Global homotopy theory

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v0.12 / August 17, 2013
Preliminary and incomplete version

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Introduction

This book introduces a context for global homotopy theory. Here 'global' refers to simultaneous and compatible actions of compact Lie groups. It has been noticed since beginnings of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class. Prominent examples of this are equivariant stable homotopy, equivariant K-theory or equivariant bordism. Various ways to formalize this idea and to obtain a category that is the home of global stable homotopy types have been explored in [**53**, Ch. II], [**39**, Sec. 5], [**14**]. We use a different approach: we work with the well-known category of orthogonal spectra, but use a much finer notion of equivalence, the *global equivalences*, then what is traditionally considered. The basic underlying observation is that every orthogonal spectrum gives rise to an orthogonal G-spectrum for every compact Lie group G, and the fact that all these individual equivariant objects come from one orthogonal spectrum implicitly encodes strong compatibility conditions as the group G varies. An orthogonal spectrum thus has G-equivariant homotopy groups for every compact Lie group, and a global equivalence is a morphism of orthogonal spectra that induces isomorphisms for all equivariant homotopy groups for all compact Lie group (compare Definition III.2.9). For the experts we should add here that the equivariant homotopy groups that we consider are based on 'complete G-universes'.

The structure on the equivariant homotopy groups of an orthogonal spectrum gives an idea of the information contained in a global homotopy type in our sense: the equivariant homotopy groups $\pi_k^G(X)$ are contravariantly functorial for continuous group homomorphisms ('restriction maps'), and they are covariantly functorial for inclusions of closed subgroups ('transfer maps'). The restriction and transfer maps enjoy various transitivity properties and interact via a double coset formula. This kind of algebraic structure has been studied under different names (e.g., 'global Mackey functor', 'inflation functor',...). From a purely algebraic perspective, there are various parameters here than one can vary, namely the class of groups to which a value is assigned and the classes of homomorphisms to which restriction maps respectively transfer maps are assigned, and lots of variations have been explored algebraically. However, the decision to work with orthogonal spectra and equivariant homotopy groups on complete universes dictates a canonical choice: we prove in Theorem III.3.5 that the algebra of natural operations between the equivariant homotopy groups of orthogonal spectra is freely generated by restriction maps along continuous group homomorphisms and transfer maps along subgroup inclusion, subject to explicitly understood relations.

We define the *global stable homotopy category* $\mathcal{GH}$ by localizing the category of orthogonal spectra at the class of global equivalences. Every global equivalence is in particular a non-equivariant stable equivalence, so there is a 'forgetful' functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ on localizations, where $\mathcal{SH}$ denotes the traditional non-equivariant stable homotopy category. By Theorem IV.5.2 this forgetful functor has a left adjoint L and a right adjoint R, both fully faithful, that participate in a recollement of triangulated categories:

$$\begin{tikzcd} \mathcal{GH}^+ \arrow[r, "i_*"] & \mathcal{GH} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{SH} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

Here $\mathcal{GH}^+$ denotes the the full subcategory of the global homotopy category spanned by the orthogonal spectra that are stably contractible in the traditional, non-equivariant sense.

The global sphere spectrum and suspension spectra are in the image of the left adjoint (Example IV.5.12). Global Borel cohomology theories are the image of the right adjoint (Example IV.5.26). The 'natural' global versions of topological K-theory, algebraic K-theory, bordism, or Eilenberg-Mac Lane spectra of global functors are not in the image of either of the two adjoints. Global topological K-theory, however, is right induced from finite cyclic groups i.e., in the image of the analogous right adjoint from an intermediate global homotopy category $\mathcal{GH}_{cyc}$ based on finite cyclic groups (Example IV.5.27).

Looking at orthogonal spectra through the eyes of global equivalences is a bit like using a prism: the latter breaks up white light into a spectrum of colors, and global equivalences split a traditional, non-equivariant homotopy type into many different global homotopy types. The first example of this phenomenon that we will encounter refines the classifying space of a compact Lie group G. On the one hand, there is the constant orthogonal space with value a non-equivariant model for BG; and there is the *global classifying* space $B_{\mathrm{gl}}G$ (see Definition I.2.7). The global classifying space is analogous to the 'geometric classifying space' of a linear algebraic group in motivic homotopy theory, compare [**62**, 4.2]. After reading Chapter I, most readers will probably agree that the global classifying space is the more interesting object. Another good example is the splitting up of the non-equivariant homotopy type of the classifying space of the infinite unitary group U. Again there is the constant orthogonal space with value BU, then the homotopy colimit, as n goes to infinity, of the global classifying spaces $B_{\mathrm{gl}}U(n)$, the Grassmannian model $\mathbf{BU}$ and finally the bar construction model $B\mathbf{U}$.

In the stable global world, every non-equivariant homotopy type has two extreme global refinements, the 'left induced' (the global analog of a constant orthogonal space, see Example IV.5.11) and the 'right induced' global homotopy type (representing Borel cohomology theories, see Example IV.5.26). Many important stable homotopy types have other natural global forms. The non-equivariant Eilenberg-Mac Lane spectrum of the integers has a 'free abelian group functor' model (see Section V.5) and another incarnation as the Eilenberg-Mac Lane spectrum of the constant global functor with value $\mathbb{Z}$ (see Remark IV.4.12). These two global refinements of the integral Eilenberg-Mac Lane spectrum agree on finite groups, but differ for compact Lie groups of positive dimensions; the author is uncertain which of the two is the 'better', or the more useful, global homotopy type. Connective topological K-theory also has two fairly natural global refinements (in addition to the left and right induced ones). The 'orthogonal subspace' model $\mathbf{ku}$ (Construction V.6.5) represents connective equivariant K-theory on the class of finite groups; on the other hand, global connective K-theory $\mathbf{ku}^c$ (Construction V.6.39) is the global synthesis of equivariant connective K-theory in the sense of Greenlees [**38**].

The global equivalences are part of a closed model structure (see Theorem IV.2.7), so the methods of homotopical algebra can be used to study the global homotopy category. This works more generally relative to a class $\mathcal{F}$ of compact Lie groups, where we define $\mathcal{F}$-equivalences by requiring that $\pi_k^G f$ is an isomorphism for all integers and all groups in $\mathcal{F}$. We call a class $\mathcal{F}$ of compact Lie groups a *global family* if it is closed under isomorphism, subgroups and quotients. For global families we establish two useful cofibration/fibration pairs that complement the $\mathcal{F}$-equivalences to stable model structures (the *flat* respectively the *projective* $\mathcal{F}$-global model structure, see Theorems IV.2.10 and IV.2.11) These model structures are useful for showing that the forgetful functor

$$(\mathcal{F}\text{-global homotopy category}) \longrightarrow (\text{stable homotopy category})$$

has both a left and a right adjoint, and both are fully faithful. Besides all compact Lie groups, interesting global families are the classes of all finite groups, or all abelian compact Lie groups. The class of trivial groups is also admissible here, but then we just recover the 'traditional' stable category. The flat $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra and satisfies the 'monoid axiom' (Proposition IV.3.10). Hence this model structures lift to modules over an orthogonal ring spectrum and to algebras over a commutative orthogonal ring spectrum (Corollary IV.3.11) The same is true for the projective $\mathcal{F}$-global model structure if the global family $\mathcal{F}$ is also closed under products.

Relation to other work. The idea of global equivariant homotopy theory is not at all new and has previously been explored in different contexts. For example, in Chapter II of [**53**], Lewis and May define *coherent families* of equivariant spectra; these consists of collections of equivariant coordinate free spectra in the sense Lewis, May and Steinberger, equipped with comparison maps involving change of groups and change of universe functors.

The approach closest to ours are the *global $\mathcal{I}_*$-functors* introduced by Greenlees and May in [**39**, Sec. 5]. These objects are 'global orthogonal spectra' in that they are indexed on pairs (G, V) consisting of a compact

Lie group and a G-representation V. The corresponding objects with commutative multiplication are called *global $\mathcal{I}_*$-functors with smash products* in [**39**, Sec. 5] and it is for these that Greenlees and May define and study multiplicative norm maps. Clearly, an orthogonal spectrum gives rise to a global $\mathcal{I}_*$-functors in the sense of Greenlees and May. In the second chapter of her thesis [**13**], A. M. Bohmann compares the approaches of Lewis-May and Greenlees-May; in the paper [**14**] she also relates these to orthogonal spectra.

Symmetric spectra in the sense of Hovey, Shipley and Smith [**44**] are another prominent model for the (non-equivariant) stable homotopy category. Much of what we do here with orthogonal spectra can also be done with symmetric spectra, if one is willing to restrict to finite groups (as opposed to general compact Lie groups). This restriction arises because only finite groups embed into symmetric groups, while every compact Lie group embeds into an orthogonal group. In his master thesis [**40**], M. Hausmann has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the $\mathcal{F}in$-global model structure to the category of symmetric with the global model structure. While some parts of the symmetric and orthogonal theories are similar, there are serious technical complications arising from the fact that for symmetric spectra the naively defined equivariant homotopy groups are not 'correct', a phenomenon that is already present non-equivariantly.

Organization. In Chapter I we set up the unstable global homotopy theory using orthogonal spaces, i.e., continuous functor from the category of finite-dimensional inner product spaces and linear isometric embeddings to spaces. We introduce global equivalences (Definition I.1.2), discuss global classifying spaces of compact Lie groups (Definition I.2.7), set up the global model structures on the category of orthogonal spaces (Theorem I.4.3) and investigate the box product of orthogonal spaces from a global equivariant perspective. In Section I.6 we add two additional perspectives on the unstable global homotopy theory: we identify the global homotopy theory orthogonal spaces with the homotopy theory of 'spaces with an action of the universal compact Lie group', and with the homotopy theory of 'orbispaces with compact Lie isotropy'. Here the 'universal compact Lie group' (which is neither compact nor a Lie group) is the topological monoid $\mathcal{L}$ of linear isometric self-embedding of $\mathbb{R}^\infty$, and we establish global model structure on the category of $\mathcal{L}$-spaces. By 'orbispaces' we mean the category of contravariant continuous functors from the global orbit category $\mathbf{O}_{\mathrm{gl}}$ to spaces. The formal comparison of these homotopy theories in contained in the Quillen equivalences of Theorems I.6.11 and I.6.18.

Chapter II is devoted to commutative orthogonal monoid spaces (a.k.a. commutative monoids with respect to the box product, or lax symmetric monoidal functors), which we want to advertise as a rigidified notion of 'global E_∞-space'. We give plenty of examples, establish the global model structure and study global group completions.

Chapter III sets the stage for stable global homotopy theory: we recall orthogonal spectra and equivariant homotopy groups along with their natural structure and global functors. We also discuss free orthogonal spectra and global Ω-spectra (Definition III.7.2), the natural concept of a 'global infinite loop object' in our setting. Two main results in Chapter III are the calculation of algebra of natural operations on equivariant homotopy groups (Theorem III.3.5) and the identification of certain morphisms between free orthogonal spectra as global equivalences (Theorem III.5.7). Section III.8 contains 'global delooping machine' that produces $\mathcal{F}in$-Ω-spectra from group-like commutative orthogonal monoid spaces.

In Chapter IV, we complement the global equivalences of orthogonal spectra by stable model structures. Here we work more generally relative to a global family $\mathcal{F}$ and consider the $\mathcal{F}$-equivalences (i.e., equivariant stable equivalences for all compact Lie groups in the family $\mathcal{F}$). We follow the familiar outline: various model structures for equivariant spaces enter into two level model structures ('projective' and 'flat') which are then Bousfield localized to two $\mathcal{F}$-global model structures (see Theorems IV.2.10 and IV.2.11). The projective and flat $\mathcal{F}$-global model structures have the same equivalences, but we need both of them to construct and study left and right adjoints to the forgetful functors associated to a change of global family (Theorem IV.5.2). We develop some basic theory around the global stable homotopy category; since it comes from a stable model structure, this category is naturally triangulated and we show that the suspension spectra of global classifying spaces form a set of compact generators (Proposition IV.4.2). As an application of Morita theory for stable model categories [**73**], we can then deduce that rationally the global homotopy category for finite groups has an algebraic model, namely the derived category of rational global functors (Theorem IV.7.5).

Chapter V focusses on *ultra-commutative ring spectra*, i.e., commutative orthogonal ring spectra under multiplicative global equivalences. We establish the global model structure for ultra-commutative ring spectra (Theorem V.1.10), and make the algebraic structure that arises explicit as *global power functors* (Definition V.1.2). This chapter also contains a detailed discussion of various important examples: Eilenberg-Mac Lane spectra (Section V.5), global K-theory (in various flavors, Section V.6), and global bordism spectra (Section V.7).

Acknowledgments. I would like to thank John Greenlees for patiently explaining various questions in equivariant stable homotopy theory to me; Johannes Ebert and Michael Joachim for tutorials on equivariant K-theory, and Markus Hausmann, Lars Hesselholt, Jacob Lurie, Steffen Sagave, Neil Strickland, Karol Szumiło, Peter Teichner and Mahmoud Zeinalian for various helpful comments and suggestions.

CHAPTER I

Unstable global homotopy theory

1. Orthogonal spaces

Before we start, let us fix some notation and conventions. By a 'space' we mean a *compactly generated weak Hausdorff space*. We denote by **U** the category of compactly generated weak Hausdorff spaces; later we will also consider *based* spaces, and **T** will denote the category of based compactly generated weak Hausdorff spaces.

An *inner product space* is a finite dimensional real vector space equipped with a scalar product, i.e., a positive definite symmetric bilinear form. We denote by **L** the category with objects the inner product spaces and morphisms the linear isometric embeddings. An example of an inner product spaces is the vector space $\mathbb{R}^n$ with the standard scalar product

$$\langle x, y \rangle \ = \ \sum_{i=1}^{n} x_i y_i \ .$$

In fact, every inner product space V is isometrically isomorphic to the inner product space $\mathbb{R}^n$, for n the dimension of V. So the full topological subcategory with objects the $\mathbb{R}^n$ is a small skeleton of **L**. The category **L** is a topological category in the sense that the morphism spaces come with a preferred topology: if $\varphi : V \longrightarrow W$ is one linear isometric embedding, then then action of the orthogonal group $O(W)$, by postcomposition, is a bijection

$$O(W)/O(W - \varphi(V)) \ \cong \ \mathbf{L}(V, W) \ , \quad A \ \longmapsto A \circ \varphi \ ,$$

where $W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V, W)$ so that this bijection is a homeomorphism, and this topology is independent of φ.

Definition 1.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathbf{U}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote by *spc* the category of orthogonal spaces.

The use of continuous functors from the category **L** to spaces has a long history in homotopy theory. The systematic use of inner product spaces (as opposed to numbers) to index objects in stable homotopy theory seems to go back to Boardman's thesis [**11**]. The category **L** (or its extension that also contains countably infinite dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [**12**], and this notation is also used in [**59**]; other sources [**54**] use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: on the one hand, our use of orthogonal spaces comes with a shift in emphasis, away from a focus on non-equivariant homotopy type, and towards viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Secondly, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global global world with the latter a corresponding stable global world.

Now we define our main new concept, the notion of 'global equivalence' between orthogonal spaces. We let G be a compact Lie group. By a *G-representation* we mean a finite dimensional orthogonal representation, i.e., a real inner product space V equipped with a continuous G-action by linear isometries. In other word, a G-representation consists of a inner product space V and a continuous homomorphism $\rho : G \longrightarrow O(V)$. For every orthogonal space Y and every G-representation V, the value $Y(V)$ inherits a G-action from the G-action on V and the functoriality of Y. For an G-equivariant linear isometric embedding $\alpha : V \longrightarrow W$ the induced map $Y(\alpha) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

Definition 1.2. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global equivalence* if the following condition holds: for every compact Lie group G, every G-representation V, every $k \geq 0$ and all continuous maps $\alpha : S^{k-1} \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $f(V)^G \circ \alpha = \beta|_{S^{k-1}} : S^{k-1} \longrightarrow Y(V)^G$, there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda_{S^{k-1}} = X(i_{V,W})^G \circ \alpha : S^{k-1} \longrightarrow X(V \oplus W)^G$ and such that $f(V \oplus W)^G \circ \lambda : D^k \longrightarrow Y(V \oplus W)^G$ is homotopic, relative to S^{k-1}, to $Y(i_{V,W}) \circ \beta$.

In other words, for every commutative square on the left

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & X(V)^G \arrow[r, "X(i_{V,W})^G"] & X(V \oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ D^k \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V \oplus W)^G \end{tikzcd}$$

there exists the lift λ on the right hand side that makes the upper left triangle commute on the nose, and the lower right triangle up to homotopy relative to the boundary S^{k-1}.

Remark 1.3. The notion of global equivalence is meant to capture the idea that for every compact Lie group G, some induced morphism

$$\mathrm{hocolim}_V\, f(V) \ : \ \mathrm{hocolim}_V\, X(V) \ \longrightarrow \ \mathrm{hocolim}_V\, Y(V)$$

is a G-weak equivalence, where 'hocolim$_V$' is a suitable homotopy colimit over all G-representations V along all equivariant linear isometric embeddings. This is a useful way to think about global equivalences, and it could be made precise by letting V run over the poset of finite dimensional subrepresentations of a complete G-universe and using the Bousfield-Kan construction of a homotopy colimit over this poset. However, the actual definition that we work with has the advantage that we do not have to make precise what we mean by 'all' G-representations and we do not have to define or manipulate homotopy colimits.

In many examples of interest, all the structure maps of an orthogonal space Y are closed embeddings. When this is the case, the actual colimit (over the subrepresentations of a complete universe) of the G-spaces $Y(V)$ serve the purpose of a 'homotopy colimit over all representations', and it can be used to detect global equivalences, compare Proposition 1.11 below.

Example 1.4. If $X = \underline{A}$ and $Y = \underline{B}$ are the constant orthogonal spaces with values the spaces A respectively B, and $f = \underline{\varphi}$ the constant morphism associated to a continuous map $\varphi : A \longrightarrow B$, then $\underline{\varphi}$ is a global equivalence if and only if for every commutative square

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d, "\mathrm{incl}"'] & A \arrow[d, "\varphi"] \\ D^k \arrow[r] \arrow[ur, dashed, "\lambda"] & B \end{tikzcd}$$

there exists the lift λ that makes the upper left triangle commute, and the lower right triangle up to homotopy relative to the boundary S^{k-1}. But this is one of the equivalent ways of characterizing weak equivalences. So $\underline{\varphi}$ is a global equivalence if and only if φ is a weak equivalence.

Definition 1.5. Let G be a compact Lie group. A *G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following two properties

- the representation $\mathcal{U}$ has non-zero G-fixed points,
- if a finite dimensional representation V embeds into $\mathcal{U}$, then a countable infinite sum $\oplus_{\mathbb{N}} V$ of copies of V also embeds into $\mathcal{U}$.

A G-universe is *complete* if every finite dimensional G-representation embeds into it.

A G-universe is characterized, up to equivariant isometry, by the set of irreducible G-representations that can be embedded into it. We let $\Lambda = \{\lambda\}$ be a complete set of pairwise non-isomorphic irreducible G-representations that embed into $\mathcal{U}$. The first conditions says that Λ contains a trivial 1-dimensional representation, and the second condition is equivalent to the requirement that

$$\mathcal{U} \cong \bigoplus_{\lambda \in \Lambda} \bigoplus_{\mathbb{N}} \lambda \ .$$

Moreover, $\mathcal{U}$ is complete if and only if Λ contains (representatives of) *all* irreducible G-representations. In the following we fix, for every compact Lie group G, a complete G-universe $\mathcal{U}_G$. We let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$.

Definition 1.6. For an orthogonal space Y and a compact Lie group G we define the *underlying G-space* as

$$Y(\mathcal{U}_G) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} Y(V) ,$$

the colimit, in the category of compactly generated weak Hausdorff G-spaces, of the G-spaces $Y(V)$.

Remark 1.7. The underlying G-space $Y(\mathcal{U}_G)$ can always be rewritten as a sequential colimit of values of Y. Indeed, we can choose a nested sequence of finite dimensional G-subrepresentations

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

whose union is all of $\mathcal{U}_G$. Since the subposet $\{V_n\}_{n \geq 0}$ is cofinal in $s(\mathcal{U}_G)$, the colimit of the functor $V \mapsto Y(V)$ over $s(\mathcal{U}_G)$ is also a colimit over the subsequence $Y(V_n)$.

If the group G is finite, then we can define a complete universe as

$$\mathcal{U}_G = \rho_G^{\infty} ,$$

a countably infinite sum of copies of the regular representation $\rho_G = \mathbb{R}[G]$ (with G as orthogonal basis). Then $\mathcal{U}_G$ is filtered by the finite sums $n\rho_G$, and we get

$$Y(\mathcal{U}_G) = \operatorname{colim}_n Y(n\rho_G) ;$$

where the colimit is taken along the inclusion $n\rho_G \longrightarrow (n+1)\rho_G$ that misses the last summand.

Proposition 1.9 below says that for certain 'closed' orthogonal spaces the global equivalences are precisely the morphisms inducing equivariant weak equivalences on underlying G-spaces for all compact Lie groups G.

Definition 1.8. A continuous map $\varphi : A \longrightarrow B$ is a *closed embedding* if it is closed and a homeomorphism of A onto the subspace $\varphi(A)$ of B. An orthogonal space Y is *closed* if it takes every linear isometric embedding $\varphi : V \longrightarrow W$ of inner product spaces to a closed embedding $Y(\varphi) : Y(V) \longrightarrow Y(W)$.

If a compact Lie group G acts on two spaces A and B and $\varphi : A \longrightarrow B$ is a G-equivariant closed embedding, then the restriction $\varphi^G : A^G \longrightarrow B^G$ to G-fixed points is also a closed embedding. In particular, for every closed orthogonal space Y and every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ of G-representations, the induced map on G-fixed points $Y(\varphi)^G : Y(V)^G \longrightarrow Y(W)^G$ is also a closed embedding.

Proposition 1.9. *Let $f : X \longrightarrow Y$ be morphism between closed orthogonal spaces. Then f is a global equivalence if and only if for every compact Lie group G the map*

$$f(\mathcal{U}_G) : X(\mathcal{U}_G) \longrightarrow Y(\mathcal{U}_G)$$

is a G-weak equivalence.

Proof. The poset $s(\mathcal{U}_G)$ has a cofinal subsequence, so all colimits over $s(\mathcal{U}_G)$ can be realized as sequential colimits. The claim is then a straightforward consequence of the fact that compact spaces such as D^k and S^{k-1} are finite with respect to sequences of closed embeddings, compare [**43**, Prop. 2.4.2]. We should recall here that points in compactly generated weak Hausdorff spaces are always closed, so the T_1-separation property holds. $\square$

We define G-equivariant path components $\pi_0^G(Y)$ as

$$\pi_0^G(Y) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)} \pi_0(Y(V)^G) \ . \tag{1.10}$$

The canonical G-maps $Y(V) \longrightarrow Y(\mathcal{U}_G)$ induce maps $\pi_0(Y(V)^G) \longrightarrow \pi_0(Y(\mathcal{U}_G)^G)$ and hence a canonical map

$$\pi_0^G(Y) \ \longrightarrow \ \pi_0(Y(\mathcal{U}_G)^G) \ .$$

Since the underlying set of $Y(\mathcal{U}_G)^G$ is a colimit, over the poset $s(\mathcal{U}_G)$, of the sets $Y(V)^G$, the canonical map is always surjective. However, some hypothesis on Y is needed for injectivity; being closed is sufficient for this:

Proposition 1.11. *Let G be a compact Lie group.*

(i) *For every closed orthogonal space Y the canonical map*

$$\pi_0^G(Y) \ \longrightarrow \ \pi_0(Y(\mathcal{U}_G)^G)$$

is bijective.

(ii) *Let $f : X \longrightarrow Y$ be a global equivalence of orthogonal spaces. Then the induced map*

$$\pi_0^G(f) \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^G(Y)$$

of equivariant homotopy sets is bijective.

Proof. Part (i) follows from the fact that for closed Y, every path in $Y(\mathcal{U}_G)^G$ is realized in $Y(V)^G$ for some finite dimensional subrepresentation V.

(ii) We let $y \in Y(V)^G$ be a G-fixed point, for some $V \in s(\mathcal{U}_G)$ that represents an element of $\pi_0^G(Y)$. Viewed as a continuous map $y : D^0 \longrightarrow Y(V)^G$ and together with the unique map from the empty space S^{-1} this specifies a lifting problem on the left:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X(V)^G \arrow[d, "f(V)^G"] \\ D^0 \arrow[r, "y"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X(V)^G \arrow[r, "X(i_{V,W})^G"] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ D^0 \arrow[r, "y"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G \end{tikzcd}$$

there is thus a G-representation W and a lift λ on the right hand side such that $f(V\oplus W)^G$ takes the image $x \in X(V\oplus W)^G$ of λ to the same path component in $Y(V\oplus W)^G$ as the point $Y(i_{V,W})(y)$. We choose a G-equivariant linear isometric embedding $\alpha : V\oplus W \longrightarrow \mathcal{U}_G$ extending the inclusion of V. Then the class in $\pi_0^G(X)$ represented by the fixed point

$$X(\alpha)(x) \ \in \ X(\alpha(V\oplus W))^G$$

is taken to $[y]$ by the map $\pi_0^G(f)$. This shows that $\pi_0^G(f)$ is surjective.

For injectivity we consider two fixed points $x, x' \in X(V)^G$, for some $V \in s(\mathcal{U}_G)$, such that $\pi_0^G(f)[x] = \pi_0^G(f)[x']$. By enlarging V, if necessary, we can assume that the images $f(V)(x), f(V)(x')$ are connected by a path $\beta : D^1 \longrightarrow Y(V)^G$ in the G-fixed points space of $Y(V)$. This data specifies a lifting problem on the left:

$$\begin{tikzcd} S^0 \arrow[r, "{x,x'}"] \arrow[d] & X(V)^G \arrow[d, "f(V)^G"] \\ D^1 \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd} \qquad \begin{tikzcd} S^0 \arrow[r, "{x,x'}"] \arrow[d] & X(V)^G \arrow[r, "X(i_{V,W})^G"] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ D^1 \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G \end{tikzcd}$$

there is thus a G-representation W and a lift λ on the right hand side such that $\lambda(-1) = x$, $\lambda(1) = x'$ and $f(V \oplus W)^G \circ \lambda$ is homotopic, relative endpoints, to $Y(i_{V,W}) \circ \beta$. As in the first part, we use a G-equivariant linear isometric embedding $\alpha : V \oplus W \longrightarrow \mathcal{U}_G$, extending the inclusion of V, to transform λ into the path

$$X(\alpha) \circ \lambda \ : \ D^1 \ \longrightarrow \ X(\alpha(V \oplus W))^G$$

that connects the images of x and x' in $X(\alpha(V \oplus W))$. This shows that $[x] = [x']$ in $\pi_0^G(X)$, so $\pi_0^G(f)$ is also injective. □

As the group varies, the homotopy sets $\pi_0^G(Y)$ have contravariant functoriality in G: every continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups induces a restriction map $\alpha^* : \pi_0^G(X) \longrightarrow \pi_0^K(X)$, as we shall now explain. We denote by α^* the restriction functor from G-spaces to K-spaces (or from G-representations to K-representations) along α, i.e., α^*Y (respectively α^*V) is the same topological space as Y (respectively the same inner product space α^*V) endowed with K-action via

$$k \cdot y \ = \ \alpha(k) \cdot y \ .$$

Given an orthogonal space Y, we note that for every G-representation V, the K-spaces $\alpha^*(Y(V))$ and $Y(\alpha^*V)$ are equal (not just isomorphic).

The restriction $\alpha^*(\mathcal{U}_G)$ is a K-universe, but if α has a non-trivial kernel, then this K-universe is not complete. When α is injective, then $\alpha^*(\mathcal{U}_G)$ is a complete K-universe, but typically different from the chosen complete K-universe $\mathcal{U}_K$. To deal with this we explain how a G-fixed point $y \in Y(V)^G$ for an arbitrary G-representation V, gives rise to an unambiguously defined element $\langle f \rangle$ in $\pi_0^G(Y)$. The point here is that V does *not* have to be a subrepresentation of the chosen universe $\mathcal{U}_G$ and the resulting class $\langle y \rangle$ does not depend on any additional choices.

To construct $\langle y \rangle$ we choose a linear isometric G-embedding $j : V \longrightarrow \mathcal{U}_G$ and look at the image $Y(j)(y)$ under the G-map

$$Y(V) \ \xrightarrow{Y(j)} \ Y(j(V)) \ .$$

Here we have used the letter j to also denote the isometry $j : V \longrightarrow j(V)$ to the image of V; since $j(V)$ is a finite dimensional G-invariant subspace of $\mathcal{U}_G$, we obtain an element

$$\langle y \rangle \ = \ [Y(j)(y)] \ \in \ \pi_0^G(Y) \ .$$

It is crucial, but not completely obvious, that $\langle f \rangle$ does not depend on the choice of embedding j:

Lemma 1.12. *Let G be a compact Lie group, V and W two G-representations and $j, j' : V \longrightarrow W$ two G-equivariant linear isometric embeddings. If the images $j(V)$ and $j'(V)$ are orthogonal, then j and j' are homotopic through G-equivariant linear isometric embeddings.*

Proof. The desired homotopy $H : V \times [0,1] \longrightarrow W$ from j to j' is given by

$$H(v,t) \ = \ \sqrt{1-t^2} \ \cdot j(v) \ + \ t \cdot j'(v) \ . \qquad \square$$

Proposition 1.13. *Let Y be an orthogonal space, G a compact Lie group, V a G-representation and $y \in Y(V)^G$ a G-fixed point.*

(i) *The class $\langle y \rangle$ in $\pi_0^G(Y)$ is independent of the choice of linear isometric embedding $j : V \longrightarrow \mathcal{U}_G$.*

(ii) *For every G-equivariant linear isometric embedding $\alpha : V \longrightarrow W$ the relation*

$$\langle Y(\alpha)(y)\rangle \ = \ \langle y\rangle \qquad \textit{holds in} \quad \pi_0^G(Y).$$

PROOF. (i) We choose a third G-equivariant linear isometric embedding $\bar{j} : V \longrightarrow \mathcal{U}_G$ whose image $\bar{j}(V)$ is orthogonal to the images $j(V)$ and $j'(V)$. We let W be the finite dimensional span generated by $j(V), j'(V)$ and $\bar{j}(V)$ inside $\mathcal{U}_G$. Then j and j' are homotopic to $\bar{j}$ through G-equivariant linear isometric embeddings into W, by Lemma 1.12. In particular, j and j' are homotopic to each other; if $H(-,t) : V \longrightarrow W$ is a continuous 1-parameter family of G-equivariant linear isometric embeddings from j to j', then

$$t \ \longmapsto \ Y(H(-,t))(y)$$

is a path in $Y(W)^G$ from $Y(j)(y)$ to $Y(j')(y)$, so $[Y(j)(y)] = [Y(j')(y)]$ in $\pi_0^G(Y)$.

(ii) If $j : W \longrightarrow \mathcal{U}_G$ is an equivariant linear isometric embedding, then so is $j\alpha : V \longrightarrow \mathcal{U}_G$. Since we can use any equivariant isometric embedding to define the class $\langle f\rangle$, we get

$$\langle Y(\alpha)(y)\rangle \ = \ [Y(j)(Y(\alpha)(y))] \ = \ [Y(j\alpha)(y)] \ = \ \langle y\rangle \ . \qquad \square$$

We can now define the *restriction map* associated to a continuous group homomorphism $\alpha : K \longrightarrow G$ by

$$\alpha^* \ : \ \pi_0^G(Y) \ \longrightarrow \ \pi_0^K(Y) \ , \quad [y] \ \longmapsto \ \langle y\rangle \ . \tag{1.14}$$

For a second continuous group homomorphism $\beta : L \longrightarrow K$ we have

$$\beta^* \circ \alpha^* \ = \ (\alpha\beta)^* \ : \ \pi_0^G(Y) \ \longrightarrow \ \pi_0^L(Y) \ .$$

Clearly, restriction along the identity homomorphism is the identity, so we have made the collection of equivariant homotopy sets $\pi_0^G(Y)$ into a contravariant functor in the group variable.

An important special case of the restriction homomorphisms are conjugation maps. Here we consider a subgroup H of G, an element $g \in G$ and denote by

$$c_g \ : \ {}^gH \ \longrightarrow \ H \ , \quad c_g(h) \ = \ g^{-1}hg$$

the conjugation homomorphism. As any group homomorphism, c_g induces a map

$$c_g^* \ : \ \pi_0^H(Y) \ \xrightarrow{c_g^*} \ \pi_0^{{}^gH}(Y) \tag{1.15}$$

of equivariant homotopy sets. For $g, \bar{g} \in G$ we have $c_{g\bar{g}} = c_{\bar{g}} \circ c_g : {}^{g\bar{g}}H \longrightarrow H$ and thus

$$c_{g\bar{g}}^* \ = \ c_g^* \circ c_{\bar{g}}^* \ : \ \pi_0^H(Y) \ \longrightarrow \ \pi_0^{{}^{g\bar{g}}H}(Y) \ .$$

A key fact is that inner automorphisms act trivially, i.e., for every $g \in G$ restriction map c_g^* is the identity on $\pi_0^G(Y)$. So the action, by the restriction maps, of the automorphism group of G on $\pi_0^G(Y)$ factors through the outer automorphism group.

Proposition 1.16. *For every orthogonal space Y, every compact Lie group G, and every $g \in G$, the restriction map $c_g^* : \pi_0^G(Y) \longrightarrow \pi_0^G(Y)$ is the identity.*

PROOF. We consider a finite dimensional G-subrepresentation V of $\mathcal{U}_G$ and a G-fixed point $y \in Y(V)^G$ that represents an element in $\pi_0^G(Y)$. Then the map $l_g : c_g^*(V) \longrightarrow \mathcal{U}$ given by left multiplication by g is a G-equivariant linear isometric embedding. So

$$c_g^*[y] \ = \ [Y(l_g^V)(y)] \ = \ [g \cdot y] \ = \ [y] \ ,$$

by the very definition of the restriction map, where $l_g^V : c_g^*(V) \longrightarrow V$. The second equation is the definition of the G-action on $Y(V)$ through the G-action on V. The third equation is the hypothesis that y is G-fixed. $\square$

We denote by Rep the category whose objects are the compact Lie groups and whose morphisms are conjugacy classes of continuous group homomorphisms. We can summarize the discussion thus far by saying that for every orthogonal space Y the restriction maps make the equivariant homotopy sets $\{\pi_0^G(Y)\}$ into a contravariant functor

$$\underline{\pi}_0(Y) \ : \ \mathrm{Rep} \ \longrightarrow (\text{sets}) \ .$$

Now we discuss orthogonal fixed point spaces, i.e., orthogonal spaces $F^G Y$ whose equivariant homotopy sets are the equivariant homotopy sets of Y 'translated' by G. The construction works best for finite groups, because then the regular representation is available, and its multiples exhausts a complete universe.

Construction 1.17. Given an orthogonal space Y and a finite group G we define a new orthogonal space $F^G Y$ by the *G-fixed point space* by

$$(F^G Y)(V) \ = \ (Y(V \otimes \rho_G))^G \ , \tag{1.18}$$

where $\otimes$ is short for the tensor product over $\mathbb{R}$, and where ρ_G is the regular representation of G. This is clearly functorial and continuous in linear isometric embeddings in V, so we have defined an orthogonal space.

As we shall see now, the equivariant homotopy sets of the orthogonal fixed point space $F^G Y$ calculate the equivariant homotopy sets of Y for product groups with G. More precisely, product with the group G is an endofunctor

$$- \times G \ : \ \mathrm{Rep} \ \longrightarrow \ \mathrm{Rep} \ ;$$

of the category of compact Lie groups and conjugacy classes of continuous homomorphisms. If M is a contravariant functor on Rep, we define the *G-translate* $\tau_G M$ as the composite

$$\mathrm{Rep}^{\mathrm{op}} \ \xrightarrow{-\times G} \ \mathrm{Rep}^{\mathrm{op}} \ \xrightarrow{\ M\ } \ (\text{sets}) \ .$$

To identify the functor $\underline{\pi}_0(F^G Y)$, we define a map

$$l^K \ : \ \pi_0^K(F^G Y) \ \longrightarrow \ \pi_0^{K\times G}(Y)$$

that is natural in the orthogonal space Y. The map is just a 'shift of perspective': a K-fixed point

$$y \ \in \ (Y(V \otimes \rho_G))^G \ = \ (F^G Y)(V)$$

is the same as a $(K \times G)$-fixed point of $Y(V \otimes \rho_G)$. So we send the class represented by y in $\pi_0^K(F^G Y)$ to the class $\langle y \rangle \in \pi_0^{K\times G}(Y)$ also represented by y.

Proposition 1.19. *For every finite group G, every compact Lie group K and every orthogonal space Y, the map $l^K : \pi_0^K(F^G Y) \longrightarrow \pi_0^{K\times G}(Y)$ is bijective. Moreover, the maps l^K are natural for restriction homomorphism in the group K.*

Proof. We use the following sequence of natural bijections:

$$\begin{aligned}
\pi_0^K(F^G Y) \ &= \ \mathrm{colim}_{V \in s(\mathcal{U}_K)} \ \pi_0\left(((F^G Y)(V))^K\right) \\
&= \ \mathrm{colim}_{V \in s(\mathcal{U}_K)} \ \pi_0\left(Y(V \otimes \rho_G)^{K\times G}\right) \\
&\cong \ \mathrm{colim}_{W \in s(\mathcal{U}_K \otimes \rho_G)} \ Y(W)^{K\times G} \ \cong \ \pi_0^{K\times G}(Y) \ .
\end{aligned}$$

Besides the definitions, we have exploited that the representations of the form $V \otimes \rho_G$, for $V \in s(\mathcal{U}_K)$, are cofinal in the poset of all $(K \times G)$-subrepresentations of the universe $\mathcal{U}_K \otimes \rho_G$. The last isomorphism is the fact that $\mathcal{U}_K \otimes \rho_G$ is a complete $(K \times G)$-universe, hence isomorphic to the chosen complete universe $\mathcal{U}_{K\times G}$. Compatibility with restriction maps is straightforward from the definition. $\square$

Remark 1.20. The G-fixed points of an orthogonal space Y receive a natural morphism

$$Y \xrightarrow{j} F^G Y$$

whose value at an inner product space V is the map

$$j(V) \ : \ Y(V) \ \longrightarrow \ (Y(V \otimes \rho_G))^G \ = \ (F^G Y)(V) \ .$$

induced by the linear isometric embedding

$$V \longrightarrow V \otimes \rho_G \ , \quad v \ \longmapsto \ \frac{1}{\sqrt{|G|}} \sum_{g \in G} v \otimes g \ .$$

The effect of j on K-equivariant homotopy sets is that of a restriction map: the composite

$$\pi_0^K(Y) \xrightarrow{\pi_0^K(j)} \pi_0^K(F^G Y) \xrightarrow{l^K} \pi_0^{K \times G}(Y)$$

agrees with the restriction map p_K^* associated to the projection $K \times G \longrightarrow K$ to the first factor.

2. Global classifying spaces

In this section we discuss free orthogonal spaces. Important special cases of this construction are the 'global classifying spaces' of compact Lie groups.

Construction 2.1. Given a compact Lie group G and a G-representation V, the functor

$$\mathrm{ev}_{G,V} \ : \ spc \ \longrightarrow \ G\mathbf{U}$$

that sends an orthogonal space Y to the G-space $Y(V)$ has a left adjoint

(2.2) $$\mathbf{L}_{G,V} \ : \ G\mathbf{U} \ \longrightarrow \ spc \ .$$

To construct the left adjoint we observe that G-acts on the right on $\mathbf{L}(V, W)$ by

$$(\varphi \cdot g)(v) \ = \ \varphi(gv)$$

for $\varphi \in \mathbf{L}(V, W)$, $g \in G$ and $v \in V$. Given a G-space A, the value of the free orthogonal space $\mathbf{L}_{G,V} A$ at an inner product space W by

$$(\mathbf{L}_{G,V} A)(W) \ = \ \mathbf{L}(V, W) \times_G A \ = \ (\mathbf{L}(V, W) \times A) / (\varphi g, a) \sim (\varphi, ga) \ .$$

We refer to $\mathbf{L}_{G,V} A$ as the *free orthogonal space* generated by A at (G, V). In this section we will analyze the homotopical properties of this construction.

The 'freeness' property of $\mathbf{L}_{V,G}$A means: for every orthogonal space Y and every continuous G-map $f : A \longrightarrow Y(V)$ there is a unique morphism $\hat{f} : \mathbf{L}_{G,V} A \longrightarrow Y$ of orthogonal spaces such that the composite

$$A \xrightarrow{[\mathrm{Id}, -]} \mathbf{L}(V, V) \times_G A = (\mathbf{L}_{G,V} A)(V) \xrightarrow{\hat{f}(V)} Y(V)$$

is f. Indeed, the morphism $\hat{f}$ is given at W as the composite

$$\mathbf{L}(V, W) \times_G A \xrightarrow{\mathrm{Id} \times_G f} \mathbf{L}(V, W) \times_G Y(V) \xrightarrow{\circ} Y(W) \ .$$

We denote by $*$ a one-point G-space, abbreviate

$$\mathbf{L}_{G,V} \ = \ \mathbf{L}_{G,V} * \ ,$$

and call this the *free orthogonal space generated by* (G, V). We define the *tautological class*

(2.3) $$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V})$$

as the path component of the G-fixed point

$$\mathrm{Id}_V \cdot G \ \in \ (\mathbf{L}(V, V)/G)^G = (\mathbf{L}_{G,V}(V))^G \ ,$$

the G-orbit of the identity of V.

Given two compact Lie groups K and G, we will frequently have occasion to consider a certain family $\mathcal{F}(K;G)$ of subgroups of $K \times G$ which we call 'graph subgroups'.

Definition 2.4. Let K and G be compact Lie groups. The family $\mathcal{F}(K;G)$ of *graph subgroups* consists of those subgroups Γ of $K \times G$ that intersect $1 \times G$ only in the neutral element $(1,1)$.

The name 'graph subgroup' is justified by the observation that $\mathcal{F}(K;G)$ consists precisely of the graphs of all 'subhomomorphisms', i.e., continuous homomorphisms $\alpha : L \longrightarrow G$ from a subgroup L of K. Clearly, the graph $\Gamma(\alpha) = \{(l,\alpha(l)) \mid l \in L\}$ of every such homomorphism belongs to $\mathcal{F}(K;G)$. Conversely, for $\Gamma \in \mathcal{F}(K;G)$ we let $L \leq K$ be the image of Γ under the projection $K \times G \longrightarrow K$. Since $\Gamma \cap (1 \times G) = \{(1,1)\}$, every element $l \in L$ then has a unique preimage $(l,\alpha(l))$ under the projection, and the assignment $l \mapsto \alpha(l)$ is a continuous homomorphism from L to G whose graph is Γ.

The next proposition identifies the fixed point spaces of a free orthogonal space $\mathbf{L}_{G,V}$ and calculates its 0-th equivariant homotopy sets. If W is another G-representation, then restriction of a linear isometry from $V \oplus W$ to V defines a morphism of orthogonal spaces

$$\rho_{G,V,W} \ : \ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \mathbf{L}_{G,V} \ . \tag{2.5}$$

Proposition 2.6. *Let K and G be compact Lie groups and V a faithful G-representation.*

(i) *The $(K \times G^{\mathrm{op}})$-space $\mathbf{L}_V(\mathcal{U}_K) = \mathbf{L}(V,\mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G^{\mathrm{op}})$ of graph subgroups.*

(ii) *The K-fixed point space $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α.*

(iii) *The map*
$$\mathrm{Rep}(K,G) \ \longrightarrow \ \pi_0^K(\mathbf{L}_{G,V}) \ , \quad [\alpha : K \longrightarrow G] \ \longmapsto \ \alpha^*(u_{G,V})$$
is bijective.

(iv) *If W another G-representation, then the map*
$$\rho_{e,V,W}(\mathcal{U}_K) \ : \ \mathbf{L}(V\oplus W,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(V,\mathcal{U}_K) \ ,$$
is a $(K \times G^{\mathrm{op}})$-homotopy equivalence and induces a K-homotopy equivalence
$$\rho_{G,V,W}(\mathcal{U}_K) \ : \ \mathbf{L}_{G,V\oplus W}(\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}_{G,V}(\mathcal{U}_K) \ ,$$
and a bijection
$$\pi_0^K(\rho_{G,V,W}) \ : \ \pi_0^K\left(\mathbf{L}_{G,V\oplus W}\right) \ \longrightarrow \ \pi_0^K\left(\mathbf{L}_{G,V}\right) \ .$$
In particular, $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$ is a global equivalence of orthogonal spaces.

Proof. Part (i) is proven in Proposition A.2.5. Part (ii) works for any universal $(K \times G^{\mathrm{op}})$-space E for the family $\mathcal{F}(K;G^{\mathrm{op}})$, for example for $E = \mathbf{L}(V,\mathcal{U}_K)$. The argument can be found in Proposition 5 of [**52**]; we repeat it for the convenience of the reader. For a continuous homomorphism $\alpha : K \longrightarrow G$ we set
$$E^\alpha \ = \ \{x \in E \mid kx = x\alpha(k) \text{ for all } k \in K\} \ .$$
Since the G-action on this universal space E is free, we obtain a homeomorphism
$$\coprod \alpha^\flat \ : \ \coprod_{\langle\alpha\rangle} E^\alpha/C(\alpha) \ \longrightarrow \ (E/G)^K \ ,$$
where the coproduct is indexed by conjugacy classes of continuous homomorphisms.

We let $\Gamma = \{(k,\alpha(k)) \mid k \in K\}$ denote the graph of α; then Γ belongs to the family $\mathcal{F}(K;G^{\mathrm{op}})$ and $E^\alpha = E^\Gamma$; so E^α is a contractible space. The action of $C(\alpha)$ on E^α is a restriction of the G-action on E, hence free. So for every homomorphism α the space $E^\alpha/C(\alpha)$ is a classifying space for the group $C(\alpha)$. This shows part (ii).

(iii) Since the classifying space of a topological group is connected, part (ii) identifies the path components of $(\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ with the conjugacy classes of continuous homomorphisms $\alpha : K \longrightarrow G$. The bijection sends the class of α to $\alpha^*(u_{G,V})$.

(iv) Every equivariant map between universal spaces for the same family of subgroups is an equivariant homotopy equivalence. Since G acts faithfully of V, and hence also on $V \oplus W$, the $(K \times G^{\mathrm{op}})$-spaces $\mathbf{L}_{V\oplus W}(\mathcal{U}_K)$ and $\mathbf{L}_V(\mathcal{U}_K)$ are universal spaces for the same family $\mathcal{F}(K; G^{\mathrm{op}})$, by part (i). So the map $\rho_{G,V,W} : \mathbf{L}_{V\oplus W}(\mathcal{U}_K) \longrightarrow \mathbf{L}_V(\mathcal{U}_K)$ is a $(K \times G^{\mathrm{op}})$-equivariant homotopy equivalence. So upon passage to G-orbits the restriction map $(\rho_{G,V,W}(\mathcal{U}_K))^K : (\mathbf{L}_{G,V\oplus W}(\mathcal{U}_K))^K \longrightarrow (\mathbf{L}_{G,V}(\mathcal{U}_K))^K$ becomes K-homotopy equivalence. The orthogonal spaces $\mathbf{L}_{G,V\oplus W}$ and $\mathbf{L}_{G,V}$ are closed, so Proposition 1.9 applies to show that $\rho_{G,V,W}$ is a global equivalence. The bijection on π_0^K follows by applying Proposition 1.11 (i). The other two statements follow directly from this. □

Definition 2.7. The *global classifying space* $B_{\mathrm{gl}}G$ of a compact Lie group G is the orthogonal space

$$B_{\mathrm{gl}}G \;=\; \mathbf{L}_{G,V} \;=\; \mathbf{L}(V,-)/G \;,$$

where V is any faithful G-representation.

The global classifying space $B_{\mathrm{gl}}G$ is well-defined up preferred zigzag of global equivalences of orthogonal spaces. Indeed, if V and $\bar{V}$ are two faithful G-representations, then $V \oplus \bar{V}$ is yet another one, and the two restriction morphisms

$$\mathbf{L}_{G,V} \;\longleftarrow\; \mathbf{L}_{G,V\oplus\bar{V}} \;\longrightarrow\; \mathbf{L}_{G,\bar{V}}$$

are global equivalences by Proposition 2.6 (v).

Remark 2.8. The term ‘global classifying space’ is justified by the fact that $B_{\mathrm{gl}}G$ ‘universally represents G-bundles’. More precisely, for every compact Lie group K the K-space $(B_{\mathrm{gl}}G)(\mathcal{U}_K)$ is a classifying space for principal (K,G)-bundles. In particular, the underlying non-equivariant homotopy type of $B_{\mathrm{gl}}G$ is the ordinary classifying space BG. We recall that a *principal* (K,G)-*bundle* is a principal G-bundle in the category of K-spaces, i.e., a G-principal bundle $p : E \longrightarrow B$ that is also a morphism of K-spaces and such that the actions of G and K on the total space E commute. There is a universal principal (K,G)-bundle $p : E(K,G) \longrightarrow B(K,G)$ whose total space is a $(K \times G^{\mathrm{op}})$-CW-complex characterized up to $(K \times G^{\mathrm{op}})$-homotopy equivalence by the property that the fixed point set $E(K,G)^H$ is contractible for every subgroup H of $K \times G^{\mathrm{op}}$ such that $H \cap (1 \times G^{\mathrm{op}}) = \{(1,1)\}$. If H intersects $1 \times G^{\mathrm{op}}$ in more than the identity, then $E(K,G)^H$ is empty since G acts freely, so another way to say this is that $E(K,G)$ is a universal space for the family $\mathcal{F}(K; G^{\mathrm{op}})$ in the sense of Definition 2.4.

Example 2.9. We make the global classifying space more explicit in the smallest non-trivial example, i.e., for the cyclic group C_2 of order 2. The sign representation σ of C_2 is faithful, so we can take $B_{\mathrm{gl}}C_2$ to be the free orthogonal space generated by (C_2, σ); its value at an inner product space W is

$$(B_{\mathrm{gl}}C_2)(W) \;=\; \mathbf{L}_{C_2,\sigma}(W) \;=\; \mathbf{L}(\sigma, W)/C_2 \;.$$

Evaluation at any of the two unit vectors in σ is a homeomorphism $\mathbf{L}(\sigma, W) \cong S(W)$ to the unit sphere of W, and the C_2-action on the left becomes the antipodal action on $S(W)$. So the map descends to a homeomorphism between $\mathbf{L}(\sigma, W)/C_2$ and $P(W)$, the projective space of W, and hence

$$(B_{\mathrm{gl}}C_2)(W) \;\cong\; P(W) \;.$$

So for a compact Lie group K, the K-space represented by $B_{\mathrm{gl}}C_2$ is $P(\mathcal{U}_K)$, the projective space of a complete K-universe. In particular, the underlying non-equivariant space is homeomorphic to $\mathbb{RP}^\infty$.

Global classifying spaces preserve products (up to global equivalence), i.e., the product of global classifying spaces for two groups G and G' is a global classifying space for the $G \times G'$:

$$(B_{\mathrm{gl}}G) \times (B_{\mathrm{gl}}G') \;\simeq_{\mathrm{gl}}\; B_{\mathrm{gl}}(G \times G') \;.$$

To see this we let V be a faithful G-representation and V' be a faithful G'-representation. Then $V \oplus V'$ is a faithful representation of the product group $G \times G'$ via the action

$$(g, g') \cdot (v, v') \;=\; (gv,\, g'v') \;.$$

Moreover, as W runs through all real inner product spaces, the restriction maps

$$\mathbf{L}(V \oplus V', W) \times_{G \times G'} (A \times A') \;\longrightarrow\; (\mathbf{L}(V, W) \times_G A) \times (\mathbf{L}(V', W) \times_{G'} A') \;, \quad [\varphi;\, a, a'] \;\longmapsto\; \big([\varphi|_V, a],\, [\varphi|_{V'}, a']\big)$$

form a morphism of orthogonal spaces

$$\mathbf{L}_{G \times G', V \oplus V'}(A \times A') \;\longrightarrow\; (\mathbf{L}_{G,V} A) \times (\mathbf{L}_{G',V'} A') \;. \tag{2.10}$$

In the special case $A = * = A'$, the next proposition then gives the desired global equivalence from $B_{\mathrm{gl}}(G \times G')$ to $(B_{\mathrm{gl}} G) \times (B_{\mathrm{gl}} G')$.

Proposition 2.11. *Let G and G' be compact Lie groups, V and V' faithful representations of G respectively G', A a G-space and A' a G'-space. Then the restriction morphism* (2.10) *is a global equivalence of orthogonal spaces.*

Proof. We let K be another compact Lie group. Before dividing out group actions the restriction map

$$\mathbf{L}(V \oplus V', \mathcal{U}_K) \;\longrightarrow\; \mathbf{L}(V, \mathcal{U}_K) \times \mathbf{L}(V', \mathcal{U}_K) \;, \quad \varphi \;\longmapsto\; (\varphi|_V,\, \varphi|_{V'})$$

is $K \times (G \times G')^{\mathrm{op}}$-equivariant, and both source and target are universal spaces for the family of those subgroups of $K \times (G \times G')^{\mathrm{op}}$ that intersect $1 \times (G \times G')^{\mathrm{op}}$ only in the identity element. So the restriction map is a $(K \times (G \times G')^{\mathrm{op}})$-equivariant homotopy equivalence. Applying $- \times_{G \times G'} (A \times A')$ then provides a K-equivariant homotopy equivalence

$$\begin{aligned} \mathbf{L}(V \oplus V', \mathcal{U}_K) \times_{G \times G'} (A \times A') \;&\longrightarrow\; (\mathbf{L}(V, \mathcal{U}_K) \times \mathbf{L}(V', \mathcal{U}_K)) \times_{G \times G'} (A \times A') \\ &\cong\; (\mathbf{L}(V, \mathcal{U}_K) \times_G A) \times (\mathbf{L}(V', \mathcal{U}_K)_{G'} A') \;, \end{aligned}$$

and that proves the claim. □

We will later prove a substantial generalization of the previous proposition: Theorem 5.9 below shows that for every pair of orthogonal spaces X and Y at least one of which is flat, a certain natural morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ from the *box product* to the cartesian product is a global equivalence. The orthogonal space $\mathbf{L}_{G \times G', V \oplus V'}(A \times A')$ is isomorphic to the box product of $\mathbf{L}_{G,V} A$ and $\mathbf{L}_{G',V'} A'$, in such a way that $\rho_{\mathbf{L}_{G,V} A, \mathbf{L}_{G',V'} A'}$ becomes the morphism (2.10).

Construction 2.12 (Cofree orthogonal spaces)**.** We let A be a topological space. We define the *cofree orthogonal space* RA as follows. For a real inner product space V we let $\mathbf{L}(V, \mathbb{R}^\infty)$ be the space of linear isometric embeddings from V to $\mathbb{R}^\infty$. The group $O(V)$ acts freely from the left by precomposition. Then we set

$$(RA)(V) \;=\; \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) \;, \tag{2.13}$$

the space of all continuous maps from $\mathbf{L}(V, \mathbb{R}^\infty)$ to A. The orthogonal group $O(V)$ acts on this mapping space through its action on $\mathbf{L}(V, \mathbb{R}^\infty)$. For a linear isometric embedding $\varphi : V \longrightarrow W$ we define the structure map $(RA)(\varphi) : (RA)(V) \longrightarrow (RA)(W)$ as the map induced on $\mathrm{map}(-, A)$ by the restriction map $\varphi^* : \mathbf{L}(W, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty)$.

We note that every orthogonal space Y comes with a natural morphism

$$\epsilon_Y \;:\; Y \;\longrightarrow\; R(Y(\mathbb{R}^\infty))$$

to the cofree orthogonal space of the underlying non-equivariant space $Y(\mathbb{R}^\infty)$. Indeed, the value of ϵ_Y at an inner product space V is simply the adjoint of the action map

$$\mathbf{L}(V, \mathbb{R}^\infty) \times Y(V) \;\longrightarrow\; Y(\mathbb{R}^\infty) \;, \quad (\varphi, y) \;\longmapsto\; Y(\varphi)(y) \;.$$

Definition 2.14. An orthogonal space Y is *cofree* if the morphism $\epsilon_Y : Y \longrightarrow R(Y(\mathbb{R}^\infty))$ is a global equivalence.

Given a compact Lie group G we write

$$BG \ = \ (B_{\mathrm{gl}}G)(\mathbb{R}^\infty) \ = \ \mathbf{L}(V, \mathbb{R}^\infty)/G$$

for the underlying non-equivariant space of the global classifying space of G; here V is a faithful G-representation. Then BG is a classifying space for G in the traditional sense.

We will now identify the G-equivariant homotopy set of the cofree orthogonal space RA with homotopy classes of maps from BG to A. We define a map

$$[BG, A] \ \longrightarrow \ \pi_0^G(RA) \tag{2.15}$$

by sending the homotopy class of a continuous map $\varphi : BG = \mathbf{L}(V, \mathbb{R}^\infty)/G \longrightarrow A$ to the class in $\pi_0^G(RA)$ of the composite

$$\mathbf{L}(V, \mathbb{R}^\infty) \ \xrightarrow{\ q\ } \ \mathbf{L}(V, \mathbb{R}^\infty)/G \ \xrightarrow{\ \varphi\ } \ A \ ,$$

where q is the quotient map. The map φq is then a G-fixed point of $\mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) = (RA)(V)$. The map (2.15) is natural for continuous maps in A.

Now we introduce the class of 'static' orthogonal spaces which, roughly speaking, don't change the equivariant homotopy type once a faithful representations has been reached. The static orthogonal spaces will later turn out to be the fibrant objects in the global model structure.

Definition 2.16. An orthogonal space X is *static* if for every compact Lie group G, every faithful G-representation V and an arbitrary G-representation W the structure map

$$X(i_{V,W}) \ : \ X(V) \ \longrightarrow \ X(V \oplus W)$$

is a G-weak equivalence.

Clearly, if X is a static orthogonal space and G a compact Lie group, then for every faithful G-representation V the canonical map

$$\pi_0(X(V)^G) \ \longrightarrow \ \pi_0^G(X) \ , \quad [f] \ \longmapsto \ \langle f \rangle$$

is bijective.

Proposition 2.17. *Let A be a topological space. The cofree orthogonal space RA is static and the map* (2.15) *is a bijection from $[BG, A]$ to $\pi_0^G(RA)$.*

PROOF. We let V be a faithful G-representation and W an arbitrary G-representation. The restriction map $\rho_{V,W} : \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty)$ is G-homotopy equivalence by Proposition 2.6 (iv) for $K = e$ a trivial group. The induced map

$$(RA)(V) = \mathrm{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) \ \xrightarrow{\mathrm{map}(\rho_{V,W}, A)} \ \mathrm{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), A) = (RA)(V \oplus W)$$

on mapping spaces is then a G-homotopy equivalence as well. This shows that RA is a static orthogonal space. So by the remark preceeding this proposition, the canonical map

$$\pi_0\left((RA)(V)^G\right) \ \longrightarrow \ \pi_0^G(RA)$$

is bijective. Precomposition with the projection $q : \mathbf{L}(V, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty)/G$ is a bijection

$$[BG, A] \ = \ [\mathbf{L}(V, \mathbb{R}^\infty)/G, A] \ \cong \ [\mathbf{L}(V, \mathbb{R}^\infty), A]^G \ = \ \pi_0\left((RA)(V)^G\right) \ . \qquad \square$$

Lashof, May and Segal show in [**48**, Thm. 2] that for *abelian* compact Lie groups G and any compact Lie group K the mapping space $\mathrm{map}(EK, BG)$ is a model for $B(K,G)$, the classifying space $B(K,G)$ for principal G-bundles over K-spaces:

$$B(K,G) \;\simeq\; \mathrm{map}(EK, BG)\ .$$

A corollary is that for all K-spaces X the isomorphism classes of principal K-G-bundles over X biject with isomorphism classes of principal G-bundles over $EK \times_K X$. In our present language, this has the reformulation that the global classifying space $B_{\mathrm{gl}}G$ is 'cofree' in a sense we now explain. Theorem 2 of [**48**] can now be rephrased as follows:

Corollary 2.18. *The global classsifying space $B_{\mathrm{gl}}G$ of every abelian compact Lie group G is cofree.*

Corollary 2.18 fails when we drop the hypothesis that the Lie group under consideration is abelian. [give specific example]

Example 2.19 (Global classifying spaces of linear groups)**.** We define global classifying spaces $B_{\mathrm{gl}}G$ more generally for *linear groups*, i.e., closed subgroups $G \leq GL_n(\mathbb{R})$ of a general linear group over $\mathbb{R}$. Such groups have the structure of Lie groups (not necessarily compact). We let

$$\mathcal{B}_n(V) \;=\; \mathrm{Mono}_{\mathbb{R}}(\mathbb{R}^n, V)$$

denote the space of injective $\mathbb{R}$-linear maps from $\mathbb{R}^n$ to V (which can be identified with the space of linearly independent n-tuples of vectors in V). The general linear group $GL_n(\mathbb{R})$ acts freely from the right on $\mathcal{B}_n(V)$ by precomposition, and we set

$$(B_{\mathrm{gl}}G)(V) \;=\; \mathcal{B}_n(V)/G\ .$$

Every linear isometric embedding $\varphi : V \longrightarrow W$ is injective, so postcomposition with φ is a $GL_n(\mathbb{R})$-equivariant map $\mathcal{B}_n(\varphi) : \mathcal{B}_n(V) \longrightarrow \mathcal{B}_n(W)$ that induces the structure map $(B_{\mathrm{gl}}G)(\varphi) : (B_{\mathrm{gl}}G)(V) \longrightarrow (B_{\mathrm{gl}}G)(W)$.

If G is compact, then this construction generalizes the global classifying space in the sense of Definition 2.7 in the following sense. Since G is compact, it leaves some scalar product on $\mathbb{R}^n$ invariant (not necessarily the standard one, though). So G is conjugate in $GL_n(\mathbb{R})$ to a subgroup of the orthogonal group $O(n)$, and we may suppose that $G \leq O(n)$. Then the tautological G-representation on $\mathbb{R}^n$ is faithful and the inclusion $\mathbf{L}(\mathbb{R}^n, V) \subset \mathcal{B}_n(V)$ descends to a continuous map

$$\mathbf{L}_{G,\mathbb{R}^n}(V) = \mathbf{L}(\mathbb{R}^n, V)/G \;\longrightarrow\; \mathcal{B}_n(V)/G \;=\; (B_{\mathrm{gl}}G)(V)\ ;$$

as V varies in $\mathbf{L}$, these maps form a morphism of orthogonal spaces $\mathbf{L}_{G,\mathbb{R}^n} \longrightarrow B_{\mathrm{gl}}G$. When K is a compact Lie group and we take the colimit over $V \in s(\mathcal{U}_G)$, then these maps become a K-weak equivalence

$$\mathbf{L}_{G,\mathbb{R}^n}(\mathcal{U}_K) \;\longrightarrow\; \mathcal{B}_n(\mathcal{U}_K)/G\ .$$

So the morphism $\mathbf{L}_{G,\mathbb{R}^n} \longrightarrow B_{\mathrm{gl}}G$ is a global equivalence.

Proposition 2.6 generalizes to the context of linear groups G as follows; here K is any compact Lie group.

(i) The $(K \times G^{\mathrm{op}})$-space $\mathcal{B}_n(\mathcal{U}_K)$ is a universal space for the family of those subgroup Γ of $K \times G^{\mathrm{op}}$ that intersect $1 \times G^{\mathrm{op}}$ in the identity element.

(ii) The K-space $(B_{\mathrm{gl}}G)(\mathcal{U}_K) = \mathcal{B}_n(\mathcal{U}_K)/G$ is a classifying space for the family of those subgroup Γ of $K \times G^{\mathrm{op}}$ that intersect $1 \times G^{\mathrm{op}}$ in the identity element.

(iii) The K-fixed point space $((B_{\mathrm{gl}}G)(\mathcal{U}_K))^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α.

(iv) We can restrict the tautological G-action on $\mathbb{R}^n$ along a continuous homomorphism $\alpha : K \longrightarrow G$ to obtain an $\mathbb{R}K$-module $\alpha^*(\mathbb{R}^n)$. The image of α is a compact subgroup of G, so there is scalar product on $\mathbb{R}^n$ (not necessarily the standard one) that is invariant under the K-action, and $\alpha^*(\mathbb{R}^n)$ becomes a K-representation. The G-orbit $(e_1, \dots, e_n) \cdot G$ of the canonical basis of $\mathbb{R}^n$ is then a K-fixed point of

$$(B_{\mathrm{gl}}G)(\alpha^*(\mathbb{R}^n)) \ = \ \mathcal{B}_n(\alpha^*(\mathbb{R}^n)) \ .$$

The image of this K-fixed point in $\pi_0^K(B_{\mathrm{gl}}G)$ depends only on the conjugacy class of α The resulting map

$$\mathrm{Rep}(K, G) \ \longrightarrow \ \pi_0^K(B_{\mathrm{gl}}G) \ , \quad [\alpha : K \longrightarrow G] \ \longmapsto \ \langle (e_1, \dots, e_n) \cdot G \rangle$$

is bijective. As K varies over all compact Lie groups, these maps form an isomorphism of Rep-functors $\mathrm{Rep}(-, G) \cong \underline{\pi}_0(B_{\mathrm{gl}}G)$.

While part (iv) gives an algebraic description of the Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}G)$, this functor is *not* generally representable (because G need not be a compact Lie group).

As the last paragraph indicates, the global homotopy type of the orthogonal space $B_{\mathrm{gl}}G$ depends on continuous homomorphisms from compact Lie groups to G. For example, if G is a linear group that is discrete and torsion free, then every continuous homomorphism from a compact Lie group to G is trivial, so by (iii) above, $B_{\mathrm{gl}}G$ is then globally equivalent to the constant orthogonal space with value BG, a classifying space (in the non-equivariant sense) of G. So to get new global homotopy types, G should have non-trivial compact Lie subgroups.

Now we are going to show that the restriction maps along a group homomorphism are the only natural operations between equivariant homotopy sets of orthogonal spaces. The strategy is to show that the functor π_0^G is representable and then to determine the equivariant homotopy sets of the representing objects. We will want to use this kind of argument several other times in this book, so we prove the representability property in a more general context of an adjoint functor pair

$$spc \ \underset{U}{\overset{\Lambda}{\rightleftarrows}} \ \mathcal{C} \quad .$$

In this section we will only consider the degenerate case $\mathcal{C} = spc$ and $\Lambda = U = \mathrm{Id}$; later we will also consider the case $\mathcal{C} = $ coms of commutative orthogonal monoid spaces with the free and forgetful functor pair $(\mathbb{P}, U)$, and also the case $\mathcal{C} = \mathcal{S}p$ of orthogonal spectra with the adjoint functor pair $(\Sigma^\infty_+, \Omega^\bullet)$, and the combination of these two cases, where $\mathcal{C}$ is the category of commutative orthogonal ring spectra. In the next proposition a morphism in $\mathcal{C}$ is called a 'global equivalence' if the right adjoint functor U takes it to a global equivalence of orthogonal spaces.

Proposition 2.20. *Let $\mathcal{C}$ be a category and*

$$spc \ \underset{U}{\overset{\Lambda}{\rightleftarrows}} \ \mathcal{C}$$

an adjoint functor pair between the category of orthogonal spaces and $\mathcal{C}$ with adjunction unit $\eta : \mathrm{Id} \longrightarrow U \circ \Lambda$. Let G be a compact Lie group, V a faithful G-representation, and Φ a functor from $\mathcal{C}$ to the category of sets that takes global equivalences to bijections. Then evaluation at the class $\eta_*(u_{G,V}) \in \pi_0^G(U(\Lambda(L_{G,V})))$ is a bijection

$$\mathrm{Nat}(\pi_0^G \circ U, \Phi) \ \longrightarrow \ \Phi(\Lambda(\mathbf{L}_{G,V})) \ , \quad \tau \ \longmapsto \ \tau(\eta_*(u_{G,V}))$$

between the set of natural transformations, from the functor $\pi_0^G \circ U$ to Φ, and the set $\Phi(\Lambda(\mathbf{L}_{G,V}))$.

Proof. To show that the evaluation map is injective we show that any natural transformation $\tau : \pi_0^G \circ U \longrightarrow \Phi$ is determined by the element $\tau(e_{G,V})$. We let X be any object of $\mathcal{C}$ and $[x] \in \pi_0^G(UX)$ a G-equivariant homotopy element. The class x is represented by a G-fixed point $x \in ((UX)(W))^G$ for some G-representation W. We can stabilize with the representation V and obtain another representative

$$(UX)(i_{V,W})(x) \ \in \ ((UX)(V \oplus W))^G$$

for the same class. This G-map is adjoint to a morphism of orthogonal spectra

$$\hat{x} \ : \ \Lambda(\mathbf{L}_{G,V\oplus W}) \ \longrightarrow \ X \ ;$$

the morphism $\hat{x}$ satisfies

$$(U\hat{x})_*(\eta_*(u_{G,V\oplus W})) \ = \ [x] \ \in \ \pi_0^G(UX) \ .$$

The restriction morphism of orthogonal spaces

$$\rho_{G,V,W} \ : \ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \mathbf{L}_{G,V}$$

is a global equivalence and sends $u_{G,V\oplus W}$ to $u_{G,V}$. The induced morphism of suspension spectra

$$\Lambda(\rho_{G,V,W}) \ : \ \Lambda(\mathbf{L}_{G,V\oplus W}) \ \longrightarrow \ \Lambda(\mathbf{L}_{G,V})$$

is thus a global equivalence in $\mathcal{C}$, and $U(\Lambda(\rho_{G,V,W}))$ sends $\eta_*(u_{G,W\oplus V})$ to $\eta_*(u_{G,V})$. The diagram

$$\begin{tikzcd}
\pi_0^G(U(\Lambda(\mathbf{L}_{G,V}))) \arrow[d, "\tau"'] & \pi_0^G(U(\Lambda(\mathbf{L}_{G,V\oplus W}))) \arrow[l, "\pi_0^G(U(\Lambda(\rho_{G,V,W}))"'] \arrow[r, "\pi_0^G(U\hat{x})"] \arrow[d, "\tau"'] & \pi_0^G(UX) \arrow[d, "\tau"] \\
\Phi(\Lambda(\mathbf{L}_{G,V})) & \Phi(\Lambda(\mathbf{L}_{G,W\oplus V})) \arrow[l, "\cong", "\Phi(\Lambda(\rho_{G,V,W}))"'] \arrow[r, "\Phi(\hat{x})"'] & \Phi(X)
\end{tikzcd}$$

commutes and the lower left horizontal maps is bijective by hypothesis.

$$\begin{aligned}\tau[x] \ &= \ \tau((\pi_0^G(U\hat{x}))(\eta_*(u_{G,V\oplus W}))) \ = \ (\Phi(\hat{x}))(\tau(\eta_*(u_{G,V\oplus W}))) \\ &= \ (\Phi(\hat{x}))(\Phi(\Lambda(\rho_{G,V,W}))^{-1}(\tau(\eta_*(u_{G,V})))) \ .\end{aligned}$$

This shows that the transformation τ is determined by the value $\tau(\eta_*(u_{G,V}))$.

It remains to construct, for every element $y \in \Phi(\Lambda(\mathbf{L}_{G,V}))$, a natural transformation $\tau : \pi_0^G \circ U \longrightarrow \Phi$ with $\tau(\eta_*(u_{G,V})) = y$. The previous paragraph dictates what to do: we represent a given class $[x] \in \pi_0^G(UX)$ by a G-fixed point $x \in ((UX)(V \oplus W))^G$ as in the previous paragraph and set

$$\tau[x] \ = \ (\Phi(\hat{x}))(\Phi(\Lambda(\rho_{G,V,W}))^{-1}(y)) \ .$$

We omit the verification that the element $\tau[x]$ only depends on the class $[x]$ (this uses that homotopy equivalences are global equivalences and hence Φ takes the same value of homotopic morphisms) and that τ is indeed natural. □

Proposition 2.21. *Let G and K be compact Lie groups. Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of orthogonal spaces is of the form α^* for a unique conjugacy class of continuous group homomorphism $\alpha : K \longrightarrow G$.*

Proof. We let V be any faithful G-representation. By Proposition 2.6 (iv) the composite

$$\mathrm{Rep}(K,G) \ \xrightarrow{[\alpha]\mapsto\alpha^*} \ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ \xrightarrow{\mathrm{ev}} \ \pi_0^K(\mathbf{L}_{G,V})$$

is bijective, where the second map is evaluation at the tautological class $u_{G,V}$. The evaluation map is bijective by Proposition 2.20 with $\mathcal{C} = spc$, $\Lambda = U = \mathrm{Id}$ and $\Phi = \pi_0^K$. So the first map is bijective as well. □

3. Strong level model structure for orthogonal spaces

In this section we start the discussion of model structures for orthogonal spaces. Ultimately we are after the global model structure, compare Theorem 4.3. Towards this aim we first discuss a 'strong level model structure' for orthogonal spaces, which we then localize. There is a functorial way to write an orthogonal space as a sequential colimit of orthogonal spaces which are made from the information below a fixed level. We refer to this as the *skeleton filtration* of an orthogonal space. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal space need not be injective.

In the following construction we denote by

$$G_m \;:\; O(m)\mathbf{U} \;\longrightarrow\; spc$$

the left adjoint to the forgetful functor that takes an orthogonal space Y to its m-th level $Y(\mathbb{R}^m)$, viewed as an $O(m)$-space. So G_m is a shorthand notation for $\mathbf{L}_{O(m),\mathbb{R}^m}$, the free functor (2.2) indexed by the tautological $O(m)$-representation.

Construction 3.1. For every orthogonal space A and $m \geq 0$ we define the following data by induction on m:

- a based $O(m)$-space $L_m A$, the *m-th latching space* of A, equipped with a natural map of based $O(m)$-spaces $\nu_m : L_m A \longrightarrow A_m$.
- an orthogonal space $\operatorname{sk}^m A$, the *m-skeleton* of A, equipped with a natural morphism $i_m : \operatorname{sk}^m A \longrightarrow A$,
- a natural morphism $j_m : \operatorname{sk}^{m-1} A \longrightarrow \operatorname{sk}^m A$ which satisfies $i_m j_m = i_{m-1}$.

We start with $\operatorname{sk}^{-1} A = \emptyset$, the empty orthogonal space. For $m \geq 0$ we define the latching space by

$$L_m A \;=\; (\operatorname{sk}^{m-1} A)_m \;, \tag{3.2}$$

the m-th level of the $(m-1)$-skeleton, and the morphism $\nu_m : L_m A = (\operatorname{sk}^{m-1} A)_m \longrightarrow A_m$ as the m-level of the previously constructed morphism $i_{m-1} : \operatorname{sk}^{m-1} A \longrightarrow A$. Then we define the m-skeleton $\operatorname{sk}^m A$ as the pushout

$$\begin{tikzcd} G_m L_m A \arrow[r, "G_m \nu_m"] \arrow[d] & G_m A_m \arrow[d] \\ \operatorname{sk}^{m-1} A \arrow[r, "j_m"'] & \operatorname{sk}^m A \end{tikzcd} \tag{3.3}$$

where the left vertical morphism is adjoint to the identity map of $L_m A = (\operatorname{sk}^{m-1} A)_m$. The morphism $\eta : G_m A_m \longrightarrow A$ which is adjoint to the identity of A_m and $i_{m-1} : \operatorname{sk}^{m-1} A \longrightarrow A$ restrict to the same morphism on $G_m L_m A$. So the universal property of the pushout provides a unique morphism $i_m : \operatorname{sk}^m A \longrightarrow A$ which satisfied $i_m j_m = i_{m-1}$ and whose restriction to $G_m A_m$ is η.

We can – and will – choose the pushout (1.3) so that

$$(\operatorname{sk}^m A)_n \;=\; A_n \qquad \text{for } n \leq m$$

and so that the morphisms $j_{m+1} : \operatorname{sk}^m A \longrightarrow \operatorname{sk}^{m+1} A$ and $i_m : \operatorname{sk}^m A \longrightarrow A$ are the identity maps in level m and below. This convention is convenient because it will later make some maps equalities which would otherwise merely be isomorphisms. This convention also forces the structure maps of the orthogonal space $\operatorname{sk}^m A$ to coincide with those of A up to level m.

The sequence of skeleta $\operatorname{sk}^m A$ stabilizes to A in a very strong sense. In every given level n, there is a point from which on all the spaces $(\operatorname{sk}^m A)_n$ are equal to A_n and the morphisms i_m and j_m are identity maps in level n. In particular, A_n is the colimit with respect to the morphisms $(i_m)_n$, of the sequence of maps $(j_m)_n$, Since colimits in the category of orthogonal spaces are created levelwise, we deduce that the space A is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Given any morphism $f : A \longrightarrow B$ of orthogonal spaces we can define a relative skeleton filtration as follows. The *relative m-skeleton* of f is the pushout

$$\mathrm{sk}^m[f] \; = \; A \cup_{\mathrm{sk}^m A} \mathrm{sk}^m B \tag{3.4}$$

where $\mathrm{sk}^m A$ is the m-skeleton of A as defined above. The relative m-skeleton comes with a unique morphism $i_m : \mathrm{sk}^m[f] \longrightarrow B$ which restricts to $f : A \longrightarrow B$ respectively to $i_m : \mathrm{sk}^m B \longrightarrow B$. Since $L_m A = (\mathrm{sk}^{m-1} A)_m$ we have

$$(\mathrm{sk}^{m-1}[f])_m \; = \; A_m \cup_{L_m A} L_m B \; ,$$

the m-th relative latching object. A morphism $j_m[f] : \mathrm{sk}^{m-1}[f] \longrightarrow \mathrm{sk}^m[f]$ is obtained from the commutative diagram

$$\begin{tikzcd}
A \arrow[d, equal] & \mathrm{sk}^{m-1} A \arrow[l] \arrow[r, "\mathrm{sk}^{m-1} f"] \arrow[d, "j_m^A"'] & \mathrm{sk}^{m-1} B \arrow[d, "j_m^B"] \\
A & \mathrm{sk}^m A \arrow[l] \arrow[r, "\mathrm{sk}^m f"'] & \mathrm{sk}^m B
\end{tikzcd}$$

by taking pushouts. The square

$$\begin{tikzcd}
G_m(A_m \cup_{L_m A} L_m B) \arrow[r, "G_m(\nu_m f)"] \arrow[d] & G_m B_m \arrow[d] \\
\mathrm{sk}^{m-1}[f] \arrow[r, "j_m[f]"'] & \mathrm{sk}^m[f]
\end{tikzcd} \tag{3.5}$$

is a pushout and the original morphism $f : A \longrightarrow B$ factors as the composite of the countable sequence

$$A = \mathrm{sk}^{-1}[f] \; \xrightarrow{j_0[f]} \; \mathrm{sk}^0[f] \; \xrightarrow{j_1[f]} \; \mathrm{sk}^1[f] \; \longrightarrow \; \cdots \; \xrightarrow{j_m[f]} \; \mathrm{sk}^m[f] \; \longrightarrow \; \cdots \; .$$

If we fix a level n, then the sequence stabilizes to the identity map of B_n from $(\mathrm{sk}^n[f])_n$ on; in particular, the compatible maps $j_m : \mathrm{sk}^m[f] \longrightarrow B$ exhibit B as the colimit of the sequence.

For a morphism $f : A \longrightarrow B$ of orthogonal spaces and $m \geq 0$ we have a commutative square of $O(m)$-spaces

$$\begin{tikzcd}
L_m A \arrow[r, "L_m f"] \arrow[d, "\nu_m"'] & L_m B \arrow[d, "\nu_m"] \\
A_m \arrow[r, "f_m"'] & B_m
\end{tikzcd}$$

We thus get a natural morphism of $O(m)$-spaces

$$\nu_m f \; : \; A_m \cup_{L_m A} L_m B \; \longrightarrow \; B_m \; .$$

Example 3.6. Let G be a compact Lie group, V a G-representation of dimension n and A a G-space. Then the free orthogonal space (2.2) generated by A in level V is 'purely n-dimensional' in the following sense. The space $(\mathbf{L}_{G,V} A)_m$ is trivial for $m < n$, and hence the latching space $L_m(\mathbf{L}_{G,V} A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(\mathbf{L}_{G,V} A) \longrightarrow (\mathbf{L}_{G,V} A)_m$ is an isomorphism. So the skeleton $\mathrm{sk}^m(\mathbf{L}_{G,V} A)$ is trivial for $m < n$ and $\mathrm{sk}^m(\mathbf{L}_{G,V} A) = \mathbf{L}_{G,V} A$ is the entire orthogonal space for $m \geq n$

Let A be an $O(n)$-space. Then as a special case of the previous paragraph, the orthogonal space $G_n A = \mathbf{L}_{O(n),\mathbb{R}^n} A$ is purely n-dimensional.

The following proposition is an immediate application of the relative skeleton filtration. It is the key ingredient to the lifting properties of the various level model structures that we will discuss soon. We recall that a pair $(i : A \longrightarrow B, f : X \longrightarrow Y)$ of morphisms in some category has the *lifting property* if for all morphism $\varphi : A \longrightarrow X$ and $\psi : B \longrightarrow Y$ such that $f\varphi = \psi i$ there exists a *lifting*, i.e., a morphism $\lambda : B \longrightarrow Y$ such that $\lambda i = \varphi$ and $f\lambda = \psi$. Instead of saying that the pair (i, f) has the lifting property we may equivalently say 'i has the left lifting property with respect to f' or 'f has the right lifting property with respect to i'.

Proposition 3.7. *Let $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ be morphisms of orthogonal spaces. If the pair $(\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B,\ f_m : X_m \longrightarrow Y_m)$ has the lifting property in the category of $O(m)$-spaces for every $m \geq 0$, then the pair (i, f) has the lifting property.*

Proof. We consider the class f-cof of all morphisms of orthogonal spaces that have the left lifting property with respect to f; this class is closed under cobase change and countable composition. Since the pair $(\nu_m i, f_m)$ has the lifting property in the category of $O(m)$-spaces, the semifree morphism $G_m(\nu_m i)$ belongs to the class f-cof by adjointness. The relative skeleton filtration (3.4) shows that i is a countable composite of cobase changes of the morphisms $\nu_m i$, so i belongs to the class f-cof. □

Now we discuss a general recipe for constructing 'level model structures' on the category of orthogonal spaces. As input we need, for every $m \geq 0$, a model structure $\mathcal{C}(m)$ on the category of $O(m)$-spaces. We call a morphism $f : X \longrightarrow Y$ of orthogonal spaces

- a *level equivalence* if $f_m : X_m \longrightarrow Y_m$ is a weak equivalence in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *level fibration* if the morphism $f_m : X_m \longrightarrow Y_m$ is a fibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *cofibration* if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is a cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$.

Proposition 3.9 below will show that if the various model structures $\mathcal{C}(m)$ satisfy the following 'consistency condition', then the level equivalences, level fibrations and cofibrations define a model structure on the category of orthogonal spaces.

In the following we use the shorthand notation

$$\mathbf{L}(m, k) \ = \ \mathbf{L}(\mathbb{R}^m, \mathbb{R}^k)$$

for the space of linear isometric embeddings from $\mathbb{R}^m$ to $\mathbb{R}^k$.

Consistency condition: For all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the model structure $\mathcal{C}(m)$ on $O(m)$-spaces, every cobase change, in the category of $O(m+n)$-spaces, of the map

$$\mathbf{L}(m, m+n) \times_{O(m)} i \ : \ \mathbf{L}(m, m+n) \times_{O(m)} A \ \longrightarrow \ \mathbf{L}(m, m+n) \times_{O(m)} B \tag{3.8}$$

is a weak equivalence in the model structure $\mathcal{C}(m+n)$.

Proposition 3.9. *Let $\mathcal{C}(m)$ be a model structure on the category of $O(m)$-spaces, for $m \geq 0$, such that the consistency condition (3.8) holds.*

(i) *The classes of level equivalences, level fibrations and cofibrations define a model structure on the category of orthogonal spaces.*
(ii) *A morphism $i : A \longrightarrow B$ of orthogonal spaces is simultaneously a cofibration and a level equivalence if and only if for all $m \geq 0$ the latching morphism $\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$.*
(iii) *Suppose that the fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $J(m)$; then the level fibrations of orthogonal spaces are detected by the set of semifree morphisms*

$$\{G_m j \mid m \geq 0,\ j \in J(m)\} \ .$$

Similarly, if the acyclic fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $I(m)$, then the level acyclic fibrations of orthogonal spaces are detected by the set of semifree morphisms

$$\{G_m i \mid m \geq 0,\, j \in I(m)\} \ .$$

(iv) *If all the model structures $\mathcal{C}(m)$ are topological, then so is the resulting level model structure of orthogonal spaces.*

PROOF. We start by showing one of the directions of part (ii): we let $i : A \longrightarrow B$ be a morphism such that the latching morphism $\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$; we show that then i is a level equivalence.

The map $i_n : A_n \longrightarrow B_n$ is the finite composite

$$A_n = (\mathrm{sk}^{-1}[i])_n \xrightarrow{(j_0[i])_n} (\mathrm{sk}^0[i])_n \xrightarrow{(j_1[i])_n} \dots \xrightarrow{(j_{n-1}[i])_n} (\mathrm{sk}^{n-1}[i])_n \xrightarrow{(j_n[i])_n} (\mathrm{sk}^n[i])_n = B_n \ ,$$

so it suffices to show that $j_k[i]$ is a level equivalence for all $k \geq 0$. The pushout square (3.5) in level $m+n$ is a pushout of $O(m+n)$-spaces

$$\begin{tikzcd} \mathbf{L}(m, m+n) \times_{O(m)} (A_m \cup_{L_m A} L_m B) \arrow[r, "\mathbf{L}(m,m+n)\times(\nu_m i)"] \arrow[d] & \mathbf{L}(m, m+n) \times_{O(m)} B_m \arrow[d] \\ (\mathrm{sk}^{m-1}[i])_{m+n} \arrow[r, "(j_m[i])_{m+n}"'] & (\mathrm{sk}^m[i])_{m+n} \end{tikzcd}$$

The consistency condition guarantees that the lower horizontal morphism is a $\mathcal{C}(m+n)$-weak equivalence.

(i) Several of the axioms are straightforward: the category of orthogonal spaces has all small limits and colimits; the level equivalences satisfy the 2-out-of-3 property; the classes of level equivalences, cofibrations and fibrations are closed under retracts.

Now we prove the factorization axiom, i.e., we show that every morphism $f : A \longrightarrow X$ of orthogonal spaces can be factored as $f = qi$ where q is a level acyclic fibration and i a cofibration; and it can be factored as $f = pj$ where p is a level fibration and i a cofibration and level equivalence. We start with the first factorization and construct an orthogonal space B and morphisms $i : A \longrightarrow B$ and $q : B \longrightarrow X$ by induction over the levels. In level 0 we choose a factorization

$$A_0 \xrightarrow{i_0} B_0 \xrightarrow{q_0} X_0$$

of f_0 in the category of spaces such that i_0 is a cofibration and q_0 is an acyclic fibration in the model structure $\mathcal{C}(0)$. Now suppose that the orthogonal space B and the morphisms i and q have already been constructed up to level $m-1$. Then we have all the data necessary to define the m-th latching object $L_m B$; moreover, the 'partial morphism' $q : B \longrightarrow X$ provides a $O(m)$-morphism $L_m B \longrightarrow X_m$ such that the square

$$\begin{tikzcd} L_m A \arrow[r, "L_m i"] \arrow[d, "\nu_m"'] & L_m B \arrow[d] \\ A_m \arrow[r, "f_m"'] & X_m \end{tikzcd}$$

commutes. We factor the resulting morphism $A_m \cup_{L_m A} L_m B \longrightarrow X_m$ in the category of $O(m)$-spaces

$$A_m \cup_{L_m A} L_m B \xrightarrow{\nu_m i} B_m \xrightarrow{q_m} X_m \tag{3.10}$$

such that ν_m is a cofibration and q_m is an acyclic fibration in the model structure $\mathcal{C}(m)$. The intermediate $O(m)$-space B_m defines the m-th level of the orthogonal space B, and the second morphism q_m is the m-th level of the morphism q. The structure morphism $\sigma_m : B_{m-1} \wedge S^1 \longrightarrow B_m$ is the composite

$$B_{m-1} \wedge S^1 \longrightarrow L_m B \longrightarrow A_m \cup_{L_m A} L_m B \xrightarrow{\nu_m i} B_m$$

and the composite of ν_m with the canonical morphism $A_m \longrightarrow A_m \cup_{L_m A} L_m B$ is the m-th level of the morphism i.

At the end of the day we have indeed factored $f = qi$ in the category of orthogonal spaces such that q is a level equivalence and level fibration. Moreover, the m-th latching morphism $\nu_m i$ comes out to be the map $\nu_m : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ in the factorization (3.10), which is a cofibration in the model structure $\mathcal{C}(m)$. So the morphism i is indeed a cofibration.

The second factorization $f = pj$ as a cofibration and level equivalence j followed by a level fibration p is similar, but instead of the factorization (3.10) we use a factorization, in the model category $\mathcal{C}(m)$, as an acyclic cofibration followed by a fibrations. Then the resulting morphism p is a level fibration and the morphism j has the property that all its latching morphisms. $\nu_m j$ are acyclic cofibrations. So j is a cofibration (by definition) and a level equivalence (by the part of (ii) established above).

It remains to show the lifting axioms. In each of the model structures $\mathcal{C}(m)$ the cofibrations have the left lifting property with respect to the acyclic fibrations; so by Proposition 3.7 the cofibrations of orthogonal spaces have the left lifting property with respect to level equivalences which are also level fibrations.

We postpone the proof of the other lifting property and prove the remaining direction of (ii) next. We let $i : A \longrightarrow B$ be a cofibration and a level equivalence. The second factorization axiom proved above provides a factorization $i = pj$ where $j : A \longrightarrow D$ is a level equivalence such that each latching morphism $\nu_m j$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$, and $p : D \longrightarrow B$ is a fibration in $\mathcal{C}(m)$. Since i and j are level equivalences, so is p. So the cofibration i has the left lifting property with respect to the level equivalence and level fibration p by the previous paragraph. In particular, a lift $\lambda : B \longrightarrow D$ in the square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] & D \arrow[d, "p"] \\ B \arrow[ur, dotted, "\lambda"] \arrow[r, equal] & B \end{tikzcd}$$

shows that the morphism i is a retract of the morphism j. So the latching morphism $\nu_n i$ is a retract of the latching morphism $\nu_n j$, hence also an acyclic cofibration in the model structure $\mathcal{C}(m)$. This proves (ii).

Now we prove the remaining half of the lifting properties. We let $i : A \longrightarrow B$ be a cofibration that is also a level equivalence. By (ii), which has just been shown, each latching morphism $\nu_m i$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$. So i has the left lifting property with respect to all level fibrations by Proposition 1.7.

Property (iii) is a straightforward consequence of the fact that the semifree functor G_m is left adjoint to evaluation at level m.

(iv) Let $f : D \longrightarrow E$ be a continuous map of spaces and $i : A \longrightarrow B$ a morphism of orthogonal spaces. Latching objects commute with colimits of orthogonal spaces, and the pairing of spaces with orthogonal spaces is levelwise, so

$$L_m(E \times A \cup_{D\times A} D \times B) \;=\; (E \times L_m A) \cup_{(D \times L_m A)} (D \times L_m B)$$

and under this identification and a rearranging of pushouts the latching map

$$\nu_m(f \times i) \;:\; (E \times A \cup_{D\times A} D \times B)_m \cup_{L_m(E\times A \cup_{D\times A} D \times B)} L_m(E \times B) \;\longrightarrow\; (E \times B)_m$$

becomes the pushout product

$$f \times (\nu_m i) \;:\; E \times (A_m \cup_{L_m A} L_m B) \cup_{D \times (A_m \cup_{L_m A} L_m B)} D \times B_m \;\longrightarrow\; E \times B_m$$

of f with the m-th latching map of i.

The proof of the pushout product property is now easy. If f is a cofibration of spaces and i a cofibration of orthogonal spaces, then $\nu_m i$ is a cofibration of $O(m)$-spaces in the model structure $\mathcal{C}(m)$. Since $\mathcal{C}(m)$ is topological, the pushout product $f \times (\nu_m i)$, and hence also $\nu_m(f \times i)$, is a cofibration in $\mathcal{C}(m)$. So $f \times i$ is a cofibration of orthogonal spaces. When in addition f is weak equivalence of i is a level equivalence, then the same reasoning (using the characterization of acyclic cofibrations in part (ii)) shows that $f \times i$ is an acyclic cofibration. □

Now we apply the general construction of level model structures to a specific situation.

Definition 3.11. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an *strong level equivalence* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-weak equivalence for every $m \geq 0$. Similarly, f is a *strong level fibration* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $O(m)$-fibration for every $m \geq 0$. The morphism f is a *flat cofibration* if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is an $O(m)$-cofibration for all $m \geq 0$.

Every inner product space V is isometrically isomorphic to $\mathbb{R}^m$ with the standard scalar product, where m is the dimension of V. So a morphism $f : X \longrightarrow Y$ of orthogonal spaces is a strong level equivalence (respectively strong level fibration) if and only if for every compact Lie group G and every faithful G-representation V the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence (respectively Serre fibration). Clearly, the class of strong level equivalences is closed under composition, retract and coproducts.

The following diagram collects various kinds of equivalences and their implications:

$$\begin{array}{ccccc} \text{homotopy equivalence} & \Longrightarrow & \text{strong level equivalence} & \Longrightarrow & \text{global equivalence} \\ & & \Downarrow & & \Downarrow \\ & & \text{level equivalence} & \Longrightarrow & \text{weak equivalence} \end{array}$$

We are ready to establish the strong level model structure.

Proposition 3.12. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the* strong level model structure*, on the category of orthogonal spaces. The strong level model structures is proper, topological and cofibrantly generated.*

Proof. We apply Proposition 3.9 as follows. We let $\mathcal{C}(m)$ be the projective model structure on the category of $O(m)$-spaces (with respect to the family of all closed subgroups of $O(m)$), compare Proposition A.1.18. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 3.9 precisely become the strong level equivalences, strong level fibrations and flat cofibrations. The consistency condition (3.8) is the special case of Proposition A.1.23 with $G = O(m+n)$, $K = O(m)$ and $A = \mathbf{L}(m, m+n)$ and the family $\mathcal{F}$ of all subgroups of $O(m+n)$.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations. We let I^{str} be the set of all morphism $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (1.19) of Section A.1. Then the set I^{str} detects the acyclic fibrations in the strong level model structure by Proposition 3.9 (iii). Similarly, we let J^{str} be the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. Again by Proposition 3.9 (iii), J^{str} detects the fibrations in the strong level model structure

Limits in the category of orthogonal spaces are constructed levelwise (i.e., evaluation at level m preserves limits). Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. The projective model structure on the category of $O(m)$-spaces is right proper for all $m \geq 0$, so right properness of the strong level model structure follows. The argument for left properness is similar, but not completely analogous. Since flat cofibrations are levelwise $O(m)$-cofibrations (Proposition 3.13) and colimits in the category of orthogonal spaces are also constructed levelwise, left properness for the strong level model structure is a consequence of left properness of the projective model structure on $O(m)$-spaces.

Since the projective model structure on $O(m)$-spaces is topological, part (iv) of Proposition 3.9 shows that the strong level model structure is topological. □

Proposition 3.13. *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spaces and K a compact Lie group.*

(i) *For every finite dimensional K-representation W the K-map $i(W) : A(W) \longrightarrow B(W)$ is a K-cofibration.*
(ii) *The K-map $i(\mathcal{U}_K) : A(\mathcal{U}_K) \longrightarrow B(\mathcal{U}_K)$ is a K-cofibration.*

Proof. Evaluation at W takes colimits of orthogonal spaces to colimits of K-spaces. Similarly, the functor

$$spc \longrightarrow K\mathbf{U} , \quad A \longmapsto A(\mathcal{U}_K)$$

preserves colimits because $A(\mathcal{U}_K)$ is a colimit of values of A, and colimits commute among themselves. So it suffices to show (i) and (ii) for the set of generating flat cofibrations [fix this...]

$$S^{n-1} \times \mathbf{L}_{G,V} \longrightarrow D^n \times \mathbf{L}_{G,V}$$

where G is a compact Lie group and V a faithful G-representation. For any K-representation U, of finite or countably infinite dimension, we have $\mathbf{L}_{G,V}(U) = \mathbf{L}(V, U)/G$, which is K-cofibrant by Proposition A.2.3 (iii). Since the morphism

$$(S^{n-1} \times \mathbf{L}_{G,V})(\mathcal{U}_K) \longrightarrow (D^n \times \mathbf{L}_{G,V})(\mathcal{U}_K)$$

is isomorphic to the map

$$S^{n-1} \times \mathbf{L}(V,\mathcal{U}_K)/G \longrightarrow D^n \times \mathbf{L}(V,\mathcal{U}_K)/G ,$$

both are cofibrations of K-spaces. □

Definition 3.14. We call an orthogonal space A *flat* if it is cofibrant in the (flat or projective) level model structure for the maximal global family of all compact Lie groups. In other words, the unique morphism from the empty orthogonal space to A is a flat cofibration. Equivalently, for every $m \geq 0$ the latching map $\nu_m : L_m A \longrightarrow A_m$ is an $O(m)$-cofibration.

Proposition 3.15. *Every flat orthogonal space is closed.*

If A is flat then for every compact Lie group G the space $A(\mathcal{U}_G)$ is G-cofibrant. In particular, the space $A(\mathcal{U}_e)$ is cofibrant, and hence its path components are open subspaces. So we can decompose the orthogonal space A into a coproduct of orthogonal subspaces according to the set $\pi_0(A) = \pi_0(A(\mathcal{U}_e))$: for every element $x \in \pi_0(A)$ and inner product space V the space

$$A^{[x]}(V) \subset A(V)$$

is the subspace of those points of $A(V)$ that represent a class in x. An equivalent way to say this is the $A^{[x]}(V)$ is the preimage, under the canonical map $A(V) \longrightarrow A(\mathcal{U}_e)$, of the path component of $A(\mathcal{U}_e)$ corresponding to X. Since A is flat and hence $A(\mathcal{U}_e)$ a cofibrant space, every path component of $A(\mathcal{U}_e)$ is open, so $A^{[x]}(V)$ is an open subspace of $A(V)$. Since $A(V)$ is the union of its open subspaces $A^{[x]}(V)$ for all $x \in \pi_0(A)$, $A(V)$ is the topological disjoint union of the subspaces $A^{[x]}(V)$.

4. Global model structure for orthogonal spaces

In this section we construct the main model structure of interest for us, the *global model structure* on the category of orthogonal spaces, see Theorem 4.3. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. The fibrations in the global model structure are defined as follows.

Definition 4.1. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every faithful G-representation V and an arbitrary G-representation W the square of G-fixed point spaces

$$\begin{tikzcd} X(V)^G \arrow[r, "X(i_{V,W})^G"] \arrow[d, "f(V)^G"'] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G \end{tikzcd} \tag{4.2}$$

is homotopy cartesian.

Equivalently, a morphism f is a global fibration if and only if f is a strong level fibration and for every compact Lie group G, every faithful G-representation V and an arbitrary G-representation W the map

$$(f(V)^G), X(i_{V,W}) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(V\oplus W)^G} X(V\oplus W)^G$$

is a weak equivalence.

Theorem 4.3 (Global model structure)**.** *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spaces. The fibrant objects in the global model structure are the static orthogonal spaces. The global model structure is proper, topological and compactly generated.*

We devote the rest of this section to the proof of Theorem 4.3. We start by establishing some closure properties of global equivalences. One of them will involve the notion of an 'h-cofibration' of orthogonal spaces; we will also need h-cofibrations in other contexts later, so we discuss them more generally now.

h-cofibrations. The *h-cofibrations* are the morphisms with the homotopy extension property. We will use this concept in various categories, for example in the category of G-spaces, orthogonal spaces, orthogonal spectra and orthogonal G-spectra. Many of the arguments involve elementary homotopy theoretic constructions and have nothing to do with the specific category at hand. So we recall some basic properties of h-cofibrations in the context of categories enriched over the category of compact Hausdorff spaces. Again, the arguments are all standard and well-known, and we include them for completeness and convenient reference.

For the discussion of h-cofibrations we work in a cocomplete category $\mathcal{C}$ that is tensored and cotensored over the category $\mathbf{U}$ of compactly generated weak Hausdorff spaces. We write '$\times$' for the pairing and X^K for the cotensor of an object X with a compact Hausdorff space K. A *homotopy* is then a morphism $H : [0,1] \times A \longrightarrow X$ defined on the pairing of the unit interval with a $\mathcal{C}$-object. For a homotopy and any $t \in [0,1]$ we denote by $H_t : A \longrightarrow X$ the composite morphism

$$A \ \cong \ \{t\} \times A \ \xrightarrow{\text{incl}\times A} \ [0,1]\times A \ \xrightarrow{H} \ X \ .$$

Definition 4.4. Let $\mathcal{C}$ be a category tensored over the category of spaces. A $\mathcal{C}$-morphism $f : A \longrightarrow B$ is an *h-cofibration* if it has the homotopy extension property, i.e., given a morphism $\varphi : B \longrightarrow X$ and a homotopy $H : [0,1] \times A \longrightarrow X$ such that $H_0 = \varphi f$, there is a homotopy $\bar{H} : [0,1] \times B \longrightarrow X$ such that $\bar{H} \circ ([0,1] \times f) = H$ and $\bar{H}_0 = \varphi$.

There is a universal test case for the homotopy extension problem, namely when X is the pushout:

$$\begin{tikzcd} A \arrow[r, "0\wedge -"] \arrow[d, "f"'] & {[0,1]\times A} \arrow[d, "H"] \\ B \arrow[r, "\varphi"'] & B \cup_f ([0,1]\times A) \end{tikzcd}$$

So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if the canonical morphism

$$B \cup_f ([0,1] \times A) \longrightarrow [0,1] \times B \tag{4.5}$$

has a retraction. Also the adjunction between $[0,1] \times -$ and $(-)^{[0,1]}$ lets us rewrite any homotopy extension data (φ, H) in adjoint form as a commutative square:

$$\begin{tikzcd} A \arrow[r, "\hat{H}"] \arrow[d, "f"'] & X^{[0,1]} \arrow[d, "\mathrm{ev}_0"] \\ B \arrow[r, "\hat{\varphi}"'] & X \end{tikzcd}$$

A solution to the homotopy extension problem is adjoint to a lifting, i.e., a morphism $\lambda : B \longrightarrow X^{[0,1]}$ such that $\lambda f = \hat{H}$ and $\mathrm{ev}_0 \lambda = \hat{\varphi}$. So a morphism $f : A \longrightarrow B$ is an h-cofibration if and only if it has the left lifting property with respect to the morphisms $\mathrm{ev}_0 : X^{[0,1]} \longrightarrow X$ for all objects in $\mathcal{C}$.

The three equivalent characterizations of h-cofibrations quickly imply various closure properties.

Corollary 4.6. *Let $\mathcal{C}$ be a cocomplete category tensored and cotensored over the category of spaces.*

(i) *The class of h-cofibrations in $\mathcal{C}$ is closed under retracts, cobase change, coproducts, sequential compositions and transfinite compositions.*
(ii) *Let $\mathcal{C}'$ be another topological model category and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ a continuous functor. Then F takes h-cofibrations in $\mathcal{C}$ to h-cofibrations in $\mathcal{C}'$.*
(iii) *If $\mathcal{C}$ is a topological model category in which every object in $\mathcal{C}$ is fibrant, then every cofibration is an h-cofibration.*

PROOF. (i) Every class of morphisms that can be characterized by the left lifting property with respect to some other class has the closure properties listed.

(ii) Let $f : A \longrightarrow B$ be a cofibration in $\mathcal{C}$ and $r : [0,1] \times B \longrightarrow B \cup_f ([0,1] \times A)$ a retraction to the canonical morphism. The following square commutes

$$\begin{tikzcd} FA \arrow[r, "0 \wedge -"] \arrow[d, "Ff"'] & {[0,1] \times FA} \arrow[d, "FH \circ \tau"] \\ FB \arrow[r, "F\varphi"'] & F(B \cup_f ([0,1] \times A)) \end{tikzcd}$$

where $\tau : [0,1] \times FA \longrightarrow F([0,1] \times A)$ is the structure map as a continuous functor. The composite

$$FB \cup_{Ff} ([0,1] \times FA) \longrightarrow F(B \cup_f ([0,1] \times A)) \xrightarrow{Fr} F([0,1] \times B) \xrightarrow{\tau^{-1}} [0,1] \times FB$$

is then a retraction to the canonical morphism for $Ff : FA \longrightarrow FB$. So Ff is an h-cofibration.

(iii) Since the model structure is topological, for every cofibration $f : A \longrightarrow B$ the canonical morphism (4.5) is an acyclic cofibration. Since every object is fibrant, this morphism has a retraction, and so f is an h-cofibration. □

Now we state and prove various useful properties of global equivalences.

Proposition 4.7. (i) *If f and g are composable morphisms of orthogonal spaces and two of the three morphisms f, g and gf are global equivalences, then so is the third.*
(ii) *Every retract of a global equivalence is a global equivalence.*
(iii) *A coproduct of global equivalences is a global equivalence.*

(iv) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is a global equivalence. If in addition f or g is an h-cofibration, then the morphism k is a global equivalence.

(v) *Let $f_n : A_n \longrightarrow A_{n+1}$ be a global equivalence and a closed embedding of orthogonal spaces, for $n \geq 0$. Then canonical morphism $f_\infty : A_0 \longrightarrow A_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is a global equivalence.*

(vi) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is a global equivalence. If in addition one of the morphisms f or h is a strong level fibration, then the morphism g is also a global equivalence.

(vii) *Every strong level equivalence is a global equivalence.*

(viii) *Every global equivalence that is also a global fibration is a strong level equivalence.*

(ix) *Every global equivalence between static orthogonal spaces is a strong level equivalence.*

Proof. Parts (i) and (ii) are straightforward. Part (iii) holds because the disc D^k is connected, so any initial lifting data for a coproduct of orthogonal spaces lands in one summand.

(iv)

(v) We let G be a compact Lie group, V a G-representation and $\alpha : S^{k-1} \longrightarrow A_0(V)^G$ and $\beta : D^k \longrightarrow A_\infty(V)^G$ continuous maps such that $f_\infty(V)^G \circ \alpha = \beta|_{S^{k-1}} : S^{k-1} \longrightarrow A_\infty(V)^G$. Since D^k is compact and $A_\infty(V)$ is a colimit of a sequence of closed embeddings, the map β factors through a map $\bar\beta : D^k \longrightarrow A_n(V)$ for some $n \geq 0$, see for example [**43**, Prop. 2.4.2]. Since the canonical map $A_n(V) \longrightarrow A_\infty(V)$ is injective, $\bar\beta$ lands in the G-fixed points and restricts to $((f_{n-1} \circ \cdots \circ f_0)(V))^G \circ \alpha$ on S^{k-1}.

The composite $f_{m-1} \circ \cdots \circ f_0 : A_0 \longrightarrow A_k$ is a global equivalence by part (i), so there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow A_0(V \oplus W)^G$ such that $\lambda|_{S^{k-1}} = A_0(i_{V,W}) \circ \alpha$ and $((f_{m-1} \circ \cdots \circ f_0)(V \oplus W))^G \circ \lambda$ is homotopic to $\bar\beta$ relative S^{k-1}. So the pair (W, λ) is also a solution for the original lifting problem, and hence $f_\infty : A_0 \longrightarrow A_\infty$ is a global equivalence.

(vi) We let G be a compact Lie group G, V a G-representation and $\alpha : S^{k-1} \longrightarrow P(V)^G$ and $\beta : D^k \longrightarrow Z(V)^G$ continuous maps such that $g(V)^G \circ \alpha = \beta|_{S^{k-1}} : S^{k-1} \longrightarrow Y(V)^G$. Since f is a global equivalence there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda|_{S^{k-1}} = X(i_{V,W})^G \circ k(V)^G \circ \alpha : S^{k-1} \longrightarrow X(V \oplus W)^G$ and such that $f(V \oplus W)^G \circ \lambda : D^k \longrightarrow Y(V \oplus W)^G$ is homotopic, relative to S^{k-1}, to $Y(i_{V,W}) \circ h(V)^G \circ \beta$. We let $H : D^k \times [0,1] \longrightarrow Y(V \oplus W)^G$ be a relative homotopy from $Y(i_{V,W}) \circ h(V)^G \circ \beta = h(V \oplus W)^G \circ Z(i_{V,W}) \circ \beta$ to $f(V \oplus W)^G \circ \lambda$. Now we distinguish two cases.

Case 1: The morphism h is a strong level fibration. Then the map $h(V\oplus W)^G : Z(V\oplus W)^G \longrightarrow Y(V\oplus W)^G$ is a Serre fibration. We can choose a lift $\bar{H}$ in the square

$$\begin{tikzcd} D^k\times 0\cup_{S^{k-1}\times 0} S^{k-1}\times[0,1] \arrow[r, "Z(i_{V,W})\circ\beta\cup K"] \arrow[d, "\sim"'] & Z(V\oplus W)^G \arrow[d, "h(V\oplus W)^G"] \\ D^k\times[0,1] \arrow[r, "H"'] \arrow[ur, dashed, "\bar{H}"] & Y(V\oplus W)^G \end{tikzcd}$$

where $K : S^{k-1}\times[0,1]\longrightarrow Z(V\oplus W)^G$ is the constant homotopy from $g(V\oplus W)^G\circ P(i_{V,W})^G\circ\alpha$ to itself. Since the square is a pullback and $h(V\oplus W)^G\circ\bar{H}(-,1) = H(-,1) = f(V\oplus W)^G\circ\lambda$, there is a unique continuous map $\bar{\lambda} : D^k\longrightarrow P(V\oplus W)^G$ that satisfies

$$g(V\oplus W)^G\circ\bar{\lambda} \ = \ \bar{H}(-,1) \qquad\text{and}\qquad k(V\oplus W)^G\circ\bar{\lambda} \ = \ \lambda\ .$$

The restriction of $\bar{\lambda}$ to S^{k-1} satisfies

$$\begin{aligned} g(V\oplus W)^G\circ\bar{\lambda}|_{S^{k-1}} \ &= \ \bar{H}(-,1)|_{S^{k-1}} \ = \ g(V\oplus W)^G\circ P(i_{V,W})\circ\alpha \qquad\text{and}\\ k(V\oplus W)^G\circ\bar{\lambda}|_{S^{k-1}} \ &= \ \lambda|_{S^{k-1}} \ = \ X(i_{V,W})^G\circ k(V)^G\circ\alpha \ = \ k(V\oplus W)^G\circ P(i_{V,W})^G\circ\alpha\ . \end{aligned}$$

The pullback property thus implies that $\bar{\lambda}|_{S^{k-1}} = P(i_{V,W})^G\circ\alpha$.

Finally, the composite $g(V\oplus W)^G\circ\bar{\lambda}$ is homotopic, relative S^{k-1} and via $\bar{H}$, to $\bar{H}(-,0) = Z(i_{V,W})^G\circ\beta$. This is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g.

Case 2: The morphism f is a strong level fibration. The argument is similar as in the first case. Now the map $f(V\oplus W)^G : X(V\oplus W)^G\longrightarrow f(V\oplus W)^G$ is a Serre fibration. We can choose a lift H' in the square

$$\begin{tikzcd} D^k\times 1\cup_{S^{k-1}\times 1} S^{k-1}\times[0,1] \arrow[r, "\lambda\cup K'"] \arrow[d, "\sim"'] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ D^k\times[0,1] \arrow[r, "H"'] \arrow[ur, dashed, "H'"] & Y(V\oplus W)^G \end{tikzcd}$$

where $K' : S^{k-1}\times[0,1]\longrightarrow X(V\oplus W)^G$ is the constant homotopy from $X(i_{V,W})^G\circ k(V)^G\circ\alpha$ to itself.

Since the square is a pullback and $f(V\oplus W)^G\circ H'(-,0) = H(-,0) = h(V\oplus W)^G\circ Z(i_{V,W})^G\circ\beta$, there is a unique continuous map $\bar{\lambda} : D^k\longrightarrow P(V\oplus W)^G$ that satisfies

$$g(V\oplus W)^G\circ\bar{\lambda} \ = \ Z(i_{V,W})^G\circ\beta \qquad\text{and}\qquad k(V\oplus W)^G\circ\bar{\lambda} \ = \ H'(-,0)\ .$$

The restriction of $\bar{\lambda}$ to S^{k-1} satisfies

$$\begin{aligned} g(V\oplus W)^G\circ\bar{\lambda}|_{S^{k-1}} \ &= \ Z(i_{V,W})^G\circ g(V)^G\circ\alpha \ = \ g(V\oplus W)^G\circ P(i_{V,W})^G\circ\alpha \qquad\text{and}\\ k(V\oplus W)^G\circ\bar{\lambda}|_{S^{k-1}} \ &= \ H'(-,0)|_{S^{k-1}} \ = \ X(i_{V,W})^G\circ k(V)^G\circ\alpha \ = \ k(V\oplus W)^G\circ P(i_{V,W})^G\circ\alpha\ . \end{aligned}$$

The pullback property thus implies that $\bar{\lambda}|_{S^{k-1}} = P(i_{V,W})^G\circ\alpha$. Since $g(V\oplus W)^G\circ\bar{\lambda} = Z(i_{V,W})\circ\beta$, this is the required lifting data, and we have verified the defining property of a global equivalence for the morphism g.

(vii) We let $f : X\longrightarrow Y$ be a strong level equivalence, G a compact Lie group, V a G-representation and $\alpha : S^{k-1}\longrightarrow X(V)^G$ and $\beta : D^k\longrightarrow Y(V)^G$ continuous maps such that $f(V)^G\circ\alpha = \beta|_{S^{k-1}} : S^{k-1}\longrightarrow Y(V)^G$. Since f is a strong level equivalence, the map $f(V)^G : X(V)^G\longrightarrow Y(V)^G$ is a weak equivalence, so there is a continuous map $\lambda : D^k\longrightarrow X(V)^G$ such that $\lambda|_{S^{k-1}} = \alpha$ and $f(V)^G\circ\lambda$ is homotopic to β relative S^{k-1}. So f is a global equivalence, where we can take $W = 0$.

(viii) We let $f : X \longrightarrow Y$ be morphism of orthogonal spaces that is both a global fibration and a global equivalence. We consider a compact Lie group G, a faithful G-representation V and a commutative square:

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "i_m"'] & X(V)^G \arrow[d, "f(V)^G"] \\ D^k \arrow[r, "\beta"'] & Y(V)^G \end{tikzcd}$$

We will exhibit a continuous map $\mu : D^k \longrightarrow X(V)^G$ such that $\mu|_{S^{k-1}} = \alpha$ and such that $f(V)^G \circ \mu$ is homotopic, relative S^{k-1}, to β. This shows that the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence, so f is a strong level equivalence.

Since f is a global equivalence there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda_{S^{k-1}} = X(i_{V,W})^G \circ \alpha : S^{k-1} \longrightarrow X(V \oplus W)^G$ and such that $f(V \oplus W)^G \circ \lambda : D^k \longrightarrow Y(V \oplus W)^G$ is homotopic, relative to S^{k-1}, to $Y(i_{V,W}) \circ \beta$. Since f is a strong level fibration, the map $f(V \oplus W)^G : X(V \oplus W)^G \longrightarrow Y(V \oplus W)^G$ is a Serre fibration, so we can improve λ into a continuous map $\lambda' : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda'_{S^{k-1}} = \lambda_{S^{k-1}} = X(i_{V,W})^G \circ \alpha$ and such that $f(V \oplus W)^G \circ \lambda'$ is equal to $Y(i_{V,W}) \circ \beta$.

Since f is a global fibration the morphism

$$(f(V)^G, X(i_{V,W})) \ : \ X(V)^G \ \longrightarrow \ Y(V)^G \times_{Y(V \oplus W)^G} X(V \oplus W)^G$$

is a weak equivalence. So we can find a continuous map $\mu : D^k \longrightarrow X(V)^G$ such that $\mu|_{S^{k-1}} = \alpha$ and $(f(V)^G, X(i_{V,W})) \circ \mu$ is homotopic, relative S^{k-1} to $(\beta, \lambda') : D^k \longrightarrow Y(V)^G \times_{Y(V \oplus W)^G} X(V \oplus W)^G$:

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "i_m"'] & X(V)^G \arrow[d, "{(f(V)^G, X(i_{V,W}))}"] \\ D^k \arrow[r, "{(\beta,\lambda')}"'] \arrow[ur, dashed, "\mu"] & Y(V)^G \times_{Y(V \oplus W)^G} X(V \oplus W)^G \end{tikzcd}$$

This is the desired map.

(ix) We let $f : X \longrightarrow Y$ be a global equivalence between static orthogonal spaces. We let G a compact Lie group, V a faithful G-representation and $\alpha : S^{k-1} \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ continuous maps such that $f(V)^G \circ \alpha = \beta|_{S^{k-1}} : S^{k-1} \longrightarrow Y(V)^G$. Since f is a global equivalence, there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda|_{S^{k-1}} = X(i_{V,W})^G \alpha$ and $f(V \oplus W)^G \circ \lambda$ is homotopic to $Y(i_{V,W})^G \circ \beta$ relative S^{k-1}. Since X is static, the map $X(i_{V,W})^G : X(V)^G \longrightarrow X(V \oplus W)^G$ is a weak equivalence, so there is a continuous map $\bar{\lambda} : D^k \longrightarrow X(V)^G$ such that $\bar{\lambda}|_{S^{k-1}} = \alpha$ and $X(i_{V,W})^G \circ \bar{\lambda}$ is homotopic to λ relative S^{k-1}. The two maps $f(V)^G \circ \bar{\lambda}$ and $\beta : D^k \longrightarrow Y(V)^G$ then agree on S^{k-1} and become homotopic, relative S^{k-1}, after composition with $Y(i_{V,W})^G : Y(V)^G \longrightarrow Y(V \oplus W)^G$. Since X is static, the map $Y(i_{V,W})^G$ is a weak equivalence, so $f(V)^G \circ \bar{\lambda}$ and $\beta : D^k \longrightarrow Y(V)^G$ are already homotopic relative S^{k-1}. This shows that the map $f(V)^G : X(V)^G \longrightarrow Y(V)^G$ is a weak equivalence, and hence f is a strong level equivalence. □

Construction 4.8. We let $j : A \longrightarrow B$ be a morphism in a topological model category. We factor j through the mapping cylinder as the composite

$$A \ \xrightarrow{c(j)} \ Z(j) = ([0,1] \times A) \cup_j B \ \xrightarrow{r(j)} \ B \ ,$$

where $c(j)$ is the 'front' mapping cylinder inclusion and $r(j)$ is the projection, which is a homotopy equivalence. In our applications we will assume that both A and B are cofibrant, and then the morphism $c(j)$ is a cofibration by the pushout produce property. We then define $\mathcal{Z}(j)$ as the set of all pushout product maps

$$i_m \Box c(j) \ : \ D^m \times A \cup_{S^{m-1}\times A} S^{m-1} \times Z(j) \ \longrightarrow \ D^m \times Z(j)$$

for $m \geq 0$, where $i_m : S^{m-1} \longrightarrow D^m$ is the inclusion.

Proposition 4.9. *Let $\mathcal{C}$ be a topological model category, $j : A \longrightarrow B$ a morphism between cofibrant objects and $f : X \longrightarrow Y$ a fibration. Then the following two conditions are equivalent:*

(i) *The square of spaces*

$$\begin{tikzcd} \mathrm{map}(B,X) \arrow[r, "\mathrm{map}(j,X)"] \arrow[d, "\mathrm{map}(B,f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(B,Y) \arrow[r, "\mathrm{map}(j,Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{4.10}$$

is homotopy cartesian.

(ii) *The morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$.*

Proof. The square (4.10) maps to the square

$$\begin{tikzcd} \mathrm{map}(Z(j),X) \arrow[r, "\mathrm{map}(c(j),X)"] \arrow[d, "\mathrm{map}(Z(j),f)"'] & \mathrm{map}(A,X) \arrow[d, "\mathrm{map}(A,f)"] \\ \mathrm{map}(Z(j),Y) \arrow[r, "\mathrm{map}(c(j),Y)"'] & \mathrm{map}(A,Y) \end{tikzcd} \tag{4.11}$$

via the map induced by $r(j) : Z(j) \longrightarrow B$ on the left part and the identity on the right part. Since $r(j)$ is a homotopy equivalence, the map of squares is a weak equivalence at all four corners. So the square (4.10) is homotopy cartesian if and only if the square (4.11) is homotopy cartesian.

Since A is cofibrant and f a fibration, $\mathrm{map}(A, f)$ is a Serre fibration. So the square (4.11) is homotopy cartesian if and only if the map

$$(\mathrm{map}(Z(j), f), \mathrm{map}(c(j), X)) \ : \ \mathrm{map}(Z(j), X) \ \longrightarrow \ \mathrm{map}(Z(j), Y) \times_{\mathrm{map}(A,Y)} \mathrm{map}(A, X) \tag{4.12}$$

is a weak equivalence. Since $c(j)$ is a cofibration and f is a fibration, the map (4.12) is always a Serre fibration. So (4.12) is a weak equivalence if and only if it is an acyclic fibration, which is equivalent to the right lifting property for the inclusions $i_m : S^{m-1} \longrightarrow D^m$ for all $m \geq 0$. By adjointness, the map (4.12) has the right lifting property with respect to the maps i_m if and only if the morphism f has the right lifting property with respect to the set $\mathcal{Z}(j)$. □

The set J^{str} was defined in the proof of Proposition 3.12 as the set morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. The set J^{str} detects the fibrations in the strong level model structure. We add another set of morphism K that detects when the squares (4.2) are homotopy cartesian. Given any compact Lie group G and G-representations V and W the restriction morphism

$$\rho_{G,V,W} \ : \ \mathbf{L}_{G,V\oplus W} \ \longrightarrow \ \mathbf{L}_{G,V}$$

was defined in (2.5). If the representation V is faithful, then this morphism is a global equivalence by Proposition 2.6 (v). We set

$$K \ = \ \bigcup_{G,V,W} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $S^{n-1} \longrightarrow D^n$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G, a faithful G-representation V and an arbitrary G-representation W. The morphism $\rho_{G,V,W}$ represents the map of G-fixed point spaces $X(i_{V,W})^G : X(V)^G \longrightarrow X(V \oplus W)^G$; by Proposition 4.9, the right lifting property with respect to the union $J^{\mathrm{str}} \cup K$ thus characterizes the global fibrations, i.e., we have shown:

Proposition 4.13. *A morphism of orthogonal spaces is a global fibration if and only if it has the right lifting property with respect to the set $J^{\mathrm{str}} \cup K$.*

Now we are ready to give the

Proof of Theorem 4.3. We refer the reader to [**30**, 3.3] for the numbering of the model category axioms. The category of orthogonal spaces is complete and cocomplete. Model category axioms MC2 (saturation) and MC3 (closure properties under retracts) are clear.

The strong level model structure shows that every morphism of orthogonal spaces can be factored as a flat cofibration followed by a strong level equivalence. Since strong level equivalences are in particular global equivalences, this provides one of the factorizations as required by MC5. For the other half of the factorization axiom MC5 we apply the small object argument to the set $J^{\mathrm{str}} \cup K$. All morphisms in J^{str} are flat cofibrations and strong level equivalences. Since $\mathbf{L}_{G,V\oplus W}$ and $\mathbf{L}_{G,V}$ are flat, the morphisms in K are also flat cofibrations, and they are global equivalences because the morphisms $\rho_{G,V,W}$ are (Proposition 2.6 (v)). The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal spaces as a composite

$$X \xrightarrow{i} W \xrightarrow{q} Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in K and q has the right lifting property with respect to $J^{\mathrm{str}} \cup K$. Since all morphisms in K are flat cofibrations and global equivalences, the morphism i is a flat cofibration and a global equivalence by the closure properties of Proposition 4.7. Moreover, q is a global fibration by Proposition 4.13.

Now we show the lifting properties MC4. By Proposition 4.7 (viii) a morphism that is both a global fibration and a global equivalence is a strong level equivalence, and hence an acyclic fibration in the strong level model structure. So every morphism that is simultaneously global fibration and a global equivalence has the right lifting property with respect to flat cofibrations. Now we let $j : A \longrightarrow B$ be a flat cofibration that is also a global equivalence and we show that it has the left lifting property with respect to all global fibrations. We factor $j = q \circ i$, via the small object argument for $J^{\mathrm{str}} \cup K$, where $i : A \longrightarrow W$ is an $(J^{\mathrm{str}} \cup K)$-cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, and hence an acyclic fibration in the strong level model structure, again by Proposition 4.7 (viii). Since j is a flat cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting with respect to global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified set of generating flat cofibrations I^{str} and generating acyclic cofibrations $J^{\mathrm{str}} \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations. So the global model structure is compactly generated.

Left properness of the global model structure follows from Proposition 4.7 (iv) and the fact that flat cofibrations are h-cofibrations (Corollary 4.6 (iii)). Right properness follows from Proposition 4.7 (vi) because global fibrations are in particular strong level fibrations.

It remains to show that the global model structure is topological. The cofibrations in the global model structure coincide with the cofibrations in the strong level model structure, so the pushout product of a cofibration of spaces with a flat cofibration is a flat cofibration by Proposition 3.12. Similarly, the pushout product of an acyclic cofibration of spaces with a flat cofibration is a strong level equivalence by Proposition 3.12, hence a global equivalence. Finally, we have to show that pushout products of cofibrations of spaces with a flat cofibration that are also global equivalences are again global equivalences. It suffices to consider a generating cofibration $i_k : S^{k-1} \longrightarrow D^k$ of spaces and a generating acyclic cofibration in the set $J^{\mathrm{str}} \cup K$. The morphisms in J^{str} are strict level equivalences, hence taken care of by Proposition 3.12 again. The pushout product of i_k and a morphism $i_m \Box c(j)$ in $\mathcal{Z}(\rho_{G,V,W})$ is isomorphic to $i_{k+m} \Box c(j)$, hence again a flat cofibration and global equivalence. □

Corollary 4.14. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spaces. Then the following conditions are equivalent.*

(i) *The morphism f is a global equivalence.*
(ii) *For some (hence any) flat approximation $f^\flat : A^\flat \longrightarrow B^\flat$ in the strong level model structure and every static orthogonal space X the induced map*

$$[f^\flat, X] \ : \ [B^\flat, X] \ \longrightarrow \ [A^\flat, X]$$

on homotopy classes of morphisms is a bijection.

Proof. Since strong level equivalences are global equivalences, the morphism f is a global equivalence if and only if the flat approximation $f^\flat : A^\flat \longrightarrow B^\flat$ is a global equivalence. Since $A^\flat$ and $B^\flat$ are flat, they are cofibrant in the global model structure. So by general model category theory, $f^\flat$ is a global equivalence if and only if the induced map $[f^\flat, X]$ is bijective for every fibrant object in the global model structure. By Theorem 4.3 these fibrant objects are precisely the static orthogonal spaces. □

Remark 4.15. We can relate the unstable global homotopy category of orthogonal space to the homotopy theory of G-spaces for a fixed compact Lie group G. We fix a faithful G-representation V. Then evaluation at V and the free functor at (G, V) are a pair of adjoint functors

$$G\mathbf{U} \ \underset{\mathrm{ev}_V}{\overset{\mathbf{L}_{G,V}}{\rightleftarrows}} \ spc$$

between the categories of G-spaces and orthogonal spaces. This adjoint pair is a Quillen pair with respect to the global model structure of orthogonal spaces and the genuine model structure of G-spaces. The adjoint total derived functors

$$\mathrm{Ho}(G\mathbf{U}) \ \underset{R(\mathrm{ev}_V)}{\overset{L(\mathbf{L}_{G,V})}{\rightleftarrows}} \ \mathrm{Ho}(spc)$$

are independent of the faithful representation V up to preferred natural isomorphism.

Every G-space is G-weakly equivalent to a G-CW-complex, and these are built from the orbits G/H. So the derived left adjoint $L(\mathbf{L}_{G,V}) : G\mathbf{U} \longrightarrow spc$ is essentially determined by its values on the coset spaces G/H. Since $\mathbf{L}_{G,V}(G/H)$ is isomorphic to $\mathbf{L}_{H,V}$, derived left adjoint takes the orbit space G/H to global classifying space of H.

5. Monoidal properties

In this section we recall the box product of orthogonal spaces and the compatibility with the model structures of the previous sections. We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spaces (X, Y) to an orthogonal space Z as a collection of based $O(V) \times O(W)$-equivariant maps

$$b_{V,W} \ : \ X(V) \times Y(W) \ \longrightarrow \ Z(V \oplus W)$$

for all inner product spaces V and W, such that the *bilinearity diagram*

$$\begin{tikzcd} & X(V)\times Y(W) \arrow[dl, "X(V)\times Y(i_{W,U})"'] \arrow[d, "b_{V,W}"] \arrow[dr, "X(i_{V,U})\times Y(W)"] & \\ X(V)\times Y(W\oplus U) \arrow[dr, "b_{V,W\oplus U}"'] & Z(V\oplus W) \arrow[d, "Z(i_{V\oplus W,U})"] & X(V\oplus U)\times Y(W) \arrow[d, "b_{V\oplus U,W}"] \\ & Z(V\oplus W\oplus U) & Z(V\oplus U\oplus W) \arrow[l, "Z(V\oplus\tau_{U,W})"] \end{tikzcd}$$

commutes for all inner product spaces U, V and W.

We can then define a box product of X and Y as a universal example of an orthogonal space with a bimorphism from X and Y. More precisely, a box product for X and Y is a pair $(X \boxtimes Y, i)$ consisting of an orthogonal space $X \boxtimes Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \boxtimes Y$, i.e., a bimorphism such that for every orthogonal space Z the map

$$spc(X\boxtimes Y, Z) \;\longrightarrow\; \mathrm{Bimor}((X,Y),Z)\ , \quad f \longmapsto fi = \{f_{p+q}\circ i_{p,q}\}_{p,q} \tag{5.1}$$

is bijective.

We have to show that a universal bimorphism out of any pair of orthogonal spaces exists; in other words: we have to construct a box product $X \boxtimes Y$ from two given orthogonal spaces X and Y. We want $X \boxtimes Y$ to be the universal recipient of a bimorphism from (X, Y), and this pretty much tells us what we have to do. For $n \geq 0$ we define the n-th level $(X \boxtimes Y)_n$ as the coequalizer, in the category of pointed $O(n)$-spaces, of two maps

$$\alpha_X,\,\alpha_Y \;:\; \coprod_{p+1+q=n} O(n)\times_{O(p)\times 1\times O(q)} X_p\times Y_q \;\longrightarrow\; \coprod_{p+q=n} O(n)\times_{O(p)\times O(q)} X_p\times Y_q\ .$$

The coproducts run over all non-negative values of p and q which satisfy the indicated relations. The map α_X takes the summand indexed by $(p, 1, q)$ to the summand indexed by $(p + 1, q)$ using the map

$$X(i_{\mathbb{R}^p,\mathbb{R}})\times Y_q \;:\; X_p\times Y_q \;\longrightarrow\; X_{p+1}\times Y_q$$

and inducing up. The other map α_Y takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p, 1 + q)$ using the map

$$X_p\times Y(j_{\mathbb{R}^q,\mathbb{R}}) \;:\; X_p\times Y_q \;\longrightarrow\; X_p\times Y_{1+q}$$

and inducing up.

The structure map $(X \times Y)_n \longrightarrow (X \times Y)_{n+1}$ is induced on coequalizers by the coproduct of the maps

$$O(n)\times_{O(p)\times O(q)} X_p\times Y_q \;\longrightarrow\; O(n+1)\times_{O(p)\times O(q+1)} X_p\times Y_{q+1}$$

induced from $X_p \times Y(i_{q,1}) : X_p \times Y_q \longrightarrow X_p \times Y_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by using the structure map of X (instead of that of Y) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \times Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p\times Y_q \xrightarrow{x\times y\mapsto 1\times x\times y} \coprod_{p+q=n} O(n)\times_{O(p)\times O(q)} X_p\times Y_q \xrightarrow{\text{projection}} (X\times Y)_{p+q}$$

forms a bimorphism – and in fact a universal one.

Very often only the object $X \boxtimes Y$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (5.1) as the *universal property* of the box product of orthogonal spaces.

The box product $X \boxtimes Y$ is a functor in both variables. It is also symmetric monoidal, i.e., there are natural associativity respectively symmetry isomorphisms

$$(X \boxtimes Y) \boxtimes Z \longrightarrow X \boxtimes (Y \boxtimes Z) \qquad \text{respectively} \qquad X \boxtimes Y \longrightarrow Y \boxtimes X$$

and unit isomorphisms $\mathbf{1} \boxtimes X \cong X \cong X \boxtimes \mathbf{1}$. The upshot is that the associativity and symmetry isomorphisms make the box product of orthogonal spaces into a symmetric monoidal product with the sphere space $\mathbf{1}$ as unit object.

The box product of orthogonal spaces is *closed* symmetric monoidal in the sense that the box product is adjoint to an internal Hom space. We recall the construction i.e., there is an adjunction isomorphism

$$\mathrm{Hom}(X \boxtimes Y, Z) \cong \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) .$$

For an orthogonal space X and an inner product space W we define the *W-th shift* $\mathrm{sh}^W X$ of X by

$$(\mathrm{sh}^W X)(V) = X(W \oplus V) .$$

The structure map $(\mathrm{sh}^W X)(\varphi)$ associated to a linear isometry $\varphi : V \longrightarrow V'$ is the structure map $X(W \oplus \varphi)$ of X. The orthogonal space $\mathrm{sh}^W X$ comes with a natural left $O(W)$-action by restriction along the monomorphism $- \oplus \mathrm{Id} : O(W) \longrightarrow O(W \oplus V)$.

For orthogonal spaces X and Y we define an orthogonal space $\mathrm{Hom}(X, Y)$ by

$$\mathrm{Hom}(X, Y)(W) = \mathrm{map}(X, \mathrm{sh}^W Y) .$$

The structure map associated to a linear isometric embedding $\psi : W \longrightarrow U$ is the map

$$\mathrm{Hom}(X, Y)(\psi) = \mathrm{map}(X, \mathrm{sh}^\psi Y) : \mathrm{map}(X, \mathrm{sh}^W Y) \longrightarrow \mathrm{map}(X, \mathrm{sh}^U Y) ,$$

where $\mathrm{sh}^\psi Y : \mathrm{sh}^W Y \longrightarrow \mathrm{sh}^U Y$ is the morphism whose value at an inner product space V is

$$Y(\psi \oplus V) : (\mathrm{sh}^W Y)(V) = Y(W \oplus V) \longrightarrow Y(U \oplus V) = (\mathrm{sh}^U Y)(V) .$$

We omit the verification that $\mathrm{Hom}(X, Y)$ is indeed an orthogonal space.

Taking function space commutes with shifting in the second variable, i.e.,

$$\mathrm{Hom}(X, \mathrm{sh}^W Y) \cong \mathrm{sh}^W (\mathrm{Hom}(X, Y))$$

by the natural isomorphism whose value at an inner product space V is

$$\begin{aligned} \mathrm{Hom}(X, \mathrm{sh}^W Y)(V) &= \mathrm{map}(X, \mathrm{sh}^V(\mathrm{sh}^W Y)) \cong \mathrm{map}(X, \mathrm{sh}^{W \oplus V} Y) \\ &= \mathrm{Hom}(X, Y)(W \oplus V) = \big(\mathrm{sh}^W \mathrm{Hom}(X, Y)\big)(V) . \end{aligned}$$

The internal function space functor $\mathrm{Hom}(X, -)$ is right adjoint to the box product $- \boxtimes X$ of orthogonal spaces. A natural isomorphism of orthogonal spaces $\mathrm{Hom}(\mathbf{L}_{e,W}, Y) \cong \mathrm{sh}^W Y$ is given at V by

$$\begin{aligned} \mathrm{Hom}(\mathbf{L}_{e,W}, Y)(V) = \mathrm{map}(\mathbf{L}_{e,W}, \mathrm{sh}^V Y) &\cong (\mathrm{sh}^V Y)(W) \\ &= Y(V \oplus W) \xrightarrow{Y(\tau_{V,W})} Y(W \oplus V) = (\mathrm{sh}^W Y)(V) \end{aligned}$$

where the second map is the adjunction bijection. This isomorphism is equivariant for the left actions of $O(W)$ induced on the source from the right $O(W)$-action on the free space $\mathbf{L}_{e,W}$. In the special case $W = 0$ we have $\mathbf{L}_{e,0} = \mathbf{1}$, which gives a natural isomorphism of orthogonal spaces $\mathrm{Hom}(\mathbf{1}, Y) \cong Y$.

Construction 5.2. Given two orthogonal spaces X and Y, we endow the equivariant homotopy set with an external pairing

$$\times : \pi_0^G(X) \times \pi_0^K(Y) \longrightarrow \pi_0^{G \times K}(X \boxtimes Y) , \tag{5.3}$$

where G and K are compact Lie groups. Suppose that V and W are representations of G respectively K and $x \in X(V)^G$ and $y \in Y(W)^G$ are fixed points that represent classes in $\pi_0^G(X)$ respectively $\pi_0^K(Y)$. We view $V \oplus W$ as a representation of the product group $G \times K$ via $(g,k)\cdot(v,w) = (gv, kw)$. We denote by $x \times y$ the image of the $(G\times K)$-fixed point (x,y) under the $(G\times K)$-map

$$i_{V,W} \ : \ X(V) \times Y(W) \ \longrightarrow \ (X \boxtimes Y)(V \oplus W)$$

that is part of the universal bimorphism. When we stabilize the representing maps by a G-representation V' respectively a K-representation W', then the relation

$$X(i_{V,V'})(x) \times Y(i_{W,W'})(y) \ = \ \alpha_*\left((X\Box Y)(i_{V\oplus W, V'\oplus W'})(x \times y)\right)$$

holds, where $\alpha : V \oplus W \oplus V' \oplus W' \longrightarrow V \oplus V' \oplus W \oplus W'$ is the equivariant isometry that interchanges W and V'; by Proposition 1.13 the classes $\langle x \times y\rangle$ and $\langle X(i_{V,V'})(x) \times Y(i_{W,W'})(y)\rangle$ coincide in $\pi_0^{G\times K}(X \boxtimes Y)$. The upshot is that the definition

$$[x] \times [y] \ = \ \langle x \times y \rangle \ \in \ \pi_0^{G\times K}(X \boxtimes Y)$$

is well-defined.

The pairing of equivariant homotopy groups have several expected properties that we summarize in the next proposition.

Proposition 5.4. *Let G, K and L be compact Lie groups and X, Y and Z orthogonal spaces.*

(i) (Unitality) *Let $1 \in \pi_0^e(\mathbf{1})$ be the unique element. The product is unital in the sense that $1\times x = x = x\times 1$ under the identifications $\mathbf{1} \boxtimes X = X = X \boxtimes \mathbf{1}$ and $e \times G \cong G \cong G \times e$.*

(ii) (Associativity) *For all classes $x \in \pi_0^G(X)$, $y \in \pi_0^K(Y)$ and $z \in \pi_0^L(Z)$ the relation*

$$x \times (y \times z) \ = \ (x \times y) \times z$$

hold in $\pi_0^{G\times K\times L}(X \boxtimes Y \boxtimes Z)$.

(iii) (Commutativity) *For all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^K(Y)$ the relation*

$$y \times x \ = \ \tau^*_{G,K}(\tau^{X,Y}_*(x \times y))$$

holds in $\pi_0^{K\times G}(Y \boxtimes X)$, where $\tau^{X,Y} : X \boxtimes Y \longrightarrow Y \boxtimes X$ is the symmetry isomorphism of the box product and $\tau_{G,K} : K \times G \longrightarrow G \times K$ interchanges the factors.

(iv) (Restriction) *For all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^K(Y)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$\alpha^*(x) \times \beta^*(y) \ = \ (\alpha \times \beta)^*(x \times y)$$

holds in $\pi_0^{\bar{G}\times\bar{K}}(X \boxtimes Y)$.

Proof. The unitality property (i), the associativity property (ii) and compatibility with restriction (iv) are straightforward from the definitions. Part (iii) exploits that the square

$$\begin{tikzcd}
X(V) \times Y(W) \arrow[r, "i_{V,W}"] \arrow[d, "\text{twist}"'] & (X \boxtimes Y)(V + W) \arrow[d, "\tau_{X,Y}\circ(\chi_{V,W})_*"] \\
Y(W) \times X(V) \arrow[r, "i_{W,V}"'] & (Y \boxtimes X)(W + V)
\end{tikzcd}$$

commutes. The image of (x,y) under the upper right composite represents $x \times y$, whereas the image under the lower left composite represents $y \times x$, so $x \times y = y \times x$. $\square$

Remark 5.5. The product of compact Lie groups and homomorphisms descends to a symmetric monoidal structure on the representation category Rep. The category of contravariant functors from Rep to sets then inherits a Day-type convolution product (with respect to cartesian product of sets). Proposition 5.4 effectively says that the maps (5.3) define a lax symmetric monoidal structure on the functor $\underline{\pi}_0$ from orthogonal spaces (under $\boxtimes$) to contravariant Rep-functors (under the convolution product).

Remark 5.6. By taking $G = K$ and restricting along the diagonal embedding $\Delta : G \longrightarrow G \times G$ we obtain an internal product as the composite

$$\pi_0^G(X) \times \pi_0^G(Y) \xrightarrow{\ \times\ } \pi_0^{G\times G}(X \boxtimes Y) \xrightarrow{\ \Delta^*\ } \pi_0^G(X \boxtimes Y) \ .$$

The external products can in fact be recovered from the internal product, as we now explain. More generally, we argue that bimorphisms of Rep-functors can be identified with another kind of structure that we call 'diagonal products'.

We let X, Y and Z be $\mathrm{Rep}^{\mathrm{op}}$-functors. A *diagonal product* is a natural transformation $X \times Y \longrightarrow Z$ of contravariant Rep-functors, where $X \times Y$ is the objectwise cartesian product. So a diagonal product consists of maps

$$\nu_G \ : \ X(G) \times Y(G) \longrightarrow Z(G)$$

for every compact Lie group G that are natural for restriction along homomorphisms $\alpha : K \longrightarrow G$.

Any bimorphism $\mu : (X, Y) \longrightarrow Z$ gives rise to a diagonal product as follows. For a group G we define ν_G as the composite

$$X(G) \times Y(G) \xrightarrow{\mu_{G,G}} Z(G \times G) \xrightarrow{\Delta_G^*} Z(G)$$

where $\Delta : G \longrightarrow G \times G$ is the diagonal. For a group homomorphism $\alpha : K \longrightarrow G$ we have $\Delta_G \circ \alpha = (\alpha \times \alpha) \circ \Delta_K$, so the diagram

$$\begin{tikzcd}
X(G) \times Y(G) \arrow[r, "\mu_{G,G}"] \arrow[d, "\alpha^* \otimes \alpha^*"'] & Z(G \times G) \arrow[r, "\Delta_G^*"] \arrow[d, "(\alpha\times\alpha)^*"] & Z(G) \arrow[d, "\alpha^*"] \\
X(K) \times Y(K) \arrow[r, "\mu_{K,K}"'] & Z(K \times K) \arrow[r, "\Delta_K^*"'] & Z(K)
\end{tikzcd}$$

commutes.

Conversely, given a diagonal product ν, we define a bimorphism as follows. For compact Lie groups G and K we define the component $\mu_{G,K}$ as the composite

$$X(G) \times Y(K) \xrightarrow{p_G^* \times p_K^*} X(G \times K) \times Y(G \times K) \xrightarrow{\nu_{G\times K}} Z(G \times K) \ ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. Given homomorphisms $\alpha : G \longrightarrow G'$ and $\beta : K \longrightarrow K'$, we have $p_{G'}(\alpha \times \beta) = \alpha p_G$ and $p_{K'}(\alpha \times \beta) = \beta p_K$, so the left part of the diagram

$$\begin{tikzcd}
X(G) \times Y(K) \arrow[r, "p_G^* \times p_K^*"] \arrow[d, "\alpha^* \otimes \beta^*"'] & X(G \times K) \times Y(G \times K) \arrow[r, "\nu_{G\times K}"] \arrow[d, "(\alpha\times\beta)^* \times (\alpha\times\beta)^*"] & Z(G \times K) \arrow[d, "(\alpha\times\beta)^*"] \\
X(G') \times Y(K') \arrow[r, "p_G^* \times p_K^*"'] & X(G' \times K') \times Y(G' \times K') \arrow[r, "\nu_{G'\times K'}"'] & Z(G' \times K')
\end{tikzcd}$$

commutes. The right part commutes by naturality of the diagonal product ν.

The last topic in this section is the compatibility of the level and global model structures with the box product of orthogonal spaces. Given two morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ of orthogonal spaces we denote by $f \Box g$ the *pushout product* morphism defined as

$$f \Box g = (f \boxtimes Y) \cup (A \boxtimes g) \ : \ A \boxtimes Y \cup_{A \boxtimes X} B \boxtimes X \longrightarrow B \boxtimes Y \ .$$

We recall that a model structure on a symmetric monoidal category satisfies the *pushout product property* if the following two conditions hold:

- for every pair of cofibrations $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ the pushout product morphism $f\Box g$ is also a cofibration;
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.

Proposition 5.7. (i) *The pushout product of two flat cofibrations is a flat cofibration.*
(ii) *The pushout product of a flat cofibration that is also a strong level equivalence with a flat cofibration is a strong level equivalence.*
(iii) *The pushout product of a flat cofibration with a flat cofibration that is also a global equivalence is again a global equivalence.*
(iv) *The strong level model structure and the global model structure of orthogonal spaces satisfy the pushout product property with respect to the box product.*
(v) *For every flat orthogonal space A the functor* $\mathrm{Hom}(A, -)$ *preserves static orthogonal spaces.*

PROOF. (i) It suffices to show the claim for a set of generating cofibrations. The flat cofibrations are generated by the morphisms

$$\mathbf{L}_{G,V} S^{n-1} \ \longrightarrow\ \mathbf{L}_{G,V} D^n$$

for G a compact Lie group, V a G-representation and $n \geq 0$. The pushout product of two such generators is isomorphic to the map

$$\mathbf{L}_{G\times K,V\oplus W}(S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1}) \ \longrightarrow\ \mathbf{L}_{G\times K,V\oplus W}(D^n \times D^m) \ ;$$

here $G \times K$ acts on $V \oplus W$ by $(g, k)\cdot(v, w) = (gv, kw)$. Since the inclusion of $S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1}$ into $D^n \times D^m$ is a cofibration of spaces, this pushout product morphism is another flat cofibration.

(ii) Again it suffices to show that the pushout product of a generating acyclic cofibration for the strong level model structure specified in (7.22). These generating acyclic cofibrations are inclusions of strong deformation retractions, so the pushout product of such a map with a generating flat cofibration is a flat cofibration (by part (i)) and a homotopy equivalence (compare part (ii)).

(iii) It suffices to show that, the pushout product of any generating cofibration with a generating acyclic cofibration is an acyclic cofibration (i.e., flat cofibration and global equivalence). The flat cofibration part is taken care of by part (i). So it suffices to show the global equivalence part.

The generating acyclic cofibrations for the global model structure come in two flavors:

(a) The generating acyclic cofibrations for the strong level model structure. The pushout product of such a morphism with any flat cofibration is a strong level equivalence by part (ii).

(b) The pushout product morphisms $c_{K,U,W}\Box i_m$, where $c_{K,U,W}$ is the mapping cylinder inclusion of the global equivalence

$$\rho_{K,U,W} \ :\ \mathbf{L}_{K,U\oplus W} \ \longrightarrow\ \mathbf{L}_{K,U} \ ,$$

for K any compact Lie group, U a faithful K-representation and W and K-representation, and $i_m : S^{m-1} \longrightarrow D^m$ is the inclusion.

The pushout product of a generating flat cofibration

$$\mathbf{L}_{G,V} S^{n-1} \ \longrightarrow\ \mathbf{L}_{G,V} D^n$$

(for G any compact Lie group and V a faithful G-representation) with $c_{K,U,W}\Box i_m^+$ is isomorphic to the pushout product of the morphism $\mathbf{L}_{G,V} \boxtimes c_{K,U,W}$ with the cofibration

$$S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1} \ \longrightarrow\ D^n \times D^m \ .$$

Since the global model structure is topological, it suffices to show that the morphism $\mathbf{L}_{G,V} \boxtimes c_{K,U,W}$ is a global equivalence. The target of $c_{K,U,W}$ maps by a homotopy equivalence to $\mathbf{L}_{K,U}$, it suffices to show that the morphism

$$\mathbf{L}_{G,V} \boxtimes \rho_{K,U,W} \ :\ \mathbf{L}_{G,V} \boxtimes \mathbf{L}_{K,U\oplus W} \ \longrightarrow\ \mathbf{L}_{G,V} \boxtimes \mathbf{L}_{K,U}$$

is a global equivalence. This morphism, in turn, is isomorphic to

$$\rho_{G\times K,V\oplus W,U} \ : \ \mathbf{L}_{G\times K,V\oplus U,W} \ \longrightarrow \ \mathbf{L}_{G\times K,V\oplus U} \ ,$$

where G acts trivially on W. This morphism is a global equivalence by Proposition 2.6 (v) because $G\times K$ acts faithfully on $V\oplus U$.

Part (iv) is the combination of (i), (ii) and (iii).

(v) Since A is flat the functor $A\boxtimes -$ is a left Quillen endofunctor for the global model structure. So the right adjoint $\mathrm{Hom}(A,-)$ is a right Quillen endofunctor for the same model structure. In particular, $\mathrm{Hom}(A,-)$ preserves fibrant objects; by Theorem 4.3, these fibrant objects are precisely the static orthogonal spaces. □

The unit object for the box product is the constant one-point orthogonal space **1**, which is flat. So with respect to the box product, the strong level model structure and the global model structure are symmetric monoidal model categories in the sense of [**43**, Def. 4.2.6]. A corollary is that the unstable global homotopy category, i.e., the localization of the category of orthogonal spaces at the class of global equivalences, inherits a closed symmetric monoidal structure, compare [**43**, Thm. 4.3.3], the 'derived box product'.

Remark 5.8. The box product of an h-cofibration with any orthogonal space is an h-cofibration. Indeed, if $f : A \longrightarrow B$ is an h-cofibration we can choose a retraction to the canonical morphism

$$B\cup_A ([0,1]\times A) \ \longrightarrow \ [0,1]\times B \ .$$

Since boxing with Y preserves colimits and products with spaces, any retraction can be boxed with Y and the provides a retraction to

$$([0,1]\times A\boxtimes Y)\cup_{A\boxtimes Y}(B\boxtimes Y) \ \longrightarrow \ [0,1]\times B\boxtimes Y \ ;$$

this shows that $f\boxtimes Y$ is an h-cofibration.

A key property of flat orthogonal spaces is established in the next theorem; in fact, this theorem is a justification for our use of the adjective 'flat'. Now we explain that the box product of orthogonal spaces is essentially a 'freed up' version of the categorical product. The constant orthogonal space with value a one-point space is both terminal object and a unit object for the symmetric monoidal product $\boxtimes$. Given two orthogonal spaces X and Y, the unique morphism $Y\longrightarrow *$ thus induces a morphism $\rho_1 : X\boxtimes Y\longrightarrow X\boxtimes *\cong X$; together with the analogous morphism $\rho_2 : X\boxtimes Y\longrightarrow Y$ the universal property of the product of orthogonal spaces (which is formed objectwise) provides a canonical morphism

$$\rho_{X,Y} = (\rho_1,\rho_2) \ : \ X\boxtimes Y \ \longrightarrow \ X\times Y$$

of orthogonal spaces. We will now show that the morphism $\rho_{X,Y}$ is a global equivalence whenever at least one factor is flat. In non-equivariant, non-global contexts, similar results have be proved by Blumberg, Cohen and Schlichtkrull for $\mathcal{L}(1)$-spaces and $*$-modules in [**10**, Prop. 4.23] and by Sagave and Schlichtkrull for $\mathcal{I}$-spaces [**67**, Prop. 2].

Theorem 5.9. *Let X and Y be orthogonal spaces at least one of which is flat.*

(i) *The morphism $\rho_{X,Y} : X\boxtimes Y\longrightarrow X\times Y$ is a global equivalence.*

(ii) *For every compact Lie group G, the map*

$$(\pi_0^G(\rho_1),\pi_0^G(\rho_2)) \ : \ \pi_0^G(X\boxtimes Y) \ \longrightarrow \ \pi_0^G(X)\times\pi_0^G(Y)$$

is bijective.

Proof. (i) Step 1: We let G and K be compact Lie groups, V a G-representation, W a K-representation, A a cofibrant G-space and B a cofibrant K-space. Then $(\mathbf{L}_{G,V}A)\boxtimes(\mathbf{L}_{K,W}B)$ is isomorphic to $\mathbf{L}_{G\times K,V\oplus W}(A\times B)$ and the morphism $\rho_{\mathbf{L}_{G,V}A,\mathbf{L}_{K,W}B}$ is a global equivalence by Proposition 2.11.

Step 2: We let (G, V, A) be as in Step 1 and let Y be any flat orthogonal space. We show that the morphism $\rho_{\mathbf{L}_{G,V}A,Y}$ is a global equivalence. To simplify notation we abbreviate $X = \mathbf{L}_{G,V}A$ and argue first, by induction on m, that $\rho_{X,\mathrm{sk}^m Y}$ is a global equivalence, where $\mathrm{sk}^m Y$ is the m-skeleton in the sense of Construction 3.1. The induction starts with $m = -1$, where there is nothing to show because $\mathrm{sk}^{-1}Y$, and hence $X \boxtimes (\mathrm{sk}^{-1}Y)$ and $X \times (\mathrm{sk}^{-1}Y)$ are empty.

For $m \geq 0$ the inclusion $\mathrm{sk}^{m-1}Y \longrightarrow \mathrm{sk}^m Y$ is a flat cofibration, hence an h-cofibration, hence so are the induced morphisms $X \boxtimes (\mathrm{sk}^{m-1}Y) \longrightarrow X \boxtimes (\mathrm{sk}^m Y)$ and $X \times (\mathrm{sk}^{m-1}Y) \longrightarrow X \times (\mathrm{sk}^m Y)$. We consider the commutative diagram

$$\begin{tikzcd}
X \boxtimes (\mathrm{sk}^{m-1} Y) \arrow[d, "\rho_{X,\mathrm{sk}^{m-1} Y}"'] & X \boxtimes (G_m(L_m Y)) \arrow[l] \arrow[r, "X \boxtimes (G_m \nu_m)"] \arrow[d, "\rho_{X,G_m(L_m Y)}"'] & X \boxtimes (G_m Y_m) \arrow[d, "\rho_{X,G_m Y_m}"] \\
X \times (\mathrm{sk}^{m-1} Y) & X \times (G_m(L_m Y)) \arrow[l] \arrow[r, "X \times (G_m \nu_m)"'] & X \times (G_m Y_m)
\end{tikzcd}$$

Since $X \boxtimes -$ and $X \times -$ preserve colimits, $X \boxtimes (\mathrm{sk}^m Y)$ is a pushout of the upper row (by (3.3)), and $X \times (\mathrm{sk}^m)$ is a pushout of the lower row. Since Y is flat, the map $\nu_m : L_m Y \longrightarrow Y_m$ is a cofibration of $O(m)$-space, hence an h-cofibration (Corollary 4.6 (iii)), so $G_m \nu_m$ is an h-cofibration of orthogonal spaces. The morphisms $X \boxtimes (G_m \nu_m)$ and $X \times (G_m \nu_m)$ are then also h-cofibrations of orthogonal spaces (Remark 5.8). The morphism $\rho_{X,\mathrm{sk}^{m-1} Y}$ is a global equivalence by induction, and the two other vertical morphism by Step 1 (since $G_m Y_m = \mathbf{L}_{O(m),\mathbb{R}^m} Y_m$ and $L_m Y$ and Y_m are both cofibrant $O(m)$-spaces). So the gluing lemma for global equivalence [ref] shows that the morphism $\rho_{X,\mathrm{sk}^m Y} : X \boxtimes (\mathrm{sk}^m Y) \longrightarrow X \times (\mathrm{sk}^m Y)$ is a global equivalence. This finishes the induction step. The passage to the colimits follows by stability of global equivalences under sequential colimits along h-cofibrations (Proposition 4.7 (v)).

Step 3: We let (G, V, A) be as before, choose a strong level equivalence $\varphi : Y^\flat \longrightarrow Y$ with a flat source and consider the commutative square:

$$\begin{tikzcd}
(\mathbf{L}_{G,V}A) \boxtimes Y^\flat \arrow[r, "\rho_{\mathbf{L}_{G,V}A,Y^\flat}"] \arrow[d, "(\mathbf{L}_{G,V}A)\boxtimes\varphi"'] & (\mathbf{L}_{G,V}A) \times Y^\flat \arrow[d, "(\mathbf{L}_{G,V}A)\times\varphi"] \\
(\mathbf{L}_{G,V}A) \boxtimes Y \arrow[r, "\rho_{\mathbf{L}_{G,V}A,Y}"'] & (\mathbf{L}_{G,V}A) \times Y
\end{tikzcd}$$

The upper morphism $\rho_{\mathbf{L}_{G,V}A,Y}$ is a global equivalence by Step 2, and the right morphism is even a strong level equivalence. We show now that the left morphism $(\mathbf{L}_{G,V}A) \boxtimes \varphi$ is a global equivalence; this implies that the lower morphism $\rho_{\mathbf{L}_{G,V}A,Y}$ is a global equivalence for every orthogonal space Y.

To prove our claim we let V be any real inner product space and exploit the natural isomorphism [ref]

$$((\mathbf{L}_{G,V}A) \boxtimes Y)(V \oplus W) \;\cong\; (O(V \oplus W) \times_G A) \times_{O(W)} Y(W) \ .$$

The $(O(V \oplus W) \times O(W)^{\mathrm{op}})$-space $O(V \oplus W) \times_G A$ is cofibrant and the right $O(W)$-action is free. Since $\varphi(W) : Y^\flat(W) \longrightarrow Y(W)$ is an $O(W)$-weak equivalence, Proposition 1.23 shows that the map $(O(V \oplus W) \times_G A) \times_{O(W)} \varphi(W)$ is an $O(V \oplus W)$-weak equivalence. Hence the same is true for the $O(V \oplus W)$-map

$$((\mathbf{L}_{G,V}A) \boxtimes \varphi)(V \oplus W) \;:\; ((\mathbf{L}_{G,V}A) \boxtimes Y^\flat)(V \oplus W) \;\longrightarrow\; ((\mathbf{L}_{G,V}A) \boxtimes Y^\flat)(V \oplus W) \ .$$

Now we let K be another compact Lie group and U a K-representation. We do not know whether $((\mathbf{L}_{G,V}A) \boxtimes \varphi)(U)$ is a K-weak equivalence in general, but the argument above show that this is true whenever U is K-isomorphic to $V \oplus W$ for some other G-representation W and with trivial K-action on W. In other words, $((\mathbf{L}_{G,V}A) \boxtimes \varphi)(U)$ is a K-weak equivalence whenever U is 'sufficiently large', namely when the dimension of U^G is at least the dimension of V. Hence $(\mathbf{L}_{G,V}A) \boxtimes \varphi$ is a global equivalence.

Step 4: We show the theorem in full generality. The argument is the same as in Step 2: we argue first, by induction on m, that the morphism $\rho_{\mathrm{sk}^m X,Y}$ is a global equivalence. This uses step 3 and that the two functors $-\boxtimes Y$ and $-\times Y$ preserve pushouts and h-cofibrations. The passage to the colimit $\rho_{X,Y}$ again uses stability of global equivalences under sequential colimits along h-cofibrations.

Part (ii) is a direct consequence of part (i) and the fact that the functor π_0^G commutes with finite products. □

By the previous theorem, the morphism $\rho_{X,Y} : X \boxtimes Y \longrightarrow X \times Y$ is a global equivalence if X or Y is flat. The cartesian product $-\times-$ of orthogonal spaces preserves global equivalences in both variables: if $\varphi : Y \longrightarrow Y'$ is a global equivalence, then we apply Proposition 4.7 (vi) to the pullback square

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d, "X\times\varphi"'] & Y \arrow[d, "\varphi"] \\ X \times Y' \arrow[r] & Y' \end{tikzcd}$$

where both horizontal maps are projections. Since projections to a factor are strong level fibrations, the base change $X \times \varphi$ of the global equivalence φ is again a global equivalence. So the previous Theorem 5.9 also implies:

Corollary 5.10. (i) *Box product with a flat orthogonal space preserves global equivalences.*
(ii) *Box product with any orthogonal space preserves global equivalences between flat orthogonal spaces.*

Finally, we will prove another important relationship between the global model structures an the smash product, namely the *monoid axiom* [**72**, Def. 3.3].

Proposition 5.11 (Monoid axiom)**.** *For every flat cofibration $j : A \longrightarrow B$ that is also a global equivalence and every orthogonal space Y the morphism*

$$j \boxtimes Y \ : \ A \boxtimes Y \ \longrightarrow \ B \boxtimes Y$$

is an h-cofibration and a global equivalence. Moreover, the class of h-cofibrations that are also global equivalences is closed under cobase change, coproducts and sequential and transfinite compositions.

Proof. Given the flatness theorem, this is a standard argument, similar to the proofs of the monoid axiom in the non-equivariant context Every flat cofibration is an h-cofibration (Proposition 4.6 (iii) applied to the strong level model structure), and h-cofibrations are closed under box product with any orthogonal space (Remark 5.8), so $j \boxtimes Y$ is an h-cofibration. Since j is a h-cofibration and global equivalence, [...] so $j\Box Y$ is a global equivalence.

Proposition 4.7 shows that the class of h-cofibrations that are also global equivalences is closed under cobase change, coproducts and sequential and transfinite compositions. □

Definition 5.12. An *orthogonal monoid space* is an orthogonal space R equipped with a unit element $1 \in R(0)$ and a multiplication morphism $\mu : R \boxtimes R \longrightarrow R$ that are unital and associative in the sense that the square

$$\begin{tikzcd} (R \boxtimes R) \boxtimes R \arrow[r, "\cong"] \arrow[d, "\mu\boxtimes R"'] & R \boxtimes (R \boxtimes R) \arrow[r, "R\boxtimes\mu"] & R \boxtimes R \arrow[d, "\mu"] \\ R \boxtimes R \arrow[rr, "\mu"'] & & R \end{tikzcd}$$

commutes. An orthogonal monoid space R is *commutative* if moreover $\mu \circ \tau_{R,R} = \mu$, where $\tau_{R,R} : R \boxtimes R \longrightarrow R \boxtimes R$ is the symmetry isomorphism of the box product.

A *morphism* of orthogonal monoid spaces is a morphism of orthogonal spaces $f : R \longrightarrow S$ such that $f \circ \mu^R = \mu^S \circ (f \boxtimes f)$ and $f(0)(1) = 1$.

One can expand the data of an orthogonal monoid space into an 'external' form as follows. The multiplication map corresponds to continuous maps $\mu_{V,W} : R(V) \times R(W) \longrightarrow R(V \oplus W)$ for all inner product spaces V and W that form a bimorphism as (V, W) varies and such that

$$\mu_{V,0}(x, 1) \ = \ x \qquad \text{and} \qquad \mu_{0,W}(1, y) \ = \ y \ .$$

Every inner product space is isometrically isomorphism to some $\mathbb{R}^m$ with the standard scalar product, so it suffices to specify the multiplication maps $\mu_{\mathbb{R}^n,\mathbb{R}^m}$ for all $n, m \geq 0$. The commutativity condition can be expressed in terms of the external multiplication as follows: the diagram

$$\begin{tikzcd}
R(V) \times R(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\text{twist}"'] & R(V \oplus W) \arrow[d, "R(\tau_{V,W})"] \\
R(W) \times R(V) \arrow[r, "\mu_{W,V}"'] & R(W \oplus V)
\end{tikzcd}$$

commutes, where $\tau_{V,W} : V \oplus W \longrightarrow W \oplus V$ interchanges the summands.

Theorem [**72**, Thm. 4.1] now applies to the global model structure and gives the following corollary.

Corollary 5.13. *Let R be an orthogonal monoid space.*

(i) *The category of orthognal R-spaces admits the* global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is a global equivalence (respectively fibration in the global model structure). If R is commutative, then this is a monoidal model category that satisfies the monoid axiom.*
(ii) *If R is commutative, then the category of R-algebras admits the* global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is a global equivalence (respectively fibration in the global model structure). Every cofibrant R-algebra is also cofibrant as an orthogonal R-space.*

Remark 5.14. Strictly speaking, Theorem 4.1 of [**72**] does not apply verbatim to the global model structures because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to sequential composites of flat cofibrations, and this suffices to run the small object argument (compare also Remark 2.4 of [**72**, Thm. 4.1]).

In Construction 2.12 we introduced the cofree functor $R : \mathbf{U} \longrightarrow spc$ from spaces to orthogonal spaces. We endow the cofree functor with a lax symmetric monoidal transformation

$$\mu_{A,A'} \ : \ (RA) \boxtimes (RA') \ \longrightarrow \ R(A \times A') \ .$$

To construct $\mu_{A,A'}$ we start from the $O(V) \times O(W)$-equivariant maps

$$\begin{aligned}
(5.15) \qquad \operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty), A) \times \operatorname{map}(\mathbf{L}(W, \mathbb{R}^\infty), A') \ &\xrightarrow{\ \times\ } \quad \operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(V, \mathbb{R}^\infty), A \times A') \\
&\xrightarrow{\operatorname{map}(\operatorname{res}_{V,W}, A \times A')} \operatorname{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), A \times A')
\end{aligned}$$

that constitute a bimorphism from (RA, RA') to $R(A \times A')$. Here

$$\operatorname{res}_{V,W} \ : \ \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty) \times \mathbf{L}(V, \mathbb{R}^\infty)$$

is the map that takes an embedding of $V \oplus W$ to the pair of its restrictions to V and W. The morphism $\mu_{A,A'}$ is associated to this bimorphism via the universal property of the box product.

Since the cofree functor R is lax symmetric monoidal with respect to the maps $\mu_{A.A'}$, it takes topological monoids to orthogonal monoid spaces, in a way preserving commutativity.

The fixed point construction and translation interact with very nicely with multiplicative structures, as we will now describe. For every finite group G, the fixed point functor F^G is lax symmetric monoidal: given orthogonal spaces Y and Z, a natural map

$$\mu^{Y,Z}_{V,W} \ : \ (F^GY)(V) \times (F^GZ)(W) \ \longrightarrow \ (F^G(Y \boxtimes Z))(V \oplus W)$$

is defined as the composite

$$\begin{aligned}(Y(V\otimes\rho_G))^G \times (Z(W\otimes\rho_G))^G \ &\xrightarrow{\ \cong\ } \ (Y(V\otimes\rho_G)\times Z(W\otimes\rho_G))^G \\ &\xrightarrow{(i_{V\otimes\rho_G,W\otimes\rho_G})^G} \ ((Y\boxtimes Z)((V\oplus W)\otimes\rho_G))^G \ .\end{aligned}$$

As V and W vary, these maps form a bimorphism, so the universal property of the box product provides a natural morphism of orthogonal spaces

$$\mu^{Y,Z} \ : \ (F^GY)\boxtimes(F^GY) \ \longrightarrow \ F^G(Y\boxtimes Z) \ .$$

A unit morphism is given by the morphism $j : 1 \longrightarrow F^G(1)$, which is in fact an isomorphism. The maps $\mu^{Y,Z}_{V,W}$, and hence the morphisms $\mu^{Y,Z}$ are suitably associative, commutative and unital, i.e., they give the G-fixed point functor a lax symmetric monoidal structure. Under the isomorphism of Proposition 4.6, the morphism $\mu^{Y,Z}$ realizes the homotopy group pairing (5.3), in the sense that the diagram

$$\begin{tikzcd}
\pi_0^K(F^GX)\times\pi_0^L(F^GY) \arrow[rr, "l^K\times l^L"] \arrow[d, "\times"'] & & \pi_0^{K\times G}(X)\times\pi_0^{L\times G}(Y) \arrow[d, "\times"] \\
\pi_0^{K\times L}((F^GX)\boxtimes(F^GY)) \arrow[d, "\pi_0^{K\times L}(\mu^{X,Y})"'] & & \pi_0^{K\times G\times L\times G}(X\boxtimes Y) \arrow[d, "\Delta^*"] \\
\pi_0^{K\times L}(F^G(X\boxtimes Y)) \arrow[rr, "l^{K\times L}"'] & & \pi_0^{K\times L\times G}(X\boxtimes Y)
\end{tikzcd}$$

commutes, where $\Delta : K\times L\times G \longrightarrow K\times G\times L\times G$ is the diagonal defined by $\Delta(k,l,g) = (k,g,l,g)$.

