j = (\partial\Delta^1 \subset \Delta^1)$.

Recall that for a quasicategory C, the core $C^{\mathrm{core}} \subseteq C$ is the maximal quasigroupoid in C (10.8). The following says that maximal quasigroupoids are preserved by certain kinds of pullbacks.

[23]E.g., the Fundamental Theoroms of Arithmetic, Algebra, Calculus, etc. But if they can have a Fundamental Theorem, why can't we?

[24]I also don't know when it was first formulated, or who first stated it.

33.3. **Proposition.** *Let*

$$\begin{tikzcd} C' \arrow[r, "u"] \arrow[d, "q"'] & C \arrow[d, "p"] \\ D' \arrow[r, "v"'] & D \end{tikzcd}$$

be a pullback square of quasicategories in which p is an inner fibration. An edge $f \in C'_1$ is an isomorphism in C' if and only if $u(f) \in C_1$ and $q(f) \in D'_1$ are isomorphisms in C and D' respectively. As a consequence, the induced map $(C')^{\mathrm{core}} \to C^{\mathrm{core}} \times_{D^{\mathrm{core}}} (D')^{\mathrm{core}}$ on cores is an isomorphism.

Proof. This is a straightforward application of Joyal lifting: as q is an inner fibration and $q(f)$ is an isomorphism, to show f is an isomorphism we must produce a lift in every lifting problem described by the left-hand square in

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_0 \arrow[r] \arrow[d, tail] & C' \arrow[r, "u"] \arrow[d] & C \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & D' \arrow[r] & D \end{tikzcd}$$

Because $u(f)$ is an isomorphism, we know a lift exists in the large rectangle by Joyal lifting, and the desired lift exists because the right-hand square is a pullback.

Recall that an n-dimensional element a of a quasicategory is in the core if and only if all of its edges a_{ij} are isomorphisms. Given this, the assertion about pullbacks of cores is immediate. ☐

33.4. **Corollary.** *Let $p\colon C \to D$ be a conservative inner fibration between quasicategories. Then the fibers of p over any vertex of D are quasigroupoids.*

In particular, if C is a quasicategory and $j\colon K \to L$ is a monomorphism which is a bijection on vertices, then the fibers of $j^*\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(K, C)$ are quasigroupoids.

Proof. For the first statement, apply (33.3) with $D' = \{d\}$ for some $d \in D_0$. Any edge $f \in C'_1$ projects to the isomorphism 1_d in D' and to a map $u(f) \in C_1$ which must be an isomorphisms, since p is conservative and $pu(f) = 1_d$. The second statement is an immediate consequence of the objectwise criterion for isomorphisms in function complexes (31.3). ☐

33.5. **Mapping spaces and homotopy classes.** The set of morphisms $x \to y$ in a quasicategory C is precisely the set of objects of $\mathrm{map}_C(x, y)$. Two such are isomorphic as objects in $\mathrm{map}_C(x, y)$ if and only if they are homotopic in C.

33.6. **Proposition.** *Let C be a quasicategory. For any two maps $f, g\colon x \to y$ in C, we have that $f \approx g$ (equivalence under the relation used to define the homotopy category hC) if and only if f and g are isomorphic as objects of the quasigroupoid $\mathrm{map}_C(x, y)$. That is,*

$$\mathrm{Hom}_{hC}(x, y) \approx \pi_0 \, \mathrm{map}_C(x, y)$$

for every pair x, y of objects of C.

Proof. Suppose $f, g \in \mathrm{map}_C(x, y)_0$ are isomorphic, so that in particular there is a morphism $f \to g$ in the quasigroupoid $\mathrm{map}_C(x, y)$. This amounts to a map $\Delta^1 \times \Delta^1 \to C$ which can be represented by a diagram of elements of C of the form:

$$\begin{tikzcd} x \arrow[r, "g"] & y \\ x \arrow[u, "1_x"] \arrow[r, "f"'] \arrow[ur, "h"] \arrow[ur, phantom, shift left=3ex, "a"] \arrow[ur, phantom, shift right=3ex, "b"] & y \arrow[u, "1_y"'] \end{tikzcd}$$

This explicitly exhibits a chain $f \sim_r h \sim_\ell g$ of homotopies, so $f \approx g$ as desired.

Conversely, if $f \approx g$, we can explicitly construct a map $H\colon f \to g$ in $\mathrm{map}_C(x,y)$: in terms of the above picture, let $h = g$, let b be an explicit choice of right-homotopy $f \sim_r g$, and let $a = g_{001}$. $\square$

33.7. **Extended mapping spaces and composition.** Given a finite list $x_0, \dots, x_n \in C_0$ of objects in a quasicategory, we have an **extended mapping space**. These are the simplicial sets defined by the pullback squares

$$\begin{tikzcd} \mathrm{map}_C(x_0,\dots,x_n) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^n, C) \arrow[d] \\ \{(x_0,\dots,x_n)\} \arrow[r] & C^{\times(n+1)} \end{tikzcd}$$

where the right-hand vertical arrow is induced by restriction along $\mathrm{Sk}_0\,\Delta^n \to \Delta^n$, using the isomorphism $\mathrm{Sk}_0\,\Delta^n \approx \coprod_{n+1} \Delta^0$, whence $\mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, C) \approx C^{\times(n+1)}$. By (33.4) the extended mapping spaces are quasigroupoids.

On the other hand, we may consider the fibers of $\mathrm{Fun}(I^n, C) \to C^{\times(n+1)}$ defined by restriction along $\mathrm{Sk}_0\,\Delta^n = \mathrm{Sk}_0\,I^n \to I^n$, where $I^n \subset \Delta^n$ is the spine. The fibers of this map are isomorphic to n-fold products of mapping spaces $\mathrm{map}_C(x_{n-1},x_n) \times \cdots \times \mathrm{map}_C(x_0,x_1)$.

33.8. **Lemma.** *The map*

$$g_n\colon \mathrm{map}_C(x_0,\dots,x_n) \to \mathrm{map}_C(x_{n-1},x_n) \times \cdots \times \mathrm{map}_C(x_0,x_1)$$

induced by restriction along the spine inclusion $I^n \subseteq \Delta^n$ is a trivial fibration. In particular, this map is a categorical equivalence between Kan complexes.

Proof. The map g_n is a base change of $p\colon \mathrm{Fun}(\Delta^n, C) \to \mathrm{Fun}(I^n, C)$. Since $I^n \subset \Delta^n$ is inner anodyne (12.11), and C is a quasicategory, the map p is a trivial fibration by enriched lifting using $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (16.8). $\square$

The inclusions $I^2 \subset \Delta^2 \supset \Delta^{\{0,2\}}$ induce restriction maps

$$\mathrm{Fun}(I^2, C) \xleftarrow{\sim} \mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Delta^{\{0,2\}}, C)$$

in which the first map is a trivial fibration. As noted earlier (17.7) by choosing a categorical inverse to the first map (e.g., a section, since it is a trivial fibration) we obtain a "composition functor" $\mathrm{Fun}(I^2, C) \to \mathrm{Fun}(\Delta^1, C)$.

For any triple (x_0, x_1, x_2) of objects of C, the above maps restrict to maps between subcomplexes:

$$\mathrm{map}_C(x_1,x_2) \times \mathrm{map}_C(x_0,x_1) \xleftarrow[\sim]{g_2} \mathrm{map}_C(x_0,x_1,x_2) \to \mathrm{map}_C(x_0,x_2).$$

As g_2 is a trivial fibration (33.8), we can carry out the same construction and so obtain a "composition" functor

$$\mathrm{comp}\colon \mathrm{map}_C(x_1,x_2) \times \mathrm{map}_C(x_0,x_1) \to \mathrm{map}_C(x_0,x_2). \tag{33.9}$$

Again, this depends on a choice of categorical inverse to g_2. However, any two categorical inverses to g_2 are naturally isomorphic (19.4), and therefore comp is defined up to natural isomorphism. That is, it is a well-defined map in $h\mathrm{Kan}$, the homotopy category of Kan complexes (22.1).

33.10. **Proposition.** *The two maps obtained by composing the sides of the square*

$$\begin{tikzcd} \mathrm{map}_C(x_2,x_3) \times \mathrm{map}_C(x_2,x_1) \times \mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{id}\times\mathrm{comp}"] \arrow[d, "\mathrm{comp}\times\mathrm{id}"'] & \mathrm{map}_C(x_2,x_3) \times \mathrm{map}_C(x_0,x_2) \arrow[d, "\mathrm{comp}"] \\ \mathrm{map}_C(x_1,x_3) \times \mathrm{map}_C(x_0,x_1) \arrow[r, "\mathrm{comp}"'] & \mathrm{map}_C(x_0,x_3) \end{tikzcd}$$

are naturally isomorphic. That is, the diagram commutes in $h\mathrm{Kan} \subset h\mathrm{qCat}$*.*

Proof. Here is a diagram of Kan complexes which actually commutes "on the nose", i.e., not merely in the homotopy category, but in $s\mathrm{Set}$. I use "$\langle x,y,z\rangle$" as shorthand for "$\mathrm{map}_C(x,y,z)$", etc.

$$\begin{tikzcd}
\langle x_2,x_3\rangle \times \langle x_1,x_2\rangle \times \langle x_0,x_1\rangle & \langle x_2,x_3\rangle \times \langle x_0,x_1,x_2\rangle \arrow[l, "\sim"'] \arrow[r] & \langle x_2,x_3\rangle \times \langle x_0,x_2\rangle \\
\langle x_1,x_2,x_3\rangle \times \langle x_0,x_1\rangle \arrow[u, "\sim"'] \arrow[d] & \langle x_0,x_1,x_2,x_3\rangle \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"'] \arrow[d] & \langle x_0,x_2,x_3\rangle \arrow[u, "\sim"'] \arrow[d] \\
\langle x_1,x_3\rangle \times \langle x_0,x_1\rangle & \langle x_0,x_1,x_3\rangle \arrow[l, "\sim"'] \arrow[r] & \langle x_0,x_3\rangle
\end{tikzcd}$$

The maps labelled "$\xrightarrow{\sim}$" are trivial fibrations, and so are categorical equivalences. All the maps in the above diagram are obtained via restriction along inclusions in

$$\begin{tikzcd}
\Delta^{\{2,3\}} \cup \Delta^{\{1,2\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[d, "\sim"] & \Delta^{\{2,3\}} \cup \Delta^{\{0,1,2\}} \arrow[d, "\sim"] & \Delta^{\{2,3\}} \cup \Delta^{\{0,2\}} \arrow[l] \arrow[d, "\sim"] \\
\Delta^{\{1,2,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] & \Delta^3 & \Delta^{\{0,2,3\}} \arrow[l] \\
\Delta^{\{1,3\}} \cup \Delta^{\{0,1\}} \arrow[r, "\sim"] \arrow[u] & \Delta^{\{0,1,3\}} \arrow[u] & \Delta^{\{0,3\}} \arrow[l] \arrow[u]
\end{tikzcd}$$

where the maps labelled "$\xrightarrow{\sim}$" are inner anodyne (being generalized inner horn inclusions (12.9)), and which therefore give rise to trivial fibrations in the previous diagram by the same argument we used to define comp. After passing to $h\mathrm{Kan}$ the categorical equivalences become isomorphisms, and the result follows. □

33.11. **Segal categories.** Thus, a quasicategory does not quite give rise to a category "enriched over Kan complexes". Although we can define a composition law, it is not uniquely determined, and is only associative "up to homotopy".

What we do get is a Segal category. A **Segal category** is a functor

$$M\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$$

such that

(1) the simplicial set $M([0])$ is discrete, i.e., $M([0]) = \mathrm{Sk}_0\, M([0])$, and
(2) for each $n \geq 1$ the "Segal map"

$$M([n]) \xrightarrow{(\langle n-1,n\rangle^*,\dots,\langle 0,1\rangle^*)} M([1]) \times_{M([0])} \cdots \times_{M([0])} M([1])$$

is a "weak equvialence" of simplicial sets.

We will define "weak equivalence" of simplicial sets below. For now, we note that a map between *Kan complexes* is a weak equivalence if and only if it is a categorical equivalence, and that if each $M([n])$ is a Kan complex, then so are the fiber products which appear in the above definition.

Given a quasicategory C, we obtain a functor $M_C\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$\begin{aligned}
M_C([0]) &:= \mathrm{Sk}_0\, C, \\
M_C([n]) &:= \mathrm{Fun}(\Delta^n, C) \times_{\mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, C)} \mathrm{Fun}(\mathrm{Sk}_0\,\Delta^n, \mathrm{Sk}_0\, C) \\
&\approx \coprod_{x_0,\dots,x_n \in C_0} \mathrm{map}_C(x_0,\dots,x_n).
\end{aligned}$$

This object encodes all the structure we used above. For instance, the zig-zag

$$M_C([1]) \times_{M_C([0]} M_C([1]) \xleftarrow{(\langle 12\rangle^*,\langle 01\rangle^*)} M_C([2]) \xrightarrow{\langle 02\rangle^*} M_C([1])$$

is a coproduct over all triples $x_0,x_1,x_2 \in C_0$ of the zig-zag (33.9) used to define "composition".

You also get a Segal category from any "simplicially enriched" category. Suppose $\mathcal{C}$ is a (small) category which is enriched over the category of simplicial sets, with object set $\operatorname{ob}\mathcal{C}$, and function objects $\mathcal{C}(x, x') \in s\mathrm{Set}$ for each x, x'. Then we can define $M_{\mathcal{C}}\colon \Delta^{\mathrm{op}} \to s\mathrm{Set}$ by

$$M_{\mathcal{C}}([0]) := \operatorname{ob}\mathcal{C},$$
$$M_{\mathcal{C}}([n]) := \coprod_{x_0,\dots,x_n \in \operatorname{ob}\mathcal{C}} \mathcal{C}(x_{n-1}, x_n) \times \cdots \times \mathcal{C}(x_0, x_1).$$

We thus obtain functors

$$\mathrm{qCat} \to \mathrm{SeCat} \leftarrow s\mathrm{Cat}$$

relating quasicategories, Segal categories, and simplicially enriched categories. Simplicially enriched categories were proposed as a model for ∞-categories by Dwyer and Kan[25], while Segal categories were proposed as a model for ∞-categories by Hirschowitz and Simpson [HS01][26]. All of these models are known to be equivalent in a suitable sense; see [Ber10] for more about these models and their comparison.

33.12. **The enriched homotopy category of a quasicategory.** Given a quasicategory C we can produce a vestigial version of a category enriched over quasigroupoids, called the **enriched homotopy category** of C and denoted $\mathcal{H}C$.[27] This object will be a category *enriched over* hKan, where hKan is the full subcategory of hqCat spanned by Kan complexes. The underlying category of the enriched category $\mathcal{H}C$ will just be the homotopy category hC of C.

We now define $\mathcal{H}C$. The objects of $\mathcal{H}C$ are just the objects of C. For any two objects $x, y \in C_0$, we have the quasigroupoid

$$\mathcal{H}C(x, y) := \mathrm{map}_C(x, y)$$

which we will regard as an object of the homotopy category hKan of Kan complexes. Composition $\mathcal{H}C(x_1, x_2) \times \mathcal{H}C(x_0, x_1) \to \mathcal{H}C(x_0, x_2)$ is the composition map defined above (33.9), which is well-defined as a morphism in hKan. Composition is associative as shown above (33.10).

The *underlying* ordinary category of $\mathcal{H}C$ is just the ordinary homotopy category hC, since

$$\mathrm{Hom}_{h\mathrm{Kan}}(\Delta^0, \mathrm{map}_C(x, y)) \approx \pi_0 \,\mathrm{map}_C(x, y) \approx \mathrm{Hom}_{hC}(x, y).$$

33.13. *Warning.* A quasicategory C *cannot* be recovered from its enriched homotopy category $\mathcal{H}C$, not even up to equivalence. Furthermore, there exist hKan-enriched categories which do not arise as $\mathcal{H}C$ for any quasicategory C. A proof is outside the scope of these notes: counterexamples may be produced (for instance) from examples of associative H-spaces which are not loop spaces, and examples spaces which admit several inequivalent loop space structures.

33.14. *Exercise.* Let C and D be quasicategories. Show that there is an isomorphism $\mathcal{H}(C \times D) \approx \mathcal{H}C \times \mathcal{H}D$ of hKan-enriched categories.

34. The fundamental theorem of quasicategory theory

34.1. **Fully faithful and essentially surjective functors between quasicategories.** Note that any functor $f\colon C \to D$ of quasicategories induces functors $\mathrm{map}_C(x, y) \to \mathrm{map}_D(f(x), f(y))$ for every pair of objects x, y in C. We say that a functor $f\colon C \to D$ between quasicategories is

- **fully faithful** if for every pair $c, c' \in C_0$, the resulting map $\mathrm{map}_C(c, c') \to \mathrm{map}_D(fc, fc')$ is a categorical equivalence, and

[25] They called them "homotopy theories" instead of "∞-categories; see [DS95, §11.6].

[26] In fact, they generalize this to "Segal n-categories", which were the first effective model for (∞, n)-categories.

[27] Lurie usually calls this "hC", though he also uses that notation for the ordinary homotopy category of C that we have already discussed. I prefer to have two separate notations.

- **essentially surjective** if for every $d \in D_0$ there exists a $c \in C_0$ together with an isomorphism $fc \to d$ in D; that is, if the induced functor $hf\colon hC \to hD$ of ordinary categories is essentially surjective.

Another way to say this: $f\colon C \to D$ is fully faithful and essentially surjective iff the induced hKan-enriched functor $\mathcal{H}f\colon \mathcal{H}C \to \mathcal{H}D$ is an equivalence of *enriched* categories.

34.2. **Proposition.** *If $f\colon C \to D$ is a categorical equivalence between quasicategories, then f is fully faithful and essentially surjective.*

Proof. To prove essential surjectivity, choose any categorical inverse g to f and natural isomorphism $\alpha\colon fg \to \mathrm{id}_D$. Then for any $d \in D_0$ we get an object $c := g(d) \in C_0$ and an isomorphism $\alpha(d)\colon f(c) \to d$ in D.

To show that f is fully faithful, choose a categorical inverse g of f. Given $x, y \in C_0$, consider the induced diagram of quasigroupoids

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r, "f"] \arrow[rr, bend left, "gf"] & \mathrm{map}_D(fx,fy) \arrow[r, "g"] \arrow[rr, bend right, "fg"'] & \mathrm{map}_C(gfx,gfy) \arrow[r, "f"] & \mathrm{map}_D(fgfx,fgfy) \end{tikzcd}$$

By the 2-out-of-6 property for categorical equivalences (22.10), it will suffice to show that the maps marked gf and fg are categorical equivalences between the respective mapping spaces. Since $gf\colon C \to C$ and $fg\colon D \to D$ are naturally isomorphic to the identity maps of C and D respectively, the claim follows from (34.3) which we prove below. ☐

34.3. **Proposition.** *If $f_0, f_1\colon C \to D$ are functors which are naturally isomorphic, then f_0 is fully faithful if and only if f_1 is.*

To prove this, we will need to use the path category construction.

34.4. **Path category.** For the proof of the fact that natural isomorphisms preserve the fully-faithful property, we will need to consider the **path category** of a quasicategory D. This is the full subcategory

$$\widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$$

spanned by the objects corresponding to functors $\Delta^1 \to D$ which represent isomorphisms in D. (A generalization of this construction will be introduced later (42.1).) The restriction maps along $\{0\} \subset \Delta^1 \supset \{1\}$ induce functors $D \xleftarrow{r_0} \widehat{D} \xrightarrow{r_1} D$. Note that a functor $\widetilde{H}\colon C \to \widehat{D}$ corresponds exactly to giving a natural isomorphism $H\colon C \times \Delta^1 \to D$ of functors $f_0, f_1\colon C \to D$, where $f_i = r_i\widetilde{H}$ (because natural isomorphisms are the same as objectwise natural isomorphisms (31.3)).

34.5. *Remark.* If D is a Kan complex (i.e., a quasigroupoid), then $\widehat{D} = \mathrm{Fun}(\Delta^1, D)$.

34.6. *Warning.* The path category $\widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$ is not the same as the core $\mathrm{Fun}(\Delta^1, D)^{\mathrm{core}} \subseteq \mathrm{Fun}(\Delta^1, D)$, unless D is a quasigroupoid: the path category is always a full subcategory, whereas the core is typically not a full subcategory.

34.7. *Exercise.* Show that there is a bijective correspondence between (i) morphisms $H\colon \Delta^1 \to \mathrm{Fun}(C, D)^{\mathrm{core}}$ (i.e., natural isomorphisms between functors $C \to D$) and (ii) morphisms $\widetilde{H}\colon C \to \widehat{D}$ to the path category.

34.8. **Lemma.** *Let D be a quasicategory. Then both restriction functors*

$$D \xleftarrow{r_0} \widehat{D} \xrightarrow{r_1} D$$

from the path category are trivial fibrations.

Proof. We need to solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \widehat{D} \arrow[r, tail] \arrow[d, "r_i"] & \mathrm{Fun}(\Delta^1, D) \arrow[dl] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & \mathrm{Fun}(\{i\}, D) & \end{tikzcd}$$

for all $n \geq 0$ and $i = 0, 1$. When $n = 0$ this is easy: any object of D is the source and target of an isomorphism in D, namely its identity map. For $n \geq 1$ it suffices to find a lifting in the adjoint lifting problem

$$\begin{tikzcd} \Delta^1 \times \{j\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{i\} \times \Delta^n) \cup_{\{i\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & D \\ & \Delta^1 \times \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

where $j = 0$ if $i = 0$ and $j = n$ if $i = 1$. In either case we know by hypothesis that f represents an isomorphism in D, so a lift exists by the "pushout-product version" of Joyal lifting (31.6).

□

34.9. **Lemma.** *Any trivial fibration $p\colon C \to D$ between quasicategories is fully faithful.* **M 4 Mar 2019**

Proof. For $x, y \in C_0$ we have a diagram of pullback squares

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r] \arrow[d, "q"] & \mathrm{Fun}(\Delta^1, C) \arrow[d, "p^{\square(\partial\Delta^1 \subset \Delta^1)}"] & \\ \mathrm{map}_D(px, py) \arrow[r] \arrow[d] & \mathrm{Fun}(\partial\Delta^1, C) \times_{\mathrm{Fun}(\partial\Delta^1, D)} \mathrm{Fun}(\Delta^1, D) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, D) \arrow[d] \\ \{(x,y)\} \arrow[r] & \mathrm{Fun}(\partial\Delta^1, C) \arrow[r] & \mathrm{Fun}(\partial\Delta^1, D) \end{tikzcd}$$

The pullback-hom $p^{\square(\partial\Delta^1 \subset \Delta^1)}$ is a trivial fibration using $\overline{\mathrm{Cell}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$, so q is a trivial fibration and thus a categorical equivalence. □

Now we can prove that fully-faithful is natural-isomorphism invariant.

Proof of (34.3). Consider a natural isomorphism $H\colon C \times \Delta^1 \to D$ between f_0 and f_1, and write $\widetilde{H}\colon C \to \widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$ for its adjoint. The lemma (34.8) implies that in the commutative diagram

$$\begin{tikzcd} & & \mathrm{Fun}(\{0\}, D) = D \\ C \arrow[r, "\widetilde{H}"] \arrow[urr, bend left, "f_0"] \arrow[drr, bend right, "f_1"'] & \widehat{D} \arrow[ur, "r_0"] \arrow[dr, "r_1"'] & \\ & & \mathrm{Fun}(\{1\}, D) = D \end{tikzcd}$$

both r_0 and r_1 are trivial fibrations. Because r_0 and r_1 are trivial fibrations, for any $x, y \in C_0$ we get a commutative diagram

$$\begin{tikzcd} & & \mathrm{map}_D(f_0(x), f_0(y)) \\ \mathrm{map}_C(x,y) \arrow[r] \arrow[urr, bend left, "f_0"] \arrow[drr, bend right, "f_1"'] & \mathrm{map}_{\widehat{D}}(\widetilde{H}(x), \widetilde{H}(y)) \arrow[ur, "\sim"] \arrow[dr, "\sim"] & \\ & & \mathrm{map}_D(f_1(x), f_1(y)) \end{tikzcd}$$

in which the maps indicated by "$\sim$" are categorical equivalences by (34.9). Using the 2-out-of-3 property of categorical equivalences (22.10), we see that the map marked f_0 is a categorical equivalence if and only if the map marked f_1 is. Thus we have shown that $f_0\colon C \to D$ is fully faithful if and only if $f_1\colon C \to D$ is fully faithful. $\square$

We've finished proving the lemma we needed for the proof that categorical equivalences are fully faithful (34.2).

We note a useful fact: to check that a functor is fully faithful, it suffices to check the defining property on *representatives of isomorphism classes* of objects.

34.10. **Proposition.** *Let $f\colon C \to D$ be a functor between quasicategories, and let $S \subset C_0$ be a subset of objects which includes a representative of every isomorphism class in C. Then f is fully faithful if and only if* $\mathrm{map}_C(c, c') \to \mathrm{map}_D(fc, fc')$ *is a categorical equivalence for all $c, c' \in S$.*

Proof. The only-if direction is immediate from the definition of fully faithful. To prove the if direction, let $x, x' \in C_0$ and choose isomorphisms $\alpha\colon x \to c$ and $\alpha\colon x \to c'$ where $c, c' \in S$. We may interpret α and α' as objects of $\widehat{C} \subseteq \mathrm{Fun}(\Delta^1, C)$. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{map}_C(x, x') \arrow[d, "f"'] & \mathrm{map}_{\widehat{C}}(\alpha, \alpha') \arrow[l, "r_0"'] \arrow[r, "r_1"] \arrow[d, "\widehat{f}"'] & \mathrm{map}_C(c, c') \arrow[d, "f"'] \\
\mathrm{map}_D(fx, fx') & \mathrm{map}_{\widehat{D}}(\widehat{f}\alpha, \widehat{f}\alpha') \arrow[l, "r_0"] \arrow[r, "r_1"'] & \mathrm{map}_D(fc, fc')
\end{tikzcd}$$

where the vertical arrows are induced by $f\colon C \to D$ and $\widehat{f}\colon \widehat{C} \to \widehat{D}$, where $\widehat{f}$ is the restriction of $\mathrm{Fun}(\Delta^1, f)\colon \mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\Delta^1, D)$ to full subcategories. The maps marked r_0 and r_1 are categorical equivalences by (34.8) and (34.9). Therefore the left-hand vertical arrow is a categorical equivalence using the hypothesis on f and 2-out-of-3 for categorical equivalences (22.10). $\square$

34.11. **The fundamental theorem for quasicategories.** The converse to (34.2) is true, but nowhere near as straightforward to prove.

G. **Deferred Proposition** (Fundamental Theorem of Quasicategory Theory)**.** *A map $f\colon C \to D$ between quasicategories is a categorical equivalence if and only if it is fully faithful and essentially surjective.*

This is a non-trivial result. It gives a necessary and sufficient condition for $f\colon C \to D$ to admit a categorical inverse, but it does not spell out how to construct such an inverse. After many preliminaries, we will give the proof in §43.

34.12. **2-out-of-6 for fully faithful essentially surjective functors.** The following result will be useful in the proof of the fundamental theorem. Recall the *2-out-of-6* and *2-out-of-3* properties of a class of morphisms (22.5), and that the class of categorical equivalences has these properties (22.10).

34.13. **Proposition.** *The class $\mathcal{C}$ of fully faithful and essentially surjective functors between quasi-categories satisfies the 2-out-of-6 property, and thus the 2-out-of-3 property.*

Proof. Any identity functor $\mathrm{id}\colon C \to C$ is manifestly fully faithful and essentially surjective.

Next note that if a functor $f\colon C \to D$ between quasicategories is fully faithful and essentially surjective, then the induced $hf\colon hC \to hD$ is an equivalence of ordinary categories. Conversely, if hf is an equivalence, then f is essentially surjective.

Suppose $C \xrightarrow{f} D \xrightarrow{g} E \xrightarrow{h} F$ is a sequence of functors between quasicategories such that gf and hg are fully faithful and essentially surjective. The induced sequence $hC \to hD \to hE \to hF$ of functors on homotopy categories has the same property, and thus all the functors between homotopy categories are equivalences. From this we conclude immediately that f, g, h, hgf are essentialy surjective.

Given objects $x, y \in C_0$, we have induced maps

$$\begin{tikzcd} \mathrm{map}_C(x,y) \arrow[r, "f"] \arrow[rr, bend left, "gf"] & \mathrm{map}_D(fx,fy) \arrow[r, "g"] \arrow[rr, bend right, "hg"'] & \mathrm{map}_E(gfx,gfy) \arrow[r, "h"] & \mathrm{map}_F(hgfx,hgfy) \end{tikzcd}$$

The hypothesis that gf and hg are fully faithful implies that the indicated arrows are categorical equivalences, and hence all arrows are by (22.10). Because f and gf are essentially surjective, the collections of objects $\{ fx \mid x \in C_0 \} \subseteq D_0$ and $\{ gfx \mid x \in C_0 \} \subseteq E_0$ include representatives of every isomorphism class of D and E respectively, and thus (34.10) implies that f, g, h, and therefore hgf, are fully faithful. □

35. The path factorization

To prove the fundamental theorem of quasicategories for a general map between quasicategories, we will reduce to the special case of isofibrations. We do this by means of the "path factorization" (or "path fibration") construction, which provides a factorization of a map into a categorical equivalence followed by an isofibration.

35.1. **The path factorization construction.** Let D be a quasicategory. Recall (34.4) the path category $\widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$ of D, equipped with restriction maps $r_i \colon \widehat{D} \to D$, $i = 0, 1$.

For a functor $f \colon C \to D$ between quasicategories, we define a factorization $C \xrightarrow{j} P(f) \xrightarrow{p} D$ by means of the commutative diagram

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[dr, equal] & P(f) \arrow[r, "t"'] \arrow[d, "s_0"] \arrow[rr, bend left, "p"] & \widehat{D} \arrow[r, "r_1"'] \arrow[d, "r_0"] & D \\ & C \arrow[r, "f"'] & D & \end{tikzcd}$$

in which the square is a pullback square. The map j is the unique one so that $s_0 j = \mathrm{id}_C$, and $tj = \widetilde{\pi} f$ where $\widetilde{\pi} \colon D \to \widehat{D} \subseteq \mathrm{Fun}(\Delta^1, D)$ is adjoint to the projection $D \times \Delta^1 \to D$.

The properties of this construction are summarized by the following.

35.2. **Proposition.** *In the path factorization of f, the simplicial set $P(f)$ is a quasicategory, the map j is a categorical equivalence, and p is an isofibration. Furthermore s_0 is a trivial fibration.*

Note that the objects of $P(f)$ are pairs (c, α) consisting of an object $c \in C_0$ and an isomorphism $\alpha \colon f(c) \to d$ in D. The map j sends an object c to $(c, 1_{f(c)})$, while p sends (c, α) to d.

35.3. *Exercise.* Show that if $f \colon C \to D$ is a functor between ordinary categories, then $P(f)$ is also an ordinary category.

Proof. From (34.8) we know that both r_0 and r_1 are trivial fibrations. Therefore the base change s_0 of r_0 is a trivial fibration, and hence an inner fibration, which implies that $P(f)$ is a quasicategory.

Since s_0 is a trivial fibration it is a categorical equivalence (20.10), and thus j is a categorical equivalence by 2-out-of-3 (22.10).

To show that p is an isofibration, observe that there is actually a pullback square of the form

$$\begin{tikzcd} P(f) \arrow[r, "t"] \arrow[d, "s=(s_0,p)"'] & \widehat{D} \arrow[d, "r=(r_0,r_1)"] \\ C\times D \arrow[r, "f\times \mathrm{id}_D"'] & D\times D \end{tikzcd}$$

(To see this, use patching of pullback squares where we regard $C\times D$ as a pullback of $C\xrightarrow{f} D\leftarrow D\times D$.) We will prove below (35.5) that r is an isofibration, whence its base-change s is also an isofibration, and since the projection $\pi\colon C\times D\to D$ is an isofibration the composite $p=\pi s$ is an isofibration as desired. (I have here used several facts about isofibrations which are left as an exercise (35.4).) $\square$

35.4. *Exercise* (Some properties of isofibrations). Prove the following facts about isofibrations.

(1) For any quasicategory C, the projection $C\to *$ is an isofibration.
(2) The composite of two isofibrations is an isofibration.
(3) Any base-change of an isofibration $p\colon C\to D$ along a map $D'\to D$ from a quasicategory is also an isofibration. (Hint: use (33.3).)
(4) For any quasicategories C and D the projection $C\times D\to D$ is an isofibration.

35.5. **Lemma.** *If D is a quasicategory, then the map $r=(r_0,r_1)\colon \widehat{D}\subseteq \mathrm{Fun}(\Delta^1,D)\to D\times D$ from the path category induced by restriction along $\partial\Delta^1\subset\Delta^1$ is an isofibration.*

Proof. First note that both maps $\widehat{D}\to\mathrm{Fun}(\Delta^1,D)\to D\times D$ are inner fibrations, whence the composite r is an inner fibration. The first map is an inner fibration because it is an inclusion of a subcategory (13.10), while the second is so by enriched lifting and $\overline{\mathrm{InnHorn}}\,\Box\,\overline{\mathrm{Cell}}\subseteq\overline{\mathrm{InnHorn}}$ (17.2).

To prove that r is an isofibration, we need to produce a lift in a diagram of the form

$$\begin{tikzcd} \{0\} \arrow[r, "g"] \arrow[d, tail] & \widehat{D} \arrow[r, tail] \arrow[d, "r"] & \mathrm{Fun}(\Delta^1,D) \arrow[d] \\ \Delta^1 \arrow[r, "f=(f_0,f_1)"'] \arrow[ur, dotted, "s"] & D\times D \arrow[r, equal] & D\times D \end{tikzcd}$$

where f represents an isomorphism in $D\times D$, or equivalently f_0 and f_1 represent isomorphisms in D. By the usual lifting-adjunction arguments, to produce a lifting $s\colon\Delta^1\to\mathrm{Fun}(\Delta^1,D)$ it is equivalent to produce an extension $\widetilde{s}$ in

$$\begin{tikzcd} (\{0\}\times\Delta^1)\cup_{\{0\}\times\partial\Delta^1}(\Delta^1\times\partial\Delta^1) \arrow[r, "(\widetilde{g},\widetilde{f})"] \arrow[d, tail] & D \\ \Delta^1\times\Delta^1 \arrow[ur, dotted, "\widetilde{s}"'] \end{tikzcd}$$

where $\widetilde{g}$ and $\widetilde{f}$ are adjoint to g and f. The map $(\widetilde{g},\widetilde{h})$ corresponds to the solid-arrow part of the diagram

$$\begin{tikzcd} \bullet \arrow[r, "f_1"] & \bullet \\ \bullet \arrow[u, "g"] \arrow[r, "f_0"'] \arrow[ur, dotted] \arrow[ur, phantom, shift left=2ex, "a"] \arrow[ur, phantom, shift right=2ex, "b"'] & \bullet \arrow[u, dotted, "h"'] \end{tikzcd}$$

We can produce an extension $\widetilde{s}$ in two steps: first extend to a, which is attached along an inner horn $\Lambda^2_1\subset\Delta^2$, then extend to b, which is attached to the previous along the horn $\Lambda^2_0\subset\Delta^2$. This extension exists by Joyal extension since f_0 is an isomorphism. (This argument is just the proof of easiest case of the pushout-product version of Joyal lifting (31.6).)

Note that g, f_0, and f_1 are isomorphisms in D, and thus h is also an isomorphism in D, since $[f_1][g] = [h][f_0]$ in hD. In particular, $\widetilde{s}|\{j\} \times \Delta^1 \colon \{j\} \times \Delta^1 \to D$ represent isomorphisms in D for $j = 0, 1$, and therefore s lands in the path subcategory $\widehat{D} \subset \mathrm{Fun}(\Delta^1, D)$ as desired. □

35.6. **Reduction of the fundamental theorem to the case of isofibrations.**

35.7. **Lemma.** *To prove the fundamental theorem* (G)*, it suffices to prove it for isofibrations.*

Proof. Let $f \colon C \to D$ be a functor which is fully faithful and essentially surjective. Consider the path factorization

$$C \xrightarrow{j} P(f) \xrightarrow{p} D$$

of f, with j a categorical equivalence and p an isofibration (35.2). Recall that the class categorical equivalences satisfies 2-out-of-3 (22.10), as does the class of functors which are fully faithful and essentially surjective (34.13). Since every categorical equivalence (such as j) is fully faithful and essentially surjective (34.2), the claim follows. □

We will prove (G) by proving it for the special case of isofibrations, following (35.7), in (43). In fact we will show that an isofibration which is fully faithful and essentially surjective is a trivial fibration.

First, we will consider the special case of quasigroupoids=Kan complexes.

36. Weak equivalence, anodyne maps, and Kan fibrations

In the next few sections, we will develop some properties related to Kan complexes and Kan fibrations. As a byproduct, we'll obtain the proof of the specialization of (G) to Kan complexes.

36.1. **Weak equivalence.** Say that a map $f \colon X \to Y$ is a **weak equivalence** of simplicial sets if and only if $\mathrm{Map}(f, G) \colon \mathrm{Map}(Y, G) \to \mathrm{Map}(X, G)$ is a categorical equivalence for every *quasigroupoid* (i.e., every *Kan complex*) G.

Every categorical equivalence is a automatically weak equivalence, but the converse does not hold; see (36.7) below. For maps between Kan complexes however, weak equivalences and categorical equivalences are the same thing.

36.2. **Proposition.** *If $f \colon X \to Y$ is a map between Kan complexes, then f is a weak equivalence if and only if it is a categorical equivalence.*

Proof. As we have observed, every categorical equivalence is a weak equivalence. For the converse the proof is straightforward, using the same ideas as the proof of (19.6). That is, for a weak equivalence $f \colon X \to Y$ between Kan complexes, the induced maps $\mathrm{Map}(f, X)$ and $\mathrm{Map}(f, Y)$ are be categorical equivalences between quasicategories. We can then use this information to produce a categorical inverse to f, exactly as in the proof of (19.6): since $f^* \colon \mathrm{Fun}(Y, X) \to \mathrm{Fun}(X, X)$ is essentially surjective, we can choose an object g of $\mathrm{Fun}(Y, X)$ such that gf and id_X are isomorphic in $\mathrm{Fun}(X, X)$, and then observe that fg and id_Y must be isomorphic in $\mathrm{Fun}(Y, Y)$ since $f^* \colon \mathrm{Fun}(Y, Y) \to \mathrm{Fun}(X, Y)$ is fully faithful and $fgf \approx f\,\mathrm{id}_X = \mathrm{id}_Y f$. □

36.3. **Proposition.** *Weak equivalences of simplicial sets satisfy the 2-out-of-6 property, and thus the 2-out-of-3 property.* **W 6 Mar 2019**

Proof. Proved exactly as for categorical equivalences (22.10). □

36.4. *Remark.* Given the analogy to categorical equivalence, a more sensible name for weak equivalence is "groupoidal equivalence". However, the term "weak equivalence" here is historically well-established.

36.5. **Simplicial homotopy equivalence.** Given maps $f, f'\colon X \to Y$ of simplicial sets, we say they are **simplicially homotopic** if there exists a chain of edges in the function complex $\mathrm{Map}(X, Y)$ connecting f to f'. That is, f and f' are simplicially homotopic if they are in the same path component (6.8) of $\mathrm{Map}(X, Y)$, i.e., they represent the same element of $\pi_0 \mathrm{Map}(X, Y)$.

Note that if $\mathrm{Map}(X, Y)$ is a Kan complex, then $f, f'\colon X \to Y$ are simplicially homotopic if and only if they are isomorphic as objects in the quasigroupoid $\mathrm{Map}(X, Y)$, in which case they can be related by a single edge $f \to f'$.

A **simplicial homotopy inverse** to a map $f\colon X \to Y$ of simplicial sets is a map $g\colon Y \to X$ such that gf is simplicially homotopic to id_X, and fg is simplicially homotopic to id_Y. A map f which admits a simplicial homotopy inverse is called a **simplicial homotopy equivalence**, and of course any simplicial homotopy inverse to f is also a simplicial homotopy equivalence.

36.6. **Proposition.** *Any simplicial homotopy equivalence is a weak equaivalence.*

Proof. First, if $f\colon X \to Y$ is a simplicial homotopy equivalence between Kan complexes, then it is clearly a categorical equivalence, because $\mathrm{Map}(X, X)$ and $\mathrm{Map}(Y, Y)$ are quasigroupoids, and so any simplicial homotopy inverse $g\colon Y \to X$ for f satisfies $gf \approx \mathrm{id}_X$ and $fg \approx \mathrm{id}_Y$ and so is a categorical inverse for f, and therefore f is a weak equivalence (36.2).

In general, suppose G is a Kan complex and consider $f^*\colon \mathrm{Map}(Y, G) \to \mathrm{Map}(X, G)$. By the same reasoning as used in the proof of (19.6), we see that f^* is a simplicial homotopy equivalence between Kan complexes, so a categorical equivalence. $\square$

36.7. *Exercise* (Simplices are weakly equivalent). Show that any two maps $f, f'\colon \Delta^m \to \Delta^n$ are simplicially homotopy equivalent. From (36.6) it follows that any map $f\colon \Delta^m \to \Delta^n$ between standard simplicies is a weak equivalence, using the 2-out-of-3 property (36.3). Any such f which is not an identity map gives an example of a weak equivalence between quasicategories which is not a categorical equivalence.

36.8. **Anodyne maps and Kan fibrations.** Let

$$\mathrm{Horn} = \left\{ \Lambda^n_j \subset \Delta^n \;\middle|\; n \geq 1,\ 0 \leq j \leq n \right\} = \mathrm{RHorn} \cup \mathrm{LHorn}$$

denote the set of all horn inclusions. A map is **anodyne** if it is in $\overline{\mathrm{Horn}}$, and is a **Kan fibration** if it is in $\mathrm{KanFib} := \mathrm{Horn}^{\boxslash}$.

Since Horn is a set, the small object argument (13.17) applies to it: any map can be factored $f = pj$ with $j \in \overline{\mathrm{Horn}}$ and $p \in \mathrm{KanFib}$.

36.9. **Proposition.** *We have that* $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$.

Proof. This amounts to showing $\mathrm{Horn} \square \mathrm{Cell} \subseteq \overline{\mathrm{Horn}}$, which is proved in [JT08, Theorem 3.2.2], or [GZ67]. We give a proof in the appendix (62). $\square$

This implies the following version of enriched lifting for Kan fibratons.

36.10. **Corollary** (Enriched lifting for Kan fibrations)**.** *If $i\colon K \to L$ is a monomorphism and $p\colon X \to Y$ is a Kan fibration, then*

$$p^{\square i}\colon \mathrm{Map}(L, X) \to \mathrm{Map}(K, X) \times_{\mathrm{Map}(K,Y)} \mathrm{Map}(L, Y)$$

is also a Kan fibration. Furthermore, if i is anodyne then $p^{\square i}$ is a trivial fibration.

As a special case, we learn that if X is a Kan complex and $i\colon K \to L$ a monomorphism, then $\mathrm{Fun}(L, X) \to \mathrm{Fun}(K, X)$ is a Kan fibration, and that it is a trivial fibration if i is anodyne. Here is an immediate consequence, reminicent of the observation that *inner* anodyne maps are categorical equivalences.