If R is an orthogonal monoid space, then the lax monoidal structure on F^G provides a product on the fixed point space F^GR, given by the composite

$$(F^GR)\boxtimes(F^GR) \ \xrightarrow{\mu^{R,R}} \ F^G(R\boxtimes R) \ \xrightarrow{F^G(\mu_R)} F^GR \ ,$$

and a unit map given by either of the two composite in the commutative square:

$$\begin{tikzcd}
1 \arrow[r, "j"] \arrow[d, "\eta"'] & F^G1 \arrow[d, "F^G\eta"] \\
R \arrow[r, "j"'] & F^GR
\end{tikzcd}$$

If the multiplication on R is commutative, then so is the induced multiplication on the fixed point spectrum F^GR.

6. Orthogonal spaces, $\mathcal{L}$-spaces and orbispaces

In this section we provide two additional perspectives on the unstable global homotopy theory of orthogonal spaces: we interpret it as the homotopy theory of 'spaces with an action of the universal compact

Lie group $\mathcal{L}$', and as the homotopy theory of 'orbispaces with compact Lie isotropy'. More formally, we establish a chain of Quillen equivalence of model categories

$$spc \quad \underset{u}{\overset{Y\mapsto Y(\mathbb{R}^\infty)}{\rightleftarrows}} \quad (\mathcal{L}\mathbf{U})_{\text{flat global}} \underset{\text{Id}}{\overset{\text{Id}}{\leftrightarrows}} \quad (\mathcal{L}\mathbf{U})_{\text{proj. global}} \underset{\Phi}{\overset{\Lambda}{\leftrightarrows}} \quad orbispc$$

Here $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ is the topological monoid of linear isometric embeddings of $\mathbb{R}^\infty$ into itself, and $\mathcal{L}\mathbf{U}$ is the category of $\mathcal{L}$-spaces, i.e., spaces (compactly generated and weakly Hausdorff, as usual), equipped with a continuous left $\mathcal{L}$-action. The right most category is the category of *orbispaces*, i.e., the category of contravariant continuous functors from the *global orbit category* $\mathbf{O}_{\text{gl}}$ to spaces.

Before we go into details we explain how one can think of the two new models. The category of $\mathcal{L}$-spaces has been much studied, for example in [**10, 12, 54, 59**]; in some of these source the symbol $\mathcal{L}$ refers to the linear isometries *operad*, so the monoid we denote $\mathcal{L}$ is then the monoid $\mathcal{L}(1)$ of unary operations. The underlying space of $\mathcal{L}$ is contractible, so the homotopy theory of $\mathcal{L}$-spaces with respect to 'underlying' weak equivalences is just another model for the homotopy theory of spaces. However, we radically shift the perspective on the homotopy theory that $\mathcal{L}$-spaces represent, and use a notion *global equivalences* of $\mathcal{L}$-spaces that is much finer than the notion of 'underlying' weak equivalence that has so far been studied. When viewed through the eyes of global equivalences, one should think of $\mathcal{L}$ as a 'universal compact Lie group' and hence of an $\mathcal{L}$-space as a 'global space' on which all compact Lie groups act simultaneously and in a compatible way. Such a statement is of course not literally correct: the topological monoid $\mathcal{L}$ is neither compact, nor a group, much less a compact Lie group. However, $\mathcal{L}$ contains a copy (in fact, many conjugate ones) of every compact Lie group in a certain way: we may choose a continuous isometric linear G-action on $\mathbb{R}^\infty$ that makes $\mathbb{R}^\infty$ into a complete G-universe. This action is a continuous injective group homomorphism $\rho : G \longrightarrow \mathcal{L}$, and we call the images $\rho(G)$ of such homomorphisms *universal subgroups* of $\mathcal{L}$ (compare Definition 6.1). Because any two complete G-universes are equivariantly isometrically isomorphic, the group $\rho(G)$ is independent, up to conjugacy by an invertible linear isometry, of the choice of ρ. So in this way every compact Lie group determines a specific conjugacy class of subgroups of $\mathcal{L}$, abstractly isomorphic to G. In this section we define two global model structures on the category of $\mathcal{L}$-spaces and establish Quillen equivalences to the global model category of orthogonal spaces, compare Theorem 6.11.

The right Quillen equivalence between $\mathcal{L}$-spaces and orbispaces is then an analog of Elmendorf's theorem [**31**] saying that the passage from G-spaces to functors on the orbit category that collects the fixed point spaces of the various closed subgroups of G is an equivalence of homotopy theories. Indeed, the global orbit category $\mathbf{O}_{\text{gl}}$ is the direct analog for the 'universal compact Lie group $\mathcal{L}$' of the orbit category of a single compact Lie group: the objects of $\mathbf{O}_{\text{gl}}$ are the universal subgroups of $\mathcal{L}$ and the morphism spaces in $\mathbf{O}_{\text{gl}}$ are defined by

$$\mathbf{O}_{\text{gl}}(K, G) \;=\; \text{map}_{\mathcal{L}}(\mathcal{L}/K, \mathcal{L}/G) \;\cong\; (\mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K)/G)^K \ .$$

As we explain in Theorem 6.18, the Quillen equivalence between $\mathcal{L}$-spaces (with the projective global model structure) and orbispaces is a special case of a generalization of Elmendorf's theorem to a context of topological monoid relative to certain collections of closed submonoids, compare Proposition A.1.16.

We should probably justify the terminology 'orbispaces' for functors on the global orbit category. For this we refer to the paper [**36**] of Gepner and Henriques, who compare the homotopy theories of 'Orb-spaces' with homotopy theories of topological stacks and of topological groupoids. The setup of [**36**] is relative to a specified class of 'allowed isotropy group', and Gepner and Henriques then construct a topological category whose objects are the allowed isotropy groups and such that the morphism space $\text{Orb}(K, G)$ from a group K to a group G has the weak homotopy type of the homotopy orbit space of G action by conjugation on the space of continuous homomorphisms from K to G. An *orbispace*, or *Orb-space*, is then a continuous functor from the category Orb to spaces. Our global orbit category $\mathbf{O}_{\text{gl}}$ is such a category for the class of compact Lie groups, whence the terminology. So a more precise, but too lengthy name would be 'orbispaces with compact Lie group isotropy'.

After this attempt to provide motivation, we go into the details.

Definition 6.1. A compact subgroup G of the topological monoid $\mathcal{L}$ is a *universal subgroup* if it admits the structure of a Lie group (necessarily unique) such that the tautological G-action makes $\mathbb{R}^\infty$ into a complete G-universe.

In the previous definition of universal subgroup we insist that the G-action on $\mathbb{R}^\infty$ makes it a *complete* G-universe, so the name 'completely universal subgroup' would be more precise (but too clumsy). When G is a universal subgroup of $\mathcal{L}$ we write $\mathbb{R}^\infty_G$ for complete G-universe given by the tautological G-action on $\mathbb{R}^\infty$. The next proposition shows that conjugacy classes of universal subgroups of $\mathcal{L}$ biject with isomorphism classes of compact Lie groups.

Proposition 6.2. *Every compact Lie group is isomorphic to a universal subgroup of $\mathcal{L}$. Every isomorphism between universal subgroups is given by conjugation by an invertible linear isometry in $\mathcal{L}$. In particular, isomorphic universal subgroups are conjugate in $\mathcal{L}$.*

Proof. Given a compact Lie group G we can choose a continuous isometric linear action of G on $\mathbb{R}^\infty$) that makes $\mathbb{R}^\infty$ into a complete G-universe. Such an action is a continuous homomorphism $\rho : G \longrightarrow \mathcal{L}$ and the image $\rho(G)$ is a universal subgroup of $\mathcal{L}$, isomorphic to G via ρ.

Now we let $G, G' \leq \mathcal{L}$ be two universal subgroup and $\alpha : G \longrightarrow G'$ and isomorphism. Then $\mathbb{R}^\infty_G$ and $\alpha^*(\mathbb{R}^\infty_{G'})$ are two complete G-universes, so there is a G-equivariant linear isometry $\psi : \mathbb{R}^\infty_G \longrightarrow \alpha^*(\mathbb{R}^\infty_{G'})$. This ψ is an invertible element of the monoid $\mathcal{L}$ that satisfies and the G-equivariance means that $\psi \circ g = \alpha(g) \circ \psi$ for all $g \in G$. Hence α coincides with conjugation by ψ. □

The topological monoid $\mathcal{L}$ contains many other compact Lie subgroups that are not universal subgroups: any continuous, faithful linear isometric action of a compact Lie group G on $\mathbb{R}^\infty$ provides such a compact Lie subgroup. However, with respect to this action, $\mathbb{R}^\infty$ may only be an incomplete G-universe, or not even a G-universe at all, in which case the image of the representation homomorphism $G \longrightarrow \mathcal{L}$ is not universal in the sense of Definition 6.1.

Definition 6.3. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is a *global equivalence* if for every universal subgroup G of $\mathcal{L}$ the induced map

$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$

is a weak homotopy equivalence. The morphism f is a *global fibration* if for every universal subgroup G of $\mathcal{L}$ the induced map

$$f^G \ : \ X^G \ \longrightarrow \ Y^G$$

is a Serre fibration.

We get many examples come from orthogonal spaces by applying the functor $Y \mapsto Y(\mathbb{R}^\infty)$. This includes constant $\mathcal{L}$-spaces obtained by equipping any space we the trivial $\mathcal{L}$-action.

Example 6.4 (Induced $\mathcal{L}$-spaces)**.** We let G be a universal subgroup of $\mathcal{L}$ and A a left G-space. Then we can form the *induced $\mathcal{L}$-space*

$$\mathcal{L} \times_G A \ = \ (\mathcal{L} \times A)/(\varphi g, a) \sim (\varphi, ga) \ .$$

The functor $\mathcal{L} \times_G -$ is left adjoint to the restriction functor from $\mathcal{L}$-spaces to G-spaces.

When A is a one-point space, this construction specializes to the 'orbit $\mathcal{L}$-space' $\mathcal{L}/G \cong \mathcal{L} \times_G *$. The following proposition shows that $\mathcal{L}/G$ is the incarnation, in the world of $\mathcal{L}$-spaces, of the global classifying space of the group G. When combined with Proposition 2.6, this shows that for every universal subgroup K of $\mathcal{L}$, the underlying K-space of $\mathcal{L}/G$ is a classifying space for principal G-bundles over K-spaces. In particular, the underlying non-equivariant space of $\mathcal{L}/G$ is a classifying space for G.

Proposition 6.5. *Let G be a universal subgroup of the monoid $\mathcal{L}$, V a faithful G-subrepresentation of $\mathbb{R}^\infty_G$ and A a G-space. Then the restriction morphism*

$$\rho_V \times_G A \ : \ \mathcal{L} \times_G A \ = \ \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty) \times_G A \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty) \times_G A \ = \ (\mathbf{L}_{G,V} A)(\mathbb{R}^\infty)$$

is a global equivalence of $\mathcal{L}$-spaces.

Proof. We let K be a second universal subgroup of the monoid $\mathcal{L}$. Then $\rho_V : \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty_K)$ is a $(K \times G^{\mathrm{op}})$-homotopy equivalence by Proposition A.2.5 (ii); so the map

$$\rho_V \times_G A \ : \ \mathbf{L}(\mathbb{R}^\infty_G, \mathbb{R}^\infty_K) \times_G A \ \longrightarrow \ \mathbf{L}(V, \mathbb{R}^\infty_K) \times_G A$$

is a K-homotopy equivalence, and the induced map of K-fixed points is a homotopy equivalence. □

In Appendix A.1 we establish projective model structures for equivariant spaces with an action of a topological monoid, relative to a collection of biclosed submonoids. The following proposition is a special case of the more general Proposition A.1.10 for the topological monoid $\mathcal{L}$ and the collection $\mathcal{C}^u$ of universal subgroups.

Proposition 6.6 (Projective global model structure)**.** *The global equivalences and global fibrations are part of a proper topological closed model category structure on the category of $\mathcal{L}$-spaces, the* projective global model structure*. The cofibrations are generated by the morphisms*

$$\mathcal{L}/G \times S^{n-1} \ \longrightarrow \ \mathcal{L}/G \times D^n$$

for all universal subgroups G of $\mathcal{L}$ and all $n \geq 0$.

Now we want to show that the global model structures of orthogonal spaces and of $\mathcal{L}$-spaces are Quillen equivalent. The adjoint functor pair

$$spc \ \underset{u}{\overset{Y \mapsto Y(\mathbb{R}^\infty)}{\rightleftarrows}} \ \mathcal{L}\mathbf{U}$$

is not a Quillen pair because the left adjoint does not take flat orthogonal spaces to cofibrant $\mathcal{L}$-spaces. We are going to fix this by considering another model structure on the category of $\mathcal{L}$-spaces, the *flat global model structure*, which has more cofibrations than the projective global model structure.

Construction 6.7. The key observation to motivate the flat global model structure for $\mathcal{L}$-spaces is that for every compact Lie group G and faithful G-representation V the $\mathcal{L}$-space $(B_{\mathrm{gl}}G)(\mathbb{R}^\infty) = \mathbf{L}(V, \mathbb{R}^\infty)/G$ is a 'homogeneous $\mathcal{L}$-space', i.e., isomorphic to the quotient of $\mathcal{L}$ by a certain submonoid. We let G be a universal subgroup of $\mathcal{L}$ and V a finite dimensional faithful G-subrepresentation of $\mathbb{R}^\infty_G$. We denote by $\mathcal{L}[G, V]$ the stabilizer of the orbit of the inclusion $V \subset \mathcal{U}_G$ in the $\mathcal{L}$-space $\mathbf{L}_{G,V}(\mathbb{R}^\infty) = \mathbf{L}(V, \mathbb{R}^\infty)/G$. So $\mathcal{L}[G, V]$ is the submonoid consisting of all $\varphi \in \mathcal{L}$ such that there is a $g \in G$ with

$$\varphi|_V \ = \ g|_V \ .$$

In particular, φ must take V to itself, and hence it takes the orthogonal complement $V^\perp$ of V to itself. The homomorphism

$$G \times \mathcal{L}(V^\perp) \ \longrightarrow \ \mathcal{L} \ , \quad (g, \psi) \ \longmapsto \ g|_V \oplus \psi$$

is thus an isomorphism with image $\mathcal{L}[G, V]$. Since V is a G-subrepresentation of $\mathbb{R}^\infty_G$, the monoid $\mathcal{L}[G, V]$ contains the universal subgroup G:

$$G \ \subset \ \mathcal{L}[G, V] \ .$$

Moreover, the $\mathcal{L}$-action on the G-orbit of the inclusion $V \subset \mathbb{R}^\infty$ descends to an isomorphism of $\mathcal{L}$-spaces

$$\rho_{G,V} \ : \ \mathcal{L}/\mathcal{L}[G, V] \ \cong \ \mathbf{L}(V, \mathbb{R}^\infty)/G \ = \ \mathbf{L}_{G,V}(\mathbb{R}^\infty) \ . \qquad (6.8)$$

We denote by

$$\mathcal{C}^\flat \ = \ \mathcal{C}^u \cup \{\mathcal{L}[G, V]\}$$

the union of all universal subgroups of $\mathcal{L}$ with set of all biclosed submonoids $\mathcal{L}[G,V]$ for all universal subgroups $G \subset \mathcal{L}$ and all finite-dimensional faithful G-subrepresentations V of $\mathbb{R}^\infty_G$. Proposition A.1.10 applied to the topological monoid $\mathcal{L}$ and the collection $\mathcal{C}^\flat$ provides the projective $\mathcal{C}^\flat$-model structure on the category of $\mathcal{L}$-spaces. A morphism $f : X \longrightarrow Y$ of $\mathcal{L}$-spaces is a weak equivalence (respectively fibration) in this projective $\mathcal{C}^\flat$-model structure if and only if it is a global equivalence (respectively global fibration) and for all universal subgroups G of $\mathcal{L}$ and all finite-dimensional faithful G-subrepresentation V of $\mathbb{R}^\infty_G$ the map

$$f^{\mathcal{L}[G,V]} \ : \ X(V)^{\mathcal{L}[G,V]} \ \longrightarrow \ Y(V)^{\mathcal{L}[G,V]}$$

is a weak homotopy equivalence (respectively Serre fibration). By the isomorphism (6.8), the $\mathcal{L}$-space $\mathbf{L}_{G,V}(\mathbb{R}^\infty)$ represents the functor $X \mapsto X^{\mathcal{L}[G,V]}$, so $\mathbf{L}_{G,V}(\mathbb{R}^\infty)$ is cofibrant in the projective $\mathcal{C}^\flat$-model structure for $\mathcal{L}$-spaces; hence the functor $(-)(\mathbb{R}^\infty)$ takes flat cofibrations of orthogonal spaces to cofibrations in the projective $\mathcal{C}^\flat$-model structure for $\mathcal{L}$-spaces. However, in the projective $\mathcal{C}^\flat$-model structure not all global equivalences of $\mathcal{L}$-spaces are weak equivalences; we fix this by localizing the projective $\mathcal{C}^\flat$-model structure to another global model structure.

Definition 6.9. An $\mathcal{L}$-space Y is *injective* if for every universal subgroup G of $\mathcal{L}$ and every finite-dimensional faithful G-subrepresentation V of $\mathbb{R}^\infty_G$ the inclusion

$$Y^{\mathcal{L}[G,V]} \ \longrightarrow \ Y^G$$

is a weak equivalence.

The following proposition provides the necessary localization functor.

Proposition 6.10. *There is an endofunctor Q of the category of $\mathcal{L}$-spaces with values in injective $\mathcal{L}$-spaces and a natural global equivalence $j_X : X \longrightarrow QX$.*

Proof. For every universal subgroup G of $\mathcal{L}$ and every finite-dimensional faithful G-subrepresentation V of $\mathbb{R}^\infty_G$ the restriction morphism

$$\rho_{G,V} = \rho_V/G \ : \ \mathcal{L}/G \ \longrightarrow \ \mathcal{L}/\mathcal{L}[G,V] = \mathbf{L}(V,\mathbb{R}^\infty)/G = \mathbf{L}_{G,V}(\mathbb{R}^\infty)$$

is a global equivalence by Proposition 6.5, but it is not a cofibration in any sense. We factor $\rho_{G,V}$ through the mapping cylinder as the composite

$$\mathcal{L}/G \ \xrightarrow{c_{G,V}} \ Z(\rho_{G,V}) = ([0,1]\times \mathcal{L}/G) \cup_{\rho_{G,V}} \mathcal{L}/\mathcal{L}[G,V] \ \xrightarrow{r_{G,V}} \ \mathcal{L}/\mathcal{L}[G,V] \ ,$$

where $c_{G,V}$ is the 'front' mapping cylinder inclusion and $r_{G,V}$ is the projection, which is a homotopy equivalence. We then define

$$K \ = \ \bigcup_{(G,V)} \ \mathcal{Z}(\rho_{G,V})$$

as the set of all pushout product maps with the inclusions $S^{m-1} \longrightarrow D^m$, compare Construction 4.8. Here (G,V) runs through all pairs consisting of a universal subgroup $G \subset \mathcal{L}$ and a faithful finite-dimensional G-subrepresentation V of $\mathbb{R}^\infty_G$. Proposition 4.9 (with Y a one-point $\mathcal{L}$-space) shows that the right lifting property with respect to the set K is equivalent to being injective.

Now we apply the countable small object argument with respect to the set K to the unique morphism from a given $\mathcal{L}$-space Y to the terminal $\mathcal{L}$-space. A countable version (i.e., with sequential colimits) of Quillen's original argument which works in our case can be found in [**30**, Prop. 7.17]. Dwyer and Spalinski assume the sources of all morphisms in the set K are sequentially small, which is not the case here. However, what is really needed is only that the sources of all morphisms in the set K are sequentially small for cobase changes of coproducts of morphisms in K, compare the more general version of the small object argument in [**43**, Thm. 2.1.14]. In our situation, all morphisms in K are h-cofibrations, hence so are all cobase changes of coproducts, and the sources of morphisms in are small with respect to sequences of h-cofibrations.

In any case, the small object argument produces an endofunctor Q on the category of $\mathcal{L}$-spaces and a natural transformation $j_Y : Y \longrightarrow QY$ with the following properties:

(i) The object QY has the right lifting property with respect to all morphisms in K, i.e., it is injective.
(ii) The morphism j_Y is a sequential composite of cobase changes of coproducts of morphisms in K.

All morphisms in K are simultaneously h-cofibrations and global equivalences; the class of h-cofibrations that are also global equivalences is closed under coproducts, cobase change and sequential composition. So the morphism $j_Y : Y \longrightarrow QY$ is an h-cofibration and a global equivalence. $\square$

Now we have all the ingredients to localize the projective $\mathcal{C}^\flat$-model structure into a second global model structure on the category of $\mathcal{L}$-spaces.

Theorem 6.11 (Flat global model structure for $\mathcal{L}$-spaces)**.** *The $\mathcal{C}^\flat$-cofibrations and global equivalences are part of a cofibrantly generated proper topological model structure on the category of $\mathcal{L}$-spaces, the* flat global model structure*. The fibrant objects in the flat global model structure are the injective $\mathcal{L}$-spaces. Moreover, the two adjoint functor pairs*

$$spc \underset{u}{\overset{Y\mapsto Y(\mathbb{R}^\infty)}{\rightleftarrows}} (\mathcal{L}\mathbf{U})_{\text{flat global}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\leftrightarrows}} (\mathcal{L}\mathbf{U})_{\text{proj. global}}$$

are Quillen equivalences with respect to the global model structures on orthogonal spaces.

Proof. We construct the flat global model structure by applying Bousfield's localization theorem [**17**, Thm. 9.3] to the projective $\mathcal{C}^\flat$-model structure. We use the localization functor Q given by Proposition 6.10. By the very definition of 'injective', a morphism $f : Y \longrightarrow Z$ between injective $\mathcal{L}$-spaces is a global equivalence if and only if for every $M \in \mathcal{C}^\flat$ the map $f^M : Y^M \longrightarrow Z^M$ is a weak equivalence. So the Q-equivalences in the sense of [**17**, Thm. 9.3] are precisely the $\mathcal{C}^\flat$-equivalences.

Now we verify the hypotheses (A1)–(A3) of Bousfield's theorem. The projective $\mathcal{C}^\flat$-level model structure is proper. If f is an $\mathcal{C}^\flat$-equivalence, then Qf is a global equivalence between injective $\mathcal{L}$-spaces, hence an $\mathcal{C}^\flat$-equivalence. This shows (A1).

The morphism j_{QX} is a global equivalence between injective $\mathcal{L}$-spaces, hence an $\mathcal{C}^\flat$-equivalence. On the other hand, $Q(j_X) : QX \longrightarrow QQX$ is an $\mathcal{C}^\flat$-equivalence since Q takes all global equivalences to $\mathcal{C}^\flat$-equivalences. So j_{QX} and $Q(j_X)$ are also $\mathcal{C}^\flat$-equivalences, and this proves axiom (A2).

In axiom (A3) we are given a pullback square

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

of $\mathcal{L}$-spaces in which f is an $\mathcal{C}^\flat$-fibration such that $j_X : X \longrightarrow QX$, $j_Y : Y \longrightarrow QY$ and Qh are $\mathcal{C}^\flat$-equivalences. We have to show that then Qk is an $\mathcal{C}^\flat$-equivalence. These hypothesis can be reformulated as follows: the $\mathcal{L}$-spaces X and Y are injective, f is an $\mathcal{C}^\flat$-fibration and h is a global equivalence. We have to show that then k is a global equivalence. But this is straightforward: for every universal subgroup G of $\mathcal{L}$ the square

$$\begin{tikzcd} V^G \arrow[r, "k^G"] \arrow[d, "g^G"'] & X^G \arrow[d, "f^G"] \\ W^G \arrow[r, "h^G"'] & Y^G \end{tikzcd}$$

is a pullback, f^G is a Serre fibration and g^G is a weak equivalence. Since the model structure of topological spaces is right proper, the map k^G is again a weak equivalence. Hence k is a global equivalence.

This proves (A3), and thus Bousfield's theorem applied to the projective $\mathcal{C}^\flat$-model structures provides a proper model structures with global equivalences as weak equivalences and with $\mathcal{C}^\flat$-cofibrations as the cofibrations. Bousfield's theorem also provides the characterization of the fibrations in this flat global model structure. For $Y = *$ the criterion specializes to the fact that X is fibrant if and only if it is $\mathcal{C}^\flat$-fibrant (an empty condition) and the morphism $j_X : X \longrightarrow QX$ is an $\mathcal{C}^\flat$-equivalence. Since QX is injective, the fibrancy is equivalent to X being injective.

The cofibrations in the flat global model structures are the same as the cofibrations in the projective $\mathcal{C}^\flat$-model structure, so the part of the pushout product property that involves only cofibrations (but not equivalences) holds because the projective $\mathcal{C}^\flat$-model structure is topological [ref... fill in the rest]

The flat cofibrations of orthogonal spaces are generated by the morphisms $i_k \times \mathbf{L}_{G,V} : S^{k-1} \times \mathbf{L}_{G,V} \longrightarrow D^k \times \mathbf{L}_{G,V}$ for all $k \geq 0$, all universal subgroups G and all faithful G-representations V. By the isomorphism (6.8), the functor $(-)(\mathbb{R}^\infty)$ takes these generating cofibrantions to the maps of $\mathcal{L}$-spaces

$$i_k \times \mathcal{L}/\mathcal{L}[G,V] \;:\; S^{k-1} \times \mathcal{L}/\mathcal{L}[G,V] \;\longrightarrow\; D^k \times \mathcal{L}/\mathcal{L}[G,V]$$

which are generating cofibrations in the flat model structure of $\mathcal{L}$-spaces. The functor $(-)(\mathbb{R}^\infty)$ also takes flat cofibrations that are global equivalences to global equivalences of $\mathcal{L}$-spaces. Again it suffices to check this for the set $J^{str} \cup K$ of generating acyclic cofibrations of the global model structure on orthogonal spaces. So morphisms in J^{str} are homotopy equivalences of orthogonal spaces, so then become homotopy equivalences of $\mathcal{L}$-spaces, which are global equivalences. [morphisms in K...]

We have now shown that the adjoint functor pair $((-)(\mathbb{R}^\infty), u)$ is a Quillen pair with respect to the flat global model structure on $\mathcal{L}$-spaces. Now we suppose that A is a flat $\mathcal{L}$-space and Z is an $\mathcal{L}$-space that is fibrant in the flat global model structure, i.e., injective. Since Z is injective, the orthogonal space $u(Z)$ is static. So any morphism $g : A \longrightarrow uZ$ of orthogonal spaces is a global equivalence if and only if its adjoint $f : A(\mathbb{R}^\infty) \longrightarrow Z$ is a global equivalence of $\mathcal{L}$-spaces (using that flat orthogonal spaces are closed, so 'global equivalence' can be tested by passing to $A(\mathbb{R}^\infty)$). This shows that the adjoint functor pair $((-)(\mathbb{R}^\infty), u)$ is a Quillen equivalence.

The flat and projective global model structure on $\mathcal{L}$-spaces have the same weak equivalences, and nested classes of cofibrations, so they are Quillen equivalent, by the identity adjoint functor pair. $\square$

Remark 6.12 (Monoidal properties)**.** With some adjustments, the Quillen equivalence between orthogonal spaces and $\mathcal{L}$-spaces of Theorem 6.11 can be made into a *monoidal* Quillen equivalence; the price to pay is that one has work with $\star$*-modules* instead of $\mathcal{L}$-spaces. Indeed, the category of $\mathcal{L}$-spaces can be endowed with the *operadic product* $\boxtimes_{\mathcal{L}}$, defined by

$$X \boxtimes_{\mathcal{L}} Y \;=\; \mathbf{L}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty) \times_{\mathcal{L}\times\mathcal{L}} (X \times Y)\ ,$$

with $\mathcal{L}$-action from postcomposition on $\mathbf{L}((\mathbb{R}^\infty)^2, \mathbb{R}^\infty)$. This operadic product is coherently associative and unital, and comes with a natural unit transformation $\lambda_X : * \boxtimes_{\mathcal{L}} X \longrightarrow X$, that, however, is not always an isomorphism. We refer the reader to [**10**, Sec. 4] for more details.

A $\star$*-module* is an $\mathcal{L}$-space for which the morphism λ_X is an isomorphism. On the full subcategory of $\star$-modules the operadic product $\boxtimes_{\mathcal{L}}$ is symmetric monoidal. The category of $*$-modules admits a (non-equivariant) model structure with weak equivalences defined after forgetting the $\mathcal{L}$-action, cf. [**10**, Thm. 4.16]; with this model structure the inclusion of $\star$-modules into $\mathcal{L}$-spaces is a Quillen equivalence. I expect that the flat global model structure of $\mathcal{L}$-spaces can be adapted to a Quillen equivalent global model structure on $\star$-modules, and that this model structure lifts to monoids and commutative monoids in the category of $\star$-modules (which are essentially A_∞-monoids and E_∞-monoids, respectively). However, I have not tried to work out the details.

The functor $(-)(\mathbb{R}^\infty)$ from orthogonal spaces to $\mathcal{L}$-spaces takes values in $\star$-modules and comes with a commutative, associative and unital isomorphism

$$X(\mathbb{R}^\infty) \boxtimes_{\mathcal{L}} Y(\mathbb{R}^\infty) \ \cong\ (X \boxtimes Y)(\mathbb{R}^\infty)\ .$$

In other words, $(-)(\mathbb{R}^\infty)$ becomes a strong symmetric monoidal functor. If the global model structure on $\star$-modules exists as expected, then this yields a symmetric monoidal Quillen equivalence between orthogonal spaces and $\star$-modules. Moreover, the induced adjoint functor pairs between monoids and commutative monoids ought to be Quillen equivalences as well.

We close this section by giving rigorous meaning to the idea that unstable global homotopy theory is the homotopy theory of 'orbispaces with compact Lie group isotropy'. In Appendix A.1 we establish a version of Elmendorf's theorem, saying that an equivariant homotopy type can be reassembled from fixed point data; out generalization works for topological monoids relative to a collection of biclosed submonoids. The identification of the global homotopy theory of $\mathcal{L}$-spaces with the homotopy theory of orbispaces is just a special case of this. Indeed, the global orbit category defined in the following construction is simply the orbit category, in the sense of Construction A.1.15, of the topological monoid $\mathcal{L}$ relative to the collection $\mathcal{C}^u$ of universal subgroups.

Construction 6.13 (Global orbit category)**.** We define a topological category $\mathbf{O}_{\mathrm{gl}}$, the *global orbit category.* The object of $\mathbf{O}_{\mathrm{gl}}$ are all universal subgroups of the monoid $\mathcal{L}$, and the space of morphisms from K to G is the space

$$\mathbf{O}_{\mathrm{gl}}(K,G) \ =\ (\mathcal{L}/G)^K \ =\ (\mathbf{L}(\mathbb{R}^\infty_G,\mathbb{R}^\infty_K)/G)^K.$$

Here the $K \times G^{\mathrm{op}}$-action on $\mathcal{L}$ is by pre- and postcomposition. Then $\mathbf{O}_{\mathrm{gl}}(K,G)$ is the space of K-fixed points of the G-orbit space. Composition in $\mathbf{O}_{\mathrm{gl}}$ is induced by composition of linear isometric embeddings. Indeed, the continuous $\mathcal{L}$-action

$$\mathcal{L}\times\mathcal{L}/G \ \longrightarrow\ \mathcal{L}/G \tag{6.14}$$

is compatible with fixed points: If $\varphi\in\mathcal{L}$ is a linear isometric embedding whose orbit φG is K-fix, then the relation

$$(\psi k)\circ(\varphi G) \ =\ \psi\circ(k\varphi G) \ =\ \psi\circ\varphi G \ =\ \psi\varphi G$$

shows that the G-orbit of $\psi\varphi$ only depends on the K-orbit of ψ. So the restriction of (6.14) to $\mathcal{L}\times(\mathcal{L}/G)^K$ factors over a well-defined map

$$(\mathcal{L}/K)\times(\mathcal{L}/G)^K \ \longrightarrow\ \mathcal{L}/G\ .$$

Finally, if the K-orbit ψK is L-fix and the G-orbit φG is K-fix, then the relation

$$l(\psi\varphi G) \ =\ (l\psi)\circ(\varphi G) \ =\ (\psi k)\circ(\varphi G) \ =\ \psi\circ(k\varphi G) \ =\ \psi\circ\varphi G \ =\ \psi\varphi G$$

shows that the G-orbit of $\psi\varphi$ is L-fix. So the composition map indeed passes to a well-defined continuous composition map

$$\mathbf{O}_{\mathrm{gl}}(L,K)\times\mathbf{O}_{\mathrm{gl}}(K,G) \ =\ (\mathcal{L}/K)^L \times (\mathcal{L}/G)^K \ \longrightarrow\ (\mathcal{L}/G)^L = \mathbf{O}_{\mathrm{gl}}(L,G)\ .$$

Remark 6.15. The global orbit category refines the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms in the sense that for all compact Lie groups G and K, the components $\pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$ biject functorially with $\mathrm{Rep}(K,G)$. Indeed, by Proposition A.2.5 the $(K\times G^{\mathrm{op}})$-space $\mathbf{L}(\mathbb{R}^\infty_G,\mathbb{R}^\infty_K)$ is a universal space for the family $\mathcal{F}(K;G^{\mathrm{op}})$ of graph subgroups. So the space $\mathbf{O}_{\mathrm{gl}}(K,G)=(\mathcal{L}/G)^K$ is a disjoint union, indexed by conjugacy classes of continuous group homomorphisms $\alpha : K \longrightarrow G$, of classifying spaces of the centralizer of the image of α, compare Proposition 2.6 (ii). In particular, the path component category $\pi_0(\mathbf{O}_{\mathrm{gl}})$ of the global orbit category is isomorphic to the category Rep of compact Lie groups and conjugacy classes of continuous homomorphisms. The preferred bijection

$$\mathrm{Rep}(K,G) \ \longrightarrow\ \pi_0(\mathbf{O}_{\mathrm{gl}}(K,G))$$

sends the conjugacy class of $\alpha : K \longrightarrow G$ to the G-orbit of any linear isometric embedding of the K-universe $\alpha^*(\mathbb{R}^\infty_G)$ into the complete K-universe $\mathbb{R}^\infty_K$.

Definition 6.16. An *orbispace* is a continuous functor $Y : \mathbf{O}_{\mathrm{gl}}^{\mathrm{op}} \longrightarrow \mathbf{U}$ from from the opposite of the global orbit category to the category of unbased spaces. We denote the category of orbispaces and natural transformations by *orbispc*.

It would be somewhat more precise (but too lengthy) to speak of 'orbispaces with compact Lie isotropy', but no confusion should arise because will not consider more general classes of allowed isotropy groups.

Construction 6.17. We introduce a *fixed point functor*

$$\Phi \ : \ \mathcal{L}\mathbf{U} \ \longrightarrow \ orbispc$$

from the category of $\mathcal{L}$-spaces to the category of orbispaces that will turn out to be a right Quillen equivalence with respect to the projective global model structure on the left hand side. Given an $\mathcal{L}$-space Y we define the value of the orbispace $\Phi(Y)$ at a universal subgroup G as the G-fixed points

$$\Phi(Y)(G) \ = \ Y^G \ \cong \ \mathrm{map}_{\mathcal{L}}(\mathcal{L}/G, Y) \ .$$

The restriction of the action map $\mathcal{L} \times Y \longrightarrow Y$ to Y^G factors over a morphism of $\mathcal{L}$-spaces

$$\mathcal{L}/G \times Y^G \ \longrightarrow \ Y$$

(with trivial $\mathcal{L}$-action on Y^G). So for a second universal subgroup K of $\mathcal{L}$, the restriction to K-fixed points is the desired map

$$\mathbf{O}_{\mathrm{gl}}(K,G) \times \Phi(Y)(G) \ = \ (\mathcal{L}/G)^K \times Y^G \ \longrightarrow \ Y^K \ = \ \Phi(Y)(K) \ .$$

As an example of this construction we note that

$$\Phi(\mathcal{L}/G) \ = \ \mathbf{O}_{\mathrm{gl}}(-,G) \ ,$$

i.e., the fixed points of the orbit $\mathcal{L}$-space $\mathcal{L}/G$ form the orbispace represented by G.

As for continuous functors out of any topological category, the category of orbispaces supports a well-known 'projective' model structure in which the weak equivalence (respectively fibrations) are those natural transformations that are weak equivalences (respectively Serre fibrations) at every object, see for example [**60**, VI Thm. 5.2]. By general arguments, the fixed point functor Φ just defined has a left adjoint Λ. The following is then the special case of Proposition A.1.16 for the topological monoid $\mathcal{L}$ with respect to the family $\mathcal{C}^u$ of universal subgroups.

Theorem 6.18. *The adjoint functor pair*

$$(\mathcal{L}\mathbf{U})_{\text{proj. global}} \quad \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}} \quad orbispc$$

is a Quillen equivalence between the category of $\mathcal{L}$-spaces with the projective global model structure and the category of orbispaces.

As Proposition A.1.16 also shows, for every cofibrant orbispace F the adjunction unit $F \longrightarrow \Phi(\Lambda F)$ is even an isomorphism.

Example 6.19 (Global Eilenberg-Mac Lane spaces)**.** We let $\underline{M}$ be a *global coefficient system*, i.e., a contravariant functor $\underline{M} : \mathrm{Rep}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ from the category Rep to the category of abelian groups. For $n \geq 0$ we let

$$K(n,-) \ : \ \mathcal{A}b \ \longrightarrow \ \mathbf{T}$$

be an Eilenberg-Mac Lane space functor, i.e., the based space $K(n,A)$ has vanishing homotopy groups except in dimension n, and there is a natural isomorphism

$$\pi_n(K(n,A)) \ \cong \ A \ .$$

We obtain an orbispace as the composite

$$\mathbf{O}_{\mathrm{gl}}^{\mathrm{op}} \xrightarrow{\pi} \mathrm{Rep}^{\mathrm{op}} \xrightarrow{\underline{M}} \mathcal{A}b \xrightarrow{K(n,-)} \mathbf{T} .$$

We write

$$K(n, \underline{M}) \ = \ \Lambda((K(n,-) \circ \underline{M} \circ \pi)^c)$$

for the $\mathcal{L}$-space obtained from a cofibrant replacement of this composite orbispace. Then for every compact Lie group G, the G-fixed point space

$$K(n, \underline{M})^G$$

is an Eilenberg-Mac Lane space for the abelian group $\underline{M}(G)$, and moreover, the entire coefficient system $\underline{\pi}_n(K(n, \underline{M}))$ is isomorphic to the original coefficient system $\underline{M}$.

Construction 6.20. The fixed point Quillen equivalence can be used to push any continuous and functorial construction for spaces to $\mathcal{L}$-spaces. In more detail, let us consider a continuous functor

$$F \ : \ \mathbf{U} \ \longrightarrow \ \mathbf{U}$$

from the category of spaces to itself. Given an $\mathcal{L}$-space Y, we take its fixed point functor ΦY and postcompose it with F. The result is the continuous composite functor

$$F \circ (\Phi Y) \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ \mathbf{U} .$$

Then we take a cofibrant replacement $(F \circ (\Phi Y))^c \longrightarrow F \circ (\Phi Y)$ in the model category of orbispaces (which can be done functorially by the small object argument). The $\mathcal{L}$-space of the replacement gives a global space

$$\bar{F}(Y) \ = \ \Lambda((F \circ (\Phi Y))^c) .$$

We obtain a chain of two weak equivalences of orbispaces

$$F \circ (\Phi Y) \ \longleftarrow \ (F \circ (\Phi Y))^c \ \xrightarrow{\eta} \ \Phi(\bar{F}(Y)) ,$$

where η is the adjunction counit.

This shows:

Proposition 6.21. *Let $F : \mathbf{U} \longrightarrow \mathbf{U}$ be a continuous functor from the category of spaces to itself. Then there is a functor $\bar{F}$ from the category of $\mathcal{L}$-spaces to itself and a natural chain of weak equivalences of orbispaces*

$$F \circ (\Phi Y) \qquad \textit{and} \qquad \Phi(\bar{F}(Y)) .$$

We emphasize that the $\mathcal{L}$-space $\bar{F}(Y)$ is *not* in general obtained by applying F to the underlying space of Y with the induced $\mathcal{L}$-action, because F need not commute with fixed points of group actions. However, there is a natural map relating these two constructions. For every universal subgroup G of $\mathcal{L}$ the map $F(\mathrm{incl}) : F(Y^G) \longrightarrow F(Y)$ has image in $F(Y)^G$, and as G varies these maps define a morphism of orbispaces

$$\iota \ : \ F \circ (\Phi Y) \ \longrightarrow \ \Phi(F(Y)) .$$

Precomposition with the cofibrant replacement and forming of adjoint is a morphism of $\mathcal{L}$-spaces

$$\bar{F}(Y) \ = \ \Lambda((F \circ (\Phi Y))^c) \ \longrightarrow \ F(Y) .$$

7. Global families

In this section we explain a variant of unstable global homotopy theory based on a *global family*, i.e., a class of compact Lie groups with certain closure properties. This gives relative versions of level and global model structures.

Definition 7.1. A *global family* is a non-empty class of compact Lie groups that is closed under isomorphism, closed subgroups and quotient groups.

Some relevant examples of global families are: all compact Lie groups; all finite groups; all abelian compact Lie groups; all finite abelian groups; all topologically cyclic groups, all finite cyclic groups; all finite p-groups; all finite p'-groups; all finite solvable p-groups. Another example is the global family $\langle G\rangle$ generated by a compact Lie group G, i.e., the class of all compact Lie groups isomorphic to a quotient of a subgroup of G. A degenerate case of a global family is the class $\langle e\rangle$ of all trivial groups. In this case our theory specializes to the non-equivariant stable homotopy theory of orthogonal spaces.

For a global family $\mathcal{F}$ and a compact Lie group G we write $\mathcal{F}\cap G$ for the family of those closed subgroups of G that belong to $\mathcal{F}$. We also write $\mathcal{F}(m)$ for $\mathcal{F}\cap O(m)$, the family of closed subgroups of $O(m)$ that belong to $\mathcal{F}$. The following definitions of $\mathcal{F}$-level equivalences of orthogonal spaces are directly relativizations of the strong level equivalences.

Definition 7.2. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is

- an *$\mathcal{F}$-level equivalence* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-equivalence for all $m \geq 0$.
- an *$\mathcal{F}$-level fibration* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is an $\mathcal{F}(m)$-projective fibration for all $m \geq 0$.
- an *injective $\mathcal{F}$-fibration* if the map $f(\mathbb{R}^m) : X(\mathbb{R}^m) \longrightarrow Y(\mathbb{R}^m)$ is a mixed $\mathcal{F}(m)$-fibration for all $m \geq 0$.
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is an $\mathcal{F}(m)$-cofibration or all $m \geq 0$.

Every inner product space V is isometrically isomorphic to $\mathbb{R}^m$ with the standard scalar product, where m is the dimension of V. So if a morphism $f : X \longrightarrow Y$ of orthogonal spaces is an $\mathcal{F}$-level equivalence (respectively $\mathcal{F}$-level fibration or injective $\mathcal{F}$-fibration), then for every for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F}\cap G)$-equivalence (respectively $(\mathcal{F}\cap G)$-projective fibration or mixed $(\mathcal{F}\cap G)$-fibration). Clearly, the class of $\mathcal{F}$-level equivalences is closed under composition, retracts and coproducts.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then an $\mathcal{F}$-level equivalence is just a strong level equivalence in the sense of Definition 3.11. Moreover, the $\mathcal{A}ll$-level fibrations coincide with the injective $\mathcal{A}ll$-fibrations, and these specialize to the strong level fibrations. For the minimal global family $\langle e\rangle$ of all trivial groups, the notion of $\langle e\rangle$-level equivalence specializes to the non-equivariant level equivalences and the $\langle e\rangle$-level fibrations are the non-equivariant level fibrations.

Proposition 7.3. *Let $\mathcal{F}$ be a global family.*

(i) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is an $\mathcal{F}$-level equivalence. If in addition f or g is an h-cofibration, then the morphism k is an $\mathcal{F}$-level equivalence.

(ii) *Let $f_n : A_n \longrightarrow A_{n+1}$ be an $\mathcal{F}$-level equivalence and a closed embedding of orthogonal spaces, for $n \geq 0$. Then canonical morphism $f_\infty : A_0 \longrightarrow A_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is an $\mathcal{F}$-level equivalence.*

(iii) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is an $\mathcal{F}$-level equivalence. If in addition one of the morphisms f or h is an $\mathcal{F}$-level fibration, then the morphism g is also an $\mathcal{F}$-level equivalence.

Now we discuss $\mathcal{F}$-level model structure on orthogonal spaces. When $\mathcal{F}$ is not the maximal global family of all compact Lie groups, there are actually two different and useful level model structures, the *projective* and the *flat* one, in which the $\mathcal{F}$-level equivalences are the weak equivalences. Both model structures have global versions, and we need both of these model structures later for showing that the forgetful functor from the global homotopy category to the global $\mathcal{F}$-homotopy category has both a left and a right adjoint.

When $\mathcal{F} = \mathcal{A}ll$ is the global family of *all* compact Lie groups, then $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$. For this maximal global family the injective $\mathcal{A}ll$-fibrations coincide with the $\mathcal{A}ll$-level fibrations, which are just the strong level fibrations in the sense of Definition 3.11. Moreover, the $\mathcal{A}ll$-cofibrations coincide with the flat cofibrations. So for the global family of all compact Lie groups the projective and flat level model structure on orthogonal spaces coincide and specialize to the strong level model structure of Proposition 3.12.

Thus we have the following implications for the various kinds of cofibrations:

$$\langle e\rangle\text{-cofibration} \implies \mathcal{F}\text{-cofibration} \implies \text{flat cofibration} \implies \text{h-cofibration}$$

When $\mathcal{F}$ is not the minimal or the maximal global family, then the first two containments are strict. A word of warning: every flat cofibration is an $O(n)$-cofibration in level n, by Proposition 3.13 (i). When $\mathcal{F}$ is not the maximal global family, however, then $\mathcal{F}$-cofibrations are *not* in general levelwise $\mathcal{F}(m)$-cofibrations. A simple example is the constant one-point orthogonal space $\mathbf{1}$, which is $\langle e\rangle$-cofibrant, hence $\mathcal{F}$-cofibrant for every global family $\mathcal{F}$. However, if $\mathcal{F}$ does not consist of all compact Lie groups, then for some $m \geq 1$, the orthogonal group $O(m)$ does not belong to $\mathcal{F}$. Since $O(m)$ is the isotropy group of the point of $\mathbf{1}(\mathbb{R}^m)$, it is not $\mathcal{F}(m)$-cofibrant.

Proposition 7.4. *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a model structure, the* projective $\mathcal{F}$-level model structure*, on the category of orthogonal spaces.*

(ii) *The $\mathcal{F}$-level equivalences, injective $\mathcal{F}$-fibrations and flat cofibrations form a model structure, the* flat $\mathcal{F}$-level model structure*, on the category of orthogonal spaces.*

(iii) *Both $\mathcal{F}$-level model structures are proper, topological and cofibrantly generated.*

Proof. We specialize Proposition 3.9 in two ways.

(i) We let $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces, compare Proposition A.1.18. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 3.9 precisely become the $\mathcal{F}$-level equivalences, projective $\mathcal{F}$-fibrations and $\mathcal{F}$-cofibrations. Since every $\mathcal{F}$-cofibration is in particular a flat cofibration, the consistency condition (3.8) is implied by the stronger consistency condition for the flat $\mathcal{F}$-level model structure, compare (ii) below.

(ii) We let $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces, compare Proposition A.1.28. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 3.9 precisely become the $\mathcal{F}$-level equivalences, injective $\mathcal{F}$-fibrations and flat cofibrations. The consistency condition (3.8) is the special case of Proposition A.1.23 with $G = O(m+n)$, $K = O(m)$ and $A = \mathbf{L}(m, m+n)$ and the family $\mathcal{F} \cap G$ of subgroups of G. The proposition applies because the closure properties of a global family under passage to subgroups and quotient implies that $(\mathcal{F} \cap G) \ltimes K \subseteq \mathcal{F} \cap K$.

(iii) Limits in the category of orthogonal spaces are constructed levelwise (i.e., evaluation at level m preserves limits). Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. Both the $\mathcal{F}(m)$-projective and the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces are right proper for all $m \geq 0$, so right properness of the two $\mathcal{F}$-level model structures follows.

The argument for left properness is similar, but not completely analogous because cofibrations are *not* defined levelwise. It suffices to show left properness for the flat $\mathcal{F}$-level model structure, since the projective $\mathcal{F}$-level model structure has the same equivalences, but fewer cofibrations. Since flat cofibrations are levelwise $O(n)$-cofibrations (Proposition 3.13 (i)) and colimits in the category of orthogonal spaces are also constructed levelwise, left properness for the flat $\mathcal{F}$-level model structure is a consequence of left properness of the $\mathcal{F}(n)$-flat model structure on $O(n)$-spaces for all n.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the two $\mathcal{F}$-level model structure. We start with the projective $\mathcal{F}$-level model structure. We let $I_{\mathcal{F}}$ be the set of all morphism $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.19) of Section A.1. Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the projective $\mathcal{F}$-level model structure by Proposition 3.9 (iii).

Similarly, we let $J^{\mathrm{proj}}_{\mathcal{F}}$ be the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. Again by Proposition 3.9 (iii), $J^{\mathrm{proj}}_{\mathcal{F}}$ detects the fibrations in the projective $\mathcal{F}$-level model structure.

The cofibrations in the flat $\mathcal{F}$-level model structure are the flat cofibrations, which are independent of $\mathcal{F}$ and coincide with the $\mathcal{A}ll$-cofibrations, where $\mathcal{A}ll$ is the global family of all compact Lie groups. So the set $I_{\mathcal{A}ll}$ (which coincides with the set I^{str} of Proposition 3.12) is a set of generating cofibrations for the flat $\mathcal{F}$-level model structure (for any global family $\mathcal{F}$).

Finally, we let $J^{\mathrm{flat}}_{\mathcal{F}}$ be the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces specified in the proof of Proposition A.1.28. One more time by Proposition 3.9 (iii), $J^{\mathrm{flat}}_{\mathcal{F}}$ detects the fibrations in the flat $\mathcal{F}$-level model structure.

Since the $\mathcal{F}(m)$-projective and $\mathcal{F}(m)$-flat model structures on $O(m)$-spaces are topological, part (iv) of Proposition 3.9 shows that the projective and flat $\mathcal{F}$-level model structures are topological. □

Now we proceed towards the construction of two *$\mathcal{F}$-global model structures* for every given global family $\mathcal{F}$, see Theorems 7.13 and 7.14 below. The weak equivalences in both of these model structures are the $\mathcal{F}$-equivalences of the following definition, the direct generalization of global equivalences in the presence of a global family.

Definition 7.5. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is an *$\mathcal{F}$-equivalence* if the following condition holds: for every compact Lie group G in $\mathcal{F}$, every G-representation V, every $k \geq 0$ and all maps $\alpha : S^{k-1} \longrightarrow X(V)^G$ and $\beta : D^k \longrightarrow Y(V)^G$ such that $f(V)^G \circ \alpha = \beta|_{S^{k-1}} : S^{k-1} \longrightarrow Y(V)^G$ there is a G-representation W and a continuous map $\lambda : D^k \longrightarrow X(V \oplus W)^G$ such that $\lambda_{S^{k-1}} = X(i_{V,W})^G \circ \alpha : S^{k-1} \longrightarrow X(V \oplus W)^G$ and such that $f(V \oplus W)^G \circ \lambda : D^k \longrightarrow Y(V \oplus W)^G$ is homotopic, relative to S^{k-1}, to $Y(i_{V,W}) \circ \beta$.

When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups, then an $\mathcal{A}ll$-equivalence is a global equivalence. In fact, the only difference between the previous definition and Definition 1.2 is that the compact Lie groups that come up are required to Lie in the global family $\mathcal{F}$. For the minimal global family $\langle e \rangle$ of all trivial groups, the notion of $\langle e \rangle$-equivalence specializes to the non-equivariant weak equivalences of orthogonal spaces, i.e., the morphisms that become (non-equivariant) weak equivalences after taking homotopy colimit over the indexing category $\mathbf{L}$. The following diagram collects various notions of equivalences and their implications:

$$\begin{tikzcd} \text{homotopy equivalence} \arrow[r, Rightarrow] & \text{strong level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{global equivalence} \arrow[d, Rightarrow] \\ & \mathcal{F}\text{-level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{F}\text{-equivalence} \arrow[d, Rightarrow] \\ & \text{level equivalence} \arrow[r, Rightarrow] & \text{weak equivalence} \end{tikzcd}$$

The next proposition contains various properties of $\mathcal{F}$-equivalences that generalize certain parts of Proposition 4.7.

Proposition 7.6. *Let $\mathcal{F}$ be a global family.*

(i) *If f and g are composable morphisms of orthogonal spaces and two of the three morphisms f, g and gf are $\mathcal{F}$-equivalences, then so is the third.*
(ii) *Every retract of an $\mathcal{F}$-equivalence is an $\mathcal{F}$-equivalence.*
(iii) *A coproduct of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalences.*
(iv) *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spaces such that f is an $\mathcal{F}$-equivalence. If in addition f or g is an h-cofibration, then the morphism k is an $\mathcal{F}$-equivalence.
(v) *Let $f_n : A_n \longrightarrow A_{n+1}$ be an $\mathcal{F}$-equivalence and a closed embedding of orthogonal spaces, for $n \geq 0$. Then canonical morphism $f_\infty : A_0 \longrightarrow A_\infty$ to the colimit of the sequence $\{f_n\}_{n\geq 0}$ is an $\mathcal{F}$-equivalence.*
(vi) *Let*

$$\begin{tikzcd} P \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Z \arrow[r, "h"'] & Y \end{tikzcd}$$

be a pullback square of orthogonal spaces in which f is an $\mathcal{F}$-equivalence. If in addition one of the morphisms f or h is an $\mathcal{F}$-level fibration, then the morphism g is also an $\mathcal{F}$-equivalence.
(vii) *Every $\mathcal{F}$-level equivalence is an $\mathcal{F}$-equivalence.*
(viii) *Every $\mathcal{F}$-equivalence that is also a global fibration is an $\mathcal{F}$-level equivalence.*
(ix) *A morphism of orthogonal spaces is an $\mathcal{F}$-equivalence if and only if it can be written as $w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*

PROOF. The proofs of (i) through (viii) are almost verbatim the same as the corresponding parts of Proposition 4.7, and we omit them.

(ix) The $\mathcal{F}$-equivalences contain the global equivalences by definition and the $\mathcal{F}$-level equivalences by part (vii), and are closed under composition by (i), so all composites $w_2 \circ w_1$ as in the proposition are $\mathcal{F}$-equivalences. On the other hand, every $\mathcal{F}$-equivalence f can be factored in the global model structure of Theorem 4.3 as $f = qj$ where j is a global equivalence and q is a global fibration. Since f and j are $\mathcal{F}$-equivalences, so is q by part (i); so q is an $\mathcal{F}$-equivalence and a global fibration, hence an $\mathcal{F}$-level equivalence by part (viii). □

Now we construct the flat and the projective $\mathcal{F}$-global model structures on the category of orthogonal spaces. When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family, then the flat and the projective $\mathcal{A}ll$-global model structure coincide and specialize to the global model structure of Theorem 4.3. Rather than constructing the $\mathcal{F}$-global model structures 'by hand', we obtain them by the general techniques of 'merging' and 'mixing' model structures, with the two $\mathcal{F}$-level model structures and the global model structure as starting points.

Construction 7.7. We introduce a technique to merge two model structures on the same category that share the same cofibrations, but whose weak equivalences may be incomparable. We let $\mathcal{A}$ be a category equipped with two model structures $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C})$ and $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C})$ that share the same class $\mathcal{C}$ of cofibrations. Then the acyclic fibrations also coincide, i.e.,

$$(\mathcal{W}_1 \cap \mathcal{F}_1) \ = \ (\mathcal{W}_2 \cap \mathcal{F}_2) \ .$$

We identify sufficient conditions for the existence of a 'merged model structure' $(\mathcal{W}, \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{C})$, again with $\mathcal{C}$ as the class of cofibrations, and with new fibrations the intersection $\mathcal{F}_1 \cap \mathcal{F}_2$ of the two original classes of fibrations. The possibility that $\mathcal{W}_1 \subset \mathcal{W}_2$ is allowed, but then $\mathcal{F}_2 \subset \mathcal{F}_1$, hence $\mathcal{F}_1 \cap \mathcal{F}_2 = \mathcal{F}_2$, and we get nothing new.

We define the *merged equivalences* as the class $\mathcal{W}$ of those $\mathcal{A}$-morphisms that can be written as a composite $w_2 \circ w_1$ with $w_2 \in \mathcal{W}_2$ and $w_1 \in \mathcal{W}_1$. We suppose that the model structures satisfy the following three conditions:

(a) The class $\mathcal{W}$ of merged equivalences is closed under retracts and has the 2-out-of-3 property.
(b) $\mathcal{W} \cap \mathcal{F}_1 \subset \mathcal{W}_2$.
(c) Let $f \in \mathcal{F}_2$ and $k \in \mathcal{W}_2$ be morphisms such that $fk \in \mathcal{F}_1$. Then $f \in \mathcal{F}_1$.

In the following proposition we refer to $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C})$ as the '1-model structure' and to $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C})$ as the '2-model structure' on $\mathcal{A}$.

Proposition 7.8 (Merging model structures)**.** *Let $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C})$ and $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C})$ be model structures on the same category $\mathcal{A}$ with the same class $\mathcal{C}$ of cofibrations that satisfy conditions* (a), (b) *and* (c) *above.*

(i) *The triple $(\mathcal{W}, \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{C})$ is a model structure on $\mathcal{A}$, the* merged model structure.
(ii) *If the 1-model structure and the 2-model structure are left proper, then so is the merged model structure.*
(iii) *Every acyclic cofibration in the merged model structure is a retract of a composite of the form $k \circ j$ with $k \in \mathcal{W}_2 \cap \mathcal{C}$ and $j \in \mathcal{W}_1 \cap \mathcal{C}$.*
(iv) *If the 1-model structure and the 2-model structure are right proper, then so is the merged model structure.*
(v) *If J_1 is a set of generating acyclic cofibrations for the 1-model structure and J_2 is a set of generating acyclic cofibrations for the 2-model structure, then $J_1 \cup J_2$ is a set of generating acyclic cofibrations for the merged model structure.*
(vi) *If the 1-model structure and the 2-model structure are cofibrantly generated, then so is the merged model structure.*
(vii) *If the 1-model structure and the 2-model structure are topological, then so is the merged model structure.*

Proof. (i) Our first claim is the relation

$$\mathcal{W} \cap \mathcal{F}_1 \cap \mathcal{F}_2 \ = \ \mathcal{W}_2 \cap \mathcal{F}_2 \ , \tag{7.9}$$

i.e., the acyclic fibrations in the model structure to be established coincide with the acyclic fibrations in the 2-model structure. (and hence also with the acyclic fibrations in the 1-model structure). Indeed, intersecting relation (b) with $\mathcal{F}_2$ gives one of the containments. On the other hand, $\mathcal{W}_2 \cap \mathcal{F}_2 = \mathcal{W}_1 \cap \mathcal{F}_1 \subset \mathcal{F}_1$ and $\mathcal{W}_2 \cap \mathcal{F}_2 \subset \mathcal{W} \cap \mathcal{F}_2$ by definition, so this gives the other containment. Because of the relation (7.9) the halves of the factorization axiom MC4 and of the lifting axiom MC5 that refer to cofibrations and acyclic fibrations are taken care of by the 2-model structure.

Now we factor any given morphism f as a morphism in $\mathcal{W}\cap\mathcal{C}$ followed by a morphism in $\mathcal{F}_1\cap\mathcal{F}_2$. We start by factoring $f = pj$ in the 1-model structure with $j \in \mathcal{W}_1\cap\mathcal{C}$ and $p \in \mathcal{F}_1$. Then we factor $p = rk$ in the 2-model structure with $k \in \mathcal{W}_2\cap\mathcal{C}$ and $r \in \mathcal{F}_2$. We claim that $f = r\circ(kj)$ is the desired factorization in new model structure. Indeed, $kj \in \mathcal{C}$ since $\mathcal{C}$ is closed under composition, and $kj \in \mathcal{W}$ by definition. The morphism r belongs to $\mathcal{F}_2$ by construction, and to $\mathcal{F}_1$ by hypothesis (c).

Before we attack the other lifting property, we prove (ii). We consider $f \in \mathcal{W}\cap\mathcal{C}$ and factor it as in the previous paragraph as $f = rkj$ with $j \in \mathcal{W}_1\cap\mathcal{C}$, $k \in \mathcal{W}_2\cap\mathcal{C}$ and $r \in \mathcal{F}_1\cap\mathcal{F}_2$. Since f and kj belong to the class $\mathcal{W}$, so does the morphism r, by the 2-out-of-3 property (a). Thus $r \in \mathcal{W}_2\cap\mathcal{F}_2$ by (7.9) and the pair (f, r) has the lifting property by the 2-model structure. A lifting in

$$\begin{tikzcd} A \arrow[r, "kj"] \arrow[d, "f"'] & W \arrow[d, "r"] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exhibits f as a retract of the morphism kj.

Now we can easily establish the remaining lifting property. Given $f \in \mathcal{W}\cap\mathcal{C}$ we write it as a retract of a composite $k\circ j$ with $k \in \mathcal{W}_2\cap\mathcal{C}$ and $j \in \mathcal{W}_1\cap\mathcal{C}$. Since k has the left lifting property with respect to $\mathcal{F}_2$ and j has the left lifting property with respect to $\mathcal{F}_1$, the composite kj has the left lifting property with respect to $\mathcal{F}_1\cap\mathcal{F}_2$; so the retract f of kj also has this lifting property.

(iii) For left properness we consider two morphisms $f \in \mathcal{W}$ and $i \in \mathcal{C}$ with common source. We write $f = w_2\circ w_1$ with $w_2 \in \mathcal{W}_2$ and $w_1 \in \mathcal{W}_1$. The cobase change w_1' of w_1 along i is in $\mathcal{W}_1$ by left properness of the 1-model structure. We let j denote the cobase change of i along w_1, which is another cofibration. Then the cobase change w_2' of w_2 along j is in $\mathcal{W}_2$ by left properness of the 2-model structure. The cobase change of $f = w_2\circ w_1$ along i can be calculated in two steps as the composite $w_2'\circ w_1'$, and is thus in the class $\mathcal{W}$. So the merged model structure is left proper.

(iv) The argument for right properness is similar to left properness. We consider two morphisms $f \in \mathcal{W}$ and $p \in \mathcal{F}_1\cap\mathcal{F}_2$ with common target and write $f = w_2\circ w_1$ with $w_2 \in \mathcal{W}_2$ and $w_1 \in \mathcal{W}_1$. The base change $\bar{w}_2$ of w_2 along p is in $\mathcal{W}_2$ by right properness of the 2-model structure. We let q denote the base change of p along w_2, which is again in $\mathcal{F}_1\cap\mathcal{F}_2$. Then the base change $\bar{w}_1$ of w_1 along q is in $\mathcal{W}_1$ by right properness of the 1-model structure. The base change of $f = w_2\circ w_1$ along p can be calculated in two steps as the composite $\bar{w}_2\circ\bar{w}_1$, and is thus in the class $\mathcal{W}$. So the merged model structure is right proper.

Part (v) is obvious from the definition of the merged fibrations as the intersection $\mathcal{F}_1\cap\mathcal{F}_2$. Part (vi) is the combination of part (v) and the fact that the merged cofibrations are the same as the cofibrations in the 1- and 2-model structure.

Part (vii) is also fairly straightforward. We let $f : K \longrightarrow L$ be a cofibration (respectively acyclic cofibration) of spaces and $i : A \longrightarrow C$ be a cofibration in $\mathcal{A}$. Then the pushout product $f\Box i$ is another cofibration (respectively acyclic cofibration) since the 1-model structure is topological. If $i = kj$ is a composite with $j \in \mathcal{W}_1\cap\mathcal{C}$ and $k \in \mathcal{W}_2\cap\mathcal{C}$, then the diagram

$$\begin{tikzcd} L\times A\cup_{K\times A} K\times B \arrow[r, "(L\times A)\cup(K\times j)"] \arrow[d, "f\Box j"'] & L\times A\cup_{K\times A} K\times C \arrow[r, "f\Box(kj)"] \arrow[d, "(L\times j)\cup(K\times C)"'] & L\times C \\ L\times B \arrow[r] & L\times B\cup_{K\times B} K\times C \arrow[ur, "f\Box k"'] & \end{tikzcd}$$

commutes and the left square is a pushout. The pushout product $f\Box j$ belongs to $\mathcal{W}_1\cap\mathcal{C}$ because the 1-model structure is topological; so its cobase change $(L\times j)\cup(K\times C)$ belongs to $\mathcal{W}_1\cap\mathcal{C}$. Similarly, the pushout product $f\Box k$ belongs to $\mathcal{W}_2\cap\mathcal{C}$ because the 2-model structure is topological. So the pushout product $f\Box i = f\Box(kj)$ belongs to $\mathcal{W}\cap\mathcal{C}$, i.e., is a merged acyclic cofibration.

As we showed in part (i), a general merged acyclic cofibration is a retract of a composite kj as in the previous paragraph. Since $f\Box i$ is then a retract of $f\Box(kj)$, $f\Box i$ is again a merged acyclic cofibration $\Box$

Now we merge the global model structure (Theorem 4.3) and the flat $\mathcal{F}$-level model structure (Proposition 7.4 (ii)) into the flat $\mathcal{F}$-global model structure. So we apply Proposition 7.8 to the category $\mathcal{A} = \mathit{spc}$ of orthogonal spaces, we let $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C})$ be the global model structure and we let $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C})$ be the flat $\mathcal{F}$-level model structure, which share the flat cofibrations. Then Proposition 7.6 (v) identifies the merged equivalences $\mathcal{W}$ with the class of $\mathcal{F}$-equivalences. We need to check the hypotheses of Proposition 7.8: The $\mathcal{F}$-equivalences are closed under retracts and have the 2-out-of-3 property by parts (i) and (ii) of Proposition 7.6. Every $\mathcal{F}$-equivalence that is also a global fibration is an $\mathcal{F}$-level equivalence by Proposition 7.6 (iv), which shows (b). Finally, hypothesis (c) is the content of the following proposition.

Proposition 7.10. *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ an injective $\mathcal{F}$-fibration of orthogonal spaces. If k is an $\mathcal{F}$-level equivalence such that fk is a global fibration, then f is a global fibration.*

Proof. An injective $\mathcal{F}$-fibration is in particular a strong level fibration, so it remains to show that the squares

$$\begin{tikzcd}
X(V)^G \arrow[r, "X(i_{V,W})^G"] \arrow[d, "f(V)^G"'] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\
Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G
\end{tikzcd} \tag{7.11}$$

are homotopy cartesian for all compact Lie groups G, all faithful G-representations V and all other G-representations W. The square

$$\begin{tikzcd}
Z(V)^G \arrow[r, "Z(i_{V,W})^G"] \arrow[d, "(fk)(V)^G"'] & Z(V\oplus W)^G \arrow[d, "(fk)(V\oplus W)^G"] \\
Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G
\end{tikzcd} \tag{7.12}$$

is homotopy cartesian because fk is a global fibration. When G belongs to $\mathcal{F}$, then the maps $k(V)^G$, $k(V\oplus W)^G$ and the identities of $Z(V)^G$ and $Z(V\oplus W)^G$ are weak equivalences, and they provide a morphism from the square (7.11) to (7.12); so the square (7.11) is also homotopy cartesian.

When G does not belong to $\mathcal{F}$, then we consider the commutative cube

$$\begin{tikzcd}
X(V)^G \arrow[rr, "X(i_{V,W})^G"] \arrow[dr] \arrow[dd, "f(V)^G"'] & & X(V\oplus W)^G \arrow[dr] \arrow[dd, "f(V\oplus W)^G" near end] & \\
& \mathrm{map}^G(E(\mathcal{F}\cap G), X(V)) \arrow[rr, "\mathrm{map}^G(E(\mathcal{F}\cap G), X(i_{V,W}))" near start] \arrow[dd, "\mathrm{map}^G(E(\mathcal{F}\cap G), f(V))" near end] & & \mathrm{map}^G(E(\mathcal{F}\cap G), X(V\oplus W)) \arrow[dd, "\mathrm{map}^G(E(\mathcal{F}\cap G), f(V\oplus W))"] \\
Y(V)^G \arrow[rr, "Y(i_{V,W})^G" near start] \arrow[dr] & & Y(V\oplus W)^G \arrow[dr] & \\
& \mathrm{map}^G(E(\mathcal{F}\cap G), Y(V)) \arrow[rr, "\mathrm{map}^G(E(\mathcal{F}\cap G), Y(i_{V,W})"'] & & \mathrm{map}^G(E(\mathcal{F}\cap G), Y(V\oplus W))
\end{tikzcd}$$

The left and right faces of the cube are homotopy cartesian because f is an injective $\mathcal{F}$-fibration. The square

$$\begin{tikzcd} X(V) \arrow[r, "X(i_{V,W})"] \arrow[d, "f(V)"'] & X(V\oplus W) \arrow[d, "f(V\oplus W)"] \\ Y(V) \arrow[r, "Y(i_{V,W})"'] & Y(V\oplus W) \end{tikzcd}$$

is $(\mathcal{F}\cap G)$-homotopy cartesian by the first part, so applying $\mathrm{map}^G(E(\mathcal{F}\cap G),-)$ takes it to a homotopy cartesian square in the non-equivariant sense. So the front face of the cube above is also homotopy cartesian. Since the front, left and right faces of the cube are homotopy cartesian, so is the back face, and so f is a global fibration. $\square$

Now we have verified all the hypotheses needed to merge the global model structure with the flat $\mathcal{F}$-level model structure. Proposition 7.8 thus applies and yields:

Theorem 7.13 (Flat $\mathcal{F}$-global model structure). *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences and flat cofibrations are part of a model structure, the* flat $\mathcal{F}$-global model structure *on the category of orthogonal spaces.*
(ii) *A morphism is a fibration in the flat $\mathcal{F}$-global model structure precisely when it is both an injective $\mathcal{F}$-fibration and a global fibration.*
(iii) *The fibrant objects in the flat $\mathcal{F}$-global model structure are the $\mathcal{F}$-injective static orthogonal spaces.*
(iv) *Every acyclic cofibration in the flat $\mathcal{F}$-global model structure is a retract of a composite $k\circ j$ with k a flat cofibration and $\mathcal{F}$-level equivalence and j a flat cofibration and global equivalence.*
(v) *The flat $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Now we establish the projective $\mathcal{F}$-global model structures on the category of orthogonal spaces. One way to prove the following theorem is to mimic the proof in the special case $\mathcal{F}=\mathcal{A}ll$, and all arguments in the proof of Theorem 4.3 go through almost verbatim. We present an alternative approach by 'mixing' the projective $\mathcal{F}$-level model structure with the flat $\mathcal{F}$-global model structure.

Theorem 7.14 (Projective $\mathcal{F}$-global model structure). *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations are part of a model structure on the category of orthogonal spaces, the* projective $\mathcal{F}$-global model structure.
(ii) *A morphism $f: X\longrightarrow Y$ of orthogonal spaces is a fibration in the projective $\mathcal{F}$-global model structure precisely when it is an $\mathcal{F}$-level fibration and for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and an arbitrary G-representation W the square of G-fixed point spaces*

$$\begin{tikzcd} X(V)^G \arrow[r, "X(i_{V,W})^G"] \arrow[d, "f(V)^G"'] & X(V\oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\ Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V\oplus W)^G \end{tikzcd} \tag{7.15}$$

is homotopy cartesian.
(iii) *The fibrant objects in the projective $\mathcal{F}$-global model structure are the* $\mathcal{F}$-static orthogonal spaces, *i.e., those orthogonal spaces X such that for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and an arbitrary G-representation W the map of G-fixed point spaces*

$$X(i_{V,W})^G \;:\; X(V)^G \;\longrightarrow\; X(V\oplus W)^G$$

is a weak equivalence.

(iv) *The projective $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

PROOF. (i) We apply Cole's 'mixing theorem' [**25**, Thm. 2.1] to the category of orthogonal spaces with $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C}_1)$ the projective $\mathcal{F}$-level model structure and with $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C}_2)$ the flat $\mathcal{F}$-global model structure. Every $\mathcal{F}$-level equivalence is an $\mathcal{F}$-equivalence and every $\mathcal{F}$-cofibration is a flat cofibration, i.e., $\mathcal{W}_1 \subseteq \mathcal{W}_2$ and $\mathcal{C}_1 \subseteq \mathcal{C}_2$. Cole's theorem provides a 'mixed' model structure $(\mathcal{W}_2, \mathcal{F}_m, \mathcal{C}_1)$ consisting of the $\mathcal{F}$-equivalences, a certain class $\mathcal{F}_m$ of 'mixed fibrations', and the $\mathcal{F}$-cofibrations.

(ii) By [**25**, Prop. 3.6] (or rather the dual formulation) a morphism $f : X \longrightarrow Y$ of orthogonal spaces is a mixed fibration if and only if it is an $\mathcal{F}$-level fibration and it can be factored as a composite $f = f' \circ \xi$ where $f' \in \mathcal{F}_2$ is a fibration in the flat $\mathcal{F}$-global model structure (i.e., an injective $\mathcal{F}$-fibration and a global fibration) and ξ is an $\mathcal{F}$-level equivalence. Our claim follows because being $\mathcal{F}$-level equivalent to a global fibration precisely means that the squares (7.15) are homotopy cartesian for those compact Lie groups G that belong to $\mathcal{F}$.

(iii) As a special case of (ii) (or by an application of the dual formulation of [**25**, Cor. 3.7]), an orthogonal space is fibrant in the mixed model structure if and only if it is $\mathcal{F}$-level fibrant (an empty condition) and $\mathcal{F}$-level equivalent to a fibrant object in the flat $\mathcal{F}$-global model structure, i.e., an $\mathcal{F}$-injective static orthogonal space. But these are precisely the $\mathcal{F}$-static orthogonal spaces.

(iv) Properness of the mixed model structure is a direct consequence of properness of the flat $\mathcal{F}$-global model structure, compare Proposition 4.1 and 4.2 of [**25**]. Similarly, the mixed model structure is topological because both the projective $\mathcal{F}$-level and the flat $\mathcal{F}$-global model structure are topological. □

For easier reference we spell out explicit sets of generating cofibrations and generating acyclic cofibrations for the flat and projective $\mathcal{F}$-model structures. In Proposition 3.12 we defined we defined J^{str} as the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. The set J^{str} detects the fibrations in the strong level model structure.

In Proposition 7.4 we introduced $I_{\mathcal{F}}$ as the set of all morphism $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.19) of Section A.1. The set $I_{\mathcal{F}}$ detects the acyclic fibrations in the projective $\mathcal{F}$-level model structure, which coincide with the acyclic fibrations in the projective $\mathcal{F}$-global model structure. In particular, the set $I_{\mathcal{A}ll}$, which was denoted I^{str} in Proposition 3.12, detects the acyclic fibrations in the strong level model structure, which are also the acyclic fibrations in the flat $\mathcal{F}$-level model structure.

Also in Proposition 7.4 we defined $J^{\mathrm{flat}}_{\mathcal{F}}$ as the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces specified in the proof of Proposition A.1.28. The set $J^{\mathrm{flat}}_{\mathcal{F}}$ detects the fibrations in the flat $\mathcal{F}$-level model structure, i.e., the injective $\mathcal{F}$-fibrations. Similarly, $J^{\mathrm{proj}}_{\mathcal{F}}$ is the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. The set $J^{\mathrm{proj}}_{\mathcal{F}}$ detects the fibrations in the projective $\mathcal{F}$-level model structure.

Similarly, We add another set of morphism $K_{\mathcal{F}}$ that detects when the squares (7.15) are homotopy cartesian for $G \in \mathcal{F}$. We set

$$K_{\mathcal{F}} \;=\; \bigcup_{G,V,W \,:\, G \in \mathcal{F}} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $S^{n-1} \longrightarrow D^n$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G in $\mathcal{F}$, a faithful G-representation V and an arbitrary G-representation W. By Proposition 4.9, the right lifting property with respect to the union $J^{\mathrm{proj}}_{\mathcal{F}} \cup K_{\mathcal{F}}$ thus characterizes the fibrations in the projective $\mathcal{F}$-global model structure. In particular, the set $J^{\mathrm{proj}}_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$, which was denoted $J^{\mathrm{str}} \cup K$ in Proposition 4.13, detects the global fibrations.

So altogether we have shown:

Proposition 7.16. *Let $\mathcal{F}$ be a global family. Then a morphism of orthogonal spaces is:*

(i) *an acyclic fibration in the flat $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{A}ll} = I^{\mathrm{str}}$;*
(ii) *a fibration in the flat $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J^{\mathrm{flat}}_{\mathcal{F}} \cup K_{\mathcal{A}ll}$,*
(iii) *an acyclic fibration in the projective $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$;*
(iv) *a fibration in the projective $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J^{\mathrm{proj}}_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

Example 7.17. In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-global homotopy theory of orthogonal spaces just another model for the (non-equivariant) homotopy theory of spaces. Indeed, the evaluation functor $\mathrm{ev}_0 : spc \longrightarrow \mathbf{U}$ is a right Quillen equivalence with respect to the projective $\langle e \rangle$-global model structure, and it is a left Quillen equivalence with respect to the flat $\langle e \rangle$-global model structure. So either of these Quillen equivalences shows that the derived functor

$$\mathrm{Ho}(\mathrm{ev}_0) \ : \ \mathrm{Ho}(spc) \ \longrightarrow \mathrm{Ho}(\mathbf{U})$$

is an equivalence of homotopy categories.

In fact, for the global family $\mathcal{F} = \langle e \rangle$, most of what we do here has already been studied before: The projective $\langle e \rangle$-global model structure and the fact that it is Quillen equivalent to the model category of spaces were established by Lind [**54**, Thm. 1.1]; in [**54**], orthogonal spaces are called '$\mathcal{I}$-spaces' and $\langle e \rangle$-global equivalences are called 'weak homotopy equivalences' and are defined as those morphisms that induce weak equivalences on homotopy colimits.

Corollary 7.18. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spaces and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*
(ii) *For some (hence any) flat approximation $f^\flat : A^\flat \longrightarrow B^\flat$ in the flat $\mathcal{F}$-level model structure and every $\mathcal{F}$-injective static orthogonal space X the induced map*

$$[f^\flat, X] \ : \ [B^\flat, X] \ \longrightarrow \ [A^\flat, X]$$

on homotopy classes of morphisms is a bijection.
(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the projective $\mathcal{F}$-level model structure and every $\mathcal{F}$-static orthogonal space Y the induced map*

$$[f^c, Y] \ : \ [B^c, Y] \ \longrightarrow \ [A^c, Y]$$

on homotopy classes of morphisms is a bijection.

Proof. (i)$\Longleftrightarrow$(ii) The morphism f is an $\mathcal{F}$-equivalence if and only if the flat approximation $f^\flat : A^\flat \longrightarrow B^\flat$ is. Since $A^\flat$ and $B^\flat$ are flat, they are cofibrant in the flat $\mathcal{F}$-global model structure. So by general model category theory $f^\flat$ is an $\mathcal{F}$-equivalence if and only if the induced map $[f^\flat, X]$ is bijective for every fibrant object in the flat $\mathcal{F}$-global model structure. By Theorem 7.13 (iii) these fibrant objects are precisely the $\mathcal{F}$-injective static orthogonal spaces.

(i)$\Longleftrightarrow$(iii) The morphism f is an $\mathcal{F}$-equivalence if and only if the $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ is an $\mathcal{F}$-equivalence. Since A^c and B^c are $\mathcal{F}$-cofibrant, they are cofibrant in the projective $\mathcal{F}$-global model structure. So by general model category theory, f^c is an $\mathcal{F}$-equivalence if and only if the induced map $[f^c, X]$ is bijective for every fibrant object in the projective $\mathcal{F}$-global model structure. By Theorem 7.14 (iii) these fibrant objects are precisely the $\mathcal{F}$-static orthogonal spaces. $\square$

Remark 7.19 (Mixed global model structures)**.** Cole's 'mixing theorem' for model structures [**25**, Thm. 2.1] allows to construct many more $\mathcal{F}$-model structures on the category of orthogonal spaces. We will concentrate on the 'mixed' $\mathcal{F}$-global model structures, but the same kind of mixing can also be performed with the $\mathcal{F}$-level model structures.

We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the projective $\mathcal{E}$-global model structure is a fibration in the projective $\mathcal{F}$-global model structure. By Cole's theorem [**25**, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the projective $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spaces. By [**25**, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts of all composite $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal space is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal space [**25**, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [**25**]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structure on the category of orthogonal spaces that are all Quillen equivalent to the model category of (non-equivariant) spaces, with respect to weak equivalences.

The next topic is the compatibility of the $\mathcal{F}$-level and $\mathcal{F}$-global model structures with the box product of orthogonal spaces. We let $\mathcal{E}$ and $\mathcal{F}$ be two global families. We denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$. So a compact Lie group H belongs to $\mathcal{E} \times \mathcal{F}$ if and only if H is isomorphic to a subgroup of a group of the form $(G \times K)/N$ for some groups $G \in \mathcal{E}$ and $K \in \mathcal{F}$, and some closed normal subgroup N of $G \times K$.

Proposition 7.20. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of an $\mathcal{E}$-cofibration that is also an $\mathcal{E}$-level equivalence with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-level equivalence.*
(iii) *The pushout product of a flat cofibration that is also an $\mathcal{E}$-level equivalence with a flat cofibration is an $\mathcal{E}$-level equivalence.*
(iv) *The flat $\mathcal{F}$-level model structure and the flat $\mathcal{F}$-global model structure satisfy the pushout product property with respect to the box product of orthogonal spaces.*
(v) *Suppose that the global family $\mathcal{F}$ is also closed under products, i.e., if $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. Then the projective $\mathcal{F}$-level model structure and the projective $\mathcal{F}$-global model structure satisfy the pushout product property with respect to the box product of orthogonal spaces.*

Proof. (i) It suffices to show the claim for a set of generating cofibrations. The $\mathcal{E}$-cofibrations are generated by the morphisms

$$\mathbf{L}_{G,V} S^{n-1} \longrightarrow \mathbf{L}_{G,V} D^n$$

for $G \in \mathcal{E}$, V a G-representation and $n \geq 0$. Similarly, the $\mathcal{F}$-cofibrations are generated by the morphisms

$$\mathbf{L}_{K,W} S^{m-1} \longrightarrow \mathbf{L}_{K,W} D^m \tag{7.21}$$

for $K \in \mathcal{F}$, W a K-representation and $m \geq 0$. The pushout product of two such generators is isomorphic to the map

$$\mathbf{L}_{G\times K, V\oplus W}(S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1}) \longrightarrow \mathbf{L}_{G\times K, V\oplus W}(D^n \times D^m) ;$$

here $G \times K$ acts on $V \oplus W$ by $(g,k)\cdot(v,w) = (gv, kw)$. Since $G \times K$ belongs to the family $\mathcal{E} \times \mathcal{F}$ and the inclusion of $S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1}$ into $D^n \times D^m$ is a cofibration of spaces, this pushout product morphism is an $(\mathcal{E} \times \mathcal{F})$-cofibration.

(ii) It suffices to show that the pushout product of any generating acyclic cofibration in the $\mathcal{E}$-projective level model structure with any generating $\mathcal{F}$-cofibration is an acyclic cofibration in the $(\mathcal{E} \times \mathcal{F})$-projective level model structure. By part (i) we already know the $(\mathcal{E} \times \mathcal{F})$-cofibration property, so it remains to show the $(\mathcal{E} \times \mathcal{F})$-level equivalence property.

The acyclic cofibrations in the $\mathcal{E}$-projective level model structure are generated by the morphisms

$$\mathbf{L}_{G,V}(\{0\} \times D^n) \longrightarrow \mathbf{L}_{G,V}([0,1] \times D^n) \tag{7.22}$$

for $G \in \mathcal{E}$, V a G-representation and $n \geq 0$. The pushout product of such a generator with a generating $\mathcal{F}$-cofibration (7.21) is isomorphic to the morphism $\mathbf{L}_{G,W} i$, where i is the inclusion

$$\{0\} \times D^n \times D^m \cup_{\{0\} \times D^n \times S^{m-1}} [0,1] \times D^n \times S^{m-1} \longrightarrow [0,1] \times D^n \times D^m \ .$$

Since i is a homotopy equivalence of spaces, the morphism $\mathbf{L}_{G \times K, V \oplus W} i$ is a homotopy equivalence of orthogonal spaces, so in particular an $(\mathcal{E} \times \mathcal{F})$-level equivalence.

(iii) Again it suffices to show that the pushout product of a generating acyclic cofibration for the $\mathcal{E}$-flat level model structure with any the generating flat cofibration (the morphisms (7.21) for all compact Lie groups K) is an acyclic cofibration for the $\mathcal{E}$-flat level model structure. The generating acyclic cofibrations for the $\mathcal{E}$-flat level model structure come in two flavors:

(a) The generating acyclic cofibrations for the $\mathcal{E}$-projective level model structure (7.22); the pushout product of such a map with a generating flat cofibration is a flat cofibration (by part (i) for $\mathcal{E} = \mathcal{F} = \mathcal{A}ll$) and a homotopy equivalence (compare part (ii)).

(b) The pushout product of the cone inclusion

$$\mathbf{L}_{K,W}(\iota) \ : \ \mathbf{L}_{K,W} EK \longrightarrow \mathbf{L}_{K,W} C(EK)$$

for $K \notin \mathcal{F}$ and W a faithful K-representation, with the inclusion $S^{n-1} \longrightarrow D^n$. The pushout product of such a map with a generating flat cofibration is the pushout product of the morphism

$$\mathbf{L}_{G \times K, V \oplus W}(p^* \iota) \ : \ \mathbf{L}_{G \times K, V \oplus W}(p^* EK) \longrightarrow \mathbf{L}_{G \times K, V \oplus W}(p^* C(EK)) \ ,$$

where $p : G \times K \longrightarrow K$ is the projection, with the relative CW-inclusion

$$S^{n-1} \times D^m \cup_{S^{n-1} \times S^{m-1}} D^n \times S^{m-1} \longrightarrow D^n \times D^m \ .$$

Since the $\mathcal{E}$-flat level model structure is topological it suffices to show that $\mathbf{L}_{G \times K, V \oplus W}(p^* \iota)_+$ is an $\mathcal{E}$-flat level acyclic cofibration. By adjointness this means showing that the $G \times K$-map $p^*(\iota) : p^*(EK) \longrightarrow p^*(C(EK))$ is any acyclic cofibration in the $\mathcal{E} \cap (G \times K)$-flat model structure on $G \times K$-spaces. The map is a relative $(G \times K)$-cofibration, so it remains to show that it is an $\mathcal{E} \cap (G \times K)$-weak equivalence. If H is a subgroup of $G \times K$ that belongs to $\mathcal{E}$, then its homomorphic image $p(H)$ also belongs to $\mathcal{E}$. Since $K \notin \mathcal{E}$, $p(H)$ must be strictly smaller than K, and hence $(p^*(EK))^H = (EK)^{p(H)}$ is contractible. Since the cone $C(EK)$ is K-equivariantly contractible, $p^*(C(EK))$ is $G \times K$-equivariantly contractible, so $p^*(\iota)$ is an $\mathcal{E} \cap (G \times K)$-weak equivalence, as claimed.

(iv) The part of the pushout product property that refers only to cofibrations is the same in the flat $\mathcal{F}$-level and the flat $\mathcal{F}$-global model structure, is independent of $\mathcal{F}$, and is true by part (i) with $\mathcal{E} = \mathcal{F} = \mathcal{A}ll$. In the flat $\mathcal{F}$-level model structure, the part of the pushout product property that also refers also to acyclic cofibrations is part (iii).

Every acyclic cofibration in the flat $\mathcal{F}$-global model structure is a retract of a composite $k \circ j$ with k a flat cofibration and $\mathcal{F}$-level equivalence and j a flat cofibration and global equivalence (Theorem 7.13), so it suffices to check these two special kinds of acyclic $\mathcal{F}$-cofibrations. The first kind was taken care of in the previous paragraph (i.e., by part (iii)). The box product of a flat cofibration with a flat cofibration and that is a global equivalence is another a global equivalence by Proposition 5.7 (iv), hence an $\mathcal{F}$-equivalence.

(v) The part of the pushout product property that refers only to cofibrations is the same in the projective $\mathcal{F}$-level and the projective $\mathcal{F}$-global model structure, and is true by part (i) with with $\mathcal{E} = \mathcal{F}$ and the hypothesis that $\mathcal{F} \times \mathcal{F} = \mathcal{F}$. The part of the pushout product property that also refers only to acyclic cofibrations is (ii) for the projective $\mathcal{F}$-level model structure. Every cofibration in the projective $\mathcal{F}$-global model structure is in particular a cofibration in the flat $\mathcal{F}$-global model structure, and the equivalences are the same in the two $\mathcal{F}$-global model structures. So the part of the pushout product property that also refers to acyclic cofibrations in the projective $\mathcal{F}$-global model structure is a special case of part (iv). □

Corollary 7.23. *Let $\mathcal{F}$ be a global family.*

(i) *For every flat orthogonal space A the functor* $\mathrm{Hom}(A,-)$ *preserves $\mathcal{F}$-injective static orthogonal spaces.*
(ii) *Suppose that $\mathcal{F}$ is also closed under products. Then for every $\mathcal{F}$-cofibrant orthogonal space A the functor* $\mathrm{Hom}(A,-)$ *preserves $\mathcal{F}$-static orthogonal spaces.*

PROOF. (i) Since A is flat the functor $A \boxtimes -$ is a left Quillen endofunctor for the flat $\mathcal{F}$-global model structure. So the right adjoint $\mathrm{Hom}(A,-)$ is a right Quillen endofunctor for the same model structure. In particular, $\mathrm{Hom}(A,-)$ preserves fibrant objects in the flat $\mathcal{F}$-global model structure. By Theorem 7.13 (iii), these fibrant objects are precisely the $\mathcal{F}$-injective $\mathcal{F}$-static orthogonal paces.

Part (ii) is similar: The functor $A \boxtimes -$ is a left Quillen endofunctor for the projective $\mathcal{F}$-global model structure, so the right adjoint $\mathrm{Hom}(A,-)$ preserves fibrant objects in the projective $\mathcal{F}$-global model structure. By Theorem 7.14 (iii), these are precisely the $\mathcal{F}$-static orthogonal spaces. □

Remark 7.24. There is also an analog of Proposition 7.20 (ii) for the projective global model structures. The pushout product of an $\mathcal{E}$-cofibration that is also an $\mathcal{E}$-equivalence with an $\mathcal{F}$-cofibration is an $(\mathcal{E}\times\mathcal{F})$-equivalence. This implies that whenever $\mathcal{F}\times\mathcal{E}=\mathcal{E}$ (so in particular $\mathcal{F}\subset\mathcal{F}\times\mathcal{E}=\mathcal{E}$), then the projective $\mathcal{E}$-global model structure is a module over the projective $\mathcal{F}$-global model structures.

The constant one-point orthogonal space $\mathbf{1}$ is the unit object for the box product of orthogonal spaces, and it is 'free', i.e., $\langle e\rangle$-cofibrant. So $\mathbf{1}$ is cofibrant in the flat and the projective $\mathcal{F}$-global model structure for every global family $\mathcal{F}$. So with respect to the box product, the flat $\mathcal{F}$-level model structure and the flat $\mathcal{F}$-global model structure are symmetric monoidal model categories in the sense of [**43**, Def. 4.2.6]. Similarly, whenever $\mathcal{F}$ is closed under products, then the projective $\mathcal{F}$-level and the projective $\mathcal{F}$-global model structures are symmetric monoidal model categories.

By Theorem 5.9, the morphism $\rho_{X,Y} : X\boxtimes Y \longrightarrow X\times Y$ is a global equivalence if X or Y is flat, hence an $\mathcal{F}$-equivalence for every global family $\mathcal{F}$.

Corollary 7.25. *Let $\mathcal{F}$ be a global family.*

(i) *Box product with a flat orthogonal space preserves $\mathcal{F}$-equivalences.*
(ii) *Box product with any orthogonal space preserves $\mathcal{F}$-equivalences between flat orthogonal spaces.*

PROOF. The product $-\times-$ of orthogonal spaces preserves $\mathcal{F}$-equivalences in both variables: if $\varphi : Y \longrightarrow Y'$ is an $\mathcal{F}$-equivalence, then we apply Proposition 7.6 (vi) to the pullback square

$$\begin{tikzcd} X\times Y \arrow[r] \arrow[d, "X\times\varphi"'] & Y \arrow[d, "\varphi"] \\ X\times Y' \arrow[r] & Y' \end{tikzcd}$$

where both horizontal maps are projections. Since projections to a factor are strong level fibrations, the base change $X\times\varphi$ of the $\mathcal{F}$-equivalence φ is again an $\mathcal{F}$-equivalence. Since the morphism $\rho_{X,Y} : X\boxtimes Y \longrightarrow X\times Y$ is a global equivalence if X or Y is flat (by Theorem 5.9), this implies (i). □

Finally, we will prove another important relationship between the global model structures an the smash product, namely the *monoid axiom* [**72**, Def. 3.3].

Proposition 7.26 (Monoid axiom)**.** *We let $\mathcal{F}$ be a global family. For every flat cofibration $j : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal space Y the morphism*

$$j\boxtimes Y \;:\; A\boxtimes Y \;\longrightarrow\; B\boxtimes Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential and transfinite compositions.

PROOF. Given the flatness theorem, this is a standard argument, similar to the proofs of the monoid axiom in the non-equivariant context Every flat cofibration is an h-cofibration (Proposition 4.6 (iii) applied to the flat $\mathcal{A}ll$-level model structure), and h-cofibrations are closed under box product with any orthogonal space (Remark 5.8), so $j \boxtimes Y$ is an h-cofibration. Since j is a h-cofibration and $\mathcal{F}$-equivalence, [...] so $j\Box Y$ is an $\mathcal{F}$-equivalence.

Proposition 7.6 shows that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential and transfinite compositions. □

Theorem [**72**, Thm. 4.1] now applies to the flat $\mathcal{F}$-global model structure of Theorem 7.13 and gives the following corollary.

Corollary 7.27. *Let R be an orthogonal monoid space and $\mathcal{F}$ a global family.*

(i) *The category of R-modules admits the* flat $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively fibration in the flat $\mathcal{F}$-global model structure). If R is commutative, then this is a monoidal model category that satisfies the monoid axiom.*

(ii) *If R is commutative, then the category of R-algebras admits the* flat $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively fibration in the flat $\mathcal{F}$-global model structure). Every cofibrant R-algebra is also cofibrant as an R-module.*

The projective $\mathcal{F}$-global model structure has the same equivalences, but fewer cofibrations, than the flat $\mathcal{F}$-global model structure. So the monoid axiom in the flat model structure implies the monoid axiom in the projective model structure. If the global family $\mathcal{F}$ is closed under products, Theorem [**72**, Thm. 4.1] then also applies to the projective $\mathcal{F}$-global model structure of Theorem 7.14 and shows:

Corollary 7.28. *Let R be an orthogonal monoid space and $\mathcal{F}$ a global family that is closed under products.*

(i) *The category of R-modules admits the* projective $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively fibration in the projective $\mathcal{F}$-global model structure). If R is commutative, then this is a monoidal model category that satisfies the monoid axiom.*

(ii) *If R is commutative, then the category of R-algebras admits the* projective $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spaces is an $\mathcal{F}$-equivalence (respectively fibration in the projective $\mathcal{F}$-global model structure). Every cofibrant R-algebra is also cofibrant as an R-module.*

Remark 7.29. Strictly speaking, Theorem 4.1 of [**72**] does not apply verbatim to the $\mathcal{F}$-flat and $\mathcal{F}$-projective global model structures because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to (suitably long) transfinite composition of flat cofibrations, and this suffices to run the small object argument (compare also Remark 2.4 of [**72**, Thm. 4.1]).

In the $\mathcal{F}$-projective global model structures, the sources of the generating cofibrations and generating acyclic cofibrations are in fact small with respect to sequential composites of h-cofibrations, so the countable version of the small object argument (as opposed to a transfinite version) suffices to lift the $\mathcal{F}$-projective global model structures. In the $\mathcal{F}$-flat global model structures one needs a transfinite small object argument because the generating acyclic cofibrations for the flat $\mathcal{F}$-level model structures involve free orthogonal spaces generated by infinite dimensional equivariant CW-complexes.

CHAPTER II

Commutative orthogonal monoid spaces

Orthogonal monoid spaces are the lax monoidal continuous functors from the category $\mathbf{L}$ to the category of spaces, compare Definition I.5.12. The orthogonal monoid spaces with strictly commutative multiplication (i.e., the lax *symmetric* monoidal continuous functors) play a special role: a strictly commutative multiplication is what is needed to 'globally deloop' an orthogonal space.

The study of orthogonal monoid spaces goes back to Boardman and Vogt [**12**], who introduce them as a 'delooping machine' (in a non-equivariant context). More precisely, they show that for every commutative orthogonal monoid space R the space $T(\mathcal{U})$ has the structure of a 'E-space' (nowadays called an E_∞-space) and, if in addition $\pi_0(T(\mathcal{U}))$ is a group, then $T(\mathcal{U})$ is an infinite loop space. Commutative orthogonal monoid spaces also appear, with an extra pointset topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [**59**, IV Def. 2.1]. We are going to study orthogonal monoid space from a global perspective, with the aim of ultimately producing a global delooping machine.

1. Global power monoids

Remark 1.1. We explain how one can think about a commutative orthogonal monoid space as a compatible collection of E_∞ G-spaces, one for every compact Lie group G, compatible under restriction. If R is a closed orthogonal space and G a compact Lie group, then the G-equivariant homotopy type encoded in R can be accessed as the 'underlying G-space'

$$R(\mathcal{U}_G) \;=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)} R(V)\ .$$

The additional structure of a commutative orthogonal monoid space on R gives rise to an action of a specific E_∞ G-operad on this G-space, namely the linear isometries operad $\mathcal{L}(\mathcal{U}_G)$ of the complete G-universe $\mathcal{U}_G$. The n-th space of this operad is the space $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ of linear isometric embedding (not necessarily equivariant) of $\mathcal{U}_G^n$ into $\mathcal{U}_G$. The group G acts on $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ by conjugation and the operad structure is by direct sum and composition of linear isometric embeddings. The space $\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)$ has the weak $(\Sigma_n\times G)$-equivariant homotopy type of a universal space for (Σ_n,G)-bundles.

A linear isometric embedding $\psi:\mathcal{U}\longrightarrow\mathcal{U}'$ between countably infinite dimensional inner product spaces induces a map

$$X(\psi)\ :\ X(\mathcal{U})\ \longrightarrow\ X(\mathcal{U}')$$

by passage to colimits over $V\in s(\mathcal{U})$ of the composite maps

$$X(V)\ \xrightarrow{X(\psi|_V)}\ X(\psi(V))\ \longrightarrow\ X(\mathcal{U}')\ .$$

This construction is continuous in both variables, i.e., the map

$$\mathrm{ev}\ :\ \mathbf{L}(\mathcal{U},\mathcal{U}')\times X(\mathcal{U})\ \longrightarrow\ X(\mathcal{U}')$$

is continuous. The operadic action map

$$\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)\times R(\mathcal{U}_G)^n\ \longrightarrow\ R(\mathcal{U}_G)$$

is then simply the composite

$$\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G) \times R(\mathcal{U}_G) \times \cdots \times R(\mathcal{U}_G) \xrightarrow{\mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G)\times\mu} \mathbf{L}(\mathcal{U}_G^n,\mathcal{U}_G) \times R(\mathcal{U}_G \oplus \cdots \oplus \mathcal{U}_G) \xrightarrow{\text{ev}} R(\mathcal{U}_G) \ .$$

Given an orthogonal monoid space R and compact Lie groups G and K, we define an internal pairing on the equivariant homotopy sets of R as the composite

$$\pi_0^G(R) \times \pi_0^K(R) \xrightarrow{\times} \pi_0^{G\times K}(R \boxtimes R) \xrightarrow{\mu_*} \pi_0^{G\times K}(R) \ . \tag{1.2}$$

We also denote by $1 \in \pi_0^e(R)$ the class of the unit $1 \in R(0)$. The following properties of these internal pairings are direct consequences of the corresponding properties of the external pairings, compare Proposition 5.4.

Corollary 1.3. *Let R be an orthogonal monoid space and G, K and L compact Lie groups.*

(i) (Unitality) *The unit $1 \in \pi_0^e(R)$ is unital in the sense that $1 \times x = x = x \times 1$ under the identifications $e \times G \cong G \cong G \times e$.*

(ii) (Associativity) *For all classes $x \in \pi_0^G(R)$, $y \in \pi_0^K(R)$ and $z \in \pi_0^L(R)$ we have $(x \times y) \times z = x \times (y \times z)$ under the identification $(G \times K) \times L \cong G \times (K \times L)$.*

(iii) (Commutativity) *If the multiplication of R is commutative, then for all classes $x \in \pi_0^G(X)$ and $y \in \pi_0^K(Y)$ the relation*

$$y \times x \ = \ \tau_{G,K}^*(x \times y)$$

holds in $\pi_0^{K\times G}(R)$, where $\tau_{G,K} : K \times G \longrightarrow G \times K$ interchanges the factors.

(iv) (Restriction) *For all classes $x \in \pi_0^G(R)$ and $y \in \pi_0^K(R)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$(\alpha^* x) \times (\beta^* y) \ = \ (\alpha \times \beta)^*(x \times y)$$

holds in the group $\pi_0^{\bar{G}\times\bar{K}}(R)$.

If the multiplication on an orthogonal monoid space R is commutative, then this does not only imply a commutativity relation of the induced products on $\underline{\pi}_0 R$; strict commutativity of the multiplication also gives rise to additional *power operations* that we discuss now. An important special case will later be the commutative orthogonal monoid space $\Omega^\bullet R$ arising from a commutative orthogonal ring spectrum R; in this situation the power operations satisfy further compatibility conditions with respect to the addition and the transfer maps on $\underline{\pi}_0(\Omega^\bullet R) = \underline{\pi}_0(R)$; altogether this structure makes altogether makes the 0-th equivariant homotopy groups of a commutative orthogonal ring spectrum into a *global power functor*.

We recall that the wreath product $\Sigma_m \wr G$ of a symmetric group Σ_m and a group G is the semidirect product

$$\Sigma_m \wr G \ = \ \Sigma_m \ltimes G^m$$

formed with respect to the action of Σ_m by permuting the factors of G^m. So the multiplication in $\Sigma_m \wr G$ is given by

$$(\sigma;\, g_1, \ldots, g_m) \cdot (\tau;\, k_1, \ldots, k_m) \ = \ (\sigma\tau;\, g_{\tau(1)}k_1, \ldots, g_{\tau(m)}k_m) \ .$$

Construction 1.4. We let R be a commutative orthogonal monoid space and G a compact Lie group. We construct natural power maps of homotopy groups

$$P^m \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^{\Sigma_m \wr G}(R) \ . \tag{1.5}$$

As the name suggests, the power operation P^m raises a representing equivariant map to the m-th power.

For every G-representation V we consider V^m as a $(\Sigma_m \wr G)$-representation with action given by

$$(\sigma;\, g_1, \ldots, g_m) \cdot (v_1, \ldots, v_m) \ = \ \left(g_{\sigma^{-1}(1)} v_{\sigma^{-1}(1)},\, \ldots,\, g_{\sigma^{-1}(m)} v_{\sigma^{-1}(m)}\right) \ .$$

We let

$$\mu_{V,\ldots,V} \ : \ R(V) \times \cdots \times R(V) \ \longrightarrow \ R(V \oplus \cdots \oplus V)$$

denote the $(V,\dots,V)$-component of the multiplication map of R, and we observe that this is $\Sigma_m \wr G$-equivariant because the multiplication on R is commutative. If $x \in R(V)^G$ is a G-fixed points representing a class in $\pi_0^G(R)$, then the image of $(x,\dots,x) \in (R(V)^G)^m$ under the map

$$(R(V)^G)^m \ = \ (R(V)^m)^G \ \xrightarrow{\mu^G_{V,\dots,V}} \ R(V^m)^G$$

is $\Sigma_n \wr G$-fixed point, so it represents an element

$$P^m[x] \ = \ \langle \mu^G_{V,\dots,V}(x,\dots,x)\rangle \ \in \ \pi_0^{\Sigma_m\wr G}(R) \ . \tag{1.6}$$

If we stabilize x to $R(i_{V,W})(x) \in R(V\oplus W)^G$, then $\mu^G_{V,\dots,V}(x,\dots,x)$ changes into

$$\mu^G_{V\oplus W,\dots,V\oplus W}(R(i_{V,W})(x),\dots,R(i_{V,W})(x)) \ = \ R(i^m_{V,W})(\mu^G_{V,\dots,V}(x,\dots,x))) \ \in \ R((V\oplus W)^m)^G \ .$$

This element represents the same class in $\pi_0^{\Sigma_m\wr G}(R)$ as $\mu^G_{V,\dots,V}(x,\dots,x)$, so the class $P^m[x]$ only depends on the class of x in $\pi_0^G(R)$. So we have constructed a well-defined power operation (1.5).

The power operations are clearly natural for homomorphisms $\varphi : R \longrightarrow R'$ of orthogonal monoid spaces, i.e., for every compact Lie group G, every $m \geq 0$ and all $x \in \pi_0^G(R)$ the relation

$$P^m(\varphi_*(x)) \ = \ \varphi_*(P^m(x))$$

holds in $\pi_0^{\Sigma_m\wr G}(R')$.

Remark 1.7. A certain 'part of' the power operation P^m can be realized by *m-power morphism* of orthogonal spaces

$$\bar P^m \ : \ R \ \longrightarrow \ F^{\Sigma_m}R \ ,$$

where $F^{\Sigma_m}R$ is the orthogonal Σ_m-fixed point space of Construction 1.17. The value of $\bar P^m$ at an inner product space V is the composite map

$$\begin{aligned} R(V) \ \xrightarrow{x\mapsto(x,\dots,x)} \ (R(V)\times\dots\times R(V))^{\Sigma_m} \ &\xrightarrow{(\mu_{V,\dots,V})^{\Sigma_m}} \ (R(V\oplus\dots\oplus V))^{\Sigma_m} \\ \cong \ (R(V\otimes\nu_m))^{\Sigma_m} \ &\xrightarrow{(R(V\otimes j))^{\Sigma_m}} \ (R(V\otimes\rho_{\Sigma_m}))^{\Sigma_m} \ = \ (F^{\Sigma_m}R)(V) \ . \end{aligned}$$

Here $j : \nu_m \longrightarrow \rho_{\Sigma_m}$ is any Σ_m-equivariant linear isometric embedding from the natural to the regular representation of Σ_m. This morphism of orthogonal spaces realizes P^m in the sense that the following square commutes:

$$\begin{tikzcd} \pi_0^G(R) \arrow[r, "P^m"] \arrow[d, "\pi_0^G(\bar P^m)"'] & \pi_0^{\Sigma_m\wr G}(R) \arrow[d, "\Delta^*"] \\ \pi_0^G(F^{\Sigma_m}R) \arrow[r, "\cong", "l^G"'] & \pi_0^{G\times\Sigma_m}(R) \end{tikzcd}$$

Here l^G is the bijection of Proposition 1.19 and $\Delta : G\times\Sigma_m \longrightarrow \Sigma_m\wr G$ is the 'diagonal' embedding defined by

$$\Delta(g,\sigma) \ = \ (\sigma;\, g,\dots,g) \ .$$

The power operations P^m satisfy various properties reminiscent of the map $x\mapsto x^m$ in a commutative monoid. We formalizes these properties into the concept of a *global power monoid*. In the definition we need certain morphisms between different wreath products, so we fix some notation for these now. An embedding of a product of wreath products is

$$\begin{aligned} \Phi_{i,j} \ : \ (\Sigma_i\wr G) \ \times \ (\Sigma_j\wr G) \ &\longrightarrow \ \Sigma_{i+j}\wr G \\ ((\sigma;\, g_1,\dots,g_i),\, (\sigma';\, g_{i+1},\dots,g_{i+j})) \ &\longmapsto \ (\sigma+\sigma';\, g_1,\dots,g_{i+j}) \ . \end{aligned} \tag{1.8}$$

Another embedding of an iterated wreath product is

$$\begin{aligned} (1.9) \qquad \Psi_{k,m} \ : \ \Sigma_k \wr (\Sigma_m \wr G) \quad &\longrightarrow \quad \Sigma_{km} \wr G \\ (\sigma;\, (\tau_1;\, h^1), \dots, (\tau_k;\, h^k)) \ &\longmapsto \ (\sigma(\tau_1 + \dots + \tau_k);\, h^1 + \dots + h^k) \ . \end{aligned}$$

Definition 1.10. A *global power monoid* is a lax symmetric monoidal functor

$$M \ : \ \mathrm{Rep}^{\mathrm{op}} \ \longrightarrow \ (\text{sets})$$

from the opposite of the category Rep of compact Lie groups and conjugacy classes of homomorphisms to the category of sets, equipped with additional maps

$$P^m \ : \ M(G) \ \longrightarrow \ M(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Unit) $P^m(1) = 1$ for the unit $1 \in M(e)$.

(ii) (Identity) $P^1 = \mathrm{Id}$ under the identification $\Sigma_1 \wr G \cong G$.

(iii) (Naturality) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and $m \geq 1$ the relation

$$P^m \circ \alpha^* \ = \ (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps $M(G) \longrightarrow M(\Sigma_m \wr K)$.

(iv) (Multiplicativity) For all compact Lie groups G, all $m \geq 1$ and all classes $x, y \in M(G)$ the relation

$$P^m(x \cdot y) \ = \ P^m(x) \cdot P^m(y)$$

holds in the group $M(\Sigma_m \wr G)$.

(v) (Restriction) For all compact Lie groups G, all $m \geq 1$, all $0 \leq i \leq m$ and all $x \in M(G)$ the relation

$$\Phi^*_{i,m-i}(P^m(x)) \ = \ P^i(x) \times P^{m-i}(x)$$

holds in $M((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G))$ where $\Phi_{i,m-i}$ is the monomorphism (1.8).

(vi) (Transitivity) For all compact Lie groups G, all $k, m \geq 1$ and all $x \in M(G)$ the relation

$$\Psi^*_{k,m}(P^{km}(x)) \ = \ P^k(P^m(x))$$

holds in $M(\Sigma_k \wr (\Sigma_m \wr G))$, where $\Psi_{k,m}$ is the monomorphism (1.9).

Remark 1.11. The relations of the power operations in a global power monoid have various other properties.

(a) For every compact Lie group G and all $m \geq 1$, the relation $P^m(1) = 1$ holds in $M(\Sigma_m \wr G)$ where $1 \in M(G)$ is the multiplicative unit. Indeed, the restriction map $p_G^* : M(e) \longrightarrow M(G)$ along the unique group homomorphism $p_G : G \longrightarrow e$ preserves the units, so this follows from the unit condition and restriction naturality.

(b) The power map is multiplicative with respect to the external product: for all compact Lie groups G and K and all $m \geq 1$, and all classes $x \in M(G)$ and $y \in M(K)$ the relation

$$P^m(x \times y) \ = \ \Delta^*(P^m(x) \times P^m(y))$$

holds in $M(\Sigma_m \wr (G \times K))$, where Δ is the 'diagonal' monomorphism

$$\begin{aligned} (1.12) \qquad \Delta \ : \ \Sigma_m \wr (G \times K) \quad &\longrightarrow \quad (\Sigma_m \wr G) \times (\Sigma_m \wr K) \\ (\sigma;\, (g_1, k_1), \dots, (g_m, k_m)) \ &\longmapsto \ ((\sigma;\, g_1, \dots, g_m),\, (\sigma;\, k_1, \dots, k_m)) \ . \end{aligned}$$

Indeed, the external product is the composite

$$M(G) \times M(K) \xrightarrow{p_G^* \times p_K^*} M(G \times K) \times M(G \times K) \ \overset{\cdot}{\longrightarrow} \ M(G \times K) \ ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. So

$$\begin{aligned} P^m(x \times y) &= P^m(p_G^*(x) \cdot p_K^*(y)) \ = \ P^m(p_G^*(x)) \cdot P^m(p_K^*(y)) \\ &= \Delta^*_{\Sigma_m \wr (G\times K)}((\Sigma_m \wr p_G)^*(P^m(x)) \times (\Sigma_m \wr p_K)^*(P^m(y))) \\ &= \Delta^*_{\Sigma_m \wr (G\times K)}(((\Sigma_m \wr p_G) \times (\Sigma_m \wr p_K))^*(P^m(x) \times P^m(y))) \ = \ \Delta^*(P^m(x) \times P^m(y)) \ . \end{aligned}$$

Here we exploit that Δ factors as the composite

$$\begin{aligned} \Sigma_m \wr (G \times K) \ &\xrightarrow{\Delta_{\Sigma_m \wr (G\times K)}} \ (\Sigma_m \wr (G \times K)) \times (\Sigma_m \wr (G \times K)) \\ &\xrightarrow{(\Sigma_m \wr p_G)\times(\Sigma_m \wr p_K)} (\Sigma_m \wr G) \times (\Sigma_m \wr K) \ . \end{aligned}$$

(c) The class $P^m(x)$ is a equivariant refinement of the m-th power x^m in the following sense. The wreath product $\Sigma_m \wr G$ contains a diagonal copy of G, via the embedding

$$\delta \ : \ G \ \longrightarrow \ \Sigma_m \wr G \ , \quad \delta(g) \ = \ (1; g, \dots, g) \ .$$

The class $P^m(x)$ restricts to x^m on this diagonal subgroup, i.e.,

$$\delta^*(P^m(x)) \ = \ x^m$$

in $M(G)$. Indeed, applying the restriction property (v) repeatedly shows that $P^m(x)$ restricts to the external m-fold power

$$x \times \dots \times x \ \in \ M(G^m)$$

on the subgroup $G^m \leq \Sigma_m \wr G$. Restricting further to the diagonal takes the m-fold external power to x^m in $M(G)$.

Proposition 1.13. *Let R be a commutative orthogonal monoid space. Then the products* (1.2) *and the power operations* (1.5) *make the functor $\underline{\pi}_0(R)$ into a global power monoid.*

Proof. The unit condition (i), the identity condition (ii), the naturality (iii), the restriction condition (v) and the transitivity condition (vi) are straightforward from the definition. Multiplicativity (iv) is a consequence of the fact that for all G-representations V and W the diagram

$$\begin{tikzcd} (R(V) \times R(W))^m \arrow[r, "\mu^m_{V,W}"] \arrow[d, "\cong"'] & R(V \oplus W)^m \arrow[r, "\mu_{V\oplus W,\dots,V\oplus W}"] & R((V \oplus W)^m) \arrow[d, "R(\alpha)"] \\ R(V)^m \times R(W)^m \arrow[r, "\mu_{V,\dots,V}\times\mu_{W,\dots,W}"'] & R(V^m) \times R(W^m) \arrow[r, "\mu_{V^m,W^m}"'] & R(V^m \oplus W^m) \end{tikzcd}$$

commutes, where the left vertical bijection and the linear isometry $\alpha : (V \oplus W)^m \longrightarrow V^m \oplus W^m$ reorder factors according to

$$((x_1, y_1), \dots, (x_m, y_m)) \ \longmapsto \ ((x_1, \dots, x_m), (y_1, \dots, y_m)) \ .$$

the summands. Given G-fixed points $x \in R(V)^G$ and $y \in R(W)^G$, the image of

$$((x, y), \dots, (x, y)) \ \in \ (R(V) \times R(W))^m$$

in $R((V \oplus W)^m)^G$ represents the class $P^m([x] \cdot [y])$, whereas its image in $R(V^m \oplus W^m)^G$ represents the class $P^m[x] \cdot P^m[y]$. Since the two representatives differ by the effect of an equivariant isometry, the two classes coincide by Proposition 1.13 (ii). $\square$

Example 1.14 (Naive units of an orthogonal monoid space)**.** Every orthogonal monoid space R contains an interesting orthogonal monoid subspace $R^{n\times}$, the *naive units* of R. The value of $R^{n\times}$ at an inner product space V is the union of those path components of $R(V)$ that are taken to an invertible element, with respect to the monoid structure on $\pi_0(R)$, under the map

$$R(V) \longrightarrow \pi_0(R(V)) \longrightarrow \pi_0(R) .$$

In other words, a point $x \in R(V)$ belongs to $R^{n\times}(V)$ if and only if there is an inner product space W and a point $y \in R(W)$ such that

$$\mu_{V,W}(x,y) \in R(V \oplus W)$$

is in the same path component as the image of the unit element $1 \in R(0)$ under the map $R(0) \longrightarrow R(V\oplus W)$. We omit the verification that the subspaces $R^{n\times}(V)$ indeed form an orthogonal monoid subspace of R as V varies. The induced map of global power monoids

$$\underline{\pi}_0(R^{n\times}) \longrightarrow \underline{\pi}_0(R)$$

is also an inclusion, and the value $\pi_0^e(R^{n\times})$ at the trivial group is, by construction, the set of invertible elements of $\pi_0^e(R)$. For a general compact Lie group G,

$$\pi_0^G(R^{n\times}) = \{ x \in \pi_0^G(R) \mid \mathrm{res}^G_e(x) \text{ is invertible in } \pi_0^e(R)\}$$

is the submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. So contrary to what one could suspect at first sight, $\pi_0^G(R^{n\times})$ may contain non-invertible elements and the orthogonal monoid space $R^\times$ is not necessarily 'group-like' (in the sense of Definition 4.1 below); this is why we use the adjective 'naive'.

Example 1.15 (Units of an orthogonal monoid space)**.** Every global monoid M has a global submonoid $M^\times$ of *units*. The value $M^\times(G)$ at a compact Lie group G consists of the set of invertible elements of $M(G)$. Since the restriction maps are multiplicative, the sets $M^\times(G)$ are closed under restriction maps. Since the external product map

$$\times \ : \ M(G) \times M(K) \longrightarrow M(G \times K)$$

is a monoid homomorphism, it takes $M^\times(G) \times M^\times(K)$ into $M^\times(G \times K)$. So the subsets $M^\times(G)$ indeed form a global submonoid of M. If $f : N \longrightarrow M$ is a homomorphism of global monoids and N is group-like (i.e., all monoids $N(G)$ are groups), then the image of f is contained in $M^\times$. So the functor $M \mapsto M^\times$ is right adjoint to the inclusion of the full subcategory of group-like global monoids. If M is even a global power monoid, then P^m takes $M^\times(G)$ into $M^\times(\Sigma_m \wr G)$ because power operations are multiplicative. So in this situation, $M^\times$ is even a global power submonoid of M.

Now we explain how the passage to algebraic units in a power monoid is realized on the level of orthogonal monoid spaces. [...]

Example 1.16. For every orthogonal space Y we denote by

$$\mathbb{P}(Y) = \coprod_{m\geq 0} Y^{\boxtimes m}/\Sigma_m$$

the free commutative orthogonal monoid space generated by Y. We look in more detail at the case $Y = B_{\mathrm{gl}}G$ of the global classifying space of a compact Lie group G. For every G-representation V we have

$$\mathbf{L}_{G,V}^{\boxtimes m} \cong \mathbf{L}_{G^m,V^m} .$$

At an inner product space W, the permutation action of Σ_m on the left hand side becomes the action on

$$\mathbf{L}_{G^m,V^m}(W) = \mathbf{L}(V^m, W)/G^m$$

by permuting the summands in V^m. So

$$\mathbf{L}_{G,V}^{\boxtimes m}/\Sigma_m \cong \mathbf{L}_{G^m,V^m}/\Sigma_m \cong \mathbf{L}_{\Sigma_m\wr G,V^m} .$$

Thus

$$\mathbb{P}(\mathbf{L}_{G,V}) \;=\; \coprod_{m\geq 0} \mathbf{L}_{\Sigma_m \wr G, V^m} \ .$$

If G acts faithfully on V and $V \neq 0$, then the action of $\Sigma_m \wr G$ on V^m is again faithful. So in terms of global classifying spaces the free commutative orthogonal monoid space generated by $B_{\mathrm{gl}}G$ is given by

$$\mathbb{P}(B_{\mathrm{gl}}G) \;=\; \coprod_{m\geq 0} B_{\mathrm{gl}}(\Sigma_m \wr G) \ . \tag{1.17}$$

The tautological class $u_G \in \pi_0^G(B_{\mathrm{gl}}G)$ is represented by the orbit of the identity of V in

$$(\mathbf{L}_{G,V}(V))^G \;=\; (\mathbf{L}(V,V)/G)^G \ ,$$

compare 2.3. So the class $P^m(u_G) \in \pi_0^{\Sigma_m \wr G}(\mathbb{P}(B_{\mathrm{gl}}G))$ is represented by the orbit of the identity of V^m in

$$(\mathbf{L}_{\Sigma_m \wr G, V^m}(V^m))^G \;=\; (\mathbf{L}(V^m,V^m)/\Sigma_m \wr G)^{\Sigma_m \wr G} \ ;$$

so with respect to the identification (1.17) we have

$$P^m(u_G) \;=\; u_{\Sigma_m \wr G} \ . \tag{1.18}$$

Now we are going to show that the restriction maps along a group homomorphism and the power operations give all natural operations between equivariant homotopy sets of commutative orthogonal monoid spaces. The strategy is the same as in Proposition 2.21: the functor π_0^G is representable, this time by the 'symmetric algebra' $\mathbb{P}(B_{\mathrm{gl}}G)$ of the global classifying space of G, so we have to determine the equivariant homotopy sets $\pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))$ of these representing objects.

Proposition 1.19. *Let G and K be compact Lie groups. Every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of commutative orthogonal monoid spaces is of the form $\alpha^* \circ P^m$ for a unique $m \geq 0$ and a unique conjugacy class of continuous group homomorphism $\alpha : K \longrightarrow \Sigma_m \wr G$.*

Proof. We let V be any faithful G-representation and write $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ for the global classifying space of G based on V and $u_G = u_{G,V}$ for the associated tautological class. We denote by $u_G^{\mathbb{P}} \in \pi_0^G(\mathbb{P}(B_{\mathrm{gl}}G))$ the image of u_G under the adjunction unit $B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ (i.e., the embedding as the summand indexed by $m = 1$). We apply the representability result of Proposition 2.20 to the category of commutative orthogonal monoid spaces, the free and forgetful adjoint functor pair

$$\mathit{spc} \underset{U}{\overset{\mathbb{P}}{\rightleftarrows}} \mathrm{coms}$$

and the functor $\pi_0^K \circ U$. We conclude that the evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{\mathrm{coms}}(\pi_0^G, \pi_0^K) \;\longrightarrow\; \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G)) \ , \quad \tau \;\longmapsto\; \tau(u_G^{\mathbb{P}})$$

to the 0-th K-equivariant homotopy group of commutative orthogonal monoid space $\mathbb{P}(B_{\mathrm{gl}}G)$. The square

$$\begin{tikzcd}
\coprod_{m\geq 0} \mathrm{Rep}(K, \Sigma_m \wr G) \arrow[r, "\coprod \mathrm{ev}_{u_{\Sigma_m \wr G}}"] \arrow[d] & \coprod_{m\geq 0} \pi_0^K(B_{\mathrm{gl}}(\Sigma_m \wr G)) \arrow[d] \\
\mathrm{Nat}^{\mathrm{coms}}(\pi_0^G, \pi_0^K) \arrow[r, "\mathrm{ev}_{u_G^{\mathbb{P}}}"'] & \pi_0^K(\mathbb{P}(B_{\mathrm{gl}}G))
\end{tikzcd}$$

commutes, where the left vertical map sends $\alpha : K \longrightarrow \Sigma_m \wr G$ to $\alpha^* \circ P^m : \pi_0^G \longrightarrow \pi_0^K$. The upper is evaluation at the tautological class $u_{G,V}$. The evaluation map is bijective by Proposition 2.6 (iv), and the right vertical map is bijective by Example 1.16. □

Construction 1.20 (Norm maps)**.** In the following we let M be a global power monoid. A formal consequence of the power operations are multiplicative *norm maps* $N_H^G : M(H) \longrightarrow M(G)$ for every subgroup H of a compact Lie group G of finite index $m = |G/H|$. We choose coset representatives, i.e., an ordered m-tuple $(g_1, \dots, g_m)$ of elements in disjoint H-cosets that satisfy

$$G \;=\; \bigcup_{i=1}^m g_i H \ .$$

The group G acts from the left on the set $\langle G : H\rangle$ of all such sets of coset representatives by

$$\gamma \cdot (g_1, \dots, g_m) \;=\; (\gamma g_1, \dots, \gamma g_m) \ .$$

The wreath product $\Sigma_m \wr H$ acts freely and transitively on $\langle G : H\rangle$ from the right by

$$\bar g \cdot (\sigma;\, h_1, \dots, h_m) \;=\; (g_1, \dots, g_m) \cdot (\sigma;\, h_1, \dots, h_m) \;=\; (g_{\sigma(1)} h_1, \dots, g_{\sigma(m)} h_m) \ .$$

The chosen basis then determines a monomorphism $\Psi_{\bar g} : G \longrightarrow \Sigma_m \wr H$ by requiring that

$$\gamma \cdot \bar g \;=\; \bar g \cdot \Psi_{\bar g}(\gamma) \ .$$

The norm map N_H^G is defined as the composite

$$M(H) \xrightarrow{\;P^m\;} M(\Sigma_m \wr H) \xrightarrow{\;\Psi_{\bar g}^*\;} M(G) \ .$$

Any other H-basis is of the form $\bar g \omega$ for a unique $\omega \in \Sigma_m \wr H$. We have $\Psi_{\bar g \omega} = c_\omega \circ \Psi_{\bar g}$, as maps $G \longrightarrow \Sigma_m \wr H$, where $c_\omega(\gamma) = \omega^{-1} \gamma \omega$. Since inner automorphisms induce the identity in any global functor, we have

$$\Psi_{\bar g}^* = \Psi_{\bar g \omega} \;:\; M(\Sigma_m \wr H) \xrightarrow{\;\Psi_{\bar g}^*\;} M(G)$$

So the norm map $N_H^G : M(H) \longrightarrow M(G)$ is independent of the choice of H-basis of G. The norm map N_H^G is natural for homomorphism of global power monoids.

As we just explained, power operations give rise to norm maps. Conversely, the power operation $P^m : M(H) \longrightarrow M(\Sigma_m \wr H)$ can be reconstructed from the norm maps as the composite

$$M(H) \xrightarrow{\;q^*\;} M((\Sigma_{m-1} \wr H) \times H) \xrightarrow{\;N^{\Sigma_m \wr H}_{(\Sigma_{m-1} \wr H) \times H}\;} M(\Sigma_m \wr H) \ ,$$

where $q : (\Sigma_{m-1} \wr H) \times H \longrightarrow H$ is the projection to the second factor. Indeed, the elements

$$(1; 1, \dots, 1),\ ((n-1\ n); 1, \dots, 1),\ \dots\ ,\ ((1\ n); 1, \dots, 1)$$

are coset representatives for the subgroup $(\Sigma_{m-1} \wr H) \times H$ of $\Sigma_m \wr H$. The associated homomorphism $\Psi : \Sigma_m \wr H \longrightarrow \Sigma_m \wr ((\Sigma_{m-1} \wr H) \times H)$ is a section to the homomorphism $\Sigma_m \wr q : \Sigma_m \wr ((\Sigma_{m-1} \wr H) \times H) \longrightarrow \Sigma_m \wr H$. So we conclude that

$$\begin{aligned} N^{\Sigma_m \wr H}_{(\Sigma_{m-1} \wr H) \times H}(q^*(x)) \;&=\; \Psi^*(P^m(q^*(x))) \;=\; \Psi^*((\Sigma_m \wr q)^*(P^m(x))) \\ &=\; ((\Sigma_m \wr q) \circ \Psi)^*(P^m(x)) \;=\; P^m(x) \ . \end{aligned}$$

So for every contravariant functor from the representation category to the category of commutative monoids, power and norm maps contain the same information, and can be reconstructed from each other.

The various properties of the power operations imply corresponding properties of the norm map.

Proposition 1.21. *The norm maps of a global power monoid M satisfy the following relations, where H is any subgroup of finite index in a compact Lie group G.*

(i) *We have*

$$N_H^G(1) \;=\; 1 \qquad \text{and} \qquad N_G^G(x) \;=\; x \ .$$

(ii) (Transitivity) *The norm maps are transitive, i.e., for subgroups $K \subseteq H \subseteq G$ of finite index the relation*

$$N_H^G \circ N_K^H \;=\; N_K^G$$

holds as maps $M(H) \longrightarrow M(G)$.

(iii) (Multiplicativity) *The norm maps are multiplicative with respect to internal product: for all classes $x, y \in M(H)$ the relation*

$$N_H^G(x) \cdot N_H^G(y) \;=\; N_H^G(x \cdot y)$$

holds in $M(G)$.

(iv) (Double coset formula) *For every subgroup K of G (not necessarily of finite index) we have:*

$$\mathrm{res}_K^G \circ N_H^G \;=\; \prod_{[g] \in K \backslash G/H} N_{K \cap {}^g H}^K \circ c_g^* \circ \mathrm{res}_{K^g \cap H}^H$$

as maps $M(H) \longrightarrow M(K)$. Here $[g]$ runs over a set of representatives of the finite set of K-H-double cosets.

(v) *For every surjective continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the relation*

$$\alpha^* \circ N_H^G \;=\; N_L^K \circ \alpha|_L^*$$

holds as maps from $M(H)$ to $M(K)$, where $L = \alpha^{-1}(H)$.

Proof. (i) We have $N_H^S(1) = \Psi_{\bar g}^*(P^m(1)) = 1$. For $G = H$ we can choose the unit 1 as the G-basis, and with this choice $\Psi_1 : G \longrightarrow \Sigma_1 \wr G$ is the preferred isomorphism that sends g to $(1; g)$. The restriction of $P^1(x)$ along this isomorphism is x, so we get $N_G^G(x) = x$.

(ii) Suppose that $k = \langle H : K \rangle$ is the index of K in H. We choose a K-basis $\bar h = (h_1, \dots, h_k)$ of H and an H-basis $\bar g = (g_1, \dots, g_m)$ of G. Then

$$\bar g \bar h \;=\; (g_1 h_1, \dots, g_1 h_k,\; g_2 h_1, \dots, g_2 h_k,\; \dots,\; g_m h_1, \dots, g_m h_k)$$

is a K-basis of G. With respect to this basis, the homomorphism $\Psi_{\bar g \bar h} : G \longrightarrow \Sigma_{km} \wr K$ equals the composite

$$G \xrightarrow{\Psi_{\bar g}} \Sigma_m \wr H \xrightarrow{\Sigma_m \wr \Psi_{\bar h}} \Sigma_m \wr (\Sigma_k \wr K) \xrightarrow{\Psi_{m,k}} \Sigma_{km} \wr K$$

where the monomorphism $\Psi_{k,m}$ was defined in (1.9). So

$$\begin{aligned} N_K^G(x) \;&=\; \Psi_{\bar g \bar h}^*(P^{km}(x)) \;=\; \Psi_{\bar g}^*((\Sigma_m \wr \Psi_{\bar h})^*(\Psi_{k,m}^*(P^{km} x))) \\ &=\; \Psi_{\bar g}^*((\Sigma_m \wr \Psi_{\bar h})^*(P^m(P^k x))) \;=\; \Psi_{\bar g}^*(P^m(\Psi_{\bar h}^*(P^k x))) \;=\; N_H^G(N_K^H(x)) \; . \end{aligned}$$

(iii) The composite

$$G \xrightarrow{\Psi_{\bar g}} \Sigma_m \wr H \xrightarrow{\Sigma_m \wr \Delta_H} \Sigma_m \wr (H \times H) \xrightarrow{\Delta} (\Sigma_m \wr H) \times (\Sigma_m \wr H)$$

equals $(\Psi_{\bar g} \times \Psi_{\bar g}) \circ \Delta_G$. So naturality and external multiplicativity of the power map yield

$$\begin{aligned} N_H^G(x) \cdot N_H^G(y) \;&=\; \Psi_{\bar g}^*(P^m(x)) \cdot \Psi_{\bar g}^*(P^m(y)) \\ &=\; \Delta_G^*(\Psi_{\bar g}^*(P^m(x)) \times \Psi_{\bar g}^*(P^m(y))) \\ &=\; \Delta_G^*((\Psi_{\bar g} \times \Psi_{\bar g})^*(P^m(x) \times P^m(y))) \\ &=\; \Psi_{\bar g}^*((\Sigma_m \wr \Delta_H)^*(\Delta^*(P^m(x) \times P^m(y)))) \\ &=\; \Psi_{\bar g}^*((\Sigma_m \wr \Delta_H)^*(P^m(x \times y))) \\ &=\; \Psi_{\bar g}^*(P^m(\Delta_H^*(x \times y)) \;=\; N_H^G(x \cdot y) \; . \end{aligned}$$

(iv) We generalize the norm construction slightly. We let A be a $(K \times H^{\mathrm{op}})$-space that consists of finitely many free H-orbits. We choose an 'H-basis', i.e., an ordered m-tuple $\bar{a} = (a_1, \dots, a_m)$ of elements in disjoint H-orbits such that

$$A \;=\; \bigcup_{i=1}^{m} a_i H \; .$$

Again the wreath product $\Sigma_m \wr H$ acts freely and transitively from the right on the set of all such H-bases of A. We obtain a continuous homomorphism $\Psi_{\bar{a}} : K \longrightarrow \Sigma_m \wr H$ by requiring that

$$k \cdot \bar{a} \;=\; \bar{a} \cdot \Psi_{\bar{a}}(k) \; .$$

In this generality, $\Psi_{\bar{a}}$ need not be injective anymore. We define a generalized norm map $\langle A|-\rangle : M(H) \longrightarrow M(K)$ as the composite

$$M(H) \xrightarrow{\;P^m\;} M(\Sigma_m \wr H) \xrightarrow{\;\Psi_{\bar{a}}^*\;} M(K) \; .$$

When $A = G$ with left G-action by translation, then $\langle G|-\rangle$ specializes to the norm map N_H^G. As in this special case, the homomorphism $\Psi_{\bar{a}}$ is independent up to conjugacy of the choice of H-basis, so the map $\langle A|-\rangle$ does not depend on the choice. Moreover, $\langle A|-\rangle$ depends only on the isomorphism class of A as a $(K \times H^{\mathrm{op}})$-space.

We let B be another $(K \times H^{\mathrm{op}})$-space that consists of finitely many free H-orbits, and $\bar{b} = (b_1, \dots, b_n)$ an H-basis of B. Then $\bar{a} + \bar{b} = (a_1, \dots, a_m, b_1, \dots, b_n)$ is an H-basis of the disjoint union $A \amalg B$. Moreover, the associated homomorphism $\Psi_{\bar{a}+\bar{b}}$ is the composite

$$K \xrightarrow{(\Psi_{\bar{a}}, \Psi_{\bar{b}})} (\Sigma_m \wr H) \times (\Sigma_n \wr H) \xrightarrow{\;\Phi_{m,n}\;} \Sigma_{m+n} \wr H \; .$$

So we conclude that

$$\begin{aligned} (1.22) \qquad \langle A \amalg B | x \rangle \;&=\; \Psi^*_{\bar{a}+\bar{b}}(P^{m+n}(x)) \;=\; (\Psi_{\bar{a}}, \Psi_{\bar{b}})^*(\Phi^*_{m,n}(P^{m+n}(x))) \\ &=\; (\Psi_{\bar{a}}, \Psi_{\bar{b}})^*(P^m(x) \times P^n(x)) \;=\; \Psi^*_{\bar{a}}(P^m(x)) \cdot \Psi^*_{\bar{b}}(P^n(x)) \;=\; \langle A|x\rangle \cdot \langle B|x\rangle \; . \end{aligned}$$

Now we let L be a subgroup of finite index of K and $\alpha : L \longrightarrow H$ a continuous homomorphism. We let $\bar{k} = (k_1, \dots, k_m)$ be an L-basis of K, so that $m = [K : L]$. Then the $(K \times H^{\mathrm{op}})$-space

$$K \times_\alpha H \;=\; (K \times H)/(kl, h) \sim (k, \alpha(l)h)$$

consists of m free H-orbits, and $\bar{a} = ([k_1, 1], \dots, [k_m, 1])$ is an H-basis of $K \times_\alpha H$. Moreover, the associated homomorphism $\Psi_{\bar{a}}$ is the composite

$$K \xrightarrow{\;\Psi_{\bar{k}}\;} \Sigma_m \wr L \xrightarrow{\;\Sigma_m \wr \alpha\;} \Sigma_m \wr H \; .$$

So we conclude that

$$(1.23) \qquad \langle K \times_\alpha H | x \rangle \;=\; \Psi^*_{\bar{a}}(P^m(x)) \;=\; \Psi^*_{\bar{k}}((\Sigma_m \wr \alpha)^*(P^m(x))) \;=\; \Psi^*_{\bar{k}}(P^m(\alpha^*(x))) \;=\; N_L^K(\alpha^*(x)) \; .$$

Now we can prove the double coset formula. We let $R \subset G$ be a set of double coset representative, so that G is the disjoint union of the K-H-double cosets KgH for $g \in R$. Moreover, KgH is isomorphic, as a $(K \times H^{\mathrm{op}})$-space, to $K \times_{c_g} H$, where $c_g : K \cap {}^gH \longrightarrow H$ is conjugation by g. So we deduce

$$\begin{aligned} \mathrm{res}^G_K(N_H^G(x)) \;&=\; \mathrm{res}^G_K \langle {}_G G_H | x \rangle \;=\; \langle {}_K G_H | x \rangle \;=_{(1.22)}\; \prod_{g \in R} \langle KgH | x \rangle \\ &=\; \prod_{g \in R} \langle K \times_{c_g} H | x \rangle \;=_{(1.23)}\; \prod_{g \in R} N^K_{K \cap {}^gH}(c_g^*(\mathrm{res}^H_{K^g \cap H}(x))) \; . \end{aligned}$$

(v) If $\bar{k} = (k_1, \dots, k_m)$ is an L-basis of K, then $\alpha(\bar{k}) = (\alpha(k_1), \dots, \alpha(k_m))$ is an H-basis of G. With respect to these bases we have

$$(\Sigma_m \wr \alpha|_L) \circ \Phi_{\bar{k}} \;=\; \Phi_{\alpha(\bar{k})} \circ \alpha \;:\; K \longrightarrow \Sigma_m \wr H \; .$$

So

$$\alpha^* \circ N_H^G \ = \ \alpha^* \circ \Phi^*_{\alpha(\bar{k})} \circ P^m \ = \ \Phi^*_{\bar{k}} \circ (\Sigma_m \wr \alpha|_L)^* \circ P^m \ = \ \Phi^*_{\bar{k}} \circ P^m \circ \alpha|_L^* \ = \ N_L^K \circ \alpha|_L^* \ . \qquad \square$$

2. Examples

In this section we discuss various examples of commutative orthogonal monoid spaces.

Example 2.1 (Orthogonal group monoid space)**.** We denote by $\mathbf{O}$ the orthogonal space that sends a real inner product space V to its orthogonal group $O(V)$. A linear isometric embedding $\varphi : V \longrightarrow W$ induces a continuous group homomorphism $\mathbf{O}(\varphi) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$ by conjugation (and the identity on the orthogonal complement of the image of φ). A commutative multiplication

$$\mu_{V,W} \ : \ \mathbf{O}(V) \times \mathbf{O}(W) \longrightarrow \ \mathbf{O}(V \oplus W)$$

on this orthogonal space is given by direct sum of orthogonal transformations. The unit element of $\mathbf{O}(V)$ is the identity of V.

If G is a compact Lie group and V a G-representation, then the G-action on the group $\mathbf{O}(V) = O(V)$ is by conjugation, so the fixed points $\mathbf{O}(V)^G$ is the group of G-equivariant orthogonal automorphisms of V. So $\mathbf{O}(\mathcal{U}_G)$ is the orthogonal group of $\mathcal{U}_G$, i.e., $\mathbb{R}$-linear isometries of $\mathcal{U}_G$ (not necessarily G-equivariant) that are the identity on the orthogonal complement of some finite dimensional subspace; the G-action is again by conjugation. Any G-equivariant isometry preserves the decomposition of $\mathcal{U}_G$ into isotypical summands, and the restriction to almost all of these isotypical summands must be the identity. The G-fixed subgroup is thus given by

$$\mathbf{O}(\mathcal{U}_G)^G \ = \ O^G(\mathcal{U}_G) \ = \ {\prod_{[\lambda]}}' O^G(\mathcal{U}_\lambda) \ ,$$

where the weak product is indexed by the isomorphism classes of irreducible orthogonal G-representations. If the compact Lie group G is finite, then there are only finitely many isomorphism classes of irreducible G-representations, so in that case the weak product coincides with the product.

Irreducible orthogonal representations come in three different flavors, and the group $O^G(\mathcal{U}_\lambda)$ has one of three different forms. If λ is an irreducible orthogonal G-representation, then the endomorphism ring $\mathrm{Hom}_{\mathbb{R}G}(\lambda, \lambda)$ is a finite dimensional skew field extension of $\mathbb{R}$, so it is isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$; the representation λ is accordingly called 'real', 'complex' respectively 'quaternionic'. We have

$$O^G(\mathcal{U}_\lambda) \ \cong \ O^G(\lambda^\infty) \ \cong \ \begin{cases} O & \text{if } \lambda \text{ is real,} \\ U & \text{if } \lambda \text{ is complex, and} \\ Sp & \text{if } \lambda \text{ is quaternionic.} \end{cases}$$

So we conclude that the G-fixed point space $\mathbf{O}(\mathcal{U}_G)^G$ is a weak product of copies of the infinite orthogonal, unitary and symplectic groups, indexed by the different types of irreducible orthogonal representations of G. Since the infinite unitary and symplectic groups are connected, only the 'real' factors contribute to $\pi_0(\mathbf{O}(\mathcal{U}_G)^G) = \pi_0^G(\mathbf{O})$, which is a weak product of copies of $\pi_0(O) = \mathbb{Z}/2$ indexed by the irreducible G-representations of real type.

There is a straightforward 'special orthogonal' analog: the property of having determinant 1 is preserves under conjugation by linear isometric embeddings and under direct sum of orthogonal operators, so the spaces $SO(V)$ form an orthogonal monoid subspace $\mathbf{SO}$ of $\mathbf{O}$.

Example 2.2 (Additive Grassmannian)**.** We define a commutative orthogonal monoid space $\mathbf{Gr}$, the *additive Grassmannian.* The value of $\mathbf{Gr}$ at an inner product space V is

$$\mathbf{Gr}(V) \ = \ \coprod_{n \geq 0} Gr_n(V) \ ,$$

the disjoint union of all Grassmannians in V. The structure map $\mathbf{Gr}(\alpha) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(W)$ induced by a linear isometric embedding $\alpha : V \longrightarrow W$ is given by $\mathbf{Gr}(\alpha)(L) = \alpha(L)$. A commutative multiplication on $\mathbf{Gr}$ is given by direct sum:

$$\mu_{V,W} \;:\; \mathbf{Gr}(V) \times \mathbf{Gr}(W) \;\longrightarrow\; \mathbf{Gr}(V \oplus W) \;, \quad (L, L') \;\longmapsto\; L \oplus L' \;.$$

The multiplicative unit is the only point $\{0\}$ in $\mathbf{Gr}(0)$. The orthogonal space $\mathbf{Gr}$ is naturally $\mathbb{N}$-graded, with degree n part given by $\mathbf{Gr}^{[n]}(V) = Gr_n(V)$. The multiplication is homogeneous in that it sends $\mathbf{Gr}^{[n]}(V) \times \mathbf{Gr}^{[m]}(W)$ to $\mathbf{Gr}^{[n+m]}(V \oplus W)$.

As a orthogonal space, $\mathbf{Gr}$ is the disjoint union of global classifying spaces of the orthogonal groups. Indeed, the homeomorphism

$$\mathbf{L}(\mathbb{R}^n, V)/O(n) \;\cong\; \mathbf{Gr}^{[n]}(V) \;, \quad \varphi \cdot O(n) \;\longmapsto\; \varphi(\mathbb{R}^n)$$

shows that the homogeneous summand $\mathbf{Gr}^{[n]}$ is isomorphic to the free orthogonal space $\mathbf{L}_{O(n),\mathbb{R}^n}$. Since the tautological action of $O(n)$ on $\mathbb{R}^n$ is faithful, this is indeed a global classifying space for $B_{\mathrm{gl}}O(n)$ for the orthogonal group. So as orthogonal spaces,

$$\mathbf{Gr} \;\cong\; \coprod_{n \geq 0} B_{\mathrm{gl}}O(n) \;. \tag{2.3}$$

Proposition 2.6 (iv) identifies the equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}O(n))$ with the set of conjugacy classes of continuous homomorphisms from G to $O(n)$; by restricting the tautological $O(n)$-representation on $\mathbb{R}^n$, this set bijects with the set $\mathbf{RO}^+(G)$ of isomorphism classes of n-dimensional G-representations. An explicit isomorphism of monoids is given as follows. We let V be a finite dimensional orthogonal G-representation. The G-fixed points of $\mathbf{Gr}(V)$ are the G-invariant subspaces of V, i.e., the G-subrepresentations. We define a map

$$\mathbf{Gr}(V)^G \;=\; \coprod_{n \geq 0} (Gr_n(V))^G \;\longrightarrow\; \mathbf{RO}^+(G)$$

from this fixed point space to the monoid of isomorphism classes of G-representations by sending $W \in \mathbf{Gr}(V)^G$ to the isomorphism class of W. Representations of compact Lie groups are discrete, so the isomorphism class of W only depends on the path component of W in $\mathbf{Gr}(V)^G$, and the resulting maps $\pi_0(\mathbf{Gr}(V)^G) \longrightarrow \mathbf{RO}^+(G)$ are compatible as V runs through the finite dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{Gr}) \;=\; \operatorname{colim}_{V \in s(\mathcal{U}_G)} \pi_0(\mathbf{Gr}(V)^G) \;\longrightarrow\; \mathbf{RO}^+(G) \;. \tag{2.4}$$

and this map is an isomorphism of monoids with respect to the direct sum of representations on the target. Moreover, the isomorphism is compatible with restriction maps, and it takes the norm maps induced by the commutative multiplication of $\mathbf{Gr}$ amounts to induction of representations on the right hand side.

We can also describe explicitly the iterated box products of copies of $\mathbf{Gr}$. For $k \geq 1$ we let $\mathbf{Gr}_{\langle k \rangle}$ be the 'additive Grassmannian of k-tuples of pairwise orthogonal subspaces'. The value of this orthogonal space at an inner product space V is thus defined as

$$\mathbf{Gr}_{\langle k \rangle}(V) \;=\; \{(L_1, \dots, L_k) \in \mathbf{Gr}(V)^k \mid L_i \perp L_j \text{ for } i \neq j\} \;.$$

A linear isometric embedding $\alpha : V \longrightarrow W$ is given by

$$\mathbf{Gr}_{\langle k \rangle}(\alpha)(L_1, \dots, L_k) \;=\; (\alpha(L_1), \dots, \alpha(L_k)) \;;$$

a commutative multiplication on $\mathbf{Gr}_{\langle k \rangle}$ is given by direct sum as for $\mathbf{Gr}$, in each of the k-factors separately. So $\mathbf{Gr}_{\langle k \rangle}$ is a commutative orthogonal monoid space; for $k = 1$ we get $\mathbf{Gr}_{\langle 1 \rangle} = \mathbf{Gr}$, the additive Grassmannian. The orthogonal monoid space $\mathbf{Gr}_{\langle k \rangle}$ is naturally $\mathbb{N}^k$-graded, with homogeneous summand of degree $n = (n_1, \dots, n_k)$ given by those tuples $(L_1, \dots, L_k)$ of orthogonal subspaces such that $\dim(L_i) = n_i$ for $i = 1, \dots, k$.

Given inner product spaces $V_1, \dots, V_k$, the map

$$\mathbf{Gr}(V_1) \times \cdots \times \mathbf{Gr}(V_k) \longrightarrow \mathbf{Gr}_{\langle k \rangle}(V_1 \oplus \cdots \oplus V_k) \ , \quad (L_1, \dots, L_k) \longmapsto (i_1(L_1), \dots, i_k(L_k))$$

is a multi-morphism from $(\mathbf{Gr}, \dots, \mathbf{Gr})$ to $\mathbf{Gr}_{\langle k \rangle}$, where $i_l : V_l \longrightarrow V_1 \oplus \cdots \oplus V_k$ is the embedding of the l-th summand. The multi-morphism is represented by a unique morphism of orthogonal spaces

$$\underbrace{\mathbf{Gr} \boxtimes \cdots \boxtimes \mathbf{Gr}}_{k} \longrightarrow \mathbf{Gr}_{\langle k \rangle} \ .$$

The multiplication morphism of the additive Grassmannian then factors as the composite

$$\mathbf{Gr} \boxtimes \mathbf{Gr} \longrightarrow \mathbf{Gr}_{\langle 2 \rangle} \xrightarrow{\oplus} \mathbf{Gr}$$

where the second map takes the direct sum of a pair of orthogonal subspaces.

Proposition 2.5. *The morphism*

$$\mathbf{Gr}^{\boxtimes k} \longrightarrow \mathbf{Gr}_{\langle k \rangle}$$

is an isomorphism of commutative orthogonal monoid spaces.

Proof. We give the proof for $k = 2$, the general case differing only by more complicated notation. The identification (2.3) of $\mathbf{Gr}$ as a disjoint union of global classifying spaces and the box product formula [...] for free orthogonal space combine into an isomorphism

$$\begin{aligned} \mathbf{Gr} \boxtimes \mathbf{Gr} \ &\cong \ \Big(\coprod_{n \geq 0} \mathbf{L}_{O(n), \mathbb{R}^n}\Big) \boxtimes \Big(\coprod_{m \geq 0} \mathbf{L}_{O(m), \mathbb{R}^m}\Big) \\ &\cong \ \coprod_{n,m \geq 0} \mathbf{L}_{O(n), \mathbb{R}^n} \boxtimes \mathbf{L}^{\mathbb{C}}_{O(m), \mathbb{R}^m} \ \cong \ \coprod_{n,m \geq 0} \mathbf{L}_{O(n) \times O(m), \mathbb{R}^n \times \mathbb{R}^m} \ . \end{aligned}$$

The orthogonal space $\mathbf{L}_{O(n) \times O(m), \mathbb{R}^n \times \mathbb{R}^m}$ is precisely the homogeneous summand of $\mathbf{Gr}_{\langle 2 \rangle}$ of bidegree (n, m), and the composite isomorphism is the morphism of the proposition. □

Example 2.6 (Periodic global BO)**.** We define a commutative orthogonal monoid space $\mathbf{BOP}$ that is a global refinement of the non-equivariant homotopy type $\mathbb{Z} \times BO$, and at the same time a global group completion of the additive Grassmannian $\mathbf{Gr}$. The orthogonal space $\mathbf{BOP}$ comes tautological vector bundles whose Thom spaces form the periodic unoriented bordism spectrum $\mathbf{MOP}$, compare Example V.7.1 below.

For a finite dimensional real inner product space V we set

$$\mathbf{BOP}(V) \ = \ \coprod_{n \geq 0} Gr_n(V^2) \ ,$$

the disjoint union of the Grassmannians of n-dimensional subspaces in $V^2 = V \oplus V$. The orthogonal group $O(V)$ acts through its diagonal action on V, i.e., via

$$\mathbf{BOP}(A)(L) \ = \ A^2(L)$$

for $A \in O(V)$ and $L \subset V^2$. The structure map $\mathbf{BOP}(i_{V,W}) : \mathbf{BOP}(V) \longrightarrow \mathbf{BOP}(V \oplus W)$ is given by

$$\mathbf{BOP}(i_{V,W})(L) \ = \ \kappa_{V,W}(L \oplus W \oplus 0)$$

where

$$\kappa_{V,W} \ : \ V^2 \oplus W^2 \ \cong \ (V \oplus W)^2$$

is the preferred isometry with $\kappa_{V,W}(v, v', w, w') = (v, w, v', w')$. We have thus defined a orthogonal space $\mathbf{BOP}$. We make $\mathbf{BOP}$ into a commutative orthogonal monoid space by endowing it with multiplication maps

$$\mu_{V,W} \ : \ \mathbf{BOP}(V) \times \mathbf{BOP}(W) \longrightarrow \mathbf{BOP}(V \oplus W) \ , \quad (L, L') \longmapsto \kappa_{V,W}(L \oplus L') \ .$$

The multiplicative unit in $\mathbf{BOP}(V)$ is the subspace $V \oplus 0$.

The orthogonal space $\mathbf{BOP}$ is naturally $\mathbb{Z}$-graded: for $k \in \mathbb{Z}$ we let

$$\mathbf{BOP}^{[k]}(V) \ \subset \ \mathbf{BOP}(V)$$

be the path component consisting of all complex subspaces $L \subset V^2$ such that $\dim(L) - \dim(V) = k$. For fixed k the spaces $\mathbf{BOP}^{[k]}(V)$ form a subfunctor of $\mathbf{BOP}$, i.e., $\mathbf{BOP}^{[k]}$ is a orthogonal subspace of $\mathbf{BOP}$. The multiplication is graded in the sense that $\mu_{V,W} : \mathbf{BOP}(V) \times \mathbf{BOP}(W) \longrightarrow \mathbf{BOP}(V \oplus W)$ takes $\mathbf{BOP}^{[k]}(V) \times \mathbf{BOP}^{[l]}(W)$ to $\mathbf{BOP}^{[k+l]}(V \oplus W)$. We write $\mathbf{BO} = \mathbf{BOP}^{[0]}$ for the homogeneous summand of $\mathbf{BOP}$ of degree 0, which is thus a commutative orthogonal monoid space in its own right.

While $\mathbf{BOP}$ and the additive Grassmannian $\mathbf{Gr}$ are both made from Grassmannians, one should beware of the different nature of their structure maps. There is a variation $\mathbf{Gr}'$ of the additive Grassmannian with values $\mathbf{Gr}'(V) = \coprod_{n \geq 0} Gr_n(V^2)$ and structure maps $\mathbf{Gr}'(\alpha)(L) = \alpha^2(L)$. This orthogonal space admits a commutative multiplication in much the same was as $\mathbf{Gr}$ and the maps

$$\mathbf{Gr}(V) \ \longrightarrow \ \mathbf{Gr}'(V) \ , \quad L \ \longmapsto \ L \oplus 0$$

form a global equivalence of orthogonal monoid spaces. A source of possible confusion is the fact that $\mathbf{Gr}'(V)$ and $\mathbf{BOP}(V)$ are *equal* as spaces, but they come with very different structure maps making them into two different global homotopy types.

The underlying non-equivariant homotopy type of $\mathbf{BO} = \mathbf{BOP}^{[0]}$ is that of a classifying space of the infinite orthogonal group; similarly, $\mathbf{BOP}$ has underlying non-equivariant homotopy type $\mathbb{Z} \times BO$, a $\mathbb{Z}$-indexed union copies of BO. More generally we will now identify the global power monoid $\underline{\pi}_0(\mathbf{BOP})$ with $\mathbf{RO}$, the 'additive' global power monoid of the complex representation ring functor. For every compact Lie group G the abelian monoid is $\mathbf{RO}$ the Grothendieck group, under direct sum, of finite dimensional G-representations. The restriction maps are induced by restriction of representations, and the power operation $P^m : \mathbf{RO}(G) \longrightarrow \mathbf{RO}(\Sigma_m \wr G)$ takes the class of a G-representation V to the class of the $(\Sigma_m \wr G)$-representation V^m. The resulting norm map $N_H^G : \mathbf{RO}(H) \longrightarrow \mathbf{RO}(G)$ of Construction 1.20, for H of finite index in G, is then the additive transfer (or induction), sending the class of an H-representation V to the class of the induced G-representation $\mathrm{map}^H(G, V)$.

We let V be a finite dimensional orthogonal G-representation. The G-fixed points of $\mathbf{BOP}(V)$ are the G-invariant subspaces of V^2, i.e., the G-subrepresentations. We define a map

$$(\mathbf{BOP}(V))^G \ = \ \coprod_{n \geq 0} \left(Gr_n(V^2)\right)^G \ \longrightarrow \ \mathbf{RO}(G)$$

from this fixed point space to the complex representation ring of G by sending $W \in \mathbf{BOP}(V)^G$ to $[W] - [V]$, the formal difference in $\mathbf{RO}(G)$ of the classes of W and V. Representations of compact Lie groups are discrete, so the image in $\mathbf{RO}(G)$ only depends on the path component of W in $\mathbf{BOP}(V)^G$, and the resulting maps $\pi_0(\mathbf{BOP}(V)^G) \longrightarrow \mathbf{RO}(G)$ are compatible as V runs through the finite dimensional G-subrepresentations of $\mathcal{U}_G$. So they assemble into a map

$$\pi_0^G(\mathbf{BOP}) \ = \ \mathrm{colim}_{V \in s(\mathcal{U})} \, \pi_0(\mathbf{BOP}(V)^G) \ \longrightarrow \ \mathbf{RO}(G) \ . \tag{2.7}$$

The next result shows that this map is an isomorphism, so the monoid $\pi_0^G(\mathbf{BOP})$ is a group and the commutative orthogonal monoid space $\mathbf{BOP}$ is group-like. In fact, $\mathbf{BOP}$ is a group completion of the additive Grassmannian $\mathbf{Gr}$, as we will see in Theorem 4.10 below.

Theorem 2.8. *For every compact Lie group G the map* (2.7) *is bijective. As G varies, these bijections form an isomorphism of global power monoids*

$$\underline{\pi}_0(\mathbf{BOP}) \ \cong \ \mathbf{RO}$$

to the additive global power monoid of the real representation ring functor. The G-fixed point space of $\mathbf{BOP}$ *is a disjoint union, indexed by* $\mathbf{RO}(G)$*, of classifying spaces of the group* $O^G(\mathcal{U}_G)$ *of* G*-equivariant orthogonal isometries of the complete* G*-universe:*

$$(\mathbf{BOP}(\mathcal{U}_G))^G \ \simeq \ \mathbf{RO}(G) \times B(O^G(\mathcal{U}_G)) \ .$$

Proof. We start by showing that the map (2.7) is bijective. A class in $\mathbf{RO}(G)$ is a formal difference $[W]-[V]$ for two orthogonal G-representations W and V. This class is not a priori in the image of (2.7), because W need not be isomorphic to a G-subrepresentation of V^2. However, we can enlarge W and V to arrange this. Indeed,

$$[W] \ - \ [V] \ = \ [W \oplus W] \ - \ [V \oplus W] \ ,$$

and $W \oplus W$ is isomorphic to the G-subrepresentation of $(0 \oplus W)^2$ of $(V \oplus W)^2$. So $(0 \oplus W)^2$ is a G-fixed point in $\mathbf{BOP}(V \oplus W)^G$ whose class in $\pi_0^G(\mathbf{BOP})$ maps to the original class in $\mathbf{R}(G)$ under (2.7). So the map (2.7) is surjective.

Now we consider two G-fixed points $W \in (\mathbf{BOP}(V))^G$ and $W' \in (\mathbf{BOP}(V'))^G$ that are taken to the same class in $\mathbf{RO}(G)$ by (2.7); we want to show that W and W' represent the same element in $\pi_0^G(\mathbf{BOP})$. By replacing W by $\mathbf{BOP}(i_{V,V'})(W) \in (\mathbf{BOP}(V \oplus V'))^G$ and W' by $\mathbf{BOP}(\tau_{V',V} \circ i_{V',V})(W') \in (\mathbf{BOP}(V \oplus V'))^G$, if necessary, we can assume without loss of generality that $V = V'$. Then

$$[W] \ - \ [V] \ = \ [W'] \ - \ [V]$$

in $\mathbf{RO}(G)$, which implies that W and W' are isomorphic as G-representations. But then W and W' lie in the same path component of the fixed point space $\mathbf{BOP}(V)^G$ [justify], so they represent the same element in $\pi_0^G(\mathbf{BOP})$.

It is straightforward to check that the bijections (2.7) are compatible with restriction maps along continuous group homomorphisms – the reason is basically that restriction maps on both sides do not change the underlying representative and only modify the group action. Similarly, compatibility with products and power operations (i.e., sums and transfers) is straightforward from the definitions.

It remains to discuss the homotopy type of the G-fixed point space $(\mathbf{BOP}(\mathcal{U}_G))^G$. As we explained in Remark 1.1, the commutative multiplication of $\mathbf{BOP}$ makes the G-space $\mathbf{BOP}(\mathcal{U}_G)$. into an E_∞ G-space; so the fixed points $\mathbf{BOP}(\mathcal{U}_G)^G$ come with the structure of a non-equivariant E_∞-space. The abelian monoid of path components $\pi_0(\mathbf{BOP}(\mathcal{U}_G)^G)$ is isomorphic to $\pi_0^G(\mathbf{BOP}) \cong \mathbf{RO}(G)$ hence an abelian group. So all path components of the space $\mathbf{BUP}(\mathcal{U}_G)^G$ are homotopy equivalent. It remains to show that the path component of the unit element in $\mathbf{BOP}(\mathcal{U}_G)^G$ is a classifying space for the group $O^G(\mathcal{U}_G)$. [...] □

The bijection (2.7) sends elements of $\pi_0^G(\mathbf{BOP}^{[k]})$ to virtual representations of dimension k, so we can also identify the global power monoid of the homogeneous degree 0 part $\mathbf{BO} = \mathbf{BOP}^{[0]}$. Indeed, the map (2.7) restricts to an isomorphism of abelian groups

$$\pi_0^G(\mathbf{BO}) \ \cong \ IO(G)$$

to the augmentation ideal $IO(G) \subset \mathbf{RO}(G)$ of the real representation ring, compatible with restriction maps and power operations. Moreover,

$$(\mathbf{BO}(\mathcal{U}_G))^G \ \simeq \ IO(G) \times B(O^G(\mathcal{U}_G)) \ .$$

As we explained in Example 2.1, the group $O^G(\mathcal{U}_G)$ is a weak product of infinite orthogonal, unitary and symplectic group, indexed by the isomorphism classes of irreducible G-representations. The classifying space construction commutes with weak products, so altogether we obtain a chain of weak equivalences

$$(\mathbf{BOP}(\mathcal{U}_G))^G \ \simeq \ \mathbf{RO}(G) \times B(O^G(\mathcal{U}_G)) \ \ \simeq \ \mathbf{RO}(G) \times {\prod_{[\lambda]}}' B(O^G(\mathcal{U}_\lambda))$$

where the weak product is indexed by the isomorphism classes of irreducible complex G-representations.

Example 2.9 (**Gr** versus **BOP**). In Example 2.2 we explained that for every compact Lie group G, the monoid $\pi_0^G(\mathbf{Gr})$ is isomorphic to the monoid $\mathbf{RO}^+(G)$, under direct sum, of isomorphism classes of complex G representations. In Theorem 2.8 we identified the monoid $\pi_0^G(\mathbf{BOP})$ with the real representation ring $\mathbf{RO}(G)$. The latter is the algebraic group completion of the former, and this group completion is realized by a morphism of orthogonal monoid spaces

$$i \ : \ \mathbf{Gr} \ \longrightarrow \ \mathbf{BOP} \ . \tag{2.10}$$

The morphism i is given at an inner product space V by the map

$$\mathbf{Gr}(V) \ = \ \coprod_n Gr_n(V) \ \longrightarrow \ \coprod_m Gr_m(V^2) = \mathbf{BOP}(V) \ , \quad L \ \longmapsto \ L \oplus V \ .$$

The morphism is homogeneous in that it takes $\mathbf{Gr}^{[n]}$ to $\mathbf{BOP}^{[n]}$, and it is straightforward from the definitions that the square

$$\begin{tikzcd} \underline{\pi}_0(\mathbf{Gr}) \arrow[r, "\underline{\pi}_0(i)"] \arrow[d, "(2.4)"', "\cong"] & \underline{\pi}_0(\mathbf{BOP}) \arrow[d, "\cong"', "(2.7)"] \\ \mathbf{RO}^+ \arrow[r] & \mathbf{RO} \end{tikzcd}$$

commutes, where the lower horizontal map is group completion. So for every compact Lie group G the induced map

$$\pi_0^G(i) \ : \ \pi_0^G(\mathbf{Gr}) \ \longrightarrow \ \pi_0^G(\mathbf{BOP})$$

is an algebraic group completion. This fact is the $\underline{\pi}_0$-shadow of a more refined relationship: as we will show in Theorem 4.10 below, the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is a group completion in the world of commutative orthogonal monoid spaces, i.e., 'homotopy-initial', in the category of commutative orthogonal monoid spaces, among morphisms from **Gr** to group-like commutative orthogonal monoid spaces.

Example 2.11 (Multiplicative Grassmannian). We define a commutative orthogonal monoid space $\mathbf{Gr}_\otimes$, the *multiplicative Grassmannian*. Given a finite dimensional complex inner product space V we let

$$\mathrm{Sym}(V) \ = \ \bigoplus_{n\geq 0} V^{\otimes n}/\Sigma_n$$

denote the symmetric algebra of V (where $\otimes$ denotes the tensor product over $\mathbb{R}$). If W is another real inner product space, then the two summand inclusions of V and W into $V \oplus W$ induce algebra homomorphisms

$$\mathrm{Sym}(V) \ \longrightarrow \ \mathrm{Sym}(V \oplus W) \ \longleftarrow \ \mathrm{Sym}(W)$$

and we use the commutative multiplication on $\mathrm{Sym}(V \oplus W)$ to combine these into a natural isomorphism

$$\mathrm{Sym}(V) \otimes \mathrm{Sym}(W) \ \cong \ \mathrm{Sym}(V \oplus W) \ . \tag{2.12}$$

This map is in fact an isomorphism of inner product spaces and it is $O(V) \times O(W)$-equivariant.

Now we are ready to define the multiplicative Grassmannian orthogonal monoid space. The value of $\mathbf{Gr}_\otimes$ at an inner product space V is

$$\mathbf{Gr}_\otimes(V) \ = \ \coprod_{n\geq 0} Gr_n(\mathrm{Sym}(V)) \ ,$$

the disjoint union of all complex Grassmannians in the symmetric algebra of V. The structure maps $\mathbf{Gr}_\otimes(\alpha) : \mathbf{Gr}_\otimes(V) \longrightarrow \mathbf{Gr}_\otimes(W)$ induced by a linear isometric embedding $\alpha : V \longrightarrow W$ is given by

$$\mathbf{Gr}_\otimes(\alpha)(L) \ = \ \mathrm{Sym}(\alpha)(L) \ ,$$

where $\mathrm{Sym}(\alpha) : \mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(W)$ is the induced map of symmetric algebras. A commutative multiplication on $\mathbf{Gr}_\otimes$ is given by tensor product, i.e.,

$$\mu_{V,W} \ : \ \mathbf{Gr}_\otimes(V) \times \mathbf{Gr}_\otimes(W) \ \longrightarrow \ \mathbf{Gr}_\otimes(V \oplus W)$$

sends $(L, L') \in \mathbf{Gr}_\otimes(V) \times \mathbf{Gr}_\otimes(W)$ to the image of $L \otimes L'$ under the isomorphism (2.12). The multiplicative unit is the point $\mathbb{R}$ in $\mathbf{Gr}_\otimes(0) = \mathbb{R}$.

As the additive Grassmannian $\mathbf{Gr}$, the multiplicative Grassmannian $\mathbf{Gr}_\otimes$ is $\mathbb{N}$-graded, with degree n part given by $\mathbf{Gr}_\otimes^{[n]}(V) = Gr_n(\mathrm{Sym}(V))$. The multiplication sends $\mathbf{Gr}_\otimes^{[n]}(V) \times \mathbf{Gr}_\otimes^{[m]}(W)$ to $\mathbf{Gr}_\otimes^{[n\cdot m]}(V \oplus W)$.