36.11. **Proposition.** *Every anodyne map is a weak equivalence.*

Proof. If $f\colon A \to B$ is anodyne, then $\mathrm{Map}(f, X)$ is a trivial fibration for every Kan complex X, and hence a categorical equivalence (20.10). $\square$

36.12. *Exercise.* Show that the inclusion $\{j\} \subseteq \Delta^n$ of any vertex into any standard n-simplex is anodyne. (Hint: for $0 \leq k \leq n$ let $F_k \subseteq \Delta^n$ be the subcomplex which is the union of all $\Delta^S \subseteq \Delta^n$ with $j \in S$ and $|S| \leq k+1$, and show that each inclusion $F_{k-1} \to F_k$ is obtained by attaching a collection of simplices to F_{k-1} along horns.)

36.13. *Exercise* (Important!). Let $f\colon X \to Y$ be any map between Kan complexes. Show that f is a Kan fibration if and only if it is an isofibration. (Hint: Joyal lifting + definition of isofibration.)

36.14. *Exercise.* Give an example of an inner fibration between Kan complexes which is not a Kan fibration.

36.15. **The walking isomorphism.** Let Iso be the "walking isomorphism", i.e., the category with two objects 0 and 1, and a unique isomorphism between them. Let $u\colon \Delta^1 \to N\mathrm{Iso}$ be the inclusion representing the unique map $0 \to 1$ in Iso. (**Should I just write "Iso" for the nerve?**)

36.16. **Proposition.** *The map $u\colon \Delta^1 \to N\mathrm{Iso}$ is anodyne, and hence a weak equivalence.*

Proof. The k-dimensional elements of $N(\mathrm{Iso})$ are in one-to-one correspondence with sequences $(x_0x_1\cdots x_k)$ with $x_i \in \{0,1\}$. For each $k \geq 0$ there are exactly two *non-degenerate* k-dimensional elements, corresponding to the alternating sequences $(0101\dots)$ and $(1010\dots)$ of length $k+1$.

Let $u_k, v_k\colon \Delta^k \to N\mathrm{Iso}$ be the non-degenerate elements $u_k = (0101\dots)$ and $v_k = (1010\dots)$ in $(N\mathrm{Iso}_k)$. Let $F_k \subset N\mathrm{Iso}$ be the smallest subcomplex containing u_k. Observe that for a simplicial operator $f\colon [d] \to [k]$ we have $u_k f = (x_0x_1\cdots x_d)$ with $x_i \equiv f(i) \bmod 2$. In particular,

- $u_k\langle 1\cdots k\rangle = v_{k-1}$,
- $u_k\langle 0\cdots k-1\rangle = u_{k-1}$,
- $u_k\langle 01\cdots\widehat{i}\cdots k-1, k\rangle$ is a degenerate element associated to u_{k-2} if $i = 1, \dots, k-1$.

From this we can see that the only non-degenerate elements of $F_k \smallsetminus F_{k-1}$ are u_k and $v_{k-1} = u_k\langle 1\cdots k\rangle$. Therefore $N\mathrm{Iso} = \bigcup F_k$, $F_1 = u(\Delta^1)$, and the commutative square

$$\begin{tikzcd} \Lambda^k_0 \arrow[r] \arrow[d, tail] & F_{k-1} \arrow[d, tail] \\ \Delta^k \arrow[r, "u_k"'] & F_k \end{tikzcd}$$

is a pushout square for all $k \geq 1$ by (15.25), since (1) it is a pullback, and (2) any element in the complement of $F_{k-1} \subset F_k$ is the image of a unique element under the map u_k.

It follows that u is anodyne. $\square$

As an immediate consequence, any map $f\colon \Delta^1 \to C$ can be extended over $N\mathrm{Iso}$ when C is a quasigroupoid. We can easily refine this to give a criterion for f to represent an isomorphism in a general quasicategory.

36.17. **Proposition.** *Let C be a quasicategory, and $f\colon \Delta^1 \to C$ a map. Then there exists $f'\colon N(\mathrm{Iso}) \to C$ with $f'u = f$ if and only if f represents an isomorphism in C.*

Proof. ($\Longrightarrow$) Clear: consider induced maps $[1] \to \mathrm{Iso} \to hC$ on homotopy categories. ($\Longleftarrow$) If f represents an isomorphism then it factors through $\Delta^1 \to C^{\mathrm{core}} \subseteq C$. Since the core is a quasigroupoid, and hence a Kan complex, an extension along the anodyne map u to a map $N\mathrm{Iso} \to C^{\mathrm{core}} \subseteq C$ exists. $\square$

36.18. *Exercise.* Let Z be the complex of (20.6), and let $F\colon \Delta^1 = \Delta^{\{1,2\}} \to Z$ be the map representing the edge $f \in Z_1$. Show that F is anodyne, and state and prove an analogue of (36.17) with Z in place of $N\mathrm{Iso}$.

36.19. *Remark.* Let $X \subset N\mathrm{Iso}$ be the subcomplex which is the union of the images of 2-dimensional elements 010 and 101[28]. The inclusion $v\colon \Delta^1 \to X$ representing the edge 01 has the same property described in (36.17): $f\colon \Delta^1 \to C$ represents an isomorphism if and only if it extends along v. The proof is easy: an extension of f to a map $f'\colon X \to C$ exactly encodes a choice of morphism g in C (i.e., $f'(\langle 10\rangle)$) together with explicit homotopies $gf \sim_r 1$ and $fg \sim_\ell 1$, (i.e., $f'(\langle 010\rangle)$ and $f'(\langle 101\rangle)$).

However, it turns out that $\Delta^1 \to X$ is *not* a weak equivalence (and therefore that X is not categorically equivalent to $N\mathrm{Iso}$). In particular, a map $X \to C$ to a quasicategory can fail to extend along $X \subset N\mathrm{Iso}$.

36.20. *Exercise.* Show that the inclusion $X \to N\mathrm{Iso}$ of the previous remark (36.19) is not anodyne, by constructing a map $X \to K(\mathbb{Z}, 2)$ which does not extend over $N\mathrm{Iso}$. (See (8.10).)

36.21. **Covering homotopy extension property.** Here is a very handy consequence of the enriched lifting properties of anodyne maps (36.10). Let $i\colon A \to B$ and $p\colon X \to Y$ be maps of simplicial sets, and recall the pullback-product map

$$p^{\square i}\colon\ \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, X).$$

A vertex (u, v) in the target of $p^{\square i}$ corresponds to a lifting problem of type $i \boxslash p$, and this lifting problem has a solution if and only if the vertex (u, v) is in the image of a vertex s in $\mathrm{Map}(B, X)$.

An edge e in $\mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$ from vertex (u_0, v_0) to vertex (u_1, v_1) corresponds to a commutative square

$$\begin{tikzcd} A \times \Delta^1 \arrow[r, "\widetilde{u}"] \arrow[d, "i\times\mathrm{id}"'] & X \arrow[d, "p"] \\ B \times \Delta^1 \arrow[r, "\widetilde{v}"'] & Y \end{tikzcd}$$

such that $\widetilde{u}|A \times \{k\} = u_k$ and $\widetilde{v}|B \times \{k\} = v_k$ for $k = 0, 1$. We think of such an edge as a "deformation" relating the two lifting problems. The covering homotopy extension property says that in certain circumstances you can solve a lifting problem by deforming it to another one which you know you can solve.

36.22. **Proposition** (Covering homotopy extension)**.** *Let $i\colon A \to B$ be a monomorphism and $p\colon X \to Y$ a Kan fibration. If two lifting problems (u_0, v_0) and (u_1, v_1) of type $i \boxslash p$ are connected by an edge in* $\mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, X)$, *then (u_0, v_0) admits a lift if and only if (u_1, v_1) admits a lift.*

Proof. Let e be such an edge. I'll show that if (u_0, v_0) admits a lift $s\colon B \to X$, then (u_1, v_1) also admits a lift. The hypotheses on i and p, togther with enriched lifting associated to $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (36.10) imply that $p^{\square i}$ is a Kan fibration, and thus in particular $(\{0\} \subset \Delta^1) \boxslash p^{\square i}$ holds, and therefore a lift t exists in the commutative square

$$\begin{tikzcd} \{0\} \arrow[r, "s"] \arrow[d, tail] & \mathrm{Map}(B, X) \arrow[d, "p^{\square i}"] \\ \Delta^1 \arrow[r, "e"'] \arrow[ur, dotted, "t"] & \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y) \end{tikzcd}$$

Then the vertex $t(1) \in \mathrm{Map}(B, X)_0$ gives the desired lift for (u_1, v_1). The proof of the reverse direction is similar, using $(\{1\} \subset \Delta^1)$ instead of $(\{0\} \subset \Delta^1)$. $\square$

[28]This is isomorphic to the complex Z' of (20.7).

36.23. **Fundamental theorem for Kan complexes: reduction to Kan fibrations.** We are going to show the following

36.24. **Theorem.** *A map $f\colon X \to Y$ between Kan complexes is a weak equivalence if and only if it is fully faithful and essentially surjective.*

Since weak equivalences between Kan complexes are the same as categorical equivalences (36.2), this is precisely the specialization of the fundamental theorem (G) to functors between Kan complexes.

36.25. **Lemma.** *To prove* (36.24)*, it suffices to prove it when f is a Kan fibration.*

Proof. We consider the path factorization for f (35.2), which takes the form $f = pj$ with $P(f)$ a quasicategory, $j\colon X \to P(f)$ a categorical equivalence, and p an isofibration. Since j is a categorical equivalence, $P(f)$ is a quasigroupoid. By a straightforward argument (36.13), any isofibration between Kan complexes is a Kan fibration, and so p is a Kan fibration. The claim follows by the same 2-out-of-3 argument used in (35.7).

(Alternate proof: since Y is a quasigroupoid, its path category satisfies $\widehat{Y} = \mathrm{Fun}(\Delta^1, Y)$. Use (36.9) to show directly that p is a Kan fibration.) □

We will prove the needed special case (that for Kan fibrations between Kan complexes, fully faithful and essentially surjective implies weak equivalence) in the next couple of sections, after analyzing Kan fibrations in more detail.

37. Kan fibrations between Kan complexes

In the next few sections, we are going to be considering various properties of Kan fibrations, with particular interest in Kan fibrations between Kan complexes.

In particualar, we are going to show that for a Kan fibration $p\colon X \to Y$ where X and Y are Kan complexes, all of the following are equivalent ((37.6), (37.9), (38.2), (39.1)):

(1) p is a trivial fibration;
(2) p is a fiberwise deformation retraction;
(3) p is a weak equivalence;
(4) p has contractible fibers;
(5) p is fully faithful and essentially surjective.

The equivalence of (3) and (5) will complete the proof of the fundamental theorem for Kan complexes (36.24). (It turns out that (1)–(4) are equivalent without the hypothesis that the objects are Kan complexes, though we will not prove this in all cases.)

37.1. **Fiberwise deformation retraction.** A map $p\colon X \to Y$ is said to be a **fiberwise deformation retraction** if there exists

- $s\colon Y \to X$ such that $ps = \mathrm{id}_Y$, and
- $k\colon X \times \Delta^1 \to X$ such that $k|X \times \{0\} = \mathrm{id}_X$, $k|X \times \{1\} = sp$, and $pk = p\pi_X$, where $\pi_X\colon X \times \Delta^1 \to X$ is projection; that is, the diagram

$$\begin{tikzcd} X \times \{0,1\} \arrow[rr, "{(\mathrm{id}_X, sp)}"] \arrow[d, hook] & & X \arrow[d, "p"] \\ X \times \Delta^1 \arrow[r, "\pi_X"'] \arrow[urr, "k"] & X \arrow[r, "p"'] & Y \end{tikzcd}$$

commutes.

Any fiberwise deformation retraction is a weak equivalence: s is a simplicial homotopy inverse to p by construction (36.6).

37.2. *Remark.* Here is one way to think about the identity $pk = p\pi_X$: it says that the map $p_* = \mathrm{Map}(X, p)\colon \mathrm{Map}(X, X) \to \mathrm{Map}(X, Y)$ sends the edge in $\mathrm{Map}(X, X)_1$ representing k to the degenerate edge associated to the vertex $p \in \mathrm{Map}(X, Y)_0$.

37.3. *Exercise.* Show that fiberwise deformation retraction can be reformulated in terms of the relative function complex of (20.9): a map $p\colon X \to Y$ is a fiberwise deformation retraction if there exists (i) a vertex $s \in \mathrm{Map}_{/Y}(Y, X)_0$ and (ii) an edge $k \in \mathrm{Map}_{/Y}(X, X)_1$ with associated vertices $k_0 = \mathrm{id}_X$ and $k_1 = sp$ in $\mathrm{Map}_{/Y}(X, X)_0$.

37.4. *Exercise.* Show that the term "fiberwise" is justified: for each $y \in Y_0$, the projection $p^{-1}(y) \to \{y\}$ of a fiber to its image is a simplicial homotopy equivalence.

37.5. *Exercise.* Show that if $p\colon X \to Y$ is a fiberwise deformation retraction as above, then any base change of p is also a fiberwise deformation retraction.

Fiberwise deformation retractions of Kan fibrations are always trivial fibrations, as can be shown with the covering homotopy extension property.

37.6. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between simplicial sets. Then p is a fiberwise deformation retraction if and only p is a trivial fibration.*

Proof. [JT08, Prop. 3.2.5]. ($\Longrightarrow$) Consider a lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

with i a monomorphism. Since p is a Kan fibration, the covering homotopy extension property (36.22) applies, so it suffices to deform the lifting problem to one we can solve. In fact, the data $(s\colon Y \to X,\ k\colon X \times \Delta^1 \to X)$ of a fiberwise deformation retraction provides us such a deformation, via the commutative rectangle:

$$\begin{tikzcd} A \times \Delta^1 \arrow[r, "a\times\mathrm{id}"] \arrow[d, tail, "i\times\mathrm{id}"] & X \times \Delta^1 \arrow[r, "k"] \arrow[d, "p\times\mathrm{id}"] & X \arrow[d, "p"] \\ B \times \Delta^1 \arrow[r, "b\times\mathrm{id}"'] & Y \times \Delta^1 \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

(Note that $\pi_Y(p \times \mathrm{id}) = p\pi_X = pk$.) Over $\{0\} \subseteq \Delta^1$ this is the original lifting problem (a, b), while over $\{1\} \subset \Delta^1$ we get a lifting problem (spa, b) since $sp = k|X \times \{1\}$ and $pspa = pa = bi$. We know a lift for (spa, b), namely $sb\colon B \to X$ (since $sbi = spa$ and $psb = b$).

($\Longleftarrow$) Left as an exercise (37.7). □

37.7. *Exercise.* Show that any trivial fibration is a fiberwise deformation retraction. (Hint: use the lifting property $\overline{\mathrm{Cell}} \boxslash p$ to produce s and k with the desired properties.)

37.8. **Trivial fibrations between Kan complexes and weak equivalences.** We know that trivial fibrations are always categorical equivalences. We now show that any Kan fibration between Kan complexes which is also a categorical equivalence (and hence a weak equivalence) is a trivial fibration.

37.9. **Proposition.** *A map $p\colon X \to Y$ between Kan complexes is a trivial fibration if and only if it is a Kan fibration and a weak equivalence.* **F 8 Mar 2019**

Proof. [JT08, Prop. 3.2.6] ($\Longrightarrow$) Clearly trivial fibrations are Kan fibrations since $\mathrm{Horn} \subseteq \overline{\mathrm{Cell}}$. We have already shown that trivial fibrations between quasicategories are always categorical equivalences (20.10), which implies that they are weak equivalences if between Kan complexes (36.2).

($\Longleftarrow$) On the other hand, suppose p is a Kan fibration and a weak equivalence. Being a weak equivalence between Kan complexes and hence a categorical equivalence, p admits a categorical inverse: there exists a functor $g\colon Y \to X$ for which there are natural isomorphisms $gp \approx \mathrm{id}_X$ and $pg \approx \mathrm{id}_Y$. We will "deform" g to a map $s\colon Y \to X$ equipped with natural isomorphisms $1_{\mathrm{id}_Y}\colon ps \to \mathrm{id}_Y$ and $k\colon sp \to \mathrm{id}_X$ which exhibit p as a fiberwise deformation retraction. Given this, we can conclude that p is a trivial fibration by (37.6).

Step 1. Choose $v\colon Y \times \Delta^1 \to Y$ representing a natural isomorphism $pg \to \mathrm{id}_Y$. Since $Y \times \{0\} \subset Y \times \Delta^1$ is anodyne by (36.9), a lift α exists in

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "g"] \arrow[d, tail] & X \arrow[d, "p"] \\ Y \times \Delta^1 \arrow[r, "v"'] \arrow[ur, dotted, "\alpha"] & Y \end{tikzcd}$$

Let $s := \alpha|Y \times \{1\}$, so $ps = \mathrm{id}_Y$. The map α exhibits a natural isomorphism $\alpha\colon g \to s$ of functors $Y \to X$. Since gp is naturally isomorphic to id_X, we have natural isomorphisms $\mathrm{id}_X \approx gp \approx sp$ of functors $X \to X$, i.e., there exists a natural isomorphism $w\colon sp \to \mathrm{id}_X$, represented by an edge $w \in \mathrm{Map}(X,X)_1$.

Step 2. We have functors $\mathrm{Fun}(X,X) \xrightarrow{p_*} \mathrm{Fun}(X,Y) \xrightarrow{s_*} \mathrm{Fun}(X,X)$ induced by postcomposition with p and s. We can apply various iterations of these functors to the natural isomorphism $w\colon sp \to \mathrm{id}_X$, some of which are pictured in the following solid arrow diagram of objects and morphisms in $\mathrm{Fun}(X,X)$ and $\mathrm{Fun}(X,Y)$:

$$\begin{tikzcd} & \mathrm{id}_X \arrow[dd, dotted, "k"] & & & & p \arrow[dd, "1_p"] & \\ sp \arrow[ur, "w"] \arrow[dr, "(sp)_*(w)"'] & & \in \mathrm{Fun}(X,X) \arrow[r, Rightarrow, "p_*"] & & p = psp \arrow[ur, "p_*(w)"] \arrow[dr, "p_*(w)=(psp)_*(w)"'] & & \in \mathrm{Fun}(X,Y) \\ & sp & & & & p & \end{tikzcd}$$

Note that $ps = \mathrm{id}_Y$ implies that $p_*(sp) = p$ and $(sp)_*(sp) = sp$, and that $(psp)_*(w) = p_*(w)$. The right hand diagram "commutes" in $\mathrm{Fun}(X,Y)$, i.e., it represents the boundary of an element $b \in \mathrm{Fun}(X,Y)_2$, namely the degenerate element $b = (p_*(w))_{011}$ associated to the edge $p_*(w)\colon psp = p \to p$.

The above picture is represented by a commutative square

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "a"] \arrow[d, tail] & \mathrm{Fun}(X,X) \arrow[d, "\mathrm{Map}(X,p)=p_*"] \\ \Delta^2 \arrow[r, "b"'] \arrow[ur, dotted, "t"] & \mathrm{Fun}(X,Y) \end{tikzcd}$$

in simplicial sets. Since p is a Kan fibration so is p_* by $\overline{\mathrm{Horn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (36.9), and therefore a lift t exists. Then $k := t|\Delta^{\{1,2\}}\colon \Delta^1 \to \mathrm{Fun}(X,X)$ is a natural isomorphism $k\colon \mathrm{id}_Y \to sp$ such that $p_*(k) = 1_p$, i.e., $pk = p\pi_X$. We have thus produced $s\colon Y \to X$ and $k\colon X \times \Delta^1 \to X$ exhibiting p as a fiberwise deformation retraction. $\square$

37.10. **Contractible Kan complexes.** The special case of (37.9) applied to $p\colon X \to *$ is already interesting.

37.11. **Corollary.** *Let X be a simplicial set. The following are equivalent.*

(1) *X is a quasicategory which is categorically equivalent to Δ^0.*
(2) *$X \to \Delta^0$ is a trivial fibration.*
(3) *Every $\partial\Delta^n \to X$ extends over $\partial\Delta^n \subset \Delta^n$.*

Such an X is necessarily a Kan complex.

Proof. We have (2) $\Leftrightarrow$ (3) by definition, and we know that (2) $\Rightarrow$ (1). Given (1), we have that X is a quasigroupoid, and hence a Kan complex (30.2), and (2) follows by the previous proposition (37.9). $\square$

We say that an X satisfying these conditions is a **contractible Kan complex**.

37.12. **Monomorphisms which are weak equivalences.** The identification (37.9) of the classes (Kan fibrations between Kan complexes which are weak equivalences) and (trivial fibrations between Kan complexes) has many important consequences. For instance, it directly implies the following characterization of the class (monomorphisms which are weak equivalences).

37.13. **Proposition.** *Let $j\colon A \to B$ be a monomorphism of simpicial sets. Then j is a weak equivalence if and only if $\mathrm{Map}(j, X)\colon \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a trivial fibration for all Kan complexes X.*

Proof. Assume X is an arbitrary Kan complex. We know that $\mathrm{Map}(j, X)$ is always a Kan fibration between Kan complexes using $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (36.9). So j is a weak equivalence iff all $\mathrm{Map}(j, X)$ are weak equivalences, iff all $\mathrm{Map}(j, X)$ are trivial fibrations by (37.9). $\square$

37.14. *Remark.* The class $\mathrm{WkEq} \cap \overline{\mathrm{Cell}}$ of monomorphisms which are weak equvialences is a weakly saturated class: (37.13) says it is the left complement of $\left\{ p^{\square\mathrm{Cell}} \;\middle|\; p\colon X \to *,\ X \in \mathrm{Kan} \right\}$. Clearly $\overline{\mathrm{Horn}} \subseteq \mathrm{WkEq} \cap \overline{\mathrm{Cell}}$ by (36.11).

It turns out that $\overline{\mathrm{Horn}} = \mathrm{WkEq} \cap \overline{\mathrm{Cell}}$, i.e., the injective weak equivalences are *precisely the same* as the anodyne maps. This is a fairly non-trivial fact, which we will not prove here. (See later discussion (**??**).) A consequence of this is that (37.9) and (37.17) still hold when we remove the condition that the objects be Kan complexes.

37.15. **More enriched lifting for Kan fibrations between Kan complexes.** Although we haven't proved that all monomorphisms which are weak equivalences are anodyne, we can show that they share some of the enriched lifting properties satisfied by anodyne maps as in (36.10).

37.16. **Proposition.** *If $j\colon A \to B$ is a monomorphism and a weak equivalence of simplicial sets, and $p\colon X \to Y$ is a Kan fibration between Kan complexes, then the pullback-hom map*

$$p^{\square j}\colon\ \mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

is a trivial fibration. In particular, $i \boxslash p$.

Proof. The pullback-hom map $p^{\square j}$ is a Kan fibration between Kan complexes, using $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ (36.9). Consider the diagram

$$\begin{tikzcd}
\mathrm{Fun}(B,X) \arrow[r, "p^{\square j}"] & \mathrm{Fun}(A,X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B,Y) \arrow[r, "q'"] \arrow[d] & \mathrm{Fun}(A,X) \arrow[d] \\
& \mathrm{Fun}(B,Y) \arrow[r, "\mathrm{Fun}(j,Y)"'] & \mathrm{Fun}(A,Y)
\end{tikzcd}$$

in which the square is a pullback. By (37.13) the maps $\mathrm{Fun}(j, Y)$ and $q'(p^{\square j}) = \mathrm{Fun}(j, X)$ are trivial fibrations. The pullback q' of $\mathrm{Fun}(j, Y)$ is also a trivial fibration, and so $p^{\square j}$ is a weak equivalence by 2-out-of-3 (36.3), and therefore a trivial fibration since it is a Kan fibration between Kan complexes (37.9). $\square$

We also obtain another characterization of Kan fibrations between Kan complexes.

37.17. **Corollary.** *A map $p\colon X \to Y$ between Kan complexes is a Kan fibration if and only if $j \boxslash p$ for all j which are monomorphisms and weak equivalences.*

Proof. ($\Longleftarrow$) Straightforward, since inner horn inclusions are monomorphisms and weak equivalences. ($\Longrightarrow$) Immediate from the previous proposition (37.16), since $p^{\square j} \in \mathrm{TrivFib}$ implies $j \boxslash p$. $\square$

38. The fiberwise criterion for trivial fibrations

We now give yet another criterion for Kan fibration to be a trivial fibration. This criterion is in terms of its fibers. The **fiber** of a map $p\colon X \to Y$ over a vertex $y \in Y_0$ is defined to be the pullback of p along $\{y\} \to Y$. We will write $p^{-1}(y) = \{y\} \times_Y X$ for the fiber of p over y.

38.1. **Fiberwise criterion for trivial fibrations.** If $p\colon X \to Y$ is a trivial fibration, then since $\mathrm{TrivFib} = \mathrm{Horn}^{\boxslash}$ we see immediately that every projection $p^{-1}(y) \to *$ from a fiber is a trivial fibration; i.e., the fibers of a trivial fibration are necessarily contractible Kan complexes. The "fiberwise criterion" asserts the converse for arbitrary Kan fibrations.

38.2. **Proposition.** *Let $p\colon X \to Y$ be a Kan fibration. Then p is a trivial fibration if and only if every fiber of p is a contractible Kan complex.*

Proof. We have just observed ($\Longrightarrow$) , so we prove ($\Longleftarrow$) . So suppose p is a Kan fibration whose fibers are contractible Kan complexes, and consider a lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

We will "deform" the lifting problem (a, b) to one of the same type which lives inside a single fiber of p. As such lifting problems have solutions by the hypothesis that the fibers of p are contractible Kan complexes, the covering homotopy extension property (36.22) implies that the original lifting problem has a solution.

Let $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ be the unique map which on vertices is given by $\gamma(k, 0) = k$ and $\gamma(k, 1) = n$, i.e., the unique natural transformation $\gamma\colon \mathrm{id}_{\Delta^n} \to \langle n \dots n\rangle$. Using that $(\partial\Delta^n \times \{0\} \subset \partial\Delta^n \times \Delta^1)$ is anodyne by $\overline{\mathrm{Horn}\square\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$ and that p is a Kan fibration, we obtain a lift u in

$$\begin{tikzcd} \partial\Delta^n \times \{0\} \arrow[rrr, "a"] \arrow[d, tail] & & & X \arrow[d, "p"] \\ \partial\Delta^n \times \Delta^1 \arrow[r, tail] \arrow[urrr, dotted, "u"] & \Delta^n \times \Delta^1 \arrow[r, "\gamma"'] & \Delta^n \arrow[r, "b"'] & Y \end{tikzcd}$$

The resulting commutative square

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "u"] \arrow[d, tail] & X \arrow[d, "p"] \\ \Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] & Y \end{tikzcd}$$

represents an edge $e \in \mathrm{Map}(\partial\Delta^n, X) \times_{\mathrm{Map}(\partial\Delta^n, Y)} \mathrm{Map}(\Delta^n, X)$ with $e_0 = (a, b)$ the original lifting problem, and $e_1 = (a', b')$ where $b' = b\gamma|\Delta^n \times \{1\}$ factors as $\Delta^n \to \{b(n)\} \rightarrowtail Y$. By the covering homotopy extension property it suffices to produce a lift in the rectangle

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, tail] & p^{-1}(b(n)) \arrow[r, tail] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] \arrow[urr, dotted] & \{b(n)\} \arrow[r, tail] & Y \end{tikzcd}$$

which amounts to producing a lift in the left-hand square, which exists because $p^{-1}(b(n))$ is a contractible Kan complex. ☐

We often apply the fiberwise criterion in the following way.

38.3. **Corollary.** *Suppose we have a pullback square of the form*

$$\begin{tikzcd} \coprod_{\alpha\in I} X'_\alpha \arrow[r] \arrow[d, "\coprod p'_\alpha"'] & X \arrow[d, "p"] \\ \coprod_{\alpha\in I} Y'_\alpha \arrow[r, "g"'] & Y \end{tikzcd}$$

such that (1) p is a Kan fibration and (2) the map g is surjective on vertices. Then p is a trivial fibration if and only if every $p'_\alpha\colon X'_\alpha \to Y'_\alpha$ is a trivial fibration. Furthermore, if all objects in the diagram are Kan complexes, then p is a weak equivalence if and only if every p'_α is a weak equivalence.

Proof. The fibers of p all appear as fibers of the p'_α by (2). Use the fiberwise criterion (38.2) for the first claim, and that together with (37.9) for the second. ☐

38.4. *Remark.* The proof of (38.2) actually shows something a little stronger: If $p\colon X \to Y$ is a Kan fibration, then for any *fixed* $n \geq 0$ we have that $(\partial\Delta^n \subset \Delta^n) \boxslash p$ if and only if $(\partial\Delta^n \subset \Delta^n) \boxslash (p^{-1}(y) \to \{y\})$ for all $y \in Y_0$.

As we have seen and will see, isofibrations between quasicategories have many properties analogous to Kan fibrations between Kan complexes. However, not every property has an analogue: there is no "fiberwise criterion" for an isofibration between quasicategories to be a trivial fibration.

38.5. *Exercise.* Give an example of an isofibration between quasicategories whose fibers are all categorically equivalent to Δ^0, but is not a categorical equivalence, and hence not a trivial fibration. (Hint: think small.)

39. Fundamental theorem for Kan complexes

In this section, we will prove quasigroupoid version of the fundamental theorem (36.24), i.e., that fully faithful and essentially surjective maps between quasigroupoids (=Kan complexes) are weak equivalences. We note that we have already reduced (36.25) to the case of Kan fibrations, which follows from the following, which implies that any Kan fibration between Kan complexes which is fully faithful and essentially surjective is a *trivial fibration*.

39.1. **Proposition.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes. Then*

(1) *p is essentially surjective if and only if $(\partial\Delta^0 \subset \Delta^0) \boxslash p$, and*
(2) *p is fully faithful if and only if $(\partial\Delta^n \subset \Delta^n) \boxslash p$ for all $n \geq 1$.*

Thus, p is both essentially surjective and fully faithful if and only if it is a trivial fibration.

Proof of (39.1) *part (1).* The property $(\partial\Delta^0 \subset \Delta^0) \boxslash p$ means exactly that $X_0 \to Y_0$ is surjective. If this holds then clearly p is essentially surjective.

Conversely, if p is essentially surjective and $y \in Y_0$, we may choose $x \in X_0$ and an isomorphism $f\colon p(x) \to y$ in Y. Then (x, f) is the data of a lifting problem of $(\{0\} \subset \Delta^1)$ against p, which has a solution $s\colon \Delta^1 \to X$ since p is a Kan fibration. The vertex $s_1 \in X_0$ satisfies $p(s_1) = y$, so we have proved that p is surjective on objects. $\square$

To prove the second part of (39.1), we first reformulate the condition for a Kan fibration between Kan complexes to be fully faithful.

39.2. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes. Then p is fully faithful if and only if $p^{\square(\partial\Delta^1\subset\Delta^1)}\colon \mathrm{Map}(\Delta^1, X) \to (X \times X) \times_{Y\times Y} \mathrm{Map}(\Delta^1, Y)$ is a trivial fibration.*

Proof. Fix $p\colon X \to Y$ a Kan fibration between Kan complexes. We can form the commutative diagram

$$\begin{tikzcd}
\coprod_{(x,y)\in X_0\times X_0} \mathrm{map}_X(x,y) \arrow[r] \arrow[d, "q_{x,y}"] & \mathrm{Map}(\Delta^1, X) \arrow[d, "q"] & \\
\coprod_{(x,y)\in X_0\times X_0} \mathrm{map}_Y(px,py) \arrow[r, "j"'] \arrow[d] & (X\times X)\times_{Y\times Y} \mathrm{Map}(\Delta^1, Y) \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1, Y) \arrow[d] \\
(\mathrm{Sk}_0 X) \times (\mathrm{Sk}_0 X) \arrow[r, "i"'] & X \times X \arrow[r, "p\times p"'] & Y \times Y
\end{tikzcd}$$

in which each square is a pullback. Observe that

- the map q is the pullback-hom $p^{\square(\partial\Delta^1\subset\Delta^1)}$, so
- q is a Kan fibration between Kan complexes (37.16), so
- $q_{x,y}$ is a Kan fibration between Kan complexes for all $(x, y) \in X_0 \times X_0$, and
- j is surjective on vertices since i is so.

Note that p is fully faithful if and only if every $q_{x,y}$ is a weak equivalence. Using the fiberwise criterion for trivial fibrations (38.3) we see that this is equivalent to $q = p^{\square(\partial\Delta^1\subset\Delta^1)} \in \mathrm{TrivFib}$, as desired. $\square$

39.3. **Transitivity triangle for pullback-homs.** We will need the following result which relates pullback-homs and composition of maps. You can think of it as an "enriched" version of the fact that $i \boxslash p$ and $j \boxslash p$ imply $ji \boxslash p$. **M 11 Mar 2019**

39.4. **Proposition** (Transitivity triangle for pullback-homs)**.** *Let $A \xrightarrow{i} B \xrightarrow{j} C$ and $p\colon X \to Y$ be maps. Then there is a factorization*

$$p^{\square(j\circ i)} = q \circ p^{\square j}$$

where q is a base-change of $p^{\square i}$.

Proof. I use "$[A, X]$" as a shorthand for "$\mathrm{Map}(A, X)$". We can form the commutative diagram

$$\begin{tikzcd}
{[C,X]} \arrow[r, "p^{\square j}"] \arrow[rr, bend left, "{[j,X]}"] \arrow[dr, "p^{\square ji}"'] \arrow[ddr, bend right, "{[C,p]}"'] & {[B,X] \times_{[B,Y]} [C,Y]} \arrow[r] \arrow[d, "q"] & {[B,X]} \arrow[d, "p^{\square i}"] \arrow[dr, bend left, "{[i,X]}"] & \\
& {[A,X] \times_{[A,Y]} [C,Y]} \arrow[r] \arrow[d] & {[A,X] \times_{[A,Y]} [B,Y]} \arrow[r] \arrow[d] & {[A,X]} \arrow[d, "{[A,p]}"] \\
& {[C,Y]} \arrow[r, "{[j,Y]}"'] & {[B,Y]} \arrow[r, "{[i,Y]}"'] & {[A,Y]}
\end{tikzcd}$$

in which all three squares are pullbacks, whence in particular q is a base-change of $p^{\square i}$. □

39.5. *Remark.* I used a special case of (39.4) in the proofs of (34.9) and (37.16).

39.6. *Exercise.* Prove the following transitivity-triangles:

(1) $(i \circ j)\square f = k \circ (i \square f)$ where k is a cobase-change of $j \square f$.
(2) $(q \circ p)^{\square i} = r \circ p^{\square i}$ where r is a base-change of $q^{\square i}$.

(Note: the large diagram in the proof of (39.2) is an example of a transitivity triangle of type (2).)

39.7. **Proof of the** (39.1)**, part (2).** Given $p\colon X \to Y$, define a class $\mathcal{C}_p$ of morphisms of simplicial sets by

$$\mathcal{C}_p := \left\{\, i \in \overline{\mathrm{Cell}} \;\middle|\; p^{\square i} \in \mathrm{TrivFib} \,\right\} = \overline{\mathrm{Cell}} \cap {}^{\boxslash}(p^{\square \mathrm{Cell}}).$$

The equality is because $p^{\square i} \in \mathrm{TrivFib}$ iff $\mathrm{Cell} \boxslash p^{\square i}$ iff $i \boxslash p^{\square \mathrm{Cell}}$. It is clear that $\mathcal{C}_p$ is a weakly saturated class, and if p is a Kan fibration then $\overline{\mathrm{Horn}} \subseteq \mathcal{C}_p$ since $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$.

39.8. *Remark.* Any $i \in \mathcal{C}_p$ automatically satisfies $i \boxslash p$, since $p^{\square i} \in \mathrm{TrivFib}$ implies that $p^{\square i}$ is surjective on vertices. That is, $\mathcal{C}_p \subseteq {}^{\boxslash}\{p\}$. Elements of $\mathcal{C}_p$ can be thought of as monomorphisms i which satisfy an "enriched" refinement of the lifting property $i \boxslash p$.

Thus, the strategy for proving the ($\Longrightarrow$) direction of (39.1)(2) is to show $\mathrm{Cell}_{\geq 1} \boxslash p$ by proving $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}_p$. Note that we have already proved (for fully faithful Kan fibrations between Kan complexes) that $(\partial\Delta^1 \subset \Delta^1) \in \mathcal{C}_p$ (39.2).

We have the following "precancellation" (or "right cancellation") property of $\mathcal{C}_p$, which is ultimately a consequence of (37.9).

39.9. **Lemma.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes. Suppose $A \xrightarrow{i} B \xrightarrow{j} C$ are monomorphisms such that $i, ji \in \mathcal{C}_p$. Then $j \in \mathcal{C}_p$.*

Proof. If $i\colon A \to B$ is any monomorphism, then $p^{\square i}\colon \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y)$ is a Kan fibration between Kan complexes, using (36.9). Thus $p^{\square i}$, $p^{\square j}$, and $p^{\square ji}$ are Kan fibrations between Kan complexes.

If $i, ji \in \mathcal{C}_p$ then $p^{\square i}$ and $p^{\square ji}$ are trivial fibrations. The transitivity triangle (39.4) gives $p^{\square ji} = q \circ p^{\square j}$, where q is a base change of $p^{\square i}$ whence q is a trivial fibration. Since trivial fibrations are weak equivalences we have that q and $p^{\square ji}$ are weak equivalences, whence $p^{\square j}$ is a weak equivalence by the 2-out-of-3 property (36.3). Therefore $p^{\square j}$ is also a trivial fibration since it is a map between Kan complexes which is a Kan fibration and weak equivalence (37.9). Thus we have proved that $j \in \mathcal{C}_p$. □

Proof of (39.1) *part (2).* ($\Longleftarrow$) The pushout-product of $(\partial\Delta^1 \subseteq \Delta^1)$ with any monomorphism gives a monomorphism which is bijective on vertices, and therefore $(\partial\Delta^1 \subseteq \Delta^1)\square\mathrm{Cell} \subseteq \overline{\mathrm{Cell}_{\geq 1}}$.

Therefore if $p\colon X \to Y$ is a Kan fibration between Kan complexes such that $\mathrm{Cell}_{\geq 1} \boxslash p$, then $\big((\partial\Delta^1 \subseteq \Delta^1)\Box\mathrm{Cell}\big) \boxslash p$, whence $\mathrm{Cell} \boxslash p^{\Box(\partial\Delta^1 \subseteq \Delta^1)}$. That is, $p^{\Box(\partial\Delta^1\subseteq\Delta^1)}$ is a trivial fibration, and therefore p is fully faithful by (39.2).

($\Longrightarrow$) Suppose $p\colon X \to Y$ is a Kan fibration between Kan complexes which is fully faithful. The reformulation of fully faithful (39.2) implies that $(\partial\Delta^1 \subseteq \Delta^1) \in \mathcal{C}_p$, where $\mathcal{C}_p = \left\{ i \in \overline{\mathrm{Cell}} \;\middle|\; p^{\Box i} \in \mathrm{TrivFib} \right\}$ is the weakly saturated class we defined above. To prove the proposition, it will suffice to show that $\mathrm{Cell}_{\geq 1} \subseteq \mathcal{C}_p$, as this certainly would imply $\mathrm{Cell}_{\geq 1} \boxslash p$.

We will argue by induction on $n \geq 1$ that $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$. The base case $n = 1$ is the hypothesis. Consider $n \geq 2$, and suppose we know that $(\partial\Delta^{n-1} \subset \Delta^{n-1}) \in \mathcal{C}_p$. We have a commutative diagram

$$\begin{tikzcd}
\partial\Delta^{n-1} \arrow[r] \arrow[d, tail, "i'"'] & \Lambda^n_1 \arrow[d, tail, "i"'] \arrow[dr, "ji"] & \\
\Delta^{n-1} \arrow[r, "\langle 023\ldots n\rangle"'] & \partial\Delta^n \arrow[r, tail, "j"'] & \Delta^n
\end{tikzcd}$$

in which the left-hand square is a pushout. By induction we have that $i' \in \mathcal{C}_p$, whence $i \in \mathcal{C}_p$ since $\mathcal{C}_p$ is weakly saturated. We observed that $\mathcal{C}_p$ contains all horn inclusions[29] since p is a Kan fibration and $\overline{\mathrm{Horn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$, and thus $ji \in \mathcal{C}_p$. Therefore $j \in \mathcal{C}_p$ as desired by "precancellation" (39.9). $\square$

39.10. **Homotopy groups.** I briefly note another criterion for weak equivalence of Kan complexes, which is essentially a form of Whitehead's theorem in homotopy theory.

Let (X, x) be a **pointed Kan complex**, i.e., a Kan complex X together with a choice of vertex $x \in X$. We define for all $n \geq 0$ a set $\pi_n(X, x)$ by the inductive formula

$$\begin{aligned}
&\pi_0(X,x) := \pi_0 X, \\
&\pi_n(X,x) := \pi_{n-1}(\mathrm{map}_X(x,x), 1_x), && \text{for } n \geq 1.
\end{aligned}$$

Note that $\pi_1(X, x) \approx \hom_{hX}(x, x)$ is equipped with a group structure via composition in hX, and thus $\pi_n(X, x)$ is a group for $n \geq 1$. (In fact, $\pi_n(X, x)$ is an abelian group for $n \geq 2$.)

39.11. *Example* (Homotopy groups of spaces). Given a topological space T, let $X = \mathrm{Sing}\, T$ be its singular complex, which is a Kan complex. Given a vertex $x \in X$ (which corresponds exactly to a point of T), we have bijections

$$\pi_n(X, x) \approx \pi_n(T, x),$$

where the right-hand side is the usual nth homotopy group of the space T at the point x. ...

39.12. **Proposition.** *Let $f\colon X \to Y$ be a functor between Kan complexes. Then f is a weak equivalence if and only if, for all $k \geq 0$ and all $x \in X_0$, the induced map*

$$\pi_k(X, x) \to \pi_k(Y, f(x))$$

is a bijection.

Maybe I'll give a proof in the appendix.

40. Properties of isofibrations

In this section, we return to isofibrations, which were defined in (29.6). The moral is that isofibrations between quasicategories play a role somewhat analogous to Kan fibrations between Kan complexes.

[29] In fact, it contains $\mathrm{WkEq} \cap \overline{\mathrm{Cell}}$ since p is a Kan fibration between Kan complexes (37.17).

40.1. **Characterizations of isofibrations.** Recall that a functor $f\colon C \to D$ between quasicategories is an isofibration if (1) it is an inner fibration, and (2) every diagram

$$\begin{tikzcd} \{j\} \arrow[r] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dotted, "g"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

with $j = 0$ admits a lift g. Furthermore, it is equivalent to require $(2')$ instead of (2), where $(2')$ is the same statement with $j = 1$.

Note that $C \to *$ is an isofibration for any quasicategory C (because identity maps are isomorphisms).

We can replace condition (2) for an isofibration with one involving the restriction of the map to cores (=maximal sub-quasigroupoids, defined in (10.8)).

40.2. **Proposition.** *A map $p\colon C \to D$ between quasicategories is an isofibration if and only if (1) it is an inner fibration, and $(2'')$ the restriction $p^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ of p to cores is a Kan fibration.*

Proof. ($\Longrightarrow$) Let p be an isofibration. Then p^{core} is seen to be an inner fibration by an elementary argument (13.11). It is also easy to see that condition (2) for an isofibration also holds for p^{core}. Thus p^{core} is an isofibration between Kan complexes, and hence a Kan fibration by a straightforward exercise using Joyal lifting (36.13).