As orthogonal spaces, the additive and multiplicative Grassmannians are globally equivalent. Indeed, for an inner product space V we let $i : V \longrightarrow \mathrm{Sym}(V)$ be the embedding as the linear summand of the symmetric algebra. Then as V varies, the maps

$$\mathbf{Gr}(V) \ = \ \coprod_{n\geq 0} Gr_n(V) \ \longrightarrow \ \coprod_{n \geq 0} Gr_n(\mathrm{Sym}(V)) \ = \ \mathbf{Gr}_\otimes(V) \ , \quad L \ \longmapsto \ i(L)$$

form a global equivalence $\mathbf{Gr} \longrightarrow \mathbf{Gr}_\otimes$. This global equivalence induces a bijection

$$\pi_0^G(\mathbf{Gr}) \ \cong \ \pi_0^G(\mathbf{Gr}_\otimes)$$

for every compact Lie group G, hence both are isomorphic to the set $\mathbf{RO}^+(G)$ of isomorphism classes of orthogonal G-representations. The commutative monoid structures and norm maps induced by the products of $\mathbf{Gr}$ respectively $\mathbf{Gr}_\otimes$ are quite different though: the monoid structure of $\pi_0^G(\mathbf{Gr})$ corresponds to direct sum of representations, and the norm maps are additive transfers; the monoid structure of $\pi_0^G(\mathbf{Gr}_\otimes)$ corresponds to tensor product of representations, and the norm maps are multiplicative transfers.

Now we define two orthogonal spaces, one with a commutative multiplication, that are 'global forms of BO', by which we mean that they are non-equivariant weakly equivalent to the classifying space of the infinite orthogonal group. Both are *not*, however, globally equivalent to the orthogonal monoid spaces $\mathbf{BO}$.

Remark 2.13. The infinite orthogonal group O is the colimit of the finite orthogonal groups $O(n)$, so non-equivariantly, the classifying space BO is the homotopy colimit of the classifying spaces $BO(n)$. This could lead one to expect a similar behavior for the global refinements, i.e., one could think that $\mathbf{BO}$ is a global homotopy colimit, in the category of orthogonal spaces, of the global classifying spaces $B_{\mathrm{gl}}O(n)$ as n goes to infinity. However, this is not the case, as we illustrate now.

The tautological action of $O(n)$ on $\mathbb{R}^n$ is faithful, so the free orthogonal space $\mathbf{L}_{O(n),\mathbb{R}^n}$ is a global classifying space $B_{\mathrm{gl}}O(n)$ for the orthogonal group $O(n)$. The homomorphism $O(n) \longrightarrow O(n+1)$ sending a linear isometry φ of $\mathbb{R}^n$ to the isometry $\varphi \oplus \mathbb{R}$ of $\mathbb{R}^{n+1}$ induces a weak morphism of global classifying spaces from $B_{\mathrm{gl}}O(n)$ to $B_{\mathrm{gl}}O(n+1)$. More precisely, we consider the zigzag of morphisms of orthogonal spaces

$$B_{\mathrm{gl}}O(n) = \mathbf{L}_{O(n),\mathbb{R}^n} \ \xleftarrow[\simeq]{\lambda_{O(n),\mathbb{R}^n,\mathbb{R}}} \ \mathbf{L}_{O(n),\mathbb{R}^{n+1}} \ \longrightarrow \ \mathbf{L}_{O(n+1),\mathbb{R}^{n+1}} = B_{\mathrm{gl}}O(n+1) \ . \tag{2.14}$$

The left restriction map is a global equivalence by Proposition 2.6 (v). By Proposition 2.6 (iv), the 0-th equivariant homotopy set $\pi_0^G(B_{\mathrm{gl}}O(n))$ bijects with the set of conjugacy classes of continuous homomorphisms from G to $O(n)$, i.e., with the set of isomorphism classes of n-dimensional orthogonal G-representations. Under this bijection, the map $\pi_0^G(B_{\mathrm{gl}}O(n)) \longrightarrow \pi_0^G(B_{\mathrm{gl}}O(n+1))$ induced by the zigzag (2.14) sends the class of a G-representation V to the class of $V \oplus \mathbb{R}$, the sum with a trivial 1-dimensional representation. So the colimit over n, of the sets $\pi_0^G(B_{\mathrm{gl}}O(n))$, bijects with the set of isomorphism classes of G-representations with trivial fixed points. This is different from the augmentation ideal of the real representation ring of G, which is the answer for $\pi_0^G(\mathbf{BO})$ given by Theorem 2.8; so $\mathbf{BO}$ is *not* a global homotopy colimit of the orthogonal spaces $B_{\mathrm{gl}}O(n)$. Indeed, $\mathbf{BO}$ has a built-in stabilization by arbitrary G-representations, whereas the homotopy colimit of the $B_{\mathrm{gl}}O(n)$ only stabilizes by trivial representations.

Example 2.15 (Bar construction model $B\mathbf{O}$)**.** Using a functorial bar construction we define yet another global refinement $B\mathbf{O}$ of the classifying space of the infinite orthogonal group. Given a topological group G, the *bar construction* is the simplicial topological group whose space of n-simplices is G^n, the n-fold cartesian power of G. For $n \geq 1$ and $0 \leq i \leq n$, the face map $d_i : G^n \longrightarrow G^{n-1}$ is given by

$$d_i(g_1, \ldots g_n) \ = \begin{cases} (g_2, \ldots, g_n) & \text{for } i = 0, \\ (g_1, \ldots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \ldots, g_n) & \text{for } 0 < i < n, \\ (g_1, \ldots, g_{n-1}) & \text{for } i = n. \end{cases}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ the degeneracy map $s_i : (BG)_{n-1} \longrightarrow (BG)_n$ is given by

$$s_i(g_1, \ldots, g_{n-1}) \ = \ (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_{n-1}) \ .$$

The geometric realization

$$BG \ = \ |[n] \mapsto G^n|$$

is a classifying space for G and $G \mapsto BG$ is functorial in continuous group homomorphisms. The bar construction commutes with products in the sense that for a pair of topological groups G and H, the canonical map

$$B(G \times H) \ \longrightarrow \ BG \times BH \tag{2.16}$$

is a homeomorphism.

We define a orthogonal space $B\mathbf{O}$ by applying the bar construction objectwise to the orthogonal monoid space $\mathbf{O}$ of Example 2.1. So the value at a complex inner product space V is

$$(B\mathbf{O})(V) \ = \ B(O(V)) \ ,$$

the bar construction of the orthogonal group of V. The structure map of a linear isometric embedding $\alpha : V \longrightarrow W$ is obtained by applying the bar construction to the continuous homomorphism $\mathbf{B}(\alpha) : \mathbf{O}(V) \longrightarrow \mathbf{O}(W)$. We make $B\mathbf{O}$ into a commutative orthogonal monoid space by endowing it with multiplication maps

$$\mu_{V,W} \ : \ (B\mathbf{O})(V) \times (B\mathbf{O})(W) \ \longrightarrow \ (B\mathbf{O})(V \oplus W)$$

defined as the composite

$$B(O(V)) \times B(O(W)) \ \xrightarrow{\cong} \ B\big(O(V) \times O(W)\big) \ \xrightarrow{B\oplus} \ B(O((V \oplus W))) \ ,$$

where the first map is inverse to the homeomorphism (2.16).

Now we let G be a compact Lie group and V a orthogonal G-representation. Then

$$((B\mathbf{O})(V))^G \ = \ (B(O(V)))^G \ = \ B(O^G(V)) \ .$$

Taking colimit over the poset $s(\mathcal{U}_G)$ gives

$$((B\mathbf{O})(\mathcal{U}_G))^G \ \cong \ \operatorname{colim}_{V \in s(\mathcal{U}_G)} \ B(O^G(V)) \ \cong \ B(O^G(\mathcal{U}_G)) \ \cong \ {\prod_{[\lambda]}}' B(O^G(\mathcal{U}_\lambda)) \ .$$

Here the last weak product is indexed by isomorphism classes of irreducible G-representations, and each of the groups $O^G(\mathcal{U}_\lambda)$ is either an infinite orthogonal, unitary or symplectic group, depending on the type of the irreducible representation. In particular, the space $((B\mathbf{O})(\mathcal{U}_G))^G$ is connected, so the equivariant homotopy set $\pi_0^G(B\mathbf{O})$ has one element for every compact Lie group G; the global power monoid structure is then necessarily trivial. In particular, $B\mathbf{O}$ is *not* globally equivalent to $\mathbf{BO}$.

Example 2.17. More variations on the commutative orthogonal monoid space $\mathbf{Gr}$ are possible. We recall from Example 2.19 the global classifying space $B_{\mathrm{gl}}(GL_n(\mathbb{Z}))$ of the linear group $GL_n(\mathbb{Z})$ (which we view as a subgroup of $GL_n(\mathbb{R})$). The disjoint union, over $n \geq 0$, of these orthogonal spaces has a commutative multiplication whose (m, n)-component is the morphism

$$B_{\mathrm{gl}}(GL_m(\mathbb{Z})) \boxtimes B_{\mathrm{gl}}(GL_n(\mathbb{Z})) \ \longrightarrow \ B_{\mathrm{gl}}(GL_{m+n}(\mathbb{Z}))$$

corresponding to the bimorphism with (V, W)-component the direct sum map

$$\begin{array}{ccc}(\mathcal{B}_m(V)/GL_m(\mathbb{Z})) \times (\mathcal{B}_n(W)/GL_n(\mathbb{Z})) & \longrightarrow & \mathcal{B}_{m+n}(V \oplus W)/GL_{m+n}(\mathbb{Z}) \\ ([v_1, \dots, v_m], [w_1, \dots, w_n]) & \longmapsto & [(v_1, 0), \dots, (v_m, 0), (0, w_1), \dots, (0, w_n)] .\end{array}$$

Item (iv) in Example 2.19 identifies the equivariant homotopy set $\pi_0^K(B_{\mathrm{gl}}(GL_n(\mathbb{Z})))$ with the set of conjugacy classes of continuous homomorphisms from K to $GL_n(\mathbb{Z})$; since $GL_n(\mathbb{Z})$ is discrete, any such homomorphism factors uniquely through the finite group $\pi_0 K$ of path components. By restricting the tautological $GL_n(\mathbb{Z})$-representation on $\mathbb{Z}^n$, this set bijects with the set of isomorphism classes of $\pi_0 K$-modules that are free of rank n as abelian groups. So the set

$$\pi_0^K\big(\coprod_{n \geq 0} B_{\mathrm{gl}}(GL_n(\mathbb{Z}))\big) \ = \ \coprod_{n \geq 0} \pi_0^K(B_{\mathrm{gl}} GL_n(\mathbb{Z}))$$

bijects with the set of isomorphism classes of $\pi_0 K$-modules that are finitely generated free as abelian groups; the addition from the commutative multiplication corresponds to the direct sum $\pi_0 K$-modules.

There are complex analogues of the commutative orthogonal monoid spaces $\mathbf{U}$, $\mathbf{Gr}$, $\mathbf{Gr}_\otimes$, $\mathbf{BOP}$, $\mathbf{BO}$ and $\mathbf{P}$ by taking complex subspaces of complex inner product spaces (as opposed to real subspaces of real inner product spaces). We could write the complex analogs down directly as orthogonal monoid spaces, but that would hide some interesting additional structure arising from complex conjugation. We will make the extra structure explicit by working with *unitary spaces* endowed with *Real structures*.

We start by defining unitary spaces and clarifying their relationship with orthogonal spaces. We denote by $\mathbf{L}^{\mathbb{C}}$ the category with objects the finite dimensional complex inner product spaces i.e., a finite dimensional complex vector space equipped with a hermitian scalar product; morphisms are the $\mathbb{C}$-linear isometric embeddings. Much like the orthogonal analog $\mathbf{L}$, the category $\mathbf{L}^{\mathbb{C}}$ is a topological category where the topology on $\mathbf{L}^{\mathbb{C}}(V, W)$ is that of the homogeneous space $U(W)/U(W - V)$.

Definition 2.18. A *unitary space* is a continuous functor $Y : \mathbf{L}^{\mathbb{C}} \longrightarrow \mathbf{U}$ to the category of spaces. A morphism of unitary spaces is a natural transformation.

Construction 2.19 (Unitary versus orthogonal spaces)**.** Unitary and orthogonal spaces have 'the same global homotopy theory'. We make that precise by relating the categories through two functors. Both functors are restriction functors along continuous functors of topological categories:

$$\mathbf{L} \ \underset{r}{\overset{(-)_{\mathbb{C}}}{\rightleftarrows}} \ \mathbf{L}^{\mathbb{C}}$$

The *complexification functor* $(-)_{\mathbb{C}}$ sends a real inner product space V to its complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ equipped with the unique hermitian inner product $\langle -, - \rangle_{\mathbb{C}}$ that satisfies

$$\langle 1 \otimes v, 1 \otimes w \rangle_{\mathbb{C}} \ = \ \langle v, w \rangle$$

for all $v, w \in V$. The complexification of every $\mathbb{R}$-linear isometric embedding preserves the hermitian inner product $\langle -, - \rangle_{\mathbb{C}}$.

The *realification functor* r sends a complex inner product space W to its underlying $\mathbb{R}$-vector space equipped with the euclidean inner product defined by

$$\langle v, w \rangle_{\mathbb{R}} \ = \ \mathrm{Re}\langle v, w \rangle \ ,$$

the real part of the given hermitian inner product. The underlying $\mathbb{R}$-linear map of a complex linear isometric embedding preserves the euclidean inner product $\langle -, - \rangle_{\mathbb{R}}$.

We can now go back and forth between unitary and orthogonal spaces by precomposing with complexification and realification; in other words, we define two functors

$$spc \ \underset{u}{\overset{c}{\rightleftarrows}} \ spc^{\mathbb{C}} \qquad \text{by} \qquad cY \ = \ Y \circ r \qquad \text{respectively} \qquad uZ \ = \ Z \circ (-)_{\mathbb{C}}$$

where Y is an orthogonal space and Z a unitary space.

For every real inner product space V the map $1 \otimes - : V \longrightarrow r(V_{\mathbb{C}})$ is an $\mathbb{R}$-linear isometric embedding, so it induces a continuous map

$$Y(1 \otimes -) \ : \ Y(V) \ \longrightarrow \ Y(r(V_{\mathbb{C}})) \ = \ (ucY)(V) \ .$$

As V varies, these maps define a natural global equivalence of orthogonal spaces

$$(1 \otimes -)_* \ : \ Y \ \longrightarrow \ ucY \ . \tag{2.20}$$

The equivariant homotopy sets $\pi_0^G(uZ)$ of the underlying orthogonal space of a unitary space Z can be calculated directly in 'complex terms' as follows. We denote by $\mathcal{U}_G^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{U}_G$ the complexification of the chosen complete G-universe. This is a complete complex G-universe, i.e., every finite dimensional complex G-representation can be embedded equivariantly into $\mathcal{U}_G^{\mathbb{C}}$. We denote by $\bar{s}(\mathcal{U}_G^{\mathbb{C}})$ the poset of finite dimensional *complex* G-subrepresentations of $\mathcal{U}_G^{\mathbb{C}}$. Complexification

$$s(\mathcal{U}_G) \ \longrightarrow \ \bar{s}(\mathcal{U}_G^{\mathbb{C}}) \ , \quad V \ \longmapsto \ V_{\mathbb{C}}$$

is a morphism of posets. The complex representations of the form $V_{\mathbb{C}}$ with $V \in s(\mathcal{U}_G)$, are cofinal in the poset $\bar{s}(\mathcal{U}_G^{\mathbb{C}})$, so we obtain a bijection on colimits

$$\begin{aligned} \pi_0^G(uZ) \ &= \ \operatorname{colim}_{V \in s(\mathcal{U})} \ \pi_0(((uZ)(V))^G) \\ &= \ \operatorname{colim}_{V \in s(\mathcal{U})} \ \pi_0(Z(V_{\mathbb{C}})^G) \ = \ \operatorname{colim}_{W \in \bar{s}(\mathcal{U}_G^{\mathbb{C}})} \ \pi_0(Z(W)^G) \ . \end{aligned}$$

When working with unitary spaces we will freely use this identification and often calculate $\pi_0^G(uZ)$ as the indicated colimit over the poset $\bar{s}(\mathcal{U}_G^{\mathbb{C}})$.

Definition 2.21. A morphism $f : Y \longrightarrow Z$ of unitary spaces is a *global equivalence* if the morphism $uf : uY \longrightarrow uZ$ is a global equivalence of orthogonal spaces.

So by definition the functor u from unitary to orthogonal space preserves global equivalences. The functor c from orthogonal to unitary spaces preserves global equivalences because of the natural global equivalence (2.20). In fact, the global equivalences of unitary spaces are part of a global model structure and both functors u and c are right Quillen equivalences with respect to the two global model structure. W will not prove that fact, however. [r and c are lax symmetric monoidal]

Now we can define the complex analogs of the commutative orthogonal monoid spaces $\mathbf{U}$, $\mathbf{Gr}$, $\mathbf{Gr}_{\otimes}$, $\mathbf{BOP}$, $\mathbf{BO}$ and $\mathbf{P}$, all of which will be commutative unitary monoid spaces.

Example 2.22 (Unitary group monoid space)**.** The commutative unitary monoid space $\mathbf{U}$ sends a complex inner product space V to its unitary group $U(V)$. The structure and multiplication maps are the complex analogs of the respective data defined $\mathbf{O}$ in Example 2.1 The unit element of $\mathbf{U}(V)$ is the identity of V. As in the real case, the special unitary groups $SU(V)$ form a unitary monoid subspace $\mathbf{SU}$ of $\mathbf{U}$.

For every compact Lie group G, the G-space So $\mathbf{U}(\mathcal{U}_G^{\mathbb{C}})$ is the unitary group of $\mathcal{U}_G^{\mathbb{C}}$ (not necessarily G-equivariant) that are the identity on the orthogonal complement of some finite dimensional subspace; the G-action is again by conjugation. Its G-fixed subgroup is

$$\mathbf{U}(\mathcal{U}_G^{\mathbb{C}})^G \ = \ U^G(\mathcal{U}_G^{\mathbb{C}}) \ = \ {\prod_{[\lambda]}}' U^G(\mathcal{U}_\lambda) \ \cong \ {\prod_{[\lambda]}}' U \ ,$$

where the weak products are indexed by the isomorphism classes of irreducible complex G-representations. The final answer is slightly simpler than its real analog because each factor $U^G(\mathcal{U}_\lambda)$ is isomorphic to the infinite unitary group U. Since the unitary group U is connected, $\mathbf{U}$ is 'globally connected' in the sense that the set $\pi_0^G(\mathbf{U})$ has only one element for every compact Lie group G.

Complexification defines a morphism of commutative orthogonal monoid spaces

$$c \ : \ \mathbf{O} \ \longrightarrow \ u\mathbf{U} \tag{2.23}$$

to the underlying orthogonal monoid space of $\mathbf{U}$. The value $c(V) : O(V) \longrightarrow U(V_{\mathbb{C}}) = (u\mathbf{U})(V)$ at a real inner product space V takes an $\mathbb{R}$-linear isometry to its complexification. Realification defines a morphism of commutative unitary monoid spaces

$$r \ : \ \mathbf{U} \ \longrightarrow \ c\mathbf{O} \ ;$$

the value $r(W) : U(W) \longrightarrow O(rW) = (c\mathbf{O})(W)$ at a complex inner product space W takes an $\mathbb{C}$-linear isometry to its underlying $\mathbb{R}$-linear map. The complexification restricts to a morphism of commutative orthogonal monoid spaces $c : \mathbf{SO} \longrightarrow u(\mathbf{SU})$ and realification restricts to a morphism of commutative unitary monoid spaces $r : \mathbf{SU} \longrightarrow c(\mathbf{SO})$.

Example 2.24 (Complex additive Grassmannian)**.** The commutative unitary monoid space $\mathbf{Gr}^{\mathbb{C}}$, the *complex additive Grassmannian.* sends a complex inner product space V is

$$\mathbf{Gr}^{\mathbb{C}}(V) \ = \ \coprod_{n\geq 0} Gr_n(V) \ ,$$

the disjoint union of all complex Grassmannians in V. As in the real analog in Example 2.2 the structure maps are given by taken images under linear isometric embeddings and the multiplication is by direct sum. The multiplicative unit is the only point $\{0\}$ in $\mathbf{Gr}^{\mathbb{C}}(0)$. The multiplication is again homogeneous for the $\mathbb{N}$-grading by with $\mathbf{Gr}^{\mathbb{C},[n]}(V) = Gr_n(V)$.

The homogeneous summand $\mathbf{Gr}^{\mathbb{C},[n]}$ is isomorphic to the free unitary space $\mathbf{L}^{\mathbb{C}}_{U(n),\mathbb{C}^n} = \mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, -)/U(n)$ and hence a global classifying space for $B_{\mathrm{gl}}U(n)$ for the unitary group. (More precisely, the underlying orthogonal space $u(\mathbf{Gr}^{\mathbb{C},[n]})$ is such a global classifying space).

$$\pi_0^G(\mathbf{Gr}^{\mathbb{C}}) \ = \ \mathrm{colim}_{V\in \bar{s}(\mathcal{U}_G^{\mathbb{C}})} \ \pi_0(\mathbf{Gr}(V)^G) \ \cong \ \mathbf{R}^+(G) \ . \tag{2.25}$$

is given by sending the class of a G-fixed point in $\mathbf{Gr}(V)^G$, i.e., a complex G-subrepresentation of V, to its isomorphism class. The isomorphism is compatible with restriction maps, and it takes the norm maps induced by the commutative multiplication of $\mathbf{Gr}$ amounts to induction of representations. So as G-varies, the maps form an isomorphism of global power monoids $\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \cong \mathbf{R}^+$.

Complexification defines a morphism of commutative orthogonal monoid spaces

$$c \ : \ \mathbf{Gr} \ \longrightarrow \ u(\mathbf{Gr}^{\mathbb{C}})$$

to the underlying orthogonal monoid space of $\mathbf{Gr}^{\mathbb{C}}$. The value $c(V) : \mathbf{Gr}(V) \longrightarrow \mathbf{Gr}(V_{\mathbb{C}}) = (u\mathbf{Gr}^{\mathbb{C}})(V)$ at a real inner product space V takes an $\mathbb{R}$-subspace L of V to its complexification $L_{\mathbb{C}} \subset V_{\mathbb{C}}$. Realification defines a morphism of commutative unitary monoid spaces

$$r \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ c(\mathbf{Gr}) \ ;$$

the value $r(W) : \mathbf{Gr}(W) \longrightarrow \mathbf{Gr}(rW) = (c\mathbf{Gr})(W)$ is the inclusion of the space of complex subspace of W into the space of real subspaces of the underlying $\mathbb{R}$-space of W. Then the square

$$\begin{tikzcd}
\underline{\pi}_0(\mathbf{Gr}) \arrow[r, "\underline{\pi}_0(c)"] \arrow[d, "(2.4)"', "\cong"] & \underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \arrow[r, "\underline{\pi}_0(r)"] \arrow[d, "\cong"', "(2.25)"] & \underline{\pi}_0(\mathbf{Gr}) \arrow[d, "(2.4)"', "\cong"] \\
\mathbf{RO}^+ \arrow[r] & \mathbf{R}^+ \arrow[r] & \mathbf{RO}^+
\end{tikzcd}$$

commutes, where the lower horizontal maps are complexification and realification of representations.

Example 2.26 (Periodic global BU)**.** The commutative unitary monoid space $\mathbf{BUP}$ that is a global refinement of the non-equivariant homotopy type $\mathbb{Z} \times BU$, and at the same time a global group completion of the complex additive Grassmannian $\mathbf{Gr}^{\mathbb{C}}$. The unitary space $\mathbf{BUP}$ comes tautological vector bundles whose Thom spaces form the periodic complex bordism spectrum $\mathbf{MP}$, compare Example V.7.16 below.

The values at a finite dimensional complex inner product space V is

$$\mathbf{BUP}(V) \;=\; \coprod_{n\geq 0} Gr_n(V^2) \;,$$

the disjoint union of the Grassmannians of complex n-dimensional subspaces in V^2. The structure maps, multiplication and units and $\mathbb{Z}$-grading are the direct complex analogues of the corresponding structure in the orthogonal monoid space $\mathbf{BOP}$ in Example 2.6. We write $\mathbf{BU} = \mathbf{BUP}^{[0]}$ for the homogeneous summand of $\mathbf{BUP}$ of degree 0, which is thus a commutative unitary monoid space in its own right.

The same proof as in the real situation in Theorem 2.8 also proves:

- For every compact Lie group G the map

$$\pi_0^G(\mathbf{BUP}) \;=\; \mathrm{colim}_{V\in s(\mathcal{U}_G^{\mathbb{C}})}\; \pi_0((\mathbf{BUP}(V))^G) \;\longrightarrow\; \mathbf{R}(G) \tag{2.27}$$

  that sends the class of a G-fixed point $W \in \mathbf{BUP}(V)^G$ to the class $[W]-[V]$ in $\mathbf{R}(G)$ is bijective. As G varies, these bijections form an isomorphism of global power monoids

$$\underline{\pi}_0(\mathbf{BUP}) \;\cong\; \mathbf{R}$$

  to the additive global power monoid of the complex representation ring functor.
- The map (2.27) restricts to an isomorphism of abelian groups

$$\pi_0^G(\mathbf{BU}) \;\cong\; I(G)$$

  to the augmentation ideal $I(G) \subset \mathbf{R}(G)$ of the complex representation ring. So the global power monoid of the homogeneous degree 0 part $\mathbf{BU} = \mathbf{BUP}^{[0]}$ is isomorphic to the augmentation ideal global functor.
- The G-fixed point space of $\mathbf{BUP}$ is a disjoint union, indexed by $\mathbf{R}(G)$, of classifying spaces of the group $U^G(\mathcal{U}_G^{\mathbb{C}})$ of G-equivariant unitary isometries of the complexified complete G-universe. Moreover, the group $U^G(\mathcal{U}_G)$ is a weak product of infinite unitary groups, indexed by the isomorphism classes of irreducible complex G-representations.

As in the real situation in Example 2.9, a morphism of graded unitary monoid spaces

$$i \;:\; \mathbf{Gr}^{\mathbb{C}} \;\longrightarrow\; \mathbf{BUP} \tag{2.28}$$

is given at a complex inner product space V by the map

$$\mathbf{Gr}^{\mathbb{C}}(V) \;=\; \coprod_n Gr_n(V) \;\longrightarrow\; \coprod_m Gr_m(V^2) = \mathbf{BUP}(V) \;, \quad L \;\longmapsto\; L\oplus V \;.$$

Then the square

$$\begin{tikzcd}
\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \arrow[r, "\underline{\pi}_0(i)"] \arrow[d, "(2.25)"', "\cong"] & \underline{\pi}_0(\mathbf{BUP}) \arrow[d, "\cong"', "(2.27)"] \\
\mathbf{R}^+ \arrow[r] & \mathbf{R}
\end{tikzcd}$$

commutes, where the lower horizontal map is group completion. So for every compact Lie group G the induced map

$$\pi_0^G(i) \;:\; \pi_0^G(\mathbf{Gr}) \;\longrightarrow\; \pi_0^G(\mathbf{BUP})$$

is an algebraic group completion. The unitary analog of Theorem 4.10 below shows that the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BUP}$ is a group completion in the world of commutative unitary monoid spaces, i.e., 'homotopy-initial', in the category of commutative unitary monoid spaces, among morphisms from $\mathbf{Gr}$ to group-like commutative unitary monoid spaces.

In much the same way as for the additive Grassmannians in Example 2.24, complexification defines a morphism of commutative orthogonal monoid spaces

$$c \ : \ \mathbf{BOP} \ \longrightarrow \ u(\mathbf{BUP})$$

and realification defines a morphism of commutative unitary monoid spaces

$$r \ : \ \mathbf{BUP} \ \longrightarrow \ c(\mathbf{BOP}) \ ;$$

the square

$$\begin{tikzcd}
\underline{\pi}_0(\mathbf{BOP}) \arrow[r, "\underline{\pi}_0(c)"] \arrow[d, "(2.4)"', "\cong"] & \underline{\pi}_0(\mathbf{BUP}) \arrow[r, "\underline{\pi}_0(r)"] \arrow[d, "\cong"', "(2.25)"] & \underline{\pi}_0(\mathbf{BOP}) \arrow[d, "(2.4)"', "\cong"] \\
\mathbf{RO} \arrow[r] & \mathbf{R} \arrow[r] & \mathbf{RO}
\end{tikzcd}$$

then commutes, where the lower horizontal maps are complexification and realification of representations.

Example 2.29 (Complex multiplicative Grassmannian)**.** We define a commutative unitary monoid space $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$, the *complex multiplicative Grassmannian*. Given a finite dimensional complex inner product space V we let

$$\mathrm{Sym}(V) \ = \ \bigoplus_{n\geq 0} V^{\otimes n}/\Sigma_n$$

denote the symmetric algebra of V (where $\otimes$ now denotes the tensor product over $\mathbb{C}$). The value of $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ at V is then

$$\mathbf{Gr}^{\mathbb{C}}_{\otimes}(V) \ = \ \coprod_{n\geq 0} Gr_n(\mathrm{Sym}(V)) \ ,$$

the disjoint union of all complex Grassmannians in the symmetric algebra of V. The structure maps, multiplication (by tensor product of subspaces) and $\mathbb{N}$-grading are defined as in the real case in Example 2.11 The multiplicative unit is the point $\mathbb{C}$ in $\mathbf{Gr}^{\mathbb{C}}_{\otimes}(0) = \mathbb{C}$.

The embeddings $i : V \longrightarrow \mathrm{Sym}(V)$ as the linear summand of the symmetric algebra induce a global equivalence of unitary spaces $\mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}_{\otimes}$. This global equivalence does *not* respect the products, however, i.e., it is *not* a morphism of unitary monoid spaces. This global equivalence induces a bijection between $\pi_0^G(\mathbf{Gr}^{\mathbb{C}})$ and $\pi_0^G(\mathbf{Gr}^{\mathbb{C}}_{\otimes})$ for every compact Lie group G, hence both are isomorphic to the set $\mathbf{R}^+(G)$ of isomorphism classes of complex G-representations. The commutative monoid structures of $\pi_0^G(\mathbf{Gr}^{\mathbb{C}})$ corresponds to direct sum of representations, and the norm maps are additive transfers; the monoid structure of $\pi_0^G(\mathbf{Gr}^{\mathbb{C}}_{\otimes})$ corresponds to tensor product of representations, and the norm maps are multiplicative transfers.

As in the previous example, complexification and the canonical isomorphism $(\mathrm{Sym}(V))_{\mathbb{C}} \cong \mathrm{Sym}(V_{\mathbb{C}})$, for a real vector space V, defines a morphism of commutative orthogonal monoid spaces

$$c \ : \ \mathbf{Gr}_{\otimes} \ \longrightarrow \ u(\mathbf{Gr}^{\mathbb{C}}_{\otimes}) \ .$$

and realification defines a morphism of commutative unitary monoid spaces

$$r \ : \ \mathbf{Gr}^{\mathbb{C}}_{\otimes} \ \longrightarrow \ c(\mathbf{Gr}_{\otimes}) \ .$$

For *abelian* compact Lie groups A there are models of the global classifying space with commutative multiplications, i.e., such that $B_{\mathrm{gl}}A$ is a commutative unitary monoid space. [ref to Boardman-Vogt [**12**]] We discuss two different constructions of multiplicative models for such global classifying spaces.

Example 2.30 (Multiplicative global classifying spaces)**.** The orthogonal subspace $\mathbf{P} = \mathbf{Gr}^{[1]}_{\otimes}$ is closed under the product of the multiplicative Grassmannian $\mathbf{Gr}_{\otimes}$ and contains the multiplicative units, hence $\mathbf{P}$ is a commutative orthogonal monoid space in its own right. Because

$$\mathbf{P}(V) \ = \ \mathbf{Gr}^{[1]}_{\otimes}(V) \ = \ P(\mathrm{Sym}(V))$$

is the projective space of the symmetric algebra of V, we use the symbol $\mathbf{P}$ and refer to it as the *orthogonal projective space*. The multiplication is given by tensor product of lines, as explained in Example 2.11. Since $\mathbf{P} = \mathbf{Gr}_{\otimes}^{[1]}$ is globally equivalent to the additive variant $\mathbf{Gr}^{[1]}$, it is a global classifying space for the circle group $O(1)$, a cyclic group of order 2,

$$\mathbf{P} \simeq \mathbf{Gr}^{[1]} \simeq B_{\mathrm{gl}}O(1) = B_{\mathrm{gl}}C_2 .$$

In other words, $\mathbf{P}$ is a strictly commutative, multiplicative model for $B_{\mathrm{gl}}C_2$.

The complex analog of this construction gives more interesting examples because the group $U(1)$ has more subgroups that $O(1)$, namely the circle group $T = U(1)$ and the finite cyclic groups $C \subset T$. For every subgroup C of $U(1)$ we define a unitary space $B_{\mathrm{gl}}^{\otimes}C$ as follows. We let $\tilde{C} \subset \mathbb{C} - \{0\}$ be the multiplicative group of all non-zero $x \in \mathbb{C}$ such that $x/|x| \in C$. The value at a complex inner product space is then

$$(B_{\mathrm{gl}}^{\otimes}C)(V) = \tilde{C}\backslash(\mathrm{Sym}(V) - \{0\}) ,$$

the quotient of the space of non-zero elements of the symmetric algebra by the action of $\tilde{C}$ by scalar multiplication. The tensor product map

$$(\mathrm{Sym}(V) - \{0\}) \times (\mathrm{Sym}(W) - \{0\}) \longrightarrow (\mathrm{Sym}(V) \otimes \mathrm{Sym}(W)) - \{0\} \cong \mathrm{Sym}(V \oplus W) - \{0\}$$

descends to a multiplication map

$$(B_{\mathrm{gl}}^{\otimes}C)(V) \times (B_{\mathrm{gl}}^{\otimes}C)(W) \longrightarrow (B_{\mathrm{gl}}^{\otimes}C)(V \oplus W)$$

on orbit spaces, and this makes $B_{\mathrm{gl}}^{\otimes}C$ into a commutative unitary monoid space. The unit element of the multiplication is the orbit of the multiplicative unit 1 in the symmetric algebra. A certain canonical morphism $\mathbf{L}_{C,\mathbb{C}} \longrightarrow B_{\mathrm{gl}}^{\otimes}C$ coming from the inclusion of the linear part of the symmetric algebra is a global equivalence of unitary spaces. So as the notation suggests, $B_{\mathrm{gl}}^{\otimes}C$ is a strictly commutative, multiplicative model for $B_{\mathrm{gl}}C$.

In the special case where $C = T = U(1)$ is the entire circle group, $(B_{\mathrm{gl}}^{\otimes}T)(V)$ is the complex projective space of $\mathrm{Sym}(V)$, so $B_{\mathrm{gl}}^{\otimes}T = \mathbf{Gr}_{\otimes}^{\mathbb{C},[1]} = \mathbf{P}^{\mathbb{C}}$ is the *unitary projective space*, the complex analog of the orthogonal monoid space $\mathbb{P}$.

Every abelian compact Lie group is a product of a finite cyclic groups and a torus, so by taking suitable box products of the unitary spaces $B_{\mathrm{gl}}^{\otimes}C$ for varying C we can realize a global classifying space $B_{\mathrm{gl}}^{\otimes}A$ for every abelian compact Lie group A as a commutative unitary monoid space. Since $B_{\mathrm{gl}}^{\otimes}A$ is a global classifying space, the natural transformation

$$\mathrm{Rep}(-, A) \longrightarrow \underline{\pi}_0(B_{\mathrm{gl}}^{\otimes}A) , \quad [\alpha] \longmapsto \alpha^*(u_A)$$

is an isomorphism of $\mathrm{Rep}^{\mathrm{op}}$-functors, where $u_A \in \pi_0^A(B_{\mathrm{gl}}^{\otimes}A)$ is the tautological class. We describe the power operations on $\underline{\pi}_0(B_{\mathrm{gl}}^{\otimes}A)$ arising from the commutative multiplication in terms of the functor $\mathrm{Rep}(A, -)$. Given a continuous group homomorphism $\alpha : K \longrightarrow A$, we define the homomorphism $P^m(\alpha) : \Sigma_m \wr K \longrightarrow A$ by

$$(P^m(\alpha))(\sigma; k_1, \dots, k_m) = \alpha(k_1) \cdot \ldots \cdot \alpha(k_m) .$$

The fact that this a homomorphism uses the commutativity of A. Since A is abelian, no conjugacy happens, and we obtain an operation

$$P^m : \mathrm{Rep}(K, A) \longrightarrow \mathrm{Rep}(\Sigma_m \wr K, A) .$$

We have

$$P^m(\alpha) = (\Sigma_m \wr K) \circ P^m(\mathrm{Id}_A) , \tag{2.31}$$

which implies naturality of these power operations. We omit the verification that these operation maps and the pointwise multiplication of homomorphisms into A make the representable functor $\mathrm{Rep}(-, A)$ into a commutative power monoid. We do show that the isomorphism is a morphism of power monoids. Indeed, the relation (2.31) and naturality of power operations in $\underline{\pi}_0(B^{\otimes}_{\mathrm{gl}}A)$ reduces this to the universal example, the generator $u_A \in \pi_0^A(B^{\otimes}_{\mathrm{gl}}A)$.

Proposition 2.32. *Let A be an abelian compact Lie group and $B^{\otimes}_{\mathrm{gl}}A$ the commutative unitary monoid space that is a global classifying space for A. Then the relation*

$$P^m(u_A) \ = \ p_m^*(u_A)$$

holds in $\pi_0^{\Sigma_m \wr A}(B^{\otimes}_{\mathrm{gl}}A)$, where $p_m : \Sigma_m \wr A \longrightarrow A$ is the homomorphism defined by $p_m(\sigma; a_1, \dots, a_m) = a_1 \cdot \ldots \cdot a_m$.

Proof. As we discussed in Example 2.29 the power operations in the complex multiplicative Grassmannian $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ are given by

$$P^m[V] \ = \ [V^{\otimes}] \ .$$

In the special case of the tautological 1-dimensional representation of the circle group $T = U(1)$ this yields

$$P^m[\mathbb{C}] \ = \ [\mathbb{C}^{\otimes}] \ = \ [p_m^*(\mathbb{C})] \ = \ p_m^*[\mathbb{C}]$$

because the action of $\Sigma_m \wr T$ on $\mathbb{C}^{\otimes}$ is via the character $p_m : \Sigma_m \wr T \longrightarrow T$. This relation takes place entirely in the global power monoid

$$\underline{\pi}_0(\mathbf{P}^{\mathbb{C}}) \ = \ \underline{\pi}_0(\mathbf{Gr}^{[1],\mathbb{C}}_{\otimes}) \ = \ \underline{\pi}_0(B^{\otimes}_{\mathrm{gl}}T) \ ,$$

so this takes care of the case $A = T$.

Now we let $C \subset T$ be a subgroup and we argue by naturality. The projections

$$(B^{\otimes}_{\mathrm{gl}}C)(V) \ = \ C\backslash S(\mathrm{Sym}(V)) \ \longrightarrow \ T\backslash S(\mathrm{Sym}(V)) \ = \ (B^{\otimes}_{\mathrm{gl}}T)(V)$$

form a morphism $i : B^{\otimes}_{\mathrm{gl}}C \longrightarrow B^{\otimes}_{\mathrm{gl}}T$ of unitary monoid spaces that satisfies

$$i_*(u_C) \ = \ \mathrm{res}^T_C(u_T) \ \in \ \pi_0^C(B^{\otimes}_{\mathrm{gl}}T) \ .$$

Then

$$\begin{aligned} i_*(P^m(u_C)) \ &= \ P^m(i_*(u_C)) \ = \ P^m(\mathrm{res}^T_C(u_T)) \\ &= \ (\Sigma_m \wr i^T_C)^*(P^m((u_T)) \ = \ (\Sigma_m \wr i^T_C)^*((p^T_m)^*(u_T)) \\ &= \ (p^T_m \circ (\Sigma_m \wr i^T_C))^*(u_T) \ = \ (i^T_C \circ p^C_m)^*(u_T) \\ &= \ (p^C_m)^*(\mathrm{res}^T_C(u_T)) \ = \ (p^C_m)^*(i_*(u_C)) \ = \ i_*((p^C_m)^*(u_C)) \ . \end{aligned}$$

Since the map $i^T_C \circ - : \mathrm{Rep}(-, C) \longrightarrow \mathrm{Rep}(-, T)$ induced by the inclusion $i : C \longrightarrow T$ is injective, this proves $P^m(u_C) = p^*_m(u_C)$.

Finally, we show that the claim of the proposition is stable under product. We let A and A' be two abelian compact Lie groups, and we set

$$B^{\otimes}_{\mathrm{gl}}(A \times A') \ = \ (B^{\otimes}_{\mathrm{gl}}A) \boxtimes (B^{\otimes}_{\mathrm{gl}}A') \ .$$

Then the class

$$u_A \times u_{A'} \ \in \ \pi_0^{A\times A'}(B^{\otimes}_{\mathrm{gl}}(A \times A'))$$

is the tautological class for $A \times A'$. So we get

$$\begin{aligned} P^m(u_A \times u_{A'}) \ &= \ \Delta^*(P^m(u_A) \times P^m(u_{A'})) \ = \ \Delta^*((p^A_m)^*(u_a) \times (p^{A'}_m)^*(u_{A'})) \\ &= \ ((p^A_m \times p^{A'}_m) \circ \Delta)^*(u_A \times u_{A'}) \ = \ (p^{A\times A'}_m)^*(u_A \times u_{A'}) \end{aligned}$$

where $\Delta : \Sigma_m \wr (A \times A') \longrightarrow (\Sigma_m \wr A) \times (\Sigma_m \wr A')$ is the diagonal morphism, see (1.12) of Chapter I, which satisfies $(p^A_m \times p^{A'}_m) \circ \Delta = p^{A\times A'}_m$. □

Construction 2.33 (Multiplicative global classifying spaces)**.** Another multiplicative model of global classifying spaces for abelian compact Lie groups is given as follows. We follow the classical bar construction (giving a non-equivariant classifying space) by the cofree functor R (see Construction I.2.12). The bar construction preserves products in the sense that for every pairs compact Lie groups G and K the natural map

$$B(G \times K) \longrightarrow BG \times BK$$

is a homeomorphism. So the composite $A \mapsto R(BA)$ is a lax symmetric monoidal functor via the morphism of orthogonal spaces

$$R(BG)\Box R(BK) \xrightarrow{\mu_{A,A'}} R(BG \times BK) \cong R(B(G \times K)) ,$$

where the first morphism was defined in (5.15) of Chapter I. The bar construction is functorial in group homomorphisms, so for an abelian compact Lie group A the composite

$$R(BA)\Box R(BA) \longrightarrow R(B(A \times A)) \xrightarrow{R(B\mu_A)} R(BA)$$

is a commutative and associative multiplication on the orthogonal space $R(BA)$, where $\mu_A : A \times A \longrightarrow A$ is the multiplication of A. Corollary 2.18 shows that for abelian A the cofree orthogonal space $R(BA)$ is a global classifying space for A.

In the last part of this section we formalize the structure that complex conjugation provides on the unitary monoid spaces $\mathbf{U}$, $\mathbf{Gr}^{\mathbb{C}}$, $\mathbf{BUP}$, $\mathbf{BU}$, $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ and $B^{\otimes}_{\mathrm{gl}}C$. The fast and simple way would be to encode complex conjugation as involutions of the underlying orthogonal monoid spaces, for example an automorphism $\psi : u(\mathbf{BUP}) \longrightarrow u(\mathbf{BUP})$ of order 2. However, this way we would forget the action of complex conjugation on unitary embeddings, and lose interesting structure (compare Remark 2.43 for more details). To keep track of this structure we introduce the notion of a *real structure* on a unitary space.

Definition 2.34. A *real structure* on a complex inner product space W is a semilinear isometric involution $\tau : W \longrightarrow W$. In other words, τ is an $\mathbb{R}$-linear automorphism of order two that also satisfies

$$\tau(\lambda \cdot w) = \bar{\lambda} \cdot \tau(w) \qquad \text{and} \qquad \langle \tau(w), \tau(w') \rangle = \overline{\langle w, w' \rangle}$$

for all $\lambda \in \mathbb{C}$ and $w, w' \in W$.

A real structure is precisely what is needed to identify a complex inner product space as the complexification of a real inner product space. Indeed, the complexification $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ of a real inner product space V has a unique hermitian scalar product characterized by $\langle 1 \otimes v, 1 \otimes w \rangle_{\mathbb{C}} = \langle v, w \rangle$ for $v, w \in V$. The map

$$\tau^V : V_{\mathbb{C}} \longrightarrow V_{\mathbb{C}} , \quad 1 \otimes v + i \otimes w \longmapsto 1 \otimes v - i \otimes w$$

is a canonical real structure on $V_{\mathbb{C}}$. Conversely, for every real structure τ on a complex inner product space W, the fixed point space

$$W^{\tau} = \{ w \in W \mid \tau(w) = w \}$$

is an $\mathbb{R}$-subspace on which the hermitian inner product takes real values, i.e., is a euclidean inner product. Moreover, the map

$$\mathbb{C} \otimes_{\mathbb{R}} (W^{\tau}) \longrightarrow W , \quad \lambda \otimes w \longmapsto \lambda w$$

is an isometric isomorphism that matches the canonical real structure on the source to the given real structure on the target. We can summarize this by saying that complexification (with the canonical real structure) is an equivalence from the category of real inner product spaces to the category of complex inner product spaces equipped with real structure.

Now we define the notion of 'real structure' on a unitary space. To make sense of the following definition we note the following: given two complex inner product spaces (V, τ) and (W, τ') and a complex linear isometric embedding $\varphi : V \longrightarrow W$, then the map $\tau' \circ \varphi \circ \tau : V \longrightarrow W$ is also a complex linear isometric embedding.

Definition 2.35. A *real structure* on a unitary space Y assigns to every real structure τ on a complex inner product space W an involution τ_* on the space $Y(W)$ with the following property. If (W, τ') is another complex inner product space with real structure and $\varphi : V \longrightarrow W$ a linear isometric embedding, then

$$\tau'_* \circ Y(\varphi) \circ \tau_* \ = \ Y(\tau' \circ \varphi \circ \tau) \ : \ Y(V) \ \longrightarrow \ Y(W) \ .$$

A *Real unitary space* is a unitary space equipped with a real structure. A *morphism* of Real unitary spaces is a morphism of unitary spaces that also commutes with the involutions of the real structure.

In the situation of the previous definition, the two maps $\varphi \circ \tau, \tau' \circ \varphi : V \longrightarrow W$ are *not* $\mathbb{C}$-linear (only semilinear), so it is not meaningful to evaluate a unitary space on $\varphi \circ \tau$ or $\tau' \circ \varphi$.

Remark 2.36. Every complex inner product space with real structure is isomorphic to the complexification of a real inner product space, and hence to $\mathbb{C}^n$ with the standard hermitian inner product and real structure by componentwise complex conjugation. So a real structure on a unitary space Y is completely determined by the involutions of $Y(\mathbb{C}^n)$ corresponding to the canonical real structure, for all $n \geq 0$.

Before we discuss the real structures on the unitary monoid spaces $\mathbf{U}$, $\mathbf{Gr}^{\mathbb{C}}$, $\mathbf{BUP}$, $\mathbf{BU}$, $\mathbf{Gr}^{\mathbb{C}}_{\otimes}$ and $B^{\otimes}_{\mathrm{gl}}C$ we discuss some of the benefits of this additional data.

Construction 2.37 (Involution and fixed points)**.** A real structure on a unitary space Y in particular determines an involution $\psi : uY \longrightarrow uY$ on the underlying orthogonal space uY as follows. For every inner product space V, we define

$$\psi_V \ = \ \tau^V_* \ ,$$

i.e., we equip the value $(uY)(V) = Y(V_{\mathbb{C}})$ with the involution τ^V_* specified for the canonical real structure τ^V. If $\varphi : V \longrightarrow W$ is a real linear isometric embedding, then its complexification satisfies $\tau^W \circ \varphi_{\mathbb{C}} \circ \tau^V = \varphi_{\mathbb{C}}$, so the induced map satisfies

$$\psi_W \circ (uY)(\varphi) \circ \psi_V \ = \ \tau^W_* \circ Y(\varphi_{\mathbb{C}}) \circ \tau^V_* \ = \ Y(\varphi_{\mathbb{C}}) \ = \ (uY)(\varphi) \ : \ (uY)(V) \ \longrightarrow \ (uY)(W) \ .$$

In other words, $(uY)(\varphi)$ is equivariant for the involutions on source and target. Altogether this makes the underlying orthogonal space uY into an orthogonal space with involution (or equivalently, a continuous functor from $\mathbf{L}$ to spaces with involution).

Since the underlying space uY of a Real unitary space has an involution ψ we can takes is (categorical) fixed points, for which we write Y^{ψ}. So Y^{ψ} is the orthogonal space whose value at a real inner product space V is given by

$$(Y^{\psi})(V) \ = \ Y(V_{\mathbb{C}})^{\tau^V_*} \ = \ \{x \in Y(V_{\mathbb{C}}) \mid \tau^V_*(x) = x\} \ .$$

This is an orthogonal subspace uY, so it comes with an inclusion morphism $Y^{\psi} \longrightarrow uY$.

Remark 2.38 (Real global equivalences)**.** Real unitary spaces admit a global homotopy theory that is somewhat finer than global homotopy theory based on orthogonal spaces, or even orthogonal spaces with involution. In other words, a Real unitary space has more 'global homotopical' information than an orthogonal space with involution. This is our main reason considering Real unitary spaces as opposed to the simpler notion of orthogonal spaces with involutions. We will not go into is in all details here, but we sketch where the extra information is located. For this purpose we recall from [**6**, Sec. 6] the concept of 'Real Lie groups' and the basic of their representation theory.

A *Real Lie group* is a compact Lie group G equipped with an involution $\tau : G \longrightarrow G$. A *Real representation* of a Real Lie group is a unitary G-representation V with a real structure $\tau : V \longrightarrow V$, such that the two real structures are compatible in the sense that

$$\tau(g \cdot v) \ = \ \tau(g) \cdot \tau(v)$$

for all $g \in G$ and $v \in V$. So making (V, τ) into a Real G-representation amounts to giving a Real homomorphism $G \longrightarrow U(V)$. A Real G-representation is *trivial* if the underlying G-action is trivial. Every Real Lie group has a faithful Real representation: if V is a faithful unitary representation, then $V \oplus \bar{V}$ is naturally a faithful Real representation, with real structure by interchanging the two summands. So every Real Lie group admits a closed, Real embedding into $U(n)$ for some n; equivalently, every Real Lie group is isomorphic to a closed subgroup of $U(n)$ that is invariant under complex conjugation.

The *Real representation ring* $\mathbf{R}_R(G)$ of a Real Lie group G is the Grothendieck group, under direct sum, of finite dimensional Real G-representations. The tensor product, over $\mathbb{C}$, of two Real G-representations becomes a Real G-representation with respect to the diagonal real structure determined by

$$\tau(v \otimes w) \ = \ \tau(v) \otimes \tau(w) \ .$$

The tensor product is biadditive and symmetric, so it induces a commutative ring structure on $\mathbf{R}_R(G)$. The unit of $\mathbf{R}_R(G)$ is the class of the trivial G-representation on $\mathbb{C}$ (with involution by complex conjugation).

The Real representation ring $\mathbf{R}_R(G)$ can be identified with a subring of the complex representation ring $\mathbf{R}(G)$ of the underlying compact Lie group. Indeed, the forgetful map is a ring homomorphism

$$\mathbf{R}_R(G) \ \xrightarrow{\ i\ } \ \mathbf{R}(G)$$

that we claim is injective. A homomorphism $\mathbf{R}(G) \longrightarrow \mathbf{R}_R(G)$ in the other direction is given by sending a complex G-representation V to the complex vector space

$$V \oplus \bar{V}$$

with G-action $g \cdot (v, v') = (gv, \tau(g)v')$ and involution $\tau(v, v') = (v', v)$. The composite

$$\mathbf{R}_R(G) \ \xrightarrow{\ i\ } \ \mathbf{R}(G) \ \longrightarrow \ \mathbf{R}_R(G)$$

is multiplication by 2, so the map i is an injective ring homomorphism.

Example 2.39. We can make every compact Lie group G into a Real Lie group G^{tr} by giving it the identity involution. Then a Real representation of G^{tr} is a complex G-representation V together with a real structure τ that is G-equivariant. So fixed point subspace V^τ is G-invariant, hence a real G-representation, and the canonical isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} (V^\tau) \ \cong \ V$$

is G-equivariant, where the G-action on the left hand side is the complexification of the G-action on V^τ. So isomorphism classes of Real representation of G^{tr} biject with isomorphism classes of real representation of G, and hence complexification is an isomorphism

$$\mathbf{RO}(G) \ \longrightarrow \ \mathbf{R}_R(G^{\mathrm{tr}}) \ .$$

The notions of complete real and complex G-universe have a Real analog for Real Lie groups.

Definition 2.40. A *complete Real universe* of a Real Lie group G is an infinite dimensional Real G-representation $\mathcal{U}_G^R$ such that every finite dimensional Real G-representation can be embedded into it.

Since the real representation ring $\mathbf{R}_R(G)$ is isomorphic to a subring of the complex representation ring $\mathbf{R}(G)$, there are at most countably many isomorphism classes of irreducible Real representations. So every Real Lie group has a complete Real universe. Moreover, any two complete Real universes are isomorphic.

A Real unitary space Y, can be evaluated on any Real representation (V, τ) of a Real Lie group G, and yields a space $Y(V)$ with continuous action of $C_2 \ltimes G$. So the G-fixed point space $Y(V)^G$ is still left with a C_2-action.

Definition 2.41. Let Y and Z be closed Real unitary spaces. A morphism $f : Y \longrightarrow Z$ of Real unitary spaces is a *Real global equivalence* if for every Real Lie group G the induced map

$$f(\mathcal{U}_G^R)^G \ : \ Y(\mathcal{U}_G^R)^G \ \longrightarrow \ Z(\mathcal{U}_G^R)^G$$

is a C_2-weak equivalence.

We observe that a Real global equivalence of Real unitary spaces in particular induces a global equivalence of underlying and fixed orthogonal spaces.

Proposition 2.42. *Let $f : Y \longrightarrow Z$ be a Real global equivalence of Real unitary spaces. Then the underlying morphism $uf : uY \longrightarrow uZ$ and the fixed point morphism $f^\psi : Y^\psi \longrightarrow Z^\psi$ are global equivalences of orthogonal spaces.*

PROOF. We let G be any compact Lie group and G^{tr} the trivial Real Lie group with identity involution. Then $\mathcal{U}^R_{G^{\mathrm{tr}}} = \mathbb{C} \otimes \mathcal{U}_G$ with canonical real structure $\psi_{\mathcal{U}_G}$ is a complete Real universe for G^{tr}. Moreover, the complexifications of the orthogonal G-subrepresentations of $\mathcal{U}_G$ form a cofinal subposet of $s(\mathcal{U}^R_G)$. So in this case

$$(Y(\mathcal{U}^R_G)^{G^{\mathrm{tr}}})^{C_2} \ = \ Y(\mathcal{U}^R_G)^{G\times C_2} \ = \ Y(\mathbb{C}\otimes\mathcal{U}_G)^{G\times C_2} \ = \ ((Y^\psi)(\mathcal{U}_G))^G \ .$$

So $f^\psi : Y^\psi \longrightarrow Z^\psi$ is a global equivalence.

On the other hand, the underlying universe of the Real universe $\mathcal{U}^R_{G^{\mathrm{tr}}}$ is a complete G-universe. So the underlying non-equivariant space of $Y(\mathcal{U}^R_G)^{G^{\mathrm{tr}}}$ is naturally homeomorphic to $((uY)(\mathcal{U}_G))^G$. So $uf : uY \longrightarrow uZ$ is a global equivalence. □

Remark 2.43. The proof of Proposition 2.42 shows kind of homotopical information we are forgetting when passing from a Real unitary space to the underlying orthogonal space with involution. The global homotopy type of the underlying orthogonal space uY and of the fixed point space Y^ψ together determine the C_2-equivariant homotopy types of the fixed point space $Y(\mathcal{U}^R_G)^G$ for all Real Lie group G with identity involution. However, we are not remembering the analogous information for non-trivial Real Lie groups.

Construction 2.44 (Real structures by complex conjugation)**.** We specify the canonical real structures on the commutative unitary monoid spaces $\mathbf{U}$, $\mathbf{Gr}^{\mathbb{C}}$, $\mathbf{BUP}$, $\mathbf{BU}$, $\mathbf{Gr}^{\mathbb{C}}_\otimes$, $\mathbf{P}^{\mathbb{C}}$ and $B^\otimes_{\mathrm{gl}}C$. In all these cases, the real structure should be thought of as 'complex conjugation'.

The underlying unitary spaces have already been defined in Examples 2.22, 2.24, 2.26, 2.29 respectively 2.30 above; so now we need to specify the real structure on them. We let W be a complex inner product space with a real structure $\tau : W \longrightarrow W$. We have to specify the involution τ_* on the values of $\mathbf{U}$, $\mathbf{Gr}^{\mathbb{C}}$, $\mathbf{BUP}$, $\mathbf{BU}$, $\mathbf{Gr}^{\mathbb{C}}_\otimes$, $\mathbf{P}^{\mathbb{C}}$ and $B^\otimes_{\mathrm{gl}}C$ at W.

The involution

$$\tau_* \ : \ \mathbf{U}(W) \ \longrightarrow \mathbf{U}(W) \ , \quad \tau_*(A) \ = \ \tau_W \circ A \circ \tau_W$$

is given by conjugation with the real structure τ. The involutions

$$\tau_* \ : \ \mathbf{Gr}^{\mathbb{C}}(W) \ \longrightarrow \mathbf{Gr}^{\mathbb{C}}(W) \ , \quad \tau_* \ : \ \mathbf{BUP}(W) \ \longrightarrow \mathbf{BUP}(W) \qquad \text{and} \qquad \tau_* \ : \ \mathbf{Gr}^{\mathbb{C}}_\otimes(W) \ \longrightarrow \mathbf{Gr}^{\mathbb{C}}_\otimes(W)$$

are given by

$$\tau_*(L) \ = \ \tau_W(L) \ , \quad \tau_*(L) \ = \ \tau^2_W(L) \qquad \text{respectively} \qquad \tau_*(L) \ = \ \mathrm{Sym}(\tau_W)(L)$$

i.e., a subspace is sent to its image under the map $\tau_W : W \longrightarrow W$, the map $\tau^2_W : W^2 \longrightarrow W^2$ or the map $\mathrm{Sym}(\tau_W) : \mathrm{Sym}(W) \longrightarrow \mathrm{Sym}(W)$. The fact that τ_W is not $\mathbb{C}$-linear may seem like a problem at first sight. However, semilinearity already guarantees that τ_W and τ^2_W take complex subspaces to complex subspaces. In the last case we also use that even semilinear maps induced maps (again semilinear) on the complex symmetric algebras.

The real structure of $\mathbf{BUP}$ is grading preserving, so it restricts to a real structure on $\mathbf{BU} = \mathbf{BUP}^{[0]}$. The same goes for the real structure of $\mathbf{Gr}^{\mathbb{C}}_\otimes$, so it restricts to a real structure on $\mathbf{P}^{\mathbb{C}} = \mathbf{Gr}^{\mathbb{C},[1]}_\otimes$. The case of $B^\otimes_{\mathrm{gl}}C$ for a subgroup C of $T = U(1)$ is a straightforward generalization of the case $\mathbf{P}^{\mathbb{C}} = B^\otimes_{\mathrm{gl}}T$. In all the cases, the real structure is compatible with the commutative multiplication, so we get commutative Real unitary monoid spaces.

We can say a bit more about the Real global homotopy types in some of the examples. For this one should imagine the definition of a *Real global classifying space* of a Real Lie groups. This is a specific Real global homotopy type with analogous properties as the orthogonal global classifying space in the world of compact Lie groups. Then the n-th homogeneous piece $\mathbf{Gr}^{\mathbb{C},[n]}$ is in fact a Real global classifying space of the unitary group $U(n)$ with respect to the preferred real structure by complex conjugation. This now has to be distinguished from the Real global classifying space of the unitary group with trivial Real structure, which is a different Real global homotopy type. Also, the morphism $\mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{Gr}^{\mathbb{C}}_{\otimes}$ arising from the embedding of the linear summand of the symmetric algebra is not only a global equivalence of unitary spaces (i.e., a a global equivalence of underlying orthogonal spaces); the morphism is even a Real global equivalence. Moreover, the Real unitary space $B^{\otimes}_{\text{gl}}A$ is a multiplicative model for a Real global classifying space of the abelian compact Lie group A, with involution given by the inverse map. Again this is different from the Real global classifying space of A with trivial Real structure.

As explained in Example 2.37, a Real unitary space Y has an underlying orthogonal space uY, equipped with an involution $\psi : uY \longrightarrow uY$. For the Real unitary spaces considered here, the orthogonal fixed point spaces are always the appropriate real (orthogonal) versions. We discuss this detail in the first example: in (2.23) we established complexification as a morphism of orthogonal spaces $c : \mathbf{O} \longrightarrow u\mathbf{U}$. Complexifications of orthogonal automorphism of V commute with the canonical real structure on $V_{\mathbb{C}}$, so the morphism c lands in the fixed point orthogonal space $\mathbf{U}^{\psi}$. More is true: every unitary transformation in $U(V_{\mathbb{C}})$ that commutes with the canonical real structure is the complexification of a unique element in $O(V)$. So the complexification morphism restricts to an isomorphism of orthogonal monoid spaces

$$c \ : \ \mathbf{O} \ \cong \ \mathbf{U}^{\psi} \ .$$

Similar arguments in the other show that the various complexification morphisms restrict to isomorphisms of orthogonal monoid spaces

$$\begin{aligned} &c \ : \ \mathbf{Gr} \ \cong \ (\mathbf{Gr}^{\mathbb{C}})^{\psi} \ , \quad c \ : \ \mathbf{BOP} \ \cong \ \mathbf{BUP}^{\psi} \ , \quad c \ : \ \mathbf{BO} \ \cong \ \mathbf{BU}^{\psi} \\ &c \ : \ \mathbf{P} \quad \cong \ (\mathbf{P}^{\mathbb{C}})^{\psi} \quad \text{respectively} \quad c \ : \ \mathbf{Gr}_{\otimes} \ \cong \ (\mathbf{Gr}^{\mathbb{C}}_{\otimes})^{\psi} \ . \end{aligned}$$

There is also a straightforward 'Real' analog of the equivariant homotopy sets of orthogonal spaces. Given a Real unitary space Y and a Real Lie group G, we define

$$\pi_0^G(Y) \ = \ \operatorname{colim}_{V \in s(\mathcal{U}_G^R)} \pi_0(Y(V)^{C_2 \ltimes G}) \ . \tag{2.45}$$

The difference with the equivariant homotopy set of an orthogonal space is that we now exhaust a Real universe $\mathcal{U}_G^R$ of the given Real Lie group, and that we take fixed points by the G-action and the involution. The structure feature of equivariant homotopy sets generalize to this context: continuous homomorphism of Real Lie groups induced restriction maps in the opposite direction, and Real global equivalences induce bijections on π_0^G for every Real Lie group G. For a compact Lie group with trivial Real structure the argument of Proposition 2.42 provides a natural isomorphism

$$\pi_0^{G^{\text{tr}}}(Y) \ \cong \ \pi_0^G(Y^{\psi}) \ .$$

In contrast to the fixed point orthogonal space, the equivariant homotopy sets of the underlying orthogonal space uY are not generally isomorphic to $\pi_0^G(Y)$ for any real Lie group G. The following example illustrates this.

Example 2.46. We consider the commutative unitary monoid space **BUP** with the preferred real structure specified in Example 2.44. We let G be a Real Lie group. The same kind of arguments as in Theorem 2.8 provides an isomorphism of monoids

$$\pi_0^G(\mathbf{BUP}) \ \cong \ \mathbf{R}_R(G)$$

to the Real representation ring of G, natural for restriction maps along continuous Real homomorphisms. This is in fact an honest generalization of Theorem 2.8 in the following sense: if we give a compact Lie group G the trivial Real structure, then the left hand side reduces to

$$\pi_0^{G^{\text{tr}}}(\mathbf{BUP}) \ = \ \pi_0^G(\mathbf{BUP}^{\psi}) \ = \ \pi_0^G(\mathbf{BOP}) \ ,$$

and the right hand side becomes isomorphic to the real representation ring $\mathbf{RO}(G)$. When the involution of G is non-trivial, then in general, the Real representation ring $\mathbf{R}_R(G)$ and the real representation ring $\mathbf{RO}(uG)$ of the underlying compact Lie group are not isomorphic – the smallest example is the cyclic group of order 3 with inverse map as involution.

3. Global model structure

In this section we construct a model structure on the category of commutative orthogonal monoid spaces with global equivalence as the weak equivalences. As is well-known from similar contexts (for example, the stable model structure for commutative orthogonal ring spectra), model structures cannot be lifted naively to multiplicative objects with strictly commutative products. For orthogonal monoid spaces, one way to see this is from properties of power operations is as follows.

Remark 3.1. The power operations for a commutative orthogonal monoid space R give obstructions for an element $\pi_0^e(R)$ to arise from the initial space $R(0)$. Indeed, any class $y \in \pi_0^e(R)$ represented by a point in $R(0)$ satisfies

$$P^m(y) \;=\; y^m \quad \text{in } \pi_0^e(R).$$

In a static orthogonal space Y all elements of $\pi_0^e(Y)$ are in the image of $\pi_0(Y(0))$; so if R is a commutative orthogonal monoid space with an element $y \in \pi_0^e(R)$ such that $P^m(y) \neq y^m$, then the underlying orthogonal space of R cannot be static. So such ring spaces also do not have 'static replacements', i.e., there does not exists a morphism of commutative orthogonal monoid spaces $R \longrightarrow R'$ that is a global equivalence with static target.

The solution, as usual, is to lift a 'positive' version of the global model structure in which the values at the trivial inner product space are homotopically meaningless and where the fibrant objects are the 'positive static' orthogonal spaces. There is a positive version of the flat $\mathcal{F}$-global model structure for every global family $\mathcal{F}$, and it lifts to a *flat $\mathcal{F}$-global model structure* on the category of commutative orthogonal monoid spaces. To simplify the discussion we only discuss the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family.

Definition 3.2. A morphism $f : A \longrightarrow B$ of orthogonal spaces is a *positive cofibration* if it is a flat cofibration and the map $f(0) : A(0) \longrightarrow B(0)$ is a homeomorphism. An orthogonal space is *positively static* if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and an arbitrary G-representation W the structure map

$$X(i_{V,W}) \;:\; X(V) \;\longrightarrow\; X(V \oplus W)$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positively static orthogonal space is static (in the absolute sense) if the structure map $X(0) \longrightarrow X(\mathbb{R})$ is a non-equivariant weak equivalence.

Proposition 3.3 (Positive global model structure)**.** *The global equivalences and positive cofibrations are part of a proper topological model structure, the* positive global model structure *on the category of orthogonal spaces. A morphism $f : X \longrightarrow Y$ of orthogonal spaces is a fibration in the positive global model structure if and only if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and an arbitrary G-representation W the square of G-fixed point spaces*

$$\begin{tikzcd}
X(V)^G \arrow[r, "X(i_{V,W})^G"] \arrow[d, "f(V)^G"'] & X(V \oplus W)^G \arrow[d, "f(V\oplus W)^G"] \\
Y(V)^G \arrow[r, "Y(i_{V,W})^G"'] & Y(V \oplus W)^G
\end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positively static orthogonal spaces.

PROOF. We start by establishing a *positive strong level model structure*. We call a morphism $f : X \longrightarrow Y$ of orthogonal spaces a *positive strong level equivalence* (respectively *positive strong level fibration*) if for every inner product space V with $V \neq 0$ the map $f(V) : X(V) \longrightarrow Y(V)$ is an $O(V)$-weak equivalence (respectively an $O(V)$-fibration). Then we claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a model structure, on the category of orthogonal spaces.

The proof is another application of the general construction method for level model structures in Proposition 3.9. Indeed, we let $\mathcal{C}(0)$ be the degenerate model structure on the category $\mathbf{U}$ of unbased spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m \geq 1$ we let $\mathcal{C}(m)$ be the projective model structure (for the family of all closed subgroups) on the category of $O(m)$-spaces, compare Proposition A.1.18. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 3.9 become the positive strong level equivalences, positive strong level fibrations and positive cofibrations. The unstable consistency condition (3.8) is now strictly weaker than for the strong level model structure, so it holds. The verification that the model structure is proper and topological is the same as for the strong level model structure in Proposition 3.12. The positive strong level model structure is cofibrantly generated; we can simply take the same sets of generating cofibrations and generating acyclic cofibrations as for the strong level model structure, *except* that we omit all morphisms freely generated in level 0.

We obtain the positive global model structure for orthogonal spaces by 'mixing' the positive strong level model structure with the global model structure of Theorem 4.3. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [**25**, Thm. 2.1], which is our first claim. By [**25**, Cor. 3.7] (or rather its dual formulation), an orthogonal space is fibrant in the positive global model structure if it is equivalent in the positive strong level model structure to a static orthogonal space; this is equivalent to being positively static. The positive global model structure is again proper (Propositions 4.1 and 4.2 of [**25**]). The proof that this model structure topological is similar as for the global model structure. □

Now we state the main result of this section. We call a morphism of commutative orthogonal monoid spaces a *global equivalence* (respectively *positive global fibration*) if the underlying morphism of orthogonal spaces is a global equivalence (respectively fibration in the positive global model structure).

Theorem 3.4 (Global model structure for commutative orthogonal monoid spaces)**.** *The global equivalences and positive global fibrations are part of a model structure, the* global model structure *on the category of commutative orthogonal monoid spaces. This global model structure is proper, topological and cofibrantly generated. Every cofibration in this model structure whose source is cofibrant as a commutative orthogonal monoid space is a positive cofibration of underlying orthogonal spaces.*

Theorem 3.4 is a special case of a lifting theorem for model structures to categories of commutative monoids that we will now formulate and prove. The general context here is a symmetric monoidal model category $\mathcal{C}$ with monoidal product $\otimes$ and unit object I. To simplify the exposition we follow the common abuse to suppress the associativity and unit isomorphisms from the notation, i.e., we pretend that the underlying monoidal structure is strict.

We let $i : A \longrightarrow B$ be a $\mathcal{C}$-morphism and arrange the n-fold smash power of i into an n-dimensional cube $K^n(i)$ in $\mathcal{C}$, i.e., a functor

$$K^n(i) \ : \ \mathcal{P}(\{1, 2, \ldots, n\}) \ \longrightarrow \ \mathcal{C}$$

from the poset category of subsets of $\{1, 2, \ldots, n\}$ and inclusions to $\mathcal{C}$. Indeed, a morphism can be viewed as a functor from the category I with two objects 0 and 1 and a unique non-identity morphism from 0 to 1.

The n-fold power $i^{\otimes n}$ then becomes a functor from I^n to $\mathcal{C}^n$, and we postcompose with the total tensor product functor $\mathcal{C}^n \longrightarrow \mathcal{C}$. More explicitly, this comes out as follows. If $S \subseteq \{1, 2, \dots, n\}$ is a subset, the vertex of the cube at S is defined to be

$$K^n(i)(S) \ = C_1 \otimes C_2 \otimes \dots \otimes C_n$$

with

$$C_i \ = \begin{cases} A & \text{if } i \notin S \\ B & \text{if } i \in S. \end{cases}$$

All morphisms in the cube $K^n(i)$ are smash products of identities and copies of the morphism $i : A \longrightarrow B$. The initial vertex of the cube is $K^n(i)(\emptyset) = A^{\otimes n}$ and the terminal vertex is $K^n(i)(\{1, \dots, n\}) = B^{\otimes n}$

We denote by $Q^n(i)$ the colimit of the punctured cube i.e., the cube $K^n(i)$ with the terminal vertex removed, and $i^{\Box n} : Q^n(i) \longrightarrow K^n(i)(\{1, \dots, n\}) = B^{\otimes n}$ is the canonical map. The morphism $i^{\Box n}$ is an iterated pushout product morphism. Indeed, for $n = 2$ the cube $K^2(i)$ is a square and looks like

$$\begin{tikzcd}
A \otimes A \arrow[r, "A\otimes i"] \arrow[d, "i\otimes A"'] & A \otimes B \arrow[d, "i\otimes B"] \\
B \otimes A \arrow[r, "B\otimes i"'] & B \otimes B
\end{tikzcd}$$

Hence

$$i^{\Box 2} \ = \ i \Box i \ = \ (B \otimes i) \cup (i \otimes B) \ : \ B \otimes A \cup_{A \otimes A} A \otimes B \ \longrightarrow \ B \otimes B \ .$$

Similarly, $i^{\Box 3}$ is the morphism from the colimit of the punctured cube to the terminal vertex of the following cube:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[rr, "A\otimes A\otimes i"] \arrow[dr, "A\otimes i\otimes A"'] \arrow[dd, "i\otimes A\otimes A"'] & & A \otimes A \otimes B \arrow[dr, "A\otimes i\otimes B"] \arrow[dd, "i\otimes A\otimes B"' near end] & \\
& A \otimes B \otimes A \arrow[rr, "A\otimes B\otimes i"' near start] \arrow[dd, "i\otimes B\otimes A" near end] & & A \otimes B \otimes B \arrow[dd, "i\otimes B\otimes B"] \\
B \otimes A \otimes A \arrow[rr, "B\otimes A\otimes i" near start] \arrow[dr, "B\otimes i\otimes A"'] & & B \otimes A \otimes B \arrow[dr, "B\otimes i\otimes B"] & \\
& B \otimes B \otimes A \arrow[rr, "B\otimes B\otimes i"'] & & B \otimes B \otimes B
\end{tikzcd}$$

For the proof of Theorem 3.4 we use results of Gorchinskiy and Guletskii [**37**], who have studied symmetric power constructions in a symmetric monoidal model structure. We observe that the symmetric group Σ_n acts on $Q^n(i)$ and $B^{\otimes n}$ by permuting the tensor factors, and the iterated pushout product morphism $i^{\Box n} : Q^n(i) \longrightarrow B^{\otimes n}$ is Σ_n-equivariant. We recall from [**37**, 3.2] the notions of *symmetrizable cofibration* and *symmetrizable acyclic cofibration.*

Definition 3.5. Let $\mathcal{C}$ be a symmetric monoidal model category. A morphism $i : A \longrightarrow B$ is a *symmetrizable cofibration* (respectively a *symmetrizable acyclic cofibration*) if the morphism

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\otimes n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (respectively an acyclic cofibration) for every $n \geq 1$.

Since the morphism $i^{\Box 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration and every symmetrizable acyclic cofibration is in particular an acyclic cofibration.

A crucial step in the proof is the homotopical analysis of certain pushouts in the category of commutative monoids. The following 'filtration' on such pushouts will be used.

Construction 3.6 (Pushout filtration)**.** We let $\mathcal{C}$ be a cocomplete symmetric monoidal category with monoidal product $\otimes$ and unit object I. We denote by $\mathbb{P} : \mathcal{C} \longrightarrow \mathcal{C}$ the functor given by

$$\mathbb{P}X \ = \ \coprod_{n\geq 0} X^{\otimes n}/\Sigma_n \ .$$

This is the free commutative monoid functor in the following sense. the object $\mathbb{P}X$ has a commutative multiplication [recall], and with respect to this multiplication $\mathbb{P}$ is a left adjoint to the forgetful functor from commutative monoid objects to $\mathcal{C}$. We will use the same symbol $\mathbb{P}$ for the free functor to commutative monoids and for the underlying endofunctor of $\mathcal{C}$.

We want to analyze pushouts, in the category of commutative monoids in $\mathcal{C}$, of diagrams of the form

$$X \xleftarrow{\ \bar{\varphi}\ } \mathbb{P}A \xrightarrow{\ \mathbb{P}i\ } \mathbb{P}B$$

where $i : A \longrightarrow B$ is any $\mathcal{C}$-morphism. By the freeness property (i.e., the fact that $\mathbb{P}$ is left adjoint to the forgetful functor), the data and the universal property of a pushout in the category of commutative monoids are equivalent to the data of a commutative monoid P, a monoid homomorphism $j : X \longrightarrow P$, and a $\mathcal{C}$-morphism $\psi : B \longrightarrow P$ such that

(i) $j\varphi = \psi i : A \longrightarrow P$ as morphisms in $\mathcal{C}$, and
(ii) for every monoid homomorphism $\alpha : X \longrightarrow M$ and every $\mathcal{C}$-morphism $f : B \longrightarrow M$ satisfying $\alpha\circ\varphi_1 = fi : A \longrightarrow M$ there is a unique monoid homomorphism $\beta : P \longrightarrow M$ such that $\beta \circ j = \alpha : X \longrightarrow M$ and $\beta \circ \psi_1 = f$: the left square in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\varphi"'] & B \arrow[d, "\psi"'] \arrow[dr, "f"] & \\
X \arrow[r, "j"] \arrow[rr, bend right, "\alpha"'] & P \arrow[r, dotted, "\beta"] & M
\end{tikzcd}$$

We construct a pushout P along with a 'relative word length filtration', i.e., as a colimit in $\mathcal{C}$ of a sequence of $\mathcal{C}$-morphism

$$X = P_0 \ \longrightarrow \ P_1 \ \longrightarrow \ P_2 \ \longrightarrow \ \dots$$

with Y as a colimit and a description of how P_{n+1} is obtained from P_n as a specific cobase change. If one thinks of Y as consisting of products of elements from X and from B, with relations coming from the elements of A and the multiplication in X, then P_n consists of those products where the total number of factors from B is less than or equal to n. For ordinary commutative monoids, this is in fact a valid description, and we will now translate this idea into the element-free form which applies to general symmetric monoidal categories.

As indicated above we set $P_0 = X$ and describe P_n inductively as a pushout in $\mathcal{C}$. We recall that $Q^n(i)$ denote the source of the n-fold pushout product morphism

$$i^{\Box n} \ : \ Q^n(i) \ \longrightarrow \ B^{\otimes n}$$

of the morphism i, a morphism of Σ_n-objects in $\mathcal{C}$. We define P_n as a pushout in $\mathcal{C}$:

$$\begin{tikzcd}
X \otimes (Q^n(i)/\Sigma_n) \arrow[r, "X\otimes(i^{\Box n}/\Sigma_n)"] \arrow[d] & X \otimes (B^{\otimes n}/\Sigma_n) \arrow[d] \\
P_{n-1} \arrow[r] & P_n
\end{tikzcd}$$

This is not a complete definition until we say what the left vertical map is. The object $Q^n(i)$ is the colimit of a certain puncture cube $K^n(i)$ and we define the map from $X \otimes (Q^n(i)/\Sigma_n) = (X \otimes Q^n(i))/\Sigma_n$ to P_{n-1} by describing how it maps $X \otimes K^n(i)(S)$ for S a proper subset of $\{1, 2, \dots, n\}$. All the smash factors of

$K^n(i)(S)$ that are equal to A are permuted next to X, using the symmetry isomorphism of the monoidal structure, and then mapped into X. Then all tensor factors of X are multiplied. This gives a morphism

$$K^n(i)(S) \longrightarrow X \otimes B^{\otimes |S|} .$$

So the right hand side maps further to $P_{|S|}$, hence to P_{n-1}, since S is a proper subset, and so has cardinality strictly less than n.

We claim that these maps on the vertices of the punctured cube $K^n(i)$ are compatible so that they assemble to a map from the colimit $Q^n(i)$. So let S be again a proper subset of $\{1, 2, \ldots, n\}$ and take $i \notin S$. We have to verify commutativity of the diagram

$$\begin{tikzcd} K^n(i)(S) \arrow[r] \arrow[d] & X \otimes B^{\otimes |S|} \arrow[r] & P_{|S|} \arrow[d] \\ K^n(i)(S \cup \{i\}) \arrow[r] & X \otimes B^{\otimes (|S|+1)} \arrow[r] & P_{|S|+1} \end{tikzcd}$$

By definition, $K^n(i)(S)$ and $K^n(i)(S \cup \{i\})$ differ at exactly one smash factor in the i-th position which is equal to A for the former and equal to B for the latter. The upper left map factors as

$$K^n(i)(S) \longrightarrow X \otimes B^{\otimes a} \otimes A \otimes B^{\otimes b} \longrightarrow X \otimes B^{\otimes |S|}$$

where a (resp. b) is the number of elements in S which are smaller (resp. larger) than i; in particular $a + b = |S|$. The right map in this factorization permutes A next to X and multiplies the two factors of X. Hence the diagram in question is the composite of two commutative squares

$$\begin{tikzcd} K^n(i)(S) \arrow[r] \arrow[d] & X \otimes B^{\otimes a} \otimes A \otimes B^{\otimes b} \arrow[r] \arrow[d] & P_{|S|} \arrow[d] \\ K^n(i)(S \cup \{i\}) \arrow[r] & X \otimes B^{\otimes (|S|+1)} \arrow[r] & P_{|S|+1} \end{tikzcd}$$

The right square commutes by the definition of $P_{|S|+1}$. This completes the verification of the compatibility, so the maps define a unique morphism $X \otimes Q^n(i) \longrightarrow P_{n-1}$. Since the multiplication of X is commutative, this map is invariant under the Σ_n-action on $Q^n(i)$, so it descends to the desired morphism $X \otimes (Q^n(i)/\Sigma_n) \longrightarrow P_{n-1}$. We have now completed the inductive definition of P_n. We set $P = \operatorname{colim}_n P_n$, the colimit being taken in $\mathcal{C}$. Then P comes equipped with $\mathcal{C}$-morphisms $X = P_0 \longrightarrow P$ and

$$B \cong I \otimes B \xrightarrow{\eta \otimes B} X \otimes B \longrightarrow P_1 \longrightarrow P$$

We define the unit of P as the composite of $X \longrightarrow P$ with the unit of X. The multiplication of P is defined from compatible maps $P_n \otimes P_m \longrightarrow P_{n+m}$ by passage to the colimit. These maps are defined by induction on $n + m$ as follows. We exploit that $P_n \otimes P_m$ is the pushout in $\mathcal{C}$ in the following diagram:

$$\begin{tikzcd} (X \otimes Q^n) \otimes (X \otimes B^{\otimes m}) \cup_{(X \otimes Q^n) \otimes (X \otimes Q^m(i))} (X \otimes B^{\otimes n}) \otimes (X \otimes Q^m) \arrow[r] \arrow[d] & (X \otimes B^{\otimes n}) \otimes (X \otimes B^{\otimes m}) \arrow[d] \\ (P_{n-1} \otimes P_m) \cup_{(P_{n-1} \otimes P_{m-1})} (P_n \otimes P_{m-1}) \arrow[r] & P_n \otimes P_m \end{tikzcd}$$

Here we abbreviated $Q^n(i)$ to Q^n, etc. The lower left corner already has a map to P_{n+m} by induction, the upper right corner is mapped there by multiplying the two factors of X followed by the map $X \otimes B^{\otimes (n+m)} \longrightarrow P_{n+m}$ from the definition of P_{n+m}. We omit the tedious verification that this in fact gives a well defined multiplication map and that the associativity, commutativity and unitality diagrams commute. Hence, P is a commutative monoid. Multiplication in P was arranged so that $X \longrightarrow P$ is a monoid homomorphism.

For (iii), suppose we are given another commutative monoid M, a monoid homomorphism map $\alpha : X \longrightarrow M$, and a $\mathcal{C}$-map $f : B \longrightarrow M$ such that the outer diagram in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\varphi_1"'] & B \arrow[d, "\psi_1"'] \arrow[dr, "f"] & \\ X \arrow[r] \arrow[rr, bend right, "\alpha"'] & P \arrow[r, dotted, "\beta"] & M \end{tikzcd}$$

commutes. We have to show that there is a unique monoid homomorphism $\beta : P \longrightarrow M$ making the entire square commute. These conditions in fact force the behavior of the composite map $X \otimes K^n(i)(S) \longrightarrow P_n \longrightarrow P \longrightarrow M$. Since P is obtained by various colimit constructions from these basic building blocks, uniqueness follows. We again omit the tedious verification that the maps $X \otimes K^n(i)(S) \longrightarrow M$ are compatible and assemble to a homomorphism $\beta : P \longrightarrow M$.

Proposition 3.7. *Let $\mathcal{C}$ be a monoidal model category satisfying the monoid axiom. Let $i : A \longrightarrow B$ be a cofibration in $\mathcal{C}$ and*

$$\begin{tikzcd} \mathbb{P}(A) \arrow[r, "\mathbb{P}(i)"] \arrow[d] & \mathbb{P}(B) \arrow[d] \\ X \arrow[r, "j"'] & P \end{tikzcd}$$

a pushout in the category of commutative monoids.

(i) *If X is cofibrant in the underlying model structure of $\mathcal{C}$ and i is a symmetrizable cofibration, then the underlying $\mathcal{C}$-morphism of j is a cofibration.*
(ii) *If i is a symmetrizable acyclic cofibration, then j is a weak equivalence.*

Proof. We use the filtration of Construction 3.6, i.e., the pushout squares in $\mathcal{C}$

$$\begin{tikzcd} X \otimes (Q^n(i)/\Sigma_n) \arrow[r, "X\otimes(i^{\Box n}/\Sigma_n)"] \arrow[d] & X \otimes (B^{\otimes n}/\Sigma_n) \arrow[d] \\ P_{n-1} \arrow[r] & P_n \end{tikzcd}$$

such that P is a colimit, in $\mathcal{C}$, of the sequence of lower horizontal morphisms $P_{n-1} \longrightarrow P_n$.

(i) The morphism $i^{\Box n}/\Sigma_n$ is a cofibration because i is symmetrizable. The upper vertical map $X \otimes (i^{\Box n}/\Sigma_n) : X \otimes (Q^n(i)/\Sigma_n) \longrightarrow X \otimes (B^{\otimes n}/\Sigma_n)$ is then a cofibration by the pushout product property since X is cofibrant. So the lower horizontal morphism $P_{n-1} \longrightarrow P_n$ is a cofibration in $\mathcal{C}$. So the morphism $j : X = P_0 \longrightarrow \operatorname{colim}_n P_n = Y$ is a cofibration as a sequential composite of cofibrations.

(ii) The morphism $i^{\Box n}/\Sigma_n : Q^n(i)/\Sigma_n \longrightarrow B^{\otimes n}/\Sigma_n$ is an acyclic cofibration because i is a symmetrizable acyclic cofibration. The morphism $j : X = P_0 \longrightarrow \operatorname{colim}_n P_n = Y$ is thus in the class

$$(\mathcal{C} \otimes \{\text{acyclic cofibrations}\})\text{-cof}_{\text{reg}} ,$$

and hence a weak equivalence by the monoid axiom. □

Theorem 3.8. *Let $\mathcal{C}$ be a monoidal model category satisfying the following additional hypotheses:*

(i) *The model structure has sets of generating cofibrations and generating acyclic cofibrations whose sources are small relative to [???]. In particular, the model structure is cofibrantly generated.*
(ii) *The monoid axiom holds.*
(iii) *Every cofibration is a symmetrizable cofibration.*

(iv) *Every acyclic cofibration is a symmetrizable acyclic cofibration.*

Then the model structure of $\mathcal{C}$ lifts to the category of commutative monoids in $\mathcal{C}$. Moreover, every cofibration of commutative monoids whose source is cofibrant in $\mathcal{C}$ is also a cofibration in $\mathcal{C}$.

Proof. We apply a standard lifting theorem for model structures to categories of algebras over a triple, Lemma 2.3 of [**72**]. Commutative monoids in $\mathcal{C}$ are the algebras over the free commutative monoid triple $\mathbb{P}$. Because the monoidal product is closed symmetric, $\otimes$ commutes with colimits. Hence, the underlying functor of $\mathbb{P}$ commutes with filtered colimits, as required for [**72**, Lemma 2.3].

We let I be a set of generating cofibrations and J be a set of generating acyclic cofibrations for the model structure of $\mathcal{C}$. Then $I_{\text{mon}} = \mathbb{P}(I)$ respectively $J_{\text{mon}} = \mathbb{P}(J)$ will be sets of generating cofibrations, respectively generating acyclic cofibrations, for the model structure on commutative monoids. The proof of Proposition 3.7 (ii) shows that every morphism in J_{mon} is a countable composite of cobase changes of morphisms in the class $\mathcal{C} \otimes \{\text{acyclic cofibrations}\}$. A transfinite composite of countable composites is again a transfinite composite. Because the forgetful functor from monoids to $\mathcal{C}$ preserves filtered colimits, the monoid axiom shows that regular J_{mon}-cofibrations are weak equivalences. So condition (1) of [**72**, Lemma 2.3] is satisfied, and this Lemma provides the desired model structure on the category of commutative monoids in $\mathcal{C}$.

Let $f : M \longrightarrow N$ be a cofibration of monoids with M cofibrant in $\mathcal{C}$. By the small object argument, the morphism f is a retract of a regular I_{mon}-cofibration. So we may assume that f itself is a regular I_{mon}-cofibration. The source of f is cofibrant in $\mathcal{C}$, so the claim follows from Proposition 3.7 (i). $\square$

The next two propositions serve to simplify the verification of the symmetrizability hypotheses (iii) and (iv) of the lifting theorem 3.8. Indeed, the next proposition reduces the verification of symmetrizability to sets of generating cofibrations and generating acyclic cofibrations; this next proposition is the combined content of Propositions 4, 5, 6 and 7 of [**37**, 3.2]. By 'closure under composition' we mean composites of two morphism, and of countable or transfinite sequences.

Proposition 3.9. *Let $\mathcal{C}$ be a symmetric monoidal model category. The classes of symmetrizable cofibrations and symmetrizable acyclic cofibrations are closed under cobase change, composition, and retracts.*

Proposition 3.10. *Let $\mathcal{C}$ be a symmetric monoidal topological model category.*

(i) *For every $n \geq 1$ the functor $\mathbb{P}^n$ preserves the homotopy relation on morphisms and it preserves homotopy equivalences.*

(ii) *Let $j : A \longrightarrow B$ be a symmetrizable acyclic cofibration between cofibrant objects. Then for every $k \geq 0$, the pushout product map*
$$i_k \Box j \ : \ D^k \times A \cup_{S^{k-1} \times A} S^{k-1} \times B \ \longrightarrow \ D^k \times B$$
is a symmetrizable acyclic cofibration.

(iii) *Let $j : A \longrightarrow B$ be a morphism between cofibrant objects such that the morphism $\mathbb{P}^n(j) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(B)$ is a weak equivalence for every $n \geq 1$. Then every morphism in the set $\mathcal{Z}(j)$ is a symmetrizable acyclic cofibration.*

Proof. (i) For every space K and every object X of $\mathcal{C}$ the morphism
$$K \times X^{\otimes n} \ \xrightarrow{\Delta \times X^{\otimes n}} \ K^n \times X^{\otimes n} \ \cong \ (K \times X)^{\otimes n}$$
is Σ_n-equivariant (with respect to the trivial Σ_n-action on K in the source) and factors over a natural morphism
$$\tilde{\Delta} \ : \ K \times \mathbb{P}^n(X) \ = \ (K \times X^{\otimes n})/\Sigma_n \ \longrightarrow \ (K \times X)^{\otimes n}/\Sigma_n \ = \ \mathbb{P}^n(K \times X) \ .$$
If $H : [0,1] \times X \longrightarrow Y$ if a homotopy from a morphism $f = H(0,-)$ to another morphism $g = H(1,-)$, then the composite
$$[0,1] \times \mathbb{P}^n(X) \ \xrightarrow{\tilde{\Delta}} \ \mathbb{P}^n([0,1] \times X) \ \xrightarrow{\mathbb{P}^n(H)} \ \mathbb{P}^n(Y)$$

is a homotopy from the morphism $\mathbb{P}^n(f)$ to $\mathbb{P}^n(g)$. So $\mathbb{P}^n$ preserves the homotopy relation, and hence also homotopy equivalences.

(ii) We argue by induction on k. For $k = 0$ the pushout product map $i_0 \Box j$ is isomorphic to j, hence a symmetrizable acyclic cofibration by hypothesis. Now we assume the claim for some $k \geq 0$, and deduce it for $k+1$. Since j is a symmetrizable acyclic cofibration between cofibrant objects, the morphism $\mathbb{P}^n(j)$ is a weak equivalence for every $n \geq 1$ [ref]. Since the functors $\mathbb{P}^n$ preserve the homotopy relation and the projections $D^k \times A \longrightarrow A$ and $D^k \times B \longrightarrow B$ are homotopy equivalences, the morphism $\mathbb{P}^n(D^k \times j)$ is a weak equivalence for every $n \geq 1$. So $D^k \times j : D^k \times A \longrightarrow D^k \times B$ is a symmetrizable acyclic cofibration by [**37**, Lemma 29]. We write $S^k = D^k_+ \cup_{S^{k-1}} D^k_-$ as the union of the upper and lower hemisphere along the equator. The upper morphism in the pushout square

$$\begin{tikzcd} D^k_+ \times A \arrow[r, "D^k_+ \times j"] \arrow[d] & D^k_+ \times B \arrow[d] \\ S^k \times A \arrow[r] & S^k \times A \cup_{D^k_+ \times A} D^k_+ \times B \end{tikzcd}$$

is a symmetrizable acyclic cofibration by the previous paragraph, hence so is the lower morphism (Proposition 3.9).

The square

$$\begin{tikzcd} D^k_- \times A \cup_{S^{k-1} \times A} S^{k-1} \times B \arrow[r, "i_k \Box j"] \arrow[d] & D^k_- \times B \arrow[d] \\ S^k \times A \cup_{D^k_+ \times A} D^k_+ \times B \arrow[r, "i' \Box j"'] & S^k \times B \end{tikzcd}$$

is a pushout. The upper morphism is a symmetrizable acyclic cofibration by the inductive hypothesis, hence so is the lower morphism (Proposition 3.9). The morphism $S^k \times j : S^k \times A \longrightarrow S^k \times B$ is thus the composite of two symmetrizable acyclic cofibrations, hence is a symmetrizable acyclic cofibration itself (Proposition 3.9). As a cobase change, the morphism

$$D^{k+1} \times A \ \longrightarrow \ D^{k+1} \times_{S^k \times A} S^k \times B$$

is a symmetrizable acyclic cofibration. The induced morphism

$$\mathbb{P}^n(D^{k+1} \times A) \ \longrightarrow \ \mathbb{P}^n(D^{k+1} \times_{S^k \times A} S^k \times B)$$

is then a weak equivalence by [**37**, Lemma 29]. Since $\mathbb{P}^n(D^{k+1} \times j) : \mathbb{P}^n(D^{k+1} \times A) \longrightarrow \mathbb{P}^n(D^{k+1} \times B)$ is a weak equivalence, so is the morphism

$$\mathbb{P}^n(i_{k+1} \Box j) \ : \ \mathbb{P}^n(D^{k+1} \times_{S^k \times A} S^k \times B) \ \longrightarrow \ \mathbb{P}^n(D^{k+1} \times B) \ .$$

One more time by [**37**, Lemma 29], this shows that $i_{k+1} \Box j$ is a symmetrizable acyclic cofibration. This completes the induction step.

(iii) Since A and B are cofibrant, the mapping cylinder inclusion

$$c(j) \ : \ A \ \longrightarrow \ [0,1] \times A \cup_j B = Z(j)$$

is a cofibration. Moreover, the projection $Z(j) \longrightarrow B$ is a homotopy equivalence, hence so is $\mathbb{P}^n(Z(j)) \longrightarrow \mathbb{P}^n(B)$ for every $n \geq 1$. Since $\mathbb{P}^n(j)$ is a weak equivalence by hypothesis, the morphism $\mathbb{P}^n(c(j)) : \mathbb{P}^n(A) \longrightarrow \mathbb{P}^n(Z(j))$ is a weak equivalence for every $n \geq 1$. So $c(j)$ is a symmetrizable acyclic cofibration by [**37**, Lemma 29]. The claim now follows by applying (i) to the morphism $c(j)$. $\square$

The next theorem says that in the category of orthogonal spaces, the all cofibrations and acyclic cofibrations in the positive global model structure on orthogonal spaces are symmetrizable with respect to the monoidal structure given by the box product.

Theorem 3.11. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spaces. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spaces are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spaces that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\boxtimes n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spaces are symmetrizable.

Proof. (i) We recall from the proof of Proposition 3.12 the set

$$I^{\text{str}} \ = \ \{ \ G_m(i_k \times O(m)/H) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spectra, where $i_k : S^{k-1} \longrightarrow D^k$ is the inclusion. The set I^{str} detects the acyclic fibrations in the strong level model structure of orthogonal spaces. In particular, every flat cofibration is a retract of an I^{str}-cell complex. By [**37**, Cor. 21] it suffices to show that the generating flat cofibrations in I^{str} are symmetrizable.

The orthogonal space $G_m(K \times O(m)/H)$ is isomorphic to $K \times L_{H,\mathbb{R}^m}$, so we show more generally that every morphism of the form

$$j \times \mathbf{L}_{G,V} \ : \ K \times \mathbf{L}_{G,V} \ \longrightarrow \ L \times \mathbf{L}_{G,V}$$

is a symmetrizable cofibration, where G is any compact Lie group, V a G-representation and $j : K \longrightarrow L$ a cofibration of spaces. The symmetrized iterated pushout product

$$(j \times \mathbf{L}_{G,V})^{\square n}/\Sigma_n \ : \ Q^n(j \times \mathbf{L}_{G,V})/\Sigma_n \ \longrightarrow \ (L \times \mathbf{L}_{G,V})^{\boxtimes n}/\Sigma_n \tag{3.12}$$

is isomorphic to

$$\mathbf{L}_{\Sigma_n \wr G, V^n}(j^{\square n}) \ : \ \mathbf{L}_{\Sigma_n \wr G, V^n}(Q^n(j)) \ \longrightarrow \ \mathbf{L}_{\Sigma_n \wr G, V^n}(L^n) \ ,$$

where

$$j^{\square n} \ : \ Q^n(j) \ \longrightarrow \ L^n$$

is the n-fold pushout product of j, with respect to the cartesian product of spaces. Here the wreath product $\Sigma_n \wr G$ acts on V^n by

$$(\sigma; g_1, \dots, g_n) \cdot (v_1, \dots, v_n) \ = \ (g_{\sigma(1)} v_{\sigma(1)}, \dots, g_{\sigma(n)} v_{\sigma(n)}) \ .$$

The map $j^{\square n}$ is Σ_n-equivariant, and it is viewed as a morphism of a $(\Sigma_n \wr G)$-spaces by restriction along the projection $(\Sigma_n \wr G) \longrightarrow \Sigma_n$. Since j is a cofibration of spaces, $j^{\square n}$ is a cofibration of Σ_n-spaces, hence a cofibration of $(\Sigma_n \wr G)$-spaces. So the morphism (3.12) is a flat cofibration.

(ii) Proposition 7.16 describes a set $J^{\text{str}} \cup K_{\mathcal{All}}$ of generating acyclic cofibrations for the global model structure on the category of orthogonal spaces. From this we obtain a set $J^{\text{str},+} \cup K^+_{\mathcal{All}}$ of generating acyclic cofibration for the *positive* global model structure of Proposition 3.3 by restricting to those morphisms in $J^{\text{str}} \cup K_{\mathcal{All}}$ that are positive cofibrations, i.e., homeomorphisms in level 0; so explicitly, we set

$$J^{\text{str},\,+} \ = \ \{ \ G_m(j_k \times O(m)/H) \mid m \geq 1, k \geq 0, H \leq O(m)\} \ ,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+_{\mathcal{All}} \ = \ \bigcup_{G,V,W \ : \ V \neq 0} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\rho_{G,V,W} : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$. Here (G, V, W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a non-zero faithful G-representation V and an arbitrary G-representation W. By [**37**, Cor. 21] it suffices to show that all morphisms in $J^{\mathrm{str},+} \cup K^+_{\mathcal{A}ll}$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m(j_k \times O(m)/H)$ in $J^{\mathrm{str},+}$. For every $n \geq 1$, the morphism

$$(G_m(j_k \times O(m)/H))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the morphism j_k is a homotopy equivalence of spaces, so $G_m(j_k \times O(m)/H)$ is a homotopy equivalence of orthogonal spaces; the morphism $\mathbb{P}^n(G_m(j_k \times O(m)/H))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition 3.10 (i). Then [**37**, Lemma 29] shows that $G_m(j_k \times O(m)/H)$ is a symmetrizable acyclic cofibration. This takes care of the set $J^{\mathrm{str},+}$.

Now we consider the morphisms in the set $K^+_{\mathcal{A}ll}$. Since G acts faithfully on the non-zero inner product space V, the action of the wreath product $\Sigma_n \wr G$ on V^n is again faithful. So the morphism

$$\rho_{\Sigma_n \wr G, V^n, W^n} \ : \ \mathbf{L}_{\Sigma_n \wr G, V^n \oplus W^n} \ \longrightarrow \ \mathbf{L}_{\Sigma_n \wr G, V^n}$$

is a global equivalence by Proposition 2.6 (v). By the natural isomorphism

$$\mathbb{P}^n(\mathbf{L}_{G,V}) \ = \ \mathbf{L}^{\boxtimes n}_{G,V}/\Sigma_n \ \cong \ \mathbf{L}_{\Sigma_n \wr G, V^n} \ ,$$

the morphism $\rho_{\Sigma_n \wr G, V^n, W^n}$ is isomorphic to $\mathbb{P}^n(\rho_{G,V,W}) : \mathbb{P}^n(\mathbf{L}_{G,V\oplus W}) \longrightarrow \mathbb{P}^n(\mathbf{L}_{G,V})$, which is thus a global equivalence. Proposition 3.10 (iii) then shows that all morphisms in $\mathcal{Z}(\rho_{G,V,W})$ are symmetrizable acyclic cofibrations. $\square$

Now put all the pieces together and prove the global model structure for commutative orthogonal monoid spaces:

Proof of Theorem 3.4. The positive global model structure of orthogonal spaces (Proposition 3.3) is monoidal and cofibrantly generated. The monoid axiom hold by Proposition 5.11. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 3.11. So Theorem 3.8 shows that the positive global model structure of orthogonal spaces lifts to the category of commutative orthogonal monoid spaces, and it provides the addendum about cofibration with cofibrant source. $\square$

4. Global group completion

For every orthogonal monoid space R and compact Lie group G, the internal product

$$\pi_0^G(R) \times \pi_0^G(R) \ \xrightarrow{\times} \ \pi_0^{G\times G}(R \boxtimes R) \ \xrightarrow{\mu_*} \ \pi_0^{G\times G}(R) \ \xrightarrow{\Delta^*} \ \pi_0^G(R)$$

makes the equivariant homotopy set $\pi_0^G(R)$ into a monoid, and this multiplication is natural with respect to restriction maps in G. If the multiplication $\mu : R \boxtimes R \longrightarrow R$ of R is commutative, then so is the internal multiplication of $\pi_0^G(R)$.

Definition 4.1. An orthogonal monoid space R is *group-like* if for every compact Lie group G the monoid $\pi_0^G(R)$ is a group.

A monoid M is a group if and only if the shearing map

$$\chi \ : \ M^2 \ \longrightarrow \ M^2 \ , \quad (x, y) \ \longmapsto \ (x, xy)$$

is bijective. Indeed, if M is a group, then the map $(x, z) \mapsto (x, x^{-1}z)$ is inverse to χ. Conversely, if the shear map is bijective, then for every $x \in M$ we have $\chi^{-1}(x, 1) = (x, y)$ with $xy = 1$. Also $\chi(x, yx) = (x, xyx) = (x, x) = \chi(x, 1)$, so $yx = 1$ and y is a two-sided inverse for x.

For orthogonal monoid spaces R, the group-like condition has a similar characterization as follows. The *shearing morphism* is the morphism of orthogonal spaces

$$\chi = (\rho_1, \mu) \ : \ R \boxtimes R \ \longrightarrow \ R \times R$$

whose first component is the projection p_1 to the first factor and whose second component is the multiplication morphism $\mu : R \boxtimes R \longrightarrow R$.

Proposition 4.2. *Let R be an orthogonal monoid space that is flat as an orthogonal space. Then R is group-like if and only if the shearing morphism $\chi : R \boxtimes R \longrightarrow R \times R$ is a global equivalence.*

Proof. Since R is flat as an orthogonal space, the morphism

$$\rho_{R,R} = (\rho_1, \rho_2) \ : \ R \boxtimes R \ \longrightarrow \ R \times R$$

is a global equivalence by Theorem 5.9 (i). The vertical maps in the commutative diagram

$$\begin{tikzcd}
\pi_0^G(R \times R) \arrow[d, "(\pi_0^G(\rho_1), \pi_0^G(\rho_2))"', "\cong"] & \pi_0^G(R \boxtimes R) \arrow[l, "\pi_0^G(\rho_{R,R})"', "\cong"] \arrow[r, "\pi_0^G(\chi)"] & \pi_0^G(R \times R) \arrow[d, "\cong"', "(\pi_0^G(\rho_1), \pi_0^G(\rho_2))"] \\
\pi_0^G(R) \times \pi_0^G(R) \arrow[rr, "(x,y) \longmapsto (x, x\cdot y)"'] & & \pi_0^G(R) \times \pi_0^G(R)
\end{tikzcd}$$

are bijective by Theorem 5.9 (ii). If the shearing morphism is a global equivalence, then the map $\pi_0^G(\chi)$ is bijective, hence so is the algebraic shearing map of the monoid $\pi_0^G(R)$. This monoid is thus a group, and so R is group-like.

Now we assume conversely that R is group-like, and we show that χ is a global equivalence. Since R is flat, so are $R \boxtimes R$ and $R \times R$ [ref], so we may show that for every compact Lie group G the continuous map

$$(\chi(\mathcal{U}_G))^G \ : \ ((R \boxtimes R)(\mathcal{U}_G))^G \ \longrightarrow \ ((R \times R)(\mathcal{U}_G))^G \ = \ R(\mathcal{U}_G)^G \times R(\mathcal{U}_G)^G$$

is a weak equivalence. Since R, and hence also $R \boxtimes R$, is group-like, choices of points in the path components provide a homotopy equivalence

$$\pi_0^G(R \boxtimes R) \times ((R \boxtimes R)(\mathcal{U}_G))_1^G \ \longrightarrow \ ((R \boxtimes R)(\mathcal{U}_G))^G$$

where the subscript $(-)_1$ denotes the part component of the identity element. The same applies $R \times R$ and yields a homotopy equivalence

$$\pi_0^G(R \times R) \times ((R \times R)(\mathcal{U}_G))_1^G \ \longrightarrow \ ((R \times R)(\mathcal{U}_G))^G \ ;$$

we choose the images of the path component representations for $R \boxtimes R$ as the representatives for $R \times R$, so that the following diagram commutes:

$$\begin{tikzcd}
\pi_0^G(R \boxtimes R) \times ((R \boxtimes R)(\mathcal{U}_G))_1^G \arrow[r, "\pi_0^G(\chi) \times (\chi(\mathcal{U}_G))_1^G"] \arrow[d, "\simeq"'] & \pi_0^G(R \times R) \times ((R \times R)(\mathcal{U}_G))_1^G \arrow[d, "\simeq"] \\
((R \boxtimes R)(\mathcal{U}_G))^G \arrow[r, "(\chi(\mathcal{U}_G))^G"'] & ((R \times R)(\mathcal{U}_G))^G
\end{tikzcd}$$

Since the map $\pi_0^G(\chi)$ is bijective, it suffices to show that the restriction

$$(\chi(\mathcal{U}_G))_1^G \ : \ ((R \boxtimes R)(\mathcal{U}_G))_1^G \ \longrightarrow \ ((R \times R)(\mathcal{U}_G))_1^G$$

to the identity components is a weak equivalence. However, when restricted to the identity components, the morphism $(\chi(\mathcal{U}_G))^G$ is homotopy to the morphism $(\rho_{R,R}(\mathcal{U}_G))^G$, which is a weak equivalence because $\rho_{R,R}$ is a global equivalence by Theorem 5.9. $\square$

We will mostly be interested in *commutative* orthogonal monoid spaces R. The flatness hypothesis of Proposition 4.2 is no serious loss of generality because it can be arranged by cofibrant replacement in the global model structure of commutative orthogonal monoid spaces of Theorem 3.4. If the multiplication of R is commutative, both p_1 and μ, and hence the shearing morphism χ, are homomorphisms of orthogonal monoid spaces.

In order to motivate the following definition, we recall that the group completion (Grothendieck construction) of an abelian monoid M is a morphism $i : M \longrightarrow M^\star$ of monoids such that $M^\star$ is an abelian group, and such that i is initial among monoid homomorphisms from M to abelian groups. One way to construct a group-completion (although not the most common one) is to define $M^\star$ as a pushout, in the category of commutative monoids, of the diagram:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d] & M \times M \arrow[d, "d"] \\ * \arrow[r] & M^\star \end{tikzcd} \tag{4.3}$$

The universal morphism $i : M \longrightarrow M^\star$ is then the composite

$$M \xrightarrow{(\mathrm{Id},*)} M \times M \xrightarrow{d} M^\star ,$$

and $d(x, y) = i(x) - i(y)$.

Construction 4.4. Let R be a commutative orthogonal monoid space. The *group completion* $R^\star$ of R is the homotopy pushout, in the category of commutative orthogonal monoid spaces, of the diagram

$$* \longleftarrow R \xrightarrow{\Delta} R \times R .$$

The group completion comes with a morphism $i : R \longrightarrow R^\star$ of orthogonal monoid spaces, defined as the composite

$$R \xrightarrow{(\mathrm{Id},*)} R \times R \longrightarrow R^\star .$$

Remark 4.5. The global model structure on the category of commutative orthogonal monoid spaces (see Theorem 3.4) is proper and topological, so a homotopy pushout of two morphisms

$$S \xleftarrow{\varphi} R \xrightarrow{\psi} S'$$

cans be constructed as a two-sided bar construction, namely as the geometric realization of the simplicial object of commutative orthogonal monoid spaces

$$B^\boxtimes(S, R, S') \;=\; \left([n] \mapsto S \boxtimes \underbrace{R \boxtimes \cdots \boxtimes R}_{n} \boxtimes S' \right) .$$

The box product arises here because it is the coproduct in the category of commutative orthogonal monoid spaces. The simplicial face maps are induced by the multiplication in R and the action of R on S respectively S' through the morphisms φ respectively ψ. The geometric realization is taken 'objectwise', i.e., for an inner product space V we have

$$\mathrm{hopushout}(\varphi, \psi)(V) \;=\; |B^\boxtimes(S, R, S')|(V) \;=\; |B^\boxtimes(S, R, S')(V)| .$$

As a special case, the group completion of a commutative orthogonal monoid space R can be constructed as $R^\star = |B^\boxtimes(*, R, R \times R)|$. The morphism $i : R \longrightarrow R^\star$ is then then composite of

$$R \xrightarrow{(\mathrm{Id},*)} R \times R \xrightarrow{i_2} * \boxtimes (R \times R) = B^\boxtimes(*, R, R \times R)_0 \xrightarrow{v} |B^\boxtimes(*, R, R \times R)| = R^\star$$

where the last map is the vertex map of a geometric realization. In the context of topological monoids, this bar construction of a group completion for 'sufficiently homotopy commutative' monoids is sketched by Segal on p. 305 of [**77**].

Proposition 4.6. *Let $B_\bullet$ be flat simplicial object of orthogonal spaces. Then for every compact Lie group G, the diagram of sets*

$$\pi_0^G(B_1) \overset{\pi_0^G(d_0)}{\underset{\pi_0^G(d_1)}{\rightrightarrows}} \pi_0^G(B_0) \xrightarrow{\pi_0^G(v)} \pi_0^G(|B_\bullet|)$$

is a coequalizer.

Proof. If $Y_\bullet$ is any simplicial space, then the diagram

$$\pi_0(Y_1) \overset{\pi_0(d_0)}{\underset{\pi_0(d_1)}{\rightrightarrows}} \pi_0(Y_0) \xrightarrow{\pi_0(v)} \pi_0(|Y_\bullet|)$$

is a coequalizer. [this seems to need a cofibrancy condition on $Y_\bullet$ to ensure that a continuous map $[0,1] \longrightarrow |Y_\bullet|$ has image in a finite skeleton; 'free degeneracies' would suffice]

This holds in particular for the simplicial space $(B_\bullet(V))^G$ for any G-representation V, i.e., the diagram

$$\pi_0((B_1(V))^G) \overset{\pi_0(d_0^G)}{\underset{\pi_0(d_1^G)}{\rightrightarrows}} \pi_0((B_0(V))^G) \xrightarrow{\pi_0(v^G)} \pi_0(|(B_\bullet(V))^G|)$$

is a coequalizer. Taking G-fixed points commutes with realization, so the right term bijects with $\pi_0(|(B_\bullet(V))|^G)$. Since colimits commute among themselves, we get the desired coequalizer by passing to colimits over $V \in s(\mathcal{U}_G)$. □

Proposition 4.7. *Let P be the homotopy pushout, in the category of commutative orthogonal monoid spaces, of two morphism*

$$S \xleftarrow{\varphi} R \xrightarrow{\psi} S' .$$

Suppose that at least two of R, S and S' are flat as orthogonal spaces. Then for every compact Lie group G, the monoid $\pi_0^G(P)$ is a cokernel, in the category of commutative monoids, of the morphism

$$\pi_0^G(R) \xrightarrow{(\pi_0^G(\varphi),\pi_0^G(\psi))} \pi_0^G(S) \times \pi_0^G(S') .$$

Proof. The definition of P as a homotopy pushout means that it is the geometric realization of a certain simplicial object, a two-sided bar construction, compare Remark 4.5. Proposition 4.6 then applies and shows that the diagram

$$\pi_0^G(S \boxtimes R \boxtimes S') \overset{\pi_0^G(d_0)}{\underset{\pi_0^G(d_1)}{\rightrightarrows}} \pi_0^G(S \boxtimes S') \xrightarrow{\pi_0^G(v)} \pi_0^G(P)$$

is a coequalizer in the category of sets, where d_0 is the composite

$$S \boxtimes R \boxtimes S' \xrightarrow{S\boxtimes\varphi\boxtimes S'} S \boxtimes S \boxtimes S' \xrightarrow{\mu_S \boxtimes S'} S \boxtimes S' ,$$

and similarly for d_1. Since d_0, d_1 and v are morphisms of commutative orthogonal monoid spaces, all three maps in this coequalizer are actually homomorphisms of commutative monoids.

Since two of the three orthogonal spaces R, S and S' are flat, Theorem 5.9 (ii) let's us replace the left part of the coequalizer by the isomorphic one

$$\pi_0^G(S) \times \pi_0^G(R) \times \pi_0^G(S') \overset{(s,r,s') \mapsto (s\cdot\varphi(r),s')}{\underset{(s,r,s') \mapsto (s,\psi(r)\cdot s')}{\rightrightarrows}} \pi_0^G(S) \times \pi_0^G(S') .$$

Any coequalizer, in the category of sets, of this pair of maps is a cokernel, in the category of commutative monoids, of the morphism in the proposition. □

Corollary 4.8. *Let R be a commutative orthogonal monoid space that is flat as an orthogonal space. Then the group completion $R^\star$ is a group-like commutative orthogonal monoid space and for every compact Lie group G the map*

$$\pi_0^G(i) \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^G(R^\star)$$

induced by the morphism $i : R \longrightarrow R^\star$ is an algebraic group completion.

Proof. The terminal orthogonal space is flat, so if also R is flat as an orthogonal space, then Proposition 4.7 shows that $\pi_0^G(R^\star)$ is a cokernel, in the category of commutative monoids, of the morphism

$$\pi_0^G(R) \ \xrightarrow{(\pi_0^G(*),\pi_0^G(\Delta))} \ \pi_0^G(*) \times \pi_0^G(R \times R) \ .$$

Since $\pi_0^G(*)$ has only one element and π_0^G commutes with products, $\pi_0^G(R^\star)$ is thus a cokernel, in the category of commutative monoids, of the morphism

$$\pi_0^G(R) \ \xrightarrow{\Delta} \ \pi_0^G(R) \times \pi_0^G(R) \ .$$

Any such cokernel, and hence the morphism $\pi_0^G(i)$, is a group completion of the abelian monoid $\pi_0^G(R)$. In particular, the monoid $\pi_0^G(R^\star)$ is a group, so $R^\star$ is group-like. □

Remark 4.9. We want to explain why Construction 4.4 deserves to be called a group completion. The analogy of the construction of $R^\star$ with the algebraic group completion (the pushout (4.3)) and the algebraic group completion property of the induced monoid map $\pi_0^G(i) : \pi_0^G(R) \longrightarrow \pi_0^G(R^\star)$ are important first evidence for this. In addition, and more importantly, the morphism $i : R \longrightarrow R^\star$ is initial, in a homotopical sense, among all morphisms from R to group-like commutative orthogonal monoid spaces. We sketch the argument for this; we do not give complete details because the property will not be used elsewhere in this book.

The homotopy-universal property we claim for the morphism $i : R \longrightarrow R^\star$ is that for every group-like commutative orthogonal monoid space S the induced continuous map

$$R\,\mathrm{map}(i,S) \ : \ R\,\mathrm{map}(R^\star,S) \ \longrightarrow \ R\,\mathrm{map}(R,S)$$

of derived mapping spaces is a weak equivalence. Here $R\,\mathrm{map}(-,-)$ means that source, respectively target, are replaced by a globally equivalent commutative orthogonal monoid space that is cofibrant, respectively fibrant, in the global model structure of Theorem 3.4, and then the actual space of morphism of commutative orthogonal monoid spaces is taken.

Applying $R\,\mathrm{map}(-,S)$ to the defining homotopy pushout square for $R^\star$ results in a homotopy pullback square of spaces

$$\begin{tikzcd} R\,\mathrm{map}(R^\star,S) \arrow[r] \arrow[d] & R\,\mathrm{map}(R\times R,S) \arrow[d] \\ * \arrow[r] & R\,\mathrm{map}(R,S) \end{tikzcd}$$

If R is flat (which is always the case after cofibrant replacement), then the morphism $\mu_{R,R} : R \boxtimes R \longrightarrow R \times R$ is a global equivalence of orthogonal monoid spaces, and $R \boxtimes R$ is a coproduct in the category of commutative orthogonal monoid spaces. So the derived mapping space $R\,\mathrm{map}(R\times R,S)$ is weakly equivalent to $R\,\mathrm{map}(R \boxtimes R,S)$ and hence to the product of two copies of $R\,\mathrm{map}(R,S)$.

The upshot is a homotopy pullback square of spaces

$$\begin{tikzcd} R\,\mathrm{map}(R^\star, S) \arrow[r] \arrow[d] & R\,\mathrm{map}(R,S)^2 \arrow[d, "+"] \\ * \arrow[r] & R\,\mathrm{map}(R,S) \end{tikzcd}$$

The derived mapping space $R\,\mathrm{map}(R,S)$ is an H-space under 'pointwise addition' of morphisms, denoted '+' above. Since S is group-like, $R\,\mathrm{map}(R,S)$ is a group-like H-space, and the homotopy fiber of the right vertical map above is weakly equivalent to $R\,\mathrm{map}(R,S)$. So $R\,\mathrm{map}(R^\star,S)$ maps to $R\,\mathrm{map}(R,S)$ by a weak equivalence, and this is in fact the map induced by the universal morphism $i : R \longrightarrow R^\star$. This shows that i has the homotopy-universal property of a group-completion.

Now we discuss an example of a global group completion that comes up naturally and will also be relevant in Section V.8 when we explain why the periodic global bordism spectrum $\mathbf{MP}$ is the universal complex oriented global ring spectrum. We showed in Theorem 2.8 that the commutative orthogonal monoid space $\mathbf{BOP}$ is group-like and that its equivariant homotopy sets $\underline{\pi}_0(\mathbf{BOP})$ realize the real representation rings additively. In Example 2.9 we introduced a morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ of commutative orthogonal monoid spaces from the additive Grassmannian and showed that $\pi_0^G(i) : \pi_0^G(\mathbf{Gr}) \longrightarrow \pi_0^G(\mathbf{BOP})$ is an algebraic group completion for every compact Lie group G. We will now refine this and show that the morphism $i : \mathbf{Gr} \longrightarrow \mathbf{BOP}$ is in fact a global group completion of commutative orthogonal monoid spaces. For this purpose we will show that a certain commutative square (4.11) of commutative orthogonal monoid spaces, with $\mathbf{BOP}$ as terminal object, is a homotopy pushout. In particular we need a morphism similar to i that realizes the *negatives* of all representations. We let $-\mathrm{Id} : \mathbf{BOP} \longrightarrow \mathbf{BOP}$ be the involution given by

$$(-\mathrm{Id})(V) \ : \ \mathbf{BOP}(V) \ = \ \coprod_{m\geq 0} Gr_m(V^2) \ \longrightarrow \ \coprod_{m\geq 0} Gr_m(V^2) = \mathbf{BOP}(V) \ , \quad L \ \longmapsto L^\perp \ ,$$

where $L^\perp = V^2 - L$ is the orthogonal complement of the subspace L in V^2. The morphism $-i$ is multiplicative and unital, and it negates the grading in that it takes $\mathbf{BOP}^{[n]}$ to $\mathbf{BOP}^{[-n]}$. If G is a compact Lie group, V a G-representation and $L \in \mathbf{BOP}(V)^G$ a G-subrepresentation of V^2, then the relation

$$[L] + [L^\perp] \ = \ [V^2] \ = \ [V] + [V]$$

holds in $\mathbf{RO}(G)$; this is equivalent to

$$[L] - [V] \ = \ -([L^\perp] - [V]) \ ,$$

and so $\underline{\pi}_0(-\mathrm{Id}) : \underline{\pi}_0(\mathbf{BOP}) \longrightarrow \underline{\pi}_0(\mathbf{BOP})$ is the inverse map. We define

$$-i \ = \ (-\mathrm{Id}) \circ i \ : \ \mathbf{Gr} \ \longrightarrow \ \mathbf{BOP}$$

and think of this as the negative of the morphism i. This morphism of orthogonal monoid spaces is explicitly given by

$$(-i)(V) \ : \ \mathbf{Gr}(V) = \coprod_{n\geq 0} Gr_n(V) \ \longrightarrow \ \coprod_{m\geq 0} Gr_m(V^2) = \mathbf{BOP}(V) \ , \quad L \ \longmapsto L^\perp \oplus 0 \ ,$$

where now $L^\perp = V - L$ is the orthogonal complement of L in V.

The composite

$$\overline{\mathbf{Gr}} \ \xrightarrow{\ D\ } \ \mathbf{Gr} \boxtimes \mathbf{Gr} \ \xrightarrow{\ \rho_{\mathbf{Gr},\mathbf{Gr}}\ } \ \mathbf{Gr} \times \mathbf{Gr}$$

of the morphism D with the global equivalence $\rho_{\mathbf{Gr},\mathbf{Gr}}$ is given by [...]. So D models the diagonal of $\mathbf{Gr}$. The composite morphism

$$\overline{\mathbf{Gr}} \ \xrightarrow{\ D\ } \ \mathbf{Gr} \boxtimes \mathbf{Gr} \ \xrightarrow{\ i\boxtimes(-i)\ } \ \mathbf{BOP} \boxtimes \mathbf{BOP} \ \xrightarrow{\ \mu_{\mathbf{BOP}}\ } \ \mathbf{BOP}$$

is constant with values the unit elements of $\mathbf{BOP}$. We write $i \vee (-i) = \mu \circ (i \boxtimes (-i)) : \mathbf{Gr} \boxtimes \mathbf{Gr} \longrightarrow \mathbf{BOP}$.

Theorem 4.10. *The commutative square*

$$\begin{tikzcd} \overline{\mathbf{Gr}} \arrow[r, "D"] \arrow[d] & \mathbf{Gr} \boxtimes \mathbf{Gr} \arrow[d, "i \vee (-i)"] \\ * \arrow[r] & \mathbf{BOP} \end{tikzcd} \tag{4.11}$$

is a homotopy pushout of commutative orthogonal monoid spaces, i.e., $\mathbf{BOP}$ *is a group completion of the additive Grassmannian* $\mathbf{Gr}$*.*

The unitary analog of the last theorem holds as well, with the same proof, mutatis mutandis: the commutative square

$$\begin{tikzcd} \overline{\mathbf{Gr}}^{\mathbb{C}} \arrow[r, "D"] \arrow[d] & \mathbf{Gr}^{\mathbb{C}} \boxtimes \mathbf{Gr}^{\mathbb{C}} \arrow[d, "i \vee (-i)"] \\ * \arrow[r] & \mathbf{BUP} \end{tikzcd} \tag{4.12}$$

is a homotopy pushout of commutative unitary monoid spaces, and $\mathbf{BUP}$ is a group completion of $\mathbf{Gr}^{\mathbb{C}}$.

Construction 4.13 (Global Bott periodicity)**.** The Bott peridicity map is traditionally seen as a homotopy equivalence between the space $\mathbb{Z} \times BU$ and the loop space of the infinite unitary group. We derived a highly structured, global form of Bott periodicity, namely that the commutative unitary monoid spaces $\mathbf{BUP}$ and $\Omega\mathbf{U}$ are multiplicatively globally equivalent.

We start by defining a morphism of commutative unitary monoid spaces

$$\beta \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \Omega\mathbf{U} \ .$$

Here $\mathbf{U}$ is the unitary monoid space of unitary groups (compare Example 2.22), and Ω means that we take loop spaces of the unitary space $\mathbf{U}$ objectwise, based at the identity. The morphism β is a modification of Behrens' coordinate free description [**8**] of the non-equivariant Bott map, which is based on ideas of MacDuff [**61**] and Aguilar and Prieto [**2**].

If W is a complex inner product space, then the Lie algebra of the unitary group $U(W)$ can be identified with the real vector space $\mathbf{sa}(W)$ of self-adjoint $\mathbb{C}$-linear endomorphisms of W. Under this identification, the exponential map

$$\exp \ : \ \mathbf{sa}(W) \ \longrightarrow \ U(W)$$

is given by the familiar formula

$$\exp(A) \ = \ e^{2\pi i A} \ = \ \sum_{n\geq 0} \frac{(2\pi i)^n}{n!} A^n \ .$$

Indeed, a self-adjoint endomorphism is diagonalizable with real eigenvalues, so the sum converges pointwise to a diagonalizable endomorphism with the same eigenspaces, but with eigenvalue λ replaced by $\exp(\lambda)$. The exponential map thus takes trace to determinant, i.e.,

$$\det(\exp(A)) \ = \ \exp(\mathrm{trace}(A))$$

holds for all self-adjoint A.

Now we let $\varphi : V \longrightarrow W$ be a linear isometric embedding of complex inner product spaces. We define an $\mathbb{R}$-linear map

$$\varphi_* \ : \ \mathbf{sa}(V) \ \longrightarrow \ \mathbf{sa}(W) \tag{4.14}$$

by sending a self-adjoint endomorphism A of V to the composite the morphism

$$W \;=\; \varphi(V)\oplus\varphi(V)^{\perp} \xrightarrow{\;^{\varphi}A\oplus 0\;} \varphi(V)\oplus\varphi(V)^{\perp} \;=\; W\ .$$

The map $\varphi_*(A)$ is a self-adjoint endomorphism of W and satisfies

$$\exp(\varphi_*(A)) \;=\; \exp(A)\oplus \mathrm{Id}_{\varphi(V)^{\perp}} \;=\; \mathbf{U}(\varphi)(\exp(A))$$

in $U(W)$. In other words, the following square commutes:

$$\begin{tikzcd}
\mathbf{sa}(V) \arrow[r, "\varphi_*"] \arrow[d, "\exp"'] & \mathbf{sa}(W) \arrow[d, "\exp"] \\
\mathbf{U}(V) \arrow[r, "\mathbf{U}(\varphi)"'] & \mathbf{U}(W)
\end{tikzcd}$$

We can now define the map $\beta(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow (\Omega\mathbf{U})(V)$. An element $L \in \mathbf{Gr}^{\mathbb{C}}(V)$ is a complex subspace L of V; the orthogonal projection p_L to the subspace L is then a self-adjoint endomorphism of $V_{\mathbb{C}}$. We define the path

$$\beta(V)(L) \;:\; [0,1] \;\longrightarrow\; U(V)$$

of unitary maps as

$$\beta(V)(L)(t) \;=\; \exp(t\cdot p_L) \;=\; (\exp(t)\cdot \mathrm{Id}_L)\oplus \mathrm{Id}_{L^{\perp}}\ .$$

Since p_L has eigenvalues in $\{0,1\}$, its exponential is diagonalizable with only eigenvalue 1; so $\exp(p_L) = \mathrm{Id}_{V_{\mathbb{C}}}$ and $\beta(V)(L)$ this is really a loop in $U(V_{\mathbb{C}})$ based at the identity. The map $\beta(V)$ is continuous in L and $O(V)$-equivariant because $p_{\varphi_{\mathbb{C}}(L)} = {}^{\varphi}(p_L)$ as endomorphisms of $V_{\mathbb{C}}$.

The maps $\beta(V)$ together form a morphism of unitary spaces. Indeed, if $\varphi : V \longrightarrow W$ is a linear isometric embedding and L a complex subspace of $V_{\mathbb{C}}$, then

$$\varphi_*(p_L) \;=\; p_{\varphi(L)}\ ,$$

and hence

$$\begin{aligned}
\mathbf{U}(\varphi)\big(\beta(V)(L)(t)\big) \;&=\; \mathbf{U}(\varphi)\big(\exp(t\cdot p_L)\big)\\
&=\; \exp(\varphi_*(t\cdot p_L)) \;=\; \exp(t\cdot p_{\varphi(L)})\\
&=\; \beta(W)(\varphi(L))(t) \;=\; \beta(W)(\mathbf{Gr}(\varphi)(L))(t)\ .
\end{aligned}$$

In other words, the square

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}}(V) \arrow[r, "\beta(V)"] \arrow[d, "\mathbf{Gr}(\varphi)"'] & \Omega\mathbf{U}(V) \arrow[d, "\Omega\mathbf{U}(\varphi)"] \\
\mathbf{Gr}^{\mathbb{C}}(W) \arrow[r, "\beta(W)"'] & \Omega\mathbf{U}(W)
\end{tikzcd}$$

commutes.

The unitary space $\mathbf{U}$ has a commutative multiplication by direct sum of unitary automorphisms. So the unitary space $\Omega\mathbf{U}$ inherits a commutative multiplication by pointwise multiplication of loops. For every inner product space V we have

$$\beta(V)(0)(t) \;=\; \exp(t\cdot p_0) \;=\; \exp(0) \;=\; \mathrm{Id}_V\ ;$$

so $\beta(V)(0)$ is the constant loop at the identity, which is the unit element of $(\Omega\mathbf{U})(V)$. Now we consider subspaces $L \in \mathbf{Gr}^{\mathbb{C}}(V)$ and $L' \in \mathbf{Gr}^{\mathbb{C}}(W)$.

$$\begin{aligned}
\beta(V\oplus W)(L\oplus L')(t) \;&=\; \exp(t\cdot(p_{L\oplus L'})) \;=\; \exp((t\cdot p_{L\oplus 0})\oplus(t\cdot p_{0\oplus L'}))\\
&=\; \exp(t\cdot p_L)\oplus\exp(t\cdot p_{L'}) \;=\; \beta(V)(L)(t)\oplus\beta(W)(L')(t)\ .
\end{aligned}$$

In other words, the square

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}}(V) \times \mathbf{Gr}^{\mathbb{C}}(W) \arrow[r, "\beta(V)\times\beta(W)"] \arrow[d, "\oplus"'] & (\Omega\mathbf{U}(V)) \times (\Omega\mathbf{U}(W)) \arrow[d, "\mu_{(\Omega\mathbf{U})_{V,W}}"] \\
\mathbf{Gr}^{\mathbb{C}}(V \oplus W) \arrow[r, "\beta(V\oplus W)"'] & \Omega\mathbf{U}(V \oplus W)
\end{tikzcd}$$

commutes, i.e., β is compatible with the commutative multiplications on both side. Since β respects multiplication and unit, it also respects the structure maps. The upshot is that β is a morphism of unitary monoid spaces.

We have

$$\det(\beta(V)(L)(t)) \ = \ \det(\exp(t \cdot p_L)) \ = \ \exp(\mathrm{trace}(t \cdot p_L)) \ = \ \exp(t \cdot \dim(L)) \ .$$

So the composite

$$\mathbf{Gr}^{\mathbb{C}}(V) \ \xrightarrow{\beta(V)} \ \Omega\mathbf{U}(V) \ \xrightarrow{\Omega \det} \ \Omega U(1)$$

sends the subspace $\mathbf{Gr}^{[n]}(V)$ to the path component of $\Omega U(1)$ consisting of loops of degree n. So the morphism β restricts to a morphism of unitary spaces

$$\beta^{[n]} \ : \ \mathbf{Gr}^{\mathbb{C},[n]} \ \longrightarrow \ \Omega^{[n]}\mathbf{U} \ .$$

Equivariant Bott periodicity for compact Lie groups can then be conveniently summarized by saying that the morphism $\beta : \mathbf{BUP} \longrightarrow \Omega\mathbf{U}$ is a global equivalence of unitary spaces.

Since the morphism $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ defined in (2.28) is also a group completion, $\Omega\mathbf{U}$ and $\mathbf{BUP}$ are globally equivalent as commutative unitary monoid spaces.

Remark 4.15. As V varies through all complex inner product spaces, the spaces $\mathbf{sa}(V)$ of self-adjoint operators form a commutative unitary monoid space $\mathbf{sa}$ with structure maps the linear maps (4.14), multiplication by direct sum of endomorphism and the zero endomorphism as unit. Since $\mathbf{sa}(V)$ is a real vector space, hence contractible, this unitary monoid space is not very interesting by itself. However, the exponential maps form a morphism $\exp : \mathbf{sa} \longrightarrow \mathbf{U}$ of commutative unitary monoid spaces and the maps

$$p(V) \ : \ \mathbf{Gr}^{\mathbb{C}}(V) = \coprod_{n\geq 0} Gr_n(V) \ \longrightarrow \ \mathbf{sa}(V) \ , \ L \ \longmapsto \ p_L$$

given by unitary projection operators embed the additive Grassmannian as a unitary monoid subspace of $\mathbf{sa}$; the image of this morphism is given by those self-adjoint operators whose only eigenvalues are 0 or 1. The square of morphisms of unitary monoid spaces

$$\begin{tikzcd}
\mathbf{Gr}^{\mathbb{C}} \arrow[r, "p"] \arrow[d] & \mathbf{sa} \arrow[d, "\exp"] \\
* \arrow[r] & \mathbf{U}
\end{tikzcd}$$

commutes, and global Bott periodicity can be interpreted as saying that this square is a homotopy pushout in the category of commutative unitary monoid spaces. Since $\mathbf{sa}$ is globally weakly contractible, yet another way to say this is that $\mathbf{U}$ is a suspension, in the category of commutative unitary monoid spaces, of the additive Grassmannian.

CHAPTER III

Stable global homotopy theory

1. Orthogonal spectra

Now we introduce the main objects for the purposes of this book, the *orthogonal spectra*, stable versions of orthogonal spaces. We use the 'coordinatized' version of orthogonal spectra for the definition, and relate it to the 'coordinate free' version in Proposition 5.3.

Definition 1.1. An *orthogonal spectrum* consists of the following data:

- a sequence of pointed spaces X_n for $n \geq 0$,
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on X_n for each $n \geq 0$,
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m}$$

defined as the composition

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge S^{m-1}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge S^{m-2}} \quad \cdots \quad X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m} \tag{1.2}$$

is $O(n) \times O(m)$-equivariant. Here the orthogonal group $O(m)$ acts on S^m since this is the one-point compactification of $\mathbb{R}^m$, and $O(n) \times O(m)$ acts on the target by restriction, along orthogonal sum, of the $O(n + m)$-action. We refer to the space X_n as the *n-th level* of the orthogonal spectrum X.

A *morphism* $f : X \longrightarrow Y$ of orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$ for all $n \geq 0$. We denote the category of orthogonal spectra by $\mathcal{S}p$.

Orthogonal spectra are used, at least implicitly, in [**59**]; commutative orthogonal ring spectra are called '$\mathscr{I}_*$-prefunctors' in [**59**, IV Def. 2.1]; the term 'orthogonal spectrum' was introduced by Mandell, May, Shipley and the author in [**57**], where the (non-equivariant, projective) stable model structure for orthogonal spectra was constructed.

The actions of the orthogonal groups on the terms of an orthogonal spectrum encode enough information so that we can evaluate an orthogonal spectrum on any inner product space. For a finite dimensional $\mathbb{R}$-vector space V we denote by S^V the one-point compactification, with basepoint at infinity. If V is endowed with a scalar product, we denote by $D(V)$ the unit ball and by $S(V)$ the unit sphere of V. We write S^n for $S^{\mathbb{R}^n}$, the one-point compactification of $\mathbb{R}^n$.

For every orthogonal spectrum X and inner product space V of dimension n we define $X(V)$, the *value of* X *on* V, as

$$X(V) \ = \ \mathbf{L}(\mathbb{R}^n, V)^+ \wedge_{O(n)} X_n \tag{1.3}$$

where $\mathbb{R}^n$ has the standard scalar product and $\mathbf{L}(\mathbb{R}^n, V)$ is the space of linear isometries from $\mathbb{R}^n$ to V. The orthogonal group $O(n)$ acts simply transitively on $\mathbf{L}(\mathbb{R}^n, V)$ by precomposition, and $X(V)$ is the coequalizer of the two $O(n)$-actions on $\mathbf{L}(\mathbb{R}^n, V)^+ \wedge X_n$. If $V = \mathbb{R}^n$ then there is a canonical homeomorphism

$$X_n \ \longrightarrow \ X(\mathbb{R}^n) \ , \quad x \ \longmapsto \ [\mathrm{Id}, x] \ .$$

In general, any choice of isometry $\varphi : \mathbb{R}^n \longrightarrow V$ (which amounts to a choice of orthonormal basis of V) gives rise to a homeomorphism

$$[\varphi, -] \ : \ X_n \ \longrightarrow \ X(V) \ , \quad x \ \longmapsto \ [\varphi, x] \ .$$

The iterated structure maps $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$ of an orthogonal spectrum X extend to *generalized structure maps*

$$\sigma_{V,W} \ : \ X(V) \wedge S^W \ \longrightarrow \ X(V \oplus W) \ . \tag{1.4}$$

To define $\sigma_{V,W}$ we set $m = \dim(W)$ and choose an isometry $\gamma : \mathbb{R}^m \longrightarrow W$. Then

$$\sigma_{V,W}([\varphi, x] \wedge w) \ = \ [\varphi \oplus \gamma, \, \sigma^m(x \wedge \gamma^{-1}(w))] \quad \text{in} \quad \mathbf{L}(\mathbb{R}^{n+m}, V \oplus W)^+ \wedge_{O(n+m)} X_{n+m} \ = \ X(V \oplus W) \ .$$

We omit the verification that the map $\sigma_{V,W}$ is well defined and independent of the choice of γ. Finally, the generalized structure maps are also associative: If we are given a third inner product space U, then the square

$$\begin{tikzcd}
X(V) \wedge S^W \wedge S^U \arrow[r, "\sigma_{V,W} \wedge S^U"] \arrow[d, "X(V) \wedge \cong"'] & X(V \oplus W) \wedge S^U \arrow[d, "\sigma_{V \oplus W, U}"] \\
X(V) \wedge S^{W+U} \arrow[r, "\sigma_{V, W \oplus U}"'] & X(V \oplus W \oplus U)
\end{tikzcd}$$

commutes.

We introduce a convenient piece of notation. For an orthogonal spectrum X and inner product spaces V and W and a based map $f : S^V \longrightarrow X(V)$ we denote by $f \diamond W : S^{V+W} \longrightarrow X(V \oplus W)$ the composite

$$S^{V+W} \ \cong \ S^V \wedge S^W \ \xrightarrow{f \wedge S^W} \ X(V) \wedge S^W \ \xrightarrow{\sigma_{V,W}} \ X(V \oplus W) \ . \tag{1.5}$$

We refer to $f \diamond W$ as the *stabilization of f by W*. The associativity property of the generalized structure maps implies the associativity property

$$(f \diamond W) \diamond U \ = \ f \diamond (W \oplus U) \ : \ S^{V+W+U} \ \longrightarrow \ X(V \oplus W \oplus U) \ .$$

We let X be an orthogonal spectrum. We consider a compact Lie group G and suppose that V is a G-representation (i.e., G acts on V by linear isometries). Then $X(V)$ becomes a G-space by the rule

$$g \cdot [\varphi, x] \ = \ [g\varphi, x] \ . \tag{1.6}$$

We want to stress that the underlying space of $X(V)$ depends, up to homeomorphism, only on the dimension of the representation V. However, the G-action on V influences the G-action on $X(V)$. If V and W are G-representations, then the generalized structure map (1.4) is G-equivariant where the group G also acts on the representation sphere S^W. The generalized structure map $\sigma_{V,W}$ is also $O(V) \times O(W)$-equivariant, so altogether it is equivariant for the semi-direct product group $G \ltimes (O(V) \times O(W))$ formed from the conjugation action of G on $O(V)$ and $O(W)$.

Remark 1.7. Given an orthogonal spectrum X and a compact Lie group G, the collection of G-spaces $X(V)$ (1.6), for V a G-representation, and the equivariant structure maps $\sigma_{V,W}$ form an *orthogonal G-spectrum* in the sense of [**58**] (indexed, for example, by the complete G-universe $\mathcal{U}_G$). Whenever we need to refer to this orthogonal G-spectrum, we use the notation

$$X\langle G \rangle \ = \ \{X(V), \sigma_{V,W}\}$$

and call it the *underlying orthogonal G-spectrum* of X.

We emphasize that only very special orthogonal G-spectra are part of a 'global family', i.e., arise in this way from an orthogonal spectrum. More precisely, for an orthogonal G-spectrum Y the following two conditions are equivalent:

(a) the G-spectrum Y is isomorphic to an orthogonal G-spectrum of the form $X\langle G\rangle$ for some orthogonal spectrum X;
(b) for every *trivial* G-representation V the G-action on $Y(V)$ is trivial.

Indeed, any orthogonal G-spectrum of the form $X\langle G\rangle$ has property (b) by the very definition (1.6) of the G-action on $X(V)$. Conversely, if Y has property (b), then we let UY be the underlying orthogonal spectrum of Y, defined by $(UY)_n = Y(\mathbb{R}^n)$, where G acts trivially on $\mathbb{R}^n$ (and hence on $Y(\mathbb{R}^n)$). If V is any G-representation of dimension n, the map

$$(UY)(V) = \mathbf{L}(\mathbb{R}^n, V)^+ \wedge_{O(n)} Y(\mathbb{R}^n) \longrightarrow Y(V) , \quad [\varphi, x] \longmapsto \varphi_*(x)$$

is then G-equivariant homeomorphism. As V varies, these maps form an isomorphism between Y and $(UY)\langle G\rangle$.

An example of an orthogonal G-spectrum that does not satisfy (a) and (b) above is the equivariant suspension spectrum of a based G-space with non-trivial G-action. In Remark 2.37 below we isolate some necessary conditions on the G-Mackey functor made from the equivariant homotopy groups of an orthogonal G-spectrum in order to be of the special form $X\langle G\rangle$.

2. Equivariant homotopy groups

As we explained in Remark 1.7, an orthogonal spectrum X has an underlying orthogonal G-spectrum $X\langle G\rangle$ for every compact Lie group G. As such, $X\langle G\rangle$ has equivariant stable homotopy groups $\pi_*^G X$ (indexed by the chosen complete G-universe $\mathcal{U}_G$), whose definition we now recall. As before we let $s(\mathcal{U}_G)$ denote the set of finite dimensional G-subrepresentations of the chosen complete G-universe $\mathcal{U}_G$, considered as a poset under inclusion. We obtain a functor from $s(\mathcal{U}_G)$ to sets by sending $V \in s(\mathcal{U}_G)$ to

$$[S^V, X(V)]^G ,$$

the set of G-equivariant homotopy classes of based G-maps from S^V to $X(V)$. For $V \subseteq W$ in $s(\mathcal{U}_G)$ the inclusion $i : V \longrightarrow W$ is sent to the map

$$[S^V, X(V)]^G \longrightarrow [S^W, X(W)]^G , \quad [f] \longmapsto [i_* f] .$$

The *0-th equivariant homotopy group* $\pi_0^G X$ is then defined as

$$\pi_0^G X = \operatorname{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G , \tag{2.1}$$

the colimit of this functor over the poset $s(\mathcal{U}_G)$. If k is an arbitrary integer, we will define the k-th equivariant homotopy group $\pi_k^G X$ in (2.8) below as the 0-th homotopy group of a suitably looped or suspended spectrum.

The groups $\pi_0^G X$ have a lot of extra structure as the group G varies. First we should recall the justification of the terminology 'homotopy *group*', i.e., explain the abelian group structure on $\pi_0^G X$. We consider a finite dimensional G-subrepresentation V of the universe $\mathcal{U}_G$ with non-zero fixed points. We choose a G-fixed unit vector $v_0 \in V$, and we let $V^\perp$ denote the orthogonal complement of v_0 in V. This induces a decomposition

$$\mathbb{R} \oplus V^\perp \cong V , \quad (t, v) \longmapsto tv_0 + v$$

that extends to a G-equivariant homeomorphism $S^1 \wedge S^{V^\perp} \cong S^V$ on one-point compactifications. From this we obtain a bijection

$$[S^V, X(V)]^G \cong [S^1, \operatorname{map}^G(S^{V^\perp}, X(V))]_* = \pi_1(\operatorname{map}^G(S^{V^\perp}, X(V))) , \tag{2.2}$$

natural in the orthogonal spectrum X. We use the bijection (2.2) to transfer the group structure on the fundamental group into a group structure on the set $[S^V, X(V)]^G$.

Now we suppose that the dimension of the fixed point space V^G is at least 2. Then the space of G-fixed unit vectors in V is connected and similar arguments as for the commutativity of higher homotopy groups show:

- the group structure on the set $[S^V, X(V)]^G$ defined by the bijection (2.2) is commutative and independent of the choice of G-fixed unit vector;
- if W is another finite dimensional G-subrepresentation of $\mathcal{U}$ containing V, then the map
$$i_* \ : \ [S^V, X(V)]^G \ \longrightarrow \ [S^W, X(W)]^G$$
is a group homomorphism.

The G-subrepresentations V of $\mathcal{U}_G$ with $\dim(V^G) \geq 2$ are cofinal in the poset $s(\mathcal{U}_G)$, so the two properties above show that the abelian group structures on $[S^V, X(V)]^G$ for $\dim(V^G) \geq 2$ assemble into a well-defined and natural abelian group structure on the colimit $\pi_0^G X$.

Next, the equivariant homotopy groups come with restriction maps. A quick way to define these, and to deduce some of their properties, is to reinterpret $\pi_0^G(X)$ as the G-equivariant homotopy set, as defined in (1.10), of a certain orthogonal space.

Construction 2.3. We define a functor
$$\Omega^\bullet \ : \ \mathcal{S}p \ \longrightarrow \ spc$$
from orthogonal spectra to orthogonal spaces. Given an orthogonal spectrum X, the value of $\Omega^\bullet X$ at an inner product space V is
$$(\Omega^\bullet X)(V) \ = \ \mathrm{map}(S^V, X(V)) \ .$$
If $\alpha : V \longrightarrow W$ is a linear isometric embedding, the induced map
$$\alpha_* \ : \ (\Omega^\bullet X)(V) \ = \ \mathrm{map}(S^V, X(V)) \ \longrightarrow \ \mathrm{map}(S^W, X(W)) \ = \ (\Omega^\bullet X)(V)$$
is by 'conjugation and extension by the identity'. In more detail: given a continuous based map $f : S^V \longrightarrow X(V)$ we define $\alpha_* f : S^W \longrightarrow X(W)$ as the composite

$$S^W \ \cong \ S^{V+(W-\alpha(V))} \ \xrightarrow{f \diamond (W-\alpha(V))} \ X(V \oplus (W - \alpha(V))) \ \cong \ X(W) \tag{2.4}$$

where each of the two unnamed homeomorphisms uses α to identify $V \oplus (W - \alpha(V))$ with W. We observe that if α is bijective (i.e., an equivariant isometry), then $\alpha_* f$ becomes the 'α-conjugate' of f, i.e., the composite
$$S^W \ \xrightarrow{\alpha^{-1}} \ S^V \ \xrightarrow{f} \ X(V) \ \xrightarrow{X(\alpha)} \ X(W) \ .$$
In particular, the orthogonal group $O(V)$ acts on $(\Omega^\bullet X)(V) = \mathrm{map}(S^V, X(V))$ by conjugation. This construction also generalizes the stabilization by a representation. Indeed, when $i : V \longrightarrow V \oplus W$ is the inclusion of the first summand, then $i_* f = f \diamond W$, the stabilization of f by W in the sense of (1.5).

The assignment $(\alpha, f) \mapsto \alpha_* f$ is continuous in both variables. If $\beta : W \longrightarrow U$ is another G-isometric embedding, then we have
$$\beta_*(\alpha_* f) \ = \ (\beta\alpha)_* f \ .$$
In other words, we have indeed defined an orthogonal space $\Omega^\bullet X$. The construction is clearly functorial in the orthogonal spectrum X; moreover, $\Omega^\bullet$ has a left adjoint 'unreduced suspension spectrum' functor Σ^∞_+ that we discuss in Construction 3.6 below.

If G acts on V by linear isometries, then the G-fixed subspace of $(\Omega^\bullet X)(V)$ is the space of G-equivariant based maps from S^V to $X(V)$:
$$((\Omega^\bullet X)(V))^G \ = \ \mathrm{map}^G(S^V, X(V)) \ .$$
The path components of this space are precisely the equivariant homotopy classes of based G-maps, i.e.,
$$\pi_0 \left(((\Omega^\bullet X)(V))^G \right) \ = \ \pi_0 \, \mathrm{map}^G(S^V, X(V)) \ = \ [S^V, X(V)]^G \ .$$
Passing to colimits over the poset $s(\mathcal{U}_G)$ gives
$$\pi_0^G(\Omega^\bullet X) \ = \ \pi_0^G X \ ,$$

i.e., the G-equivariant homotopy group of the orthogonal spectrum X equals the G-equivariant homotopy set (as previously defined in (1.10)) of the orthogonal space $\Omega^\bullet X$.

So by specializing the restriction maps for orthogonal spaces we obtain restriction maps

$$\alpha^* \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^K(X)$$

for every continuous group homomorphism $\alpha : K \longrightarrow G$. These restriction maps between the equivariant homotopy groups of an orthogonal spectrum are again contravariantly functorial and depend only on the conjugacy class of the homomorphism α (by Proposition 1.16). Moreover, α^* is additive, i.e., a group homomorphism.

Remark 2.5 (Global homotopy types are split G-spectra)**.** We emphasize that the equivariant homotopy groups $\pi_*^G(X)$ of an orthogonal spectrum X come equipped with restriction maps along arbitrary continuous group homomorphisms, not necessarily injective. This is in contrast to stable equivariant homotopy theory for a fixed compact Lie group, where one can only restrict to subgroups, or along conjugation maps by elements of the ambient group.

In particular, the unique group homomorphism $p : G \longrightarrow e$ to the trivial group induces a homomorphism $p^* : \pi_*^e(X) \longrightarrow \pi_*^G(X)$ that splits the restriction map $\mathrm{res}^G_e : \pi_*^G(X) \longrightarrow \pi_*^e(X)$ (induced by the inclusion of the trivial subgroup into G). This is the algebraic shadow of the fact that the G-equivariant homotopy types that underlie global homotopy types (i.e., are represented by orthogonal spectra) are 'split' in the sense that there is a morphism from the underlying non-equivariant spectrum to the genuine G-fixed point spectrum that splits the restriction map.

Now we recall some important properties of the equivariant homotopy groups of an orthogonal spectrum, such as the long exact sequences associated to mapping cones and homotopy fibers, and the stability under suspension and loop.

Construction 2.6. If A is pointed space and X an orthogonal spectrum, we can define two new orthogonal spectra $A\wedge X$ and X^A by smashing with A or taking maps from A levelwise; the structure maps and actions of the orthogonal groups do not interact with A. In more detail we set

$$(A \wedge X)_n \ = \ A \wedge X_n \qquad \text{respectively} \qquad (X^A)_n \ = \ X_n^A \ = \ \mathrm{map}(A, X_n)$$

for $n \geq 0$. The group $O(n)$ acts through its action on X_n. The structure map is given by the composite

$$(A \wedge X)_n \wedge S^1 \ = \ A \wedge X_n \wedge S^1 \ \xrightarrow{\mathrm{Id}\wedge \sigma_n} \ A \wedge X_{n+1} \ = \ (A \wedge X)_{n+1}$$

respectively by the composite

$$X_n^A \wedge S^1 \ \longrightarrow \ (X_n \wedge S^1)^A \ \xrightarrow{\sigma_n^A} \ X_{n+1}^A$$

where the first is an assembly map that sends $\varphi \wedge t \in X_n^A \wedge S^1$ to the map sending $a \in A$ to $\varphi(a) \wedge t$. The second is application of $\mathrm{map}(A, -)$ to the structure map of X. For example, the spectrum $A \wedge \mathbb{S}$ is equal to the suspension spectrum $\Sigma^\infty A$. For the values on an inner product space V we have

$$(A \wedge X)(V) \ \cong \ A \wedge X(V) \qquad \text{respectively} \qquad (X^A)(V) \ \cong \ X(V)^A \ .$$

Just as the functors $A \wedge -$ and $\mathrm{map}(A, -)$ are adjoint on the level of based spaces, the two functors just introduced are an adjoint pair on the level of orthogonal spectra. The adjunction

$$\text{(2.7)} \qquad \hat{} \ : \ \mathcal{S}p(X, Y^A) \ \xrightarrow{\cong} \ \mathcal{S}p(A \wedge X, Y)$$

takes a morphism $f : X \longrightarrow Y^A$ to the morphism $\hat{f} : A \wedge X \longrightarrow Y$ whose n-th level $\hat{f}_n : A \wedge X_n \longrightarrow Y_n$ is given by $\hat{f}_n(a \wedge x) = f_n(x)(a)$.

An important special case of the this construction is when $A = S^1$ is a 1-sphere, where the constructions specialize to the *loop spectrum* $\Omega X = X^{S^1}$ is defined by

$$(\Omega X)_n \ = \ \Omega(X_n) \ = \ \mathrm{map}(S^1, X_n) \ ,$$

the based mapping space from S^1 to the n-th level of X. The *suspension* $S^1 \wedge X$ is defined by

$$(S^1 \wedge X)_n \ = \ S^1 \wedge X_n \ ,$$

the box product of the sphere S^1 with the n-th level of X. We obtain an adjunction between $S^1 \wedge -$ and Ω as the special case $A = S^1$ of (2.7).

We can now define the *integer graded equivariant homotopy groups* $\pi_k^G X$ of an orthogonal spectrum X. If k is a positive integer, then we set

$$\pi_k^G(X) \ = \ \pi_0^G(\Omega^k X) \qquad \text{and} \qquad \pi_{-k}^G(X) \ = \ \pi_0^G(S^k \wedge X) \ . \tag{2.8}$$

Definition 2.9. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all compact Lie groups G and all integers k.

The global equivalences are the weak equivalences of the *global model structure* on the category of orthogonal spectra, see Theorem IV.2.7. So the methods of homotopical algebra are available for studying global equivalences and the associated global homotopy category. In later sections we will also consider a relative notion of global equivalence, the '$\mathcal{F}$-equivalence', based on a *global family* $\mathcal{F}$ of compact Lie groups. There we require that the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all compact Lie groups G that belong to the global family $\mathcal{F}$.

We let

$$\eta \ : \ X \ \longrightarrow \ \Omega(S^1 \wedge X) \qquad \text{and} \qquad \epsilon \ : \ S^1 \wedge \Omega X \ \longrightarrow \ X$$

denote the unit respectively counit of the adjunction (2.7). Then both η and ϵ induce isomorphisms of 0-th homotopy groups

$$\eta_* \ : \ \pi_0^G(X) \ \longrightarrow \ \pi_0^G(\Omega(S^1 \wedge X)) \qquad \text{respectively} \qquad \epsilon_* \ : \ \pi_0^G(S^1 \wedge \Omega X) \ \longrightarrow \ \pi_0^G(X)$$

for every orthogonal spectrum X and every compact Lie group G [ref] In other words, for every orthogonal spectrum X the unit and counit of the adjunction $(S^1 \wedge -, \Omega)$ are global equivalences.

Next we define the *suspension isomorphism*

$$S^1 \wedge - \ : \ \pi_k^G(X) \ \longrightarrow \ \pi_{1+k}^G(S^1 \wedge X) \ . \tag{2.10}$$

For $k \geq 0$ the suspension isomorphism is the map

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \ \longrightarrow \ \pi_0^G(\Omega^{1+k}(S^1 \wedge X)) = \pi_{1+k}^G(S^1 \wedge X)$$

induced on π_0^G by the composite

$$\Omega^k X \ \xrightarrow{\eta} \ \Omega(S^1 \wedge \Omega^k X) \ \xrightarrow{\text{assembly}} \ \Omega^{1+k}(S^1 \wedge X) \ .$$

For $k \leq -1$ the suspension isomorphism is the map

$$\pi_k^G(X) = \pi_0^G(S^{-k} \wedge X) \ \longrightarrow \ \pi_0^G(S^{-(1+k)} \wedge (S^1 \wedge X)) = \pi_{1+k}^G(S^1 \wedge X)$$

induced on π_0^G by the isomorphism [which?]

$$S^{-k} \wedge X \ \longrightarrow \ S^{-(1+k)} \wedge (S^1 \wedge X) \ .$$

We define the *loop isomorphism*

$$\alpha \ : \ \pi_k^G(\Omega X) \ \longrightarrow \ \pi_{1+k}^G(X) \tag{2.11}$$

as the composite

$$\pi_k^G(\Omega X) \ \xrightarrow{S^1 \wedge -} \ \pi_{1+k}^G(S^1 \wedge \Omega X) \ \xrightarrow{\epsilon_*} \ \pi_{1+k}^G(X) \ .$$

Construction 2.12 (Mapping cone and homotopy fiber)**.** The *(reduced) mapping cone* $C(f)$ of a morphism of based spaces $f : A \longrightarrow B$ is defined by

$$C(f) \;=\; ([0,1] \wedge A) \cup_f B \;. \tag{2.13}$$

Here the unit interval $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge A$ is the reduced cone of A. The mapping cone comes with an inclusion $i : B \longrightarrow C(f)$ and a projection $p : C(f) \longrightarrow S^1 \wedge A$; the projection sends B to the basepoint and is given on $[0,1] \wedge A$ by $p(x,a) = \mathbf{t}(x) \wedge a$ where $\mathbf{t} : [0,1] \longrightarrow S^1$ is given by $\mathbf{t}(x) = \frac{2x-1}{x(1-x)}$. What is relevant about the map $\mathbf{t}$ is not the precise formula, but that it passes to a homeomorphism between the quotient space $[0,1]/\{0,1\}$ and the circle S^1, and that it satisfies $\mathbf{t}(1-x) = -\mathbf{t}(x)$.

The *mapping cone* $C(f)$ of a morphism of orthogonal spectra $f : X \longrightarrow Y$ is now defined levelwise by

$$(Cf)_n \;=\; C(f_n) \;=\; ([0,1] \wedge X_n) \cup_f Y_n \;, \tag{2.14}$$

the reduced mapping cone of $f_n : X_n \longrightarrow Y_n$. The orthogonal group $O(n)$ acts on $C(f)_n$ through the given action on X_n and Y_n and trivially on the interval. The inclusions $i_n : Y_n \longrightarrow C(f)_n$ and projections $p_n : C(f)_n \longrightarrow S^1 \wedge X_n$ assemble into morphisms of orthogonal spectra $i : Y \longrightarrow Cf$ and $p : Cf \longrightarrow S^1 \wedge X$.

We define a *connecting homomorphism* $\delta : \pi^G_{1+k}(Cf) \longrightarrow \pi^G_k(X)$, where G is any compact Lie group, as the composite

$$\pi^G_{1+k}(Cf) \xrightarrow{\pi^G_{1+k}(p)} \pi^G_{1+k}(S^1 \wedge X) \xrightarrow{S^{-1}\wedge -} \pi^G_k(X) \;, \tag{2.15}$$

where the first map is the effect of the projection $p : Cf \longrightarrow S^1 \wedge X$ on homotopy groups, and the second map is the inverse of the suspension isomorphism $S^1 \wedge - : \pi^G_k(X) \longrightarrow \pi^G_{1+k}(S^1 \wedge X)$, compare (2.10).

The *homotopy fiber* is the construction 'dual' to the mapping cone. The homotopy fiber of a morphism $f : A \longrightarrow B$ of based spaces is the fiber product

$$F(f) \;=\; * \times_B B^{[0,1]} \times_B A \;=\; \{(\lambda, a) \in B^{[0,1]} \times A \mid \lambda(0) = *,\ \lambda(1) = f(a)\} \;,$$

i.e., the space of paths in B starting at the basepoint and equipped with a lift of the endpoint to A. As basepoint of the homotopy fiber we take the pair consisting of the constant path at the basepoint of B and the basepoint of A. The homotopy fiber comes with maps

$$\Omega B \xrightarrow{\;i\;} F(f) \xrightarrow{\;p\;} A \;;$$

the map p is the projection to the second factor and the value of the map i on a based loop $\omega : S^1 \longrightarrow B$ is

$$i(\omega) = (\omega \circ \mathbf{t}, *) \;.$$

The homotopy fiber $F(f)$ of the morphism $f : X \longrightarrow Y$ of orthogonal spectra is the orthogonal spectrum defined by

$$F(f)_n \;=\; F(f_n) \;, \tag{2.16}$$

the homotopy fiber of $f_n : X_n \longrightarrow Y_n$. The group $O(n)$ acts on $F(f)_n$ through the given action on X_n and Y_n and trivially on the interval. Put another way, the homotopy fiber is the pullback in the cartesian square of orthogonal spectra:

$$\begin{tikzcd}
F(f) \arrow[r, "p"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\
Y^{[0,1]} \arrow[r, "{\omega \mapsto (\omega(0),\omega(1))}"'] & Y \times Y
\end{tikzcd}$$

The inclusions $i_n : \Omega Y_n \longrightarrow F(f)_n$ and projections $p_n : F(f)_n \longrightarrow X_n$ assemble into morphisms of orthogonal spectra $i : \Omega Y \longrightarrow F(f)$ and $p : F(f) \longrightarrow X$. We define a *connecting homomorphism* δ :

$\pi^G_{1+k}(Y) \longrightarrow \pi^G_k(F(f))$, for a compact Lie group G, as the composite

$$\pi^G_{1+k}(Y) \xrightarrow{\epsilon_*^{-1}} \pi^G_{1+k}(S^1 \wedge \Omega Y) \xrightarrow{S^{-1}\wedge -} \pi^G_k(\Omega Y) \xrightarrow{\pi^G_k(i)} \pi^G_k(F(f)) , \tag{2.17}$$

where the second map is the inverse of the suspension isomorphism (2.10).

Mapping cone and homotopy fiber commute with evaluation at a representation: for every compact Lie group G and every G-representation V, the G-spaces $C(f)(V)$ respectively $F(f)(V)$ are naturally G-homeomorphic to the mapping cone respectively homotopy fiber of the G-map $f(V) : X(V) \longrightarrow Y(V)$.

Proposition 2.18. *For every morphism $f : X \longrightarrow Y$ of orthogonal spectra and every compact Lie group G the long sequences of abelian groups*

$$\cdots \longrightarrow \pi^G_k(X) \xrightarrow{\pi^G_k(f)} \pi^G_k(Y) \xrightarrow{\pi^G_k(i)} \pi^G_k(Cf) \xrightarrow{\delta} \pi^G_{k-1}X \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi^G_k(X) \xrightarrow{\pi^G_k(f)} \pi^G_k(Y) \xrightarrow{\delta} \pi^G_{k-1}(F(f)) \xrightarrow{\pi^G_{k-1}(p)} \pi^G_{k-1}(X) \longrightarrow \cdots$$

are exact.

Corollary 2.19. *Let $f : A \longrightarrow B$ be an h-cofibration of orthogonal spectra. Then for every compact Lie group G the long sequence of equivariant homotopy groups*

$$\cdots \longrightarrow \pi^G_k(A) \xrightarrow{\pi^G_k(f)} \pi^G_k(B) \xrightarrow{\pi^G_k(q)} \pi^G_k(B/A) \xrightarrow{\delta} \pi^G_{k-1}(A) \longrightarrow \cdots$$

is exact.

Corollary 2.20. *Let G be a compact Lie group and let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "k"'] & D \end{tikzcd}$$

be a pushout square of orthogonal spectra such that the induced map $\pi^G_* f : \pi^G_* A \longrightarrow \pi^H_* B$ of G-equivariant homotopy groups is an isomorphism. If in addition f or g is an h-cofibration, then the induced map $\pi^G_* k : \pi^G_* C \longrightarrow \pi^G_* D$ is also an isomorphism of G-equivariant homotopy groups.

Proof. If f is an h-cofibration, then its long exact homotopy group sequence (Corollary 2.19) shows that all G-equivariant homotopy groups of the cokernel B/A are trivial. Since the square is a pushout, the induced morphism from B/A to any cokernel D/C of k is an isomorphism, so the groups $\pi^G_*(D/C)$ are all trivial. As a cobase change of the h-cofibration f, the morphism k is again an h-cofibration, so its long exact homotopy group sequence shows that $\pi^G_* k$ is an isomorphism.

If g is an h-cofibration, then so is its cobase change h. Moreover, any cokernel C/A of g maps by an isomorphism to any cokernel D/B of h, since the square is a pushout. The square induces compatible maps between the two long exact homotopy group sequences of g and h, and the five lemma then shows that $\pi^G_* k$ is an isomorphism. □

Corollary 2.21. *Let G be a compact Lie group and k an integer.*

(i) *For every family of orthogonal spectra $\{X^i\}_{i\in I}$ the canonical map*

$$\bigoplus_{i\in I} \pi^G_k(X^i) \longrightarrow \pi^G_k\left(\bigvee_{i\in I} X^i\right)$$

is an isomorphism.

(ii) *For every finite indexing set I and every family $\{X^i\}_{i\in I}$ of orthogonal spectra the canonical map*
$$\pi_k^G\left(\prod_{i\in I} X^i\right) \longrightarrow \prod_{i\in I} \pi_k^G(X^i)$$
is an isomorphism.

(iii) *For every finite family of orthogonal spectra the canonical morphism from the wedge to the product is a global equivalence.*

We warn the reader that the equivariant homotopy group functor π_0^G does not in general commute with infinite products. The issue is that π_0^G involves a filtered colimit, and these not not always commute with infinite products. However, this defect is cured when we pass to the global stable homotopy category, i.e., π_0^G takes 'derived' infinite products to products. We refer the reader to Remark IV.4.3 below for more details.

Corollary 2.22. *The class of h-cofibrations that are simultaneously global equivalences is closed under cobase change, coproducts and sequential and transfinite compositions.*

PROOF. The class of h-cofibrations is closed under coproducts, cobase change and composition (finite, sequential or transfinite), compare Corollary 4.6 (i). The class of global equivalences is closed under coproducts because equivariant homotopy groups take wedges to sums (Corollary 2.21 (i)). The cobase change of an h-cofibration that is also a global equivalence is another global equivalence by Corollary 2.20. The class of h-cofibrations that are also global equivalences is closed under sequential and transfinite composition because equivariant homotopy groups commute with colimits along sequential or transfinite composition of h-cofibrations. □

Since the group $\pi_k^G(\Omega^m X)$ is naturally isomorphic to $\pi_{m+k}^G(X)$, looping preserves global equivalences. Similarly, the group $\pi_{m+k}^G(S^m \wedge X)$ is naturally isomorphic to $\pi_k^G(X)$, so suspension preserves global equivalences. The next proposition generalizes this.