($\Longleftarrow$) If p^{core} is a Kan fibration, then it is immediate that property (2) of an isofibration holds. $\square$

In particular, isofibrations between Kan complexes are precisely Kan fibrations. (This can be proved directly using Joyal lifting, as in (36.13).)

We have another "lifting criterion" for isofibrations involving the walking isomorphism.

40.3. **Proposition.** *An map p between quasicategories is an isofibration iff (1) it is an inner fibration, and $(2''')$ $(\{0\} \subset N(\mathrm{Iso})) \boxslash p$.*

Proof. ($\Longleftarrow$) Straightforward, using the fact (36.17) that every $f\colon \Delta^1 \to D$ representing an isomorphism factors through a map $N(\mathrm{Iso}) \to D$.

($\Longrightarrow$) Solve a lifting problem $(a\colon \{0\} \to C,\ b\colon N\mathrm{Iso} \to D)$ of type $(\{0\} \subset N\mathrm{Iso}) \boxslash p$ by solving two lifting problems in sequence

$$\begin{tikzcd} \{0\} \arrow[r, "a"] \arrow[d, tail] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ \Delta^1 \arrow[r, "b'u"'] \arrow[ur, dotted, "t"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd} \qquad \begin{tikzcd} \Delta^1 \arrow[r, "t"] \arrow[d, tail, "u"'] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ N\mathrm{Iso} \arrow[r, "b'"'] \arrow[ur, dotted, "s"] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

$b'\colon N\mathrm{Iso} \to D^{\mathrm{core}}$ is the factorization of b through the core, and $u\colon \Delta^1 \to N\mathrm{Iso}$ represents the morphism $0 \to 1$ in Iso. Since p is an isofibration, there exists a lift t in the left-hand square which represents an isomorphism in C. Both t and b land in the relevant cores, and so it suffices to produce a lift in the right-hand diagram, which exists because u is anodyne (36.16) and p^{core} is a Kan fibration by (40.2). $\square$

In other words, the isofibrations are precisely the maps *between quasicategories* which are contained in $\big(\mathrm{InnHorn} \cup \{\{0\} \subset N(\mathrm{Iso})\}\big)^{\boxslash}$.

40.4. *Remark.* We have deliberately excluded maps between non-quasicategories from the definition of isofibration. The correct generalization of isofibration to arbitrary simplicial sets is called "categorical fibration", and will be discussed later (44).

40.5. **Enriched lifting for isofibrations.** We are now ready to prove the following proposition, which will be the key tool in what follows. It is analogous to the first statement of (36.10), except that Kan fibrations between Kan complexes are replaced by isofibrations between quasicategories.

40.6. **Proposition.** *Let $p\colon C \to D$ be an isofibration between quasicategories, and $i\colon K \to L$ any monomorphism of simplicial sets. Then the induced pullback-hom map*

$$p^{\square i}\colon \operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C) \times_{\operatorname{Fun}(K,D)} \operatorname{Fun}(L, D)$$

is an isofibration.

We pause to note an important special, namely when $D = *$: for any monomorphism i and quasicategory C, the restriction map $\operatorname{Fun}(L, C) \to \operatorname{Fun}(K, C)$ is an isofibration.

Proof. Fix an isofibration $p\colon C \to D$ between quasicategories. First note that since p is an inner fibration, any map $p^{\square i}$ with i a monomorphism is also an inner fibration by $\overline{\mathrm{Horn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (17.2). For the same reason, the target of $p^{\square i}$ is a quasicategory when i is a monomorphism.

Let

$$\mathcal{C}_p := \{\, i \in \overline{\mathrm{Cell}} \;\mid\; p^{\square i} \in \mathrm{IsoFib}\,\},$$

the class of monomorphisms such that $p^{\square i}$ is an isofibration. First note that $\mathcal{C}_p$ is weakly saturated. To see this, let $S := \mathrm{InnHorn} \cup \{\{0\} \subset N\mathrm{Iso}\}$, so that $p^{\square i} \in \mathrm{IsoFib}$ if and only if $S \boxslash p^{\square i}$ by (40.3). We have that $S \boxslash p^{\square i}$ if and only if $i \boxslash p^{\square S}$ by an adjunction of lifting problems (16.5), and therefore $\mathcal{C}_p$ is equal to the left complement of the set $p^{\square S}$, and so is weakly saturated.

Therefore, to show that $\mathcal{C}_p$ contains all monomorphisms, it suffices show that it contains $(\partial\Delta^n \subset \Delta^n)$ for $n \geq 0$.

First note that when $n = 0$ we have that $p^{\square(\partial\Delta^0 \subset \Delta^0)} = p$, which is an isofibration by hypothesis[30].

Now consider $p^{\square(\partial\Delta^{n-1} \subset \Delta^n)}$ for $n \geq 1$. As this is an inner fibration between quasicategories, to show that it is an isofibration it suffices to solve the lifting problem

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, tail] & \operatorname{Fun}(\Delta^n, C) \arrow[d, "p^{\square i}"] \\ \Delta^1 \arrow[r, "f"'] \arrow[ur, dotted] & \operatorname{Fun}(\partial\Delta^n, C) \times_{\operatorname{Fun}(\partial\Delta^n, D)} \operatorname{Fun}(\Delta^n, D) \end{tikzcd} \iff \begin{tikzcd} (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n) \arrow[r, "g"] \arrow[d, tail] & C \arrow[d, "p"] \\ \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & D \end{tikzcd}$$

where f represents an isomorphism in the target.

The edge $f' := (g|\Delta^1 \times \{0\})$ is the same as the composite

$$\Delta^1 \xrightarrow{f} \operatorname{Fun}(\partial\Delta^n, C) \times_{\operatorname{Fun}(\partial\Delta^n, D)} \operatorname{Fun}(\Delta^n, C) \to \operatorname{Fun}(\partial\Delta^n, C) \to \operatorname{Fun}(\{0\}, C) = C.$$

Since f represents an isomorphism in the fiber product, it follows that f' represents an isomorphism in C. Therefore a lift exists in the right-hand diagram by the pushout-product version of Joyal lifting (31.6), since $n \geq 1$. $\square$

We are especially interested in the restriction of $p^{\square i}$ to cores.

40.7. **Corollary.** *Let $p\colon C \to D$ be an isofibration between quasicategories, and $i\colon K \to L$ any monomorphism of simplicial sets. Then the restriction of the pullback-hom map $p^{\square i}$ to cores, which has the form*

$$(p^{\square i})^{\mathrm{core}}\colon \operatorname{Fun}(L, C)^{\mathrm{core}} \to \operatorname{Fun}(K, C)^{\mathrm{core}} \times_{\operatorname{Fun}(K,D)^{\mathrm{core}}} \operatorname{Fun}(L, D)^{\mathrm{core}},$$

is a Kan fibration between Kan complexes.

[30]This step is the only place in the proof where we actually use the fact that p is an isofibration, and not merely an inner fibration! In fact, if p is merely an inner fibration, but $K_0 = L_0$, then $p^{\square i}$ is a isofibration. This proof is closely related to that of (31.3).

Proof. For the statement about the form of the target of the map $(p^{\square i})^{\mathrm{core}}$, note that because p is an inner fibration, we have that $\mathrm{Fun}(i, D)\colon \mathrm{Fun}(L, D) \to \mathrm{Fun}(K, D)$ is an inner fibration by enriched lifting for $\overline{\mathrm{InnHorn}}\square\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$, and thus the core of the pullback is the pullback of cores by (33.3).

The claim follows from (40.6) and the fact that an isofibration induces a Kan fibration on cores (40.2). $\square$

41. Trivial fibrations between quasicategories

Now we can prove a generalization of (37.9), which identified trivial fibrations between Kan complexes as the Kan fibrations which are weak equivalences. In the generalization, the role of Kan fibrations is replaced with isofibrations. The proof of the generalization will make essential use of the Kan fibration case. **W 13 Mar**

41.1. **Proposition.** *Let $p\colon C \to D$ be a map between quasicategories. Then p is a trivial fibration if and only if it is an isofibration and a categorical equivalence.*

Proof. [Joy08a, Theorem 5.15]. ($\Longrightarrow$) If p is a trivial fibration, it is an inner fibration and $(\{0\} \subset N(\mathrm{Iso})) \boxslash p$, so it is an isofibration (40.3). We have already shown that p is a categorical equivalence (20.10).

($\Longleftarrow$) Conversely, suppose p is isofibration and categorical equivalence, and that $i\colon K \to L$ is a monomorphism. To show $i \boxslash p$ we show that the pullback-hom map $p^{\square i}$ is surjective on vertices. In fact, it is enough to show that its restriction

$$(p^{\square i})^{\mathrm{core}}\colon \mathrm{Fun}(L, C)^{\mathrm{core}} \to \mathrm{Fun}(K, C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L, D)^{\mathrm{core}}$$

to cores is surjective on vertices, since the core of a quasicategory contains all its objects. We will actually show that $(p^{\square i})^{\mathrm{core}}$ is a trivial fibration for any monomorphism i.

By (40.7) the map $(p^{\square i})^{\mathrm{core}}$ is a Kan fibration between Kan complexes, since p is an isofibration and i is a monomorphism. The same reasoning applies to $\mathrm{Fun}(L, p)^{\mathrm{core}}$ and $\mathrm{Fun}(K, p)^{\mathrm{core}}$, the restriction of the maps

$$\mathrm{Fun}(L, p)\colon \mathrm{Fun}(L, C) \to \mathrm{Fun}(L, D) \qquad \text{and} \qquad \mathrm{Fun}(K, p)\colon \mathrm{Fun}(K, C) \to \mathrm{Fun}(K, D)$$

to cores. We know that $\mathrm{Fun}(L, p)$ and $\mathrm{Fun}(K, p)$ are categorical equivalences since p is (19.8), and therefore so are $\mathrm{Fun}(L, p)^{\mathrm{core}}$ and $\mathrm{Fun}(K, p)^{\mathrm{core}}$ (31.10). Therefore $\mathrm{Fun}(L, p)^{\mathrm{core}}$ and $\mathrm{Fun}(K, p)^{\mathrm{core}}$ are trivial fibrations, being Kan fibrations and weak equivalences between Kan complexes (37.9). We have a commutative diagram[31]

$$\begin{tikzcd}
\mathrm{Fun}(L,C)^{\mathrm{core}} \arrow[r, "(p^{\square i})^{\mathrm{core}}"] \arrow[dr, "\mathrm{Fun}(L,p)^{\mathrm{core}}"'] & \mathrm{Fun}(K,C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L,D)^{\mathrm{core}} \arrow[r] \arrow[d, "q"] & \mathrm{Fun}(K,C)^{\mathrm{core}} \arrow[d, "\mathrm{Fun}(K,p)^{\mathrm{core}}"] \\
& \mathrm{Fun}(L,D)^{\mathrm{core}} \arrow[r] & \mathrm{Fun}(K,D)^{\mathrm{core}}
\end{tikzcd}$$

in which the map q is a base-change of $\mathrm{Fun}(K, p)^{\mathrm{core}}$ and hence is a trivial fibration, as in $\mathrm{Fun}(L, p)^{\mathrm{core}}$. Now use that trivial fibrations are categorical equivalences (20.10) and 2-out-of-3 for categorical equivaleces (22.10) to conclude that $(p^{\square i})^{\mathrm{core}}$ is a categorical equivalence. As $(p^{\square i})^{\mathrm{core}}$ is also a Kan fibration between Kan complexes, we conclude that it is a trivial fibration by (37.9). $\square$

41.2. *Exercise.* Let C be a quasicategory and let $\pi\colon C \to hC$ be the tautological map to its homotopy category. Show that

(1) π is an isofibration, and
(2) $(\partial\Delta^n \subset \Delta^n) \boxslash \pi$ for $n = 0, 1, 2$.

[31] This is an example of a transitivity triangle of type (39.6)(2), or rather the restriction of such a triangle to cores.

Conclude that π is a categorical equivalence if and only if $(\partial\Delta^n \subset \Delta^n) \boxslash \pi$ for all $n \geq 3$.

41.3. **Invariance of slice categories under categorical equivalence.** That isofibrations which are categorical equivalences are trivial fibrations has a number of useful consequences. For instance, we can now show that a categorical equivalence between quasicategories induces equivalences of its slice categories.

41.4. **Proposition.** *Let $f\colon C \to D$ be a categorical equivalence of quasicategories. For any map $q\colon K \to C$ of simplicial sets, the induced maps $C_{q/} \to D_{fq/}$ and $C_{/q} \to D_{/fq}$ on slice categories are also categorical equivalences.*

Proof. I'll prove the slice-under case; the slice-over case is exactly the same. Consider the path factorization (35.2) of f, which gives a commutative diagram

$$\begin{tikzcd} C \arrow[r, "j"] \arrow[rr, bend left, "f"] \arrow[dr, equal] & P(f) \arrow[r, "p"'] \arrow[d, "s_0"] & D \\ & C & \end{tikzcd}$$

where j is a categorical equivalence, p is an isofibration, and s_0 a trivial fibration. The hypothesis that f is a categorical equivalence implies that p is a categorical equivalence by 2-out-of-3 (22.10), and therefore that p is a trivial fibration by (41.1).

Recall that if f is a trivial fibration, then so is the induced map $C_{q/} \to D_{fq/}$ by $\overline{\mathrm{Cell}} \circledast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ on slices (27.13). Taking slices in the above diagram gives

$$\begin{tikzcd} C_{q/} \arrow[r, "\widetilde{j}"] \arrow[rr, bend left, "\widetilde{f}"] \arrow[dr, equal] & P(f)_{jq/} \arrow[r, "\widetilde{p}"'] \arrow[d, "\widetilde{s}_0"] & D_{fq/} \\ & C_{q/} & \end{tikzcd}$$

in which both $\widetilde{p}$ and $\widetilde{s}_0$ are trivial fibrations and thus categorical equivalences (20.10). Applying the 2-out-of-3 property shows that $\widetilde{f}$ is a categorical equivalence as desired.

□

41.5. **Invariance of colimits and limits under categorical equivalence.** We can now prove (E). First we prove the following criterion for a cone to be a limit or colimit, which relaxes the condition "trivial fibration" of (28.7) to "categorical equivalence".

41.6. **Proposition.** *Let C be a quasicategory. A map $\widehat{p}\colon K^{\triangleright} \to C$ is a colimit cone in C if and only if the restriction map $C_{\widehat{p}/} \to C_{p/}$ is a categorical equivalence. Likewise, a map $\widehat{p}\colon K^{\triangleleft} \to C$ is a limit cone in C if and only if the restriction map $C_{/\widehat{p}} \to C_{/p}$ is a categorical equivalence.*

Proof. We prove the case of colimits. Consider $\widehat{p}\colon K^{\triangleright} \to C$, and write $p := \widehat{p}|K$. Let $\pi\colon C_{\widehat{p}/} \to C_{p/}$ be the evident restriction map on slices. We know that π is a left fibration between quasicategories (27.15) because $\overline{\mathrm{Cell}} \circledast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}}$, and thus is an isofibration (29.10).

We also know that $\widehat{p}$ is a colimit cone if and only if π is a trivial fibration (28.7). So the claim is immediate from (41.1). □

41.7. **Proposition** (Deferred Proposition (E))**.** *Let $f\colon C \to D$ be a categorical equivalence between quasicategories. A map $\widehat{p}\colon K^{\triangleright} \to C$ is a colimit cone in C if and only if $f\widehat{p}$ is a colimit cone in D, and a map $\widehat{q}\colon K^{\triangleleft} \to C$ is a limit cone in C if and only if $f\widehat{q}$ is a colimit cone in D,* *Proof.* We prove the case of colimits. Consider the commutative diagram

$$\begin{tikzcd} C_{\widehat{p}/} \arrow[r, "f''"] \arrow[d, "\pi"'] & D_{f\widehat{p}/} \arrow[d, "\pi'"] \\ C_{p/} \arrow[r, "f'"'] & D_{fp/} \end{tikzcd}$$

Since f is a categorical equivalence, f' and f'' are also categorical equivalences by (41.4). Therefore by 2-out-of-3 for categorical equivalences (22.10) π is a categorical equivalence if and only if π' is, and the claim follows from (41.6).

□

We can also show that the existence of colimit and limit cones is reflected by categorical equivalences.

41.8. **Proposition.** *Let $f\colon C \to D$ be a categorical equivalence between quasicategories. A map $u\colon K \to C$ admits a colimit cone in C if and only if fu admits a colimit cone in D, and f admits a limit cone in C if and only if fu admits a colimit cone in D.*

Proof. We prove the case of colimits. Let $\widehat{w}\colon K^{\triangleright} \to D$ be a colimit cone of $w = fu$. Use a path factorization to construct a commutative diagram of solid arrows

$$\begin{tikzcd} K \arrow[rr, tail] \arrow[d, "u"'] & & K^{\triangleright} \arrow[d, "\widehat{w}"] \arrow[dl, dotted, "\widehat{v}"'] \\ C \arrow[r, "i"] \arrow[dr, equal] & P(f) \arrow[r, "p"] \arrow[d, "s_0"] & D \\ & C & \end{tikzcd}$$

in which p and s_0 are trivial fibrations and i a categorical equivalence. Since p is a trivial fibration a lift $\widehat{v}$ exists, which by (41.7) applied to p must be a colimit cone of $v = iu$. Therefore again by (41.7) $\widehat{u} = s_0\widehat{v}$ must be a colimit cone of $s_0v = s_0iu = u$, as desired. □

41.9. **Monomorphisms which are categorical equivalences.** We can now prove a generalization of (37.13), which characterized the injective weak equivalences.

41.10. **Proposition.** *Let $j\colon K \to L$ be a monomorphism of simplicial sets. Then j is a categorical equivalence if and only if* $\mathrm{Map}(j, C)\colon \mathrm{Map}(L, C) \to \mathrm{Map}(K, C)$ *is a trivial fibration for all quasicategories C.*

Proof. Straightforward using the fact that $\mathrm{Map}(j, C)$ is an isofibration for any inclusion (40.6), and that isofibrations which are categorical equivalences are trivial fibrations (41.1). □

41.11. *Remark.* The class $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ of monomorphisms which are categorical equivalences is a weakly saturated class: (41.10) says it is the left complement of $\{\, p^{\square \mathrm{Cell}} \mid p\colon C \to *,\ C \in \mathrm{qCat} \,\}$. Clearly $\overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq} \cap \overline{\mathrm{Cell}}$ by (20.14).

However, $\overline{\mathrm{InnHorn}} \neq \mathrm{CatEq} \cap \overline{\mathrm{Cell}}$. For instance, every inner anodyne map is a bijection on vertices, but $\{0\} \to N\mathrm{Iso}$ which is not bijective on vertices is an injective categorical equivalence. This is a significant way in which the theory of quasicategories is not entirely parallel with the theory of Kan complexes; compare (37.14). More on this later (**??**).

41.12. **Monomorphisms which are categorical equivalences lift against isofibrations.** Now we can identify those elements of the right complement of $\overline{\text{Cell}} \cap \text{CatEq}$ which are maps between quasicategories.

41.13. **Proposition.** *A map $p\colon C \to D$ with D a quasicategory is an isofibration if and only if $j \boxslash p$ for every $j\colon K \to L$ which is both a monomorphism and a categorical equivalence.*

Proof. ($\Longleftarrow$) Immediate from the the characterization of isofibrations as maps between quasicategories in the right complement of $\text{InnHorn} \cup \{\{0\} \subset N\text{Iso}\}$ (40.3).

($\Longrightarrow$) Suppose p is an isofibration. We have a commutative diagram

$$\begin{tikzcd}
\text{Fun}(L,C) \arrow[r, "p^{\square j}"'] \arrow[rr, bend left, "\text{Fun}(j,C)"] \arrow[dr, bend right] & \text{Fun}(K,C) \times_{\text{Fun}(K,D)} \text{Fun}(L,D) \arrow[r, "q"'] \arrow[d] & \text{Fun}(K,C) \arrow[d] \\
& \text{Fun}(L,D) \arrow[r, "\text{Fun}(j,D)"'] & \text{Fun}(K,D)
\end{tikzcd}$$

in which $p^{\square i}$, $\text{Fun}(j,C)$, and $\text{Fun}(j,D)$ are isofibrations by (40.6), and $\text{Fun}(j,C)$ and $\text{Fun}(j,D)$ are categorical equivalences since i and j are. Therefore $\text{Fun}(j,C)$ and $\text{Fun}(j,D)$, and hence the base-change q, are trivial fibrations by (41.1), whence $p^{\square j}$ is a categorical equivalence by 2-out-of-3 (22.10) and so a trivial fibration by (41.1). It follows that $p^{\square j}$ is surjective on vertices, i.e., $j \boxslash p$ as desired. $\square$

42. Localization of quasicategories

42.1. **Functors into the core of a quasicategory.** Let C be a quasicategory, and X a simplicial set. Let

$$\text{Fun}^{(X)}(X,C) \subseteq \text{Fun}(X,C)$$

denote the full subcategory spanned by objects which are functors $f\colon X \to C$ with the property that $f(X) \subseteq C^{\text{core}}$.

42.2. *Example.* When $X = \Delta^1$, then $\text{Fun}^{(\Delta^1)}(\Delta^1, C) = \widehat{C} \subseteq \text{Fun}(\Delta^1, C)$, the path category introduced in (34.4).

Note that $\text{Fun}^{(X)}(X,C)$ is a quasicategory, but not necessarily a quasigroupoid: for instance, morphisms in $\text{Fun}^{(X)}(X,C)$ correspond to $f\colon X \times \Delta^1 \to C$ such that $f(X \times \{0\}) \subseteq C^{\text{core}}$ and $f(X \times \{1\}) \subseteq C^{\text{core}}$, but need not satisfy $f(X \times \Delta^1) \subseteq C^{\text{core}}$ (i.e., they are not required to be natural *isomorphisms* of functors). So $\text{Fun}(X, C^{\text{core}}) \subseteq \text{Fun}^{(X)}(X,C)$, but they are not generally equal (unless C is already a quasigroupoid).

We have a convenient characterization of maps into $\text{Fun}^{(X)}(X,C)$.

42.3. **Proposition.** *For any quasicategory C and simplicial sets X, the evident bijection* $\text{Hom}(S, \text{Fun}(X,C)) \approx \text{Hom}(S, \text{Fun}(X,C))$ *restricts to a bijection*

$$\left\{ S \dashrightarrow \text{Fun}^{(X)}(X,C) \right\} \longleftrightarrow \left\{ X \dashrightarrow \text{Fun}(S,C)^{\text{core}} \right\}.$$

Proof. Consider $f\colon S \to \text{Fun}(X,C)$, and write $f'\colon X \to \text{Fun}(S,C)$ and $f''\colon S \times X \to C$ for its adjoints. Observe the following.

(1) The map f factors through $\text{Fun}^{(X)}(X,C) \subseteq \text{Fun}(X,C)$ if and only if for each vertex $s \in S_0$ the induced map $f(s)\colon X \to C$ factors through $C^{\text{core}} \subseteq C$. This amounts to saying that for each edge $g \in X_1$, each map $f(s)$ sends g to an isomorphism in C.

(2) The map f' factors through $\mathrm{Fun}(S,C)^{\mathrm{core}} \subseteq \mathrm{Fun}(S,C)$ if and only if for each edge $g \in X_1$ the the image $f'(g) \in \mathrm{Fun}(S,C)_1$ represents an isomorphism in $\mathrm{Fun}(S,C)$. By the objectwise criterion (31.1), this amounts to saying that $f'(g)$ sends each vertex $s \in S_0$ to an isomorphism in C.

It is thus apparent that conditions (1) and (2) are equivalent: both are amount to the requirement that $\Delta^0 \times \Delta^1 \xrightarrow{s\times g} S \times X \xrightarrow{f''} C$ represent an isomorphism in C for every $s \in S_0$ and $g \in X_1$. □

For any map $i\colon X \to Y$ of simplicial sets, the induced map $\mathrm{Fun}(i,C)$ restricts to a map $\mathrm{Fun}^{(Y)}(Y,C) \to \mathrm{Fun}^{(X)}(X,C)$ between full subcategories.

42.4. **Proposition.** *Let $i\colon X \to Y$ be any map of simplical sets which is a monomorphism and a weak equivalence (e.g., an anodyne map). Then for any quasicategory C, the restriction map*

$$i^*\colon \mathrm{Fun}^{(Y)}(Y,C) \to \mathrm{Fun}^{(X)}(X,C)$$

is a trivial fibration, and thus in particular a categorical equivalence between quasicategories.

Proof. We need to solve lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "u"] \arrow[d, tail] & \mathrm{Fun}^{(Y)}(Y,C) \arrow[d, "i^*"] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "s"] & \mathrm{Fun}^{(X)}(X,C) \end{tikzcd}$$

for all $n \geq 0$. Using (42.3) we can replace this with the adjoint lifting problem

$$\begin{tikzcd} X \arrow[r, "\widetilde{v}"] \arrow[d, tail, "i"'] & \mathrm{Fun}(\Delta^n,C)^{\mathrm{core}} \arrow[d, "p^{\mathrm{core}}"] \\ Y \arrow[r, "\widetilde{u}"'] \arrow[ur, dotted, "\widetilde{s}"] & \mathrm{Fun}(\partial\Delta^n,C)^{\mathrm{core}} \end{tikzcd}$$

where p^{core} is induced by the restriction map $p\colon \mathrm{Fun}(\Delta^n,C) \to \mathrm{Fun}(\partial\Delta^n,C)$. By (40.6) the map p is an isofibration, and thus p^{core} is a Kan fibration (40.2). The hypothesis that $i \in \overline{\mathrm{Cell}} \cap \mathrm{WkEq}$ implies that a lift must exist, by (37.16). □

42.5. **Groupoid completion.** Given any simplicial set X, there exists a quasigroupoid X_{Kan} together with a map $X \to X_{\mathrm{Kan}}$ which is a monomorphism and a weak equivalence, e.g., an anodyne map construted using the small object argument applied to the set Horn (13.17). We will call any such choice a **groupoid completion** of X. This terminology is justified by the following.

42.6. **Proposition.** *Let $i\colon X \to X_{\mathrm{Kan}}$ be any groupoid completion as above. Then for any quasicategory C, restriction along i induces to a trivial fibration*

$$p\colon \mathrm{Fun}(X_{\mathrm{Kan}},C) \to \mathrm{Fun}^{(X)}(X,C).$$

In particular, any map $f\colon X \to C^{\mathrm{core}} \subseteq C$ extends to a functor $g\colon X_{\mathrm{Kan}} \to C$, and any two such extensions are naturally isomorphic.

Proof. To show that p is a trivial fibration, apply (42.4), together with the easy observation that $\mathrm{Fun}^{(X)}(X,C) = \mathrm{Fun}(X,C)$ when X is a Kan complex. □

Although the groupoid completion isn't unique, it is unique up to categorical equivalence.

42.7. *Exercise.* Let $f_i\colon X \to X_i$ be groupoid completions of X, for $i = 1,2$. Show that there exists a categorical equivalence $g\colon X_1 \to X_2$ such that $gf_1 = f_2$, and that any two such are naturally isomorphic. (Hint: proof of (20.16) and (41.10).)

We can apply this construction when X is a quasicategory, or even when X is the nerve of an ordinary category, and obtain interesting new quasigroupoids.

42.8. *Example.* It turns out that every simplicial set is weakly equivalent to the nerve of some ordinary category, and in fact to the nerve of some poset [Tho80]. Thus, for every Kan complex K, there exists an ordinary category A and a weak equivalence $NA \to K$, which therefore induces categorical equivalences $\mathrm{Fun}(K, C) \approx \mathrm{Fun}^{(NA)}(NA, C)$ for every quasicategory C.

We note that there is also a classical groupoid completion construction, which given an ordinary category A produces an ordinary groupoid A_{Gpd} by "formally inverting all maps". We have that $h((NA)_{\mathrm{Kan}}) \approx N(A_{\mathrm{Gpd}})$, but in general $(NA)_{\mathrm{Kan}}$ is not weakly equivalent to $N(A_{\mathrm{Gpd}})$.

42.9. *Exercise.* Let A be the poset of *proper and non-empty* subsets of $\{0, 1, 2, 3\}$. Show that A_{Gpd} is equivalent to the one-object category, but that $(NA)_{\mathrm{Kan}}$ is not equivalent to the one-object category. (In the second case, you can prove non-equivalence by showing $\pi_0 \mathrm{Fun}(NA, K(\mathbb{Z}, 2)) \approx \mathbb{Z}$, using the Eilenberg-MacLane object of (8.7).)

42.10. **Localization of quasicategories.** There is a more general construction, which applies to a simplicial set X equipped with a subcomplex $W \subseteq X$. Let

$$\mathrm{Fun}^{(W)}(X, C) \subseteq \mathrm{Fun}(X, C)$$

denote the full subcategory spanned by objects $f\colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$. (Note that this condition only depends on knowing the edges in W.) Clearly $\mathrm{Fun}^{(W)}(X, C)$ is the primage of $\mathrm{Fun}^{(W)}(W, C)$ along the restriction map $\mathrm{Fun}(X, C) \to \mathrm{Fun}(W, C)$.

Given a subcomplex $W \subseteq X$, we may define a **localization** of C with respect to W. This is any map $X \to X_{(W)}$ constructed as follows.

(1) Choose a groupoid completion $i\colon W \to W_{\mathrm{Kan}}$ of W.
(2) Choose an inner anodyne map $j\colon X \cup_W W_{\mathrm{Kan}} \to X_{(W)}$ to a quasicategory $X_{(W)}$.

If $W = X$ then $X \to X_{(X)}$ is an example of a groupoid completion of X as discussed above.

42.11. **Proposition.** *For any localization $X \to X_{(W)}$ as defined above, and any quasicategory C, the restriction map $\mathrm{Fun}(X_{(W)}, C) \to \mathrm{Fun}(X, C)$ induces a trivial fibration*

$$\mathrm{Fun}(X_{(W)}, C) \to \mathrm{Fun}^{(W)}(X, C).$$

In particular, any map $f\colon X \to C$ such that $f(W) \subseteq C^{\mathrm{core}}$ extends to a functor $g\colon X_{(W)}$, and any two such extensions are naturally isomorphic.

Proof. Consider

$$\begin{tikzcd}
\mathrm{Fun}(X_{(W)}, C) \arrow[r, "j^*"] & \mathrm{Fun}(X \cup_W W_{\mathrm{Kan}}, C) \arrow[r, "p"] \arrow[d] & \mathrm{Fun}^{(W)}(X, C) \arrow[r, tail] \arrow[d] & \mathrm{Fun}(X, C) \arrow[d] \\
& \mathrm{Fun}(W_{\mathrm{Kan}}, C) \arrow[r, "i^*"'] & \mathrm{Fun}^{(W)}(W, C) \arrow[r, tail] & \mathrm{Fun}(W, C)
\end{tikzcd}$$

in which both squares are pullbacks. The map j^* is a trivial fibration since g is inner anodyne, using $\overline{\mathrm{InnHorn}} \square \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$, while i^* is a trivial fibration as we have shown (42.6), whence p is a trivial fibration. $\square$

42.12. **Quasicategories from relative categories.** A **relative category** is a pair $W \subseteq C$ consisting of an *ordinary category* C and a subcategory W containing all the objects of C. The above construction gives, for any relative category, a map

$$C \to C_{(W)},$$

unique up to categorical equivalence. We may call $C_{(W)}$ the **localization** of C with respect to W.

It turns out that many quasicategories of interest arise as such localizations. **All quasicategories, up to categorical equivalence, right? Give references.**

43. Proof of the fundamental theorem

We are ready to finish the proof of (G), The Fundamental Theorem of Quasicategories. **F 15 Mar 2019**

43.1. **Proposition** (Deferred Proposition (G)). *If $f\colon C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then f is a categorical equivalence.*

We will deduce this from the following.

43.2. **Proposition.** *If $p\colon C \to D$ is an isofibration which is fully fathiful and essentially surjective, then $(p^{\square i})^{\mathrm{core}}\colon \mathrm{Fun}(L, C)^{\mathrm{core}} \to \mathrm{Fun}(K, C)^{\mathrm{core}} \times_{\mathrm{Fun}(K,D)^{\mathrm{core}}} \mathrm{Fun}(L, D)^{\mathrm{core}}$ is a trivial fibration for every monomorphism $i\colon K \to L$.*

Proof that (43.2) *implies* (43.1). To prove the fundamental theorem it suffices to prove that any isofibration $p\colon C \to D$ which is fully faithful and essentially surjective is a trivial fibration (35.7). In that case, (43.2) implies that for any monomorphism i the map $(p^{\square i})^{\mathrm{core}}$ is surjective on vertices, and thus $\mathrm{Cell} \boxslash p$, i.e., p is a trivial fibration. $\square$

43.3. **Recognizing isofibrations which induce trivial fibration on cores.** We start with the following proposition, which characterizes the isofibrations which induce trivial fibrations on cores in terms of a lifting property.

43.4. **Proposition.** *There exists a set of maps S such that for any isofibration $p\colon C \to D$ between quasicategories, we have $S \boxslash p$ iff $p^{\mathrm{core}} \in \mathrm{TrivFib}$.*

Proof. For each cell inclusion $i_n\colon \partial\Delta^n \to \Delta^n$, form $i'_n\colon (\partial\Delta^n)_{\mathrm{Kan}} \to (\Delta^n)_{\mathrm{Kan}}$ as in (43.5) below. Then we can take $S = \{\, i'_n \mid n \geq 0 \,\}$. $\square$

43.5. **Lemma.** *Let $i\colon K \to L$ be a monomorphism of simplical sets. Then there exists a monomorphism i' such that, for any isofibration $p\colon C \to D$, we have that $i \boxslash p^{\mathrm{core}}$ if and only if $i' \boxslash p$.*

Proof. First, note that since p is an isofibration then p^{core} is a Kan fibration (40.2).

Given a monomorphism i, use the small object argument (13.17) to construct a diagram

$$\begin{tikzcd} K \arrow[r, tail, "u"] \arrow[d, "i"'] & K_{\mathrm{Kan}} \arrow[d, tail, "j"'] \arrow[dr, tail, "i'"] & \\ L \arrow[r, tail, "v"'] & P \arrow[r, tail, "k"'] & L_{\mathrm{Kan}} \end{tikzcd}$$

in which the square is pushout, the maps u and k (and hence v) are anodyne, and the objects K_{Kan} and L_{Kan} are Kan complexes. (So $K \to K_{\mathrm{Kan}}$ and $L \to L_{\mathrm{Kan}}$ are examples of groupoid completions as in (42.5).) Then i' is also a monomorphism, and we show that $i \boxslash p^{\mathrm{core}}$ if and only if $i' \boxslash p$.

($\Longrightarrow$) Suppose $i \boxslash p^{\mathrm{core}}$. Since K_{Kan} and L_{Kan} are Kan complexes, any maps from them to quasicategories must factor through cores. Thus, any lifting problem of type $i' \boxslash p$ factors as

$$\begin{tikzcd} K_{\mathrm{Kan}} \arrow[r] \arrow[d, tail, "i'"'] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d, "p^{\mathrm{core}}"] & C \arrow[d, "p"] \\ L_{\mathrm{Kan}} \arrow[r] \arrow[ur, dotted] & D^{\mathrm{core}} \arrow[r, tail] & D \end{tikzcd}$$

so to prove $i' \boxslash p$ it suffices to show $i' \boxslash p^{\mathrm{core}}$. This is clear, since $j \boxslash p^{\mathrm{core}}$ since j is a cobase-change of i, and $k \boxslash p^{\mathrm{core}}$ since k is anodyne and p^{core} is a Kan fibration.

($\Longleftarrow$) Suppose $i' \boxslash p$. Consider a lifting problem

$$\begin{tikzcd}
K \arrow[r, "a"] \arrow[d, tail, "i"'] & C^{\mathrm{core}} \arrow[d, "p^{\mathrm{core}}"] \\
L \arrow[r, "b"'] \arrow[ur, dotted] & D^{\mathrm{core}}
\end{tikzcd}$$

We extend this square to a diagram

$$\begin{tikzcd}
K \arrow[r, tail, "u"] \arrow[d, tail, "i"'] \arrow[rrr, bend left, "a"] & K_{\mathrm{Kan}} \arrow[r, equal] \arrow[d, tail, "j"] & K_{\mathrm{Kan}} \arrow[r, tail, "a'"] \arrow[d, tail, "i'"] & C^{\mathrm{core}} \arrow[r, tail] \arrow[d] & C \arrow[d, "p"] \\
L \arrow[r, tail, "v"'] \arrow[rrr, bend right, "b"'] & P \arrow[r, tail, "k"'] & L_{\mathrm{Kan}} \arrow[r, "b''"'] \arrow[ur, dotted, "t'"] \arrow[urr, dotted, "t"] & D^{\mathrm{core}} \arrow[r, tail] & D
\end{tikzcd}$$

as follows. Because $C^{\mathrm{core}} \subseteq C$ is a Kan complex and $u\colon K \to K_{\mathrm{Kan}}$ is a groupoid completion, the map a factors through some $a'\colon K_{\mathrm{Kan}} \to C^{\mathrm{core}}$ (**??**), and there is a unique map $b'\colon P \to D^{\mathrm{core}}$ from the pushout such that $b'v = b$. Similarly, b' factors through some $b''\colon L_{\mathrm{Kan}} \to D^{\mathrm{core}}$. Now note that a lift t exists by hypothesis, and since L_{Kan} is a Kan complex it factors through a lift t'. The composite $s := t'kv\colon L \to C^{\mathrm{core}}$ is the desired lift. □

43.6. **Restricting fully faithful and essentially surjective functors to cores.** Any categorical equivalence between quasicategories restricts to an equivalence between their cores (31.10). There is a similar statement about functors between categories which are fully faithful and essentially surjective.

43.7. **Lemma.** *If $f\colon C \to D$ is a fully faithful and essentially surjective functor between quasicategories, then $f^{\mathrm{core}}\colon C^{\mathrm{core}} \to D^{\mathrm{core}}$ is a categorical equivalence (in fact, a weak equivalence).*

Proof. In view of the fundamental theorem for quasigroupoids (36.24), it suffices to show that f^{core} is fully faithful and essentially surjective. To show essential surjectivity is entirely straightforward, as C^{core} has all the objects and isomorphisms of C.

Note that f induces an equivalence of homotopy categories $hC \to hD$, and therefore is conservative.

Given $x, x' \in C_0$, the functor f induces functors

$$f'\colon \mathrm{map}_C(x,x') \to \mathrm{map}_D(fx,fx') \qquad \text{and} \qquad (f^{\mathrm{core}})'\colon \mathrm{map}_{C^{\mathrm{core}}}(x,x') \to \mathrm{map}_{D^{\mathrm{core}}}(fx,fx').$$

By hypothesis f' admits a categorical inverse g. We will show that g restricts to a functor $g'\colon \mathrm{map}_{D^{\mathrm{core}}}(fx,fx') \to \mathrm{map}_{C^{\mathrm{core}}}(x,x')$ which is a categorical inverse to $(f^{\mathrm{core}})'$.

First note that $\mathrm{map}_{C^{\mathrm{core}}}(x,x') \subseteq \mathrm{map}_C(x,x')$ and $\mathrm{map}_{D^{\mathrm{core}}}(fx,fx') \subseteq \mathrm{map}_D(fx,fx')$ are full subcategories, by the lemma below (43.8).

Thus, to show that g restricts to a g', it suffices to check what it does on objects. Suppose $v \in \mathrm{map}_D(fx,fx')_0$ represents an isomorphism in D. Since we have a natural isomorphism $fg \approx \mathrm{id}_{\mathrm{map}_D(fx,fx')}$ of functors, we have that $fg(v)$ and v are isomorphic as objects of $\mathrm{map}_D(fx,fx')$, i.e., they represent the same element in $\hom_{hD}(fx,fx')$. Therefore $fg(v)\colon fx \to fx'$ is also an isomorphism in D. It follows that $g(v)\colon x \to x'$ is an isomorphism in C since $f\colon C \to D$ is conservative.

Thus we have a g', and it is straightforward to check that it is a categorical inverse to $(f^{\mathrm{core}})'$, as these are restrictions to full subcategories of g and f' respectively. □

43.8. **Lemma.** *For C a quasicategory and $x, y \in C_0$, the subcomplex $\mathrm{map}_{C^{\mathrm{core}}}(x,y) \subseteq \mathrm{map}_C(x,y)$ is a full subcategory.*

Proof. Let $a\colon \Delta^n \to \mathrm{map}_C(x,y)$ be such that a sends each vertex to an isomorphism in C. This a adjoint to a map $\widetilde{a}\colon \Delta^n\times\Delta^1 \to C$ which sends each $\{i\}\times\Delta^1$ and each $\Delta^{i,j}\times\{k\}$ to isomorphisms in C, from which it is easy to see that $\widetilde{a}$ sends every edge to an isomorphism in C, i.e., the image of $\widetilde{a}$ is in C^{core} and thus a factors through $\mathrm{map}_{C^{\mathrm{core}}}(x,x')$. ☐

43.9. **Proof of** (43.2)**.** Fix an isofibration $p\colon C\to D$ between quasicategories which is fully faithful and essentially surjective. Consider the class

$$\mathcal{C}_p := \left\{\, i\in\overline{\mathrm{Cell}} \;\middle|\; (p^{\square i})^{\mathrm{core}} \in \mathrm{TrivFib}\,\right\}.$$

The statement of (43.2) amounts to showing that $\mathcal{C}_p$ contains every monomorphism.

43.10. **Lemma.** *The class $\mathcal{C}_p$ is weakly saturated.*

Proof. First note that for any monomorphism i, the map $p^{\square i}$ is an isofibration since p is (40.6). Using the set of maps S provided by (43.4), for a monomorphism i we have that $(p^{\square i})^{\mathrm{core}}\subseteq \mathrm{TrivFib}$ iff $S\boxslash(p^{\square i})$ iff $i\boxslash(p^{\square S})$. Thus $\mathcal{C}_p$ is the intersection of ${}^{\boxslash}(p^{\square S})$ with $\overline{\mathrm{Cell}}$, and so is weakly saturated. ☐

Next we observe that $\mathcal{C}_p$ has a "precancellation" property.

43.11. **Lemma.** *Let $p\colon C\to D$ be an isofibration between quasicategories. If $i\colon K\to K'$ and $j\colon K'\to K''$ are monomorphisms, then $i, ji\in\mathcal{C}_p$ implies $j\in\mathcal{C}_p$.*

Proof. We use the transitivity triangle (39.4) for i,j and p, which asserts that $p^{\square ji} = q\circ p^{\square j}$ where q is a base-change of $p^{\square i}$. Restricting to cores gives a factorization $(p^{\square ji})^{\mathrm{core}} = q^{\mathrm{core}}\circ(p^{\square i})^{\mathrm{core}}$. Furthermore q^{core} is a base-change of $(p^{\square i})^{\mathrm{core}}$ as (33.3) applies since $p^{\square i}$ is an inner fibration between quasicategories (16.8).