Proposition 2.23. *For every cofibrant based space A the functor $A\wedge -$ preserves global equivalences. For every finite based CW-complex A the functor* $\mathrm{map}(A,-)$ *preserves global equivalences.*

The *shift* $\mathrm{sh}\, X$ of an orthogonal spectrum X is given in level n by

$$(\mathrm{sh}\, X)_n \;=\; X_{1+n}\ . \tag{2.24}$$

The orthogonal group $O(n)$ acts through the monomorphism $\mathrm{Id}\oplus - : O(n) \longrightarrow O(1+n)$. The n-th structure map of $\mathrm{sh}\, X$ is the $(1+n)$-th structure map of X. For an inner product space V we have
$$(\mathrm{sh}\, X)(V) \;\cong\; X(\mathbb{R}\oplus V)$$
by a natural $O(V)$-equivariant homeomorphism. As an example, the shift of a suspension spectrum is another suspension spectrum:
$$\mathrm{sh}(\Sigma^\infty A) \;\cong\; \Sigma^\infty(A\wedge S^1)\ .$$

The suspension and the shift of an orthogonal spectrum X are related by a natural morphism $\lambda_X : S^1\wedge X \longrightarrow \mathrm{sh}\, X$. In level n, this is defined as the composite

$$S^1\wedge X_n \xrightarrow{\text{twist}} X_n\wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow{\chi_{n,1}} X_{1+n} = (\mathrm{sh}\, X)_n\ . \tag{2.25}$$

Proposition 2.26. *For every orthogonal spectrum X the morphism*
$$\lambda \;:\; S^1\wedge X \;\longrightarrow\; \mathrm{sh}\, X \qquad \textit{and its adjoint} \qquad \tilde\lambda \;:\; X \;\longrightarrow\; \Omega(\mathrm{sh}\, X)$$
are global equivalences.

Construction 2.27. Besides a natural abelian group structure, the equivariant homotopy groups of an orthogonal *spectrum* have another piece of structure not present in the equivariant homotopy sets of an orthogonal *space*: for every orthogonal spectrum X and every subgroup H of a compact Lie group G, there is a *transfer map*

$$\mathrm{tr}_H^G \ : \ \pi_0^H(X) \ \longrightarrow \ \pi_0^G(X)$$

whose construction we recall in this section. The key properties of the transfer maps are

- transfers are transitive;
- transfer commutes with restriction along epimorphisms;
- restriction of a transfer to a subgroup satisfies a double coset formula (3.15).

We show in Theorem 3.5 below that restriction and transfer maps generate all natural operations between the 0-dimensional equivariant homotopy group functors for orthogonal spectra.

We let G be a compact Lie group and H a subgroup of G. The *transfer map* is a natural group homomorphism

$$\mathrm{tr}_H^G \ : \ \pi_0^H(X) \ \longrightarrow \ \pi_0^G(X) \ .$$

To construct the transfer we choose a G-equivariant smooth embedding

$$i \ : \ G/H \ \longrightarrow \ V$$

into some finite dimensional G-subrepresentation V of the universe $\mathcal{U}_G$. The equivariant tubular neighborhood theorem [ref] provides a G-equivariant embedding

$$j \ : \ \nu(i) \ \longrightarrow \ V$$

of the total space of the normal bundle $\nu(i)$ of the embedding i as a tubular neighborhood; in particular, the composite of j with the zero section $G/H \longrightarrow \nu(i)$ is i.

The total space of any equivariant vector bundle ξ over G/H is canonically isomorphic to $G \times_H \xi_H$, where the H-representation ξ_H is the fiber over the preferred coset H. If we denote by $L(H) = T_H(G/H)$ the special fiber of the tangent bundle of G/H, then the tangent bundle of G/H is of the form

$$T(G/H) \ \cong \ G \times_H L(H) \ .$$

We let W denote the orthogonal complement of $L(H)$ in the tangent space of V at $i(H)$ (which is V itself). Then

$$\nu(i) \ \cong \ G \times_H W \ .$$

The Thom-Pontryagin collapse then becomes a G-map

$$S^V \ \xrightarrow{\text{collapse}} \ T\nu(i) \ \cong \ G \ltimes_H S^W \ , \tag{2.28}$$

where $T\nu(i)$ is the Thom space of the normal bundle i. We compose with the effect of the inclusion $W \longrightarrow V$ to obtain a G-map

$$\mathrm{tr} \ : \ S^V \ \xrightarrow{\text{collapse}} \ T\nu \ \cong \ G \ltimes_H S^W \ \longrightarrow \ G \ltimes_H S^V \ \cong \ G/H^+ \wedge S^V \ , \tag{2.29}$$

the *transfer map*. The last homeomorphism sends $[g, v] \in G \ltimes_H S^V$ to $gH \wedge gv \in G/H^+ \wedge S^V$.

Now we consider an element in $\pi_0^H X$, for some orthogonal spectrum X, represented by a based H-map $S^U \longrightarrow X(U)$ for some finite dimensional H-representation U. By increasing U and stabilizing with the orthogonal complement we can assume, if necessary, that U is underlying a G-representation; the map f, however is only H-equivariant, but it has a G-equivariant extension $\hat{f} : S^U \wedge G/H^+ \longrightarrow X(U)$ defined by $\hat{f}(u) = g \cdot f(g^{-1} \cdot u)$. [or: $\hat{f} : G \ltimes_H S^U \longrightarrow X(U)$, $[g, u] \mapsto g \cdot f(u)$.]

The transfer

$$\mathrm{tr}_H^G[f] \ \in \ \pi_0^G(X)$$

is then represented by the composite

$$S^{U+V} \xrightarrow{S^U \wedge \mathrm{tr}} S^U \wedge G/H^+ \wedge S^V \xrightarrow{\hat{f} \wedge S^V} X(U) \wedge S^V \xrightarrow{\sigma_{U,V}} X(U \oplus V) . \tag{2.30}$$

There is some work involved in showing that the class $\mathrm{tr}_H^G[f]$ only depends on the class $[f]$, and not on the choice of representation V, the embedding i or the tubular neighborhood.

Example 2.31. If H has infinite index in its normalizer, then the transfer map just constructed is trivial. Indeed, the inclusion of the normalizer $N = N_G H$ of H into G induces a smooth embedding

$$W_G H = N/H \longrightarrow G/H$$

and thus a monomorphism of tangent spaces

$$T_H(W_G H) \longrightarrow T_H(G/H) = L(H) .$$

If $n \in N$ is an element of the normalizer and $h \in H$, then

$$h \cdot nH = n \cdot (n^{-1} h n) H = nH ,$$

so $W_G H$ is H-fixed inside G/H. Consequently, the tangent space $T_H(W_G H)$ is contained in the H-fixed space $L(H)^H$.

If H has infinite index in its normalizer, then the Weyl group $W_G H = N/H$ and the tangent space $T_H(W_G H)$ have positive dimension. In particular, $L(H)$ has non-zero H-fixed points. The point 0 in $S^{L(H)}$ can thus be moved through H-fixed points to the basepoint at infinity. The H-map $S^W \longrightarrow S^V$ induced by the inclusion $W \longrightarrow W \oplus L(H) = V$ is then H-equivariantly homotopic to the trivial map. The transfer map $\mathrm{tr} : S^V \longrightarrow G/H^+ \wedge S^V$ is thus G-equivariantly null-homotopic, and the entire transfer tr_H^G is the zero map.

Remark 2.32. In the construction of the transfer we are forgetting information by using the including the orthogonal complement W of $L(H)$ into the representation. The previous example shows that this can sometimes make the entire transfer trivial. We can instead remember more and obtain the *dimension shifting* transfer. We discuss the extreme case of this now, namely the case where the subgroup H is normal in the Lie group G. As we saw in Example 2.31, the action of the group H on the tangent representation $L(H)$ is then altogether trivial. The Thom-Pontryagin collapse map (2.28) then becomes a G-map

$$S^V \xrightarrow{\text{collapse}} G \ltimes_H S^W \cong G/H^+ \wedge S^W .$$

We look at the transfer element for the inclusion of the trivial subgroup of the circle group $T = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. The inclusion $i : T \longrightarrow \mathbb{C}$ is T-equivariant if we let T acts on $\mathbb{C}$ by weight 1. This embedding identifies the tangent space of T at 1 with the imaginary part $i\mathbb{R} \in \mathbb{C}$, so $W = \mathbb{R}$ is the real part.

The normal bundle of this embedding is trivial, with a trivialization given by

$$T \times \mathbb{R} \longrightarrow \nu(i) , \quad (\lambda, x) \longmapsto [\lambda, x \cdot \lambda] .$$

Under this identification, an equivariant tubular neighborhood is given by

$$j : T \times \mathbb{R} \cong \nu(i) \longrightarrow \mathbb{C} , \quad (\lambda, x) \longmapsto \lambda \cdot \varphi(x)$$

where φ is some homeomorphism $\varphi : \mathbb{R} \cong (0, \infty)$ with $\varphi(0) = 1$. The Thom-Pontryagin collapse

$$c : S^{\mathbb{C}(1)} \longrightarrow T_+ \wedge S^1$$

sends the fixed points 0 and ∞ to the basepoint, and it sends a point $z \in S^{\mathbb{C}} - \{0, \infty\} = \mathbb{C} - \{0\}$ to $(z/|z|, \varphi^{-1}(|z|))$. We can form the composite

$$S^{\mathbb{C}(1)} \wedge S^1 \xrightarrow{c \wedge S^1} T_+ \wedge S^1 \wedge S^1 \xrightarrow{\alpha} S^{\mathbb{C}(1)}$$

where α is essentially the action map, given by

$$\alpha(\lambda, x, y) = \lambda \cdot (x + iy) .$$

This composite represents an element

$$\mathrm{Tr}_e^T(1) \ \in \ \pi_1^T \mathbb{S} \tag{2.33}$$

in the first T-equivariant stable stem.

Proposition 2.34. *Let G be a compact Lie groups and $H \leq G$ a subgroup. Then the transfer map $\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$ is additive and natural for morphisms of orthogonal spectra. The transfer maps are transitive, i.e., for every subgroup K of H we have*

$$\mathrm{tr}_H^G \circ \mathrm{tr}_K^H \ = \ \mathrm{tr}_K^G \ : \ \pi_0^K(X) \ \longrightarrow \ \pi_0^G(X) \ .$$

Proof. The transfer maps form a natural transformation between the additive functors π_0^H and π_0^G, so the are automatically additive. [transitivity] □

Proposition 2.35. *Let X be an orthogonal spectrum and K and G compact Lie groups. For every surjective group homomorphism $\alpha : K \longrightarrow G$ and every subgroup H of G the relation*

$$\alpha^* \circ \mathrm{tr}_H^G \ = \ \mathrm{tr}_L^K \circ (\alpha|_L)^*$$

holds as maps $\pi_0^H(X) \longrightarrow \pi_0^K(X)$, where $L = \alpha^{-1}(H)$ and $\alpha|_L : L \longrightarrow H$ is the restriction of α.

The composition of a transfer map with restriction to a subgroup can be rewritten according to the double coset formula (3.15) that we will discuss below.

Remark 2.36 (Finite index transfers)**.** We look at the transfer in the special case where H has finite index in G, (i.e., when H and G have the same dimension), because then some things simplify. In this case G/H is a finite set and a tubular neighborhood of $i(G/H)$ in V is just a collection of small balls. Moreover, $L(H) = 0$, $W = V$ and $\nu(i) = G \times_H V$. For every sufficiently small real number $\delta > 0$ the δ-balls around the image of i still embed, i.e., the G-map

$$j \ : \ G \times_H D(V) \ \longrightarrow \ V \ , \quad [g, x] \ \longmapsto \ g \cdot (i(H) + \delta \cdot x)$$

is an embedding. So we get a G-equivariant Thom-Pontryagin collapse map

$$c \ : \ S^V \ \longrightarrow \ \frac{G \times_H D(V)}{G \times_H S(V)}$$

by

$$c(x) \ = \ \begin{cases} j^{-1}(x) & \text{if } x \in \mathrm{Im}(j) \text{, and} \\ \infty & \text{else.} \end{cases}$$

We compose with a G-homeomorphism

$$\frac{G \times_H D(V)}{G \times_H S(V)} \ \cong \ G \ltimes_H S^V \ , \quad [g, v] \ \longmapsto \ g \cdot \frac{v}{1 - |v|}$$

(that stretches the interval by a homeomorphism $[0,1] \cong [0, \infty]$). Altogether this yields a G-equivariant transfer map

$$\mathrm{tr} \ : \ S^V \ \longrightarrow \ G \ltimes_H S^V \ .$$

Remark 2.37. In Remark 1.7 we observed that only very special kinds of orthogonal G-spectra are part of a 'global family', i.e., isomorphic to an orthogonal G-spectrum of the form $X\langle G \rangle$ for some orthogonal spectrum X. The previous obstructions were in terms of pointset level conditions, and now we can also formulate obstructions to 'being global' in terms of the Mackey functor homotopy groups of an orthogonal G-spectrum.

For every orthogonal G-spectrum Y the homotopy groups $\{\pi_0^H(Y)\}_H$ form a G-Mackey functor as H varies through the closed subgroups of G. This means that these groups for different H are related by restriction maps (to subgroups, *not* along arbitrary homomorphisms), by conjugation maps and by transfer maps. This data satisfies certain compatibility relations, among them the double coset formula.

For every orthogonal spectrum X and compact Lie groups $H \leq G$ the H-homotopy groups of the G-spectrum $X\langle G\rangle$ are, by definition, the H-homotopy groups $\pi_*^H(X)$. So these groups depend only on H, and not on the ambient group G. One obstruction to Y being part of a 'global family' is that the G-Mackey functor structure can be extended to a 'global functor' (in the sense of Definition 3.2 below). In particular, the G-Mackey functor homotopy groups can be complemented by restriction maps along arbitrary group homomorphisms between the subgroups of G. As the extreme case this includes a restriction map $p^* : \pi_*^e(X) \longrightarrow \pi_*^G(X)$ associated to the unique homomorphism $p : G \longrightarrow e$, splitting the restriction map $\pi_*^G(X) \longrightarrow \pi_*^e(X)$. So one obstruction to being global is that this restriction map from $\pi_*^G(X)$ to $\pi_*^e(X)$ needs to be a split epimorphism. Another consequence is that every global functor takes isomorphic values on a pair of isomorphic subgroups of G; for G-Mackey functors this is true when the subgroups are conjugate in G, but not in general when they are merely abstractly isomorphic.

Another obstruction can be given in terms of the conjugation maps. For $g \in G$ the Mackey functor conjugation map $c_g^* : \pi_0^H(X\langle G\rangle) \longrightarrow \pi_0^{{}^gH}(X\langle G\rangle)$ of the G-spectrum $X\langle G\rangle$ is the conjugation map (1.15). In particular, for G-spectra of the form $X\langle G\rangle$, the map c_g^* depends only on the conjugation homomorphism $c_g : {}^gH \longrightarrow H$ and not on the element g that conjugates. In other words, if g centralizes H, then c_g^* is the identity of $\pi_0^H(X\langle G\rangle)$ – a condition that does not hold in general G-Mackey functors.

The most extreme case of this is when $H = e$ is the trivial subgroup of G. Every element of G centralizes e, so for G-spectra of the form $X\langle G\rangle$, the conjugation maps c_g^* of $\pi_*^e(X\langle G\rangle)$ are all identity maps. For example, for finite groups G we can let M be a left $\mathbb{Z}G$-module; a G-Mackey functor $\underline{M}$ is then given by $\underline{M}(H) = M^H$ with inclusions as restriction maps, and with conjugation and transfer maps defined by

$$c_g^*(m) \ = \ gm \qquad \text{respectively} \qquad \mathrm{tr}_H^G(m) \ = \sum_{gH\in G/H} gm \ .$$

The Eilenberg-Mac Lane G-spectrum $H\underline{M}$ realizes the Mackey functor $\underline{M}$ as on 0-th homotopy groups; so as soon as the G-action on M is non-trivial, the Eilenberg-Mac Lane G-spectrum $H\underline{M}$ does not come from a global homotopy type.

3. Global functors

This section is devoted to the category $\mathcal{GF}$ of *global functors*, the natural home of the collection of equivariant homotopy groups $\{\pi_0^G(X)\}$ of an orthogonal spectrum. The category $\mathcal{GF}$ of global functors is a symmetric monoidal abelian category with enough injectives and projective that plays the same role for global homotopy theory that is played by the category of abelian groups in ordinary homotopy theory, or by the category of G-Mackey functors for G-equivariant homotopy theory.

An abstract way to motivate the appearance of global functors is as follows. One can consider the globally connective (respectively globally coconnective) orthogonal spectra, i.e., those where all equivariant homotopy groups vanish in negative dimensions (respectively in positive dimensions). It turns out that the full subcategories of globally connective respectively globally coconnective spectra define a non-degenerate t-structure on the triangulated global stable homotopy category, and the heart of this t-structure is (equivalent to) the abelian category of global functors; we refer the reader to Corollary IV.4.10 below for details.

Construction 3.1 (Burnside category)**.** We define the pre-additive *global Burnside category* $\mathbf{A}$. The objects of $\mathbf{A}$ are all compact Lie groups; morphisms from a group G to K are defined as

$$\mathbf{A}(G,K) \ = \ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ ,$$

the set of natural transformations of functors, from orthogonal spectra to sets, between the equivariant homotopy group functors π_0^G and π_0^K. It is not a priori clear that the natural transformations from π_0^G to

π_0^K form a set (as opposed to a proper class), but this follows from Proposition 3.13 below. Composition in the category $\mathbf{A}$ is composition of natural transformations.

The functor π_0^K is abelian group valued, so the set $\mathbf{A}(G,K)$ is an abelian group under objectwise addition of transformations. Composition is additive in each variable, so $\mathbf{A}(G,K)$ is a pre-additive category.

The Burnside category $\mathbf{A}$ is skeletally small: isomorphic compact Lie groups are also isomorphic as objects in the category $\mathbf{A}$, and every compact Lie group is isomorphic to a closed subgroup of an orthogonal group $O(n)$. So the set of all closed subgroups of $O(n)$ for all $n \geq 0$ is a skeleton of the category $\mathbf{A}$.

Definition 3.2. A *global functor* is an additive functor from the Burnside category $\mathbf{A}$ to the category of abelian groups. A morphism of global functors is a natural transformation.

We discuss various explicit examples of interesting global functors in Example 3.24.

Example 3.3. The definition of the global Burnside category $\mathbf{A}$ is made so that the collection of equivariant homotopy groups of an orthogonal spectrum are tautologically a global functor. Explicitly, the global homotopy group functor $\underline{\pi}_0 X$ of an orthogonal spectrum X is defined on objects by

$$(\underline{\pi}_0 X)(G) \;=\; \pi_0^G(X)$$

and on morphisms by evaluating natural transformations at X. It is less obvious that conversely every global functor is isomorphic to the global functor $\underline{\pi}_0 X$ of some orthogonal spectrum X; we refer the reader to Remark IV.4.12 below for the construction of Eilenberg-Mac Lane spectra from global functors.

As a category of additive functors out of a skeletally small pre-additive category, the category $\mathcal{GF}$ of global functors has some immediate properties that we collect in the following proposition.

Proposition 3.4. *The category $\mathcal{GF}$ of global functors is an abelian category with enough injectives and projectives.*

Proof. Any category of additive functors out of a skeletally small additive category is abelian with notions of exactness defined objectwise [ref]. A set of projective generators is given by the represented global functors $\mathbf{A}(G,-)$ where G runs through a set of representatives of the isomorphism classes of compact Lie groups. A set of injective cogenerators is given similarly by the global functors

$$\mathrm{Hom}(\mathbf{A}(-,K),\mathbb{Q}/\mathbb{Z}) \;:\; \mathbf{A} \;\longrightarrow\; \mathcal{A}b\ ,$$

where K runs through a set of representatives of the isomorphism classes of compact Lie groups. $\square$

As we shall explain in Proposition 4.26 below, the category $\mathcal{GF}$ has a symmetric monoidal closed product $\Box$ that arises as a convolution product for a certain symmetric monoidal structure on the global Burnside category $\mathbf{A}$.

Our definition of the Burnside category is made so that every orthogonal spectrum X gives rise to a homotopy group global functor without further ado, but it is not clear from the definition how to describe the morphism groups of $\mathbf{A}$ explicitly. Our next aim is to show that each morphism group $\mathbf{A}(G,K)$ is a free abelian group with an explicit basis given by certain composites of a restriction and a transfer morphism. For a pair (L,α) consisting of a subgroup L of K and a continuous group homomorphism $\alpha : L \longrightarrow G$ we define

$$[L,\alpha] \;\in\; \mathbf{A}(G,K)$$

as the natural transformation whose value at an orthogonal spectrum is the composite

$$\pi_0^G(X) \xrightarrow{\;\alpha^*\;} \pi_0^L(X) \xrightarrow{\;\mathrm{tr}_L^K\;} \pi_0^K(X)$$

of restriction along α with transfer from L to K.

If L has infinite index in its normalizer, then the transfer map tr_L^K, and hence also the element $[L,\alpha]$, is zero by Example 2.31. The *conjugate* of (L,α) by a pair $(\kappa,\gamma) \in K\times G$ of group elements is the pair $({}^\kappa L, c_\gamma\circ\alpha\circ c_\kappa)$ consisting of the conjugate subgroup ${}^\kappa L$ and the composite homomorphism

$$ {}^\kappa L \xrightarrow{c_\kappa} L \xrightarrow{\alpha} G \xrightarrow{c_\gamma} G \ . $$

Since inner automorphisms induce the identity on equivariant homotopy groups (compare Proposition 1.16),

$$ \mathrm{tr}_{{}^\kappa L}^K \circ (c_\gamma\circ\alpha\circ c_\kappa)^* \ = \ \mathrm{tr}_{{}^\kappa L}^K \circ c_\kappa^*\circ\alpha^*\circ c_\gamma^* \ = \ c_\kappa^*\circ \mathrm{tr}_L^K\circ\alpha^*\circ c_\gamma^* \ = \ \mathrm{tr}_L^K\circ\alpha^* \ . $$

So the transformation $[L,\alpha]$ only depends on the conjugacy class of (L,α), i.e.,

$$ [{}^\kappa L, c_\gamma\circ\alpha\circ c_\kappa] \ = \ [L,\alpha] \ \in \ \mathbf{A}(G,K) \ . $$

Theorem 3.5. *Let K and G be compact Lie groups. The morphism group $\mathbf{A}(G,K)$ in the global Burnside category is a free abelian group with basis the transformations $[L,\alpha]$, where (L,α) runs over all conjugacy classes of pairs consisting of*

- *a subgroup $L\leq K$ whose Weyl group $W_K L$ is finite, and*
- *a continuous group homomorphism $\alpha : L\longrightarrow G$.*

The proof of Theorem 3.5 has two ingredients: We identify natural transformations from π_0^G to π_0^K with the value $\pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)$, and then we calculate the latter equivariant homotopy group using the tom Dieck splitting for equivariant homotopy groups of suspension spectra.

The 0-th equivariant homotopy groups of orthogonal *spectra* have two extra pieces of structure, compared to orthogonal *spaces*: an abelian group structure and transfers. As we shall now explain, every orthogonal space has a suspension spectrum, and at the level of 0-th equivariant homotopy sets, the suspension spectrum ‘freely builds in’ the extra structure that is available stably.

Construction 3.6 (Suspension spectra)**.** To every orthogonal space Y we can associate an unreduced *suspension spectrum* $\Sigma^\infty_+ Y$ whose value on an inner product space is given by

$$ (\Sigma^\infty_+ Y)(V) \ = \ Y(V)_+\wedge S^V \ ; $$

here the orthogonal group $O(V)$ acts diagonally and the structure map

$$ \sigma_{V,W} \ : \ (Y(V)_+\wedge S^V)\wedge S^W \ \longrightarrow \ Y(V\oplus W)_+\wedge S^{V\oplus W} $$

is the combination of the map $Y(i_V);Y(V)\longrightarrow Y(V\oplus W)$ and the canonical homeomorphism $S^V\wedge S^W\cong S^{V\oplus W}$. If Y is the constant orthogonal space with value a topological space A, then $\Sigma^\infty_+ Y = \Sigma^\infty_+ A$ specializes to the suspension spectrum of A with a disjoint basepoint added. The functor

$$ \Sigma^\infty_+ \ : \ spc \ \longrightarrow \ \mathcal{S}p $$

is left adjoint to the functor $\Omega^\bullet$ of Construction 2.3.

Let Y be an orthogonal space and G a compact Lie group. We define a map

$$ \sigma^G \ : \ \pi_0^G(Y) \ \longrightarrow \ \pi_0^G(\Sigma^\infty_+ Y) \tag{3.7} $$

as the effect of the adjunction unit $Y\longrightarrow \Omega^\bullet(\Sigma^\infty_+ Y)$ on the G-equivariant homotopy set π_0^G, using the identification $\pi_0^G(\Omega^\bullet(\Sigma^\infty_+ Y))\cong\pi_0^G(\Sigma^\infty_+ Y)$. More explicitly: if V is a finite dimensional G-subrepresentation of the complete G-universe $\mathcal{U}_G$ and $y\in Y(V)^G$ a G-fixed point, then $\sigma^G[y]$ is represented by the G-map

$$ S^V \xrightarrow{y\wedge -} Y(V)_+\wedge S^V \ = \ (\Sigma^\infty_+ Y)(V) \ . $$

As G-varies, the maps σ^G are compatible with restriction, since they arise from a morphism of orthogonal spaces.

For a subgroup H of G, the normalizer $N_G H$ acts on H by conjugation, and hence on $\pi_0^H(Y)$ by restriction along the conjugation maps. Restriction along inner automorphisms are the identity, so the action of N_GH factors over an action of the Weyl group $W_GH = N_GH/H$ on $\pi_0^H Y$. After passing to the stable classes along the map $\sigma^H : \pi_0^H(Y) \longrightarrow \pi_0^H(\Sigma^\infty_+ Y)$, we can then transfer from H to G. For an element $g \in N_GH$ and a class $x \in \pi_0^H(Y)$ we have

$$\mathrm{tr}_H^G(\sigma^H(c_g^* x)) \ = \ \mathrm{tr}_H^G(c_g^*(\sigma^H(x))) \ = \ c_g^*(\mathrm{tr}_H^G(\sigma^H(x))) \ = \ \mathrm{tr}_H^G(\sigma^H(x))) \ .$$

because transfer commutes with restriction along epimorphisms. So transferring from H to G in the global functor $\underline{\pi}_0(\Sigma^\infty_+ Y)$ annihilates the action of the Weyl group on $\pi_0^H(Y)$.

Proposition 3.8. *Let Y be an orthogonal space. Then for every compact Lie group K the equivariant homotopy group $\pi_0^K(\Sigma^\infty_+ Y)$ of the suspension spectrum of Y is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_L^K(\sigma^L(x))$$

where L runs through all conjugacy classes of subgroups of K with finite Weyl group and x runs through a set of representatives of the W_KL-orbits of the set $\pi_0^L(Y)$.

Proof. By definition,

$$\begin{aligned}\pi_0^K(\Sigma^\infty_+ Y) \ &= \ \mathrm{colim}_{V\in s(\mathcal{U}_K)}\, [S^V,\, Y(V)_+ \wedge S^V]^K \\ &\cong \ \mathrm{colim}_{V\in s(\mathcal{U}_K)}\, [S^V,\, Y(\mathcal{U}_K)_+ \wedge S^V]^K \ = \ \pi_0^K(\Sigma^\infty_+ Y(\mathcal{U}_K)) \ .\end{aligned}$$

The tom Dieck splitting [**95**, Satz 2] provides an isomorphism

$$\bigoplus_{(L)} \pi_0^{WL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K)^L)) \ \cong \ \pi_0^K(\Sigma^\infty_+ Y(\mathcal{U}_K)) \ ,$$

where the sum is indexed over all conjugacy classes of subgroups L and $WL = W_KL$ is the Weyl group of L in K. By [**95**, Sec. 4] the group $\pi_0^{WL}((EWL \times Y(\mathcal{U}_K)^L)_+)$ vanishes if the Weyl group WL is infinite; so only the summands with finite Weyl group contribute to π_0.

On the other hand, if the Weyl group WL is finite, then the group $\pi_0^{WL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K)^L))$ is free abelian with a basis given by the set

$$WL\backslash\pi_0(Y(\mathcal{U}_K)^L) \ , \tag{3.9}$$

the WL-orbit set of the path components of $Y(\mathcal{U}_K)^L$. The tom Dieck splitting is given on the L-summand by the composite

$$\begin{aligned}\pi_0^{WL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K)^L)) \ &\xrightarrow{\ p^*\ } \ \pi_0^{NL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K)^L)) \\ &\xrightarrow{\ i_*\ } \ \pi_0^{NL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K))) \\ &\ \cong \quad \pi_0^K(\Sigma^\infty_+(K \times_{NL} (EWL \times Y(\mathcal{U}_K)))) \\ &\longrightarrow \ \pi_0^K(\Sigma^\infty_+ Y(\mathcal{U}_K)) \ .\end{aligned}$$

Here $NL = N_KL$ is the normalizer of L in K, $p : NL \longrightarrow WL$ is the projection, $i : Y(\mathcal{U}_K)^L \longrightarrow Y(\mathcal{U}_K)$ the inclusion (which is NL-equivariant), the third map is the Wirthmüller isomorphism [**fix this...**] and the last map is induced by the projection $K \times_{NL} (EWL \times Y(\mathcal{U}_K)) \longrightarrow Y(\mathcal{U}_K)$. So the basis (3.9) of the group $\pi_0^{WL}(\Sigma^\infty_+(EWL \times Y(\mathcal{U}_K)^L))$ is taken to the elements $\mathrm{tr}_L^K(\sigma^L(x))$ as x runs through a set of representatives of the W_KL-orbits of the set $\pi_0^L(Y)$. □

We let G be a compact Lie group and V a G-representation. We recall from (2.3) of Chapter I the tautological class

$$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V})$$

in the G-equivariant homotopy set of the free orthogonal space $\mathbf{L}_{G,V}$. The *stable tautological class* is

$$e_{G,V} \ = \ \sigma^G(u_{G,V}) \ \in \ \pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V}) \ . \tag{3.10}$$

Explicitly, $e_{G,V}$ is the homotopy class of the G-map

$$S^V \longrightarrow (\mathbf{L}(V,V)/G)^+ \wedge S^V = (\Sigma^\infty_+ \mathbf{L}_{G,V})(V) , \quad v \longmapsto (\mathrm{Id}_V \cdot G) \wedge v .$$

Corollary 3.11. *Let G and K be compact Lie groups and V a faithful G-representation. Then the homotopy group $\pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$ is a free abelian group with basis given by the classes*

$$\mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

as (L,α) runs over a set of representatives of all $K\times G$-conjugacy classes of pairs consisting of a subgroup L of K of finite index in its normalizer and a continuous homomorphism $\alpha : L \longrightarrow G$.

Proof. The map

$$\mathrm{Rep}(K,G) \longrightarrow \pi_0^K(\mathbf{L}_{G,V}) , \quad [\alpha : K \longrightarrow G] \longmapsto \alpha^*(u_{G,V})$$

is bijective according to Proposition I.2.6. Proposition 3.8 thus says that $\pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$ is a free abelian group with a basis given by the elements

$$\mathrm{tr}_L^K(\sigma^L(\alpha^*(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(\sigma^G(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(e_{G,V}))$$

where L runs through all conjugacy classes of subgroups of K with finite Weyl group and α runs through a set of representatives of the $W_K L$-orbits of the set $\mathrm{Rep}(L,G)$. The claim follows because $K\times G$-conjugacy classes of such pairs (L,α) biject with pairs consisting of a conjugacy class of subgroups (L) and a $W_K L$-equivalence class in $\mathrm{Rep}(L,G)$. □

Example 3.12. We discuss a specific example of Corollary 3.11, with $G = A_3$ the alternating group on three letters and $K = \Sigma_3$ the symmetric group on 3 letters. The group Σ_3 has four conjugacy classes of subgroups, with representatives $\Sigma_3, A_3, (12)$ and e. The groups $\Sigma_3, (12)$ and e admit only trivial homomorphisms to A_3, whereas the alternating group A_3 also has two automorphisms. None of the three endomorphisms of A_3 are conjugate, so the set $\mathrm{Rep}(A_3, A_3)$ has three elements. However, the Weyl group $W_{\Sigma_3} A_3$ has two elements, and its action identifies the two automorphisms of A_3. So while $\pi_0^{A_3}(B_{\mathrm{gl}} A_3) \cong \mathrm{Rep}(A_3, A_3)$ has three elements, it only contributes two generators to the stable group $\pi_0^{\Sigma_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$. A basis for the free abelian group $\pi_0^{\Sigma_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$ is thus given by the classes

$$p^*_{\Sigma_3}(1) , \qquad \mathrm{tr}_{A_3}^{\Sigma_3}(e_{A_3}) , \qquad \mathrm{tr}_{A_3}^{\Sigma_3}(p^*_{A_3} 1) , \qquad \mathrm{tr}_{(12)}^{\Sigma_3}(p^*_{(12)} 1) \qquad \text{and} \qquad \mathrm{tr}_e^{\Sigma_3}(p^*_e 1) .$$

Here the faithful A_3-representation V is unspecified and we write $e_{A_3} = e_{A_3,V} \in \pi_0^{A_3}(\Sigma^\infty_+ B_{\mathrm{gl}} A_3)$ for the stable tautological class. Moreover $p_H : H \longrightarrow e$ denotes the unique homomorphism to the trivial group and $1 = \mathrm{res}_e^{A_3}(e_{A_3})$ is the restriction of the class e to the trivial group.

Now comes the last ingredient for the calculation of the morphism group $\mathbf{A}(G,K)$. We let V be any faithful G-representation and apply the representability result of Proposition I.2.20 to the category $\mathcal{C} = \mathcal{S}p$ of orthogonal spectra and the adjoint functor pair

$$spc \underset{\Omega^\bullet}{\overset{\Sigma^\infty_+}{\rightleftarrows}} \mathcal{S}p$$

and the functor $\pi_0^K \circ \Omega^\bullet$ from orthogonal spectra to sets. In this case Proposition I.2.20 specializes to:

Proposition 3.13. *Let G and K be compact Lie groups and V a faithful G-representation. Then evaluation at the stable tautological class is a bijection*

$$\mathbf{A}(G,K) = \mathrm{Nat}^{\mathcal{S}p}(\pi_0^G, \pi_0^K) \longrightarrow \pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V}) , \quad \tau \longmapsto \tau(e_{G,V})$$

to the 0-th K-equivariant homotopy group of the orthogonal spectrum $\Sigma^\infty_+ \mathbf{L}_{G,V}$. In other words, the morphism

$$\mathbf{A}(G,-) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ \mathbf{L}_{G,V})$$

classified by the stable tautological class e_G is an isomorphism of global functors.

Now we are ready for the

Proof of Theorem 3.5 . We let V be any faithful G-representation. By Proposition 3.11 the composite

$$\mathbb{Z}\{[L,\alpha] \mid \ |W_K L| < \infty,\, \alpha : L \longrightarrow G\} \ \longrightarrow \ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ \xrightarrow{\ \mathrm{ev}\ } \ \pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$$

is an isomorphism, where the first map takes $[L,\alpha]$ to $\mathrm{tr}_L^K \circ \alpha^*$ and the second map is evaluation at the stable tautological class $e_{G,V}$. The evaluation map is an isomorphism by Proposition 3.13, so the first map is an isomorphism, as claimed. □

Example 3.14 (Sphere spectrum)**.** The *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, where the orthogonal group acts as the one-point compactification of its natural action on $\mathbb{R}^n$. The map $\sigma_n : S^n \wedge S^1 \longrightarrow S^{n+1}$ is the canonical homeomorphism. For every compact Lie group G and every G-representation V, the map

$$\mathbb{S}(V) \ = \ \mathbf{L}(\mathbb{R}^n, V)^+ \wedge_{O(n)} S^n \ \longrightarrow \ S^V \ , \quad [\varphi, x] \ \longmapsto \ \varphi(x)$$

is a G-equivariant homeomorphism to the representation sphere of V. The equivariant homotopy groups of the sphere spectrum are the equivariant stable stems.

We discuss the 0-th equivariant stable stems in some detail. The sphere spectrum is isomorphic to the suspension spectrum of the free global space $\mathbf{L}_{e,0}$ generated by the trivial representation of the trivial group,

$$\mathbb{S} \ \cong \ \Sigma^\infty_+ \mathbf{L}_{e,0} \ .$$

The trivial representation is faithful as a representation of the trivial group, so $\mathbf{L}_{e,0} = B_{\mathrm{gl}} e$ is a global classifying space for the trivial group. For this particular faithful representation, the class $1 \in \pi_0(\mathbb{S})$ represented by the identity of S^0 is the stable tautological class $e_{e,0}$ (compare (3.10)). By Proposition 3.13, the action on the unit $1 \in \pi_0(\mathbb{S})$ is an isomorphism of global functors

$$\mathbb{A} \ = \ \mathbf{A}(e, -) \ \longrightarrow \ \underline{\pi}_0(\mathbb{S})$$

from the Burnside ring global functor $\mathbb{A}$ to the 0-th homotopy global functor of the sphere spectrum. For finite groups, this is originally due to Segal [**76**], and for general compact Lie groups to tom Dieck, as a corollary to his splitting theorem (see Satz 2 and Satz 3 of [**95**]).

For finite groups, the ‘global 1-stem’, i.e., the global functor $\underline{\pi}_1(\mathbb{S})$, can also be described in terms of finite G-sets. At a finite group G, the equivariant 1-stem $\pi_1^G(\mathbb{S})$ is naturally isomorphic to the quotient of the free abelian group generated by isomorphism class of automorphisms of finite G-sets, with relations coming from composition and disjoint union of automorphisms. The proof uses that the equivariant stems can be calculated as the algebraic K-theory groups of the category of finite G-sets [ref].

The multiplication provides a map

$$\mathbb{A} \otimes \mathbb{Z}/2\{\eta\} = \mathbb{A} \otimes \pi_1^{\mathrm{s}} \ \longrightarrow \ \underline{\pi}_1(\mathbb{S})$$

that is injective by the tom Dieck splitting, where $\mathbb{A}$ is the Burnside ring global functor. The cokernel at a group G is isomorphic to the direct sum of the groups $(W_G H)_{\mathrm{ab}}$, the abelianized Weyl group of H, where H runs through the conjugacy classes of subgroups of G. In this description, however, the restriction and transfer maps are not so easily visible.

Theorem 3.5 is almost a complete calculation of the Burnside category, but one important piece of information is still missing: how does one express the composite of two operations, each given in the basis of Theorem 3.5, as a sum of basis elements? Since restrictions are contravariantly functorial and transfers are transitive, the key question is how to express a transfer followed by a restriction in terms of the specified basis. Every group homomorphism is the composite of an epimorphism and a subgroup inclusion, and restriction along epimorphisms commute with transfers according to Proposition 2.35. So the remaining issue is to rewrite the composite

$$\pi_0^H(X) \xrightarrow{\mathrm{tr}_H^G} \pi_0^G(X) \xrightarrow{\mathrm{res}_K^G} \pi_0^K(X)$$

of a transfer map and a restriction map, where H and K are two subgroups of a compact Lie group G. The answer is given by the *double coset formula*:

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G \ = \ \sum_{[M]} \chi^\sharp(M) \cdot \mathrm{tr}_{K\cap {}^gH}^K \circ c_g^* \circ \mathrm{res}_{K^g\cap H}^H \ . \tag{3.15}$$

The double coset formula was proved by Feshbach for Borel cohomology theories [**34**, Thm. II.11] and later generalized to equivariant cohomology theories by Lewis and May [**53**, IV §6].

The formula (3.15) needs some explanation. The *double coset space* $K\backslash G/H$ is the quotient space of G by the left K and right H-action by translation. In contrast to a homogeneous space G/H, the double coset space is in general not a smooth manifold. However, it can be written as the union of certain subspaces (in general neither open nor closed) that are manifolds of varying dimensions.

Definition 3.16. Let K and H be subgroups of a compact Lie group G. Two points KgH and $K\bar{g}H$ of the double coset space $K\backslash G/H$ are of the *same orbit type* if the groups $K\cap {}^gH$ and $K\cap {}^{\bar{g}}H$ are conjugate inside K.

The set of point of $K\backslash G/H$ of fixed orbit type is a manifold (with the subspace topology of the compact space $K\backslash G/H$), the *orbit type manifold.* Orbit type manifolds are thus indexed by conjugacy classes of subgroups of K: for a subgroup L of K we set

$$M_{(L)} \ = \ \{KgH \in K\backslash G/H \mid K\cap {}^gH \text{ is conjugate to } L \text{ in } K\} \ .$$

The orbit type manifolds need not be connected, but each $M_{(L)}$ has only finitely many path components. Also, $M_{(L)}$ need not be closed inside $K\backslash G/H$; but if one orbit type manifold $M_{(L)}$ lies in the closure of another one $M_{(\bar{L})}$, then $\bar{L}$ is subconjugate to L in K. Only finitely many of the orbit type manifolds are non-empty, so the double coset formula is a finite sum.

The sum in the double coset formula (3.15) runs over all connected components M of orbit type manifolds $M_{(L)}$, and the group element $g\in G$ that occurs is such that $KgH\in M$. Moreover, $\chi^\sharp(M) = \chi(\bar{M})-\chi(\bar{M}-M)$ is the 'internal Euler characteristic' of M, where $\bar{M}$ is the closure of M inside $K\backslash G/H$. For some of the summands, the group $K\cap {}^gH$ may have infinite index in its normalizer in K, and then the corresponding summands on the right hand side vanishes.

If H has finite index in G, then the double coset formula (3.15) simplifies. For any other subgroup K of G the intersection $K\cap {}^gH$ then also has finite index in K, so only finite index transfers are involved in the double coset formula. Since G/H is finite, so is the set $K\backslash G/H$ of double cosets, and all orbit type manifold components are points. So all internal Euler characteristics that occur are 1 and the double coset formula specializes to

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G \ = \sum_{[g]\in K\backslash G/H} \mathrm{tr}_{K\cap {}^gH}^K \circ c_g^* \circ \mathrm{res}_{K^g\cap H}^H \ ; \tag{3.17}$$

the sum runs over a set of representatives of the K-H-double cosets.

Since our context and exposition are quite different from those of the main references for the double coset formula [**34, 53**], we explain how to deduce the double coset formula in the form (3.15) from a double coset formula of Lewis and May [**53**, IV Cor. 6.4].

Proof of the double coset formula (3.15). We let

$$\rho_K^G \ : \ \Sigma_+^\infty B_{\mathrm{gl}}K \ \longrightarrow \ \Sigma_+^\infty B_{\mathrm{gl}}G$$

be a morphism of orthogonal spectra that sends the tautological stable class $e_K \in \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}K)$ to the element $\mathrm{res}^G_K(e_G) \in \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}G)$. For example, we can take any faithful G-representation V, set $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ and $B_{\mathrm{gl}}K = \mathbf{L}_{K,\mathrm{res}^G_K V}$, and take the map induced by the quotient map $\mathbf{L}_{K,\mathrm{res}^G_K V} \longrightarrow \mathbf{L}_{G,V}$. Similarly, we can take a representing G-map (2.30) for the homotopy class $\mathrm{tr}_H^G(e_H) \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}H)$ and adjoin it to a morphism of orthogonal spectra

$$\tau_H^G \ : \ \Sigma_+^\infty B_{\mathrm{gl}}G \ \longrightarrow \ \Sigma_+^\infty B_{\mathrm{gl}}H$$

that sends the tautological stable class $e_G \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}G)$ to $\mathrm{tr}_H^G(e_H)$. Then the composite

$$\Sigma_+^\infty B_{\mathrm{gl}}K \ \xrightarrow{\ \rho_K^G\ } \ \Sigma_+^\infty B_{\mathrm{gl}}G \ \xrightarrow{\ \tau_H^G\ } \ \Sigma_+^\infty B_{\mathrm{gl}}H$$

sends e_K to $\mathrm{res}^G_K(\mathrm{tr}_H^G(e_H)) \in \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}H)$. For any compact Lie group Π, the underlying orthogonal K-spectrum of $\Sigma_+^\infty B_{\mathrm{gl}}\Pi$ has the K-homotopy type of $\Sigma_+^\infty B(K,\Pi)$, the suspension spectrum of a classifying K-space for (K,Π)-bundles (by Remark I.2.8). So in the stable homotopy category of orthogonal K-spectra (indexed on a complete universe), the composite above becomes

$$\Sigma_+^\infty B(K,K) \ \xrightarrow{\ \rho_K^G\ } \ \Sigma_+^\infty B(K,G) \ \xrightarrow{\ \tau_H^G\ } \ \Sigma_+^\infty B(K,H) \ .$$

Corollary IV 6.4 of [**53**] shows that in the stable homotopy category of orthogonal K-spectra, the latter composite equals the sum, indexed of the orbit type manifold components M of $K\backslash G/H$, of $\chi^\sharp(M)$ times the composite

$$\Sigma_+^\infty B(K,K) \ \xrightarrow{\ \tau^K_{K\cap {}^gH}\ } \ \Sigma_+^\infty B(K,K\cap {}^gH) \ \xrightarrow{\ \rho^{{}^gH}_{K\cap {}^gH}\ } \ \Sigma_+^\infty B(K,{}^gH) \ \xrightarrow{\ \gamma_g\ } \ \Sigma_+^\infty B(K,H) \ .$$

Here γ_g is the underlying morphisms of orthogonal K-spectra of a morphism $\gamma_g : \Sigma_+^\infty B_{\mathrm{gl}}({}^gH) \longrightarrow \Sigma_+^\infty B_{\mathrm{gl}}H$ that sends $e_{{}^gH}$ to $c_g^*(e_H) \in \pi_0^{{}^gH}(\Sigma_+^\infty B_{\mathrm{gl}}H)$. The last composite thus sends e_K to $\mathrm{tr}^K_{K\cap {}^gH}(\mathrm{res}^{{}^gH}_{K\cap {}^gH}(c_g^*(e_H))) \in \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}H)$, so Corollary IV 6.4 of [**53**] shows that

$$\mathrm{res}^G_K(\mathrm{tr}_H^G(e_H)) \ = \ \sum_{[M]} \ \chi^\sharp(M)\cdot \mathrm{tr}^K_{K\cap {}^gH}(c_g^*(\mathrm{res}^H_{K^g\cap H}(e_H)))$$

in the group $\pi_0^K(\Sigma_+^\infty B(K,H)) = \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}}H)$.

For varying orthogonal spectra X, the two sides of the double coset formula (3.15) are natural transformations from the functor π_0^H to the functor π_0^K. Since the functor π_0^H is represented by the pair $(\Sigma_+^\infty B_{\mathrm{gl}}H, e_H)$ (by Proposition 3.13), this special case of the double coset formula is a universal example, and the double coset formula holds in general. □

Example 3.18. A special case of the double coset formula is when $K = e$, i.e., when we restrict a transfer all the way to the trivial subgroup of G. In this case there is only one orbit type manifold, namely the entire (double) coset space G/H and the sum in the double coset formula (3.15) is indexed over the path components of G/H. All path component of G/H are homeomorphic, so they have the same Euler characteristic, and all the summands are the same. So the double coset formula becomes

$$\mathrm{res}^G_e \circ \mathrm{tr}^G_H \ = \ \chi(G/H)\cdot \mathrm{res}^H_e \ .$$

(For an orthogonal G-spectrum that is not global, the conjugation action of $\pi_0 G = \pi_0(W_G e)$ on $\pi_0^e(X)$ may be non-trivial, and the formula cannot in general be simplified in the same way).

Example 3.19. We calculate the double coset formula (3.15) for the maximal torus

$$H \ = \ T^2 \ = \ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \ : \ \lambda,\mu \in T \right\}$$

of $G = U(2)$. We take $K = \Sigma_2 \wr T = N_{U(2)}T^2$, the normalizer of the maximal torus, generated by T^2 and the involution $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$; this is Example VI.2 in [**34**]. We calculate the double coset space $\Sigma_2 \wr T\backslash U(2)/T^2$ by identifying the orbit space of $T^2\backslash U(2)/T^2$ and the residual action of the symmetric group Σ_2 on this space. A homeomorphism from $T^2\backslash U(2)/T^2$ to the unit interval $[0,1]$ is induced by

$$h \;:\; U(2) \;\longrightarrow\; [0,1] \;, \quad A \;\longmapsto\; |a_{11}|^2 \;,$$

the square of the length of the upper left entry a_{11} of A. Lengths are non-negative and every column of a unitary matrix is a unit vector, so the map really lands in the interval $[0,1]$. The vector $h(A)$ only depends on the double coset of the matrix A, so the map h factors over the double coset space $T^2\backslash U(2)/T^2$. For every $x \in [0,1]$ the vector $(\sqrt{x}, \sqrt{1-x}) \in \mathbb{C}^2$ has length 1, so it can be complemented to an orthonormal basis, and so it occurs as the first column of a unitary matrix; the map h is thus surjective. On the other hand, if $h(A) = h(B)$ for two unitary matrices A and B, then left multiplication by an element in T^2 makes the first row of B equal to the first row of A. Right multiplication by an element of $1 \times T$ then makes the matrices equal. So A and B represent the same element in the double coset space. The induced map

$$\bar{h} \;:\; T^2\backslash U(2)/T^2 \;\longrightarrow\; [0,1]$$

is thus bijective, hence a homeomorphism. The action of Σ_2 on the orbit space $T^2\backslash U(2)/T^2$ permutes the two rows of a matrix; under the homeomorphism $\bar{h}$, this action thus corresponds to the involution of $[0,1]$ sending x to $1-x$. Altogether this specifies a homeomorphism

$$\Sigma_2 \wr T\backslash U(2)/T^2 \;\cong\; \Sigma_2\backslash[0,1] \;\cong\; [0,1/2]$$

that sends a double coset $(\Sigma_2 \wr T) \cdot A \cdot T^2$ to the minimum of $|a_{11}|^2$ and $|a_{21}|^2$, the squared lengths of the two entries in the first column of A. The orbit type decomposition is as

$$\{0\} \cup (0,1/2) \cup \{1/2\} \;.$$

So as representatives of the orbit types we can choose

$$\begin{aligned} g_0 = 1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for } \{0\}, \\ g_t &= \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{for } (0,1/2), \text{ and} \\ g_1 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{for } \{1/2\}. \end{aligned}$$

For the intersections (which are representatives of the conjugacy classes of those subgroups of $\Sigma_2 \wr T$ with non-empty orbit type manifolds) we get

$$\begin{aligned} (\Sigma_2 \wr T) \cap ({}^{g_0}T^2) \;&=\; T^2 \\ (\Sigma_2 \wr T) \cap ({}^{g_t}T^2) \;&=\; \Delta(T) \\ (\Sigma_2 \wr T) \cap ({}^{g_1}T^2) \;&=\; \Sigma_2 \times \Delta(T) \;. \end{aligned}$$

Here $\Delta(T) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \;:\; \lambda, \mu \in T \right\}$ is the diagonal copy of T. The group $\Delta(T)$ is normal in $\Sigma_2 \wr T$, so it has infinite index in its normalizer, and the corresponding transfer map does not contribute to the double coset formula. The group $\Sigma_2 \times \Delta(T)$ is its own normalizer in $\Sigma_2 \wr T$. The internal Euler characteristics of the orbit type manifolds are

$$\chi^\sharp(\{0\}) \;=\; 1 \;, \quad \chi^\sharp((0,1/2)) \;=\; -1 \quad \text{and} \quad \chi^\sharp(\{1/2\}) \;=\; 1 \;.$$

So the double coset formula (3.15) has two non-trivial summands, and it specializes to

$$\mathrm{res}^{U(2)}_{\Sigma_2 \wr T} \circ \mathrm{tr}^{U(2)}_{T^2} \;=\; \mathrm{tr}^{\Sigma_2 \wr T}_{T^2} \;+\; \mathrm{tr}^{\Sigma_2 \wr T}_{\Sigma_2 \times \Delta(T)} \circ \mathrm{res}^{{}^gT^2}_{\Sigma_2 \times \Delta(T)} \circ c_g^* \;:\; \pi_0^{T^2} X \;\longrightarrow\; \pi_0^{\Sigma_2 \wr T} X \;, \tag{3.20}$$

where $g = g_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Example 3.21. We generalize Example 3.19 and identify the double coset space for the unitary group $G = U(n)$ with respect to the subgroup $H = T \times U(n-1)$ and K the normalizer of the diagonal maximal torus, so that $K \cong \Sigma_n \wr T$. Again we first identify the orbit space of $T^n \backslash U(n)/T \times U(n-1)$ and the residual action of the symmetric group Σ_n on this space. A homeomorphism from this double coset space to the $(n-1)$-simplex

$$\Delta^{n-1} \ = \ \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum x_i = 1\}$$

is induced by

$$h \ : \ U(n) \ \longrightarrow \ \Delta^{n-1} \ , \quad A \ \longmapsto \ (|a_{11}|^2, \dots, |a_{n1}|^2) \ ,$$

the tuple of squares of the lengths of the entries in the first column of A. By arguments as in the previous example the induced map

$$\bar{h} \ : \ T^n \backslash U(n)/(T \times U(n-1)) \ \longrightarrow \ \Delta^{n-1}$$

is bijective, hence a homeomorphism. The action of Σ_n on the orbit space $T^n \backslash U(n)/T \times U(n-1)$ is via left multiplication by permutation matrices, so it permutes the rows of a matrix. Under the homeomorphism $\bar{h}$, this action thus corresponds to the coordinate permutation action on the simplex. A fundamental domain of this Σ_n-action on Δ^{n-1} is any top dimensional simplex of the barycentric subdivision; so the quotient $\Sigma_n \backslash \Delta^{n-1}$ is homeomorphic to the $(n-1)$-simplex

$$B \ = \ \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \geq x_2 \geq \dots \geq x_n \geq 0, \sum x_i = 1\} \ ,$$

the simplex spanned by the vertices $b_p = \frac{1}{p}(1, \dots, 1, 0, \dots, 0)$ for $1 \leq p \leq n$, where b_p has p 1's. Altogether this specifies a homeomorphism

$$\Sigma_n \wr T \backslash U(n)/T \times U(n-1) \ \cong \ \Sigma_n \backslash \Delta^{n-1} \ \cong \ B$$

that sends a double coset $(\Sigma_n \wr T) \cdot A \cdot (T \times U(n-1))$ to the vector of squared lengths of the first column of A, ordered by decreasing size.

Remark 3.22. Theorem 3.5 and the double coset formula (3.15) tell us what data is necessary to specify a global functor $M : \mathbf{A} \longrightarrow \mathcal{A}b$. For this, one needs to give the values $M(G)$ at all compact Lie groups G, restriction maps $\alpha^* : M(G) \longrightarrow M(L)$ for all continuous group homomorphisms $\alpha : L \longrightarrow G$ and transfer maps $\mathrm{tr}_L^K : M(L) \longrightarrow M(K)$ for all subgroup inclusions $L \leq K$. This data has to satisfy the same kind of relations that the restriction and transfer maps for equivariant homotopy groups have to satisfy, namely:

- the restriction maps are contravariantly functorial;
- inner automorphisms induce the identity;
- transfers are transitive and tr_K^K is the identity;
- the transfer tr_L^K is zero if the Weyl group $W_K L$ is infinite;
- transfer along an inclusion $H \leq G$ interacts with restriction along an epimorphism $\alpha : K \longrightarrow G$ according to
$$\alpha^* \circ \mathrm{tr}_H^G \ = \ \mathrm{tr}_L^K \circ (\alpha|_L)^* \ : \ M(H) \ \longrightarrow \ M(K) \ ,$$
where $L = \alpha^{-1}(H)$;
- for all pairs of subgroups H and K of G, the double coset formula (3.15) holds.

This explicit description allows us to relate our notion of global functor to other 'global' versions of Mackey functors, which are typically introduced by specifying generating operations and relations between them. For example, our category of global functors is equivalent to the category of *functors with regular Mackey structure* in the sense of Symonds [**92**, §3]. As we shall explain in the next remark, global functors defined on finite groups are equivalent to *inflation functors* in the sense of Webb [**97**].

Remark 3.23. The full subcategory $\mathbf{A}_{\mathcal{F}in}$ of the Burnside category $\mathbf{A}$ spanned by *finite* groups has a different, more algebraic description, as we shall now recall. This alternative description is often taken as the definition in algebraic treatments of global functors. The category of 'global functors on finite groups', i.e., additive functors from $\mathbf{A}_{\mathcal{F}in}$ to abelian groups, is thus equivalent to the category of *inflation functors* in the sense of [**97**, p.271].

We define the additive *combinatorial Burnside category* A^c. The objects of A^c are all finite groups. The abelian group $A^c(G,K)$ of morphisms from a group G to K is the Grothendieck group of finite K-G-bisets where the right G-action is free. Composition

$$\circ \ : \ A^c(K,L) \times A^c(G,K) \ \longrightarrow \ A^c(G,L)$$

is induced by the balanced product over K, i.e., it is the biadditive extension of

$$(S,T) \ \longmapsto \ S \times_K T \ .$$

Here S has a left L-action and a commuting free right K-action, whereas T has a left K-action and a commuting free right G-action. The balanced product $S \times_K T$ than inherits a left L-action from S and a free right G-action from T. Since the balanced product is associative up to isomorphism, this defines an additive category. In the special case $G = e$ for the trivial group as target we obtain $A^c(e,K) = A(K)$, the Burnside ring of finite K-sets.

We define additive maps

$$\Psi_{K,G} \ : \ \mathbf{A}(G,K) \ \longrightarrow \ A^c(G,K)$$

that will turn out to give an additive equivalence of categories (restricted to finite groups). The map $\Psi_{K,G}$ sends a basis element $[L,\alpha]$ to the class of the K-G-biset

$$K \times_{(L,\alpha)} G \ = \ K \times G/(kl,g) \sim (k, \alpha(l)g)$$

whose right G-action is free. It is straightforward to check that every transitive G-free K-G-biset is isomorphic to one of this form, and that $K \times_{(L,\alpha)} G$ is isomorphic, as a K-G-biset, to $K \times_{(L',\alpha')} G$ if and only if (L,α) is conjugate to (L',α'). So the map $\Psi_{K,G}$ sends the basis of $\mathbf{A}(G,K)$ of Theorem 3.5 to a basis of $A^c(G,K)$, and it is thus an isomorphism.

We claim that the maps $\Psi_{K,G}$ form a functor as G and K vary through all finite groups; this then shows that Ψ is an additive equivalence of categories from the full subcategory of $\mathbf{A}_{\mathcal{F}in}$ to A^c. [...finish...] Since only finite groups are involved, the double coset space $\alpha(L)\backslash G/H$ is a finite set of points and all internal Euler characteristics are 1. The compatibility with compositions thus amounts to the relation

$$\Psi_{G,M}[H,\beta] \circ \Psi_{K,G}[L,\alpha] \ = \sum_{\alpha(L)\backslash G/H} \Psi_{K,M}[\bar{L}, \ \beta \circ c_g \circ i^{\,g}{}^{H}_{\alpha(L)\cap {}^gH} \circ \alpha|_{\bar{L}}] \ .$$

This, in turn, follows from the fact that the K-M-orbits of the K-M-biset

$$(K \times_{(L,\alpha)} G) \times_G (G \times_{(H,\beta)} M) \ \cong \ K \times_{(L,\alpha)} G \times_{(H,\beta)} M$$

are in bijection with the double coset in $\alpha(L)\backslash G/H$, and the orbit indexed by $\alpha(L)gH$ is isomorphic, as a K-M-biset to

$$K \times_{(\bar{L}, \ \beta \circ c_g \circ i^{\,g}{}^{H}_{\alpha(L)\cap {}^gH} \circ \alpha|_{\bar{L}})} M \ .$$

Example 3.24. The presentation of the Burnside category in Theorem 3.5 allows us to give some explicit examples of global functors.

(i) The *Burnside ring global functor* is the representable global functor $\mathbb{A} = \mathbf{A}(e,-)$ of morphisms out of the trivial group e. By Theorem 3.5, the value $\mathbb{A}(K) = \mathbf{A}(e,K)$ at a compact Lie group K is a free abelian group with basis the set of conjugacy classes of subgroups $L \leq K$ with finite Weyl group. When K is finite, then the Weyl group condition is vacuous and $\mathbb{A}(K)$ this is canonically isomorphic to the Burnside ring of K, by sending the operation $[L, p_L] = \mathrm{tr}_K^L \circ p_L^* \in \mathbb{A}(K)$ to the class of the K-set K/L (where $p_L : L \longrightarrow e$ is the unique homomorphism). As we discussed in Example 3.14, the Burnside ring global functor $\mathbf{A}(e,-)$ is realized by the sphere spectrum $\mathbb{S}$. More generally, the representable functors $\mathbf{A}(G,-)$ are other examples of global functors, and we have seen in Proposition 3.13 that the represented global functor $\mathbf{A}(G,-)$ is realized by the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ of the compact Lie group G.

(ii) Given an abelian group M, the *constant global functor* $\underline{M}$ is given by $\underline{M}(G) = M$ and all restriction maps are identity maps. The transfer $\mathrm{tr}_H^G : \underline{M}(H) \longrightarrow \underline{M}(G)$ is multiplication by the Euler characteristic of the homogeneous space G/H. In particular, if H is a subgroup of finite index of G, then tr_H^G is multiplication by the index $[G : H]$.

There is a well-known point set level model of an Eilenberg-Mac Lane spectrum HM, that we recall in Section V.5 below. However, contrary to what one may expect, the 0-th homotopy group global functor $\underline{\pi}_0(HM)$ is *not* isomorphic to the constant global functor $\underline{M}$. More precisely, the restriction map $p_G^* : \pi_0^e(HM) \longrightarrow \pi_0^G(HM)$ is an isomorphism for finite groups G, but not for general compact Lie groups, see Proposition V.5.5.

(iii) The *representation ring global functor* $\mathbf{R}$ assigns to a compact Lie group G the complex representation ring $\mathbf{R}(G)$, i.e., the Grothendieck group of finite dimensional complex G-representations, with product induced by tensor product of representations. The restriction maps $\alpha^* : \mathbf{R}(G) \longrightarrow \mathbf{R}(K)$ are induced by restriction of representations along a continuous homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [**80**, § 2]. If H is a subgroup of finite index of G, then this induction sends the class of an H-representations to the induced G-representation $\mathrm{map}^G(H, V)$ (which is then isomorphic to $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$); in general, induction sends actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for $\mathbf{R}$ was proved by Snaith [**87**, Thm. 2.4]. We look more closely at the representation global functor in Remark V.3.11 in the section devoted to global K-theory. The global functor $\mathbf{R}$ ought to be realized by the global K-theory spectrum $\mathbf{KU}$ (see Construction V.6.34), but we have not verified that.

(iv) Given any generalized cohomology theory E (in the non-equivariant sense), we can define a global functor $\underline{E}$ by setting

$$\underline{E}(G) \;=\; E^0(BG) \;,$$

the 0-th E-cohomology of the classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer maps for a subgroup inclusion $H \leq G$ comes from the stable transfer map

$$\Sigma_+^\infty BG \;\longrightarrow\; \Sigma_+^\infty BH \;.$$

The double coset formula was proved in this context by Feshbach [**34**, Thm. II.11]. We will show in Proposition III.7.3. that the global functor $\underline{E}$ is realized by an orthogonal spectrum RE.

(v) The Borel cohomology construction of part (iv) has a 'relative' generalization where we start with compact Lie group K and a generalized K-equivariant cohomology theory Z (represented by an orthogonal K-spectrum Z) – the previous paragraph is the special case where K is a trivial group. Then we can define a global functor $\underline{Z}$ by setting

$$\underline{Z}(G) \;=\; Z^0(B(K, G)) \;,$$

the 0-th K-equivariant Z-cohomology of a classifying K-space $B(K, G)$ for (K, G)-bundles. The double coset formula in this context is due to Lewis and May [**53**, IV Cor. 6.4]. We will show in Proposition IV.6.12 that this global functor $\underline{Z}$ is realized by an orthogonal spectrum RZ, where R is a right adjoint of the derived forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$ from the global stable homotopy category to the G-equivariant stable homotopy category.

Remark 3.25. If we fix a group G and let H run through all subgroups of a compact Lie group G, then the collection H-equivariant homotopy groups $\pi_0^H(X)$ of an orthogonal spectrum X forms a *Mackey functor* for the group G, with respect to the restriction to subgroups, conjugation and transfer maps. However, the G-Mackey functors that arise this way are special, and not all G-Mackey functors arise this way. because the action of the Weyl group of H on $\pi_0^H(X)$ factors through the outer automorphism group of H. For example, the values $\pi_0^e(X)$ for the trivial group have no additional action. To illustrate this we compare Mackey functors for the group $C_3 = \{1, \tau, \tau^2\}$ with three elements to additive functors on the full subcategory of **A** spanned by the group e and C_3. Generating operations can be displayed as follows:

$$\begin{tikzcd} F(C_3) \arrow[loop above, "\alpha^*"] \arrow[d, "\mathrm{res}"] \\ F(e) \arrow[u, bend left=60, "p^*"] \arrow[u, bend right=60, "\mathrm{tr}"'] \end{tikzcd} \qquad\qquad \begin{tikzcd} F(C_3) \arrow[d, shift right, "\mathrm{res}"'] \\ F(e) \arrow[u, shift right, "\mathrm{tr}"'] \arrow[loop below, "\tau"] \end{tikzcd}$$

global functor on e and C_3 $\qquad\qquad$ C_3-Mackey functor

Here $\mathrm{res} = \mathrm{res}_e^{C_3}$ and $\mathrm{tr} = \mathrm{tr}_e^{C_3}$ are the restriction and transfer maps that are present in both cases. A global functor also comes with restriction maps along the epimorphism $p : C_3 \longrightarrow e$ and along the automorphism $\alpha : C_3 \longrightarrow C_3$ with $\alpha(\tau) = \tau^2$, and the relations are

$$\mathrm{res} \circ p^* \ = \ \mathrm{Id}_{F(e)} \qquad \text{and} \qquad \mathrm{res} \circ \mathrm{tr} \ = \ 3 \cdot \mathrm{Id}_{F(e)}$$

as well as $\alpha^* \circ \alpha^* = \mathrm{Id}_{F(C_3)}$, $\alpha^* \circ p^* = p^*$, $\mathrm{res} \circ \alpha^* = \mathrm{res}$ and $\alpha^* \circ \mathrm{tr} = \mathrm{tr}$. In contrast, C_3-Mackey functors have an additional action of C_3 (the Weyl group of e in C_3) on $F(e)$;

$$\mathrm{res} \circ \mathrm{tr} \ = \ \mathrm{Id}_{F(e)} + \tau + \tau^2 \ .$$

4. Products

In this section we recall the smash product of orthogonal spectra and study the products on homotopy groups and global functors that it gives rise to.

The tensor product of abelian groups is the universal target for a biadditive map. The smash product of orthogonal spectra has an analogous universal property that we now discuss. We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of orthogonal spectra (X, Y) to an orthogonal spectrum Z as a collection of based $O(p) \times O(q)$-equivariant maps

$$b_{p,q} \ : \ X_p \wedge Y_q \ \longrightarrow \ Z_{p+q}$$

for $p, q \geq 0$, such that the *bilinearity diagram*

$$\begin{tikzcd} & X_p \wedge Y_q \wedge S^1 \arrow[r, "X_p \wedge \mathrm{twist}"] \arrow[dl, "X_p \wedge \sigma_q"'] \arrow[d, "b_{p,q} \wedge S^1"] & X_p \wedge S^1 \wedge Y_q \arrow[d, "\sigma_p \wedge Y_q"] \\ X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge S^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\ & Z_{p+q+1} & Z_{p+1+q} \arrow[l, "1 \times \chi_{1,q}"] \end{tikzcd}$$

commutes for all $p, q \geq 0$.

We can then define a smash product of X and Y as a universal example of an orthogonal spectrum with a bimorphism from X and Y. More precisely, a smash product for X and Y is a pair $(X \wedge Y, i)$ consisting of an orthogonal spectrum $X \wedge Y$ and a universal bimorphism $i : (X, Y) \longrightarrow X \wedge Y$, i.e., a bimorphism such that for every orthogonal spectrum Z the map

$$\mathcal{S}p(X \wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q} \tag{4.1}$$

is bijective.

We have to show that a universal bimorphism out of any pair of orthogonal spectra exists; in other words: we have to construct a smash product $X \wedge Y$ from two given orthogonal spectra X and Y. We want $X \wedge Y$ to be the universal recipient of a bimorphism from (X, Y), and this pretty much tells us what we have to do. For $n \geq 0$ we define the n-th level $(X \wedge Y)_n$ as the coequalizer, in the category of pointed $O(n)$-spaces, of two maps

$$\alpha_X,\, \alpha_Y \ : \ \bigvee_{p+1+q=n} O(n)^+ \wedge_{O(p)\times 1\times O(q)} X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ \bigvee_{p+q=n} O(n)^+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \ .$$

The wedges run over all non-negative values of p and q which satisfy the indicated relations. The map α_X takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p + 1, q)$ using the map

$$\sigma_p^X \wedge \mathrm{Id} \ : \ X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ X_{p+1} \wedge Y_q$$

and inducing up. The other map α_Y takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p, 1 + q)$ using the composite

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{Id}\wedge \mathrm{twist}} X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{Id}\wedge \sigma_q^Y} X_p \wedge Y_{q+1} \xrightarrow{\mathrm{Id}\wedge \chi_{q,1}} X_p \wedge Y_{1+q}$$

and inducing up.

The structure map $(X \wedge Y)_n \wedge S^1 \longrightarrow (X \wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$O(n)^+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \wedge S^1 \ \longrightarrow \ O(n + 1)^+ \wedge_{O(p)\times O(q+1)} X_p \wedge Y_{q+1}$$

induced from $\mathrm{Id}\wedge\sigma_q^Y : X_p \wedge Y_q \wedge S^1 \longrightarrow X_p \wedge Y_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by moving S^1 past Y_q, using the structure map of X (instead of that of Y) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X \wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \xrightarrow{x\wedge y\mapsto 1\wedge x\wedge y} \bigvee_{p+q=n} O(n)^+ \wedge_{O(p)\times O(q)} X_p \wedge Y_q \xrightarrow{\text{projection}} (X \wedge Y)_{p+q}$$

forms a bimorphism – and in fact a universal one.

Very often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (4.1) as the *universal property* of the smash product of orthogonal spectra.

The smash product $X \wedge Y$ is a functor in both variables. It is also symmetric monoidal, i.e., there are natural associativity respectively symmetry isomorphisms

$$(X \wedge Y) \wedge Z \ \longrightarrow \ X \wedge (Y \wedge Z) \qquad \text{respectively} \qquad X \wedge Y \ \longrightarrow \ Y \wedge X$$

and unit isomorphisms $\mathbb{S} \wedge X \cong X \cong X \wedge \mathbb{S}$. The upshot is that the associativity and symmetry isomorphisms make the smash product of orthogonal spectra into a symmetric monoidal product with the sphere spectrum $\mathbb{S}$ as unit object.

The smash product of orthogonal spectra is *closed* symmetric monoidal in the sense that the smash product is adjoint to an internal Hom spectrum. We recall the construction i.e., there is an adjunction isomorphism

$$\mathrm{Hom}(X \wedge Y, Z) \ \cong \ \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \ .$$

For an orthogonal spectrum X and $n \geq 0$ we define the *n-th shift* $\mathrm{sh}^n X$ of X by

$$(\mathrm{sh}^n X)_m \ = \ X_{n+m}$$

The orthogonal group $O(m)$ acts through the monomorphism $\mathrm{Id} \oplus - : O(m) \longrightarrow O(n+m)$. The m-th structure map of $\mathrm{sh}^n X$ is the $(n+m)$-th structure map of X. The orthogonal spectrum $\mathrm{sh}^n X$ comes with a natural left $O(n)$-action by restriction along the monomorphism $- \oplus \mathrm{Id} : O(n) \longrightarrow O(n+m)$.

For orthogonal spectra X and Y we define an orthogonal spectrum $\mathrm{Hom}(X, Y)$ in level n by

$$\mathrm{Hom}(X,Y)_n \;=\; \mathrm{map}(X, \mathrm{sh}^n Y) \tag{4.2}$$

with left $O(n)$-action through the $O(n)$-action on $\mathrm{sh}^n Y$. The structure map $\sigma_n : \mathrm{Hom}(X,Y)_n \wedge S^1 \longrightarrow \mathrm{Hom}(X,Y)_{n+1}$ is the composite

$$\mathrm{map}(X, \mathrm{sh}^n Y) \wedge S^1 \xrightarrow{\text{assembly}} \mathrm{map}(X, S^1 \wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X, \lambda_{\mathrm{sh}^n Y})} \mathrm{map}(X, \mathrm{sh}^{n+1} Y)\ ;$$

here the first map is of 'assembly type', i.e., it takes $f \wedge t$ to the map which sends $x \in X$ to $t \wedge f(x)$ (for $f : X \longrightarrow \mathrm{sh}^n Y$ and $t \in S^1$), and $\lambda_{\mathrm{sh}^n Y} : S^1 \wedge \mathrm{sh}^n Y \longrightarrow \mathrm{sh}(\mathrm{sh}^n Y) = \mathrm{sh}^{n+1} Y$ is the natural morphism defined in (2.25).

In order to verify that this indeed gives an orthogonal spectrum we describe the iterated structure map. Let us denote by $\lambda_Y^{(m)} : S^m \wedge Y \longrightarrow \mathrm{sh}^m Y$ the morphism (2.25) for $V = \mathbb{R}^m$. Then for all $k, m \geq 0$ the diagram

$$\begin{tikzcd}
S^k \wedge S^m \wedge Y \arrow[r, "\mathrm{Id} \wedge \lambda_Y^{(m)}"] \arrow[d, "\cong"'] & S^k \wedge \mathrm{sh}^m Y \arrow[r, "\lambda^{(k)}_{\mathrm{sh}^m Y}"] & \mathrm{sh}^k(\mathrm{sh}^m Y) \arrow[d, equal] \\
S^{k+m} \wedge Y \arrow[r, "\chi_{k,m} \wedge \mathrm{Id}"'] & S^{m+k} \wedge Y \arrow[r, "\lambda_Y^{(m+k)}"'] & \mathrm{sh}^{m+k} Y
\end{tikzcd}$$

commutes. This implies that the iterated structure map of the spectrum $\mathrm{Hom}(X, Y)$ equals the composite

$$\mathrm{map}(X, \mathrm{sh}^n Y) \wedge S^m \xrightarrow{\text{assembly}} \mathrm{map}(X, S^m \wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X, \lambda^{(m)}_{\mathrm{sh}^n Y})} \mathrm{map}(X, \mathrm{sh}^{n+m} Y)$$

and is thus $O(n) \times O(m)$-equivariant. The first map is again of 'assembly type', i.e., for $f : X \longrightarrow \mathrm{sh}^n Y$ and $t \in S^m$ it takes $f \wedge t$ to the map which sends $x \in X$ to $t \wedge f(x)$.

Taking function spectrum commutes with shifting in the second variable, i.e.,

$$\mathrm{Hom}(X, \mathrm{sh}^m Y) \;=\; \mathrm{sh}^m\left(\mathrm{Hom}(X,Y)\right)$$

(with equality, not just isomorphism). Indeed, in level n we have

$$\begin{aligned}
\mathrm{Hom}(X, \mathrm{sh}^m Y)_n \;&=\; \mathrm{map}(X, \mathrm{sh}^n(\mathrm{sh}^m Y)) \;=\; \mathrm{map}(X, \mathrm{sh}^{m+n} Y) \\
&=\; \mathrm{Hom}(X,Y)_{m+n} \;=\; \left(\mathrm{sh}^m \mathrm{Hom}(X,Y)\right)_n\ .
\end{aligned}$$

The orthogonal group actions and structure maps coincide as well.

The internal function spectrum functor $\mathrm{Hom}(X, -)$ is right adjoint to the smash product $- \wedge X$ of orthogonal spectra. A natural isomorphism of orthogonal spectra $\mathrm{Hom}(F_m, Y) \cong \mathrm{sh}^m Y$ is given at level n by

$$\mathrm{Hom}(F_m, Y)_n = \mathrm{map}(F_m, \mathrm{sh}^n Y) \;\cong\; (\mathrm{sh}^n Y)_m \;=\; Y_{n+m} \xrightarrow{\chi_{n,m}} Y_{m+n} = (\mathrm{sh}^m Y)_n$$

where the second map is the adjunction bijection. This isomorphism is equivariant for the left actions of $O(m)$ induced on the source from the right $O(m)$-action on a free spectrum. In the special case $V = 0$ we have $F_0 S^0 = \mathbb{S}$, which gives a natural isomorphism of orthogonal spectra $\mathrm{Hom}(\mathbb{S}, Y) \cong Y$.

Remark 4.3. In the coordinate free picture, bimorphisms corresponds to a [continuous ?] natural transformation

$$b \;:\; X \triangle Y \;\longrightarrow\; Z \circ \oplus$$

of continuous functors from the product category $\mathbf{O} \times \mathbf{O}$ to spaces. Here $X \triangle Y$ is the external smash product, i.e., the composite

$$\mathbf{O} \times \mathbf{O} \xrightarrow{X \wedge Y} \mathbf{T} \times \mathbf{T} \xrightarrow{\wedge} \mathbf{T}$$

and $\oplus : \mathbf{O} \times \mathbf{O} \longrightarrow \mathbf{O}$ is the symmetric monoidal product on the Thom space category given on objects by the direct sum of inner product spaces and on morphisms by the Thomification of the vector bundle map

$$\oplus \ : \ \xi(V, \bar{V}) \times \xi(W, \bar{W}) \ \longrightarrow \ \xi(V \oplus W, \bar{V} \oplus \bar{W}) \ , \quad ((\varphi, v), (\psi, w)) \ \longmapsto \ (\varphi \oplus \psi, v \oplus w) \ .$$

A bimorphism $b : (X, Y) \longrightarrow Z$ extends to $O(V) \times O(W)$-equivariant maps

$$b_{V,W} \ : \ X(V) \wedge Y(W) \ \longrightarrow \ Z(V \oplus W)$$

for all finite dimensional real inner product space V and W, such that the

$$\begin{tikzcd}
\mathbf{O}(V,\bar{V}) \wedge \mathbf{O}(W,\bar{W}) \wedge X(V) \wedge Y(W) \arrow[rr, "\oplus \wedge b_{V,W}"] \arrow[d, "\mathrm{Id} \wedge \tau \wedge \mathrm{Id}"'] & & \mathbf{O}(V \oplus W, \bar{V} \oplus \bar{W}) \wedge Z(V \oplus W) \arrow[dd, "\circ"] \\
\mathbf{O}(V,\bar{V}) \wedge X(V) \wedge \mathbf{O}(W,\bar{W}) \wedge Y(W) \arrow[d, "\circ \wedge \circ"'] & & \\
X(\bar{V}) \wedge Y(\bar{W}) \arrow[rr, "b_{\bar{V},\bar{W}}"'] & & Z(\bar{V} \oplus \bar{W})
\end{tikzcd}$$

Construction 2.3 we associated an orthogonal space $\Omega^\bullet X$ to every orthogonal spectrum X. This functor is compatible with the smash product of orthogonal spectra and the box product of orthogonal spaces, in the sense of a lax symmetric monoidal map

$$(\Omega^\bullet X) \boxtimes (\Omega^\bullet Y) \ \longrightarrow \ \Omega^\bullet (X \wedge Y) \tag{4.4}$$

that we review now. This morphism of orthogonal spaces is associated to a bimorphism from $(\Omega^\bullet X, \Omega^\bullet Y)$ to $\Omega^\bullet (X \wedge Y)$ with (V, W)-component the composite

$$\begin{aligned} \mathrm{map}(S^V, X(V)) \times \mathrm{map}(S^W, Y(W)) \ &\xrightarrow{\wedge} \ \mathrm{map}(S^{V \oplus W}, X(V) \wedge Y(W)) \\ &\xrightarrow{\mathrm{map}(S^{V \oplus W}, i_{V,W})} \ \mathrm{map}(S^{V \oplus W}, (X \wedge Y)(V \oplus W)) \ . \end{aligned}$$

The morphism is unital, associative and symmetric.

Construction 4.5. Given two orthogonal spectra X and Y, we endow the global homotopy functors with an external pairing

$$\times \ : \ \pi_k^G(X) \ \times \ \pi_l^K(Y) \ \longrightarrow \ \pi_{k+l}^{G \times K}(X \wedge Y) \ , \tag{4.6}$$

where G and K are compact Lie groups. For $k = l = 0$ the pairing is defined as the composite

$$\begin{aligned} \pi_0^G(X) \times \pi_0^K(Y) \ = \ \pi_0^G(\Omega^\bullet X) \times \pi_0^K(\Omega^\bullet Y) \ &\xrightarrow{\times} \ \pi_0^{G \times K}(\Omega^\bullet X \boxtimes \Omega^\bullet Y)) \\ &\xrightarrow{(4.4)} \ \pi_0^{G \times K}(\Omega^\bullet (X \wedge Y)) \ = \ \pi_0^{G \times K}(X \wedge Y) \ . \end{aligned}$$

We can unravel this definition to arrive at the following explicit recipe: we let $f : S^V \longrightarrow X(V)$ and $g : S^W \longrightarrow Y(W)$ represent classes in $\pi_0^G(X)$ respectively $\pi_0^K(Y)$, where V and W are representations of G respectively K. We view $V \oplus W$ as a representation of the product group $G \times K$ via $(g, k) \cdot (v, w) = (gv, kw)$. The class $[f] \times [g]$ in $\pi_0^{G \times K}(X \wedge Y)$ is then represented by the $(G \times K)$-equivariant composite

$$S^{V+W} \cong S^V \wedge S^W \ \xrightarrow{f \wedge g} \ X(V) \wedge Y(W) \ \xrightarrow{i_{V,W}} \ (X \wedge Y)(V \oplus W) \ .$$

The pairing of equivariant homotopy groups has several expected properties that we summarize in the next proposition.

Theorem 4.7. *Let G, K and L be compact Lie groups and X, Y and Z orthogonal spectra.*

(i) (Biadditivity) *The product $\times : \pi_k^G(X) \times \pi_l^K(Y) \longrightarrow \pi_{k+l}^{G\times K}(X \wedge Y)$ is biadditive.*

(ii) (Unitality) *Let $1 \in \pi_0^e(\mathbb{S})$ denote the class represented by the identity of S^V for any inner product space V. The product is unital in the sense that $1 \times x = x = x \times 1$ under the identifications $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$ and $e \times G \cong G \cong G \times e$.*

(iii) (Associativity) *For all classes $x \in \pi_k^G(X)$, $y \in \pi_l^K(Y)$ and $z \in \pi_j^L(Z)$ the relation*

$$x \times (y \times z) \ = \ (x \times y) \times z$$

hold in $\pi_{k+l+j}^{G\times K\times L}(X \wedge Y \wedge Z)$.

(iv) (Commutativity) *For all classes $x \in \pi_k^G(X)$ and $y \in \pi_l^K(Y)$ the relation*

$$y \times x \ = \ (-1)^{kl} \cdot \tau_{G,K}^*(\tau_*^{X,Y}(x \times y))$$

holds in $\pi_{l+k}^{K\times G}(Y \wedge X)$, where $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetry isomorphism of the smash product and $\tau_{G,K} : K \times G \longrightarrow G \times K$ interchanges the factors.

(v) (Restriction) *For all classes $x \in \pi_k^G(X)$ and $y \in \pi_l^K(Y)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$\alpha^*(x) \times \beta^*(y) \ = \ (\alpha \times \beta)^*(x \times y)$$

holds in $\pi_{k+l}^{\bar{G}\times\bar{K}}(X \wedge Y)$.

(vi) (Transfer) *For all subgroup inclusions $H \leq G$ and $L \leq K$ the square*

$$\begin{tikzcd}
\pi_k^H(X) \times \pi_l^L(Y) \arrow[r, "\times"] \arrow[d, "\mathrm{tr}_H^G \times \mathrm{tr}_L^K"'] & \pi_{k+l}^{H\times L}(X \wedge Y) \arrow[d, "\mathrm{tr}_{H\times L}^{G\times K}"] \\
\pi_k^G(X) \times \pi_l^K(Y) \arrow[r, "\times"'] & \pi_{k+l}^{G\times K}(X \wedge Y)
\end{tikzcd}$$

commutes.

Proof. The associativity property (iii) and compatibility with restriction (v) are straightforward from the definitions.

(iv) Indeed, for representing G-maps f and g as above the diagram

$$\begin{tikzcd}
S^{k+l+V+W} \arrow[r, "\mathrm{Id} \wedge \tau_{l,V} \wedge \mathrm{Id}"] \arrow[d, "\chi_{k,l} \times \chi_{V,W}"'] & S^{k+V} \wedge S^{l+W} \arrow[r, "f \wedge g"] \arrow[d, "\mathrm{twist}"'] & X(V) \wedge Y(W) \arrow[r, "i_{V,W}"] \arrow[d, "\mathrm{twist}"'] & (X \wedge Y)(V+W) \arrow[d, "\tau_{X,Y} \circ (\chi_{V,W})_*"] \\
S^{l+k+W+V} \arrow[r, "\mathrm{Id} \wedge \chi_{k,W} \wedge \mathrm{Id}"'] & S^{l+W} \wedge S^{k+V} \arrow[r, "g \wedge f"'] & Y(W) \wedge X(V) \arrow[r, "i_{W,V}"'] & (Y \wedge X)(W+V)
\end{tikzcd}$$

commutes. The two vertical coordinate permutations induce the signs $(-1)^{kl+nm}$ respectively (after one suspension) $(-1)^{nm}$ on homotopy groups. Since the upper horizontal composite represents $x \times y$ and the lower composite represents $y \times x$, this proves the relation $x \times y = (-1)^{kl} y \times x$.

The unitality (ii) is a consequence that $f \times S^V = f \diamond V$ is the stabilization of f by V, and $\langle f \rangle \times 1 = \langle f \times S^V \rangle = \langle f \diamond V \rangle = \langle f \rangle$. Commutativity (iv) gives the other unit relation.

(i) and (vi) For the biadditivity and transfer property we fix a G-map $f : S^V \longrightarrow X(V)$ that represents an element in $\pi_0^G X$. This gives rise to a morphism of orthogonal spectra

$$\hat{f} \ : \ Y \ \longrightarrow \ \Omega^V \operatorname{sh}^V(X \wedge Y) \ ,$$

defined on W as the adjoint of the composite

$$\hat{f}(W) \ : \ S^V \wedge Y(W) \xrightarrow{f\wedge Y(W)} X(V) \wedge Y(W) \xrightarrow{i_{V,W}} (X \wedge Y)(V \oplus W) \ = (\mathrm{sh}^V(X \wedge Y))(W) \ .$$

Then by definition the composite

$$\pi_l^K Y \xrightarrow{\hat{f}_*} \pi_l^K(\Omega^V \mathrm{sh}^V(X \wedge Y)) \cong \pi_l^{G\times K}(X \wedge Y)$$

is multiplication by the class $\langle f \rangle$. The map $\hat{f}_*$ is additive and compatible with transfers since it arises from a homomorphism of orthogonal spectra. The isomorphism between $\pi_l^K(F^G(X \wedge Y))$ and $\pi_l^{G\times K}(X \wedge Y)$ is additive and compatible with transfers by Proposition V.4.6. To see that $-\times y : \pi_k^G(X) \longrightarrow\cong \pi_{k+l}^{G\times K}(X\wedge Y)$ is additive and compatible with transfers we can either run an analogous argument or use commutativity (iv) to deduce it from the previous paragraph. □

Remark 4.8. By taking $G = K$ and restricting along the diagonal embedding $\Delta : G \longrightarrow G \times G$ we obtain an internal product as the composite

$$\pi_k^G(X) \times \pi_l^G(Y) \xrightarrow{\times} \pi_{k+l}^{G\times G}(X \wedge Y) \xrightarrow{\Delta^*} \pi_{k+l}^G(X \wedge Y) \ .$$

This composite satisfies a reciprocity law with respect to the transfer maps, i.e.,

$$\begin{aligned} \mathrm{tr}_H^G(x) \cdot y \ &= \ \Delta^*(\mathrm{tr}_H^G(x) \times y) \ = \ \Delta^*(\mathrm{tr}_{H\times G}^{G\times G}(x \times y)) \\ &= \ \mathrm{tr}_H^G(\Delta_H^*(\mathrm{res}^{H\times G}_{\Delta(G)\cap(H\times G)}(x \times y))) \ = \ \mathrm{tr}_H^G(x \cdot \mathrm{res}_H^G(y)) \end{aligned}$$

The third equation is the double coset formula for the diagonal subgroup $\Delta(G)$ and $H \times G$ inside the group $G \times G$, using that the double coset space $\Delta(G)\backslash G \times G/G \times H$ is a single point. Moreover, $\Delta_H : H \longrightarrow \Delta(G) \cap (H \times G)$ is the diagonal identification.

Definition 4.9. An *orthogonal ring spectrum* R consists of the following data:

- a sequence of pointed spaces R_n for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on R_n for each $n \geq 0$
- $O(n) \times O(m)$-equivariant *multiplication maps* $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ for $n, m \geq 0$, and
- $O(n)$-equivariant *unit maps* $\iota_n : S^n \longrightarrow R_n$ for all $n \geq 0$.

This data is subject to the following conditions:

(Associativity) The square

$$\begin{tikzcd} R_n \wedge R_m \wedge R_p \arrow[r, "\mathrm{Id}\wedge\mu_{m,p}"] \arrow[d, "\mu_{n,m}\wedge\mathrm{Id}"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\ R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"'] & R_{n+m+p} \end{tikzcd}$$

commutes for all $n, m, p \geq 0$.

(Unit) The two composites

$$R_n \cong R_n \wedge S^0 \xrightarrow{R_n\wedge\iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \cong S^0 \wedge R_n \xrightarrow{\iota_0\wedge R_n} R_0 \wedge R_n \xrightarrow{\mu_{0,n}} R_n$$

are the identity for all $n \geq 0$.

(Multiplicativity) The composite

$$S^{n+m} \ \cong \ S^n \wedge S^m \xrightarrow{\iota_n\wedge\iota_m} R_n \wedge R_m \xrightarrow{\mu_{n,m}} R_{n+m}$$

equals the unit map $\iota_{n+m} : S^{n+m} \longrightarrow R_{n+m}$. (where the first map is the canonical homeomorphism sending $(x, y) \in S^{n+m}$ to $x \wedge y$ in $S^n \wedge S^m$).

(Centrality) The diagrams

$$\begin{tikzcd} R_m \wedge S^n \arrow[r, "R_m \wedge \iota_n"] \arrow[d, "\text{twist}"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"] & R_{m+n} \arrow[d, "\chi_{m,n}"] \\ S^n \wedge R_m \arrow[r, "\iota_n \wedge R_m"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

commutes for all $m, n \geq 0$. Here $\chi_{m,n} \in O(m+n)$ denotes the permutation matrix of the shuffle permutation which moves the first m elements past the last n elements, keeping each of the two blocks in order; in formulas,

$$\chi_{m,n}(i) = \begin{cases} i+n & \text{for } 1 \leq i \leq m, \\ i-m & \text{for } m+1 \leq i \leq m+n. \end{cases} \tag{4.10}$$

An orthogonal ring spectrum R is *commutative* if the square

$$\begin{tikzcd} R_m \wedge R_n \arrow[r, "\mu_{m,n}"] \arrow[d, "\text{twist}"'] & R_{m+n} \arrow[d, "\chi_{m,n}"] \\ R_n \wedge R_m \arrow[r, "\mu_{n,m}"'] & R_{n+m} \end{tikzcd}$$

commutes for all $m, n \geq 0$. Note that this commutativity diagram implies the centrality condition above.

A *morphism* $f : R \longrightarrow S$ of orthogonal ring spectra consists of $O(n)$-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps in the sense that $f_{n+m}\mu_{n,m} = \mu_{n,m}(f_n \wedge f_m)$ and $f_n\iota_n = \iota_n$.

Commutative orthogonal ring spectra already appear, with an extra pointset topological hypothesis and under the name $\mathscr{I}_*$-prefunctor, in [**59**, IV Def. 2.1].

Remark 4.11. (i) The higher-dimensional unit maps $\iota_n : S^n \longrightarrow R_n$ for $n \geq 2$ are determined by the unit map $\iota_1 : S^1 \longrightarrow R_1$ and the multiplication as the composite

$$S^n = S^1 \wedge \ldots \wedge S^1 \xrightarrow{\iota_1 \wedge \ldots \wedge \iota_1} R_1 \wedge \ldots \wedge R_1 \xrightarrow{\mu_{1,\ldots,1}} R_n .$$

The centrality condition implies that this map is Σ_n-equivariant, but we require that ι_n is even $O(n)$-equivariant.

(ii) As the terminology suggests, the orthogonal ring spectrum R has an underlying orthogonal spectrum. We keep the spaces R_n and orthogonal group actions and define the missing structure maps $\sigma_n : R_n \wedge S^1 \longrightarrow R_{n+1}$ as the composite $\mu_{n,1} \circ (R_n \wedge \iota_1)$. Associativity implies that the iterated structure map $\sigma^m : R_n \wedge S^m \longrightarrow R_{n+m}$ equals the composite

$$R_n \wedge S^m \xrightarrow{R_n \wedge \iota_m} R_n \wedge R_m \xrightarrow{\mu_{n,m}} R_{n+m} .$$

So the iterated structure map is $O(n) \times O(m)$-equivariant, and we have in fact obtained an orthogonal spectrum.

Remark 4.12. Orthogonal ring spectra are 'the same' as monoid objects in the symmetric monoidal category of orthogonal spectra with respect to the smash product. In more detail, the 'explicit' definition of an orthogonal ring spectrum which we just gave encodes the same data as the 'implicit' definition of an orthogonal spectrum R together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ (where $\mathbb{S}$ is the sphere spectrum) which are suitably associative and unital. The 'explicit' and 'implicit' definitions of orthogonal ring spectra coincide in the sense that they define isomorphic categories.

The monoidal morphism (4.4) is unital, associative and symmetric. In particular the orthogonal space $\Omega^\bullet R$ associated to an orthogonal ring spectrum becomes an orthogonal monoid space via the composite

$$(\Omega^\bullet R) \boxtimes (\Omega^\bullet R) \longrightarrow \Omega^\bullet(R \wedge R) \xrightarrow{\Omega^\bullet \mu} \Omega^\bullet R ,$$

and this passage respects commutativity of multiplications. The bimorphism corresponding to the induced product on $\Omega^\bullet R$ thus has as (V,W)-component the composite

$$\begin{aligned} \mathrm{map}(S^V, R(V)) \times \mathrm{map}(S^W, R(W)) &\xrightarrow{-\wedge-} \mathrm{map}(S^{V\oplus W}, R(V) \wedge R(W)) \\ &\xrightarrow{\mathrm{map}(S^{V\oplus W}, \mu_{V,W})} \mathrm{map}(S^{V\oplus W}, R(V \oplus W)) . \end{aligned}$$

Example 4.13 (Ring spectra from orthogonal monoid spaces)**.** The suspension spectrum functor (see Construction 3.6) takes the box product of orthogonal spaces to the smash product of orthogonal spectra. In more detail: we let V and W be inner product spaces. Then the maps

$$\begin{aligned} (\Sigma^\infty_+ X)(V) \wedge (\Sigma^\infty_+ Y)(W) \ &= \ (X(V)_+ \wedge S^V) \wedge (Y(W)_+ \wedge S^W) \ \cong \ (X(V) \times Y(W))_+ \wedge S^{V\oplus W} \\ &\xrightarrow{i_{X,Y} \wedge S^{V\oplus W}} (X \boxtimes Y)(V \oplus W)_+ \wedge S^{V\oplus W} \ = \ (\Sigma^\infty_+ (X \boxtimes Y))(V \oplus W) \end{aligned}$$

form a bimorphism, so they correspond to a morphism of orthogonal spectra

$$(\Sigma^\infty_+ X) \wedge (\Sigma^\infty_+ Y) \longrightarrow \Sigma^\infty_+ (X \boxtimes Y) .$$

These morphisms are isomorphisms, and together with the isomorphism $\Sigma^\infty_+ * \cong \mathbb{S}$ adjoint to the unique based isomorphism $*_+ \cong S^0$, this makes Σ^∞_+ into a strong symmetric monoidal functor.

We can thus produce many examples of orthogonal ring spectra as the suspension spectra of orthogonal monoid spaces. We consider an orthogonal monoid space R. Then the suspension spectrum $\Sigma^\infty_+ R$ of the underlying orthogonal space becomes an orthogonal ring spectrum via the multiplication map

$$(\Sigma^\infty_+ R) \wedge (\Sigma^\infty_+ R) \ \cong \ \Sigma^\infty_+ (R \boxtimes R) \xrightarrow{\Sigma^\infty_+ \mu_R} \Sigma^\infty_+ R .$$

We have $(\Sigma^\infty_+ R)^{\mathrm{op}} = \Sigma^\infty_+ (R^{\mathrm{op}})$. If the multiplication of R is commutative, then so is the resulting multiplication on $\Sigma^\infty_+ R$.

This construction contains spherical monoid ring spectra: if M is a topological monoid, then the constant orthogonal space with value M inherits an associative and unital product from M which is commutative if M is. The suspension spectrum with such a constant multiplicative functor is the monoid ring spectrum $\mathbb{S}[M]$.

Given an orthogonal ring spectrum R and compact Lie groups G and K, we define an internal pairing on the equivariant homotopy groups of R as the composite

$$\pi^G_k(R) \times \pi^K_l(R) \xrightarrow{\times} \pi^{G\times K}_{k+l}(R \wedge R) \xrightarrow{\mu_*} \pi^{G\times K}_{k+l}(R) . \tag{4.14}$$

We also define a unit element $1 \in \pi^e_0(R)$ as the image of the unit element $1 \in \pi^e_0(\mathbb{S})$ under the unit morphism $\eta : \mathbb{S} \longrightarrow R$.

Corollary 4.15. *Let R be an orthogonal ring spectrum and G, K and L compact Lie groups.*

(i) (Biadditivity) *The product $\times : \pi^G_k(R) \times \pi^K_l(R) \longrightarrow \pi^{G\times K}_{k+l}(R)$ is biadditive.*

(ii) (Unitality) *The unit $1 \in \pi^e_0(R)$ satisfies $1 \times x = x = x \times 1$ under the identifications $e \times G \cong G \cong G \times e$.*

(iii) (Associativity) *For all classes $x \in \pi^G_k(R)$, $y \in \pi^K_l(R)$ and $z \in \pi^L_j(R)$ we have $(x \times y) \times z = x \times (y \times z)$ under the identification $(G \times K) \times L \cong G \times (K \times L)$.*

(iv) (Restriction) *For all classes $x \in \pi^G_k(R)$ and $y \in \pi^K_l(R)$ and all continuous homomorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$(\alpha^* x) \times (\beta^* y) \ = \ (\alpha \times \beta)^* (x \times y)$$

holds in the group $\pi_{k+l}^{\bar{G}\times\bar{K}}(R)$.

(v) (Transfer) *For all subgroup inclusions* $H \leq G$ *and* $L \leq K$ *and all classes* $x \in \pi_k^H(R)$ *and* $y \in \pi_l^L(R)$ *the relation*

$$(\mathrm{tr}_H^G\, x) \times (\mathrm{tr}_L^K\, y) \ = \ \mathrm{tr}_{H\times L}^{G\times K}(x \times y)$$

holds in the group $\pi_{k+l}^{G\times K}(R)$.

Remark 4.16. If the multiplication on R is commutative, then the homotopy group pairing is commutative in the sense that for all compact Lie groups G and K and all classes $x \in \pi_k^G(R)$ and $y \in \pi_l^K(R)$ the relation

$$y \times x \ = \ (-1)^{kl} \cdot \tau_{G,K}^*(x \times y)$$

holds in the group $\pi_{l+k}^{K\times G}(R)$, where $\tau_{G,K} : G \times K \longrightarrow K \times G$ interchanges the factors. However, strict commutativity of the multiplication not only implies this commutativity relation, but it gives rise to a huge amount of extra structure in the form of power operations and norm maps. We will return to this in more detail in Section V.1.

Example 4.17 (Units of a ring spectrum)**.** In Example II.1.15 we defined the units of an orthogonal monoid space. When R is an orthogonal ring spectrum, the units of the orthogonal monoid space $\Omega^\bullet R$ are an interesting orthogonal monoid subspace

$$GL_1(R) \ = \ (\Omega^\bullet R)^\times \ ,$$

the *units* of the ring spectrum R. So by definition, the value of $GL_1(R)$ at an inner product space V is the space of all continuous based maps $S^V \longrightarrow R(V)$ that represent an invertible element in the multiplicative monoid $\pi_0^e(R)$. For a compact Lie group G,

$$\pi_0^G(R^\times) \ = \ \{x \in \pi_0^G(R) \mid \mathrm{res}_e^G(x) \text{ is invertible in } \pi_0^e(R)\}$$

is the multiplicative submonoid of $\pi_0^G(R)$ of elements that become invertible when restricted to the trivial group. So one should beware that $\pi_0^G(R^\times)$ may contain non-invertible elements, and then the orthogonal monoids space $GL_1(R)$ is *not* group-like!

When the multiplication of R is commutative, then $GL_1(R)$ is a commutative orthogonal monoid space. In the non-equivariant context, $GL_1(R)$ is then an infinite loop space, i.e., weakly equivalent to the 0-th space of an Ω-spectrum $gl_1(R)$. The naive generalization of this fact to the global context cannot work, however, i.e., $GL_1(R)$ cannot in general be globally equivalent to the orthogonal space $\Omega^\bullet X$ for an orthogonal spectrum X. A global equivalence between $GL_1 R$ and $\Omega^\bullet X$ that translates the multiplication of $GL_1 R$ in any reasonable way to the loop product $\Omega^\bullet X$ would turn the multiplication and norm maps in $\underline{\pi}_0(GL_1 R)$ (written multiplicatively) into the addition and transfer maps in $\underline{\pi}_0(\Omega^\bullet X) = \underline{\pi}_0(X)$ (written additively). This yields two obvious obstructions: firstly, $GL_1(R)$ need not be group-like in the sense of Definition II.4.1, i.e., the commutative monoid $\pi_0^G(GL_1 R)$ need not be a group when G is non-trivial. Secondly, even if the commutative monoids $\pi_0^G(GL_1 R)$ are all groups, then $\underline{\pi}_0(GL_1 R)$ has norm maps only for subgroup inclusions of finite index, whereas $\underline{\pi}_0(X)$ has transfer maps for arbitrary closed inclusions of compact Lie groups. It seems unreasonable to expect that the infinite index transfers suddenly appear out of nowhere, so even globally group-like commutative orthogonal monoid spaces should not always have global deloopings.

Construction 4.18 (Monoidal structure of the Burnside category)**.** We can use the pairing of equivariant homotopy groups to define a symmetric monoidal structure on the Burnside category $\mathbf{A}$. We define a biadditive functor

$$\times \ : \ \mathbf{A} \times \mathbf{A} \ \longrightarrow \ \mathbf{A}$$

on objects by the product of Lie groups. To define the monoidal product on morphisms we let $\tau \in \mathbf{A}(G,K)$ and $\tau' \in \mathbf{A}(G',K')$ be two natural transformations. We choose a faithful G-representation V,and a faithful

G'-representation V', which have associated stable tautological classes (3.10)

$$e_{G,V} \in \pi_0^G(\Sigma_+^\infty F_{G,V}) \quad \text{and} \quad e_{G',V'} \in \pi_0^{G'}(\Sigma_+^\infty F_{G',V'}) .$$

We can evaluate the transformations on these tautological classes and pair them to obtain an element

$$\tau(e_{G,V}) \times \tau'(e_{G',V'}) \in \pi_0^{K\times K'}\left(\Sigma_+^\infty F_{G,V} \wedge \Sigma_+^\infty F_{G',V'}\right)$$

We let $\mu : \Sigma_+^\infty F_{G,V} \wedge \Sigma_+^\infty F_{G',V'} \longrightarrow \Sigma_+^\infty F_{G\times G',V\oplus V'}$ be the preferred morphism [...]. The representation $V \oplus V'$ of the product group $G \times G'$ is again faithful. By Proposition 3.13 there is thus a unique natural transformation $\tau \times \tau' \in \mathbf{A}(G \times G,\, K \times K')$ such that

$$(\tau \times \tau')(e_{G\times G',V'\oplus V'}) = \mu_*(\tau(e_{G,V}) \times \tau'(e_{G',V'})) \in \pi_0^{K\times K'}\left(\Sigma_+^\infty F_{G\times G',V\oplus V'}\right).$$

The associativity, commutativity and unitality isomorphisms for the product of groups

$$(G\times G')\times G'' \cong G\times(G'\times G'') , \quad G\times G' \cong G'\times G \qquad \text{respectively} \quad G\times e \cong G \cong e\times G .$$

induces isomorphism in the global Burnside category by passing to restriction isomorphisms. We take these isomorphisms in $\mathbf{A}$ as the associativity, commutativity and unitality constraints for the symmetric monoidal structure in $\mathbf{A}$. The various naturality properties of the external product pairing $\times$ and the morphisms $\mu : \Sigma_+^\infty F_{G,V} \wedge \Sigma_+^\infty F_{G',V'} \longrightarrow \Sigma_+^\infty F_{G\times G',V\oplus V'}$ formally imply:

Theorem 4.19. *The functor $\times : \mathbf{A} \times \mathbf{A} \longrightarrow \mathbf{A}$ is a biadditive symmetric monoidal structure on the global Burnside category. On the generating morphisms the monoidal product is given by*

$$[L,\alpha] \times [L'\alpha'] = [L\times L', \alpha\times\alpha'] .$$

Remark 4.20. The restriction to finite groups of the monoidal structure on the global Burnside category has an interpretation in terms of the cartesian product of bisets: under the equivalence of categories $\mathbf{A}^{\text{fin}} \cong A^{\text{c}}$ (compare Remark 3.23), it corresponds to the monoidal structure

$$A^{\text{c}}(G,K) \times A^{\text{c}}(G',K') \longrightarrow A^{\text{c}}(G\times G', K\times K') , \quad ([S],[S']) \longmapsto [S\times S'] .$$

Here S is a right free K-G-biset S and S' is a right free K'-G'-biset S'; the cartesian product $S \times S'$ is then a right free $(K \times K')$-$(G \times G')$-biset.

It was shown by Day [**27**] that the category of functors between two symmetric monoidal categories inherits a symmetric monoidal structure given by a kind of convolution product. This also works for enriched functors over some symmetric monoidal base category, and the smash product of orthogonal spectra is an example of this construction (where the base is the category of based spaces under smash product). Global functors are another example, as they are additive functors from the symmetric monoidal Burnside category $\mathbf{A}$ to the symmetric monoidal category of abelian groups, under tensor product. We now make the convolution product explicit for global functors.

We let F, F' and F'' be global functors. We denote by $F \otimes F' : \mathbf{A} \times \mathbf{A} \longrightarrow \mathcal{A}b$ the objectwise tensor product given on objects by

$$(F \otimes F')(G,G') = F(G) \otimes F'(G') .$$

A *bimorphism* is a natural transformation

$$F \otimes F' \longrightarrow F'' \circ \times$$

of biadditive functors on the category $\mathbf{A} \times \mathbf{A}$. So explicitly, a bimorphism is a collection of group homomorphisms

$$b_{G,\bar{G}} : F(G) \otimes F'(\bar{G}) \longrightarrow F''(G \times \bar{G})$$

for all compact Lie groups G and G', such that for all group homomorphisms $\alpha : K \longrightarrow G$ and $\bar\alpha : \bar K \longrightarrow \bar G$ and for all subgroups $H \leq G$ and $\bar H \leq \bar G$ the diagram

$$\begin{tikzcd}
F(H) \otimes F'(\bar H) \arrow[r, "b_{H,\bar H}"] \arrow[d, "\mathrm{tr}_H^G \otimes \mathrm{tr}_{\bar H}^{\bar G}"'] & F''(H \times \bar H) \arrow[d, "\mathrm{tr}_{H\times\bar H}^{G\times\bar G}"] \\
F(G) \otimes F'(\bar G) \arrow[r, "b_{G,\bar G}"] \arrow[d, "\alpha^*\otimes\bar\alpha^*"'] & F''(G \times \bar G) \arrow[d, "(\alpha\times\bar\alpha)^*"] \\
F(K) \otimes F'(\bar K) \arrow[r, "b_{K,\bar K}"'] & F''(K \times \bar K)
\end{tikzcd}$$

commutes. Equivalently: for every compact Lie group G the maps $\{b_{G,K}\}_K$ for a morphism of global functors $F(G) \otimes F'(-) \longrightarrow F''(G \times -)$ and for every compact Lie group K the maps $\{b_{G,K}\}_G$ for a morphism of global functors $F(-) \otimes F'(K) \longrightarrow F''(- \times K)$.

A *box product* of F and F' as a universal example of a global functor with a bimorphism from F and F'. More precisely, a box product a pair $(F\Box F', i)$ consisting of a global functor $F\Box F'$ and a universal bimorphism $i : F \otimes F' \longrightarrow (F\Box F') \circ \times$, i.e., a bimorphism such that for every global functor F'' the map

$$\mathcal{GF}(F\Box F', F'') \longrightarrow \mathrm{Bimor}((F, F'), F'') \ , \quad f \longmapsto fi \tag{4.21}$$

is bijective.

The construction of a box product follows the familiar pattern. We want $F\Box F'$ to be the universal recipient of a bimorphism, and this pretty much tells us what we have to do. We choose a set $\mathcal G$ of representative of the isomorphism classes of compact Lie groups and set

$$(F\Box F')(K) \ = \ \left(\bigoplus_{G,\bar G\in\mathcal G} \mathbf{A}(G\times\bar G, K) \otimes F(G) \otimes F'(\bar G) \right) / A \ ,$$

where A is the subgroup of the sum generated by the elements

$$(\beta \circ (\tau\times\bar\tau)) \otimes x \otimes y \ - \ \beta \otimes F(\tau)(x) \otimes F'(\bar\tau)(y) \ ,$$

for all $\beta \in \mathbf{A}(G\times\bar G, K)$, $\tau \in \mathbf{A}(H,G)$, $\bar\tau \in \mathbf{A}(\bar H, \bar G)$ $x \in F(H)$ and $y \in F'(\bar H)$. The construction of $(F\Box F')(K)$ is clearly functorial in K. We define a bimorphism $F \otimes F' \longrightarrow (F\Box F) \circ \times$ at $(H, \bar H)$ as the composite

$$F(H) \otimes F'(\bar H) \ \longrightarrow \ \bigoplus_{G,\bar G\in\mathcal G} \mathbf{A}(G\times\bar G, H\times\bar H) \otimes F(G) \otimes F'(\bar G) \ \xrightarrow{\text{proj}} \ (F\Box F')(H\times\bar H)$$

where the first map injects into the summand indexed by $(H, \bar H)$ by tensoring with the identity of $H \times \bar H$ in $\mathbf{A}(H\times\bar H, H\times\bar H)$. One should check that the maps form a bimorphism – and in fact a universal one.

Very often only the global functor $F\Box F'$ will be referred to as the box product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (4.21) as the *universal property* of the box product of global functors.

Example 4.22. Given orthogonal spectra X and Y the external product maps

$$\times \ : \ \pi_0^G(X) \otimes \pi_0^K(Y) \ \longrightarrow \ \pi_0^{G\times K}(X \wedge Y)$$

for a bimorphism of global functors by Theorem 4.7. So the universal property of the box product produces a morphism of global functors

$$\underline{\pi}_0(X) \Box \underline{\pi}_0(Y) \ \longrightarrow \ \underline{\pi}_0(X \wedge Y) \ . \tag{4.23}$$

We show in Proposition 4.15 below that whenever X and Y are globally connective and at least one of then is flat, then the smash product $X \wedge Y$ is again globally connective and the morphism (4.23) is an isomorphism of global functors.

Since the box product enjoys a universal property it is unique up to preferred isomorphism. Also, the universal property guarantees that given any collection of choices of box product $F\Box F'$ for all pairs of global functors, $F\Box F'$ is an additive functor in both variables and there are preferred associativity and commutativity isomorphism

$$(F\Box F')\Box F'' \;\cong\; F\Box(F'\Box F'') \qquad \text{and} \qquad F\Box F' \;\cong\; F'\Box F \ .$$

Moreover, there a preferred unit isomorphisms

$$F\Box\mathbb{A} \;\cong\; F \;\cong\; \mathbb{A}\Box F$$

where $\mathbb{A} = \mathbf{A}(e,-)$ is the Burnside ring global functor; together this structure makes the category of global functors into a symmetric monoidal category, with the Burnside ring global functor $\mathbb{A}$ as unit object. Moreover, the box product of representable global functors is again representable: the maps

$$\times \ : \ \mathbf{A}(G,H)\otimes\mathbf{A}(\bar{G},\bar{H}) \;\longrightarrow\; \mathbf{A}(G\times\bar{G},H\times\bar{H})$$

for a bimorphism from $(\mathbf{A}(G,-),\mathbf{A}(\bar{G},-))$ to $\mathbf{A}(G\times\bar{G},-)$

given by the monoidal structure of $\mathbf{A}$ induces a morphism of global functors

$$\mathbf{A}(G,-)\,\Box\,\mathbf{A}(\bar{G},-) \;\longrightarrow\; \mathbf{A}(G\times\bar{G},-)$$

that is an isomorphism.

The box product of global functors is *closed* symmetric monoidal in the sense that there are adjoint internal Hom functors. We recall the construction i.e., there is an adjunction isomorphism

$$\mathcal{GF}(F\Box F',F'') \;\cong\; \mathcal{GF}(F,\mathrm{Hom}(F',F'')) \ .$$

For a global functor F and a compact Lie group G we define the *translated* functor $\tau_G F$ as the composite

$$\mathbf{A} \xrightarrow{\;-\times G\;} \mathbf{A} \xrightarrow{\;F\;} \mathcal{A}b \ . \tag{4.24}$$

Since the monoidal product on the Burnside category is additive in both variables, the shift construction is also functorial in G: for every pair of compact Lie groups G and K a morphism of global functors

$$\mathbf{A}(G,K)\otimes(\tau_G F) \;\longrightarrow\; \tau_K F$$

is given at a group H by the composite

$$\begin{aligned}\mathbf{A}(G,K)\otimes(\tau_G F)(H) \;&=\mathbf{A}(G,K)\otimes F(H\times G)\\ &\xrightarrow{(H\times -)\otimes\mathrm{Id}} \mathbf{A}(H\times G,H\times K)\otimes F(H\times G)\\ &\xrightarrow{\quad\text{act}\quad} \;F(H\times K) \;=\; (\tau_K F)(H) \ .\end{aligned}$$

We can now define an internal Hom global functor $\mathrm{Hom}(F',F'')$ at a compact Lie group G by

$$\mathrm{Hom}(F',F'')(G) \;=\; \mathcal{GF}(F',\tau_G F'') \ ,$$

the group of morphism of global functors from F' to F''. This is a global functor in G via the additive maps

$$\begin{aligned}\mathbf{A}(G,K)\otimes\mathrm{Hom}(F',F'')(G) \;&=\; \mathbf{A}(G,K)\otimes\mathcal{GF}(F',\tau_G F'')\\ &\longrightarrow\; \mathcal{GF}(F',\mathbf{A}(G,K)\otimes\tau_G F'')\\ &\longrightarrow\; \mathcal{GF}(F',\tau_K F'') \;=\; \mathrm{Hom}(F',F'')(K) \ .\end{aligned}$$

The internal Hom of global functors commutes with translation in the second variable. Indeed, for compact Lie groups G and K the associativity isomorphism of the monoidal structure on $\mathbf{A}$ provides a natural isomorphism

$$(\tau_G(\tau_K F))(H) \;=\; (\tau_K F)(H \times G) \;=\; F((H\times G)\times K) \;\cong\; F(H\times(G\times K)) \;=\; (\tau_{G\times K}F)(H)\ .$$

As H varies, this is an isomorphism of global functors $\tau_G(\tau_K F)\cong \tau_{G\times K}F$. We can thus define an isomorphism of Hom global functors

$$\mathrm{Hom}(F',\tau_K F'') \;\cong\; \tau_K\left(\mathrm{Hom}(F',F'')\right)$$

at a group G by

$$\begin{aligned}\mathrm{Hom}(F',\tau_K F''X)(G) \;&=\; \mathcal{GF}(F,\tau_G(\tau_K F'')) \;\cong\; \mathcal{GF}(X,\tau_{G\times K}F'')\\ &=\; \mathrm{Hom}(F',F'')(G\times K) \;=\; (\tau_K\,\mathrm{Hom}(F',F''))(G)\ .\end{aligned}$$

The internal Hom $\mathrm{Hom}(F',-)$ is right adjoint to the box product $-\Box F'$ of global functors. A natural isomorphism $\mathrm{Hom}(\mathbf{A}(G,-),F)\cong\tau_G F$ of global functors is given at a group K by

$$\mathrm{Hom}(\mathbf{A}(G,-),F)(K)=\mathcal{GF}(\mathbf{A}(G,-),\tau_K F) \;\cong\; (\tau_K F)(G) \;=\; F(G\times K) \xrightarrow{F(\tau_{G,K})} F(K\times G)=(\tau_G F)(K)$$

where the second map is the enriched Yoneda isomorphism. In the special case $G=e$ of the trivial group this gives a natural isomorphism of global functors $\mathrm{Hom}(\mathbb{A},F)\cong \mathrm{sh}_e F\cong F$.

Remark 4.25. Theorem 3.5 describes explicit free generators for the morphism group in the global Burnside category. Using this, the box product $F\Box F'$ admits the following somewhat more explicit description. Again we choose a set $\mathcal{G}$ of representative of the isomorphism classes of compact Lie groups. Then $(F\Box F')(K)$ is a quotient of the abelian group

$$\bigoplus_{G,\bar G\in\mathcal{G};\ [L,\alpha]} F(G)\otimes F'(\bar G)\ ,$$

where the sum is indexed over $(G,\bar G)\in\mathcal{G}^2$ and over all conjugacy classes of pairs (L,α) consisting of a subgroup L of K and a continuous homomorphism $\alpha: L\longrightarrow G\times\bar G$. From this sum we have to divide out the subgroup generated by the following two kinds of elements:

- $$[x\otimes y]_{[L,(\tau\times\bar\tau)\alpha]} \;-\; [F(\tau^*)(x)\otimes F'(\bar\tau^*)(y)]_{[L,\alpha]}\ ,$$
  for all $[L,\alpha]$ and all pairs of continuous group homomorphisms $\tau: G\longrightarrow H$, $\bar\tau:\bar G\longrightarrow \bar H$ $x\in F(H)$ and $y\in F'(\bar H)$.
- $$\sum_{[M,\beta]}\lambda_{[M,\beta]}\cdot[x\otimes y]_{[M,\beta]} \;-\; [F(\mathrm{tr}_H^G)(x)\otimes F'(\mathrm{tr}_{\bar H}^{\bar G})(y)]_{[L,\alpha]}\ ,$$
  for all $[L,\alpha]$ and subgroup pairs $H\le G$ and $\bar H\le\bar G$, $x\in F(H)$ and $y\in F'(\bar H)$. Here
  $$\alpha^*\circ\mathrm{tr}_{H\times\bar H}^{G\times\bar G} \;=\; \sum_{[M,\beta]}\lambda_{[M,\beta]}\cdot\mathrm{tr}_M^{H\times\bar H}\circ\beta^*$$
  is the expansion of the composite $\alpha^*\circ\mathrm{tr}_{H\times\bar H}^{G\times\bar G}$ in the preferred basis of the group $\mathbf{A}(H\times\bar H,L)$ that ultimately goes back to the double coset formula.

The construction of $(F\Box F')(K)$ is clearly functorial in K. We define a bimorphism $F\otimes F'\longrightarrow (F\Box F)\circ\times$ at $(H,\bar H)$ as the composite

$$F(H)\otimes F'(\bar H) \;\longrightarrow\; \bigoplus_{G,\bar G\in\mathcal{G}}\mathbf{A}(G\times\bar G,H\times\bar H)\otimes F(G)\otimes F'(\bar G) \;\xrightarrow{\mathrm{proj}}\; (F\Box F')(H\times\bar H)$$

where the first map injects into the summand indexed by $(H, \bar{H})$ by tensoring with the identity of $H \times \bar{H}$ in $\mathbf{A}(H \times \bar{H}, H \times \bar{H})$. One should check that the maps form a bimorphism – and in fact a universal one.

Proposition 4.26. *The box product of global functors is symmetric monoidal and biadditive.*

For two global functors F, F and a compact Lie group G, the value $(F\Box F')(G)$ depends only on the multiplicative global family generated by G. [The box product is right exact]

Remark 4.27. While the box product of global functors has many properties familiar from the tensor product of modules over a commutative ring, there one aspect where these constructions are fundamentally different: *projectives are not generally flat* in the category of global functors. In other words, for most projective global functors P the functor $-\Box P$ does *not* send monomorphisms to monomorphisms. This kind of phenomenon has been analyzed in great detail by Lewis in [**51**]; global functors on finite groups are explicitly considered in [...] of [**51**].

We give an explicit example to illustrate the phenomenon. By [...] the value of a Box product $F\Box F'$ at the cyclic group C_2 of order 2 is given by

$$(F\Box F')(C_2) \;=\; \left(F(C_2)\otimes F'(C_2) \;\oplus\; F(e)\otimes F'(e)\right)/M$$

where $M \subseteq F(C_2)\otimes F'(C_2)\oplus F(e)\otimes F'(e)$ is the abelian subgroup generated by the elements

$$(\mathrm{tr}(x)\otimes y',\; -x\otimes \mathrm{res}(y')) \qquad \text{and} \qquad (y\otimes \mathrm{tr}(x'),\; -\mathrm{res}(y)\otimes x')$$

for all $x \in F(e), x' \in F'(e), y \in F(C_2)$ and $y' \in F'(C_2)$, and where $\mathrm{tr} = \mathrm{tr}_e^{C_2}$ and $\mathrm{res} = \mathrm{res}_e^{C_2}$ are the transfer and restriction maps relating the values at e and C_2. We specialize to $F = \tilde{\mathbf{A}}_{C_2}$, the reduced representable global functor corresponding to the group C_2, i.e., the direct summand of $\mathbf{A}(C_2, -)$ split off by the idempotent

$$\epsilon \;=\; \mathrm{Id}_{C_2} - z^* \;\in\; \mathbf{A}(C_2, C_2)$$

where $z : C_2 \longrightarrow C_2$ is the trivial group homomorphism. Since $\tilde{\mathbf{A}}_{C_2}(e) = 0$ and $\tilde{\mathbf{A}}_{C_2}(C_2)$ is free abelian of rank 1, generated by the idempotent ϵ, the description of the box product with $\tilde{\mathbf{A}}_{C_2}$ at C_2 simplifies to

$$(F\Box\tilde{\mathbf{A}}_{C_2})(C_2) \;=\; \left(F(C_2)\otimes \mathbb{Z}\{\epsilon\}\right)/M \;,$$

where now $M \subseteq F(C_2)\otimes\mathbb{Z}\{\epsilon\}$ is the subgroup generated by the elements $\mathrm{tr}(x)\otimes\epsilon$ for all $x \in F(e)$. In other words, the group $(F\Box\tilde{\mathbf{A}}_{C_2})(C_2)$ is naturally isomorphic to the cokernel of the transfer map $\mathrm{tr} : F(e) \longrightarrow F(C_2)$. For the constant global global functor $\underline{B}$ associated to an abelian group B we have $(\underline{B}\Box\tilde{\mathbf{A}}_{C_2})(C_2) \cong B/2B$, so $-\Box\tilde{\mathbf{A}}_{C_2}$ does not preserve exact sequences of global functors, and the projective global functor $\tilde{\mathbf{A}}_{C_2}$ is not flat.

Because projectives are not generally flat, one has to be careful when deriving the box product. The box product $F\Box-$ is additive and right exact, and the category of global functors has enough projectives, so there are left derived functors $L_i(F\Box-)$ for $i \geq 0$ with the usual properties, such as $L_0(F\Box-) \cong F\Box-$. Similarly, there are left derived functors $L_i(-\Box F')$ of the functor $-\Box F'$. However, unlike with usual Tor groups, the global functors $L_i(F\Box-)(F')$ and $L_i(-\Box F')(F)$ are not generally isomorphic. Indeed, the classical proof that Tor groups of module over ring can be calculated by projectively resolving the left factor, or the right factor, or both, uses that projective modules are flat.

We now remark that bimorphisms of global functors can be identified with another kind of structure that we call 'diagonal products'.

Definition 4.28. Let X, Y and Z be global functors. A *diagonal product* is a natural transformation $X\otimes Y \longrightarrow Z$ of contravariant functors from groups to abelian groups that satisfies reciprocity, where $X\otimes Y$ is the objectwise tensor product.

More explicitly, a diagonal product consists of additive maps

$$\nu_G \ : \ X(G) \otimes Y(G) \ \longrightarrow \ Z(G)$$

for every compact Lie group G that are natural for restriction along homomorphisms $\alpha : K \longrightarrow G$ and satisfy the reciprocity relation

$$\mathrm{tr}_H^G(x \cdot_H \mathrm{res}_H^G(y)) \ = \ \mathrm{tr}_H^G(x) \cdot_G y$$

for all subgroups H of G and all classes $x \in X(H)$ and $y \in Y(G)$, where we have written $a \cdot_G b$ for $\nu_G(a \otimes b)$. [same in other variable; commutative product...] Any bimorphism $\mu : (X, Y) \longrightarrow Z$ gives rise to a diagonal product as follows. For a group G we define ν_G as the composite

$$X(G) \otimes Y(G) \xrightarrow{\mu_{G,G}} Z(G \times G) \xrightarrow{\Delta_G^*} Z(G)$$

where $\Delta : G \longrightarrow G \times G$ is the diagonal. For a group homomorphism $\alpha : K \longrightarrow G$ we have $\Delta_G \circ \alpha = (\alpha \times \alpha) \circ \Delta_K$, so the diagram

$$\begin{tikzcd}
X(G) \otimes Y(G) \arrow[r, "\mu_{G,G}"] \arrow[d, "\alpha^* \otimes \alpha^*"'] & Z(G \times G) \arrow[r, "\Delta_G^*"] \arrow[d, "(\alpha \times \alpha)^*"] & Z(G) \arrow[d, "\alpha^*"] \\
X(K) \otimes Y(K) \arrow[r, "\mu_{K,K}"'] & Z(K \times K) \arrow[r, "\Delta_K^*"'] & Z(K)
\end{tikzcd}$$

commutes. **fix this !!!** If H is a subgroup of G, then the map

$$G \times_H \Delta_H^*(H \times G) \ \longrightarrow \ \Delta_G^*(G \times G) \ , \quad [\gamma, (h, g)] \ \longmapsto \ (\gamma h, \gamma g)$$

is an isomorphism of left-G, right-$(H \times G)$-bisets (with inverse sending (g, g') to $[g, (1, g^{-1}g')]$). This proves the 'relation'

$$(\mathrm{Id}_H \otimes r_H^G) \circ \Delta_H \circ t_H^G \ = \ (t_H^G \otimes \mathrm{Id}_G) \circ \Delta_G$$

in the group $A(G, H \times G)$ (where $r_H^G = [G] \in A(H, G)$, $\Delta_H = [H \times H] \in A(H, H \times H)$, $t_H^G = [G] \in A(G, H)$ and $\Delta_G = [G \times G] \in A(G, G \times G)$). Hence the lower right part of the diagram

$$\begin{tikzcd}
 & X(H) \otimes Y(G) \arrow[rr, "(t_H^G)^* \otimes Y(G)"] \arrow[dl, "X(H) \otimes (r_H^G)^*"'] \arrow[d, "\mu_{H,G}"] & & X(G) \otimes Y(G) \arrow[d, "\mu_{G,G}"] \arrow[dd, bend left, "\nu_G"] \\
X(H) \otimes Y(H) \arrow[dr, "\mu_{H,H}"'] \arrow[drr, bend right, "\nu_H"'] & Z(H \times G) \arrow[rr, "(t_H^G \otimes \mathrm{Id}_G)^*"] \arrow[d, "(\mathrm{Id}_H \otimes r_H^G)^*"] & & Z(G \times G) \arrow[d, "\Delta_G^*"] \\
 & Z(H \times H) \arrow[r, "\Delta_H^*"'] & Z(H) \arrow[r, "(t_H^G)^*"'] & Z(G)
\end{tikzcd}$$

commutes. The rest of the diagram commutes by naturality of the external product μ. So for $x \in X(H)$ any $y \in Y(G)$ this proves the reciprocity relation

$$\mathrm{tr}_H^G(x \cdot \mathrm{res}_H^G(y)) \ = \ \mathrm{tr}_H^G(\nu_H(x \otimes \mathrm{res}_H^G(y)) \ = \ \nu_G(\mathrm{tr}_H^G(x) \otimes y) \ = \ \mathrm{tr}_H^G(x) \cdot y$$

for the diagonal product ν.

Conversely, given a diagonal product ν, we define a bimorphism as follows. For compact Lie groups G and K we define the component $\mu_{G,K}$ as the composite

$$X(G) \otimes Y(K) \xrightarrow{p_G^* \otimes p_K^*} X(G \times K) \otimes Y(G \times K) \xrightarrow{\nu_{G \times K}} Z(G \times K) \ ,$$

where $p_G : G \times K \longrightarrow G$ and $p_K : G \times K \longrightarrow K$ are the projections. Given homomorphisms $\alpha : G \longrightarrow G'$ and $\beta : K \longrightarrow K'$, we have $p_{G'}(\alpha \times \beta) = \alpha p_G$ and $p_{K'}(\alpha \times \beta) = \beta p_K$, so the left part of the diagram

$$\begin{tikzcd} X(G) \otimes Y(K) \arrow[r, "p_G^* \otimes p_K^*"] \arrow[d, "\alpha^* \otimes \beta^*"'] & X(G \times K) \otimes Y(G \times K) \arrow[r, "\nu_{G\times K}"] \arrow[d, "(\alpha\times\beta)^* \otimes (\alpha\times\beta)^*"] & Z(G \times K) \arrow[d, "(\alpha\times\beta)^*"] \\ X(G') \otimes Y(K') \arrow[r, "p_G^* \otimes p_K^*"'] & X(G' \times K') \otimes Y(G' \times K') \arrow[r, "\nu_{G' \times K'}"'] & Z(G' \times K') \end{tikzcd}$$

commutes. The right part commutes by naturality of the diagonal product ν.

For naturality with respect to transfer we let H by a subgroup of G, and we consider classes $x \in X(H)$ and $y \in Y(K)$. Then we have

$$\begin{aligned} \operatorname{tr}_{H\times K}^{G\times K}(x \times y) \ &= \ \operatorname{tr}_{H\times K}^{G\times K}(p_H^*(x) \cdot p_K^*(y)) \\ &= \ \operatorname{tr}_{H\times K}^{G\times K}(p_H^*(x) \cdot \operatorname{res}_{H\times K}^{G\times K}(\bar{p}_K^*(y))) \\ &= \ \operatorname{tr}_{H\times K}^{G\times K}(p_H^*(x)) \cdot \bar{p}_K^*(y) \\ &= \ p_G^*(\operatorname{tr}_H^G(x)) \cdot p_K^*(y) \ = \ \operatorname{tr}_H^G(x) \times y \ . \end{aligned}$$

Here $p : H \times K \longrightarrow K$ and $\bar{p} : G \times K \longrightarrow K$ are the projections to K, and the third equality is reciprocity. The argument for transfer naturality in the K-variable is similar.

5. Free spectra

Given a compact Lie group G and a G-representation V, the functor

$$\operatorname{ev}_{G,V} \ : \ \mathcal{S}p \ \longrightarrow \ G\mathbf{T}$$

that sends an orthogonal spectrum to the G-space $Y(V)$ has a left adjoint

$$F_{G,V} \ : \ G\mathbf{T} \ \longrightarrow \ \mathcal{S}p \ .$$

In this section we construct the adjoint explicitly and analyze the homotopical properties of free orthogonal spectra.

In Definition 1.1 we introduced orthogonal spectra in a way that resembles a presentation by 'generators and relations'. For the discussion of free spectra it is convenient to recall the 'coordinate free' description of orthogonal spectra as continuous functors on a certain topological category. We let V and W inner product spaces. Over the space $\mathbf{L}(V,W)$ of linear isometric embeddings sits a certain 'orthogonal complement' vector bundle with total space

$$\xi(V,W) \ = \ \{\, (\alpha, w) \in \mathbf{L}(V,W) \times W \mid w \perp \alpha(V) \,\} \ .$$

The structure map $\xi(V,W) \longrightarrow \mathbf{L}(V,W)$ is the projection to the first factor. So $\xi(V,W)$ is a vector subbundle of the trivial vector bundle $\mathbf{L}(V,W) \times W$, and the fiber over $\alpha : V \longrightarrow W$ is the orthogonal complement $W - \alpha(V)$ of the image of α.

We let $\mathbf{O}(V,W)$ be the Thom space of the bundle $\xi(V,W)$, i.e., the one-point compactification of the total space of $\xi(V,W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V,W)$ differently as follows. If the dimension of W is smaller than the dimension of V, then the space $\mathbf{L}(V,W)$ is empty and $\mathbf{O}(V,W)$ consists of a single point. Otherwise we can choose a linear isometric embedding $\varphi : V \longrightarrow W$, and we let $V^\perp = W - \varphi(V)$ denote the orthogonal complement of its image. Then the maps

$$\begin{aligned} O(W)/O(V^\perp) \quad &\longrightarrow \ \mathbf{L}(V,W) \ , \quad A \cdot O(V^\perp) \ \longmapsto \ A \cdot \varphi \qquad \text{and} \\ O(W)^+ \wedge_{O(V^\perp)} S^{V^\perp} \ &\longrightarrow \ \mathbf{O}(V,W) \ , \quad [A, w] \ \longmapsto \ A \cdot (\varphi, w) \end{aligned}$$

are homeomorphisms. Put yet another way: if $\dim V = n$ and $\dim W = n+m$, then $\mathbf{L}(V,W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V,W)$ is homeomorphic to $O(n+m)^+ \wedge_{O(m)} S^m$. The vector bundle $\xi(V,W)$ becomes trivial upon product with the trivial bundle V, via the trivialization

$$\xi(V,W) \times V \;\cong\; \mathbf{L}(V,W) \times W \;, \quad ((\varphi,w),v) \;\longmapsto\; (\varphi,\, w+\varphi(v)) \;.$$

When we pass to Thom spaces on both sides this becomes the *untwisting homeomorphism:*

$$\mathbf{O}(V,W) \wedge S^V \;\cong\; \mathbf{L}(V,W)_+ \wedge S^W \;. \tag{5.1}$$

The Thom spaces $\mathbf{O}(V,W)$ are the morphism spaces of a based topological category. Given a third inner product space U, there is a bundle map

$$\xi(U,V) \times \xi(V,W) \;\longrightarrow\; \xi(U,W) \;, \quad ((\beta,v),\, (\varphi,w)) \;\longmapsto\; (\varphi\beta,\, \varphi(v)+w)$$

which covers the composition map $\mathbf{L}(U,V) \times \mathbf{L}(V,W) \longrightarrow \mathbf{L}(U,W)$. Passage to Thom spaces gives a based map

$$\circ \;:\; \mathbf{O}(U,V) \wedge \mathbf{O}(V,W) \;\longrightarrow\; \mathbf{O}(U,W)$$

which is clearly associative, and is the composition in the category $\mathbf{O}$. The unit of V is $1_V = (\mathrm{Id}_V, 0)$ in $\mathbf{O}(V,V)$.

Now we let X be an orthogonal spectrum. We define a continuous based *action map*

$$\begin{aligned} \circ \;:\; X(V) \wedge \mathbf{O}(V,W) \;&\longrightarrow\; X(W) \\ x \;\wedge\; (\varphi,w) \;&\longmapsto\; X(\tilde\varphi)(\sigma_{V,W-\varphi(V)}(x \wedge w)) \end{aligned} \tag{5.2}$$

where $\tilde\varphi : V \oplus (W - \varphi(V)) \longrightarrow W$ is the isometry defined by

$$\tilde\varphi(v,w) \;=\; \varphi(v) + w \;.$$

We obtain a map

$$\kappa \;:\; S^W \;\longrightarrow\; \mathbf{O}(V, V \oplus W) \;, \quad w \;\longmapsto\; (i_V, (0,w)) \;,$$

as the inclusion of the fiber over $i_V : V \longrightarrow V \oplus W$, the inclusion of the first summand. The generalized structure map $\sigma_{V,W}$ originally defined in (1.4) then coincides with the composite

$$X(V) \wedge S^W \xrightarrow{X(V)\wedge\kappa} X(V) \wedge \mathbf{O}(V, V \oplus W) \xrightarrow{\;\circ\;} X(W) \;.$$

The action maps are associative: The action is also associative in the sense that the square

$$\begin{tikzcd} X(U) \wedge \mathbf{O}(U,V) \wedge \mathbf{O}(V,W) \arrow[r, "\circ\wedge\mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\circ"'] & X(V) \wedge \mathbf{O}(V,W) \arrow[d, "\circ"] \\ X(U) \wedge \mathbf{O}(U,W) \arrow[r, "\circ"'] & X(W) \end{tikzcd}$$

commutes for every triple of inner product spaces. So for every orthogonal spectrum X, the action maps (5.2) make the collection of based spaces $\{X(V)\}_V$ into a based continuous functor $X : \mathbf{O} \longrightarrow \mathcal{T}$.

Proposition 5.3. *The assignment $X \mapsto \{X(V)\}_V$ is in fact an equivalence of categories from the category of orthogonal spectra to the category of (based, continuous) functors from $\mathbf{O}$ to $\mathcal{T}$.*

So loosely speaking, orthogonal spectra 'are' enriched functors from $\mathbf{O}$ to based spaces.

Now we let G be a compact Lie group and V an orthogonal G-representation. Now we define the free functor

$$F_{G,V} \;:\; G\mathbf{T} \;\longrightarrow\; \mathcal{S}p \tag{5.4}$$

that is left adjoint of the evaluation functor at the G-representation V. The free orthogonal spectrum $F_{G,V}A$ generated by a based G-space A in level V is given in level n by

$$(F_{G,V}A)_n \;=\; \mathbf{O}(V,\mathbb{R}^n)\wedge_G A \;,$$

with with $O(n)$-action through $\mathbb{R}^n$. The right G-action on $\mathbf{O}(V,\mathbb{R}^n)$ is via the right G-action on $\mathbf{L}(V,W)$, i.e.,

$$(\varphi,x)\cdot g \;=\; (\varphi g, x)\;,$$

for $\varphi\in\mathbf{L}(V,\mathbb{R}^n)$ and $x\in\mathbb{R}^n-\varphi(V)$, where $(\varphi g)(v)=\varphi(gv)$. We note that $F_{G,V}A$ consists of a single point in all levels below the dimension of V. The structure map $(F_{G,V}A)_n\wedge S^1\longrightarrow(F_{G,V}A)_{n+1}$ is given by

$$\mathbf{O}(V,\mathbb{R}^n)\wedge_G A\wedge S^1 \xrightarrow{\;\mathrm{Id}\wedge\kappa\;} (\mathbf{O}(V,\mathbb{R}^n)\wedge\mathbf{O}(\mathbb{R}^n,\mathbb{R}^{n+1}))\wedge_G A \xrightarrow{\;\circ\;} \mathbf{O}(V,\mathbb{R}^{n+1})\wedge_G A\;.$$

The ‘freeness’ property of $F_{V,G}$A means: for every orthogonal spectrum X and every based G-map $f: A\longrightarrow X(V)$ there is a unique morphism $\hat f: F_{G,V}A\longrightarrow X$ of orthogonal spectra such that the composite

$$A\xrightarrow{\;\mathrm{Id}\wedge -\;}\mathbf{O}(V,V)\wedge_G A=(F_{G,V}A)(V)\xrightarrow{\;\hat f(V)\;} X(V)$$

is f. Indeed, the morphism $\hat f$ is given in level n as the composite

$$\mathbf{O}(V,\mathbb{R}^n)\wedge_G A\xrightarrow{\;\mathrm{Id}\wedge f\;}\mathbf{O}(V,\mathbb{R}^n)\wedge_G X(V)\xrightarrow{\;\circ\;} X_n\;.$$

The smash product of two free orthogonal spectra is again a free orthogonal spectra. In more detail, we consider

- two compact Lie groups G and K,
- a G-representation V and a K-representation W, and
- a based G-space A and a based K-space B.

Here $V\oplus W$ is a $(G\times K)$-representation and $A\wedge B$ is a $(G\times K)$-space via

$$(g,k)\cdot(v,w)\;=\;(gv,kw)\qquad\text{respectively}\qquad (g,k)\cdot(a\wedge b)\;=\;ga\wedge kb\;.$$

We claim that the smash product $F_{G,V}A\wedge F_{K,W}B$ is canonically isomorphic to the free spectrum $F_{G\times K,V\oplus W}(A\wedge B)$. Indeed, a morphism

$$F_{G,V}A\wedge F_{K,W}B\longrightarrow F_{G\times K,V\oplus W}(A\wedge B) \tag{5.5}$$

is obtained by the universal property (4.1) from the bimorphism with (p,q)-component

$$\begin{aligned}(F_{G,V}A)_p\wedge(F_{K,W}B)_q \;&=\; (\mathbf{O}(V,\mathbb{R}^p)\wedge_G A)\wedge(\mathbf{O}(W,\mathbb{R}^q)\wedge_K B)\\ &\xrightarrow{\;\oplus\;}\mathbf{O}(V\oplus W,\mathbb{R}^{p+q})\wedge_{G\times K}(A\wedge B)\;=\;((F_{G\times K,V\oplus W})(A\wedge B))_{p+q}\;.\end{aligned}$$

In the other direction, a morphism $F_{G\times K,V\oplus W}(A\wedge B)\longrightarrow F_{G,V}A\wedge F_{G,W}B$ is freely generated by the $(G\times K)$-map

$$A\wedge B\;\longrightarrow\;(F_{G,V}A)(V)\wedge(F_{K,W}B)(W)\xrightarrow{\;i_{V,W}\;}(F_{G,V}A\wedge F_{K,W}B)(V\oplus W)\;.$$

These two maps are inverse to each other.

In Proposition 2.6 (v) we have seen that for every compact Lie group G, every faithful G-representation V and every other G-representation W the restriction morphism of orthogonal spaces $\rho_{G,V,W}:\mathbf{L}_{G,V\oplus W}\longrightarrow\mathbf{L}_{G,V}$ is a global equivalence. One consequence is that the free orthogonal space $\mathbf{L}_{G.V}$ has a well-defined global homotopy type, independent of which faithful G-representation is used.

Another consequence is that the induced morphism

$$\Sigma^\infty_+\rho_{G,V,W}\;:\;\Sigma^\infty_+\mathbf{L}_{G,V\oplus W}\;\longrightarrow\;\Sigma^\infty_+\mathbf{L}_{G,V}$$

of suspension spectra is a global equivalence of orthogonal spectra. For an inner product space U, the untwisting isomorphisms (5.1) descend to homeomorphisms on G-orbit spaces

$$\mathbf{O}(V,U) \wedge_G S^V \;\cong\; \mathbf{L}(V,U)/G_+ \wedge S^U \ .$$

As U varies, these form an isomorphism of orthogonal spectra

$$F_{G,V} S^V \;\cong\; \Sigma^\infty_+ \mathbf{L}_{G,V} \ .$$

So suspension spectra of free orthogonal spaces are free orthogonal spectra. One can think of $F_{G,V\oplus W}S^W$ and $F_{G,V}$ as global Thom spectra of certain 'virtual global vector bundles' over the orthogonal spaces $F_{G,V\oplus W}$ respectively $F_{G,V}$.

We will now prove a generalization of the fact that $\Sigma^\infty_+ \rho_{G,V,W}$ is a global equivalence for certain 'twisted forms' of suspension spectra of free global spaces, i.e., certain free orthogonal spectra that are 'global Thom spectra'. Given a compact Lie group G and G-representations V and W we can define a restriction morphism of orthogonal spectra

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^W \ \longrightarrow \ F_{G,V} \tag{5.6}$$

at an inner product space U as

$$\lambda_{G,V,W}(U) \ : \ \mathbf{O}(V\oplus W,U) \wedge_G S^W \ \longrightarrow \ \mathbf{O}(V,U)/G \ , \quad [(\varphi,u)\wedge w] \ \longmapsto \ [\varphi|_V,\, u+\varphi(w)] \ .$$

Moreover, the bundle over $F_{G,V\oplus W}$ is obtained from the bundle over $F_{G,V}$ by pullback along the global equivalence $\lambda_{G,V,W} : F_{G,V\oplus W} \longrightarrow F_{G,V}$. This ought to motivate that the morphism $\lambda_{G,V,W}$ is a global equivalence of orthogonal spectra. However, we will not define the concept of 'global vector bundles' and instead give a more direct proof of the global equivalence.

Theorem 5.7. *Let G be a compact Lie group, V a faithful G-representation and W any G-representation. Then the morphism*

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^W \ \longrightarrow \ F_{G,V}$$

is a global equivalence of orthogonal spectra.

Proof. We let K be another compact Lie group and $U \in s(\mathcal{U})$ a finite dimensional K-subrepresentation of the complete K-universe $\mathcal{U}_K$. In a first step we produce a K-representation $U' \in s(\mathcal{U})$ with $U \subseteq U'$ and a continuous $(K\times G^{\mathrm{op}})$-equivariant map

$$h \ : \ \mathbf{L}(V,U) \ \longrightarrow \ \mathbf{L}(V\oplus W,U')$$

such that in the diagram

$$\begin{tikzcd} \mathbf{L}(V\oplus W,U) \arrow[r, "i_*"] \arrow[d, "\mathrm{res}"'] & \mathbf{L}(V\oplus W,U') \arrow[d, "\mathrm{res}"] \\ \mathbf{L}(V,U) \arrow[r, "i_*"'] \arrow[ur, dashed, "h"] & \mathbf{L}(V,U') \end{tikzcd} \tag{5.8}$$

the lower right triangle commutes, and the upper left triangle commutes up to $(K\times G^{\mathrm{op}})$-equivariant fiberwise homotopy over $\mathbf{L}(V,U')$, where $i : U \longrightarrow U'$ is the inclusion.

Since G acts faithfully on V (and hence on $V\oplus W$), both $\mathbf{L}(V,\mathcal{U}_K)$ and $\mathbf{L}(V\oplus W,\mathcal{U}_K)$ are universal spaces for the same family of subgroups of $K\times G^{\mathrm{op}}$, namely the family of those $\Gamma \leq K\times G^{\mathrm{op}}$ that intersect $1\times G^{\mathrm{op}}$ only in the identity element, compare Proposition 2.6 (i). Moreover, if Γ is in this family, then it is the graph of a homomorphism $\alpha : L \longrightarrow G$ defined on some subgroup L of K. The Γ-fixed points of $\mathbf{L}(V,\mathcal{U}_K)$ are then given by

$$\mathbf{L}(V,\mathcal{U}_K)^\Gamma \ = \ \mathbf{L}^L(\alpha^* V, i^*\mathcal{U}_K) \ ,$$

the space of L-equivariant linear isometric embeddings from α^*V. The same is true for $V\oplus W$ instead of V, and so the Γ-fixed point map $\mathrm{res}^\Gamma : \mathbf{L}(V\oplus W,\mathcal{U}_K)^\Gamma \longrightarrow \mathbf{L}(V,\mathcal{U}_K)^\Gamma$ is the restriction map

$$\mathbf{L}^L(\alpha^*V\oplus\alpha^*W, i^*\mathcal{U}_K) \longrightarrow \mathbf{L}^L(\alpha^*V, i^*\mathcal{U}_K)$$

to the summand α^*V. This map is a locally trivial fiber bundle, hence a Serre fibration. We conclude that the restriction map $\mathrm{res} : \mathbf{L}(V\oplus W,\mathcal{U}_K) \longrightarrow \mathbf{L}(V\oplus W,\mathcal{U}_K)$ is both a $(K\times G^{\mathrm{op}})$-weak equivalence and a $(K\times G^{\mathrm{op}})$-fibration.

Non-equivariantly, the space $\mathbf{L}(V,U)$ is homeomorphic to the homogeneous space $O(n+m)/O(n)$, where $n=\dim V$ and $n+m=\dim U$, hence a smooth manifold. The $(K\times G^{\mathrm{op}})$-action on $\mathbf{L}(V,U)$ is smooth, so by Illman's theorem [**45**,], $\mathbf{L}(V,U)$ can be given the structure of a $(K\times G^{\mathrm{op}})$-CW-complex. The $(K\times G^{\mathrm{op}})$-map $i_* : \mathbf{L}(V,U) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)$ thus admits a $(K\times G^{\mathrm{op}})$-equivariant lift $h : \mathbf{L}(V,U) \longrightarrow \mathbf{L}(V\oplus W,\mathcal{U}_K)$ such that $\mathrm{res}\circ h = i_*$. Since the space $\mathbf{L}(V,U)$ is compact and $\mathbf{L}(V\oplus W,\mathcal{U}_K)$ is the filtered union of the closed subspaces $\mathbf{L}(V\oplus W,U')$ for $U'\in s(\mathcal{U}_K)$, the lift h lands in the subspace $\mathbf{L}(V\oplus W,U')$ for suitably large $U'\in s(\mathcal{U}_K)$, and we may assume that $U\subseteq U'$.

The two maps

$$h\circ \mathrm{res}\ ,\ i_*\ :\ \mathbf{L}(V\oplus W,U) \longrightarrow \mathbf{L}(V\oplus W,U')$$

become equal after applying $\mathrm{res} : \mathbf{L}(V\oplus W,U') \longrightarrow \mathbf{L}(V,U')$, hence the composites with $i_* : \mathbf{L}(V\oplus W,U') \longrightarrow \mathbf{L}(V\oplus W,\mathcal{U}_K)$ become equal after applying the $(K\times G^{\mathrm{op}})$-equivariant acyclic fibration $\mathrm{res} : \mathbf{L}(V\oplus W,\mathcal{U}_K) \longrightarrow \mathbf{L}(V,\mathcal{U}_K)$. Since $\mathbf{L}(V\oplus W,U)$ also admits the structure of a $(K\times G^{\mathrm{op}})$-CW-complex, there is a fiberwise $(K\times G^{\mathrm{op}})$-equivariant homotopy between $h\circ\mathrm{res}$ and i_* in $\mathbf{L}(V\oplus W,\mathcal{U}_K)$. Again by compactness, the homotopy has image in $\mathbf{L}(V\oplus W,U'')$ for suitably large $U''\in s(\mathcal{U}_K)$. So after increasing U', if necessary, we have proved the claim subsumed in the diagram (5.8).

Now we lift the data produced in the first step to the Thom spaces of the orthogonal complement bundles. The diagram (5.8) is covered by morphisms of $(K\times G^{\mathrm{op}})$-vector bundles:

(5.9)
$$\begin{tikzcd}
\xi(V\oplus W,U)\times W\times(U'-U) \arrow[r, "\bar{i}"] \arrow[d, "\overline{\mathrm{res}}"'] & \xi(V\oplus W,U')\times W \arrow[d, "\overline{\mathrm{res}}"] \\
\xi(V,U)\times(U'-U) \arrow[r, "\bar{i}"'] \arrow[ur, dashed, "\bar{h}"] & \xi(V,U')
\end{tikzcd}$$

The maps on the total spaces of the bundles are defined as follows: The right vertical morphism is defined by

$$\overline{\mathrm{res}}\ :\ \xi(V\oplus W,U')\times W \longrightarrow \xi(V,U')\ ,\quad ((\varphi,u'),\,w)\ \longmapsto\ (\varphi|_V,\,u'+\varphi(w))\ .$$

The left vertical morphism is defined in the same way, but with U instead of U' and multiplied by the identity of $U'-U$. The lower horizontal morphism is defined by

$$\bar{i}\ :\ \xi(V,U)\times(U'-U) \longrightarrow \xi(V,U')\ ,\quad ((\varphi,u),\,u')\ \longmapsto\ (\varphi,\,u+u')\ .$$

The upper horizontal morphism is defined in the same way, but with $V\oplus W$ instead of V and multiplied by the identity of W. These four outer morphisms in (5.9) are all fiberwise linear isomorphisms; so each of these four bundle maps expresses the source bundle as a pullback of the target bundle. In particular, the square

(5.10)
$$\begin{tikzcd}
\xi(V\oplus W,U')\times W \arrow[r, "\overline{\mathrm{res}}"] \arrow[d] & \xi(V,U') \arrow[d] \\
\mathbf{L}(V\oplus W,U') \arrow[r, "\mathrm{res}"'] & \mathbf{L}(V,U')
\end{tikzcd}$$

is a pullback; so the composite

$$\xi(V,U) \times (U'-U) \;\longrightarrow\; \mathbf{L}(V,U) \;\xrightarrow{h}\; \mathbf{L}(V \oplus W, U')$$

and the map of total spaces $\bar{i} : \xi(V,U) \times (U'-U) \longrightarrow \xi(V,U')$ assemble into a map

$$\bar{h} \;:\; \xi(V,U) \times (U'-U) \;\longrightarrow\; \xi(V \oplus W, U')$$

that covers h and is a fiberwise linear isomorphism.

In (5.9) (as in (5.8)) the outer square and the lower right triangle commute, but the upper left triangle does *not* commute. We will now show that the upper left triangle commutes up to homotopy of $(K \times G^{\mathrm{op}})$-equivariant bundle maps. For this purpose we let

$$H \;:\; \mathbf{L}(V \oplus W, U) \times [0,1] \;\longrightarrow\; \mathbf{L}(V \oplus W, U')$$

by a $(K \times G^{\mathrm{op}})$-equivariant homotopy from the map i_* to $h \circ \mathrm{res}$, such that $\mathrm{res} \circ H : \mathbf{L}(V \oplus W, U) \times [0,1] \longrightarrow \mathbf{L}(V, U')$ is the constant homotopy from map $\mathrm{res} \circ i_* = i_* \circ \mathrm{res}$ to itself. Again because the square (5.10) is a pullback, the composite

$$\xi(V \oplus W, U) \times W \times (U'-U) \times [0,1] \;\longrightarrow\; \mathbf{L}(V \oplus W, U) \times [0,1] \;\xrightarrow{H}\; \mathbf{L}(V \oplus W, U')$$

and the map of total spaces

$$\xi(V \oplus W, U) \times W \times (U'-U) \times [0,1] \;\xrightarrow{\mathrm{proj}}\; \xi(V \oplus W, U) \times W \times (U'-U) \;\xrightarrow{\overline{\mathrm{res}} \circ \bar{i} = \bar{i} \circ \overline{\mathrm{res}}}\; \xi(V, U')$$

assemble into a map

$$\bar{H} \;:\; \xi(V \oplus W, U) \times W \times (U'-U) \times [0,1] \;\longrightarrow\; \xi(V \oplus W, U') \times W$$

that covers the homotopy H. This lift $\bar{H}$ is a $(K \times G^{\mathrm{op}})$-equivariant homotopy of vector bundle morphisms, and for every $t \in [0,1]$, the relation

$$\overline{\mathrm{res}} \circ \bar{H}(-,t) \;=\; \overline{\mathrm{res}} \circ \bar{i} \;=\; \overline{\mathrm{res}} \circ (\bar{h} \circ \overline{\mathrm{res}})$$

holds by definition of $\bar{H}$. For $t = 0$ this shows that $\bar{H}$ starts with $\bar{i} : \xi(V \oplus W, U) \times W \times (U'-U) \longrightarrow \xi(V \oplus W, U') \times W$; for $t = 1$ this shows that $\bar{H}$ ends with $\bar{h} \circ \overline{\mathrm{res}}$, one more time because (5.10) is a pullback. We conclude that $\bar{H}$ makes the upper left triangle in (5.9) commute up to equivariant homotopy of vector bundle maps.

Passing to Thom spaces in (5.9) gives a diagram of $(K \times G^{\mathrm{op}})$-equivariant based maps

$$\begin{tikzcd}
\mathbf{O}(V \oplus W, U) \wedge S^W \wedge S^{U'-U} \arrow[rr, "\sigma_{U,U'-U}"] \arrow[d, "\lambda_{V,W}(U) \wedge S^{U'-U}"'] & & \mathbf{O}(V \oplus W, U') \wedge S^W \arrow[d, "\lambda_{V,W}(U')"] \\
\mathbf{O}(V,U) \wedge S^{U'-U} \arrow[rru, dashed, "\bar{h}"] \arrow[rr, "\sigma_{U,U'-U}"'] & & \mathbf{O}(V,U')
\end{tikzcd}$$

Again, lower right triangle commutes, and the upper left triangle commutes up to $(K \times G^{\mathrm{op}})$-equivariant based homotopy. We pass to G-orbit spaces and obtain a diagram of based G-spaces

$$\begin{tikzcd}
(F_{G,V \oplus W} S^W)(U) \wedge S^{U'-U} \arrow[rr, "\sigma_{U,U'-U}"] \arrow[d, "\lambda_{G,V,W}(U) \wedge S^{U'-U}"'] & & (F_{G,V \oplus W} S^W)(U') \arrow[d, "\lambda_{G,V,W}(U')"] \\
(F_{G,V})(U) \wedge S^{U'-U} \arrow[rru, "\bar{h}/G"] \arrow[rr, "\sigma_{U,U'-U}"'] & & (F_{G,V})(U')
\end{tikzcd}$$

whose lower right triangle commutes, and whose upper left triangle commutes up to K-equivariant based homotopy. Since we had started with an arbitrary K-subrepresentation $U \in s(\mathcal{U}_K)$, this implies that for every based K-space A the map on colimits

$$\begin{aligned}(F_{G,V\oplus W}S^W)^0(A) \ &= \ \operatorname{colim}_{U\in s(\mathcal{U}_K)}\, [A\wedge S^U, (F_{G,V\oplus W}S^W)(U)]^K \\ &\longrightarrow \ \operatorname{colim}_{U\in s(\mathcal{U}_K)}\, [A\wedge S^U, (F_{G,V})(U)]^K \ = \ (F_{G,V})^0(A)\end{aligned}$$

induced by the morphism $\lambda_{G,V,W}$ is bijective. So $\lambda_{G,V,W}$ is a global equivalence. □

6. Geometric fixed points

It is often convenient that equivariant equivalences can be detected by looking at a different kind of homotopy groups, the 'geometric fixed point homotopy groups'. In this section we recall the definition and prove some properties that we need later.

We recall the *geometric fixed point* homotopy groups $\Phi^G_* X$ of an orthogonal spectrum X with respect to a compact Lie group G. As before we let $s(\mathcal{U}_G)$ denote the set of finite dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$, considered as a poset under inclusion. We obtain a functor from $s(\mathcal{U}_G)$ to sets by

$$V \ \longmapsto \ [S^{V^G}, X(V)^G] \ ,$$

the set of (non-equivariant) homotopy classes of based maps from the fixed point sphere S^{V^G} to the fixed point space $X(V)^G$. For $V \subseteq W$ in $s(\mathcal{U}_G)$ is sent to the map

$$[S^{V^G}, X(V)^G] \ \longrightarrow \ [S^{W^G}, X(W)^G]$$

that takes the homotopy class of $\varphi : S^{V^G} \longrightarrow X(V)^G$ to the homotopy class of the composite

$$S^{W^G} \cong S^{V^G} \wedge S^{(V^\perp)^G} \xrightarrow{\varphi\wedge \mathrm{Id}} X(V)^G \wedge S^{(V^\perp)^G} \ \cong \ (X(V)\wedge S^{V^\perp})^G \xrightarrow{\sigma^G_{V,V^\perp}} X(V\oplus V^\perp)^G \cong X(W)^G$$

Definition 6.1. Let X be an orthogonal spectrum and G a compact Lie groups. The 0-th *geometric fixed point homotopy group* is then defined as

$$\Phi^G_0(X) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)}\, [S^{V^G}, X(V)^G] \ . \tag{6.2}$$

If k is an arbitrary integer, we define the k-th equivariant homotopy group $\pi^G_0 X$ as the 0-th homotopy group of a suitably looped or suspended spectrum, analogous to (2.8).

The construction comes with a *geometric fixed point map*

$$\Phi \ : \ \pi^G_0(X) \ \longrightarrow \ \Phi^G_0(X) \ , \quad [f : S^V \longrightarrow X(V)] \ \longmapsto \ [f^G : S^{V^G} \longrightarrow X(V)^G] \tag{6.3}$$

from the G-equivariant homotopy groups to the geometric fixed point homotopy groups.

Example 6.4 (Geometric fixed points of suspension spectra)**.** If A is any based space (CW-complex), then the geometric fixed points $\Phi^G_*(\Sigma^\infty A)$ are given by

$$\Phi^G_k(\Sigma^\infty A) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)}\, [S^{k+V^G}, A\wedge S^{V^G}] \ .$$

Moreover, the composite

$$\pi^e_k(\Sigma^\infty A) \xrightarrow{p^*} \pi^G_k(\Sigma^\infty A) \xrightarrow{\Phi} \Phi^G_k(\Sigma^\infty A)$$

is an isomorphism, where p^* is restriction along the unique group homomorphism $p : G \longrightarrow e$.

Similarly, we let Y is any orthogonal space and look at the geometric fixed points of the suspension spectrum $\Sigma^\infty_+ Y$; these are given by

$$\Phi^G_k(\Sigma^\infty_+ Y) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)}\, [S^{k+V^G}, Y(V)^G_+ \wedge S^{V^G}] \ \cong \ \pi^{\mathrm{st}}_k(Y(\mathcal{U}_G)^G_+) \ ,$$

the non-equivariant stable homotopy groups of the G-fixed points $Y(\mathcal{U}_G)^G$. We will sometimes refer to this isomorphism by saying that 'geometric fixed points commute with suspension spectra'.

The collection of equivariant homotopy groups $\{\pi_0^G(X)\}_G$ come with restriction and transfer maps, and this data together forms a global functor. The geometric fixed point groups have fewer natural operations: the geometric fixed point map annihilates all transfers from proper subgroups H of G; and geometric fixed points do *not* have restrictions to subgroups. However, geometric fixed points still have restriction maps *along epimorphisms*. (This is one more time a piece of structure that exists is the global world, but not in G-equivariant stable homotopy theory for a fixed group G.)

Proposition 6.5. *Let X be an orthogonal spectrum, G and compact Lie group and H a proper subgroup of G. Then the geometric fixed point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ annihilates the image of the transfer homomorphism $\mathrm{tr}_H^G : \pi_0^H(X) \longrightarrow \pi_0^G(X)$.*

Proof. By the very definition in (2.30), the transfer $\mathrm{tr}_H^G(x)$ of an element $x \in \pi_0^H(X)$ is represented by a based G-map that factors through a G-space of the form $S^U \wedge G/H^+ \wedge S^V$. So $\Phi(\mathrm{tr}_H^G(x))$ is represented by a based map that factors through

$$(S^U \wedge G/H^+ \wedge S^V)^G \;=\; (S^U)^G \wedge ((G/H)^G)^+ \wedge (S^V)^G \ .$$

If H is a proper subgroup of G, then G/H has no G-fixed points, and so $\Phi(\mathrm{tr}_H^G(x))$ is represented by the constant map at the basepoint. Thus $\Phi(\mathrm{tr}_H^G(x)) = 0$. □

Construction 6.6. We let X be an orthogonal spectrum and define *restriction maps*

$$\alpha^* \ : \ \Phi_0^G(X) \ \longrightarrow \ \Phi_0^K(X)$$

on geometric fixed point homotopy groups for every *epimorphism* $\alpha : K \longrightarrow G$.

We choose a K-equivariant linear isometric embedding $\psi : \alpha^*(\mathcal{U}_G) \longrightarrow \mathcal{U}_K$ of the restriction along α of the complete G-universe into the complete K-universe. We let $f : S^{V^G} \longrightarrow X(V)^G$ be a based map representing an element in $\Phi_0^G X$, for some $V \in s(\mathcal{U}_G)$. Since α is surjective, $V^G = (\alpha^* V)^K$ and $X(V)^G = (\alpha^* X(V))^K = X(\alpha^* V)^K$. We use ψ to identify $\alpha^* V$ with $\psi(V)$ as K-representations, and hence also $(\alpha^* V)^K$ with $\psi(V)^K$. This turns f into a based map

$$S^{\psi(V)^K} \ \cong \ S^{(\alpha^* V)^K} \ = \ S^{V^G} \ \xrightarrow{\ f\ } \ X(V)^G \ = \ X(\alpha^* V)^K \ \cong \ X(\psi(V))^K \ .$$

This latter map represents the element $\alpha^*[f]$ in $\Phi_0^K(X)$. The element $\alpha^*[f]$ depends only on the class of f in $\Phi_0^G(X)$, so the restriction map α^* is a well defined homomorphism. Any two equivariant embeddings of $\alpha^*(\mathcal{U}_G)$ into $\mathcal{U}_K$ are homotopic through K-equivariant linear isometric embeddings, so the restriction map is independent of the choice of ψ.

The surjectivity of α is essential to obtain a restriction map α^*, and geometric fixed points do not have natural restriction maps to subgroups. These restriction maps between the geometric fixed point homotopy groups are clearly natural in the orthogonal spectrum. The next proposition lists the other naturality properties, which can be summarized by saying the geometric fixed point homotopy groups and restriction maps from a contravariant functor on the category of compact Lie groups and conjugacy classes of continuous epimorphisms, and the geometric fixed point map (6.3) is a natural transformation of functors on this category.

Proposition 6.7. *Let X be an orthogonal spectrum.*

(i) *For a trivial group, equivariant and geometric fixed point groups coincide and the geometric fixed point map $\Phi : \pi_0^e(X) \longrightarrow \Phi_0^e(X) = \pi_0^e(X)$ is the identity.*

(ii) *For every pair of composable continuous epimorphisms $\alpha : K \longrightarrow G$ and $\beta : L \longrightarrow K$ we have*

$$\beta^* \circ \alpha^* \ = \ (\alpha\beta)^* \ : \ \Phi_0^G(X) \ \longrightarrow \ \Phi_0^L(X) \ .$$

(iii) *For every compact Lie group G and every element $g \in G$ the restriction map $c_g^* : \Phi_0^G(X) \longrightarrow \Phi_0^G(X)$ associated to the conjugation homomorphism $c_g : G \longrightarrow G$ is the identity.*

(iv) *For every surjective continuous homomorphism $\alpha : K \longrightarrow G$ of compact Lie groups the following square commutes:*

$$\begin{tikzcd} \pi_0^G(X) \arrow[r, "\Phi"] \arrow[d, "\alpha^*"'] & \Phi_0^G(X) \arrow[d, "\alpha^*"] \\ \pi_0^K(X) \arrow[r, "\Phi"'] & \Phi_0^K(X) \end{tikzcd}$$

Construction 6.8. Like the equivariant homotopy groups, the geometric fixed point homotopy groups also come with pairings. Given two orthogonal spectra X and Y, and compact Lie groups G and K, we define an external pairing

$$\times \ : \ \Phi_k^G(X) \ \times \ \Phi_l^K(Y) \ \longrightarrow \ \Phi_{k+l}^{G\times K}(X \wedge Y) \ , \tag{6.9}$$

Again we define the pairing first for $k = l = 0$. Suppose that V and W are representations of G respectively K and $f : S^{V^G} \longrightarrow (X(V))^G$ and $g : S^{W^K} \longrightarrow (Y(W))^K$ are based maps representing classes in $\Phi_0^G X$ respectively $\Phi_0^K Y$. We view $V \oplus W$ as a representation of the product group $G \times K$ via $(g,k)\cdot(v,w) = (gv, kw)$. We denote by $f \times g$ the $(G \times K)$-equivariant composite

$$\begin{aligned} S^{(V+W)^{G\times K}} \cong S^{V^G} \wedge S^{W^K} \ &\xrightarrow{f\wedge g} \ (X(V))^G \wedge (Y(W))^K \\ &\cong \ (X(V) \wedge Y(W))^{G\times K} \ \xrightarrow{i_{V,W}^{G\times K}} \ ((X \wedge Y)(V \oplus W))^{G\times K} \ . \end{aligned}$$

[embed $V \times W$ into $\mathcal{U}_{G\times K}$] Similar arguments as for the pairing on equivariant homotopy groups in Construction 4.5 show that the definition

$$[f] \times [g] \ = \ \langle f \times g \rangle \ \in \ \Phi_0^{G\times K}(X \wedge Y)$$

is well-defined.

The pairing of geometric fixed point homotopy groups have properties analogous to that of equivariant homotopy groups (compare Theorem (4.7)).