We have that $(p^{\square ji})^{\mathrm{core}}$, $(p^{\square j})^{\mathrm{core}}$, $(p^{\square i})^{\mathrm{core}}$, and hence q^{core} are Kan fibrations (40.2). Since $ji, i\in\mathcal{C}_p$, we have that $(p^{\square ji})^{\mathrm{core}}$, $(p^{\square i})^{\mathrm{core}}$ and hence q are trivial fibrations and thus weak equivalences between Kan complexes, whence $p^{\square j}$ is also a weak equivalence by 2-out-of-3, and therefore $p^{\square j}$ is a trivial fibration since it is a Kan fibration between Kan complexes (37.9). ☐

43.12. **Lemma.** *If $p\colon C\to D$ is an isofibration which is fully faithful and essentially surjective, then $(\partial\Delta^0\subset\Delta^0)\in\mathcal{C}_p$ and $(\partial\Delta^1\subset\Delta^1)\in\mathcal{C}_p$.*

Proof. First we show $(\partial\Delta^0\subset\Delta^1)\in\mathcal{C}_p$, which amounts to the claim that $p^{\mathrm{core}}\in\mathrm{TrivFib}$. Recall that since p is fully faithful and essentially surjective, p^{core} is a weak equivalence (43.7). Since p is an isofibration, p^{core} is a Kan fibration (40.2) between Kan complexes, and therefore p^{core} is a trivial fibration (37.9).

Now we show $(\partial\Delta^1\subset\Delta^1)\in\mathcal{C}_p$. We form a diagram as in the proof of (39.1), except that we restrict to cores. This has the form

$$\begin{tikzcd}
\displaystyle\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_C(c,c') \arrow[r] \arrow[d, "q_{c,c'}"] & \mathrm{Fun}(\Delta^1,C)^{\mathrm{core}} \arrow[d, "q"] & \\
\displaystyle\coprod_{(c,c')\in C_0\times C_0} \mathrm{map}_D(pc,pc') \arrow[r, "j"'] \arrow[d] & (C\times C)^{\mathrm{core}}\times_{(D\times D)^{\mathrm{core}}}\mathrm{Fun}(\Delta^1,D)^{\mathrm{core}} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1,D)^{\mathrm{core}} \arrow[d] \\
(\mathrm{Sk}_0\,C)\times(\mathrm{Sk}_0\,C) \arrow[r, "i"'] & (C\times C)^{\mathrm{core}} \arrow[r, "p\times p"'] & (D\times D)^{\mathrm{core}}
\end{tikzcd}$$

in which the squares are all pullbacks (since we are pulling back quasicategories along inner fibrations, so (33.3) applies). The mapping spaces are already quasigroupoids, and so are equal to their cores already. The functors q and $q_{c,c'}$ are Kan fibrations between Kan complexes (40.2), and j is surjective on vertices. If p is fully faithful, then the $q_{c,c'}$ are weak equivalences, and an argument much as in the proof of (39.1) implies that $q = (p^{\Box(\partial\Delta^1 \subset \Delta^1)})$ is a trivial fibration as desired, i.e., using the fiberwise criterion (38.3). $\Box$

Proof of (43.2). Given an isofibration p which is fully faithful and essentially surjective, we need to show that $\overline{\text{Cell}} \subseteq \mathcal{C}_p$. As $\mathcal{C}_p$ is weakly saturated (43.10) it suffices to show $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for all $n \geq 0$.

We have that $p^{\Box\text{InnHorn}} \subseteq \text{TrivFib}$ since p is an inner fibration by $\overline{\text{InnHorn}}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$ (16.8). Therefore $(p^{\Box\text{InnHorn}})^{\text{core}} \in \text{TrivFib}$, by an elementary argument (43.13). Thus we have shown $\text{InnHorn} \subseteq \mathcal{C}_p$.

We have already proved $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for $n = 0$ and $n = 1$ (43.12). To prove $(\partial\Delta^n \subset \Delta^n) \in \mathcal{C}_p$ for $n \geq 2$ assuming $(\partial\Delta^{n-1} \subset \Delta^n) \in \mathcal{C}_p$, we use the factorization

$$\Lambda^n_1 \xrightarrow{i} \partial\Delta^n \xrightarrow{j} \Delta^n,$$

where the horn inclusion i is a cobase-change of $\partial\Delta^{n-1} \subset \Delta^n$. Thus $i \in \mathcal{C}_p$ and $ji \in \mathcal{C}_p$, whence $j \in \mathcal{C}_p$ by the precancellation property of $\mathcal{C}_p$ (43.11).

$\Box$

43.13. *Exercise.* Let $p\colon C \to D$ be a trivial fibration between quasicategories. Show that $p^{\text{core}}\colon C^{\text{core}} \to D^{\text{core}}$ is also a trivial fibration. (Hint: this is straightforward to prove with a direct argument; it is also a consequence of the Kan complex case of the fundamental theorem together with (43.7).)

Part 5. **Model categories**

44. Categorical fibrations

A map $p\colon X \to Y$ of simplicial sets is a **categorical fibration** if and only if $j \boxslash p$ for all j which are monomorphisms and categorical equivalences. I'll write CatFib for the class of categorical fibrations.

Categorical fibrations generalize isofibrations. In fact, a map $p\colon C \to D$ with D a quasicategory is a categorical fibration if and only if it is an isofibration, as we proved in (41.13).

44.1. **Proposition.** *A map $p\colon X \to Y$ of simplicial sets is a trivial fibration if and only if it is a categorical fibration and a categorical equivalence.*

Proof. ($\Longrightarrow$) We know trivial fibrations are categorical equivalences (20.10), and it is clear that they are categorical fibrations by definition.

($\Longleftarrow$) If p is a categorical fibration and a categorical equivalence, factor p as $X \xrightarrow{j} Z \xrightarrow{q} Y$ with j a monomorphism and q a trivial fibration, by the small object argument applied to $\overline{\text{Cell}}\Box\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$. By 2-out-of-3 j is a categorical equivalence (22.10), so $j \boxslash p$ by hypothesis. Thus the "retract trick" (13.19) exhibits p as a retract of q, whence p is also a trivial fibration. $\Box$

44.2. **Proposition.** *If $p\colon X \to Y$ is a categorical fibration and $j\colon K \to L$ is a monomorphism, then*

$$q = p^{\Box j}\colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$$

is a categorical fibration. Furthermore, if either j or p is also a categorical equivalence, then so is q.

Proof. To show that q is a categorical fibration, consider $i\colon A \to B$ a monomorphism which is a categorical equivalence. We have $i \boxslash q$ iff $(i\Box j) \boxslash p$, so since p is a categorical fibration it suffices to show that the monomorphism $i\Box j$ is a categorical equivalence, i.e., to show $\operatorname{Map}(i\Box j, C)$ is a categorical equivalence for every quasicategory C. This map is isomorphic to $r^{\Box j}$ where $r = \operatorname{Map}(i, C)$. Note that $r := \operatorname{Map}(i, C)$ is an isofibration (40.6) and a categorical equivalence, and therefore a trivial fibration (41.1). Thus $r^{\square j}$ is also a trivial fibration using $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$, and hence $\text{Map}(i\square j, C)$ is a categorical equivalence as desired.

If p is also a categorical equivalence, then it is a trivial fibration by (44.1), so q is a trivial fibration by $\overline{\text{Cell}}\square\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$.

If j is also a categorical equivalence, then for any monomorphism i, we have $i \boxslash q$ iff $(i\square j) \boxslash p$ iff $j \boxslash (p^{\square i})$. But $p^{\square i}$ is a categorical fibration by what we have just proved, so the result holds. $\square$

44.3. **Categorical fibrations and the small object argument.** Clearly, $\text{CatFib} = \left(\overline{\text{Cell}} \cap \text{CatEq}\right)^{\boxslash}$ is a right complement to a class of maps. We would like to know that CatFib is the right complement to a *set* of maps; then we could use the small object argument to factor any map into an injective categorical equivalence followed by a categorical fibration.

Unfortunately, it's apparently not known how to write down an explicit set of maps S so that $S^{\boxslash} = \text{CatFib}$. What is known is that such a set *exists*.

44.4. **Proposition.** *There exists a set S of maps of simplicial sets such that $\overline{S} = \overline{\text{Cell}} \cap \text{CatEq}$, whence $S^{\boxslash} = \text{CatFib}$.*

In particular, we learn that every map can be factored into an injective categorical equivalence followed by a categorical fibration. We give a proof in the appendix, together with the proof of an analogous fact for weak equivalences.

44.5. **Proposition.** *There exists a set T of maps of simplicial sets such that $\overline{T} = \overline{\text{Cell}} \cap \text{WkEq}$.*

45. The Joyal model structure on simplicial sets

45.1. **Model categories.** A **model category** (in the sense of Quillen) is a category $\mathcal{M}$ with three classes of maps: W, Cof, Fib, which I will call **weak equivalences**, **cofibrations**, and **fibrations** respectively, satisfying the following axioms.

- $\mathcal{M}$ has all small limits and colimits.
- W satisfies the 2-out-of-3 property.
- $(\text{Cof} \cap \text{W}, \text{Fib})$ and $(\text{Cof}, \text{Fib} \cap \text{W})$ are weak factorization systems (13.20).

An object X is **cofibrant** if the map from the initial object is a cofibration, and **fibrant** if the map to the terminal object is a fibration. A map in $\text{Cof} \cap W$ is called a **trivial cofibration**, and a map in $\text{Fib} \cap W$ is called a **trivial fibration**.

45.2. *Warning.* Do not confuse the general notion of "weak equivalence" in an arbitrary model category with the specific notion of "weak equivalence of simplicial sets" defined in (36.1). **I should call just call the model category ones "equivalences".**

45.3. *Remark.* The third axiom implies that Cof, $\text{Cof} \cap \text{W}$, Fib, and $\text{Fib} \cap \text{W}$ are closed under retracts.

45.4. *Exercise.* Show that in a model category (as defined above), the class of weak equivalences is closed under retracts. *Hint:* if f is a retract of a weak equivalence g, construct a factorization of f which is itself a retract of a factorization of g[32].

45.5. *Exercise* (Slice model categories). Let $\mathcal{M}$ be a model category, and let X be an object of $\mathcal{M}$. Show that the slice categories $\mathcal{M}_{X/}$ and $\mathcal{M}_{/X}$ admit model category structures, in which the weak equivalences, cofibrations, and fibrations are precisely the maps whose images under $\mathcal{M}_{/X} \to \mathcal{M}$ or $\mathcal{M}_{X/} \to \mathcal{M}$ are weak equivalences, cofibrations, and fibrations in $\mathcal{M}$.

45.6. *Exercise* (Goodwillie). Classify all model category structures on the category of sets. (There are exactly nine. *Hint:* use (13.22).)

[32]In many formulations of model categories, closure of weak equivalences under retracts is taken as one of the axioms. The formulation we use is described in Riehl, "A concise definition of a model category" [Rie09], which gives a solution to this exercise.

45.7. **The Joyal model structure.**

45.8. **Theorem** (Joyal)**.** *The category of simplicial sets admits a model structure, in which*

- W = *categorical equivalences (*CatEq*),*
- Cof = *monomorphims (*$\overline{\text{Cell}}$*),*
- Fib = *categorical fibrations (*CatFib*).*

Furthermore, the fibrant objects are precisely the quasicategories, and the fibrations with target a fibrant object are precisely the isofibrations.

Proof. Categorical equivalences satisfy 2-out-of-3 by (22.10). We have that

- Cof = $\overline{\text{Cell}}$ by (15.26).
- Fib ∩ W = CatFib ∩ CatEq = TFib = $\text{Cell}^{\boxslash}$ by (44.1),
- Cof ∩ W = $\overline{\text{Cell}}$ ∩ CatEq = $\overline{S}$ for some set S (44.4),
- Fib = CatFib = $(\text{Cof} \cap \text{W})^{\boxslash} = S^{\boxslash}$ by definition,

so both (Cof ∩ W, Fib) and (Cof, Fib ∩ W) are weak factorization systems via the small object argument (13.17). Thus, we get a model category.

We have shown (41.13) that the categorical fibrations $p\colon C \to D$ with D a quasicategory are precisely the isofibrations. Applied when $D = *$, this implies that quasicategories are exactly the fibrant objects, and thus that fibrations with fibrant target are precisely the isofibrations. □

45.9. *Remark.* It is a fact that a model category structure is uniquely determined by its cofibrations and fibrant objects **(reference? Joyal)**. Thus, the Joyal model structure is the unique model structure on simplicial sets with Cof = monomorphisms and with fibrant objects the quasicategories.

45.10. **Cartesian model categories.** Recall that the category of simplicial sets is *cartesian closed*. A **cartesian model category** is a model category which is cartesian closed, with the following properties. Suppose $i\colon A \to B$ and $j\colon K \to L$ are cofibrations and $p\colon X \to Y$ is a fibration. Then

- $$i\Box j\colon (A \times L) \cup_{A\times K} (B \times K) \to B \times L$$
  is a cofibration, and is in addition a weak equivalence if either i or j is also a weak equivalence, and
- $$p^{\Box j}\colon \operatorname{Map}(L, X) \to \operatorname{Map}(K, X) \times_{\operatorname{Map}(K,Y)} \operatorname{Map}(L, Y)$$
  is a fibration, and is in addition a weak equivalence if either j or p is also a weak equivalence.

In fact, we only need to specify *one* of the above two properties, as they imply each other.

45.11. **Proposition.** *The Joyal model structure is cartesian.*

Proof. This is just (44.2). □

46. The Kan-Quillen model structure on simplicial sets

A map $p\colon X \to Y$ is a **groupoidal fibration** if and only if $j \boxslash p$ for all j which are monomorphisms and weak equivalences. I write GpdFib for the class of categorical fibrations. By (44.5) we see there exists a set of maps T such that GpdFib = $T^{\boxslash}$. **M 25 Mar 2019**

46.1. **The Kan-Quillen model structure.**

46.2. **Theorem.** *The category of simplicial sets admits a model structure, in which*

- W = *weak equivalences (*WkEq*),*
- Cof = *monomorphims (*$\overline{\text{Cell}}$*),*
- Fib = *groupoidal fibrations (*GpdFib*).*

Furthermore, the fibrant objects are precisely the Kan complexes, and the fibrations with target a fibrant object are precisely the Kan fibrations.

Proof. Weak equivalences satisfy 2-out-of-3 by (36.3). We have that

- $\mathrm{Cof} = \overline{\mathrm{Cell}}$ by definition,
- $\mathrm{Fib} \cap \mathrm{W} = \mathrm{GpdFib} \cap \mathrm{WkEq} = \mathrm{TFib} = \mathrm{Cell}^{\boxslash}$ by (44.1),
- $\mathrm{Cof} \cap \mathrm{W} = \overline{\mathrm{Cell}} \cap \mathrm{WkEq} = \overline{T}$ for some set T, by (44.5),
- $\mathrm{Fib} = \mathrm{GpdFib} = (\mathrm{Cof} \cap \mathrm{W})^{\boxslash} = T^{\boxslash}$ by definition,

so both $(\mathrm{Cof} \cap \mathrm{W}, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{W})$ are weak factorization systems via the small object argument (13.17). Thus, we get a model category.

We have already proved that Kan fibrations between Kan complexes have the lifting property of groupoidal fibrations (37.17), so the statements about fibrant objects and fibrations to fibrant objects follow just as in the categorical case. □

46.3. **Proposition.** *The Quillen model structure is cartesian.*

Proof. We must show that $p^{\Box j}$ is a groupoidal fibration if j is a monomorphsm and p a groupoidal fibration, and also that it is a weak equivalence if either j or p is. This is proved by an argument nearly identical to the proof of (44.2). □

46.4. **Kan fibrations are groupoidal fibrations.** The proof of the Quillen model structure we gave above relied on (44.5) to produce a set T such that $\overline{T} = \mathrm{Cof} \cap \mathrm{WkEq}$. In fact, more is true. It turns out that we can take $T = \mathrm{Horn}$, so that $\mathrm{GpdFib} = \mathrm{KanFib}$. It was in this form that the model structure was first constructed by Quillen.

We will not give a proof of this here. The non-trivial part is to show that $\mathrm{KanFib} \subseteq \mathrm{GpdFib}$; note that we already know that a Kan fibration *between Kan complexes* is a groupoidal fibration by (37.17). This proposition is usually proved via an argument (due to Quillen) based on the theory of *minimal fibrations*. See for instance Quillen's original argument [Qui67, §II.3] or [GJ09, Ch. 1].

These arguments work by showing that KanFib is the weak cosaturation of the class of *Kan fibrations between Kan complexes*. In fact one can even show that every Kan fibration is a *base change* of a Kan fibration between Kan complexes, see [KLV12].

The observation that the Kan-Quillen model structure can be constructed without first showing $\mathrm{GpdFib} = \mathrm{KanFib}$, and thus (46.2) in the form I have stated it, is due to Cisinski [?].

47. Model categories and homotopy colimits

We are going to exploit these model category structures now. The main purpose of model categories is to give tools for showing that a given construction preserves certain kinds of equivalence.

47.1. **Creating new model categories.** Given a model category $\mathcal{M}$, many other categories related to it can also be equipped with model category structures, such as functor categories $\mathrm{Fun}(C, \mathcal{M})$ where C is a small category. We won't consider general formulations of this here, but rather will set up some special cases.

As an example, we consider the case of $C = [1] = \{0 \xrightarrow{01} 1\}$.

47.2. **Proposition.** *There exists a model structure on $\mathcal{N} := \mathrm{Fun}([1], \mathcal{M})$ in which a map $\alpha\colon X \to X'$ is*

- *a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1$*
- *a cofibration if both $\alpha(0)$ and the map $(\alpha(1), X(01))\colon X(1) \cup_{X(0)} X'(0) \to X'(1)$ are cofibrations in $\mathcal{M}$, and*
- *a fibration if $\alpha(i)$ is a fibration in $\mathcal{M}$ for $i = 0, 1$.*

Proof. It is clear that $\mathcal{N}$ has small limits and colimits, and that weak equivalences in it have the 2-out-of-3 property. It remains to show that $(\mathrm{Cof} \cap W, \mathrm{Fib})$ and $(\mathrm{Cof}, \mathrm{Fib} \cap W)$ are weak factorization systems, where W, Cof, Fib are the of maps in $\mathcal{N}$ defined in the statement of the proposition.

We start with the following observation about lifting in $\mathcal{N} = \mathrm{Fun}([1], \mathcal{M})$: given maps $j\colon A \to B$ and $p\colon X \to Y$ in $\mathcal{N}$, we can solve a lifting problem (u, v) of type $j \boxslash p$ in $\mathcal{N}$ by solving a sequence of two lifting problems in $\mathcal{M}$, namely

$$\begin{tikzcd} A(0) \arrow[r, "u(0)"] \arrow[d, "j(0)"'] & X(0) \arrow[d, "p(0)"] \\ B(0) \arrow[r, "v(0)"'] \arrow[ur, dotted, "s(0)"] & Y(0) \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A(1) \cup_{A(0)} B(0) \arrow[r, "{(u(1),\, X(01)\circ s(0))}"] \arrow[d, "{(j(1),\, B(01))}"'] & X(1) \arrow[d, "p(1)"] \\ B(1) \arrow[r, "v(1)"'] \arrow[ur, dotted, "s(1)"] & Y(1) \end{tikzcd}$$

where the second problem depends on the solution $s(0)$ to the first problem. Then the maps $s(0)$ and $s(1)$ fit together to give a map $s\colon B \to X$ in $\mathcal{N}$ which solve the original lifting problem.

Given this, it is not hard to prove that $\mathrm{Cof} \cap W \boxslash \mathrm{Fib}$ and $\mathrm{Cof} \boxslash \mathrm{Fib} \cap W$, using the definitions and the fact that $\mathcal{M}$ is a model category. The trickiest point is to observe that if $j\colon A \to B$ is both a cofibration and a weak equivalence in $\mathcal{N}$, then $(j(1), B(01))$ is a trivial cofibration in $\mathcal{M}$: this uses 2-out-of-3 for weak equivalences in $\mathcal{M}$ and the fact that $A(1) \to A(1) \cup_{A(0)} B(0)$ must be a trivial cofibration in $\mathcal{M}$, being a cobase-change of $j(0)$.

Next, observe that to describe a factorization of a map $f\colon X \to Y$ in $\mathcal{N}$ into $f = pi$ with $j\colon X \to U$ and $p\colon U \to Y$, it suffices to describe a sequence of two factorizations in $\mathcal{M}$, namely $f(0) = p(0) \circ j(0)$ and $h = p(1) \circ g$, as in

$$X(0) \xrightarrow{j(0)} U(0) \xrightarrow{p(0)} Y(0) \qquad \text{and} \qquad \begin{tikzcd} X(0) \arrow[r, "j(0)"] \arrow[d, "X(01)"'] & U(0) \arrow[rr, "p(0)"] \arrow[d, "\eta'"] & & Y(0) \arrow[d, "Y(01)"] \\ X(1) \arrow[r, "\eta"'] & X(1) \cup_{X(0)} U(0) \arrow[r, "g"] \arrow[rr, bend right, "h"'] & U(1) \arrow[r, "p(1)"] & Y(1) \end{tikzcd}$$

where $h = (f(1), Y(01) \circ p(0))$, so that $j(1) = g \circ \eta$ and $U(01) = g \circ \eta'$.

To factor $f = pj$ in $\mathcal{N}$ with $j \in \mathrm{Cof} \cap W$ and $p \in \mathrm{Fib}$, it suffices to successively choose factorizations of $f(0)$ and h of this type. Likewise, to factor $f = pj$ in $\mathcal{N}$ with $j \in \mathrm{Cof}$ and $p \in \mathrm{Fib} \cap W$, it suffices to successively choose factorizations of $f(0)$ and h of this type.

It remains to show that $\mathrm{Cof} \cap W = {}^{\boxslash}\mathrm{Fib}$, $\mathrm{Fib} = \mathrm{Cof} \cap W^{\boxslash}$, $\mathrm{Cof} = {}^{\boxslash}\mathrm{Fib} \cap W$, and $\mathrm{Cof}^{\boxslash} = \mathrm{Fib} \cap W$. This is an immediate consequence of the "retract trick" (13.18), together with the easily checked fact that Cof, $\mathrm{Cof} \cap W$, Fib, and $\mathrm{Fib} \cap W$ are closed under retracts, which can be proved directly using the definition and the fact that the analogous classes in $\mathcal{M}$ are closed under retracts (45.3). ☐

The opposite of a model category is also a model category, by switching the roles of fibrations and cofibrations. Therefore, there is another model structure on $\mathrm{Fun}([1], \mathcal{M}) = (\mathrm{Fun}([1], \mathcal{M}^{\mathrm{op}}))^{\mathrm{op}}$.

47.3. **Reedy lemma.** The Reedy lemma gives an explicit criterion for a functor to preserve weak equivalences between large classes of objects.

47.4. **Proposition** (Reedy lemma)**.** *Let $F\colon \mathcal{M} \to \mathcal{N}$ be a functor between model categories.*

(1) *If F takes trivial cofibrations to weak equivalences, then F takes weak equivalences between cofibrant objects to weak equivalences.*
(2) *If F takes trivial fibrations to weak equivalences, then F takes weak equivalences between fibrant objects to weak equivalences.*

Proof. I prove (1); the proof of (2) is formally dual.

Let $f\colon X \to Y$ be a weak equivalence between cofibrant objects in $\mathcal{M}$. Form the commutative diagram

$$\begin{tikzcd}
\varnothing \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dr, "b"] \arrow[drr, bend left, "\mathrm{id}_Y"] & & \\
X \arrow[r] \arrow[rr, bend right, "a"'] \arrow[rrr, bend right=40, "f"'] & X \amalg Y \arrow[r, "i"] & C \arrow[r, "p"] & Y
\end{tikzcd}$$

where the square is a pushout, and we have chosen a factorization of $(f, \mathrm{id}_Y)\colon X \amalg Y \to Y$ as pi, a cofibration followed by a weak equivalence (e.g., a trivial fibration). Because X and Y are cofibrant, the maps $X \to X \amalg Y \leftarrow Y$ are cofibrations. Using this and the 2-of-3 property for weak equivalences, we see that a and b are trivial cofibrations. Applying F gives

$$\begin{tikzcd}
& F(Y) \arrow[d, "F(b)"'] \arrow[dr, "\mathrm{id}"] & \\
F(X) \arrow[r, "F(a)"'] & F(C) \arrow[r, "F(p)"'] & F(Y)
\end{tikzcd}$$

in which $F(b)$ and $F(a)$ are weak equivalences by hypothesis, whence $F(b)$ is a weak equivalence by 2-of-3, and therefore $F(f) = F(p)F(b)$ is a weak equivalence, as desired. ☐

Relate this to the path fibration construction.

47.5. **Quillen pairs.** Given an adjoint pair of functors $F\colon \mathcal{M} \leftrightarrows \mathcal{N} : G$ between model categories, we see from the properties of weak factorization systems that

- F preserves cofibrations if and only if G preserves trivial fibrations, and
- F preserves trivial cofibrations if and only if G preserves fibrations.

If both of these are true, we say that (F, G) is a **Quillen pair**.

Note that if (F, G) is a Quillen pair, then the Reedy lemma (47.4)(1) applies to F, while (47.4)(2) applies to G.

47.6. **Good colimits.** We can apply the above to certain examples of colimit functors, which we will refer to generically as "good colimits". There are three types of these: arbitrary coproducts of cofibrant objects, countable sequential colimits of cofibrant objects along cofibrations, and pushouts of cofibrant objects along a cofibration. We will show that "good colimits are weak equivalence invariant".

47.7. *Exercise.* Let S be a small discrete category (i.e., all maps are identities). Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(S, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence, cofibration, or fibration iff each $\alpha_s\colon X_s \to X'_s$ is one in $\mathcal{M}$.

Then show that $\mathrm{colim}\colon \mathrm{Fun}(S, \mathcal{M}) \leftrightarrows \mathcal{M} : \mathrm{const}$ is a Quillen pair, and use this to prove the next proposition.

47.8. **Proposition** (Good coproducts)**.** *Given a collection $f_s\colon X_s \to X'_s$ of weak equivalences between cofibrant objects in $\mathcal{M}$, the induced map $\amalg f_s\colon \coprod X_s \to \coprod X'_s$ is a weak equivalence.*

47.9. *Exercise.* Let ω be the category

$$0 \to 1 \to 2 \to \cdots$$

with objects indexed by natural numbers. Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(\omega, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if each $\alpha(i)$ is a weak equivalence in $\mathcal{M}$,

- a cofibration if (i) $\alpha(0)$ is a cofibration in $\mathcal{M}$, and $X'(i) \cup_{X(i)} X(i+1) \to X'(i+1)$ is a cofibration in $\mathcal{M}$for all $i \geq 0$, and
- a fibration if each $\alpha(i)$ is a fibration in $\mathcal{M}$.

Then show that colim: $\mathrm{Fun}(\omega, \mathcal{M}) \leftrightarrows \mathcal{M}$: const is a Quillen pair, and use this to prove the next proposition.

47.10. **Proposition** (Good sequential colimits)**.** *Give a natural transformation $\alpha\colon X \to X'$ of functors $\omega \to \mathcal{M}$ such that all maps $\alpha(i)\colon X(i) \to X'(i)$ are weak equivalences, all objects $X(i)$ and $X(i')$ are cofibrant, and the maps $X(i) \to X(i+1)$ and $X'(i) \to X'(i+1)$ are cofibrations, the induced map* $\mathrm{colim}_\omega X \to \mathrm{colim}_\omega X'$ *is a weak equivalence.*

47.11. *Exercise.* Recall that Λ^2_0 is a category:

$$1 \xleftarrow{01} 0 \xrightarrow{12} 2.$$

Show that if $\mathcal{M}$ is a model category, then $\mathrm{Fun}(\Lambda^2_0, \mathcal{M})$ is a model category in which $\alpha\colon X \to X'$ is

- a weak equivalence if $\alpha(i)\colon X(i) \to X'(i)$ is a weak equivalence in $\mathcal{M}$ for $i = 0, 1, 2$ (i.e., an **objectwise weak equivalence**),
- a cofibration if $\alpha(0)$, $\alpha(1)$, and the evident map $X(2) \cup_{X(0)} X'(0) \to X'(2)$ are cofibrations in $\mathcal{M}$, and
- a fibration if $\alpha(1)$, $\alpha(2)$, and the evident map $X(1) \to X'(1) \times_{X'(0)} X(0)$ are fibrations in $\mathcal{M}$.

Then show that colim: $\mathrm{Fun}(\Lambda^2_0, \mathcal{M}) \leftrightarrows \mathcal{M}$: const is a Quillen pair, and use this to prove the next proposition.

47.12. **Proposition** (Good pushouts)**.** *Given a natural transformation $\alpha\colon X \to X'$ of functors $\Lambda^2_0 \to \mathcal{M}$, i.e., a diagram*

$$\begin{tikzcd} X(1) \arrow[d, "\sim"'] & X(0) \arrow[l] \arrow[r, "X(02)"] \arrow[d, "\sim"'] & X(2) \arrow[d, "\sim"'] \\ X'(1) & X'(0) \arrow[l] \arrow[r, "X'(02)"] & X'(2) \end{tikzcd}$$

in which the vertical maps are weak equivalences, all objects $X(i)$ and $X'(i)$ are cofibrant, and the maps $X(02)$ and $X'(02)$ are cofibrations, the induced map $\mathrm{colim}_{\Lambda^2_0} X \to \mathrm{colim}_{\Lambda^2_0} X'$ *is a weak equivalence.*

In the Joyal and Kan-Quillen model structures on sSet, all objects are automatically cofibrant, which makes the above propositions especially handy.

We will call any colimit diagram in a model category, satisfying the hypotheses of one of (47.8), (47.12), (47.10) a **good colimit**. Thus, we see that good colimits are homotopy invariant. These "good colimits" are examples of what are called *homotopy colimits*.

Since the opposite of a model category is also a model category, all of the results of this section admit dual formulations, leading to the observation that **good limits** are homotopy invariant.

47.13. *Exercise.* State and prove the dual versions of all the results in this section.

48. Every quasigroupoid is equivalent to its opposite

Recall that any ordinary groupoid G is isomorphic to its own opposite: define a functor $G \to G^{\mathrm{op}}$ which is the identity on objects, and which sends any morphism f to its inverse. Such a straightforward functor is not possible for quasigroupoids. However, we will show that every quasigropoid is categorically equivalent to its opposite, via a zig-zag of categorical equivalences.

48.1. **A zig-zag relating a quasigroupoid and its opposite.** For an object $[n] \in \Delta$ of the simplicial indexing category, let Iso^n denote the ordinary category whose set of objects is the set $[n] = \{0, 1, \dots, n\}$, and such that $\mathrm{Hom}_{\mathrm{Iso}^n}(x, y) = \{*\}$ for all pairs $x, y \in [n]$. For instance, $\mathrm{Iso}^0 = \Delta^0$, while $\mathrm{Iso}^1 = \mathrm{Iso}$ the "walking isomorphism" of (**??**). For all $n \geq 0$ we have functors

$$[n] \xrightarrow{\alpha_n} \mathrm{Iso}^n \xleftarrow{\beta_n} [n]^{\mathrm{op}}$$

which are uniquely determined by what they do on objects: $\alpha_n(x) = x = \beta_n(x)$ for $x \in [n]$.

Taking nerves gives maps of simplicial sets

$$\Delta^n \xrightarrow{\alpha_n} N\mathrm{Iso}^n \xleftarrow{\beta_n} (\Delta^n)^{\mathrm{op}}.$$

Note that k-dimensional elements of $N\mathrm{Iso}^n$ correspond bijectively to the set of all functions $[k] \to [n]$ (*not* monotone functions), while α_n and β_n are monomorphisms whose images are the subsets of weakly monotone (increasing or decreasing) functions.

The assignment $[n] \mapsto N\mathrm{Iso}^n$ defines a functor $\Delta \to s\mathrm{Set}$, and the maps α_n and β_n fit together to define natural transformations of functors $\Delta \to s\mathrm{Set}$, which we display as

$$\Delta^\bullet \xrightarrow{\alpha} N\mathrm{Iso}^\bullet \xleftarrow{\beta} (\Delta^\bullet)^{\mathrm{op}}.$$

We now define a functor $\mathcal{G}\colon s\mathrm{Set} \to s\mathrm{Set}$, which sends a simplicial set X to the simplicial set $\mathcal{G}X$ with

$$(\mathcal{G}X)_n := \mathrm{Hom}_{s\mathrm{Set}}(N\mathrm{Iso}^n, X).$$

The natural transformations α and β give rise to maps

$$X \xleftarrow{\alpha^*} \mathcal{G}X \xrightarrow{\beta^*} X^{\mathrm{op}}$$

natural in the simplicial set X.

We will prove the following.

48.2. **Proposition.** *If C is a quasicategory, then $\mathcal{G}C$ is a Kan complex, and the maps α^* and β^* restrict to give trivial fibrations*

$$C^{\mathrm{core}} \xleftarrow{\alpha^*} \mathcal{G}C \xrightarrow{\beta^*} (C^{\mathrm{core}})^{\mathrm{op}}.$$

In particular, if C is a Kan complex then $C \xleftarrow{\alpha^*} \mathcal{G}C \xrightarrow{\beta^*} C^{\mathrm{op}}$ is a zig-zag of categorical equivalences.

48.3. **Adjoint transformations and lifting problems.** Consider adjoint pairs $F\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ and $F'\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ of functors between two categories $\mathcal{A}$ and $\mathcal{B}$. There is a bijective correspondence

$$\{\text{natural transformations } \gamma_*\colon F \to F'\} \longleftrightarrow \{\text{natural transformations } \gamma^*\colon G' \to G\}.$$

48.4. *Exercise.* Construct this correspondence, by verifying that $\eta_* \mapsto \eta^*$ defined by

$$G'(Y) \xrightarrow{\eta G'(Y)} GFG'(Y) \xrightarrow{G\eta_* G'(Y)} GF'G'(Y) \xrightarrow{G\epsilon'(Y)} G(Y)$$

and $\eta^* \mapsto \eta_*$ defined by

$$F(X) \xrightarrow{F\eta'(X)} FG'F'(X) \xrightarrow{F\gamma^* F'(X)} FGF'(X) \xrightarrow{\epsilon F(X)} F'(X)$$

are inverse to each other, where $\eta\colon \mathrm{Id} \to GF$, $\eta'\colon \mathrm{Id} \to G'F'$, $\epsilon\colon FG \to \mathrm{Id}$, $\epsilon'\colon F'G' \to \mathrm{Id}$ are units and counits of the adjunctions.

We assume $\mathcal{A}$ has pushouts and $\mathcal{B}$ has pullbacks. Given such a related pair $\gamma = (\gamma_*, \gamma^*)$ of natural transformations, and given maps $i\colon K \to L$ in $\mathcal{A}$ and $p\colon X \to Y$ in $\mathcal{B}$, we define maps

$$\gamma_\square(i) := (F'(i), \gamma_*(L))\colon F'(K) \amalg_{F(K)} F(L) \to F'(L) \qquad \text{in } \mathcal{B}$$

and

$$\gamma^\square(p) := (G'(p), \gamma^*(X))\colon G'(X) \to G'(Y) \times_{G(Y)} G(X) \qquad \text{in } \mathcal{A}.$$

Note that if $K = \varnothing$ is the initial object, then $\gamma_\Box(i) = \gamma_*(i) \colon F(L) \to F'(L)$. Likewise, if $Y = *$ is the terminal object, then $\gamma^\Box(p) = \gamma^*(p) \colon G'(X) \to G(X)$.

48.5. **Proposition.** *We have $\gamma_\Box(i) \boxslash p$ if and only if $i \boxslash \gamma^\Box(p)$.*

Proof. Straightforward. $\square$

Let $\mathcal{F}$ be the left adjoint to $\mathcal{G}$. The transformations α^* and β^* give rise to natural maps

$$X \xrightarrow{\alpha_*} \mathcal{F}X \xleftarrow{\beta_*} X^{\mathrm{op}}.$$

We will prove the following below.

48.6. **Proposition.** *For any monomorphism $i \colon K \to L$ of simplicial sets, the maps $\alpha_\Box(i)$ and $\beta_\Box(i)$ are both monomorphisms and weak equivalences of simplicial sets.*

Given this we obtain the following consequence

48.7. **Corollary.** *For any groupoidal fibration $p \colon X \to Y$ of simplicial sets, the maps $\alpha^\Box(p)$ and $\beta^\Box(p)$ are trivial fibrations.*

In particular, if X is a Kan complex, then the maps $X \xleftarrow{\alpha^*} \mathcal{G}X \xrightarrow{\beta^*} X^{\mathrm{op}}$ are trivial fibrations.

48.8. **Proposition.** *Let $\gamma \colon F \to F'$ be a natural transformation of functors $F, F' \colon s\mathrm{Set} \to \mathrm{Set}$, and suppose that both F and F' admit right adjoints. Suppose further that*

(1) *$\gamma(\Delta^n) \colon F(\Delta^n) \to F'(\Delta^n)$ is injective for all $n \geq 0$, and*
(2) *both $\mathrm{Ker}\big[(\langle 0\rangle, \langle 1\rangle) \colon F(\Delta^0) \rightrightarrows F(\Delta^1)\big]$ and $\mathrm{Ker}\big[(\langle 0\rangle, \langle 1\rangle) \colon F'(\Delta^0) \rightrightarrows F'(\Delta^1)\big]$ are empty.*

Then $\gamma_\Box F(i)$ is a monomorphism for every monomorphism i of simplicial sets.

48.9. **Singular and realization functors.** Given a category $\mathcal{A}$, a **cosimplicial object** in $\mathcal{A}$ is a functor $C \colon \Delta \to \mathcal{A}$, where Δ is the category of simplicial operators. Often I will write C^n instead of $C([n])$ for the values of this functor, and so write $C^\bullet$ for the whole functor. In many of our examples, $\mathcal{A} = s\mathrm{Set}$, in which case $C^\bullet$ is a "cosimplicial simplicial set".

Given any cosimplicial object $C^\bullet$ in a category $\mathcal{A}$ with all small colimits, we have induced functors

$$\mathrm{Re}_C \colon s\mathrm{Set} \rightleftarrows \mathcal{A} : \mathrm{Si}_C\,.$$

The **singular functor** Si_C is defined on an object $A \in \mathcal{A}$ by $\mathrm{Si}_C\, A = \mathrm{Hom}_{\mathcal{A}}(C(-), A)$, i.e.,

$$(\mathrm{Si}\, A)_n = \mathrm{Hom}_{\mathcal{A}}(C^n, A),$$

with simplicial operators induced by the fact that C is a functor on Δ.

48.10. *Exercise.* Show that for $X \in s\mathrm{Set}$ we must have

$$\mathrm{Re}_C\, X \approx \operatorname{colim}\Bigg[\coprod_{f \colon [m]\to[n]} \coprod_{x \in X_n} C^m \rightrightarrows \coprod_{[p]} \coprod_{x \in X_p} C^p\Bigg].$$

(Part of the exercise is to figure out what the two maps are.)

48.11. *Example.* Let $\Delta^\bullet_{\mathrm{top}} \colon \Delta \to \mathrm{Top}$ be the functor taking $[n]$ to the topological n-simplex. Then $\mathrm{Re}_{\Delta_{\mathrm{top}}}$ and $\mathrm{Si}_{\Delta_{\mathrm{top}}}$ are geometric realization and singular complex respectively.

48.12. *Example.* Let $(*)^\bullet \colon \Delta \to \mathrm{Set}$ be the functor which sends each $[n]$ to the one-element set. Then $\mathrm{Re}_* \colon s\mathrm{Set} \to \mathrm{Set}$ is naturally isomorphic to π_0, and $\mathrm{Si}_* \colon \mathrm{Set} \to s\mathrm{Set}$ sends each set S to the discrete simplicial set S^{disc}.

48.13. *Example.* Let $\Delta^\bullet \colon \Delta \to s\mathrm{Set}$ be the Yoneda embedding, sending $[n] \mapsto \Delta^n$. Then Re_Δ and Si_Δ are both naturally isomorphic to the identity functor on $s\mathrm{Set}$.

48.14. *Example.* Let $(\Delta^\bullet)^{\mathrm{op}} \colon \Delta \to s\mathrm{Set}$ be the composite functor $\Delta^\bullet \circ \mathrm{op}$, using $\mathrm{op} \colon \Delta \to \Delta$. Then both $\mathrm{Re}_{\Delta^{\mathrm{op}}}$ and $\mathrm{Si}_{\Delta^{\mathrm{op}}}$ are both naturally isomorphic to the functor $(-)^{\mathrm{op}} \colon s\mathrm{Set} \to s\mathrm{Set}$ sending a simplicial set to its opposite.

A map $\eta \colon C^\bullet \to D^\bullet$ of cosimplicial objects $\Delta \to \mathcal{A}$ induces natural transformations

$$\mathrm{Re}_\eta \colon \mathrm{Re}_C \to \mathrm{Re}_D \quad \text{and} \quad \mathrm{Si}_\eta \colon \mathrm{Si}_D \to \mathrm{Si}_C\,.$$

The map $\mathrm{Si}_\eta A \colon \mathrm{Si}_D A \to \mathrm{Si}_C$ is given by $\mathrm{Hom}_{\mathcal{A}}(D^n, A) \to \mathrm{Hom}_{\mathcal{A}}(C^n, A)$ in each dimension.

. . . I'm starting over. Old stuff below. . . .

We can also connect any simplicial set to its opposite by a zig-zag of weak equivalences.

48.15. **Proposition.** *There exists a functor $\mathcal{G}' \colon s\mathrm{Set} \to s\mathrm{Set}$ together with maps*

$$X \xrightarrow{\alpha_*} \mathcal{G}'X \xleftarrow{\beta_*} X^{\mathrm{op}},$$

natural in the simplicial set X, such that for every X both α_* and β_* are weak equivalences.

48.16. **Singular and realization functors.** Given a category $\mathcal{A}$, a **cosimplicial object** in $\mathcal{A}$ is a functor $C \colon \Delta \to \mathcal{A}$, where Δ is the category of simplicial operators. Often I will write C^n instead of $C([n])$ for the values of this functor, and so write $C^\bullet$ for the whole functor. In many of our examples, $\mathcal{A} = s\mathrm{Set}$, in which case $C^\bullet$ is a "cosimplicial simplicial set".

Given any cosimplicial object $C^\bullet$ in a category $\mathcal{A}$ with all small colimits, we have induced functors

$$\mathrm{Re}_C \colon s\mathrm{Set} \rightleftarrows \mathcal{A} : \mathrm{Si}_C\,.$$

The **singular functor** Si_C is defined on an object $A \in \mathcal{A}$ by $\mathrm{Si}_C A = \mathrm{Hom}_{\mathcal{A}}(C(-), A)$, i.e.,

$$(\mathrm{Si}\, A)_n = \mathrm{Hom}_{\mathcal{A}}(C^n, A),$$

with simplicial operators induced by the fact that C is a functor on Δ.

48.17. *Exercise.* Show that for $X \in s\mathrm{Set}$ we must have

$$\mathrm{Re}_C X \approx \mathrm{colim}\left[\coprod_{f \colon [m] \to [n]} \coprod_{x \in X_n} C^m \rightrightarrows \coprod_{[p]} \coprod_{x \in X_p} C^p\right].$$

(Part of the exercise is to figure out what the two maps are.)

48.18. *Example.* Let $\Delta^\bullet_{\mathrm{top}} \colon \Delta \to \mathrm{Top}$ be the functor taking $[n]$ to the topological n-simplex. Then $\mathrm{Re}_{\Delta_{\mathrm{top}}}$ and $\mathrm{Si}_{\Delta_{\mathrm{top}}}$ are geometric realization and singular complex respectively.