Theorem 6.10. *Let G, K and L be compact Lie groups and X, Y and Z orthogonal spectra.*

(i) *(Biadditivity) The product $\times : \Phi_k^G(X) \times \Phi_l^K(Y) \longrightarrow \Phi_{k+l}^{G\times K}(X \wedge Y)$ is biadditive.*

(ii) *(Unitality) Let $1 \in \Phi_0^e(\mathbb{S}) = \pi_0^e(\mathbb{S})$ denote the class represented by the identity of S^V for any inner product space V. The product is unital in the sense that $1 \times x = x = x \times 1$ under the identifications $\mathbb{S} \wedge X = X = X \wedge \mathbb{S}$ and $e \times G \cong G \cong G \times e$.*

(iii) *(Associativity) For all classes $x \in \Phi_k^G(X)$, $y \in \Phi_l^K(Y)$ and $z \in \Phi_j^L(Z)$ the relation*

$$x \times (y \times z) \ = \ (x \times y) \times z$$

holds in $\Phi_{k+l+j}^{G\times K\times L}(X \wedge Y \wedge Z)$.

(iv) *(Commutativity) For all classes $x \in \Phi_k^G(X)$ and $y \in \Phi_l^K(Y)$ the relation*

$$y \times x \ = \ (-1)^{kl} \cdot \tau_{G,K}^*(\tau_*^{X,Y}(x \times y))$$

holds in $\Phi_{l+k}^{K\times G}(Y \wedge X)$, where $\tau^{X,Y} : X \wedge Y \longrightarrow Y \wedge X$ is the symmetry isomorphism of the smash product and $\tau_{G,K} : K \times G \longrightarrow G \times K$ interchanges the factors.

(v) *(Restriction) For all classes $x \in \Phi_k^G(X)$ and $y \in \Phi_l^K(Y)$ and all continuous epimorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$\alpha^*(x) \times \beta^*(y) \ = \ (\alpha \times \beta)^*(x \times y)$$

holds in $\Phi_{k+l}^{\bar{G}\times\bar{K}}(X \wedge Y)$.

(vi) *(Geometric fixed point map) For all classes $x \in \pi_k^G(X)$ and $y \in \pi_l^K(Y)$ the relation*

$$\Phi(x) \times \Phi(y) \;=\; \Phi(x \times y)$$

holds in $\Phi_{k+l}^{G\times K}(X \wedge Y)$.

Proof. The associativity property (iii) and compatibility with restriction (iv) are straightforward from the definitions. □

Given an orthogonal ring spectrum R and compact Lie groups G and K, we define an internal pairing on the geometric fixed point homotopy groups of R as the composite

$$\Phi_k^G(R) \times \Phi_l^K(R) \xrightarrow{\;\times\;} \Phi_{k+l}^{G\times K}(R \wedge R) \xrightarrow{\;\mu_*\;} \Phi_{k+l}^{G\times K}(R)\ .$$

Corollary 6.11. *Let R be an orthogonal ring spectrum and G, K and L compact Lie groups.*

(i) *(Biadditivity) The product $\times : \Phi_k^G(R) \times \Phi_l^K(R) \longrightarrow \Phi_{k+l}^{G\times K}(R)$ is biadditive.*

(ii) *(Unitality) The unit $1 \in \Phi_0^e(R) = \pi_0^e(R)$ is unital in the sense that $1 \times x = x = x \times 1$ under the identifications $e \times G \cong G \cong G \times e$.*

(iii) *(Associativity) For all classes $x \in \Phi_k^G(R)$, $y \in \Phi_l^K(R)$ and $z \in \Phi_j^L(R)$ we have $(x\times y)\times z = x\times(y\times z)$ under the identification $(G \times K) \times L \cong G \times (K \times L)$.*

(iv) *(Commutativity) If the multiplication of R is commutative, then for all classes $x \in \Phi_k^G(R)$ and $y \in \Phi_l^K(R)$ the relation*

$$y \times x \;=\; (-1)^{kl} \cdot \tau_{G,K}^*(x \times y)$$

holds in $\Phi_{l+k}^{K\times G}(R)$, where $\tau_{G,K} : K \times G \longrightarrow G \times K$ interchanges the factors.

(v) *(Restriction) For all classes $x \in \Phi_k^G(R)$ and $y \in \Phi_l^K(R)$ and all continuous epimorphisms $\alpha : \bar{G} \longrightarrow G$ and $\beta : \bar{K} \longrightarrow K$ the relation*

$$(\alpha^* x) \times (\beta^* y) \;=\; (\alpha \times \beta)^*(x \times y)$$

holds in the group $\Phi_{k+l}^{\bar{G}\times\bar{K}}(R)$.

(vi) *(Geometric fixed point map) For all classes $x \in \pi_k^G(R)$ and $y \in \pi_l^K(R)$ the relation*

$$\Phi(x) \times \Phi(y) \;=\; \Phi(x \times y)$$

holds in $\Phi_{k+l}^{G\times K}(R)$.

Now we can given another interpretation of the geometric fixed point homotopy groups as the equivariant homotopy groups of the smash product of X with a certain universal G-space. This makes the link to other definitions of the geometric fixed point spectra. We denote by $\mathcal{P}_G$ the family of proper subgroups of G. We denote by $E\mathcal{P}_G$ a universal space for the family $\mathcal{P}_G$; so $E\mathcal{P}_G$ is a G-CW-complex without G-fixed points and such that the fixed point space $(E\mathcal{P}_G)^H$ is contractible for every proper subgroup H of G. These properties determine $E\mathcal{P}_G$ uniquely up to G-homotopy equivalence.

We denote by $\tilde{E}\mathcal{P}_G$ the reduced mapping cone of the based G-map $E\mathcal{P}_G^+ \longrightarrow S^0$ that sends $E\mathcal{P}_G$ to the non-basepoint of S^0. So $\tilde{E}\mathcal{P}_G$ is the unreduced suspension of the universal space $E\mathcal{P}_G$. The G-fixed points of $E\mathcal{P}_G$ are empty and fixed points commute with mapping cones, so the map $S^0 \longrightarrow (\tilde{E}\mathcal{P}_G)^G$ is an isomorphism. For all proper subgroups H of G the map $(E\mathcal{P}_G)^H \longrightarrow (S^0)^H = S^0$ is a weak equivalence, so the mapping cone $(\tilde{E}\mathcal{P}_G)^H$ is contractible.

Example 6.12. We let $\mathcal{U}_G^\perp = \mathcal{U}_G - (\mathcal{U}_G)^G$ be the orthogonal complement of the G-fixed points in the complete G-universe $\mathcal{U}_G$. We claim that the unit sphere $S(\mathcal{U}_G^\perp)$ of this complement is a universal space $E\mathcal{P}_G$ and hence the representation sphere

$$S^{\mathcal{U}_G^\perp} \;=\; \tilde{E}\mathcal{P}_G$$

of the orthogonal complement is a model for $\tilde{E}\mathcal{P}_G$.

Indeed, the unit sphere $S(\mathcal{U}_G^\perp)$ can be given a G-CW-structure and it has no G-fixed points. So if H is a subgroup of G that occurs as isotropy group of a point in $\mathcal{U}_G^\perp$, then $H \neq G$, i.e., H belongs to the family $\mathcal{P}_G$. On the other hand, for every proper subgroup H of G there is a G-representation V with $V^G = 0$ but $V^H \neq 0$. Since $\mathcal{U}_G^\perp$ contains infinitely many isomorphic copies of V, the H-fixed points

$$(S(\mathcal{U}_G^\perp))^H \;=\; S((\mathcal{U}_G^\perp)^H)$$

form an infinite dimensional sphere, and hence are contractible.

Lemma 6.13. *Let G be a compact Lie group. For every finite dimensional G-representation V and every based G-space Y the map*

$$(-)^G \;:\; \operatorname{map}^G(S^V, \tilde{E}\mathcal{P}_G \wedge Y) \;\longrightarrow\; \operatorname{map}(S^{V^G}, Y^G)$$

that takes a G-map $f : S^V \longrightarrow \tilde{E}\mathcal{P}_G \wedge Y$ to the induced map on G-fixed points

$$f^G \;:\; (S^V)^G \;\longrightarrow\; (\tilde{E}\mathcal{P}_G \wedge Y)^G = (\tilde{E}\mathcal{P}_G)^G \wedge Y^G \cong Y^G$$

is a weak equivalence and Serre fibration.

Proof. Since S^V admits the structure of a G-CW-complex, the inclusion of fixed points $(S^V)^G \longrightarrow S^V$ is a G-cofibration and induces a Serre fibration of equivariant mapping spaces

$$\operatorname{map}^G(S^V, \tilde{E}\mathcal{P}_G \wedge Y) \;\longrightarrow\; \operatorname{map}^G(S^{V^G}, \tilde{E}\mathcal{P}_G \wedge Y)\ .$$

Since every G-map from S^{V^G} lands in the G-fixed points of $\tilde{E}\mathcal{P}_G \wedge Y$ and because $(\tilde{E}\mathcal{P}_G \wedge Y)^G = Y^G$, the target space is the non-equivariant mapping space $\operatorname{map}(S^{V^G}, Y^G)$. The G-space S^V is built from its fixed points $(S^V)^G = S^{V^G}$ by attaching G-cells $G/H^+ \wedge D^n$ whose isotropy H is a proper subgroup. Since the H-fixed points of $\tilde{E}\mathcal{P}_G \wedge Y$ are contractible for all proper subgroups H of G, the fibration is also a weak equivalence. □

Proposition 6.14. *For every orthogonal G-spectrum X and every integer k, the geometric fixed point map*

$$\Phi \;:\; \pi_k^G(\tilde{E}\mathcal{P}_G \wedge X\langle G\rangle) \;\longrightarrow\; \Phi_k^G(X)$$

is an isomorphism.

A consequence of the previous proposition is the following *isotropy separation sequence.* The mapping cone sequence of based G-CW-complexes

$$(E\mathcal{P}_G)_+ \;\longrightarrow\; S^0 \;\longrightarrow\; \tilde{E}\mathcal{P}_G$$

becomes a mapping cone sequence of G-spectra

$$(E\mathcal{P}_G)_+ \wedge Y \;\longrightarrow\; Y \;\longrightarrow\; \tilde{E}\mathcal{P}_G \wedge Y \tag{6.15}$$

after smashing with any given G-spectrum Y. In the example $Y = X\langle G\rangle$ for an orthogonal spectrum X, taking equivariant homotopy groups gives a long exact sequence

$$\cdots \longrightarrow \pi_k^G((E\mathcal{P}_G)_+ \wedge X\langle G\rangle) \;\longrightarrow\; \pi_k^G(X) \;\xrightarrow{\;\Phi\;}\; \Phi_k^G(X) \;\longrightarrow\; \pi_{k-1}^G((E\mathcal{P}_G)_+ \wedge X\langle G\rangle) \;\longrightarrow\; \cdots \tag{6.16}$$

where we exploited the identification of Proposition 6.14.

Proposition 6.17. *Let G be a compact Lie group. For a morphism $f : X \longrightarrow Y$ of orthogonal G-spectra the following are equivalent:*

(i) *The morphism $f : X \longrightarrow Y$ is an equivalence of orthogonal G-spectra, i.e.,. for every subgroup H of G and every integer k the map*

$$f_* \;:\; \pi_k^H(X) \;\longrightarrow\; \pi_k^H(Y)$$

is an isomorphism.

(ii) *For every subgroup H of G and every integer k the map*

$$f_* \ : \ \Phi_k^H(X) \ \longrightarrow \ \Phi_k^H(Y)$$

is an isomorphism.

Proof. (i)$\Longrightarrow$(ii) If $f\langle G\rangle$ is is an equivalence of orthogonal G-spectra, then so is $\tilde{E}\mathcal{P}_G \wedge f\langle G\rangle$. Indeed, we can use a cell induction on a G-CW-structure on $\tilde{E}\mathcal{P}_G$, which reduces the claim to showing that $G/H_+ \wedge f\langle G\rangle$ by is an equivalence of orthogonal G-spectra for all subgroups H of G; this part uses the Wirthmüller isomorphism. Proposition 6.14 then implies that $f_* : \Phi_k^H(X) \longrightarrow \Phi_k^H(Y)$ is an isomorphism for all k.

(ii)$\Longrightarrow$(i) We show by induction on the size of the group G (i.e., of the dimension of G and order of $\pi_0 G$). If G is the trivial group, then the geometric fixed point map $\Phi : \pi_k^e(X) \longrightarrow \Phi_k^e(X)$ does not do anything, and is an isomorphism.

If G is a non-trivial group we know by induction hypothesis that the map $f\langle H\rangle : X\langle H\rangle \longrightarrow Y\langle H\rangle$ is an equivalence of orthogonal H-spectra for every proper subgroup H of G. Since $E\mathcal{P}_G$ has no G-fixed points, this lets us conclude that $(E\mathcal{P}_G)_+ \wedge f\langle G\rangle$ is an equivalence of orthogonal G-spectra. Since $f_* : \Phi_*^G(X) \longrightarrow \Phi_*^G(Y)$ is also an isomorphism, the isotropy separation sequence lets us conclude that $f_* : \pi_*^G(X) \longrightarrow \pi_*^G(Y)$ is an isomorphism. □

7. Global Borel cohomology

We let E be any orthogonal spectrum and k any integer. Then we obtain a global functor $\underline{E}^k$ by setting

$$\underline{E}^k(G) \ = \ E^k(BG) \ ,$$

the 0-th E-cohomology of the classifying space of the group G. The contravariant functoriality in group homomorphisms $\alpha : K \longrightarrow G$ comes from the covariant functoriality of the classifying space construction. The transfer maps for a subgroup inclusion $H \subset G$ comes from the transfer map

$$BG \ \xrightarrow{\text{tr}} \ EG \times_G H \ \simeq \ BH$$

for the $[G : H]$-sheeted covering space $EG/H \longrightarrow EG/G = BG$. The global functors of the form $\underline{E}^k$ have special properties, namely they extend to additive functors on the *completed Burnside category* $\hat{A}$. [explain]

The global functor $\underline{E}^k$ is realized as the homotopy group global functor of a specific global homotopy type that we construct now. For this purpose we may represent the cohomology theory by an orthogonal Ω-spectrum E (in the traditional non-equivariant sense). We define new orthogonal spectrum bE as follows. For a real inner product space V we let $\mathbf{L}(V, \mathbb{R}^\infty)$ be the Stiefel manifold of linear isometric embeddings from V to $\mathbb{R}^\infty$. The group $O(V)$ acts freely from the left by precomposition. Then we set

$$(bE)(V) \ = \ \text{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V)) \ , \tag{7.1}$$

the space of all (not necessarily equivariant) continuous maps from $\mathbf{L}(V, \mathbb{R}^\infty)$ to $E(V)$. The orthogonal group $O(V)$ acts on this mapping space by conjugation, through its actions on $\mathbf{L}(V, \mathbb{R}^\infty)$ and on $E(V)$. We define structure maps $\sigma_{V,W} : (bE)(V) \wedge S^W \longrightarrow (bE)(V \oplus W)$ as the composite

$$\begin{aligned}
\text{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V)) \wedge S^W \quad &\xrightarrow{\text{assembly}} \quad \text{map}(\mathbf{L}(V, \mathbb{R}^\infty), E(V) \wedge S^W) \\
&\xrightarrow{\text{map}(\text{res}_V, \sigma^E_{V,W})} \text{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty), E(V \oplus W))
\end{aligned}$$

where $\text{res}_V : \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty)$ is the map that restrict an isometric embedding from $V \oplus W$ to V.

Applying the global Borel construction to Ω-spectra yields *global Ω-spectra*, an important class of orthogonal spectra that keep track of compatible equivariant infinite loop spaces for all compact Lie groups. These should be thought of as objects that are simultaneously G-Ω-spectra for all compact Lie groups G at once, in a compatible way.

Definition 7.2. An orthogonal spectrum X is a *global Ω-spectrum* if for every compact Lie group G, every faithful G-representation V and an arbitrary G-representation W the adjoint generalized structure map

$$\tilde{\sigma}_{V,W} \; : \; X(V) \longrightarrow \operatorname{map}(S^W, X(V \oplus W))$$

is a G-weak equivalence.

For a global Ω-spectrum X and a compact Lie group G the associated orthogonal G-spectrum $X\langle G\rangle$ is 'eventually an Ω-G-spectrum' in the sense that the Ω-G-spectrum condition of [**58**, III Def. 3.1] holds for all 'sufficiently large' (i.e., faithful) G-representations. However, if G is a non-trivial group, then the associated orthogonal G-spectrum $X\langle G\rangle$ is in general *not* an Ω-G-spectrum since there is no control over the G-homotopy type of the values at non-faithful representations.

The global Ω-spectra will turn out to be the fibrant objects in the global model structure on orthogonal spectra, see Theorem IV.2.7. This also means that global Ω-spectra abound, because every orthogonal spectrum admits a global equivalence to a global Ω-spectrum.

Proposition 7.3. *Let E be an orthogonal Ω-spectrum.*

(i) *The orthogonal spectrum bE is a global Ω-spectrum.*

(ii) *For every compact Lie group G and every based G-CW-complex A, the map [which?]*

$$(bE)^0_G(A) \longrightarrow E^0(EG^+ \wedge_G A)$$

is an isomorphism.

(iii) *For every compact Lie group G and every integer k the map*

$$\pi_k^G(bE) \longrightarrow E^{-k}(BG)$$

is an isomorphism. As G varies, these maps constitute an isomorphism of global functors $\underline{\pi}_k(bE) \cong \underline{E}^{-k}$.

Proof. (i) We let G be a compact Lie group and V and W two G-representations such that V is faithful. Since E is an Ω-spectrum, the adjoint structure map

$$\tilde{\sigma}^E_{V,W} \; : \; E(V) \longrightarrow \Omega^W E(V \oplus W)$$

is a non-equivariant weak equivalence. The spaces $\mathbf{L}(V, \mathbb{R}^\infty)$ and $\mathbf{L}(V \oplus W, \mathbb{R}^\infty)$ are non-equivariantly contractible; because V, and hence also $V \oplus W$, is a faithful G-representation, the induced G-action on $\mathbf{L}(V, \mathbb{R}^\infty)$ and $\mathbf{L}(V \oplus W, \mathbb{R}^\infty)$ is free. So the induced map

$$\operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, \tilde{\sigma}_{V,W}) \; : \; (bE)(V) = \operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, E(V)) \longrightarrow \operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, \Omega^W E(V \oplus W))$$

is a G-weak equivalence. Moreover, the restriction map $\operatorname{res}_V : \mathbf{L}(V \oplus W, \mathbb{R}^\infty) \longrightarrow \mathbf{L}(V, \mathbb{R}^\infty)$ is a weak equivalence between free and non-equivariantly contractible G-CW-complexes, hence it induces another G-weak equivalence

$$\begin{aligned} \operatorname{map}(\operatorname{res}_V, \Omega^W E(V \oplus W)) \; : \; & \operatorname{map}(\mathbf{L}(V, \mathbb{R}^\infty)_+, \Omega^W E(V \oplus W)) \\ & \longrightarrow \operatorname{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty)_+, \Omega^W E(V \oplus W)) \end{aligned}$$

on mapping spaces. The target of this last map is G-homeomorphic to

$$\operatorname{map}(S^W, \operatorname{map}(\mathbf{L}(V \oplus W, \mathbb{R}^\infty)_+, E(V \oplus W))) \; = \; \Omega^W (bE)(V \oplus W) \; ;$$

under this homeomorphism, the composite of the two G-weak equivalences becomes the adjoint structure map

$$\tilde{\sigma}^{bE}_{V,W} \; : \; (bE)(V) \longrightarrow \Omega^W (bE)(V \oplus W) \; .$$

So we have shown that $\tilde{\sigma}^{bE}_{V,W}$ is a G-weak equivalence, and that means that bE is a global Ω-spectrum.

(ii) As for any global Ω-spectrum, the stabilization map

$$[A \wedge S^V, (bE)(V)]^G \longrightarrow (bE)^0_G(A)$$

is bijective whenever the G-representation V is faithful. Since E is an Ω-spectrum, the adjoint structure map $\tilde{\sigma}_V : E_0 \longrightarrow \Omega^V R(V)$ is a non-equivariant weak equivalence, so it induces a G-weak equivalence

$$\mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),\tilde{\sigma}_V) \ : \ \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),E_0) \ \longrightarrow \ \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),\Omega^V R(V))$$

on mapping spaces out of $\mathbf{L}(V,\mathbb{R}^\infty)$. The right hand side is G-homeomorphic to

$$\mathrm{map}(S^V,\mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),R(V)) \ = \ \Omega^V(bE)(V) \ .$$

Taking homotopy classes of based G-maps out of A gives a bijection

$$E^0(EG^+\wedge_G A) = [\mathbf{L}(V,\mathbb{R}^\infty)^+\wedge_G A, E_0] \ \cong [A,\mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),E_0)]^G \ \longrightarrow [A,\Omega^V(bE)(V)]^G \ \cong \ (bE)^0_G(A) \ .$$

Here we have exploited that G acts trivially on E_0 and that $\mathbf{L}(V,\mathbb{R}^\infty)$ is a universal G-space.

(iii) This is the special case of (ii) where $A = S^k$ with trivial G-action. □

As we shall show in Example IV.5.26 below, the functor b realizes, in a certain precise way, the right adjoint to the forgetful functor from the global homotopy category to the traditional stable homotopy category.

The endofunctor R on the category of orthogonal spectra has some more convenient features. Firstly, it comes with a natural transformation

$$i_E \ : \ E \ \longrightarrow \ bE \tag{7.4}$$

whose value at an inner product space V is sends a point $e \in E(V)$ to the constant map $\mathbf{L}(V,\mathbb{R}^\infty) \longrightarrow E(V)$ with value e. Said another way, the map $E(V) \longrightarrow \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),E(V)) = (bE)(V)$ is induced by the unique map $\mathbf{L}(V,\mathbb{R}^\infty) \longrightarrow *$. Since $\mathbf{L}(V,\mathbb{R}^\infty)$ is contractible, the morphism $i_E : E \longrightarrow bE$ is a non-equivariant level equivalence, hence a non-equivariant stable equivalence.

Secondly, we can endow the functor b with a lax symmetric monoidal transformation

$$\mu_{E,E'} \ : \ bE\wedge bE' \ \longrightarrow \ b(E\wedge E') \ .$$

To construct $\mu_{E,E'}$ we start from the $O(V)\times O(W)$-equivariant maps

$$\begin{aligned}\mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty),E(V))\wedge\mathrm{map}(\mathbf{L}(W,\mathbb{R}^\infty),E'(W)) \ \xrightarrow{\ \wedge\ } \ & \mathrm{map}(\mathbf{L}(V,\mathbb{R}^\infty)\times\mathbf{L}(V,\mathbb{R}^\infty),E(V)\wedge E'(W)) \\ \xrightarrow{\mathrm{map}(\mathrm{res}_{V,W},i_{V,W})} \ & \mathrm{map}(\mathbf{L}(V\oplus W,\mathbb{R}^\infty),(E\wedge E')(V\oplus W))\end{aligned}$$

that constitute a bimorphism from (bE,bE') to $b(E\wedge E')$. Here $\mathrm{res}_{V,W} : \mathbf{L}(V\oplus W,\mathbb{R}^\infty) \longrightarrow \mathbf{L}(V,\mathbb{R}^\infty)\times\mathbf{L}(V,\mathbb{R}^\infty)$ is the map that takes an embedding of $V\oplus W$ to the pair of its restrictions to V and W. The morphism $\mu_{E,E'}$ is associated to this bimorphism via the universal property of the smash product.

Since the functor b is lax symmetric monoidal with respect to the maps $\mu_{E.E'}$, it takes orthogonal ring spectra to orthogonal ring spectra, in a way preserving commutativity. Since the transformation i_E is monoidal, it becomes a homomorphism of orthogonal ring spectra when E is an orthogonal ring spectrum.

Remark 7.5. The 'global Borel cohomology' functor $b : \mathcal{S}p \longrightarrow \mathcal{S}p$ is lax symmetric monoidal; in particular is preserves ring spectrum structures and module structures. In particular, the orthogonal spectrum $\hat{\mathbb{S}} = b\mathbb{S}$ comes with a commutative ring spectrum structure; we call this the *completed sphere spectrum*. Moreover, for every orthogonal spectrum E the map

$$b\mathbb{S}\wedge bE \ \longrightarrow \ b(\mathbb{S}\wedge E) \ \cong \ bE$$

makes the orthogonal spectrum bE into a module spectrum over the completed sphere spectrum. Consequently, for every group G the equivariant homotopy group

$$\pi_k^G(bE) \ \cong \ E^{-k}(BG)$$

is naturally a module over the ring commutative $\pi_0^G(\hat{\mathbb{S}})$. For finite groups G, Carlsson's theorem [**24**] (also known as the Segal conjecture), identifies the latter ring with the completion of the Burnside ring at the augmentation ideal:

$$\pi_0^G(\hat{\mathbb{S}}) \;\cong\; \pi^0(BG) \;\cong\; \mathbb{A}(G)_I^\wedge \;.$$

Remark 7.6. Various 'completion' maps (also called 'bundling map') fit in here as follows. For every orthogonal Ω-spectrum E the morphism $i_E : E \longrightarrow bE$ is a kind of 'global completion map'. For every compact Lie group G it induces a map of G-equivariant homotopy groups

$$\pi_0^G(E) \;\longrightarrow\; \pi_0^G(bE) \;\cong\; E^0(BG) \;.$$

When $E = \mathbb{S}$ is the sphere spectrum the map

$$\mathbb{A}(G) \cong \pi_0^G(\mathbb{S}) \;\longrightarrow\; \pi^0(BG)$$

is the completion map of the Segal conjecture, compare the previous remark.

The sphere spectrum is the global classifying space $B_{\mathrm{gl}}e$ of the trivial group; more generally, for the global classifying space $B_{\mathrm{gl}}K$ of a finite group K the 'forgetful' map

$$\mathbf{A}(K,G) \cong \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}}K) \;\longrightarrow\; [\Sigma_+^\infty BG,\, \Sigma_+^\infty BK]$$

is again completion at the augmentation ideal of the Burnside ring $A(G)$ (using that $\mathbf{A}(K,G)$ is a module over $A(G)$).

For the global K-theory spectrum and any compact Lie group G, the map

$$\mathbf{R}(G) \cong \pi_0^G(\mathbf{KU}) \;\longrightarrow\; K^0(BG)$$

is the map of the Atiyah-Segal completion theorem [**6**], and the target is isomorphic to the completion of the representation ring at the augmentation ideal. For the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$ (compare Section V.5), the global functor $G \mapsto H^0(BG;\mathbb{Z})$ is constant with value $\mathbb{Z}$; the map

$$\pi_0^G(H\mathbb{Z}) \;\longrightarrow\; H^0(BG;\mathbb{Z})$$

is surjective and an isomorphism modulo torsion for all compact Lie finite groups whose identity path component is commutative (compare Example 5.6).

Remark 7.7. A global Ω-spectrum X is a very rich kind of structure, because it encodes compatible equivariant infinite loop spaces for all compact Lie groups. Indeed, for every compact Lie group G and faithful G-representation V, the G-space

$$X[G] \;=\; \Omega^V X(V) \tag{7.8}$$

is a 'genuine' equivariant infinite loop space, i.e., deloopable in the direction of every representation. Indeed, for every G-representation W, the G-map

$$\Omega^V(\tilde{\sigma}_{V,W}) \;:\; X[G] \;=\; \Omega^V X(V) \;\longrightarrow\; \Omega^W(\Omega^V X(V \oplus W))$$

is a G-weak equivalence, so the global Ω-spectrum X provides a W-deloop $\Omega^V X(V \oplus W)$. The G-space $X[G]$ is also independent, up to G-weak equivalence, of the choice of faithful G-representation. Indeed, if $\bar{V}$ is another faithful G-representation, then the G-maps

$$\Omega^V X(V) \;\xrightarrow{\Omega^V(\tilde{\sigma}_{V,\bar{V}})}\; \Omega^V(\Omega^{\bar{V}} X(V \oplus \bar{V})) \;\cong\; \Omega^{\bar{V}}(\Omega^V, X(\bar{V} \oplus V)) \;\xleftarrow{\Omega^{\bar{V}}(\tilde{\sigma}_{\bar{V},V})}\; \Omega^{\bar{V}} X(\bar{V})$$

are G-weak equivalences.

As G varies, the equivariant infinite loop spaces $X\langle G\rangle$ are closely related to each other. For example, if H is a subgroup of G, then any faithful G-representation is also faithful as an H-representation. So $X[H]$ is H-weakly equivalent to the restriction of the G-equivariant infinite loop space $X[G]$.

Remark 7.9. Let X be a global Ω-spectrum. Specialized to the trivial group, the condition in Definition 7.2 says that X is in particular a non-equivariant Ω-spectrum in the sense that the adjoint structure map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence of (non-equivariant) spaces.

If X is a global Ω-spectrum, then so is the shifted spectrum $\operatorname{sh} X$ and the function spectrum $\operatorname{map}(K, X)$ for every based CW-complex K. Indeed, if V is a faithful G-representation, then $\mathbb{R} \oplus V$, the sum with a trivial 1-dimension representation, is also faithful. So the adjoint structure map

$$\tilde{\sigma}_{\mathbb{R}\oplus V,W} \ : \ X(\mathbb{R} \oplus V) \ \longrightarrow \ \Omega^W(X(\mathbb{R} \oplus V \oplus W))$$

is a G-weak equivalence for every G-representation W. This map is naturally G-homeomorphic to the adjoint structure map

$$(\operatorname{sh} X)(V) \ \longrightarrow \ \Omega^W((\operatorname{sh} X)(V \oplus W))$$

of the shifted spectrum.

The argument for mapping spectra is similar. The mapping space functor $\operatorname{map}(K, -)$ takes the G-weak equivalence $\tilde{\sigma}_{V,W} : X(V) \longrightarrow \Omega^W X(V \oplus W)$ to a G-weak equivalence

$$\operatorname{map}(K, X(V)) \ \xrightarrow{\operatorname{map}(K,\tilde{\sigma}_{V,W})} \ \operatorname{map}(K, \Omega^W X(V \oplus W)) \ ,$$

and this map is G-homeomorphic to the adjoint structure map

$$\operatorname{map}(K, X)(V) \ \longrightarrow \ \Omega^W(\operatorname{map}(K, X)(V \oplus W))$$

of the mapping spectrum $\operatorname{map}(K, X)$.

8. Global deloopings

In this section we explain that commutative orthogonal monoid spaces can be 'globally delooped'. More precisely, we construct a *global delooping* functor

$$\mathbf{B} \ : \ \text{coms} \ \longrightarrow \ \mathcal{S}p$$

from commutative orthogonal monoid spaces to orthogonal spectra, and a natural transformation of orthogonal spaces

$$\xi \ : \ R \ \longrightarrow \ \Omega^\bullet(\mathbf{B}R)$$

that is trying to be a global equivalence, in a sense we explain now. There are three obstructions to the existence of genuine global deloopings, one technical and two genuinely mathematical:

- **[Flatness]** The technical part is that the construction of $\mathbf{B}R$ involves iterated box products of R with itself, which we only know to be globally homotopy invariant, by Theorem I.5.9, when R is flat as an orthogonal space. This is a merely technical point because we can always replace R by a globally equivalent commutative orthogonal monoid space that is flat (for example by a cofibrant replacement in the global model structure of Theorem II.3.4).
- **[Group-likeness]** The first genuinely mathematical point is very familiar from non-equivariant delooping theorems, and from equivariant delooping theorems for one fixed group: since $\mathbf{B}R$ is an orthogonal spectrum, the abelian monoid $\pi_0^G(\mathbf{B}R)$ is a group, i.e., it has additive inverses. So if we want a global equivalence from R to $\Omega^\bullet(\mathbf{B}R)$ that is in any sense additive, then we must require that the abelian monoid $\pi_0^G(R)$ must have inverses for every compact Lie group G; in other words, R needs to be group-like in the sense of Definition II.4.1.
- **[Infinite index norm maps]** The other genuinely mathematical issue is a new global phenonmenon and comes from the absense of *infinite index* norm maps in global power monoids. In more detail: by Proposition 8.13 the morphism of $\mathrm{Rep}^{\mathrm{op}}$-functors $\underline{\pi}_0(\xi) : \underline{\pi}_0(R) \longrightarrow \underline{\pi}_0(\mathbf{B}R)$ takes norm maps for inclusions $H \leq G$ of finite index to transfer maps. Since $\mathbf{B}R$ is an orthogonal spectrum, the right hand side $\underline{\pi}_0(\mathbf{B}R)$ has more general transfer maps, namely for arbitrary inclusions of closed subgroups. In general there are no corresponding 'infinite index norm maps' on the left

hand side, and the global delooping functor is, morally speaking, 'freely building in infinite index transfers'. A rigorous statement in this direction is Theorem 8.27 (ii) saying that algebraically, $\underline{\pi}_0(\xi)$ is precisely doing this on the level of $\underline{\pi}_0$.

Because of the above caveats, the best one can hope for is a $\mathcal{F}in$-global delooping for flat group-like commutative orthogonal monoid spaces, and that is precisely the property of the morphism $\xi : R \longrightarrow \Omega^\bullet(\mathbf{B}R)$, compare Theorem 8.27.

The global delooping functor is the composite of two constructions

$$\text{coms} \xrightarrow{\mathcal{H}} \mathbf{\Gamma}\text{-}spc \xrightarrow{-(\mathbb{S})} \mathcal{S}p$$

passing through the intermediate category of $\mathbf{\Gamma}$-orthogonal spaces. The functor $\mathcal{H}$ that turns a commutative orthogonal monoid space into a $\mathbf{\Gamma}$-orthogonal space will be reviewed in Construction 8.20. The second functor that turns a $\mathbf{\Gamma}$-orthogonal space into an orthogonal spectrum is a slight generalization of the well-known construction of 'evaluation of a $\mathbf{\Gamma}$-space on spheres' which goes back all the way to Segal's paper [**77**].

Example 8.1 ($\mathbf{\Gamma}$-spaces and $\mathbf{\Gamma}$-orthogonal spaces)**.** We let $\mathbf{\Gamma}$ denote the category of finite based sets; morphisms are all based maps. A $\mathbf{\Gamma}$-*space* is a functor $F : \mathbf{\Gamma} \longrightarrow \mathbf{U}$ to the category of spaces which is pointed (i.e., the value $F(*)$ at any one-point set is a one-point space). A morphism of $\mathbf{\Gamma}$-spaces is a natural transformation of functors. We may then view a $\mathbf{\Gamma}$-space as a functor to *pointed* spaces, where $F(A)$ is pointed by the image of the map $F(*) \longrightarrow F(A)$ induced by the unique morphism $* \longrightarrow A$ in $\mathbf{\Gamma}$.

A $\mathbf{\Gamma}$-space F can be extended to a continuous functor on the category of based spaces by a coend construction. If K is a based space, the value of the extended functor on K is given by

$$F(K) \;=\; \int^{n^+\in\mathbf{\Gamma}} F(n^+) \times K^n \;=\; \left(\coprod_{n\geq 0} F(n^+) \times K^n \right) / \sim \; , \tag{8.2}$$

where $n^+ = \{0, 1, \dots, n\}$ with basepoint 0, and we use that $K^n = \mathrm{map}(n^+, K)$ is contravariantly functorial in n^+. In more detail $F(K)$ is obtained from the disjoint union of the spaces $F(n^+) \times K^n$ by modding out the equivalence relation generated by

$$(F(\alpha)(x);\, k_1, \dots, k_n) \;\sim\; (x;\, k_{\alpha(1)}, \dots, k_{\alpha(m)})$$

for all morphisms $\alpha : m^+ \longrightarrow n^+$ in $\mathbf{\Gamma}$, all $x \in F(m^+)$, and all $(k_1, \dots, k_n)$ in K^n. Here $k_{\alpha(i)}$ is to be interpreted as the basepoint of K whenever $\alpha(i) = 0$. We write $[x;\, k_1, \dots, k_n]$ for the equivalence class in $F(K)$ of a tuple $(x,\, k_1, \dots, k_n) \in F(n^+) \times K^n$. The assignment $(F, K) \mapsto F(K)$ is functorial in the $\mathbf{\Gamma}$-space F and the based space K.

We will not distinguish notationally between the original $\mathbf{\Gamma}$-space and its extension. The extended functor is continuous and comes with a continuous, based *assembly map*

$$\alpha \;:\; F(K) \wedge L \;\longrightarrow\; F(K \wedge L) \;, \quad \alpha([x;\, k_1, \dots, k_n] \wedge l) \;=\; [x;\, k_1 \wedge l, \dots, k_n \wedge l] \; . \tag{8.3}$$

The assembly map is natural in both variables and associative and unital.

Now we add another degree of freedom to the story. A $\mathbf{\Gamma}$-*orthogonal space* is a continuous functor $F : \mathbf{L} \times \mathbf{\Gamma} \longrightarrow \mathbf{U}$ from $\mathbf{L} \times \mathbf{\Gamma}$ to the category of spaces which is pointed i.e., the value $F(V, *)$ a one-point space for every inner product space V. It is often convienient to break the symmetry between the two indexing categories $\mathbf{\Gamma}$ and $\mathbf{L}$ and view a $\mathbf{\Gamma}$-orthogonal space either as a $\mathbf{\Gamma}$-object of orthogonal spaces, i.e., a functor $\mathbf{\Gamma} \longrightarrow spc$, or as a functor from the linear isometries category $\mathbf{L}$ to the category of $\mathbf{\Gamma}$-spaces. We will often switch back and forth between these points without further comment.

The category of $\mathbf{\Gamma}$-orthogonal spaces 'contains' the category of $\mathbf{\Gamma}$-spaces as a full subcategory (by making them constant in the linear isometries direction). It does *not* 'contain' the category of orthogonal spaces in the same way, because a $\mathbf{\Gamma}$-orthogonal space has to be pointed; since if it is also constant in the $\mathbf{\Gamma}$-direction, then it must be entirely trivial.

Construction 8.4 (Evaluation on spheres)**.** Every $\mathbf{\Gamma}$-orthogonal space F gives rise to an orthogonal spectrum $F(\mathbb{S})$ by 'evaluating F on spheres'. More precisely, the value at an inner product space V is

$$F(\mathbb{S})(V) \ = \ F(V, S^V) \ , \tag{8.5}$$

the evaluation of the $\mathbf{\Gamma}$-space $F(V,-)$ at the V-sphere. The generalized structure map $\sigma_{V,W}$ is the diagonal composite in the commutative square:

$$\begin{tikzcd} F(\mathbb{S})(V) \wedge S^W \arrow[r, equal] & F(V,S^V) \wedge S^W \arrow[r, "\text{assembly}"] \arrow[d, "F(i_{V,W},S^V)\wedge S^W"'] & F(V, S^{V\oplus W}) \arrow[d, "F(i_{V,W},S^{V\oplus W})"] & \\ & F(V\oplus W, S^V)\wedge S^W \arrow[r, "\text{assembly}"'] & F(V\oplus W, S^{V\oplus W}) \arrow[r, equal] & F(\mathbb{S})(V\oplus W) \end{tikzcd}$$

(where we suppressed the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$). The $O(V)$-action on $F(\mathbb{S})(V)$ is diagonal, via the actions on V and on S^V.

Remark 8.6. One could break up the construction of $F(\mathbb{S})$ into two steps: for every inner product space V we could evaluate the $\mathbf{\Gamma}$-space $F(V,-)$ on all sphere and obtain an orthogonal spectrum $F(V,-)(\mathbb{S})$. As V-varies, this gives a functor $\mathbf{L} \longrightarrow \mathcal{S}p$ (i.e., an 'orthogonal orthogonal spectrum'). Such an object can be diagonalized to obtain an orthogonal spectrum.

The philosophy behind the next definition is that just as a special $\mathbf{\Gamma}$-space encodes an E_∞-multiplication, a $\mathbf{\Gamma}$-orthogonal space with a suitably specialness condition encodes a 'global E_∞-multiplication'. The right notion of specialness, however, is not the naive analog of Segal's condition [**77**] where a certain morphism $F(n^+) \longrightarrow F(1^+)^n$ is a global equivalence of orthogonal spaces; rather, the good specialness notion is a global refinement of the specialness for equivariant $\mathbf{\Gamma}$-spaces by Shimakawa [**81**, Def. 1.3]. We emphasize that this is traditionally only considered for *finite* groups, but the following definition uses all compact Lie groups.

Definition 8.7. A $\mathbf{\Gamma}$-orthogonal space F is *globally special* if for every compact Lie group G and every finite G-set S the map

$$(P_S)^G \ : \ (F(S^+)(\mathcal{U}_G))^G \ \longrightarrow \ \mathrm{map}^G(S, (F(1^+)(\mathcal{U}_G)))$$

is a weak equivalence.

Remark 8.8 (Globally versus equivariantly special)**.** Indeed, a $\mathbf{\Gamma}$-orthogonal space F is globally special if and only if for every compact Lie group a certain equivariant $\mathbf{\Gamma}$-spaces made from F is special in Shimakawa's sense: we can evaluate each of the orthogonal spaces $F(n^+)$ at the complete G-universe $\mathcal{U}_G$ and obtain a $\mathbf{\Gamma}$-G-space $F(\mathcal{U}_G)$. For finite groups G, the global specialness in Definition 8.7 precisely says that the $\mathbf{\Gamma}$-G-space $F(\mathcal{U}_G)$ is special (in Shimakawa's equivariant sense).

For a general compact Lie group G, we let G° denote the path component of the identity element and let $\bar{G} = G/G^\circ$ the finite group of components of G. By taking G°-fixed points objectwise we obtain a $\mathbf{\Gamma}$-$\bar{G}$-space $F(\mathcal{U}_G)^{G^\circ}$ with

$$\left(F(\mathcal{U}_G)^{G^\circ}\right)(n^+) \ = \ (F(n^+)(\mathcal{U}_G))^{G^\circ} \ .$$

If G acts continuously on a finite set S, then the identity component G° must act trivially, and the G-action factors uniquely over an action of the finite group $\bar{G}$. So

$$(F(S^+)(\mathcal{U}_G))^G \ = \ ((F(\mathcal{U}_G)^{G^\circ})(S^+))^{\bar{G}} \quad \text{and} \quad \mathrm{map}^G(S, (F(1^+)(\mathcal{U}_G))) \ = \ \mathrm{map}^{\bar{G}}(S, (F(\mathcal{U}_G)^{G^\circ})(1^+)) \ .$$

We conclude that F is globally special if and only if the $\mathbf{\Gamma}$-$\bar{G}$-space $(F(\mathcal{U}_G))^{G^\circ}$ is special, in the sense of [**81**, Def. 1.3], for every compact Lie group G.

For the next result we need to explain how the equivariant homotopy set $\underline{\pi}_0(F(1^+))$ of the underlying orthogonal space inherits extra algebraic structure.

Construction 8.9 (Global power monoid of a globally special **Γ**-orthogonal space)**.** We let F be a globally special **Γ**-orthogonal space. We also suppose that for every $n \geq 1$ the orthogonal space $F(n^+)$ is closed. We will make the equivariant homotopy sets $\underline{\pi}_0(uF)$ of the underlying orthogonal space $uF = F(1^+)$ into to a global power monoid. We let G be a compact Lie group and let it act trivially on the set $\{1, 2\}$. Then the map

$$(P_{\{1,2\}})^G \ : \ (F(2^+)(\mathcal{U}_G))^G \ \longrightarrow \ \mathrm{map}^G(\{1,2\}, (F(1^+)(\mathcal{U}_G))) \ = \ ((uF)(\mathcal{U}_G))^G \times ((uF)(\mathcal{U}_G))^G$$

is a weak equivalence, so it induces a bijection

$$\pi_0((P_{\{1,2\}})^G) \ : \ \pi_0^G(F(2^+)) \ \xrightarrow{\cong} \ \pi_0^G(uF) \times \pi_0^G(uF) \ .$$

on path components. The composite map

$$+ \ : \ \pi_0^G(uF) \times \pi_0^G(uF) \ \cong \ \pi_0^G(F(2^+)) \ \xrightarrow{\pi_0^G(F(\nabla^+))} \ \underline{\pi}_0^G(uF) \tag{8.10}$$

is then a commutative monoid structure on the set $\pi_0^G(uF)$, where $\nabla : \{1, 2\} \longrightarrow \{1\}$ is the unique map. The monoid structure is natural for restriction maps in G, and the units are the images of the morphism $\underline{\pi}_0^G(*) = \pi_0^G(F(0^+)) \longrightarrow \underline{\pi}_0^G(F(1^+)) = \underline{\pi}_0(uF)$ induced by the unique based map $0^+ \longrightarrow 1^+$.

Now we let H be a closed subgroup of G of finite index. Then G/H is a finite G-set, so the map

$$(P_{G/H})^G \ : \ (F(G/H^+)(\mathcal{U}_G))^G \ \longrightarrow \ \mathrm{map}^G(G/H, (F(1^+)(\mathcal{U}_G))) \ \cong \ ((uF)(\mathcal{U}_G))^H$$

is a weak equivalence, so it induces a bijection

$$\pi_0((P_{G/H})^G) \ : \ \pi_0((F(G/H^+)(\mathcal{U}_G))^G) \ \xrightarrow{\cong} \ \pi_0(((uF)(\mathcal{U}_G))^H) \ \cong \ \pi_0^H(uF)$$

on path components. The second isomorphism uses that the underlying H-representation of $\mathcal{U}_G$ is a complete H-universe. One should beware that the source of the last isomorphism is in general *not* isomorphic to $\pi_0^G(F(G/H^+))$, because G acts non-trivially on G/H. Still, the unique map $\nabla : G/H \longrightarrow \{1\}$ is G-equivariant, so it induces a G-map $(F(\nabla^+)(\mathcal{U}_G))^G : (F(G/H^+)(\mathcal{U}_G))^G \longrightarrow (F(1^+)(\mathcal{U}_G))^G$. We define the norm map of the global power monoid $\underline{\pi}_0(uF)$ as the composite map

$$N_H^G \ : \ \pi_0^H(uF) \ \cong \ \pi_0((F(G/H^+)(\mathcal{U}_G))^G) \ \xrightarrow{(F(\nabla^+)(\mathcal{U}_G))^G} \ (F(1^+)(\mathcal{U}_G))^G \ = \ \pi_0^G(uF) \ . \tag{8.11}$$

As we explained in Construction II.1.20, norm maps can be turned into power operations (and, given various other pieces of structure, are equivalent pieces of data). We omit the verification that the additions and the power operations arising from the norm maps just defined make the equivariant homotopy sets $\underline{\pi}_0(uF)$ into a global power monoid in the sense of Definition II.1.10.

We let F be a **Γ**-orthogonal space that is 'naively special', i.e., the morphisms $P_n : F(n^+) \longrightarrow F(1^+)^n$ are global equivalences. Then Construction 8.9 still applies to make that the equivariant homotopy sets $\underline{\pi}_0(uF)$ into commutative monoids. Moreover, the weak morphism

$$F(1^+) \times F(1^+) \ \xleftarrow[\sim]{P_2} \ F(2^+) \ \xrightarrow{F\nabla^+} \ F(1^+)$$

makes the orthogonal space $uF = F(1^+)$ into a commutative monoid, with respect to the derived box product $\boxtimes^L$, in the unstable global homotopy category. However, the stronger global specialness condition is needed in order to define the norm maps (8.11), and if one want $F(1^+)$ to be isomorphic (as a homotopy-commutative homotopy monoid) to a strictly commutative orthogonal monoid space.

Given a **Γ**-orthogonal space F and an inner product space V, the assembly map $F(V, 1^+) \wedge S^V \longrightarrow F(V, S^V) = F(\mathbb{S})(V)$ of the **Γ**-space $F(V)$ is adjoint to a based map

$$\xi(V) \ : \ F(V, 1^+) \ \longrightarrow \ \mathrm{map}(S^V, F(V, S^V)) \ .$$

As V varies, these maps define a natural morphism of orthogonal spaces

$$\xi \ : \ uF \ = \ F(1^+) \ \longrightarrow \ \Omega^\bullet(F(\mathbb{S})) \ . \tag{8.12}$$

Proposition 8.13. *Let F be a globally special $\mathbf{\Gamma}$-orthogonal space. Then the maps*

$$\pi_0^G(\xi) \ : \ \pi_0^G(uF) \ \longrightarrow \ \pi_0^G(\Omega^\bullet(F(\mathbb{S}))) = \pi_0^G(F(\mathbb{S}))$$

are additive and take the norm maps (8.11) *on the left hand side to transfer maps on the right hand side. In other words, $\underline{\pi}_0(\xi)$ is a morphism of global power monoids to the additive global power monoid $\underline{\pi}_0(F(\mathbb{S}))$.*

Proof. We consider the $\mathbf{\Gamma}$-orthogonal space F_{2^+} defined by $F_{2^+}(A) = F(2^+ \wedge A)$. Since F is globally special, so is F_{2^+}. The projections and the fold map from 2^+ to 1^+ induce morphisms of $\mathbf{\Gamma}$-orthogonal spaces

$$F \times F \ \xleftarrow{(F_{p_1}, F_{p_2})} \ F_{2^+} \ \xrightarrow{F_{\nabla^+}} \ F \ .$$

In particular, the sets $\pi_0^G(uF_{2^+})$ are abelian monoids in their own right.

We consider elements $x, y \in \pi_0^G(uF)$ and let $[x,y]$ be the unique element of $\pi_0^G(uF_{2^+}) = \pi_0^G(F(2^+))$ that satisfies

$$p_1[x,y] \ = \ x \qquad \text{and} \qquad p_2([x,y]) \ = \ y \ .$$

Then $x + y = \nabla_*[x,y]$, by definition.

$$p_1(\xi[x,y]) \ = \ \xi(p_1[x,y]) \ = \ \xi(x)$$

and similarly $p_2(\xi[x,y]) = \xi(y)$. Since the map

$$((p_1)_*, (p_2)_*) \ : \ \pi_0^G(F_{2^+}(\mathbb{S})) \ \longrightarrow \ \pi_0^G(F(\mathbb{S})) \times \pi_0^G(F(\mathbb{S}))$$

is bijective [???] and a group homomorphism, this implies

$$\xi(x) \ + \ \xi(y) \ = \ \nabla_*(\xi[x,y]) \ = \ \xi(\nabla_*[x,y]) \ = \ \xi(x+y) \ .$$

The compatibility with transfers [...]. □

Definition 8.14. A globally special $\mathbf{\Gamma}$-orthogonal space F is *group-like* if for every compact Lie group G the abelian monoid $\pi_0^G(uF)$ is a group.

Construction 8.15. Now we make precise the slogan that evaluation on spheres 'freely adds infinite index transfers', at least algebraically on the level of $\underline{\pi}_0$. We let $\mathbf{A}^{\mathrm{res}}$ be the (non-full) preadditive subcategory of the Burnside category $\mathbf{A}$ with objects all compact Lie groups, but with $\mathbf{A}^{\mathrm{res}}(G,K)$ the subgroup of $\mathbf{A}(G,K)$ generated by the operations $\mathrm{tr}_L^K \circ \alpha^*$ for all subgroups $L \leq K$ *of finite index* and all continuous homomorphisms $\alpha : L \longrightarrow K$. Then the data of an additive functor $\mathbf{A}^{\mathrm{res}} \longrightarrow \mathcal{A}b$ is exactly the data of a group-like commutative global power monoid (written additively). We refer to such an additive functor as a *restricted global functor*. Restriction along the inclusion $\mathbf{A}^{\mathrm{res}} \longrightarrow \mathbf{A}$ is an exact functor $u : \mathcal{GF} \longrightarrow \mathcal{GF}^{\mathrm{res}}$ between abelian categories that has a left adjoint

$$\Lambda \ : \ \mathcal{GF}^{\mathrm{res}} \ \longrightarrow \ \mathcal{GF} \ .$$

This left adjoint can be thought of as 'freely adding infinite index transfers' (for subgroups with finite Weyl groups, of course).

We let F be a group-like globally special $\mathbf{\Gamma}$-orthgonal space. Then $\underline{\pi}_0(uF)$ is a restricted global functor and the morphism

$$\underline{\pi}_0(\xi) \ : \ \underline{\pi}_0(uF) \ \longrightarrow \ \underline{\pi}_0(\Omega^\bullet(F(\mathbb{S}))) \ = \ \underline{\pi}_0^G(F(\mathbb{S}))$$

is a morphism of restricted global functors by Proposition 8.13. Since the target is underlying a genuine global functor, the morphism is adjoint to a morphism of global functors

$$\hat{\xi} \ : \ \Lambda(\underline{\pi}_0(uF)) \ \longrightarrow \ \underline{\pi}_0(F(\mathbb{S})) \ . \tag{8.16}$$

Theorem 8.17. *Let F be a flat and group-like globally special $\mathbf{\Gamma}$-orthogonal space.*

(i) *The morphism $\xi : uF \longrightarrow \Omega^\bullet(F(\mathbb{S}))$ is a $\mathcal{F}in$-global equivalence of orthogonal spaces.*

(ii) *The morphism $\hat{\xi} : \Lambda(\underline{\pi}_0(F(1^+))) \longrightarrow \underline{\pi}_0(F(\mathbb{S}))$ is an isomorphism of global functors.*

Proof. (i) We let G be a finite group and consider a G-lifting problem for the morphism in question, i.e., a commutative square

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F(V,1^+)^G \arrow[r, equal] \arrow[d, "(\xi(V))^G"] & ((uF)(V))^G \arrow[d, "(\xi(V))^G"] \\ D^k \arrow[r, "\beta"'] & \mathrm{map}^G(S^V, F(V,S^V)) \arrow[r, equal] & ((\Omega^\bullet(F(\mathbb{S})))(V))^G \end{tikzcd}$$

where we assume without loss of generality that $V \in s(\mathcal{U}_G)$.

The $\mathbf{\Gamma}$-G-space $F(\mathcal{U}_G)$ is special by hypothesis [and flat...], and it is very special because F is globally very special. So by Shimakawa's theorem [**81**], the adjoint assembly map

$$F(\mathcal{U}_G, 1^+) \ = \ F(\mathcal{U}_G)(S^0) \ \longrightarrow \ \mathrm{map}(S^V, F(\mathcal{U}_G)(S^V))$$

is a G-weak equivalence for every G-representation V. So there is a continuous map $\lambda : D^k \longrightarrow F(\mathcal{U}_G, 1^+)^G$ that the diagram

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F(V,1^+)^G \arrow[r] & F(\mathcal{U}_G,1^+)^G \arrow[d, "\sim"] \\ D^k \arrow[urr, dashed, "\lambda"] \arrow[r, "\beta"'] & \mathrm{map}^G(S^V, F(V,S^V)) \arrow[r] & \mathrm{map}^G(S^V, F(\mathcal{U}_G,S^V)) \end{tikzcd}$$

commutes strictly in the upper left triangle part, and up to relative homotopy in the lower right part. Since D^k is compact and $F(1^+)$ is a closed orthogonal space, the map λ has image in $F(\bar{W}, 1^+)^G$ for some $\bar{W} \in s(\mathcal{U}_G)$ with $V \subset \bar{W}$. Since $F(1^+)$ is closed the map $F(W, 1^+)^G \longrightarrow F(\mathcal{U}_G, 1^+)^G$ induced by the inclusion $\bar{W} \subset \mathcal{U}_G$ is injective, so the diagram

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F(V,1^+)^G \arrow[r] & F(\bar{W},1^+)^G \arrow[d] \\ D^k \arrow[urr, dashed, "\lambda"] \arrow[r, "\beta"'] & \mathrm{map}^G(S^V, F(V,S^V)) \arrow[r] & \mathrm{map}^G(S^V, F(\mathcal{U}_G,S^V)) \end{tikzcd}$$

commutes strictly in the upper left triangle part, and up to relative homotopy in the lower right part. The relative homotopy is a continuous map from the compact space $D^k \times [0,1]$, so it has image in $\mathrm{map}^G(S^V, F(W, S^V))$ for some $W \in s(\mathcal{U}_G)$ with $\bar{W} \subset W$. Then the diagram

$$\begin{tikzcd} S^{k-1} \arrow[r, "\alpha"] \arrow[d, "\mathrm{incl}"'] & F(V,1^+)^G \arrow[r] & F(W,1^+)^G \arrow[r, equal] \arrow[d, "(\xi(W))^G"] & ((uF)(W))^G \arrow[d, "(\xi(W))^G"] \\ D^k \arrow[urr, dashed, "\lambda"] \arrow[r, "\beta"'] & \mathrm{map}^G(S^V, F(V,S^V)) \arrow[r] & \mathrm{map}^G(S^W, F(W,S^W)) \arrow[r, equal] & ((\Omega^\bullet(F(\mathbb{S})))(W))^G \end{tikzcd}$$

commutes strictly in the upper left triangle part, and up to relative homotopy in the lower right part. So we found a solution to the original homotopy lifting problem, and the morphism ξ is a $\mathcal{F}in$-global equivalence. □

Remark 8.18. One can also identify the effect of the construction $F \mapsto F(\mathbb{S})$ on $\underline{\pi}_0$ for a globally special, but not necessarily group-like, flat orthogonal $\mathbf{\Gamma}$-space F. The global functor $\underline{\pi}_0(F(\mathbb{S}))$ is obtained by

- first group completing the global power monoid $\underline{\pi}_0(uF)$, making it a restricted global functor,
- and then applying the left adjoint Λ to make it into a global functor.

In other words, the morphism $\underline{\pi}_0(\xi) : \underline{\pi}_0(uF) \longrightarrow \underline{\pi}_0(F(\mathbb{S}))$ is initial among morphisms of global power monoids to a global power monoid underlying a global functor.

Remark 8.19. We let G be a compact Lie group that 'has no non-trivial infinite index transfers', i.e., such that every subgroup H with finite Weyl group in G already has finite index in G. This class includes all finite groups, all tori, and more generally all products of tori and finite groups. We claim that for every such group the connected component G° of the identity is abelian (hence a torus of the same dimension as G). Indeed, any maximal torus $T \subset G^\circ$ of the connected component has finite Weyl group in G°, hence also in G. So if $\mathbf{A}^{\mathrm{res}}(e,G) = \mathbf{A}(e,G) = \mathbb{A}(G)$, then $\mathrm{tr}_T^G \in \mathbf{A}^{\mathrm{res}}(e,G)$, and so T must have finite index in G, which forces $T = G^\circ$.

On the other hand, the group $O(2)$ is an example of a Lie group whose identity component is abelian, but that does have non-trivial infinite index transfers: the dihedral subgroups of $O(2)$ have index 2 in their normalizers.

Now we introduce and investigate the functor $\mathcal{H}$ that turns a commutative orthogonal monoid space R into a $\mathbf{\Gamma}$-orthogonal space. The main result is Corollary 8.26 below, showing that when R is flat, then $\mathcal{H}R$ is globally special.

Construction 8.20. We let R be a commutative orthogonal monoid space. We define a $\mathbf{\Gamma}$-object $\mathcal{H}R$ in the category of orthogonal spaces by

$$(\mathcal{H}R)(n^+) \;=\; \underbrace{R \boxtimes \cdots \boxtimes R}_{n} \;,$$

which is also the n-fold coproduct in the pointed category of commutative orthogonal monoid spaces. So the inclusion and fold maps in this pointed category make $\mathcal{H}R$ into a $\mathbf{\Gamma}$-object in the category of commutative orthogonal monoid spaces, from which we forget to a $\mathbf{\Gamma}$-object in orthogonal spaces.

We do not claim originality for the previous construction, which is a variation of similar constructions in related contexts. If R is a constant orthogonal monoid space associated to a commutative topological monoid M, then the associated $\mathbf{\Gamma}$-orthogonal space is also constant in the linear isometries direction, and it specializes to the $\mathbf{\Gamma}$-object $[n] \mapsto M^{\times n}$ that goes back, at least, to Segal's paper [**77**]. In [**67**, p. 667] a similar construction is discussed for commutative I-space monoids. The global equivariant properties, however, of the construction have not been investigated so far.

Remark 8.21. The combined global delooping functor

$$\mathbf{B} \;=\; (-)(\mathbb{S}) \circ \mathcal{H} \;:\; \mathit{coms} \;\longrightarrow\; \mathcal{S}p$$

thus comes out as follows: the value of the orthogonal spectrum $\mathbf{B}R$ at an real inner product space V is

$$(\mathbf{B}R)(V) \;=\; (\mathcal{H}R)(V, S^V) \;,$$

the value of the $\mathbf{\Gamma}$-$O(V)$-space $(\mathcal{H}R)(V)$ at the V-sphere, with diagonal $O(V)$-action. There is another way to look at this space. The category of commutative orthgonal monoid spaces is tensored and cotensored over the category $\mathbf{T}$ of based spaces: for a bases space A and a commutative orthogonal monoid space R, the cotensor R^A is simply the orthogonal space $\mathrm{map}(A,R)$ with multiplication

$$\mathrm{map}(A,R) \boxtimes \mathrm{map}(A,R) \;\longrightarrow\; \mathrm{map}(A \times A, R \boxtimes R) \xrightarrow{\mathrm{map}(\Delta_A, \mu_R)} \mathrm{map}(A,R) \;.$$

The functor $\mathrm{map}(A,-) : \mathit{coms} \longrightarrow \mathit{coms}$ has a left adjoint $A \otimes - : \mathit{coms} \longrightarrow \mathit{coms}$. For $n \geq 0$, the cotensor $\mathrm{map}(n^+, R)$ is an n-fold product of R, so $n^+ \otimes R$ is an n-fold coproduct copies of R. Because the coproduct of commutative orthgonal monoid spaces is given by the boxproduct, this amounts to

$$n^+ \otimes R \;=\; \underbrace{R \boxtimes \cdots \boxtimes R}_{n} \;=\; \mathcal{H}R(n^+) \ .$$

The construction is extended to general based spaces A by a coend, i.e.,

$$A \otimes R \;=\; \int^{n^+ \in \mathbf{\Gamma}} (n^+ \otimes R) \times A^n \;\;=\;\; \left(\coprod_{n \geq 0} F(n^+) \times K^n \right) / \sim \ ,$$

So [...] $A \otimes R = (\mathcal{H}R)(A)$ and hence

$$(\mathbf{B}R)(V) \;=\; (S^V \otimes R)(V) \ ,$$

the value of the orthogonal space $S^V \otimes R$ at V.

Remark 8.22. The multiplication and the power operations on the homotopy sets $\underline{\pi}_0(R)$ of a commutative orthogonal monoid space R can be recovered from the $\mathbf{\Gamma}$-orthogonal space $\mathcal{H}R$. Indeed, P^m factors as

$$\pi_0^G(R) \xrightarrow{\bar{P}^m} \pi_0^{\Sigma_m \wr G}(R^{\boxtimes m}) \;=\; \pi_0^{\Sigma_m \wr G}((\mathcal{H}R)(m^+)) \xrightarrow{\nabla_*} \pi_0^{\Sigma_m \wr G}((\mathcal{H}R)(1^+)) = \pi_0^{\Sigma_m \wr G}(R) \ ,$$

where $\nabla : m^+ \longrightarrow 1^+$ is the fold map. In the intermediate term we view $R^{\boxtimes m}$ as a Σ_m-orthogonal space by pulling back the permutation action of Σ_m along the projection $\Sigma_m \wr G \longrightarrow \Sigma_m$. The map $\tilde{P}^m$ sends the class of a G-fixed point in $R(V)^G$ to the image of $(x, \dots, x) \in R(V)^m$ under the $(\Sigma_m \wr G)$-equivariant map

$$i_{V,\dots,V} \;:\; R(V)^m \;\longrightarrow\; (R^{\boxtimes m})(V^m) \ .$$

Example 8.23. We will discuss various examples of the global delooping construction in this book. When applied to the constant orthogonal monoid space associated to a commutative topological monoid, then the global delooping gives the traditional Eilenberg-Mac Lane spectrum; we discuss this example in detail in Section V.5. The global delooping $\mathbf{BGr}$ of the additive Grassmannian $\mathbf{Gr}$ gives an orthogonal spectrum globally equivalent to connective global K-theory $\mathbf{ku}$, compare Proposition V.6.14 below. The global delooping is invariant under group completions, so the global delooping $\mathbf{BBUP}$ of periodic global BU is also globally equivalent to $\mathbf{ku}$. [discuss also $\mathbf{B}(\mathbf{Gr}_{\otimes}^{[1]})$ and more generally $\mathbf{B}(B^{\otimes}_{\mathrm{gl}} A)$ for an abelian compact Lie group, and $gl_1(R) = \mathbf{B}\left((\Omega^\bullet R)^\times\right)$ for an ultra-commutative ring spectrum R]

We need some preparatory results.

Lemma 8.24. *Let G be a compact Lie group and E a universal G-space for some family $\mathcal{F}$ of subgroups of G. Then the $(\Sigma_n \wr G)$-space E^n is a universal space for the family of those subgroup $\Gamma \leq (\Sigma_n \wr G)$ such that*

$$p_i(\Gamma \cap (1 \ltimes G^n)) \;\in\; \mathcal{F}$$

for all $i = 1, \dots, n$, where $p_i : G^n \longrightarrow G$ is the projection to the i-th factor.

The next theorem is a refinement of the global invariance property of the box product of orthogonal spaces. Indeed, for the trivial homomorphism $\alpha : G \longrightarrow \Sigma_n$, part (i) of the next theorem reduces to Theorem I.5.9 (i)

Theorem 8.25. *Let X be a flat orthogonal space and K a compact Lie group. For every $n \geq 1$ and every continuous homomorphism $\alpha : K \longrightarrow \Sigma_n$ the map*

$$(\rho_{X,\dots,X}(\mathcal{U}_K))^\Gamma \;:\; (X^{\boxtimes n}(\mathcal{U}_K))^\Gamma \;\;\longrightarrow\;\; (X^{\times n}(\mathcal{U}_K))^\Gamma$$

is a weak equivalence, where $\Gamma = \{(\alpha(k), k) \mid k \in K\} \leq \Sigma_n \times K$ is the graph of α.

PROOF. We start with the key special case where $X = \mathbf{L}_{G,V}$ is a free orthogonal space generated by a faithful G-representation V. If $V = 0$, then G must be a trivial group, and then X is a constant orthogonal space. In this case the morphism $\rho_{X,\dots,X} : X^{\boxtimes n} \longrightarrow X^{\times n}$ is an isomorphism, and the claim is clear. So we assume for the rest of this argument that V is non-zero.

We record some algebraic facts that we will need in the course of the proof. We denote by $K \ltimes_\alpha G^n$ the semidirect product of K acting on G^n by permuting the coordinates according to the homomorphism $\alpha : K \longrightarrow \Sigma_n$. We let

$$\alpha_\star : K \ltimes_\alpha G^n \longrightarrow \Sigma_n \wr (K \times G) \qquad \text{and} \qquad \alpha_\sharp : K \ltimes_\alpha G^n \longrightarrow K \times (\Sigma_n \wr G)$$

denote the monomorphisms defined by

$$\begin{aligned} \alpha_\star(k;\, g_1, \dots, g_n) \ &= \ (\alpha(k);\, (k, g_1), \dots, (k, g_n)) \quad \text{respectively} \\ \alpha_\sharp(k;\, g_1, \dots, g_n) \ &= \ \ (k, (\alpha(k);\, g_1, \dots, g_n)) \ . \end{aligned}$$

For a subgroup $\Lambda \leq K \ltimes_\alpha G^n$ the relation

$$p_i \left(\alpha_\star(\Lambda) \cap (1 \ltimes (K \times G)^n)\right) \cap (1 \times G) \ = \ p_i(\Lambda \cap (1 \ltimes G^n))$$

holds as subgroups of G, where the maps p_i on the left and the right denote the (different!) projections $(K \times G)^n \longrightarrow K \times G$ respectively $G^n \longrightarrow G$ to the i-th factor.

We recall that $\mathcal{F}(K; G^{\mathrm{op}})$ denotes the family of 'graph subgroups' of $K \times G^{\mathrm{op}}$, i.e., subgroups $\Gamma \leq K \times G^{\mathrm{op}}$ such that $\Gamma \cap (1 \times G^{\mathrm{op}}) = (1, 1)$. So the following three conditions are equivalent:

(a) The intersection $\Lambda \cap (1 \ltimes_\alpha (G^{\mathrm{op}})^n)$ consists only of the neutral element.

(b) The subgroup $\alpha_\star(\Lambda)$ of $\Sigma_n \wr (K \times G^{\mathrm{op}})$ satisfies the condition

$$p_i(\alpha_\star(\Lambda) \cap (1 \ltimes (K \times G^{\mathrm{op}})^n)) \ \in \ \mathcal{F}(K; G^{\mathrm{op}})$$

for all $i = 1, \dots, n$.

(c) The subgroup $\alpha_\sharp(\Lambda)$ of $K \times (\Sigma_n \wr G^{\mathrm{op}})$ belongs to $\mathcal{F}(K; \Sigma_n \wr G^{\mathrm{op}})$.

By Proposition I.2.6 (i) the space $\mathbf{L}_V(\mathcal{U}_K)$ is a universal $(K \times G^{\mathrm{op}})$-space for the family $\mathcal{F}(K; G^{\mathrm{op}})$. By Lemma 8.24 the n-th power

$$(\mathbf{L}_V^{\times n})(\mathcal{U}_K) \ = \ (\mathbf{L}_V(\mathcal{U}_K))^{\times n}$$

is a universal $(\Sigma_n \wr (K \times G^{\mathrm{op}}))$-space for the family of those subgroups $\Delta \leq \Sigma_n \wr (K \times G^{\mathrm{op}})$ such that

$$p_i(\Delta \cap (1 \ltimes (K \times G^{\mathrm{op}})^n)) \ \in \ \mathcal{F}(K; G^{\mathrm{op}})$$

for all $i = 1, \dots, n$. By the equivalence of conditions (a) and (b) above we conclude that the $(K \ltimes_\alpha (G^{\mathrm{op}})^n)$-space

$$\alpha_\star^*((\mathbf{L}_V(\mathcal{U}_K))^{\times n})$$

is a universal space for the family of those subgroups $\Lambda \leq K \ltimes_\alpha (G^{\mathrm{op}})^n$ such that $\Lambda \cap (1 \ltimes_\alpha (G^{\mathrm{op}})^n)$ consists only of the neutral element.

Now we investigate the space $(\mathbf{L}_V^{\boxtimes n})(\mathcal{U}_K)$ that comes to us with an action of $K \times (\Sigma_n \wr G^{\mathrm{op}})$. The orthogonal space $\mathbf{L}_V^{\boxtimes n}$ is $(\Sigma_n \wr G^{\mathrm{op}})$-equivariantly isomorphic to $\mathbf{L}_{V^n}$ [ref]; since G acts faithfully on V, and V is non-zero, the induced action of $\Sigma_n \wr G^{\mathrm{op}}$ on V^n is faithful. The $K \times (\Sigma_n \wr G)^{\mathrm{op}}$-space

$$(\mathbf{L}_V^{\boxtimes n})(\mathcal{U}_K) \ \cong \ \mathbf{L}_{V^n}(\mathcal{U}_K)$$

is thus a universal space for the family $\mathcal{F}(K; \Sigma_n \wr G^{\mathrm{op}})$, one more time by Proposition I.2.6 (i). By the equivalence of conditions (a) and (c) above we conclude that the $(K \ltimes_\alpha (G^{\mathrm{op}})^n)$-space

$$\alpha_\sharp^*((\mathbf{L}_V^{\boxtimes n}(\mathcal{U}_K)))$$

is also a universal space for the family of those subgroups $\Lambda \leq K \ltimes_\alpha (G^{\mathrm{op}})^n$ such that $\Lambda \cap (1 \ltimes_\alpha (G^{\mathrm{op}})^n)$ consists only of the neutral element.

The value of the morphism

$$\rho_{\mathbf{L}_V,\dots,\mathbf{L}_V} \ : \ \mathbf{L}_V^{\boxtimes n} \ \longrightarrow \ \mathbf{L}_V^{\times n}$$

at $\mathcal{U}_K$ is a $(K \ltimes_\alpha (G^{\mathrm{op}})^n)$-equivariant map between universal $(K \ltimes_\alpha (G^{\mathrm{op}})^n)$-space for the same family of subgroups. So this map is a $(K \ltimes_\alpha (G^{\mathrm{op}})^n)$-equivariant homotopy equivalence. We can thus pass to orbit spaces by the right action of G^n and conclude that the induced map

$$\rho_{\mathbf{L}_V,\dots,\mathbf{L}_V}(\mathcal{U}_K)/G^n \ : \ \mathbf{L}_V^{\boxtimes n}(\mathcal{U}_K)/G^n \ \longrightarrow \ \mathbf{L}_V^{\times n}(\mathcal{U}_K)/G^n$$

is a K-equivariant homotopy equivalence. Since

$$\mathbf{L}_V^{\boxtimes n}/G^n \ \cong \ (\mathbf{L}_V/G)^{\boxtimes n} \ = \ \mathbf{L}_{G,V}^{\boxtimes n} \qquad \text{and} \qquad \mathbf{L}_V^{\times n}/G^n \ \cong \ (\mathbf{L}_V/G)^{\times n} \ = \ \mathbf{L}_{G,V}^{\times n} \ ,$$

this means that the value of the morphism

$$\rho_{\mathbf{L}_{G,V},\dots,\mathbf{L}_{G,V}} \ : \ \mathbf{L}_{G,V}^{\boxtimes n} \ \longrightarrow \ \mathbf{L}_{G,V}^{\times n}$$

at $\mathcal{U}_K$ is a K-equivariant homotopy equivalence. This proves property (i) in the special case $X = \mathbf{L}_{G,V}$.

To work our way up from free orthogonal spaces to general flat orthogonal spaces we consider a pushout square of orthogonal spaces

$$\begin{tikzcd} A \times \mathbf{L}_{G,V} \arrow[r, "i \times \mathbf{L}_{G,V}"] \arrow[d] & B \times \mathbf{L}_{G,V} \arrow[d] \\ X \arrow[r, "j"'] & Y \end{tikzcd}$$

where $i : A \longrightarrow B$ is a cofibration between cofibrant spaces, V is a [non-zero] faithful G-representation and X is a flat orthogonal space [with $X(0) = \emptyset$]. We assume that property (i) holds for the orthogonal space X, and we show that they also hold for Y.

We let $\otimes$ be any symmetric monoidal product on the category of orthogonal spaces, compatible with the objectwise product with spaces in the sense of coherent natural isomorphisms $A \times (X \otimes Y) \cong (A \times X) \otimes Y$. The two cases we have in mind are $\otimes = \boxtimes$ and $\otimes = \times$. The morphism $j^{\otimes n} : X^{\otimes n} \longrightarrow Y^{\otimes n}$ is the composite of n morphisms of Σ_n-orthogonal spaces

$$X^{\otimes n} = F_0^{\otimes} \ \longrightarrow \ F_1^{\otimes} \ \longrightarrow \ \dots \ \longrightarrow \ F_n^{\otimes} = Y^{\otimes n} \ ,$$

the k-th of which takes part in a pushout square of Σ_n-orthogonal spaces

$$\begin{tikzcd} (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times Q^k) \times \mathbf{L}_{G,V}^{\otimes n} \arrow[r, "i \times \mathbf{L}_{G,V}^{\otimes n}"] \arrow[d] & (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times B^k) \times \mathbf{L}_{G,V}^{\otimes n} \arrow[d] \\ F_{k-1} \arrow[r] & F_k \end{tikzcd}$$

Here $i^{\square k} : Q^k \longrightarrow B^k$ is the k-fold self-pushout product of copies of the map $i : A \longrightarrow B$.

Now we use this Σ_n-equivariant filtration for the two symmetric monoidal products $\boxtimes$ and $\times$, and we show by induction on k that the morphism

$$F_k^{\boxtimes} \ \longrightarrow \ F_k^{\times}$$

is a Σ_n-global equivalence. For $k = 0$ this is the hypothesis that part (i) of the proposition holds for X. In the inductive step the map is obtained by passage to horizontal pushouts in the commutative diagram

$$\begin{tikzcd} (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times B^k) \times \mathbf{L}_{G,V}^{\boxtimes n} \arrow[d] & (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times Q^k) \times \mathbf{L}_{G,V}^{\boxtimes n} \arrow[l] \arrow[r] \arrow[d] & F_{k-1}^{\boxtimes} \arrow[d] \\ (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times B^k) \times \mathbf{L}_{G,V}^{\times n} & (\Sigma_n \times_{\Sigma_{n-k} \times \Sigma_k} A^{n-k} \times Q^k) \times \mathbf{L}_{G,V}^{\times n} \arrow[l] \arrow[r] & F_{k-1}^{\times} \end{tikzcd}$$

The left and middle vertical morphisms are Σ_n-global equivalences by (a refinement of) the special case treated above, shand the right vertical morphisms is a Σ_n-global equivalences by induction. Since the upper and lower left horizontal maps are h-cofibrations, the induced morphism of horizontal pushouts is another Σ_n-global equivalence.

Now we can prove part (i) of the proposition in general. We let X be any flat orthogonal space. We show first, by induction on m, that the claim holds for the m-skeleton of X, i.e., that the morphism

$$\rho_{\mathrm{sk}^m X,\dots,\mathrm{sk}^m X} \ : \ (\mathrm{sk}^m X)^{\boxtimes n} \ \longrightarrow \ (\mathrm{sk}^m X)^{\times n}$$

is a Σ_n-global equivalence. The induction starts with $m = 0$; since $\mathrm{sk}^0 X$ is a constant orthogonal space with value $X(0)$, the morphism $\rho_{\mathrm{sk}^0 X,\dots,\mathrm{sk}^0 X}$ is even an isomorphism (between constant orthogonal space). Now we assume part (i) for $\mathrm{sk}^{m-1} X$ for some $m \geq 1$, and we deduce it for $\mathrm{sk}^m X$. Since X is flat the latching map $\nu_m : L_m X \longrightarrow X(\mathbb{R}^m)$ is an $O(m)$-cofibration, hence homotopy equivalence, relative to $L_m X$, to a relative $O(m)$-CW-inclusion. So we can assume without loss of generality that ν_m is a relative $O(m)$-CW-inclusion with relative skeleta F^i. We show the claim for the pushout

$$\begin{tikzcd} G_m(L_m X) \arrow[r] \arrow[d] & G_m F^i \arrow[d] \\ \mathrm{sk}^{m-1} X \arrow[r] & P_i \end{tikzcd}$$

by induction on i. The induction starts with $i = -1$, where $P^i = \mathrm{sk}^{m-1} X$. We suppose for simplicity that F^i is obtained from F^{i-1} by attaching a single equivariant cell of orbit type $H \leq O(m)$. There is thus a pushout

$$\begin{tikzcd} S^{i-1} \times \mathbf{L}_{H,\mathbb{R}^m} \arrow[r] \arrow[d] & D^i \times \mathbf{L}_{H,\mathbb{R}^m} \arrow[d] \\ P_{i-1} \arrow[r] & P_i \end{tikzcd}$$

Here we have exploited that

$$G_m(O(m)/H \times A) \ \cong \ A \times \mathbf{L}_{H,\mathbb{R}^m}$$

for every space A. We have seen above that the property (i) of the proposition is inherited by such pushouts, so this shows the inductive step.

Now we have to pass to colimits over sequences of h-cofibrations twice, once along the sequece of P_i's, and once along the sequence of skeleta $\mathrm{sk}^k X$. □

We let R be a commutative orthogonal monoid space. If the underlying orthogonal space of R is flat, then Theorem I.5.9 shows that the morphism

$$\rho_{R,\dots,R} \ : \ (\mathcal{H}R)(n^+) \ = \ R^{\boxtimes n} \ \longrightarrow \ R^{\times n} \ = \ ((\mathcal{H}R)(1^+))^n$$

is a global equivalence; so $\mathcal{H}R$ is a special $\mathbf{\Gamma}$-orthogonal space in the naive sense. However, more is true:

Corollary 8.26. *Let R be a commutative orthogonal monoid space that is flat as an orthogonal space. Then the $\mathbf{\Gamma}$-orthogonal space $\mathcal{H}R$ is flat and globally special.*

Proof. We let G be a compact Lie group and S a finite G-set; we can assume without loss of generality that $S = \{1, \dots, n\}$ for some $n \geq 0$, so that the G-action is encoded in a continuous homomorphism $\alpha : G \longrightarrow \Sigma_n$. The map

$$P_{\{1,\dots,n\}} \ : \ ((\mathcal{H}R)(n^+)(\mathcal{U}_G))^G \ \longrightarrow \ \mathrm{map}^G(n, ((\mathcal{H}R)(1^+)(\mathcal{U}_G)))$$

then becomes the map

$$(\rho_{R,\dots,R}(\mathcal{U}_G))^G \ : \ ((R^{\boxtimes n}(\mathcal{U}_G))^G \ \longrightarrow \ (R(\mathcal{U}_G)^{\times n})^G \ .$$

This map is a weak equivalence by Theorem 8.25. □

The next statement is one of the key result that justifies the name 'global delooping functor'. When specialized to the $\mathbf{\Gamma}$-orthogonal space $\mathcal{H}R$, the morphism (8.12) becomes a morphism of orthogonal spaces

$$\xi \;:\; R \longrightarrow \Omega^\bullet(\mathbf{B}R)$$

and the morphism (8.16) becomes a morphism of global functors

$$\hat{\xi} \;:\; \Lambda(\underline{\pi}_0(R)) \longrightarrow \underline{\pi}_0(\mathbf{B}R)\ .$$

The following theorem is then a special case of Theorem 8.17, applied to the $\mathbf{\Gamma}$-orthogonal space $\mathcal{H}R$, exploiting Corollary 8.26. Nevertheless we label it a 'theorem' because of its importance.

Theorem 8.27. *Let R be a group-like commutative orthogonal monoid space that is flat as an orthogonal space.*

(i) *The morphism*

$$\xi \;:\; R \longrightarrow \Omega^\bullet(\mathbf{B}R)$$

is a $\mathcal{F}in$-global equivalence of orthogonal spaces.

(ii) *The morphism*

$$\hat{\xi} \;:\; \Lambda(\underline{\pi}_0(R)) \longrightarrow \underline{\pi}_0^G(\mathbf{B}R)$$

is an isomorphism of global functors.

In the remaining part of this section we prove further useful properties of the global delooping functor.

Proposition 8.28. *Let R and S be commutative orthogonal monoid spaces that are flat as orthogonal spaces. Then the morphisms*

$$\mathbf{B}(R)\vee\mathbf{B}(S) \xrightarrow{\mathbf{B}i_1+\mathbf{B}i_2} \mathbf{B}(R\boxtimes S) \xrightarrow{(\mathbf{B}\rho_1,\mathbf{B}\rho_2)} \mathbf{B}(R)\times\mathbf{B}(S)$$

are global equivalences of orthogonal spectra.

Proof. We can apply Theorem I.5.9 to the flat orthogonal spaces $R^{\boxtimes n}$ and $S^{\boxtimes n}$ and conclude that the morphism

$$\rho_{R^{\boxtimes n},S^{\boxtimes n}} \;:\; (R^{\boxtimes n})\boxtimes(S^{\boxtimes n}) \longrightarrow (R^{\boxtimes n})\times(S^{\boxtimes n})$$

is a global equivalence of orthogonal spaces. Up to a reshuffling of factors by symmetry and associativity isomorphism, this is the morphism

$$((\mathcal{H}\rho_1)(n^+),(\mathcal{H}\rho_2)(n^+)) \;:\; \mathcal{H}(R\boxtimes S)(n^+) \longrightarrow (\mathcal{H}R)(n^+)\times(\mathcal{H}S)(n^+)$$

which is thus also a global equivalence of orthogonal spaces. All orthogonal space in sight are closed, so for every compact Lie group G the G-map

$$((\mathcal{H}\rho_1)(n^+),(\mathcal{H}\rho_2)(n^+)) \;:\; \mathcal{H}(R\boxtimes S)(\mathcal{U}_G,n^+) \longrightarrow (\mathcal{H}R)(\mathcal{U}_G,n^+)\times(\mathcal{H}S)(\mathcal{U}_G,n^+)$$

is a G-weak equivalence. □

Proposition 8.29. *Consider a pushout square of commutative orthogonal monoid spaces on the left*

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{B}A & \xrightarrow{\mathbf{B}i} & \mathbf{B}B \\ \downarrow & & \downarrow \\ \mathbf{B}R & \longrightarrow & \mathbf{B}S \end{array}$$

such that $i : A \longrightarrow B$ is a cofibration in the global model structure of Theorem II.3.4 . Then the square on the right hand side is a global homotopy pushout square of orthogonal spectra.

PROOF. Use the twosided bar construction model for the homotopy pushout and use the previous proposition. □

Theorem 8.30. *Let $i : R \longrightarrow R^\star$ be a group completion between commutative orthogonal monoid spaces that are flat as orthogonal spaces. Then the induced morphism*

$$\mathbf{B}i \ : \ \mathbf{B}R \ \longrightarrow \ \mathbf{B}(R^\star)$$

is a global equivalence of orthogonal spectra.

PROOF. Apply the previous proposition to the homotopy pushout square that defines a group completion. □

We let X be a flat orthogonal space. Then the free commutative orthogonal monoid space $\mathbb{P}X$ generated by X is also flat as an orthogonal space. The adjoint $\hat{\xi} : \Sigma^\infty_+ X \longrightarrow \mathbf{B}(\mathbb{P}X)$ of the composite

$$X \ \xrightarrow{\eta} \ \mathbb{P}X \ \xrightarrow{\xi} \ \Omega^\bullet(\mathbf{B}(\mathbb{P}X))$$

is a morphism of orthogonal spaces. The following is a global version of the Barratt-Priddy-Quillen theorem:

Theorem 8.31. *For every based flat orthogonal space X such that $X(0)$ consists only of the basepoint, the morphism*

$$\hat{\xi} \ : \ \Sigma^\infty X \ \longrightarrow \ \mathbf{B}(\tilde{\mathbb{P}}X)$$

is a global equivalence of orthogonal spectra.

Construction 8.32. We define a homomorphism out of the group $\pi_0^G(\mathbf{B}R)$ that will provide a retraction to $\langle - \rangle : \pi_0^G(R) \longrightarrow \pi_0^G(\mathbf{B}R)$; in particular, the homomorphism (6.17) is injective for every compact Lie group. We let $G^\circ \leq G$ be the connected component of the identity and $\bar{G} = G/G^\circ$ the finite group of path components of G. We define an equivariant homotopy group $\Psi^G(\mathbf{B}R)$ as a mixture of a genuine equivariant homotopy group for the finite group $\bar{G}$ and a geometric fixed point homotopy group for the connected group G°:

$$\Psi^G(\mathbf{B}R) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)} \, [S^{V^{G^\circ}}, ((\mathbf{B}R)(V))^{G^\circ}]^{\bar{G}} \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)} \, [S^{V^{G^\circ}}, ((\mathcal{H}R)(V, S^V))^{G^\circ}]^{\bar{G}} \ ;$$

as usual, $s(\mathcal{U}_G)$ is the poset of finite dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$. We define a partial geometric fixed point map

$$\psi^G \ : \ \pi_0^G(\mathbf{B}R) \ \longrightarrow \ \Psi^G(\mathbf{B}R) \ , \quad [f : S^V \longrightarrow \mathbf{ku}(V)] \ \longmapsto \ [f^{G^\circ} : S^{V^{G^\circ}} \longrightarrow ((\mathbf{B}R)(V))^{G^\circ}] \ .$$

If the group G is finite, then G° is trivial, $\bar{G} = G$, $\pi_0^G(\mathbf{B}R) = \Psi^G(\mathbf{B}R)$ and ψ^G is the identity. On the other extreme, if G is connected, then $\bar{G}$ is trivial, $\Psi^G(\mathbf{B}R) = \Phi^{G^\circ}(\mathbf{B}R)$ is the geometric fixed point group defined in (6.2) and ψ^G is the geometric fixed point map defined in (6.3).

Proposition 8.33. *Let G be a connected compact Lie group and F a $\mathbf{\Gamma}$-G-space. Then for every based G-space K the map*

$$F^G(K^G) \ \longrightarrow \ (F(K))^G$$

induced by the fixed point inclusions $F^G \longrightarrow F$ and $K^G \longrightarrow K$ is a homeomorphism.

PROOF. We consider a point of $F(K)$ represented by a tuple $(x; k_1, \dots, k_n) \in F(n^+) \times K^n$. We assume that the number n has been chosen minimally, so that the entries k_i are pairwise distinct and different from the basepoint of K. If the point $[x; k_1, \dots, k_n]$ of $F(K)$ is G-fixed, then for every group element g the tuple $(gx; gk_1, \dots, gk_n)$ is equivalent to the original tuple, so there is a unique permutation $\sigma(g) \in \Sigma_n$ such that

$$(gx; gk_1, \dots, gk_n) \ = \ (F(\sigma(g))(x); k_{\sigma(g)(1)}, \dots, k_{\sigma(g)(n)}) \ .$$

Since G acts continuously on K, the map $\sigma : G \longrightarrow \Sigma_n$ is a continuous homomorphism. The kernel of σ is then a closed subgroup of finite index in G; since G is connected, this kernel must be all of G. We conclude that $\sigma(g) = 1$ for all $g \in G$, i.e., the points $k_1, \dots, k_n$ and the element $x \in F(n^+)$ are all G-fixed. □

Corollary 8.34. *Let R be a group-like commutative orthogonal monoid space that is flat as an orthogonal space. Then for every compact Lie group G the composite*

$$\pi_0^G(R) \longrightarrow \pi_0^G(\mathbf{B}R) \xrightarrow{\psi^G} \Psi^G(\mathbf{B}R)$$

is an isomorphism. Moreover, the kernel of ψ^G is generated by transfers from subgroups of G of strictly smaller dimension.

PROOF.

$$\begin{aligned}
\Psi^G(\mathbf{B}R) \ &= \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, ((\mathcal{H}R)(V,S^V))^{G^\circ}]^{\bar G}\\
&\cong \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, ((\mathcal{H}R)(\mathcal{U}_G,S^V))^{G^\circ}]^{\bar G}\\
&\cong \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, ((\mathcal{H}R)(\mathcal{U}_G))^{G^\circ}(S^{V^{G_0}})]^{\bar G}\\
&\cong \ \pi_0\left(\left(((\mathcal{H}R)(\mathcal{U}_G,1^+))^{G^\circ})\right)^{\bar G}\right)\\
&\cong \ \pi_0\left(((R(\mathcal{U}_G))^{G^\circ})^{\bar G}\right) \ \cong \ \pi_0\left((R(\mathcal{U}_G))^G\right) \ \cong \ \pi_0^G(R)
\end{aligned}$$

The third isomorphism is Proposition 8.33 applied to the $\mathbf{\Gamma}$-G°-space $((\mathcal{H}R)(\mathcal{U}_G))^{G^\circ}$. The fourth isomorphism is the fact that $(\mathcal{H}R)(\mathcal{U}_G))$ is a very special $\mathbf{\Gamma}$-$\bar G$-space. □

Corollary 8.35. *Let B be a flat group-like globally special $\mathbf{\Gamma}$-orthogonal space. Then for every compact Lie group G the composite*

$$\pi_0^G(B(1^+)) \longrightarrow \pi_0^G(B(\mathbb{S})) \xrightarrow{\psi^G} \Psi^G(B(\mathbb{S}))$$

is an isomorphism. Moreover, the kernel of ψ^G is generated by transfers from subgroups of G of strictly smaller dimension.

PROOF.

$$\begin{aligned}
\Psi^G(B(\mathbb{S})) \ &= \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, (B(V,S^V))^{G^\circ}]^{\bar G}\\
&\cong \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, (B(\mathcal{U}_G,S^V))^{G^\circ}]^{\bar G}\\
&\cong \ \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, B(\mathcal{U}_G)^{G^\circ}(S^{V^{G_0}})]^{\bar G}\\
&\cong \ \pi_0\left(\left(B(\mathcal{U}_G,1^+)^{G^\circ}\right)^{\bar G}\right) \ \cong \ \pi_0\left(B(\mathcal{U}_G,1^+)^G\right) \ \cong \ \pi_0^G(B(1^+)) \ .
\end{aligned}$$

The third isomorphism is Proposition 8.33 applied to the $\mathbf{\Gamma}$-G°-space $B(\mathcal{U}_G)$. The fourth isomorphism is the fact that $B(\mathcal{U}_G)^{G^\circ}$ is a very special $\mathbf{\Gamma}$-$\bar G$-space. □

[verify algebraically that for every restricted global functor F the morphism $F \longrightarrow \Lambda(R)$ is objectwise a split monomorphism]

whose only eigenvalues are 0 or 1. For every flat commutative orthogonal monoid space the commutative square

$$\begin{tikzcd}
R \arrow[r, "p"] \arrow[d] & \mathscr{C}(R;[0,1]) \arrow[d, "\exp"] \\
* \arrow[r] & \mathscr{C}(R;U(1))
\end{tikzcd}$$

should be a homotopy pushout in the category of commutative unitary monoid spaces. In the special case $R = \mathbf{Gr}^{\mathbb{C}}$, this specializes to the homotopy pushout square in Remark II.4.15. A consequence is that the morphism $R \longrightarrow \Omega\mathscr{C}(R;U(1))$ arising from the linear contraction of the interval $[0,1]$ to 0 is a global group completion.

CHAPTER IV

The global homotopy category

1. Level model structures for orthogonal spectra

In this section and the next we establish the level and global model structures on the category of orthogonal spectra. Many arguments are parallel to the unstable analogs in the respective sections of Chapter I, so there is a certain amount of repetition.

There is a functorial way to write an orthogonal spectrum as a sequential colimit of spectra which are made from the information below a fixed level. We refer to this as the *skeleton filtration* of an orthogonal spectrum. The word 'filtration' should be used with caution because the maps from the skeleta to the orthogonal spectrum need not be injective.

In the following construction we denote by

$$G_m \ : \ O(m)\mathbf{T} \ \longrightarrow \ \mathcal{S}p$$

the left adjoint to the forgetful functor that takes an orthogonal spectrum X to its m-th level X_m, viewed as an $O(m)$-space. Since X_m is naturally isomorphic to $X(\mathbb{R}^m)$, the value of X at the tautological $O(m)$-representation, G_m is a shorthand notation for $F_{O(m),\mathbb{R}^m}$, the free functor (see (5.4) of Chapter III) indexed by the tautological $O(m)$-representation.

Construction 1.1. For every orthogonal spectrum A and $m \geq 0$ we define the following data by induction on m:

- a based $O(m)$-space $L_m A$, the *m-th latching space* of A, equipped with a natural map of based $O(m)$-spaces $\nu_m : L_m A \longrightarrow A_m$.
- an orthogonal spectrum $\mathrm{sk}^m A$, the *m-skeleton* of A, equipped with a natural morphism $i_m : \mathrm{sk}^m A \longrightarrow A$,
- a natural morphism $j_m : \mathrm{sk}^{m-1} A \longrightarrow \mathrm{sk}^m A$ which satisfies $i_m j_m = i_{m-1}$.

We start with $\mathrm{sk}^{-1} A = *$, the trivial spectrum. For $m \geq 0$ we define the latching space by

$$L_m A \ = \ (\mathrm{sk}^{m-1} A)_m \ , \tag{1.2}$$

the m-th level of the $(m-1)$-skeleton, and the morphism $\nu_m : L_m A = (\mathrm{sk}^{m-1} A)_m \longrightarrow A_m$ as the m-level of the previously constructed morphism $i_{m-1} : \mathrm{sk}^{m-1} A \longrightarrow A$. Then we define the m-skeleton $\mathrm{sk}^m A$ as the pushout

$$\begin{tikzcd} G_m L_m A \arrow[r, "G_m \nu_m"] \arrow[d] & G_m A_m \arrow[d] \\ \mathrm{sk}^{m-1} A \arrow[r, "j_m"'] & \mathrm{sk}^m A \end{tikzcd} \tag{1.3}$$

where the left vertical morphism is adjoint to the identity map of $L_m A = (\mathrm{sk}^{m-1} A)_m$. The morphism $\eta : G_m A_m \longrightarrow A$ which is adjoint to the identity of A_m and $i_{m-1} : \mathrm{sk}^{m-1} A \longrightarrow A$ restrict to the same morphism on $G_m L_m A$. So the universal property of the pushout provides a unique morphism $i_m : \mathrm{sk}^m A \longrightarrow A$ which satisfied $i_m j_m = i_{m-1}$ and whose restriction to $G_m A_m$ is η.

We can – and will – choose the pushout (1.3) so that

$$(\mathrm{sk}^m A)_n \;=\; A_n \qquad \text{for } n \le m$$

and so that the morphisms $j_{m+1} : \mathrm{sk}^m A \longrightarrow \mathrm{sk}^{m+1} A$ and $i_m : \mathrm{sk}^m A \longrightarrow A$ are the identity maps in level m and below. This convention is convenient because it will later make some maps equalities which would otherwise merely be isomorphisms. This convention also forces the structure maps of the orthogonal spectrum $\mathrm{sk}^m A$ to coincide with those of A up to level m.

The sequence of skeleta $\mathrm{sk}^m A$ stabilizes to A in a very strong sense. In every given level n, there is a point from which on all the spaces $(\mathrm{sk}^m A)_n$ are equal to A_n and the morphisms i_m and j_m are identity maps in level n. In particular, A_n is the colimit with respect to the morphisms $(i_m)_n$, of the sequence of maps $(j_m)_n$, Since colimits in the category of orthogonal spectra are created levelwise, we deduce that the spectrum A is a colimit, with respect to the morphisms i_m, of the sequence of morphisms j_m.

Given any morphism $f : A \longrightarrow B$ of orthogonal spectra we can define a relative skeleton filtration as follows. The *relative m-skeleton* of f is the pushout

$$\mathrm{sk}^m[f] \;=\; A \cup_{\mathrm{sk}^m A} \mathrm{sk}^m B \tag{1.4}$$

where $\mathrm{sk}^m A$ is the m-skeleton of A as defined above. The relative m-skeleton comes with a unique morphism $i_m : \mathrm{sk}^m[f] \longrightarrow B$ which restricts to $f : A \longrightarrow B$ respectively to $i_m : \mathrm{sk}^m B \longrightarrow B$. Since $L_m A = (\mathrm{sk}^{m-1} A)_m$ we have

$$(\mathrm{sk}^{m-1}[f])_m \;=\; A_m \cup_{L_m A} L_m B \ ,$$

the m-th relative latching object. A morphism $j_m[f] : \mathrm{sk}^{m-1}[f] \longrightarrow \mathrm{sk}^m[f]$ is obtained from the commutative diagram

$$\begin{tikzcd}
A \arrow[d, equal] & \mathrm{sk}^{m-1} A \arrow[l] \arrow[r, "\mathrm{sk}^{m-1} f"] \arrow[d, "j_m^A"'] & \mathrm{sk}^{m-1} B \arrow[d, "j_m^B"] \\
A & \mathrm{sk}^m A \arrow[l] \arrow[r, "\mathrm{sk}^m f"'] & \mathrm{sk}^m B
\end{tikzcd}$$

by taking pushouts. The square

$$\begin{tikzcd}
G_m(A_m \cup_{L_m A} L_m B) \arrow[r, "G_m(\nu_m f)"] \arrow[d] & G_m B_m \arrow[d] \\
\mathrm{sk}^{m-1}[f] \arrow[r, "j_m[f]"'] & \mathrm{sk}^m[f]
\end{tikzcd} \tag{1.5}$$

is a pushout and the original morphism $f : A \longrightarrow B$ factors as the composite of the countable sequence

$$A = \mathrm{sk}^{-1}[f] \xrightarrow{j_0[f]} \mathrm{sk}^0[f] \xrightarrow{j_1[f]} \mathrm{sk}^1[f] \longrightarrow \cdots \xrightarrow{j_m[f]} \mathrm{sk}^m[f] \longrightarrow \cdots .$$

If we fix a level n, then the sequence stabilizes to the identity map of B_n from $(\mathrm{sk}^n[f])_n$ on; in particular, the compatible maps $j_m : \mathrm{sk}^m[f] \longrightarrow B$ exhibit B as the colimit of the sequence.

For a morphism $f : A \longrightarrow B$ of orthogonal spectra and $m \ge 0$ we have a commutative square of $O(m)$-spaces

$$\begin{tikzcd}
L_m A \arrow[r, "L_m f"] \arrow[d, "\nu_m"'] & L_m B \arrow[d, "\nu_m"] \\
A_m \arrow[r, "f_m"'] & B_m
\end{tikzcd}$$

We thus get a natural morphism of $O(m)$-spaces

$$\nu_m f \ : \ A_m \cup_{L_m A} L_m B \ \longrightarrow \ B_m \ .$$

Example 1.6. Let G be a compact Lie group, V a G-representation of dimension n and A a based G-space. Then the free orthogonal spectrum (5.4) generated by A in level V is 'purely n-dimensional' in the following sense. The space $(F_{G,V}A)_m$ is trivial for $m < n$, and hence the latching space $L_m(F_{G,V}A)$ is trivial for $m \leq n$. For $m > n$ the latching map $\nu_m : L_m(F_{G,V}A) \longrightarrow (F_{G,V}A)_m$ is an isomorphism. So the skeleton $\mathrm{sk}^m(F_{G,V}A)$ is trivial for $m < n$ and $\mathrm{sk}^m(F_{G,V}A) = F_{G,V}A$ is the entire spectrum for $m \geq n$

Let A be a based $O(n)$-space. Then as a special case of the previous paragraph, the orthogonal spectrum $G_n A = F_{O(n),\mathbb{R}^n}A$ is purely n-dimensional.

The following proposition is an immediate application of the relative skeleton filtration. It is the key ingredient to the lifting properties of the various level model structures that we will discuss soon. We recall that a pair $(i : A \longrightarrow B, f : X \longrightarrow Y)$ of morphisms in some category has the *lifting property* if for all morphism $\varphi : A \longrightarrow X$ and $\psi : B \longrightarrow Y$ such that $f\varphi = \psi i$ there exists a *lifting*, i.e., a morphism $\lambda : B \longrightarrow Y$ such that $\lambda i = \varphi$ and $f\lambda = \psi$. Instead of saying that the pair (i, f) has the lifting property we may equivalently say 'i has the left lifting property with respect to f' or 'f has the right lifting property with respect to i'.

Proposition 1.7. *Let $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ be morphisms of orthogonal spectra. If the pair $(\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B,\ f_m : X_m \longrightarrow Y_m)$ has the lifting property in the category of $O(m)$-spaces for every $m \geq 0$, then the pair (i, f) has the lifting property.*

PROOF. We consider the class f-cof of all morphisms of orthogonal spectra that have the left lifting property with respect to f; this class is closed under cobase change and countable composition. Since the pair $(\nu_m i, f_m)$ has the lifting property in the category of $O(m)$-spaces, the semifree morphism $G_m(\nu_m i)$ belongs to the class f-cof by adjointness. The relative skeleton filtration (1.4) shows that i is a countable composite of cobase changes of the morphisms $\nu_m i$, so i belongs to the class f-cof. □

Now we discuss a general recipe for constructing 'level model structures' on the category of orthogonal spectra. As input we need, for every $m \geq 0$, a model structure $\mathcal{C}(m)$ on the category of based $O(m)$-spaces. We call a morphism $f : X \longrightarrow Y$ of orthogonal spectra

- a *level equivalence* if $f_m : X_m \longrightarrow Y_m$ is a weak equivalence in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *level fibration* if the morphism $f_m : X_m \longrightarrow Y_m$ is a fibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$;
- a *cofibration* if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is a cofibration in the model structure $\mathcal{C}(m)$ for all $m \geq 0$.

Proposition 1.9 below will show that if the various model structures $\mathcal{C}(m)$ satisfy the following 'stable consistency condition', then the level equivalences, level fibrations and cofibrations define a model structure on the category of orthogonal spectra.

Stable consistency condition: For all $m, n \geq 0$ and every acyclic cofibration $i : A \longrightarrow B$ in the model structure $\mathcal{C}(m)$ on based $O(m)$-spaces, every cobase change, in the category of based $O(m+n)$-spaces, of the map

$$(1.8)\qquad O(m+n) \ltimes_{O(m)\times O(n)} (i \wedge S^n) \ : \ O(m+n) \ltimes_{O(m)\times O(n)} (A \wedge S^n) \longrightarrow \ O(m+n) \ltimes_{O(m)\times O(n)} (B \wedge S^n)$$

is a weak equivalence in the model structure $\mathcal{C}(m+n)$.

The next result is the stable analog of the general method for level model structure on orthogonal spaces (Proposition I.3.9). The proof is completely parallel, and we omit it.

Proposition 1.9. *Let $\mathcal{C}(m)$ be a model structure on the category of based $O(m)$-spaces, for $m \geq 0$, such that the stable consistency condition* (1.8) *holds.*

(i) *The classes of level equivalences, level fibrations and cofibrations define a model structure on the category of orthogonal spectra.*
(ii) *A morphism $i : A \longrightarrow B$ of orthogonal spectra is simultaneously a cofibration and a level equivalence if and only if for all $m \geq 0$ the latching morphism $\nu_m i : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is an acyclic cofibration in the model structure $\mathcal{C}(m)$.*
(iii) *Suppose that the fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $J(m)$; then the level fibrations of orthogonal spectra are detected by the set of semifree morphisms*

$$\{G_m j \mid m \geq 0,\, j \in J(m)\}\ .$$

Similarly, if the acyclic fibrations in the model structure $\mathcal{C}(m)$ are detected by a set of morphisms $I(m)$, then the level acyclic fibrations of orthogonal spectra are detected by the set of semifree morphisms

$$\{G_m i \mid m \geq 0,\, j \in I(m)\}\ .$$

(iv) *If all the model structures $\mathcal{C}(m)$ are topological, then so is the resulting level model structure of orthogonal spectra.*

Definition 1.10. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *flat cofibration* if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is an $O(m)$-cofibration for all $m \geq 0$. An orthogonal spectrum A is *flat* if the morphism from the trivial spectrum to it is a flat cofibration, i.e., for every $m \geq 0$ the latching map $\nu_m : L_m A \longrightarrow A_m$ is an $O(m)$-cofibration.

Proposition 1.11. *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spectra. Then for every $n \geq 0$, the morphism $i_n : A_n \longrightarrow B_n$ is an $O(n)$-cofibration.*

Proof. The morphism i_n is the finite composite

$$A_n = (\mathrm{sk}^{-1}[i])_n \xrightarrow{(j_0[i])_n} (\mathrm{sk}^0[i])_n \xrightarrow{(j_1[i])_n} \dots \xrightarrow{(j_{n-1}[i])_n} (\mathrm{sk}^{n-1}[i])_n \xrightarrow{(j_n[i])_n} (\mathrm{sk}^n[i])_n = B_n\ ,$$

so it suffices to show that for all $m \geq 0$, the morphism $j_m[i] : \mathrm{sk}^{m-1}[i] \longrightarrow \mathrm{sk}^m[i]$ is levelwise an equivariant cofibration for the relevant orthogonal group.

The pushout square (1.5) in level $m+n$ is a pushout of $O(m+n)$-spaces:

$$\begin{tikzcd}
O(m+n) \ltimes_{O(m)\times O(n)} ((A_m \cup_{L_m A} L_m B) \wedge S^n) \arrow[r] \arrow[d, "O(m+n)\ltimes(\nu_m i \wedge S^n)"'] & (\mathrm{sk}^{m-1}[i])_{m+n} \arrow[d, "(j_m[i])_{m+n}"] \\
O(m+n) \ltimes_{O(m)\times O(n)} (B_m \wedge S^n) \arrow[r] & (\mathrm{sk}^m[i])_{m+n}
\end{tikzcd}$$

Since S^n can be given the structure of a based $O(n)$-CW-complex, the functor $O(m+n) \ltimes_{O(m)\times O(n)} (- \wedge S^n)$ takes $O(m)$-cofibrations to $O(m+n)$-cofibrations; so the left vertical morphism, and hence also $(j_m[i])_{m+n}$, is an $O(m+n)$-cofibration. $\square$

We let $\mathcal{F}$ be a global family in the sense of Definition I.7.1, i.e., a non-empty class of compact Lie groups that is closed under isomorphisms, closed subgroups and quotient groups. As in the unstable situation in Section I.7, we now develop two $\mathcal{F}$-level model structures the *projective* and the *flat* one, on the category of orthogonal spectra, in which the $\mathcal{F}$-level equivalences are the weak equivalences. Both model structures have global versions, and we need both of these model structures later for showing that the forgetful functor from the global homotopy category to the global $\mathcal{F}$-homotopy category has both a left and a right adjoint.

We recall that $\mathcal{F}(m)$ denotes the family of those subgroups of the orthogonal group $O(m)$ that belong to the global family $\mathcal{F}$.

Definition 1.12. Let $\mathcal{F}$ be a global family. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is

- an *$\mathcal{F}$-level equivalence* if the map $f_m : X_m \longrightarrow Y_m$ is an $\mathcal{F}(m)$-equivalence for all $m \geq 0$.
- an *$\mathcal{F}$-level fibration* if the map $f_m : X_m \longrightarrow Y_m$ is an $\mathcal{F}(m)$-projective fibration for all $m \geq 0$.
- an *injective $\mathcal{F}$-fibration* if the map $f_m : X_m \longrightarrow Y_m$ is a mixed $\mathcal{F}(m)$-fibration for all $m \geq 0$.
- an $\mathcal{F}$-cofibration if the latching morphism $\nu_m f : X_m \cup_{L_m X} L_m Y \longrightarrow Y_m$ is an $\mathcal{F}(m)$-cofibration or all $m \geq 0$.

In other words, $f : X \longrightarrow Y$ is an $\mathcal{F}$-level equivalence if for every $m \geq 0$ and every subgroup H of $O(m)$ that belongs to the family $\mathcal{F}$ the map $f_m^H : X_m^H \longrightarrow Y_m^H$ is a weak equivalence. Clearly, the class of $\mathcal{F}$-level equivalences is closed under composition, retracts and coproducts.

Let G be any group from the family $\mathcal{F}$ and V a faithful G-representation of dimension m. We let $\alpha : \mathbb{R}^m \longrightarrow V$ be a linear isometry and define a homomorphism $c_\alpha : G \longrightarrow O(m)$ by 'conjugation by α', i.e., we set

$$(c_\alpha(g))(x) \;=\; \alpha^{-1}(g \cdot \alpha(x))$$

for $g \in G$ and $x \in \mathbb{R}^m$. We restrict the $O(m)$-action on X_m to a G-action along the homomorphism c_α. Then the map

$$c_\alpha^*(X_m) \;\longrightarrow\; X(V) \;, \quad x \longmapsto [\alpha, x]$$

is a G-equivariant homeomorphism, natural in X and restrict to a natural homeomorphism

$$X_m^{\bar{G}} \;\longrightarrow\; X(V)^G \;, \quad x \longmapsto [\alpha, x]$$

where $\bar{G} = c_\alpha(G)$ is the image of c_α. This implies:

Proposition 1.13. *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ a morphism of orthogonal spectra.*

(i) *The morphism f is an $\mathcal{F}$-level equivalence if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is an $(\mathcal{F} \cap G)$-weak equivalence.*
(ii) *The morphism f is an $\mathcal{F}$-level fibration if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is a projective $(\mathcal{F} \cap G)$-fibration.*
(iii) *The morphism f is a injective $\mathcal{F}$-fibration if and only if for every compact Lie group G and every faithful G-representation V the map $f(V) : X(V) \longrightarrow Y(V)$ is a mixed $(\mathcal{F} \cap G)$-fibration.*

When $\mathcal{F} = \langle e \rangle$ is the minimal global family consisting of all trivial groups, then the $\langle e \rangle$-level equivalences (respectively $\langle e \rangle$-level fibrations) are the level equivalences (respectively level fibrations) of orthogonal spectra in the sense of Definition 6.1 of [**57**]. Hence the $\langle e \rangle$-cofibrations are the 'q-cofibrations' in the sense of [**57**, Def. 6.1]. For the minimal global family, the projective $\langle e \rangle$-level model structure thus specializes to the level model structure of [**57**, Thm. 6.5]. On the other hand, the flat $\langle e \rangle$-level model structure is the $\mathbb{S}$-level model structure of Stolz [**89**, Prop. 1.3.5].

Thus we have the following implications for the various kinds of cofibrations:

$$\langle e \rangle\text{-cofibration} \implies \mathcal{F}\text{-cofibration} \implies \text{flat cofibration} \implies \text{h-cofibration}$$

When $\mathcal{F}$ is not the minimal or the maximal global family, then the first two containments are strict.

In the special case $G = O(m+n)$, $K = O(m)$ and $A = O(m+n) \ltimes_{O(n)} S^n$, the next proposition amounts to the stable consistency condition (1.8) for the projective and flat $\mathcal{F}$-model structures. The next proposition for based equivariant spaces is largely analogous to version for unbased spaces in Proposition A.1.23. There is one key difference, though: in the unbased situation, the functor $A \times_K -$ takes K-weak equivalences between arbitrary unbased K-spaces to G-weak equivalences. In the based situation, the functor $A \wedge_K -$ takes K-weak equivalences between *cofibrant* based K-spaces to G-weak equivalences, but not all K-weak equivalences between arbitrary based K-spaces.

Proposition 1.14. *Let G and K be compact Lie groups, $\mathcal{F}$ a global family and A a cofibrant based $(G\times K^{\mathrm{op}})$-space such that the right K-action is free away from the basepoint. Then the functor*

$$A \wedge_K - \;:\; K\mathbf{T} \longrightarrow G\mathbf{T}$$

takes K-cofibrations that are also $(\mathcal{F}\cap K)$-equivalences to G-cofibrations that are also $(\mathcal{F}\cap G)$-equivalences.

Proof. We denote by $\mathcal{F}[G;K]$ the family of those subgroups $\Gamma \leq G \times K^{\mathrm{op}}$ that intersect $\{1\} \times K^{\mathrm{op}}$ only in the identity element. We show more generally that the pushout product, with respect to the functor

$$- \wedge_K - \;:\; (G \times K^{\mathrm{op}})\mathbf{T} \times K\mathbf{T} \longrightarrow G\mathbf{T} ,$$

of an $\mathcal{F}[G;K]$-cofibration with an acyclic cofibration in the $(\mathcal{F} \cap K)$-flat model structure is an acyclic cofibration in the $(\mathcal{F} \cap G)$-flat model structure. The claim is a special case because the $\mathcal{F}[G;K]$-cofibrant based $(G \times K^{\mathrm{op}})$-spaces are precisely those cofibrant based $(G \times K^{\mathrm{op}})$-spaces on which the right K-action is free.

We deduce this claim in the *based* context from the corresponding result about *unbased* equivariant spaces. Since the functor $- \wedge_K -$ preserves colimits in both variables, it suffices to check the claim for the generating based $\mathcal{F}[G;K]$-cofibrations and the generating based acyclic cofibrations in the $(\mathcal{F} \cap K)$-flat model structure. All these generators are obtained from generating cofibrations (respectively acyclic cofibrations) in the corresponding model categories of unbased space by adding disjoint basepoints. For unbased $(G \times K^{\mathrm{op}})$-spaces B and unbased K-spaces X, the based G-spaces $(B_+) \wedge_K (X_+)$ and $(B \times_K X)_+$ are naturally isomorphic, so for the generators the claim follows from the corresponding unbased statement in Proposition A.1.23, using also that $(\mathcal{F} \cap G) \ltimes K \subseteq \mathcal{F} \cap K$. □

Now we are ready to establish the two $\mathcal{F}$-level model structures.

Proposition 1.15. *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-level equivalences, $\mathcal{F}$-level fibrations and $\mathcal{F}$-cofibrations form a model structure, the* projective $\mathcal{F}$-level model structure*, on the category of orthogonal spectra.*
(ii) *The $\mathcal{F}$-level equivalences, injective $\mathcal{F}$-fibrations and flat cofibrations form a model structure, the* flat $\mathcal{F}$-level model structure*, on the category of orthogonal spectra.*
(iii) *Both $\mathcal{F}$-level model structures are proper, topological and cofibrantly generated.*

Proof. The first two parts are special cases of Proposition 1.9, in the following way.

(i) We let $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-projective model structure on the category of based $O(m)$-spaces, compare Proposition A.1.18. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 1.9 precisely become the $\mathcal{F}$-level equivalences, projective $\mathcal{F}$-fibrations and $\mathcal{F}$-cofibrations. Since every $\mathcal{F}$-cofibration is in particular a flat cofibration, the stable consistency condition (1.8) follows from the stronger consistency condition for the flat $\mathcal{F}$-level model structure in (ii) below.

(ii) We let $\mathcal{C}(m)$ be the $\mathcal{F}(m)$-flat model structure on the category of based $O(m)$-spaces, compare Proposition A.1.28. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition 1.9 precisely become the $\mathcal{F}$-level equivalences, injective $\mathcal{F}$-fibrations and flat cofibrations. The stable consistency condition (1.8) is the special case of Proposition 1.14 with $G = O(m+n)$, $K = O(m)$ and $A = O(m+n)^+ \wedge_{O(n)} S^n$.

(iii) Limits in the category of orthogonal spectra are constructed levelwise (i.e., evaluation at level m preserves limits). Since weak equivalences and fibrations are also defined levelwise, right properness is inherited levelwise. Both the $\mathcal{F}(m)$-projective and the $\mathcal{F}(m)$-flat model structure on the category of based $O(m)$-spaces are right proper for all $m \geq 0$, so right properness of the two $\mathcal{F}$-level model structures follows.

The argument for left properness is similar, but not completely analogous because cofibrations are *not* defined levelwise. It suffices to show left properness for the flat $\mathcal{F}$-level model structure, since the projective $\mathcal{F}$-level model structure has the same equivalences, but fewer cofibrations. Since flat cofibrations are levelwise $O(n)$-cofibrations (Proposition 1.11) and colimits in the category of orthogonal spectra are also constructed levelwise, left properness for the flat $\mathcal{F}$-level model structure is a consequence of left properness of the $\mathcal{F}(n)$-flat model structure on $O(n)$-spaces for all n.

We describe explicit sets of generating cofibrations and generating acyclic cofibrations for the two $\mathcal{F}$-level model structure. We start with the projective $\mathcal{F}$-level model structure. We let $I_{\mathcal{F}}$ be the set of all morphism $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.19) of Section A.1. Then the set $I_{\mathcal{F}}$ detects the acyclic fibrations in the projective $\mathcal{F}$-level model structure by Proposition 1.9 (iii).

Similarly, we let $J_{\mathcal{F}}^{\mathrm{proj}}$ be the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. Again by Proposition 1.9 (iii), $J_{\mathcal{F}}^{\mathrm{proj}}$ detects the fibrations in the projective $\mathcal{F}$-level model structure

The cofibrations in the flat $\mathcal{F}$-level model structure are the flat cofibrations, which are independent of $\mathcal{F}$ and coincide with the $\mathcal{A}ll$-cofibrations, where $\mathcal{A}ll$ is the global family of all compact Lie groups. So the set $I_{\mathcal{A}ll}$ as defined above is a set of generating cofibrations for the flat $\mathcal{F}$-level model structure (for any global family $\mathcal{F}$).

Finally, we let $J_{\mathcal{F}}^{\mathrm{flat}}$ be the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces specified in the proof of Proposition A.1.28. One more time by Proposition 1.9 (iii), $J_{\mathcal{F}}^{\mathrm{flat}}$ detects the fibrations in the flat $\mathcal{F}$-level model structure.

Since the $\mathcal{F}(m)$-projective and $\mathcal{F}(m)$-flat model structures on $O(m)$-spaces are topological, part (iv) of Proposition 1.9 shows that the projective and flat $\mathcal{F}$-level model structures are topological. $\square$

For easier reference we spell out the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family of all compact Lie groups. In this case $\mathcal{A}ll(m)$ is the family of all closed subgroups of $O(m)$, and the $\mathcal{A}ll$-cofibrations specialize to the flat cofibrations. We introduce special names for the $\mathcal{A}ll$-level equivalences and the $\mathcal{A}ll$-level fibrations (which coincide with the injective $\mathcal{A}ll$-fibrations), analogous to the unstable situation in Section I.3.

Definition 1.16. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *strong level equivalence* (respectively *strong level fibration*) if for every $m \geq 0$ the map $f_m : X_m \longrightarrow Y_m$ is an $O(m)$-weak equivalence. (respectively an $O(m)$-fibration).

For the global family $\mathcal{A}ll$ of all compact Lie groups the projective and flat level model structure on orthogonal spaces coincide and specialize to the following strong level model structure:

Proposition 1.17. *The strong level equivalences, strong level fibrations and flat cofibrations form a model structure, the* strong level model structure*, on the category of orthogonal spectra. The strong level model structure is proper, topological and cofibrantly generated.*

2. Global model structures for orthogonal spectra

In this section we construct the main model structure of interest for us, the *global model structure* on the category of orthogonal spectra, see Theorem 2.7. The weak equivalences in this model structure are the global equivalences and the cofibrations are the flat cofibrations. More generally, we consider a global family $\mathcal{F}$ and define two *$\mathcal{F}$-global model structures*, see Theorems 2.10 and 2.11 below. The weak equivalences in both of these model structures are the $\mathcal{F}$-equivalences, i.e., those morphisms inducing isomorphisms of G-equivariant homotopy groups for all $G \in \mathcal{F}$. We need both $\mathcal{F}$-global model structures later for showing that the forgetful functor from the global stable homotopy category to the $\mathcal{F}$-global homotopy category has both a left and a right adjoint. Corollary 3.5 below shows that the flat $\mathcal{F}$-global model structure is monoidal with respect to the smash product of orthogonal spectra; the projective $\mathcal{F}$-global model structure is monoidal under the additional hypothesis that $\mathcal{F}$ is closed under products.

Our strategy is the same as in the unstable situation in Section I.7: in a first step we construct the global model structure 'by hand'; in a second step we *merge* the global model structure and the flat $\mathcal{F}$-level model structure into the flat $\mathcal{F}$-global model structure; in a third step we *mix* the projective $\mathcal{F}$-level model structure and the flat $\mathcal{F}$-global model structure into the projective $\mathcal{F}$-global model structure. When $\mathcal{F} = \mathcal{A}ll$ is the maximal global family, then the flat and the projective $\mathcal{A}ll$-global model structure coincide and specialize to the global model structure.

Definition 2.1. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global fibration* if it is a strong level fibration and for every compact Lie group G, every faithful G-representation V and an arbitrary G-representation W the square

$$\begin{tikzcd} X(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"] \arrow[d, "f(V)^G"'] & \operatorname{map}^G(S^W, X(V \oplus W)) \arrow[d, "\operatorname{map}^G(S^W, f(V\oplus W))"] \\ Y(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"'] & \operatorname{map}^G(S^W, Y(V \oplus W)) \end{tikzcd} \tag{2.2}$$

is homotopy cartesian.

We observe that an orthogonal spectrum is a global Ω-spectrum precisely when the unique morphism to any trivial spectrum is a global fibration. In other words, global Ω-spectra are the 'globally fibrant' objects.

We state a useful criterion for checking when a morphism is a global fibration. We recall from Construction III.2.12 the homotopy fiber $F(f)$ of a morphism $f : X \longrightarrow Y$ of orthogonal spectra along with the natural map $i : \Omega Y \longrightarrow F(f)$. We apply this for the shifted morphism $\operatorname{sh} f : \operatorname{sh} X \longrightarrow \operatorname{sh} Y$ and precomposite with the morphism $\tilde{\lambda}_Y$ (see (2.24) and (2.25) of Chapter III) and denote by $\xi(f)$ the resulting composite

$$Y \xrightarrow{\tilde{\lambda}_Y} \Omega(\operatorname{sh} Y) \xrightarrow{i} F(\operatorname{sh} f) \ .$$

We note that the orthogonal spectrum $F(\mathrm{Id}_{\operatorname{sh} X})$ has a preferred contraction, so part (ii) of the next proposition is a way the make precise that the sequence

$$X \xrightarrow{f} Y \xrightarrow{\xi(f)} F(\operatorname{sh} f)$$

is a 'strong level homotopy fiber sequence'.

Proposition 2.3. *Let $f : X \longrightarrow Y$ be a strong level fibration of orthogonal spectra. Then the following two conditions are equivalent.*

(i) *The morphism f is a global fibration.*
(ii) *The strict fiber of f is a global Ω-spectrum and the commutative square*

$$\begin{tikzcd} X \arrow[r, "\xi(\mathrm{Id}_X)"] \arrow[d, "f"'] & F(\mathrm{Id}_{\operatorname{sh} X}) \arrow[d] \\ Y \arrow[r, "\xi(f)"'] & F(\operatorname{sh} f) \end{tikzcd}$$

is homotopy cartesian in the strong level model structure.

Proof. (i)$\Longrightarrow$(ii) Since the square (2.2) is homotopy cartesian and the vertical maps are Serre fibrations, the induced map on strict fibers $f^{-1}(*)(V) \longrightarrow \operatorname{map}^G(S^W, f^{-1}(*)(V \oplus W))$ is a weak equivalence. So the strict fiber is a global Ω-spectrum.

The square of condition (ii) factors as the composite of two commutative squares:

$$\begin{tikzcd}
X \arrow[r, "\tilde{\lambda}_X"] \arrow[d, "f"'] & \Omega(\mathrm{sh}\, X) \arrow[r, "i"] \arrow[d, "\Omega(\mathrm{sh}\, f)"] & F(\mathrm{Id}_{\mathrm{sh}\, X}) \arrow[d] \\
Y \arrow[r, "\tilde{\lambda}_Y"'] & \Omega(\mathrm{sh}\, Y) \arrow[r, "i"'] & F(\mathrm{sh}\, f)
\end{tikzcd}$$

Specializing the global fibration property (2.2) for $W = \mathbb{R}$ with trivial G-action shows that the left square is homotopy cartesian in the strong level model structure. The right square is homotopy cartesian on general grounds. So the composite square is homotopy cartesian in the strong level model structure.

(ii)$\Longrightarrow$(i) We let G be a compact Lie group, V a faithful G-representation, W any G-representation and consider the commutative diagram

$$\begin{tikzcd}
X(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"] \arrow[d, "f(V)^G"'] & \mathrm{map}^G(S^W, X(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^W, f(V \oplus W))"] \\
Y(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"'] \arrow[d, "(\xi(f)(V))^G"'] & \mathrm{map}^G(S^W, Y(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^W, (\xi(f)(V \oplus W)))"] \\
F(f)(V \oplus \mathbb{R})^G \arrow[r, "\sim", "\tilde{\sigma}^G_{V,W}"'] & \mathrm{map}^G(S^W, F(f)(V \oplus W \oplus \mathbb{R}))
\end{tikzcd}$$

Since f is a strong level fibration the natural morphism from the strict fiber to the homotopy fiber is a strong level equivalence. So the lower horizontal map is a weak equivalence because the strict fiber, and hence the homotopy fiber, is a global Ω-spectrum. Moreover, the two vertical columns are homotopy fiber sequences by hypothesis, so the upper square is homotopy cartesian, both with respect to the strong level model structure. $\square$

Definition 2.4. Let $\mathcal{F}$ be a global family.

- A morphism $f : X \longrightarrow Y$ of orthogonal spectra is an $\mathcal{F}$-*equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all G in $\mathcal{F}$ and all integers k.
- An orthogonal spectrum X is an $\mathcal{F}$-Ω-*spectrum* if for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and an arbitrary G-representation W the adjoint generalized structure map

$$\tilde{\sigma}_{V,W} \ : \ X(V) \ \longrightarrow \ \mathrm{map}(S^W, X(V \oplus W))$$

is a G-weak equivalence.

When $\mathcal{F} = \mathcal{A}\mathit{ll}$ is the maximal global family of all compact Lie groups, then an $\mathcal{A}\mathit{ll}$-equivalence is just a global equivalence in the sense of Definition III.2.9. Also, an $\mathcal{A}\mathit{ll}$-Ω-spectrum is the same as a global Ω-spectrum in the sense of Definition III.7.2. When $\mathcal{F} = \langle e \rangle$ is the minimal global family of all trivial groups, then the $\langle e \rangle$-equivalences are just the traditional, non-equivariant, stable equivalences of orthogonal spectra, also known as 'π_*-isomorphisms'. The $\langle e \rangle$-Ω-spectra are just the traditional, non-equivariant, Ω-spectra.

The following diagram collects various notions of equivalences and their implications that play a role in this book:

$$\begin{tikzcd}
\text{homotopy equivalence} \arrow[r, Rightarrow] & \text{strong level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{global equivalence} \arrow[d, Rightarrow] \\
& \mathcal{F}\text{-level equivalence} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathcal{F}\text{-equivalence} \arrow[d, Rightarrow] \\
& \text{level equivalence} \arrow[r, Rightarrow] & \text{stable equivalence}
\end{tikzcd}$$

The following proposition collects various useful relations between $\mathcal{F}$-equivalence, $\mathcal{F}$-level equivalences and $\mathcal{F}$-Ω-spectra.

Proposition 2.5. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-level equivalence of orthogonal spectra is an $\mathcal{F}$-equivalence.*

(ii) *Let X be an $\mathcal{F}$-Ω-spectrum. Then for every G in $\mathcal{F}$, every faithful G-representation V and every $k \geq 0$ the stabilization map*

$$[S^{k+V}, X(V)]^G \longrightarrow \pi_k^G(X) , \quad [f] \longmapsto \langle f \rangle$$

is bijective.

(iii) *Let X be an $\mathcal{F}$-Ω-spectrum such that $\pi_k^G(X) = 0$ for every integer k and all G in $\mathcal{F}$. Then for every group G in $\mathcal{F}$ and every faithful G-representation V the space $X(V)$ is G-weakly contractible.*

(iv) *Every $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra is an $\mathcal{F}$-level equivalence.*

(v) *Every $\mathcal{F}$-equivalence that is also a global fibration is an $\mathcal{F}$-level equivalence.*

Proof. (i) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-level equivalence, and we need to show that the map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all integers k and all G in the global family $\mathcal{F}$. We start with the case $k = 0$. We let G be a group from the family $\mathcal{F}$ and V a finite dimensional G-subspace of the complete G-universe $\mathcal{U}_G$ such that G acts faithfully on V. By Proposition 1.13 (i) the map $f(V) : X(V) \longrightarrow Y(V)$ is a G-weak equivalence. Since the representation sphere S^V can be given a G-CW-structure, the induced map on G-homotopy classes

$$[S^V, f(V)]^G \ : \ [S^V, X(V)]^G \longrightarrow [S^V, Y(V)]^G$$

is bijective. The faithful G-representations are cofinal in the poset $s(\mathcal{U}_G)$, so taking the colimit over $V \in s(\mathcal{U}_G)$ shows that $\pi_0^G(f) : \pi_0^G(X) \longrightarrow \pi_0^G(Y)$ is an isomorphism. For dimensions $k > 0$ we exploit that $(\Omega^k X)(V)$ is naturally G-homeomorphic to $\Omega^k X(V)$, so the k-fold loop of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence. For dimensions $k < 0$ we exploit that $(\mathrm{sh}^k X)(W)$ is naturally G-homeomorphic to $X(\mathbb{R}^k \oplus W)$, so every shift of an $\mathcal{F}$-level equivalence is again an $\mathcal{F}$-level equivalence.

(ii) We start with the special case where V is a G-invariant subspace of the complete G-universe $\mathcal{U}_G$ used to define $\pi_k^G(X)$. Since V is faithful, so is every G-representation that contains V. So the directed system whose colimit is $\pi_*^G(X)$ consists of isomorphism 'above V'. So the canonical map

$$[S^{k+V}, X(V)]^G \longrightarrow \pi_k^G(X)$$

is an isomorphism.

If V is an arbitrary G-representation we can choose a G-equivariant linear isometry $\alpha : V \longrightarrow \bar{V}$ to a finite dimensional G-subrepresentation of the complete universe $\mathcal{U}_G$. The map in question is the composite

$$[S^{k+V}, X(V)]^G \longrightarrow [S^{k+\bar{V}}, X(\bar{V})]^G \longrightarrow \pi_k^G(X)$$

of 'conjugation by α' and the canonical map that was shown to be bijective is the previous paragraph.

(iii) We adapt an argument of Lewis-May-Steinberger [**53**, I 7.12] to our context. Every $\mathcal{F}$-Ω-spectrum is in particular a non-equivariant Ω-spectrum; every non-equivariant Ω-spectrum with trivial homotopy groups is levelwise weakly contractible, so this takes care of trivial groups.

Now we let G be a non-trivial group in $\mathcal{F}$. We argue by a nested induction over the 'size' of G: we induct over the dimension of G and, for fixed dimension, over the cardinality of the finite set $\pi_0 G$ of components of G. Every proper subgroup H of G either has strictly smaller dimension than G, or the same dimension but fewer path components. So we know by induction that the fixed point space $X(V)^H$ is weakly contractible for every proper subgroup H of G. So it remains to analyze the G-fixed points of $X(V)$.

We let $W = V - V^G$ be the orthogonal complement of the fixed subspace V^G of V. Then W is a faithful G-representation with trivial fixed points. The cofiber sequence of G-CW-complexes

$$S(W)_+ \longrightarrow D(W)_+ \longrightarrow S^W$$

induces a fiber sequence of equivariant mapping spaces

$$\mathrm{map}^G(S^W, X(V)) \longrightarrow \mathrm{map}^G(D(W)_+, X(V)) \longrightarrow \mathrm{map}^G(S(W)_+, X(V)) .$$

Since $W^G = 0$, the G-fixed points $S(W)^G$ are empty, so any CW-structure on $S(W)$ uses only equivariant cells of the form $G/H \times D^n$ for proper subgroups H of G. But for proper subgroups H, all the fixed point spaces $X(V)^H$ are weakly contractible by induction. Hence the space $\mathrm{map}^G(S(W)_+, X(V))$ is weakly contractible. Since the unit disc $D(W)$ is equivariantly contractible, the space $\mathrm{map}^G(D(W)_+, X(V))$ is homotopy equivalent to $X(V)^G$, and we conclude that evaluation at the fixed point $0 \in W$ is a weak equivalence

$$\mathrm{map}^G(S^W, X(V)) \simeq X(V)^G .$$

We have $X(V) = X(V^G \oplus W) = (\mathrm{sh}^{V^G} X)(W)$. The stabilization map

$$\pi_k \,\mathrm{map}^G(S^W, X(V)) = [S^{k+W}, (\mathrm{sh}^{V^G} X)(W)]^G \longrightarrow \pi_k^G(\mathrm{sh}^{V^G} X) \cong \pi^G_{k-\dim(V^G)} X$$

is bijective for all $k \geq 0$, by part (ii). Since we assumed that the G-equivariant homotopy groups of X vanish, the space $\mathrm{map}^G(S^W, X(V))$ is path connected and has vanishing homotopy groups, i.e., it is weakly contractible. So $X(V)^G$ is weakly contractible, and this completes the proof of (iii).

(iv) Let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between $\mathcal{F}$-Ω-spectra. We let F denote the homotopy fiber of f, and we let G be a group from the family $\mathcal{F}$. For every G-representation V the G-space $F(V)$ is then G-homeomorphic to the homotopy fiber of $f(V) : X(V) \longrightarrow Y(V)$. So F is again an $\mathcal{F}$-Ω-spectrum. The long exact sequence of homotopy groups (see Proposition III.2.18) implies that $\pi_*^H(F) = 0$ for all H in $\mathcal{F}$.

If G acts faithfully on V, then by the $\mathcal{F}$-Ω-spectrum property, the space $X(V)$ is G-weakly equivalent to $\Omega X(V \oplus \mathbb{R})$ and similarly for Y. So the map $f(V)$ is G-weakly equivalent to

$$\Omega f(V \oplus \mathbb{R}) \; : \; \Omega X(V \oplus \mathbb{R}) \longrightarrow \Omega X(V \oplus \mathbb{R}) .$$

Hence we have a homotopy fiber sequence of G-spaces

$$X(V) \xrightarrow{f(V)} Y(V) \longrightarrow F(V \oplus \mathbb{R}) .$$

Since F is an $\mathcal{F}$-Ω-spectrum with vanishing equivariant homotopy groups for groups in $\mathcal{F}$, the space $F(V\oplus\mathbb{R})$ is G-weakly contractible by part (iii). So $f(V)$ is a G-weak equivalence. The morphism f is then an $\mathcal{F}$-level equivalence by the criterion of Proposition 1.13.

(v) We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence and a global fibration. Then the strict fiber $f^{-1}(*)$ of f is a global Ω-spectrum with trivial G-equivariant homotopy groups for all $G \in \mathcal{F}$. So $f^{-1}(*)$ is $\mathcal{F}$-level equivalent to the trivial spectrum by part (iii). Since f is a strong level fibration, the embedding $f^{-1}(*) \longrightarrow F(f)$ of the strict fiber into the homotopy fiber $F(f)$ is a strong level equivalence, so the homotopy fiber $F(f)$ is $\mathcal{F}$-level equivalent to the trivial spectrum. The homotopy cartesian square of Proposition 2.3 (ii) then shows that that f is an $\mathcal{F}$-level equivalence. □

Remark 2.6. We also give an alternative proof of part (iii) of the previous proposition by reduction to equivariant homotopy theory for a fixed Lie group. We let X be an $\mathcal{F}$-Ω-spectrum and $G \in \mathcal{F}$. The associated orthogonal G-spectrum $X\langle G\rangle$ need not be an Ω-G-spectrum; however, for every faithful G-representation V the shift $\mathrm{sh}^V(X\langle G\rangle)$ is an Ω-G-spectrum. Since $\mathcal{F}$ is closed under subgroups, $X\langle G\rangle$ has vanishing equivariant homotopy groups for G and all its closed subgroups. The shift $\mathrm{sh}^V(X\langle G\rangle)$ then also has vanishing equivariant homotopy groups for G and all its closed subgroups. So $\mathrm{sh}^V(X\langle G\rangle)$ is levelwise G-weakly contractible by [**58**, III Lemma 9.1]. In particular, $X(V)$ is G-weakly contractible for every faithful G-representation V.

Now we can establish the global model structure on the category of orthogonal spectra. The strategy of proof is exactly the same as for the global model structure on orthogonal spaces in Theorem I.4.3.

Theorem 2.7 (Global model structure)**.** *The global equivalences, global fibrations and flat cofibrations form a model structure, the* global model structure *on the category of orthogonal spectra. The fibrant objects in the global model structure are the global Ω-spectra. The global model structure is proper, topological and compactly generated.*

Proof. The category of orthogonal spectra is complete and cocomplete (MC1), the global equivalences satisfy the 2-out-of-3 property (MC2) and the classes of global equivalences, global fibrations and flat cofibrations are closed under retracts (MC3). The strong level model structure (Proposition 1.17) shows that every morphism of orthogonal spectra can be factored as a flat cofibration followed by a strong level equivalence. Since strong level equivalences are in particular global equivalences, this provides one of the factorizations as required by MC5.

For the other half of the factorization axiom MC5 we exhibit morphisms that detect the stable fibrations. The set J^{str} was defined in the proof of Proposition 1.17 as the set morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. The set J^{str} detects the fibrations in the strong level model structure. We add another set of morphism K that detects when the squares (2.2) are homotopy cartesian. Given any compact Lie group G and G-representations V and W we recall from (5.6) of Chapter III the morphism

$$\lambda_{G,V,W} \ : \ F_{G,V\oplus W} S^W \ \longrightarrow \ F_{G,V} \ .$$

If the representation V is faithful, then this morphism is a global equivalence by Theorem III.5.7. We set

$$K \ = \ \bigcup_{G,V,W} \mathcal{Z}(\lambda_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $S^{m-1} \longrightarrow D^m$ with the mapping cylinder inclusions of the morphisms $\lambda_{G,V,W}$ (compare Construction I.4.8); here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G, a faithful G-representation V and an arbitrary G-representation W. The morphism $\lambda_{G,V,W}$ represents the map $(\tilde{\sigma}_{V,W})^G : X(V)^G \longrightarrow \mathrm{map}^G(S^W, X(V \oplus W))^G$; by Proposition I.4.9, the right lifting property with respect to the union $J^{\mathrm{str}} \cup K$ thus characterizes the global fibrations.

We apply the small object argument to the set $J^{\mathrm{str}} \cup K$. All morphisms in J^{str} are flat cofibrations and strong level equivalences; strong level equivalences are global equivalences by Proposition 2.5 (i). Since $F_{G,V\oplus W} S^W$ and $F_{G,V}$ are flat, the morphisms in K are also flat cofibrations, and they are global equivalences because the morphisms $\lambda_{G,V.W}$ are. The small object argument provides a functorial factorization of every morphism $\varphi : X \longrightarrow Y$ of orthogonal spectra as a composite

$$X \ \xrightarrow{\ i\ } \ W \ \xrightarrow{\ q\ } \ Y$$

where i is a sequential composition of cobase changes of coproducts of morphisms in K and q has the right lifting property with respect to $J^{\mathrm{str}} \cup K$; in particular, the morphism q is a global fibration. All morphisms in K are global equivalences and flat cofibrations, hence also h-cofibrations (by Proposition I.4.6 applied to the strong level model structure). By Corollary 2.22, the class of h-cofibrations that are simultaneously global equivalences is closed under coproducts, cobase change and sequential composition. So the morphism i is a flat cofibration and a global equivalence.

Now we show the lifting properties MC4. By Proposition 2.5 (v) a morphism that is both a global fibration and a global equivalence is a strong level equivalence, and hence an acyclic fibration in the strong level model structure. So every morphism that is simultaneously a global fibration and a global equivalence has the right lifting property with respect to flat cofibrations. Now we let $j : A \longrightarrow B$ be a flat cofibration that is also a global equivalence and we show that it has the left lifting property with respect to all global fibrations. We factor $j = q \circ i$, via the small object argument for $J^{\mathrm{str}} \cup K$, where $i : A \longrightarrow W$ is a $(J^{\mathrm{str}} \cup K)$-cell complex and $q : W \longrightarrow B$ a global fibration. Then q is a global equivalence since j and i are, so q is an acyclic fibration in the strong level model structure, again by Proposition 2.5 (v). Since j is a flat cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & W \arrow[d, "\sim"', "q"] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus j is a retract of the morphism i that has the left lifting with respect to global fibrations. But then j itself has this lifting property. This finishes the verification of the model category axioms. Alongside we have also specified sets of generating flat cofibrations I^{str} and generating acyclic cofibrations $J^{\mathrm{str}} \cup K$. Sources and targets of all morphisms in these sets are small with respect to sequential colimits of flat cofibrations. So the global model structure is compactly generated.

Left properness of the global model structure follows from Corollary 2.20 and the fact that flat cofibrations are h-cofibrations. Right properness [...] [topological] □

Corollary 2.14 below (in the special case $\mathcal{F} = \mathcal{A}ll$) provides another characterization of global equivalences: a morphism $f : A \longrightarrow B$ of orthogonal spectra is a global equivalence if and only if for some (hence any) flat approximation $f^c : A^c \longrightarrow B^c$ in the strong level model structure and every global Ω-spectrum X the induced map

$$[f^c, X] \ : \ [B^c, X] \ \longrightarrow \ [A^c, X]$$

on homotopy classes of morphisms is bijective.

Now we merge the global model structure (Theorem 2.7) and the flat $\mathcal{F}$-level model structure (Proposition 1.15 (ii)) of orthogonal spectra into the flat $\mathcal{F}$-global model structure. So we apply Proposition I.7.8 to the category $\mathcal{A} = \mathcal{S}p$ of orthogonal spectra, we let $(\mathcal{W}_1, \mathcal{F}_1, \mathcal{C})$ be the global model structure and we let $(\mathcal{W}_2, \mathcal{F}_2, \mathcal{C})$ be the flat $\mathcal{F}$-level model structure, which share the flat cofibrations. The next proposition identifies the merged equivalences $\mathcal{W}$ with the class of $\mathcal{F}$-equivalences:

Proposition 2.8. *Let $\mathcal{F}$ be a global family. A morphism of orthogonal spectra is an $\mathcal{F}$-equivalence if and only if it can be written as $w_2 \circ w_1$ for an $\mathcal{F}$-level equivalence w_2 and a global equivalence w_1.*

Proof. The $\mathcal{F}$-equivalences contain the global equivalences by definition and the $\mathcal{F}$-level equivalences by Proposition 2.5 (i), so all composites $w_2 \circ w_1$ as in the proposition are $\mathcal{F}$-equivalences. On the other hand, every $\mathcal{F}$-equivalence f can be factored in the global model structure of Theorem 2.7 as $f = qj$ where j is a global equivalence and q is a global fibration. Since f and j are $\mathcal{F}$-equivalences, so is q; so q is an $\mathcal{F}$-equivalence and a global fibration, hence an $\mathcal{F}$-level equivalence by Proposition 2.5 (v). □

In order to merge the two model structures we need to check the hypotheses of Proposition I.7.8: The $\mathcal{F}$-equivalences are closed under retracts and have the 2-out-of-3 property, which shows (a). Every $\mathcal{F}$-equivalence that is also a global fibration is an $\mathcal{F}$-level equivalence by Proposition 2.5 (v), which shows (b). Finally, hypothesis (c) is the content of the part (ii) of the following proposition.

An orthogonal spectrum X is *$\mathcal{F}$-injective* if it is fibrant in the flat $\mathcal{F}$-level model structure, i.e., for every compact Lie group G and every faithful G-representation V the map

$$X(V) \longrightarrow \mathrm{map}(E(\mathcal{F} \cap G), X(V))$$

is a G-weak equivalence. The terminology is motivated by the fact that $\mathcal{F}$-injective spectra are characterized by the extension property for flat cofibration that are simultaneously $\mathcal{F}$-level equivalences.

Proposition 2.9. *Let $\mathcal{F}$ be a global family.*

(i) *Every $\mathcal{F}$-injective $\mathcal{F}$-Ω-spectrum is a global Ω-spectrum.*
(ii) *Let $f : X \longrightarrow Y$ be an injective $\mathcal{F}$-fibration of orthogonal spectra and $k : Z \longrightarrow Y$ an $\mathcal{F}$-level equivalence. If fk is a global fibration, then f is also a global fibration.*

Proof. (i) We let X be an $\mathcal{F}$-injective $\mathcal{F}$-Ω-spectrum. Then for every compact Lie group G, all faithful G-representations V and all other G-representations W, the adjoint structure map $\tilde{\sigma}_{V,W} : X(V) \longrightarrow \mathrm{map}(S^W, X(V \oplus W))$ is an $(\mathcal{F} \cap G)$-equivalence of G-spaces. So the right map in the square

$$\begin{tikzcd}
X(V)^G \arrow[r] \arrow[d, "(\tilde{\sigma}_{V,W})^G"'] & \mathrm{map}^G(E(\mathcal{F} \cap G), X(V)) \arrow[d, "\mathrm{map}^G(E(\mathcal{F}\cap G),\tilde{\sigma}_{V,W})"] \\
\mathrm{map}^G(S^W, X(V \oplus W)) \arrow[r] & \mathrm{map}^G(E(\mathcal{F} \cap G), \mathrm{map}(S^W, X(V \oplus W)))
\end{tikzcd}$$

is a weak equivalence. Both horizontal maps are G-weak equivalences because X is $\mathcal{F}$-injective, so the left map $(\tilde{\sigma}_{V,W})^G$ is a weak equivalence, i.e., X is a global Ω-spectrum.

(ii) An injective $\mathcal{F}$-fibration is in particular a strong level fibration, so we can check the criterion provided by Proposition 2.3. Since fk is a global fibration, its fiber $(fk)^{-1}(*)$ is a global Ω-spectrum. Since k is an $\mathcal{F}$-level equivalence and both f and fk are strong level fibrations, the morphism $(fk)^{-1}(*) \longrightarrow f^{-1}(*)$ induced by k on strict fibers is another $\mathcal{F}$-level equivalence. So the fiber $(fk)^{-1}(*)$ is an $\mathcal{F}$-Ω-spectrum. Since f is an injective $\mathcal{F}$-fibration, the strict fiber $f^{-1}(*)$ is $\mathcal{F}$-injective, and hence a global Ω-spectrum by part (i).

Now we contemplate the commutative diagram:

$$\begin{tikzcd}
X(V) \arrow[r] \arrow[d, "f(V)"'] & \mathrm{map}(E(\mathcal{F} \cap G), X(V)) \arrow[d, "\mathrm{map}(E(\mathcal{F}\cap G),f(V))"'] & \mathrm{map}(E(\mathcal{F} \cap G), Z(V)) \arrow[l, "\mathrm{map}(E(\mathcal{F}\cap G),k(V))"', "\sim_G"] \arrow[d, "\mathrm{map}(E(\mathcal{F}\cap G),(fk)(V))"] \\
Y(V) \arrow[r] \arrow[d, "\xi(f)(V)"'] & \mathrm{map}(E(\mathcal{F} \cap G), Y(V)) \arrow[r, equal] \arrow[d, "\mathrm{map}(E(\mathcal{F}\cap G),\xi(f)(V))"'] & \mathrm{map}(E(\mathcal{F} \cap G), Y(V)) \arrow[d, "\mathrm{map}(E(\mathcal{F}\cap G),\xi(fk)(V))"] \\
F(f)(V \oplus \mathbb{R}) \arrow[r, "\sim_G"'] & \mathrm{map}(E(\mathcal{F} \cap G), F(f)(V \oplus \mathbb{R})) & \mathrm{map}(E(\mathcal{F} \cap G), F(fk)(V \oplus \mathbb{R})) \arrow[l, "\sim_G"']
\end{tikzcd}$$

Since fk is a global fibration, the sequence $Z(V) \longrightarrow Y(V) \longrightarrow F(fk)(V)$ is a G-homotopy fiber sequence by Proposition 2.3. Applying $\mathrm{map}(E(\mathcal{F}\cap G), -)$ preserves this property, so the right column in the diagram is a G-homotopy fiber sequence. Since k is an $\mathcal{F}$-level equivalence, the G-map $k(V \oplus \mathbb{R}) : Z(V \oplus \mathbb{R}) \longrightarrow X(V \oplus \mathbb{R})$ is an $(\mathcal{F}\cap G)$-equivalence, hence so is the G-map $F(fk)(V \oplus \mathbb{R}) \longrightarrow F(f)(V \oplus \mathbb{R})$, Applying $\mathrm{map}(E(\mathcal{F}\cap G), -)$ thus turns it into a G-weak equivalence. Similarly, the upper right map $\mathrm{map}(E(\mathcal{F} \cap G), k(V))$ is a G-weak equivalence. We conclude that the middle column is a G-homotopy fiber sequence.

Since f is a strong level fibration, its homotopy fiber $F(f)$ is strongly level equivalent to the strict fiber $f^{-1}(*)$. Since the latter is $\mathcal{F}$-injective, the homotopy fiber $F(f)$ is $\mathcal{F}$-injective. Thus the lower left map is a G-weak equivalence. Since f is an injective $\mathcal{F}$-fibration, the upper left square of G-homotopy cartesian. Since the middle column is a G-homotopy fiber sequence, so is the left column. We have thus verified the criterion of Proposition 2.3 (ii), and so f is a global fibration. □

Now we have verified all the hypotheses needed to merge the global model structure with the flat $\mathcal{F}$-level model structure. Proposition I.7.8 thus applies and yields:

Theorem 2.10 (Flat $\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences and flat cofibrations are part of a model structure, the* flat $\mathcal{F}$-global model structure *on the category of orthogonal spectra.*
(ii) *A morphism is a fibration in the flat $\mathcal{F}$-global model structure precisely when it is both an injective $\mathcal{F}$-fibration and a global fibration.*
(iii) *The fibrant objects in the flat $\mathcal{F}$-global model structure are the $\mathcal{F}$-injective global Ω-spectra.*
(iv) *Every acyclic cofibration in the flat $\mathcal{F}$-global model structure is a retract of a composite $k \circ j$ with k a flat cofibration and $\mathcal{F}$-level equivalence and j a flat cofibration and global equivalence.*
(v) *The flat $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

Now we obtain the projective $\mathcal{F}$-global model structure on the category of orthogonal spectra by 'mixing' the projective $\mathcal{F}$-level model structure with the flat $\mathcal{F}$-global model structure. The proof is another application of Cole's mixing theorem [**25**, Thm. 2.1], and it is almost literally the same as in the unstable situation in Theorem I.7.14, hence we omit it.

Theorem 2.11 (Projective $\mathcal{F}$-global model structure)**.** *Let $\mathcal{F}$ be a global family.*

(i) *The $\mathcal{F}$-equivalences and $\mathcal{F}$-cofibrations are part of a model structure on the category of orthogonal spectra, the* projective $\mathcal{F}$-global model structure*.*
(ii) *A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a fibration in the projective $\mathcal{F}$-global model structure precisely when it is an $\mathcal{F}$-level fibration and for every compact Lie group G in $\mathcal{F}$, every faithful G-representation V and an arbitrary G-representation W the square*

$$\begin{tikzcd} X(V)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"] \arrow[d, "f(V)^G"'] & \mathrm{map}^G(S^W, X(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^W, f(V\oplus W))"] \\ Y(V)^G \arrow[r, "(\tilde{\sigma}_{V,W})^G"'] & \mathrm{map}^G(S^W, Y(V \oplus W)) \end{tikzcd} \tag{2.12}$$

is homotopy cartesian.
(iii) *The fibrant objects in the projective $\mathcal{F}$-global model structure are the $\mathcal{F}$-Ω-spectra.*
(iv) *The projective $\mathcal{F}$-global model structure is cofibrantly generated, proper and topological.*

In the case $\mathcal{F} = \langle e \rangle$ of the minimal global family of trivial groups, the $\langle e \rangle$-equivalences are the (non-equivariant) stable equivalences of orthogonal spectra, and the two $\langle e \rangle$-global model structures have been studied before: The projective $\langle e \rangle$-global model structure coincides with the stable model structure established by Mandell, May, Shipley and the author in [**57**, Thm. 9.2]. The flat $\langle e \rangle$-global model structure is the $\mathbb{S}$-model structure of Stolz [**89**, Prop. 1.3.10].

For easier reference we spell out explicit sets of generating cofibrations and generating acyclic cofibrations for the flat and projective $\mathcal{F}$-model structures. In Proposition 1.17 we defined J^{str} as the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the projective model structure on the category of $O(m)$-spectra specified in (1.20) of Section A.1. The set J^{str} detects the fibrations in the strong level model structure.

In Proposition 1.15 we introduced $I_{\mathcal{F}}$ as the set of all morphism $G_m i$ for $m \geq 0$ and for i in the set of generating cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spectra specified in (1.19) of Section A.1. The set $I_{\mathcal{F}}$ detects the acyclic fibrations in the projective $\mathcal{F}$-level model structure, which coincide with the acyclic fibrations in the projective $\mathcal{F}$-global model structure. In particular, the set $I_{\mathcal{A}ll}$ detects the acyclic fibrations in the strong level model structure, which are also the acyclic fibrations in the flat $\mathcal{F}$-level model structure.

Also in Proposition 1.15 we defined $J^{\mathrm{flat}}_{\mathcal{F}}$ as the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-flat model structure on the category of $O(m)$-spaces specified in the proof of Proposition A.1.28. The set $J^{\mathrm{flat}}_{\mathcal{F}}$ detects the fibrations in the flat $\mathcal{F}$-level model structure, i.e., the injective $\mathcal{F}$-fibrations. Similarly, $J^{\mathrm{proj}}_{\mathcal{F}}$ is the set of all morphism $G_m j$ for $m \geq 0$ and for j in the set of generating acyclic cofibrations for the $\mathcal{F}(m)$-projective model structure on the category of $O(m)$-spaces specified in (1.20) of Section A.1. The set $J^{\mathrm{proj}}_{\mathcal{F}}$ detects the fibrations in the projective $\mathcal{F}$-level model structure.

We add another set of morphism $K_{\mathcal{F}}$ that detects when the squares (2.12) are homotopy cartesian for $G \in \mathcal{F}$. We set

$$K_{\mathcal{F}} \ = \bigcup_{G,V,W \ : \ G\in\mathcal{F}} \mathcal{Z}(\rho_{G,V,W}) \ ,$$

the set of all pushout products of sphere inclusions $S^{n-1} \longrightarrow D^n$ with the mapping cylinder inclusions of the morphisms $\rho_{G,V,W}$; here the union is over a set of representatives of the isomorphism classes of triples (G, V, W) consisting of a compact Lie group G in $\mathcal{F}$, a faithful G-representation V and an arbitrary G-representation W. By Proposition 4.9, the right lifting property with respect to the union $J^{\mathrm{proj}}_{\mathcal{F}} \cup K_{\mathcal{F}}$ thus characterizes the fibrations in the projective $\mathcal{F}$-global model structure. In particular, the set $J^{\mathrm{proj}}_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$, which was denoted $J^{\mathrm{str}} \cup K$ in Proposition 4.13, detects the global fibrations.

So altogether we have shown:

Proposition 2.13. *Let $\mathcal{F}$ be a global family. Then a morphism of orthogonal spectra is:*

(i) *an acyclic fibration in the flat $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{A}ll} = I^{\mathrm{str}}$;*

(ii) *a fibration in the flat $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J^{\mathrm{flat}}_{\mathcal{F}} \cup K_{\mathcal{A}ll}$,*

(iii) *an acyclic fibration in the projective $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $I_{\mathcal{F}}$;*

(iv) *a fibration in the projective $\mathcal{F}$-global model structure if and only if it has the right lifting property with respect to the set $J^{\mathrm{proj}}_{\mathcal{F}} \cup K_{\mathcal{F}}$.*

The proof of the following corollary is, again, almost literally the same as that of its unstable analog, Corollary I.7.18; so one more time we omit the proof.

Corollary 2.14. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spectra and $\mathcal{F}$ a global family. Then the following conditions are equivalent.*

(i) *The morphism f is an $\mathcal{F}$-equivalence.*

(ii) *For some (hence any) flat approximation $f^\flat : A^\flat \longrightarrow B^\flat$ in the flat $\mathcal{F}$-level model structure and every $\mathcal{F}$-injective global Ω-spectrum X the induced map*

$$[f^\flat, X] \ : \ [B^\flat, X] \ \longrightarrow \ [A^\flat, X]$$

on homotopy classes of morphisms is a bijection.

(iii) *For some (hence any) $\mathcal{F}$-cofibrant approximation $f^c : A^c \longrightarrow B^c$ in the projective $\mathcal{F}$-level model structure and every $\mathcal{F}$-global Ω-spectrum Y the induced map*

$$[f^c, Y] \ : \ [B^c, Y] \ \longrightarrow \ [A^c, Y]$$

on homotopy classes of morphisms is a bijection.

Remark 2.15 (Mixed global model structures)**.** Cole's 'mixing theorem' for model structures [**25**, Thm. 2.1] allows to construct many more $\mathcal{F}$-model structures on the category of orthogonal spectra. We consider two global families such that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is an $\mathcal{F}$-equivalence and every fibration in the projective $\mathcal{E}$-global model structure is a fibration in the projective $\mathcal{F}$-global model structure. By Cole's theorem [**25**, Thm. 2.1] the $\mathcal{F}$-equivalences and the fibrations of the projective $\mathcal{E}$-global model structure are part of a model structure, the *$\mathcal{E}$-mixed $\mathcal{F}$-global model structure* on the category of orthogonal spectra. By [**25**, Prop. 3.2] the cofibrations in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure are precisely the retracts of all composite $h \circ g$ in which g is an $\mathcal{F}$-cofibration and h is simultaneously an $\mathcal{E}$-equivalence and an $\mathcal{E}$-cofibration. In particular, an orthogonal space is cofibrant in the $\mathcal{E}$-mixed $\mathcal{F}$-global model structure if it is $\mathcal{E}$-cofibrant and $\mathcal{E}$-equivalent to an $\mathcal{F}$-cofibrant orthogonal space [**25**, Cor. 3.7]. The $\mathcal{E}$-mixed $\mathcal{F}$-global model structure is again proper (Propositions 4.1 and 4.2 of [**25**]).

When $\mathcal{F} = \langle e \rangle$ is the minimal family of trivial groups, this provides infinitely many $\mathcal{E}$-mixed model structure on the category of orthogonal spectra that are all Quillen equivalent.

3. Monoidal properties

In this section we look at the interaction of the smash product of orthogonal spectra with the level and global model structures. Given two morphisms $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ of orthogonal spectra we denote by $f \Box g$ the *pushout product* morphism defined as

$$f \Box g = (f \wedge Y) \cup (A \wedge g) \; : \; A \wedge Y \cup_{A \wedge X} B \wedge X \;\longrightarrow\; B \wedge Y \; .$$

We let $\mathcal{E}$ and $\mathcal{F}$ be two global families. As before we denote by $\mathcal{E} \times \mathcal{F}$ the smallest global family that contains all groups of the form $G \times K$ for $G \in \mathcal{E}$ and $K \in \mathcal{F}$.

Proposition 3.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two global families.*

(i) *The pushout product of an $\mathcal{E}$-cofibration with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-cofibration.*
(ii) *The pushout product of an $\mathcal{E}$-cofibration that is also an $\mathcal{E}$-level equivalence with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-level equivalence.*
(iii) *The pushout product of a flat cofibration that is also an $\mathcal{F}$-level equivalence with a flat cofibration is an $\mathcal{F}$-level equivalence.*
(iv) *The pushout product of a flat cofibration that is also a global equivalence with a flat cofibration is again a global equivalence.*

Proof. (i) It suffices to show the claim for a set of generating cofibrations. The $\mathcal{E}$-cofibrations are generated by the morphisms

$$F_{G,V}(S^{n-1}_+) \;\longrightarrow\; F_{G,V}(D^n_+)$$

for $G \in \mathcal{E}$, V a G-representation and $n \geq 0$. Similarly, the $\mathcal{F}$-cofibrations are generated by the morphisms

$$F_{K,W}(S^{m-1}_+) \;\longrightarrow\; F_{K,W}(D^m_+) \tag{3.2}$$

for $K \in \mathcal{F}$, W a K-representation and $m \geq 0$. The pushout product of two such generators is isomorphic to the map

$$F_{G \times K, V \oplus W}(S^{n-1} \times D^m \cup_{S^{n-1} \times S^{m-1}} D^n \times S^{m-1})_+ \;\longrightarrow\; F_{G \times K, V \oplus W}(D^n \times D^m)_+ \; ;$$

here $G \times K$ acts on $V \oplus W$ by $(g,k) \cdot (v,w) = (gv, kw)$. Since $G \times K$ belongs to the family $\mathcal{E} \times \mathcal{F}$ and the inclusion of $S^{n-1} \times D^m \cup_{S^{n-1} \times S^{m-1}} D^n \times S^{m-1}$ into $D^n \times D^m$ is a cofibration of spaces, this pushout product morphism is an $(\mathcal{E} \times \mathcal{F})$-cofibration.

(ii) It suffices to show that the pushout product of any generating acyclic cofibration in the $\mathcal{E}$-projective level model structure with any generating $\mathcal{F}$-cofibration is an acyclic cofibration in the $(\mathcal{E} \times \mathcal{F})$-projective level model structure. By part (i) we already know the $(\mathcal{E} \times \mathcal{F})$-cofibration property, so it remains to show the $(\mathcal{E} \times \mathcal{F})$-level equivalence property.

The acyclic cofibrations in the $\mathcal{E}$-projective level model structure are generated by the morphisms

$$F_{G,V}(\{0\} \times D^n)_+ \;\longrightarrow\; F_{G,V}([0,1] \times D^n)_+ \tag{3.3}$$

for $G \in \mathcal{E}$, V a G-representation and $n \geq 0$. The pushout product of such a generator with a generating $\mathcal{F}$-cofibration (3.2) is isomorphic to the morphism $F_{G,W}(i_+)$, where i is the inclusion

$$\{0\} \times D^n \times D^m \cup_{\{0\}\times D^n \times S^{m-1}} [0,1] \times D^n \times S^{m-1} \longrightarrow [0,1] \times D^n \times D^m .$$

Since i is a homotopy equivalence of spaces, the morphism $F_{G\times K, V\oplus W}(i_+)$ is a homotopy equivalence of orthogonal spectra, so in particular an $(\mathcal{E} \times \mathcal{F})$-level equivalence.

(iii) Again it suffices to show that the pushout product of a generating acyclic cofibration for the $\mathcal{F}$-flat level model structure with any the generating flat cofibration (the morphisms (3.2) for all compact Lie groups K) is an acyclic cofibration for the $\mathcal{F}$-flat level model structure. The generating acyclic cofibrations for the $\mathcal{F}$-flat level model structure come in two flavors:

(a) The generating acyclic cofibrations for the $\mathcal{F}$-projective level model structure (3.3); the pushout product of such a map with a generating flat cofibration is a flat cofibration (by part (i) for $\mathcal{E} = \mathcal{F} = \mathcal{A}ll$) and a homotopy equivalence (compare part (ii)).

(b) The pushout product of the cone inclusion

$$F_{K,W}(\iota_+) \ : \ F_{K,W}EK_+ \longrightarrow F_{K,W}C(EK)_+$$

for $K \notin \mathcal{F}$ and W a faithful K-representation, with the inclusion $S^{n-1}_+ \longrightarrow D^n_+$. The pushout product of such a map with a generating flat cofibration is the pushout product of the morphism

$$F_{G\times K, V\oplus W}(p^*\iota)_+ \ : \ F_{G\times K, V\oplus W}(p^*EK)_+ \longrightarrow F_{G\times K, V\oplus W}(p^*C(EK))_+ ,$$

where $p : G \times K \longrightarrow K$ is the projection, with the relative CW-inclusion

$$S^{n-1} \times D^m \cup_{S^{n-1}\times S^{m-1}} D^n \times S^{m-1} \longrightarrow D^n \times D^m .$$

Since the $\mathcal{F}$-flat level model structure is topological it suffices to show that $F_{G\times K,V\oplus W}(p^*\iota)_+$ is an $\mathcal{F}$-flat level acyclic cofibration. By adjointness this means showing that the $G \times K$-map $p^*(\iota) : p^*(EK) \longrightarrow p^*(C(EK))$ is any acyclic cofibration in the $\mathcal{F} \cap (G \times K)$-flat model structure on $G \times K$-spaces. The map is a $(G \times K)$-cofibration, so it remains to show that it is an $\mathcal{F} \cap (G \times K)$-weak equivalence. If H is a subgroup of $G \times K$ that belongs to $\mathcal{F}$, then its homomorphic image $p(H)$ also belongs to $\mathcal{F}$. Since $K \notin \mathcal{F}$, $p(H)$ must be strictly smaller than K, and hence $(p^*(EK))^H = (EK)^{p(H)}$ is contractible. Since the cone $C(EK)$ is K-equivariantly contractible, $p^*(C(EK))$ is $G \times K$-equivariantly contractible, so $p^*(\iota)$ is an $\mathcal{F} \cap (G \times K)$-weak equivalence, as claimed.

(iv) We work in the global model structure, i.e., the projective $\mathcal{A}ll$-global model structure (or, what is the same, the flat $\mathcal{A}ll$-global model). It suffices to show that, in this model structure, the pushout product of any generating flat cofibration with a generating acyclic cofibration is an acyclic cofibration (i.e., flat cofibration and global equivalence). The flat cofibration part is taken care of by part (i) for $\mathcal{E} = \mathcal{F} = \mathcal{A}ll$. So it suffices to show the global equivalence part.

The generating acyclic cofibrations for the global model structure come in two flavors:

(a) The generating acyclic cofibrations for the $\mathcal{A}ll$-projective level model structure (or, what is the same, the flat $\mathcal{A}ll$-level model). The pushout product of such a morphism with any flat cofibration is a strong level equivalence by part (iii) with $\mathcal{E} = \mathcal{A}ll$.

(b) The pushout product morphisms $c_{K,U,W} \Box i_m^+$, where $c_{K,U,W}$ is the mapping cylinder inclusion of the global equivalence

$$\lambda_{K,U,W} \ : \ F_{K,U\oplus W} S^W \longrightarrow F_{K,U} ,$$

for K any compact Lie group, U a faithful K-representation and W and K-representation, and $i_m : S^{m-1} \longrightarrow D^m$ is the inclusion.

The pushout product of a generating flat cofibration

$$F_{G,V}(S^{n-1}_+) \longrightarrow F_{G,V}(D^n_+)$$

(for G any compact Lie group and V a faithful G-representation) with $c_{K,U,W}\Box i_m^+$ is isomorphic to the pushout product of the morphism $F_{G,V}\wedge c_{K,U,W}$ with the relative CW-inclusion

$$S^{n-1}\times D^m\cup_{S^{n-1}\times S^{m-1}} D^n\times S^{m-1} \longrightarrow D^n\times D^m \ .$$

Since the global model structure is topological, it suffices to show that the morphism $F_{G,V}\wedge c_{K,U,W}$ is a global equivalence. The target of $c_{K,U,W}$ maps by a homotopy equivalence to $F_{K,U}$, it suffices to show that the morphism

$$F_{G,V}\wedge\lambda_{K,U,W} \ : \ F_{G,V}\wedge F_{K,U\oplus W}S^W \longrightarrow F_{G,V}\wedge F_{K,U}$$

is a global equivalence. This morphism, in turn, is isomorphic to

$$\lambda_{G\times K,V\oplus W,U} \ : \ F_{G\times K,V\oplus U,W}S^W \longrightarrow F_{G\times K,V\oplus U} \ ,$$

where G acts trivially on W. This morphism is a global equivalence by Theorem III.5.7 because $G\times K$ acts faithfully on $V\oplus U$. □

The sphere spectrum $\mathbb{S}$ is the unit object for the smash product of orthogonal spectra, and it is 'free', i.e., $\langle e\rangle$-cofibrant. So $\mathbb{S}$ is cofibrant in the flat $\mathcal{F}$ and the projective $\mathcal{F}$-level model structure for every global family $\mathcal{F}$. So with respect to the smash product, the flat $\mathcal{F}$-level model structure is a *symmetric monoidal model category* in the sense of [**43**, Def. 4.2.6]. Similarly, whenever $\mathcal{F}$ is closed under products, then the projective $\mathcal{F}$-level model structure is a symmetric monoidal model category.