48.19. *Example.* Let $(*)^\bullet \colon \Delta \to \mathrm{Set}$ be the functor which sends each $[n]$ to the one-element set. Then $\mathrm{Re}_* \colon s\mathrm{Set} \to \mathrm{Set}$ is naturally isomorphic to π_0, and $\mathrm{Si}_* \colon \mathrm{Set} \to s\mathrm{Set}$ sends each set S to the discrete simplicial set S^{disc}.

48.20. *Example.* Let $\Delta^\bullet \colon \Delta \to s\mathrm{Set}$ be the Yoneda embedding, sending $[n] \mapsto \Delta^n$. Then Re_Δ and Si_Δ are both naturally isomorphic to the identity functor on $s\mathrm{Set}$.

48.21. *Example.* Let $(\Delta^\bullet)^{\mathrm{op}} \colon \Delta \to s\mathrm{Set}$ be the composite functor $\Delta^\bullet \circ \mathrm{op}$, using $\mathrm{op} \colon \Delta \to \Delta$. Then both $\mathrm{Re}_{\Delta^{\mathrm{op}}}$ and $\mathrm{Si}_{\Delta^{\mathrm{op}}}$ are both naturally isomorphic to the functor $(-)^{\mathrm{op}} \colon s\mathrm{Set} \to s\mathrm{Set}$ sending a simplicial set to its opposite.

A map $\eta \colon C^\bullet \to D^\bullet$ of cosimplicial objects $\Delta \to \mathcal{A}$ induces natural transformations

$$\mathrm{Re}_\eta \colon \mathrm{Re}_C \to \mathrm{Re}_D \quad \text{and} \quad \mathrm{Si}_\eta \colon \mathrm{Si}_D \to \mathrm{Si}_C\,.$$

The map $\mathrm{Si}_\eta A \colon \mathrm{Si}_D A \to \mathrm{Si}_C$ is given by $\mathrm{Hom}_{\mathcal{A}}(D^n, A) \to \mathrm{Hom}_{\mathcal{A}}(C^n, A)$ in each dimension.

We construct the promised functor $\mathcal{G} \colon s\mathrm{Set} \to s\mathrm{Set}$ and natural transformations $(-) \leftarrow \mathcal{G}(-) \to (-)^{\mathrm{op}}$ via singular functors.

For an object $[n]$ of Δ, let Iso^n denote the ordinary category whose set of objects is $[n]$, and such that $\mathrm{Hom}_{\mathrm{Iso}^n}(x, y) \approx \{*\}$ for all pairs $x, y \in [n]$. For instance, Iso^1 is exactly the "walking isomorphism" Iso discussed earlier (**??**). For brevity I will also write Iso^n for the simplicial set which is its nerve. For all $n \geq 0$ we have functors

$$\Delta^n \xrightarrow{\alpha_n} \mathrm{Iso}^n \xleftarrow{\beta_n} (\Delta^n)^{\mathrm{op}},$$

which are uniquely determined by what they do on objects: $\alpha_n(x) = x = \beta_n(x)$ for $x \in [n]$. These fit together to define maps

$$\Delta^\bullet \xrightarrow{\alpha} \mathrm{Iso}^\bullet \xleftarrow{\beta} (\Delta^\bullet)^{\mathrm{op}}$$

of cosimplicial objects in $s\mathrm{Set}$.

We take the functor $\mathcal{G}$ of (48.2) to be $\mathrm{Si}_{\mathrm{Iso}}$. We take the functor $\mathcal{G}'$ of (48.15) to be $\mathrm{Re}_{\mathrm{Iso}}$. To prove it, we will need the following statements.

(1) For any Kan complex X, the object $\mathrm{Si}_{\mathrm{Iso}} X$ is a Kan complex.
(2) For any Kan complex X, the maps $\mathrm{Si}_\alpha X \colon \mathrm{Si}_{\mathrm{Iso}} X \to \mathrm{Si}_\Delta X \approx X$ and $\mathrm{Si}_\beta X \colon \mathrm{Si}_{\mathrm{Iso}} X \to \mathrm{Si}_{\Delta^{\mathrm{op}}} \approx X^{\mathrm{op}}$ are trivial fibrations.
(3) For any simplicial set X, the maps $\mathrm{Re}_\alpha \colon X \approx \mathrm{Re}_\Delta X \to \mathrm{Re}_{\mathrm{Iso}} X$ and $\mathrm{Re}_\beta \colon X^{\mathrm{op}} \approx \mathrm{Re}_{\Delta^{\mathrm{op}}} X \to \mathrm{Re}_{\mathrm{Iso}} X$ are weak equivalences.

Note that if C is a quasicategory, $\mathrm{Hom}(\mathrm{Iso}^n, C^{\mathrm{core}}) \approx \mathrm{Hom}(\mathrm{Iso}^n, C)$, and thus $\mathrm{Si}_{\mathrm{Iso}}(C^{\mathrm{core}}) \approx \mathrm{Si}_{\mathrm{Iso}} C$. With this we recover the full statement of (48.2).

48.22. **Singular and realization functors in lifting problems.** Fix a map $\eta \colon C^\bullet \to D^\bullet$ of cosimplicial objects $\Delta \to \mathcal{A}$. Given a map $f \colon K \to L$ of simplicial sets, we obtain a map

$$\mathrm{Re}^{\square}_\eta f := (\mathrm{Re}_\eta K, \mathrm{Re}_D f) \colon \mathrm{Re}_C L \cup_{\mathrm{Re}_C K} \mathrm{Re}_D K \to \mathrm{Re}_D L$$

in $\mathcal{A}$. Likewise, given a map $g \colon X \to Y$ in $\mathcal{A}$, we obtain a map

$$\mathrm{Si}^{\square}_\eta g := (\mathrm{Si}_\eta X, \mathrm{Si}_D g) \colon \mathrm{Si}_D X \to \mathrm{Si}_C X \times_{\mathrm{Si}_C Y} \mathrm{Si}_D Y.$$

of simplicial sets.

48.23. *Remark.* Given any natural transformation $\lambda \colon F \to G$ of functors, and map $f \colon X \to Y$, we get induced maps

$$F(Y) \cup_{F(X)} G(X) \to G(Y), \qquad F(X) \to F(Y) \times_{G(Y)} G(X).$$

These can be thought of as a variant of the "box" construction we've considered elsewhere (27.6), but associated to the "evaluation pairing" $\mathrm{Fun}(s\mathrm{Set}, s\mathrm{Set}) \times s\mathrm{Set} \to s\mathrm{Set}$ rather than a functor $s\mathrm{Set} \times s\mathrm{Set} \to s\mathrm{Set}$.

48.24. *Exercise* (Important). Show that for any $\eta \colon C^\bullet \to D^\bullet$ in $\mathrm{Fun}(\Delta, \mathcal{A})$, $f \colon K \to L$ in $s\mathrm{Set}$, and $g \colon X \to Y$ in $\mathcal{A}$, we have that $(\mathrm{Re}^{\square}_\eta f) \boxslash g$ if and only if $f \boxslash (\mathrm{Si}^{\square}_\eta g)$.

48.25. **Proposition.** *Let $S = \{s_i \colon A_i \to B_i\}$ be a set of maps in $s\mathrm{Set}$, and let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in $\mathcal{A}$. Consider a map $\eta \colon C^\bullet \to D^\bullet$ of cosimplicial objects in $\mathcal{A}$. If $\mathrm{Re}^{\square}_\eta s_i \in \mathcal{L}$ for every $s_i \in S$, then*

(1) *for all $i \colon K \to L$ in $\overline{S}$, we have that $\mathrm{Re}^{\square}_\eta i \colon \mathrm{Re}_C L \cup_{\mathrm{Re}_C K} \mathrm{Re}_D K \to \mathrm{Re}_D L$ is in $\mathcal{L}$, and*
(2) *for all $p \colon X \to Y$ in $\mathcal{R}$, we have that $\mathrm{Si}^{\square}_\eta p \colon \mathrm{Si}_D X \to \mathrm{Si}_C X \times_{\mathrm{Si}_C Y} \mathrm{Si}_D Y$ is in ${}^{\boxslash}S$.*

In particular, if $E^\bullet \colon \Delta \to \mathcal{A}$ is a cosimplicial object such that $\mathrm{Re}_E s_i \in \mathcal{L}$ for all $s_i \in S$, then $\mathrm{Re}_E(\overline{S}) \subseteq \mathcal{L}$ and $\mathrm{Si}_E(\mathcal{R}) \subseteq {}^{\boxslash}S$.

Proof. In view of (48.24) and the properties of the weak factorization systems $(\overline{S}, {}^{\boxslash}S)$ and $(\mathcal{L}, \mathcal{R})$, we have that $\mathrm{Re}^{\square}_\eta i \in \mathcal{L}$ iff $(\mathrm{Re}^{\square}_\eta i) \boxslash \mathcal{R}$ iff $i \boxslash (\mathrm{Si}^{\square}_\eta \mathcal{R})$. Thus, the class of i for which statement (1) holds is weakly saturated, and the claim follows. Likewise, $\mathrm{Si}^{\square}_\eta p \in {}^{\boxslash}S$ iff $S \boxslash (\mathrm{Si}^{\square}_\eta p)$ iff $(\mathrm{Re}^{\square}_\eta S) \boxslash p$, so statement (2) follows by what we have already proved.

The final statement corresponds to: the special case of (1) with $D^\bullet = E^\bullet$ and $C^\bullet$ the initial cosimplicial object, and the special case of (2) with $C^\bullet = E^\bullet$ and $D^\bullet$ the terminal cosimplicial object. $\square$

We want to apply this to the transformations $\Delta^\bullet \xrightarrow{\alpha} \mathrm{Iso}^\bullet \xleftarrow{\beta} (\Delta^\bullet)^{\mathrm{op}}$ of cosimplicial objects in $\mathcal{A} = s\mathrm{Set}$, with respect to the weak factorization system $(\overline{\mathrm{Cell}}, \mathrm{TrivFib})$.

It turns out that for maps of cosimplicial objects in sSet, there is a fantastic criterion for verifying that $\mathrm{Re}^{\square}_{\eta}\,\mathrm{Cell} \subseteq \overline{\mathrm{Cell}}$. Given a cosimplicial object $C^\bullet$, let $\mathrm{Ker}\, C := \lim\bigl[\langle 0\rangle, \langle 1\rangle \colon C^0 \rightrightarrows C^1\bigr]$, the equalizer of the pair of arrows.

48.26. **Proposition.** *Let $\eta\colon C^\bullet \to D^\bullet$ be a map of cosimplicial objects of sSet. Then $\mathrm{Re}^{\square}_{\eta}$ takes elements of* Cell *to monomorphisms if and only if (i) each $\eta^n\colon C^n \to D^n$ is a monomorphism and (ii) the induced map $\mathrm{Ker}\,\eta\colon \mathrm{Ker}\, C \to \mathrm{Ker}\, D$ is an isomorphism.*

I will prove this at the end of the section. We note now that the proposition clearly applies to both transformations $\Delta^\bullet \xrightarrow{\alpha} \mathrm{Iso}^\bullet \xleftarrow{\beta} (\Delta^{\mathrm{op}})^\bullet$, since $\mathrm{Ker}\,\Delta$, $\mathrm{Ker}\,\Delta^{\mathrm{op}}$, and $\mathrm{Ker}\,\mathrm{Iso}$, are all empty.

48.27. **Skeletal induction.** The next step is to show that we have weak equivalences $X \to \mathrm{Re}_{\mathrm{Iso}} X \leftarrow X^{\mathrm{op}}$. To do this, we will use the following strategy.

48.28. **Proposition** (Skeletal induction)**.** *Let $\mathcal{C}$ be a class of simplicial sets with the following properties.*

1. *If $X \in \mathcal{C}$, then every object isomorphic to X is in $\mathcal{C}$.*
2. *Every $\Delta^n \in \mathcal{C}$.*
3. *The class $\mathcal{C}$ is closed under* good colimits. *That is:*
   - (a) *any coproduct of objects of $\mathcal{C}$ is in $\mathcal{C}$;*
   - (b) *any pushout of a diagram $X_0 \leftarrow X_1 \to X_2$ of objects in $\mathcal{C}$ along a monomorphism $X_1 \to X_2$ is in $\mathcal{C}$;*
   - (c) *any colimit of a countable sequence $X_0 \to X_1 \to X_2 \to \cdots$ of objects in $\mathcal{C}$, such that each $X_k \to X_{k+1}$ is a monomorphism, is in $\mathcal{C}$.*

Then $\mathcal{C}$ is the class of all simplicial sets.

Proof. This is a straightforward consequence of the skeletal filtration (15.24). To show $X \in \mathcal{C}$, it suffices to show each $\mathrm{Sk}_n X \in \mathcal{C}$ by (3c). So we show that all n-skeleta are in $\mathcal{C}$ by induction on n, with base case $n = -1$ (the empty simplicial set), which is really a special case of (3a). Since $\mathrm{Sk}_{n-1} X \subseteq \mathrm{Sk}_n X$ is a pushout along a coproduct of maps $\partial\Delta^n = \mathrm{Sk}_{n-1}\Delta^n \to \Delta^n$, this follows using (2), (3a), (3b), and the inductive hypothesis, which tells us that $\partial\Delta^n \in \mathcal{C}$. $\square$

48.29. **Equivalences between realization functors.**

48.30. **Proposition.** *Let $\eta\colon C^\bullet \to D^\bullet$ be a map of cosimplicial objects in some model category $\mathcal{M}$. Suppose that*

- (a) $\mathrm{Re}_\eta X\colon \mathrm{Re}_C X \to \mathrm{Re}_D X$ *is a cofibration in $\mathcal{M}$ for every simplicial set X, and*
- (b) $\mathrm{Re}_\eta \Delta^n\colon \mathrm{Re}_C \Delta^n \to \mathrm{Re}_D \Delta^n$ *is an equivalence in $\mathcal{M}$ for all $n \geq 0$.*

Then $\mathrm{Re}_\eta X$ is a weak equivalence in $\mathcal{M}$ for every simplicial set X.

Proof. Let $\mathcal{C}$ be the class of all X such that $\mathrm{Re}_\eta X\colon \mathrm{Re}_C X \to \mathrm{Re}_D X$ is an equivalence in $\mathcal{M}$. We verify the hypotheses of skeletal filtration (15.24) for $\mathcal{C}$. Property (1) is obvious, while property (2) is (b).

Since Re_η is also colimit preserving, (a) implies that Re_η takes good colimit diagrams in sSet to good colimit diagrams in $\mathcal{M}$. Therefore, for every good colimit of functors $X\colon I \to s\mathrm{Set}$ such that every $X(i) \in \mathcal{C}$, the induced map $\mathrm{Re}_C \operatorname{colim} X(i) \to \mathrm{Re}_D \operatorname{colim} X(i)$ is a weak equivalence by (47.8), (47.12), (47.10). This proves property (3). $\square$

48.31. **Proposition.** *The maps* $X \xrightarrow{\mathrm{Re}_\alpha} \mathrm{Re}_{\mathrm{Iso}} X \xleftarrow{\mathrm{Re}_\beta} X^{\mathrm{op}}$ *are weak equivalences of simplicial sets for all simplicial sets* X.

Proof. We apply (48.30) to α and β The maps $\Delta^\bullet \xrightarrow{\alpha} \mathrm{Iso} \xleftarrow{\beta} (\Delta^\bullet)^{\mathrm{op}}$ satisfy the hypothesis of (48.26), and so by (48.25) both Re_α and Re_β take monomorphisms to monomorphisms. Thus condition (a) of (48.30) holds.

To prove property (b) for α, recall that $\mathrm{Re}_\alpha \Delta^1 \colon \Delta^1 \to \mathrm{Iso}^1$ is anodyne (36.16). We can use this to show that $\mathrm{Re}_\alpha \Delta^n$ is anodyne for all $n \geq 0$, and thus a weak equivalence. In fact, we have maps

$$\Delta^n \xrightarrow{s} (\Delta^1)^{\times n} \xrightarrow{r} \Delta^n \qquad \text{and} \qquad \mathrm{Iso}^n \xrightarrow{s} (\mathrm{Iso}^1)^{\times n} \xrightarrow{r} \mathrm{Iso}^n,$$

which in either case are the unique maps which on vertices send

$$s(k) = (\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{n-k}), \qquad r(k_1, \dots, k_n) = \max\{\, j \mid k_j = 1 \,\}.$$

Clearly $rs = \mathrm{id}$ in both cases, and we easily see that $\mathrm{Re}_\alpha \Delta^n$ is a retract of $(\mathrm{Re}_\alpha \Delta^1)^{\times n}$, which is anodyne since the product of anodyne maps is anodyne. Since anodyne maps are always weak equivalences (36.11), this gives what we want.

The case of (b) for β is proved by a similar argument, or by noticing that $\mathrm{Re}_\beta \Delta^n \colon (\Delta^n)^{\mathrm{op}} \to \mathrm{Iso}^n$ is isomorphic to $(\mathrm{Re}_\alpha \Delta^n)^{\mathrm{op}} \colon (\Delta^n)^{\mathrm{op}} \to (\mathrm{Iso}^n)^{\mathrm{op}}$, and noting that the opposite of an anodyne map is anodyne since $\mathrm{Horn}^{\mathrm{op}} = \mathrm{Horn}$. □

We thus obtain the desired result.

48.32. **Corollary.** *Every simplicial set is weakly equivalent to its opposite* X^{op}.

Proof. Both maps in $X \xrightarrow{\mathrm{Re}_\alpha X} \mathrm{Re}\, X \approx \mathrm{Re}\, X^{\mathrm{op}} \xleftarrow{\mathrm{Re}_\beta X} X^{\mathrm{op}}$ are weak equivalences (48.31). □

48.33. **Proposition.** *For each monomorphism* $i \colon K \to L$, *the maps* $\mathrm{Re}^{\square}_\alpha i \colon L \cup_K (\mathrm{Re}_{\mathrm{Iso}} K) \to \mathrm{Re}_{\mathrm{Iso}} L$ *and* $\mathrm{Re}^{\square}_\beta i \colon L^{\mathrm{op}} \cup_{K^{\mathrm{op}}} (\mathrm{Re}_{\mathrm{Iso}} K) \to \mathrm{Re}_{\mathrm{Iso}} L$ *are monomorphisms and weak equivalences.*

Proof. We already have that $\mathrm{Re}^{\square}_\alpha$ and $\mathrm{Re}^{\square}_\beta$ are monomorphisms from (48.26). This also gives that $\mathrm{Re}_\alpha K$ is a monomorphism. Therefore

$$\begin{tikzcd} K \arrow[r, "\mathrm{Re}_\alpha"] \arrow[d] & \mathrm{Re}_{\mathrm{Iso}} K \arrow[d] \\ L \arrow[r] & L \cup_K (\mathrm{Re}_{\mathrm{Iso}} K) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Re}_{\mathrm{Iso}} K \arrow[r, "\mathrm{id}"] \arrow[d] & \mathrm{Re}_{\mathrm{Iso}} K \arrow[d] \\ \mathrm{Re}_{\mathrm{Iso}} L \arrow[r, "\mathrm{id}"] & \mathrm{Re}_{\mathrm{Iso}} L \end{tikzcd}$$

are good pushouts. The evident map from the left square to the right square is a weak equivalence at the upper left ($\mathrm{Re}_\alpha K$), upper right ($\mathrm{id}_{\mathrm{Re}_{\mathrm{Iso}} K}$), and lower left corners ($\mathrm{Re}_\alpha L$) by (48.31), so the result follows from the invariance of good pushouts (47.12). A similar argument applies with β in place of α. □

48.34. **Corollary.** *If* $p \colon X \to Y$ *is a Kan fibration between Kan complexes, then* $\mathrm{Si}^{\square}_\alpha p$ *and* $\mathrm{Si}^{\square}_\beta p$ *are trivial fibrations. In particular, if* X *is a Kan complex, then* $X \xleftarrow{\mathrm{Si}_\alpha} \mathrm{Si}_{\mathrm{Iso}} X \xrightarrow{\mathrm{Si}_\beta} X$ *are trivial fibrations between Kan complexes.*

Proof. Let $p \colon X \to Y$ be a Kan fibration between Kan complexes, and $i \colon K \to Y$ a monomorphism. By (48.33) we have that $\mathrm{Re}^{\square}_\alpha i$ and $\mathrm{Re}^{\square}_\beta i$ are monomorphisms and weak equivalences, and therefore $\mathrm{Re}^{\square}_\alpha i \boxslash p$ and $\mathrm{Re}^{\square}_\beta i \boxslash p$ by (**??**), whence $i \boxslash \mathrm{Si}^{\square}_\alpha p$ and $i \boxslash \mathrm{Si}^{\square}_\beta p$.

□

48.35. **Cosimplicial sets.** Let $C^\bullet\colon \Delta \to \mathrm{Set}$ be a cosimplicial object in sets, i.e., a **cosimplicial set**. Say that an element $x \in C^n$ is **codegenerate** if it is in the image of the map $C_k \to C_n$ induced by some *non-surjective* simplicial operator $f\colon [k] \to [n]$. If this is not the case we say x is **non-codegenerate**. Note that since any non-surjective $f\colon [k] \to [n]$ factors as $f = f^{\mathrm{inj}} f^{\mathrm{surj}}$ with f^{inj} not an idenity map, an $x \in C^n$ is codegenerate if and only if x is in the image of some *injective non-identity* simplicial operator.

Note that all elements of C^0 are necessarily non-codegenerate.

48.36. **Lemma.** *For a cosimplicial set $C^\bullet\colon \Delta \to \mathrm{Set}$ and an element $x \in C^n$. Consider the set S of all triples $(k,\, f\colon [k] \to [n],\, y \in C^k)$ such that (i) $fy = x$, (ii) f is injective, and (iii) y is non-codegenerate. Then*

1. *The set S is non-empty.*
2. *for all $(k, f, y), (k', f', y') \in S$, we have that $k = k'$ and $y = y'$. Furthermore*
3. *either $f = f'$, or $k = k' = 0$ and $y \in \operatorname{Ker} C$.*

Proof. To see that S is non-empty, note that there certainly exist (k, f, y) such that (i) and (ii) hold, e.g., $(n, 1_{[n]}, x)$. Any such (k, f, y) with k minimal must also satisfy (iii).

Given $(f\colon [k] \to [n]) \in \Delta^{\mathrm{inj}}$, let $R(f) = \left\{ r\colon [n] \to [k] \;\middle|\; rf = \mathrm{id}_{[k]} \right\}$, the set of retractions of f. Note that $R(f)$ is always non-empty.

Suppose (k, f, y), $(k', f,' y') \in S$ such that $fy = x = f'y'$. Choose arbitrary $r \in R(f)$ and $r' \in R(f')$. Then $y = rfy = rf'y'$ and $y' = r'f'y' = r'fy$. Since y and y' are non-codegenerate, both $r'f\colon [k] \to [k']$ and $rf'\colon [k'] \to [k]$ must be surjective, whence $k = k'$ and thus $r'f = \mathrm{id} = rf'$ and so $y = y'$.

We also have $r' \in R(f)$ and $r \in R(f')$. Since the retractions were arbitrary, we see that $R(f) = R(f')$. When $k \geq 1$, it is easy to see that this implies that $f = f'$; however, this is not the case when $k = 0$.

Suppose $k = 0$. We have $f = \langle i \rangle$ and $f' = \langle j \rangle$ for some $i, j \in [n]$; without loss of generality we can assume $i \leq j$, by switching f and f' if necessary. If $f \neq f'$ then we can choose $g \in R(\langle ij \rangle\colon [1] \to [n])$, whence $gf = \langle 0 \rangle$ and $gf' = \langle 1 \rangle$ as maps $[0] \to [1]$. Then

$$\langle 0 \rangle y = gfy = gx = gf'y = \langle 1 \rangle y,$$

i.e., $y \in \operatorname{Ker} C$. ☐

48.37. **Lemma.** *Let $F\colon s\mathrm{Set} \to \mathrm{Set}$ be a colimit preserving functor such that*

$$\lim\bigl[(\langle 0 \rangle, \langle 1 \rangle)\colon F(\Delta^0) \rightrightarrows F(\Delta^1)\bigr] = \varnothing.$$

Then the map $F(\partial\Delta^n) \to F(\Delta^n) = C^n$ obtained by applying F to the cell inclusion $\partial\Delta^n \subset \Delta^n$ is a monomorphism, with image precisely the codegenerate elements of C^n.

Proof. Let $\mathcal{A}$ be the poset of non-empty proper subsets of $[n]$. For each $S \in \mathcal{A}$ the inclusion $S \subset [n]$ admits an order preserving retraction, whence the induced map $F(\Delta^S) \to F(\Delta^n)$ admits a retraction and thus is injective.

For each $S \in \mathcal{A}$ let $C^S \subseteq C^n$ denote the image of the evident map $F(\Delta^S) \to F(\Delta^n)$. It is clear that $C^{\mathrm{codeg}} = \bigcup_{S \in \mathcal{A}} C^S$ is precisely the subset of codegenerate elements.

We have a commutative diagram

$$\begin{tikzcd}
\operatorname{colim}_{S\in\mathcal{A}} F(\Delta^S) \arrow[r, "\sim"] \arrow[d, "\sim"'] & F(\partial\Delta^n) \arrow[r] \arrow[d] & F(\Delta^n) \arrow[d, equal] \\
\operatorname{colim}_{S\in\mathcal{A}} C^S \arrow[r, "\pi"'] & C^{\mathrm{codeg}} \arrow[r, tail] & C^n
\end{tikzcd}$$

in which the indicated maps are bijections. The top left map is a bijection because F preserves colimits and $\partial\Delta^n \approx \operatorname{colim}_{S\in\mathcal{A}} \Delta^S$. Thus to prove the claim it suffices to show that π is a bijection.

By (**??**) it suffices to show that if $x \in C^S \cap C^{S'}$, then there exists $T \subseteq S \cap S'$ such that $x \in C^T$. This follows from the previous proposition. $\square$

48.38. **Proposition.** *Let $\gamma\colon F \to F'$ be a natural transformation of colimit preserving functors $s\mathrm{Set} \to \mathrm{Set}$. Suppose that*

- $\lim\big[(\langle 0\rangle, \langle 1\rangle)\colon F(\Delta^0) \rightrightarrows F(\Delta^1)\big] = \varnothing = \lim\big[(\langle 0\rangle, \langle 1\rangle)\colon F'(\Delta^0) \rightrightarrows F'(\Delta^1)\big]$ *and*
- *for each $n \geq 0$ the map $\gamma(\Delta^n)\colon F(\Delta^n) \to F'(\Delta^n)$ is injective.*

Then $\gamma_\square(i)$ is a monomorphism for all monomorphisms i of simplicial sets.

Older material below.

48.39. *Remark.* Here is one possible proof (in some sense, the most natural proof). Note that there is a *homeomorphism* of geometric realizations $\|X\| \approx \|X^{\mathrm{op}}\|$. Then use the fact that geometric realization induces an equivalence $h(s\mathrm{Set}, \mathrm{WkEq}) \approx h(\mathrm{Top}, \mathrm{WkEq})$. Of course, we haven't actually proved this fact about homotopy categories yet.

Although $\Delta^\bullet$ and $\Delta^\bullet \circ \mathrm{op}$ are not isomorphic as functors $\Delta \to s\mathrm{Set}$, it is the case that $\mathrm{Iso}^\bullet \approx \mathrm{Iso}^\bullet \circ \mathrm{op}$, using the isomorphisms of categories $\mathrm{Iso}^n \to \mathrm{Iso}^n$ given on objects by $x \mapsto n - x$. Putting all this together, we obtain natural transformations

$$X \xrightarrow{\eta_X} \mathrm{Re}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Re}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xleftarrow{\eta_{X^{\mathrm{op}}}} X^{\mathrm{op}}, \qquad X \xleftarrow{\epsilon_X} \mathrm{Si}_{\mathrm{Iso}^\bullet} X \approx \mathrm{Si}_{\mathrm{Iso}^\bullet} X^{\mathrm{op}} \xrightarrow{\epsilon_{X^{\mathrm{op}}}} X^{\mathrm{op}}.$$

We'll show that that η_X, and hence $\eta_{X^{\mathrm{op}}}$, are always weak equivalences, and that ϵ_X, and hence $\epsilon_{X^{\mathrm{op}}}$, are weak equivalences whenever X is a Kan complex. In the following, $\mathrm{Re} = \mathrm{Re}_{\mathrm{Iso}^\bullet}$ and $\mathrm{Si} = \mathrm{Si}_{\mathrm{Iso}^\bullet}$.

48.40. **Lemma.** *For each monomorphism $K \to L$, the induced map $(\mathrm{Re}\, K) \amalg_K L \to \mathrm{Re}\, L$ is a monomorphism. In particular,*

- Re *preserves monomorphisms and* Si *preserves trivial fibrations, and*
- $\eta_L\colon L \to \mathrm{Re}\, L$ *is always a monomorphism.*

Proof. Formally, it is enough to check the case of $\partial\Delta^n \subset \Delta^n$. To see this, check that the lifting problems

$$\begin{tikzcd} (\mathrm{Re}\, K) \cup_K L \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Re}\, L \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \iff \begin{tikzcd} K \arrow[r] \arrow[d, hook] & \mathrm{Si}\, X \arrow[d] \\ L \arrow[r] \arrow[ur, dotted] & (\mathrm{Si}\, Y) \times_Y X \end{tikzcd}$$

are equivalent. This means we need to show that all monomorphisms are contained in the weakly saturated class ${}^{\boxslash}\mathcal{C}$, where $\mathcal{C}$ is the class of all the maps $(\mathrm{Si}\, p, \epsilon_X)\colon \mathrm{Si}\, X \to (\mathrm{Si}\, Y) \times_Y X$ such that $p \in \mathrm{TrivFib}$, which means we only need to show that Cell is contained in it.

This is a calculation: $\mathrm{Re}(\partial\Delta^n) \to \mathrm{Re}(\Delta^n) = N(\mathrm{Iso}^n)$ is isomorphic to inclusion of the subcomplex $K \subseteq N(\mathrm{Iso}^n)$ whose k-dimensional elements are sequences $x_0 \to \cdots \to x_k$ such that $\{x_0, \dots, x_k\} \neq \{0, \dots, n\}$. To show this, use the fact that Δ^n is a colimit of its $(n-1)$-dimensional faces along their intersections, and that Re preserves colimits. The image of the element $(0, 1, \dots, n)$ in $\mathrm{Re}(\Delta^n)$ intersects K exactly in its boundary, so $(\mathrm{Re}\,\partial\Delta^n) \cup_{\partial\Delta^n} \Delta^n \to \mathrm{Re}\,\Delta^n$ is a monomorphism as desired. $\square$

49. Initial and terminal objects, revisited

We defined initial and terminal objects in a quasicategory in terms of a certain lifting property (26.1). Thesed definitions are manifestly equivalent to the following characterizations: x is an initial object of C iff the left fibration $\pi\colon C_{x/} \to C$ is a trivial fibration, and a terminal object iff the right fibration $\pi'\colon C_{/x} \to C$ is a trival fibration (26.3). Later we observed that we could replace this with: π or π' a categorical equivalence (41.6).

When C is the nerve of an ordinary category, these reduce to the usual definitions of initial and terminal object. In this case, there is an equivalent characterization: x is initial if and only if $\mathrm{Hom}_C(x, y)$ is a singleton set for all objects y of C, and terminal if and only if $\mathrm{Hom}_C(y, x)$ is a singleton set for all y.

We would like to generalize this to the case of quasicategories.

H. **Deferred Proposition.** *An object x of a quasicategory is initial if and only if $\mathrm{map}_C(x, c)$ is contractible for all objects c of C, and terminal if and only if $\mathrm{map}_C(c, x)$ is contractible for all objects c of C.*

To prove this, you need to be able to relate mapping spaces of a quasicategory to the join/slice constructions that we used to define initial and terminal. We will establish such a relation in the next few sections.

49.1. **Right and left mapping spaces.** Let x, y be objects of a quasicategory C. We define the **right mapping space** $\mathrm{map}^R_C(x, y)$ and **left mapping space** $\mathrm{map}^L_C(x, y)$ by pullback diagrams

$$\begin{tikzcd} \mathrm{map}^R_C(x,y) \arrow[r] \arrow[d] & C_{x/} \arrow[d, "\pi"] \\ \Delta^0 \arrow[r, "y"'] & C \end{tikzcd} \qquad \begin{tikzcd} \mathrm{map}^L_C(x,y) \arrow[r] \arrow[d] & C_{/y} \arrow[d, "\pi"] \\ \Delta^0 \arrow[r, "x"'] & C \end{tikzcd}$$

where the maps labelled π are the evident forgetful maps.

For instance, an n-dimensional element of $\mathrm{map}^R_C(x, y)$ is precisely a map $a\colon \Delta^{n+1} \to C$ such that $a|\Delta^{\{0,\dots,n\}}$ represents the vertex x, and $a|\{n + 1\} = y$. In particular, a vertex of $\mathrm{map}^R_C(x, y)$ is a morphism $x \to y$ in C, while an edge of $\mathrm{map}^R_C(x, y)$ is a 2-dimensional element in C exhibiting the $\sim_r$ relation between two maps, which we used to define the homotopy category in §9.

Recall (27.15) that when C is a quasicategory, the maps $C_{x/} \to C$ and $C_{/y} \to C$ are left fibrations and right fibrations respectively. Thus both $\mathrm{map}^R_C(x, y)$ and $\mathrm{map}^L_C(x, y)$ are Kan complexes, using the following.

49.2. *Exercise.* Show that if $X \to \Delta^0$ is a left fibration or a right fibration, then X is a Kan complex. (Hint: Joyal lifting.)

Furthermore, by the above remarks relating edges in the right and left mapping spaces to the homotopy relation, we have bijections

$$\pi_0 \mathrm{map}^R_C(x, y) \approx \pi_0 \mathrm{map}^L_C(x, y) \approx \mathrm{Hom}_{hC}(x, y) \approx \pi_0 \mathrm{map}_C(x, y).$$

We will show below that both $\mathrm{map}^R_C(x, y)$ and $\mathrm{map}^L_C(x, y)$ are actually *weakly equivalent* to the standard mapping space $\mathrm{map}_C(x, y)$. In the meantime, we characterize initial objects in terms of contractibility of map^L_C, and terminal objects in terms of contractibility of map^R_C.

49.3. **Box products and right and left anodyne maps.** Recall that $\overline{\mathrm{InnHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$ (16.8) and $\overline{\mathrm{Horn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$. We have an analogous fact for left or right anodyne maps.

49.4. **Proposition.** *We have that $\overline{\mathrm{LHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$ and $\overline{\mathrm{RHorn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{RHorn}}$.*

Proof. This is a calculation. See the appendix (62). $\square$

Note that the above proposition actually implies $\overline{\mathrm{Horn}}\Box\overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Horn}}$.

49.5. **Fiberwise criterion for trivial fibrations, revisited.** Recall the fiberwise criterion for trivial fibrations (38.2): a Kan fibration p is a trivial fibration if and only if the fibers of p are contractible Kan complexes. In fact, this still holds if we only know p is a left or right fibration.

49.6. **Proposition.** *Suppose $p\colon X \to Y$ is a right fibration or left fibration of simplicial sets. Then p is a trivial fibration if and only if it has contractible fibers.*

Proof. [Lur09, 2.1.3.4]. Let's consider the case of $p\colon X \to Y$ a left fibration. The direction ($\Longrightarrow$) is immediate, so we only need to prove ($\Longleftarrow$) .

We attempt to carry out the argument of the proof of (38.2), and show that $(\partial\Delta^n \subset \Delta^n) \boxslash p$ for all $n \geq 0$. The case of $n = 0$ is immediate, since the fibers of p must be non-empty, since they are contractible, so we can assume $n \geq 1$.

The proof of (38.2) works by "deforming" a general lifting problem of type $(\partial\Delta^n \subset \Delta^n) \boxslash p$ to one which lives in a single fiber of p. In that proof we used the hypothesis that p is a Kan fibration in two places:

(a) To construct the desired "deformation" of a lifting problem (a, b) of type $i \boxslash p$ to one contained in a fiber, by solving a particular lifting problem of type
$$\big((\partial\Delta^n \times \{0\}) \subset (\partial\Delta^n \times \Delta^1)\big) \boxslash p.$$

(b) To apply the covering homotopy extension property (36.22), so that a lifting in the deformed problem (a', b') implies a lifting in the original problem

For case (a), the inclusion $(\partial\Delta^n \times \{0\}) \subset (\partial\Delta^n \times \Delta^1)$ is left anodyne by (49.4), so the lifting problem still has a solution when p is merely a left fibration.

For case (b), the covering homotopy exension property in the form we need doesn't apply if p is merely a left fibration. So we use a different argument which gives what we need.

From the argument of (a), we have an edge we have constructed an edge $e = (u, b\gamma)$ in $\mathrm{Map}(\partial\Delta^n, X) \times_{\mathrm{Map}(\partial\Delta^n, Y)} \mathrm{Map}(\Delta^n, X)$ so that the vertex $e_0 = (a, b)$ is the original lifting problem. The map $\gamma\colon \Delta^n \times \Delta^1 \to \Delta^n$ sends $\gamma(\Delta^n \times \{1\}) = \{n\}$, so the vertex $e_1 = (a', b')$ has $b' = b\gamma$ so that $b'(\Delta^n) \subseteq \{b(n)\} \subseteq Y$. Thus the lifting problem (a', b') factors through the fiber over $b(n) \in Y_0$, and so admits a solution $t\colon \Delta^n \to X$ by the hypothesis that the fibers of p are contractible.

Thus we have a solid arrow commutative diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \arrow[r, "{(u,t)}"] \arrow[d, hook] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "b\gamma"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

If we can produce a lift s, then the restriction $s|\Delta^n \times \{0\}$ is the desired solution to the original lifting problem (a, b).

Form the diagram

$$\begin{tikzcd}
(\partial\Delta^n \times \Delta^1) \cup_{\partial\Delta^n \times \{1\}} (\Delta^n \times \{1\}) \arrow[r, "g"] \arrow[d, hook] \arrow[rr, bend left, "{(u,t)}"] & C \arrow[r, "h"'] \arrow[d, "p'"'] & X \arrow[d, "p"] \\
\Delta^n \times \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dotted, "s'"'] & \Delta^n \arrow[r, "b"'] & Y
\end{tikzcd}$$

where the right-hand square is a pullback. Observe that (i) p' is a left fibration, and hence an inner fibration, between quasicategories, and that (ii) γ sends the edge $\{n\} \times \Delta^1$ to the degenerate edge $\langle nn\rangle$ in Δ^n. Therefore g sends the edge $\{n\} \times \Delta^1$ into the fiber of p' over $n \in (\Delta^n)_0$, which is isomorphic to the fiber of p over $b(n)$, which is by hypothesis a contractible Kan fibration. Thus, $g|\{n\} \times \Delta^1$ represents an isomorphism in the quasicategory C. Therefore the pushout-product version of Joyal lifting (31.6) gives a lift s', and so $s := hs'$ is the desired lift. □

49.7. **Corollary.** *An object x of a quasicategory C is initial if and only if $\mathrm{map}^R_C(x, c)$ is contractible for all objects c of C, and is final if and only if $\mathrm{map}^L_C(c, x)$ is contractible for all objects c of C.*

Proof. The fibers of the left fibration $C_{x/} \to C$ are precisely the right mapping spaces $\mathrm{map}^R_C(x, c)$. By what we just proved (49.6) these fibers are all contractible if and only if $C_{x/} \to C$ is a trivial fibration, which we have noted (26.3) is equivalent to x being initial in C. □

50. The alternate join and slice

We now want to compare the right and left mapping spaces, which are fibers of the projections $C_{x/} \to C$ and $C_{/x} \to C$, to the ordinary mapping spaces, which are fibers of $\mathrm{Fun}(\Delta^1, C) \to \mathrm{Fun}(\partial\Delta^1, C)$. We do this using constructions called the "alternate join" and "alternate slice" [Lur09, §4.2.1].

50.1. **Some alternate versions of the slice construction.** Consider a morphism $f\colon K \to C$ to a quasicategory. An object of the slice category $C_{f/}$ is a morphism $f'\colon K^{\triangleright} \to C$ such that $f'|K = f$. When C is an ordinary category, then you can show that morphisms $\alpha\colon f' \to f''$ in the slice $C_{f/}$ are in bijective correspondence with natural transformations of functors $K^{\triangleright} \to C$ such that $\alpha|K$ is the identity map of f. However, this is not generally the case for an arbitrary quasicategory C.

Given $f\colon K \to C$, we can instead consider

$$C^{f/}_{\mathrm{semi}} := \mathrm{Fun}(K^{\triangleright}, C) \times_{\mathrm{Fun}(K,C)} \{f\},$$

the fiber of the restriction map $\mathrm{Fun}(K^{\triangleright}, C) \to \mathrm{Fun}(K, C)$ over the vertex f. This is a quasicategory by (16.8). I will call this the **semi-alternate slice** under f. Note that the objects of $C^{f/}_{\mathrm{semi}}$ are in bijective correspondence with those of $C_{f/}$, and that the morphisms of $C^{f/}$ are in fact natural transformations $\alpha\colon f' \to f''$ such that $\alpha|K$ is the identity of f. We can similarly form

$$C^{/f}_{\mathrm{semi}} := \mathrm{Fun}(K^{\triangleleft}, C) \times_{\mathrm{Fun}(K,C)} \{f\}.$$

Another possibility is to consider

$$C^{f/} := \mathrm{Fun}((K \times \Delta^1)/(K \times \{1\}), C) \times_{\mathrm{Fun}(K\times\{0\},C)} \{f\},$$

where $(K \times \Delta^1)/(K \times \{1\})$ is the pushout of $K \times \Delta^1 \hookleftarrow K \times \{1\} \to *$, and we use restriction along the evident inclusion $K \times \{0\} \to K \times \Delta^1 \to K \times \Delta^1/K \times \{1\}$. This is also a quasicategory, and is called the **alternate slice** under f. We can similarly form

$$C^{/f} := \mathrm{Fun}((K \times \Delta^1)/(K \times \{0\}), C) \times_{\mathrm{Fun}(K\times\{1\},C)} \{f\}.$$

The quotients $(K \times \Delta^1)/(K \times \{1\})$ and $(K \times \Delta^1)/(K \times \{0\})$ may themselves be regarded as alternate versions of the cones $K^{\triangleright}$ and $K^{\triangleleft}$.

What we will show is that for any map $f\colon K \to C$ to a quasicategory there are natural commutive diagrams

$$\begin{tikzcd} C_{f/} \arrow[r] \arrow[dr, "p"'] & C^{f/}_{\mathrm{semi}} \arrow[r] \arrow[d, "p'"] & C^{f/} \arrow[dl, "p''"] \\ & C & \end{tikzcd} \qquad \begin{tikzcd} C_{/f} \arrow[r] \arrow[dr, "q"'] & C^{/f}_{\mathrm{semi}} \arrow[r] \arrow[d, "q'"] & C^{/f} \arrow[dl, "q''"] \\ & C & \end{tikzcd}$$

where the maps p', p'', q', and q'' are induced by restriction along the inclusion of a "cone point" (e.g., the cone points of $K^{\triangleright}$ or $K^{\triangleleft}$, or the collapsed point of $K \times \Delta^1/K \times \{1\}$ or $K \times \Delta^1/K \times \{0\}$. Furthermore, we will have that

(1) the horizontal maps are all categorical equivalences, and
(2) the maps p, p', and p'' are left fibrations, and the maps q, q', and q'' are right fibrations. Hence all are isofibrations (29.10).