Remark 3.4. In the special case where $\mathcal{E}=\mathcal{F}=\langle e\rangle=\mathcal{E}\times\mathcal{F}$ are the trivial global families, the $\langle e\rangle$-level equivalences are the traditional (non-equivariant) level equivalences. The $\langle e\rangle$-cofibrations are called 'q-cofibrations' in [**57**]. Part (ii) of the previous proposition then specializes to Lemma 6.6 of [**57**].

We recall that a model structure on the category of orthogonal spectra satisfies the *pushout product property* if the following two conditions hold:

- for every pair of cofibrations $f : A\longrightarrow B$ and $g : X\longrightarrow Y$ the pushout product morphism $f\Box g$ is also a cofibration;
- if in addition f or g is a weak equivalence, then so is the pushout product morphism $f\Box g$.

Here, and in the next corollary, the monoidal product on the category of orthogonal spectra is the smash product. Proposition 3.1 has all that is needed to verify the pushout product property for flat $\mathcal{F}$-level and $\mathcal{F}$-global model structure. If the global family $\mathcal{F}$ is 'multiplicative', i.e., closed under products, then the same holds for the two projective $\mathcal{F}$-model structures.

Corollary 3.5. *Let $\mathcal{F}$ be a global family.*

(i) *The flat $\mathcal{F}$-level model structure satisfies the pushout product property.*
(ii) *The flat $\mathcal{F}$-global model structure satisfies the pushout product property.*
(iii) *For every flat orthogonal spectrum A the functor $A\wedge -$ preserves $\mathcal{F}$-equivalences between flat orthogonal spectra, and the functor $\mathrm{Hom}(A,-)$ preserves $\mathcal{F}$-injective global Ω-spectra.*

Suppose that the global family $\mathcal{F}$ is multiplicative, i.e., if $\mathcal{F}\times\mathcal{F}=\mathcal{F}$. Then the following properties hold.

(vi) *The projective $\mathcal{F}$-level model structure satisfies the pushout product property.*
(vii) *The projective $\mathcal{F}$-global model structure satisfies the pushout product property.*
(viii) *For every $\mathcal{F}$-cofibrant orthogonal spectrum A the functor $\mathrm{Hom}(A,-)$ preserves $\mathcal{F}$-Ω-spectra.*

Proof. (i) This is Proposition 3.1 (i) with $\mathcal{E}=\mathcal{F}=\mathcal{A}ll$, respectively Proposition 3.1 (iii) with $\mathcal{E}=\mathcal{A}ll$ and $\mathcal{F}$ as given.

(ii) The cofibrations are the same in the flat $\mathcal{F}$-level and the flat $\mathcal{F}$-global model structure, so the cofibration part is taken care of by part (i). The acyclic cofibrations in the flat $\mathcal{F}$-global model structure are generated by the acyclic cofibrations in the flat $\mathcal{F}$-level model structure and the acyclic cofibrations in the global model structure, compare Theorem 2.10 (iv). The pushout product of the former with any flat cofibration is an $\mathcal{F}$-level equivalence by Proposition 3.1 (iii), hence an $\mathcal{F}$-equivalence. The pushout product of the latter with any flat cofibration is a global equivalence by Proposition 3.1 (iv), hence an $\mathcal{F}$-equivalence.

(iii) The first claim is Ken Brown's lemma, applied to the fact that $A \wedge -$ preserves acyclic cofibrations in the flat $\mathcal{F}$-global model structure, by part (ii). Also by part (ii), the functor $A \wedge -$ is a left Quillen endofunctor for the flat $\mathcal{F}$-global model structure. So the right adjoint $\mathrm{Hom}(A, -)$ is a right Quillen endofunctor for the same model structure. In particular, $\mathrm{Hom}(A, -)$ preserves fibrant objects in the flat $\mathcal{F}$-global model structure. By Theorem 2.10 (iii), these fibrant objects are precisely the $\mathcal{F}$-injective global Ω-spectra.

(iv) This is Proposition 3.1 (i) and (ii) with with $\mathcal{E} = \mathcal{F}$ and the hypothesis that $\mathcal{F} \times \mathcal{F} = \mathcal{F}$.

(v) The cofibrations are the same in the projective $\mathcal{F}$-level and the projective $\mathcal{F}$-global model structure, so the cofibration part is taken care of by part (iv). Every cofibration in the projective $\mathcal{F}$-global model structure is in particular a cofibration in the flat $\mathcal{F}$-global model structure, and the equivalences are the same in the two $\mathcal{F}$-global model structures. So the $\mathcal{F}$-equivalence part of the pushout product property is a special case of part (ii).

Part (vi) is analogous to part (iii): By part (v), the functor $A \wedge -$ is a left Quillen endofunctor for the projective $\mathcal{F}$-global model structure, so the right adjoint $\mathrm{Hom}(A, -)$ preserves fibrant objects in the projective $\mathcal{F}$-global model structure. By Theorem 2.11 (iii), these are precisely the $\mathcal{F}$-Ω-spectra. □

Remark 3.6. There is also an analog of Proposition 3.1 (ii) for the projective global model structure. The pushout product of an $\mathcal{E}$-cofibration that is also an $\mathcal{E}$-equivalence with an $\mathcal{F}$-cofibration is an $(\mathcal{E} \times \mathcal{F})$-equivalence. This implies that whenever $\mathcal{F} \times \mathcal{E} = \mathcal{E}$ (so in particular $\mathcal{F} \subset \mathcal{F} \times \mathcal{E} = \mathcal{E}$), then the projective $\mathcal{E}$-global model structure is a module over the projective $\mathcal{F}$-global model structures.

The sphere spectrum $\mathbb{S}$ is the unit object for the smash product of orthogonal spectra, and it is 'free', i.e., $\langle e \rangle$-cofibrant. So $\mathbb{S}$ is cofibrant in the flat and the projective $\mathcal{F}$-global model structure for every global family $\mathcal{F}$. So with respect to the smash product, the flat $\mathcal{F}$-global model structure is a symmetric monoidal model category in the sense of [**43**, Def. 4.2.6]. Similarly, whenever $\mathcal{F}$ is closed under products, then the projective $\mathcal{F}$-global model structure is a symmetric monoidal model category. A corollary is that the homotopy category $\mathcal{GH}_{\mathcal{F}}$, i.e., the localization of the category of orthogonal spectra at the class of $\mathcal{F}$-equivalences, inherits a closed symmetric monoidal structure, compare [**43**, Thm. 4.3.3]. The *derived smash product*

$$\wedge^{\mathbb{L}}_{\mathcal{F}} \ : \ \mathcal{GH}_{\mathcal{F}} \times \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{GH}_{\mathcal{F}} \ , \tag{3.7}$$

i.e., the induced symmetric monoidal product on $\mathcal{GH}_{\mathcal{F}}$, is any total left derived functor of the smash product. Its value at a pair (X, Y) of orthogonal spectra can be calculated as

$$X \wedge^{\mathbb{L}}_{\mathcal{F}} Y \ = \ X^\flat \wedge Y^\flat \ ,$$

where $X^\flat \longrightarrow X$ and $Y^\flat \longrightarrow Y$ are cofibrant replacements in the flat $\mathcal{F}$-global model structure, i.e., $\mathcal{F}$-equivalences with flat sources. Such 'flat resolution' can even be chosen up to $\mathcal{F}$-level equivalence, and by the flatness theorem below (Theorem 3.9) it actually suffices to choose a flat resolution for only one of the factors.

Corollary 3.8. *For every global family $\mathcal{F}$, the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$ is closed symmetric monoidal under the derived smash product* (3.7)*.*

A key property of flat orthogonal spectra is established in the next theorem; in fact, this theorem is a justification for our use of the adjective 'flat'.

Theorem 3.9 (Flatness theorem)**.** *Let $\mathcal{F}$ be a global family.*

(i) *Smashing with a flat orthogonal spectrum preserves $\mathcal{F}$-equivalences.*
(ii) *Smashing with any orthogonal spectrum preserves $\mathcal{F}$-equivalences between flat orthogonal spectra.*

One can deduce Theorem 3.9 from a flatness result of Stolz for orthogonal G-spectra. Indeed, every flat orthogonal spectrum is $\mathbb{S}$-cofibrant in the sense of [**89**, Def. 2.3.4], compare Corollary 6.8 below. So given an $\mathcal{F}$-equivalence $f : X \longrightarrow Y$ between orthogonal spectra, and a group $G \in \mathcal{F}$, we view A, X and Y as orthogonal G-spectra with trivial G-action. Proposition 2.3.29 of [**89**] shows that $A \wedge f : A \wedge X \longrightarrow A \wedge Y$ is a π_*-isomorphism of orthogonal G-spectra. So in particular, $\pi_*^G(A \wedge f)$ is an isomorphism. (We use here, among other things, that orthogonal G-spectra are just orthogonal spectra with G-action and that the smash product of orthogonal G-spectra is the smash product of the underlying orthogonal spectra with diagonal G-action.)

Since Stolz' thesis is not published we provide an independent proof of the flatness theorem without explicit reference to the $\mathbb{S}$-model structure of orthogonal G-spectra. Not surprisingly, we will use several arguments that are familiar from the proofs of the corresponding statements in the non-equivariant context ([**57**, Prop. 12.3], [**89**, Prop. 1.3.11]) and the G-equivariant context ([**58**, III Prop. 7.3], [**89**, Prop. 2.3.29]).

Proof of Theorem 3.9. (i) We go through a sequence of seven steps, proving successively more general special cases of the theorem.

Step 1: We let C be an orthogonal spectrum that is levelwise strongly contractible, i.e., for every $m \geq 0$ the space C_m is $O(m)$-weakly contractible. We let G and K be compact Lie groups, V a K-representation and A a cofibrant based $(G \times K^{\mathrm{op}})$-space such that the right K-action is free (away from the basepoint). We claim that the naive G-fixed points of the orthogonal G-spectrum $A \triangleright_{K,V} C$ are (non-equivariantly) stably contractible:

$$\pi_*\left((A \triangleright_{K,V} C)^G\right) \;=\; 0\ .$$

Any cofibrant based $(G \times K^{\mathrm{op}})$-space is equivariantly homotopy equivalent to a $(G \times K^{\mathrm{op}})$-CW-complex, so it is no loss of generality to assume a $(G \times K^{\mathrm{op}})$-CW-structure on A with skeleta A^n. We show first by induction on n that the naive G-fixed points of the orthogonal G-spectrum $A^n \triangleright_{K,V} C$ are (non-equivariantly) stably contractible. Since A^{-1} consists of the basepoint only, there is nothing to show. Now we suppose $n \geq 0$ and assume the claim for A^{n-1}. The inclusion $A^{n-1} \longrightarrow A^n$ is an h-cofibration of $(G \times K^{\mathrm{op}})$-spaces, hence the induced morphism $(A^{n-1} \triangleright_{K,V} C)^G \longrightarrow (A^n \triangleright_{K,V} C)^G$ is an h-cofibration of orthogonal spectra. By the long exact homotopy group sequence (Corollary III.2.19) it suffices to show the claim for the quotient:

$$\begin{aligned}(A^n \triangleright_{K,V} C)^G/(A^{n-1} \triangleright_{K,V} C)^G \;&\cong\; ((A^n/A^{n-1}) \triangleright_{K,V} C)^G \\ &\cong\; \bigvee_{j \in J} S^n \wedge ((G \times K^{\mathrm{op}}/\Gamma_j)^+ \triangleright_{K,V} C)^G\end{aligned}$$

Here J is an indexing set of the n-cells of the equivariant CW-structure and Γ_j is a subgroup of $G \times K^{\mathrm{op}}$. Since stable homotopy groups take wedges to sums and reindex upon suspension, we may show that each summand $((G \times K^{\mathrm{op}}/\Gamma_j)^+ \triangleright_{K,V} C)^G$ is stably contractible. Since the right K-action on A is free, the stabilizer group Γ_j is the graph of a homomorphism $\alpha_j : H_j \longrightarrow K^{\mathrm{op}}$ from some subgroup H_j of G (namely the projection of Γ_j to G). So we can rewrite

$$((G \times K^{\mathrm{op}}/\Gamma_j)^+ \triangleright_{K,V} C)^G \;\cong\; (G \triangleright_{H_j, \alpha_j^* V} C)^G\ .$$

If H_j is a proper subgroup of G, then the naive fixed point spectrum $(G \triangleright_{H_j, \alpha_j^* V} C)^G$ is trivial. If $H_j = G$, then the m-th level of this naive fixed point spectrum is

$$C(\alpha^*(V) \oplus \mathbb{R}^m)^G$$

which is weakly contractible because C is levelwise strongly contractible. In either case the naive fixed point spectrum $(G \triangleright_{H_j, \alpha_j^* V} C)^G$ is stably contractible. This finishes the inductive proof that the spectra $(A^n \triangleright_{K,V} C)^G$ are stably contractible. The spectrum $(A \triangleright_{K,V} C)^G$ is the sequential colimit, along h-cofibrations, of the spectra $(A^n \triangleright_{K,V} C)^G$; stable homotopy groups commute with such colimits, so $(A \triangleright_{K,V} C)^G$ is stably contractible as well.

Step 2: We let C be an orthogonal spectrum that is levelwise strongly contractible. We let G and K be compact Lie groups, V a K-representation and A a cofibrant based $(G \times K^{\mathrm{op}})$-space such that the right K-action is free (away from the basepoint). We show that for every finite based G-cell complex Z and every based continuous G-map $g : Z \longrightarrow A \wedge_K C(V)$ there is an $m \geq 0$ such that the composite

$$Z \wedge S^m \xrightarrow{g \wedge S^m} A \wedge_K C(V) \wedge S^m \xrightarrow{A \wedge_K \sigma_{V,\mathbb{R}^m}} A \wedge_K C(V \oplus \mathbb{R}^m)$$

is G-equivariantly null-homotopic. We prove the claim by induction over the number of G-cells of Z. There is nothing to show when $Z = *$, so we suppose that the claim holds for a based G-space Y and Z is obtained from Y by attaching an equivariant cell $D^k \times G/H$ along an attaching G-map $\alpha : S^{k-1} \times G/H \longrightarrow Y$. The induction hypothesis, applied to the restriction of g to Y, provides an $m \geq 0$ such that the composite

$$Y \wedge S^m \xrightarrow{g_Y \wedge S^m} A \wedge_K C(V) \wedge S^m \xrightarrow{A \wedge_K \sigma_{V,\mathbb{R}^m}} A \wedge_K C(V \oplus \mathbb{R}^m)$$

is G-equivariantly null-homotopic. The equivariant homotopy extension property of the pair $(Z \wedge S^m, Y \wedge S^m)$ provides a based G-map $g' : Z \wedge S^m \longrightarrow A \wedge_K C(V \oplus \mathbb{R}^m)$ that is G-homotopic to $(A \wedge_K \sigma_{V,\mathbb{R}^m}) \circ (g \wedge S^m)$ and sends $Y \wedge S^m$ to the basepoint. So g' factors through a based G-map

$$Z/Y \wedge S^m \longrightarrow A \wedge_K C(V \oplus \mathbb{R}^m) .$$

Since Z/Y is G-homeomorphic to $(G/H)^+ \wedge S^k$, the last map can be translated into a G-map

$$(G/H)^+ \wedge S^{k+m} \longrightarrow A \wedge_K C(V \oplus \mathbb{R}^m)$$

whose composite with $H \wedge - : S^{k+m} \longrightarrow (G/H)^+ \wedge S^{k+m}$ lands in the H-fixed points of the target and is thus a based map

$$h : S^{k+m} \longrightarrow (A \wedge_K C(V \oplus \mathbb{R}^m))^H = (A \triangleright_{K,V} C)^H_m .$$

This maps represents an element in $\pi_k((A \triangleright_{K,V} C)^H)$, the stable homotopy group of the naive H-fixed point spectrum of $A \triangleright_{K,V} C$. The underlying $(H \times K^{\mathrm{op}})$-space of A is a cofibrant by Proposition A.1.22 (i), so this stable homotopy group is zero by Step 1. Hence for some $l \geq 0$ the composite

$$S^{k+m+l} \xrightarrow{h \wedge S^l} (A \wedge_K C(V \oplus \mathbb{R}^m))^H \wedge S^l = (A \wedge_K C(V \oplus \mathbb{R}^m) \wedge S^l)^H \xrightarrow{(A \wedge_K \sigma_{V \oplus \mathbb{R}^m, \mathbb{R}^l})^H} (A \wedge_K C(V \oplus \mathbb{R}^{m+l}))^H$$

is (non-equivariantly) null-homotopic. Hence the G-map

$$(G/H)^+ \wedge S^{k+m+l} \longrightarrow A \wedge_K C(V \oplus \mathbb{R}^m) \wedge S^l \xrightarrow{A \wedge_K \sigma_{V \oplus \mathbb{R}^m, \mathbb{R}^l}} A \wedge_K C(V \oplus \mathbb{R}^{m+l})$$

is G-equivariantly null-homotopic. The same is thus true after precomposition with the quotient map $Z \wedge S^{m+l} \longrightarrow Z/Y \wedge S^{m+l} \cong (G/H)^+ \wedge S^{k+m+l}$, and so the composite

$$Z \wedge S^{m+l} \xrightarrow{g' \wedge S^l} A \wedge_K C(V \oplus \mathbb{R}^m) \wedge S^l \xrightarrow{A \wedge_K \sigma_{V \oplus \mathbb{R}^m, \mathbb{R}^l}} A \wedge_K C(V \oplus \mathbb{R}^{m+l})$$

is G-equivariantly null-homotopic. Since g' is equivariantly homotopic to $(A \wedge_K \sigma_{V,\mathbb{R}^m}) \circ (g \wedge S^m)$, this finishes the induction, and hence Step 2.

Step 3: We let C be an orthogonal spectrum that is levelwise strongly contractible. We let K be a compact Lie group, W a K-representation and B a cofibrant based K-space. We show that the orthogonal spectrum $(F_{K,W} B) \wedge C$ is globally contractible. So we let G be another compact Lie group and show that $\pi_k^G((F_{K,W} B) \wedge C) = 0$ for $k \geq 0$; the argument for negative dimensions is similar. We let V be a G-representation and $f : S^{V+k} \longrightarrow ((F_{K,W} B) \wedge C)(V)$ a based G-map representing an arbitrary element of the group $\pi_k^G((F_{K,W} B) \wedge C)$. Then the stabilization

$$W \diamond f : S^{W+V+k} \longrightarrow ((F_{K,W} B) \wedge C)(W \oplus V)$$

represents the same class in $\pi_k^G((F_{K,W} B) \wedge C)$, where we let G act trivially on W. The target of $W \diamond f$ is $O(W \oplus V)$-equivariantly homeomorphic to

$$(O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} C(V) .$$

The space $O(W \oplus V)^+ \wedge_K B$ has a left G-action by translation through the composite $G \longrightarrow O(V) \longrightarrow O(W \oplus V)$; the space also has a free right action of $K \times O(V)$ by right translation on $O(W \oplus V)$. Since $O(W \oplus V)$ is a smooth manifold and the $G \times (K \times O(V))^{\mathrm{op}}$-action is smooth, Illman's theorem [**45**, Cor. 7.2] provides a finite $G \times (K \times O(V))^{\mathrm{op}}$-CW-structure on $O(W \oplus V)$; so $O(W \oplus V)^+ \wedge_K B$ is cofibrant as a based $(G \times O(V)^{\mathrm{op}})$-space, with free right $O(V)$-action.

Since the representation sphere S^{W+V+k} admits the structure of a finite G-CW-complex, Step 2 thus provides an $m \geq 0$ such that the composite

$$\begin{aligned} S^{W+V+k+m} &\xrightarrow{W \diamond f \wedge S^m} ((F_{K,W}B) \wedge C)(W \oplus V) \wedge S^m = (O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} C(V) \wedge S^m \\ &\xrightarrow{(O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} \sigma_{V,\mathbb{R}^m}} (O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} C(V \oplus \mathbb{R}^m) \end{aligned}$$

is G-equivariantly null-homotopic. The generalized structure map

$$\sigma_{W \oplus V, \mathbb{R}^m} \ : \ ((F_{K,W}B) \wedge C)(W \oplus V) \wedge S^m \ \longrightarrow \ ((F_{K,W}B) \wedge C)(W \oplus V \oplus \mathbb{R}^m)$$

of the spectrum $(F_{K,W}B) \wedge C$ factors as the composite

$$\begin{aligned} (O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} C(V) \wedge S^m &\xrightarrow{(O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} \sigma_{V,\mathbb{R}^m}} (O(W \oplus V)^+ \wedge_K B) \wedge_{O(V)} C(V \oplus \mathbb{R}^m) \\ &\longrightarrow (O(W \oplus V \oplus \mathbb{R}^m)^+ \wedge_K B) \wedge_{O(V \oplus \mathbb{R}^m)} C(V \oplus \mathbb{R}^m) \ . \end{aligned}$$

So also the stabilization (in the G-spectrum $((F_{K,W}B) \wedge C)\langle G \rangle$) of $W \diamond f$ with $\mathbb{R}^m$ is equivariantly null-homotopic. Since $W \diamond f \diamond \mathbb{R}^m$ represents the same element as f in $\pi_k^G((F_{K,W}B) \wedge C)$ and V and f were arbitrary, this shows that $\pi_k^G((F_{K,W}B) \wedge C) = 0$.

Step 4: We let C be an orthogonal spectrum that is levelwise strongly contractible and A any flat orthogonal spectrum. We show that then $A \wedge C$ is globally contractible. We show first, by induction on m, that $(\mathrm{sk}^m A) \wedge C$ is globally contractible, where $\mathrm{sk}^m A$ is the m-skeleton in the sense of Construction 1.1. The induction starts with $m = -1$, where $\mathrm{sk}^{-1} A$ is the trivial spectrum, and there is nothing to show. For $m \geq 0$ the inclusion $\mathrm{sk}^{m-1} A \longrightarrow \mathrm{sk}^m A$ is an h-cofibration, hence so is the induced morphism $(\mathrm{sk}^{m-1} A) \wedge C \longrightarrow (\mathrm{sk}^m A) \wedge C$. By induction and the long exact sequence of equivariant homotopy groups (Corollary III.2.19) it suffices to show that the quotient

$$\begin{aligned} (\mathrm{sk}^m A \wedge C)/(\mathrm{sk}^{m-1} A \wedge C) \ &\cong \ (\mathrm{sk}^m A / \mathrm{sk}^{m-1} A) \wedge C \\ &\cong \ G_m(A_m/L_m A) \wedge C \ \cong \ (F_{O(m),\mathbb{R}^m}(A_m/L_m A)) \wedge C \end{aligned}$$

is globally contractible. Since A is flat, $A_m/L_m A$ is a cofibrant based $O(m)$-space, and $(F_{O(m),\mathbb{R}^m}(A_m/L_m A)) \wedge C$ is globally contractible by Step 3. This finishes the inductive proof that the spectra $(\mathrm{sk}^m A) \wedge C$ are globally contractible. The spectrum $A \wedge C$ is the sequential colimit, along h-cofibrations, of the spectra $(\mathrm{sk}^m A) \wedge C$; equivariant homotopy groups commute with such colimits, so $A \wedge C$ is globally contractible as well.

Step 5: We let C be a globally contractible orthogonal spectrum and A a flat orthogonal spectrum. We show that then $A \wedge C$ is globally contractible. For this purpose we choose an acyclic cofibration $j : C \longrightarrow C'$ in the global model structure to a globally fibrant object. The orthogonal spectrum C' is also globally contractible, and a global Ω-spectrum, hence strongly level contractible by Proposition 2.5 (iii). So $A \wedge C'$ is globally contractible by Step 4. Since A is a flat orthogonal spectrum, the pushout product property (Proposition 3.1 (iv)) shows that $A \wedge j : A \wedge C \longrightarrow A \wedge C'$ is a global equivalence; so $A \wedge C$ is globally contractible.

Step 6: We let $\mathcal{F}$ be a global family, C an $\mathcal{F}$-globally contractible orthogonal spectrum and A any flat orthogonal spectrum. We show that then the orthogonal spectrum $A \wedge C$ is $\mathcal{F}$-globally contractible. By factoring the unique morphism $C \longrightarrow *$ in the global model structure we find a flat cofibration $j : C \longrightarrow C'$ such that C' is globally contractible and a global Ω-spectrum. The spectrum $A \wedge C'$ is then globally contractible by Step 5. Since C and C' are both $\mathcal{F}$-globally contractible, the morphism j is an acyclic cofibration in the flat $\mathcal{F}$-global model structure. Since A is flat, the pushout product property in the flat $\mathcal{F}$-global model structure (Corollary 3.5 (ii)) shows that $A \wedge j : A \wedge C \longrightarrow A \wedge C'$ is again an $\mathcal{F}$-equivalence. Since $A \wedge C'$ is globally contractible, $A \wedge C$ is $\mathcal{F}$-globally contractible.

Step 7: Now we prove the flatness theorem in full generality. We let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence. Then the mapping cone $C(f)$ is $\mathcal{F}$-globally contractible by the long exact homotopy group sequence (Proposition III.2.18). So the smash product $A \wedge C(f)$ is $\mathcal{F}$-globally contractible by Step 6. Since $A \wedge C(f)$ is isomorphic to the mapping cone of $A \wedge f : A \wedge X \longrightarrow A \wedge Y$, the morphism $A \wedge f$ is an $\mathcal{F}$-equivalence, again by the long exact homotopy groups sequence.

(ii) This is a direct consequence of (i): we let $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence between flat orthogonal spectra and A any orthogonal spectrum. We choose an $\mathcal{F}$-equivalence $\varphi : A^\flat \longrightarrow A$ with flat source. In the commutative square

$$\begin{tikzcd} A^\flat \wedge X \arrow[r, "A^\flat \wedge f"] \arrow[d, "\varphi \wedge X"'] & A^\flat \wedge Y \arrow[d, "\varphi \wedge Y"] \\ A \wedge X \arrow[r, "A \wedge f"'] & A \wedge Y \end{tikzcd}$$

the two vertical morphisms are $\mathcal{F}$-equivalences by part (i). The upper horizontal morphism is an $\mathcal{F}$-equivalence by Corollary 3.5 (ii), so the lower morphism $A \wedge f$ is an $\mathcal{F}$-equivalence. □

Finally, we will prove another important relationship between the global model structures an the smash product, namely the *monoid axiom* [**72**, Def. 3.3].

Proposition 3.10 (Monoid axiom)**.** *We let $\mathcal{F}$ be a global family. For every flat cofibration $j : A \longrightarrow B$ that is also an $\mathcal{F}$-equivalence and every orthogonal spectrum Y the morphisms*

$$j \wedge Y \ : \ A \wedge Y \ \longrightarrow \ B \wedge Y$$

is an h-cofibration and an $\mathcal{F}$-equivalence. Moreover, the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential and transfinite compositions.

Proof. Given the flatness theorem, this is a standard argument, similar to the proofs of the monoid axiom in the non-equivariant context ([**57**, Prop. 12.5], [**89**, Prop. 1.3.10]) and the G-equivariant context ([**58**, III Prop. 7.4], [**89**, Prop. 2.3.27]).

Every flat cofibration is an h-cofibration (Corollary I.4.6 (iii)), and h-cofibrations are closed under smashing with any orthogonal spectrum [ref], so $j \wedge Y$ is an h-cofibration. Since j is a h-cofibration and $\mathcal{F}$-equivalence, its cokernel B/A is $\mathcal{F}$-stably contractible by the long exact homotopy group sequence (Corollary III.2.19). But B/A is also flat as a cokernel of a flat cofibration, so the spectrum $(B/A) \wedge Y$ is $\mathcal{F}$-stably contractible by Theorem 3.9 (ii). Since $j \wedge Y$ is an h-cofibration with cokernel isomorphic to $(B/A) \wedge Y$, the long exact homotopy group sequence then shows that $j \wedge Y$ is an $\mathcal{F}$-equivalence.

The proof that the class of h-cofibrations that are also $\mathcal{F}$-equivalences is closed under cobase change, coproducts and sequential and transfinite compositions is the same as for the special case $\mathcal{F} = \mathcal{A}ll$ (i.e., for global equivalences) in Corollary III.2.22. □

Theorem [**72**, Thm. 4.1] now applies to the flat $\mathcal{F}$-global model structure and gives the following corollary.

Corollary 3.11. *Let R be an orthogonal ring spectrum and $\mathcal{F}$ a global family.*

(i) *The category of R-modules admits the* flat $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the flat $\mathcal{F}$-global model structure). If R is commutative, then this is a monoidal model category that satisfies the monoid axiom.*

(ii) *If R is commutative, then the category of R-algebras admits the* flat $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the flat $\mathcal{F}$-global model structure). Every cofibrant R-algebra is also cofibrant as an R-module.*

The projective $\mathcal{F}$-global model structure has the same equivalences, but fewer cofibrations, than the flat $\mathcal{F}$-global model structure. So the monoid axiom in the flat model structure implies the monoid axiom in the projective model structure. If the global family $\mathcal{F}$ is closed under products, Theorem [**72**, Thm. 4.1] then also applies to the projective $\mathcal{F}$-global model structure and shows:

Corollary 3.12. *Let R be an orthogonal ring spectrum and $\mathcal{F}$ a multiplicative global family.*

(i) *The category of R-modules admits the* projective $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the projective $\mathcal{F}$-global model structure). If R is commutative, then this is a monoidal model category that satisfies the monoid axiom.*

(ii) *If R is commutative, then the category of R-algebras admits the* projective $\mathcal{F}$-global model structure *in which a morphism is an equivalence (respectively fibration) if the underlying morphism of orthogonal spectra is an $\mathcal{F}$-equivalence (respectively fibration in the projective $\mathcal{F}$-global model structure). Every cofibrant R-algebra is also cofibrant as an R-module.*

Remark 3.13. Strictly speaking, Theorem 4.1 of [**72**] does not apply verbatim to the $\mathcal{F}$-flat and $\mathcal{F}$-projective global model structures because the hypothesis that every object is small (with respect to some regular cardinal) is not satisfied. However, in our situation the sources of the generating cofibrations and generating acyclic cofibrations are small with respect to (suitably long) transfinite composition of flat cofibrations, and this suffices to run the small object argument (compare also Remark 2.4 of [**72**, Thm. 4.1]).

In the $\mathcal{F}$-projective global model structures, the sources of the generating cofibrations and generating acyclic cofibrations are in fact small with respect to sequential composites of h-cofibrations, so the countable version of the small object argument (as opposed to a transfinite version) suffices to lift the $\mathcal{F}$-projective global model structures. In the $\mathcal{F}$-flat global model structures one needs a transfinite small object argument because the generating acyclic cofibrations for the flat $\mathcal{F}$-level model structures involve free orthogonal spectra generated by infinite dimensional equivariant CW-complexes.

4. Triangulated global homotopy categories

The global homotopy category $\mathcal{GH}$ is the homotopy category of a stable model structure, so it is naturally a triangulated category, for example by [**43**, Sec. 7.1] of [**74**, Thm. A.12]. The shift functor is modeled by the pointset level suspension of orthogonal spectra, and the distinguished triangles can be defined from mapping cone sequences, or, equivalently. from homotopy fiber sequences. In this section we notice that the suspension spectra of the global classifying spaces $B_{\mathrm{gl}}G$ are compact generators for the global stable homotopy category. This has various formal, bur rather useful, consequences, such as Brown representability, a t-structures, Postnikov sections and the existence of Eilenberg-Mac Lane spectra for global functors. We also show that the functor $\underline{\pi}_0$ takes the smash products of globally connective spectra to the box product of global functors. The same applies, with the obvious modifications, to the $\mathcal{F}$-global stable homotopy category for any global family $\mathcal{F}$.

Definition 4.1. Let $\mathcal{T}$ be a triangulated category which has infinite sums. An object C of $\mathcal{T}$ is *compact* (sometimes called *finite* or *small*) if for every family $\{X_i\}_{i\in I}$ of objects the canonical map

$$\bigoplus_{i\in I}[C, X_i] \longrightarrow [C, \bigoplus_{i\in I} X_i]$$

is an isomorphism. A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a set of *weak generators* if the following condition holds: if X is an object such that the groups $[\Sigma^k G, X]$ are trivial for all $k \in \mathbb{Z}$ and all $G \in \mathcal{G}$, then X is a zero object. The triangulated category $\mathcal{T}$ is *compactly generated* if it has a set of compact weak generators.

If G is from a global family $\mathcal{F}$, then the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ takes $\mathcal{F}$-equivalences to isomorphisms. So the universal property of a localization provides a unique factorization

$$\pi_0^G \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ \mathcal{A}b$$

through the $\mathcal{F}$-global stable homotopy category. We will abuse notation and use the same symbol for the equivariant homotopy group functor on the category of orthogonal spectra and for its 'derived' functor defined on $\mathcal{GH}_{\mathcal{F}}$. This abuse of notation is mostly harmless, but there is one point where it can create confusion, namely in the context of infinite products; we refer the reader to Remark 4.3 for this issue.

We recall from Definition I.2.7 that the global classifying space $B_{\mathrm{gl}}G$ of a compact Lie group G is the free orthogonal space $\mathbf{L}_{G,V} = \mathbf{L}(V,-)/G$ for some faithful G-representation V. The choice of faithful representation is omitted from the notation because the global homotopy type of $B_{\mathrm{gl}}G$ does not depend on it. The suspensions spectrum of $B_{\mathrm{gl}}G$ comes with a stable tautological class

$$e_G \ = \ e_{G,V} \in \ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}}G) \ ,$$

defined in (3.10) of Chapter III.

In the proof of the next proposition we will start using the shorthand notation

$$[\![X,Y]\!]_{\mathcal{F}} \ = \ \mathcal{GH}_{\mathcal{F}}(X,Y)$$

for the abelian group of morphisms from X to Y in the triangulated $\mathcal{F}$-global stable homotopy category.

Proposition 4.2. *Let $\mathcal{F}$ be a global family and G a compact Lie group in $\mathcal{F}$.*

(i) *The pair $(\Sigma^\infty_+ B_{\mathrm{gl}}G, e_G)$ represents the functor $\pi_0^G : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{A}b$.*
(ii) *The orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is a compact object of the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$.*
(iii) *As G varies through a set of representatives of isomorphism classes of groups in $\mathcal{F}$, the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators for the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$.*

In particular, the $\mathcal{F}$-global homotopy category $\mathcal{GH}_{\mathcal{F}}$ is compactly generated.

Proof. (i) We need to show that for every orthogonal spectrum X the map

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \ \longrightarrow \ \pi_0^G X \ , \quad f \longmapsto f_*(e_G)$$

is bijective. Since both sides take $\mathcal{F}$-equivalences in X to bijections, we can assume that X is an $\mathcal{F}$-Ω-spectrum, and hence fibrant in the flat $\mathcal{F}$-global model structure. The orthogonal spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is flat, and hence cofibrant in the flat $\mathcal{F}$-global model structure. So the localization functor induces a bijection

$$\mathcal{S}p(\Sigma^\infty_+ B_{\mathrm{gl}}G, X)/\text{homotopy} \ \longrightarrow \ [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}}$$

from the set of homotopy classes of morphisms of orthogonal spectra to the set of morphisms in $\mathcal{GH}_{\mathcal{F}}$.

We let V be the faithful G-representation that is implicit in the definition of the global classifying space $B_{\mathrm{gl}}G$. By the freeness property of $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$, morphisms from $\Sigma^\infty_+ B_{\mathrm{gl}}G$ to X biject with based G-map $S^V \longrightarrow X(V)$, and similarly for homotopies. The composite

$$[S^V, X(V)]^G \ \xrightarrow{\cong} \ [\![\Sigma^\infty_+ B_{\mathrm{gl}}G, X]\!]_{\mathcal{F}} \ \xrightarrow{f \mapsto f_*(e_G)} \ \pi_0^G X$$

is the stabilization map, and hence bijective by Proposition 2.5 (ii). Since the left map and the composite are bijective, so is the evaluation map at the stable tautological class.

(ii) Since a wedge of $\mathcal{F}$-equivalences is an $\mathcal{F}$-equivalence (by Corollary III.2.21 (i)), the wedge of any family $\{X_i\}_{i\in I}$ of orthogonal spectra is a coproduct in $\mathcal{GH}_{\mathcal{F}}$. We have a commutative square

$$\begin{tikzcd}
\bigoplus_{i\in I}[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, X_i]\!]_{\mathcal{F}} \arrow[r] \arrow[d] & [\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, \bigoplus_{i\in I} X_i]\!]_{\mathcal{F}} \arrow[d] \\
\bigoplus_{i\in I}\pi_0^G(X_i) \arrow[r] & \pi_0^G\left(\bigvee_{i\in I} X_i\right)
\end{tikzcd}$$

in which the vertical maps are evaluation at the stable tautological class, which are isomorphisms by part (i). The lower horizontal map is an isomorphism by Corollary III.2.21 (i), hence so is the upper horizontal map. This shows that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is compact as an object of the triangulated category $\mathcal{GH}_{\mathcal{F}}$.

(iii) If X is an orthogonal spectrum such that graded abelian group $[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\, X]\!]_*$ is trivial for every group G in $\mathcal{F}$, then X is $\mathcal{F}$-equivalent to the trivial orthogonal spectrum; so X is a zero object in $\mathcal{GH}_{\mathcal{F}}$. This proves that the spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ form a set of weak generators $\mathcal{GH}_{\mathcal{F}}$ as G varies over $\mathcal{F}$. □

Remark 4.3 (Equivariant homotopy groups of infinite products)**.** In Corollary III.2.21 (ii) we showed that for every compact Lie group G the functor $\pi_0^G : \mathcal{S}p \longrightarrow \mathcal{A}b$ preserves *finite* products. However, it is *not* true that π_0^G, as a functor on the category of orthogonal spectra, preserves infinite products in general.

On the other hand, the 'derived' functor $\pi_0^G : \mathcal{GH} \longrightarrow \mathcal{A}b$ is representable, by the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$, so it preserves infinite products. This is no contradiction because an infinite product of orthogonal spectra is *not* in general a product in the global homotopy category. To calculate a product in $\mathcal{GH}$ of a family $\{X_i\}_{i\in I}$ of orthogonal spectra, one has to choose stable equivalences $f_i : X_i \longrightarrow X_i^{\mathrm{f}}$ to global Ω-spectra. For an infinite indexing set, the morphism

$$\prod_{i\in I} f_i \;:\; \prod_{i\in I} X_i \;\longrightarrow\; \prod_{i\in I} X_i^{\mathrm{f}}$$

may fail to be a global equivalence, and then the target, but not the source, of this map is a product in $\mathcal{GH}$ of the family $\{X_i\}_{i\in I}$. So when considering infinite products it is important to be aware of our abuse of notation and to remember that the symbol π_0^G has two different meanings.

Let $\mathcal{T}$ be a triangulated category with sums. A *localizing subcategory* of $\mathcal{T}$ is a full subcategory $\mathcal{X}$ which is closed under sums and under extensions in the following sense: if

$$A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$$

if a distinguished triangle in $\mathcal{T}$ such that two of the objects A, B or C belong to $\mathcal{X}$, then so does the third.

Corollary 4.4. *Let $\mathcal{F}$ be a global family. Every localizing subcategory of the $\mathcal{F}$-global homotopy category which contains the spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}G$ for every group G of $\mathcal{F}$ is all of $\mathcal{GH}_{\mathcal{F}}$.*

We recall that a contravariant functor E from a triangulated category $\mathcal{T}$ to the category of abelian groups is called *cohomological* if it takes sums in $\mathcal{T}$ to products of abelian groups and if for every distinguished triangle (f, g, h) in $\mathcal{T}$ the sequence of abelian groups

$$E(\Sigma A) \xrightarrow{E(h)} E(C) \xrightarrow{E(g)} E(B) \xrightarrow{E(f)} E(A)$$

is exact. In a compactly generated triangulated category, every cohomological functor is representable, see for example [**63**, Thm. 3.1]. Since the global stable homotopy categories are compactly generated, we are entitled to the conclusion:

Corollary 4.5 (Brown representability)**.** *Let $\mathcal{F}$ be a global family. Then every cohomological functor on $\mathcal{GH}_{\mathcal{F}}$ is representable.*

The preferred set of generators $\{\Sigma^\infty_+ B_{\mathrm{gl}}G\}$ of the global stable homotopy category has another special property, it is 'positive' in the following sense: for all compact Lie groups G and K and all $n > 0$ the group

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}}G,\ \Sigma^n\Sigma^\infty_+ B_{\mathrm{gl}}K]\!] \ \cong\ \pi^G_{-n}(\Sigma^\infty_+ B_{\mathrm{gl}}K) \ =\ 0$$

is trivial, because the underlying orthogonal G-spectrum of $\Sigma^\infty_+ B_{\mathrm{gl}}K$ is the suspension spectrum of a G-space. A set of positive compact generators in this sense has strong implications, as we shall now explain. Even though our main interest is in the global stable homotopy category, we will work in general triangulated categories for some time, because we feel that this makes the arguments more transparent.

A 't-structure' as introduced by Beilinson, Bernstein and Deligne in [**9**, Def. 1.3.1] axiomatizes the situation in the derived category of an abelian category given by cochain complexes whose cohomology vanishes in positive respectively negative dimensions.

Definition 4.6. A *t-structure* on a triangulated category $\mathcal{T}$ is a pair $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ of full subcategories satisfying the following three conditions, where $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$ and $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$:

(a) For all $X \in \mathcal{T}^{\leq 0}$ and all $Y \in \mathcal{T}^{\geq 1}$ we have $\mathcal{T}(X, Y) = 0$.
(b) $\mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1}$ and $\mathcal{T}^{\geq 0} \supset \mathcal{T}^{\geq 1}$.
(c) For every object X of $\mathcal{T}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{T}^{\leq 0}$ and $B \in \mathcal{T}^{\geq 1}$.

A t-structure is *non-degenerate* if $\bigcap_{n\leq 0} \mathcal{T}^{\leq n} = \{0\}$ and $\bigcap_{n\geq 0} \mathcal{T}^{\geq n} = \{0\}$.

The original definition of t-structures is motivated by derived categories cochain complexes as the main examples, and the subcategory $\mathcal{T}^{\leq 0}$ behaves like complexes with trivial cohomology in positive degrees. We are mainly interested is spectra where a homological (as opposed to *co*homological) grading is more common. So we turn a t-structure into homological notation by setting

$$\mathcal{T}_{\geq n} = \mathcal{T}^{\leq -n} \qquad \text{and} \qquad \mathcal{T}_{\leq n} = \mathcal{T}^{\geq -n} \ .$$

In this homological notation $\mathcal{T}_{\geq n} = \mathcal{T}_{\geq 0}[n]$ and $\mathcal{T}_{\leq n} = \mathcal{T}_{\leq 0}[n]$, and the conditions for a pair $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0}) = (\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ to be a t-structure become:

(a') For all $X \in \mathcal{T}_{\geq 0}$ and all $Y \in \mathcal{T}_{\leq -1}$ we have $\mathcal{T}(X, Y) = 0$.
(b') $\mathcal{T}_{\geq 0} \subset \mathcal{T}_{\geq -1}$ and $\mathcal{T}_{\leq 0} \supset \mathcal{T}_{\leq -1}$.
(c') For every object X of $\mathcal{T}$ there is a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

such that $A \in \mathcal{T}_{\geq 0}$ and $B \in \mathcal{T}_{\leq -1}$.

Some of the basic results of Beilinson, Bernstein and Deligne about t-structures are (in our homological notation):

- for every $n \in \mathbb{Z}$, the inclusion $\mathcal{T}_{\geq n} \longrightarrow \mathcal{T}$ has a right adjoint $\tau_{\geq n} : \mathcal{T} \longrightarrow \mathcal{T}_{\geq n}$, and the inclusion $\mathcal{T}_{\leq n} \longrightarrow \mathcal{T}$ has a left adjoint $\tau_{\leq n} : \mathcal{T} \longrightarrow \mathcal{T}_{\leq n}$ [**9**, Prop. 1.3.3].
- given any choices a adjoints as in the previous item, then for all $a \leq b$ there is a preferred natural isomorphism of functors between $\tau_{\geq a} \circ \tau_{\leq b}$ and $\tau_{\leq b} \circ \tau_{\geq a}$ [**9**, Prop. 1.3.5].
- The *heart*

$$\mathcal{H} \ =\ \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0} \ =\ \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0} \ ,$$

viewed as a full subcategory of $\mathcal{T}$, is an abelian category and $\tau_{\leq 0} \circ \tau_{\geq 0} : \mathcal{T} \longrightarrow \mathcal{H}$ is a homological functor [**9**, Thm. 1.3.6]. Two composable morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ form a short exact sequence in $\mathcal{H}$ if and only if there is a morphism $\delta : C \longrightarrow A[1]$ (necessarily unique) such that the triangle

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C \xrightarrow{\ \delta\ } A[1]$$

is distinguished.

In general the heart of a t-structure need not have enough injectives, nor enough projectives, nor infinite sums. The following proposition gives a sufficient condition for when the heart of the t-structure generated by a set $\mathcal{C}$ of compact objects has all these properties, and can even be identified with a module category. Here we denote by $\mathrm{End}(\mathcal{C})$ the 'endomorphism category' of $\mathcal{C}$, i.e., the full preadditive subcategory of $\mathcal{T}$ with object set $\mathcal{C}$. By an $\mathrm{End}(\mathcal{C})$-module we mean an additive functor

$$M \ : \ \mathrm{End}(\mathcal{C})^{\mathrm{op}} \ \longrightarrow \ \mathcal{A}b$$

from the opposite category of $\mathrm{End}(\mathcal{C})$. So when $\mathcal{C} = \{C\}$ consists of a single object, then the $\mathrm{End}(\mathcal{C})$-modules are just the right modules over the endomorphism ring $\mathrm{End}(C) = \mathcal{T}(C, C)$. The tautological functor

$$\mathcal{T} \ \longrightarrow \ \text{mod-}\,\mathrm{End}(\mathcal{C}) \tag{4.7}$$

takes an object X to $\mathcal{T}(-, X)|_{\mathrm{End}(\mathcal{C})}$, the restriction of the contravariant Hom-functor to the full subcategory $\mathrm{End}(\mathcal{C}) \subset \mathcal{T}$.

We let $\mathcal{T}$ be a triangulated category with infinite sums and we let $\mathcal{C}$ be a set of compact objects of $\mathcal{T}$. We denote by $\langle \mathcal{C} \rangle_+$ the smallest class of objects of $\mathcal{T}$ that contains $\mathcal{C}$, is closed under sums (possibly infinite) and is *closed under cones* in the following sense: if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle such that A and B belong to the class, then so does C. Any non-empty class of objects that is closed under cones contains all zero objects (because a zero object is a cone of any identity morphism) and is closed under suspension (because $A[1]$ is a cone of the morphism from A to a zero object).

The following results are well known in the triangulated category community, but I was unable to find a reference in the generality that we need. For a single compact object (as opposed to a set of compact objects), part (i) of the following proposition can be found in Lemma 6.1 of [**3**]; part (ii) is essentially Theorem A.1 of [**3**].

Proposition 4.8. *Let $\mathcal{T}$ be a triangulated category with infinite sums and let $\mathcal{C}$ be a set of compact weak generators of $\mathcal{T}$ such that the group $\mathcal{T}(C[n], C')$ is trivial for all $C, C' \in \mathcal{C}$ and all $n < 0$.*

(i) *The class $\langle \mathcal{C} \rangle_+$ coincides with the class of those objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $n < 0$.*
(ii) *We denote by $\langle \mathcal{C} \rangle_{\leq 0}$ the class of objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n > 0$. Then the pair $(\langle \mathcal{C} \rangle_+, \langle \mathcal{C} \rangle_{\leq 0})$ is a non-degenerate t-structure on the category $\mathcal{T}$.*
(iii) *The heart $\mathcal{H} = \langle \mathcal{C} \rangle_+ \cap \langle \mathcal{C} \rangle_{\leq 0}$ of the t-structure consists of those objects X of $\mathcal{T}$ such that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n \neq 0$.*
(iv) *Every sum in $\mathcal{T}$ of objects in the heart $\mathcal{H} = \langle \mathcal{C} \rangle_+ \cap \langle \mathcal{C} \rangle_{\leq 0}$ belongs to $\mathcal{H}$. In particular, the heart has infinite sums and the inclusion $\mathcal{H} \longrightarrow \mathcal{T}$ preserves them.*
(v) *The set $\mathcal{P} = \{\tau_{\leq 0} C \mid C \in \mathcal{C}\}$ belongs to the heart $\mathcal{H}$ and is a set of small projective generators of the heart.*
(vi) *The restriction of the tautological functor* (4.7) *to the heart is an equivalence of categories*

$$\mathcal{H} \ \xrightarrow{\ \cong\ } \ \text{mod-}\,\mathrm{End}(\mathcal{C}) \ .$$

Proof. We start by proving one half of property (i). We let $\mathcal{X}$ denote the class of $\mathcal{T}$-objects X such that $\mathcal{T}(C[n], X) = 0$ for all $n < 0$. Then $\mathcal{C} \subset \mathcal{X}$ by the positivity hypothesis. Since the objects in $\mathcal{C}$ are compact, the class $\mathcal{X}$ is closed under arbitrary sums. Finally, the class $\mathcal{X}$ is closed under cones: Given a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

with $X, Y \in \mathcal{X}$, the exact sequence

$$\mathcal{T}(C[n], Y) \ \longrightarrow \ \mathcal{T}(C[n], Z) \ \longrightarrow \ \mathcal{T}(C[n-1], X)$$

shows that Z again belongs to $\mathcal{X}$. Altogether this shows that $\langle \mathcal{C} \rangle_+ \subseteq \mathcal{X}$.

Now we prove (ii), i.e., we verify the axioms of a t-structure. For axiom (a) we consider the class $\mathcal{A}$ of $\mathcal{T}$-objects A such that $\mathcal{T}(A,X) = 0$ for all $X \in \mathcal{T}_{\leq 0}$. Then $\mathcal{C}[1] \subset \mathcal{A}$ by definition of $\langle \mathcal{C} \rangle_{\leq 0}$. Since $\mathcal{T}(-,X)$ takes sums to products, the class $\mathcal{A}$ is closed under arbitrary sums. Moreover, the class $\mathcal{A}$ is closed under cones: given a distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

with $A, B \in \mathcal{A}$, then for every $X \in \langle \mathcal{C} \rangle_{\leq 0}$ the exact sequence

$$\mathcal{T}(A[1], X) \longrightarrow \mathcal{T}(C, X) \longrightarrow \mathcal{T}(B, X)$$

and the fact that $\mathcal{A}[1] \subset \mathcal{A}$ shows that C again belongs to $\mathcal{A}$. Altogether this shows that $\langle \mathcal{C} \rangle_{+}[1] \subseteq \mathcal{A}$, i.e., property (a) holds. Property (b) is a direct consequence of the definitions. It remains to construct the distinguished triangle required by axiom (c). By induction on n we construct objects A_n in $\langle \mathcal{C} \rangle_{+}$ and morphisms $i_n : A_n \longrightarrow A_{n+1}$ and $u_n : A_n \longrightarrow X$ such that $u_{n+1} i_n = u_n$. We start with

$$A_0 = \bigoplus_{C \in \mathcal{C}, k \geq 0, x \in \mathcal{T}(C[k], X)} C[k] \ .$$

Then A_0 belongs to $\langle \mathcal{C} \rangle_{+}$ and the canonical map $u_0 : A_0 \longrightarrow X$ (i.e., the morphism x on the summand indexed by x) induces a surjection $\mathcal{T}(C[k], u_0) : \mathcal{T}(C[k], A_0) \longrightarrow \mathcal{T}(C[k], X)$ for all $C \in \mathcal{C}$ and $n \geq 0$.

In the inductive step we suppose that A_n and $u_n : A_n \longrightarrow X$) have already been constructed. We define

$$D_n = \bigoplus_{C \in \mathcal{C}, k \geq 0, x \in \ker(\mathcal{T}(C[k], u_n))} C[k] \ ,$$

which comes with a tautological morphism $\tau : D_n \longrightarrow A_n$, again given by x on the summand indexed by x. We choose a distinguished triangle

$$D_n \xrightarrow{\ \tau\ } A_n \xrightarrow{\ i_n\ } A_{n+1} \longrightarrow D_n[1] \ .$$

Since $D_n, A_n \in \langle \mathcal{C} \rangle_{+}$, we also have $A_{n+1} \in \langle \mathcal{C} \rangle_{+}$. Since $u_n \tau = 0$ (by definition), we can choose a morphism $u_{n+1} : A_{n+1} \longrightarrow X$ such that $u_{n+1} i_n = u_n$. This completes the inductive construction.

Now we choose a homotopy colimit $(A, \{\varphi_n : A_n \longrightarrow A\}_n)$, of the sequence of morphisms $i_n : A_n \longrightarrow A_{n+1}$. Since all the objects A_n are in $\langle \mathcal{C} \rangle_{+}$, so is A. Since a homotopy colimit in $\mathcal{T}$ is a weak colimit, we can choose a morphism $u : A \longrightarrow X$ such that $u \varphi_n = u_n$ for all $n \geq 0$. We choose a distinguished triangle

$$A \xrightarrow{\ u\ } X \xrightarrow{\ v\ } B \longrightarrow A[1] \ .$$

We claim $B \in \langle \mathcal{C} \rangle_{\leq -1}$, i.e., we show that $\mathcal{T}(C[k], B) = 0$ for all $k \geq 0$ and $C \in \mathcal{C}$. The map

$$\mathcal{T}(C[k], u_0) = \mathcal{T}(C[k], u) \circ \mathcal{T}(C[k], \varphi_0) : \mathcal{T}(C[k], A_0) \longrightarrow \mathcal{T}(C[k], X)$$

is surjective, hence $\mathcal{T}(C[k], u) : \mathcal{T}(C[k], A) \longrightarrow \mathcal{T}(C[k], X)$ is also surjective. To show that $\mathcal{T}(C[k], u)$ is injective we let $\alpha : C[k] \longrightarrow A$ be an element such that $u\alpha = 0$. Since C is compact, there is an $n \geq 0$ and a morphism $\alpha' : C[k] \longrightarrow A_n$ such that $\alpha = \varphi_n \alpha'$. Then $u_n \alpha' = u \varphi_n \alpha' = u\alpha = 0$. So α' indexes one of the summands of D_n. So α' factors through the tautological morphism $\tau : D_n \longrightarrow A_n$ as $\alpha' = \tau \alpha''$, and hence

$$\alpha = \varphi_n \alpha' = \varphi_{n+1} i_n \tau \alpha'' = 0$$

since the morphisms i_n and τ are adjacent in a distinguished triangle, and so have trivial composite. Hence $\mathcal{T}(C[k], u)$ is also injective, hence bijective. The exact sequence

$$\mathcal{T}(C[k], A) \xrightarrow{\ u_*\ } \mathcal{T}(C[k], X) \xrightarrow{\ v_*\ } \mathcal{T}(C[k], B) \xrightarrow{\ w_*\ } \mathcal{T}(C[k-1], A) \xrightarrow{\ u_*\ } \mathcal{T}(C[k-1], X)$$

then shows that $\mathcal{T}(C[k], B) = 0$ for all $k \geq 1$. Because $A \in \langle \mathcal{C} \rangle_{+}$ we also have $\mathcal{T}(C[-1], A) = 0$ by the already established part of (i). So for $k = 0$ the exact sequence above shows that also $\mathcal{T}(C, B) = 0$, and hence $B \in \langle \mathcal{C} \rangle_{\leq -1}$. So $(\langle \mathcal{C} \rangle_{+}, \langle \mathcal{C} \rangle_{\leq 0})$ is a t-structure on $\mathcal{T}$.

Now we show that the t-structure is non-degenerate. If an object X of $\mathcal{T}$ belongs to $\langle \mathcal{C} \rangle_{+}[m]$, then in particular $\mathcal{T}(\mathcal{C}[n], X) = 0$ for all $n \leq m$ by the already established part of (i). So if X lies in the intersection

$\bigcap_{m\geq 0}\langle\mathcal{C}\rangle_+[m]$, then X is a zero object because $\mathcal{C}$ is a set of weak generators. If X belongs to $\langle\mathcal{C}\rangle_{\leq m}$ for all $m \leq 0$, then also $\mathcal{T}(\mathcal{C}[n], X) = 0$ for all $n \in \mathbb{Z}$ by the definition of $\langle\mathcal{C}\rangle_{\leq 0}$. So again X is a zero object because $\mathcal{C}$ is a set of weak generators. This shows that the t-structure is non-degenerate and finishes the proof of (ii).

Now we prove the other half of part (i). We suppose that $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n < 0$. Part (ii) provides a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

with $A \in \langle\mathcal{C}\rangle_+$ and $B \in \langle\mathcal{C}\rangle_{\leq -1}$. For $n \geq 0$ the group $\mathcal{T}(C[n], B)$ is trivial because $B \in \langle\mathcal{C}\rangle_{\leq -1}$. For $n < 0$ the group $\mathcal{T}(C[n], X)$ is trivial by hypothesis and the group $\mathcal{T}(C[n-1], A)$ is trivial because $A \in \langle\mathcal{C}\rangle_+$ and the already established part of (i). The exact sequence

$$\mathcal{T}(C[n], X) \longrightarrow \mathcal{T}(C[n], B) \longrightarrow \mathcal{T}(C[n-1], A)$$

shows that the group $\mathcal{T}(C[n], B)$ is also trivial. So no shift of any object of $\mathcal{C}$ has a non-trivial morphism to B, and thus B is a zero object because $\mathcal{C}$ is a set of weak generators. But then the morphism $A \longrightarrow X$ must be an isomorphism, so $X \in \langle\mathcal{C}\rangle_+$.

Part (iii) is the combination of part (i) and the definition of $\langle\mathcal{C}\rangle_{\leq 0}$.

(iv) We let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{H}$. Then for every $C \in \mathcal{C}$ and every integer $n \neq 0$ we have

$$\mathcal{T}(C[n], \bigoplus X_i) \cong \bigoplus \mathcal{T}(C[n], X_i) = 0$$

because $\mathcal{C}$ is compact; so by (iii) the sum of the X_i also belongs to the heart. Since the heart is a full subcategory of $\mathcal{T}$, this sum is also a coproduct in the abelian category $\mathcal{H}$.

(v) For every $C \in \mathcal{C}$ the object $\tau_{\leq 0}C$ belongs to $\langle\mathcal{C}\rangle_{\leq 0}$ by definition. Since

$$\tau_{\leq 0}C = \tau_{\leq 0}(\tau_{\geq 0}C) = \tau_{\geq 0}(\tau_{\leq 0}C) ,$$

the object $\tau_{\leq 0}C$ also belongs to $\langle\mathcal{C}\rangle_+$, hence to the heart $\mathcal{H}$.

Now we let $\{X_i\}_{i\in I}$ be a family of objects in the heart. Then their sum in $\mathcal{T}$ is also a sum in the heart by (iv). The vertical maps in the commutative square

$$\begin{tikzcd}
\bigoplus \mathcal{H}(\tau_{\leq 0}C, X_i) \arrow[r] \arrow[d, "\cong"'] & \mathcal{H}(\tau_{\leq 0}C, \bigoplus X_i) \arrow[d, "\cong"] \\
\bigoplus \mathcal{T}(C, X_i) \arrow[r] & \mathcal{T}(C, \bigoplus X_i)
\end{tikzcd}$$

are isomorphisms by the universal property of the truncation morphism $C \longrightarrow \tau_{\leq 0}C$, and the lower horizontal map is an isomorphism because C is compact. So the upper horizontal map is an isomorphism, and that shows that the object $\tau_{\leq 0}C$ is small in the abelian category $\mathcal{H}$.

We let $X \in \mathcal{H}$ be any object of the heart. Then $\mathcal{T}(C[n], X) = 0$ for all $C \in \mathcal{C}$ and all $n \neq 0$ by (iii). So if in addition also $\mathcal{H}(\tau_{\leq 0}C, X) = 0$, then

$$\mathcal{T}(C, X) \cong \mathcal{H}(\tau_{\leq 0}C, X) = 0 ,$$

and hence $X = 0$ because $\mathcal{C}$ is a set of weak generators for the triangulated category $\mathcal{T}$. Altogether this shows that $\mathcal{P}$ is a set of generators for the abelian category $\mathcal{H}$.

It remains to show that for all $C \in \mathcal{C}$ the object $\tau_{\leq 0}C$ is projective in the abelian category $\mathcal{H}$. For this we consider any epimorphism $f : X \longrightarrow Y$ in the heart $\mathcal{H}$, so that there is a distinguished triangle in $\mathcal{T}$

$$F \longrightarrow X \xrightarrow{f} Y \longrightarrow F[1] .$$

such that the object F again belongs to $\mathcal{H}$. In the commutative diagram

$$\begin{tikzcd}
\mathcal{H}(\tau_{\le 0}C, X) \arrow[r, "\mathcal{H}(\tau_{\le 0}C,f)"] \arrow[d, "\cong"'] & \mathcal{H}(\tau_{\le 0}C, Y) \arrow[d, "\cong"'] & \\
\mathcal{T}(C,X) \arrow[r, "\mathcal{T}(C,f)"'] & \mathcal{T}(C,Y) \arrow[r] & \mathcal{T}(C[-1],F) = 0
\end{tikzcd}$$

the lower row is then exact, where the vertical maps are induced by the adjunction unit $C \longrightarrow \tau_{\le 0}C$. Moreover, the group $\mathcal{T}(C[-1],F)$ is trivial by (iii) because $F \in \mathcal{H}$. So the morphism $\mathcal{T}(C,f)$ is surjective, hence so if $\mathcal{H}(\tau_{\le 0}C,f)$, and this shows that the object $\tau_{\le 0}C$ is projective.

(vi) We let $\mathcal{A}$ be any abelian category with infinite sums and a set $\mathcal{P}$ of small projective generators. It is well-known that then the functor

$$\mathcal{A}(\mathcal{P},-) \ : \ \mathcal{A} \ \longrightarrow \ \text{mod-}\operatorname{End}(\mathcal{P})$$

is an equivalence of categories; a proof in the case of a single generator can for example be found in [**7**, II Thm. 1.3] or [**71**, Thm. 2.5]. This general fact applies to the heart $\mathcal{H}$ of the t-structure by (v). The claim follows because the preadditive category $\operatorname{End}(\mathcal{P})$ generated by the set $\mathcal{P}$ in the heart $\mathcal{H}$ is isomorphic to the preadditive category generated by the original set $\mathcal{C}$ in the triangulated category $\mathcal{T}$. A preferred isomorphism from $\operatorname{End}(\mathcal{C})$ to $\operatorname{End}(\mathcal{P})$ is given on objects by $C \mapsto \tau_{\le 0}C$, and on morphisms by the isomorphisms

$$\mathcal{T}(C,C') \xrightarrow{\mathcal{T}(C,\eta_{C'})} \mathcal{T}(C,\tau_{\le 0}C') \xleftarrow{\mathcal{T}(\eta_C,\tau_{\le 0}C')} \mathcal{H}(\tau_{\le 0}C,\tau_{\le 0}C')$$

induced by the adjunction units $\eta_C : C \longrightarrow \tau_{\le 0}C$ and $\eta_{C'} : C' \longrightarrow \tau_{\le 0}C'$. □

We specialize Proposition 4.8 to the $\mathcal{F}$-global stable homotopy category for a global family $\mathcal{F}$. By Proposition 4.2 (iii) the set

$$\mathcal{C}_{\mathcal{F}} \ = \ \{\Sigma^\infty_+ B_{\mathrm{gl}}G\}_{[G]\in\mathcal{F}}$$

is a set of compact weak generators for the triangulated category $\mathcal{GH}_{\mathcal{F}}$, where $[G]$ indicates that we choose a set of representatives for the isomorphism classes of compact Lie groups in $\mathcal{F}$. This generating set is 'positive' in the sense of Proposition 4.8.

Definition 4.9. Let $\mathcal{F}$ be a global family. An orthogonal spectrum X is *$\mathcal{F}$-connective* if the homotopy groups $\pi^G_n(X)$ is trivial for every group G in $\mathcal{F}$ and every $n < 0$. An orthogonal spectrum X is *$\mathcal{F}$-coconnective* if the homotopy groups $\pi^G_n(X)$ is trivial for every group G in $\mathcal{F}$ and every $n > 0$.

Since the spectrum $(\Sigma^\infty_+ B_{\mathrm{gl}}G)[n]$ represents the functor π^G_n (by Proposition 4.2 (i)), the class $\langle\mathcal{C}_{\mathcal{F}}\rangle_{\le 0}$ is precisely the class of $\mathcal{F}$-coconnective spectra. So Proposition 4.8 specializes to:

Corollary 4.10. *Let $\mathcal{F}$ be a global family.*

(i) *The class $\langle\Sigma^\infty_+ B_{\mathrm{gl}}G : [G] \in \mathcal{F}\rangle_+$ coincides with the class of $\mathcal{F}$-connective spectra.*

(ii) *The classes of $\mathcal{F}$-connective spectra and $\mathcal{F}$-coconnective spectra form a non-degenerate t-structure on $\mathcal{GH}_{\mathcal{F}}$ whose heart consists of those orthogonal spectra X such that $\pi^G_n(X) = 0$ for all $G \in \mathcal{F}$ and all $n \ne 0$.*

(iii) *The functor*

$$\underline{\pi}_0 \ : \ \mathcal{H} \ \longrightarrow \ \mathcal{GF}_{\mathcal{F}}$$

is an equivalence of categories from the heart of the t-structure to the category of $\mathcal{F}$-global functors.

Proof. Parts (i) and (ii) are special cases of Proposition 4.8. For part (iii) it suffices to show that the full preadditive subcategory $\operatorname{End}(\mathcal{C}_{\mathcal{F}}) \subset \mathcal{GH}_{\mathcal{F}}$ with object set $\{\Sigma^\infty_+ B_{\mathrm{gl}}G\}_{[G]\in\mathcal{F}}$ is anti-equivalent to the full subcategory of the Burnside category $\mathbf{A}$ with object class $\mathcal{F}$, in such a way that the tautological functor corresponds to the functor $\underline{\pi}_0$. Then Proposition 4.8 (iv) provides the claim. The equivalence $\mathrm{End}(\mathcal{C}_{\mathcal{F}})^{\mathrm{op}} \longrightarrow \mathbf{A}_{\mathcal{F}}$ is given by the inclusion on objects, and on morphisms by the isomorphisms

$$[\![\Sigma^\infty_+ B_{\mathrm{gl}} G, \Sigma^\infty_+ B_{\mathrm{gl}} K]\!]_{\mathcal{F}} \ \cong\ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} K) \ \cong\ \mathbf{A}(K, G)$$

specified in Proposition 4.2 (i) respectively Proposition III.3.13. □

Remark 4.11 (Postnikov sections)**.** For the standard t-structure on the global homotopy category (i.e., Corollary 4.10 for $\mathcal{F} = \mathcal{A}ll$) the truncation functor

$$\tau_{\leq n} \ :\ \mathcal{GH} \ \longrightarrow\ \mathcal{GH}_{\leq n}\ ,$$

left adjoint to the inclusion, provides a 'global Postnikov section': For every orthogonal spectrum X the spectrum $\tau_{\leq n} X$ satisfies $\underline{\pi}_k(\tau_{\leq n} X) = 0$ for $k > n$ and the adjunction unit $X \longrightarrow X_{\leq n}$ induces an isomorphism on the global functor $\underline{\pi}_k$ for every $k \leq n$.

Remark 4.12 (Eilenberg-Mac Lane spectra)**.** In the case $\mathcal{F} = \mathcal{A}ll$ of the maximal global family, part (iii) of Corollary 4.10 in particular provides an Eilenberg-Mac Lane spectrum for every global functor M, i.e., an orthogonal spectrum HM such that $\underline{\pi}_k(HM) = 0$ for all $k \neq 0$ and such that the global functor $\underline{\pi}_0(HM)$ is isomorphic to M, and these properties characterize HM up to preferred isomorphism in $\mathcal{GH}$. Moreover, a choice of inverse to the equivalence $\underline{\pi}_0$ of Corollary 4.10 (iii), composed with the inclusion of the heart, provides an Eilenberg-Mac Lane functor

$$H \ :\ \mathcal{GF} \ \longrightarrow\ \mathcal{GH}$$

to the global homotopy category.

A general fact about t-structures, proved in [**9**, Rem. 3.1.17 (i)], is that for all objects A, B in the heart $\mathcal{H}$ not only do morphism in the heart coincide with morphism in the ambient triangulated category $\mathcal{T}$ (simply because the heart is defined as a full subcategory), but also the Yoneda extension group $\mathrm{Ext}^1_{\mathcal{H}}(A, B)$ is isomorphic to the group $\mathcal{T}(A, B[1])$, naturally in both variables. In the case of the global homotopy category, this specializes to the following property: for every short exact sequence of global functors

$$0 \ \longrightarrow\ A \ \xrightarrow{\ i\ }\ B \ \xrightarrow{\ j\ }\ C \ \longrightarrow\ 0$$

there is a unique morphism $\delta : HC \longrightarrow \Sigma(HA)$ in the global stable homotopy category such that the diagram

$$HA \ \xrightarrow{\ Hi\ }\ HB \ \xrightarrow{\ Hj\ }\ HC \ \xrightarrow{\ \delta\ }\ \Sigma(HA)$$

is a distinguished triangle. Moreover, the assignment

$$\mathrm{Ext}^1_{\mathcal{GF}}(C, A) \ \longrightarrow\ [\![HC, \Sigma(HA)]\!]\ , \quad [i, j] \longmapsto \delta$$

is a natural group isomorphism. One should beware, though, that for $n \geq 2$ there is no such simple relationship between the group $\mathrm{Ext}^n_{\mathcal{GF}}(C, A)$ and the morphism group $[\![HC, \Sigma^n(HA)]\!]$.

Example 4.13 (Non-standard t-structures)**.** In addition to the 'standard' t-structure specified in Corollary 4.10 we can use Proposition 4.8 to exhibit non-degenerate t-structures on $\mathcal{GH}_{\mathcal{F}}$ with different (i.e., inequivalent) hearts. We illustrate this in the simplest non-trivial case for the global family $\mathcal{F} = \langle C_2 \rangle$ generated by a cyclic group of order 2. The 'standard' set of compact generators for $\mathcal{GH}_{\langle C_2 \rangle}$ is then $\mathcal{C}_2 = \{\mathbb{S}, \Sigma^\infty_+ B_{\mathrm{gl}} C_2\}$. Corollary 4.10 identifies the heart of the standard t-structure with the category of $\langle C_2 \rangle$-global functors or, equivalently, with the category of $\mathrm{End}(\mathcal{C}_2)$-modules. The information contained in an $\mathrm{End}(\mathcal{C}_2)$-module M consists of two abelian groups $M(C_2)$ and $M(e)$ and morphisms $p^*, \mathrm{tr} : M(e) \longrightarrow M(C_2)$ and $\mathrm{res} : M(C_2) \longrightarrow M(e)$ that satisfy the relations

$$\mathrm{res} \circ p^* \ =\ \mathrm{Id}_{M(e)} \qquad \text{and} \qquad \mathrm{res} \circ \mathrm{tr} \ =\ 2 \cdot \mathrm{Id}_{M(e)}\ ,$$

compare Remark III.3.25.

There is some redundancy in this 'standard' presentation of the heart of the standard t-structure, because

$$M(C_2) \ = \ p^*(M(e)) \ \oplus \ \ker(\mathrm{res}^{C_2}_e) \tag{4.14}$$

naturally splits off a copy of $M(e)$. We get a more economical presentation from a modified set of generators. We denote by $\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2$ is the *reduced* suspension spectrum of the global classifying space, i.e., of the mapping cone of the morphism

$$\mathbf{L}_{e,V} \ \longrightarrow \ \mathbf{L}_{G,V} \ ,$$

where V is any faithful G-representation. Since the previous morphism has a retraction in $\mathcal{GH}$, we obtain a splitting

$$\Sigma^\infty_+ B_{\mathrm{gl}} C_2 \ \cong \ \mathbb{S} \oplus \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2$$

in the triangulated category $\mathcal{GH}$, reflecting the splitting (4.14). The set $\mathcal{C}_2' = \{\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2\}$ is thus another set of compact generators for $\mathcal{GH}_{\langle C_2 \rangle}$. However, $\langle \mathcal{C}_2 \rangle_+ = \langle \mathcal{C}_2' \rangle_+$ and so the t-structure generated by $\mathcal{C}_2'$ is again the standard t-structure, and the heart has not changed. However, the endomorphism category $\mathrm{End}(\mathcal{C}_2')$ is not equivalent to $\mathrm{End}(\mathcal{C}_2)$: the group $[\![\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!]$ is trivial, and the groups $[\![\mathbb{S}, \mathbb{S}]\!]$, and $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \mathbb{S}]\!]$ and $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!]$ are free abelian of rank 1. So an $\mathrm{End}(\mathcal{C}_2')$-module N consists of two abelian groups $N(C_2)$ and $N(e)$ and a morphism $\mathrm{tr} : N(e) \longrightarrow N(C_2)$, not subject to any relations. Even though $\mathrm{End}(\mathcal{C}_2')$ is not equivalent to $\mathrm{End}(\mathcal{C}_2)$, the category of $\mathrm{End}(\mathcal{C}_2')$-modules is equivalent to the category of $\mathrm{End}(\mathcal{C}_2)$-modules, as it must be, since both are equivalent to $\mathcal{GF}_{\langle C_2 \rangle}$.

A small modification of $\mathcal{C}_2'$, however, returns a different non-degenerate t-structure on $\mathcal{GH}_{\langle C_2 \rangle}$ with a heart that is not equivalent to the standard heart. We set $\mathcal{C}_2'' = \{\Sigma\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2\}$, i.e., the sphere spectrum is suspended once, which is another set of compact generators for $\mathcal{GH}_{\langle C_2 \rangle}$. The positivity condition is still satisfied because

$$[\![(\Sigma\mathbb{S})[-1], \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2 \ ]\!] \ \cong \ [\![\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2 \ ]\!] \ \cong \ \pi_0^e(\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2) \ = 0 \ .$$

So Proposition 4.8 applies and shows that $(\langle \mathcal{C}_2'' \rangle_+, \langle \mathcal{C}_2'' \rangle_{\leq 0})$ is a non-degenerate t-structure on $\mathcal{GH}_{\langle C_2 \rangle}$ whose heart $\mathcal{H}'$ consists of all orthogonal spectra X such that $\pi_k^e(X) = 0$ for $k \neq 1$, and $\pi_k^{C_2}(X) = 0$ for $k \neq 0$. The group $[\![\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2, \Sigma\mathbb{S}]\!]$ is trivial and the group

$$[\![\Sigma\mathbb{S}, \Sigma^\infty \bar{B}_{\mathrm{gl}} C_2]\!] \ \cong \ \pi_1^e(\Sigma^\infty \bar{B}_{\mathrm{gl}} C_2)$$

is cyclic of order 2. So an $\mathrm{End}(\mathcal{C}_2'')$-module P consists of two abelian groups $P(C_2)$ and $P(e)$ and a morphism $g : N(C_2) \longrightarrow N(e)$ such that $2g = 0$. This module category, and hence the heart of this non-standard t-structure, is not equivalent to the standard heart. The following diagram displays the three module categories schematically:

$$\begin{array}{ccc}
\text{mod-}\mathrm{End}(\mathcal{C}_2) & \text{mod-}\mathrm{End}(\mathcal{C}_2') & \text{mod-}\mathrm{End}(\mathcal{C}_2'') \\
\begin{tikzcd} M(C_2) \arrow[d, "\mathrm{res}"'] \\ M(e) \arrow[u, bend left=60, "p^*"] \arrow[u, bend right=60, "\mathrm{tr}"'] \end{tikzcd} &
\begin{tikzcd} N(C_2) \\ N(e) \arrow[u, bend right, "\mathrm{tr}"'] \end{tikzcd} &
\begin{tikzcd} P(C_2) \arrow[d, "g"'] \\ N(e) \end{tikzcd} \\
\mathrm{res} \circ p^* = \mathrm{Id} \ , \quad \mathrm{res} \circ \mathrm{tr} \ = \ 2 \cdot \mathrm{Id} & & 2g = 0
\end{array}$$

We recall from (3.7) that the derived smash product $\wedge^{\mathbb{L}}$ is the symmetric monoidal product on the global stable homotopy category obtained as the total left derived functor of the smash product of orthogonal spectra. The box product of global functors was introduced in (4.21) of Chapter III. A canonical morphism of global functors

$$(\underline{\pi}_0 X) \,\square\, (\underline{\pi}_0 Y) \;\longrightarrow\; \underline{\pi}_0(X \wedge Y)$$

was defined in (4.23) of Chapter III, and when applied to flat resolutions of X and Y this becomes the morphism of the following proposition.

Proposition 4.15. *For all globally connective spectra X and Y the orthogonal spectrum $X \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism*

$$(\underline{\pi}_0 X) \,\square\, (\underline{\pi}_0 Y) \;\longrightarrow\; \underline{\pi}_0(X \wedge^{\mathbb{L}} Y)$$

is an isomorphism of global functors.

Proof. We fix a compact Lie group K and let $\mathcal{Y}$ be the class of globally connective orthogonal spectra Y such that $\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism

$$\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K) \,\square\, \underline{\pi}_0 Y \;\longrightarrow\; \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} Y)$$

is an isomorphism of global functors.

The class $\mathcal{Y}$ is closed under sums and we claim that it is also closed under cones. We let

$$A \;\longrightarrow\; B \;\longrightarrow\; C \;\longrightarrow\; \Sigma A$$

be a distinguished triangle in $\mathcal{GH}$ such that A and B belong to $\mathcal{Y}$. Since A is globally connective the global functor $\underline{\pi}_{-1}A$ vanishes; since A belongs to $\mathcal{Y}$ the global functor $\underline{\pi}_{-1}(\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} A)$ vanishes. Since $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square -$ is right exact, the upper row in the commutative diagram

$$\begin{tikzcd}
\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 A) \arrow[r] \arrow[d] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 B) \arrow[r] \arrow[d] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K)\square(\underline{\pi}_0 C) \arrow[r] \arrow[d] & 0 \\
\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} A) \arrow[r] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} B) \arrow[r] & \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K \wedge^{\mathbb{L}} C) \arrow[r] & 0
\end{tikzcd}$$

is exact. The lower row is exact because smashing with $\Sigma^\infty_+ B_{\mathrm{gl}}K$ preserves distinguished triangles. The two left vertical maps are isomorphism because A and B belong to the class $\mathcal{Y}$. So the right vertical map is an isomorphism and C belongs to $\mathcal{Y}$ as well. Moreover, we have

$$\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}K) \,\square\, \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}G) \;\cong\; \mathbf{A}(K,-) \,\square\, \mathbf{A}(G,-) \;\cong\; \mathbf{A}(K \times G,-) \;\cong \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}(K \times G))$$

by Proposition III.3.13 and

$$(\Sigma^\infty_+ B_{\mathrm{gl}}K) \wedge (\Sigma^\infty_+ B_{\mathrm{gl}}G) \;\cong\; \Sigma^\infty_+(B_{\mathrm{gl}}K \boxtimes B_{\mathrm{gl}}G) \;\cong\; \Sigma^\infty_+ B_{\mathrm{gl}}(K \times G)$$

by Proposition I.2.11. This shows that the class $\mathcal{Y}$ is closed under sums and cones and contains the suspension spectra $\Sigma^\infty_+ B_{\mathrm{gl}}G$ for all compact Lie groups G. Corollary 4.10 (i) (for the global family $\mathcal{F} = \mathcal{A}ll$) then shows that $\mathcal{Y}$ is the class of all globally connective orthogonal spectra. This proves the proposition in the special case $X = \Sigma^\infty_+ B_{\mathrm{gl}}K$

Now we perform the same argument in the other variable. We fix a globally connective spectrum Y and let $\mathcal{X}$ denote be the class of globally connective orthogonal spectra X such that $X \wedge^{\mathbb{L}} Y$ is globally connective and the natural morphism of the proposition is an isomorphism of global functors. The class $\mathcal{X}$ is again closed under sums and cones, by the same arguments as above. Moreover, for every compact Lie group K the suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}K$ belongs to the class $\mathcal{X}$ by the previous paragraph. Again Corollary 4.10 (i) shows that $\mathcal{X}$ is the class of all globally connective orthogonal spectra. $\square$

5. Change of families

In this section we compare the global stable homotopy categories for two different global families $\mathcal{F}$ and $\mathcal{E}$, where we suppose that $\mathcal{F} \subseteq \mathcal{E}$. Then every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so we get a 'forgetful' functor on the homotopy categories

$$U = U^{\mathcal{E}}_{\mathcal{F}} \;:\; \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$$

form the universal property of localizations. The global model structures are stable, so the two global homotopy categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have a preferred triangulated structure, and the forgetful functor is canonically an exact functor of triangulated categories. We will show in this section that this forgetful functor has both a left and a right adjoint, both fully faithful, and we will characterize the $\mathcal{E}$-global homotopy types in the image of the adjoints. The categories $\mathcal{GH}_{\mathcal{E}}$ and $\mathcal{GH}_{\mathcal{F}}$ have infinite sums and infinite products, and the forgetful functor preserves both (because it has adjoints on either side).

We recall that a *left Quillen functor* is a functor between model categories that has a right adjoint and preserves cofibrations and acyclic cofibrations. Similarly, a *right Quillen functor* is a functor between model categories that has a left adjoint and preserves fibrations and acyclic fibrations. Any right adjoint of a left Quillen functor is a right Quillen functor, and similarly with 'left' and 'right' exchanged.

Proposition 5.1. *We let $\mathcal{F}$ and $\mathcal{E}$ be two global families such that $\mathcal{F} \subseteq \mathcal{E}$.*

(i) *Every fibration in the projective $\mathcal{E}$-global model structure is also a fibration in the projective $\mathcal{F}$-global model structure. Hence the identity functor of the category of orthogonal spectra is a right Quillen functor from the projective $\mathcal{E}$-global to the projective $\mathcal{F}$-global model structure.*

(ii) *Every $\mathcal{F}$-equivalence between $\mathcal{F}$-cofibrant orthogonal spectra is a global equivalence.*

(iii) *The identity functor of the category of orthogonal spectra is a left Quillen functor from the flat $\mathcal{E}$-global to the flat $\mathcal{F}$-global model structure.*

(iv) *Every $\mathcal{F}$-equivalence between $\mathcal{F}$-injective global Ω-spectra is a global equivalence.*

Proof. Part (i) is immediate from the characterization of fibrations in the projective global model structures in Theorem 2.11 (ii).

(ii) For $\mathcal{E} = \mathcal{A}ll$ the maximal global family, the adjoint statement to part (i) is that every acyclic cofibration in the projective $\mathcal{F}$-model structure is a global equivalence. Ken Brown's lemma (see the proof of [**21**, I.4 Lemma 1] or [**43**, Lemma 1.1.12]) then implies that every $\mathcal{F}$-equivalence between $\mathcal{F}$-cofibrant orthogonal spectra is a global equivalence.

(iii) The cofibrations in the flat global model structures are independent of the global family (they are always the flat cofibrations), and every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence. So every acyclic cofibration in the flat $\mathcal{E}$-global model structure are also acyclic cofibration in the flat $\mathcal{F}$-global model structure.

(iv) For $\mathcal{E} = \mathcal{A}ll$ the maximal global family, the adjoint statement to part (iii) implies that every acyclic fibration in the flat $\mathcal{F}$-model structure is an acyclic fibration in the global model structure, so in particular a global equivalence. Ken Brown's lemma applies again and shows that every $\mathcal{F}$-equivalence between $\mathcal{F}$-injective global Ω-spectra is a global equivalence. □

The smash product of orthogonal spectra can be derived to symmetric monoidal products $\wedge^{\mathbb{L}}_{\mathcal{E}}$ on $\mathcal{GH}_{\mathcal{E}}$ and $\wedge^{\mathbb{L}}_{\mathcal{F}}$ on $\mathcal{GH}_{\mathcal{F}}$ (see Corollary 3.8). The forgetful functor is strongly monoidal with respect to these derived smash products. Indeed, the derived smash product in $\mathcal{GH}_{\mathcal{E}}$ can be calculated by flat approximation up to $\mathcal{E}$-equivalence; every $\mathcal{E}$-equivalence is also an $\mathcal{F}$-equivalence, so these flat approximation can also be used to calculate the derived smash product in $\mathcal{GH}_{\mathcal{F}}$.

Theorem 5.2. *We let $\mathcal{F}$ and $\mathcal{E}$ be two global families such that $\mathcal{F} \subseteq \mathcal{E}$.*

(i) *The forgetful functor*

$$U \;:\; \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$$

has a left adjoint L and a right adjoint R, and both adjoints are fully faithful.

(ii) *The right adjoint has a preferred lax symmetric monoidal structure* $(RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB) \longrightarrow R(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$. *The left adjoint has a preferred lax symmetric comonoidal structure* $L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B) \longrightarrow (LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$.

(iii) *If the family $\mathcal{F}$ is multiplicative, then the lax comonoidal structure on the left adjoint is strong, i.e., a natural isomorphism between* $L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$ *and* $(LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$.

Proof. (i) The existence of the two adjoints is general model category theory: exploiting that the identity functor is both a left and a right Quillen functor (by part (i) respectively (iii)) with respect to different model structures with the same equivalences that are relevant here (i.e., the $\mathcal{E}$-equivalences in the source and the $\mathcal{F}$-equivalences in the target); a standard reference is [**43**, Lemma 1.3.10].

In order to show that the left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ is fully faithful we may show that for every orthogonal spectrum A, the adjunction unit $A \longrightarrow U(LA)$ is an isomorphism in the $\mathcal{F}$-global homotopy category. Since the right Quillen functor (i.e., the identity) preserves all weak equivalences, this is automatic. In order to show that the right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ is fully faithful we may similarly show that for every orthogonal spectrum X, the adjunction counit $U(RX) \longrightarrow X$ is an isomorphism in the $\mathcal{E}$-global homotopy category. As in the previous case this is automatic, now because the left Quillen functor (i.e., the identity again) preserves all weak equivalences.

(ii) The lax monoidal structure of the right adjoint R is a formal consequence of the strong monoidal structure of the forgetful functor U. Indeed, for every pair of orthogonal spectra A and B this strong monoidal structure and the adjunction counits provide a morphism

$$U\big((RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB)\big) \;\cong\; (URA) \wedge^{\mathbb{L}}_{\mathcal{F}} (URB) \xrightarrow{\epsilon_A \wedge^{\mathbb{L}} \epsilon_B} A \wedge^{\mathbb{L}}_{\mathcal{F}} B$$

whose adjoint $(RA) \wedge^{\mathbb{L}}_{\mathcal{E}} (RB) \longrightarrow R(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$ is associative, commutative and unital.

Similarly, the strong symmetric monoidal structure on U and the adjunction units provide a morphism

$$A \wedge^{\mathbb{L}}_{\mathcal{F}} B \xrightarrow{\eta_A \wedge^{\mathbb{L}} \eta_B} U(LA) \wedge^{\mathbb{L}}_{\mathcal{F}} U(LB) \;\cong\; U\big((LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)\big)$$

whose adjoint $\lambda_{A,B} : L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B) \longrightarrow (LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB)$ is associative, commutative and unital.

(iii) We need to show that the morphism $\lambda_{A,B}$ is an isomorphism in $\mathcal{GH}_{\mathcal{E}}$ whenever $\mathcal{F} \times \mathcal{F} \subset \mathcal{F}$. Indeed, we can assume that A and B are $\mathcal{F}$-cofibrant so that $LA = A$ and $LB = B$. Then A and B are in particular flat, so the derived smash product in $\mathcal{GH}_{\mathcal{F}}$ can be calculated as the pointset level smash product $A \wedge B$. Since $\mathcal{F}$ is multiplicative, the pointset level smash product $A \wedge B$ is again $\mathcal{F}$-cofibration by Proposition 3.1 (i). So the value of the left adjoint L on $A \wedge B$ is also given by $A \wedge B$. □

Remark 5.3 (Recollements)**.** Theorem 5.2 implies that for all pairs of nested global families $\mathcal{F} \subseteq \mathcal{E}$ the diagram of triangulated categories and exact functors

$$\begin{tikzcd} \mathcal{GH}(\mathcal{E};\mathcal{F}) \arrow[r, "i_*"] & \mathcal{GH}_{\mathcal{E}} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "U"] & \mathcal{GH}_{\mathcal{F}} \arrow[l, bend right, "L"'] \arrow[l, bend left, "R"] \end{tikzcd}$$

is a *recollement* in the sense of [**9**, Sec. 1.4]. Here $\mathcal{GH}(\mathcal{E};\mathcal{F})$ denotes the '$\mathcal{E}$-global homotopy category with support outside $\mathcal{F}$', i.e., the full subcategory of $\mathcal{GH}_{\mathcal{E}}$ of spectra all of whose $\mathcal{F}$-equivariant homotopy groups vanish. The functor $i_* : \mathcal{GH}(\mathcal{E};\mathcal{F}) \longrightarrow \mathcal{GH}_{\mathcal{E}}$ is the inclusion, and i^* (respectively $i^!$) is a left adjoint (respectively right adjoint) of i_*.

Remark 5.4. In Theorem 5.2 the left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ of the forgetful functor $U : \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$ is obtained as the total left derived functor of the identity functor on orthogonal spectra with respect to change of model structure from the projective $\mathcal{F}$- to the projective $\mathcal{E}$-global model structure. So one can calculate the value of the left adjoint L on an orthogonal spectrum X by choosing any cofibrant replacement in the projective $\mathcal{F}$-global model structure, i.e., an $\mathcal{F}$-equivalence $X_{\mathcal{F}} \longrightarrow X$ such that $X_{\mathcal{F}}$ is $\mathcal{F}$-cofibrant. (One can even do this in the projective $\mathcal{F}$-level model structure and thereby arrange that the map is even an $\mathcal{F}$-level equivalence.) The global homotopy type (so in particular the $\mathcal{E}$-global homotopy type) of $X_{\mathcal{F}}$ is then well-defined.

Similarly, one can calculate the value of the right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ on an orthogonal spectrum X by choosing any fibrant replacement in the flat $\mathcal{F}$-global model structure, i.e., an $\mathcal{F}$-equivalence $X \longrightarrow X^{\mathcal{F}}$ with $X^{\mathcal{F}}$ an $\mathcal{F}$-injective global Ω-spectrum. The global homotopy type of $X^{\mathcal{F}}$ is then well-defined by part (iv) of Proposition 5.1.

It seems worth spelling out the extreme case when $\mathcal{F} = \langle e \rangle$ is the minimal global family of trivial groups and when $\mathcal{E} = \mathcal{A}ll$ is the maximal global family of all compact Lie groups:

- An orthogonal spectrum X is $\langle e \rangle$-cofibrant if for every $m \geq 0$ the latching morphism $\nu_m : L_m X \longrightarrow X_m$ is an $O(m)$-cofibration and $O(m)$ acts freely on the complement of the image. These are precisely the orthogonal spectra called 'q-cofibrant' in [**57**]. Then every non-equivariant stable equivalence between such $\langle e \rangle$-cofibrant orthogonal spectra is a global equivalence.
- Let X be a (non-equivariant) Ω-spectrum such that for every $m \geq 0$ the map $\eta_{X_m} : X_m \longrightarrow \mathrm{map}(EO(m), X_m)$ is an $O(m)$-weak equivalence. Equivalently, X is an Ω-spectrum such that the natural map $i_X : X \longrightarrow bX$ to the Borel theory construction (see (7.4) of Chapter III) is a strong level equivalence. Then X is a global Ω-spectrum. Moreover, every non-equivariant stable equivalence between two such 'cofree' Ω-spectra is a global equivalence.

Remark 5.5. We let $\mathcal{F} \subseteq \mathcal{E}$ be a pair of nested global families. By Theorem 5.2 (ii) the right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ to the forgetful functor comes with a lax symmetric monoidal structure, so it takes '$\mathcal{F}$-global homotopy ring spectra' (i.e., monoid objects in $\mathcal{GH}_{\mathcal{F}}$ under $\wedge^{\mathbb{L}}_{\mathcal{F}}$) to '$\mathcal{E}$-global homotopy ring spectra', preserving commutativity of multiplications. In the extreme case $\mathcal{F} = \langle e \rangle$ and $\mathcal{E} = \mathcal{A}ll$ this can also be deduced from the fact that the Borel theory functor b on the category of orthogonal spectra is lax symmetric monoidal on the pointset level, and models the right adjoint $R : \mathcal{SH} = \mathcal{GH}_{\langle e \rangle} \longrightarrow \mathcal{GH}$; we refer the reader to Section III.7 for details.

The left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$, in contrast, will not in general preserve homotopy multiplications. If the global family $\mathcal{F}$ is multiplicative, however, then the symmetric comonoidal structure on L is an isomorphism, so we can turn it around, i.e., consider the inverse isomorphisms $(LA) \wedge^{\mathbb{L}}_{\mathcal{E}} (LB) \longrightarrow L(A \wedge^{\mathbb{L}}_{\mathcal{F}} B)$; this gives a symmetric monoidal structure on the left adjoint, so it can be used to upgrade L to a functor on homotopy ring spectra.

Remark 5.6 ($\mathcal{F}in$-global homotopy via symmetric spectra)**.** We denote by $\mathcal{F}in$ the global family of finite groups. The $\mathcal{F}in$-global homotopy category $\mathcal{GH}_{\mathcal{F}in}$ has another very natural model, namely the category of symmetric spectra in the sense of Hovey, Shipley and Smith [**44**]. In his master thesis [**40**], M. Hausmann has established a global model structure on the category of symmetric spectra, and he showed that the forgetful functor is a right Quillen equivalence from the category of orthogonal spectra with the flat $\mathcal{F}in$-global model structure to the category of symmetric with the global model structure. Symmetric spectra cannot model global homotopy types for all compact Lie groups, basically because compact Lie groups of positive dimensions do not have any faithful permutation representations.

Now we develop criteria that characterize global homotopy types in the essential image of one of the adjoints to a forgetful change-of-family functor. The following terminology is convenient here.

Definition 5.7. Let $\mathcal{F}$ be a global family. An orthogonal spectrum *left induced* from $\mathcal{F}$ if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$. Similarly, an orthogonal spectrum *right induced* from $\mathcal{F}$ if it is in the essential image of the right adjoint $R_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$.

We start with a criterion, for certain 'reflexive' global families, that characterizes the left induced homotopy types in terms of geometric fixed points.

Definition 5.8. A global family $\mathcal{F}$ is *reflexive* if for every compact Lie group K there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among continuous homomorphisms from K to groups in $\mathcal{F}$.

In other words, $\mathcal{F}$ is reflexive if and only if the inclusion into the category of all compact Lie groups has a left adjoint. As always with adjoints, the universal pair (uK, p) is then unique up to unique isomorphism under K. Moreover, the universal homomorphism $p : K \longrightarrow uK$ is necessarily surjective. Indeed, the image of p is a closed subgroup of uK, hence also in the global family $\mathcal{F}$. So if the image of p were strictly smaller than K, then p would not be initial among morphisms into groups from $\mathcal{F}$. Some examples of reflexive global families are the minimal global family $\langle e \rangle$ of trivial groups, the global family $\mathcal{F}in$ of finite groups and the global family $\mathcal{A}b$ of abelian compact Lie groups. The maximal family of all compact Lie groups is also reflexive, but in this case the following proposition has no content.

A reflexive global family $\mathcal{F}$ is in particular multiplicative. Indeed, for $G, K \in \mathcal{F}$ the projection maps $p_G : G \times K \longrightarrow G$ $p_K : G \times K \longrightarrow K$ factor through continuous homomorphisms $q_G : u(G \times K) \longrightarrow G$ respectively $q_K : u(G \times K) \longrightarrow K$. The composite

$$G \times K \xrightarrow{\ u\ } u(G \times K) \xrightarrow{(q_G, q_K)} G \times K$$

is then the identity, so the universal homomorphism $p : G \times K \longrightarrow u(G \times K)$ is injective. Since $u(G \times K)$ belongs to $\mathcal{F}$, so does $G \times K$.

Proposition 5.9. *Let $\mathcal{F}$ be a reflexive global family. Then an orthogonal spectrum X is left induced from $\mathcal{F}$ if and only if for every compact Lie group K the universal morphism $p : K \longrightarrow uK$ induces isomorphisms*

$$p^* \ : \ \Phi_*^{uK}(X) \ \longrightarrow \ \Phi_*^K(X)$$

between the geometric fixed point homotopy groups with respect to uK and K.

Proof. We let $\mathcal{X}$ be the full subcategory of $\mathcal{GH}$ consisting of the orthogonal spectra X such that for every compact Lie group K the restriction map $p^* : \Phi_*^{uK}(X) \longrightarrow \Phi_*^K(X)$ is an isomorphism. We need to show that $\mathcal{X}$ coincides with the class of spectra left induced from $\mathcal{F}$.

Geometric fixed point homotopy groups commute with sums and take exact triangles to long exact sequences. So $\mathcal{X}$ is closed under sums and triangles, i.e., it is a localizing subcategory of the global homotopy category. Now we claim that for every group G in $\mathcal{F}$ the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ belongs to $\mathcal{X}$. Since $p : K \longrightarrow uK$ is initial among morphisms into group from $\mathcal{F}$, precomposition with p is a bijection between the sets of conjugacy classes of homomorphisms into G; moreover, for each homomorphism $\alpha : uK \longrightarrow G$, then centralizer of α agrees with the centralizer of $\alpha p : K \longrightarrow G$. Proposition I.2.6 (iii) identifies the fixed points of the orthogonal space $B_{\mathrm{gl}}G$ as a disjoint union, over conjugacy classes of homomorphisms, of centralizers of images. So the restriction map along p is a weak equivalence of fixed points spaces

$$p^* \ : \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{uK}))^{uK} \ = \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{uK}))^{K} \ \simeq \ ((B_{\mathrm{gl}}G)(\mathcal{U}_{K}))^{K} \ .$$

Geometric fixed points commute with suspension spectra (see Example III.6.4), in the sense of an isomorphism

$$\Phi_*^K(\Sigma_+^\infty B_{\mathrm{gl}}G) \ \cong \ \pi_*^e(\Sigma_+^\infty \left((B_{\mathrm{gl}}G)(\mathcal{U}_K)\right)^K) \ ,$$

natural for epimorphisms in K. So together this implies the claim for the suspension spectrum of $B_{\mathrm{gl}}G$.

Now we have shown that $\mathcal{X}$ is a localizing subcategory of the global stable homotopy category that contains the suspension spectra of global classifying spaces of all groups in $\mathcal{F}$. The left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is fully faithful and $\mathcal{GH}_{\mathcal{F}}$ is generated by the suspension spectra of the global classifying spaces in $\mathcal{F}$ (by Proposition 4.2). So $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is an equivalence onto the full triangulated subcategory generated by the suspension spectra $\Sigma_+^\infty B_{\mathrm{gl}}G$ for all $G \in \mathcal{F}$. So the image of L is contained in $\mathcal{X}$.

Now suppose that conversely X is an orthogonal spectrum in $\mathcal{X}$. The adjunction counit $\epsilon_X : L(UX) \longrightarrow X$ is an $\mathcal{F}$-equivalence, so it induces isomorphisms of geometric fixed point groups for all groups in $\mathcal{F}$. By the hypothesis of X and naturality of the restriction maps p^*, η_X induces isomorphism of geometric fixed point groups for all compact Lie groups. So η_X is a global equivalence, and in particular X is left induced from $\mathcal{F}$. $\square$

Remark 5.10. The same proof as in Proposition 5.9 yields the following relative version of the proposition. We let $\mathcal{F} \subset \mathcal{E}$ be global families and assume the $\mathcal{F}$ is *reflexive relative to* $\mathcal{E}$, i.e., for every compact Lie group K from the family $\mathcal{E}$ there is a compact Lie group uK, belonging to $\mathcal{F}$, and a continuous homomorphism $p : K \longrightarrow uK$ that is initial among homomorphisms to groups in $\mathcal{F}$. Then an orthogonal spectrum X is in the essential image of the relative left adjoint $L : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every compact Lie group K in $\mathcal{E}$ the universal morphism $p : K \longrightarrow uK$ induces isomorphisms

$$p^* \ : \ \Phi_*^{uK}(X) \ \longrightarrow \ \Phi_*^K(X)$$

between the geometric fixed point homotopy groups of uK and K.

Example 5.11. The minimal global family $\mathcal{F} = \langle e \rangle$ of trivial groups is reflexive, and the unique morphism $K \longrightarrow e$ to any trivial group is universal. So Proposition 5.9 characterizes the global homotopy types in the essential of the left adjoint $L : \mathcal{SH} = \mathcal{GH}_{\langle e \rangle} \longrightarrow \mathcal{GH}$ from the non-equivariant stable homotopy category to the global stable homotopy category: an orthogonal spectrum X is left induced from the trivial family if and only if for every compact Lie group K to unique morphism $p : K \longrightarrow e$ induces an isomorphism

$$p^* \ : \ \Phi_*^e(X) \ \longrightarrow \ \Phi_*^K(X) \ .$$

The geometric fixed point homotopy groups $\Phi_*^e(X)$ with respect to the trivial group isomorphic to $\pi_*^e(X)$, the stable homotopy groups of the underlying non-equivariant spectrum. So the global homotopy types in the essential of the left adjoint $L : \mathcal{SH} \longrightarrow \mathcal{GH}$ are precisely the orthogonal spectra with ‘constant geometric fixed points’.

Here are some specific examples of left induced global homotopy types.

Example 5.12 (Suspension spectra)**.** The orthogonal sphere spectrum $\mathbb{S}$ and the suspension spectrum of every based space are left induced from the trivial global family $\langle e \rangle$. Indeed, geometric fixed points commute with suspension spectra in the following sense: the G-geometric fixed point spectrum of the suspension spectrum of any based G-space A is stably equivalent to the suspension spectrum of the G-fixed point space A^G. So when A has trivial G-action,

$$\Phi_*^G(\Sigma^\infty A) \ \cong \ \pi_*(\Sigma^\infty A) \ .$$

So the suspension spectrum $\Sigma^\infty A$ has ‘constant geometric fixed points’, and it is left induced from the trivial family by the criterion of Example 5.11.

Example 5.13 (Global classifying spaces)**.** If $\mathcal{F}$ is any global family and G a compact Lie group from $\mathcal{F}$, then the suspension spectrum of the global classifying space $B_{\mathrm{gl}}G$ is left induced from $\mathcal{F}$. To see this, it suffices to show that $\Sigma_+^\infty B_{\mathrm{gl}}G$ is $\mathcal{F}$-cofibrant, i.e., has the left lifting property with respect to morphisms that are both $\mathcal{F}$-level equivalences and $\mathcal{F}$-level fibrations. We recall that $B_{\mathrm{gl}}G = \mathbf{L}_{G,V} \ = \ \mathbf{L}(V,-)/G$ is a free orthogonal space, where V is any faithful G-representation. So morphisms $\Sigma_+^\infty B_{\mathrm{gl}}G \longrightarrow X$ of orthogonal spectra biject with based morphisms $S^V \longrightarrow X(V)$ of G-spaces; since S^V can be given the structure of a based G-CW-complex, it has the left lifting property with respect to G-weak equivalences that are also G-fibrations, and the claim follows by adjointness.

Example 5.14 ($\mathbf{\Gamma}$-spaces)**.** We let F be a $\mathbf{\Gamma}$-*space* i.e., a functor $F : \mathbf{\Gamma} \longrightarrow \mathbf{U}$ to the category of spaces which is pointed (i.e., the value $F(*)$ at any one-point set is a one-point space). We view F as a $\mathbf{\Gamma}$-orthogonal space by making it constant in the variable $\mathbf{L}$. Then we obtain an orthogonal spectrum $F(\mathbb{S})$ by evaluating F

on spheres, compare Construction III.8.4. Since F is constant in the linear isometries direction, this means that the value at an inner product space V is

$$F(\mathbb{S})(V) \;=\; F(S^V) \tag{5.15}$$

and the generalized structure map $\sigma_{V,W} : F(\mathbb{S})(V)\wedge S^W \longrightarrow F(\mathbb{S})(V\oplus W)$ is an assembly map (8.3) followed by the effect of F on the canonical homeomorphism $S^V \wedge S^W \cong S^{V\oplus W}$. The $O(V)$-action on $F(\mathbb{S})(V)$ is via the action on S^V and the continuous functoriality of F.

Proposition 5.16. *Let F be a $\mathbf{\Gamma}$-space and G a compact Lie group.*

(i) *The projection $p : G \longrightarrow \pi_0 G$ to the group of path components induces an isomorphism*

$$p^* \;:\; \Phi_*^{\pi_0 G}(F(\mathbb{S})) \;\longrightarrow\; \Phi_*^G(F(\mathbb{S}))$$

of geometric fixed point homotopy groups of the orthogonal spectrum $F(\mathbb{S})$.

(ii) *The orthogonal spectrum $F(\mathbb{S})$ obtained by evaluation of F on spheres is left induced from the global family $\mathcal{F}in$ of finite groups.*

Proof. (i) We can calculate G-fixed points by first taking fixed points with respect to the normal subgroup G° (the path component of the identity) and then fixed points with respect to the quotient $\bar{G} = \pi_0 G = G/G^\circ$. So the G-geometric fixed points of the orthogonal spectrum $F(\mathbb{S})$ can be rewritten as

$$\begin{aligned}\Phi_k^G(F(\mathbb{S})) \;&=\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\,[S^{k+V^G},\, F(S^V)^G]\\ &\cong\; \operatorname{colim}_{V\in s(\mathcal{U}_G)}\,[(S^{k+V^{G^\circ}})^{\bar{G}},\, F(S^{V^{G^\circ}})^{\bar{G}}]\\ &\cong\; \operatorname{colim}_{W\in s(\mathcal{U}_{\bar{G}})}\,[S^{k+W^{\bar{G}}},\, F(S^W)^{\bar{G}}] \;=\; \Phi_k^{\bar{G}}(F(\mathbb{S}))\;.\end{aligned}$$

The second step uses the homeomorphism $F((S^V)^{G^\circ}) \cong (F(S^V))^{G^\circ}$ of Proposition 8.33, where G acts trivially on F. The third step uses that $(\mathcal{U}_G)^{G^\circ}$ is a complete universe for the finite group $\bar{G}$ and as V runs through $s(\mathcal{U}_G)$, the G°-fixed points V^{G° exhaust $(\mathcal{U}_G)^{G^\circ}$. The composite isomorphism is given by the restriction map p^*.

(ii) The global family $\mathcal{F}in$ of finite groups is reflexive, and for every compact Lie group K the projection $K \longrightarrow \pi_0 K$ to the finite group of path components is universal (with respect to $\mathcal{F}in$). Part (ii) verifies the geometric fixed point criterion, so by Proposition 5.9 the orthogonal spectrum $F(\mathbb{S})$ is left induced from the global family of finite groups. □

By Theorem 5.2 the forgetful functor $U : \mathcal{GH} \longrightarrow \mathcal{GH}_{\mathcal{F}}$ has a left adjoint

$$L_{\mathcal{F}} \;:\; \mathcal{GH}_{\mathcal{F}} \;\longrightarrow\; \mathcal{GH}\;.$$

The composite $U\circ L_{\mathcal{F}}$ is naturally isomorphic to the identity functors, so for groups G in $\mathcal{F}$ the G-equivariant homotopy groups of a left induced spectrum $L_{\mathcal{F}}X$ are 'the same' as for X. For a compact Lie group K that does not belong to $\mathcal{F}$, the K-equivariant homotopy groups $\pi_0^K(L_{\mathcal{F}}X)$ are not in general functors ini the $\mathcal{F}$-equivariant homotopy groups of X. However, if X happens to be $\mathcal{F}$-connective, then $L_{\mathcal{F}}X$ is globally connective and in the lowest non-trivial dimension, the global functor $\underline{\pi}_0(L_{\mathcal{F}}X)$ only depends on the $\mathcal{F}$-global homotopy groups of X, as we shall now explain.

Construction 5.17 (Induced global functors)**.** For every global family $\mathcal{F}$ the restriction functor

$$\operatorname{res}_{\mathcal{F}} \;:\; \mathcal{GF} \;\longrightarrow\; \mathcal{GF}_{\mathcal{F}}$$

from the category of global functors to the category of $\mathcal{F}$-global functors has both a left and a right adjoint, and they are given by enriched Kan extensions. We will now discuss the left adjoint

$$\operatorname{ind}_{\mathcal{F}} \;:\; \mathcal{GF}_{\mathcal{F}} \;\longrightarrow\; \mathcal{GF}$$

in some detail. For an $\mathcal{F}$-global functor M, the *induced* global functor $\operatorname{ind}_{\mathcal{F}} M$ is a cokernel of the morphism

$$\begin{array}{rcl} \bigoplus_{[K],[L]\in\mathcal{F}} \mathbf{A}(L,-)\otimes\mathbf{A}(K,L)\otimes M(K) & \longrightarrow & \bigoplus_{[K]\in\mathcal{F}} \mathbf{A}(K,-)\otimes M(K) \\ f\otimes g\otimes m & \longmapsto & f\otimes M(g)(m)-(f\circ g)\otimes m\ . \end{array}$$

Here the sums are indexed by representatives of the isomorphism classes of groups in $\mathcal{F}$.

The adjoint functor pair $(\operatorname{ind}_{\mathcal{F}}, \operatorname{res}_{\mathcal{F}})$ behaves a lot like extension and restriction of scalars along a ring homomorphism. For example, the right adjoint $\operatorname{res}_{\mathcal{F}}$ is exact, and the left adjoint takes projectives to projectives. Moreover, the left adjoint induction functor $\operatorname{ind}_{\mathcal{F}}$ is right exact: a short exact sequence

$$0\longrightarrow M\longrightarrow M'\longrightarrow M''\longrightarrow 0$$

of $\mathcal{F}$-global functors gives rise to a commutative diagram of global functors

$$\begin{tikzcd} \bigoplus_{[K],[L]\in\mathcal{F}} \mathbf{A}(L,-)\otimes\mathbf{A}(K,L)\otimes M(K) \arrow[r] \arrow[d] & \bigoplus_{[K]\in\mathcal{F}} \mathbf{A}(K,-)\otimes M(K) \arrow[d] \\ \bigoplus_{[K],[L]\in\mathcal{F}} \mathbf{A}(L,-)\otimes\mathbf{A}(K,L)\otimes M'(K) \arrow[r] \arrow[d] & \bigoplus_{[K]\in\mathcal{F}} \mathbf{A}(K,-)\otimes M'(K) \arrow[d] \\ \bigoplus_{[K],[L]\in\mathcal{F}} \mathbf{A}(L,-)\otimes\mathbf{A}(K,L)\otimes M''(K) \arrow[r] \arrow[d] & \bigoplus_{[K]\in\mathcal{F}} \mathbf{A}(K,-)\otimes M''(K) \arrow[d] \\ 0 & 0 \end{tikzcd}$$

with exact columns. The induced sequence of horizontal cokernels

$$\operatorname{ind}_{\mathcal{F}} M\longrightarrow \operatorname{ind}_{\mathcal{F}} M'\longrightarrow \operatorname{ind}_{\mathcal{F}} M''\longrightarrow 0$$

is thus also exact.

For every object $X\in\mathcal{GH}_{\mathcal{F}}$ in the $\mathcal{F}$-global homotopy category, the adjunction unit $\eta_X : X\longrightarrow U(L_{\mathcal{F}}X)$ is an isomorphism in $\mathcal{GH}_{\mathcal{F}}$, so it induces an isomorphism of $\mathcal{F}$-global functors

$$\underline{\pi}_0^{\mathcal{F}}(\eta_X)\ :\ \underline{\pi}_0^{\mathcal{F}}(X)\ \longrightarrow\ \underline{\pi}_0^{\mathcal{F}}(U(L_{\mathcal{F}}X))\ =\ \operatorname{res}_{\mathcal{F}}(\underline{\pi}_0(L_{\mathcal{F}}X))\ .$$

Adjoint to this is a morphism of global functors

$$\operatorname{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}}(X))\ \longrightarrow\ \underline{\pi}_0(L_{\mathcal{F}}X)\ . \tag{5.18}$$

Theorem 5.19. *Let $\mathcal{F}$ be a global family. Then for every $\mathcal{F}$-connective orthogonal spectrum X the orthogonal spectrum $L_{\mathcal{F}}X$ is globally connective and the natural morphism* (5.18) *is an isomorphism of global functors.*

Proof. We let $\mathcal{X}$ be the class of $\mathcal{F}$-connective orthogonal spectra such that $L_{\mathcal{F}}X$ is globally connective and the map (5.18) is an isomorphism. Since the functors $\operatorname{ind}_{\mathcal{F}}$, $\underline{\pi}_0^{\mathcal{F}}$, $\underline{\pi}_0$ and $L_{\mathcal{F}}$ all preserve sums, the class $\mathcal{X}$ is closed under sums. We claim that $\mathcal{X}$ is also closed under cones. We let

$$A\ \longrightarrow\ B\ \longrightarrow\ C\ \longrightarrow\ \Sigma A$$

be a distinguished triangle in $\mathcal{GH}_{\mathcal{F}}$ such that A and B belong to $\mathcal{X}$. Then the sequence of $\mathcal{F}$-global functors

$$\underline{\pi}_0^{\mathcal{F}}(A)\ \longrightarrow\ \underline{\pi}_0^{\mathcal{F}}(B)\ \longrightarrow\ \underline{\pi}_0^{\mathcal{F}}(C)\ \longrightarrow\ \underline{\pi}_0^{\mathcal{F}}(\Sigma A)=0$$

is exact. Since the left adjoint $\operatorname{ind}_{\mathcal{F}}$ is right exact, the upper row in the commutative diagram

$$\begin{tikzcd} \operatorname{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}}(A)) \arrow[r] \arrow[d, "\cong"'] & \operatorname{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}}(B)) \arrow[r] \arrow[d, "\cong"'] & \operatorname{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}}(C)) \arrow[r] \arrow[d] & 0 \\ \underline{\pi}_0(L_{\mathcal{F}}A) \arrow[r] & \underline{\pi}_0(L_{\mathcal{F}}B) \arrow[r] & \underline{\pi}_0(L_{\mathcal{F}}C) \arrow[r] & 0 \end{tikzcd}$$

is exact. We claim that the lower row is exact as well. Indeed, $L_{\mathcal{F}}$ is an exact functor of triangulated categories, so it produces a distinguished triangle

$$L_{\mathcal{F}}A \longrightarrow L_{\mathcal{F}}B \longrightarrow L_{\mathcal{F}}C \longrightarrow \Sigma(L_{\mathcal{F}}A)$$

in $\mathcal{GH}$. The long exact homotopy group sequence (Proposition III.2.18) shows that $L_{\mathcal{F}}C$ is again globally connective and, using that $\underline{\pi}_0(\Sigma(L_{\mathcal{F}}A)) = \underline{\pi}_{-1}(L_{\mathcal{F}}A) = 0$, shows the exactness of the lower row. Since the map (5.18) is an isomorphism for A and B, the map is also an isomorphism for C. So the class $\mathcal{X}$ is closed under cones. Now we show that the generators $\Sigma^\infty_+ B_{\mathrm{gl}}G$ belong to the class $\mathcal{X}$ for all $G \in \mathcal{F}$. Since $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ is isomorphic to the represented global functor $\mathbf{A}(G,-)$ (see Proposition III.3.13), the $\mathcal{F}$-global functor $\underline{\pi}_0^{\mathcal{F}}(\Sigma^\infty_+ B_{\mathrm{gl}}G)$ is isomorphic to the represented $\mathcal{F}$-global functor $\mathbf{A}_{\mathcal{F}}(G,-)$. The left adjoint $\mathrm{ind}_{\mathcal{F}}$ takes this to $\mathbf{A}(G,-)$,

$$\mathrm{ind}_{\mathcal{F}}(\mathbf{A}_{\mathcal{F}}(G,-)) \cong \mathbf{A}(G,-) ,$$

because both sides represent evaluation of a global functor at the group G. Altogether,

$$\begin{aligned}\mathrm{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}}(\Sigma^\infty_+ B_{\mathrm{gl}}G)) &\cong \mathrm{ind}_{\mathcal{F}}(\mathbf{A}_{\mathcal{F}}(G,-)) \cong \mathbf{A}(G,-)\\ &\cong \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}G) \cong \underline{\pi}_0(L_{\mathcal{F}}(\Sigma^\infty_+ B_{\mathrm{gl}}G)) .\end{aligned}$$

The last isomorphism uses that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ is left induced from $\mathcal{F}$ whenever $G \in \mathcal{F}$, compare Example IV.5.13. The composite isomorphism is induced by (5.18), so this shows that $\Sigma^\infty_+ B_{\mathrm{gl}}G$ belongs to $\mathcal{X}$. Corollary 4.10 (i) then shows that $\mathcal{X}$ is the class of all $\mathcal{F}$-connective orthogonal spectra. □

For specific global families, there is often a more explicit description of the induced global functor $\mathrm{ind}_{\mathcal{F}} M$ in terms of M, giving more explicit content to Theorem 5.19. We spell this our in two special cases.

Example 5.20. In the extreme case $\mathcal{F} = \langle e\rangle$ of the minimal global family, $\langle e\rangle$-global functors are (equivalent to) abelian groups and the left adjoint

$$\mathcal{A}b \cong \mathcal{GF}_{\langle e\rangle} \xrightarrow{\mathrm{ind}_{\langle e\rangle}} \mathcal{GF}$$

sends an abelian group M to the global functor $M \otimes \mathbb{A}$ where $\mathbb{A} = \mathbf{A}(e,-)$ is the Burnside ring global functor. So Theorem 5.19 specializes to the fact that for every orthogonal spectrum X that is connective in the traditional, non-equivariant sense, the orthogonal spectrum $L_{\langle e\rangle}X$ is globally connective and the natural morphism (5.18)

$$\pi_0^e(X) \otimes \mathbb{A} \longrightarrow \underline{\pi}_0(L_{\langle e\rangle}X)$$

given is an isomorphism of global functors. This can be paraphrased by saying that for every compact Lie group G the action map

$$\pi_0^e(X) \otimes \mathbb{A}(G) \xrightarrow{\cong} \pi_0^G(L_{\langle e\rangle}X)$$

is an isomorphism.

Example 5.21. We take a look at the global family of finite groups; similar arguments work more generally for reflexive global families. We let M be a $\mathcal{F}in$-global functor and give a more explicit description of the value of the induced global functor $\mathrm{ind}_{\mathcal{F}in} M$ at a compact Lie group G. For this purpose we consider the following commutative diagram:

$$\begin{tikzcd} \bigoplus_{(J)\leq(H)\leq G,\ J^\circ=H^\circ,\ |WH|<\infty} M(\pi_0 J) \arrow[r, "d'"] \arrow[d, "\alpha"] & \bigoplus_{(H)\leq G,\ |WH|<\infty} M(\pi_0 H) \arrow[d, "\beta"] \\ \bigoplus_{[K],[L]\in\mathcal{F}in} \mathbf{A}(L,G)\otimes\mathbf{A}(K,L)\otimes M(K) \arrow[r, "d"'] & \bigoplus_{[K]\in\mathcal{F}in} \mathbf{A}(K,G)\otimes M(K) \end{tikzcd}$$

The upper horizontal homomorphism is given by

$$d'(x_{(J,H)}) = \{x\}_J - \{\mathrm{tr}_{\pi_0 J}^{\pi_0 H}(x)\}_H .$$

The lower horizontal homomorphism is given by

$$d(f \otimes g \otimes x) \ = \ (f \circ g) \otimes m \ - \ f \otimes M(g)(x) \ .$$

The left vertical map α maps the summand $M(\pi_0 K)$ indexed by $K \leq H$ to the summand indexed by the pair $(\pi_0 K, \pi_0 H)$ via the homomorphism

$$(\mathrm{tr}_H^G \circ q_H^*) \otimes \mathrm{tr}_{\pi_0 K}^{\pi_0 H} \otimes - \ : \ M(\pi_0 K) \ \longrightarrow \ \mathbf{A}(\pi_0 H, G) \otimes \mathbf{A}(\pi_0 K, \pi_0 H) \otimes M(\pi_0 K) \ .$$

The right vertical map β maps the summand $M(\pi_0 H)$ indexed by H to the summand indexed by $\pi_0 H$ via the homomorphism

$$(\mathrm{tr}_H^G \circ q_H^*) \otimes - \ : \ M(\pi_0 K) \ \longrightarrow \ \mathbf{A}(\pi_0 H, G) \otimes M(\pi_0 K) \ .$$

We denote by R the image of the homomorphism d'.

Proposition 5.22. *For every $\mathcal{F}in$-global functor M and every compact Lie group G the map*

$$\left(\bigoplus_{(H)\leq G,\ |WH|<\infty} M(\pi_0 H) \right) /R \ \longrightarrow \ (\mathrm{ind}_{\mathcal{F}in} M)(G) \ , \quad \langle x \rangle_H \ \longmapsto \ \mathrm{tr}_H^G(q_H^*(x))$$

is an isomorphism.

Proof. By Theorem III.3.5 the lower right corner is generated, as an abelian group, by the elements $(\mathrm{tr}_H^G \circ \alpha^*) \otimes x$ for all conjugacy classes of pairs (H, α) of subgroups H of G with finite Weyl group and continuous homomorphisms $\alpha : H \longrightarrow K$. Since K is finite, the homomorphism factors uniquely as $\alpha = \bar{\alpha} \circ q_H$ for some homomorphism $\bar{\alpha} : \pi_0 H \longrightarrow K$, where $q_H : H \longrightarrow \pi_0 H$ is the projection. Then

$$(\mathrm{tr}_H^G \circ \alpha^*) \otimes x \ = \ \beta \left(\mathrm{tr}_H^G \circ q_H^*) \otimes (\bar{\alpha}^*(x) \right) \ + \ d((\mathrm{tr}_H^G \circ q_H^*) \otimes \bar{\alpha}^* \otimes x) \ ,$$

so the map induced on horizontal cokernels is surjective.

To show that the map induced on cokernels is also injective we consider the diagram

$$\begin{tikzcd}
\bigoplus_{[K],[L]\in\mathcal{F}in} \mathbf{A}(L,G) \otimes \mathbf{A}(K,L) \otimes M(K) \arrow[r, "d"] \arrow[d, "r"] & \bigoplus_{[K]\in\mathcal{F}in} \mathbf{A}(K,G) \otimes M(K) \arrow[d, "s"] \\
\bigoplus_{(J)\leq(H)\leq G,\ J^\circ = H^\circ,\ |WH|<\infty} M(\pi_0 K) \arrow[r, "d'"'] & \bigoplus_{(H)\leq G,\ |WH|<\infty} M(\pi_0 H)
\end{tikzcd}$$

The map r is given as follows. A $\mathbb{Z}$-basis of the group $\mathbf{A}(L,G) \otimes \mathbf{A}(K,L)$ is given by the classes

$$(\mathrm{tr}_H^G \circ q_H^* \circ \bar{\alpha}^*) \otimes (\mathrm{tr}_N^L \circ \varphi^*)$$

for $\alpha : H \longrightarrow L$, $N \leq L$ and $\varphi : N \longrightarrow K$. We express $\bar{\alpha}^* \circ \mathrm{tr}_N^L \in \mathbf{A}(N, \pi_0 H)$ in the preferred basis as

$$\bar{\alpha}^* \circ \mathrm{tr}_N^L \ = \ \sum \mathrm{tr}_{\pi_0(J_i)}^{\pi_0 H} \circ \psi_i^*$$

for suitable subgroups $J_i \leq H$ with $J_i^\circ = H^\circ$ and homomorphisms $\psi_i : \pi_0(J_i) \longrightarrow N$. Then map r is then given by

$$r \left((\mathrm{tr}_H^G \circ q_H^* \circ \bar{\alpha}^*) \otimes (\mathrm{tr}_N^L \circ \varphi^*) \otimes x \right) \ = \ \sum \langle (\psi_i^* \circ \varphi^*)(x) \rangle_{(J_i, H)}$$

Here the summand $(\psi_i^* \circ \varphi^*)(x)$ is to be interpreted as 0 if J_i has infinite Weyl group in G.

Now we claim that the following relations hold:

$$s \circ d \ = \ d' \circ r \qquad \text{and} \qquad s \circ \beta = \mathrm{Id} \ .$$

The first relation shows that s induces a well defined map on the horizontal cokernel. The second relation shows that the map induced by β on cokernels is injective. The verification of the second relation straightforward; for the second relation this is more involved:

$$\begin{aligned} d'\left(r\left((\mathrm{tr}_H^G \circ q_H^* \circ \bar{\alpha}^*)\otimes(\mathrm{tr}_N^L \circ \varphi^*)\otimes x\right)\right) &= d'\left(\sum \langle(\psi_i^*\circ\varphi^*)(x)\rangle_{(J_i,H)}\right) \\ &= \sum\langle\psi_i^*\circ\varphi^*)(x)\rangle_{J_i} - \sum\langle(\mathrm{tr}_{\pi_0(J_i)}^{\pi_0 H}\circ\psi_i^*\circ\varphi^*)(x))\rangle_H \\ &= \left(\sum\langle\psi_i^*\circ\varphi^*)(x)\rangle_{J_i}\right) - \langle(\bar{\alpha}^*\circ\mathrm{tr}_N^L\circ\varphi^*)(x))\rangle_H \\ &= \left(\sum s((\mathrm{tr}_{J_i}^G\circ q_{J_i}^*\circ\psi_i^*\circ\varphi^*)\otimes x\right) - s((\mathrm{tr}_H^G\circ q_H^*\circ\bar{\alpha}^*\circ\mathrm{tr}_N^L\circ\varphi^*)\otimes x) \\ &= s\left(\sum(\mathrm{tr}_H^G\circ q_H^*\circ\mathrm{tr}_{\pi_0(J_i)}^{\pi_0 H}\circ\psi_i^*\circ\varphi^*)\otimes x\right) - \left((\mathrm{tr}_H^G\circ q_H^*\circ\bar{\alpha}^*)\otimes(\mathrm{tr}_N^L(\varphi^*(x)))\right) \\ &= s\left((\mathrm{tr}_H^G\circ q_H^*\circ\bar{\alpha}^*\circ\mathrm{tr}_N^L\circ\varphi^*)\otimes x - (\mathrm{tr}_H^G\circ q_H^*\circ\bar{\alpha}^*)\otimes(\mathrm{tr}_N^L(\varphi^*(x)))\right) \\ &= s\left(d\left((\mathrm{tr}_H^G\circ q_H^*\circ\bar{\alpha}^*)\otimes(\mathrm{tr}_N^L\circ\varphi^*)\otimes x\right)\right) \end{aligned}$$

We use that $\mathrm{tr}_H^G\circ q_H^*\circ\mathrm{tr}_{\pi_0 J}^{\pi_0 H} = \mathrm{tr}_H^G\circ\mathrm{tr}_J^H\circ q_J^* = \mathrm{tr}_J^G\circ q_J^*$ □

Here the sums are indexed by representatives of the conjugacy classes of subgroups of G with finite Weyl group. We write $t_H^G(x)$ for a class $x \in M(\pi_0 H)$ in the summand indexed by H. The subgroup R is generated by the elements

$$t_H^G\left(\mathrm{tr}_{\pi_0 K}^{\pi_0 H}(y)\right) - t_K^G(y)$$

for all nested subgroups $K \leq H \leq G$ such that $K^\circ = H^\circ$, i.e., K and H have the same connected component, and all $y \in M(\pi_0 K)$; we note that in this situation the group $\pi_0 K$ is a subgroup of $\pi_0 H$. Here the possibility that K has infinite Weyl group in G is allowed, in which case we set $t_K^G(y)$.

Now we look more closely at right induced global homotopy types.

Proposition 5.23. *Let $\mathcal{F}$ be a global family and $f : X \longrightarrow Y$ be an $\mathcal{F}$-equivalence of orthogonal spectra. Then for every compact Lie group G and every based G-CW-complex A all of whose isotropy groups belong to $\mathcal{F}$, the map*

$$f_G^*(A) \ : \ X_G^*(A) \ \longrightarrow \ Y_G^*(A)$$

is an isomorphism.

Proof. The natural isomorphism

$$X_G^k(G/H \wedge S^n) \ \cong \ X_H^k(S^n) \ \cong \ \pi_{n-k}^H X$$

and the hypothesis that f is an $\mathcal{F}$-equivalence show that the claim is true for $A = G/H^+ \wedge S^n$ whenever $H \in \mathcal{F}$.

Now treat the special case where A is compact, i.e., built from finitely many G-cells. If $A = *$ consists of the basepoint only, then both sides of the map $f_G^*(A)$ are trivial, so the map is an isomorphism. Otherwise A can be obtained from a G-CW-subcomplex B by attaching an n-cell $G/H \times D^n$ with isotropy $H \in \mathcal{F}$. Then B/A is G-homeomorphic to $G/H^+ \wedge S^n$. The claim holds for B (by induction) and for B/A (by the first paragraph), so it holds for A by the long exact sequence of G-equivariant cohomology groups and the 5-lemma. For an arbitrary G-CW-complex A we use the short exact Milnor sequence. □

Proposition 5.24. *Let $\mathcal{F}$ be a global family. An orthogonal spectrum X is right induced from $\mathcal{F}$ if and only if for every compact Lie group G and every cofibrant based G-space A the map*

$$X_G^*(A) \ \longrightarrow \ X_G^*(A \wedge E(\mathcal{F}\cap G)_+)$$

induced by the projection $A \wedge E(\mathcal{F}\cap G)_+ \longrightarrow A$ is an isomorphism. Here $\mathcal{F}\cap G$ is the family of those subgroups of G that belong to $\mathcal{F}$ and $E(\mathcal{F}\cap G)$ is a universal G-space for the family $\mathcal{F}\cap G$.

Proof. Every orthogonal spectrum in the image of $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}$ is globally equivalent to an $\mathcal{F}$-injective global Ω-spectrum Y (see Remark 5.4). So for every compact Lie group G, every faithful G-representation V and every based G-CW-complex A we have

$$\begin{aligned} Y^k_G(A) \ &\cong\ [S^{V+k}\wedge A, Y(V)]^G \ \cong\ [S^{V+k}\wedge A, \mathrm{map}(E(\mathcal{F}\cap G), Y(V))]^G \\ &\cong\ [S^{V+k}\wedge A\wedge E(\mathcal{F}\cap G)_+, Y(V)]^G \ \cong\ Y^k_G(A\wedge E(\mathcal{F}\cap G)_+)\ . \end{aligned}$$

The first isomorphism uses that Y is a global Ω-spectrum, the second isomorphism uses that Y is $\mathcal{F}$-injective.

Now we suppose conversely that for all G, the G-cohomology theory $X^*_G(-)$ is insensitive to smashing with $E(\mathcal{F}\cap G)$. We claim that then the adjunction unit $\eta_X : X \longrightarrow R(UX)$ is a global equivalence, and hence X is right induced from $\mathcal{F}$. To prove the claim we consider the commutative square:

$$\begin{tikzcd} X^k_G(*) \arrow[r, "(\eta_X)_*"] \arrow[d, "\cong"'] & (RUX)^k_G(*) \arrow[d, "\cong"] \\ X^k_G(E(\mathcal{F}\cap G)_+) \arrow[r, "(\eta_X)_*"'] & (RUX)^k_G(E(\mathcal{F}\cap G)_+) \end{tikzcd}$$

The left vertical map is an isomorphism by hypothesis, the right vertical map is an isomorphism by the previous paragraph. The universal space $E(\mathcal{F}\cap G)$ is a G-CW-complex with all isotropy groups in $\mathcal{F}$, so the lower horizontal map is an isomorphism by Proposition 5.23. We conclude that the upper horizontal map is an isomorphism for all compact Lie groups G and all integers k; since $X^k_G(*)$ is naturally isomorphic to $\pi^G_{-k}(X)$, this shows that η_X is a global equivalence. □

Remark 5.25. Again there is a relative generalization of Proposition 5.23, with essentially the same proof. We $\mathcal{F}\subseteq\mathcal{E}$ be two nested global families. Then an orthogonal spectrum X is in the essential image of the relative right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ if and only if for every group G in $\mathcal{E}$ and every G-CW-complex A the map

$$X^*_G(A) \ \longrightarrow\ X^*_G(A\wedge E(\mathcal{F}\cap G)_+)$$

induced by the projection $A\wedge E(\mathcal{F}\cap G)_+ \longrightarrow A$ is an isomorphism. Here $\mathcal{F}\cap G$ is the family of those subgroups of G that belong to $\mathcal{F}$ and $E(\mathcal{F}\cap G)$ is a universal G-space for the family $\mathcal{F}\cap G$.

Here are specific examples of right induced global homotopy types.

Example 5.26 (Global Borel theories)**.** We let X be any orthogonal spectrum that is a non-equivariant Ω-spectrum. In Section III.7 we introduced the orthogonal spectrum bX that represent the global Borel cohomology theory associated to X. The name is derived from the fact that for every compact Lie group G and every G-CW-complex A the groups

$$(bX)^*_G(A) \ \cong\ X^*(EG_+\wedge A)$$

are isomorphic to the X-Borel cohomology of A, compare Proposition III.7.3 (ii).

We claim that the orthogonal spectrum bX is right induced from the trivial global family. Indeed, we showed in Proposition III.7.3 (i) that bX is a global Ω-spectrum. But bX is also $\langle e\rangle$-injective because

$$(bX)_m \ =\ \mathrm{map}(\mathbf{L}(\mathbb{R}^m,\mathbb{R}^\infty), X_m)$$

and $\mathbf{L}(\mathbb{R}^m,\mathbb{R}^\infty)$ is a universal free $O(m)$-space $EO(m)$. So the non-equivariant level equivalence $i_X : X \longrightarrow bX$ defined in (7.4) of Chapter III is a π_*-isomorphism to an $\langle e\rangle$-injective Ω-spectrum, and hence bX models the value RX of the right adjoint by Remark 5.4.

A general orthogonal spectrum Y is right induced from the trivial family if and only if the adjunction unit $Y \longrightarrow R_{\langle e\rangle}(U_{\langle e\rangle}Y)$ is a global equivalence; by the previous paragraph that happens precisely when Y is the global Borel theory of its underlying non-equivariant cohomology theory. We conclude: the global homotopy types in the essential image of the right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor are precisely the global Borel theories. The morphism $i_X : X \longrightarrow bX$ models the adjunction unit $\eta_X : X \longrightarrow R(UX)$, so Remark III.7.6 explains how this adjunction unit can be viewed as a 'global completion map'

Example 5.27 (Global K-theory)**.** Our language allows a reformulation of the generalization, due to Adams, Haeberly, Jackowski and May [**1**], of the Atiyah-Segal completion theorem: the periodic global complex topological K-theory spectrum **KU** is right induced from finite cyclic groups.

If G is a compact Lie group then a virtual G-representation that restricts to zero on every finite cyclic subgroup is already zero. In other words, the intersection of the kernels of all restriction maps $\mathrm{res}^G_C : R(G) \longrightarrow R(C)$ for all finite cyclic subgroups C of G, is trivial. We let *cyc* denote the global family of finite cyclic groups. Then by [**1**, Cor. 2.1], the projection $A \times E(cyc \cap G) \longrightarrow A$ induces an isomorphism

$$K^*_G(A) \cong K^*_G(A \times E(cyc \cap G))$$

on equivariant K-groups for every finite G-CW-complex A, where $E(cyc \cap G)$ is a universal G-space for the family of finite cyclic subgroups of G. The Milnor short exact sequence lets us extend this to infinite G-CW-complex, so the criterion provided by Proposition 5.24 for being right induced from the global family *cyc* is satisfied.

Remark 5.28. We close this section by remarking that the right adjoint $R : \mathcal{GH}_{\mathcal{F}} \longrightarrow \mathcal{GH}_{\mathcal{E}}$ to the forgetful functor $U : \mathcal{GH}_{\mathcal{E}} \longrightarrow \mathcal{GH}_{\mathcal{F}}$ does not in general preserve infinite sums, and the left adjoint L does not preserve infinite products. So the class of left induced spectra is *not* closed under products in the ambient category and the class of right induced spectra is *not* closed under coproducts in the ambient category. The reader may want to recall from Remark 4.3 the subtleties involved with infinite products in $\mathcal{GH}$.

We illustrate this by specific examples in the extreme case $\mathcal{F} = \langle e \rangle$ and $\mathcal{E} = \mathcal{A}ll$. In the non-equivariant stable homotopy category the canonical map

$$\bigoplus_{i \geq 0} \Sigma^i H\mathbb{F}_2 \longrightarrow \prod_{i \geq 0} \Sigma^i H\mathbb{F}_2$$

from the coproduct to the product of infinitely many suspended copies of the mod-2 Eilenberg-Mac Lane spectrum is an isomorphism. Since the right adjoint and equivariant homotopy groups preserve products, the canonical map

$$\pi_0^G \left(R \left(\bigoplus\nolimits_{i \geq 0} \Sigma^i H\mathbb{F}_2 \right) \right) \xrightarrow{\cong} \prod\nolimits_{i \geq 0} \pi_0^G(R(\Sigma^i H\mathbb{F}_2)) \cong \prod\nolimits_{i \geq 0} H^i(BG, \mathbb{F}_2) \ .$$

is an isomorphism of abelian groups. In the special case $G = C_2$, the cyclic group of order 2, the group above then becomes an infinite product of copies of $\mathbb{F}_2$. On the other hand,

$$\pi_0^{C_2} \left(\bigoplus\nolimits_{i \geq 0} R(\Sigma^i H\mathbb{F}_2) \right) \cong \bigoplus\nolimits_{i \geq 0} \pi_0^{C_2}(R(\Sigma^i H\mathbb{F}_2)) \cong \bigoplus\nolimits_{i \geq 0} H^i(BC_2, \mathbb{F}_2)$$

is a countable direct sum of copies of $\mathbb{F}_2$. So the canonical map

$$\bigoplus_{i \geq 0} R\left(\Sigma^i H\mathbb{F}_2\right) \longrightarrow R\left(\bigoplus_{i \geq 0} \Sigma^i H\mathbb{F}_2 \right)$$

is not a global equivalence.

A similar, but slightly more involved, argument shows that the left adjoint does not preserve products. As before the canonical map

$$\bigoplus_{i < 0} \Sigma^i H\mathbb{F}_2 \longrightarrow \prod_{i < 0} \Sigma^i H\mathbb{F}_2$$

is an isomorphism in $\mathcal{SH}$, where now sum and product are taken over all $i < 0$ (as opposed to $i \geq 0$). Since the left adjoint and equivariant homotopy groups preserves coproducts, the canonical map

$$\bigoplus_{i<0} \pi_0^G \left(L(\Sigma^i H\mathbb{F}_2)\right) \longrightarrow \pi_0^G \left(L\left(\prod_{i<0} \Sigma^i H\mathbb{F}_2\right)\right) \tag{5.29}$$

is an isomorphism of abelian groups. Again we specialize to $G = C_2$, the cyclic group of order 2. If X is an non-equivariant homotopy type, then LX has 'constant geometric fixed points' in the sense of Remark 5.11. So the geometric fixed point map $\Phi : \pi_0^{C_2}(LX) \longrightarrow \Phi_0^{C_2}(LX) \cong \pi_0^e(LX)$ has a section and the isotropy separation sequence (see (6.16) of Chapter III) splits. So the C_2-equivariant homotopy groups decompose as

$$\pi_0^{C_2}(LX) \ \cong \ \pi_0^{C_2}(EC_2^+ \wedge LX) \oplus \Phi_0^{C_2}(LX) \ \cong \ \pi_0^e(BC_2^+ \wedge X) \oplus \ \pi_0^e(X) \ .$$

The second step uses the Adams isomorphism and the fact that any global homotopy type has trivial G-action upon restriction to a trivial G-universe. When $X = \Sigma^i H\mathbb{F}_2$ for negative i, then the second summand is trivial and hence

$$\pi_0^{C_2}(L(\Sigma^i H\mathbb{F}_2)) \ \cong \ \pi_0^e(BC_2^+ \wedge \Sigma^i H\mathbb{F}_2) \ \cong \ H_{-i}(BC_2, \mathbb{F}_2) \ .$$

So the group (5.29) is a countably infinite sum of copies of $\mathbb{F}_2$. On the other hand,

$$\pi_0^{C_2}\left(\prod_{i<0} L(\Sigma^i H\mathbb{F}_2)\right) \ \cong \ \prod_{i<0} \pi_0^{C_2}(L(\Sigma^i H\mathbb{F}_2)) \ \cong \ \prod_{i<0} \pi_0^e\left(BC_2^+ \wedge \Sigma^i H\mathbb{F}_2\right) \ \cong \ \prod_{i<0} H_{-i}(BC_2, H\mathbb{F}_2) \ ,$$

again by the split isotropy separation sequence. This is an infinite product of copies of $\mathbb{F}_2$, so the canonical map

$$L\left(\prod_{i<0} \Sigma^i H\mathbb{F}_2\right) \ \longrightarrow \ \prod_{i<0} L(\Sigma^i H\mathbb{F}_2)$$

is not a global equivalence.

6. Global versus G-equivariant stable homotopy

In this section we fix a compact Lie group G and relate the global homotopy category to the G-equivariant stable homotopy category. Our notion of equivariant homotopy groups, and hence the entire global theory, is based on complete universes, so the natural target for the comparison is the G-equivariant stable homotopy indexed on a complete G-universe. There are various models for this homotopy category around, for example [**53, 58**]. The model that is most convenient for our purposes is the category of orthogonal G-spectra with the stable '$\mathbb{S}$-model structure' established by Stolz [**89**, Thm. 2.3.27].

Definition 6.1. An *orthogonal G-spectrum* is an orthogonal spectrum equipped with a continuous G-action through automorphisms of orthogonal spectra.

We write $G\mathcal{S}p$ for the category of orthogonal G-spectra and G-equivariant morphisms. If we unravel the definitions, we obtain that an orthogonal G-spectrum consists of pointed spaces X_n for $n \geq 0$, a based left $O(n) \times G$-action on X_n and based structure maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ that are G-equivariant with respect to the given G-actions on X_n and X_{n+1} and the trivial G-action on the sphere S^1. Of course, this data is again subject to the condition that the iterated structure maps $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$ are $O(n) \times O(m)$-equivariant. The iterated structure map σ^m is then automatically G-equivariant with respect to the given G-actions on X_n and X_{n+m} and the trivial G-action on S^m.

Readers familiar with other accounts of equivariant stable homotopy theory may wonder immediately why no orthogonal representations of the group G show up in the definition of equivariant spectra (as for example in [**53, 58**]). The reason is that they are secretly already present: just as we can evaluate an orthogonal spectrum on a G-representation, we can more generally evaluate an orthogonal G-spectrum on a G-representation.

For every orthogonal G-spectrum X and every G-representation V of dimension n we define $X(V)$, the *value of* X *on* V, as

$$X(V) \ = \ \mathbf{L}(\mathbb{R}^n, V)^+ \wedge_{O(n)} X_n \ . \tag{6.2}$$

In other words: as far as the underlying space $X(V)$ is concerned we forget the G-action on X and V and employ the same procedure as for orthogonal spectra in (1.3) of Chapter III. Now we let G act on $X(V)$ diagonally, through its actions on X_n *and* on V: for $g \in G$, $\varphi \in \mathbf{L}(\mathbb{R}^n, V)$ and $x \in X_n$ we set

$$g \cdot [\varphi, x] \ = \ [l_g \circ \varphi, gx] \ ,$$

where $l_g : V \longrightarrow V$ is is left multiplication by G. Again generalized structure maps $\sigma_{V,W} : X(V) \wedge S^W \longrightarrow X(V \oplus W)$, for G-representations V and W, are equivariant. So the definition of the 0-th equivariant homotopy group

$$\pi_0^G X \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \, [S^V, X(V)]^G \ , \tag{6.3}$$

makes just as much sense as in Section III.2. Again, for an arbitrary integer k, the k-th equivariant homotopy group $\pi_k^G X$ is the 0-th homotopy group of a suitably looped or suspended spectrum, as in (2.8) of Chapter III.

Remark 6.4. Let us clarify the relationship between our current definition of an orthogonal G-spectrum and the one used by Mandell and May in [**58**]. The two concepts are *not* the same, but the two categories are equivalent – in fact, the equivalence of categories appears explicitly (albeit in somewhat intransparent notation) in [**58**, V, Thm. 1.5]. Moreover, under this equivalence of categories, our present notion of equivariant homotopy groups corresponds to that of [**58**, III Def. 3.2].

For us, an orthogonal G-spectrum is simply an orthogonal spectrum with action by the group G; let us denote, for the course of this remark, the category of orthogonal spectra with G-action by G-$\mathcal{S}p^O$. in particular, our equivariant spectra do not initially assign values to general G-representations, but they can be defined by the formula (6.2).

The definition of an orthogonal G-spectrum used by Mandell and May refers to a universe U, i.e., a certain infinite dimensional real inner product space with G-action by linear isometries. However, one upshot of this discussion is that, up to equivalence of categories, the equivariant orthogonal spectra of [**58**] are nevertheless independent of the universe. Mandell and May denote by $\mathscr{V}(U)$ the class of all finite dimensional G-representations that admit a G-equivariant, isometric embedding into the universe U. An $\mathscr{I}_G$-spectrum Y, or orthogonal G-spectrum, in the sense of [**58**, II Def. 2.6], consists of the following data:

(i) a based G-space $Y(V)$ for every G-representation V in the class $\mathscr{V}(U)$,

(ii) a continuous based G-map

$$\mathbf{L}(V, W)^+ \wedge Y(V) \ \longrightarrow \ Y(W) \tag{6.5}$$

for every pair of G-representations V and W in $\mathscr{V}(U)$ of the same dimension (where Mandell and May write $\mathscr{I}_G^{\mathscr{V}}(V, W)$ for $\mathbf{L}(V, W)$),

(iii) continuous based G-maps

$$\sigma_{V,W} \ : \ Y(V) \wedge S^W \ \longrightarrow \ Y(V \oplus W)$$

for all pairs of G-representation V and W in $\mathscr{V}(U)$.

This data is subject to the following conditions:

- the action maps (6.5) of the isometries on the values of Y have to be unital and associative;

- the action maps (6.5) of the isometries on the values of Y and on representations spheres have to be compatible with the structure maps $\sigma_{V,W}$, i.e., the squares

$$\begin{tikzcd} \mathbf{L}(V,V')^+ \wedge \mathbf{L}(W,W')^+ \wedge Y(V) \wedge S^W \arrow[r, "\oplus\wedge\sigma_{V,W}"] \arrow[d] & \mathbf{L}(V\oplus W, V'\oplus W')^+ \wedge Y(V\oplus W) \arrow[d] \\ Y(V') \wedge S^{W'} \arrow[r, "\sigma_{V',W'}"'] & Y(V'\oplus W') \end{tikzcd} \tag{6.6}$$

  commute.
- the morphism $\sigma_{V,0} : Y(V) \wedge S^0 \longrightarrow Y(V \oplus 0)$ is the composite of the natural isomorphisms $Y(V) \wedge S^0 \cong Y(V)$ and $Y(V) \cong Y(V \oplus 0)$, and am certain associativity diagram commutes.

A morphism of $f : Y \longrightarrow Z$ of $\mathscr{I}_G$-spectra consists of a based continuous G-map $f(V) : Y(V) \longrightarrow Z(V)$ for every V in $\mathscr{V}(U)$, strictly compatible with the action by the isometries and the structure maps $\sigma_{V,W}$. We denote the category of $\mathscr{I}_G$-spectra by $\mathscr{I}_G$-$\mathcal{S}p$.

The definition of $\mathscr{I}_G$-spectra above can be cast into an isomorphic, but more compact form, as enriched functors on a topological G-category $\mathscr{J}_G$, compare Theorem II.4.3 of [**58**]. In the formulation as enriched functors on $\mathscr{J}_G$, the structure on the collection of G-spaces $Y(V)$ consists of continuous based G-map

$$\mathscr{J}_G^{\mathscr{V}}(V,W) \wedge Y(V) \longrightarrow Y(W)$$

for every pair of G-representations V and W in $\mathscr{V}(U)$ (of possibly different dimensions), where $\mathscr{J}_G^{\mathscr{V}}(V,W)$ is the Thom G-space of the orthogonal complement bundle over the G-space $\mathbf{L}(V,W)$. This formulation combines the actions (6.5) of the linear isometries and the structure maps $\sigma_{V,W}$ into a single piece of structure, and also simplifies the compatibility conditions.

The inverse equivalences of categories

$$G\mathcal{S}p \underset{\mathbb{U}}{\overset{\mathbb{P}}{\rightleftarrows}} \mathscr{I}_G\mathcal{S}p$$

are given by

- forgetting the values on all representations *except* at the inner product spaces $\mathbb{R}^n$ with trivial G-action, respectively
- creating the values on arbitrary G-representations via the formula (6.2).

A functor

$$\tau : \mathcal{S}p \longrightarrow G\mathcal{S}p$$

from orthogonal spectra to orthogonal G-spectra is given by endowing an orthogonal spectrum with the trivial G-action. For orthogonal G-spectra in the image of this functor, equivariant homotopy groups (as defined in (6.3)) specialize to the equivariant homotopy groups of Section III.2. So every global equivalence of orthogonal spectra becomes G-equivalence after applying τ, i.e., when viewed as a morphism of orthogonal G-spectra (between G-spectra with trivial G-action).

We denote by G-$\mathcal{SH}$ the *G-equivariant stable homotopy category*, i.e., any localization of the category $G\mathcal{S}p$ of orthogonal G-spectra at the class of π_*-isomorphisms. Since the trivial action functor takes global equivalences of orthogonal spectra to G-equivalence of orthogonal G-spectra, we get a 'forgetful' functor on the homotopy categories

$$U = U_G : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$$

form the universal property of localizations. We issue a warning: the functor τ is fully faithful on the pointset level, but its homotopical 'derived analog' U is typically *not* fully faithful. A hint is the fact that the equivariant homotopy groups of a global homotopy type, restricted to G and its subgroups, have more structure than is available for a general G-homotopy type, and satisfy certain relations that do not hold for general orthogonal G-spectra, compare Remark III.2.37.

We shall see that the forgetful functor U has adjoints on either side, by Theorem 6.9 below. Moreover, U is canonically an exact functor of triangulated categories. The categories $\mathcal{GH}$ and G-SH have infinite sums and infinite products, and the forgetful functor preserves both. A convenient way to show the existence of an adjoint to a derived functor is by exhibiting suitable model structures for which the original functor is a (left or right) Quillen functor. To make this work in our situation, we appeal to the *stable* $\mathbb{S}$-*model structure* on orthogonal G-spectra established by Stolz [**89**, Thm. 2.3.27]. Stolz' model structure on orthogonal G-spectra is different from, but Quillen equivalent to, the model structure established by Mandell and May in [**58**, III Thm. 4.2]. The two model structures share the same equivalences (the equivariant π_*-isomorphisms), and the relationship between Stolz' model structure and the Mandell-May model structure is a lot like the relationship between our flat $\mathcal{F}$-global model structure and the projective $\mathcal{F}$-global model structure.

We need a characterization of the $\mathbb{S}$-cofibrations by a latching condition that does not appear explicitly in [**89**]. Since an orthogonal G-spectrum is an orthogonal spectrum with G-action, the latching spaces $L_m A$ of the underlying orthogonal spectrum (compare Construction 1.1) inherits a continuous G-action. So we can, and will, view $L_m A$ as an $O(m) \times G$-space, and then the latching map $\nu_m : L_m A \longrightarrow A_m$ is $(O(m) \times G)$-equivariant. Similarly, for a morphism $f : A \longrightarrow B$ of orthogonal G-spectra, the latching map $\nu_m f : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is $(O(m) \times G)$-equivariant.

Lemma 6.7. *A morphism $f : A \longrightarrow B$ of orthogonal G-spectra is an $\mathbb{S}$-cofibration in the sense of* [**89**, Def. 2.3.4] *if and only if for every $m \geq 0$ the latching map $\nu_m f : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is an $(O(m) \times G)$-cofibration.*

Proof. Following [**89**, Remark 2.3.3] we denote by $\mathcal{F}ree_m$ the family of those subgroups $\Gamma \leq O(m) \times G$ such that $\Gamma \cap (O(m) \times 1)$ consists only of the unit element. Equivalently, Γ belongs to $\mathcal{F}ree_m$ if and only if it is the graph of a homomorphism to $O(m)$ defined on some subgroup of G.

We let P denote the class of morphisms $\varphi : X \longrightarrow Y$ of orthogonal G-spectra such that for every $m \geq 0$ the morphism $\varphi_m : X_m \longrightarrow Y_m$ is an acyclic fibration in the $\mathcal{F}ree$-flat model structure on $O(m) \times G$-spaces, in the sense of Proposition 1.28. The $\mathbb{S}$-cofibrations are defined in [**89**, Def. 2.3.4] by the left lifting property with respect to the class of morphisms P. By the analog of Proposition 1.7 in the category of orthogonal G-spectra (as opposed to the category of orthogonal spectra), this left lifting property for a morphism $f : A \longrightarrow B$ is equivalent to the requirement that for every $m \geq 0$ the latching map $\nu_m f : A_m \cup_{L_m A} L_m B \longrightarrow B_m$ is cofibration in $\mathcal{F}ree$-flat model structure on $O(m) \times G$-spaces. But the cofibrations in this model structure are precisely the $(O(m) \times G)$-cofibrations, so this proves the lemma. □

Corollary 6.8. *Let $f : A \longrightarrow B$ be a morphism of orthogonal spectra and G a compact Lie group. Then the following are equivalent.*

(i) *The morphism f is a flat cofibration.*
(ii) *The morphism $\tau f : \tau A \longrightarrow \tau B$ between orthogonal G-spectra with trivial G-action is an $\mathbb{S}$-cofibration.*

Theorem 2.3.27 of [**89**] shows that the $\mathbb{S}$-cofibrations and π_*-isomorphisms for a stable, cofibrantly generated and proper model structure on the category of orthogonal G-spectra. The fibrant objects in this model structure are those orthogonal G-spectra X that satisfy the following two conditions:

(a) X is a 'strong G-Ω-spectrum', i.e., for every subgroup H of G and every pair of G-representations V and W the adjoint structure map $\tilde{\sigma}_{V,W} : X(V) \longrightarrow \mathrm{map}(S^W, X(V \oplus W))$ is an H-weak equivalence.
(b) Let $\mathcal{F}ree_m$ be the family of those subgroups $\Gamma \leq O(m) \times G$ that intersect $O(m) \times 1$ only in the unit element, and let $E\mathcal{F}ree_m$ be a universal $O(m) \times G$-space for this family. Then for every $m \geq 0$ the map
$$X_m \longrightarrow \mathrm{map}(E\mathcal{F}, X_m)$$
adjoint to the projection is a $G \times O(m)$-weak equivalence.

The 'equivariant' smash product of orthogonal G-spectra is simply the smash product of the underlying non-equivariant orthogonal spectra with diagonal G-action. So the trivial action functor $\tau : \mathcal{S}p \longrightarrow G\mathcal{S}p$

is strong symmetric monoidal. The smash product of orthogonal spectra and of orthogonal G-spectra can be derived to symmetric monoidal products on $\mathcal{GH}$ and on G-$\mathcal{SH}$ (see Corollary 3.8). The forgetful functor is strongly monoidal with respect to these derived smash products. Indeed, the derived smash product in $\mathcal{GH}$ can be calculated by flat approximation up to global equivalence; by Corollary 6.8, a flat orthogonal spectrum, endowed with trivial G-action, is cofibrant in the monoidal stable $\mathbb{S}$-model structure on orthogonal G-spectra.

We will show now that the forgetful functor has both a left and a right adjoint. When $G = e$ is a trivial group, this reduces to the change of family functor of Theorem 5.2, with $\mathcal{E} = \mathcal{A}ll$ and $\mathcal{F} = \langle e \rangle$.

Theorem 6.9. *Let G be a compact Lie group.*

(i) *The forgetful functor*
$$U \ : \ \mathcal{GH} \ \longrightarrow \ G\text{-}\mathcal{SH}$$
has a left adjoint and a right adjoint. The left adjoint has a preferred lax symmetric comonoidal structure. The right adjoint has a preferred lax symmetric monoidal structure.

(ii) *For every G-Ω-spectrum Z that is fibrant in the flat stable model structure, the categorical fixed point spectrum Z^G is a global Ω-spectrum.*

Proof. (i) The trivial action functor $\tau : \mathcal{S}p \longrightarrow G\mathcal{S}p$ takes flat cofibrations to $\mathbb{S}$-cofibrations (Corollary 6.8), and it takes global equivalences to π_*-isomorphisms of orthogonal G-spectra. So τ is a left Quillen functor from the global model structure on orthogonal spectra to the stable $\mathbb{S}$-model structure on orthogonal G-spectra. The existence of the derived right adjoint on the level of homotopy categories is now a formal consequence. Also, the same formal argument as in part (ii) of Theorem 5.2 shows how the lax monoidal structure of the right adjoint R is obtained from the strong monoidal structure of the forgetful functor U.

The existence of the left adjoint is slightly less direct, because the trivial action functor is *not* a right Quillen functor. Indeed, if it were a right Quillen functor, then it would preserve fibrant objects. However, a global Ω-spectrum is typically *not* a G-Ω-spectrum when given the trivial G-action.

However, a global Ω-spectrum is 'eventually' a G-Ω-spectrum, i.e., starting at faithful representations. This allows us to modify τ into a right Quillen functor as follows. We choose a faithful G-representation V and let
$$\Omega^V \operatorname{sh}^V \ : \ \mathcal{S}p \ \longrightarrow \ G\mathcal{S}p$$
denote the functor that takes an orthogonal spectrum X to the orthogonal G-spectrum $\Omega^V \operatorname{sh}^V X$ with n-th level
$$(\Omega^V \operatorname{sh}^V X)_n \ = \ \operatorname{map}(S^V, X(\mathbb{R}^n \oplus V)) \ .$$
We emphasize that th G-action on $\Omega^V \operatorname{sh}^V X$ is non-trivial, despite the fact that we started with an orthogonal spectrum without a G-action. For every orthogonal spectrum X, the maps
$$X_n \ \longrightarrow \ \operatorname{map}(S^V, X(\mathbb{R}^n \oplus V)) \ = \ (\Omega^V \operatorname{sh}^V X)_n$$
adjoint to the generalized structure maps
$$\sigma_{\mathbb{R}^n,V} \ : \ X_n \wedge S^V \ = \ X(\mathbb{R}^n) \wedge S^V \ \longrightarrow \ X(\mathbb{R}^n \oplus V)$$
form a morphism of orthogonal G-spectra $\eta_X \ : \ X \ \longrightarrow \ \Omega^V \operatorname{sh}^V X$ that is a natural G-equivalence. In particular, the functor $\Omega^V \operatorname{sh}^V$ also takes global equivalences of orthogonal spectra to G-equivalences of orthogonal G-spectra, and the derived functor of $\Omega^V \operatorname{sh}^V$ is naturally isomorphic to the forgetful functor $U : \mathcal{GH} \longrightarrow G\mathcal{SH}$.

We claim that the functor $\Omega^V \operatorname{sh}^V$ is a right Quillen functor from the global model structure on orthogonal spectra to the flat stable model structure on orthogonal G-spectra. [show] So again by general model category theory, the derived functor of $\Omega^V \operatorname{sh}^V$, and hence also the forgetful functor U, has a left adjoint. The same formal argument as in part (ii) of Theorem 5.2 shows how turn the strong monoidal structure of the forgetful functor U into a lax comonoidal structure $L(A \wedge^{\mathbb{L}} B) \longrightarrow (LA) \wedge^{\mathbb{L}} (LB)$ of the left adjoint. In contrast to Theorem 5.2 (iii), however, this morphism is usually *not* and isomorphism, so we cannot turn it around into a monoidal structure on L.

(ii) As we argued in part (i), the trivial action functor is a left Quillen functor from the global model structure on orthogonal spectra to the flat stable model structure on orthogonal G-spectra. So its right adjoint, the categorical fixed point functor

$$(-)^G \ : \ G\mathcal{S}p \ \longrightarrow \ \mathcal{S}p$$

is a right Quillen functor in the other direction. In particular, taking categorical fixed points takes fibrant objects to fibrant objects. Thee fibrant objects in the global model structure of orthogonal spectra are precisely the global Ω-spectra, so the claim follows. □

Remark 6.10. We proved part (ii) of the previous theorem in a model categorical way, but it can also be proved directly from the characterization of the fibrant objects in the flat stable model structure on orthogonal G-spectra. [fill in]

Remark 6.11. Theorem 6.9 looks similar to the change-of-family Theorem 5.2, but there is one important difference: if the group G is non-trivial, then neither of the two adjoints to the forgetful functor $U : \mathcal{GH} \longrightarrow G\text{-}\mathcal{SH}$ is fully faithful.

The left adjoint $L : G\text{-}\mathcal{SH} \longrightarrow \mathcal{GH}$ is an exact functor of triangulated categories that preserves infinite sums. The G-equivariant stable homotopy category is compactly generated by the unreduced suspension spectra of all the coset spaces G/H, for all subgroups H of G. So L is essentially determined by its values on these generators. The sequence of bijections

$$\mathcal{GH}(L(\Sigma^\infty_+ G/H), X) \ \cong \ G\text{-}\mathcal{SH}(\Sigma^\infty_+ G/H, UX) \ \cong \ \pi_0^H X \ \cong \ \mathcal{GH}(\Sigma^\infty_+ B_{\mathrm{gl}} H, X)$$

shows that the left adjoint L takes the unreduced suspension spectrum of the coset space G/H to the suspension spectrum of the global classifying space of H. In the special case $H = G$ the spectrum $\Sigma^\infty_+ G/H$ is the equivariant sphere spectrum $\mathbb{S}_G$, and we obtain that

$$L(\mathbb{S}_G) \ \cong \ \Sigma^\infty_+ B_{\mathrm{gl}} G \ .$$

Now

$$G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \ \cong \ \pi_0^G \mathbb{S} \ \cong \ A(G)$$

is the Burnside ring, whereas

$$\mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G)) \ \cong \ \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G) \ \cong \ \mathbf{A}(G, G)$$

is the double Burnside ring. Then map $L : G\text{-}\mathcal{SH}(\mathbb{S}_G, \mathbb{S}_G) \longrightarrow \mathcal{GH}(L(\mathbb{S}_G), L(\mathbb{S}_G))$ corresponds to the ring homomorphism from the Burnside ring to the double Burnside ring; this ring map is never surjective unless G is trivial, so the left adjoint is not full.

Theorem 6.9 constructs a right adjoint $R : G\text{-}\mathcal{SH} \longrightarrow \mathcal{GH}$ to the forgetful functor, and this right adjoint assigns a global homotopy type to every G-homotopy type. An obvious question is how to describe the K-equivariant cohomology theory represented by RZ, for another compact Lie group K, in terms of the G-cohomology theory represented by Z. When G is the trivial group, the right adjoint R specializes to the change of family functor $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ and Example 5.26 identifies RZ as the global Borel theory associated to the cohomology theory represented by Z. For general G the answer is a 'relative' version of a global Borel theory.

Proposition 6.12. *Let G and K be compact Lie groups and Z be an orthogonal G-spectrum. Then for every based K-CW-complex,*

$$(RZ)^*_K(A) \ \cong \ Z^*_G(E(G,K)^+ \wedge_K A) \ ,$$

where $E(G,K)$ is a universal space for (G,K)-bundles, i.e., a universal space for the family of those subgroups of $G \times K$ that intersect $1 \times K$ only in the unit element.

Remark 6.13. The discussion in this section could be done relative to a global family $\mathcal{F}$, as long as $\mathcal{F}$ contains the compact Lie group G under consideration (and hence also all its subgroups). Indeed, if $\mathcal{F}$ contains G, then every $\mathcal{F}$-equivalence of orthogonal spectra is a G-equivalence with respect to the trivial G-actions. Hence the trivial action functor descends to a 'forgetful' functor on the homotopy categories

$$U_G^{\mathcal{F}} \ : \ \mathcal{GH}_{\mathcal{F}} \ \longrightarrow \ G\text{-}\mathcal{SH}$$

by the universal property of localizations. The same arguments as in Theorem 6.9 show the existence of both adjoints to this forgetful functor, with the same kind of monoidal properties.

Theorem 6.9 discusses the maximal case of the global family $\mathcal{F} = \mathcal{A}ll$ of all compact Lie groups. The minimal case is the global family $\langle G \rangle$ generated by G, i.e., the class of compact Lie groups that are isomorphic to a quotient of a closed subgroup of G. All the forgetful functors $U_G^{\mathcal{F}}$ then factor as composites

$$\mathcal{GH}_{\mathcal{F}} \ \xrightarrow{U_{\langle G \rangle}^{\mathcal{F}}} \ \mathcal{GH}_{\langle G \rangle} \ \xrightarrow{U_G^{\langle G \rangle}} \ G\text{-}\mathcal{SH}$$

of a change-of-family functor and a family-to-group functor. The various adjoints then compose accordingly.

We emphasize that whenever G is non-trivial, then the global homotopy category $\mathcal{GH}_{\langle G \rangle}$ associated to the global family generated by G is different from the G-equivariant stable homotopy category G-$\mathcal{SH}$. In other words, if G is non-trivial, then the forgetful family-to-group functor $U_G^{\langle G \rangle} : \mathcal{GH}_{\langle G \rangle} \longrightarrow G\text{-}\mathcal{SH}$ is *not* an equivalence, and neither of its adjoints is fully faithful.

7. Rational finite global homotopy theory

In this section we discuss another model for the unstable $\mathcal{F}in$-global homotopy theory, i.e., unstable global homotopy theory relative to the global family $\mathcal{F}in$ of finite groups. This alternative model is given by the category of $\mathcal{I}$-spaces, i.e., functors on the category $\mathcal{I}$ of finite sets and injections. Just like the category of orthogonal spaces, $\mathcal{I}$-spaces are much studied objects in homotopy theory, but previously the focus has always been as a model for non-equivariant homotopy theory.

The point here is not so much to have yet another model category that is Quillen equivalent orthogonal spaces with one of the $\mathcal{F}in$-global model structure. Rather, the emphasis is on new examples: the category $\mathcal{I}$ is 'smaller' and more combinatorial than the category $\mathbf{L}$, and every orthogonal space can be turned into an $\mathcal{I}$-space by precomposition with the linearization functor $\mathcal{I} \longrightarrow \mathbf{L}$. However, there are many interesting $\mathcal{I}$-spaces that do not admit functoriality in linear isometric embeddings, and these yield unstable $\mathcal{F}in$-global homotopy types that are not easily constructed as orthogonal spaces.

Definition 7.1 ($\mathcal{I}$-spaces)**.** We let $\mathcal{I}$ denote the category of finite sets and injective maps. An *$\mathcal{I}$-space* is a functor from the category $\mathcal{I}$ to the category $\mathbf{U}$ of spaces.

The category $\mathcal{I}$ and the category $\mathcal{I}$-spaces have also played a prominent role in homotopy theory [references]; the non-equivariant homotopy theory of $\mathcal{I}$-spaces has been laid out in great detail by Sagave and Schlichtkrull in [**66**].

The methods of Sections I.3 and I.4 can be adapted to construct a global model structure on the category of $\mathcal{I}$-spaces that is another model for the unstable $\mathcal{F}in$-global homotopy theory. We briefly indicate the main steps, resulting in an analog of the global model structure of orthogonal spaces Theorem I.4.3

The first step is the straightforward analog of the strong level model structure of Proposition I.3.12: A morphism $f : Y \longrightarrow Z$ of $\mathcal{I}$-spaces is a *strong level equivalence* (respectively *strong level fibration*) if for every finite set A the map $f(A) : Y(A) \longrightarrow Z(A)$ is a Σ_A-weak equivalence (respectively Σ_A-fibration). The strong level equivalences, strong level fibrations and the flat cofibrations of [**66**, Def. 3.9], (the direct analog of the flat cofibrations of orthogonal spaces) then form a 'strong level model structure' on the category of $\mathcal{I}$-spaces.