...

Given an object x in C, consider the map

$$q\colon \operatorname{Fun}(\Delta^1, C) \times_{\operatorname{Fun}(\{0\},C)} \{x\} \to \operatorname{Fun}(\{1\}, C) = C$$

induced by restriction along $\{0\} \subset \Delta^1$. Note that the fiber of q over some object c of C is precisely the quasigroupoid $\operatorname{map}_C(x, c)$. The domain of q is an example of what we will call the "alternate slice" construction, for which we (following Lurie) will use the (unmemorable) notation $C^{x/}$.

50.2. *Exercise.* Show that if C is an ordinary category, then $C^{x/}$ is *isomorphic* to the usual slice category $C_{x/}$, and q is *isomorphic* to the usual projection $p\colon C_{x/} \to C$.

For a general quasicategory, q is *not* isomorphic p. What is true is that there is a commutative diagram

$$\begin{tikzcd} C_{x/} \arrow[rr, "f"] \arrow[dr, "p"'] & & C^{x/} = \operatorname{Fun}(\Delta^1, C) \times_{\operatorname{Fun}(\{0\},C)} \{x\} \arrow[dl, "q"] \\ & C & \end{tikzcd}$$

The map f sends an element $a\colon \Delta^k \to C_{/x}$, which corresponds to $\widetilde{a}\colon \Delta^{k+1} \to C$ such that $\widetilde{a}_0 = x$, to an element in $C^{x/}$ corresponding to $\widetilde{a}r\colon \Delta^k \times \Delta^1 \to C$, where $r\colon \Delta^k \times \Delta^1 \to \Delta^{k+1}$ is the unique map given on vertices by $r(i, 0) = 0$, $r(i, 1) = i + 1$.

The characterization (H) of initial objects in terms of contractible mapping spaces thus amounts to the claim that p is a trivial fibration if and only if q has contractible fibers. In fact, we'll prove that

- both p and q are left fibrations,
- f is a categorical equivalence.

Because p and q are left fibrations, they are trivial fibrations iff their fibers are contractible (49.6). Because f is a categorical equivalence, p is a categorical equivalence if and only if q is by 2-out-of-3 (22.10). The result follows because p and q are in particular isofibrations (29.10), and an isofibration is a trivial fibration if and only if it is a categorical equivalence (41.1).

In other words, we can regard $C^{x/}$ as an alternate version of the slice construction, so we call it the "alternate slice". It is related to an alternate version of the join, denoted $X \diamond Y$ and called the "alternate join", which we define first.

50.3. **The alternate join.** Given simplicial sets X and Y, define the **alternate join** by the pushout diagram

$$\begin{tikzcd} (X \times \{0\} \times Y) \amalg (X \times \{1\} \times Y) \arrow[r, tail] \arrow[d] & X \times \Delta^1 \times Y \arrow[d] \\ (X \times \{0\} \times \Delta^0) \amalg (\Delta^0 \times \{1\} \times Y) \arrow[r, tail] & X \diamond Y \end{tikzcd}$$

where the maps on top and left are induced by the evident inclusion and projection maps.

The alternate join comes with a natural comparison map

$$X \diamond Y \to X \star Y,$$

defined as follows. Using the recipe of (23.14) for constructing maps to a join, we get a map $X \times \Delta^1 \times Y \to X \star Y$ corresponding to the triple $(g, f^{(0)}, f^{(1)})$, where $g\colon X \times \Delta^1 \times Y \to \Delta^1$, $f^{(0)}\colon X \times \{0\} \times Y \to X$, and $f^{(1)}\colon X \times \{1\} \times Y \to Y$ are the evident projections. A similar procedure produces compatible maps to $X \star Y$ from the other vertices of the pushout square defining $X \diamond Y$. Note that the comparison map induces a bijection on vertices.

50.4. *Example.* We have

$$X \diamond \Delta^0 \approx (X \times \Delta^1)/(X \times \{1\}), \qquad \Delta^0 \diamond Y \approx (\Delta^1 \times Y)/(\{0\} \times Y),$$

simplical sets obtained by collapsing subcomplexes to a single point. These come with evident maps $X \diamond \Delta^0 \to X^{\triangleright}$ and $\Delta^0 \diamond Y \to Y^{\triangleleft}$.

Like the true join, $X \diamond \varnothing \approx X \approx \varnothing \diamond X$, and the functors $X \diamond -\colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ and $- \diamond Y\colon s\mathrm{Set} \to s\mathrm{Set}_{Y/}$ commute with colimits.

50.5. *Warning.* When X and Y are non-empty, $X \times \Delta^1 \times Y \to X \diamond Y$ is surjective, but this is not the case when either X or Y are empty.

50.6. *Exercise.* Show that for $p, q \geq 0$, the composite

$$\Delta^p \times \Delta^1 \times \Delta^q \twoheadrightarrow \Delta^p \diamond \Delta^q \to \Delta^p \star \Delta^q \xrightarrow{\sim} \Delta^{p+1+q}$$

is the unique map which is given on vertices by

$$(x, t, y) \;\mapsto\; (1-t)x + t(p+1+y), \qquad x \in [p],\, t \in [1],\, y \in [q].$$

Unlike the true join, the alternate join is not monoidal: $(X \diamond Y) \diamond Z \not\approx X \diamond (Y \diamond Z)$ in general. Also, the alternate join of two quasicategories is not usually a quasicategory.

The alternate join is a categorically invariant construction.

50.7. **Proposition.** *The alternate join $\diamond$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \diamond Y \to X \diamond Y'$ and $Y \diamond Z \to Y' \diamond Z$.*

Proof. The $\diamond$ product is constructed using a “good” pushout, i.e., a pushout along a cofibration (=monomorphism). The result follows because both products and good pushouts preserve categorical equivalences (47.12). $\square$

50.8. **Equivalence of join and alternate join.** The key result of this section is the following.

50.9. **Proposition.** *The canonical comparison map $X \diamond Y \to X \star Y$ is a categorical equivalence for all simplicial sets X and Y.*

This will imply the categorical invariance of the usual join.

50.10. **Corollary.** *The join $\star$ preserves categorical equivalences in either variable. That is, if $Y \to Y'$ is a categorical equivalence, then so are $X \star Y \to X \star Y'$ and $Y \star Z \to Y' \star Z$.*

Proof. Immediate using (50.9), the invariance of the alternate join under categorical equivalence (50.7), and the 2-out-of-3 property of categorical equivalences (22.10). $\square$

The proof is based on the following general strategy.

50.11. **Proposition.** *Let $\alpha\colon F \to F'$ be a natural transformation between functors $s\mathrm{Set} \to \mathcal{M}$, where $\mathcal{M}$ is some model category. If*

1. *F and F' preserve colimits,*
2. *F and F' take monomorphisms to cofibrations,*
3. *F and F' take inner anodyne maps to to weak equivalences in $\mathcal{M}$, and*
4. *$\alpha(\Delta^1)\colon F(\Delta^1) \to F'(\Delta^1)$ is a weak equivalence in $\mathcal{M}$,*

then $\alpha(X)\colon F(X) \to F'(X)$ is a weak equivalence in $\mathcal{M}$ for all simplicial sets X.

Proof. [Lur09, 4.2.1.2] Consider the class of simplicial sets $\mathcal{C} := \{\, X \mid \alpha(X) \text{ is a weak equivalence} \,\}$. We use skeletal induction (48.28) to show that $\mathcal{C}$ contains all simplicial sets.

It is clear that $\mathcal{C}$ is closed under isomorphic objects. Because F and F' preserve colimits (1) and cofibrations (2), they take good colimit diagrams in $s\mathrm{Set}$ to good colimit diagrams in $\mathcal{M}$. Since good colimits are weak equivalence invariant (47.8), (47.12), (47.10), we see that $\mathcal{C}$ is closed under forming good colimits. It remains to show that $\Delta^n \in \mathcal{C}$ for all n.

We have $\Delta^1 \in \mathcal{C}$ by (4). Since Δ^0 is a retract of Δ^1, we get that $\Delta^0 \in \mathcal{C}$ since weak equivalences in $\mathcal{M}$ are closed under retracts (45.4).

The spines I^n can be built from Δ^0 and Δ^1 by a sequence of good pushouts (glue on one 1-simplex at a time), so the $I^n \in \mathcal{C}$. The inclusions $I^n \subset \Delta^n$ are inner anodyne (12.11), so by (3) and the 2-out-of-3 property of weak equivalences in $\mathcal{M}$ it follows that $\Delta^n \in \mathcal{C}$. □

We will apply this idea to functors $s\mathrm{Set} \to s\mathrm{Set}_{X/}$, where the slice category $s\mathrm{Set}_{X/}$ inherits its model structure from the Joyal model structure on $s\mathrm{Set}$ (45.5).

Proof of (50.9). The functors $X \diamond (-), X \star (-), (-) \diamond X, (-) \star X \colon s\mathrm{Set} \to s\mathrm{Set}_{X/}$ satisfy the first three properties required of the functors in the previous proposition (50.11). That is, they (1) preserve colimits, (2) take monomorphisms to monomorphisms, and (3) take inner anodyne maps to categorical equivalences. Condition (3) for $\diamond$ follows from (50.7), while condition (3) for $\star$ this follows from (27.13) since $\overline{\mathrm{InnHorn}} \subseteq \overline{\mathrm{LHorn}} \cap \overline{\mathrm{RHorn}}$.

Thus, to show $X \diamond Y \to X \star Y$ is a categorical equivalence for a fixed X and arbitrary Y, it suffices by the previous proposition to show that $X \diamond \Delta^1 \to X \star \Delta^1$ is a categorical equivalence. The same argument lets us reduce to the case when $X = \Delta^1$, i.e., to showing that a single map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ is a categorical equivalence.

We will show $\overline{f}$ is a categorical equivalence by producing a map $\overline{g} \colon \Delta^1 \star \Delta^1 \to \Delta^1 \diamond \Delta^1$ such that $\overline{f}\overline{g} = \mathrm{id}_{\Delta^1 \star \Delta^1}$ and $\overline{g}\overline{f}$ is preisomorphic to the identity map of $\Delta^1 \diamond \Delta^1$, via (20.8) .

Since $\Delta^1 \diamond \Delta^1$ is a quotient of a cube, we start with maps involving the cube. Write vertices in $(\Delta^1)^{\times 3}$ as sequences (x, t, y) where $x, t, y \in \{0, 1\}$. Let

$$f \colon (\Delta^1)^{\times 3} \to \Delta^1 \star \Delta^1 = \Delta^3$$

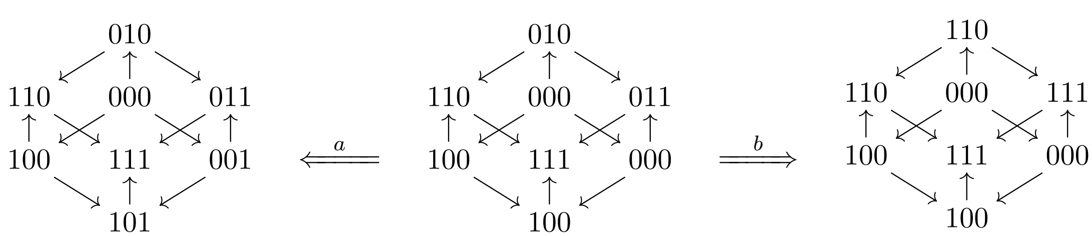
be the map which on vertices sends

$$(x, t, y) \mapsto (1 - t)x + t(2 + y) = \begin{cases} x & \text{if } t = 0, \\ 2 + y & \text{if } t = 1. \end{cases}$$

On passage to quotients this gives the comparison map $\overline{f} \colon \Delta^1 \diamond \Delta^1 \to \Delta^1 \star \Delta^1$ of the proposition (50.6).

Let $g \colon \Delta^3 \to (\Delta^1)^{\times 3}$ be the map classifying the element $\langle (000), (100), (110), (111) \rangle$, and let $\overline{g} \colon \Delta^3 \to \Delta^1 \diamond \Delta^1$ be the composite with the quotient map. We have $fg = \mathrm{id}_{\Delta^3} = \overline{f}\overline{g}$.

Let $h \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_0$ and $a, b \in \mathrm{Map}((\Delta^1)^{\times 3}, (\Delta^1)^{\times 3})_1$ be as indicated in the following picture.



id $\qquad$ h $\qquad$ gf

These pass to elements $\overline{h}$, $\overline{a}$, $\overline{b}$ in $\mathrm{Map}(\Delta^1 \diamond \Delta^1, \Delta^1 \diamond \Delta^1)$. The edges $\overline{a}$ and $\overline{b}$ are preisomorphisms, as one sees that for each vertex $v \in (\Delta^1 \diamond \Delta^1)$, the induced maps $\Delta^1 \times \{v\} \subset \Delta^1 \times (\Delta^1 \diamond \Delta^1) \xrightarrow{\overline{a} \text{ or } \overline{b}} \Delta^1 \diamond \Delta^1$ represent degenerate edges of $\Delta^1 \diamond \Delta^1$. Thus $\overline{f}\overline{g}$ and $\overline{g}\overline{f}$ are preisomorphic to identity maps, and hence $\overline{f}$ is a categorical equivalence as desired. □

50.12. **Alternate slice.** Given $p\colon S \to X$ and $q\colon T \to X$, we define the **alternate slices** $X^{p/}$ and $X^{/q}$ via the bijective correspondences

$$\left\{ \begin{tikzcd} S \diamond \varnothing \arrow[d] \arrow[dr, "p"] & \\ S \diamond K \arrow[r, dotted] & X \end{tikzcd} \right\} \iff \{K \dashrightarrow X^{p/}\}, \qquad \left\{ \begin{tikzcd} \varnothing \diamond T \arrow[d] \arrow[dr, "q"] & \\ K \diamond T \arrow[r, dotted] & X \end{tikzcd} \right\} \iff \{K \dashrightarrow X^{/q}\}.$$

just as we defined ordinary slices using joins. These constructions give right adjoints to the alternate join functors:

$$S \diamond (-)\colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{S/} : (p \mapsto X^{p/}), \qquad (-) \diamond T\colon s\mathrm{Set} \rightleftarrows s\mathrm{Set}_{T/} : (q \mapsto X^{/q}).$$

Alternate slices are "functorial" in exactly the sense that ordinary slices are (24.13): a sequence of maps $T \xrightarrow{j} S \xrightarrow{p} X \xrightarrow{f} Y$ induces $X^{p/} \to Y^{fpj/}$ and $X^{/p} \to Y^{/fpj}$.

50.13. *Exercise.* Show that there are pullback squares of the form

$$\begin{tikzcd} X^{p/} \arrow[r] \arrow[d] & \mathrm{Map}(S \times \Delta^1, X) \arrow[d] \\ X \arrow[r, "(\widetilde{p}, c)"'] & \mathrm{Map}(S \times \{0\}, X) \times \mathrm{Map}(S \times \{1\}, X) \end{tikzcd} \qquad \begin{tikzcd} X^{/p} \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1 \times S, X) \arrow[d] \\ X \arrow[r, "(c, \widetilde{p})"'] & \mathrm{Map}(\{0\} \times S, X) \times \mathrm{Map}(\{1\} \times S, X) \end{tikzcd}$$

where $\widetilde{p}\colon X \to \mathrm{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\mathrm{proj}} S \xrightarrow{p} X$, and $c\colon X \to \mathrm{Map}(S, X)$ is adjoint to $X \times S \xrightarrow{\mathrm{proj}} X \xrightarrow{\mathrm{id}} X$.

Using the adjunction relation between joins and slices, and alternate joins and slices, the natural comparison map $X \diamond Y \to X \star Y$ induces natural comparison maps on alternate slices. That is, given $p\colon S \to X$ and $q\colon T \to Y$ we have natural comparison maps

$$X_{p/} \to X^{p/} \qquad \text{and} \qquad Y_{/q} \to Y^{/q}.$$

50.14. **Joins, slices, and function complexes.** Recall the function complex $\mathrm{Map}(X, Y) \in s\mathrm{Set}$, defined for any pair of simplicial sets X, Y. Recall also (20.15) the relative function complex under S, which for objects $p\colon S \to X$ and $q\colon S \to Y$ in $s\mathrm{Set}_{S/}$ is a simplicial set

$$\mathrm{Map}_{S/}(X, Y) := \mathrm{Map}(X, Y) \times_{\mathrm{Map}(S,Y)} \{q\}$$

with bijective correspondences

$$\left\{ K \dashrightarrow \mathrm{Map}_{S/}(X, Y) \right\} \iff \left\{ \begin{tikzcd} K \times S \arrow[r, "\mathrm{proj}"] \arrow[d, "\mathrm{id} \times p"'] & S \arrow[d, "q"] \\ K \times X \arrow[r, dotted] & Y \end{tikzcd} \right\}$$

natural in the simplicial set K. The set of vertices of $\mathrm{Map}_{S/}(X, Y)$ is precisely the set $\mathrm{Hom}_{s\mathrm{Set}_{S/}}(X, Y)$ of morphisms in the category $s\mathrm{Set}_{S/}$.

Given X and $p\colon S \to Y$, we have join/slice adjunctions

$$\mathrm{Hom}_{s\mathrm{Set}}(X, Y_{p/}) \approx \mathrm{Hom}_{s\mathrm{Set}_{S/}}(S \star X, Y), \qquad \mathrm{Hom}_{s\mathrm{Set}}(X, Y_{/p}) \approx \mathrm{Hom}_{s\mathrm{Set}_{S/}}(X \star S, Y).$$

We now construct maps

$$\mathrm{Map}(X, Y_{p/}) \to \mathrm{Map}_{S/}(S \star X, Y), \qquad \mathrm{Map}(X, Y_{/p}) \to \mathrm{Map}_{S/}(X \star S, Y).$$

which are natural in both X and p, and which on vertices are exactly the join/slice adjunctions. We will call these the **enriched adjunction maps** for join/slice; they are not isomorphisms in general.

I write this out in the case of slice-over, by constructing a transformation

$$\left\{ K \dashrightarrow \mathrm{Map}(X, Y_{p/}) \right\} \Longrightarrow \left\{ K \dashrightarrow \mathrm{Map}_{S/}(S \star X, Y) \right\}$$

natural in the simplicial set K. Applying the product/function complex adjunction, and the join/slice adjunction, this amounts to defining natural maps

$$\left\{ \begin{tikzcd} S \star \varnothing \arrow[d] \arrow[dr, "p"] & \\ S \star (K \times X) \arrow[r, dashed] & Y \end{tikzcd} \right\} \Longrightarrow \left\{ \begin{tikzcd} K \times (S \star \varnothing) \arrow[r, "\mathrm{proj}"] \arrow[d] & S \arrow[d, "p"] \\ K \times (S \star X) \arrow[r, dashed] & Y \end{tikzcd} \right\}$$

Thus it suffices to produce natural maps

$$K \times (S \star X) \to S \star (K \times X)$$

which in the case that $X = \varnothing$ reduce to the projection map $K \times S \to S$. We take this to be the map corresponding by (23.14) to the triple $(g, f_{\{0\}}, f\{1\})$ so that g is the composite

$$K \times (S \star X) \to K \times (\Delta^0 \star \Delta^0) \to \Delta^0 \star \Delta^0 = \Delta^1,$$

and

$$f^{(0)} = \mathrm{proj}\colon K \times (S \star \varnothing) \to S, \qquad f^{(1)} = \mathrm{id}\colon K \times (\varnothing \star X) \to K \times X.$$

It is now straightforward to derive explicit formulas for the desired transformation (by specializing to $K = \Delta^n$), and to show that is is natural.

50.15. *Exercise.* Construct a natural "distributivity" map $K \times (X \star Y) \to (K \times X) \star (K \times Y)$.

50.16. **Alternate joins, alternate slices, and function complexes.** We can carry out the same procedure for alternate joins and slices, to obtain maps

$$\mathrm{Map}(X, Y^{p/}) \to \mathrm{Map}_{S/}(S \diamond X, Y), \qquad \mathrm{Map}(X, Y^{/p}) \to \mathrm{Map}_{S/}(X \diamond S, Y)$$

which are natural in both X and p, and which on vertices are exactly the alternate join/slice adjunctions. We will call these the **enriched adjunction maps** for alternate join/slice.

Tracing through the same steps as in the previous section, we see that (in the first case) we need natural maps

$$K \times (S \diamond X) \to S \diamond (K \times X)$$

which when $X = \varnothing$ reduce to the projection map $K \times S \to S$. In this case it is entirely straightforward to construct such a map, since both objects are naturally quotients of the product $K \times S \times \Delta^1 \times X \approx S \times \Delta^1 \times K \times X$. In fact, examination of the constructions shows that the evident diagram

$$\begin{tikzcd} K \times S \arrow[r, "\mathrm{proj}"] \arrow[d] & S \arrow[d] \\ K \times (S \diamond X) \arrow[r] & S \diamond (K \times X) \end{tikzcd}$$

is a pushout square. (*Exercise:* prove this.) Given this consideration, we see that we have actually defined natural *isomorphisms*

$$\mathrm{Map}(X, Y^{p/}) \xrightarrow{\approx} \mathrm{Map}_{S/}(S \diamond X, Y), \qquad \mathrm{Map}(X, Y^{/p}) \xrightarrow{\approx} \mathrm{Map}_{S/}(X \diamond S, Y).$$

Furthermore, these natural isomorphisms are compatible with the transformations for join/slice.

50.17. **Proposition.** *The evident diagrams*

$$\begin{tikzcd} \mathrm{Map}(X, Y_{p/}) \arrow[r] \arrow[d] & \mathrm{Map}_{S/}(S \star X, Y) \arrow[d] \\ \mathrm{Map}(X, Y^{p/}) \arrow[r, "\approx"'] & \mathrm{Map}_{S/}(S \diamond X, Y) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Map}(X, Y_{/p}) \arrow[r] \arrow[d] & \mathrm{Map}_{S/}(X \star S, Y) \arrow[d] \\ \mathrm{Map}(X, Y^{/p}) \arrow[r, "\approx"'] & \mathrm{Map}_{S/}(X \diamond S, Y) \end{tikzcd}$$

commute.

Proof. This amounts to showing that the evident diagram

$$\begin{tikzcd} K \times (S \diamond X) \arrow[r] \arrow[d] & S \diamond (K \times X) \arrow[d] \\ K \times (S \star X) \arrow[r] & S \star (K \times X) \end{tikzcd}$$

commutes, which we leave to the reader. □

Below we will show that if $Y = C$ is a quasicategory, then all of the maps in these diagrams are categorical equivalences. As a consequence, we will obtain categorical equivalences $C_{p/} \to C^{p/}$ and $C_{/p} \to C^{/p}$.

51. Equivalence of the two join and slice constructions

51.1. **The enriched adjunction map for joins/slices preserves isomorphism classes of objects.** We now consider the natural maps

$$\mathrm{Map}(X, C_{p/}) \to \mathrm{Map}_{S/}(S \star X, C), \qquad \mathrm{Map}(X, C_{/p}) \to \mathrm{Map}_{S/}(X \star S, C)$$

in the case when $p\colon S \to C$ is a map to a quasicategory C. In this case both sources and targets of the natural maps in question are themselves quasicategories, and both induce bijections on sets of objects. Eventually we will show that these functors are categorical equivalencs. Right now we will just prove that these functors induce bijections on *isomorphism classes of objects*.

51.2. **Proposition.** *For X a simplicial set and $p\colon S \to C$ a map to a quasicategory, the enriched adjunction map for join/slice induces bijections*

$$\pi_0\big(\mathrm{Map}(X, C_{p/})^{\mathrm{core}}\big) \xrightarrow{\sim} \pi_0\big(\mathrm{Map}_{S/}(S{\star}X, C)^{\mathrm{core}}\big), \quad \pi_0\big(\mathrm{Map}(X, C_{/p})^{\mathrm{core}}\big) \xrightarrow{\sim} \pi_0\big(\mathrm{Map}_{S/}(X{\star}S, C)^{\mathrm{core}}\big),$$

Proof. We give the proof in the slice-under case. Since the enriched adjunction map gives a bijection on objects, it suffices to prove injectivity on sets of isomorphism classes.

Let $f_0, f_1\colon X \to C_{p/}$ be objects of $\mathrm{Map}(X, C_{p/})$, which correspond to objects $\widetilde{f}_0, \widetilde{f}_1\colon S \star X \to C$ of $\mathrm{Map}_{S/}(S \star X, C)$, with $\widetilde{f}_j|S = p$. If $\widetilde{f}_0$ and $\widetilde{f}_1$ are isomorphic objects of $\mathrm{Map}_{S/}(S \star X, C)$, then there exists a map $N\mathrm{Iso} \to \mathrm{Map}_{S/}(S \star X, C)$ representing such an isomorphism (36.17). The data of such a map amounts to a an arrow $\widetilde{f}$ fitting in the commutative diagram

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d] & C \arrow[r] & \mathrm{Map}(N\mathrm{Iso}, C) \arrow[d] \\ S \star X \arrow[urr, "\widetilde{f}"] \arrow[rr, "(\widetilde{f}_0, \widetilde{f}_1)"'] & & C \times C \end{tikzcd}$$

where $C \to \mathrm{Map}(N\mathrm{Iso}, C) \to C \times C$ are induced by restriction along $* \leftarrow N\mathrm{Iso} \leftarrow \{0,1\}$. Write $D = \mathrm{Map}(N\mathrm{Iso}, C)$ and $\overline{p}\colon S \to D$ for the map along the top of the rectangle. Applying the join/slice adjunction, we see that we have a diagram

$$\begin{tikzcd} & D_{\overline{p}/} \arrow[d] \\ X \arrow[ur, "f"] \arrow[r, "(f_0,f_1)"'] & C_{p/} \times C_{/p} \end{tikzcd}$$

That is, we have produced an object f in $\mathrm{Map}(X, D_{\overline{p}/})$ which under the two evident projections $\pi_0, \pi_1 \colon \mathrm{Map}(X, D_{\overline{p}/}) \to \mathrm{Map}(X, C)$ is sent to f_0 and f_1 respectively.

We have that both projections $D \to C$ are trivial fibrations, whence so are both projections $D_{\overline{p}/} \to C_{p/}$ since $\overline{\mathrm{Cell}} \circledast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}}$ (27.16), and hence both projections π_0 and π_1. Considering the induced commutative diagram

$$\begin{tikzcd} & & \mathrm{Map}(X, C_{p/}) \\ \mathrm{Map}(X, C_{p/}) \arrow[r] \arrow[urr, bend left, "\mathrm{id}"] \arrow[drr, bend right, "\mathrm{id}"'] & \mathrm{Map}(X, D_{\overline{p}/}) \arrow[ur, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ & & \mathrm{Map}(X, C_{p/}) \end{tikzcd}$$

we see that every arrow in this diagram is a categorical equivalence, and therefore both π_0 and π_1 induce *the same bijection* on isomorphism classes on objects. Since f_0 and f_1 are respectively the images under π_0 and π_1 of the same object $f \in \mathrm{Map}(X, D_{\overline{p}/})_0$, we conclude that f_0 and f_1 are isomorphic objects, as desired. □

51.3. **Equivalence of slice and alternate slice.**

51.4. **Proposition.** *For any quasicategory C and map $p \colon S \to C$, the comparison maps $C_{p/} \to C^{p/}$ and $C_{/p} \to C^{/p}$ are categorical equivalences.*

Proof. [Lur09, 4.2.1.5] We do the first case. We use the following fact: if $f \colon A \to B$ is a functor between quasicategories, then f is a categorical equivalence if and only if the induced maps $\pi_0\big(\mathrm{Fun}(X, A)^{\mathrm{core}}\big) \to \big(\mathrm{Fun}(X, B)^{\mathrm{core}}\big)$ are bijections for all simplicial sets X. **I probably did this before somewhere.**

Recall (50.17) that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Map}(X, C_{p/}) \arrow[r] \arrow[d] & \mathrm{Map}_{S/}(S \star X, C) \arrow[d] \\ \mathrm{Map}(X, C^{p/}) \arrow[r, "\approx"'] & \mathrm{Map}_{S/}(S \diamond X, C) \end{tikzcd}$$

in which the bottom map is an isomorphism. By (51.2) the top map is a bijection on isomorphism classes of objects By (50.9) $S \diamond X \to S \star X$ is a categorical equivalence, and thus the right-hand map is a categorical equivalence, and hence a bijection on isomorphism classes of objects. It follows that the left-hand map is a bijection on isomorphism classes of objects, and the proposition is proved. □

51.5. **Corollary.** *For any quasicategory C and map $p \colon S \to C$, the enriched adjunction maps $\mathrm{Fun}(X, C_{p/}) \to \mathrm{Map}_{S/}(S \star X, C)$ and $\mathrm{Fun}(X, C_{/p}) \to \mathrm{Map}_{S/}(X \star S, C)$ are categorical equivalences.*

51.6. **Alternate pushout-join.** Just as we defined the "pushout-join" $\circledast$, we can define the "alternate pushout-join" $\boxtimes$: given $f \colon A \to B$ and $g \colon K \to L$, we obtain

$$f \boxtimes g \colon (B \diamond K) \cup_{A \diamond K} (A \diamond L) \to B \diamond L.$$

51.7. **Proposition.** *We have that*

$$\overline{\mathrm{RHorn}} \boxtimes \overline{\mathrm{Cell}} \quad\cup\quad \overline{\mathrm{Cell}} \boxtimes \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}.$$

Proof. We'll show that $\overline{\mathrm{RHorn}} \boxtimes \overline{\mathrm{Cell}} \subseteq \mathrm{Cell} \cap \mathrm{CatEq}$. It is straightforward to show that the $\boxtimes$-product of two monomorphisms is a monomorphism. Thus, it suffices to show that for $f\colon A \to B$ right anodyne and any inclusion $g\colon K \to L$, the map $f \boxtimes g$ is a categorical equivalence. We know that $\overline{\mathrm{RHorn}} \circledast \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}} \subseteq \mathrm{CatEq}$ (27.13), so $f \circledast g$ is a categorical equivalence. Furthermore, in

$$\begin{tikzcd}
(B \diamond K) \cup_{A \diamond K} (A \diamond L) \arrow[r] \arrow[d] & B \diamond L \arrow[d] \\
(B \star K) \cup_{A \star K} (A \star L) \arrow[r] & B \star L
\end{tikzcd}$$

the vertical maps are categorical equivalences; this uses the result proved above (50.9), as well as the fact that since f and g are monomorphisms, the domains of $f \boxtimes g$ and $f \circledast g$ are constructed from good pushouts. ☐

Question: is $\overline{\mathrm{LHorn}} \boxtimes \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}}$?

51.8. **Proposition.** *Given $K \xrightarrow{j} L \xrightarrow{p} C$, if C is a quasicategory and j is a monomorphism, then $C^{p/} \to C^{pj/}$ is a left fibration, and $C^{/p} \to C^{/pj}$ is a right fibration. In particular, both maps are isofibrations.*

Proof. The first statement follows from $\overline{\mathrm{LHorn}} \boxtimes \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$ and $\overline{\mathrm{Cell}} \boxtimes \overline{\mathrm{RHorn}} \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$. The second statement is immediate as left and right fibrations between quasicategories are always isofibrations (29.10). ☐

51.9. **Equivalence of various mapping spaces.** Finally we can prove our original goal.

51.10. **Proposition.** *For any quasicategory C and object $x \in C_0$, the natural comparison maps $\mathrm{map}^R_C(x,y) \to \mathrm{map}_C(x,y) \leftarrow \mathrm{map}^L_C(x,y)$ are weak equivalences.*

Proof. In

$$\begin{tikzcd}
C_{x/} \arrow[rr, "f"] \arrow[dr, "p"'] & & C^{x/} \arrow[dl, "q"] \\
& C &
\end{tikzcd}$$

the map f is a categorical equivalence (51.4) and p and q are isofibrations by (29.10) and (51.8) respectively, and hence categorical fibrations. It follows that the induced maps on fibers $\mathrm{map}^R_C(x,c) \to \mathrm{map}_C(x,c)$ are categorical equivalences and hence weak equivalences, since the pullbacks describing the pullbacks are good pullbacks (with respect to the Joyal model structure). ☐

51.11. **Slices as fibers. Rewrite this in terms of the enriched adjunction maps.**

The alternate slice $C^{f/}$ has another convenient characterization: it is the fiber over f of a map between functor categories.

51.12. **Proposition.** *For a map $f\colon S \to X$ of simplicial sets, the alternate slice $X^{f/}$ is isomorphic to the fiber of the restriction map*

$$\mathrm{Map}(S \diamond \Delta^0, X) \to \mathrm{Map}(S, X).$$

over f.

Proof. Let F be the fiber of the restriction map. There is an evident correspondence

$$K \dashrightarrow F \qquad \Longleftrightarrow \qquad \begin{tikzcd} S \times K \arrow[r, "\pi"] \arrow[d] & S \arrow[d, "f"] \\ (S \diamond \Delta^0) \times K \arrow[r, dashed] & C \end{tikzcd}$$

The claim follows by showing that the evident quotient map $S \times \Delta^1 \times K \to (S \diamond \Delta^0) \times K$ extends to an isomorphism

$$S \diamond K \xrightarrow{\sim} ((S \diamond \Delta^0) \times K) \cup_{S \times K} S$$

compatible with the standard inclusions of S. $\square$

We can also consider the fiber of the inclusion $S \subset S \star \Delta^0$ into the standard cone. This gives yet another version of the slice.

51.13. **Corollary.** *Let C be a quasicategory, and let $F(f) :=$ the fiber of* $\mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$ *over f. Then there is a chain of categorical equivalences*

$$F(f) \to C^{f/} \leftarrow C_{f/}.$$

Furthermore, $F(f)$ and $C_{f/}$ have the same set of 0-dimensional elements, and both arrows above coincide on 0*-dimensional elements.*

Proof. The second equivalence is just (51.4). For the first equivalence, note that

$$\begin{tikzcd} \mathrm{Fun}(S \star \Delta^0, C) \arrow[r] \arrow[d] & \mathrm{Fun}(S \diamond \Delta^0, C) \arrow[d] \\ \mathrm{Fun}(S, C) \arrow[r, equal] & \mathrm{Fun}(S, C) \end{tikzcd}$$

the top horizontal map is a categorical equivalence using (50.9), while the vertical maps are both categorical fibrations. Therefore the induced map on fibers over f is a categorical equivalence, since the pullback squares in question are good.

The vertices of $F(f)$ and $C_{f/}$ are exactly the set $\{S^{\triangleright} \to C\}$. Both inclusions $F(f)_0 \to (C^{f/})_0 \leftarrow (C_{f/})_0$ are induced by restriction along the standard comparison map $S \diamond \Delta^0 \to S \star \Delta^0$. $\square$

Part 6. **The quasicategory of ∞-categories**

Category theory naturally provides an example of itself. That is, the collection of categories and functors between them is itself a category. In particular, we write Cat for the (large) category whose objects are (small) categories, and whose morphisms are functors between them.

By analogy, one may expect that quasicategory theory provides an an example of itself, e.g., that the collection of quasicategories and functors between forms part of the data of a *quasicategory.* In fact, that is the case: we write qCat for the (large) category whose objects are (small) quasicategories, and whose morphisms are functors between them. Being an ordinary category, qCat is *a fortiori* a quasicategory.

This is unsatisfying. We would hope to have a richer object, i.e., a quasicategory which we might name Cat_∞, whose objects and functors are as in qCat, but which in addition has some kind of non-trivial "higher structure".

Such "quasicategory of ∞-categories" Cat_∞ does exist. However, its description is nowhere as evident or natural as Cat. In fact, there are many possible constructions of Cat_∞ which are not isomorphic to each other (though they are categorically equivalent to each other). We will describe a particular construction of Cat_∞ (due to Lurie, and which is in some sense the standard construction) below.

In order to understand the role that Cat_∞ plays, we will start by thinking about ordinary categories. We will see that even in the classical case, what is wanted is not the ordinary category Cat, but rather a certain quasicategorical thickening of it, which we will Cat_1.

52. The quasicategory of categories

52.1. **The category of categories.** We write Cat for the category of categories, by which we mean the category of *small* categories (i.e., categories whose collections of objects and morphisms are sets). Sometimes I will need to talk about a larger category CAT of possibly non-small categories (so that Cat is an object of CAT).

To understand Cat (and CAT), let us think about maps to it. Given a category C, we obtain the category $\mathrm{Fun}(C, \mathrm{Cat})$ of functors from C to the category of categories. Explicitly:

- An object of $\mathrm{Fun}(C, \mathrm{Cat})$ is a functor $F\colon C \to \mathrm{Cat}$, which assigns
  - to each object $c \in \mathrm{ob}\, C$ a category $F(c)$, and
  - each morphism $\alpha\colon c \to c'$ in C a functor $F(\alpha)\colon F(c) \to F(c')$, such that
  - $F(\mathrm{id}_c)$ is the identity functor of $F(c)$, and $F(\beta\alpha) = F(\beta) \circ F(\alpha)$ for all composable arrows $c \xrightarrow{\alpha} c' \xrightarrow{\beta} c''$ in C.
- A morphism $\gamma\colon F \to F'$ in $\mathrm{Fun}(C, \mathrm{Cat})$ is a natural transformation of functors, which assigns
  - to each object $c \in \mathrm{ob}\, C$ a functor $\gamma(c)\colon F(c) \to F'(c)$ such that
  - for each morphism $\alpha\colon c \to c'$ in C we have an equality $F'(\alpha)\gamma(c) = \gamma(c')F(\alpha)$ of functors $F(c) \to F'(c')$.

The same description applies to $\mathrm{Fun}(C, \mathrm{CAT})$.

A functor $C \to \mathrm{Cat}$ may be thought of as a "family of categories parameterized by C".

52.2. *Example.* Let Ring be the category of associative rings and homomorphisms; it is a large category. We may define a functor $\mathcal{M}\colon \mathrm{Ring}^{\mathrm{op}} \to \mathrm{CAT}$ as follows.

- For a ring $R \in \mathrm{ob}\,\mathrm{Ring}$, let $\mathcal{M}(R) := \mathrm{Mod}_R$ the category of left R-modules.
- For a homorphism $\alpha\colon R \to R'$ of rings, let $\mathcal{M}(\alpha)\colon \mathrm{Mod}_{R'} \to \mathrm{Mod}_R$ be the restriction-along-α functor, which on objects sends an R'-module M to the R-module with same underlying abelian group and with $rx = \alpha(r)x$ for $r \in R$ and $x \in M$.

It is straightforward to see that this indeed defines a functor $\mathrm{Ring}^{\mathrm{op}} \to \mathrm{CAT}$.

There however many examples of "families of categories parameterized by a category" which can not be easily described by a functor to Cat or CAT.

52.3. *Example.* As in (52.2) let Ring be the category of associative rings and homomorphisms. We may attempt to define a functor $\mathcal{M}'\colon \mathrm{Ring} \to \mathrm{CAT}$ as follows.

- For a ring $R \in \mathrm{ob}\,\mathrm{Ring}$, let $\mathcal{M}'(R) := \mathrm{Mod}_R$ the category of left R-modules.
- For a homomorphism $\alpha\colon R \to R'$, let $\mathcal{M}'(\alpha)\colon \mathrm{Mod}_R \to \mathrm{Mod}_{R'}$ be the extension-of-scalars-along-α functor, which on objects sends an R-module M to the R'-module $R' \otimes_R M$.

However, the above data does not define a functor. Given morphisms $R \xrightarrow{\alpha} R' \xrightarrow{\beta} R''$, we need to have an *equality* of functors $\mathcal{M}'(\beta) \circ \mathcal{M}'(\alpha) = \mathcal{M}'(\beta\alpha)$. However, we only have a *natural isomorphism* of functors, given by isomorphisms $\mathcal{M}'(\beta, \alpha)\colon R'' \otimes_{R'} (R' \otimes_R M) \xrightarrow{\sim} R'' \otimes_R M$ which are natural in the R-module M.

The data of (52.3) does not define a functor, but rather a **emph**. I won't define a pseudofunctor. Instead, I'll describe a quasicategory Cat_1 with the property that functors $C \to \mathrm{Cat}_1$ are pseudofunctors.

52.4. **The quasicategory of categories.** We define the (large) simplicial set Cat_1 as follows. The n-dimensional elements of Cat_1 are data $(C_i, f_{ij}, \zeta_{ijk})$ where

- C_i is a small category for $0 \leq i \leq n$,
- $f_{ij} \colon C_i \to C_j$ is a functor, and
- $\zeta_{ijk} \colon f_{ik} \to f_{jk} f_{ij}$ is a natural isomorphism of functors $C_i \to C_k$,

such that

- $f_{ii} = \mathrm{id}_{C_i}$ for all $0 \leq i \leq n$,
- $\zeta_{iij} = \mathrm{id}_{f_{ij}}$ and $\zeta_{ijj} = \mathrm{id}_{f_{ij}}$ for all $0 \leq i \leq j \leq n$, and
- the square

$$\begin{tikzcd} f_{i\ell} \arrow[r, "\zeta_{ij\ell}"] \arrow[d, "\zeta_{ik\ell}"'] & f_{j\ell} f_{ij} \arrow[d, "\zeta_{jk\ell} f_{ij}"] \\ f_{k\ell} f_{ik} \arrow[r, "f_{k\ell}\zeta_{ijk}"'] & f_{k\ell} f_{jk} f_{ij} \end{tikzcd}$$

of natural isomorphisms of functors $C_i \to C_\ell$ commutes for all $0 \leq i \leq j \leq k \leq n$.

Part 7. Old stuff

Note. From this point forward, these notes are not an organized narrative, but rather a collection of bits and pieces that might be worked into something useful at some point.

53. Coherent nerve

53.1. **The coherent nerve.** The coherent nerve $\mathcal{N}$ is a construction which turns a simplicially enriched category into a simplicial set, and in particular turns a Kan-enriched category into a quasicategory. It was invented by Cordier [Cor82]. The coherent nerve is constructed as right adjoint of a "realization/singular" pair

$$\mathfrak{C} \colon s\mathrm{Set} \rightleftarrows s\mathrm{Cat} : \mathcal{N}.$$

Given a finite totally ordered set S, define

$$\mathcal{P}(S) := \{\, A \subseteq S \mid \{\min, \max\} \subseteq A \subseteq S \,\}.$$

This is a poset, ordered by set containment; here $\min, \max$ denote the least and greatest elements of S (possibly the same). If S is empty, so is $P(S)$.

Let $\mathfrak{C}(\Delta^n)$ denote the simplicially enriched category defined as follows.

- The objects are elements of $[n] = \{0, \dots, n\}$.
- For objects $x, y \in [n]$, the function complex is
$$\mathrm{Map}_{\mathfrak{C}(\Delta^n)}(x, y) := N\mathcal{P}([x, y]), \qquad [x, y] := \{\, t \in [n] \mid x \leq t \leq y \,\},$$
which is set to be empty if $x > y$.
- Composition is induced by the union operation on subsets:
$$(T, S) \mapsto T \cup S \colon \mathcal{P}([y, z]) \times \mathcal{P}([x, y]) \to \mathcal{P}([x, z]).$$

Every $f \colon [m] \to [n]$ in Δ gives rise to an enriched functor $\mathfrak{C}(f) \colon \mathfrak{C}(\Delta^m) \to \mathfrak{C}(\Delta^n)$, which on objects operates as f does on elements of the ordered sets, and is induced on morphisms by

$$S \mapsto f(S) \colon \mathcal{P}([x, y]) \to \mathcal{P}([fx, fy])$$

We obtain (after identifying Δ with its image in $s\mathrm{Set}$) a functor $\mathfrak{C} \colon \Delta \to s\mathrm{Cat}$.