Bousfield localization with the obvious class of static $\mathcal{I}$-spaces as local objects then provides a global model structure on $\mathcal{I}$-spaces. In more detail: an $\mathcal{I}$-space Z is *static* if for every finite group G, every faithful finite G-set A and every other finite G-set B the structure map $Z(i_{A,B}) : Z(A) \longrightarrow Z(A \amalg B)$ is a G-weak equivalence. A morphism of $f : Y \longrightarrow Z$ of $\mathcal{I}$-spaces is a *global equivalence* if for some (hence any) flat approximation $f^\flat : Y^\flat \longrightarrow Z^\flat$ in the strong level model structure and every static $\mathcal{I}$-space X the induced map

$$[f^\flat, Z] \ : \ [Z^\flat, X] \ \longrightarrow \ [Y^\flat, X]$$

on homotopy classes of morphism is bijective; this definition should be compared with the characterization of global equivalences of orthogonal spaces in Corollary I.4.14.

A morphism $f : X \longrightarrow Y$ of $\mathcal{I}$-spaces is a *global fibration* if it is a strong level fibration and for every finite group G, every faithful finite G-set a and every other finite G-set B the square of G-fixed point spaces

$$\begin{tikzcd} X(A)^G \arrow[r, "X(i_{A,B})^G"] \arrow[d, "f(A)^G"'] & X(A \amalg B)^G \arrow[d, "f(A\amalg B)^G"] \\ Y(A)^G \arrow[r, "Y(i_{A,B})^G"'] & Y(A \amalg B)^G \end{tikzcd}$$

is homotopy cartesian.

Theorem 7.2 (Global model structure for $\mathcal{I}$-spaces)**.** *The global equivalences, global fibrations and flat cofibrations form the* global model structure *on the category of $\mathcal{I}$-spaces. The fibrant objects in the global model structure are the static $\mathcal{I}$-spaces. The global model structure is proper, topological, compactly generated, and it satisfies the pushout product and monoid axioms with respect to the box product of $\mathcal{I}$-spaces.*

The free $\mathbb{R}$-vector space $\mathbb{R}A$ generated by a finite set A gets an inner product by declaring the basis A to be orthonormal. Injective maps then linearize to isometric embeddings, i.e., linearization is a functor $\mathbb{R}\langle - \rangle : \mathcal{I} \longrightarrow \mathbf{L}$. Precomposition with the functor then defines a forgetful functor

$$U \ : \ spc \ \longrightarrow \ \mathcal{I}\mathbf{U} \ , \ X \ \longmapsto \ X \circ \mathbb{R}\{-\}$$

from orthogonal spaces to $\mathcal{I}$-spaces. This forgetful functor has left adjoint L (and also a right adjoint), given by an enriched Kan extension. Now we can make precise in which way the global homotopy theory of $\mathcal{I}$-spaces model $\mathcal{F}in$-global homotopy theory:

Theorem 7.3. *The adjoint functor pair*

$$\begin{tikzcd} \mathcal{I}\mathbf{U} \arrow[r, shift left, "L"] & spc \arrow[l, shift left, "U"] \end{tikzcd}$$

is a Quillen equivalence between the global model structure on $\mathcal{I}$-spaces and the projective $\mathcal{F}in$-global model structure on orthogonal spaces.

We let $\mathcal{F}in$ denote the global family of finite groups. By the results of the previous sections, the associated global homotopy category $\mathcal{GH}_{\mathcal{F}in}$ indexed on finite groups is a compactly generated triangulated category with a symmetric monoidal derived smash product. We call an object X of the category $\mathcal{GH}_{\mathcal{F}in}$ *rational* if the equivariant homotopy groups $\pi_k^G(X)$ are uniquely divisible (i.e., $\mathbb{Q}$-vector spaces) for all finite groups G. In this section we will give an algebraic model of *rational global stable homotopy category* indexed on finite groups, i.e., the full subcategory $\mathcal{GH}_{\mathbb{Q}}^{\mathcal{F}in}$ of rational spectra in $\mathcal{GH}_{\mathcal{F}in}$. Theorem 7.5 below shows that the homotopy types in $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ are determined by a chain complex of global functors, up to quasi-isomorphism. More precisely, we construct an equivalence of triangulated categories from $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ to the unbounded derived category of rational global functors on finite groups.

We let G and K be compact Lie groups. We recall from Proposition III.3.13 that the evaluation map

$$\mathbf{A}(G, K) \ \longrightarrow \ \pi_0^K(\Sigma^\infty_+ B_{\mathrm{gl}} G) \ , \quad \tau \longmapsto \tau(e_G)$$

is an isomorphism, where $e_G \in \pi_0^G(\Sigma_+^\infty B_{\mathrm{gl}} G)$ is the stable tautological class. (More precisely, the definition of the global classifying space $B_{\mathrm{gl}} G$ involves an implicit choice of faithful G-representation V that is omitted from the notation, and e_G is the class denote $e_{G,V}$ in (3.10) of Chapter III). Like for every suspension spectrum, the group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}} G)$ is trivial for $k < 0$.

Proposition 7.4. *Let G and K be finite groups. Then for every $k > 0$, the equivariant homotopy group $\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}} G)$ is finite.*

Proof. The tom Dieck splitting [**95**, Satz 2] provides an isomorphism

$$\pi_k^K(\Sigma_+^\infty B_{\mathrm{gl}} G) \;\cong\; \bigoplus_{(L)} \pi_k^{WL}\left(\Sigma_+^\infty (EWL \times B(K,G)^L)\right) \ ,$$

where the sum is indexed over all conjugacy classes of subgroups L of K and $WL = W_K L$ is the Weyl group of L in K. We have also used the fact that the underlying K-space $(B_{\mathrm{gl}} G)(\mathcal{U}_K)$ of the global classifying space of G is a classifying K-space $B(K,G)$ for (K,G)-bundles, compare Proposition 2.6 and Remark 2.8 of Chapter I. Since K is finite, there are only finitely many conjugacy classes of subgroups, so we may show that each summand of the tom Dieck splitting is finite.

We fix a fixed subgroup L of K. Since the action of WL on EWL, and hence on $EWL \times B(K,G)^L$ is free, the WL-equivariant stable homotopy groups of $EWL \times B(K,G)^L$ are isomorphic to the non-equivariant stable homotopy groups of the orbit space [ref?]:

$$\pi_k^{WL}\left(\Sigma_+^\infty (EWL \times B(K,G)^L)\right) \;\cong\; \pi_k\left(\Sigma_+^\infty (EWL \times_{WL} B(K,G)^L)\right) \ .$$

Since the group WL is finite and $B(K,G)$ is a K-CW-complex of finite type, and so the space $EWL \times_{WL} B(K,G)^L$ is a CW-complex of finite type. Hence every stable homotopy group of $EWL \times_{WL} B(K,G)^L$ is finitely generated.

Proposition I.2.6 (iii) identifies the L-fixed points of the K-space $B(K,G)$ as

$$B(K,G)^L \;\simeq \coprod_{[\alpha]\in \mathrm{Rep}(L,G)} BC_G(\alpha) \ ,$$

where the disjoint union is indexed by conjugacy classes of homomorphisms from L to G and $C_G(\alpha)$ is the centralizer of the image of $\alpha : L \longrightarrow G$. Since G is finite, so are all the centralizers $C_G(\alpha)$, and we conclude that the rational homology of the space $B(K,G)^L$ vanishes in positive dimensions. Since the group WL is finite, the rational homology of the homotopy orbit space $EWL \times_{WL} B(K,G)^L$ is isomorphic to quotient of the rational homology of $B(K,G)^L$ by the WL-action. So the rational homology of the space $EWL \times_{WL} B(K,G)^L$ also vanishes in positive dimensions. Rationally, stable homotopy groups are isomorphic to homology groups, so the rationalized stable homotopy groups of the unreduced suspension spectrum of $EWL \times_{WL} B(K,G)^L$ all vanish in positive dimensions. Since these stable homotopy groups are also finitely generated, they must be finite. □

Now we can establish an algebraic model for the rational $\mathcal{F}in$-global homotopy category.

Theorem 7.5. *There is a chain of Quillen equivalences between the category of orthogonal spectra with the rational $\mathcal{F}in$-global model structure and the category of chain complexes of rational global functors on finite groups. In particular, this induces an equivalence of triangulated categories*

$$\mathcal{GH}_{\mathcal{F}in}^{\mathbb{Q}} \;\longrightarrow\; \mathcal{D}\left(\mathcal{GF}_{\mathcal{F}in}^{\mathbb{Q}}\right) \ .$$

The equivalence can be chosen so that the homotopy group global functor on the left hand side correspond to the homology group global functor on the right hand side.

Proof. We prove this as a special case of the 'generalized tilting theorem' of Brooke Shipley and the author. Indeed, by Proposition 4.2 suspensions spectra of the global classifying spaces $B_{\mathrm{gl}} G$ are compact generators of the global homotopy category $\mathcal{GH}_{\mathcal{F}in}$ as G varies through all finite groups. So the rationalizations $(\Sigma^\infty_+ B_{\mathrm{gl}}G)_{\mathbb{Q}}$ are compact generators of the rational global homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$. If k is any integer, then the morphism vector spaces between two such objects are given by

$$\begin{aligned}\llbracket(\Sigma^\infty_+ B_{\mathrm{gl}}K)_{\mathbb{Q}},\ (\Sigma^\infty_+ B_{\mathrm{gl}}G)_{\mathbb{Q}}\rrbracket_k \ &\cong\ \pi_k^K((\Sigma^\infty_+ B_{\mathrm{gl}}G)_{\mathbb{Q}})\ \cong\ \mathbb{Q}\otimes\pi_k^K(\Sigma^\infty_+ B_{\mathrm{gl}}G)\\ &\cong\ \begin{cases}\mathbb{Q}\otimes\mathbf{A}(G,K) & \text{for } k=0, \text{ and}\\ \quad 0 & \text{for } k\neq 0.\end{cases}\end{aligned}$$

Here we have used Proposition 7.4.

The rational $\mathcal{F}in$-global model structure on orthogonal spectra is simplicial, compactly generated, proper and stable; so we can apply the Tilting Theorem [**73**, Thm. 5.1.1]. This theorem yields a chain of Quillen equivalences between orthogonal spectra in the rational $\mathcal{F}in$-global model structure and the category of chain complexes of $\mathbb{Q}\otimes\mathbf{A}_{\mathcal{F}in}$-modules, i.e., additive functors from the rationalized Burnside category $\mathbb{Q}\otimes\mathbf{A}_{\mathcal{F}in}$ to abelian groups. This functor category is equivalent to the category of additive functors from $\mathbf{A}_{\mathcal{F}in}$ to $\mathbb{Q}$-vector spaces, and this proves the theorem. $\square$

Remark 7.6. There is an important homological difference between global functors on finite groups and Mackey functors for one fixed finite group. Indeed, for a finite group G, the rationalized Burnside ring $\mathbb{Q}\otimes A(G)$ has plenty of idempotents that can be used to split a rational Mackey functor for the group G into smaller pieces. The end result of this is that the category of rational Mackey functors over G is equivalent to a product, indexed over conjugacy classes (H) of subgroups of G, of the module categories over the rational groups rings $\mathbb{Q}[W_G H]$ of the Weyl group. In particular, the abelian category of rational G-Mackey functors is semisimple, every object is projective and injective and the derived category is equivalent, by taking homology, to the category of graded rational Mackey functors over G.

There is no analog of this for rational $\mathcal{F}in$-global functors. For example, the surjection

$$\mathbf{A}(e,-)\ \longrightarrow\ \underline{\mathbb{Z}}$$

from the Burnside ring global functor to the constant global functor for the group $\mathbb{Z}$ does *not* split rationally on finite groups. The new phenomenon is that any splitting would have to be natural for arbitrary restriction maps along homomorphisms $\alpha: K\longrightarrow G$ that are no necessarily injective.

Let us be even more specific. In the constant global functor $\underline{\mathbb{Z}}$ we have

$$2\cdot p^*(1)=\mathrm{tr}_e^{C_2}(1)\qquad\text{in}\qquad \underline{\mathbb{Z}}(C_2)=\mathbb{Z},$$

where $p: C_2\longrightarrow e$ is the unique group homomorphism. So for any morphism of global functors $\varphi:\underline{\mathbb{Z}}\longrightarrow N$ the image $\varphi(e)(1)$ of the unit element under the map $\varphi(e):\mathbb{Z}\longrightarrow N(e)$ must satisfy

$$\mathrm{tr}_e^{C_2}(\varphi(e)(1))\ =\ 2\cdot p^*(\varphi(e)(1))\ .$$

But in the Burnside ring $A(e)$, and also in its rationalization, 0 is the only element x that satisfies $\mathrm{tr}_e^{C_2}(x)=2\cdot p^*(x)$; so every morphism of global functors from $\underline{\mathbb{Z}}$ to $\mathbb{Q}\otimes\mathbf{A}(e,-)$ is zero.

While the idempotents in the rational Burnside rings of finite groups do not suffice to make the category $\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}$ semisimple, they can still be used to 'get rid of transfers' and thereby replace $\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}$ by an equivalent, but simpler category. Let Out denote the category of finite groups and conjugacy classes of surjective group homomorphisms. To a $\mathcal{F}in$-global functor $F:\mathbf{A}_{\mathcal{F}in}\longrightarrow\mathcal{A}b$ we can associate a contravariant functor $\tau F:\mathrm{Out}^{\mathrm{op}}\longrightarrow\mathcal{A}b$, the *reduced functor* as follows. On objects we set

$$(\tau F)(G)\ =\ F(G)/tF(G)\ ,$$

the quotient of the group $F(G)$ by the subgroup $tF(G)$ generated by the images of all transfer maps $\mathrm{tr}_H^G: F(H)\longrightarrow F(G)$ for all proper subgroups $H\leq G$. If $\alpha: K\longrightarrow G$ is a surjective group homomorphism and $H\leq G$ a proper subgroup, then $L=\alpha^{-1}(H)$ is a proper subgroup of K and the relation

$$\alpha^*\circ\mathrm{tr}_H^G\ =\ \mathrm{tr}_L^K\circ(\alpha|_L)^*$$

as maps $\pi_0^H X \longrightarrow \pi_0^K X$ shows that the restriction map $\alpha^* : F(G) \longrightarrow F(K)$ passes to a homomorphism $\alpha^* : (\tau F)(G) \longrightarrow (\tau F)(K)$ of quotient groups. We will now recall that the reduction functor

$$\tau \ : \ \mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in} \ \longrightarrow \ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$$

is rationally an equivalence of categories, compare Proposition 7.10 below. By construction, the projection maps $F(G) \longrightarrow \tau F(G)$ form a natural transformation from the restriction of the global functor F to the category Out to τF.

Example 7.7. Let M be an abelian group and $\underline{M}$ the constant global functor with value M. This global functor is indeed constant with respect to all restriction maps, and the transfer maps tr_H^G are given by multiplication by the index of H in G. So we have $\tau\underline{M}(G) = M/cM$ where c is the greatest common divisor of indices of all proper subgroups of G. If G is not a p-group for any prime p, then this greatest common divisor is 1. If G is a non-trivial p-group, then G has a proper subgroup of index p. So we have

$$(\tau\underline{M})(G) \ = \ \begin{cases} M & \text{if } G = e, \\ M/pM & \text{if } G \text{ is a non-trivial } p\text{-group, and} \\ 0 & \text{else.} \end{cases}$$

The restriction maps in $\tau\underline{M}$ are quotient maps.

Example 7.8. The Burnside ring $\mathbb{A}(K) = \mathbf{A}(e, K)$ of a finite group K is freely generated, as an abelian group, by the transfers $\mathrm{tr}_L^K(1)$ where L runs through representatives of the conjugacy classes of subgroups of K. So $(\tau\mathbb{A})(K)$ is free abelian of rank 1, generated by the class of the multiplicative unit 1. All restriction maps preserve the unit, so the reduced functor $\tau\mathbb{A}$ of the global Burnside ring functor is isomorphic to the constant functor $\mathrm{Out}^{\mathrm{op}} \longrightarrow \mathcal{A}b$ with value $\mathbb{Z}$,

$$\tau\mathbb{A} \ \cong \ \underline{\mathbb{Z}} \ .$$

More generally we consider a compact Lie group G, and we will present $\tau(\mathbf{A}_G)$ as an explicit quotient of a sum of representable Out-functors. For every subgroup H of G the restriction map res_H^G is a morphism in $\mathbf{A}(G, H)$. The Yoneda lemma provides a unique morphism

$$\mathrm{Out}_H \ \longrightarrow \tau(\mathbf{A}_G)$$

of Out-functors from the representable functor $\mathrm{Out}_H = \mathrm{Out}(-, H)$ to $\tau(\mathbf{A}_G)$ that sends the identity of H to the class of res_H^G in $\tau(\mathbf{A}_G)(H)$. For every element $g \in G$ the conjugation isomorphism $c_g : {}^gH \longrightarrow H$ given by $c_g(\gamma) = g^{-1}\gamma g$ induces an isomorphism

$$c_g^* \circ - \ : \ \mathbf{A}_G(H) \ \longrightarrow \ \mathbf{A}_G({}^gH)$$

by postcomposition. We have

$$c_g^* \circ [\mathrm{res}_H^G] \ = \ [c_g^* \circ \mathrm{res}_H^G] \ = \ [\mathrm{res}_{{}^gH}^G \circ c_g^*] \ = \ [\mathrm{res}_{{}^gH}^G]$$

in $(\tau\mathbf{A}_G)({}^gH)$. The Yoneda lemma translates this into the fact that the triangle of Out-functors

$$\begin{tikzcd} \mathrm{Out}_H \arrow[d, "c_g\circ -"'] \arrow[dr] & \\ \mathrm{Out}_{{}^gH} \arrow[r] & \tau(\mathbf{A}_G) \end{tikzcd}$$

commutes. So the direct sum of the transformations $\mathrm{Out}_H \longrightarrow \tau\mathbf{A}_G$ factors over a natural transformation

$$\left(\bigoplus_{H \leq G} \mathrm{Out}_H \right) / G \ \longrightarrow \ \tau(\mathbf{A}_G) \ .$$

The source of this morphism can be rewritten if we choose representatives of the conjugacy classes of subgroups in H:

$$\bigoplus_{(H)} \mathrm{Out}_H / N_G H \ \longrightarrow\ \tau(\mathbf{A}_G)\ . \tag{7.9}$$

Now the sum is indexed by conjugacy classes of subgroups of G.

The abelian group $\mathbf{A}_G(K) = \mathbf{A}(G,K)$ is freely generated by the elements $\mathrm{tr}_L^K \circ \alpha^*$ where (L,α) runs through representatives of the conjugacy classes of pairs consisting of a subgroup L of K and a continuous homomorphism $\alpha : L \longrightarrow G$. So $(\tau \mathbf{A}_G)(K)$ is a free abelian group with basis the classes of α^* for all conjugacy classes of homomorphisms $\alpha : K \longrightarrow G$, modulo [...].

Proposition 7.10. *The functor*

$$\tau \ :\ \mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in} \ \longrightarrow\ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$$

is an equivalence of categories.

Proof. The functor $\tau : \mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in} \longrightarrow \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$ is additive and commutes with infinite sums. For every finite group G, the value $\tau(\mathbf{A}_G)$ on the representable global functor $\mathbf{A}_G$ is described by the isomorphism (7.9) as finite sum of quotients of representable Out-functors. The Weyl group $W_G H$ is finite, so rationally the quotient $\mathrm{Out}_H / N_G H$ is in fact a direct summand of the representable Out-functor Out_H. Since Out_H is a finitely presented projective object in the abelian category $\mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$ so is its summand $\mathrm{Out}_H / N_G H$. So rationally, $\tau(\mathbf{A}_G)$ is a finitely presented projective Out-functor.

The summand of $\tau(A_G)$ corresponding to $H = G$ in the decomposition (7.9) is the represented Out-functor Out_G. So if we let G run through a set of representatives of all isomorphism classes of finite groups, then the Out-functors $\tau(A_G)$ are a set of generators of the abelian category Out_G.

Now Morita theory kicks in: the additive functor $\tau : \mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in} \longrightarrow \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q})$ between abelian categories preserves infinite sums and takes the set of finitely presentable projective generators $\mathbf{A}_G$ of the source category (where G runs through representatives of all isomorphism classes of finite groups) to a set of finitely presentable projective generators of the target category. Every functor with these properties is an equivalence of categories, so for example □

Remark 7.11. Since global functors are a functor category and the functor

$$\tau \ :\ \mathcal{GF}_{\mathcal{F}in} \ \longrightarrow\ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathcal{A}b)$$

commutes with colimits, general nonsense implies that τ has a right adjoint

$$\rho \ :\ \mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathcal{A}b) \ \longrightarrow\ \mathcal{GF}_{\mathcal{F}in}\ .$$

Given an Out-functor Y, the value of the global functor ρY at a finite group G is given by

$$(\rho Y)(G) \ =\ \mathrm{Hom}_{\mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathcal{A}b)}(\tau(\mathbf{A}_G), Y)\ ,$$

the abelian group of natural transformations from the functor $\tau \mathbf{A}_G$ to Y. The adjoint functors (τ, ρ) are *not* equivalences integrally, but the unit and counit of the adjunctions become isomorphism after tensoring with $\mathbb{Q}$. So rationally, the right adjoint ρ also becomes an inverse to the equivalence τ.

The rational equivalence τ of abelian categories prolongs to an equivalence of derived categories by applying τ dimensionwise to chain complexes. The combination with the equivalence of triangulated categories of Theorem 7.5 is then a chain of two exact equivalence of triangulated categories

$$\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in} \ \xrightarrow{\ \cong\ }\ \mathcal{D}\left(\mathcal{GF}^{\mathbb{Q}}_{\mathcal{F}in}\right) \ \xrightarrow[\mathcal{D}(\tau)]{\ \cong\ }\ \mathcal{D}(\mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q}))\ . \tag{7.12}$$

The next proposition shows that this composite equivalence is an algebraic model for the geometric fixed point homotopy groups.

For every orthogonal spectrum X and every compact Lie group G, the geometric fixed point map $\Phi : \pi_0^G(X) \longrightarrow \Phi_0^G(X)$ (see (6.3) of Chapter III) annihilates all transfers from proper subgroups by Proposition III.6.5. So the geometric fixed point map factors over a homomorphism

$$\bar{\Phi} \ : \ \tau(\underline{\pi}_0(X))(G) \ \longrightarrow \ \Phi_0^G(X) \ .$$

The geometric fixed point maps are compatible with restriction along epimorphisms (Proposition III.6.7 (iv)), so as G varies among finite groups, the reduced geometric fixed point maps form a morphism of Out-functors.

Proposition 7.13. *For every orthogonal spectrum X, every finite group G and every integer k the map*

$$\bar{\Phi} \ : \ \tau(\underline{\pi}_k(X))(G) \ \longrightarrow \ \Phi_k^G(X)$$

of becomes an isomorphism after tensoring with $\mathbb{Q}$. So for varying finite groups G, these maps form a rational isomorphism of Out*-functors $\tau(\underline{\pi}_k(X)) \cong \underline{\Phi}_k(X)$.*

Proof. We let $\mathcal{X}$ be the class of orthogonal spectra for which the map

$$\mathbb{Q} \otimes \bar{\Phi} \ : \ \mathbb{Q} \otimes \tau(\underline{\pi}_*(X)) \ \longrightarrow \ \mathbb{Q} \otimes \underline{\Phi}_*(X)$$

of graded rationalized Out-functors is an isomorphism. We let K be a compact Lie group. The tom Dieck splitting implies that the reduced geometric fixed point map

$$\bar{\Phi} \ : \ \tau(\underline{\pi}_*(\Sigma_+^\infty B_{\mathrm{gl}}K)) \ \longrightarrow \ \underline{\Phi}_*(\Sigma_+^\infty B_{\mathrm{gl}}K)$$

is an isomorphism (even integrally), so the class $\mathcal{X}$ contains the suspension spectra of all global classifying spaces.

Equivariant homotopy groups and geometric fixed point homotopy groups take wedges to sums and the functor τ preserves sums. So the class $\mathcal{X}$ is closed and sums in the global homotopy category. Equivariant and geometric fixed point homotopy groups take distinguished triangles in $\mathcal{GH}$ to long exact sequences. The functor τ is *not* exact integrally, but it becomes exact after tensoring with $\mathbb{Q}$ (because it is rationally an equivalence of abelian categories). So $\mathcal{X}$ is a localizing subcategory of the global stable homotopy category. Corollary 4.4 (for the global family $\mathcal{A}ll$) then shows that $\mathcal{X}$ consists of all orthogonal spectra, $\square$

As a corollary we obtain that the combined equivalence κ of triangulated categories (7.12) from the rational finite global homotopy category $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$ to the derived category of the abelian category $\mathcal{D}(\mathcal{F}(\mathrm{Out}^{\mathrm{op}}, \mathbb{Q}))$ comes with a natural isomorphism

$$\Phi_*^G(X) \ \cong \ H_*(\kappa(X)) \ ,$$

for every object X of $\mathcal{GH}^{\mathbb{Q}}_{\mathcal{F}in}$, between the geometric fixed point homotopy groups and the homology Out-functors of $\kappa(X)$.

Example 7.14. As before we let $\mathbf{R}$ denote the complex representation ring global functor. Artin's theorem says that the representation ring $\mathbf{R}(G)$ is rationally generated by all representations induced from cyclic subgroups of G. So if G is not itself cyclic, then $\mathbf{R}_{\mathbb{Q}}(G) = \mathbb{Q} \otimes \mathbf{R}(G)$ is generated by transfers from proper subgroups, and hence $\tau \mathbf{R}_{\mathbb{Q}}(G) = 0$.

Now suppose that $G = C_n = \mathbb{Z}/n\mathbb{Z}$ is cyclic of order n with generator γ. We let z_n denote the representation of C_n on $\mathbb{C}$ where γ acts by multiplication by the primitive n-th root of unity $\zeta_n = e^{2\pi i/n}$. Then $\mathbf{R}(C_n) = \mathbb{Z}[z]/(z^n - 1)$. If $n = p^h$ for a prime p, we have we have

$$\mathrm{ind}_{p^{h-1}}^{p^h}(z^i_{p^{h-1}}) \ = \ z^i \cdot (1 + z^q + z^{2q} + \cdots + z^{(p-1)q}) \ ,$$

where $q = p^{h-1}$. So for $G = C_{p^h}$ the subgroup generated by transfers from proper subgroups is the ideal of $\mathbf{R}(C_n) = \mathbb{Z}[z]/(z^n - 1)$ generated by $1 + z^q + z^{2q} + \cdots + z^{(p-1)q}$. This factor ring is isomorphic to $\mathbb{Z}(\zeta_{p^h})$, the ring of integers in the cyclotomic number field $\mathbb{Q}(\zeta_{p^h})$. Now suppose that $n = kl$ where k and l are coprime. Then $C_n = C_k \times C_l$ for uniquely determined cyclic subgroups C_k and C_l of order k respectively l. So we have $\mathbf{R}(C_n) = \mathbf{R}(C_k \times C_l) \cong \mathbf{R}(C_k) \otimes \mathbf{R}(C_l)$, the isomorphism given by tensor product of representations. Moreover, the maximal subgroups of C_n are of the form $H \times C_l$ or $C_k \times H'$ for maximal subgroups H of C_k or H' of C_l. So

$$(\tau\mathbf{R}_{\mathbb{Q}})(C_n) \;=\; \tau\mathbf{R}_{\mathbb{Q}}(C_k \times C_l) \;\cong\; \tau\mathbf{R}_{\mathbb{Q}}(C_k) \otimes \tau\mathbf{R}_{\mathbb{Q}}(C_l)\ .$$

So altogether we conclude that

$$\tau\mathbf{R}_{\mathbb{Q}}(G) \;\cong\; \begin{cases} \mathbb{Q}(\zeta_n) & \text{if } G = C_n, \text{ and} \\ \quad 0 & \text{if } G \text{ is not cyclic.} \end{cases}$$

The isomorphism sends the primitive n-th root of unity ζ_n to the residue class of the representation z. The projection map $p : C_{ni} \longrightarrow C_n$ induces

$$p^* \;:\; \mathbb{Q}(\zeta_n) \;\longrightarrow\; \mathbb{Q}(\zeta_{ni})\ , \quad p^*(\zeta_n) = \zeta_{ni}^i\ .$$

The degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ is $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$, which is also the order of $\mathrm{Aut}(C_n) = \mathrm{Out}(C_n)$. So $(\tau\mathbf{R}_{\mathbb{Q}})(C_n)$ should be free of rank 1 over over the group ring $\mathbb{Q}[\mathrm{Out}(C_n)]$.

CHAPTER V

Ultra-commutative ring spectra

1. Power operations and global model structure

In Section II.1 we introduced power operations on the equivariant homotopy sets of commutative orthogonal monoid spaces. For every orthogonal ring spectrum R the orthogonal space $\Omega^\bullet R$ inherits a commutative multiplication, making it a commutative orthogonal monoid space (compare Remark III.4.12). Moreover, $\pi_0^G(\Omega^\bullet R) = \pi_0^G(R)$, so this endows the 0-th stable equivariant homotopy groups $\pi_0^G(X)$ with power operations

$$P^m \ : \ \pi_0^G(R) \ \longrightarrow \ \pi_0^{\Sigma_m \wr G}(R) \ , \tag{1.1}$$

natural for homomorphisms of orthogonal ring spectra. Since we will use these power operations a lot, we take the time to expand the definition: the operation P^m takes the class represented by a based G-map $f : S^V \longrightarrow R(V)$, for some G-representation V, to the class of the $(\Sigma_m \wr G)$-map

$$S^{V^m} = (S^V)^{(m)} \ \xrightarrow{f^{(m)}} \ R(V)^{(m)} \ \xrightarrow{\mu_{V,\dots,V}} R(V^m) \ ,$$

where $\mu_{V,\dots,V}$ is the iterated, $(\Sigma_m \wr G)$-equivariant multiplication map of R.

In this section we study the power operations for commutative orthogonal ring spectra in some detail. We package the resulting algebraic structure on the global functor $\underline{\pi}_0 R$ as a *global power functor*, see Definition 1.2. For a different perspective and a detailed algebraic study of global power functors (restricted to finite groups), including the relationship to the concepts of λ-rings, τ-rings and β-rings, we draw the reader's attention to Ganter's paper [**35**].

Some of the main results in this section are the following: for every commutative orthogonal ring spectrum R the global functor $\underline{\pi}_0(R)$ is naturally a global power functor (Theorem 1.4) and the natural operations on $\underline{\pi}_0(R)$ are generated by restriction, transfer and power operations (so we are not missing any additional structure). Theorem 3.16 below shows that every global power functor is realized by a commutative orthogonal Eilenberg-Mac Lane ring spectrum.

For orthogonal spectra (as opposed to orthogonal spaces), the equivariant homotopy set π_0^G has two pieces of additional structure, namely an addition and transfer maps. We clarify next how the power operations of commutative orthogonal ring spectra interact with the additional structure. For $m \geq 2$ the power operation P^m is *not* additive, but it satisfies various properties reminiscent of the map $x \mapsto x^m$ in a commutative ring. We formalize these properties into the concept of a *global power functor*. Conditions (i) through (iv) in the following definition express the fact that a global power functor has an underlying global power monoid, in the sense of Definition II.1.10, if we forget the additive groups and transfer maps.

Definition 1.2. A *global power functor* is a commutative global functor R equipped with a unital, associative and commutative multiplication

$$\mu \ : \ R \Box R \ \longrightarrow \ R$$

and maps

$$P^m \ : \ R(G) \ \longrightarrow \ R(\Sigma_m \wr G)$$

for all compact Lie groups G and $m \geq 1$, called *power operations*, that satisfy the following relations.

(i) (Unit) $P^m(1) = 1$ for the unit $1 \in R(e)$.
(ii) (Identity) $P^1 = \mathrm{Id}$ under the identification $\Sigma_1 \wr G \cong G$.
(iii) (Restriction) For every continuous homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups and $m \geq 1$ the relation

$$P^m \circ \alpha^* \;=\; (\Sigma_m \wr \alpha)^* \circ P^m$$

holds as maps $R(G) \longrightarrow R(\Sigma_m \wr K)$.
(iv) (Multiplicativity) For all compact Lie groups G, all $m \geq 1$ and all classes $x, y \in R(G)$ the relation

$$P^m(x \cdot y) \;=\; P^m(x) \cdot P^m(y)$$

holds in the group $M(\Sigma_m \wr G)$.
(v) (Restriction) For all compact Lie groups G, all $m \geq 1$, all $1 \leq i \leq m$ and all $x \in R(G)$ the relation

$$\Phi^*_{i,m-i}(P^m(x)) \;=\; P^i(x) \cdot P^{m-i}(x)$$

holds in $R((\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G))$ where $\Phi_{i,m-i}$ is the monomorphism (1.8).
(vi) (Transitivity) For all compact Lie groups G, all $k, m \geq 1$ and all $x \in R(G)$ the relation

$$\Psi^*_{k,m}(P^{km}(x)) \;=\; P^k(P^m(x))$$

holds in $R(\Sigma_k \wr (\Sigma_m \wr G))$, where $\Psi_{k,m} : \Sigma_m \wr (\Sigma_m \wr G) \longrightarrow \Sigma_{km} \wr G$ is the monomorphism (1.9).
(vii) (Additivity) For all compact Lie groups G, all $m \geq 1$, and all $x, y \in R(G)$ the relation

$$P^m(x + y) \;=\; \sum_{i=0}^{m} \mathrm{tr}_{i,m-i}(P^i(x) \times P^{m-i}(y))$$

holds in $R(\Sigma_m \wr G)$, where $\mathrm{tr}_{i,m-i}$ is the transfer associated to the embedding $\Phi_{i,m-i} : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_m \wr G$ defined in (1.8). Here $P^0(x)$ is the multiplicative unit 1.
(viii) (Transfer) For every subgroup H of a compact Lie group G and every $m \geq 1$ the relation

$$P^m \circ \mathrm{tr}_H^G \;=\; \mathrm{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G} \circ P^m$$

holds as maps $R(H) \longrightarrow R(\Sigma_m \wr G)$.

The relations of the power operations in a global power functor have various other properties. For every $n \geq 1$ and all H we have $P^n(0) = 0$. Indeed, because $0 + 0 = 0$, the additivity relation gives

$$P^n(0) \;=\; \sum_{i+j=n} \mathrm{tr}_{i,j}(P^i(0) \cdot P^{m-i}(0)) \ .$$

Since $P^0(0) = 1$, subtracting $P^n(0)$ gives

$$P^n(0) \;=\; \sum_{i=1}^{n-1} \mathrm{tr}_{i,j}(P^i(0) \cdot P^{m-i}(0)) \ .$$

Starting from $P^1(0) = 0$ this shows inductively that $P^n(0) = 0$.

Remark 1.3 (Global power functors versus global Tambara functors)**.** As we explained in Construction II.1.20, the power operations lead to norm maps $N_H^G : R(H) \longrightarrow R(G)$, also called 'multiplicative transfers', for every subgroup H of finite index in G, and conversely the power operations can be reconstructed from the norm maps. So the information in a global power functor could be packaged in an equivalent but different way using norm maps instead of power operations. The algebraic structure that arises then is the global analog of a *TNR-functor* in the sense of Tambara [**93**], nowadays also called a *Tambara functor*; here the acronym stands form 'Transfer, Norm and Restriction'.

For a fixed compact Lie group G, Brun [**22**, Sec. 7.2] has constructed norm maps on the 0-th equivariant homotopy group Mackey functor of every commutative orthogonal G-ring spectrum, and he showed that this structure is a TNR-functor. So the global power functor structure on $\underline{\pi}_0(R)$ for a commutative orthogonal ring spectrum, obtained in the following Theorem 1.4, could also be deduced by using Brun's TNR-structure for the underlying orthogonal G-ring spectrum $R\langle G\rangle$ for every compact Lie group G and then turning the norm maps into power operations as explained in Construction II.1.20. However, Brun's construction is rather indirect and this would hide the simple and explicit nature of the power operations.

Our reason for favoring power operations over norm maps is that they satisfy explicit and intuitive formulas with respect to the rest of the structure (restriction, transfer, sum, product,...). The norm maps also satisfy universal formulas when applied to sums and transfers, but these formulas are harder to describe and to remember and, in the author's opinion, less intuitive.

Much of the next result is contained, at least implicitly, in Greenlees' and May's construction of norm maps [**39**, Sec. 7-9], simply because a commutative orthogonal ring spectrum is an example of a '$\mathcal{GI}_*$-FSP' in the sense of [**39**, Def. 5.5].

Theorem 1.4. *Let R be a commutative orthogonal ring spectrum. Then the products ((4.14) of Chapter III) and the power operations (1.1) make the global functor $\underline{\pi}_0(R)$ into a global power functor.*

PROOF. The properties (i) through (vi) only involve the multiplication, power operations and restriction maps, so they are special cases of Proposition II.1.13 for the commutative orthogonal monoid space $\Omega^\bullet R$. It remains to show the behavior of power operations on sums and transfers.

The additivity formula is based on the following fact. We let $p : S^1 \longrightarrow S^1 \vee S^1$ be a pinch map, i.e., p represents the product in $\pi_1(S^1 \vee S^1, *)$ of the inclusions $\iota_1, \iota_2 : S^1 \longrightarrow S^1 \vee S^1$ of the two wedge summands. The m-th smash power of $S^1 \vee S^1$ decomposes Σ_m-equivariantly as

$$\bigvee_{i=0}^{m} \Sigma_m \ltimes_{\Sigma_i \times \Sigma_{m-i}} S^i \wedge S^{m-i} \;\cong\; (S^1 \wedge S^1)^{\wedge m} \ ,$$

where the restriction of this homeomorphism to the i-th summand is the Σ_m-equivariant extension of the map

$$\iota_1^{\wedge i} \wedge \iota_2^{\wedge(m-i)} \ : \ S^i \wedge S^{m-i} \ \longrightarrow \ (S^1 \vee S^1)^{\wedge m} \ .$$

Then the composite

$$S^m \xrightarrow{\ p^{\wedge m}\ } (S^1 \vee S^1)^{\wedge m} \xrightarrow{\ \mathrm{proj}_i\ } \Sigma_m \ltimes_{\Sigma_i \times \Sigma_{m-i}} S^i \wedge S^{m-i}$$

is in the Σ_m-equivariant homotopy class of the transfer map I.(2.29) for the embedding of $\Sigma_i \times \Sigma_{m-i}$ into Σ_m. More generally we can let G be any compact Lie group and V any G-representation. Then the analogous $(\Sigma_m \wr G)$-map

$$(S^{V\oplus\mathbb{R}})^{\wedge m} \xrightarrow{\ (S^V \wedge p)^{\wedge m}\ } (S^V \wedge (S^1 \vee S^1))^{\wedge m} \xrightarrow{\ \mathrm{proj}_i\ } \Sigma_m \ltimes_{\Sigma_i \times \Sigma_{m-i}} (S^{V\oplus\mathbb{R}})^{\wedge i} \wedge (S^{V\oplus\mathbb{R}})^{m-i}$$

is in the equivariant homotopy class of the transfer for the embedding $\Phi_{i,m-i} : (\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow \Sigma_m \wr G$.

Now we can prove the additivity relation. We let $f : S^V \longrightarrow R(V)$ and $g : S^V \longrightarrow R(V)$ represent the classes $x, y \in \pi_0^G(R)$. Then the composite

$$S^{V\oplus\mathbb{R}} \cong S^V \wedge S^1 \xrightarrow{\ S^V \wedge p\ } S^V \wedge (S^1 \vee S^1) \xrightarrow{\ (f\wedge S^1)\vee(g\wedge S^1)\ } R(V) \wedge S^1 \xrightarrow{\ \sigma_{V,\mathbb{R}}\ } R(V \oplus \mathbb{R})$$

represents the sum $x + y$. The m-th power of this composite factors as

$$\begin{tikzcd}
S^{(V\oplus\mathbb{R})^m} \arrow[r, "(S^V\wedge p)^{\wedge m}"] \arrow[dr, "\bigvee \mathrm{tr}^{\Sigma_m\wr G}_{(\Sigma_i\wr G)\times(\Sigma_{m-i}\wr G)}"'] & (S^V \wedge (S^1 \vee S^1))^{\wedge m} \arrow[r, "((f\wedge S^1)\vee(g\wedge S^1))^{\wedge m}"] & (R(V) \wedge S^1)^{\wedge m} \arrow[r, "\sigma_{V,\mathbb{R}}^{\wedge m}"] & R(V\oplus\mathbb{R})^{\wedge m} \\
& \bigvee_{i=0}^m \Sigma_m \ltimes_{\Sigma_i\times\Sigma_{m-i}} (S^{V\oplus\mathbb{R}})^{\wedge i} \wedge (S^{V\oplus\mathbb{R}})^{m-i} \arrow[u, "\cong"'] \arrow[ur, "\bigvee \mathrm{extn}(f\wedge S^1)^{\wedge i}\wedge(g\wedge S^1)^{\wedge(m-i)}"'] & &
\end{tikzcd}$$

We sketch the argument for the transfer relation. We let $f : S^W \longrightarrow R(W)$ be an H-map that represents a class in $\pi_0^H(R)$. By increasing W, if necessary, we can assume that $W = i^*V$ is the underlying H-representation of a G-representation V that admits a G-equivariant embedding

$$j \ : \ G/H \ \longrightarrow \ V \ .$$

The transfer $\mathrm{tr}_H^G\langle f\rangle \in \pi_0^G(R)$ is then represented by the composite G-map

$$S^V \xrightarrow{\mathrm{tr}_H^G[j]} G \ltimes_H i^*(S^V) \xrightarrow{\tilde{f}} R(V) \ ,$$

where $\tilde{f}$ is the G-equivariant extension of the H-map $f : i^*(S^V) \longrightarrow i^*(R(V)) = R(i^*V)$ given by $\tilde{f}[g,x] = g \cdot f(x)$ and

$$\mathrm{tr}_H^G[j] \ : \ S^V \ \longrightarrow \ G \ltimes_H i^*(S^V)$$

is the G-map constructed by the Thom-Pontryagin collapse around the G-embedding $j : G/H \longrightarrow V$, compare Construction III.2.27. The map

$$j^m \ : \ (\Sigma_m \wr G)/(\Sigma_m \wr H) \ \longrightarrow \ V^m \ , \quad (\sigma; g_1, \dots, g_m) \cdot (\Sigma_m \wr H) \ \longmapsto \ (j(g_1H), \dots, j(g_mH))$$

is then a $(\Sigma_m \wr G)$-equivariant embedding, and we use it to define the transfer $\mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}$. The square

$$\begin{tikzcd}
S^{V^m} \arrow[r, "\mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}{[j^m]}"] \arrow[d, "\cong"'] & (\Sigma_m \wr G) \ltimes_{\Sigma_m\wr H} i^*(S^{V^m}) \arrow[d, "\cong"] & (\sigma; g_1, \dots, g_m) \ltimes (v_1, \dots, v_m) \arrow[d, mapsto] \\
(S^V)^{(m)} \arrow[r, "(\mathrm{tr}_H^G{[j]})^{(m)}"'] & (G \ltimes_H i^*(S^V))^{(m)} & (g_1 \ltimes v_1, \dots, g_m \ltimes v_m)
\end{tikzcd}$$

then commutes up to $(\Sigma_m \wr G)$-equivariant homotopy, where the upper transfer map is based on the $(\Sigma_m \wr G)$-equivariant embedding j^m. So also the square

$$\begin{tikzcd}
S^{V^m} \arrow[r, "\mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}"] \arrow[d, "\cong"'] & (\Sigma_m \wr G) \ltimes_{\Sigma_m\wr H} i^*(S^{V^m}) \arrow[rr, "(\Sigma_m\wr G)\ltimes_{\Sigma_m\wr H} f^{(m)}"] \arrow[d, "\cong"'] & & (\Sigma_m \wr G) \ltimes_{\Sigma_m\wr H} R(i^*V)^{(m)} \arrow[d, "\tilde{\mu}_{i^*V,\dots,i^*V}"] \\
(S^V)^{(m)} \arrow[r, "(\mathrm{tr}_H^G)^{(m)}"'] \arrow[rrr, bend right, "P^m(\mathrm{tr}_H^G f)"'] & (G \ltimes_H i^*(S^V))^{(m)} \arrow[r, "\tilde{f}^{(m)}"'] & R(V)^{(m)} \arrow[r, "\mu_{V,\dots,V}"'] & R(V^m)
\end{tikzcd}$$

commutes. Here $\tilde{\mu}_{i^*V,\dots,i^*V} : (\Sigma_m \wr G) \ltimes_{\Sigma_m\wr H} R(i^*V)^{(m)} \longrightarrow R(V^m)$ is the $(\Sigma_m \wr G)$-equivariant extension of the $(\Sigma_m \wr H)$-map $\mu_{i^*V,\dots,i^*V} : R(i^*V)^{(m)} \longrightarrow R(i^*(V^m)) = i^*(R(V^m))$. Since $\tilde{\mu}_{i^*V,\dots,i^*V} \circ ((\Sigma_m \wr G) \ltimes_{\Sigma_m\wr H} f^{(m)})$ is the $(\Sigma_m \wr G)$-equivariant extension of the $(\Sigma_m \wr H)$-map $P^m(f)$, the composite around the upper right corner represents $\mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(P^m\langle f\rangle)$. This proves the relation $P^m(\mathrm{tr}_H^G\langle f\rangle) = \mathrm{tr}_{\Sigma_m\wr H}^{\Sigma_m\wr G}(P^m\langle f\rangle)$. $\square$

Now we are going to show that the restriction maps, transfer maps and power operations generate all natural operations between the 0-th equivariant homotopy groups of ultra-commutative ring spectra. The strategy is the one that we have employed several times before: the functor π_0^G from commutative orthogonal ring spectra to sets is representable, namely by $\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)$, the unreduced suspension spectrum of the free commutative ring space generated by $B_{\mathrm{gl}}G$. (The orthogonal ring spectrum $\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)$ is isomorphic to the free commutative orthogonal ring spectrum on the unreduced suspension spectrum of $B_{\mathrm{gl}}G$). So we have to determine the equivariant homotopy groups $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$, which just means assembling various results already proved.

Proposition 1.5. *Let G and K be compact Lie groups. The group of natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of ultra-commutative ring spectra is a free abelian group with basis the operations*

$$\mathrm{tr}_L^K \circ \alpha^* \circ P^m \ : \ \pi_0^G \ \longrightarrow \pi_0^K$$

for all $m \geq 0$ and all conjugacy classes of pairs (L, α) consisting of a subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow \Sigma_m \wr G$.

Proof. We let V be any faithful G-representation and write $B_{\mathrm{gl}}G = \mathbf{L}_{G,V}$ for the global classifying space of G based on V and $u_G = u_{G,V}$ for the associated tautological class. We denote by $\hat{e}_G \in \pi_0^G(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ the class obtained by pushing forward the tautological class u_G along the adjunction unit $B_{\mathrm{gl}}G \longrightarrow \mathbb{P}(B_{\mathrm{gl}}G)$ and then applying the map σ^G (compare (3.7) of Chapter I) that makes an unstable homotopy class into a stable homotopy class of the suspension spectrum.

We apply the representability result of Proposition I.2.20 to the category of commutative orthogonal ring spectra, the adjoint functor pair

$$spc \ \underset{U \circ \Omega^\bullet}{\overset{\Sigma_+^\infty \circ \mathbb{P}}{\rightleftarrows}} \ \text{com ring sp}$$

and the functor $\Phi = \pi_0^K$. We conclude that the evaluation at the tautological class is a bijection

$$\mathrm{Nat}^{\text{com ring sp}}(\pi_0^G, \pi_0^K) \ \longrightarrow \ \pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)) \ , \quad \tau \ \longmapsto \ \tau(\hat{e}_G)$$

to the 0-th K-equivariant homotopy group of commutative orthogonal ring spectrum $\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G)$.

In Proposition II.1.19 we have seen that the set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$ bijects with the set

$$\coprod_{m \geq 0} \mathrm{Rep}(L, \Sigma_m \wr G)$$

by evaluation $\alpha : K \longrightarrow \Sigma_m \wr G$ to $\alpha^*(P^m(u_G))$. Proposition III.3.8 then implies that the group $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group, and it specifies a basis consisting of the elements

$$\mathrm{tr}_L^K(\sigma^L(x))$$

where L runs through all conjugacy classes of subgroups of K with finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0^L(\mathbb{P}(B_{\mathrm{gl}}G))$. So together this shows that $\pi_0^K(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}}G))$ is a free abelian group with basis the classes

$$\mathrm{tr}_L^K(\sigma^L(\alpha^*(P^m(u_G)))) \ = \ \mathrm{tr}_L^K(\alpha^*(P^m(\sigma^G(u_G)))) \ = \ \mathrm{tr}_L^K(\alpha^*(P^m(\hat{u}_G)))$$

for all $m \geq 0$ and all $(K \times G)$-conjugacy classes of pairs (L, α) consisting of a subgroup L of K with finite Weyl group and a homomorphism $\alpha : L \longrightarrow \Sigma_m \wr G$. □

In the remaing part of this section we construct a model structure on the category of commutative orthogonal ring spectra with global equivalences as the weak equivalences. The strategy is the same is in the unstable situation in Section II.3: we establish a 'positive' version of the global model structure and lift it to commutative monoid objects with the help of the general lifting theorem II.3.8.

Definition 1.6. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive cofibration* if it is a flat cofibration and the map $f_0 : X_0 \longrightarrow Y_0$ is a homeomorphism. An orthogonal spectrum is a *positive global Ω-spectrum* if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and an arbitrary G-representation W the adjoint structure map

$$\tilde{\sigma}_{V,W} \ : \ X(V) \ \longrightarrow \ \mathrm{map}(S^W, X(V \oplus W))$$

is a G-weak equivalence.

If G is a non-trivial compact Lie group, then any faithful G-representation is automatically non-trivial. So a positive global Ω-spectrum is a global Ω-spectrum (in the absolute sense) if the adjoint structure map $\tilde{\sigma}_0 : X_0 \longrightarrow \Omega X_1$ is a non-equivariant weak equivalence.

Proposition 1.7 (Positive global model structure). *The global equivalences and positive cofibrations are part of a proper topological model structure, the* positive global model structure *on the category of orthogonal spectra. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a fibration in the positive global model structure if and only if for every compact Lie group G, every faithful G-representation V with $V \neq 0$ and an arbitrary G-representation W the square*

$$\begin{tikzcd}
X(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"] \arrow[d, "f(V)^G"'] & \mathrm{map}^G(S^W, X(V \oplus W)) \arrow[d, "\mathrm{map}^G(S^W, f(V\oplus W))"] \\
Y(V)^G \arrow[r, "\tilde{\sigma}^G_{V,W}"'] & \mathrm{map}^G(S^W, Y(V \oplus W))
\end{tikzcd}$$

is homotopy cartesian. The fibrant objects in the positive global model structure are the positive global Ω-spectra.

Proof. We start by establishing a *positive strong level model structure.* A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *positive strong level equivalence* (respectively *positive strong level fibration*) if for every inner product space V with $V \neq 0$ the map $f(V) : X(V) \longrightarrow Y(V)$ is an $O(V)$-weak equivalence (respectively an $O(V)$-fibration). Then we claim that the positive strong level equivalences, positive strong level fibrations and positive cofibrations form a model structure, on the category of orthogonal spectra.

The proof is another application of the general construction method for level model structures in Proposition IV.1.9. Indeed, we let $\mathcal{C}(0)$ be the degenerate model structure on the category $\mathbf{T}$ of based spaces in which every morphism is a weak equivalence and a fibration, but only the isomorphisms are cofibrations. For $m \geq 1$ we let $\mathcal{C}(m)$ be the projective model structure (for the family of all closed subgroups) on the category of based $O(m)$-spaces. With respect to these choices of model structures $\mathcal{C}(m)$, the classes of level equivalences, level fibrations and cofibrations in the sense of Proposition IV.1.9 precisely become the positive strong level equivalences, positive strong level fibrations and positive cofibrations. The stable consistency condition (1.8) in Chapter IV is now strictly weaker than for the strong level model structure, so it holds. The verification that the model structure is proper and topological is the same as in Proposition IV.1.15.

The positive strong level model structure is cofibrantly generated: we can simply take the same sets of generating cofibrations and generating acyclic cofibrations as for the $\mathcal{A}ll$-level model structure in Proposition IV.1.15, *except* that we omit all morphisms freely generated in level 0.

We obtain the positive global model structure for orthogonal spectra by 'mixing' the positive strong level model structure with the global model structure of Theorem IV.2.7. Every positive strong level equivalence is a global equivalence and every positive cofibration is a flat cofibration. The global equivalences and the positive cofibrations are part of a model structure by Cole's mixing theorem [**25**, Thm. 2.1], which is our first claim. By [**25**, Cor. 3.7] (or rather its dual formulation), an orthogonal spectrum is fibrant in the positive global model structure if it is equivalent in the positive strong level model structure to a global Ω-spectrum; this is equivalent to being a positive global Ω-spectrum. The positive global model structure is again proper by Propositions 4.1 and 4.2 of [**25**]. The proof that this model structure topological is similar as for the global model structure. $\square$

We recall from Definition II.3.5 that a morphism $i : A \longrightarrow B$ of orthogonal spectra is a *symmetrizable cofibration* (respectively a *symmetrizable acyclic cofibration*) if the morphism

$$i^{\Box n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n = \mathbb{P}^n(B)$$

is a cofibration (respectively an acyclic cofibration) for every $n \geq 1$. Since the morphism $i^{\square 1}/\Sigma_1$ is the original morphism i, every symmetrizable cofibration is in particular a cofibration, and similarly for acyclic cofibrations.

The next theorem says that in the category of orthogonal spectra, all cofibrations and acyclic cofibrations in the positive global model structure on orthogonal spectra are symmetrizable with respect to the monoidal structure given by the smash product.

Theorem 1.8. (i) *Let $i : A \longrightarrow B$ be a flat cofibration of orthogonal spectra. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n$$

is a flat cofibration. In other words, all cofibrations in the global model structure of orthogonal spectra are symmetrizable.

(ii) *Let $i : A \longrightarrow B$ be a positive flat cofibration of orthogonal spectra that is also a global equivalence. Then for every $n \geq 1$ the morphism*

$$i^{\square n}/\Sigma_n \ : \ Q^n(i)/\Sigma_n \ \longrightarrow \ B^{\wedge n}/\Sigma_n$$

is a global equivalence. In other words, all acyclic cofibrations in the positive global model structure of orthogonal spectra are symmetrizable.

Proof. (i) We recall from Proposition IV.2.13 the set

$$I_{\mathcal{A}ll} \ = \ \{ \ G_m((i_k \times O(m)/H)_+) \mid m, k \geq 0, H \leq O(m)\}$$

of generating flat cofibrations of orthogonal spectra, where $i_k : S^{k-1} \longrightarrow D^k$ is the inclusion. The set $I_{\mathcal{A}ll}$ detects the acyclic fibrations in the strong level model structure of orthogonal spectra. In particular, every flat cofibration is a retract of an $I_{\mathcal{A}ll}$-cell complex. By [**37**, Cor. 21] it suffices to show that the generating flat cofibrations in $I_{\mathcal{A}ll}$ are symmetrizable.

The orthogonal spectrum $G_m((K \times O(m)/H)_+)$ is isomorphic to $K \ltimes F_{H,\mathbb{R}^m} = (K_+) \wedge F_{H,\mathbb{R}^m}$; so we show more generally that every morphism of the form

$$j \ltimes F_{G,V} \ : \ K \ltimes F_{G,V} \ \longrightarrow \ L \ltimes F_{G,V}$$

is a symmetrizable cofibration, where G is any compact Lie group, V a G-representation and $j : K \longrightarrow L$ a cofibration of unbased spaces. The symmetrized iterated pushout product

$$(j \ltimes F_{G,V})^{\square n}/\Sigma_n \ : \ Q^n(j \ltimes F_{G,V})/\Sigma_n \ \longrightarrow \ (L \ltimes F_{G,V})^{\wedge n}/\Sigma_n \tag{1.9}$$

is isomorphic to

$$F_{\Sigma_n \wr G, V^n}(j_+^{\square n}) \ : \ F_{\Sigma_n \wr G, V^n}(Q^n(j)_+) \ \longrightarrow \ F_{\Sigma_n \wr G, V^n}(L_+^n) \ ,$$

where

$$j^{\square n} \ : \ Q^n(j) \ \longrightarrow \ L^n$$

is the n-fold pushout product of j, with respect to the cartesian product of spaces. The map $j^{\square n}$ is Σ_n-equivariant, and it is viewed as a morphism of $(\Sigma_n \wr G)$-spaces by restriction along the projection $(\Sigma_n \wr G) \longrightarrow \Sigma_n$. Since j is a cofibration of spaces, $j^{\square n}$ is a cofibration of Σ_n-spaces, hence a cofibration of $(\Sigma_n \wr G)$-spaces. So the morphism (1.9) is a flat cofibration.

(ii) Proposition IV.2.13 (ii) describes a set $J^{\text{flat}}_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ of generating acyclic cofibrations for the global model structure on the category of orthogonal spectra. From this we obtain a set $J^+ \cup K^+$ of generating acyclic cofibration for the *positive* global model structure of Proposition 1.7 by restricting to those morphisms in $J^{\text{flat}}_{\mathcal{A}ll} \cup K_{\mathcal{A}ll}$ that are positive cofibrations, i.e., homeomorphisms in level 0; so explicitly, we set

$$J^+ \ = \ \{ \ G_m((j_k \times O(m)/H)_+) \mid m \geq 1, k \geq 0, H \leq O(m)\} \ ,$$

where $j_k : D^k \times \{0\} \longrightarrow D^k \times [0,1]$ is the inclusion, and

$$K^+ = \bigcup_{G,V,W\ :\ V\neq 0} \mathcal{Z}(\lambda_{G,V,W}) ,$$

the set of all pushout products of sphere inclusions i_k with the mapping cylinder inclusions of the global equivalences $\lambda_{G,V,W} : F_{G,V\oplus W}S^W \longrightarrow F_{G,V}$. Here (G,V,W) runs through a set of representatives of the isomorphism classes of triples consisting of a compact Lie group G, a non-zero faithful G-representation V and an arbitrary G-representation W. By [**37**, Cor. 21] it suffices to show that all morphisms in $J^+ \cup K^+$ are symmetrizable acyclic cofibrations.

We start with a morphism $G_m((j_k \times O(m)/H)_+)$ in J^+. For every $n \geq 1$, the morphism

$$(G_m((j_k \times O(m)/H)_+))^{\Box n}/\Sigma_n$$

is a flat cofibration by part (i), and a homeomorphism in level 0 because $m \geq 1$. Moreover, the morphism j_k is a homotopy equivalence of spaces, so $G_m((j_k \times O(m)/H)_+)$ is a homotopy equivalence of orthogonal spectra; the morphism $\mathbb{P}^n(G_m((j_k \times O(m)/H)_+))$ is then again a homotopy equivalence for every $n \geq 1$, by Proposition II.3.10 (i). Then [**37**, Lemma 29] shows that $G_m((j_k \times O(m)/H)_+)$ is a symmetrizable acyclic cofibration. This takes care of the set J^+.

Now we consider the morphisms in the set K^+. Since G acts faithfully on the non-zero inner product space V, the action of the wreath product $\Sigma_n \wr G$ on V^n is again faithful. So the morphism

$$\lambda_{\Sigma_n\wr G,V^n,W^n} \ :\ F_{\Sigma_n\wr G,V^n\oplus W^n}\left(S^{W^n}\right) \ \longrightarrow\ F_{\Sigma_n\wr G,V^n}$$

is a global equivalence by Theorem III.5.7. The vertical morphisms in the commutative square

$$\begin{tikzcd}
F_{\Sigma_n\wr G,V^n\oplus W^n}\left(S^{W^n}\right) \arrow[r, "\lambda_{\Sigma_n\wr G,V^n,W^n}"] \arrow[d, "\cong"'] & F_{\Sigma_n\wr G,V^n} \arrow[d, "\cong"] \\
\mathbb{P}^n(F_{G,V\oplus W}S^W) \arrow[r, "\mathbb{P}^n(\lambda_{G,V,W})"'] & \mathbb{P}^n(F_{G,V})
\end{tikzcd}$$

are isomorphisms; so the morphism $\mathbb{P}^n(\lambda_{G,V,W})$ is a global equivalence. Proposition II.3.10 (iii) then shows that all morphisms in $\mathcal{Z}(\lambda_{G,V,W})$ are symmetrizable acyclic cofibrations. $\Box$

Now we put all the pieces together and prove another main result of this section. We call a morphism of ultra-commutative ring spectra a *global equivalence* (respectively *positive global fibration*) if the underlying morphism of orthogonal spectra is a global equivalence (respectively fibration in the positive global model structure).

Theorem 1.10 (Global model structure for ultra-commutative ring spectra)**.** *The global equivalences and positive global fibrations are part of a model structure, the* global model structure *on the category of ultra-commutative ring spectra. This global model structure is proper, topological and cofibrantly generated. Every cofibration in this model structure whose source is cofibrant as an ultra-commutative ring spectrum is a positive cofibration of underlying orthogonal spectra.*

Proof. The positive global model structure of orthogonal spectra (Proposition 1.7) is monoidal and cofibrantly generated. The monoid axiom holds by Proposition IV.3.10. Cofibrations and acyclic cofibrations are symmetrizable by Theorem 1.8. So Theorem II.3.8 shows that the positive global model structure of orthogonal spectra lifts to the category of commutative orthogonal ring spectra, and it provides the addendum about cofibrations with cofibrant source. $\Box$

There is a positive version of the flat $\mathcal{F}$-global model structure for every global family $\mathcal{F}$, and it lifts to a *flat $\mathcal{F}$-global model structure* on the category of ultra-commutative ring spectra.

2. Algebra of global power functors

This section is mostly of an algebraic nature. In Definition 1.2 we introduced global power functors, and Proposition 1.5 shows that the power operations are precisely the additional structure that the global functor $\underline{\pi}_0(R)$ of an ultra-commutative ring spectrum R has. We will show in Theorem 3.16 below that every global power functor can be realized by an ultra-commutative ring spectrum, so we have good reasons for studying the category of global power functors.

In Remark 1.3 we indicated that power operations on a global Green functor can be traded for norm maps, and vice versa. This observation can be stated as an equivalence of categories between global power functors and a certain category of 'global Tambara functors' or 'global TNR functors'. We shall not pursue this further. Instead, we will show in this section that the category of global power functors is both monadic and comonadic over the category of global Green functors. A formal consequence of this fact is that the category of global power functors has all limits and colimits, and they created in the underlying category of global Green functors.

We describe global power functors as the coalgebra over a certain cotriple exp on the category of global Green functors. When restricted to finite groups, most of the results about this cotriple are contained in the PhD thesis of J. Singer [**83**], a former student of the author. For finite groups (as opposed to compact Lie groups), this comonadic description has independently been obtained by Ganter [**35**].

We start by introducing the functor exp of *exponential sequences* that takes a global Green functor R to another global Green functor $\exp(R)$. We recall that a *global Green functor* is a commutative monoid in the category $\mathcal{GF}$ of global functors under the monoidal structure given by the box product, compare (4.21) of Chapter III. As we explain after Definition III.4.28, this commutative multiplication on a global Green functor R can be made more explicit in two equivalent ways:

- as a commutative ring structure on the group $R(G)$ for every compact Lie groups, subject to the requirement that all restrictions maps are ring homomorphisms and the transfer maps satisfy Frobenius reciprocity;
- as a unit element $1 \in R(e)$ and biadditive, commutative, associative and unital external pairings $\times : R(G) \times R(H) \longrightarrow R(G \times H)$ that are morphisms of global functors in each variable separately.

Construction 2.1. We let R be a global Green functor and G a compact Lie group. We let

$$\exp(R; G) \ \subset \ \prod_{n \geq 0} R(\Sigma_n \wr G)$$

be the set of *exponential sequences*, i.e., of those families

$$(x_n)_n \ \in \ \prod_{n \geq 0} R(\Sigma_n \wr G)$$

that satisfy $x_0 = 1$ in $R(\Sigma_0 \wr G) = R(e)$ and

$$\Phi^*_{i,j}(x_{i+j}) \ = \ x_i \times x_j$$

in $R(\Sigma_i \wr G) \times (\Sigma_j \wr G)$ for all $i, j \geq 1$, where $\Phi_{i,j} : (\Sigma_i \wr G) \times (\Sigma_j \wr G) \longrightarrow \Sigma_{i+j} \wr G$ is the monomorphism defined in (1.8) of Chapter I. We define an addition on the set $\exp(R; G)$ by

$$\big((x_n) + (y_n)\big)_m \ = \ \sum_{i=0}^{m} \operatorname{tr}_{i,m-i}(x_i \times y_{m-i}) \ ,$$

where $\mathrm{tr}_{i,m-i} : R(\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G) \longrightarrow R(\Sigma_m \wr G)$ is the transfer associated to the monomorphism $\Phi_{i,m-i}$. For all $i+j=m$ we have

$$\begin{aligned}
\Phi^*_{i,j}((x+y)_m) &= \sum_{k+l=m} \Phi^*_{i,j}(\mathrm{tr}_{k,l}(x_k \times y_l)) \\
&= \sum_{k+l=m} \sum_{[\gamma]\in\Sigma_i\times\Sigma_j\backslash\Sigma_m/\Sigma_k\times\Sigma_l} \mathrm{tr}^{\Sigma_i\times\Sigma_j}_{(\Sigma_i\times\Sigma_j)\cap^\gamma(\Sigma_k\times\Sigma_l)} \circ c_\gamma \circ \mathrm{res}^{\Sigma_k\times\Sigma_l}_{(\Sigma_i\times\Sigma_j)^\gamma\cap(\Sigma_k\times\Sigma_l)}(x_k \times y_l) \\
&= \sum_{a+b=i,\ c+d=j} \mathrm{tr}^{\Sigma_{a+b}\times\Sigma_{c+d}}_{\Sigma_a\times\Sigma_b\times\Sigma_c\times\Sigma_d}((\mathrm{res}^{\Sigma_{a+c}}_{\Sigma_a\times\Sigma_c} x_{a+c}) \times (\mathrm{res}^{\Sigma_{b+d}}_{\Sigma_b\times\Sigma_d} y_{b+d})) \\
&= \sum_{a+b=i,\ c+d=j} \mathrm{tr}^{\Sigma_{a+b}\times\Sigma_{c+d}}_{\Sigma_a\times\Sigma_b\times\Sigma_c\times\Sigma_d}(x_a \times y_b \times x_c \times y_d) \\
&= \sum_{a+b=i, c+d=j} \mathrm{tr}_{a,b}(x_a \times y_b) \times \mathrm{tr}_{c,d}(x_c \times y_d) \ = \ (x+y)_i \times (x+y)_j
\end{aligned}$$

The second equation is the double coset formula, and the third equation uses that the permutations [...] form a set of representatives of the $(\Sigma_i \times \Sigma_j)$-$(\Sigma_k \times \Sigma_l)$-double cosets. This shows that the sequence $x+y$ is exponential whenever both x and y are exponential.

The sequence $\underline{0}$ with

$$\underline{0}_n \ = \ \begin{cases} 1 & \text{for } n=0, \text{ and} \\ 0 & \text{for } n \geq 1 \end{cases}$$

is exponential and is a neutral element for the sum operation on $\exp(R;G)$. The sum on $\exp(R;G)$ is associative and commutative [show; does this need exponential?], so it makes $\exp(R;G)$ into an abelian monoid. The relation $x+y=\underline{0}$ is equivalent to $x_0 \times y_0 = 1$ in $R(e)$ and

$$x_0 \times y_m \ = \ -\sum_{i=1}^m \mathrm{tr}_{i,m-i}(x_i \times y_{m-i})$$

in $R(\Sigma_m \wr G)$ for all $m \geq 1$. So any given sequence x has an inverse with respect to $+$ if and only if x_0 is a unit in the ring $R(e)$. In particular, the monoid $\exp(R;G)$ is a an abelian group with respect to $+$.

Now suppose that $\alpha : K \longrightarrow G$ is a group homomorphism. We define a map

$$\alpha^* = \exp(R;\alpha) \ : \ \exp(R;G) \ \longrightarrow \ \exp(R;K)$$

by

$$\alpha^*(x) \ = \ ((\Sigma_n \wr \alpha)^*(x_n))_{n\geq 0} \ .$$

Then $\alpha^* : \exp(R;G) \longrightarrow \exp(R;K)$ respects the sum of exponential sequences and makes the assignment $\exp(R) = \exp(R;-)$ into a $\mathrm{Rep}^{\mathrm{op}}$-functor, i.e., a contravariant functor from compact Lie groups and conjugacy classes of continuous homomorphism to abelian groups.

If H is a subgroup of G, then we define a transfer morphism

$$\mathrm{tr}_H^G = \exp(R;\mathrm{tr}_H^G) \ : \ \exp(R;H) \ \longrightarrow \ \exp(R;G)$$

by

$$\mathrm{tr}_H^G((x_n)_n) \ = \ (\mathrm{tr}^{\Sigma_n\wr G}_{\Sigma_n\wr H}(x_n))_n \ .$$

Indeed, this construction preserves the property of being exponential [...]. Also, the transfer map $\exp(R;\mathrm{tr}_H^G)$ respects the sum of exponential sequences [...] and is transitive for nested sequences of subgroups. The double coset formula [...] If the Weyl group of H in G is infinite, then [...] For a group epimorphism $\alpha : K \longrightarrow G$ we have $\alpha^* \circ \mathrm{tr}_H^G = \mathrm{tr}_L^K \circ (\alpha|_L)^{-1}$, where $L = \alpha^{-1}(H)$. So altogether we have defined a global functor $\exp(R)$.

But this is not the end of the story. Indeed, $\exp(R)$ inherits the structure of global Green functor as follows. For finite groups G and K we define the multiplication

$$\times \;:\; \exp(R;G) \times \exp(R;K) \;\longrightarrow\; \exp(R;G\times K) \tag{2.2}$$

by

$$(x\times y)_n \;=\; \Delta^*(x_n \times y_n)\ ,$$

where $\Delta : \Sigma_n\wr(G\times K) \longrightarrow (\Sigma_n\wr G)\times(\Sigma_n\wr K)$ is the 'diagonal' monomorphism defined in (1.12) of Chapter I. The square of group homomorphisms

$$\begin{tikzcd}
(\Sigma_i \wr (G\times K)) \times (\Sigma_{n-i} \wr (G\times K)) \arrow[r, "\Phi_{i,n-i}"] \arrow[d, "D"'] & \Sigma_n \wr (G\times K)) \arrow[d, "\Delta"] \\
(\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G) \times (\Sigma_i \wr K) \times (\Sigma_{n-i} \wr K) \arrow[r, "\Phi_{i,n-i}\times\Phi_{i,n-i}"'] & (\Sigma_n \wr G) \times (\Sigma_n \wr K)
\end{tikzcd}$$

commutes where

$$\begin{aligned} D((\sigma;\,(g_1,k_1),\dots(g_i,k_i)),\,(\tau;\,(g_{i+1},k_{i+1}),\dots(g_n,k_n)) \\ = ((\sigma;\,g_1,\dots,g_i),\,(\tau;\,g_{i+1},\dots,g_n),\,(\sigma;\,k_1,\dots,k_i),\,(\tau;\,k_{i+1},\dots,k_n))\ . \end{aligned}$$

So the relation

$$\begin{aligned} \Phi^*_{i,n-i}((x\times y)_n) \;&=\; (\Delta\circ\Phi_{i,n-i})^*(x_n\times y_n) \\ &=\; ((\Phi_{i,n-i}\times\Phi_{i,n-i})\circ D)^*(x_n\times y_n) \\ &=\; D^*(\Phi^*_{i,n-i}(x_n)\times\Phi^*_{i,n-i}(y_n)) \\ &=\; D^*(x_i\times x_{n-i}\times y_i\times y_{n-i}) \\ &=\; \Delta^*(x_i\times y_i)\times\Delta^*(x_{n-i}\times y_{n-i}) \\ &=\; (x\times y)_i\times(x\times y)_{n-i} \end{aligned}$$

shows that the product $x\times y$ is exponential if both factors are.

Proposition 2.3. *For every global Green functor R the external products* (2.2) *and the unit element*

$$\underline{1} \;=\; (1)_{n\geq 0} \;\in\; \exp(R;e)$$

make the global functor $\exp(R)$ *into a global Green functor.*

Proof. The product is biadditive because the original product of the Green functor R is biadditive and because the restriction Δ^* is additive. The product is associative and commutative because the original product of the Green functor R and the diagonal maps are associative and comuutative. [Unital] This shows that for every Green functor R, the global functor $\exp(R)$ is another Green functor. □

A natural transformation of Green functors

$$\eta_R \;:\; \exp(R) \;\longrightarrow\; R$$

is given by $\eta(x) = x_1$, using the identification $G\cong\Sigma_1\wr G$ via $g\mapsto(1;g)$. A natural transformation of Green functors

$$\kappa_R \;:\; \exp(R) \;\longrightarrow\; \exp(\exp(R))$$

is given at a compact Lie group G by $\kappa(x) = (\kappa_n(x))_n$, where $\kappa_n(x)\in\exp(R;\Sigma_n\wr G)$ is the exponential sequence whose k-th component is

$$(\kappa_n(x))_k \;=\; \Psi^*_{k,n}(x_{kn}) \;\in\; R(\Sigma_k\wr(\Sigma_n\wr G))\ ,$$

where $\Psi_{k,n} : \Sigma_k \wr (\Sigma_n \wr G) \longrightarrow \Sigma_{kn} \wr G$ is the monomorphism defined in (1.9) of Chapter I. The square of group homomorphisms

$$\begin{tikzcd}
(\Sigma_i \wr (\Sigma_n \wr G)) \times (\Sigma_{k-i} \wr (\Sigma_n \wr G)) \arrow[r, "\Phi_{i,k-i}"] \arrow[d, "\Psi_{i,n}\times\Psi_{k-i,n}"'] & \Sigma_k \wr (\Sigma_n \wr G) \arrow[d, "\Psi_{k,n}"] \\
(\Sigma_{in} \wr G) \times (\Sigma_{(k-i)n} \wr G) \arrow[r, "\Phi_{in,(k-i)n}"'] & \Sigma_{kn} \wr G
\end{tikzcd}$$

commutes; so the relation

$$\begin{aligned}
\Phi^*_{i,k-i}((\kappa_n(x))_k) \ &= \ (\Psi_{k,n} \circ \Phi_{i,k-i})^*(x_{kn}) \\
&= \ (\Phi_{in,(k-i)n} \circ (\Psi_{i,n} \times \Psi_{k-i,n}))^*(x_{kn}) \\
&= \ (\Psi_{i,n} \times \Psi_{k-i,n})^*(x_{in} \times x_{(k-i)n}) \\
&= \ \Psi^*_{i,n}(x_{in}) \times \Psi^*_{k-i,n}(x_{(k-i)n}) \ = \ (\kappa_n(x))_i \times (\kappa_n(x))_{k-i}
\end{aligned}$$

shows that $\kappa_n(x)$ is indeed an exponential sequence whenever x and y are exponential. We claim that for all $1 \leq i \leq n - i$, the relation

$$\Phi^*_{i,n-i}(\kappa_n(x)) \ = \ \kappa_i(x) \times \kappa_{n-i}(x) \quad \text{holds in } \exp(R; (\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)), \tag{2.4}$$

so that $\kappa(x) = (\kappa_n(x))_{n\geq 0}$ is indeed an exponential sequence for the global Green functor $\exp(R)$. For this we pick $k \geq 1$ and note that the square of group homomorphisms

$$\begin{tikzcd}
\Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)) \arrow[r, "\Sigma_k\wr\Phi_{i,n-i}"] \arrow[d, "\Delta"'] & \Sigma_k \wr (\Sigma_n \wr G) \arrow[dd, "\Psi_{k,n}"] \\
(\Sigma_k \wr (\Sigma_i \wr G)) \times (\Sigma_k \wr (\Sigma_{n-i} \wr G)) \arrow[d, "\Psi_{k,i}\times\Psi_{k,n-i}"'] & \\
(\Sigma_{ki} \wr G) \times (\Sigma_{k(n-i)} \wr G) \arrow[r, "\Phi_{ki,n(n-i)}"'] & \Sigma_{kn} \wr G
\end{tikzcd}$$

commutes; so the relation

$$\begin{aligned}
(\Phi^*_{i,n-i}(\kappa_n(x)))_k \ &= \ (\Sigma_k \wr \Phi_{i,n-i})^*((\kappa_n(x))_k) \\
&= \ (\Psi_{k,n} \circ (\Sigma_k \wr \Phi_{i,n-i}))^*(x_{kn}) \\
&= \ (\Phi_{ki,k(n-i)} \circ (\Psi_{k,i} \times \Psi_{k,n-i}) \circ \Delta)^*(x_{kn}) \\
&= \ ((\Psi_{k,i} \times \Psi_{k,n-i}) \circ \Delta)^*(x_{ki} \times x_{k(n-i)}) \\
&= \ \Delta^*(\Psi^*_{k,i}(x_{ki}) \times \Psi^*_{k,n-i}(x_{k(n-i)})) \\
&= \ \Delta^*((\kappa_i(x))_k \times (\kappa_{n-i}(x))_k) \ = \ (\kappa_i(x) \times \kappa_{n-i}(x))_k
\end{aligned}$$

holds in $R(\Sigma_k \wr ((\Sigma_i \wr G) \times (\Sigma_{n-i} \wr G)))$, and this shows (2.4).

The following is Satz 2.17 in [**83**].

Proposition 2.5. *The natural transformations*

$$\eta \ : \ \exp \ \longrightarrow \ \mathrm{Id} \qquad \textit{and} \qquad \kappa \ : \ \exp \ \longrightarrow \ \exp\circ\exp$$

make the functor exp *into a cotriple on the category of global Green functors.*

Proof. [additive, natural for restriction and transfers, compatible with product] We have to show that for every Green functor R the diagrams of natural transformations of Green functors

$$\begin{tikzcd}
\exp(R) \arrow[r, "\kappa_R"] \arrow[d, "\kappa_R"'] & \exp(\exp(R)) \arrow[d, "\kappa_{\exp(R)}"] \\
\exp(\exp(R)) \arrow[r, "\exp(\kappa_R)"'] & \exp(\exp(\exp(R)))
\end{tikzcd}
\qquad
\begin{tikzcd}
& \exp(R) \arrow[dl, equal] \arrow[d, "\kappa_R"] \arrow[dr, equal] & \\
\exp(R) & \exp(\exp(R)) \arrow[l, "\exp(\eta_R)"] \arrow[r, "\eta_{\exp(R)}"'] & \exp(R)
\end{tikzcd}$$

commute. □

Now suppose that R is a global Green functor and $P : R \longrightarrow \exp(R)$ a natural transformation of global Green functors. For every compact Lie group G, a sequence of operations $P^n : R(G) \longrightarrow R(\Sigma_n \wr G)$ is defined by

$$P(x) \ = \ (P^n(x))_n \ ,$$

i.e., $P^n(x)$ is the n-th component of the exponential sequence $P(x)$.

Theorem 2.6. *Let R be a global Green functor and $P : R \longrightarrow \exp(R)$ a natural transformation that makes R into a coalgebra over the coptriple $(\exp, \eta, \kappa)$. Then the operations $P^n : R(G) \longrightarrow R(\Sigma_n \wr G)$ make R into a global power functor. Altogether the assignment*

$$(\exp\text{-coalgebras}) \ \longrightarrow \ (\text{global power functors}) \ , \quad (R, P) \ \longmapsto \ (R, \{P^n\}_{n\geq 0})$$

is an isomorphism of categories.

Proof. The fact that $P : R \longrightarrow \exp(R)$ takes values in exponential sequences is equivalent to the restriction condition of the power operations. The fact that $P : R \longrightarrow \exp(R)$ is a transformation of Green functors encoded simultaneously the unit, contravariant naturality, transfer, multiplicativity and additivity relations of a global power functor. The identity relation is equivalent to the counit condition of a coalgebra, i.e., that the composite

$$R \ \xrightarrow{P} \ \exp(R) \ \xrightarrow{\eta_R} \ R$$

is the identity. The transitivity relation is equivalent to the coassociativity condition of a coalgebra, i.e., that the square

$$\begin{tikzcd}
R \arrow[r, "P"] \arrow[d, "P"'] & \exp(R) \arrow[d, "\kappa_R"] \\
\exp(R) \arrow[r, "\exp(P)"'] & \exp(\exp(R))
\end{tikzcd}$$

commutes. □

The interpretation of global power functors as coalgebra over a cotriple has some useful consequences that are not so easy to see directly from the original definition in terms of power operations and explicit relations. In general, colimits in any category of coalgebras are created in the underlying category [ref]. In our situation that means:

Corollary 2.7. *Colimits in the category of global power functors exist and are created in the underlying category of global Green functors.*

3. Examples

In this section we discuss various examples of and constructions with global power functors, and how they are realized topologically by examples of or constructions with ultra-commutative ring spectra.

Example 3.1 (Burnside ring global functor)**.** The Burnside ring global functor $\mathbb{A} = \mathbf{A}(e, -)$ is initial in the category of global Green functors, and initial objects are examples of colimits. So Corollary 2.7 implies that $\mathbb{A}$ has a unique structure of global power functor, and with this structure it is an initial global power functor. Indeed, there is a unique morphism $P : \mathbb{A} \longrightarrow \exp(\mathbb{A})$ of global Green functors (since $\mathbb{A}$ is initial), and the coalgebra diagrams commute (again since $\mathbb{A}$ is initial), and the coalgebra diagrams commute. With these power operations, $\mathbb{A}$ is also an initial global power functor.

We can make the power operations in the Burnside ring functor more explicit. Indeed, the group $\mathbb{A}(G)$ is free abelian with a basis given by the elements $t_H = \operatorname{tr}_H^G(p_H^*(1))$ for every conjugacy class of subgroups $H \leq G$ with finite Weyl group, where $p_H : H \longrightarrow e$ is the unique homomorphism. On these generators, the naturality properties of a global power functor force the power operations to be given by

$$\begin{aligned} P^m(t_H) \ &= \ P^m(\operatorname{tr}_H^G(p_H^*(1))) \ = \ \operatorname{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}((\Sigma_m \wr p_H)^*(P^m(1))) \\ &= \ \operatorname{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}((\Sigma_m \wr p_H)^*(p_{\Sigma_m}^*(1))) \ = \ \operatorname{tr}_{\Sigma_m \wr H}^{\Sigma_m \wr G}(p_{\Sigma_m \wr H}^*(1)) \ = \ t_{\Sigma_m \wr H} \ . \end{aligned} \tag{3.2}$$

This determines the power operations in general by the additivity property, and shows the uniqueness.

When restricted to *finite* groups, the ring $\mathbb{A}(G)$ is isomorphic to the Grothendieck group of finite G-sets, and in this description the power operations are given by raising a finite G-set to a power, i.e., the power map

$$P^m \ : \ \mathbb{A}(G) \ \longrightarrow \ \mathbb{A}(\Sigma_m \wr G)$$

takes the class of a finite G-set S to the class of the $(\Sigma_m \wr G)$-set S^m. Indeed, for the additive generator $[G/H] = t_H$ of $\mathbb{A}(G)$ this is the relation (3.2), and for general finite G-sets it follows from the additivity formula for power operations and the fact that for two finite G-sets S and T the power $(S \coprod T)^m$ is $(\Sigma_m \wr G)$-equivariantly isomorphic to the coproduct

$$\coprod_{i=0}^{m} (\Sigma_m \wr G) \times_{(\Sigma_i \wr G) \times (\Sigma_{m-i} \wr G)} S^i \times T^{m-i} \ .$$

The canonical power operations in the Burnside ring global functor corresponds to the homotopy theoretic power operations for the global sphere spectrum. Indeed, since $\mathbb{A}$ is initial in both the category of global Green functors and in the category of global power functors, the isomorphism multiplicative isomorphism of global functors is automatically compatible with power operations. In other words, we can conclude that the square

$$\begin{tikzcd} \mathbb{A}(G) \arrow[r, "P^m"] \arrow[d, "\cong"'] & \mathbb{A}(\Sigma_m \wr G) \arrow[d, "\cong"] \\ \pi_0^G(\mathbb{S}) \arrow[r, "P^m"'] & \pi_0^{\Sigma_m \wr G}(\mathbb{S}) \end{tikzcd}$$

commutes for all G and m without even having to go back to the definition of the operations in $\underline{\pi}_0(\mathbb{S})$; the vertical maps are the action on the generator $1 \in \pi_0^e(\mathbb{S})$.

Example 3.3 (Coproducts)**.** We let R and S be two global Green functors. Global Green functors are the commutative monoids, with respect to the box product, in the category of global functors. So the box product $R \Box S$ is the coproduct in the category of global Green functors, with multiplication defined as the composite

$$(R \Box S) \Box (R \Box S) \ \xrightarrow{R \Box \tau_{S,R} \Box S} \ R \Box R \Box S \Box S \ \xrightarrow{\mu_R \Box \mu_S} \ R \Box S \ .$$

If $P : R \longrightarrow \exp(R)$ and $P' : S \longrightarrow \exp(S)$ are global power structures on R and S, then $R\Box S$ has preferred power operations specified by the morphism of global Green functors

$$R\Box S \xrightarrow{P\Box P'} \exp(R)\Box\exp(S) \longrightarrow \exp(R\Box S)$$

where the second morphism is the canonical one from the coproduct of exp to the values of exp at a coproduct. With these power operations, $R\Box S$ becomes a coproduct of R and S in the category of global power functors.

This abstract definition of the power operations on $R\Box S$ can be made more explicit. Indeed, the power operations on $R\Box S$ are determined by the fomula

$$P^n(x \times y) \ = \ \Delta^*(P^n(x) \times P^n(y))$$

for all compact Lie groups G and K and classes $x \in R(G)$ and $y \in S(K)$, and by the relations of the power operations. Here $\Delta : \Sigma_n \wr (G \times K) \longrightarrow (\Sigma_n \wr G) \times (\Sigma_n \wr K)$ is the diagonal monomorphism (see (1.12) of Chapter I).

The coproduct of global power functors is realized by the coproduct of ultra-commutative ring spectra in the following sense. If E and F are ultra-commutative ring spectra, then the ring spectra morphisms $E \longrightarrow E \wedge F$ and $F \longrightarrow E \wedge F$ induced morphisms of global power functors $\underline{\pi}_0(E) \longrightarrow \underline{\pi}_0(E \wedge F)$ and $\underline{\pi}_0(F) \longrightarrow \underline{\pi}_0(E \wedge F)$, and together they define a morphism of the coproduct of global power functors

$$\underline{\pi}_0(E)\,\Box\,\underline{\pi}_0(F) \ \longrightarrow \ \underline{\pi}_0(E \wedge F) \ .$$

If E and F are globally connective and at least one of them is flat as an orthogonal spectrum, then this is an isomorphism of global functors by Proposition IV.4.15, hence an isomorphism of global power functors.

Example 3.4 (Representable global functors)**.** The previous example generalizes to the representable global functor $\mathbf{A}(A, -)$ for every *abelian* compact Lie group, which has a preferred structure of a global power functor. The product of $\mathbf{A}(A, -)$ is the composite

$$\mathbf{A}(A, -)\Box\mathbf{A}(A, -) \ \cong \ \mathbf{A}(A \times A, -) \ \xrightarrow{\mathbf{A}(\mu^*, -)} \ \mathbf{A}(A, -) \ ,$$

where $\mu : A \times A \longrightarrow A$ is the commutative multiplication on A and $\mu^* \in \mathbf{A}(A, A \times A)$ is the associated restriction morphism. This representable global functor is freely generated by the identity 1_A in $\mathbf{A}(A, A)$, and the power operations are all determined by naturality from the effect on this generator, which is given by

$$P^m(1_A) \ = \ p_m^* \ \in \mathbf{A}(A, \Sigma_m \wr A) \ , \tag{3.5}$$

the restriction map of the continuous homomorphism

$$p_m \ : \ \Sigma_m \wr A \ \longrightarrow \ A \ , \quad (\sigma; a_1, \dots, a_m) \ \longmapsto \ a_1 \cdot \ldots \cdot a_m \ .$$

We could now justify purely algebraically that this multiplication and power operations make the representable global functor $\mathbf{A}(A, -)$ into a global power functor. Instead, we will show how $\mathbf{A}(A, -)$ is realized by an ultra-commutative ring spectrum, which then implies all the relations. As a global power functor, $\mathbf{A}(A, -)$ is in fact freely generated by the class 1_A subject only to the relations (3.5). In other words, for every global power functor R the evaluation map

$$(\text{global power functors})(\mathbf{A}(A, -), R) \ \longrightarrow \ \{x \in R(A) \mid P^m(x) = p_m^*(x) \text{ for all } m \geq 1\}$$

is bijective.

In Example II.2.30 we provided a global classifying space $B_{\mathrm{gl}}^{\otimes} A$ for the abelian group A with a commutative multiplication. The associated unreduced suspension spectrum $\Sigma^\infty_+ B_{\mathrm{gl}}^{\otimes} A$ is then an ultracommutative ring spectrum. The global functor $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}^{\otimes} A)$ is isomorphic to $\mathbf{A}(A, -)$ by evaluation at the stable tautological class $e_A \in \pi_0^A(\Sigma^\infty_+ B_{\mathrm{gl}}^{\otimes} A)$ (by Theorem III.3.13). The class e_A is the stabilization of the unstable tautological class $u_A \in \pi_0^A(B_{\mathrm{gl}}^{\otimes} A)$, whose m-th power is $P^m(u_A) = p_m^*(u_A)$ (by Proposition II.2.32). The stabilization map $\sigma : \underline{\pi}_0(B_{\mathrm{gl}}^{\otimes} A) \longrightarrow \underline{\pi}_0(\Sigma_+^\infty B_{\mathrm{gl}}^{\otimes} A)$ commutes with power operations, so this shows that

$$P^m(e_A) \ = \ p_m^*(e_A) \qquad \text{in } \pi_0^{\Sigma_m \wr A}(\Sigma_+^\infty B_{\mathrm{gl}}^{\otimes} A).$$

Construction 3.6 (Free global power functors)**.** For a compact Lie group K we construct a *free global power functor* C_K generated by K. The underlying global functor is

$$C_K \ = \ \bigoplus_{m \geq 0} \mathbf{A}(\Sigma_m \wr K, -) \ ,$$

the direct sum of the global functors represented by the wreath products $\Sigma_m \wr K$, including the trivial group $\Sigma_0 \wr K = e$. The multiplication $\mu : C_K \Box C_K \longrightarrow C_K$ that makes this into a global Green functor restricted to the (m,n)-summand is the morphism

$$\mathbf{A}(\Sigma_m \wr K, -) \Box \mathbf{A}(\Sigma_n \wr K, -) \ \longrightarrow \ \mathbf{A}((\Sigma_m \wr K) \times (\Sigma_n \wr K), -) \ \xrightarrow{\mathbf{A}(\Phi_{m,n}^*, -)} \ \mathbf{A}(\Sigma_{m+n} \wr K, -) \ \longrightarrow \ C_K \ ;$$

here $\Phi_{m,n}^*$ is the restriction map associated to the embedding (1.8)

$$\Phi_{m,n} \ : \ (\Sigma_m \wr K) \times (\Sigma_n \wr K) \ \longrightarrow \ \Sigma_{m+n} \wr K \ .$$

The multiplication is associative because

$$\Phi_{m+n,k} \circ (\Phi_{m,n} \times (\Sigma_k \wr K)) = \Phi_{m,n+k} \circ ((\Sigma_m \wr K) \times \Phi_{n,k}) \ : \ (\Sigma_m \wr K) \times (\Sigma_n \wr K) \times (\Sigma_k \wr K) \ \longrightarrow \ \Sigma_{m+n+k} \wr K \ .$$

The multiplication is commutative because the group homomorphisms

$$\Phi_{m,n} \ , \ \Phi_{n,m} \circ \tau_{\Sigma_m \wr K, \Sigma_n \wr K} \ : \ (\Sigma_m \wr K) \times (\Sigma_n \wr K) \longrightarrow \ \Sigma_{m+n} \wr K$$

are conjugate, so they represent the same morphism in $\mathbf{A}((\Sigma_m \wr K) \times (\Sigma_n \wr K), \ \Sigma_{m+n} \wr K)$. The unit is the inclusion $\mathbf{A}(e, -) \longrightarrow C_K$ of the summand indexed by $m = 0$.

The global Green functor C_K can be made into a global power functor in a unique way such that the relation

$$P^m(1_K) \ = \ 1_{\Sigma_m \wr K}$$

holds in the m-th summand of $C_K(\Sigma_m \wr K)$, where $1_K \in \mathbf{A}(K, K)$ and $1_{\Sigma_m \wr K} \in \mathbf{A}(\Sigma_m \wr K, \Sigma_m \wr K)$ are the identity maps. Indeed, C_K is generated as a global functor by the classes $1_{\Sigma_m \wr K}$ for all $k \geq 0$, so there is at most one such global power structure, and every morphism of global power functors out of C_K is determined by its values on the class 1_K. The existence of a global power structure on C_K with this property could be justified purely algebraically, but we show it by realizing C_K by an ultracommutative ring spectrum.

The unreduced suspension spectrum

$$\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}} K) \ \cong \ \bigvee_{m \geq 0} \Sigma_+^\infty B_{\mathrm{gl}}(\Sigma_m \wr K)$$

of the free commutative orthogonal monoid space (compare Example II.1.16) generated by a global classifying space of K is an ultra-commutative ring spectrum. According to Proposition III.3.13, its 0-th homotopy group global functor is given additively by

$$\underline{\pi}_0(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}} K)) \ \cong \ \bigoplus_{m \geq 0} \underline{\pi}_0(\Sigma_+^\infty B_{\mathrm{gl}}(\Sigma_m \wr K)) \ \cong \ \bigoplus_{m \geq 0} \mathbf{A}(\Sigma_m \wr K, -) \ .$$

Under this isomorphism, the stable tautological class $e_K \in \pi_0^K(\Sigma_+^\infty B_{\mathrm{gl}} K)$ maps to the generator $1_K \in \mathbf{A}(K, K)$. The class e_K is the stabilization of the unstable tautological class $u_K \in \pi_0^K(B_{\mathrm{gl}} K)$, whose m-th power is $P^m(u_K) = u_{\Sigma_m \wr K}$ (see (1.18) of Chapter I). The stabilization map $\sigma : \underline{\pi}_0(B_{\mathrm{gl}} K) \longrightarrow \underline{\pi}_0(\Sigma_+^\infty B_{\mathrm{gl}} K)$ commutes with power operations, so this shows that $P^m(e_K) = e_{\Sigma_m \wr K}$ in $\pi_0^{\Sigma_m \wr K}(\Sigma_+^\infty \mathbb{P}(B_{\mathrm{gl}} K))$.

We warn the reader that the previous example of the free global power functor C_K is *not* the symmetric algebra, with respect to the box product of global functors, of the represented global functor $\mathbf{A}(K,-)$. The issue is that the global functors

$$\mathbf{A}(K,-)^{\Box m}/\Sigma_m \;\cong\; \mathbf{A}(K^m,-)/\Sigma_m \qquad \text{and} \qquad \mathbf{A}(\Sigma_m \wr K,-)$$

are typically not isomorphic. The restriction map $\mathrm{res}^{\Sigma_m \wr K}_{K^m} \in \mathbf{A}(\Sigma_m \wr K, K^m)$ induces a morphism of represented global functors

$$- \circ \mathrm{res}^{\Sigma_m \wr K}_{K^m} \;:\; \mathbf{A}(K^m,-) \longrightarrow \mathbf{A}(\Sigma_m \wr K,-)$$

that equalizes the Σ_m-action on the source because every permutation of the factors of K^m becomes an inner automorphism in $\Sigma_m \wr K$. So the morphism factors over a morphism of global functors

$$\mathbf{A}(K^m,-)/\Sigma_m \;\longrightarrow\; \mathbf{A}(\Sigma_m \wr K,-)$$

which, however, is generally *not* an isomorphism (already for $K = e$ and $m = 2$). The box product symmetric algebra

$$\bigoplus_{m \geq 0} \mathbf{A}(K,-)^{\Box m}/\Sigma_m \;\cong\; \bigoplus_{m\geq 0} \mathbf{A}(K^m,-)/\Sigma_m$$

also has a universal property: it is the free global Green functor generated by K; however, this box product symmetric algebra does not seem to have natural power operations, so it is of little interest for our topological applications.

Example 3.7 (Limits)**.** The category of global power functors has limits, and the are defined 'groupwise'. First we observe that the category of global Green functors has limits, calculated groupwise. [???] Moreover, the functor exp of exponential sequences commutes with limits since the 'exponential' property can be checked componentwise. If is then a formal consequence that limits are created on the underlying global power functors.

A products is a special case of a limit, and the product of global power functors is realized by the product of ultra-commutative ring spectra. If E and F are ultra-commutative ring spectra, then so is the product $E \times F$ of the underlying orthogonal spectra, and the canonical map

$$\underline{\pi}_0(E \times F) \;\longrightarrow\; \underline{\pi}_0(E) \times \underline{\pi}_0(F)$$

is an isomorphism of global power functors.

Example 3.8 (Constant global power functors and Eilenberg-Mac Lane spectra)**.** We let B be a commutative ring. Then the constant global functor $\underline{B}$ (Example III.3.24 (ii)) becomes a global power functor via the multiplication of B and the power operations

$$P^m \;:\; \underline{B}(G) \;= B \;\longrightarrow\; B = \underline{B}(\Sigma_m \wr G)\;, \quad b \;\longmapsto\; b^m$$

given by powers in the ring B. In fact, since the operation P^m has to be an equivariant refinement of the m-th power map in $R(G)$ (compare Remark II.1.11), this is the only possibility to define power operations on a constant global functor.

Since $\underline{B}$ is constant, every morphism $R \longrightarrow \underline{B}$ of global functors is determined by the map $R(e) \longrightarrow \underline{B}(e) = B$. Moreover, every ring homomorphism $\psi : R(e) \longrightarrow B$ extends uniquely to a morphism of global power functors $\hat{\psi} : R \longrightarrow \underline{B}$ by defining its value at a compact Lie group as the composite

$$R(G) \;\xrightarrow{\;p_G^*\;}\; R(e) \;\xrightarrow{\;\psi\;}\; B\;.$$

In other words, the functor

$$(\text{commutative rings}) \;\longrightarrow\; (\text{global power functors})\;, \quad B \;\longmapsto\; \underline{B}$$

is right adjoint to the functor that takes a global power functor R to the ring $R(e)$.

The author does not know an explicit pointset level model for an ultra-commutative ring spectrum that realizes the constant global power functor $\underline{B}$. The Eilenberg-Mac Lane spectrum HB, discussed in Section V.5 below, is an ultra-commutative ring spectrum and tries to realize $\underline{B}$: the global power functor $\underline{\pi}_0(HB)$ is indeed constant *on finite groups*, but the restriction maps are not generally isomorphisms, compare Proposition V.5.5. The morphism of global power functors $\underline{\pi}_0(HB) \longrightarrow \underline{B}$ adjoint to the identification $\pi_0^e(HB) \cong B$ is thus an isomorphism at finite groups (and some other compact Lie groups), but *not* an isomorphism in general.

Example 3.9 (Monoid rings)**.** Let R be a global Green functor and M a commutative monoid. We denote by $R[M]$ the *monoid ring functor*; its value at a finite group G is given by

$$(R[M])(G) \ = \ R(G)[M] \ ,$$

the monoid ring of M over $R(G)$. The structure as global functor is induced from the structure of R and constant in M. The multiplication and unit are induced from the multiplication and units of R and M.

Now suppose that R is even a global power functor. Then $R[M]$ inherits a natural structure as global power functor; indeed, we define the power operation

$$P^n \ : \ R(G)[M] \ \longrightarrow \ R(\Sigma_n \wr G)[M]$$

by $P^n(r \cdot m) = P^n(r) \cdot m^n$ for $r \in R(G)$ and $m \in M$, and then we extend this by additivity to general elements in $R(G)[M]$.

We let E be an ultra-commutative ring spectrum and M a commutative monoid. Then the monoid ring spectrum $E[M] = E \wedge \Sigma^\infty_+ M$ is another ultra-commutative ring spectrum with

$$(\pi_0 E)[M] \ \longrightarrow \ \pi_0(E[M])$$

an isomorphism of global power functors [show].

Example 3.10 (Representation ring global functor)**.** As G varies over all compact Lie groups, the complex representation rings $\mathbf{R}(G)$ form the *representation ring global functor* $\mathbf{R}$. This is classical in the restricted realm of finite groups, but somewhat less familiar for compact Lie groups in general. The restriction maps $\alpha^* : \mathbf{R}(G) \longrightarrow \mathbf{R}(K)$ are induced by restriction of representations along a homomorphism $\alpha : K \longrightarrow G$. The transfer maps $\mathrm{tr}_H^G : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ along a closed subgroup inclusion $H \leq G$ are given by the *smooth induction* of Segal [**80**, § 2]. If H is a subgroup of finite index of G, then this induction sends the class of an H-representation to the induced G-representation $\mathrm{map}^G(H, V)$ (which is then isomorphic to $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$); in general, induction sends actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for $\mathbf{R}$ was proved by Snaith [**87**, Thm. 2.4]. The representation rings also have well-known power operations

$$P^m \ : \ \mathbf{R}(G) \ \longrightarrow \ \mathbf{R}(\Sigma_m \wr G) \ ;$$

on the class of a G-representation V, the power operation is represented by the tensor power,

$$P^m[V] \ = \ [V^{\otimes m}]$$

using the canonical action of $\Sigma_m \wr G$ on $V^{\otimes m}$. By additivity, this determines the power operations in general. The representation ring global functor $\mathbf{R}$ ought to be realized by the periodic global K-theory spectrum $\mathbf{KU}$ (compare Construction 6.34 below), but the author has not verified that.

Remark 3.11. By [**80**, Prop. 3.11 (ii)] the representation ring $\mathbf{R}(G)$ is generated as an abelian group by all (virtual) representations that are induced from 1-dimensional representations; this uses the general smooth induction for not necessarily finite index subgroups. In our language that can be expressed by saying that the representation ring global functor $\mathbf{R}$ is 'cyclic' in the sense that it is generated by a single element, the class $x \in \mathbf{R}(T)$ of the tautological 1-dimensional representation of the circle group $T = U(1)$. Equivalently, the morphism of global functors

$$\mathbf{A}(T, -) \ \longrightarrow \ \mathbf{R}$$

classified by the element x is an epimorphism. We recall the argument: we let $i : T \times U(n-1) \longrightarrow U(n)$ be the block sum inclusion and $p_1 : T \times U(n-1) \longrightarrow T$ the projection to the first factor. The character formula [**80**, p. 119] for induced representations shows that the element

$$i_!(p_1^*(x)) \in \mathbf{R}(U(n))$$

has the same character as the tautological n-dimensional representation of $U(n)$. Since characters determine complex representations of compact Lie groups, $i_!(p_1^*(x))$ equals the class of the tautological representation τ_n of $U(n)$. Any unitary representation of a compact Lie group G of dimension n is isomorphic to $\alpha^*(\tau_n)$ for a continuous homomorphism $\alpha : G \longrightarrow U(n)$; so the class of such a representation equals

$$\alpha^*(i_!(p_1^*(x))) \in \mathbf{R}(G) .$$

So the global functor $\mathbf{R}$ is generated by the single class $x = \tau_1$.

Since the class $x \in \mathbf{R}(T)$ satisfies the relation $P^m(x) = p_m^*(x)$ for all $m \geq 1$, the evaluation at x is even a morphism of global power functors, by the universal property of $\mathbf{A}(T, -)$ spelled out in Example 3.4. We can also exhibit additional 'global relations' for the representation ring functor, i.e., elements that generate the kernel as a global functor, at least on all finite groups. In the representation ring global functor we have

$$\mathrm{tr}_{T\times U(n-1)}^{U(n)}(p_1^*(x)) = \tau_n \in \mathbf{R}(U(n)) ,$$

the class of the tautological $U(n)$-representation on $\mathbb{C}^n$, where again $x = \tau_1 \in \mathbf{R}(T)$ and $p_1 : T\times U(n-1) \longrightarrow T$ is the projection to the first factor. So

$$\mathrm{res}_{\Sigma_n \wr T}^{U(n)}\left(\mathrm{tr}_{T\times U(n-1)}^{U(n)}(p_1^*(x))\right) = \mathrm{res}_{\Sigma_n\wr T}^{U(n)}(\tau_n) \in \mathbf{R}(\Sigma_n \wr T) .$$

is the class of the tautological $(\Sigma_n \wr T)$-representation on $\mathbb{C}^n$. On the other hand,

$$\mathrm{tr}_{T\times(\Sigma_{n-1}\wr T)}^{\Sigma_n\wr T}\left(\mathrm{res}_{T\times(\Sigma_{n-1}\wr T)}^{T\times U(n-1)}(p_1^*(x))\right) = \mathrm{tr}_{T\times(\Sigma_{n-1}\wr T)}^{\Sigma_n\wr T}(q_1^*(x))$$

is the same class [...], where $q_1 : T \times (\Sigma_{n-1} \wr T) \longrightarrow T$ is the projection to the first factor. In other words,

$$\mathrm{res}_{\Sigma_n \wr T}^{U(n)}\left(\mathrm{tr}_{T\times U(n-1)}^{U(n)}(p_1^*(x))\right) = \mathrm{tr}_{T\times(\Sigma_{n-1}\wr T)}^{\Sigma_n\wr T}(p_1^*(x)) \quad \text{in } \mathbf{R}(\Sigma_n \wr T).$$

The class

$$\rho_n = \mathrm{res}_{\Sigma_n\wr T}^{U(n)} \circ \mathrm{tr}_{T\times U(n-1)}^{U(n)} \circ p_1^* - \mathrm{tr}_{T\times(\Sigma_{n-1}\wr T)}^{\Sigma_n\wr T} \circ q_1^* \in \mathbf{A}(T, \Sigma_n \wr T)$$

is thus in the kernel of the evaluation map $\mathbf{A}(T, \Sigma_n \wr T) \longrightarrow \mathbf{R}(\Sigma_n \wr T)$.

Example 3.12. The composite $\mathrm{res}_{\Sigma_n\wr T}^{U(n)} \circ \mathrm{tr}_{T\times U(n-1)}^{U(n)}$ can be expressed, using the double coset formula, in the basis of $\mathbf{A}(T, \Sigma_n \wr T)$ given by transfers after restrictions (compare Theorem III.3.5); this gives the expansion of the element ρ_n in this basis. We do this for $n = 2$: the double coset formula (3.20) of Chapter III shows that

$$\mathrm{res}_{\Sigma_2\wr T}^{U(2)} \circ \mathrm{tr}_{T^2}^{U(2)} - \mathrm{tr}_{T^2}^{\Sigma_2\wr T} = \mathrm{tr}_{\Sigma_2\times\Delta(T)}^{\Sigma_2\wr T} \circ \mathrm{res}_{\Sigma_2\times\Delta(T)}^{{}^gT^2} \circ c_g^*$$

where $g = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$. The composite

$$\Sigma_2 \times \Delta(T) \xrightarrow{\mathrm{incl}} {}^gT^2 \xrightarrow{c_g} T^2 \xrightarrow{p_1} T$$

is the projection $q : \Sigma_2 \times \Delta(T) \longrightarrow T$, so we conclude that

$$\rho_2 = \mathrm{tr}_{\Sigma_2\times\Delta(T)}^{\Sigma_2\wr T} \circ \mathrm{res}_{\Sigma_2\times\Delta(T)}^{{}^gT^2} \circ c_g^* \circ p_1^* = \mathrm{tr}_{\Sigma_2\times\Delta(T)}^{\Sigma_2\wr T} \circ q^* \in \mathbf{A}(T, \Sigma_2 \wr T) .$$

Proposition 3.13. *Let J denote the global subfunctor of $\mathbf{A}(T,-)$ generated by the classes $\rho_n \in \mathbf{A}(T, \Sigma_n \wr T)$ for all $n \geq 2$. Then the morphism of global functors*

$$\mathbf{A}(T,-)/J \longrightarrow \mathbf{R}$$

that sends $u \cdot J(T)$ to $x \in \mathbf{R}(T)$ is an isomorphism at all finite groups.

Proof. We let G be any compact Lie group, $H \leq G$ a subgroup of finite index m and $\alpha : H \longrightarrow T$ a character. We define a homomorphism $\lambda : G \longrightarrow \Sigma_m \wr T$ as follows. We choose a bijection $\psi : G/H \cong \{1,\dots,m\}$ (i.e., we enumerate the cosets of H in G) and use it to turn the left translation action of G on itself into a homomorphism $\Psi : G \longrightarrow \Sigma_m \wr H$ [...]. The homomorphism $\lambda : G \longrightarrow \Sigma_m \wr T$ is the composite

$$G \xrightarrow{\ \Psi\ } \Sigma_m \wr H \xrightarrow{\Sigma_m \wr \alpha} \Sigma_m \wr T \ .$$

Then λ satisfies

$$\lambda^* \langle \mathbb{C}^m \rangle \ = \ \mathrm{tr}_H^G(\alpha^*(x))$$

in $\mathbf{R}(G)$, where $\mathbb{C}^m$ is the tautological $\Sigma_m \wr T$-representation. We claim that in addition the relation

$$\mathrm{tr}_H^G \circ \alpha^* \ = \ \lambda^* \circ \mathrm{tr}_{T \times (\Sigma_{m-1} \wr T)}^{\Sigma_m \wr T} \circ q_1^*$$

holds in $\mathbf{A}(T,G)$. To see this we set $C = \alpha(H)$ and factor α as

$$H \xrightarrow{\ \bar{\alpha}\ } C \xrightarrow{\ i\ } T \ .$$

Then

$$\begin{aligned}
\mathrm{tr}_H^G \circ \alpha^* \ &= \ \Psi^* \circ \mathrm{tr}_{H \times (\Sigma_{m-1} \wr H)}^{\Sigma_m \wr H} \circ q_H^* \circ \alpha^* \\
&= \ \Psi^* \circ \mathrm{tr}_{H \times (\Sigma_{m-1} \wr H)}^{\Sigma_m \wr H} \circ (\bar{\alpha} \times (\Sigma_{m-1} \wr \bar{\alpha}))^* \circ q_C^* \\
&= \ \Psi^* \circ (\Sigma_m \wr \bar{\alpha})^* \circ \mathrm{tr}_{C \times (\Sigma_{m-1} \wr C)}^{\Sigma_m \wr C} \circ q_C^* \\
&= \ \Psi^* \circ (\Sigma_m \wr \bar{\alpha})^* \circ \mathrm{tr}_{C \times (\Sigma_{m-1} \wr C)}^{\Sigma_m \wr C} \circ \mathrm{res}_{C \times (\Sigma_{m-1} \wr C)}^{T \times (\Sigma_{m-1} \wr T)} \circ q_1^* \\
&= \ \Psi^* \circ (\Sigma_m \wr \bar{\alpha})^* \circ \mathrm{res}_{\Sigma_m \wr C}^{\Sigma_m \wr T} \circ \mathrm{tr}_{T \times (\Sigma_{m-1} \wr T)}^{\Sigma_m \wr T} \circ q_1^* \\
&= \ \Psi^* \circ (\Sigma_m \wr \alpha)^* \circ \mathrm{tr}_{T \times (\Sigma_{m-1} \wr T)}^{\Sigma_m \wr T} \circ q_1^* \\
&= \ \lambda^* \circ \mathrm{tr}_{T \times (\Sigma_{m-1} \wr T)}^{\Sigma_m \wr T} \circ q_1^* \ .
\end{aligned}$$

The fifth relation is the double coset formula, using that

$$\Sigma_m \wr C \setminus \Sigma_m \wr T \,/\, T \times (\Sigma_{m-1} \wr T)$$

has only one element and

$$(\Sigma_m \wr C) \cap (T \times (\Sigma_{m-1} \wr T)) \ = \ C \times (\Sigma_{m-1} \wr C) \ .$$

Now we consider a finite sequence

$$(H_1, \alpha_1) \ , \dots \ , (H_n, \alpha_n)$$

of subhomomorphisms $G \longrightarrow T$. We set $m_i = [G : H_i]$ and $d = \sum_{i=1}^n m_i = \sum_{i=1}^n [G : H_i]$. We let $\lambda_i : G \longrightarrow \Sigma_{m_i} \wr T$ be the homomorphism defined in the previous paragraph and

$$\lambda \ : \ G \ \longrightarrow \ \Sigma_d \wr T$$

the composite

$$G \xrightarrow{(\lambda_i)} \prod_{i=1}^n \Sigma_{m_i} \wr T \xrightarrow{\ i\ } \Sigma_d \wr T \ .$$

Then λ satisfies

$$\lambda^*\langle\mathbb{C}^d\rangle \;=\; \bigoplus_{i=1}^n \mathrm{tr}_{H_i}^G(\alpha_i^*(x))$$

in $\mathbf{R}(G)$, and

$$\begin{aligned}\lambda^*\circ \mathrm{tr}_{T\times(\Sigma_{d-1}\wr T)}^{\Sigma_d\wr T}\circ q_1^* \;&=\; (\lambda_1,\dots,\lambda_n)^*\circ \mathrm{res}^{\Sigma_d\wr T}_{\prod \Sigma_{m_i}\wr T}\circ \mathrm{tr}_{T\times(\Sigma_{d-1}\wr T)}^{\Sigma_d\wr T}\circ q_1^*\\ &=\; (\lambda_1,\dots,\lambda_n)^*\circ\left(\sum_{i=1}^n \mathrm{proj}_i^*\circ \mathrm{tr}_{T\times(\Sigma_{m_i-1}\wr T)}^{\Sigma_{m_i}\wr T}\circ q_1^*\right)\\ &=\; \sum_{i=1}^n \lambda_i^*\circ \mathrm{tr}_{T\times(\Sigma_{m_i-1}\wr T)}^{\Sigma_{m_i}\wr T}\circ q_1^* \;=\; \sum_{i=1}^n \mathrm{tr}_{H_i}^G\circ\alpha_i^*\end{aligned}$$

in $\mathbf{A}(T,G)$, where

$$\mathrm{proj}_i \;:\; \prod_{i=1}^n \Sigma_{m_i}\wr T \;\longrightarrow\; \Sigma_{m_i}\wr T$$

is the projection to the i-th factor.

Now we consider an element $z\in\mathbf{A}(T,G)$ in the kernel of the map $\bar{x}:\mathbf{A}(T,G)\longrightarrow\mathbf{R}(G)$. We suppose that z lies in the subgroup generated by the operations $\mathrm{tr}_H^G\circ\alpha^*$ for which H has finite index in G (and not just in its normalizer). We write z as

$$z \;=\; \left(\sum_{i=1}^n \mathrm{tr}_{H_i}^G\circ\alpha_i^*\right) - \left(\sum_{j=1}^m \mathrm{tr}_{K_j}^G\circ\beta_j^*\right)$$

for suitable subhomomorphisms [...]. Then

$$d \;=\; \sum_{i=1}^n [G:H_i] \;=\; \sum_{j=1}^m [G:K_j]\ .$$

We let $\lambda,\rho : G\longrightarrow\Sigma_d\wr T$ be two homomorphisms constructed as in the previous paragraph from the pairs (H_i,α_i) respectively the pairs (K_j,β_j). In particular, these homomorphisms satisfy

$$\lambda^*\langle\mathbb{C}^d\rangle \;=\; \bigoplus_{i=1}^n \mathrm{tr}_{H_i}^G(\alpha_i^*(x)) \qquad \text{respectively} \qquad \rho^*\langle\mathbb{C}^d\rangle \;=\; \bigoplus_{j=1}^m \mathrm{tr}_{K_j}^G(\beta_j^*(x))\ .$$

The assumption that z lies in the kernel of $\mathbf{A}(T,G)\longrightarrow\mathbf{R}(G)$ means that $\lambda^*\langle\mathbb{C}^d\rangle=\rho^*\langle\mathbb{C}^d\rangle$ in $\mathbf{R}(G)$, so the underlying G-representations are isomorphic. So the homomorphisms λ and ρ become conjugate after composition with the inclusion $\Sigma_d\wr T\longrightarrow U(d)$. So for every global functor F the restriction homomorphisms

$$\lambda^*\ ,\rho^* \;:\; F(\Sigma_d\wr T) \;\longrightarrow\; F(G)$$

coincide on all classes that are in the image of the restriction map $F(U(d))\longrightarrow F(\Sigma_d\wr T)$. Since we have divided out the class $\rho_d\in\mathbf{A}(A,\Sigma_d\wr T)$, the class $\mathrm{tr}_{T\times(\Sigma_{d-1}\wr T)}^{\Sigma_d\wr T}\circ q_1^*$ is in the image of the restriction map

$$\mathrm{res}^{U(d)}_{\Sigma_d\wr T} \;:\; \mathbf{A}(T,U(d))/J(U(d)) \;\longrightarrow\; \mathbf{A}(T,\Sigma_d\wr T)/J(\Sigma_d\wr T)\ .$$

So in the global functor $\mathbf{A}(T,-)/J$, the restriction maps along λ and ρ coincide on the class $\mathrm{tr}_{T\times(\Sigma_{n-1}\wr T)}^{\Sigma_n\wr T}\circ q_1^*$. So we obtain that

$$\begin{aligned}\sum_{i=1}^n \mathrm{tr}_{H_i}^G\circ\alpha_i^* \ &= \ \lambda^*\circ\mathrm{tr}_{T\times(\Sigma_{d-1}\wr T)}^{\Sigma_d\wr T}\ q_1^* \\ &= \ \rho^*\circ\mathrm{tr}_{T\times(\Sigma_{d-1}\wr T)}^{\Sigma_w r T}\circ q_1^* \ = \ \sum_{j=1}^m \mathrm{tr}_{K_j}^G\circ\beta_j^*\end{aligned}$$

in the group $\mathbf{A}(T,G)/J(G)$. In other words, the original element z in $\mathbf{A}(T,G)$ maps to zero in $\mathbf{A}(T,G)/J(G)$, so $z\in J(G)$. Since z was an arbitrary element of the kernel of $\mathbf{A}(T,G)\longrightarrow\mathbf{R}(G)$, we conclude that this kernel coincides with $J(G)$. □

Example 3.14. The right adjoint $R:\mathcal{SH}\longrightarrow\mathcal{GH}$ to the forgetful functor from the global to the non-equivariant stable homotopy category is modeled on the pointset level by the Borel functor $b:\mathcal{S}p\longrightarrow\mathcal{S}p$ discussed in Section III.7. The global homotopy type of bE is that of a Borel cohomology theory, and in particular,

$$\pi_0^G(bE)\ \cong\ E^0(BG)\ ,$$

natural in G for transfers and restriction maps. The functor b is lax symmetric monoidal, so it takes a commutative orthogonal ring spectrum R to an ultra-commutative ring spectrum bR; the power operations

$$P^m\ :\ \pi_0^G(bE)\ \longrightarrow\ \pi_0^{\Sigma_m\wr G}(bE)$$

correspond to the classical power operations [ref?]

$$P^m\ :\ E^0(BG)\ \longrightarrow\ E^0(B(\Sigma_m\wr G))\ .$$

Remark 3.15 (G_∞ ring spectra)**.** In the non-equivariant situation, power operations on a ring valued cohomology theory already arise from a weaker structure than a commutative (or equivalently E_∞ multiplication): all that is needed is an H_∞-structure, compare [**23**, I.§4]. We recall that an *H_∞-structure* is an algebra structure over the monad

$$L\mathbb{P}\ :\ \mathcal{SH}\ \longrightarrow\ \mathcal{SH}$$

on the stable homotopy category that can be obtained be suitably deriving the 'symmetric algebra' monad

$$\mathbb{P}\ :\ \mathcal{S}p\ \longrightarrow\ \mathcal{S}p$$

on the category of orthogonal spectra (whose algebras are commutative orthogonal ring spectra).

This suggests a global analog of an H_∞-structure that, for lack of better name, we call a *G_∞-structure.* Indeed, the 'symmetric algebra' monad $\mathbb{P}$ takes global equivalences between flat orthogonal spectra to global equivalences between flat orthogonal spectra [ref], so it can also be derived with respect to global equivalences. The result is a monad

$$\mathbb{G}\ =\ L_{\mathrm{gl}}\mathbb{P}\ :\ \mathcal{GH}\ \longrightarrow\ \mathcal{GH}$$

on the global stable homotopy category, whose algebras we call G_∞ ring spectra.

The right adjoint $R:\mathcal{SH}\longrightarrow\mathcal{GH}$ takes non-equivariant H_∞ multiplications into global G_∞ multiplications [explain], so we arrive at a square of forgetful functors between categories of structured ring spectra with different degrees of commutativity:

$$\begin{tikzcd}\mathrm{Ho}(\text{ultra-commutative ring spectra}) \arrow[r] \arrow[d] & (G_\infty\text{ ring spectra}) \arrow[d] \\ \mathrm{Ho}(\text{commutative ring spectra}) \arrow[r] & (H_\infty\text{ ring spectra})\end{tikzcd}$$

We emphasize that, like H_∞ ring spectra, the category of G_∞ ring spectra is not the homotopy category of any natural model category.

Now we show that every global power functor is realized by an ultra-commutative ring spectrum. More is true: the next theorem effectively constructs a right adjoint functor

$$H \ : \ \text{(global power functors)} \ \longrightarrow \ \mathrm{Ho}^{\text{gl. conn.}}\text{(ultra-com ring spectra)}$$

to the functor $\underline{\pi}_0$ such that the adjunction counit is an isomorphism $\underline{\pi}_0(HR) \cong R$ of global power functors. The analogous result in equivariant stable homotopy theory for a fixed finite group has been obtained by Ullman [**96**] by the same method of proof.

Theorem 3.16. *Let R be a global power functor.*

(i) *There is an ultra-commutative ring spectrum HR such that $\underline{\pi}_k(HR) = 0$ for all $k \neq 0$ and an isomorphism of global power functors*

$$\underline{\pi}_0(HR) \ \cong \ R \ .$$

(ii) *For every globally connective ultra-commutative ring spectrum T, the functor $\underline{\pi}_0$ restricts to a bijection*

$$\underline{\pi}_0 \ : \ \mathrm{Ho}(\text{ultra-com ring spectra})(T, HR) \ \cong \ \text{(global power functors)}(\underline{\pi}_0(T), R) \ .$$

Proof. We sketch the proof (and intend to add full details later). We choose an index set I, compact Lie groups G_i and elements $x_i \in R(G_i)$, for $i \in I$, that altogether generate R as a global power functor. We start with the free commutative orthogonal ring spectrum

$$T \ = \ \Sigma^\infty_+ \mathbb{P}(\amalg_{i\in I} B_{\mathrm{gl}} G_i) \ .$$

By an elaboration of Example 3.6, the orthogonal spectrum T is globally connective and $\underline{\pi}_0(T)$ is the free global power functor generated by the classes e_i (for $i \in I$), that are obtained from the stable tautological classes $e_{G_i} \in \pi_0^G(\Sigma^\infty_+ B_{\mathrm{gl}} G_i)$ by pushing forward along the inclusion

$$\Sigma^\infty_+ B_{\mathrm{gl}} G_i \ \longrightarrow \ T$$

as the linear part of the i-th summand. So there is a unique morphism of global power functors

$$\epsilon \ : \ \underline{\pi}_0(T) \ \longrightarrow \ R$$

sending e_i to x_i, and this morphism is an epimorphism.

Now we kill the kernel of ϵ and all higher homotopy groups of T by attaching free commutative ring spectrum cells. In more detail, we choose an index set J, compact Lie groups G_j and elements $y_j \in \pi_0^{G_j}(T)$, for $j \in J$, in the kernel of ϵ that altogether generate this kernel as a global power ideal. We represent each class y_j as a morphism of orthogonal spectra

$$f_j \ : \ \Sigma^\infty_+ B_{\mathrm{gl}} G_j \ \longrightarrow \ T$$

that sends the stable tautological class e_{G_j} to y_j; this involves an implicit choice of faithful G_j-representation. We form the wedge of all these morphisms and freely extend that to a morphism of commutative orthogonal ring spectra

$$F \ : \ \mathbb{P}(\bigvee_{j\in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j) \ \longrightarrow \ T \ .$$

The unique morphisms from $B_{\mathrm{gl}} G_j$ to the constant one-point orthogonal space $*$ induces a morphism of orthogonal spectra $\Sigma^\infty_+ B_{\mathrm{gl}} G_j \longrightarrow \Sigma^\infty_+ * = \mathbb{S}$, which frees up to morphism of commutative orthogonal ring spectra from the source of F to $\mathbb{P}(\Sigma^\infty_+ *)$ to the sphere spectrum. We let T_1 be the homotopy pushout, in the category of ultra-commutative ring spectra, of the diagram

$$\mathbb{S} \ \longleftarrow \ \mathbb{P}(\bigvee_{j\in J} \Sigma^\infty_+ B_{\mathrm{gl}} G_j) \ \xrightarrow{F} \ T \ .$$

The resulting morphism of ring spectra $T \longrightarrow T_1$ induces an epimorphism of global power functors

$$\underline{\pi}_0(T) \longrightarrow \underline{\pi}_0(T_1)$$

that annihilates all the classes y_j. So it factors over an isomorphism of global power functors $R \cong \underline{\pi}_0(T_1)$.

The orthogonal spectrum T_1 is still globally connective. Now we successively kill all the positive dimensional homotopy group global functors $\underline{\pi}_k$ by similar commutative ring spectrum cell attachments and then define HR as the colimit of the resulting sequence of commutative orthogonal ring spectra. □

4. Fixed point spectra

In this section we consider fixed point spectra, i.e., orthogonal spectra $F^G X$ whose non-equivariant stable homotopy groups are isomorphic to the G-equivariant stable homotopy groups of X. The construction is the spectrum analog of the orthogonal fixed point spaces discussed in Construction I.1.17.

Construction 4.1. Given an orthogonal spectrum X and a finite group G we define a new orthogonal spectrum $F^G X$, the *G-fixed point spectrum*, at an inner product space V by

$$(F^G X)(V) \;=\; \operatorname{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) \;. \tag{4.2}$$

Here the source of the mapping space uses the *reduced* regular representation $\bar\rho_G$, whereas the target uses the regular representation ρ_G. The orthogonal group $O(V)$ acts by conjugation, through the two actions on V; this $O(V)$-action commutes with the G-action, so it restricts to an $O(V)$-action on the space of G-equivariant maps.

The structure map $(F^G X)(V)\wedge S^W \longrightarrow (F^G X)(V\oplus W)$ is the composite

$$\begin{aligned}
\operatorname{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G))\wedge S^W &\xrightarrow{\text{assembly}} \operatorname{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)\wedge S^W)\\
&\xrightarrow{-\wedge S^{W\otimes\bar\rho_G}} \operatorname{map}^G(S^{V\otimes\bar\rho_G}\wedge S^{W\otimes\bar\rho_G}, X(V\otimes\rho_G)\wedge S^W\wedge S^{W\otimes\bar\rho_G})\\
&\xrightarrow{\ \cong\ } \operatorname{map}^G(S^{(V\oplus W)\otimes\bar\rho_G}, X(V\otimes\rho_G)\wedge S^{W\otimes\rho_G})\\
&\xrightarrow{(\sigma_{V\otimes\rho_G, W\otimes\rho_G})_*} \operatorname{map}^G(S^{(V\oplus W)\otimes\bar\rho_G}, X((V\oplus W)\otimes\rho_G))
\end{aligned}$$

where among other things we have used the equivariant isometry

$$W\oplus(W\otimes\bar\rho_G)\cong W\otimes\rho_G\;,\quad (w,x)\;\longmapsto\;\frac{w}{\sqrt{|G|}}\otimes\sum\nolimits_{g\in G} g \;+\; x\;. \tag{4.3}$$

The functor $\Omega^\bullet$ from orthogonal spectra to orthogonal spaces commutes with the fixed point constructions in a sense we explain now. For every inner product space W we obtain a homeomorphism

$$\begin{aligned}
(\Omega^\bullet(F^G X))(W) \;&=\; \operatorname{map}(S^W, \operatorname{map}^G(S^{W\otimes\bar\rho_G}, X(W\otimes\rho_G)))\\
&\cong\; \operatorname{map}(S^{W\otimes\rho_G}, X(W\otimes\rho_G))) \;=\; (F^G(\Omega^\bullet X))(W)\;,
\end{aligned}$$

by adjointness of $\operatorname{map}^G(S^{W\otimes\bar\rho_G},-)$ and $-\wedge S^{W\otimes\bar\rho_G}$ and the $(O(W)\times G)$-equivariant isometry (4.3). As W varies, these maps form a natural isomorphism of orthogonal spaces

$$\Omega^\bullet(F^G X) \;\cong\; F^G(\Omega^\bullet X)\;. \tag{4.4}$$

The composite

$$\pi_0^K(F^G X) \;=\; \pi_0^K(\Omega^\bullet(F^G X)) \;\cong\; \pi_0^K(F^G(\Omega^\bullet X)) \xrightarrow{\ l^K\ } \pi_0^{K\times G}(\Omega^\bullet X) \;=\; \pi_0^{K\times G}(X) \tag{4.5}$$

is then a natural isomorphism of $\mathrm{Rep}^{\mathrm{op}}$-functors by Proposition I.1.19. In other words, the K-equivariant homotopy groups of the fixed point spectrum $F^G X$ calculate the $(K\times G)$-homotopy groups of X, compatible with restriction maps in K. Unraveling the definitions show that the composite (4.5) sends the class of a K-equivariant map

$$f\;:\;S^V \;\longrightarrow\; \operatorname{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) = (F^G X)(V)\;,$$

for some K-representation V, to the class $l^K[f] \in \pi_0^{K\times G}(X)$ represented by the $(K\times G)$-equivariant composite

$$S^{V\otimes\rho_G} \cong S^V \wedge S^{V\otimes\bar\rho_G} \xrightarrow{\hat f} X(V\otimes\rho_G)$$

where $\hat f$ is the adjoint of f.

But more is true: the passage from X to $F^G X$ corresponds to translation by G on the level of homotopy global functors. We recall from (4.24) of Chapter III that the G-translate $\tau_G R$ is defined as the composite

$$\mathbf{A} \xrightarrow{-\times G} \mathbf{A} \xrightarrow{R} \mathcal{A}b \ .$$

In particular, $\tau_G R$ is given on objects by

$$(\tau_G R)(K) = R(K\times G) \ .$$

Proposition 4.6. *For every finite group G and every compact Lie group K, the bijection*

$$l^K : \pi_0^K(F^G X) \longrightarrow \pi_0^{K\times G}(X)$$

is additive, hence an isomorphism of abelian groups. Moreover, as K varies, these maps commute with transfers, so they form an isomorphism of global functors

$$l : \underline{\pi}_0(F^G X) \cong \tau_G(\underline{\pi}_0 X) \ , \tag{4.7}$$

natural in the orthogonal spectrum X.

Proof. □

The G-fixed points of an orthogonal spectrum X receive a natural morphism of orthogonal spectra

$$X \xrightarrow{j} F^G X$$

whose V-th level

$$j(V) : X(V) \longrightarrow \mathrm{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G)) = (F^G X)(V)$$

is adjoint to the G-map

$$X(V)\wedge S^{V\otimes\bar\rho_G} \xrightarrow{\sigma_{V,V\otimes\bar\rho_G}} X(V\oplus(V\otimes\bar\rho_G)) \cong X(V\otimes\rho_G) \ .$$

In the same sense as for orthogonal spaces, the effect of j on K-equivariant homotopy groups agrees with the restriction map $p_K^* : \pi_0^K(X) \longrightarrow \pi_0^{K\times G}(X)$.

The fixed point constructions and translation interact nicely with multiplicative structures, as we will now describe. For every finite group G, the fixed point functor F^G is lax symmetric monoidal: given orthogonal spectra X and Y, a natural map

$$\mu_{V,W}^{X,Y} : (F^G X)(V)\wedge(F^G Y)(W) \longrightarrow (F^G(X\wedge Y))(V\oplus W)$$

is defined as the composite

$$\begin{aligned}
&\mathrm{map}^G(S^{V\otimes\bar\rho_G}, X(V\otimes\rho_G))\wedge\mathrm{map}^G(S^{W\otimes\bar\rho_G}, Y(W\otimes\rho_G))\\
&\qquad\xrightarrow{\wedge} \mathrm{map}^G(S^{V\otimes\bar\rho_G}\wedge S^{W\otimes\bar\rho_G}, X(V\otimes\rho_G)\wedge Y(W\otimes\rho_G))\\
&\qquad\xrightarrow{(i_{V\otimes\rho_G,W\otimes\rho_G})_*} \mathrm{map}^G(S^{(V\oplus W)\otimes\bar\rho_G}, (X\wedge Y)((V\oplus W)\otimes\rho_G)) \ .
\end{aligned}$$

As V and W vary, these maps form a bimorphism, so the universal property of the smash product provides a morphism of orthogonal spectra

$$\mu^{X,Y} : (F^G X)\wedge(F^G Y) \longrightarrow F^G(X\wedge Y) \ .$$

A unit morphism is given by $j : \mathbb{S}\longrightarrow F^G\mathbb{S}$. The maps $\mu_{V,W}^{X,Y}$, and hence the morphisms $\mu^{X,Y}$ are suitably associative, commutative and unital, i.e., they give the G-fixed point functor a lax symmetric monoidal structure. Under the isomorphism of Proposition 4.6, the morphism $\mu^{X,Y}$ realizes the homotopy group pairing (4.6) defined in Chapter III, in the sense that the diagram

$$\begin{tikzcd}
\pi_0^K(F^G X) \times \pi_0^L(F^G Y) \arrow[r, "l^K \times l^L"] \arrow[d, "\times"'] & \pi_0^{K\times G}(X) \times \pi_0^{L\times G}(Y) \arrow[d, "\times"] \\
\pi_0^{K\times L}((F^G X) \wedge (F^G Y)) \arrow[d, "\pi_0^{K\times L}(\mu^{X,Y})"'] & \pi_0^{K\times G\times L\times G}(X \wedge Y) \arrow[d, "\Delta^*"] \\
\pi_0^{K\times L}(F^G(X\wedge Y)) \arrow[r, "l^{K\times L}"'] & \pi_0^{K\times L\times G}(X\wedge Y)
\end{tikzcd} \tag{4.8}$$

commutes, where $\Delta : K \times L \times G \longrightarrow K \times G \times L \times G$ is the diagonal defined by $\Delta(k, l, g) = (k, g, l, g)$.

If R is an orthogonal ring spectrum, then the lax monoidal structure on F^G provides a product on the fixed point spectrum $F^G R$, given by the composite

$$(F^G R) \wedge (F^G R) \xrightarrow{\mu^{R,R}} F^G(R \wedge R) \xrightarrow{F^G(\mu_R)} F^G R ,$$

and the unit map is $(F^G\eta) \circ j_{\mathbb{S}} = j_R \circ \eta : \mathbb{S} \longrightarrow F^G R$. If the multiplication of R is commutative, then so is the induced multiplication of the fixed point spectrum $F^G R$. In this case the global functors $\underline{\pi}_0(R)$ and $\underline{\pi}_0(F^G R)$ also come with power operations, and the isomorphism l matches them in the sense that the following diagram

$$\begin{tikzcd}
\pi_0^K(F^G R) \arrow[r, "l^K"] \arrow[dd, "P^m"'] & \pi_0^{K\times G}(R) \arrow[d, "P^m"] \\
& \pi_0^{\Sigma_m \wr (K\times G)}(R) \arrow[d, "\Delta^*"] \\
\pi_0^{\Sigma_m \wr K}(F^G R) \arrow[r, "l^{\Sigma_m \wr K}"'] & \pi_0^{(\Sigma_m \wr K)\times G}(R)
\end{tikzcd} \tag{4.9}$$

commutes, where $\Delta : (\Sigma_m \wr K) \times G \longrightarrow \Sigma_m \wr (K \times G)$ is the partial diagonal given by

$$((\sigma; k_1, \dots, k_m), g) \longmapsto (\sigma; (k_1, g), \dots, (k_m, g)) .$$

Now we discuss the algebraic analog of the fixed point construction F^G, namely translation of global functors by a compact Lie group G (not necessarily finite). In fact, if we want the isomorphism of global functors $l : \underline{\pi}_0(F^G R) \cong \tau_G(\underline{\pi}_0(R))$ to preserve multiplications and power operations, then the construction is dictated by the diagrams (4.8) and (4.9).

Construction 4.10. We let R be a global Green functor. We define an external multiplication on $\tau_G R$ as the composite

$$\begin{aligned}(\tau_G R)(K) \otimes (\tau_G R)(L) \ = \ R(K \times G) \otimes R(L \times G) \ &\xrightarrow{\times} \ R(K \times G \times L \times G) \\ &\xrightarrow{\Delta^*} \ R(K \times L \times G) \ = \ (\tau_G)(K \times L) .\end{aligned}$$

Here $\Delta : K \times L \times G \longrightarrow K \times G \times L \times G$ is the partial diagonal given by $\Delta(k, l, g) = (k, g, l, g)$. This makes $\tau_G R$ into a global Green functor. Moreover, the resulting internal product is such that $(\tau_G R)(G)$ has the same ring structure as $R(K \times G)$.

Now we suppose that R is even a global power functor; we endow the global Green functor $\tau_G R$ with induced power operations: the n-th operation $P^n : (\tau_G R)(K) \longrightarrow (\tau_G R)(\Sigma_n \wr K)$ is simply the composite

$$R(K \times G) \xrightarrow{P^m} R(\Sigma_m \wr (K \times G)) \xrightarrow{\Delta^*} R(K \times (\Sigma_m \wr G)) .$$

The commutative diagrams (4.8) and (4.9) show:

Corollary 4.11. *For every ultra-commutative ring spectrum E and every finite group G, the morphism* (4.7)

$$l \ : \ \underline{\pi}_0(F^G E) \ \longrightarrow \ \tau_G(\underline{\pi}_0 E)$$

is an isomorphism of global power functors.

Example 4.12 (Endomorphism ring spectra)**.** The orthogonal function spectrum $\mathrm{Hom}(X,Y)$ of two orthogonal spectra was defined in (4.2) of Chapter III; its V-th level is given by $\mathrm{Hom}(X,Y)(V) = \mathrm{map}(X, \mathrm{sh}^V Y)$. We can define associative and unital composition morphisms

$$\circ \ : \ \mathrm{Hom}(Y,Z) \wedge \mathrm{Hom}(X,Y) \ \longrightarrow \ \mathrm{Hom}(X,Z)$$

for any other orthogonal spectrum Z as follows. The composition morphism corresponds to the bimorphism with (V,W)-component $\circ_{V,W} : \mathrm{Hom}(Y,Z)(V) \wedge \mathrm{Hom}(X,Y)(W) \longrightarrow \mathrm{Hom}(X,Z)(V \oplus W)$ defined as the composite

$$\begin{aligned} \mathrm{map}(Y, \mathrm{sh}^V Z) \wedge \mathrm{map}(X, \mathrm{sh}^W Y) &\xrightarrow{\mathrm{sh}^W \wedge \mathrm{Id}} \mathrm{map}(\mathrm{sh}^W Y, \mathrm{sh}^W(\mathrm{sh}^V Z)) \wedge \mathrm{map}(X, \mathrm{sh}^W Y) \\ &\cong \mathrm{map}(\mathrm{sh}^W Y, \mathrm{sh}^{V\oplus W} Z) \wedge \mathrm{map}(X, \mathrm{sh}^W Y) \xrightarrow{\ \circ\ } \mathrm{map}(X, \mathrm{sh}^{V \oplus W} Z) \end{aligned}$$

where the isomorphism $\mathrm{sh}^W(\mathrm{sh}^V Z) \cong \mathrm{sh}^{V\oplus W} Z$ is given at an inner product space U by

$$(\mathrm{sh}^W(\mathrm{sh}^V Z))(U) = Z(V \oplus (W \oplus U)) \ \cong \ Z((V \oplus W) \oplus U) = (\mathrm{sh}^{V \oplus W} Z)(U)$$

and the last map is composition.

The *endomorphism ring spectrum* $\mathrm{Hom}(X,X)$ of an orthogonal spectrum is the special case when $X = Y$. The composition maps make the orthogonal function spectrum $\mathrm{Hom}(X,X)$ into an orthogonal ring spectrum.

Proposition 4.13. *Let X and Y be orthogonal spectra and G a compact Lie group.*

(i)
$$F^G Y \ \cong \ R\,\mathrm{Hom}(B_{\mathrm{gl}} G, Y)$$
and hence
$$[\![X, F^G Y]\!] \ \cong \ [\![(B_{\mathrm{gl}} G)_+ \wedge X, Y]\!] \ .$$

(ii) *If X is flat and Y is a global Ω-spectrum, then*
$$\pi_0^G \, \mathrm{Hom}(X,Y) \ \cong \ [\![(B_{\mathrm{gl}} G)_+ \wedge X, Y]\!] \ \cong [\![X, F^G Y]\!] \ .$$

5. Eilenberg-Mac Lane spectra

In Remark IV.4.12 we have constructed an Eilenberg-Mac Lane spectrum HM associated to a global functor M, an orthogonal spectrum satisfying

$$\underline{\pi}_k(HM) \ \cong \ \begin{cases} M & \text{for } k = 0, \text{ and} \\ 0 & \text{for } k \neq 0. \end{cases}$$

If M has the extra multiplicative structure of a global power functor, then Theorem 3.16 shows that HM can be realized as an ultra-commutative ring spectrum. In all these constructions, the global homotopy type of HM is determined by the algebraic input data up to preferred isomorphism (in the appropriate homotopy category), but the constructions are abstract versions of 'killing homotopy groups' and do not yield explicit pointset level models for Eilenberg-Mac Lane spectra. In this section we discuss a well-known pointset level construction that yields Eilenberg-Mac Lane spectra – at least when retricted to finite groups.

Construction 5.1. Let M be an abelian group. The *Eilenberg-Mac Lane spectrum* HM is defined at an inner product space V by

$$(HM)(V) \ = \ M[S^V] \ , \tag{5.2}$$

the reduced M-linearization of the V-sphere. The orthogonal group $O(V)$ acts through the action on S^V and the generalized structure map $\sigma_{V,W} : (HM)(V) \wedge S^W \longrightarrow (HM)(V \oplus W)$ is given by

$$M[S^V] \wedge S^W \ \longrightarrow \ M[S^{V\oplus W}] \ , \quad \left(\sum\nolimits_i a_i \cdot v_i\right) \wedge w \ \longmapsto \ \sum\nolimits_i a_i \cdot (v_i \wedge w) \ .$$

The underlying non-equivariant space of $M[S^V]$ is an Eilenberg-Mac Lane space of type (M, n), where $n = \dim(V)$. Hence the underlying non-equivariant homotopy type of HM is that of an Eilenberg-Mac Lane spectrum for M.

As we shall now discuss, the equivariant homotopy groups of HM are in general *not* concentrated in dimension 0, and hence HM is not the Eilenberg-Mac Lane spectrum on a global functor. However, the on *finite* groups, the equivariant behaviour of HM is as expected. We recall from Definition IV.5.7 that an orthogonal spectrum *left induced* from the global family $\mathcal{F}in$ of finite groups if it is in the essential image of the left adjoint $L_{\mathcal{F}} : \mathcal{GH}_{\mathcal{F}in} \longrightarrow \mathcal{GH}$ from the $\mathcal{F}in$-global homotopy category. We let $\underline{M}$ denote the constant global functor with value M, compare Example III.3.24 (ii). By the Dold-Thom theorem, the non-equivariant homotopy group $\pi_0^e(HM)$ is naturally isomorphic to M. The restriction map

$$\mathrm{res}^G_e \ : \ \pi_k^G(HM) \ \longrightarrow \ \pi_0^e(HM) \cong M \ = \ \underline{M}(G)$$

form a morphism of global functors $\underline{\pi}_0(HM) \longrightarrow \underline{M}$. Since HM is globally connective, there is a unique morphism

$$\rho \ : \ HM \ \longrightarrow \ H\underline{M}$$

in the global homotopy category that realizes the morphism on $\underline{\pi}_0$.

Proposition 5.3. *For every abelian group M the Eilenberg-Mac Lane spectrum HM is left induced from the global family $\mathcal{F}in$ of finite groups and HM is a $\mathcal{F}in$-global Ω-spectrum. The morphism $\rho : HM \longrightarrow H\underline{M}$ is a $\mathcal{F}in$-global equivalence.*

Proof. The orthogonal spectrum HM is obtained by evaluation of a $\mathbf{\Gamma}$-space $\underline{M}$ on spheres, where $\underline{M}(S) = M[S]$ is the reduced M-linearization of a finite basedset S. So the first claim is a special case of Proposition IV.5.16 (ii).

Dos Santos shows in [**68**] that for every finite group G and every G-representation V the G-space $HM(V) = M[S^V]$ is an equivariant Eilenberg-Mac Lane space of type (M, V), i.e., the G-space $\mathrm{map}(S^V, M[S^V])$ has homotopically discrete fixed points for all subgroups of G and the natural map

$$M \ \longrightarrow \ [S^V, M[S^V]]^H = \pi_0 \, \mathrm{map}^H(S^V, M[S^v])$$

sending $m \in M$ to the homotopy class of $m \cdot - : S^V \longrightarrow M[S^V]$ is an isomorphism. This shows that HM is a $\mathcal{F}in$-Ω-spectrum for the constant global functor $\underline{M}$.

The result is also a special case of the earlier work of Segal and Shimakawa's on $\mathbf{\Gamma}_G$-spaces [**79, 81**]. Indeed, we can view the $\mathbf{\Gamma}$-space $\underline{M}$ as a $\mathbf{\Gamma}_G$-space by letting G act trivially. For every finite G-set S, the maps

$$P_S \ : \ \underline{M}(S) \ \longrightarrow \ \underline{M}^S$$

are then homeomorphisms, so in particular G-homotopy equivalences, and $\underline{M}$ is a very special $\mathbf{\Gamma}$-G-space in the sense of Shimakawa [**81**, Def. 1.3]. Since $\pi_0(\underline{M}(1^+)$ is a group (as opposed to a monoid only), Shimakawa's Theorem B proves that the adjoint structure maps $\tilde{\sigma}_{V,W} : M[S^V] \longrightarrow \mathrm{map}(S^W, M[S^{V\oplus W}])$ are G-weak equivalences. □

Slightly more is true: the proof of Proposition 5.3 shows that for finite groups G, the orthogonal G-spectrum $HM\langle G\rangle$ is a full fledged G-Ω-spectrum, i.e., the adjoint structure maps are equivariant weak equivalences starting at arbitrary G-representations (and not just at faithful representations).

These properties mentioned in the previous proposition do *not* generalize to compact Lie groups of positive dimensions, i.e., contrary to what one may expect at first, $HM\langle G\rangle$ is not generally a G-Ω-spectrum, not all restriction maps are isomorphisms, and the groups $\pi_*^G(HM)$ need not be concentrated in dimension 0.

Example 5.4. The equivariant Ω-spectrum property of HM already fails for the circle group $T = U(1)$. We consider the tautological T-representation on $\mathbb{C}$, i.e., the action by scalar multiplication. Then the map

$$M \longrightarrow \left(M[S^{\mathbb{C}}]\right)^T \ , \quad m \longmapsto m\cdot[0]$$

is an isomorphism by Proposition III.8.33. where we denote by $[0]\in S^{\mathbb{C}}$ the zero vector. The composite map

$$M[S^0] \xrightarrow{\tilde\sigma^T_{0,\mathbb{C}}} \operatorname{map}^T(S^{\mathbb{C}},\, M[S^{\mathbb{C}}]) \xrightarrow{\operatorname{ev}_{[0]}} \left(M[S^{\mathbb{C}}]\right)^T$$

is thus a homeomorphism. On the other hand, the evaluation at $[0]$ is induced by the fixed point inclusion $(S^{\mathbb{C}})^T \longrightarrow S^{\mathbb{C}}$ and it is a Serre fibration; its fiber over the basepoint is the mapping space

$$\operatorname{map}^T(S^{\mathbb{C}}/S^0,\, M[S^{\mathbb{C}}]) \ \cong\ \operatorname{map}^T(T_+\wedge S^1,\, M[S^{\mathbb{C}}]) \ \cong\ \Omega\left(M[S^{\mathbb{C}}]\right) \ .$$

Non-equivariantly $M[S^{\mathbb{C}}]$ is an Eilenberg-Mac Lane space of type $(M,2)$, so the fiber is an Eilenberg-Mac Lane space of type $(M,1)$, and hence the adjoint structure map

$$\tilde\sigma_{0,\mathbb{C}} \ :\ M[S^0] \longrightarrow \operatorname{map}(S^{\mathbb{C}}, M[S^{\mathbb{C}}])$$

is *not* a T-weak equivalence.

The equivariant homotopy group $\pi_0^G(H\mathbb{Z})$ may be larger than a single copy of the integers, and we are now going to give a presentation of $\pi_0^G(H\mathbb{Z})$. The key ingredient is that we can identify the geometric fixed point homotopy group $\Phi_0^G(H\mathbb{Z})$ of a compact Lie group G with the geometric fixed point homotopy group $\Phi_0^{\pi_0 G}(H\mathbb{Z})$ of the finite group $\pi_0 G$ of components. This, again, is a consequence of the fact that the orthogonal spectrum $H\mathbb{Z}$ is obtained from a $\mathbf{\Gamma}$-space by evaluation on spheres. The following proposition may seem like a mere technical result at first, but it has the conceptual reformulation that all orthogonal spectra obtained from $\mathbf{\Gamma}$-spaces by evaluation on spheres a 'left induced' from the family of finite groups.

Proposition 5.5. *Let G be a compact Lie group.*

(i) *The group $\pi_0^G(H\mathbb{Z})$ is freely generated by the classes $\operatorname{tr}_H^G(1)$ where H runs through a set of representatives of the conjugacy classes of subgroups of G with finite Weyl group, modulo the relations*

$$[H:J]\cdot \operatorname{tr}_H^G(1) \ =\ \begin{cases} \operatorname{tr}_J^G(1) & \text{if } W_G J \text{ is finite, respectively} \\ \quad 0 & \text{if } W_G J \text{ is infinite,}\end{cases}$$

for all nested sequences of subgroup $J\le H\le G$ such that $W_G H$ is finite and J has finite index in H.

(ii) *The quotient of the group $\pi_0^G(H\mathbb{Z})$ by its torsion subgroup is a free abelian group with basis given by the classes $\operatorname{tr}_{N_G H}^G(1)$ where H runs through a set of representatives of the conjugacy classes of those connected subgroups H of G whose Weyl group is finite.*

Proof. (i) Since the orthogonal spectrum $H\mathbb{Z}$ is globally connective and left induced from the global family of finite groups, the morphism (5.18) of Chapter IV

$$\operatorname{ind}_{\mathcal{F}}(\underline{\pi}_0^{\mathcal{F}in}(H\mathbb{Z})) \ \longrightarrow\ \underline{\pi}_0(H\mathbb{Z})$$

is an isomorphism by Theorem IV.5.19. Moreover, Proposition IV.5.22 allows us to calculate the values of the induced global functor on the left hand side from the values of $\underline{\pi}_0(H\mathbb{Z})$ on finite groups, where it is the constant $\mathcal{F}in$-global functor with value $\mathbb{Z}$.

When we specialize Proposition IV.5.22 to $M = \underline{\mathbb{Z}}$, the constant $\mathcal{F}in$-global functor with value $\mathbb{Z}$, we arrive at the following presentation. For every compact Lie group G we then obtain an isomorphism

$$\mathbb{Z}\{t_H \ : \ (H) \leq G, \ |WH| < \infty\}/R \ \xrightarrow{\cong} \ (\mathrm{ind}_{\mathcal{F}in}\,\underline{\mathbb{Z}})(G)$$

where the generator t_H on the left hand side maps to the class $\mathrm{tr}_H^G(q_H^*(1)) = \mathrm{tr}_H^G(1)$ on the right hand side. The subgroup R of relations is generated by the classes

$$[H : J] \cdot t_H \ - \ t_J$$

for all pairs of nested subgroups $J \leq H \leq G$ such that $W_G H$ is finite and $J^\circ = H^\circ$ (so that $\pi_0 J$ is a subgroup of $\pi_0 H$ and $[H : J] = [\pi_0 H : \pi_0 J]$). If J has infinite Weyl group in G, then the relation is to be read as $[\pi_0 H : \pi_0 J] \cdot t_H = 0$.

(ii) We simplify the presentation given by part (i). The group $\pi_0^G(H\mathbb{Z})$ is additively generated by the transfers $t_H = \mathrm{tr}_H^G(1)$ for all subgroups H of G whose Weyl group is finite, and one in each conjugacy class suffices. The relations specified in Proposition 5.5 show that many of these generators are redundant.

The class t_H behaves in one of two ways, depending on the Weyl group $W_G(H^\circ)$ of the connected component H° of the identity element. We emphasize that H° is the identity path component of H, which may be strictly smaller than $H \cap G^\circ$. Since H° is normal in H, the group H is always contained in the normalizer $N_G(H^\circ)$ of its identity component.

- If the Weyl group $W_G(H^\circ)$ is infinite, then $[H : H^\circ] \cdot t_H = 0$, so the class t_H is a torsion element of $\pi_0^G(H\mathbb{Z})$.
- If the Weyl group $W_G(H^\circ)$ is finite, then H°, and hence also H, has finite index in the normalizer $N_G(H^\circ)$. So in this case

$$t_H \ = \ [N_G(H^\circ) : H] \cdot t_{N_G(H^\circ)} \ .$$

We conclude that the classes $t_{N_G(H^\circ)}$ generate $\pi_0^G(H\mathbb{Z})$ modulo torsion, as H° runs through the *connected* subgroups of G with finite Weyl group. Moreover, all relations given by part (i) have been taken into account, so the classes $t_{N_G(H^\circ)}$ for a $\mathbb{Z}$-basis of $\pi_0^G(H\mathbb{Z})$ modulo torsion. □

Example 5.6. For every finite group G, the group $\pi_0^G(H\mathbb{Z})$ is free of rank 1, generated by the multiplicative unit 1, and the restriction map $p_G^* : \pi_0^e(H\mathbb{Z}) \longrightarrow \pi_0^G(H\mathbb{Z})$ is an isomorphism. Proposition 5.5 allows us to identify the class of compact Lie group for which share this property.

We let G be a compact Lie group whose identity component G° is abelian. Then every connected subgroup H of G is abelian as well. If such a connected subgroup H is strictly contained in G°, then H has strictly smaller dimension and its normalizer contains G°, so the Weyl group $W_G H$ is infinite. So the transfer from the proper connected subgroup of G do not contribute to $\pi_0^G(H\mathbb{Z})$. So the only connected subgroup that contributes to the basis of $\pi_0^G(H\mathbb{Z})$ modulo torsion described in Proposition 5.5 is G°, and so $\pi_0^G(H\mathbb{Z})$ has rank 1.

If G° is not abelian, we let T be a maximal torus of G°. The Weyl group $W_{G^\circ}(T)$ is finite, and hence $W_G T$ is finite as well. The classes 1 and $\mathrm{tr}_{N_G T}^G(1)$ are then linearly independent in $\pi_0^G(H\mathbb{Z})$ by Proposition 5.5, so the rank of this group is bigger than 1. We conclude that $\pi_0^G(H\mathbb{Z})$ is free abelian of rank 1 if and only if G° is abelian.

The smallest example for which $\pi_0^G(H\mathbb{Z})$ has rank bigger than 1 is thus $G = SU(2)$. Here there are three conjugacy classes of connected subgroups: the trivial subgroup, the conjugacy class of the 1-dimensional maximal torus

$$T \ = \ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ | \ \lambda \in U(1) \right\} \ ,$$

and the full group $SU(2)$. Among these, the maximal torus T and $SU(2)$ have finite Weyl groups, so the classes 1 and $\mathrm{tr}_{N_{SU(2)}T}^{SU(2)}(1)$ are a $\mathbb{Z}$-basis for $\pi_0^{SU(2)}(H\mathbb{Z})$ modulo torsion.

Example 5.7 (Infinite symmetric product)**.** There is no essential difference if we consider the infinite symmetric product Sp^∞ (i.e., the reduced free abelian monoid) instead of the reduced free abelian group $\mathbb{Z}[-]$ generated by representation spheres: the levelwise inclusions of the free abelian monoid into the free abelian groups provide a morphism of commutative orthogonal ring spectra

$$Sp^\infty(\mathbb{S}) \;=\; \{Sp^\infty(S^n)\}_{n\geq 0} \;\longrightarrow\; \{\mathbb{Z}[S^n]\}_{n\geq 0} \;=\; H\mathbb{Z}\ , \tag{5.8}$$

and this morphism is a global equivalence by part (ii) of the following proposition. The infinite symmetric product of any based space X has a natural filtration

$$X = Sp^1(X) \subseteq Sp^2(X) \subseteq \dots \subseteq Sp^n(X) \subseteq \dots$$

by the finite symmetric products. By evaluation on spheres, this provides a filtration

$$\mathbb{S} = Sp^1(\mathbb{S}) \subseteq Sp^2(\mathbb{S}) \subseteq \dots \subseteq Sp^n(\mathbb{S}) \subseteq \dots$$

of $Sp^\infty(\mathbb{S})$ by orthogonal subspectra. We study the equivariant and global properties of this filtration in [**75**].

In his unpublished preprint [**79**], Segal argues that that for every finite group G and every G-representation V with $V^G \neq 0$ the map

$$Sp^\infty(S^V) \;\longrightarrow\; \mathbb{Z}[S^V]$$

is a G-weak equivalence. A published proof of this fact appears as Proposition A.6 in Dugger's paper [**28**]. This generalizes to compact Lie groups:

Proposition 5.9. *Let G be a compact Lie group.*

(i) *For every G-representation V such that $V^G \neq 0$, the natural map $Sp^\infty(S^V) \longrightarrow \mathbb{Z}[S^V]$ is a G-weak equivalence.*
(ii) *The morphism* (5.8) *is a global equivalence of orthogonal spectra from $Sp^\infty(\mathbb{S})$ to $H\mathbb{Z}$.*

Proof. (i) It suffices to show (by application to all closed subgroups of G) that the fixed point map $(Sp^\infty(S^V))^G \longrightarrow (\mathbb{Z}[S^V])^G$ is a non-equivariant weak equivalence. We let $G^\circ \leq G$ denote the connected component of the identity element and $\bar{G} = G/G^\circ$ the finite group of components of G. To calculate G-fixed points we can first take G°-fixed points and then $\bar{G}$-fixed points. Proposition III.8.33 applied to the $\mathbf{\Gamma}$-spaces Sp^∞ and $\mathbb{Z}[-]$ (with trivial G-action) provides homeomorphisms

$$(Sp^\infty(S^V))^G \;\cong\; (Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \qquad \text{and} \qquad (\mathbb{Z}[S^V])^G \cong\; (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}}\ .$$

Since V^{G° is an orthogonal representation of the finite group $\bar{G}$ the map

$$(Sp^\infty(S^{V^{G^\circ}}))^{\bar{G}} \;\longrightarrow\; (\mathbb{Z}[S^{V^{G^\circ}}])^{\bar{G}}\ ,$$

is a weak equivalence by [**28**, Prop. A.6]. Strictly speaking, Dugger's proposition is stated only for geometric realizations of $\bar{G}$-simplicial sets; since every $\bar{G}$-CW-complex, such as the representation sphere $S^{V^{G^\circ}}$, is $\bar{G}$-homotopy equivalence to the realization of a $\bar{G}$-simplicial sets, we can also apply it in our situation. Part (ii) is then immediate from (i). □

6. Global K-theory

In this section we define and discuss various global flavors of topological K-theory. We start with a *connective global K-theory* **ku**, an elaboration of a model of non-equivariant connective K-theory by Segal [**78**], constructed from certain equivariant $\mathbf{\Gamma}$-spaces of 'orthogonal subspaces in the symmetric algebra'. Then we recall a model **KU** for *periodic global K-theory.* A certain homotopy pullback of the periodic theory **KU**, its associated global Borel theory, and the global Borel theory of connective K-theory define *global connective K-theory*, a global version of Greenlees 'equivariant connective K-theory' $\mathbf{ku}^c$ [**38**]. One should note the different order of the adjectives 'global' and 'connective', indicating that **ku** and $\mathbf{ku}^c$ are quite different global homotopy types (with the same underlying non-equivariant homotopy type).

Construction 6.1. We let $\mathcal{U}$ be a complex vector space of countable dimension (finite or infinite) equipped with a hermitian inner product. We recall a certain $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ of 'orthogonal subspaces in $\mathcal{U}$', due to Segal [**78**, Sec. 1]. The $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ is special whenever $\mathcal{U}$ in infinite dimensional; so the associated orthogonal spectrum $\mathscr{C}(\mathcal{U})(\mathbb{S})$, obtained by evaluating $\mathscr{C}$ on spheres (compare (8.5) of Chapter IV), is a positive Ω-spectrum and a (non-equivariant) model for connective complex topological K-theory.

For a finite based set A we let $\mathscr{C}(\mathcal{U}, A)$ be the space of tuples (V_a), indexed by the non-basepoint elements of A, of finite dimensional, pairwise orthogonal subspaces of $\mathcal{U}$. The topology on $\mathscr{C}(\mathcal{U}, A)$ is that of a disjoint union of subspaces of a product of Grassmannians. The basepoint of $\mathscr{C}(\mathcal{U}, A)$ is the tuple where each V_a is the zero subspace. For a based map $\alpha : A \longrightarrow B$ the induced map $\mathscr{C}(\mathcal{U}, \alpha) : \mathscr{C}(\mathcal{U}, A) \longrightarrow \mathscr{C}(\mathcal{U}, B)$ sends (V_a) to (W_b) where

$$W_b \;=\; \bigoplus_{\alpha(a)=b} V_a \ .$$

Then $\mathscr{C}(\mathcal{U})$ is a $\mathbf{\Gamma}$-space whose underlying space is

$$\mathscr{C}(\mathcal{U}, 1^+) \;=\; \coprod_{n\geq 0} \mathrm{Gr}_n(\mathcal{U})$$

the disjoint union of the different Grassmannians of $\mathcal{U}$. Of course, if $\mathcal{U}$ is finite dimensional, then $\mathrm{Gr}_n(\mathcal{U})$ is empty when n exceeds the dimension of $\mathcal{U}$.

Every $\mathbf{\Gamma}$-space can be evaluated on a space by a coend construction, compare (8.2) of Chapter IV. We write $\mathscr{C}(\mathcal{U}, K) = \mathscr{C}(\mathcal{U})(K)$ for the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ on a space K. There are at least two useful ways to interpret the space $\mathscr{C}(\mathcal{U}, K)$, for a compact based space K: as unordered configurations labeled by vector spaces, and as a space of C^*-algebra homomorphisms. For a based topological space K, elements of $\mathscr{C}(\mathcal{U}, K)$ are 'labeled configurations'. A point is represented by a tuple

$$(V_1, \dots, V_n;\, k_1, \dots, k_n)$$

where $k_1, \dots, k_n$ are points of K and $(V_1, \dots, V_n) \in \mathscr{C}(\mathcal{U}, n^+)$ is an n-tuple of finite dimensional, pairwise orthogonal subspaces of $\mathcal{U}$, for some n. The topology is such that, informally speaking, the labels are summed up whenever two points collide and a label disappears whenever a point approaches the basepoint of K. When K is compact, the space $\mathscr{C}(\mathcal{U}, K)$ can be described differently, compare again [**78**, Sec. 1], namely as the space

$$\bigcup_{U\subset\mathcal{U}} \mathrm{hom}_{C^*}(C_0(K), \mathrm{End}_{\mathbb{C}}(U)) \ ;$$

on K that vanish at the point at infinity, and $\mathrm{hom}_{C^*}(-,-)$ is the space of C^*-algebra homomorphisms, with the subspace topology of the compact open topology on the space of all continuous maps; the basepoint is the zero map. The union runs over all finite dimensional subspaces U of the universe $\mathcal{U}$, and is by extensions of an endomorphism on the orthogonal complement. A homeomorphism

$$\mathscr{C}(\mathcal{U}, K) \;\longrightarrow\; \bigcup_{U\subset\mathcal{U}} \mathrm{hom}_{C^*}(C_0(K), \mathrm{End}_{\mathbb{C}}(U))$$

is given by sending a configuration $[V_1,\dots,V_n;\,k_1,\dots,k_n]$ in $\mathscr{C}(\mathcal{U},K)$ to the homomorphism that takes a function $\varphi\in C_0(K)$ to

$$\sum_{i=1}^n \varphi(k_i)\cdot p_{V_i}\ ,$$

where $p_{V_i}:\mathcal{U}\longrightarrow\mathcal{U}$ is the orthogonal projection onto the subspace V_i.

If $\mathcal{U}$ is infinite dimensional, then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ is special, compare Theorem 6.8 (i) below (with G the trivial group). The orthogonal spectrum $\mathscr{C}(\mathcal{U})(\mathbb{S})$ is then a positive Ω-spectrum by the general theory. In particular, the space $\mathscr{C}(\mathcal{U})(\mathbb{S})_1=\mathscr{C}(\mathcal{U},S^1)$ is an infinite loop space. In fact, $\mathscr{C}(\mathcal{U},S^1)$ is a familiar space, namely the infinite unitary group $U(\mathcal{U})$, i.e., the group of linear self-isometries of $\mathcal{U}$ that are the identity on the orthogonal complement of some finite dimensional subspace. To construct a homeomorphism from $\mathscr{C}(\mathcal{U},S^1)$ to $U(\mathcal{U})$ we identify S^1 with the unit circle $U(1)$ in the complex numbers, sending the basepoint in S^1 to 1. Given a tuple $(E_1,\dots,E_n)$ of pairwise orthogonal subspaces of $\mathcal{U}$ and a point $(\lambda_1,\dots,\lambda_n)\in U(1)^n$ we let $\psi(E_1,\dots,E_n;\,\lambda_1,\dots,\lambda_n)$ be the isometry of $\mathcal{U}$ that is multiplication by λ_i on V_i and the identity on the orthogonal complement of $\bigoplus_{i=1}^n V_i$. In other words: E_i is the eigenspace of $\psi(E_1,\dots,E_n;\,\lambda_1,\dots,\lambda_n)$ for eigenvalue λ_i As n varies, these maps are compatible with the equivalence relation and so they assemble into a continuous map

$$\mathscr{C}(\mathcal{U},U(1))\ =\ \int^{n^+\in\mathbf{\Gamma}}\mathscr{C}(\mathcal{U},n^+)\times U(1)^n\ \longrightarrow\ U(\mathcal{U})\ . \tag{6.2}$$

This map is bijective because every unitary transformation is diagonalizable with eigenvalues in $U(1)$ and pairwise orthogonal eigenspaces.