Given a simplicially enriched category C, its **coherent nerve** (or **simplicial nerve**) is the simplicial set $\mathcal{N}C$ defined by

$$(\mathcal{N}C)_n = \mathrm{Hom}_{s\mathrm{Cat}}(\mathfrak{C}(\Delta^n), C).$$

...

53.2. **Quasicategories from simplicial nerves.**

53.3. **Proposition.** *If $\mathcal{C}$ is a category enriched over Kan complexes, then $\mathcal{N}(\mathcal{C})$ is a quasicategory.*

Proof. □

54. Correspondences

A **correspondence** is defined to be an inner fibration $p\colon M \to \Delta^1$. A map of correspondences is a morphism in the slice category $s\mathrm{Set}_{/\Delta^1}$.

54.1. **Correspondences of ordinary categories.** If M is an ordinary category, then any functor $p\colon M \to \Delta^1$ is an inner fibration. Given such a functor, we can identify the following data:

- categories $C := p^{-1}(\{0\})$ and $D := p^{-1}(\{1\})$, the preimages of the vertices, and
- for each pair of objects $c \in \operatorname{ob} C$, $d \in \operatorname{ob} D$, a set
$$\mathcal{M}(c,d) := \operatorname{Hom}_M(c,d),$$
which
- fit together to give a functor
$$\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

Conversely, given the data of categories C and D, and a functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$, we can construct a category M with functor $p\colon M \to \Delta^1$ in the evident way, with
$$\operatorname{ob} M := \operatorname{ob} C \amalg \operatorname{ob} D, \qquad \operatorname{mor} M := \operatorname{mor} C \amalg \left(\coprod_{c,d} \mathcal{M}(c,d)\right) \amalg \operatorname{mor} D.$$

Under the above, maps $f\colon M \to M'$ between correspondences which are categories are sent to data consisting of: functors $u\colon C \to C'$ and $v\colon D \to D'$, and natural transformations
$$\mathcal{M} \to \mathcal{M}' \circ (u \times v) \quad \text{of functors} \quad C^{\mathrm{op}} \times D \to \mathrm{Set}.$$

54.2. *Example.* If C and D are categories, then the functor $C \star D \to \Delta^0 \star \Delta^0 \approx \Delta^1$ is an example of a correspondence. The corresponding functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ is the one with $\mathcal{M}(c,d) = \{*\}$ for all objects.

54.3. *Example.* Let $F\colon C \to D$ be a functor between categories. Then we get a functor $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ defined by
$$\mathcal{M}(c,d) := \operatorname{Hom}_D(F(c), D),$$
and thus an associated correspondence $p\colon M \to \Delta^1$.

Similarly, let $G\colon D \to C$ be a functor between categories. Then we get a functor $\mathcal{M}'\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ defined by
$$\mathcal{M}'(c,d) := \operatorname{Hom}_C(c, G(d)),$$
and thus an associated correspondence $p'\colon M' \to \Delta^1$.

54.4. *Example.* Suppose $F\colon C \rightleftarrows D : G$ is an adjoint pair of functors. If we form $\mathcal{M}$ and $\mathcal{M}'$ as in the previous example, we see that the adjunction gives a natural isomorphism $\mathcal{M} \approx \mathcal{M}'$ of functors $C^{\mathrm{op}} \times D \to \mathrm{Set}$. The associated correspondences $M \to \Delta^1$ and $M' \to \Delta^1$ are isomorphic.

55. Cartesian and cocartesian morphisms

In the following, we fix an inner fibration $p\colon M \to S$. We will often assume that S (and thus M) is a quasicategory.

Consider an edge $f\colon x \to y$ in M. We say that the edge represented by $f\colon \Delta^1 \to M$ is **p-cartesian** if a lift exists in every diagram of the form

$$\begin{tikzcd} \Delta^{\{n-1,n\}} \arrow[r, tail] \arrow[rr, bend left, "f"] & \Lambda^n_n \arrow[r] \arrow[d, tail] & M \arrow[d, "p"] \\ & \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

for all $n \geq 2$.

There is a dual notion of a **p-cocartesian edge**, where Λ^n_n is replaced by Λ^n_0, and we use the leading edge of the simplex instead of the trailing edge.

We have already seen examples of this property.

- Let $p\colon C \to *$ where C is a quasicategory By the Joyal extension theorem (29.2), we have that an edge in C is p-cartesian if and only if it is p-cocartesian if and only if it is an isomorphism.
- Let $p\colon M \to S$ be an inner fibration between quasicategories, and suppose $f \in M_1$ is an edge such that $p(f)$ is an isomorphism in S. By the Joyal lifting theorem (29.13), f is p-cartesian if and only if it is p-cocartesian if and only if f is an isomorphism in M.
- If $p\colon M \to S$ is a right fibration, then every edge in M is p-cartesian. Likewise, if p is a left fibration, then every edge in M is p-cocartesian.

Thus, Joyal's theorem completely describe cartesian/cocartesian edges over an *isomorphism* in a quasicategory.

We have an equivalent formulation: f is p-cartesian if and only if

$$M_{/f} \to M_{/y} \times_{S_{/p(y)}} S_{/pf}$$

is a trivial fibration.

55.1. **Cartesian edges and correspondence.** Let $p\colon M \to \Delta^1$ be a correspondence, with M an ordinary category. We write $C := p^{-1}(\{0\})$, $D := p^{-1}(\{1\})$, and $\mathcal{M}\colon C^{\mathrm{op}} \times D \to \mathrm{Set}$ for the associated functor.

Suppose $f\colon c \to d$ is an edge such that $p(f) = \langle 01 \rangle$.

55.2. **Lemma.** *The edge f is p-catesian if and only if, for each $u\colon x \to d$ with $p(u) = \langle 01 \rangle$, there exists a unique $v\colon x \to c$ such that $fv = u$.*

In particular, if f is p-cartesian, then composition

$$f_*\colon \mathrm{Hom}_M(x, c) \to \mathrm{Hom}_M(x, d)$$

is a bijection for all $x \in \mathrm{ob}\, C$. Equivalently, the map

$$\mathrm{Hom}_C(x, c) \to \mathcal{M}(x, d), \qquad v \mapsto fv$$

is a bijection, so $\mathcal{M}(-, d)\colon C^{\mathrm{op}} \to \mathrm{Set}$ is represented by c.

55.3. **Box criterion for cartesian edges.**

55.4. **Proposition.** [Lur09, 2.4.1.8] *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Then f is p-cartesian if and only if a lift exists in every diagram of the form*

$$\begin{tikzcd} \Delta^1 \times \{n\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{1\} \times \Delta^n) \cup_{\{1\}\times\partial\Delta^n} (\Delta^1 \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & M \arrow[d, "p"] \\ & \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

for all $n \geq 1$.

Proof. The if part is just like the proof of the box version of Joyal lifting. □

We reformulate this criterion. Consider the box power map

$$q := p^{\square(\{1\}\subset\Delta^1)}\colon \operatorname{Map}(\Delta^1, M) \to \operatorname{Map}(\Delta^1, S) \times_{\operatorname{Map}(\{1\},S)} \operatorname{Map}(\{1\}, M).$$

Then the above proposition says that f is p-cartesian iff a lift exists in every diagram

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & \operatorname{Map}(\Delta^1, M) \arrow[d, "q"] \\ \Delta^n \arrow[r, "b"'] \arrow[ur, dotted] & \operatorname{Map}(\Delta^1, S) \times_{\operatorname{Map}(\{1\},S)} \operatorname{Map}(\{1\}, M) \end{tikzcd}$$

such that $n \geq 1$ and $a(n) = f \in \operatorname{Map}(\Delta^1, M)_0$.

55.5. **Uniqueness of lifts to Cartesian edges.** Let $U \subseteq \operatorname{Map}(\Delta^1, M)$ be the full subsimplicial set spanned by the vertices which represent p-cartesian edges. Likewise, let $V \subseteq \operatorname{Map}(\Delta^1, S) \times_{\operatorname{Map}(\{1\},S)} \operatorname{Map}(\{1\}, M)$ denote the essential image of U under q, i.e., the full subsimplicial set spanned by the vertices $q(U_0)$. Obviously, the map q restricts to a map $q'\colon U \to V$.

Note in particular that V_0 is the subset of $\{\, (g, y) \in S_1 \times M_0 \mid g_1 = p(y) \,\}$ such that there *exists* a Cartesian edge $f \in M_1$ with $f_1 = y$ and $p(f) = g$, and the preimage of (g, y) under $q'\colon U \to V$ is the set of all choices of lifts. The following in particular asserts a kind of uniqueness for choices of lifts.

55.6. **Proposition.** *The map $q'\colon U \to V$ is a trivial fibration.*

Proof. Consider

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, tail] & U \arrow[r, tail] \arrow[d, "q|U"] & \operatorname{Map}(\Delta^1, M) \arrow[d, "q"] \\ \Delta^n \arrow[r] & V \arrow[r, tail] & \operatorname{Map}(\Delta^1, S) \times_{\operatorname{Map}(\{1\},S)} \operatorname{Map}(\{1\}, M) \end{tikzcd}$$

If $n \geq 1$, then a lift $s\colon \Delta^n \to \operatorname{Map}(\Delta^1, M)$ exists by the previous proposition, since $a(n) \in U_0 \subseteq \operatorname{Map}(\Delta^1, M)_0$ represents a p-cartesian edge. Because $(\partial\Delta^n)_0 = (\Delta^n)_0$ when $n \geq 1$, we see that s maps into the full subcomplex U.

If $n = 0$, this amounts to $U_0 \to V_0$ being surjective, which holds by definition. □

55.7. **Cartesian fibration.** A **cartesian fibration** is a map $p\colon M \to S$ which is an inner fibration, and is such that for all $(g, y) \in S_1 \times M_0$ with $g_1 = p(y)$, there exists a p-cartesian edge f with $p(f) = g$ and $f_1 = y$.

55.8. *Example.* Every left or right fibration is a cartesian fibration, since all edges are cartesian.

By the above, we see that an inner fibration $p\colon M \to S$ is a cartesian fibration if and only if $V = \mathrm{Map}(\Delta^1, S) \times_{\mathrm{Map}(\{1\},S)} \mathrm{Map}(\{1\}, M)$.

55.9. **Cartesian correspondences.** Given a map $p\colon M \to S$, for any element $a \in S_k$ write

$$M_a := \mathrm{Map}_{/S}(\Delta^k, M) = \mathrm{Map}_{/S}(a, p).$$

Note that if $a = bf$ for some simplicial operator $f\colon [k] \to [l]$, we obtain an induced restriction map

$$f^*\colon M_b \to M_a.$$

Given a correspondence $p\colon M \to \Delta^1$, we obtain

$$C = M_{\langle 0\rangle} \xleftarrow{\langle 0\rangle^*} M_{\langle 01\rangle} \xrightarrow{\langle 1\rangle^*} M_{\langle 1\rangle} = D.$$

Note that these are all quasicategories. The objects of $M_{\langle 01\rangle}$ are precisely the edges in M lying over $\langle 01\rangle$.

55.10. **Proposition.** *Let $p\colon M \to S$ be a cartesian fibration, and let $M^{\mathrm{cart}}_{\langle 01\rangle} \subseteq M_{\langle 01\rangle}$ be the full subcategory spanned by elements corresponding to cartesian edges. Then $M^{\mathrm{cart}}_{\langle 01\rangle} \to M_{\langle 1\rangle}$ is a trivial fibration.*

Proof. Every square in

$$\begin{tikzcd}
M^{\mathrm{cart}}_{\langle 01\rangle} \arrow[rr] \arrow[d, tail] & & U \arrow[d, tail, "i"] & \\
M_{\langle 01\rangle} \arrow[rr] \arrow[d] & & \mathrm{Map}(\Delta^1, M) \arrow[d, "q"] & \\
M_{\langle 1\rangle} \arrow[r] \arrow[d] & \mathrm{Map}(\Delta^1,\Delta^1) \times_{\mathrm{Map}(\{1\},\Delta^1)} \mathrm{Map}(\{1\}, M) \arrow[r, phantom, ""] & \arrow[r] & \mathrm{Map}(\{1\}, M) \arrow[d] \\
\{\mathrm{id}_{\Delta^1}\} \arrow[rr] & & \mathrm{Map}(\Delta^1,\Delta^1) \arrow[r] & \mathrm{Map}(\{1\},\Delta^1)
\end{tikzcd}$$

is a pullback. The result follows because $qi = q'$ is a trivial fibration. □

More generally, given an inner fibration $p\colon M \to S$ and an element $a \in S_k$, the objects of the quasicategory M_a correspond to k-dimensional elements $b \in M_k$ such that $p(b) = a$. Let $M^{\mathrm{cart}}_a \subseteq M_a$ denote the full subcategory spanned by objects corresponding to $b \in M_k$ such that all edges of b are p-cartesian.

55.11. **Proposition.** *Let $p\colon M \to S$ be an inner fibration, and $f \in M_1$ an edge. Consider*

$$\begin{tikzcd}
\Delta^k \times \{n\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\Lambda^k_j \times \Delta^n) \cup_{\Lambda^k_j \times \partial\Delta^n} (\Delta^k \times \partial\Delta^n) \arrow[r] \arrow[d, tail] & M \arrow[d, "p"] \\
& \Delta^1 \times \Delta^n \arrow[r] \arrow[ur, dotted] & S
\end{tikzcd}$$

where $\Lambda^k_j \subset \Delta^k$ is a right horn inclusion, and f represents a p-cartesian edge. Then a lift exists whenever $n, k \geq 1$, and also when $k \geq 2$, $n = 0$.

Proof. This should also be like the box version of Joyal lifting. Note that if $k = 0$, we recover the definition of p-cartesian edge. ☐

This admits a reformulation: if f is p-cartesian, then if $0 < j \leq k$ and $a(n) = f$ is p-cartesian there is a lift in

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & \mathrm{Map}(\Delta^k, M) \arrow[d, "q"] \\ \Delta^n \arrow[r, "b"'] \arrow[ur, dotted] & \mathrm{Map}(\Delta^k, S) \times_{\mathrm{Map}(\Lambda^k_j, S)} \mathrm{Map}(\Lambda^k_j, M) \end{tikzcd}$$

when $k \geq 1$ and $n \geq 1$, or for all $n \geq 0$ if $k \geq 2$.

55.12. **Cartesian fibrations and right fibrations.**

55.13. **Proposition.** [Lur09, 2.4.2.4] *A map $p\colon M \to S$ is a right fibration iff it is a cartesian fibration whose fibers are Kan complexes.*

Proof. We have already seen that a right fibration is a cartesian fibration, and has Kan complexes as fibers.

Now suppose p is cartesian fibration with Kan complex fibers. Let $f\colon x \to y$ be an edge in M. Since p is cartesian, there exists a p-cartesian edgee $f'\colon x' \to y$ over $p(f)$. Since p is cartesian fibration and f' a cartesian edge, there exists $a \in M_2$ with $a_{02} = f$ and $a_{12} = f'$ and $p(a) = (p(f))_{001}$. Thus $g := a_{01}$ is an edge in the fiber over $(p(f))_0$, so is an isomorphism in that fiber. ☐

55.14. **Mapping space criterion for cartesian edges.**

55.15. **Proposition.** [Lur09, 2.4.4.3] *Let $p\colon C \to D$ be an inner fibration between quasicategories, and $f\colon x \to y$ a morphism in C. The following are equivalent.*

(1) *f is p-cartesian.*
(2) *For every $c \in C_0$, the diagram*

$$\begin{tikzcd} \mathrm{map}_C(c, x) \arrow[r, "f_*"] \arrow[d] & \mathrm{map}_C(c, y) \arrow[d] \\ \mathrm{map}_D(p(c), p(x)) \arrow[r, "p(f)_*"'] & \mathrm{map}_D(p(c), p(y)) \end{tikzcd}$$

is a homotopy pullback.

56. Limits and colimits as functors

Suppose J and C are categories. We say that C has all J-colimits if every functor $F\colon J \to C$ has a colimit in J. It is a standard observation that if F is such a functor, then we can assemble a functor

$$\mathrm{colim}_J\colon \mathrm{Fun}(J, C) \to C.$$

In fact, we can regard this functor as a composite of functors

$$\mathrm{Fun}(J, C) \xrightarrow{s} \mathrm{Fun}(J^{\triangleright}, C) \xrightarrow{\text{eval. at } v} C,$$

where s is some section of the restriction functor $\mathrm{Fun}(J^{\triangleright}, C) \to \mathrm{Fun}(J, C)$ which takes values in colimit cones.

Even when C does not have all J-colimits, we can assert the following. Consider the diagram

$$\begin{tikzcd} \mathrm{Fun}^{\text{colim cone}}(J^{\triangleright}, C) \arrow[r, tail] \arrow[d, "p"'] & \mathrm{Fun}(J^{\triangleright}, C) \arrow[d] \\ \mathrm{Fun}^{\exists \text{ colim}}(J, C) \arrow[r, tail] & \mathrm{Fun}(J, C) \end{tikzcd}$$

in which the objects on the left are the evident full subcategories of the corresponding objects on the right, i.e., the ones consisting of colimit cones, and of functors which admit colimits. Then p is an equivalence of categories, and in fact is a trivial fibration. Therefore there is a contractible groupoid of sections of p, and any section s gives rise to a colimit functor

$$\mathrm{Fun}^{\exists \text{ colim}}(J, C) \xrightarrow{s} \mathrm{Fun}^{\text{colim cone}}(J^{\triangleright}, C) \xrightarrow{\text{eval. at } v} C.$$

We want to prove the analogous statement for quasicategories. Thus, given a quasicategory C and a simplicial set S, let $\mathrm{Fun}^{\text{colim cone}}(S^{\triangleright}, C) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$ denote the full subcategory spanned by $S^{\triangleright} \to C$ which are colimit cones, and let $\mathrm{Fun}^{\exists \text{ colim}}(S, C) \subseteq \mathrm{Fun}(S, C)$ denote the full subcategory spanned by $S \to C$ for which a colimit exists.

56.1. **Proposition.** *The induced projection $q\colon \mathrm{Fun}^{colim\ cone}(S^{\triangleright}, C) \to \mathrm{Fun}^{\exists\ colim}(S, C)$ is a trivial fibration.*

We refer to this as the *functoriality of colimits*. We will prove it below.

The strategy is to show (1) that q is an isofibration, and (2) that q is fully faithful and essentially surjective. Then (43.1) applies to show that q is a categorical equivalence, and so a trivial fibration by (44.1).

Parts of this are already clear. For instance, q is certainly an inner fibration, since $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$ is one, and q is the restriction of p to full subcategories. Likewise, q is manifestly essentially surjective.

56.2. **Conical maps.** In what follows, C will be a quasicategory and S a simplicial set, and we write

$$V = V(S) := \mathrm{Fun}(S^{\triangleright}, C), \qquad U = U(S) := \mathrm{Fun}(S, C).$$

Let $p\colon V \to U$ be the evident restriction map.

Let's say that a morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V is **conical** if its evalutation $\widehat{\alpha}(v)\colon \widehat{f}(v) \to \widehat{g}(v)$ at the cone point of $S^{\triangleright}$ is an isomorphism in C.

"Conical" here is really equivalent to "p-Cartesian morphism" where $p\colon \mathrm{Fun}(S^{\triangleright}, C) \to \mathrm{Fun}(S, C)$**. This whole section needs to have that observation baked in.**

What follows are two propositions involving conical maps. We will prove them soon. The first says that any morphism in U can be lifted to a *conical* morphism in V with prescribed target.

56.3. **Proposition.** *Fix a quasicategory C and a simplicial set S. Suppose given*

- *a functor $\widehat{g}\colon S^{\triangleright} \to C$, and*
- *a natural transformation $\alpha\colon f \Rightarrow g$ of functors $S \to C$ such that $g = \widehat{g}|S$.*

Then there exists a conical morphism $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V such that $\widehat{\alpha}|S = \alpha$.

$$\begin{tikzcd} \{1\} \arrow[r, "\widehat{g}"] \arrow[d, tail] & \mathrm{Fun}(S^{\triangleright}, C) = V \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"'] \arrow[ur, dotted, "\widehat{\alpha}", "\textit{conical}"'] & \mathrm{Fun}(S, C) = U \end{tikzcd}$$

The second says that morphisms in V can be "transported" along conical maps.

56.4. **Proposition.** *Fix a quasicategory C, simplicial set S, and a map $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$ in V, and let $\alpha\colon f \to g$ denote $\widehat{\alpha}|S$. For any object $\widehat{h}$ of V, consider the square*

$$\begin{tikzcd} \mathrm{map}_V(\widehat{h}, \widehat{f}) \arrow[r, "\widehat{\alpha}\circ"] \arrow[d] & \mathrm{map}_V(\widehat{h}, \widehat{g}) \arrow[d] \\ \mathrm{map}_U(h, f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(h, g) \end{tikzcd}$$

where $h = \widehat{h}|S$, and the horizontal maps are induced by postcomposition with $\widehat{\alpha}$ and α respectively. If $\widehat{\alpha}$ is conical, then the above square is a homotopy pullback square.

We will explain and prove these two propositions soon. For the time being, you should convince yourself that if C is the nerve of an ordinary category, then both propositions are entirely straightforward to prove.

56.5. **Proof of functoriality of colimits, using properties of conical maps.** Recall that $\widehat{f}\colon S^{\triangleright} \to C$ extending $f\colon S \to C$ is a colimit cone if and only if it corresponds to an initial object of $C_{f/}$. Using the categorical equivalences

$$F(f) \to C^{f/} \leftarrow C_{f/}$$

where $F(f) \subseteq V$ is the fiber of $p\colon V \to U$ over f, we see that it is equivalent to say that $\widehat{f}$ is initial in $F(f)$.

The following gives a criterion for being a colimit cone in terms of the whole functor category $V = \mathrm{Fun}(S^{\triangleright}, C)$, rather than just in terms of the fiber over some f.

56.6. **Proposition.** *A functor $\widehat{f}\colon S^{\triangleright} \to C$ is a colimit cone if and only if*

$$p'\colon \mathrm{map}_V(\widehat{f}, \widehat{g}) \to \mathrm{map}_U(f, g)$$

is a weak equivalence for every $\widehat{g}\colon S^{\triangleright} \to C$, $g = \widehat{g}|S = p(\widehat{g})$.

Proof. Since $p\colon V \to U$ is a categorical fibration, the induced maps p' on mapping spaces are Kan fibrations. Thus, p' is a weak equivalence if and only if its fibers are contractible.

($\Longleftarrow$) Suppose every p' is a weak equivalence. Then in particular p' is a weak equivalence for any $\widehat{g}\colon S^{\triangleright} \to C$ such that $\widehat{g}|S = f$. In this case, the fiber of p' over $1_f \in \mathrm{map}_U(f, f)$ is precisely the mapping space $\mathrm{map}_{F(f)}(\widehat{f}, \widehat{g})$ in the fiber quasicategory $F(f) \subseteq \mathrm{Fun}(S^{\triangleright}, C)$, and this fiber is contractible. Therefore, $\widehat{f}$ is an initial object of $F(f)$, and therefore $\widehat{f}$ is initial in $C_{f/}$ by the above discussion. We have shown that $\widehat{f}$ is a colimit cone.

($\Longrightarrow$) Suppose $\widehat{f}$ is a colimit cone. Therefore for $\widehat{f}'$ such that $\widehat{f}'|S = f$ the fiber of $\mathrm{map}_V(\widehat{f}, \widehat{f}') \to \mathrm{map}_U(f, f)$ over 1_f is contractible. We need to show that the fiber of $p'\colon \mathrm{map}_V(\widehat{f}, \widehat{g}) \to \mathrm{map}_U(f, g)$ over a general $\alpha \in \mathrm{map}_U(f, g)$ is contractible.

Given such an α, choose a conical map $\widehat{\alpha}\colon \widehat{f}' \to \widehat{g}$ with $\widehat{\alpha}|S = \alpha$ (56.3), and consider the resulting square

$$\begin{tikzcd} \mathrm{map}_V(\widehat{f}, \widehat{f}') \arrow[r, "\widehat{\alpha}\circ"] \arrow[d, "p'"'] & \mathrm{map}_V(\widehat{f}, \widehat{g}) \arrow[d, "p''"] \\ \mathrm{map}_U(f, f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(f, g) \end{tikzcd}$$

$$1_f \longmapsto \alpha$$

Since $\widehat{\alpha}$ is conical, the square is a homotopy pullback square (56.4). Therefore, the fiber of p'' over α is weakly equivalent to the fiber of p' over 1_f, which is contractible since $\widehat{f}$ is a colimit cone. $\square$

Proof of (56.1). First we show that $q\colon \mathrm{Fun}^{\text{colim cone}}(S^{\rhd}, C) \to \mathrm{Fun}^{\exists\text{ colim}}(S, C)$ is an isofibration; we have already observed that it is an inner fibration. Given an isomorphism $\alpha\colon f \to g$ between objects in $\mathrm{Fun}^{\exists\text{ colim}}(S, C) \subseteq U$ and a choice of colimit cone $\widehat{g}$ over g, chose a conical lift $\widehat{\alpha}\colon \widehat{f} \to \widehat{g}$. The arrow $\widehat{\alpha}\colon S^{\rhd} \times \Delta^1 \to C$ restricts to an isomorphism at each vertex of $S^{\rhd}$, and so is a natural isomorphism by the objectwise criterion for natural isomorphisms. Thus $\widehat{f}$ is also a colimit cone by (56.6), so $\widehat{\alpha}$ is an isomorphism in $\mathrm{Fun}^{\text{colim cone}}(S^{\rhd}, C)$.

We have already observed that q is essentially surjective (in fact, it is surjective on vertices). That q is fully faithful is immediate from (56.6). □

56.7. **Proof of properties of conical maps.**

Proof of (56.3). Recall the situation: we are given a natural transformation $\alpha\colon f \Rightarrow g$ of functors $S \to C$, and a lift $\widehat{g}\colon S^{\rhd} \to C$ of the target to the cone, and we want to find a *conical* lift of α:

$$\begin{tikzcd} \{1\} \arrow[r, "\widehat{g}"] \arrow[d, tail] & \mathrm{Fun}(S^{\rhd}, C) \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"'] \arrow[ur, dotted, "\widehat{\alpha}", "\text{conical}"'] & \mathrm{Fun}(S, C) \end{tikzcd}$$

We make use of a natural map

$$\kappa\colon S^{\rhd} \times K \to (S \times K)^{\rhd}.$$

Note that this map sends $\{v\} \times K$ to the cone point $\{v\}$. Consider the composite

$$\lambda\colon (S \times \Delta^1) \cup_{S\times\{1\}} (S^{\rhd} \times \{1\}) \to S^{\rhd} \times \Delta^1 \xrightarrow{\kappa} (S \times \Delta^1)^{\rhd}$$

where the first map is the box-product $(S \subset S^{\rhd})\square(\{1\} \subset \Delta^1)$. By inspection, we see that the composite map can be identified with the box-join

$$(S \times \{1\} \subseteq S \times \Delta^1) \circledast (\varnothing \subseteq \Delta^0).$$

Since $\mathrm{RHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{RHorn}}$ (49.4) we have that $(S \times \{1\} \subseteq S \times \Delta^1)$ is right anodyne. Likewise, since $\mathrm{RHorn} \circledast \mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (27.13), we conclude that λ is inner anodyne. Therefore, an extension $\overline{\alpha}$ exists in

$$\begin{tikzcd} (S \times \Delta^1) \cup_{S\times\{1\}} (S^{\rhd} \times \{1\}) \arrow[r, "(\alpha,\widehat{g})"] \arrow[d, tail] & C \\ S^{\rhd} \times \Delta^1 \arrow[d, "\kappa"'] & \\ (S \times \Delta^1)^{\rhd} \arrow[uur, dotted, "\overline{\alpha}"'] & \end{tikzcd}$$

We set $\widetilde{\alpha} := \overline{\alpha} \circ \kappa$. It is clear that $\widetilde{\alpha}$ is conical: $\widehat{\alpha}(v)$ is the identity map of $\overline{\alpha}(v)$. □

For the proof of (56.4), let's first note that, as stated, it actually doesn't make sense! This proposition asserts that for conical $\widehat{\alpha}$, the diagram

$$\begin{tikzcd} \mathrm{map}_V(\widehat{h}, \widehat{f}) \arrow[r, "\widehat{\alpha}\circ"] \arrow[d] & \mathrm{map}_V(\widehat{h}, \widehat{g}) \arrow[d] \\ \mathrm{map}_U(h, f) \arrow[r, "\alpha\circ"'] & \mathrm{map}_U(h, g) \end{tikzcd}$$

is a homotopy pullback. However, the horizontal maps ("postcomposition" with α and $\widehat{\alpha}$) are only defined as a homotopy class of maps in $h\mathrm{Kan}$. For instance, "$\alpha\circ$" is the homotopy class defined by the zig-zag around the left and top of the diagram

$$\begin{tikzcd}
\mathrm{map}_U(h,f,g)_\alpha \arrow[r] \arrow[d, "\sim"'] & \mathrm{map}_U(h,f,g) \arrow[r, "\mathrm{comp}"] \arrow[d, "\sim"] & \mathrm{map}_U(h,g) \\
\mathrm{map}_U(h,f)\times\{\alpha\} \arrow[r] & \mathrm{map}_U(h,f)\times\mathrm{map}(f,g) &
\end{tikzcd}$$

where the left-hand square is a pullback. The correct statement of (56.4) is that in

$$\begin{tikzcd}
\mathrm{map}_V(\widehat{h},\widehat{f}) \arrow[d] & \mathrm{map}_V(\widehat{h},\widehat{f},\widehat{g})_{\widehat{\alpha}} \arrow[l, "\sim"'] \arrow[r] \arrow[d] & \mathrm{map}_V(\widehat{h},\widehat{g}) \arrow[d] \\
\mathrm{map}_U(h,f) & \mathrm{map}_U(h,f,g)_\alpha \arrow[l, "\sim"] \arrow[r] & \mathrm{map}_U(h,g)
\end{tikzcd}$$

the right-hand square is a homotopy pullback.

We can refine this a little further. Fix a map $e\colon \Delta^{\{1,2\}} \to C$. For a simplicial set S, let $K \subseteq S^{\triangleright}\times\Delta^2$ be a subcomplex containing the edge $\{v\}\times\Delta^{\{1,2\}}$, and define $\mathrm{Map}(K,C)_e$ by the pullback square

$$\begin{tikzcd}
\mathrm{Map}(K,C)_e \arrow[r, tail] \arrow[d] & \mathrm{Map}(K,C) \arrow[d] \\
\{e\} \arrow[r, tail] & \mathrm{Map}(\{v\}\times\Delta^{\{1,2\}},C)
\end{tikzcd}$$

To prove our proposition, it suffices to show that for every isomorphism e in C, the map

$$\mathrm{Map}(S^{\triangleright}\times\Delta^2,C)_e \to \mathrm{Map}((S^{\triangleright}\times\Lambda^2_2)\cup_{S\times\Lambda^2_2}(S\times\Delta^2),C)_e$$

is a trivial fibration. Equivalently, we must produce a lift in each diagram of the form

$$\begin{tikzcd}
& (S^{\triangleright}\times\partial\Delta^m)\cup_{S\times\partial\Delta^m}(S\times\Delta^m) \arrow[r] \arrow[d, tail] & \mathrm{Map}(\Delta^2,C) \arrow[d] & \\
\{v\}\times\Delta^{\{1,2\}} \arrow[r, tail] \arrow[drrr] & S^{\triangleright}\times\Delta^m \arrow[r] \arrow[ur, dotted] & \mathrm{Map}(\Lambda^2_2,C) \arrow[r] & \mathrm{Map}(\Delta^{\{1,2\}},C) \\
& & & \{e\} \arrow[u, tail]
\end{tikzcd}$$

We reduce to producing a lift in

$$\begin{tikzcd}
\{n\}\times\Delta^{\{1,2\}} \arrow[r, tail] \arrow[rr, bend left, "e"] & (\Delta^n\times\Lambda^2_2)\cup_{\partial\Delta^n\times\Lambda^2_2}(\partial\Delta^n\times\Delta^2) \arrow[r] \arrow[d, tail] & C \\
& \Delta^n\times\Delta^2 \arrow[ur, dotted] &
\end{tikzcd}$$

where e is an isomorphism in C. This is precisely the box-version of Joyal extension.

57. More stuff

I'm not sure what this is needed for.

Recall that the join constructions $K\star -$ and $-\star K$ are colimit preserving functors $s\mathrm{Set}\to s\mathrm{Set}_{K/}$ to the category of simplicial sets *under* K. In particular, viewed as functors $s\mathrm{Set}\to s\mathrm{Set}$ to plain simplicial sets, they preserve pushouts, and transfinite compositions.

57.1. **Proposition.** *If $\mathcal{A}$ is a class of maps in $s\mathrm{Set}$, then $K \star \overline{\mathcal{A}} \subseteq \overline{K \star \mathcal{A}}$ and $\overline{\mathcal{A}} \star K \subseteq \overline{\mathcal{A} \star K}$.*

Proof. Check that $K \star -\colon s\mathrm{Set} \to s\mathrm{Set}$ preserves isomorphisms, transfinite composition, pushouts, and retracts. ☐

57.2. *Remark.* Given $f\colon X \to Y$ and K, we have a factorization of $K \star f$ as

$$K \star X \to (K \star X) \amalg_{\varnothing \star X} (\varnothing \star Y) \xrightarrow{(\varnothing \subseteq K) \circledast f} K \star Y.$$

57.3. **Proposition.** *We have $\Delta^0 \star \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{LHorn}}$ and $\overline{\mathrm{Cell}} \star \Delta^0 \subseteq \overline{\mathrm{RHorn}}$.*

57.4. **Proposition.** *Let C be a quasicategory and x an object of C. Then x is an initial object iff $\{x\} \to C$ is left anodyne, and x is a terminal object iff $\{x\} \to C$ is right anodyne.*

Proof. ($\Longrightarrow$) Let x be terminal, and consider $j\colon \{x\} \to C$. Since $j^{\triangleright}$ is right anodyne, it suffices to show that j is a retract of $j^{\triangleright}$. To do this, we construct a map r fitting into

$$\begin{tikzcd}
\{x\} \arrow[r, tail] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}"] & \{x\}^{\triangleright} \arrow[r] \arrow[d, "j^{\triangleright}"'] & \{x\} \arrow[d, "j"'] \\
C \arrow[r, tail] \arrow[rr, bend right, "\mathrm{id}"'] & C^{\triangleright} \arrow[r, dotted, "r"] & C
\end{tikzcd}$$

This amounts to solving the lifting problem

$$\begin{tikzcd}
C \cup \{x\}^{\triangleright} \arrow[r, "{(\mathrm{id},1_x)}"] \arrow[d, tail] & C \\
C^{\triangleright} \arrow[ur, dotted]
\end{tikzcd}
\qquad \Longleftrightarrow \qquad
\begin{tikzcd}
\{x\} \arrow[r, "1_x"] \arrow[d, tail] & C_{x/} \arrow[d] \\
C \arrow[r, equal] \arrow[ur, dotted] & C
\end{tikzcd}$$

Since x is terminal, $C_{x/} \to C$ is a trivial fibration (**??**), so a lift exists.

($\Longleftarrow$) Suppose $j\colon \{x\} \to C$ is right anodyne. Since $C_{/x} \to C$ is a right fibration, a lift exists in

$$\begin{tikzcd}
\{x\} \arrow[r, "1_x"] \arrow[d, tail] & C_{x/} \arrow[d] \\
C \arrow[r, equal] \arrow[ur, dotted] & C
\end{tikzcd}$$

which is equivalent to x being terminal. ☐

57.5. **Corollary.** *Let $p\colon D \to C$ be a right fibration between quasicategories, and let x be an object of C. Then the induced map*

$$\mathrm{Map}(C_{/x}, D) \to \mathrm{Map}(\{1_x\}, D) \times_{\mathrm{Map}(\{1_x\}, C)} \mathrm{Map}(C_{/x}, C)$$

is a trivial fibration. In particular, the map

$$\mathrm{Map}_C(C_{/x}, D) \to \mathrm{Map}_C(\{1_x\}, D)$$

induced by restriction over the projection map $(C_{/x} \to C) \in \mathrm{Map}(C_{/x}, C)$ is a trivial fibration between Kan complexes.

58. Straightening and unstraightening

Let $\mathfrak{D}_{\Delta^n}\colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ be the simplicialy enriched functor defined as follows.

- For each object $x \in \{0, \dots, n\}$, set
$$\mathfrak{D}_{\Delta^n}(x) := N\mathcal{P}_\ell(x),$$
the nerve of the poset
$$\mathcal{P}_\ell(x) := \big\{S \mid \{x\} \subseteq S \subseteq [x,n]\big\}$$
of subsets of the interval $[x,n] = \{x, \dots, n\}$ which contain the left endpoint.
- The structure of enriched functor is induced by the union operation on subsets:
$$(T,S) \mapsto T \cup S\colon \mathcal{P}(x,y) \times \mathcal{P}_\ell(y) \to \mathcal{P}_\ell(x).$$
- For each map $\delta\colon \Delta^m \to \Delta^n$, we define a natural transformation
$$\mathfrak{D}_\delta\colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$$
of simplicially enriched functors $\mathfrak{C}(\Delta^m)^{\mathrm{op}} \to s\mathrm{Set}$, which at each object x of $\mathfrak{C}(\Delta^m)^{\mathrm{op}}$ is a map $\mathfrak{D}_{\Delta^m}(x) \to \mathfrak{D}_{\Delta^n}(\delta x)$ induced by the map of posets
$$S \mapsto \delta(S)\colon \mathcal{P}_\ell(x) \to \mathcal{P}_\ell(\delta x).$$

58.1. *Remark.* The functor $\mathcal{D}_{\Delta^n}\colon \mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$ is isomorphic to the representable functor $\mathrm{Map}_{\mathfrak{C}((\Delta^n)^{\triangleright})}(-,v)$, where v represents the cone point of $(\Delta^n)^{\triangleright}$. Likewise, the natural transformation $\mathfrak{D}_\delta\colon \mathfrak{D}_{\Delta^m} \to \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}}$ coincides with the transformation
$$\mathrm{Map}_{\mathfrak{C}((\Delta^m)^{\triangleright})}(-,v) \xrightarrow{\mathfrak{C}(\delta^{\triangleright})} \mathrm{Map}_{\mathfrak{C}((\Delta^n)^{\triangleright})}(\delta(-),v)$$
induced by $\delta^{\triangleright}\colon (\Delta^m)^{\triangleright} \to (\Delta^n)^{\triangleright}$.

Fix a simplical set S, and consider a simplicially enriched functor $F\colon \mathfrak{C}(S)^{\mathrm{op}} \to s\mathrm{Set}$. We define a morphism
$$\mathrm{Un}_S(F)\colon X \to S$$
of simplicial sets, called the **unstraightening** of F over S, as follows.

- An n-dimensional element of $\mathrm{Un}_S F$ is a pair
$$f\colon \Delta^n \to S, \qquad t\colon \mathfrak{D}_{\Delta^n} \to F \circ \mathfrak{C}(f),$$
where f is a map of simplicial sets, and t is a map of simplicially enriched functors $\mathfrak{C}(\Delta^n)^{\mathrm{op}} \to s\mathrm{Set}$.
- To a map $\delta\colon \Delta^m \to \Delta^n$ we have an induced map $(\mathrm{Un}_S F)_n \to (\mathrm{Un}_S F)_m$, which sends an n-dimensional element (f,t) to the pair
$$\Delta^m \xrightarrow{\delta} \Delta^n \xrightarrow{f} S, \qquad \mathfrak{D}_{\Delta^m} \xrightarrow{\mathfrak{D}_\delta} \mathfrak{D}_{\Delta^n} \circ \mathfrak{C}(\delta)^{\mathrm{op}} \xrightarrow{t \circ \mathfrak{C}(\delta)^{\mathrm{op}}} F \circ \mathfrak{C}(f) \circ \mathfrak{C}(\delta)^{\mathrm{op}}.$$

59. Cartesian fibrations

Let $p\colon C \to D$ a functor between ordinary categories. A morphism $f\colon x' \to x$ in C is called **p-Cartesian** if for every object c of C the evident commutative square
$$\begin{tikzcd}
\mathrm{Hom}_C(c,x') \arrow[r, "f\circ"] \arrow[d, "p"'] & \mathrm{Hom}_C(c,x) \arrow[d, "p"] \\
\mathrm{Hom}_D(p(c),p(x')) \arrow[r, "p(f)\circ"'] & \mathrm{Hom}_D(p(c),p(x))
\end{tikzcd}$$
is a pullback square of sets.

Given an object $x \in \operatorname{ob} C$ and a morphism $g\colon y' \to p(x)$ in D, a **Cartesian lift of** g **at** x is a p-Cartesian morphism $f\colon x' \to x$ such that $p(f) = g$.

We say that $p\colon C \to D$ is a **Cartesian fibration** of categories if every pair $(x \in \operatorname{ob} C, g\colon y' \to p(x) \in D)$ admits a Cartesian lift.

Here are some observations, whose verification we leave to the reader. Fix a functor $p\colon C \to D$.

- Every isomorphism in C is p-Cartesian.
- Every Cartesian lift of an isomorphism in D is itself an isomorphism.
- If $f\colon x' \to x$ is p-Cartesian, then for any $g\colon x'' \to x'$ in C, we have that g is p-Cartesian if and only if gf is p-Cartesian.
- Any two Cartesian lifts of g at x are "canonically isomorphic".

  Explicitly, fix $g\colon y' \to y$ in D and an object x in C such that $y = p(x)$. If $f_1\colon x'_1 \to x$ and $f_2\colon x'_2 \to x$ are any two Cartesian lifts of g, then there exists a unique map $u\colon x'_1 \to x'_2$ such that $p(u) = 1_{y'}$ and $f_2 u = f_1$; the map u is necessarily an isomorphism.
- The map p is a right fibration if and only if it is a Cartesian fibration and every morphism in C is p-Cartesian.

Now suppose that $p\colon C \to D$ is a Cartesian fibration. For an object y of D, we write $C_y := p^{-1}(y)$ for the fiber of C over y.

- The map p is an isofibration.
- For each morphism $g\colon y' \to y$ in D and object x in C with $p(x) = y$, fix a choice of Cartesian lift $\widetilde{g}_x$ of g at x. Using this data, we obtain functors

$$g^!\colon C_y \to C'_y$$

  so that for morphism $\alpha\colon x_1 \to x_2$ in C_y, the map $g^!(\alpha)$ in $C_{y'}$ is the unique one fitting into

$$\begin{tikzcd} x'_1 \arrow[r, "\widetilde{g}_{x_1}"] \arrow[d, "g^!(\alpha)"'] & x_1 \arrow[d, "\alpha"] \\ x'_2 \arrow[r, "\widetilde{g}_{x_2}"'] & x_2 \end{tikzcd}$$

  The functor $g^!$ depends on the choices of Cartesian lifts of g. Any two set of choices of lifts give rise to isomorphic functors.
- For each pair of morphisms $y'' \xrightarrow{h} y' \xrightarrow{g} y$, we obtain a natural isomorphism of functors

$$\gamma\colon h^! \circ g^! \xRightarrow{\sim} (hg)^!\colon C_y \to C_{y''}.$$

  This natural transformation is given by the unique maps γ_x in $C_{y''}$ fitting into

$$\begin{tikzcd} x'' \arrow[r, "\widetilde{h}_{x'}"] \arrow[d, "\gamma_x"'] & x' \arrow[d, "\widetilde{g}_x"'] \\ x''' \arrow[r, "\widetilde{(gh)}_x"'] & x \end{tikzcd}$$

  Similarly, there is a natural isomorphism $\mathrm{id} \xRightarrow{\sim} (1_y)_!\colon C_y \to C_y$. The data of the functors $g^!$ together with these natural isomorphisms define a **pseudofunctor** $D^{\mathrm{op}} \to \mathrm{Cat}$, which on objects sends $y \mapsto C_y$.
- We can produce an actual functor $F\colon D^{\mathrm{op}} \to \mathrm{Cat}$ with $F(y)$ equivalent to C_y as follows. Given functors $p'\colon C' \to D$ and $p\colon C \to D$, let $\mathrm{Fun}_D(C', C)$ denote the category of fiberwise functors and natural transformations; i.e., the fiber of $p\circ\colon \mathrm{Fun}(C', C) \to \mathrm{Fun}(C', D)$ over q.

Let $\mathrm{Fun}^{+}_{D}(C',C) \subseteq \mathrm{Fun}_D(C',C)$ denote the full subcategory of functors $f\colon C' \to C$ which take p'-Cartesian morphisms to p-Cartesian morphims.

We obtain a functor $F\colon D^{\mathrm{op}} \to \mathrm{Cat}$, given on objects by

$$F(y) := \mathrm{Fun}^{+}_{D}(D_{/y}, C).$$

One can show that restriction to $\{1_y\} \subseteq D_{/y}$ defines an equivalence of categories $F(y) \to C_y$.

- Given D, there is a 2-category $\mathcal{F}_D$, whose objects are Cartesian fibrations $p\colon C \to D$; for any two objects $p\colon C \to D$ and $p'\colon C' \to D$ we take $\mathrm{Fun}^{+}_{D}(C',C)$ as the category of morphisms from p' to p. One can show that $\mathcal{F}_D$ is 2-equivalent to the 2-category $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat})$.

Part 8. Appendices

60. Appendix: Generalized horns

A **generalized horn**[33] is a subcomplex $\Lambda^n_S \subset \Delta^n$ of the standard n-simplex, where $S \subseteq [n]$ and

$$(\Lambda^n_S)_k := \{\, f\colon [k] \to [n] \;\mid\; S \not\subseteq f([k]) \,\}.$$

In other words, a generalized horn is a union of some codimension 1 faces of the n-simplex:

$$\Lambda^n_S = \bigcup_{s\in S} \Delta^{[n]\smallsetminus s}.$$

In particular,

$$\Lambda^n_{[n]} = \partial\Delta^n, \quad \Lambda^n_{[n]\smallsetminus j} = \Lambda^n_j, \quad \Lambda^n_{\{j\}} = \Delta^{[n]\smallsetminus j}, \quad \Lambda^n_{\varnothing} = \varnothing.$$

In general $S \subseteq T$ implies $\Lambda^n_S \subseteq \Lambda^n_T$.

60.1. **Proposition** (Joyal [Joy08a, Prop. 2.12]). *Let $S \subsetneq [n]$ be a* proper *subset.*

1. $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{Horn}}$ *if $S \neq \varnothing$.*
2. $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{LHorn}}$ *if $n \in S$.*
3. $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{RHorn}}$ *if $0 \in S$.*
4. $(\Lambda^n_S \subset \Delta^n) \in \overline{\mathrm{InnHorn}}$ *if S is not an "interval"; i.e., if there exist $a < b < c$ with $a, c \in S$ and $b \notin S$.*

Proof. We start with an observation. Consider $S \subsetneq [n]$ and $t \in [n] \smallsetminus S$. Observe the diagram

$$\begin{tikzcd}
\Delta^{[n]\smallsetminus t} \cap \Lambda^n_S \arrow[r, tail] \arrow[d, tail] & \Delta^{[n]\smallsetminus t} \arrow[d, tail] & \\
\Lambda^n_S \arrow[r, tail] & \Lambda^n_{S\cup t} \arrow[r, tail] & \Delta^n
\end{tikzcd}$$

in which the square is a pushout, and the top arrow is isomorphic to the generalized horn $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$. Thus, $(\Lambda^n_S \subset \Delta^n)$ is contained in the weak saturation of any set contaning the two inclusions

$$\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t} \quad \text{and} \quad \Lambda^n_{S\cup t} \subset \Delta^n.$$

Each of the statements of the proposition is proved by an evident induction on the size of $[n] \smallsetminus S$, using the above observation. I'll do case (4), as the other cases are similar. If $S \subset [n]$ is not an interval, there exists some $s < u < s'$ with $s, s' \in S$ and $u \notin S$. If $[n] \smallsetminus S = \{u\}$ then we already have an inner horn. If not, then choose $t \in [n] \smallsetminus (S \cup \{u\})$, in which case $S \cup t$ is not an interval in $[n]$, and S is not an interval in $[n] \smallsetminus t$. Therefore both $\Lambda^{[n]\smallsetminus t}_S \subset \Delta^{[n]\smallsetminus t}$ and $\Lambda^n_{S\cup t} \subset \Delta^n$ are inner anodyne by the inductive hypothesis. The proofs of the other cases are similar. $\square$

[33] This notion is from [Joy08a, §2.2.1]. However, I have changed the sense of the notation: our Λ^n_S is Joyal's $\Lambda^{[n]\smallsetminus S}$. I find my notation easier to follow, but note that it does conflict with the standard notation for horns. Maybe I should use something like $\Lambda^{n,S}$?

60.2. **Proposition** (Joyal [Joy08a, Prop. 2.13]). *For all $n \geq 2$, we have that $(I^n \subset \Delta^n) \in \overline{\mathrm{InnHorn}}$.*

Proof. We can factor the inclusion spine inclusion as $h_n = g_n f_n$:

$$I^n \xrightarrow{f_n} \Delta^{\{1,\dots,n\}} \cup I^n \xrightarrow{g_n} \Delta^n.$$

We show by induction on n that $f_n, g_n, h_n \in \overline{\mathrm{InnHorn}}$, noting that the case $n = 2$ is immediate.

To show that $f_n \in \overline{\mathrm{InnHorn}}$, consider the pushout square

$$\begin{tikzcd} I^{\{1,\dots,n\}} \arrow[r, tail] \arrow[d, tail] & \Delta^{\{1,\dots,n\}} \arrow[d, tail] \\ I^n \arrow[r, tail, "f_n"'] & \Delta^{\{1,\dots,n\}} \cup I^n \end{tikzcd}$$

in which the top arrow is isomorphic to h_{n-1}, which is in $\overline{\mathrm{InnHorn}}$ by induction.

To show that $g_n \in \overline{\mathrm{InnHorn}}$, consider the diagram

$$\begin{tikzcd} \Delta^{\{1,\dots,n-1\}} \cup I^{\{0\dots,n-1\}} \arrow[r, tail, "g_{n-1}"] \arrow[d, tail] & \Delta^{\{0,\dots,n-1\}} \arrow[d, tail] & \\ \Delta^{\{1,\dots,n\}} \cup I^n \arrow[r, tail] & \Delta^{\{1,\dots,n\}} \cup \Delta^{\{0,\dots,n-1\}} \arrow[r, tail] & \Delta^n \end{tikzcd}$$

in which the square is a pushout, the top horizontal arrow is isomorphic to g_{n-1}, an element of $\overline{\mathrm{InnHorn}}$ by induction, and the bottom right horizontal arrow is equal to $\Lambda^n_{\{0,n\}} \subset \Delta^n$, which is in $\overline{\mathrm{InnHorn}}$ by (60.1)(4).

□

61. Leftover examples

61.1. **The Morita quasicategory.** This is an example of a quasicategory in which *objects* are associative rings, *morphisms* between two rings are bimodules for the pair of rings, and *2-dimensional elements* are given by certain isomorphisms of bimodules.

Define a simplicial set C, so that C_n is a set whose elements are data $x := (A_i, M_{ij}, f_{ijk})$, where

- for each $i \in [n]$, A_i is an associative ring,
- for each $i < j$ in $[n]$, M_{ij} is an (A_i, A_j)-bimodule,
- for each $i < j < k$ in $[n]$, $f_{ijk} \colon M_{ij} \otimes_{A_j} M_{jk} \to M_{ik}$ is an isomorphism of (A_i, A_k)-bimodules, such that
- for each $i < j < k < \ell$, the diagram

$$\begin{tikzcd} M_{ij} \otimes M_{jk} \otimes M_{k\ell} \arrow[r, "\mathrm{id} \otimes f_{jk\ell}"] \arrow[d, "f_{ijk} \otimes \mathrm{id}"'] & M_{ij} \otimes M_{j\ell} \arrow[d, "f_{ij\ell}"] \\ M_{ik} \otimes M_{k\ell} \arrow[r, "f_{ik\ell}"'] & M_{i\ell} \end{tikzcd} \tag{61.2}$$

  commutes.

Here is a picture of the data of an n-simplex for $n \in \{0,1,2,3\}$:

$$
A_0
\qquad
\begin{tikzcd}
A_0 \arrow[d, "M_{01}"'] \\
A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[dd, "M_{01}"'] \arrow[dr, "M_{02}"] & \\
\arrow[r, phantom, "f_{012}"] & A_2 \\
A_1 \arrow[ur, "M_{12}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[dd, bend right, "M_{01}"'] \arrow[dr, "M_{02}"] \arrow[drr, bend left, "M_{03}"] & \arrow[d, phantom, "f_{023}"] & \\
\arrow[r, phantom, "f_{012}"] & A_2 \arrow[r, "M_{23}"] & A_3 \\
A_1 \arrow[ur, "M_{12}"] \arrow[urr, bend right, "M_{13}"'] & \arrow[u, phantom, "f_{123}"] &
\end{tikzcd}
$$

For an simplicial operator $\delta\colon [m] \to [n]$, we define $x\delta := (A_{\delta(i)}, M_{\delta(i)\delta(j)}, f_{\delta(i)\delta(j)\delta(k)})$. When δ is injective this stands as it is, but if δ is not injective, we must set $M_{ij} := A_{\delta(i)}$ when $\delta(i) = \delta(j)$, and set f_{ijk} to the canonical isomorphism $A_{\delta(i)} \otimes_{A_{\delta(j)}} M_{\delta(j)\delta(k)} \to M_{\delta(i)\delta(k)}$ if $\delta(i) = \delta(j)$ or $M_{\delta(i)\delta(j)} \otimes_{A_{\delta(j)}} A_{\delta(k)} \to M_{\delta(i)\delta(k)}$ if $\delta(j) = \delta(k)$.

I claim that C is a quasicategory. Fillers for $\Lambda^2_1 \subset \Delta^2$ always exist: a map $\Lambda^2_1 \to C$ is a choice of $(A_0, M_{01}, A_1, M_{12}, A_2)$, and an extension to Δ^2 can be given by setting M_{02} to be the tensor product, and f_{012} the identity map. Note that there can be more than one choice: even keeping M_{02} the same, there is a choice of isomorphism f_{012}.

Fillers for $\Lambda^3_1 \subset \Delta^3$ and $\Lambda^3_2 \subset \Delta^3$ always exist, and are unique: finding a filler amounts to choosing isomorphisms $f_{023} = f_{ik\ell}$ (for Λ^3_1) or $f_{013} = f_{ij\ell}$ (for Λ^3_2) making (61.2) commute, and such choices are unique. Similarly, all fillers in higher dimensions $\Lambda^n_j \subset \Delta^n$ with $n \geq 4$ exist and are unique.

61.3. **Nerve of a crossed module.** A **crossed module** is data (G, H, ϕ, ρ), consisting of groups G and H, and homomorphisms $\phi\colon H \to G$ and $\rho\colon G \to \mathrm{Aut}H$, such that

$$\phi(\rho(g)(h)) = g\phi(h)g^{-1}, \qquad \rho(\phi(h))(h') = hh'h^{-1}, \qquad \text{for all } g \in G,\ h, h' \in H.$$

(For instance: $G = H =$ the cyclic group of order 4, with $\phi(x) = x^2$ and ρ the non-trivial action.) From this we can construct a quasicategory (in fact, a "quasigroupoid") much as in the last example: an n-simplex is data (g_{ij}, h_{ijk}) with $g_{ij} \in G$, $h_{ijk} \in H$, satisfying identities

$$g_{ij}g_{jk} = \phi(h_{ijk})g_{ik}, \qquad h_{ijk}h_{ik\ell} = \rho(g_{ij})(h_{jk\ell})h_{ij\ell}.$$

61.4. **Spans.** (See [Bar14, §§2–3], where this is called the *effective Burnside ∞-category.*) For each object $[n]$ of Δ, define $[n]^{\mathrm{tw}}$ to be the category with

- **objects** pairs (i, j) with $0 \leq i \leq j \leq n$, and
- a unique **morphism** $(i, j) \to (i', j')$ whenever $i' \leq i \leq j \leq j'$.

The construction $[n] \mapsto [n]^{\mathrm{tw}}$ defines a functor $\Delta \to \mathrm{Cat}$. (The category $[n]^{\mathrm{tw}}$ is called the *twisted arrow category* of $[n]$; in fact you can define a twisted arrow category C^{tw} for any category C.)

Let C be a category which has pullbacks; for an explicit example, think of the category of finite sets. Let $\mathcal{R}(C)$ be the simplicial set defined so that

$$\mathcal{R}(C)_n := \{\text{functors } ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C\}.$$

Elements of $\mathcal{R}(C)_0$ are just objects of C. Elements of $\mathcal{R}(C)_1$, $\mathcal{R}(C)_2$, $\mathcal{R}(C)_3$ are respectively diagrams in C of shape

$$
\begin{tikzcd}[column sep=small]
& X_{01} \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & X_{02} \arrow[dl] \arrow[dr] & & \\
& X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11} & & X_{22}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & & X_{03} \arrow[dl] \arrow[dr] & & & \\
& & X_{02} \arrow[dl] \arrow[dr] & & X_{13} \arrow[dl] \arrow[dr] & & \\
& X_{01} \arrow[dl] \arrow[dr] & & X_{12} \arrow[dl] \arrow[dr] & & X_{23} \arrow[dl] \arrow[dr] & \\
X_{00} & & X_{11} & & X_{22} & & X_{33}
\end{tikzcd}
$$

Let $\mathcal{A}(C)_n \subseteq \mathcal{R}(C)_n$ denote the subset whose n-dimensional elements are functors $X\colon ([n]^{\mathrm{tw}})^{\mathrm{op}} \to C$ such that for every $i' \leq i \leq j \leq j'$ the square

$$\begin{tikzcd} X_{i'j'} \arrow[r] \arrow[d] & X_{ij'} \arrow[d] \\ X_{i'j} \arrow[r] & X_{ij} \end{tikzcd}$$

is a pullback in C. Then $\mathcal{A}(C)$ is a subcomplex, and in fact is a quasicategory. This is another example in which extensions along inner horns $\Lambda^n_j \subset \Delta^n$ exist for $n \geq 2$, and are unique for $n \geq 3$.

62. Appendix: Box product lemmas

Here is where I'l prove various statements mentioned in the text.

- $\mathrm{LHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{LHorn}}$ (49.4), proved in (62.1) below.
- $\mathrm{RHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{RHorn}}$ (49.4), proved in (62.1) below.
- $\mathrm{Horn}\square\mathrm{Cell} \subseteq \overline{\mathrm{Horn}}$, is a consequence of the above, since $\mathrm{Horn} = \mathrm{LHorn} \cup \mathrm{RHorn}$ and $\overline{\mathrm{LHorn}} \cup \overline{\mathrm{RHorn}} \subseteq \overline{\mathrm{Horn}}$.
- $\mathrm{InnHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{InnHorn}}$ (16.10), proved in (62.3) below.

62.1. **Left and right horns.** We prove the case of $\mathrm{LHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{LHorn}}$ here. Given this $\mathrm{RHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{RHorn}}$ follows since $\mathrm{op}\colon s\mathrm{Set} \to s\mathrm{Set}$ carries LHorn to RHorn and preserves Cell.

Joyal [Joy08a, 2.25][34] observes that $(\Lambda^n_k \subset \Delta^n)$ is a retract of $(\Lambda^n_k \subset \Delta^n)\square(\{0\} \subset \Delta^1)$ when $0 \leq k < n$. The retraction is

$$\Delta^n \xrightarrow{s} \Delta^n \times \Delta^1 \xrightarrow{r} \Delta^n$$

defined by $s(x) = (x, 1)$ and

$$r(x,0) = \begin{cases} x & \text{if } x \leq k, \\ k & \text{if } x \geq k, \end{cases} \qquad r(x,1) = x.$$

Note that $r(\Delta^{[n]\smallsetminus j} \times \Delta^1) = \Delta^{[n]\smallsetminus j}$ if $j \neq k$, and $r(\Delta^n \times \{0\}) = \Delta^{\{0,\dots,k\}} \subseteq \Delta^{[n]\smallsetminus (k+1)}$, so this gives the desired retraction.

The existence of the retraction reduces showing $\mathrm{LHorn}\square\mathrm{Cell} \subseteq \overline{\mathrm{LHorn}}$ to proving

$$(\{0\} \subset \Delta^1)\square\mathrm{Cell} \subseteq \overline{\mathrm{LHorn}},$$

since $(\Lambda^n_k \subset \Delta^n) \in \mathrm{Cell}$ and thus $(\Lambda^n_k \subset \Delta^n)\square\mathrm{Cell} \subseteq \overline{\mathrm{Cell}}$.

62.2. **Lemma.** *We have that* $(\{0\} \subset \Delta^1)\square\mathrm{Cell} \subseteq \overline{\mathrm{LHorn}}$.

Proof. …Let $K = (\{0\} \times \Delta^n) \cup (\Delta^1 \times \partial\Delta^n)$, so that $(\{0\} \subset \Delta^1)\square(\partial\Delta^n \subset \Delta^n)$ is the inclusion $K \to \Delta^1 \times \Delta^n$. We will show that we can build $\Delta^1 \times \Delta^n$ from K by an explicit sequence of steps, where in each case we attach an $(n+1)$-sequence along a left horn.

For each $0 \leq a \leq n$ let τ_a be the $(n+1)$-dimensional element of $\Delta^1 \times \Delta^n$ defined by

$$\tau_a = \langle (0,0), \dots, (0,a),\ (1,a), \dots, (1,n) \rangle.$$

We obtain an ascending filtration of $\Delta^1 \times \Delta^n$ by starting with K and attaching simplices in the following order:

$$\tau_n, \tau_{n-1}, \dots, \tau_1, \tau_0.$$

The τs range through all non-degenerate $(n+1)$-dimensional elements of $\Delta^1 \times \Delta^n$, so $K \cup \bigcup \tau_a = \Delta^1 \times \Delta^n$. (Here I am using the same notation for elements $\tau_a \in (\Delta^1 \times \Delta^n)_{n+1}$ and for the corresponding subcomplex of $\Delta^1 \times \Delta^n$ which is isomorphic to Δ^{n+1}.)

The claim is that each attachment is along a specified horn inclusion. More precisely, for $a \in [n]$ the simplex τ_a is attached to $K \cup \bigcup_{k>a} \tau_k$ along the horn at the vertex $(0,a)$ in τ_a, i.e., via a

[34] Lurie [Lur09, 2.1.2.6] states this incorrectly.

$\Lambda^{n+1}_a \subset \Delta^n$ horn inclusion. Note that if when $a > 0$ this is an inner horn, while when $a = 0$ this is the inclusion $\Lambda^{n+1}_0 \subset \Delta^n$; in either case, it is a left horn. Given the claim, it follows that $(\{0\} \subset \Delta^1)\Box(\partial\Delta^n \subset \Delta^n) \in \overline{\text{LHorn}}$ as desired.

The proof of the claim amounts to the following list of elementary observations about τ_a:

- Every codimension-one face is contained in $\Delta^1 \times \partial\Delta^n$ except: the face opposite vertex $(0, a)$, and the face opposite vertex $(1, a)$.
- The face opposite vertex $(1, a)$ is contained in $\{0\} \times \Delta^n$ if $a = n$, or is a face of τ_{a+1} if $a < n$.
- The face opposite vertex $(0, a)$ is not contained in $\Delta^1 \times \partial\Delta^n$, nor in $\{0\} \times \Delta^n$. Nor is it contained in any τ_i with $i > a$ (beacuse the vertex $(1, a)$ is in this face but not in τ_i with $i > a$).

Taken together these show that $\tau_a \cap (K \cup \bigcup_{k>a} \tau_k)$ is the ath horn in the $(n + 1)$-simplex τ_a.

□

62.3. **Inner horns.** Here is an argument for the key case for inner horns.

Consider $\Delta^n \xrightarrow{s} \Delta^2 \times \Delta^n \xrightarrow{r} \Delta^n$, the unique maps which are given on vertices by

$$s(y) = \begin{cases} (0, y) & \text{if } y < j, \\ (1, y) & \text{if } y = j, \\ (2, y) & \text{if } y > j, \end{cases} \qquad r(x, y) = \begin{cases} y & \text{if } x = 0 \text{ and } y < j, \\ y & \text{if } x = 2 \text{ and } y > j, \\ j & \text{otherwise.} \end{cases}$$

These explicitly exhibit $(\Lambda^n_j \subset \Delta^n)$ as a retract of $(\Lambda^2_1 \subset \Delta^2)\Box(\Lambda^n_j \subset \Delta^n)$, so

$$\text{InnHorn} \subseteq \{\Lambda^2_1 \subset \Delta^2\}\Box\overline{\text{Cell}}.$$

We have (17.5) that $\overline{\text{Cell}}\Box\overline{\text{Cell}} \subseteq \overline{\text{Cell}}$, so the above implies that $\text{InnHorn}\Box\overline{\text{Cell}} \subseteq \{\Lambda^2_1 \subset \Delta^2\}\Box\overline{\text{Cell}}$. Thus the assertions "$\text{InnHorn}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" and "$\{\Lambda^2_0 \subset \Delta^2\}\Box\overline{\text{Cell}} \subseteq \overline{\text{InnHorn}}$" are equivalent. Thus both assertions follow from the following.

62.4. **Lemma.** *For all $n \geq 0$ we have that $(\Lambda^2_1 \subset \Delta^2)\Box(\partial\Delta^n \subset \Delta^n) \in \overline{\text{InnHorn}}$.*

Proof. [Lur09, 2.3.2.1].

For each $0 \leq a \leq b < n$, let σ_{ab} be the $(n + 1)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\sigma_{ab} = \langle (0, 0), \ldots, (0, a),\ (1, a), \ldots, (1, b),\ (2, b + 1), \ldots, (2, n)\rangle.$$

For each $0 \leq a \leq b \leq n$, let τ_{ab} be the $(n + 2)$-simplex of $\Delta^2 \times \Delta^n$ defined by

$$\tau_{ab} = \langle (0, 0), \ldots, (0, a),\ (1, a), \ldots, (1, b),\ (2, b), \ldots, (2, n)\rangle.$$

The set $\{\tau_{ab}\}$ consists of all the non-degenerate $(n + 2)$-dimensional elements. Note that σ_{ab} is a face of τ_{ab} and of $\tau_{a,b+1}$, but not a face of any other τ.

We attach simplices to $K := (\Lambda^2_1 \times \Delta^n) \cup (\Delta^2 \times \partial\Delta^n)$ in the following order:

$$\sigma_{00},\ \ \sigma_{01}, \sigma_{11},\ \ \sigma_{02}, \sigma_{12}, \sigma_{22},\ \ \ldots\ \ \sigma_{0,n-1}, \ldots, \sigma_{n-1,n-1},$$

followed by

$$\tau_{00},\ \ \tau_{01}, \tau_{11},\ \ \tau_{02}, \tau_{12}, \tau_{22},\ \ \ldots\ \ \tau_{0,n}, \ldots, \tau_{n,n}.$$

The τs range through all the non-degenerate $(n + 2)$-dimensional elements of $\Delta^2 \times \Delta^n$, so that $K \cup \bigcup \sigma_{a,b} \cup \bigcup \tau_{a,b} = \Delta^2 \times \Delta^n$.

The claim is that each attachment is along an inner horn inclusion. More precisely, each σ_{ab} gets attached along the horn at the vertex $(1, a)$ in σ_{ab}, i.e., via a $\Lambda^{n+1}_{a+1} \subset \Delta^{n+1}$ horn inclusion, which is always inner since $a \leq b < n$. Likewise, each τ_{ab} gets attached along the horn at vertex $(1, a)$ in τ_{ab}, i.e., via a $\Lambda^{n+2}_{a+1} \subset \Delta^{n+2}$ horn inclusion, which is always inner since $a \leq b \leq n$.

The proof of the claim amounts to the following lists of elementary observations.

For $\sigma_{a,b}$:

- Every codimension-one face is contained in $\Delta^2\times\partial\Delta^n$, except the following: the face opposite vertex $(0,a)$, and the face opposite vertex $(1,a)$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda^2_0\times\Delta^n$ if $a=0$, or a face of $\sigma_{a-1,b}$ if $a>0$.
- The face of $\sigma_{a,b}$ opposite vertex $(1,a)$ is not contained in $\Delta^2\times\partial\Delta^n$, nor in $\Lambda^2_0\times\Delta^n$, nor in any $\sigma_{i,b}$ with $i<a$ (because of the vertex $(0,a)$), nor in any $\sigma_{i,j}$ with $i\leq j<b$ (because of the vertex $(1,b)$ if $a<b$, or the vertex $(0,a)$ if $a=b$).

For $\tau_{a,b}$ when $a<b$:

- Every codimension-one face is contained in $\Delta^2\times\partial\Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a)$, the face opposite vertex $(1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$, while the face opposite vertex $(1,b)$ is $\sigma_{a,b-1}$.
- The face opposite vertex $(0,a)$ is either contained in $\Lambda^2_1\times\Delta^n$ if $a=0$, or is a face of $\tau_{a-1,b}$ if $a>0$.
- The face opposite vertex $(1,a)$ is not contained in $\Delta^2\times\partial\Delta^n$, nor in $\Lambda^2_1\times\Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(1,b)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i<b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i\leq j<b$ (because of the vertex $(1,b)$).

For $\tau_{a,b}$ when $a=b$:

- Every codimension-one face is contained in $\Delta^2\times\partial\Delta^n$ except the following: the face opposite vertex $(0,a)$, the face opposite vertex $(1,a)=(1,b)$, and the face opposite vertex $(2,b)$.
- The face opposite vertex $(2,b)$ is $\sigma_{a,b}$.
- The face opposite vertex $(0,a)$ is contained in $\Lambda^2_1\times\Delta^n$ if $a=0$, or is a face of $\tau_{a-1,b}$ if $a>0$.
- The face opposite vertex $(1,a)=(1,b)$ is not contained in $\Delta^2\times\partial\Delta^n$, nor in $\Lambda^2_1\times\Delta^n$, nor in any $\sigma_{i,j}$ (because of the vertices $(0,a)$ and $(2,b)$), nor in any $\tau_{i,b}$ with $i<b$ (because of the vertex $(0,a)$), nor in any $\tau_{i,j}$ with $i\leq j<b$ (because of the vertex $(0,a)$).

$\square$

62.5. **A pushout-product version of Joyal lifting.** We now give a proof of (31.6): we will prove the case of $(i,j)=(0,0)$, i.e., given $p\colon C\to D$ an inner fibration of quasicategories, $n\geq 1$, and

$$\begin{tikzcd}
\Delta^1\times\{0\} \arrow[r, tail] \arrow[rr, bend left, "f"] & (\{0\}\times\Delta^n)\cup_{\{0\}\times\partial\Delta^n}(\Delta^1\times\partial\Delta^n) \arrow[r] \arrow[d, tail] & C \arrow[d, "p"] \\
& \Delta^1\times\Delta^n \arrow[r] \arrow[ur, dotted] & D
\end{tikzcd}$$

such that f represents an isomorphism in C, we will construct a lift. (Note that if $n=0$ such a lift does not generally exist.)

We refer to the proof of (62.2), where we observed that we can build $\Delta^1\times\Delta^n$ from $K=(\{0\}\times\Delta^n)\cup(\Delta^1\times\partial\Delta^n)$ by successively attaching a sequence $\tau_n,\dots,\tau_0$ of $(n+1)$-simplices along horns; in particular, τ_a is attached to $K\cup\bigcup_{k>a}\tau_k$ along a horn inclusion isomorphic to $\Lambda^{n+1}_a\subset\Delta^{n+1}$.

Given this, we thus construct the desired lift by inductively choosing a lift defined on each τ_a relative to the given lift on its Λ^{n+1}_a-horn. When $a>0$ such a lift exists because p is an inner fibration and τ_a is attached along an inner horn, while when $a=0$ a lift exists by Joyal lifting (29.13), as $\Delta^1\times\{0\}$ is the leading edge of τ_0.

63. Appendix: Weak equivalences and homotopy groups

63.1. **Reduction to Kan fibrations.** Say that a map $f\colon X \to Y$ between Kan complexes is a **π_*-equivalence** if for all $k \geq 0$ and all $x \in X_0$, the induced map $\pi_k(X,x) \to \pi_k(Y, f(x))$ is a bijection. It is clear that every weak equivalence of Kan complexes is a π_*-equivalence.

63.2. **Proposition.** *The class of π_*-equivalences satisfies 2-out-of-6, and thus satisfies 2-out-of-3.*

Proof. This is much like the proof that functors which are essentially surjective and fully faithful share this property. One ingredient is to prove that if $f_0, f_1\colon X \to Y$ are functors which are naturally isomorphic, then f_0 is a π_*-equivalenec if and only if f_1 is. Another ingredient is the observation that to check that f is a π_*-equivalence, it suffices to check $\pi_k(X,x) \to \pi_k(Y, f())$ for $x \in S$ where $S \subseteq X_0$ is a set of representatives of $\pi_0 X$. □

Given this, it is straightforward to reduce to the case that f is a Kan fibration between Kan complexes which is a π_*-equivalence, using the path factorization construction. In this case, we actually prove that f is a trivial fibration, using the following.

63.3. **Proposition.** *Let $p\colon X \to Y$ be a Kan fibration between Kan complexes, and consider $n \geq 0$. Then $\mathrm{Cell}_{\leq n} \boxslash p$ if and only if, for all $0 \leq k < n$ and all $x \in X_0$, the induced map*

$$\pi_k(X,x) \to \pi_k(Y, p(x))$$

is a bijection, and is a surjection for $k = n$.

Note that the $n = 0$ case is immediate.

63.4. **Some lemmas.** Fix $p\colon X \to Y$ a Kan fibration between Kan complexes. Define

$$\mathcal{C}_p := \left\{ i \in \overline{\mathrm{Cell}} \;\middle|\; i \boxslash p \right\}, \qquad \mathcal{D}_p := \left\{ i \in \overline{\mathrm{Cell}} \;\middle|\; p^{\square i} \in \mathrm{TrivFib} \right\}.$$

These are weakly saturated classes, and $\mathcal{D}_p \subseteq \mathcal{C}_p$. Note that $\mathcal{D}_p = \bigcap_{n \geq 0} \mathcal{C}_{p^{\square(\partial\Delta^n \subset \Delta^n)}}$.

63.5. **Lemma.** *Given any commutative square*

$$\begin{tikzcd} A \arrow[r, "\sim"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r, "\sim"'] & B' \end{tikzcd}$$

such that the horizontal maps are monomorphisms and weak equivalences, we have that $i \in \mathcal{C}_p$ if and only if $i' \in \mathcal{C}_p$, and $i \in \mathcal{D}_p$ if and only if $i'\mathcal{D}_p$.

Proof. By covering homotopy extension, $i \in \mathcal{C}_p$ is equivalent to

$$\pi_0 \mathrm{Map}(B, X) \to \pi_0(\mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, y))$$

being surjective. This implies the first statement, and the second statement follows from the first. □

63.6. **Proposition.** *Let $T \subseteq \mathrm{Map}(B, Y)_0$ be a set of vertices which includes at least one representative from each path component. Then to show $i \boxslash p$, it suffices to solve all lifting problems (u, v) such that $v \in T$.*

Let $S \subseteq \mathrm{Map}(A, X)_0$ be a set of vertices which includes at least one representative from each path component. Then to show $i \boxslash p$, it suffices to solve all lifting problems (u, v) such that $u \in S$.

Note that $\pi_n(X, x) = \pi_0 \mathrm{Map}_*(\square^n/\partial\square^n, X)$. Use above to show this is the same as $\pi_0 \mathrm{Map}_*(\Delta^n/\partial\Delta^n, X)$, using weak equivalences $\partial\square^n \to \partial\Delta^n$. We also have weak equivalences $\partial\Delta^n \to \Delta^{n-1}/\partial\Delta^{n-1}$, whence $\pi_{n-1}(X, x) \approx \pi_0 \mathrm{Map}_*(\partial\Delta^n, X)$.

63.7. **Proposition.** *Let $n \geq 0$. If $\pi_n(X, x) \approx *$ for all $x \in X_0$, then $(\partial\Delta^{n+1} \subseteq \Delta^n) \boxslash (X \to *)$.*

Proof. Using a weak equivalence $\partial\Delta^{n+1} \approx \Delta^n/\partial\Delta^n$, we see that $\pi_n(X, x) \approx *$ implies that any $\partial\Delta^{n+1} \to X$ is homotopic to a constant map. Any lifting problem (u, v) of this type admits a lift since $\partial\Delta^{n+1}/\partial\Delta^{n+1} \to \Delta^{n+1}/\partial\Delta^{n+1}$ admits a retraction. □

Suppose p is surjective on π_n, and iso on π_k for $k < n$. By induction on n this gives $\mathrm{Cell}_{\leq n-1} \subseteq \mathcal{C}_p$.

Show that with these hypotheses (e.g., injectivity on π_{n-1}), any lifting problem $(\partial\Delta^n \subset \Delta^n) \Rightarrow p$ can be deformed to one which factors through $(* \subset \Delta^n/\partial\Delta^n) \Rightarrow p$. Then surjectivity on π_n gives the solution.

64. A set which generates $\mathrm{CatEq} \cap \overline{\mathrm{Cell}}$

The idea is to show that CatFib is the right complement of the class of all injective categorical equivalences $K \to L$ for which the number of elements in K and L is bounded by some explicit cardinal κ. We obtain S by choosing one representative for each isomorphism class in this class; then S is a set because of the cardinality bound.

We will define a **detection functor** $F\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], \mathrm{Set})$ on categories of morphisms. This will have the following properties:

- For each map $f\colon X \to Y$, the map $F(f)$ is a monomorphism of sets.
- A map $f\colon X \to Y$ is a categorical equivalence if and only if $F(f)$ is a bijection.
- The functor F commutes with κ-filtered colimits for some regular cardinal κ.
- The functor F takes κ-small simplicial sets to κ-small sets.

We define F as the composite of several intermediate steps.

Step 1: Recall that the small object argument gives a *functorial* way to factor a map f as $f = pi$, with $i \in \overline{S}$ and $p \in S^{\boxslash}$. "Funtorial factorization" means that we get a section of the functor $\mathrm{Fun}([2], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$ defining composition.

We can apply this using $S = \mathrm{InnHorn}$. Thus, given any simplicial set, we *functorially* obtain an inner anodyne map $X \to X_{\mathrm{qCat}}$ to a quasicategory X_{qCat}. As a result, we have a functor $f \mapsto f_{\mathrm{qCat}}\colon \mathrm{Fun}([1], s\mathrm{Set}) \to \mathrm{Fun}([1], s\mathrm{Set})$, with the property that f is a categorical equivalence if and only if f_{qCat} is, and both source and target of f_{qCat} are quasicategories.

Step 2: Form the path fibration $Q(f)\colon P(f_{\mathrm{qCat}}) \to Y_{\mathrm{qCat}}$ of f_{qCat}. The map $Q(f)$ is thus an isofibration between quasicategories, and is a trivial fibration if and only if f is a categorical equivalence.

Step 3: Write $Q(f)\colon X' \to Y'$. Define $E(f)$ to be the map of sets

$$E(f)\colon \coprod_n \mathrm{Hom}(\Delta^n, X') \to \coprod_n \mathrm{Hom}(\partial\Delta^n, X') \times_{\mathrm{Hom}(\partial\Delta^n, Y')} \mathrm{Hom}(\Delta^n, Y').$$

Thus, f is a categorical equivalence if and only if $E(f)$ is *surjective.*

Step 4: Write $E(f)\colon E_0(f) \to E_1(f)$, and define $F(f)$ by

$$F(f)\colon \operatorname{colim}\bigl[E_0(f) \times_{E_1(f)} E_0(f) \rightrightarrows E_0(f)\bigr] \to E_1(f).$$

In other words, $F(f)$ is the map from the image of $E(f)$ to $E_1(f)$. Thus, $F(f)$ is always a monomorphism, and f is a categorical equivalence if and only if $F(f)$ is a bijection.

There exists a regular cardinal κ such that F commutes with κ-filtered colimits, and takes κ-small simplicial sets to κ-small sets. (In fact, we can take $\kappa = \omega^+$, the sucessor to the countable cardinal).

Using the detection functor, we can prove the following key lemma.

64.1. **Lemma.** *Let $f\colon X \subseteq Y$ be an inclusion which is a categorical equivalence. Every κ-small subcomplex $A \subseteq Y$ is contained in a κ-small subcomplex $B \subseteq Y$ with the property that $B \cap X \subseteq B$ is a categorical equivalence.*

Proof. For a subcomplex $A \subseteq Y$ let f_A denote the inclusion $A \cap X \subseteq A$. The collection of all κ-small subcomplexes of Y is κ-filtered. Thus

$$\operatorname{colim}_{\kappa\text{-small } A \subseteq Y} F(f_A) = F(f),$$

which we have assumed is an isomorphism. Thus for any κ-small $A \subseteq$ there must exist a κ-small $A' \supset A$ such that a lift exists in

$$\begin{tikzcd} F_0(f_A) \arrow[r] \arrow[d, hook] & F_0(f_{A'}) \arrow[d, hook] \\ F_1(f_A) \arrow[r] \arrow[ur, dotted] & F_1(f_{A'}) \end{tikzcd}$$

This is because $F_1(f_A)$ is a κ-small set, so any lift $F_1(f_A) \to F_0(F)$ factors through some stage of the κ-filtered colimit.

We use transfinite induction to obtain a sequence $\{A_i\}$ indexed by $i < \kappa$, where at limit ordinals we take a colimit. Set $B := \operatorname{colim} A_i$. Because κ is regular $|B| < \kappa$, and we have that $F(f_B)$ is an isomorphism by construction. $\square$

Consider the collection of monomorphisms $i\colon A \to B$ such that i is a categorical equivalence and $|B| < \kappa$. Choose a set S of such spanning all isomorphism classes of such maps; this is a set because of the cardinality bound. Clearly $S \subseteq \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

64.2. **Proposition.** *We have* $\overline{S} = \overline{\mathrm{Cell}} \cap \mathrm{CatEq}$.

Proof. [Joy08a, D.2.16]. Given an injective categorical equivalence $X \subseteq Y$, we consider the following poset $\mathcal{P}$. The objects of $\mathcal{P}$ are subobjects $P \subseteq Y$ such that $X \subseteq P$ so that the inclusion $X \to P$ is contained in $\overline{S}$. The morphisms of $\mathcal{P}$ are inclusions $P \to Q$ of subobjects of Y which are contained in $\overline{S}$. Because $\overline{S}$ is weakly saturated, the hypotheses of Zorn's lemma apply to give a maximal element M of $\mathcal{P}$. Since $X \subseteq Y$ is assumed to be a categorical equivalence, 2-out-of-3 gives that $M \subseteq Y$ is a categorical equivalence.

If $M = Y$ we are done, so suppose $M \neq Y$. Then there exists a κ-small $A \subseteq Y$ not contained in M, which by the above lemma can be chosen so that $A \cap M \subseteq A$ is a categorical equivalence, and thus an element of S. The pushout $M \subseteq A \cup M$ of this map is thus in $\overline{S}$ contradicting the maximality of M. $\square$

References

[Bar14] Clark Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)* (2014), available at `arXiv:1404.0108`.

[Ber10] Julia E. Bergner, *A survey of $(\infty, 1)$-categories*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 69–83.

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, Vol. 347, Springer-Verlag, Berlin-New York, 1973.

[Cam19] Alexander Campbell, *A counterexample in quasicategory theory* (2019), available at `arXiv:1904.04965`.

[Cor82] Jean-Marc Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers Topologie Géom. Différentielle **23** (1982), no. 1, 93–112 (French). Third Colloquium on Categories, Part VI (Amiens, 1980).

[DS95] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[Fri12] Greg Friedman, *Survey article: an elementary illustrated introduction to simplicial sets*, Rocky Mountain J. Math. **42** (2012), no. 2, 353–423, available at `arXiv:0809.4221`.

[GZ67] P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[GJ09] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009. Reprint of the 1999 edition [MR1711612].

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[HS01] André Hischowitz and Carlos Simpson, *Descente pour les n-champs* (2001), available at `arXiv:math/9807049`.

[Joy02] A. Joyal, *Quasi-categories and Kan complexes*, J. Pure Appl. Algebra **175** (2002), no. 1-3, 207–222. Special volume celebrating the 70th birthday of Professor Max Kelly.

[Joy08a] André Joyal, *The theory of quasi-categories and its applications*, 2008, Notes for course at CRM, Barcelona, February 2008. At `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf`.

[Joy08b] ———, *Notes on quasi-categories*, 2008. At `http://www.math.uchicago.edu/~may/IMA/Joyal.pdf`.

[JT08] André Joyal and Myles Tierney, *Notes on simplicial homotopy theory*, 2008, Notes for course at CRM, Barcelona, February 2008. At `http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf`.

[KLV12] Chris Kapulkin, Peter LeFanu Lumsdaine, and Vladimir Voevodsky, *Univalence in simplicial sets* (2012), available at `arXiv:1203.2553`.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur12] ———, *Higher algebra* (2012), available at `http://www.math.harvard.edu/~lurie/`.

[Qui67] Daniel G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin, 1967.

[Rie16] Emily Riehl, *Category theory in context*, Dover Publications, 2016. Available at `http://www.math.jhu.edu/~eriehl/context.pdf`.

[Rie14] ———, *Categorical homotopy theory*, New Mathematical Monographs, vol. 24, Cambridge University Press, Cambridge, 2014.

[Rie09] ———, *A concise definition of a model category* (2009), preprint, available at `Availableat\url{http://www.math.jhu.edu/~eriehl/modelcat.pdf}`.

[RV15] Emily Riehl and Dominic Verity, *The 2-category theory of quasi-categories*, Adv. Math. **280** (2015), 549–642.

[TS14] The Stacks Project Authors, *Stacks Project*, 2014. At `http://stacks.math.columbia.edu`.

[Tho80] R. W. Thomason, *Cat as a closed model category*, Cahiers Topologie Géom. Différentielle **21** (1980), no. 3, 305–324.

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