We let $\mathcal{U}$ and $\mathcal{U}'$ be two hermitian vector spaces (of countable dimension, but possibly finite dimensional). We endow the tensor product $\mathcal{U}\otimes\mathcal{U}'$ with a scalar product by declaring

$$\langle u\otimes u',\,v\otimes v'\rangle\ =\ \langle u,v\rangle\cdot\langle u',v'\rangle \tag{6.3}$$

on elementary tensors and extending biadditively. If V,W are orthogonal subspaces of $\mathcal{U}$ and V' is a subspaces of $\mathcal{U}'$, then $V\otimes V'$ and $W\otimes V'$ are orthogonal subspaces of $\mathcal{U}\otimes\mathcal{U}'$ (and similarly in the second variable). For finite based sets A and A' we can thus define a multiplication map

$$\mathscr{C}(\mathcal{U},A)\wedge\mathscr{C}(\mathcal{U}',A')\ \longrightarrow\ \mathscr{C}(\mathcal{U}\otimes\mathcal{U}',A\wedge A')\ ,\quad (V_a)\wedge(W_b)\ \longmapsto\ (V_a\otimes W_b)_{a\wedge b}\ . \tag{6.4}$$

These multiplication maps are associative, and commutative in the sense that the following square commute:

$$\begin{CD}
\mathscr{C}(\mathcal{U},A)\wedge\mathscr{C}(\mathcal{U}',A') @>>> \mathscr{C}(\mathcal{U}\otimes\mathcal{U}',A\wedge A')\\
@V{\tau_{\mathscr{C}(\mathcal{U},A),\mathscr{C}(\mathcal{U}',A')}}VV @VV{\mathscr{C}(\tau_{\mathcal{U},\mathcal{U}'},\tau_{A,A'})}V\\
\mathscr{C}(\mathcal{U}',A')\wedge\mathscr{C}(\mathcal{U},A) @>>> \mathscr{C}(\mathcal{U}'\otimes\mathcal{U},A'\wedge A)
\end{CD}$$

Construction 6.5 (Connective global K-theory)**.** Given a finite dimensional complex inner product space V we let

$$\mathrm{Sym}(V)\ =\ \bigoplus_{n\geq 0}V^{\otimes n}/\Sigma_n$$

denote the symmetric algebra of V (where $\otimes$ denotes the tensor product over $\mathbb{C}$). The symmetric algebra inherits a preferred hermitian inner product from V as follows. The tensor product of two hermitian inner product spaces inherits a scalar product characterized by (6.3). By iteration the n-fold tensor product $V^{\otimes n}$ inherits a scalar product, hence so does every subspace of $V^{\otimes n}$. In particular, the subspace $(V^{\otimes n})^{\Sigma_n}$ of symmetric tensors becomes a hermitian vector space. The composite

$$(V^{\otimes n})^{\Sigma_n}\ \xrightarrow{\text{incl}}\ V^{\otimes n}\ \xrightarrow{\text{proj}}\ (V^{\otimes n})/\Sigma_n\ =\ \mathrm{Sym}^n(V) \tag{6.6}$$

is an isomorphism, and we endow $\mathrm{Sym}^n(V)$ with the scalar product that makes this isomorphism an isometry. To illustrate the resulting inner product on $\mathrm{Sym}^n(V)$ we look at the special case $n = 2$. We let $v, w \in V$ be orthonormal, i.e., unit vectors with $\langle v, w\rangle = 0$. Then the three vectors

$$v \otimes v\ , \quad \frac{1}{\sqrt{2}} \cdot (v \otimes w + w \otimes v) \quad \text{and} \quad w \otimes w$$

are orthonormal in $(V \otimes V)^{\Sigma_2}$. Their images under the composite (6.6) are the three vectors

$$v^2\ , \quad \sqrt{2} \cdot vw \quad \text{and} \quad w^2\ ,$$

so this triple is orthogonal in $\mathrm{Sym}^2(V)$.

Having explained the scalar product on $\mathrm{Sym}^n(V)$, we use the orthogonal direct sum inner product on the symmetric algebra $\mathrm{Sym}(V) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(V)$. With these definitions, the action of $U(V)$ on V extends to a unitary action on each homogeneous summand $\mathrm{Sym}^n(V)$, and hence on $\mathrm{Sym}(V)$. If W is another complex inner product space, then the two summand inclusions of V and W into $V \oplus W$ induced algebra homomorphisms

$$\mathrm{Sym}(V) \longrightarrow \mathrm{Sym}(V \oplus W) \longleftarrow \mathrm{Sym}(W)$$

and we use the commutative multiplication on $\mathrm{Sym}(V \oplus W)$ to combine these into a natural isomorphism

$$\mathrm{Sym}(V) \otimes_{\mathbb{C}} \mathrm{Sym}(W) \cong \mathrm{Sym}(V \oplus W)\ .$$

This map is in fact an isomorphism of hermitian vector spaces and it is $U(V) \times U(W)$-equivariant.

We can now define a commutative orthogonal ring spectrum $\mathbf{ku}$, the *connective global K-theory spectrum*. The value of $\mathbf{ku}$ on a real inner product space V is

$$\mathbf{ku}(V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)\ , \tag{6.7}$$

the value of the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ on the one-point compactification of V, where $V_{\mathbb{C}}$ is the complexification of V. We let the orthogonal group $O(V)$ act diagonally, via the action on the sphere S^V and the action on the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$. Explicitly, given a orthogonal automorphism $\varphi : V \longrightarrow V$, pairwise orthogonal subspaces $X_1, \dots, X_n$ of $\mathrm{Sym}(V_{\mathbb{C}})$ and elements $v_1, \dots, v_n$ of S^V, we set

$$\varphi \cdot [X_1, \dots, X_n; v_1, \dots, v_n] = [(\varphi_{\mathbb{C}})_*(X_1), \dots, (\varphi_{\mathbb{C}})_*(X_n); \varphi(v_1), \dots, \varphi(v_n)]\ .$$

Using the tensor product pairing (6.4) we define an $O(V) \times O(W)$-equivariant multiplication map

$$\begin{aligned} \mu_{V,W}\ :\ \mathbf{ku}(V) \wedge \mathbf{ku}(W) \ =\ & \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \wedge \mathscr{C}(\mathrm{Sym}(W_{\mathbb{C}}), S^W) \\ \longrightarrow\ & \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathrm{Sym}(W_{\mathbb{C}}), S^V \wedge S^W) \\ \cong\ & \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V\oplus W}) = \mathbf{ku}(V \oplus W)\ . \end{aligned}$$

The maps $\mu_{V,W}$ are associative and commutative. A $O(V)$-equivariant unit map is given by

$$\iota_V\ :\ S^V \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V)\ , \quad v \longmapsto [\mathbb{C}\cdot 1; v]\ ,$$

where $\mathbb{C} \cdot 1$ is the line in $\mathrm{Sym}(V_{\mathbb{C}})$ spanned by the unit element 1 of the symmetric algebra, i.e., the homogeneous summand of degree 0 in the symmetric algebra.

The space $\mathbf{ku}_0 = \mathscr{C}(\mathbb{C}\cdot 1, S^0)$ consists of all subspaces of $\mathrm{Sym}(0) = \mathbb{C}\cdot 1$, so it has two points, the basepoint 0 and the point $\mathbb{C}\cdot 1$. The unit map $\iota_0 : S^0 \longrightarrow \mathbf{ku}_0$ is thus an isomorphism; the unitality condition for the maps $\mu_{n,0}$ and $\mu_{0,n}$ holds. This structure makes $\mathbf{ku}$ into a commutative orthogonal ring spectrum.

Now we will justify that for every finite group G the associated orthogonal G-spectrum represents connective G-equivariant topological K-theory. Every linear isometric embedding $u : \mathcal{U} \longrightarrow \bar{\mathcal{U}}$ of complex inner product spaces induces a morphism of $\mathbf{\Gamma}$-space $\mathscr{C}(u) : \mathscr{C}(\mathcal{U}) \longrightarrow \mathscr{C}(\bar{\mathcal{U}})$ by applying u elementwise to a tuple of orthogonal subspaces. So if a compact Lie group G acts on $\mathcal{U}$ by linear isometries (e.g., if $\mathcal{U}$ is a

G-universe), then the $\mathbf{\Gamma}$-space $\mathscr{C}(\mathcal{U})$ inherits a G-action, so it becomes a $\mathbf{\Gamma}$-G-space. We let $G^\circ \leq G$ be the connected component of the identity and $\bar{G} = G/G^\circ$ the finite group of path components of G.

Theorem 6.8. *Let G be a compact Lie group.*

(i) *For every complete complex G-universe $\mathcal{U}$ the $\mathbf{\Gamma}$-$\bar{G}$-space $\mathscr{C}(\mathcal{U})^{G^\circ}$ is special.*
(ii) *If G is finite, then for every complete complex G-universe $\mathcal{U}$ the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathcal{U})$ is special.*
(iii) *For every G-equivariant linear isometric embedding $u : \mathcal{U} \longrightarrow \bar{\mathcal{U}}$ between complete G-universes and every based G-space K the map*

$$\mathscr{C}(u, K) \ : \ \mathscr{C}(\mathcal{U}, K) \ \longrightarrow \ \mathscr{C}(\bar{\mathcal{U}}, K)$$

is a G-homotopy equivalence.

Proof. (i)
(ii) We need to show that for every finite G-set S the map

$$P_S \ : \ \mathscr{C}(\mathcal{U}, S^+) \ \longrightarrow \ \mathrm{map}(S, \mathscr{C}(\mathcal{U}, 1^+))$$

is a G-weak equivalence. This means that for all subgroups H of G the restriction to H-fixed points is a weak equivalence in the non-equivariant sense. Since the underlying H-universe of a complete G-universe is again complete, it suffices to treat the case $H = G$, i.e., we need to show that for every finite G-set S the map

$$(P_S)^G \ : \ (\mathscr{C}(\mathcal{U}, S^+))^G \ \longrightarrow \ \mathrm{map}^G(S, \mathscr{C}(\mathcal{U}, 1^+)) \tag{6.9}$$

is a weak equivalence.

Every finite G-set is isomorphic to a disjoint union of G-sets of the form G/H_i for various subgroups H_i of G. So we may assume that

$$S \ = \ \coprod_{i=1}^n G/H_i \ .$$

Then the space $\mathrm{map}^G(S, \mathscr{C}(\mathcal{U}, 1^+))$ is homeomorphic, by evaluation at the special cosets $H_1, \dots, H_n$, to the product

$$\prod_{i=1}^n (\mathscr{C}(\mathcal{U}, 1^+))^{H_i} \ .$$

For any subgroup H of G the space $(\mathscr{C}(\mathcal{U}, 1^+))^H$ is the space of finite dimensional H-subrepresentations of $\mathcal{U}$. Since $\mathcal{U}$ is a complete universe (over G, hence also over H), every finite dimensional H-representation embeds into $\mathcal{U}$. So the path components of $(\mathscr{C}(\mathcal{U}, 1^+))^H$ are indexed by isomorphism classes of finite dimensional H-representations, and the entire space is homeomorphic to

$$(\mathscr{C}(\mathcal{U}, 1^+))^H \ = \ \coprod_{[W]} \mathbf{L}^H(W, \mathcal{U})/O^H(W) \ ,$$

where the disjoint union is indexed by isomorphism classes of finite dimensional complex H-representations, and $O^H(W) = \mathbf{L}^H(W, W)$ is the group of H-equivariant linear self-isometries of W. The space $\mathrm{map}^G(S, \mathscr{C}(\mathcal{U}))$ is thus homeomorphic to

$$\mathrm{map}^G(S, \mathscr{C}(\mathcal{U})) \ \cong \ \coprod_{[W_1], \dots, [W_n]} \mathbf{L}^{H_1}(W_1, \mathcal{U})/O^{H_1}(W_1) \times \cdots \times \mathbf{L}^{H_n}(W_n, \mathcal{U})/O^{H_n}(W_n) \ ;$$

this union is indexed over all tuples $[W_1], \dots, [W_n]$ with $[W_i]$ an isomorphism class of finite dimensional H_i-representation.

The source is the space of tuples $(V_{gH_1})_{G/H_1}, \dots, (V_{gH_n})_{G/H_n}$ of finite dimensional, pairwise orthogonal subspace of $\mathcal{U}$ such that $\gamma(V_{gH_i}) = V_{\gamma g H_i}$ for all $\gamma, g \in G$ and $i = 1, \dots, n$. The component V_{H_i} is then a H_i-subrepresentation of $\mathcal{U}$, and the rest of the data is determined by these V_{H_i}, so the map (6.9) is a subspace inclusion. The condition for a tuple $(V_{H_1}, \dots, v_{H_n})$ of H_i-subrepresentations of $\mathcal{U}$ to lie in the subspace is that the induced map

$$\mathrm{ind}_{H_1}^G W_1 \oplus \cdots \oplus \mathrm{ind}_{H_n}^G W_n, \longrightarrow \mathcal{U}$$

adjoint to the inclusions $W_i \longrightarrow \mathcal{U}$ be an isometric embedding. The source of the map (6.9) is thus homeomorphic to the space

$$(\mathscr{C}(\mathcal{U}, S^+))^G \cong \coprod_{[W_1],\dots,[W_n]} \mathbf{L}^G(\mathrm{ind}_{H_1}^G W_1 \oplus \cdots \oplus \mathrm{ind}_{H_n}^G W_n, \mathcal{U})/(O^{H_1}(W_1) \times \cdots \times O^{H_n}(W_n))$$

For a fixed tuple $[W_1], \dots, [W_n]$ the map is induced by the restriction map

$$\mathbf{L}^G(\mathrm{ind}_{H_1}^G W_1 \oplus \cdots \oplus \mathrm{ind}_{H_n}^G W_1, \mathcal{U}) \longrightarrow \mathbf{L}^{H_1}(W_1,\mathcal{U}) \times \cdots \times \mathbf{L}^{H_n}(W_n,\mathcal{U})$$

by passage to orbit spaces. Since $\mathcal{U}$ is a complete universe, the source space and all the spaces $\mathbf{L}^{H_i}(W_i,\mathcal{U})$ are contractible. Moreover, the action of the product group $O^{H_1}(W_1) \times \cdots \times O^{H_n}(W_n)$ by precomposition is free on source and target. So the restriction map is an equivariant map between free universal free $(O^{H_1}(W_1) \times \cdots \times O^{H_n}(W_n))$-spaces, and so the induced map on orbit spaces is a weak equivalence. The map (6.9) is a disjoint union of such weak equivalences, so it is a weak equivalence. This completes the verification of the specialness condition for transitive G-sets.

(iii) We start with the special case where $\bar{\mathcal{U}} = \mathcal{U}$. The space of G-equivariant linear isometric embeddings from $\mathcal{U}$ to itself is contractible. A homotopy from u to the identity then induces a G-homotopy from $\mathscr{C}(u, K)$ to the identity of the G-space $\mathscr{C}(\mathcal{U}, K)$. In the general case we choose a G-equivariant linear isometric embedding $v : \bar{\mathcal{U}} \longrightarrow \mathcal{U}$ in the opposite direction. By the previous paragraph the two G-maps

$$\mathscr{C}(vu, K) : \mathscr{C}(\mathcal{U}, K) \longrightarrow \mathscr{C}(\mathcal{U}, K) \quad \text{and} \quad \mathscr{C}(uv, K) : \mathscr{C}(\bar{\mathcal{U}}, K) \longrightarrow \mathscr{C}(\bar{\mathcal{U}}, K)$$

are G-homotopic to the respective identity maps. □

Proposition 6.10. *For every finite group G and every G-representation V (possibly infinite dimensional), the $\mathbf{\Gamma}$-G-space $\mathscr{C}(V)$ is cofibrant.*

The orthogonal spectrum **ku** is trying to be a 'positive global Ω-spectrum'. However, the global Ω-spectrum condition on the adjoint structure maps only hold in certain situations. For every finite group G the underlying orthogonal G-spectrum $\mathbf{ku}\langle G \rangle$ the G-stable homotopy type of connective G-equivariant complex K-theory spectrum.

Theorem 6.11. *Let G be a finite group and V a G-representation such that $\mathrm{Sym}(V_{\mathbb{C}})$ is a complete complex G-universe. Then for every complex G-representation W the adjoint structure map*

$$\tilde{\sigma}_{V,W} : \mathbf{ku}(V) \longrightarrow \mathrm{map}(S^W, \mathbf{ku}(V \oplus W))$$

is a G-weak equivalence.

Proof. The adjoint structure map $\tilde{\sigma}_{V,W}$ factors as the composite

$$\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \xrightarrow{\mathscr{C}(\mathrm{Sym}(i_{\mathbb{C}}), S^V)} \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^V) \xrightarrow{\tilde{\alpha}} \mathrm{map}(S^W, \mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W})) ;$$

here $i : V \longrightarrow V \oplus W$ is the inclusion of the first summand and $\tilde{\alpha}$ is the adjoint of the assembly map for the $\mathbf{\Gamma}$-G-space $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))$. By the hypothesis on V the map $\mathrm{Sym}(i_{\mathbb{C}}) : \mathrm{Sym}(V_{\mathbb{C}}) \longrightarrow \mathrm{Sym}((V \oplus W)_{\mathbb{C}})$ is an equivariant linear isometric embedding between complete complex G-universes. So the first map is a G-homotopy equivalence by Theorem 6.8 (iii). Since $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}))$ is a special $\mathbf{\Gamma}$-G-space (see Theorem 6.8 (ii)), the second map is a G-weak equivalence by Shimakawa's theorem [**81**, Thm. B]. Proposition 6.10 enters here to compare the value $\mathscr{C}(\mathrm{Sym}((V \oplus W)_{\mathbb{C}}), S^{V \oplus W})$ (which is a categorical coend) with the bar construction used by Shimakawa (which is the corresponding homotopy coend). □

Remark 6.12. The previous Theorem 6.11 raises the question of how characterize G-representations V such that the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification is a complete complex G-universe; I do not know of a good criterion for this. A necessary condition is clearly that V is non-zero and G acts faithfully on V.

A class of representations that qualify in this context are non-empty faithful permutation representations. Indeed, we let A be a faithful finite G-set, which means in particular that the group G is finite, and we let $\mathbb{R}A$ denote the associated permutation representation. Then $(\mathbb{R}A)_{\mathbb{C}} = \mathbb{C}A$ is the complex permutation representation and its symmetric algebra $\mathrm{Sym}(\mathbb{C}A)$ is also a complex permutation representation, namely of the infinite G-set $\mathbb{N}^A = \mathrm{map}(A, \mathbb{N})$ of functions from A to $\mathbb{N}$. Since G acts faithfully on A, every *injective* map $A \longrightarrow \mathbb{N}$ generates a free G-orbit in the G-set $\mathbb{N}^A$. There are infinitely many injections from A to $\mathbb{N}$ with pairwise disjoint images, and these generate infinitely many distinct free G-orbits in $\mathbb{N}^A$. So $\mathrm{Sym}(\mathbb{C}A) = \mathbb{C}[\mathbb{N}^A]$ contains infinitely many copies of the complex regular G-representation, and is thus a complete G-universe.

Another class that ought to qualify are the G-representations into which G embeds equivariantly, compare [**46**, Lemma 5.5]. Indeed, a choice of G-equivariant embedding $G \longrightarrow W$ and a choice of G-equivariant tubular neighborhood allows us to embed the Hilbert space $L^2(G;\mathbb{C}) \otimes \mathcal{H}$ G-equivariantly into $L^2(W;\mathbb{C})$. Since $L^2(G;\mathbb{C}) \otimes \mathcal{H}$ contains every finite dimensional G-representation infinitely often, so does $L^2(W;\mathbb{C})$. But $L^2(W;\mathbb{C})$ is the Hilbert space completion of $\mathrm{Sym}(W^*_{\mathbb{C}})$: elements of $\mathrm{Sym}(W^*_{\mathbb{C}})$ are polynomial functions on W, which we can map to L^2-functions by scaling with the function $v \mapsto \exp(-|v|^2)$. This gives a G-equivariant isometric embedding $\mathrm{Sym}(W^*_{\mathbb{C}}) \longrightarrow L^2(W;\mathbb{C})$ with dense image. So if there was any complex G-representation that did not embed equivariantly into $\mathrm{Sym}(W^*_{\mathbb{C}})$, then it would also not embed into $L^2(W;\mathbb{C})$, a contradiction.

Construction 6.13 (ku globally deloops BUP). Now we argue that the unitary spectrum $\mathbf{ku}$ is a global delooping of the orthogonal space $\mathbf{BUP}$, the periodic global version of BU introduced in Example II.2.26.

We start from a description of the $\mathbf{\Gamma}$-orthogonal space $\mathcal{H}\mathbf{Gr}^{\mathbb{C}}$ associated to the additive Grassmannian, introduced in Example II.2.24. Proposition II.2.5 provides an isomorphism of orthogonal spaces

$$(\mathcal{H}\mathbf{Gr}^{\mathbb{C}})(n^+) \ = \ (\mathbf{Gr}^{\mathbb{C}})^{\boxtimes n} \ \cong \ \mathbf{Gr}^{\mathbb{C}}_{\langle n\rangle}$$

to the additive Grassmannian of n-tuples of pairwise orthogonal subspaces. For an inner product space V, the $\mathbf{\Gamma}$-space $(\mathcal{H}\mathbf{Gr}^{\mathbb{C}}))(V)$ is thus isomorphic to the $\mathbf{\Gamma}$-space $\mathscr{C}(V_{\mathbb{C}})$ of orthogonal subspaces in $V_{\mathbb{C}}$. The embedding $i : V_{\mathbb{C}} \longrightarrow \mathrm{Sym}(V_{\mathbb{C}})$ as the linear summand thus induces continuous, $O(V)$-equivariant embeddings of $\mathbf{\Gamma}$-spaces

$$(\mathcal{H}\mathbf{Gr}^{\mathbb{C}})(V) = \mathscr{C}(V_{\mathbb{C}}) \xrightarrow{\mathscr{C}(i)} \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}})) \ .$$

Evaluating this at the V-sphere yields a based continuous map

$$\xi(V) \ : \ (\mathbf{BGr}^{\mathbb{C}}))(V) \ = \ (\mathcal{H}\mathbf{Gr}^{\mathbb{C}})(V, S^V) = \mathscr{C}(V_{\mathbb{C}}, S^V) \xrightarrow{\mathscr{C}(i,S^V)} \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V) \ .$$

As V varies, these maps form a morphism of orthogonal spectra $\xi : \mathbf{BGr}^{\mathbb{C}} \longrightarrow \mathbf{ku}$.

Proposition 6.14. *The morphism*

$$\xi \ : \ \mathbf{BGr}^{\mathbb{C}} \ \longrightarrow \ \mathbf{ku}$$

is a global equivalence of orthogonal spectra.

A corollary of the previous proposition is that the connective global K-theory spectrum $\mathbf{ku}$ is a global delooping of the group-like commutative orthogonal monoid space $\mathbf{BUP}$. Indeed, the morphism $i : \mathbf{Gr}^{\mathbb{C}} \longrightarrow \mathbf{BUP}$ defined in Example II.2.26 is a group completion by the unitary analog of Theorem II.4.10. Both $\mathbf{Gr}^{\mathbb{C}}$ and $\mathbf{BUP}$ are flat as orthogonal spaces, so the morphism $\mathbf{B}i : \mathbf{BGr}^{\mathbb{C}} \longrightarrow \mathbf{BBUP}$ is a global equivalence by Theorem 8.30, and we obtain a chain of global equivalences of orthogonal spectra

$$\mathbf{B}(\mathbf{BUP}) \xleftarrow{\mathbf{B}i} \mathbf{B}(\mathbf{Gr}^{\mathbb{C}}) \xrightarrow{\xi} \mathbf{ku} \ . \tag{6.15}$$

A consequence is a chain of $\mathcal{F}in$-global equivalences of orthogonal spaces

$$\mathbf{BUP} \longrightarrow \Omega^\bullet(\mathbf{B}(\mathbf{BUP})) \xleftarrow{\Omega^\bullet \mathbf{B}i} \Omega^\bullet(\mathbf{BGr}^{\mathbb{C}}) \xrightarrow{\Omega^\bullet \xi} \Omega^\bullet(\mathbf{ku}) \; ; \tag{6.16}$$

the middle are right morphisms are even $\mathcal{A}ll$-global equivalences.

Recognizing $\mathbf{ku}$ as the delooping of $\mathbf{BUP}$ lets us draw some consequences about the global functor $\underline{\pi}_0(\mathbf{ku})$, i.e., the 0-th equivariant homotopy groups of the connective global K-theory spectrum. The global equivalence ξ induces a morphism of $\mathrm{Rep}^{\mathrm{op}}$-functors

$$\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \longrightarrow \underline{\pi}_0(\mathbf{BGr}^{\mathbb{C}}) \xrightarrow{\underline{\pi}_0(\xi)} \underline{\pi}_0(\mathbf{ku}) \; ;$$

The right morphism $\underline{\pi}_0(\xi)$ is an isomorphisms of global functors, and the left morphism is additive and transfer preserving, and an isomorphism at finite groups. The monoid $\pi_0^G(\mathbf{Gr}^{\mathbb{C}})$ is isomorphic to the monoid $\mathbf{R}^+(G)$ of isomorphism classes of complex G-representation under direct sum, via the map (2.25). Unraveling all the definitions exhibits the combined homomorphism from $\mathbf{R}^+(G)$ to $\pi_0^G(\mathbf{ku})$ as the map

$$\langle - \rangle \; : \; \mathbf{R}^+(G) \longrightarrow \pi_0^G(\mathbf{ku})$$

with the following explicit construction. Given an isomorphism class of complex G-representations, we choose a real inner product space V and a G-subrepresentation $W \subset V_{\mathbb{C}}$ in the given isomorphism class. Then $\langle W \rangle \in \pi_0^G(\mathbf{ku})$ is the to the homotopy class of the based G-map

$$[W; -] \; : \; S^V \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \; = \; \mathbf{ku}(V) \; , \quad v \longmapsto \quad [W; v] \; .$$

Here $V_{\mathbb{C}}$ is identified with the linear summand in the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$. The equivariant stable homotopy class of the G-map $[W; -]$ only depends on the isomorphism class of W.

Since $\langle - \rangle$ is a homomorphism of abelian monoids and the target $\pi_0^G(\mathbf{ku})$ is a group, there is a unique extension to a group homomorphism

$$\langle - \rangle \; : \; \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku}) \tag{6.17}$$

from the group completion, for which we use the same symbol. Before group completion the maps $\langle - \rangle$ were compatible with restriction and finite index transfers, so we conclude:

Corollary 6.18. *As G varies over all compact Lie groups, the maps* (6.17)

$$\langle - \rangle \; : \; \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$$

are additive, compatible with restriction maps and compatible with transfers for subgroup inclusions of finite index. Moreover, the map is an isomorphism whenever the group G is finite.

The maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are *not* compatible with transfers along subgroup inclusions of infinite index, so they do *not* form a morphism of global functors, see also Remark 6.21 below. This originates from the fact that the maps $\underline{\pi}_0(\mathbf{Gr}^{\mathbb{C}}) \longrightarrow \underline{\pi}_0(\mathbf{BGr}^{\mathbb{C}})$ are a morphism of global power monoids, but that notion only involves norm maps along finite index inclusions. In fact, for a general commutative orthogonal monoid space R, the global power monoid $\underline{\pi}_0(R)$ does not admit any 'infinite index norm maps'.

Both $\mathbf{R}(G)$ and $\pi_0^G(\mathbf{ku})$ have additional 'multiplicative' functoriality in G, i.e., they are global power functors in the sense of Definition V.1.2. The maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ also preserve this 'multiplicative structure', but that is not apparent from what we showed so far. Indeed, the commutative orthogonal monoid spaces $\mathbf{Gr}^{\mathbb{C}}$ and $\mathbf{BUP}$ ought to be 'commutative orthogonal ring spaces', but we do not have a convenient formalism that encode this and such that the global delooping of a 'commutative orthogonal ring space' is an ultracommutative ring spectrum.

In the statement of the following proposition, all representations are *complex* representations of finite dimension.

Proposition 6.19. *Let G and K be compact Lie groups.*

(i) *Let $\mathbb{C}$ denote the trivial 1-dimensional G-representation. Then $\langle \mathbb{C} \rangle = 1$ in the group $\pi_0^G(\mathbf{ku})$.*

(ii) *Let V be a G-representation and W a K-representation. Then*

$$\langle V \rangle \times \langle W \rangle \ = \ \langle V \otimes W \rangle$$

in the group $\pi_0^{G\times K}(\mathbf{ku})$.

(iii) *Let V and W be two G-representations. Then*

$$\langle V \rangle \cdot \langle W \rangle \ = \ \langle V \otimes W \rangle$$

in the group $\pi_0^G(\mathbf{ku})$, i.e., the map (6.17) is a ring homomorphism.

Proof. (i) By naturality for the restriction along the homomorphism $G \longrightarrow e$ it suffices to show the case $G = e$ of the trivial group. Then the elements 1 and $\langle \mathbb{C} \rangle$ are represented by the two maps

$$\eta_{\mathbb{C}} \ , \ [\mathbb{C}; -] \ : \ S^{\mathbb{R}} \ \longrightarrow \ \mathbf{ku}(\mathbb{R}) \ = \ \mathscr{C}(\mathrm{Sym}(\mathbb{C}), S^{\mathbb{R}})$$

that send x to $[i_0(\mathbb{C}); x]$ respectively $[i_1(\mathbb{C}^2); x]$, where

$$i_0, i_1 \ : \ \mathbb{C} \ \longrightarrow \ \mathrm{Sym}(\mathbb{C}^2)$$

are two different embeddings as the constant respectively the linear part of the symmetric algebra. Since the images of the i_0 and i_1 are orthogonal, they are homotopic, through linear isometric embeddings $\mathbb{C} \longrightarrow \mathrm{Sym}(\mathbb{C})$ (by the complex analog of Lemma I.1.12). If $\{H_t\}_{t\in[0,1]}$ is such a homotopy between the linear isometric embeddings, then $x \mapsto [H_t(\mathbb{C}); x]$ is a homotopy between $\eta_{\mathbb{C}}$ and $[\mathbb{C}; -]$.

(ii) The class $\langle V \rangle \times \langle W \rangle$ is represented by the $(G \times K)$-map

$$[V; -] \times [W; -] \ : \ S^{V+W} \ \longrightarrow \ \mathbf{ku}(V \oplus W) \ .$$

The stabilization

$$[V; -] \times [W; -] \diamond (V \otimes W) \ : \ S^{V+W+V\otimes W} \ \longrightarrow \ \mathbf{ku}(V \oplus W \oplus V \otimes W) \ .$$

by $V \otimes W$ is then another representative for $\langle V \rangle \times \langle W \rangle$. On the other hand, $\langle V \otimes W \rangle$ is represented by the $(G \times K)$-map

$$V \diamond W \diamond [V \otimes W; -] \ : \ S^{V+W+V\otimes W} \ \longrightarrow \ \mathbf{ku}(V \oplus W \oplus (V \otimes W)) \ .$$

These two maps send an element $x \in V \oplus W \oplus (V \otimes W)$ to

$$[i_0(V \otimes W);\, x] \qquad \text{respectively} \qquad [i_1(V \otimes W);\, x]$$

where $i_0, i_1 : V \otimes W \longrightarrow \mathrm{Sym}((V \oplus W \oplus V \otimes W)^2)$ are two different $(G \times K)$-equivariant linear isometric embeddings: i_0 includes $V \otimes W$ via the composite

$$V \otimes W \ \to \ (V \oplus W)^{\otimes 2}/\Sigma_2 \ \xrightarrow{\text{incl}} \mathrm{Sym}((V \oplus W)^2) \ \xrightarrow{\mathrm{Sym}(i^2_{V\oplus W})} \ \mathrm{Sym}((V \oplus W \oplus (V \otimes W))^2) \ ,$$

whereas i_1 is the composite

$$V \otimes W \ \xrightarrow{\text{incl}} \ \mathrm{Sym}((V \otimes W)^2) \ \xrightarrow{\mathrm{Sym}(i^2_{V\otimes W})} \ \mathrm{Sym}((V \oplus W \oplus (V \otimes W))^2) \ .$$

The images of i_0 and i_1 land in the quadratic respectively linear part of the symmetric algebra; so the images of i_0 and i_1 are orthogonal. Lemma I.1.12 provides a $(G\times K)$-equivariant homotopy $i_t : (V\otimes W) \longrightarrow \mathrm{Sym}(V \oplus W \oplus V \otimes W)$, $t \in [0, 1]$, from i_0 to i_1 through linear isometric embeddings. Then

$$S^{V+W+V\otimes W} \ \longrightarrow \ \mathbf{ku}(V \oplus W \oplus (V \otimes W)) \ , \quad x \ \longmapsto \ [i_t(V \otimes W);\, x]$$

is a $(G\times K)$-equivariant based homotopy between the maps $[V; -]\cdot[W; -]\diamond(V \otimes W)$ and $V \diamond W \diamond [V \otimes W; -]$. We conclude that the classes $\langle V \rangle \cdot \langle W \rangle$ and $\langle V \otimes W \rangle$ represented by these maps are equal in $\pi_0^{G\times K}(\mathbf{ku})$.

The multiplicativity statement (iii) follows from the external multiplicativity (ii) and naturality because $V\otimes W = \Delta^*(V\otimes W)$ and $\langle V \rangle\cdot\langle W \rangle = \Delta^*(\langle V \rangle \times \langle W \rangle)$ where $\Delta : G \longrightarrow G\times G$ is the diagonal homomorphism. □

Remark 6.20. The representation rings also admit power operations $P^m : \mathbf{R}(G) \longrightarrow \mathbf{R}(\Sigma_m \wr G)$ that come from sending a G-representation V to the m-fold power $V^{\otimes m}$ with the natural action by the wreath product group $\Sigma_m \wr G$. (Raising to a power is not an additive operation, so one has to explain in more detail how to define P^m on virtual representations).

As we explained in Section V.1 the homotopy group global functor $\underline{\pi}_0(R)$ of a commutative orthogonal ring spectrum R also comes with power operations $P^m : \pi_0^G(R) \longrightarrow \pi_0^{\Sigma_m \wr G}(R)$. The basic idea is very simple: we raise any representative $f : S^V \longrightarrow R(V)$ of an element of $\pi_0^G(R)$ to the m-th power and multiply in R; then we observe that the resulting map $f^m : S^{V^m} \longrightarrow R(V^m)$ is $(\Sigma_m \wr G)$-equivariant by the strict commutativity of the multiplication of R. This applies to the commutative orthogonal ring spectrum $\mathbf{ku}$, and we will show later that the maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are also compatible with power operations.

A formal consequence of the power operations are norm maps ('multiplicative transfers') $N_H^G : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ and $N_H^G : \pi_0^H(\mathbf{ku}) \longrightarrow \pi_0^G(\mathbf{ku})$ for subgroup inclusions $H \subset G$ of finite index, see Construction II.1.20. The maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are thus also compatible with the norm maps.

Remark 6.21. Segal [**80**, §2] discusses a 'smooth transfer' $i_! : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ where $i : H \longrightarrow G$ is the inclusion of a closed subgroup of a compact Lie group G. One could hope that the maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ take the smooth transfer to the homotopy theoretic transfer

$$\mathrm{tr}_H^G \ : \ \pi_0^H(\mathbf{ku}) \ \longrightarrow \ \pi_0^G(\mathbf{ku}) \ .$$

This (false!) expectation is suggested by several facts:

- If H has finite index in G, then the smooth transfer $i_! : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ is given by inducing an H-representation to a G-representation; the maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ are compatible with this by Corollary 6.18.
- The smooth transfer vanishes whenever H has infinite index in its normalizer, and the same is true for the homotopy theoretic transfer by Example 2.31 .
- The smooth transfer $i_! : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ multiplies the rank of a virtual representation by the Euler characteristic of the coset space G/H; the analog for the homotopy theoretic transfer holds by Example 3.18.
- The induction satisfies reciprocity, i.e., $i_! : \mathbf{R}(H) \longrightarrow \mathbf{R}(G)$ is $\mathbf{R}(G)$-linear, and the analog is true for $\underline{\pi}_0(\mathbf{ku})$ because $\mathbf{ku}$ is an ultra-commutative ring spectrum.

Contrary to this obvious guess, though, the maps $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ do *not* in general transform the smooth transfer into the homotopy theoretic transfers in $\underline{\pi}_0(\mathbf{ku})$. In other words, the collection of ring homomorphisms $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ comes very close to being a morphism of global functors, but it fails to commute with infinite index transfers. We suspect however that these two transfers do correspond for *periodic* global K-theory $\mathbf{KU}$, discussed below in Constuction 6.34, i.e., that $\underline{\pi}_0(\mathbf{KU})$ is isomorphic, as a global power functor, to the representation ring global functor.

Example 6.22. To illustrate the issue we discuss a specific example and recall the smooth transfer for the embedding $i : T \longrightarrow SU(2)$ given by

$$i(\lambda) \ = \ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ .$$

So i identifies T with the diagonal maximal torus of $SU(2)$. The representation ring $\mathbf{R}(T) = \mathbb{Z}[x, x^{-1}]$ is a Laurent polynomial ring generated by the class x of the tautological T-representation on $\mathbb{C}$. The representation ring $\mathbf{R}(SU(2)) = \mathbb{Z}[s]$ is a polynomial ring generated by the class s of the tautological 2-dimensional representation. The restriction map

$$i^* \ : \ \mathbf{R}(SU(2)) \ \longrightarrow \ \mathbf{R}(T)$$

identifies $\mathbf{R}(SU(2))$ with the polynomial subring of $\mathbf{R}(T)$ generated by $i^*(s) = x + x^{-1}$.

We use Segal's character formula [**80**, p. 119] to calculate the character of the smooth transfer $i_!(1)$ of the trivial 1-dimensional representation. The normalizer of T in $SU(2)$ is generated by the image of i and the matrix $t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Conjugation by t is the involution of T sending λ to $\lambda^{-1} = \bar{\lambda}$. So the space of conjugacy classes

$$T/N_{SU(2)}T \;\cong\; SU(2)/\mathrm{conj}$$

is the quotient of T by $\lambda \simeq \bar{\lambda}$. An element $\lambda \in T$ is regular if and only if λ has infinite order. In particular, a regular λ is different from 1 and -1, and the fixed set of a regular λ is thus $F_\lambda = \{T, tT\}$ (independent of λ). So

$$\chi_{i_!(1)}(\lambda) \;=\; \chi_1(\lambda) \;+\; \chi_1(t^{-1}\lambda t) \;=\; 2 \;.$$

Since the character is continuous and determines the representation, we have $i_!(1) = 2$.

We claim that in contrast to this relation in $\mathbf{R}(SU(2))$, the elements $\mathrm{tr}_T^{SU(2)}(1)$ and 2 are linearly independent (so in particular different) in $\pi_0^{SU(2)}(\mathbf{ku})$. We can detect this through the *dimension homomorphism*

$$\dim \;:\; \mathbf{ku} \;\longrightarrow\; H\mathbb{Z} \;, \tag{6.23}$$

a multiplicative morphism of orthogonal ring spectra from connective global K-theory to the Eilenberg-Mac Lane spectrum (5.2) of the integers. For every complex inner product space $\mathcal{U}$ and every based space K a based map $\dim : \mathscr{C}(\mathcal{U}, K) \longrightarrow \mathbb{Z}[K]$ is given by the dimension function, i.e., a configuration of vector spaces

$$[X_1, \dots, X_n; k_1, \dots, k_n] \;\in\; \mathscr{C}(\mathcal{U}, K)$$

is sent to the configuration of dimensions

$$\sum_{i=1}^n \dim(X_i) \cdot k_i \;\in\; \mathbb{Z}[K] \;.$$

Taking $\mathcal{U} = \mathrm{Sym}(V_\mathbb{C})$ and $K = S^V$ for a real inner product space V gives a map of $O(V)$-spaces

$$\dim(V) \;:\; \mathbf{ku}(V) \;=\; \mathscr{C}(\mathrm{Sym}(V_\mathbb{C}), S^V) \;\longrightarrow\; \mathbb{Z}[S^V] = (H\mathbb{Z})(V) \;.$$

The dimension is additive on direct sums and multiplicative on tensor products, so as V varies the maps $\dim(V)$ form a morphism of orthogonal ring spectra. Proposition 5.5 shows that the elements $\mathrm{tr}_T^{SU(2)}(1)$ and 2 are linearly independent in $\pi_0^{SU(2)}(H\mathbb{Z})$, so they must also be linearly independent in $\pi_0^{SU(2)}(\mathbf{ku})$.

Construction 6.24. We define a homomorphism out of the group $\pi_0^G(\mathbf{ku})$ that will provide a retraction to $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$; in particular, the homomorphism (6.17) is injective for every compact Lie group. We let $G^\circ \leq G$ be the connected component of the identity and $\bar{G} = G/G^\circ$ the finite group of path components of G. We define an equivariant homotopy group $\Psi^G(\mathbf{ku})$ as a mixture of a genuine equivariant homotopy group for the finite group $\bar{G}$ and a geometric fixed point homotopy group for the connected group G°:

$$\Psi^G(\mathbf{ku}) \;=\; \mathrm{colim}_{V \in s(\mathcal{U}_G)} \; [S^{V^{G^\circ}}, \mathbf{ku}(V)^{G^\circ}]^{\bar{G}} \;;$$

as usual, $s(\mathcal{U}_G)$ is the poset of finite dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$. We define a partial geometric fixed point map

$$\psi^G \;:\; \pi_0^G(\mathbf{ku}) \;\longrightarrow\; \Psi^G(\mathbf{ku}) \;, \quad [f : S^V \longrightarrow \mathbf{ku}(V)] \;\longmapsto\; [f^{G^\circ} : S^{V^{G^\circ}} \longrightarrow \mathbf{ku}(V)^{G^\circ}] \;.$$

If the group G is finite, then G° is trivial, $\bar{G} = G$, $\pi_0^G(\mathbf{ku}) = \Psi^G(\mathbf{ku})$ and ψ^G is the identity. On the other extreme, if G is connected, then $\bar{G}$ is trivial, $\Psi^G(\mathbf{ku}) = \Phi^{G^\circ}(\mathbf{ku})$ is the geometric fixed point group defined in (6.2) of Chapter III and ψ^G is the geometric fixed point map defined in (6.3) of Chapter III.

We already know that the map $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is bijective when the group G is finite. The next proposition shows that the map is in fact always injective, and that it is bijective if every subgroup of G of strictly smaller dimension has infinite Weyl group. This happens, for example, for all abelian compact Lie groups.

Proposition 6.25. *For every compact Lie group G the composite*

$$\mathbf{R}(G) \xrightarrow{\langle\ \rangle} \pi_0^G(\mathbf{ku}) \xrightarrow{\psi^G} \Psi^G(\mathbf{ku})$$

is an isomorphism and the kernel of ψ^G is the subgroup of $\pi_0^G(\mathbf{ku})$ generated by transfers from subgroups of G of smaller dimension.

Proof. In a first step we identify the group $\Psi^G(\mathbf{ku})$ with the 0-th $\bar{G}$-equivariant homotopy group of the $\bar{G}$-spectrum associated to the $\mathbf{\Gamma}$-$\bar{G}$-space $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))^{G^\circ}$ for any sufficiently large G-representation V. Indeed,

$$\begin{aligned} \Psi^G(\mathbf{ku}) &= \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, (\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V))^{G^\circ}]^{\bar{G}} \\ &= \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))^{G^\circ}(S^{V^{G^\circ}})]^{\bar{G}} \\ &\cong \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^{V^{G^\circ}}, \mathscr{C}(\mathcal{U}_G)^{G^\circ}(S^{V^{G^\circ}})]^{\bar{G}} \\ &\cong \mathrm{colim}_{W\in s(\mathcal{U}_G^{G^\circ})}[S^W, \mathscr{C}(\mathcal{U}_G)^{G^\circ}(S^W)]^{\bar{G}} \\ &\cong \pi_0^{\bar{G}}\left(\mathscr{C}(\mathcal{U}_G)^{G^\circ}(\mathbb{S}_{\bar{G}})\right) , \end{aligned}$$

where the second equation is Proposition III.8.33; the next isomorphism uses that $\mathrm{Sym}(V_{\mathbb{C}})$ a complete G-universe, hence isomorphic to $\mathcal{U}_G$, for sufficiently large V [fix this...] the next isomorphism uses that as V runs over $s(\mathcal{U}_G)$, the space V^{G° runs over all $\bar{G}$-invariant subspaces of $(\mathcal{U}_G)^{G^\circ}$; the last isomorphism uses that $(\mathcal{U}_G)^{G^\circ}$ is a complete universe for the group $\bar{G}$.

The $\mathbf{\Gamma}$-$\bar{G}$-space $\mathscr{C}(\mathcal{U}_G)^{G^\circ}$ is special by Theorem 6.8 (i). It is a general fact about special $\mathbf{\Gamma}$-$\bar{G}$-spaces F [ref to Shimakawa] that the canonical map

$$\pi_0^{\bar{G}}(F(S^0)) = \pi_0(F(S^0)^{\bar{G}}) \longrightarrow \pi_0^{\bar{G}}(F(\mathbb{S}))$$

sending the component of a $\bar{G}$-fixed point $y \in F(1^+)^{\bar{G}}$ to the homotopy class of y, interpreted as a based $\bar{G}$-map

$$S^0 \longrightarrow F(S^0) ,$$

is additive with respect to the monoid structure on $\pi_0^{\bar{G}}(F(S^0))$ coming from the $\mathbf{\Gamma}$-space structure, and that it becomes an isomorphism after taking the Grothendieck group on the left hand side. We apply this to the special $\mathbf{\Gamma}$-$\bar{G}$-space $F = (\mathscr{C}(\mathcal{U}_G))^{G^\circ}$. In this case

$$\left(\mathscr{C}(\mathcal{U}_G)^{G^\circ}(S^0)\right)^{\bar{G}} = \mathscr{C}(\mathcal{U}_G, S^0)^G$$

is the set of finite dimensional G-subrepresentations of a complete G-universe, so the monoid $\pi_0^G\left(\mathscr{C}(\mathcal{U}_G)^{G^\circ}(S^0)\right)$ is isomorphic to the monoid $\pi(G)$ of isomorphism classes of finite dimensional G-representations. We conclude that the group $\pi_0^{\bar{G}}(\mathscr{C}(\mathcal{U}_G)^{G^\circ}(\mathbb{S}))$, and hence the group $\Psi^G(\mathbf{ku})$, is isomorphic to the representation ring $\mathbf{R}(G)$.

In fact, the composite

$$\mathbf{R}^+(G) \xrightarrow{\langle - \rangle} \pi_0^G(\mathbf{ku}) \longrightarrow \Psi^G(\mathbf{ku})$$

sends the isomorphism class of a G-representation V to the class of the map

$$S^{V^{G^\circ}} \xrightarrow{[V;-]^{G^\circ}} \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)^{G^\circ} .$$

Under the homeomorphism $\mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)^{G^\circ} \cong \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))^{G^\circ}(S^{V^{G^\circ}})$ this becomes the map

$$S^{V^{G^\circ}} \xrightarrow{[V^{G^\circ};-]} \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))^{G^\circ}(S^{V^{G^\circ}}) \ .$$

This is the stabilization, by the $\bar{G}$-representation V^{G°, of the point $V^{G^\circ} \in \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))^{G^\circ}(1^+)$. So the composite is an isomorphism, as claimed. □

Remark 6.26. We can identify the class of Lie groups for which the map $\langle - \rangle : \mathbf{R}(G) \longrightarrow \pi_0^G(\mathbf{ku})$ is bijective. This class includes all finite groups, all tori, and all products of a finite group and a torus.

For example, we let $G = T^n$ be an n-dimensional torus. This is a connected Lie group, so the retraction $\pi_0^{T^n}(\mathbf{ku}) \longrightarrow \Psi^{T^n}(\mathbf{ku})$ coincides with the geometric fixed point map $\Phi : \pi_0^{T^n}(\mathbf{ku}) \longrightarrow \Phi_0^{T^n}(\mathbf{ku})$ defined in (6.3) of Chapter III. Since the spectrum $\mathbf{ku}$ is globally connective, the kernel of the geometric fixed point map is generated by all transfer from proper subgroups of T^n [ref]. However, every proper subgroup of T^n has smaller dimension and is normal, so all these transfers are zero. We conclude that for a torus the retraction map is also injective, and hence the map $\langle - \rangle : \mathbf{R}(T^n) \longrightarrow \pi_0^{T^n}(\mathbf{ku})$ is an isomorphism.

The special unitary group $SU(2)$ is the simplest example where the map $\langle - \rangle$ is not surjective, We consider the diagonal maximal torus $T \leq SU(2)$ given by

$$T \ = \ \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \ : \ \lambda \in \mathbb{C}, |\lambda| = 1 \right\} \ .$$

Then the transfer

$$\mathrm{tr}_T^{SU(2)}(1) \ \in \ \pi_0^{SU(2)}(\mathbf{ku})$$

of the unit $1 \in \pi_0^T(\mathbf{ku})$ is in the kernel of the geometric fixed point map, see Proposition III.6.5. Moreover, the transfer is non-zero on $\pi_0^{SU(2)}(\mathbf{ku})$ because the Weyl group of T is finite, and so the element $\mathrm{tr}_T^{SU(2)}(1)$ is detected in the group $\pi_0^{SU(2)}(H\mathbb{Z})$ using the dimension homomorphism (6.23) $\dim : \mathbf{ku} \longrightarrow H\mathbb{Z}$ and Proposition 5.5.

Construction 6.27 (Complex conjugation on $\mathbf{ku}$)**.** The ultra-commutative ring spectrum $\mathbf{ku}$ comes with an involution by 'complex conjugation' that preserves all the structure available. Indeed, for every real inner product space V the complex symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ of the complexification is canonically $O(V)$-equivariantly isomorphic to $\mathbb{C} \otimes_{\mathbb{R}} \mathrm{Sym}_{\mathbb{R}}(V)$, the complexification of the real symmetric algebra of V. So $\mathrm{Sym}(V_{\mathbb{C}})$ comes with an involution $\psi_{\mathrm{Sym}(V)}$ that is semilinear and preserves the orthogonality relation. Applying this involution elementwise to tuples of orthogonal subspaces given an involution $\mathscr{C}(\psi_{\mathrm{Sym}(V)}) : \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}})) \longrightarrow \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$ of the $\mathbf{\Gamma}$-space and hence a homeomorphism

$$\psi(V) \ = \ \mathscr{C}(\psi_{\mathrm{Sym}(V)}, S^V) \ : \ \mathbf{ku}(V) = \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) = \mathbf{ku}(V)$$

of order 2. As V varies, the maps $\psi(V)$ form an automorphism

$$\psi \ : \ \mathbf{ku} \ \longrightarrow \ \mathbf{ku} \tag{6.28}$$

of the ultra-commutative ring spectrum $\mathbf{ku}$. The complex conjugation (6.28) of the orthogonal spectrum $\mathbf{ku}$ deloops the complex conjugation of the orthogonal monoid space $\mathbf{Gr}^{\mathbb{C}}$ that we discussed in construction II.2.44 in the sense of the following commutative diagram of orthogonal spectra:

$$\begin{tikzcd} \mathbf{BGr}^{\mathbb{C}} \arrow[r, "\xi"] \arrow[d, "\mathbf{B}\psi"'] & \mathbf{ku} \arrow[d, "\psi"] \\ \mathbf{BGr}^{\mathbb{C}} \arrow[r, "\xi"'] & \mathbf{ku} \end{tikzcd}$$

Here ξ is the global equivalence discussed in Proposition 6.14.

Construction 6.29 (Rank filtration)**.** The connective global K-theory spectrum $\mathbf{ku}$ also comes with an exhaustive multiplicative filtration

$$\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}^{[2]} \longrightarrow \dots \longrightarrow \mathbf{ku}^{[m]} \longrightarrow \dots \ \longrightarrow \ \mathbf{ku}$$

by orthogonal subspectra whose first piece $\mathbf{ku}^{[1]}$ is a multiplicative model for the global classifying space of the circle group $T = U(1)$. For $m \geq 0$ and $\mathcal{U}$ a complex inner product space we define a sub-$\mathbf{\Gamma}$-space $\mathscr{C}^{[m]}(\mathcal{U}) \subseteq \mathscr{C}(\mathcal{U})$ of tuples of orthogonal vector spaces whose total dimension is at most m. So a tuple $(V_a)_{a\in A} \in \mathscr{C}(\mathcal{U}, A)$ belongs to $\mathscr{C}^{[m]}(\mathcal{U}, A)$ if and only if the sum of the dimensions of the vector spaces V_a is at most m. We define a orthogonal subspectrum $\mathbf{ku}^{[m]}$ of $\mathbf{ku}$ by

$$\mathbf{ku}^{[m]}(V) \ = \ \mathscr{C}^{[m]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ \subset \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}(V) \ .$$

The dimension function is multiplicative on tensor products, so if $\mathcal{U}$ and $\mathcal{U}'$ are two hermitian vector spaces and A and A' are finite based sets, then the multiplication map

$$\mathscr{C}(\mathcal{U}, A) \wedge \mathscr{C}(\mathcal{U}', A') \ \longrightarrow \ \mathscr{C}(\mathcal{U} \otimes \mathcal{U}', \ A \wedge A') \ , \quad (V_a) \wedge (W_b) \ \longmapsto \ (V_a \otimes W_b)_{a \wedge b}$$

takes $\mathscr{C}^{[m]}(\mathcal{U}, A) \wedge \mathscr{C}^{[n]}(\mathcal{U}', A')$ to $\mathscr{C}^{[mn]}(\mathcal{U} \otimes \mathcal{U}', \ A \wedge A')$. The multiplication of $\mathbf{ku}$ thus restricts to associative and unital pairings

$$\mathbf{ku}^{[m]} \wedge \mathbf{ku}^{[n]} \ \longrightarrow \ \mathbf{ku}^{[mn]} \ .$$

For $m = 1$ this gives $\mathbf{ku}^{[1]}$ the structure of a commutative orthogonal ring spectrum and it gives $\mathbf{ku}^{[n]}$ a module structure over $\mathbf{ku}^{[1]}$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ is multiplicative, i.e., a morphism of orthogonal ring spectra.

A configuration of points in K labeled by vector spaces of total dimension 1 has to be concentrated on at most one point. So the map

$$\mathbf{P}(V)_+ \wedge S^V \ = \ P(\mathrm{Sym}(V_{\mathbb{C}}))_+ \wedge S^V \ \longrightarrow \ \mathscr{C}^{[1]}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \ = \ \mathbf{ku}^{[1]}(V) \ , \quad L \wedge v \ \longmapsto \ [L; v]$$

is a homeomorphism. As V varies through real inner product spaces, these maps form an isomorphism of orthogonal ring spectra

$$\Sigma^\infty_+ \mathbf{P} \ \cong \ \mathbf{ku}^{[1]}$$

from the unreduced suspension spectrum of the orthogonal projective space. Since $\mathbf{P}$ is a multiplicative model of the global classifying space of the circle group, the ultra-commutative ring spectrum $\mathbf{ku}^{[1]}$ is globally a suspension spectrum of a global classifying space of the circle group.

By Proposition III.3.13, the homotopy group global functor $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}} T)$, and hence also the global functor $\underline{\pi}_0(\mathbf{ku}^{[1]})$, is the represented global functor $\mathbf{A}(T, -)$. The inclusion $\mathbf{ku}^{[1]} \longrightarrow \mathbf{ku}$ is a morphism of commutative orthogonal ring spectra, so it induces a morphism of global power functors $\mathbf{A}(T, -) \cong \underline{\pi}_0(\mathbf{ku}^{[1]}) \longrightarrow \underline{\pi}_0(\mathbf{ku})$. The global functor $\underline{\pi}_0(\mathbf{ku})$ is closely related to the complex representation rings. We have commented on the algebraic properties of the analogous morphism of global power functors $\mathbf{A}(T, -) \longrightarrow \mathbf{R}$ in Remark V.3.11 above.

Remark 6.30. The underlying non-equivariant spectrum of $\Sigma^\infty_+ \mathbf{P}$ is the suspension spectrum of $\mathbb{C}P^\infty$, and on underlying non-equivariant spectra, the morphism

$$\Sigma^\infty_+ \mathbf{P} \ \cong \ \mathbf{ku}^{[1]} \ \longrightarrow \ \mathbf{ku} \tag{6.31}$$

is a rational stable equivalence. However, the morphism (6.31) is *not* a rational global equivalence, which can be seen already on the level of $\underline{\pi}_0$, where the induced map is isomorphic to the morphism of global functors $\mathbf{A}(T, -) \longrightarrow \mathbf{R}$ that sends the generator $\mathrm{Id}_T \in \mathbf{A}(T, T)$ to the class of the tautological T-representation on $\mathbb{C}$. This morphism is surjective, but not injective (not even rationally), and we investigated the kernel in Remark V.3.11. The simplest example is the non-zero element

$$\mathrm{tr}_e^{C_2} \circ \mathrm{res}^T_e - z^* - \mathrm{res}^T_{C_2} \ \in \ \mathbf{A}(T, C_2)$$

of infinite order, where $z : T \longrightarrow C_2$ is the trivial homomorphism. This element maps trivially to $\mathbf{R}(C_2)$ because the regular representation of C_2 splits as the sum of the 1-dimensional trivial and sign representations.

Remark 6.32 (Real connective global K-theory)**.** There is a straightforward real analog $\mathbf{ko}$ of the complex connective global K-theory spectrum $\mathbf{ku}$. The value of $\mathbf{ko}$ on a real inner product space V is

$$\mathbf{ko}(V) \ = \ \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ ,$$

where now $\mathrm{Sym}(V)$ is the symmetric algebra, over the real numbers, of the real inner product space V, and $\mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V))$ is the $\mathbf{\Gamma}$-space of tuples of pairwise orthogonal, finite dimensional (real) subspaces of $\mathrm{Sym}(V)$. As V varies, the spaces $\mathbf{ko}(V)$ again come with the structure of a commutative orthogonal ring spectrum. Proposition 6.25 has a direct real analog, saying that for every compact Lie group G the composite

$$\mathbf{RO}(G) \ \xrightarrow{\langle\ \rangle} \ \pi_0^G(\mathbf{ko}) \ \xrightarrow{\psi^G} \ \Psi^G(\mathbf{ko})$$

is an isomorphism, where $\mathbf{RO}(G)$ is the real representation ring of G. Moreover, the kernel of ψ^G is the subgroup of $\pi_0^G(\mathbf{ko})$ generated by transfers from subgroups of G of smaller dimension. In particular, the map $\langle\ \rangle : \mathbf{RO}(G) \longrightarrow \pi_0^G(\mathbf{ko})$ is an isomorphism for finite groups G.

Complexification defines a morphism of ultra-commutative ring spectra

$$c \ : \ \mathbf{ko} \ \longrightarrow \ \mathbf{ku} \ . \tag{6.33}$$

In more detail: if V is a real inner product space, then a morphism of $\mathbf{\Gamma}$-spaces

$$\mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V)) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}))$$

is defined by sending a configuration $[X_1, \dots, X_n; v_1, \dots, v_n]$ to the 'complexified' configuration $[(X_1)_{\mathbb{C}}, \dots, (X_n)_{\mathbb{C}}; v_1, \dots, v_n]$ Evaluating at the V-sphere gives a continuous, $O(V)$-equivariant based map

$$c(V) \ : \ \mathbf{ko}(V) = \mathscr{C}_{\mathbb{R}}(\mathrm{Sym}(V), S^V) \ \longrightarrow \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}})) = \mathbf{ku}(V) \ .$$

As V varies, these maps form the morphism (6.33) of ultra-commutative ring spectra. The complexification of a real subspace of $\mathrm{Sym}(V)$ is invariant under the complex conjugation involution $\psi_{\mathrm{Sym}(V)}$ of $\mathrm{Sym}(V_{\mathbb{C}})$, so the image of the complexification morphism is invariant under the complex conjugation involution ψ of $\mathbf{ku}$ defined in (6.28),

$$\psi \circ c \ = \ c \ : \ \mathbf{ko} \ \longrightarrow \ \mathbf{ku} \ .$$

Even more is true: a complex subspace of $\mathrm{Sym}(V_{\mathbb{C}})$ is ψ-invariant if and only if it is the complexification of its real part (the $+1$ eigenspaces of the restriction of $\psi_{\mathrm{Sym}(V)}$). This means that $\mathbf{ko}$ 'is' the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$; more formally, the complexification morphism is an isomorphism

$$c \ : \ \mathbf{ko} \ \cong \ \mathbf{ku}^{\psi}$$

to the ψ-fixed orthogonal ring subspectrum of $\mathbf{ku}$.

Construction 6.34 (Periodic global K-theory)**.** Now we describe *periodic global K-theory*, an ultra-commutative ring spectrum whose G-homotopy type realizes G-equivariant periodic K-theory. We use the pointset level construction of M. Joachim [**46**] for equivariant K-theory spectra. In the Greenlees-May context of global homotopy theory, it was first observed by Bohmann ([**13**, Sec. 4.3], [**14**, Sec. 4]) that Joachim's model is 'global' (although in a slightly different sense).

Given a real inner product space V we denote by $\mathrm{Cl}\,V$ the complexified Clifford algebra of V, i.e., the quotient of the complex tensor algebra of V by the ideal generated by $v \otimes v - |v|^2 \cdot 1$; the Clifford algebra is $\mathbb{Z}/2$-graded with the even (respectively odd) part generated by an even (respectively odd) number of vectors from V. We denote by $\mathcal{H}_V$ the Hilbert space completion of the symmetric algebra, $\mathbb{Z}/2$-graded by even and odd symmetric powers. Then $\mathcal{K}_V$ denotes the $\mathbb{Z}/2$-graded C^*-algebra of compact operators on the $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}_V$. We let s denote the C^*-algebra of continuous functions on $\mathbb{R}$ vanishing at infinity; this C^*-algebra is $\mathbb{Z}/2$-graded by even and odd functions. Joachim's construction associates to a real inner product space V the based space

$$\mathbf{KU}(V) \;=\; \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,V \otimes \mathcal{K}_V)$$

of $\mathbb{Z}/2$-graded homomorphism of C^*-algebras (with subspace topology of the compact open topology). The orthogonal group $O(V)$ acts on the Clifford algebra $\mathrm{Cl}\,V$ by functoriality, on the compact operators $\mathcal{K}_V$ (by conjugation) and hence diagonally on $\mathrm{Cl}\,V \otimes \mathcal{K}_V$ by automorphisms of C^*-algebras. This induces the $O(V)$-action on the mapping space $\mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,V \otimes \mathcal{K}_V)$.

The multiplication of $\mathbf{KU}$ uses extra structure on the algebra s, namely a comultiplication $\Delta : s \longrightarrow s \otimes s$ in the category of $\mathbb{Z}/2$-graded C^*-algebras. To define it, we use the vector space identification $s \otimes s \cong C_0(\mathbb{R}^2)$ with the C^*-algebra of functions on $\mathbb{R}^2$ vanishing at infinity, sending $f \otimes g$ to the function $f \cdot g$.

The above isomorphism of $s \otimes s$ with $C_0(\mathbb{R}^2)$ is not multiplicative. The point is that $s \otimes s$ is the graded tensor product of $\mathbb{Z}/2$-graded algebras, so in $s \otimes s$ two odd elements anti-commute. In contrast, the algebra $C_0(\mathbb{R}^2)$ is commutative. While we are at it, we emphasize that s itself is *not* commutative as $\mathbb{Z}/2$-graded algebra, because odd functions commute (instead of anti-commuting).

The comultiplication is defined on an even function f by

$$\Delta(f)(x,y) \;=\; f(\sqrt{x^2+y^2})$$

and on an odd function g by

$$\Delta(g)(x,y) \;=\; \frac{x+y}{\sqrt{x^2+y^2}} \cdot g(\sqrt{x^2+y^2}) \ ,$$

where $(x,y) \in \mathbb{R}^2$. The algebra s is generated by the functions

$$u^+(t) = \frac{1}{1+t^2} \qquad \text{and} \qquad u^-(t) = \frac{t}{1+t^2} \ ,$$

and the comultiplication is completely determined by the values on these, which are

$$\Delta(u^+)(x,y) = \frac{1}{1+x^2+y^2} \qquad \text{respectively} \qquad \Delta(u^-)(x,y) = \frac{x+y}{1+x^2+y^2} \ .$$

A key fact is that the comultiplication Δ is coassociative and cocommutative.

The multiplication map

$$\mu_{V,W} \;:\; \mathbf{KU}(V) \wedge \mathbf{KU}(W) \;\longrightarrow\; \mathbf{KU}(V \oplus W)$$

is now defined as the composite

$$\begin{aligned}
\mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,V \otimes \mathcal{K}_V) \wedge \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,W \otimes \mathcal{K}_W) \ &\xrightarrow{\ \otimes\ } \ \mathrm{Hom}_{C^*}(s \otimes s, (\mathrm{Cl}\,V \otimes \mathcal{K}_V) \otimes (\mathrm{Cl}\,W \otimes \mathcal{K}_W)) \\
&\xrightarrow{\ \cong\ } \ \mathrm{Hom}_{C^*}(s \otimes s, \mathrm{Cl}(V \oplus W) \otimes \mathcal{K}_{V \oplus W}) \\
&\xrightarrow{\ \Delta^*\ } \ \mathrm{Hom}_{C^*}(s, \mathrm{Cl}(V \oplus W) \otimes \mathcal{K}_{V \oplus W}) \ .
\end{aligned}$$

Next we define $O(V)$-equivariant continuous based maps

$$j(V) \;:\; \mathbf{ku}(V) \;=\; \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V) \;\longrightarrow\; \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,V \otimes \mathcal{K}_V) \;=\; \mathbf{KU}(V)$$

that form a morphism of ultra-commutative ring spectra. A new ingredient here is a map

$$\mathrm{fc} \;:\; S^V \;\longrightarrow\; \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\,V)$$

often referred to as 'functional calculus'. Given $v \in V$ the associated C^*-homomorphism $\mathrm{fc}(v) : s \longrightarrow \mathrm{Cl}\,V$ sends an even function f to

$$\mathrm{fc}(v)(f) \;=\; f(|v|) \cdot 1$$

and an odd function g to

$$\mathrm{fc}(v)(g) \;=\; g(|v|) \cdot v/|v| \ .$$

For example, the generating functions $u^+(t) = (1+t^2)^{-1}$ and $u^-(t) = t(1+t^2)^{-1}$ are sent to

$$\mathrm{fc}(v)(u^+) \ = \ \frac{1}{1+|v|^2} \qquad \text{respectively} \qquad \mathrm{fc}(v)(u^-) \ = \ \frac{v}{1+|v|^2} \ .$$

If the norm of v tends to infinity, then $\mathrm{fc}(v)$ tends to the constant $*$-homomorphism with value 0 in $\mathrm{Hom}_{C^*}(s, \mathrm{Cl}\, V)$, which is the basepoint. So fc indeed extends to a continuous map on the one-point compactification S^V. The map fc is evidently $O(V)$-equivariant.

We consider a configuration

$$[E_1, \dots, E_n;\, v_1, \dots, v_n] \ \in \ \mathscr{C}(\mathrm{Sym}(V_{\mathbb{C}}), S^V)$$

of pairwise orthogonal, finite dimensional subspaces on S^V. The associated C^*-homomorphism

$$j(V)[E_1, \dots, E_n;\, v_1, \dots, v_n] \ : \ s \ \longrightarrow \ \mathrm{Cl}\, V \otimes \mathcal{K}_V$$

is then defined on a function $f \in C_0(\mathbb{R})$ by

$$j(V)[E_1, \dots, E_n;\, v_1, \dots, v_n](f) \ = \ \sum_{i=1}^n \ \mathrm{fc}(v_i)(f) \otimes p_{E_i} \ ,$$

where p_E denotes the orthogonal projection onto a subspace E. This is an operator of finite rank, so in particular a compact operator, i.e., an element of the C^*-algebra $\mathcal{K}_V$.

The unit maps

$$\eta_V \ : \ S^V \ \longrightarrow \ \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\, V \otimes \mathcal{K}_V) \ = \ \mathbf{KU}(V)$$

are then defined as the composite

$$S^V \ \xrightarrow{\ \eta_V\ } \ \mathbf{ku}(V) \ \xrightarrow{\ j(V)\ } \ \mathbf{KU}(V)$$

of the unit map for the connective global K-theory spectrum and the map $j(V)$ just defined. Unraveling the definitions show that

$$\eta_V(v)(f) \ = \ \mathrm{fc}(v)(f) \otimes p_{\mathbb{C}} \ .$$

For the time being, we omit the verifications needed to show that the maps $j(V)$ form a morphism of ultra-commutative ring spectra $j : \mathbf{ku} \longrightarrow \mathbf{KU}$.

Joachim shows in Theorem 4.4 of [**46**] that for every compact Lie group G, the orthogonal spectrum $\mathbf{KU}$ represents G-equivariant K-theory. In particular, the ring $\pi_0^G(\mathbf{KU})$ is isomorphic to the complex representation ring $\mathbf{R}(G)$. It would be more than surprising if the composite map

$$\mathbf{R}(G) \ \xrightarrow{\ \langle - \rangle\ } \ \pi_0^G(\mathbf{ku}) \ \xrightarrow{\ \pi_0^G(j)\ } \ \pi_0^G(\mathbf{KU})$$

were not an isomorphism, but we have not checked that yet. The Bott class $u \in \pi_2^e(\mathbf{KU})$ is a unit and restricts along the unique homomorphism $G \longrightarrow e$ to a unit in $\pi_2^G(\mathbf{KU})$ for every group G. So $\underline{\pi}_*(\mathbf{KU})$ is 2-periodic, $\underline{\pi}_1(\mathbf{KU}) = 0$ and

$$\underline{\pi}_0(\mathbf{KU}) \ = \ \mathbf{R}$$

is the complex representation ring global functor.

Remark 6.35. Our definition of $\mathbf{KU}(V)$ is not exactly the same as in [**46**], because we use the Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$, instead of the Hilbert space $L^2(V)$ of $\mathbb{C}$-valued square integrable functions on V. These two Hilbert spaces are naturally isomorphic, as follows. We use the inner product on V to identify it with its dual space V^*, and hence the symmetric algebra $\mathrm{Sym}(V_{\mathbb{C}})$ with $\mathrm{Sym}(V_{\mathbb{C}}^*)$. Elements of $\mathrm{Sym}(V_{\mathbb{C}}^*)$ are complex valued polynomial functions on V. We make them square integrable by multiplying with the rapidly decaying function $v \mapsto \exp(-\langle v, v \rangle)$. This provides an $O(V)$-equivariant linear isometric embedding

$$\mathrm{Sym}(V_{\mathbb{C}}^*) \ \longrightarrow \ L^2(V) \ , \quad f \ \longmapsto \ f \cdot \exp(-\langle -, - \rangle)$$

with dense image. Altogether this exhibits $L^2(V)$ as a Hilbert space completion of $\mathrm{Sym}(V_{\mathbb{C}})$.

Remark 6.36 (Naive localization of $\Sigma^\infty_+\mathbf{P}$)**.** We consider the composite morphism of ultra-commutative ring spectra

$$\Sigma^\infty_+\mathbf{P} \cong \mathbf{ku}^{[1]} \xrightarrow{\text{incl}} \mathbf{ku} \xrightarrow{j} \mathbf{KU} . \tag{6.37}$$

Explicitly, the V-th component of this morphism is the map

$$\begin{aligned} (\Sigma^\infty_+\mathbf{P})(V) = P(\mathrm{Sym}(V_{\mathbb{C}}))_+ \wedge S^V \ &\longrightarrow\ \mathrm{Hom}_{C^*}(s, \mathrm{Cl}\, V \otimes \mathcal{K}_V) \ = \ \mathbf{KU}(V) \\ L \wedge v \quad &\longmapsto \qquad \{f \mapsto \mathrm{fc}(v)(f) \otimes p_L\} \ . \end{aligned}$$

The source is non-equivariantly an unreduced suspension spectrum of the infinite projective space $P(\mathbb{C}^\infty)$. By a theorem of Snaith the underlying morphism of non-equivariant spectra from $\Sigma^\infty_+\mathbf{P}$ to $\mathbf{KU}$ becomes a stable equivalence after inverting the Bott element in $\pi_2^e(\Sigma^\infty_+ P(\mathbb{C}^\infty))$ (see [**84**, II §9] or [**85**]). Non-equivariantly, inverting the Bott element can be realized on the level of spectra by forming the homotopy colimit

$$\Sigma^\infty_+\mathbf{P} \ \xrightarrow{\beta\cdot -}\ S^{-2} \wedge \Sigma^\infty_+\mathbf{P} \ \xrightarrow{\beta\cdot -}\ S^{-4} \wedge \Sigma^\infty_+\mathbf{P} \ \longrightarrow\ \dots$$

and such a homotopy colimit makes perfect sense in the global stable homotopy category, and we denote it by

$$(\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}] \ .$$

The subscript 'naive' is supposed to remind is that while this construction is useful and interesting non-equivariantly, it is too naive in the global context. Since the non-equivariant Bott class becomes is invertible in $\pi_2^e(\mathbf{KU})$, the morphism (6.37) extends to a morphism $(\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}] \longrightarrow \mathbf{KU}$ in the global stable homotopy category $\mathcal{GH}$. At this point we need to issue a warning.

The naive localization $(\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}]$ does *not* admit the structure of an ultra-commutative ring spectrum, and for non-trivial compact Lie groups, the map $(\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}] \longrightarrow \mathbf{KU}$ does *not* induce an equivariant equivalence of underlying G-orthogonal spectra, not even rationally. Contrary to this, the main result of [**88**] claims that for *abelian* compact Lie groups, the naive localization is equivariantly equivalent to $\mathbf{KU}$; however, that claim is incorrect, as we will now show by calculating the C_2-geometric fixed point homotopy groups of the naive localization rationally.

The fixed points of the projective space of a complete C_2-universe are given by

$$\mathbf{P}(\mathcal{U}_{C_2})^{C_2} \ = \ P(\mathcal{U}_1) \amalg P(\mathcal{U}_\sigma) \ ,$$

the disjoint union of projective spaces of the two isotypical summands, the fixed space $\mathcal{U}_1 = (\mathcal{U}_{C_2})^{C_2}$ and the anti-fixed space $\mathcal{U}_\sigma$, where σ is the sign representation. On the level of π_0, the component $P(\mathcal{U}_1)$ is the multiplicative unit and the square of the component $P(\mathcal{U}_\sigma)$. So the rationalized geometric C_2-fixed points are given by

$$\mathbb{Q} \otimes \Phi_*^{C_2}(\Sigma^\infty_+\mathbf{P}) \ = \ \mathbb{Q} \otimes \pi_*^e(\Sigma^\infty_+\mathbf{P}(\mathcal{U}_{C_2})^{C_2}) \ = \ H_*(\mathbf{P}(\mathcal{U}_{C_2})^{C_2}, \mathbb{Q}) \ = \ \mathbb{Q}[\bar{\beta}][C_2^*] \ ,$$

and this is an isomorphism of graded $\mathbb{Q}$-algebras. Here C_2^* is the dual group and $\bar{\beta} \in \Phi_2^{C_2}(\Sigma^\infty_+\mathbf{P})$ is the image of the non-equivariant Bott class under the geometric fixed point map ((6.3) of Chapter III). Geometric fixed point commute with desuspension and sequential homotopy colimits, hence also with naive localization. So

$$\mathbb{Q} \otimes \Phi_*^{C_2}((\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}]) \ = \ (\mathbb{Q} \otimes \Phi_*^{C_2}(\Sigma^\infty_+\mathbf{P}))[1/\bar{\beta}] \ \cong\ \mathbb{Q}[\bar{\beta}, \bar{\beta}^{-1}][C_2^*] \ .$$

In particular, the $\mathbb{Q} \otimes \Phi_0^{C_2}((\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}])$ is a 2-dimensional $\mathbb{Q}$-vector space. On the other hand,

$$\mathbb{Q} \otimes \Phi_0^{C_2}(\mathbf{KU}) \ \cong\ \pi_0^{C_2}(\mathbf{KU})/\mathrm{Im}(\mathrm{tr}_e^{C_2})$$

is isomorphic to the cokernel of the rationalized transfer map $\mathrm{tr}_e^{C_2} : \mathbf{R}(e) \longrightarrow \mathbf{R}(C_2)$ between the representation rings, hence a 1-dimensional $\mathbb{Q}$-vector space. So the natural morphism from $(\Sigma^\infty_+\mathbf{P})[1/\beta_{\text{naive}}]$ to $\mathbf{KU}$ is not a C_2-equivariant homotopy equivalence.

Remark 6.38 (Global localization of $\Sigma^\infty_+\mathbf{P}$)**.** The reason that the naive localization of $\Sigma^\infty_+\mathbf{P}$ is too naive to be close to $\mathbf{KU}$ is that it only inverts the Bott classes of *trivial* representations. Such a localization of a non-equivariant homotopy class by a naive homotopy colimit will typically *not* invert the norms of restrictions of the given class, in which case it cannot be modeled by an ultra-commutative ring spectrum. On the other, global K-theory $\mathbf{KU}$ is complex stable, i.e., the Bott classes of *all* complex representations are invertible. Of these, only the Bott classes of sums of 1-dimensional representations are present in $\underline{\pi}_*(\Sigma^\infty_+\mathbf{P})$, so the best thing we can do is to invert all of those. For *abelian* compact Lie group, this takes care of all complex representation, but not in general. On the other, if we perform the localization globally and in the most structured way possible, we only need to invert a single class, the Bott class

$$\beta_{T,\mathbb{C}} \;\in\; (\Sigma^\infty_+\mathbf{P})^T(S^{\mathbb{C}})$$

of the tautological 1-dimensional representation of the circle group. If we do the localization in the category of ultra-commutative ring spectra then the result

$$(\Sigma^\infty_+\mathbf{P})[1/\beta_{T,\mathbb{C}}]$$

is an ultra-commutative ring spectrum, so all products, norms and restrictions of $\beta_{T,\mathbb{C}}$ have also become invertible. In particular, for *abelian* compact Lie groups, the Bott classes of all complex representations are invertible in this global localization. In fact, we expect that the morphism of ultra-commutative ring spectra

$$(\Sigma^\infty_+\mathbf{P})[1/\beta_{T,\mathbb{C}}] \;\longrightarrow\; \mathbf{KU}$$

that extends the morphism (6.37) is an $\mathcal{A}b$-global equivalence, where $\mathcal{A}b$ is the global family of abelian compact Lie groups. Since $\mathbf{KU}$ is right induced from the global family *cyc* of finite cyclic groups (see Example IV.5.27), it is also right induced from the larger global family $\mathcal{A}b$. So $\mathbf{KU}$ is globally equivalent, as an ultra-commutative ring spectrum, to

$$R_{\mathcal{A}b}\left((\Sigma^\infty_+\mathbf{P})[1/\beta_{T,\mathbb{C}}]\right) \;,$$

where $R_{\mathcal{A}b}$ is the right adjoint to the forgetful functor from the global homotopy category of ultra-commutative ring spectra to its $\mathcal{A}b$-global analog.

Construction 6.39 (Global connective K-theory)**.** Now we define *global connective K-theory* $\mathbf{ku}^c$, a commutative orthogonal ring spectrum whose associated G-homotopy type is that of *G-equivariant connective K-theory* in the sense of Greenlees [**38**]. This is *not* a connective equivariant theory, i.e., the equivariant homotopy groups $\pi_k^G(\mathbf{ku}^c)$ do *not* generally vanish in negative dimensions, as soon as the group G is non-trivial. Hence the order of the adjectives 'global' and 'connective' matters, i.e., 'global connective' K-theory is different from 'connective global' K-theory. Two of the key advantages of $\mathbf{ku}^c$ over $\mathbf{ku}$ are that $\mathbf{ku}^c$ is equivariantly (and in fact globally) complex orientable, and that $\mathbf{ku}^c$ satisfies a completion theorem in the sense that for every compact Lie group G, the completion of the graded ring $\pi_*^G(\mathbf{ku}^c)$ at the augmentation ideal is the connective $\mathbf{ku}$-cohomology of the classifying space BG.

Our construction of $\mathbf{ku}^c$ is a direct 'globalization' of Greenlees' definition in [**38**, Def. 3.1]. We define $\mathbf{ku}^c$ as the homotopy pullback in the square of ultra-commutative ring spectra

$$\begin{tikzcd} \mathbf{ku}^c \arrow[r] \arrow[d] & b(\mathbf{ku}) \arrow[d, "bj"] \\ \mathbf{KU} \arrow[r, "i_{\mathbf{KU}}"'] & b(\mathbf{KU}) \end{tikzcd} \tag{6.40}$$

Here $\mathbf{KU}$ is periodic global K-theory (see Construction 6.34), and the Borel theory functor b and the natural transformation $i : \mathrm{Id} \longrightarrow b$ are defined in Section III.7. So more explicitly, we set

$$\mathbf{ku}^c \;=\; \mathbf{KU} \times_{b(\mathbf{KU})} b(\mathbf{KU})^{[0,1]} \times_{b(\mathbf{KU})} b(\mathbf{ku}) \;.$$

As a homotopy pullback, the square (6.40) does *not* commute, but the construction comes with a preferred homotopy between the two composites around the square.

Since the spectra $\mathbf{KU}$, $b(\mathbf{KU})$ and $b(\mathbf{ku})$ are commutative orthogonal ring spectra and the two morphism $i_{\mathbf{KU}} : \mathbf{KU} \longrightarrow b(\mathbf{KU})$ and $bj : b(\mathbf{ku}) \longrightarrow b(\mathbf{KU})$ are morphisms of orthogonal ring spectra, the homotopy pullback is canonically a commutative orthogonal ring spectrum and the two morphism from $\mathbf{ku}^c$ to $\mathbf{KU}$ and $b(\mathbf{ku})$ are morphisms of ring spectra.

Naturality of the morphism $i_{\mathbf{ku}} : \mathbf{ku} \longrightarrow b(\mathbf{ku})$ provides a morphism of commutative orthogonal ring spectra

$$\mathbf{ku} \longrightarrow \mathbf{ku}^c$$

from connective global K-theory to global connective K-theory. For finite groups, this morphisms induces an isomorphism on homotopy global functors in non-negative dimensions. The construction of $\mathbf{ku}^c$ endows it with a ring spectrum homomorphism $\mathbf{ku}^c \longrightarrow \mathbf{KU}$, and this morphism becomes a global equivalence after inverting the Bott class $\beta \in \pi_2^e(\mathbf{ku}^c)$.

The composite

$$\mathbf{ku}^c \longrightarrow b(\mathbf{ku}) \xrightarrow{b(\dim)} b(H\mathbb{Z})$$

is a morphism of commutative orthogonal ring spectra, where the dimension homomorphism $\dim : \mathbf{ku} \longrightarrow H\mathbb{Z}$ was defined in (6.23). We claim that there is a global homotopy cofiber sequence

$$S^2 \wedge \mathbf{ku}^c \xrightarrow{\beta} \mathbf{ku}^c \longrightarrow b(H\mathbb{Z}) \ . \tag{6.41}$$

Indeed, the sequence

$$S^2 \wedge \mathbf{ku} \xrightarrow{\beta} \mathbf{ku} \longrightarrow H\mathbb{Z}$$

is a non-equivariant homotopy fiber sequence by Bott periodicity. The Borel theory functor b takes this non-equivariant homotopy fiber sequence to the global homotopy fiber sequence

$$S^2 \wedge b(\mathbf{ku}) \xrightarrow{b(\beta)} b(\mathbf{ku}) \longrightarrow b(H\mathbb{Z}) \ .$$

Since the spectra $\mathbf{KU}$ and $b(\mathbf{KU})$ are Bott periodic, the two sequences

$$S^2 \wedge b(\mathbf{KU}) \xrightarrow{b(\beta)} b(\mathbf{KU}) \longrightarrow *$$

and

$$S^2 \wedge \mathbf{KU} \xrightarrow{\beta} \mathbf{KU} \longrightarrow *$$

are global homotopy fiber sequences. Passing to homotopy pullbacks gives the desired global homotopy fiber sequence (6.41).

The global homotopy cofiber sequence (6.41) and the isomorphism

$$\pi_k^G(b(H\mathbb{Z})) \cong H^{-k}(BG, \mathbb{Z})$$

of Proposition III.7.3 (iii) give rise to a long exact sequence of global functors

$$\cdots \longrightarrow \underline{\pi}_{k+1}(b(H\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{k-2}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_k(\mathbf{ku}^c) \xrightarrow{\dim_*^{\mathrm{gl}}} \underline{\pi}_k(b(H\mathbb{Z})) \longrightarrow \cdots \ .$$

This allows to calculate some of the homotopy global functors $\underline{\pi}_k(\mathbf{ku}^c)$ of global connective K-theory, thereby reproducing Greenlees' calculations of equivariant connective K-theory in [**38**, Prop. 2.6] from our present global perspective.

The group $H^{-k}(BG, \mathbb{Z})$ vanishes for $k > 0$. So multiplication by the Bott class

$$\beta \cdot - \ : \ \underline{\pi}_{k-2}(\mathbf{ku}^c) \longrightarrow \underline{\pi}_k(\mathbf{ku}^c)$$

is an isomorphism for $k > 0$ and a monomorphism for $k = 0$. In particular, we conclude that

$$\underline{\pi}_k(\mathbf{ku}^c) \;\cong\; \begin{cases} \mathbf{R} & \text{for } k \geq 0 \text{ and } k \text{ even, and} \\ 0 & \text{for } k \geq -1 \text{ and } k \text{ odd.} \end{cases}$$

The global functor $G \mapsto \pi_0^G(b(H\mathbb{Z})) = H^0(BG, \mathbb{Z})$ is constant with value $\mathbb{Z}$ and the morphism $\dim_*^{\mathrm{gl}} : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(H\mathbb{Z}))$ is isomorphic to the augmentation morphism $\dim : \mathbf{R} \longrightarrow \underline{\mathbb{Z}}$ of global functors. This is an epimorphism, so the sequence of global functors

$$0 \longrightarrow \underline{\pi}_{-2}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_0(\mathbf{ku}^c) \xrightarrow{\dim_*^{\mathrm{gl}}} \underline{\pi}_0(b(H\mathbb{Z})) \longrightarrow 0$$

is short exact and

$$\underline{\pi}_{-2}(\mathbf{ku}^c) \;\cong\; JU = \ker(\dim : \mathbf{R} \longrightarrow \underline{\mathbb{Z}})$$

is the augmentation ideal global functor. Again since the map $\dim_*^{\mathrm{gl}} : \underline{\pi}_0(\mathbf{ku}^c) \longrightarrow \underline{\pi}_0(b(H\mathbb{Z}))$ surjective, the global functor $\underline{\pi}_{-3}(\mathbf{ku}^c)$ injects into $\underline{\pi}_{-1}(\mathbf{ku}^c)$ which is trivial by the above. So we conclude that also

$$\underline{\pi}_{-3}(\mathbf{ku}^c) \;=\; 0\ .$$

The group $\pi_{-1}^G(b(H\mathbb{Z})) \cong H^1(BG, \mathbb{Z})$ is isomorphic to $\mathrm{Hom}(\pi_1(BG), \mathbb{Z})$ by the universal coefficient theorem; since G is compact, the group $\pi_1(BG) \cong \pi_0 G$ is finite, and so $H^1(BG, \mathbb{Z})$ vanishes for all compact Lie groups G; so the global functor $\underline{\pi}_{-1}(b(H\mathbb{Z}))$ vanishes. The long exact sequence splits off an exact sequence of global functors

$$0 \longrightarrow \underline{\pi}_{-4}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_{-2}(\mathbf{ku}^c) \xrightarrow{\dim_*^{\mathrm{gl}}} \underline{\pi}_{-2}(b(H\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{-5}(\mathbf{ku}^c) \longrightarrow 0\ . \tag{6.42}$$

The square

$$\begin{tikzcd} \pi_{-2}^G(\mathbf{ku}^c) \arrow[r, "\dim_*^{\mathrm{gl}}"] \arrow[d, "\cong"'] & \pi_{-2}^G(b(H\mathbb{Z})) \arrow[d, "\cong"] \\ JU(G) \arrow[r, "c_1 \circ \det"'] & H^2(BG, \mathbb{Z}) \end{tikzcd}$$

commutes, where the lower horizontal map sends a virtual representation $[V] - [V']$ of dimension zero to the first Chern class of the determinant line bundle $\det(V) \otimes \det(V')^{-1}$ [show]. Every class in $H^2(BG, \mathbb{Z})$ is the Chern class of a line bundle induced bundle by a 1-dimensional representation V of G. The class of $[V] - 1$ in $JU(G)$ then hits the given cohomology class. So the map $c_1 \circ \det : JU(G) \longrightarrow H^2(BG, \mathbb{Z})$, and hence the map $\dim_*^{\mathrm{gl}} : \pi_{-2}^G(\mathbf{ku}^c) \longrightarrow \pi_{-2}^G(b(H\mathbb{Z}))$ is surjective. The exact sequence (6.42) above then shows that

$$\underline{\pi}_{-4}(\mathbf{ku}^c) \;\cong\; JSU(G) \;=\; \{x \in JU(G) \mid \det(x) = 0\}$$

and that $\underline{\pi}_{-5}(\mathbf{ku}^c) = 0$. The next piece below in the long exact sequence is:

$$0 \longrightarrow \underline{\pi}_{-3}(b(H\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{-6}(\mathbf{ku}^c) \xrightarrow{\beta\cdot} \underline{\pi}_{-4}(\mathbf{ku}^c) \xrightarrow{\dim_*^{\mathrm{gl}}} \underline{\pi}_{-4}(b(H\mathbb{Z})) \xrightarrow{\partial} \underline{\pi}_{-7}(\mathbf{ku}^c) \longrightarrow 0$$

Using the already established isomorphism, this translates into an exact sequence

$$0 \longrightarrow H^2(BG, \mathbb{Z}) \xrightarrow{\partial} \pi_{-6}^G(\mathbf{ku}^c) \longrightarrow JSU(G) \xrightarrow{\dim_*^{\mathrm{gl}}} H^4(BG, \mathbb{Z}) \xrightarrow{\partial} \pi_{-7}^G(\mathbf{ku}^c) \longrightarrow 0$$

The group $H^2(BG, \mathbb{Z})$ classifies central extensions of the Lie group G by the circle group $U(1)$. [what is the map $JSU(G) \longrightarrow H^4(BG, \mathbb{Z})$?] This shows that the global functor $\underline{\pi}_{-6}(\mathbf{ku}^c)$ can have additive torsion and $\underline{\pi}_{-7}(\mathbf{ku}^c)$ may be non-zero. In fact, the global functor $\underline{\pi}_k(\mathbf{ku}^c)$ is supposedly non-zero for every integer $k \leq -6$. After this point things become less explicit.

7. Global bordism

We discuss two commutative orthogonal ring spectra that are global refinements of the unoriented bordism spectrum: the *Grassmannian model* **MO** and the *bar construction model* $\bar{M}O$, along with complex analogs **MU** and $\bar{M}U$ and periodic versions **MOP** and **MP**. The same, or closely related, strictly commutative ring spectrum models for these homotopy types have been discussed in various places, see for example [**59**], [**39**, Ex. 5.8], [**91**, App. A] or [**22**, Sec. 8].

Example 7.1 (Periodic unoriented global bordism)**.** We define an ultra-commutative ring spectrum **MOP** of *periodic unoriented global bordism.* Non-equivariantly, and ignoring the multiplication, **MOP** is a wedge of all even suspensions of the bordism spectrum MO whose non-equivariant stable homotopy groups are isomorphic to the ring of cobordism closed smooth manifolds. The equivariant homotopy type $\mathbf{MO}\langle G\rangle$, for a compact Lie group G, is the real unoriented analog of tom Dieck's homotopical equivariant bordism [**94**].

For an inner product space V we recall from Example II.2.6 the 'full Grassmannian'

$$Gr(V^2) \;=\; \coprod_{n\geq 0} Gr_n^{\mathbb{R}}(V^2) \;=\; \mathbf{BOP}(V)$$

of $V^2 = V\oplus V$. A point is thus a real sub-vector space of V^2, of any dimension; this space is topologized as the disjoint union of the Grassmannians of k-dimensional subspaces of V^2 for $k = 0, 1, \ldots, 2\cdot\dim(V)$. Over the space $Gr(V^2)$ sits a tautological vector bundle (of non-constant rank!): the total space of this bundle consist of pairs $(U, x) \in Gr(V^2) \times V^2$ such that $x \in U$. We define $\mathbf{MOP}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. (Since the base space of the bundle is compact, the one point compactification is homeomorphic to the quotient space of the unit disc bundle by the sphere bundle.)

A linear isometry $\alpha : V \longrightarrow W$ acts by

$$\mathbf{MO}(\alpha) \;:\; \mathbf{MO}(V) \;\longrightarrow\; \mathbf{MO}(W)\;, \quad (U, x) \;\longmapsto\; (\alpha^2(U), \alpha^2(x))\;,$$

i.e., by applying the linear isometric embedding $\alpha^2 : V^2 \longrightarrow W^2$ to the subspace and the vector in it. The generalized structure maps are given by

$$\sigma_{V,W} \;:\; \mathbf{MO}(V)\wedge S^W \;\longrightarrow\; \mathbf{MO}(V\oplus W)\;, \quad (U,x)\wedge w \;\longmapsto\; (\kappa_{V,W}(U\oplus W\oplus 0), \kappa_{V,W}(x, w, 0))\;,$$

where $\kappa_{V,W} : V^2\oplus W^2 \longrightarrow (V\oplus W)^2$ is the linear isometry given by $\kappa_{V,W}(v, v', w, w') = (v, w, v', w')$. Multiplication maps

$$\mu_{V,W} \;:\; \mathbf{MOP}(V)\wedge\mathbf{MOP}(W) \;\longrightarrow\; \mathbf{MOP}(V\oplus W) \tag{7.2}$$

are defined by sending $(U, x) \wedge (U'x')$ to $(U + U', (x, x'))$ where $U + U'$ is the image of $U \oplus U'$ under the preferred isometry $V^2 \oplus W^2 \cong (V\oplus W)^2$ sending $((v, v'),(w, w'))$ to $((v, w),(v', w'))$. These multiplication maps are $O(V) \times O(W)$-equivariant, and associative and commutative, in the sense that the following diagram commutes

$$\begin{tikzcd}
\mathbf{MO}(V)\wedge\mathbf{MO}(W) \arrow[r, "\mu_{V,W}"] \arrow[d, "\text{twist}"'] & \mathbf{MO}(V\oplus W) \arrow[d, "\tau_{V,W}\cdot"] \\
\mathbf{MO}(W)\wedge\mathbf{MO}(V) \arrow[r, "\mu_{W,V}"'] & \mathbf{MO}(W\oplus V)
\end{tikzcd}$$

where the right vertical map is the action of the linear isometry $\tau_{V,W} : V\oplus W \longrightarrow W\oplus V$ that interchanges the summands. Unit maps are defined by

$$S^V \;\longrightarrow\; \mathbf{MOP}(V)\;, \quad v \;\longmapsto\; (V\oplus 0,\, (v, 0))\;.$$

The spectrum **MOP** is a $\mathbb{Z}$-graded orthogonal ring spectrum; the summand $\mathbf{MOP}^{[k]}(V)$ of degree k is defined as the Thom space of the tautological $(\dim(V)+k)$-plane bundle over $Gr^{\mathbb{R}}_{\dim(V)+k}(V^2)$; then $\mathbf{MOP}(V)$

is the one-point union of the Thom spaces $\mathbf{MOP}^{[k]}(V)$ for $-\dim(V) \leq k \leq \dim(V)$. So we have a wedge decomposition

$$\mathbf{MOP} \;=\; \bigvee_{k\in\mathbb{Z}} \mathbf{MOP}^{[k]} \tag{7.3}$$

as orthogonal spectra. The multiplication map (7.2) is graded in the sense that its restriction to $\mathbf{MOP}^{[k]}(V) \wedge \mathbf{MOP}^{[l]}(W)$ has image in $\mathbf{MOP}^{[k+l]}(V \oplus W)$.

We denote by

$$\mathbf{MO} \;=\; \mathbf{MOP}^{[0]} \tag{7.4}$$

the homogeneous summand of degree 0 and refer to it as the *unoriented global bordism spectrum*. The unit map $S^V \longrightarrow \mathbf{MOP}(V)$ has image in $\mathbf{MOP}^{[0]}$, and $\mathbf{MO}$ is closed under the product of $\mathbf{MOP}$, so $\mathbf{MO}$ is an ultra-commutative subring spectrum of $\mathbf{MOP}$. As we explain in Remark 7.9, the summand $\mathbf{MOP}^{[k]}$ is globally equivalent to a k-fold suspension of $\mathbf{MO}$. So altogether, $\mathbf{MOP}$ is globally equivalent to the wedge of all (de-)suspensions of $\mathbf{MO}$.

Remark 7.5. Certain variations of the construction of $\mathbf{MOP}$ and $\mathbf{MO}$ are possible, and have been used at other places in the literature. Indeed, if U is any real vector space, finite or infinite dimensional, and $e_0 \in U$ a non-zero vector, we obtain a commutative orthogonal ring spectrum $\mathbf{MO}_{U,e_0}$ in exactly the same way as above, with value at V given by the Thoms space over the tautological vector bundle over $Gr(V \otimes U)$. The chosen vector e_0 enters in the definition of the unit ans structure maps. For $U = \mathbb{R}^2$ and $e_0 = (1, 0)$, this construction specializes to the ring spectrum $\mathbf{MO}$ as above. Moreover, any linear isometric embedding $\psi : U \longrightarrow U'$ such that $\psi(e_0) = u_0'$ induces a morphism of orthogonal ring spectra $\psi_* : \mathbf{MO}_{U,e_0} \longrightarrow \mathbf{MO}_{U',u_0'}$.

In the minimal case, i.e., when the vector space U is 1-dimensional, then $\mathbf{MO}_{U,e_0}$ is isomorphic to the sphere spectrum $\mathbb{S}$. In all other cases we obtain the same global homotopy type as $\mathbf{MO}$: for any linear isometric embedding $\psi : U \longrightarrow U'$ with $\psi(e_0) = u_0'$ and such that the dimension of U is at least 2, the morphism of orthogonal ring spectra $\psi_* : \mathbf{MO}_{U,e_0} \longrightarrow \mathbf{MO}_{U',u_0'}$ is a global equivalence.

The global bordism spectrum $\mathbf{MOP}$ has the special property that it is *globally orientable*. In fact, it is the universal example, in a sense that we plan to make precise later, of a commutative globally oriented cohomology theory. We start by recalling certain 'periodicity' classes for every compact Lie group G and G-representation V of dimension n. The Grassmannian $Gr_{n+n}(V^2)$ has only one point (the full space V^2), and so $\mathbf{MOP}^{[n]}(V)$ 'is' the one-point compactification of V^2. The G-equivariant identification

$$\bar{\sigma}_{G,V} \;:\; S^{V^2} \;\cong\; \mathbf{MOP}^{[n]}(V)\ , \quad x \;\longmapsto\; (V^2, x)$$

represents a class

$$\sigma_{G,V} \;\in\; \mathbf{MOP}^{[n]}_G(S^V) \;=\; \operatorname{colim}_{U\in s(\mathcal{U}_G)} [S^{U+V}, \mathbf{MOP}^{[n]}(U)]^G\ ,$$

the G-equivariant $\mathbf{MOP}^{[n]}$-cohomology of S^V.

We uppose that $\varphi : V \longrightarrow W$ is an isomorphism of G-representations. Then φ compactifies to a G-homeomorphism $S^\varphi : S^V \longrightarrow S^W$ and hence induces an isomorphism

$$(S^\varphi)^* \;:\; \mathbf{MOP}^{[n]}_G(S^W) \;\longrightarrow\; \mathbf{MOP}^{[n]}_G(S^V)\ .$$

Proposition 7.6. *The classes $\sigma_{G,V}$ have the following properties.*

(i) *For every isomorphism $\varphi : V \longrightarrow W$ of G-representations, the induced isomorphism $(S^\varphi)^*$ takes the class $\sigma_{G,W}$ to the class $\sigma_{G,V}$.*

(ii) *The class $\sigma_{G,0}$ of the trivial 0-dimensional G-representation is the unit element $1 \in \pi_0^G(\mathbf{MO}) = \mathbf{MOP}^{[0]}_G(S^0)$*

(iii) *For every continuous homomorphism* $\alpha : K \longrightarrow G$ *the relation*

$$\alpha^*(\sigma_{G,V}) \;=\; \sigma_{K,\alpha^* V}$$

holds in $\mathbf{MOP}^{[n]}_K(S^{\alpha^* V})$.

(iv) *For all* G*-representations* V *and all* K*-representations* W *the relation*

$$\sigma_{G,V} \times \sigma_{K,W} \;=\; \sigma_{G\times K,\, V\oplus W} \qquad \textit{holds in } \mathbf{MOP}^{[n+m]}_{G\times K}(S^{V\oplus W})\ .$$

(v) *For every integer* k *the map*

$$\cdot\sigma_{G,V} \;:\; \pi_k^G(\mathbf{MOP}) \;\longrightarrow\; \mathbf{MOP}^G_k(S^V)\ , \quad x \;\longmapsto\; x\cdot \sigma_{G,V}$$

is an isomorphism.

Proof. (i) The square

$$\begin{tikzcd}
S^V \wedge S^V \arrow[r, "\tilde\sigma_{G,V}", "\cong"'] \arrow[d, "S^\varphi\wedge S^\varphi"'] & \mathbf{MOP}^{[n]}(V) \arrow[d, "\mathbf{MOP}^{[n]}(\varphi)"] \\
S^W \wedge S^W \arrow[r, "\cong", "\tilde\sigma_{G,W}"'] & \mathbf{MOP}^{[n]}(W)
\end{tikzcd}$$

commutes. The class $\sigma_{G,V}$ is represented by the G-map $\tilde\sigma_{G,V}$, where as $(S^\varphi)^*(\sigma_{G,W})$ is represented by the composite around the lower left corner of the square. Since these two maps differ by the effect of the structure map $\mathbf{MOP}^{[n]}(\varphi)$, they define the same element in the colimit $\mathbf{MOP}^{[n]}(S^V)$ (by Proposition I.1.13 (ii)).

Parts (ii), (iii) and (iv) are straightforward from the definitions.

(v) We define a map

$$(7.7) \qquad \mathbf{MOP}^G_k(S^V) \;=\; \operatorname{colim}_{U\in s(\mathcal{U}_G)}\,[S^{k+U+V}, \mathbf{MOP}(U)]^G \;\longrightarrow \\ \operatorname{colim}_{U\in s(\mathcal{U}_G)}\,[S^{k+U}, \mathbf{MOP}(U)]^G \;=\; \pi_k^G(\mathbf{MOP})$$

in the other direction by multiplying with the G-fixed point

$$(\{0\}, 0) \;\in\; \mathbf{MOP}^{[-n]}(V)\ ,$$

where $\{0\}$ is the trivial subspace of V^2, the unique point in $Gr_0(V^2)$. In other words, the class of a based G-map $f : S^{k+U+V} \longrightarrow \mathbf{MOP}(U)$ is sent to the composite

$$S^{k+U+V} \xrightarrow{f(-)\wedge(\{0\},0)} \mathbf{MOP}(U)\wedge \mathbf{MOP}^{[-n]}(V) \xrightarrow{\mu_{U,V}} \mathbf{MOP}(U\oplus V)\ .$$

This is compatible with stabilization in U, so it passes to a well-defined homomorphism on colimits.

The composite of $-\cdot\sigma_{G,V}$ with the map (7.7) is right multiplication by the class of the map

$$(7.8) \qquad \begin{array}{ccccc} S^{V^2} & \xrightarrow{\tilde\sigma_{G,C}(-)\wedge(\{0\},0)} & \mathbf{MOP}^{[n]}(V)\wedge\mathbf{MOP}^{[-n]}(V) & \xrightarrow{\mu_{V,V}} & \mathbf{MOP}^{[0]}(V\oplus V) \\ (v,v') & \longmapsto & (V\oplus 0\oplus V\oplus 0,\ (v,0,v',0)) & & \end{array}$$

The space $\mathbf{L}^{\mathbb{C}}(V^2, (V\oplus V)^2)^G$ of G-equivariant complex linear embeddings is path-connected, so the embedding $(v,v')\mapsto(v,0,v',0)$ is homotopic, through G-equivariant linear isometric embeddings, to the embedding $(v,v')\mapsto(v,v',0,0)$. [fix this...] A homotopy between these two embeddings induces a G-homotopy between the composite (7.8) and the unit map $\eta_{V^2} : S^{V^2} \longrightarrow \mathbf{MOP}^{[0]}(V^2)$. So right multiplication by the composite (7.8) is G-homotopic to stabilization, hence it becomes the identity on G-equivariant stable homotopy groups. This shows that the composite of right multiplication by $\sigma_{G,V}$ and the is the identity. The argument that the composite of (7.7) with $-\cdot\sigma_{G,V}$ is the identity of $\mathbf{MOP}^G_k(S^V)$ is similar. □

The external product formula (iii) and the naturality (iv) of the previous proposition together imply an 'internal' product formula by restriction along the diagonal homomorphism $\Delta : G \longrightarrow G \times G$: for all complex G-representations V and W the relation

$$\sigma_{G,V} \cdot \sigma_{G,W} \;=\; \sigma_{G,V\oplus W}$$

holds in $\mathbf{MOP}_G^{[n+m]}(S^{V\oplus V})$.

Remark 7.9. Part (v) of the previous Proposition 7.21 can be rephrased entirely in terms of the (non-periodic) global bordism spectrum $\mathbf{MO}$ as follows. In terms of the $\mathbb{Z}$-grading of $\mathbf{MOP}$, the class $\sigma_{G,V}$ is homogeneous of degree $n = \dim(V)$. So right multiplication by $\sigma_{G,V}$ is homogeneous of degree n, and restricts to an isomorphism

$$\cdot\sigma_{G,V} \;:\; \pi_*^G(\mathbf{MOP}^{[k]}) \;\longrightarrow\; (\mathbf{MOP}^{[k+n]})_*^G(S^V)$$

for every integer k. For $k = -n$ this specializes to an isomorphism

$$\cdot\sigma_{G,V} \;:\; \pi_*^G(\mathbf{MOP}^{[-n]}) \;\longrightarrow\; \mathbf{MO}_*^G(S^V)$$

In the special case of the trivial G-action on $\mathbb{R}^n$, this becomes an isomorphism

$$\cdot\sigma_{G,\mathbb{R}^n} \;:\; \pi_*^G(\mathbf{MOP}^{[-n]}) \;\longrightarrow\; \mathbf{MO}_*^G(S^n) \;=\; \pi_{*+n}^G(\mathbf{MO}) \;.$$

This isomorphism is induced by a morphism $\mathbf{MOP}^{[n]} \longrightarrow \Sigma^n\mathbf{MO}$ in the global homotopy category, so $\mathbf{MOP}^{[-n]}$ is globally equivalent to the n-fold suspension of $\mathbf{MO}$.

We can combine the last two isomorphism into a single isomorphism

$$\mathbf{MO}_*^G(S^V) \;\cong\; \mathbf{MO}_*^G(S^n) \;=\; \pi_{*+n}^G(\mathbf{MO}) \;,$$

which again is given by right multiplication by a certain class in $\mathbf{MO}_{-n}^G(S^V)$, using the $\pi_*^G(\mathbf{MO})$-module structure of $\mathbf{MO}_*^G(S^V)$.

Remark 7.10. It is well-known that $2 = 0$ in the homotopy ring $\pi_0^e(\mathbf{MO})$ of the underlying non-equivariant unoriented bordism spectrum and hence also in the periodic version. In fact, in the global functor $\underline{\pi}_0(\mathbf{MOP})$ various equivariant generalizations of the relation $-1 = 1$ hold, as we now explain. For every G-representation V the involution $\tau_{V,V}$ of $V \oplus V$ given by $\tau_{V,V}(v, v') = (v', v)$ compactifies to a G-equivariant based G-selfmap $S^{\tau_{V,V}}$ of $S^{V\oplus V}$. We denote by

$$\epsilon_V \;=\; \langle S^{\tau_{V,V}} \rangle \;\in \pi_0^G(\mathbb{S})$$

the class represented by this G-map in the equivariant 0-stem. This element always satisfies $\epsilon_V^2 = 1$, but as the sign representation of C_2 shows, ϵ_V is generally different from both 1 and -1. If V is underlying a *complex* G-representation, then $\epsilon_V = 1$ because the space $\mathbf{L}^{\mathbb{C}}(V^2, V^2)$ is path connected, so $S^{\tau_{V,V}}$ is G-homotopic to the identity.

Because $\mathbf{MOP}$ is ultra-commutative,

$$\sigma_{G,V} \cdot \sigma_{G,W} \;=\; (S^{\tau_{V,W}})^*(\sigma_{G,W} \cdot \sigma_{G,V}),$$

where $\tau_{V,W} : V \oplus W \longrightarrow W \oplus V$ interchanges the summands. For $V = W$ this implies that

$$\sigma_{G,V}^2 \;=\; (S^{\tau_{V,V}})^*(\sigma_{G,V}^2) \;=\; \epsilon_V \cdot \sigma_{G,V}^2$$

in $\mathbf{MOP}_G^{[n+m]}(S^{V\oplus V})$. Since right multiplication be $\sigma_{G,V}$ is injective by Proposition 7.6 (v) above, this implies that

$$\epsilon_V \;=\; 1 \qquad \text{in} \quad \pi_0^G(\mathbf{MOP}^{[0]}) = \pi_0^G(\mathbf{MO}).$$

When $V = \mathbb{R}^n$ with trivial G-action, $\epsilon_V = (-1)^n$; since n can be odd, the relation specializes to $2 = 0$ in $\pi_0^G(\mathbf{MO})$.

The global power functor $\underline{\pi}_0(\mathbf{MOP})$ of the periodic complex bordism spectrum $\mathbf{MOP}$ appears to be a very interesting object, but a complete algebraic description does not seem to be known. We summarize the structure that $\underline{\pi}_0(\mathbf{MOP})$ has and recall some of the known calculations.

As we explained, the spectrum $\mathbf{MOP}$ comes with the structure of an ultra-commutative ring spectrum, and this makes the collection of equivariant homotopy groups $\underline{\pi}_0(\mathbf{MOP})$ into a global power functor. Since $2 = 0$ in $\pi_0^e(\mathbf{MOP})$, the global power functor $\underline{\pi}_0(\mathbf{MOP})$ takes values in $\mathbb{F}_2$-vector spaces, and the equivariant generalizations of this relation described in Remark 7.10 hold as well.

The orthogonal spectrum underlying $\mathbf{MOP}$ comes with a $\mathbb{Z}$-grading, i.e., a wedge decomposition (7.3) into summands $\mathbf{MOP}^{[k]}$. The grading is multiplicative, and hence the m-th power operation takes the summand $\mathbf{MOP}^{[k]}$ to the summand $\mathbf{MOP}^{[mk]}$. The geometric splitting induces a direct sum decomposition

$$\underline{\pi}_0(\mathbf{MOP}) \;=\; \bigoplus_{k\in\mathbb{Z}} \underline{\pi}_0(\mathbf{MOP}^{[k]})$$

that makes $\underline{\pi}_0(\mathbf{MOP})$ into a commutative $\mathbb{Z}$-graded ring, and the power operations restrict to maps

$$P^m \;:\; \underline{\pi}_0(\mathbf{MOP}^{[k]}) \;\longrightarrow\; \underline{\pi}_0(\mathbf{MOP}^{[mk]}) \;.$$

The $\mathbb{Z}$-graded global power functor $\underline{\pi}_0(\mathbf{MOP})$ comes with Euler classes

$$e(V) \;=\; i^*(\sigma_{G,V}) \;\in\; \pi_0^G(\mathbf{MOP}^{[n]})$$

for every representation V of a compact Lie group G, where $n = \dim(V)$; the Euler class is obtained by restricting the periodicity class $\sigma_{G,V} \in \mathbf{MOP}_G^{[n]}(S^V)$ along the inclusion $i : S^0 \longrightarrow S^V$. The Euler class is thus represented by the G-map

$$S^V \;\longrightarrow\; \mathbf{MOP}^{[n]}(V) \;, \quad v \;\longmapsto\; (V^2, (v,0)) \;. \tag{7.11}$$

If V has non-trivial G-fixed points, then the inclusion $i : S^0 \longrightarrow S^V$ is G-equivariantly null-homotopic, so $e(V) = 0$ whenever $V^G \neq 0$.

The following properties of Euler classes can either be proved directly from the explicit definition (7.25) above, or deduced from the properties of the stability classes listed in Proposition 7.21.

(i) If V and W are isomorphic G-representations, then $e(V) = e(W)$.

(ii) The Euler class $e(0)$ of the trivial 0-dimensional G-representation is the unit element $1 \in \pi_0^G(\mathbf{MOP}^{[0]}) = \pi_0^G(\mathbf{MO})$

(iii) For every continuous homomorphism $\alpha : K \longrightarrow G$ the relation

$$\alpha^*(e(V)) \;=\; e(\alpha^* V)$$

holds in $\pi_0^K(\mathbf{MOP}^{[n]})$.

(iv) For all G-representations V and all K-representations W the relation

$$e(V) \times e(W) \;=\; e(V \oplus W)$$

holds in $\pi_0^{G\times K}(\mathbf{MOP}^{[n+m]})$.

(v) For all G-representations V and W the relation

$$e(V) \cdot e(W) \;=\; e(V \oplus W)$$

holds in $\pi_0^G(\mathbf{MOP}^{[n+m]})$.

(vi) For every n-dimensional G-representation V and $m \geq 1$, the relation

$$P^m(e(V)) \;=\; e(V^m)$$

holds in $\pi_0^{\Sigma_m \wr G}(\mathbf{MOP}^{[mn]})$.

(vii) If H is a closed subgroup of finite index in G, then for every n-dimensional complex H-representation W the relation

$$N_H^G(e(W)) \ = \ e(\mathrm{tr}_H^G(V))$$

holds in $\pi_0^G(\mathbf{MOP}^{[[G:H]\cdot n]})$.

By the naturality property (iii), all Euler classes are determined by the Euler classes of the tautological $O(n)$-representation on $\mathbb{R}^n$.

We let σ denote the sign representation of the cyclic group $C = C_2$ on $\mathbb{R}$. Then for every compact Lie group G the sequence

$$0 \ \longrightarrow \ \pi_0^{C\times G}(\mathbf{MOP}) \ \xrightarrow{e(p^*\sigma)\cdot -} \ \pi_0^{C\times G}(\mathbf{MOP}) \ \xrightarrow{\mathrm{res}_G^{C\times G}} \ \pi_0^G(\mathbf{MOP}) \ \longrightarrow \ 0$$

is exact, where $p : C \times G \longrightarrow C$ is the projection to the first factor. The sequence is split by the restriction morphism along the projection of $T \times G$ to the second factor. This implies that the completion of the commutative $\mathbb{Z}$-graded $\mathbb{F}_2$-algebra $\pi_0^{C\times G}(\mathbf{MOP})$ at the ideal generated by the Euler class $e(p^*\sigma)$ is a power series algebra in $e(p^*\sigma)$ over $\pi_0^G(\mathbf{MOP})$. We let $p_1, p_2 : C \times C \longrightarrow C$ be the two projections and define $x, y \in \pi_0^{C^2}(\mathbf{MOP}^{[2]})$ by $x = e(p_1^*(\sigma))$ and $y = e(p_2^*(\sigma))$. The completion of the ring $\pi_0^{C^2}(\mathbf{MOP})$ at the ideal generated by x and y is then a power series algebra over the underlying ring $\pi_0^e(\mathbf{MOP})$ in two variables:

$$\pi_0^{C^2}(\mathbf{MOP})^\wedge_{(x,y)} \ = \ \pi_0^e(\mathbf{MOP})[[x,y]] \ .$$

We let $\mu : C \times C \longrightarrow C$ be the multiplication map. Then the image of the class $e(\mu^*(\sigma)) \in \pi_0^{C^2}(\mathbf{MOP}^{(2)})$ in the completion is a power series F in x and y:

$$e(\mu^*(\sigma)) \ = \ F(x,y) \ \in \ \pi_0^{C^2}(\mathbf{MOP})^\wedge_{(x,y)} \ = \ \pi_0^e(\mathbf{MOP})[[x,y]] \ .$$

By Quillen's theorem, $F(x,y)$ is a universal 1-dimensional commutative formal group law with $[2]_F(x) = 0$, and the underlying commutative ring $\pi_0^e(\mathbf{MOP})$ of the $\mathbb{Z}$-graded global power functor $\underline{\pi}_0(\mathbf{MOP})$ is isomorphic to universal ring. Abstractly, $\pi_0^e(\mathbf{MOP})$ is thus a polynomial algebra over $\mathbb{F}_2$ in infinitely many generators, one in $\pi_0^e(\mathbf{MOP}^{[k]})$ for every $k \leq 0$ with $k \neq 2^i - 1$.

Unitary spaces and spectra. We also recall and discuss *complex* global bordism; this is an important global homotopy type that is most naturally indexed on *complex* (as opposed to real) inner product spaces. The appropriate framework for these examples are *unitary* spectra, the obvious analog of orthogonal spectra with unitary groups instead of orthogonal groups, and with complex inner product spaces instead of real inner product space. Along with unitary spectra it is natural to consider also unitary spaces. We quickly go through the necessary definitions.

Definition 7.12. A *unitary spectrum* consists of

- a sequence of based spaces Y_n for $n \geq 0$,
- a based continuous left action of the unitary group $U(n)$ on Y_n for each $n \geq 0$,
- based maps $\sigma_n : Y_n \wedge S^{\mathbb{C}} \longrightarrow Y_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ Y_n \wedge S^{\mathbb{C}^m} \ \longrightarrow \ Y_{n+m}$$

is $U(n)\times U(m)$-equivariant. Here the unitary group $U(m)$ acts on $S^{\mathbb{C}^m}$ since this is the one-point compactification of $\mathbb{C}^m$, and $U(n)\times U(m)$ acts on the target by restriction, along unitary sum, of the $U(n+m)$-action.

A *morphism* $f : Y \longrightarrow Y'$ of unitary spectra consists of $U(n)$-equivariant based maps $f_n : Y_n \longrightarrow Y_n'$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^{\mathbb{C}})$ for all $n \geq 0$. We denote the category of unitary spectra by $\mathcal{S}p^U$.

A unitary spectrum Y can be evaluated on a complex inner product space. The formula is analogous to the orthogonal case: if V has (complex) dimension n, then we define $Y(V)$, the *value of* Y *on* V, as

$$Y(V) \;=\; \mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V)^+ \wedge_{U(n)} Y_n$$

where $\mathbb{C}^n$ has the standard scalar product and $\mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V)$ is the space of complex linear isometries from $\mathbb{C}^n$ to V. Then Y_n is canonically homeomorphic to $Y(\mathbb{C}^n)$ and the iterated structure maps $\sigma^m : Y_n \wedge S^{\mathbb{C}^m} \longrightarrow Y_{n+m}$ of an unitary spectrum Y extend to associative and unital generalized structure maps

$$\sigma_{V,W} \;:\; Y(V) \wedge S^W \;\longrightarrow\; Y(V \oplus W) \tag{7.13}$$

for all pairs of complex inner product spaces V and W. If G is a compact Lie group G and V is a complex G-representation (i.e., G acts on V by complex linear isometries), then $Y(V)$ becomes a G-space by the rule

$$g \cdot [\varphi, x] \;=\; [g\varphi, x] \;.$$

If W is another complex G-representation, then the generalized structure map (7.13) is G-equivariant.

There are obvious unitary versions of the adjoint functors $\Omega^\bullet$ and Σ^∞_+ discussed in Constructions III.2.3 respectively III.3.6.

Construction 7.14. We relate unitary and orthogonal spectra via a functor

$$\Phi \;:\; \mathcal{S}p^U \;\longrightarrow \mathcal{S}p \;.$$

For a unitary spectrum Y and a real inner product space V we set

$$(\Phi Y)(V) \;=\; \mathrm{map}(S^{iV}, Y(V_{\mathbb{C}}))$$

where $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ is the complexification of V and $iV = i\mathbb{R} \otimes_{\mathbb{R}} V \subset V_{\mathbb{C}}$ its imaginary part. If W is another real inner product space, the generalized structure map $\sigma_{V,W} : (\Phi Y)(V) \wedge S^W \longrightarrow (\Phi Y)(V \oplus W)$ is adjoint the composite

$$\begin{aligned} \mathrm{map}(S^{iV}, Y(V_{\mathbb{C}})) \wedge S^W \wedge S^{i(V\oplus W)} \;&\cong\; \mathrm{map}(S^{iV}, Y(V_{\mathbb{C}})) \wedge S^{iV} \wedge S^{W_{\mathbb{C}}} \\ &\xrightarrow{\mathrm{eval} \wedge S^{W_{\mathbb{C}}}} Y(V_{\mathbb{C}}) \wedge S^{W_{\mathbb{C}}} \;\xrightarrow{\sigma_{V_{\mathbb{C}}, W_{\mathbb{C}}}}\; Y((V \oplus W)_{\mathbb{C}}) \;. \end{aligned}$$

where we have implicitly used the identifications

$$S^W \wedge S^{i(V\oplus W)} \;\cong\; S^W \wedge S^{iV} \wedge S^{iW} \;\xrightarrow{\tau \wedge S^{iW}}\; S^{iV} \wedge S^W \wedge S^{iW} \;\cong\; S^{iV} \wedge S^{W_{\mathbb{C}}}$$

and $V_{\mathbb{C}} \oplus W_{\mathbb{C}} \cong (V \oplus W)_{\mathbb{C}}$.

Remark 7.15. The functor Φ is lax symmetric monoidal, i.e., there is a natural transformation

$$\Phi(Y) \wedge \Phi(Z) \;\longrightarrow\; \Phi(Y \wedge Z)$$

for unitary spectra Y and Z, where the smash product of the right hand side is the obvious unitary analog of the smash product of orthogonal spectra. The maps are associative and commutative, and unital with respect to the morphism $\mathbb{S} \longrightarrow \Phi(\mathbb{S}_{\mathbb{C}})$, where $\mathbb{S}_{\mathbb{C}}$ is the unitary sphere spectrum.

The functor Φ also has a left adjoint, but we won't use this left adjoint here, so we will not go into details.

The equivariant homotopy groups of the orthogonal spectrum ΦY can be described more directly as the 'complex equivariant homotopy groups' of the unitary spectrum Y as follows. For a compact Lie group G we denote by $\mathcal{U}_G^{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{U}_G$ the complexification of the chosen complete G-universe. This is a complete complex G-universe, i.e., every finite dimensional complex G-representation can be embedded equivariantly into $\mathcal{U}_G^{\mathbb{C}}$.

We denote by $\bar{s}(\mathcal{U}_G^{\mathbb{C}})$ the poset of finite dimensional *complex* G-subrepresentations of $\mathcal{U}_G^{\mathbb{C}}$. Complexification

$$s(\mathcal{U}_G) \;\longrightarrow\; \bar{s}(\mathcal{U}_G^{\mathbb{C}}) \;, \quad V \;\longmapsto\; V_{\mathbb{C}}$$

is a morphism of posets. For every $V \in s(\mathcal{U}_G)$ we consider the bijective map

$$[S^V, (\Phi Y)(V)]^G \longrightarrow [S^{V_{\mathbb{C}}}, Y(V_{\mathbb{C}})]^G$$

that takes the class of a G-map

$$f : S^V \longrightarrow (\Phi Y)(V) = \mathrm{map}(S^{iV}, Y(V_{\mathbb{C}}))$$

to the class of its adjoint

$$\hat{f} : S^{V_{\mathbb{C}}} \cong S^V \wedge S^{iV} \longrightarrow Y(V_{\mathbb{C}}) .$$

The complex representations of the form $V_{\mathbb{C}}$ with $V \in s(\mathcal{U}_G)$, are cofinal in the poset $\bar{s}(\mathcal{U}_G^{\mathbb{C}})$, so we obtain a bijection on colimits

$$\pi_0^G(\Phi Y) = \mathrm{colim}_{V \in s(\mathcal{U})} [S^V, (\Phi Y)(V)]^G \xrightarrow{\cong} \mathrm{colim}_{W \in \bar{s}(\mathcal{U}_G^{\mathbb{C}})} [S^W, Y(W)]^G .$$

Example 7.16 (Periodic complex global bordism)**.** We recall the commutative unitary ring spectrum $\mathbf{MP}$, of *periodic complex global bordism*. Non-equivariantly, and ignoring the multiplication, $\mathbf{MP}$ is a wedge of all even suspensions of the complex bordism spectrum MU. The observation that this homotopy type has a commutative multiplication goes back to Strickland [**91**, Appendix A], and Strickland's model has been turned into a unitary ring spectrum by Brun [**22**, Thm. 8.1]. In most parts, we just have to replace *real* vector spaces in the construction of $\mathbf{MOP}$ in Example 7.1 by *complex* vector spaces throughout. We will be brief the construction and properties of $\mathbf{MP}$ are analogous to those of $\mathbf{MOP}$, and we give more details where something essential changes.

For a complex inner product space V we consider the 'full complex Grassmannian'

$$Gr(V^2) = \coprod_{n \geq 0} Gr_n(V^2)$$

of $V^2 = V \oplus V$. A point is thus a complex sub-vector space of V^2, of any dimension; if $V = U_{\mathbb{C}}$ is the complexification of a real inner product space U, then this is space is $\mathbf{BUP}(U)$ as defined in Example II.2.26.

Over the space $Gr(V^2)$ sits a tautological hermitian vector bundle (of non-constant rank!): the total space of this bundle consist of pairs $(U, x) \in Gr(V^2) \times V^2$ such that $x \in U$. We define $\mathbf{MP}(V)$ as the Thom space of this tautological vector bundle, i.e., the one-point compactification of the total space. The structure maps $\sigma_{V,W} : \mathbf{MP}(V) \wedge S^W \longrightarrow \mathbf{MP}(V \oplus W)$, multiplication maps

$$\mu_{V,W} : \mathbf{MP}(V) \wedge \mathbf{MP}(W) \longrightarrow \mathbf{MP}(V \oplus W) ,$$

unit maps

$$\eta_V : S^V \longrightarrow \mathbf{MP}(V)$$

and the $\mathbb{Z}$-grading, i.e., the wedge decomposition

$$\mathbf{MP} = \bigvee_{k \in \mathbb{Z}} \mathbf{MP}^{[k]} \tag{7.17}$$

as unitary spectra, are defined as for $\mathbf{MOP}$. The multiplication map $\mu_{V,W}$ is graded in the sense that its restriction to $\mathbf{MP}^{[k]}(V) \wedge \mathbf{MP}^{[l]}(W)$ has image in $\mathbf{MP}^{[k+l]}(V \oplus W)$. We denote by

$$\mathbf{MU} = \mathbf{MP}^{[0]} \tag{7.18}$$

the homogeneous summand of degree 0 and refer to it as the *complex global bordism spectrum.* The unit map $S^V \longrightarrow \mathbf{MP}(V)$ has image in $\mathbf{MP}^{[0]}$, and $\mathbf{MU}$ is closed under the product of $\mathbf{MP}$, so $\mathbf{MU}$ is an unitary subring spectrum of $\mathbf{MP}$. As we explain in Remark 7.24, the summand $\mathbf{MP}^{[k]}$ is globally equivalent to a $2k$-fold suspension of $\mathbf{MU}$. So altogether, $\mathbf{MP}$ is globally equivalent to the wedge of all *even* (de-)suspensions of $\mathbf{MU}$.

Remark 7.19. For a compact Lie group G, the G-equivariant spectrum $\mathbf{MU}\langle G\rangle$ is a model for tom Dieck's homotopical equivariant bordism [**94**]. When the group G is non-trivial, this is different from the geometric theory of bordism of G-manifolds.

The periodic unitary global bordism spectrum $\mathbf{MP}$ has the special property that it is *complex orientable.* In fact, it is the universal example, in a sense that we plan to make precise later, of a commutative complex oriented global cohomology theory. We start with the definition of certain 'complex periodicity' classes for every compact Lie group G and complex G-representation V of complex dimension n. The Grassmannian $Gr_{n+n}(V^2)$ has only one point (the full space V^2), and so $\mathbf{MP}^{[n]}(V)$ 'is' the one-point compactification of V^2. The G-equivariant identification

$$\bar{\sigma}_{G,V} \ : \ S^{V^2} \ \cong \ \mathbf{MP}^{[n]}(V) \ , \quad x \ \longmapsto \ (V^2, x)$$

represents a class

$$\sigma_{G,V} \ \in \ \mathbf{MP}_G^{[n]}(S^V) \ = \ \mathrm{colim}_{U\in s(\mathcal{U}_G^{\mathbb{C}})} \, [S^{U+V}, \mathbf{MP}^{[n]}(U)]^G \ , \tag{7.20}$$

the G-equivariant $\mathbf{MP}^{[n]}$-cohomology of S^V.

Suppose that V and W are two isomorphic complex G-representations. Then a choice of G-isomorphism $\varphi : V \longrightarrow W$ induces a G-homeomorphism $S^\varphi : S^V \longrightarrow S^W$ and hence an isomorphism

$$(S^\varphi)^* \ : \ \mathbf{MP}_G^{[n]}(S^W) \ \longrightarrow \ \mathbf{MP}_G^{[n]}(S^V)$$

of G-equivariant $\mathbf{MP}^{[n]}$-cohomology groups. Since we deal with complex representations, the space $\mathbf{L}_{\mathbb{C}}^G(V,W)$ of G-equivariant $\mathbb{C}$-linear isometries is path connected, so different choices of isomorphisms lead to homotopic G-homeomorphism between S^V and S^W, and so the isomorphism $(S^\varphi)^*$ is independent of the isomorphism $\varphi : V \longrightarrow W$. In the special case where G acts trivially on V we obtain a preferred isomorphism

$$\pi_{2n}^G(\mathbf{MP}^{[n]}) \ \cong \ \mathbf{MP}_G^{[n]}(S^{\mathbb{C}^n}) \ \xrightarrow{(S^\varphi)^*} \ \mathbf{MP}_G^{[n]}(S^V) \ .$$

Proposition 7.21. *The classes $\sigma_{G,V}$ have the following properties.*

(i) *If V and W are isomorphic complex G-representations, then the preferred isomorphism*

$$\mathbf{MP}_G^{[n]}(S^W) \ \cong \ \mathbf{MP}_G^{[n]}(S^V)$$

takes the class $\sigma_{G,W}$ to the class $\sigma_{G,V}$.

(ii) *The class $\sigma_{G,0}$ of the trivial 0-dimensional G-representation is the unit element $1 \in \pi_0^G(\mathbf{MU}) = \mathbf{MP}_G^{[0]}(S^0)$*

(iii) *For every continuous homomorphism $\alpha : K \longrightarrow G$ the relation*

$$\alpha^*(\sigma_{G,V}) \ = \ \sigma_{K,\alpha^* V}$$

holds in $\mathbf{MP}_K^{[n]}(S^{\alpha^* V})$.

(iv) *For all complex G-representations V and all complex K-representations W the relation*

$$\sigma_{G,V} \times \sigma_{K,W} \ = \ \sigma_{G\times K,\, V\oplus W} \qquad \text{holds in } \mathbf{MP}_{G\times K}^{[n+m]}(S^{V\oplus W}) \ .$$

(v) *For every integer k the map*

$$\cdot\sigma_{G,V} \ : \ \pi_k^G(\mathbf{MP}) \ \longrightarrow \ \mathbf{MP}_k^G(S^V) \ , \quad x \ \longmapsto \ x \cdot \sigma_{G,V}$$

is an isomorphism.

PROOF. (i) We let $\varphi : V \longrightarrow W$ be an isomorphism of complex G-representations. Then the square

$$\begin{tikzcd} S^V \wedge S^V \arrow[r, "\bar{\sigma}_{G,V}", "\cong"'] \arrow[d, "S^\varphi \wedge S^\varphi"'] & \mathbf{MP}^{[n]}(V) \arrow[d, "\mathbf{MP}^{[n]}(\varphi)"] \\ S^W \wedge S^W \arrow[r, "\cong", "\bar{\sigma}_{G,W}"'] & \mathbf{MP}^{[n]}(W) \end{tikzcd}$$

commutes. The class $\sigma_{G,V}$ is represented by the G-map $\tilde{\sigma}_{G,V}$, where as $(S^\varphi)^*(\sigma_{G,W})$ is represented by the composite around the lower left corner of the square. Since these two maps differ by the effect of the structure map $\mathbf{MP}^{[n]}(\varphi)$, they define the same element in the colimit $\mathbf{MP}^{[n]}(S^V)$ (by Proposition I.1.13 (ii)).

Parts (ii), (iii) and (iv) are straightforward from the definitions.

(v) We define a map

$$\text{(7.22)} \qquad \mathbf{MP}^G_k(S^V) \; = \; \mathrm{colim}_{U \in s(\mathcal{U}^{\mathbb{C}}_G)} \, [S^{k+U+V}, \mathbf{MP}(U)]^G \; \longrightarrow \; \mathrm{colim}_{U \in s(\mathcal{U}^{\mathbb{C}}_G)} \, [S^{k+U}, \mathbf{MP}(U)]^G \; = \; \pi^G_k(\mathbf{MP})$$

in the other direction by multiplying with the G-fixed point

$$(\{0\}, 0) \; \in \; \mathbf{MP}^{[-n]}(V) \; ,$$

where $\{0\}$ is the trivial subspace of V^2, the unique point in $Gr_0(V^2)$. In other words, the class of a based G-map $f : S^{k+U+V} \longrightarrow \mathbf{MP}(U)$ is sent to the composite

$$S^{k+U+V} \xrightarrow{f(-)\wedge(\{0\},0)} \mathbf{MP}(U) \wedge \mathbf{MP}^{[-n]}(V) \xrightarrow{\mu_{U,V}} \mathbf{MP}(U \oplus V) \; .$$

This is compatible with stabilization in U, so it passes to a well-defined homomorphism on colimits.

The composite of $- \cdot \sigma_{G,V}$ with the map (7.22) is right multiplication by the class of the map

$$\text{(7.23)} \qquad \begin{array}{ccccc} S^{V^2} & \xrightarrow{\bar{\sigma}_{G,C}(-)\wedge(\{0\},0)} & \mathbf{MP}^{[n]}(V) \wedge \mathbf{MP}^{[-n]}(V) & \xrightarrow{\mu_{V,V}} & \mathbf{MP}^{[0]}(V \oplus V) \\ (v, v') & \longmapsto & (V \oplus 0 \oplus V \oplus 0, \; (v, 0, v', 0)) & & \end{array}$$

The space $\mathbf{L}^{\mathbb{C}}(V^2, (V \oplus V)^2)^G$ of G-equivariant complex linear embeddings is path-connected, so the embedding $(v, v') \mapsto (v, 0, v', 0)$ is homotopic, through G-equivariant linear isometric embeddings, to the embedding $(v, v') \mapsto (v, v', 0, 0)$. A homotopy between these two embeddings induces a G-homotopy between the composite (7.23) and the unit map $\eta_{V^2} : S^{V^2} \longrightarrow \mathbf{MP}^{[0]}(V^2)$. So right multiplication by the composite (7.23) is G-homotopic to stabilization, hence it becomes the identity on G-equivariant stable homotopy groups. This shows that the composite of right multiplication by $\sigma_{G,V}$ and the is the identity. The argument that the composite of (7.22) with $- \cdot \sigma_{G,V}$ is the identity of $\mathbf{MP}^G_k(S^V)$ is similar. $\square$

The external product formula (iii) and the naturality (iv) of the previous proposition together imply an 'internal' product formula by restriction along the diagonal homomorphism $\Delta : G \longrightarrow G \times G$: for all complex G-representations V and W the relation

$$\sigma_{G,V} \cdot \sigma_{G,W} \; = \; \sigma_{G,V \oplus W}$$

holds in $\mathbf{MP}^{[n+m]}_G(S^{V \oplus W})$.

Remark 7.24. Part (v) of the previous Proposition 7.21 can be rephrased entirely in terms of the (non-periodic) global bordism spectrum $\mathbf{MU}$ as follows, and this is the more traditional way to express the equivariant complex periodicity property of tom Dieck's G-equivariant homotopical bordism.

In terms of the $\mathbb{Z}$-grading of $\mathbf{MP}$, the class $\sigma_{G,V}$ is homogeneous of degree $n = \dim_{\mathbb{C}}(V)$. So right multiplication by $\sigma_{G,V}$ is homogeneous of degree n, and restricts to an isomorphism

$$\cdot \sigma_{G,V} \; : \; \pi^G_*(\mathbf{MP}^{[k]}) \; \longrightarrow \; (\mathbf{MP}^{[k+n]})^G_*(S^V)$$

for every integer k. For $k = -n$ this specializes to an isomorphism

$$\cdot\sigma_{G,V} \ : \ \pi_*^G(\mathbf{MP}^{[-n]}) \ \longrightarrow \ \mathbf{MU}_*^G(S^V)$$

In the special case of the trivial G-action on $\mathbb{C}^n$, this becomes an isomorphism

$$\cdot\sigma_{G,\mathbb{C}^n} \ : \ \pi_*^G(\mathbf{MP}^{[-n]}) \ \longrightarrow \ \mathbf{MU}_*^G(S^{\mathbb{C}^n}) \ = \ \pi_{*+2n}^G(\mathbf{MU}) \ .$$

This isomorphism is induced by a morphism $\mathbf{MP}^{[n]} \longrightarrow \Sigma^{2n}\mathbf{MU}$ in the global homotopy category, so $\mathbf{MP}^{[-n]}$ is globally equivalent to the $2n$-fold suspension of $\mathbf{MU}$.

We can combine the last two isomorphism into a single isomorphism

$$\mathbf{MU}_*^G(S^V) \ \cong \ \mathbf{MU}_*^G(S^{\mathbb{C}^n}) \ = \ \pi_{*+2n}^G(\mathbf{MU}) \ ,$$

which again is given by right multiplication by a certain class $t(V) \in \mathbf{MU}_{-2n}^G(S^V)$, using the $\pi_*^G(\mathbf{MU})$-module structure of $\mathbf{MU}_*^G(S^V)$. This is, in somewhat different notation, tom Dieck's Thom class from [**94**, §1].

The unitary spectrum $\mathbf{MP}^{[n]}$ is globally equivalent to the $2n$-fold suspension of $\mathbf{MU}$, so via this identification the class $\sigma_{G,V}$ can also be viewed as an element of the group $\mathbf{MU}_G^{2n}(S^V)$, the reduced, G-equivariant $\mathbf{MU}$-cohomology of S^V of dimension $2n$.

The global power functor $\underline{\pi}_0(\mathbf{MP})$ of the periodic complex bordism spectrum $\mathbf{MP}$ appears to be a very interesting object, but a complete algebraic description does not seem to be known. We summarize the structure that $\underline{\pi}_0(\mathbf{MP})$ has and recall some of the known calculations. Some known general facts are that for every *abelian* compact Lie group A the groups $\pi_*^A(\mathbf{MP})$ are concentrated in even degrees; since they are also 2-periodic, for abelian compact Lie groups all the information is concentrated in the ring $\pi_0^A(\mathbf{MP})$. The non-equivariant homotopy groups $\pi_0^e(\mathbf{MP})$ are a polynomial ring in countably many generators (although there is no known set of generators that is both explicit and convenient). For cyclic groups of prime order, Kriz [**47**] has described $\pi_0^{C_p}(\mathbf{MP})$ as a pullback of two explicit ring homomorphisms. For the cyclic group of order 2, Strickland [**90**] has turned this into an explicit presentation of $\pi_0^{C_2}(\mathbf{MP})$ as an algebra over $\pi_0^e(\mathbf{MP})$.

Tom Dieck [**94**] calculated the geometric fixed point homotopy rings $\Phi_0^G(\mathbf{MP})$ and showed that the geometric fixed point map

$$\Phi_0^G \ : \ \pi_0^G(\mathbf{MP}) \ \longrightarrow \ \Phi_0^G(\mathbf{MP})$$

is localization at the set of Euler classes of all complex representations (compare also Theorem 8.20 below). Löffler showed [**55**, Satz 2.7] that the geometric fixed point map is injective if and only G is a torus.

As we explained, the spectrum $\mathbf{MP}$ comes with the structure of a commutative unitary ring spectrum, and this makes the collection of equivariant homotopy groups $\underline{\pi}_0(\mathbf{MP})$ into a global power functor. On top of this, the unitary spectrum underlying $\mathbf{MP}$ comes with a $\mathbb{Z}$-grading, i.e., a wedge decomposition (7.17) into summands $\mathbf{MP}^{[k]}$. The grading is multiplicative, and hence the m-th power operation takes the summand $\mathbf{MP}^{[k]}$ to the summand $\mathbf{MP}^{[mk]}$. On the algebraic side, the geometric splitting induces a direct sum decomposition

$$\underline{\pi}_0(\mathbf{MP}) \ = \ \bigoplus_{k\in\mathbb{Z}} \underline{\pi}_0(\mathbf{MP}^{[k]})$$

that makes $\underline{\pi}_0(\mathbf{MP})$ into a commutative $\mathbb{Z}$-graded ring, and the power operations restrict to maps

$$P^m \ : \ \underline{\pi}_0(\mathbf{MP}^{[k]}) \ \longrightarrow \ \underline{\pi}_0(\mathbf{MP}^{[mk]}) \ .$$

The $\mathbb{Z}$-graded global power functor $\underline{\pi}_0(\mathbf{MP})$ comes with Euler classes

$$e(V) \ = \ i^*(\sigma_{G,V}) \ \in \ \pi_0^G(\mathbf{MP}^{[n]})$$

for every complex representation V of a compact Lie group G, originally considered by tom Dieck [**94**, §1]; here n is the complex dimension of V and the Euler class is obtained by restricting the complex periodicity class $\sigma_{G,V} \in \mathbf{MP}_G^{[n]}(S^V)$ along the inclusion $i : S^0 \longrightarrow S^V$

The Euler class is thus represented by the G-map

$$S^V \longrightarrow \mathbf{MP}^{[n]}(V) , \quad v \longmapsto (V^2, (v, 0)) . \tag{7.25}$$

If V has non-trivial G-fixed points, then the inclusion $i : S^0 \longrightarrow S^V$ is G-equivariantly null-homotopic, so $e(V) = 0$ whenever $V^G \neq 0$.

The following properties of Euler classes can either be proved directly from the explicit definition (7.25) above, or deduced from the properties of the periodicity classes listed in Proposition 7.21.

(i) If V and W are isomorphic complex G-representations, then $e(V) = e(W)$.

(ii) The Euler class $e(0)$ of the trivial 0-dimensional G-representation is the unit element $1 \in \pi_0^G(\mathbf{MP}^{[0]}) = \pi_0^G(\mathbf{MU})$

(iii) For every continuous homomorphism $\alpha : K \longrightarrow G$ the relation

$$\alpha^*(e(V)) \ = \ e(\alpha^* V)$$

holds in $\pi_0^K(\mathbf{MP}^{[n]})$.

(iv) For all complex G-representations V and K-representation W the relation

$$e(V) \times e(W) \ = \ e(V \oplus W)$$

holds in $\pi_0^{G\times K}(\mathbf{MP}^{[n+m]})$.

(v) For all complex G-representations V and W the relation

$$e(V) \cdot e(W) \ = \ e(V \oplus W)$$

holds in $\pi_0^G(\mathbf{MP}^{[n+m]})$.

(vi) For every n-dimensional complex G-representation V and $m \geq 1$, the relation

$$P^m(e(V)) \ = \ e(V^m)$$

holds in $\pi_0^{\Sigma_m \wr G}(\mathbf{MP}^{[mn]})$.

(vii) If H is a closed subgroup of finite index in G, then for every n-dimensional complex H-representation W the relation

$$N_H^G(e(W)) \ = \ e(\mathrm{tr}_H^G(V))$$

holds in $\pi_0^G(\mathbf{MP}^{[[G:H]\cdot n]})$.

By the naturality property (iii), all Euler classes are determined by the Euler classes of the tautological $U(n)$-representation on $\mathbb{C}^n$.

We let $T = U(1)$ denote the circle group and z the tautological complex T-representation on $\mathbb{C}$. We let $z^n = z^{\otimes n}$ denote its n-fold tensor power. Then for every $n \geq 1$ the sequence

$$0 \longrightarrow \pi_0^T(\mathbf{MP}) \xrightarrow{e(z^n)\cdot -} \pi_0^T(\mathbf{MP}) \xrightarrow{\mathrm{res}^T_{C_n}} \pi_0^{C_n}(\mathbf{MP}) \longrightarrow 0$$

is exact, where $C_n \leq T$ is the cyclic subgroup of order n. Moreover, for every compact Lie group G the sequence

$$0 \longrightarrow \pi_0^{T\times G}(\mathbf{MP}) \xrightarrow{e(p^*z)\cdot -} \pi_0^{T\times G}(\mathbf{MP}) \xrightarrow{\mathrm{res}^{T\times G}_G} \pi_0^G(\mathbf{MP}) \longrightarrow 0$$

is exact, where $p : T \times G \longrightarrow T$ is the projection to the first factor. The sequence is split by the restriction morphism along the projection of $T \times G$ to the second factor. This implies that the completion of the commutative graded ring $\pi_0^{T\times G}(\mathbf{MP})$ at the ideal generated by the Euler class $e(p^*z)$ is a power series ring in $e(p^*z)$ over the ring $\pi_0^G(\mathbf{MP})$. We let $p_1, p_2 : T \times T \longrightarrow T$ be the two projections and define

$x, y \in \pi_0^{T^2}(\mathbf{MP}^{[2]})$ by $x = e(p_1^*(z))$ and $y = e(p_2^*(z))$. The completion of the ring $\pi_0^{T^2}(\mathbf{MP})$ at the ideal generated by x and y is then a power series ring over the underlying ring $\pi_0(\mathbf{MP})$ in two variables:

$$\pi_0^{T^2}(\mathbf{MP})^\wedge_{(x,y)} \ = \ \pi_0(\mathbf{MP})[[x, y]] \ .$$

We let $\mu : T \times T \longrightarrow T$ be the multiplication map. Then the image of the class $e(\mu^*(z)) \in \pi_0^{T^2}(\mathbf{MP}^{(2)})$ in the completion is a power series F in x and y:

$$e(\mu^*(z)) \ = \ F(x, y) \ \in \ \pi_0^{T^2}(\mathbf{MP})^\wedge_{(x,y)} \ = \ \pi_0^e(\mathbf{MP})[[x, y]] \ .$$

By Quillen's theorem, $F(x, y)$ is a universal 1-dimensional commutative formal group law and the underlying commutative ring $\pi_0^e(\mathbf{MP})$ of the $\mathbb{Z}$-graded global power functor $\underline{\pi}_0(\mathbf{MP})$ is isomorphic to the Lazard ring. Abstractly, $\pi_0^e(\mathbf{MP})$ is thus a polynomial ring in infinitely many generators, one in $\pi_0^e(\mathbf{MP}^{[k]})$ for every $k \leq 0$.

Construction 7.26 (Geometric fixed points of **MP**)**.** We calculate the geometric fixed point homotopy groups $\Phi_*^G(\mathbf{MP})$ of periodic complex global bordism. [this should be in tom Dieck's [**94**]...] Much of this can be done in a similar way for the unoriented version **MOP**, but the final answer is more complicated because irreducible real representations come in different types (real, complex or quaternionic), depending on their endomorphism algebras.

If V is a complex G-representation, we write V^G for the subspace of G-fixed points and $V^\perp = V - V^G$ for the orthogonal complement if V^G. Then 0 is the only G-fixed point of $V^\perp$. If $(U, x) \in \mathbf{MP}(V)$ is a G-fixed point of the Thom space, then U must be a G-invariant subspace of V^2 and x must be a G-fixed element of U. Since U is G-invariant, it must be the internal direct sum $U = U^G \oplus U^\perp$, where $U^G = U \cap (V^2)^G = U \cap (V^G)^2$ and $U^\perp = U \cap (V^2)^\perp = U \cap (V^\perp)^2$. Since $U^\perp$ has no non-trivial G-fixed points, the element x must lie in U^G. This can be summarized by saying that the map

$$\mathbf{MP}(V^G) \wedge \mathbf{BUP}(V^\perp)^G_+ \ \longrightarrow \ (\mathbf{MP}(V))^G \ , \quad (U, x) \wedge W \ \longmapsto \ (U \oplus W, x)$$

is a homeomorphism.

Now we specialize to the case of a *finite* group G, because then the description of $\Phi_*^G(\mathbf{MP})$ can be derived from a 'decomposition' of the geometric fixed point spectrum $\Phi^G(\mathbf{MP})$ [define...]. As usual, we let ρ_G denote the regular representation of G and $\bar{\rho}_G = (\rho_G)^\perp$ the reduced regular representation. If W is any complex inner product space (with trivial G-action), then the above homeomorphism for $V = W \otimes \rho_G$ specializes to an $U(W)$-equivariant homeomorphism

$$\mathbf{MP}(W) \wedge \mathbf{BUP}(W \otimes \bar{\rho}_G)^G_+ \ \longrightarrow \ (\mathbf{MP}(W \otimes \rho_G))^G \ .$$

Implicitely we have identified $(W \otimes \rho_G)^G \cong W \otimes (\rho_G)^G \cong W$ and $(W \otimes \rho_G)^\perp \cong W \otimes (\rho_G)^\perp = W \otimes \bar{\rho}_G$. As W varies, these homeomorphisms form an isomorphism of unitary spectra

$$\mathbf{MP} \wedge \bar{F}^G(\mathbf{BUP}) \ \cong \ \Phi^G(\mathbf{MP}) \ .$$

Here $\bar{F}^G Y$ is the *reduced G-fixed point space* of the unitary space Y, whose value at a complex inner product space W is $(\bar{F}^G Y)(W) = Y(W \otimes \bar{\rho}_G)^G$. Similarly as in Example II.2.26

$$(\bar{F}^G(\mathbf{BUP}))(\mathbb{C}^\infty) \ \simeq \ J(G) \times B(U^G((\mathcal{U}_G^{\mathbb{C}})^\perp)) \ ,$$

where $J(G)$ is the kernel of the homomorphism

$$\mathbf{R}(G) \ \longrightarrow \ \mathbb{Z} \ , \quad [V] \ \longmapsto \ \dim_{\mathbb{C}}(V^G) \ ;$$

equivalently, $J(G)$ is the subgroup of the real representation ring $\mathbf{R}(G)$ generated by the *non-trivial* irreducible complex G-representations. Moreover, the group $U^G((\mathcal{U}_G^{\mathbb{C}})^\perp)$ decomposes as the weak product, indexed over isomorphism classes of non-trivial irreducible G-representations, of copies of the infinite unitary group. As far as non-equivariant homotopy groups go, this implies an isomorphism

$$\Phi^G_*(\mathbf{MP}) \;\cong\; \mathbf{MP}_*\left((\bar{F}^G(\mathbf{BUP})(\mathbb{C}^\infty)\right) \;\cong\; \mathbb{Z}[J(G)] \otimes \mathbf{MP}_*\left(\prod_{\lambda\neq 1}{}' BU\right)$$
$$\cong\; \mathbb{Z}[J(G)] \otimes \bigotimes_{\pi^e_*(\mathbf{MP})}^{\lambda\neq 1} \mathbf{MP}_*(BU)\ .$$

Since the ring spectrum $\mathbf{MP}$ is canonically complex oriented (in the non-equivariant sense), each tensor factor $\mathbf{MP}_*(BU)$ is a polynomial algebra over $\mathbf{MP}_*$ on preferred generators $b_{\lambda,i}$ for $i \geq 1$. Since $J(G)$ is a free abelian group on the classes of the non-trivial irreducible representations, the ring $\mathbb{Z}[J(G)]$ is a Laurent series ring in generators $b_{\lambda,0}$ for all $\lambda \neq 1$. In fact, we can take

$$b_{\lambda,0} \;=\; \Phi^G(e(\lambda))\ ,$$

the image of the Euler class $e(\lambda)$ under the geometric fixed point homomorphism. Altogether we conclude that

$$\Phi^G_*(\mathbf{MP}) \;=\; \mathbf{MP}_*[b^{\pm 1}_{\lambda,0}, b_{\lambda,i} \mid i \geq 1, \lambda \neq 1]\ .$$

8. Globally oriented ring spectra

In this section we will show that the global periodic bordism spectrum $\mathbf{MP}$ is universal among globally complex oriented ring spectra. The non-equivariant analog of this goes back to Quillen [ref?]. The complex bordism spectrum MU comes with a preferred complex orientation $t \in MU^2(P(\mathbb{C}^\infty))$ (in the non-equivariant sense), and for every commutative homotopy ring spectrum E the map

$$\mathrm{Ring}(MU, E) \;\longrightarrow\; \mathrm{Or}(E)\ , \quad \varphi \;\longmapsto\; \varphi_*(t)$$

is a bijection from the set of morphisms of homotopy ring spectra to the set of complex orientations of E. Araki [**4**] proved an analog of Quillen's theorem for Real oriented cohomology theories, where the role of the non-equivariant complex bordism spectrum is taken by the *Real* bordism spectrum MR. In motivic homotopy theory, an analog of Quillen's theorem has been obtained by Panin, Pimenov and Röndigs [**64**, Thm. 2.7], with complex bordism replaced by Voevodsky's motivic ring spectrum MGl. In equivariant homotopy theory of compact Lie groups, I am not aware of a direct generalization in full generality, but there are several results in this direction.

Cole, Greenlees and Kriz show in [**26**, Thm. 1.2] that for *abelian* compact Lie groups A and any A-equivariant multiplicative cohomology theory E^*_A equipped with complex stability isomorphisms and an orientation class in the equivariant cohomology of the projective space $P(\mathcal{U}^{\mathbb{C}}_A)$ of a complete complex A-universe, there is unique morphism $MU_A \longrightarrow E_A$ of A-equivariant homotopy ring spectra that takes the preferred complex stability classes and orientation of MU_A to the corresponding data for E_A.

In all the previously mentioned results, orientations are defined as cohomology classes of some infinite projective space (non-equivariant, equivariant or motivic), and the result depends on the calculation of the cohomology of appropriate Grassmannians. The corresponding calculation is not currently known for non-abelian compact Lie groups. Okonek shows in [**65**, Lemma 1.6] that a G-equivariant multiplicative cohomology theory E^*_G that is equipped with complex stability isomorphisms and Thom classes receives a unique transformation of equivariant cohomology theories from $\mathbf{MU}^*_G$ that respects Thom classes. For non-abelian compact Lie groups, assuming Thom classes for arbitrary equivariant vector bundles is stronger than fixing an equivariant complex orientation, though, because in general there may not be a projective bundle theorem.

Definition 8.1. A *global ring spectrum* is a commutative monoid in the global homotopy category with respect to the derived smash product.

In other words, a global ring spectrum is an orthogonal spectrum E together with morphisms

$$\eta \ : \ \mathbb{S} \longrightarrow E \qquad \text{and} \qquad \mu \ : \ E \wedge^L E \longrightarrow E$$

in the global homotopy category $\mathcal{GH}$ that are suitably commutative associative and unital. Many important examples of global ring spectra arise from ultra-commutative ring spectra (i.e., commutative monoids in the category of orthogona spectra with respect to the derived smash product).

We recall the definition of the equivariant cohomology theory represented by an orthogonal spectrum E. If G is a compact Lie group and A a based G-space, then we set

$$E_G(A) \ = \ \operatorname{colim}_{V \in s(\mathcal{U}_G)} [A \wedge S^V, E(V)]^G \ .$$

In the special case $A = S^n$ with trivial G-action we have $E_G(S^n) = \pi_n^G(E)$.

Definition 8.2. A *preorientation* of a global ring spectrum E consists of a collection of classes

$$\sigma_{G,V} \ \in \ E_G(S^V)$$

for every pair consisting of a compact Lie group G and a complex G-representation V, that satisfy the following conditions:

(i) For every isomorphism $\varphi : V \longrightarrow W$ of complex G-representations, the induced isomorphism $(S^\varphi)^* : E_G(S^W) \longrightarrow E_G(S^V)$ takes the class $\sigma_{G,W}$ to the class $\sigma_{G,V}$.

(ii) The class $\sigma_{G,0}$ of the trivial 0-dimensional G-representation is the unit element $1 \in \pi_0^G(E) = E_G(S^0)$

(iii) For every continuous group homomorphism $\alpha : K \longrightarrow G$ the relation

$$\alpha^*(\sigma_{G,V}) \ = \ \sigma_{K,\alpha^* V}$$

holds in $E_K(S^{\alpha^* V})$.

(iv) For all complex G-representations V and all complex K-representations W the relation

$$\sigma_{G,V} \times \sigma_{K,W} \ = \ \sigma_{G\times K,\, V\oplus W} \qquad \text{holds in } E_{G\times K}(S^{V\oplus W}) \ .$$

Remark 8.3. A preorientation of a global ring spectrum E is completely determined by the classes

$$\sigma_n = \sigma_{U(n),\mathbb{C}^n} \ \in \ E_{U(n)}(S^{\mathbb{C}^n})$$

assigned to the tautological $U(n)$-representation on $\mathbb{C}^n$, for $n \geq 1$. Indeed, if V is an n-dimensional complex G-representation, then there exists a continuous homomorphism $\alpha : G \longrightarrow U(n)$ and an isomorphism of G-representations $\varphi : V \cong \alpha^*(\mathbb{C}^n)$. Thus

$$\sigma_{G,V} \ = \ (S^\varphi)^*(\alpha^*(\sigma_n)) \tag{8.4}$$

by the naturality and isomorphism property. The multiplicativity property (iv) of a global preorientation forces the classes σ_n to satisfy the relation

$$\operatorname{res}^{U(n+m)}_{U(n)\times U(m)}(\sigma_{n+m}) \ = \ \sigma_n \times \sigma_m \tag{8.5}$$

in the group $E_{U(n)\times U(m)}(S^{\mathbb{C}^{n+m}})$ for all $n, m \geq 1$. Conversely, if the classes $\{\sigma_n\}_{n\geq 1}$ are given and satisfy the relation (8.5), then defining $\sigma_{G,V}$ by (8.4) gives a global preorientation.

A global preorientation is a global orientation if the classes $\sigma_{G,V}$ are invertible in a sense that we now define. Equivalently, $\sigma_{G,V}$ is a unit in the $\mathbf{RO}(G)$-graded homotopy ring of the global ring spectrum.

Definition 8.6. A global preorientation of a global ring spectrum E is a *global orientation* if the map

$$\cdot\sigma_{G,V} \ : \ \pi_k^G(E) \ \longrightarrow \ E_k^G(S^V) \ , \quad x \ \longmapsto \ x \cdot \sigma_{G,V}$$

is an isomorphism for every compact Lie group G, every complex G-representation V and every integer k.

A global orientation of a global ring spectrum E gives rise to compatible equivariant complex orientations of the equivariant cohomology theories E^*_G for all compact Lie groups G. The class $\sigma_{G,V}$ will then turn out to be the Thom class of V, considered as a G-equivariant vector bundle over a point.

Remark 8.7. Global ring spectra that are globally oriented in the sense of Definition 8.6 are in particular 2-periodic. Indeed, the class $u = \sigma_{e,\mathbb{C}}$ of the 1-dimensional representation of the trivial group lies in $E_e(S^{\mathbb{C}}) = \pi_2(E)$ and has the property that for all integers k the multiplication map

$$\cdot u \ : \ \pi^e_k(E) \ \longrightarrow \ E^G_k(S^{\mathbb{C}}) = \pi^e_{k+2}(E)$$

is an isomorphism. So u is a unit of degree 2 in the $\mathbb{Z}$-graded homotopy ring $\pi_*(E)$ of the underlying non-equivariant ring spectrum.

Remark 8.8 (Functoriality of (pre-)orientations)**.** We let $\psi : E \longrightarrow F$ be a morphism of global ring spectra. If $\sigma = \{\sigma_{G,V}\}$ is a preorientation of E, then the classes $\psi(\sigma) = \{\psi_*(\sigma_{G,V})\}$ form a preorientation of F, by the naturality and multiplicativity properties of the induced maps $\psi_* : E^G(S^V) \longrightarrow F^G(S^V)$.

If the preorientation σ is even an orientation, then so is the induced preorientation $\psi(\sigma)$. Indeed [...]

Construction 8.9 (Global orientations give complex orientations)**.** We recall how a global orientation of a global ring spectrum induces a complex orientation (in the non-equivariant sense) of the underlying non-equivariant homotopy ring spectrum. We recall that a complex orientation of a commutative homotopy ring spectrum F (i.e., a commutative monoid in the non-equivariant stable homotopy category) is a reduced F-cohomology class $t \in F^2(P(\mathbb{C}^\infty))$ of degree 2 of the infinite dimensional complex projective space such that

$$\iota^*(t) \ = \ S^2 \wedge 1 \ \in \ F^2(S^2)$$

is the preferred generator of the free $\pi_0(F)$-module $F^2(S^2)$. Here $\iota : S^2 = S^{\mathbb{C}} \longrightarrow P(\mathbb{C}^\infty)$ is the continuous map given by $\iota(\lambda) = [\lambda, 1, 0, 0 \dots]$, $P(\mathbb{C}^\infty)$ is based at the point $[1, 0, 0, 0, \dots]$ and $S^2 \wedge - : \pi_0(F) \longrightarrow F^2(S^2)$ is the suspension isomorphism.

Now we let $\sigma = \{\sigma_{G,V}\}$ be a global orientation of a global ring spectrum E. We recall from [...] the 'bundling' homomorphism

$$E_T(A) \ \longrightarrow \ E(S(\mathbb{C}^\infty) \ltimes_T A) \ ,$$

natural for based T-spaces A, where $T = U(1)$ is the circle group. The composite

$$E_T(A) \ \xrightarrow{\ b\ } \ E(S(\mathbb{C}^\infty) \ltimes_T A) \ \xrightarrow{\ j^*\ } \ E(A)$$

is the restriction map $\mathrm{res}^T_e : E_T(A) \longrightarrow E(A)$, where j is the natural embedding defined by

$$j \ : \ A \ \longrightarrow \ S(\mathbb{C}^\infty) \ltimes_T A \ , \quad a \ \longmapsto \ (1, 0, 0, \dots) \ltimes a \ .$$

The mapping cone sequence of based T-spaces

$$T^+ \ \xrightarrow{\ p\ } \ S^0 \ \xrightarrow{\ i\ } \ S^{\mathbb{C}}$$

gives rise to a commutative diagram of E-cohomology groups

$$\begin{tikzcd}
E_T(S^{\mathbb{C}}) \arrow[r, "i^*"] \arrow[d, "b"] \arrow[dd, bend right=60, "\mathrm{res}^T_e"'] & E_T(S^0) \arrow[r, "p^*"] \arrow[d, "b"] & E_T(T^+) \arrow[d, "b", "\cong"'] \\
E(S(\mathbb{C}^\infty) \ltimes_T S^{\mathbb{C}}) \arrow[r, "(S(\mathbb{C}^\infty) \ltimes_T i)^*"'] \arrow[d, "j^*"] & E(P(\mathbb{C}^\infty)_+) \arrow[r, "(S(\mathbb{C}^\infty) \ltimes_T p)^*"'] \arrow[d, "j^*"] & E(S(\mathbb{C}^\infty)_+) \arrow[d, "j^*"] \\
E(S^{\mathbb{C}}) \arrow[r, "i^*"'] & E(S^0) \arrow[r, "p^*"'] & E(T_+)
\end{tikzcd}$$

We define

$$t \ = \ b(i^*(\sigma_{T,\mathbb{C}})) \cdot \sigma_{e,\mathbb{C}}^{-1} \ \in \ E^2(P(\mathbb{C}^\infty)_+)$$

and claim that this is a complex orientation of E.

Since $i \circ p : T^+ \longrightarrow S^{\mathbb{C}}$ is T-equivariantly based null-homotopic,

$$(S(\mathbb{C}^\infty) \ltimes_T p)^* \circ b \circ i^* \ = \ b \circ p^* \circ i^* \ = \ b \circ (ip)^*$$

is the zero map, and hence

$$(S(\mathbb{C}^\infty) \ltimes_T p)^*(t) \ = \ (S(\mathbb{C}^\infty) \ltimes_T p)^*(b(i^*(\sigma_{T,\mathbb{C}}))) \cdot \sigma_{e,\mathbb{C}}^{-1} \ = \ 0 \ .$$

So the restriction of t to any point of $P(\mathbb{C}^\infty)$ is zero, i.e., t is a reduced E-cohomology class. Indeed, since the sphere $S(\mathbb{C}^\infty)$ is contractible, the map $(S(\mathbb{C}^\infty) \ltimes_T i)^* : E^2(S(\mathbb{C}^\infty) \ltimes_T S^{\mathbb{C}}) \longrightarrow E^2(P(\mathbb{C}^\infty)_+)$ identifies the E-cohomology of $S(\mathbb{C}^\infty) \ltimes_T S^{\mathbb{C}}$ with the reduced E-cohomology of the projective space $P(\mathbb{C}^\infty)$.

The composite

$$S^{\mathbb{C}} \ \xrightarrow{\ \iota\ } \ P(\mathbb{C}^\infty) \ \xrightarrow{\ s\ } \ S(\mathbb{C}^\infty) \ltimes_T S^{\mathbb{C}}$$

is homotopic (in the unbased sense) to the map j [show], where s is the 'zero section' given by

$$s[\lambda_1, \lambda_2, \dots] = (\lambda_1, \lambda_2, \dots) \ltimes 0 \ .$$

So

$$\iota^*(t) \ = \ \iota^*(b(i^*(\sigma_{T,\mathbb{C}}))) \cdot \sigma_{e,\mathbb{C}}^{-1} \ = \ j^*(b(\sigma_{T,\mathbb{C}})) \cdot \sigma_{e,\mathbb{C}}^{-1} \ = \ \mathrm{res}^T_e(\sigma_{T,\mathbb{C}}) \cdot \sigma_{e,\mathbb{C}}^{-1} \ = \ S^2 \wedge 1 \ .$$

Remark 8.10. If a global ring spectrum E is globally orientable, then the set of all global orientations can be parametrized by a certain group $\mathbb{G}(\underline{\pi}_0(E))$ or, equivalently, by the set of morphism of global ring spectra from $\Sigma^\infty_+ \mathbf{BUP}$ to E, as we now explain.

We let $\sigma = \{\sigma_{G,V}\}$ and $\bar\sigma = \{\bar\sigma_{G,V}\}$ be two global orientations of a global ring spectrum E. As before we write $\sigma_n = \sigma_{U(n),\mathbb{C}^n}$ and $\bar\sigma_n = \bar\sigma_{U(n),\mathbb{C}^n}$ for the classes that determine all other orientation classes. Then by the orientation property, for every $n \geq 1$ there is a unique class $\lambda_n \in \pi_0^{U(n)}(E)$ such that

$$\sigma_n \cdot \lambda_n \ = \ \bar\sigma_n \ .$$

Reversing the roles of σ and $\bar\sigma$ shows that λ_n is a unit in the ring $\pi_0^{U(n)}(E)$. The classes λ_n satisfy

$$\begin{aligned}
(\sigma_n \times \sigma_m) \cdot \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\lambda_{n+m}) \ &= \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\sigma_{n+m}) \cdot \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\lambda_{n+m}) \\
&= \ \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\sigma_{n+m}\lambda_{n+m}) \\
&= \ \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\bar\sigma_{n+m}) \ = \ \bar\sigma_n \times \bar\sigma_m \\
&= \ (\sigma_n\lambda_n) \times (\sigma_m\lambda_m) \ = \ (\sigma_n \times \sigma_m) \cdot (\lambda_n \times \lambda_m) \ .
\end{aligned}$$

Since $\sigma_n \times \sigma_m$ is invertible, this shows that

$$\mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\lambda_{n+m}) \ = \ \lambda_n \times \lambda_m \ . \tag{8.11}$$

Conversely, given a global orientation σ and units $\lambda_n \in \pi_0^{U(n)}(E)$, for $n \geq 1$, that satisfy the restriction formulas (8.11), then the classes $\sigma_n\lambda_n$ detemine another global preorientation of E according to (8.4), and this preorientation is a global orientation. Altogether this shows that for a globally orientable ring spectrum E the set of global orientations is a torsor over the multiplicative group

$$\mathbb{G}(\underline{\pi}_0(E)) \ = \ \{(\lambda_n)_{n\geq 1} \in \prod_{n\geq 1} \left(\pi_0^{U(n)}(E)\right)^\times \ \mid \ \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\lambda_{n+m}) = \lambda_n \times \lambda_m\} \ .$$

A x class in $\pi_0^{U(n)}(E)$ is represented by a unique morphism $\hat{x} : \Sigma^\infty_+ B_{\mathrm{gl}}U(n) \longrightarrow E$ in the stable global homotopy category such that

$$\hat{x}(e_{U(n)}) \ = \ x \ .$$

Here $B_{\mathrm{gl}}U(n) = \mathbf{L}_{U(n),\mathbb{C}^n}$ is the global classifying space for $U(n)$ based on the tautological representation on $\mathbb{C}^n$ and $e_n = e_{U(n),\mathbb{C}^n} \in \pi_0^{U(n)}(\Sigma^\infty_+ B_{\mathrm{gl}}U(n))$ is the stable tautological class (see (3.10) of Chapter III).

In Example II.2.24 we discussed the complex additive Grassmannian $\mathbf{Gr}^{\mathbb{C}}$, a certain commutative orthogonal monoid space made from Grassmannians; the underlying orthogonal space is a disjoint union

$$\mathbf{Gr}^{\mathbb{C}} \;=\; \coprod_{n\geq 0} B_{\mathrm{gl}}U(n)$$

of global classifying spaces of the unitary groups. The monoid structure is such that

$$u_n \times u_m \;=\; \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(u_{n+m})$$

holds in $\pi_0^{U(n)\times U(m)}(\mathbf{Gr}^{\mathbb{C}})$. Given $(\lambda_n)_{n\geq 1} \in \mathbb{G}(\underline{\pi}_0(E))$ we can this consider the morphism

$$\hat\lambda \;=\; \coprod \hat\lambda_n \;:\; \coprod_{n\geq 0} \Sigma^\infty_+ B_{\mathrm{gl}}U(n) = \;\Sigma^\infty_+\mathbf{Gr}^{\mathbb{C}} \;\longrightarrow\; E$$

in the stable global homotopy category that takes the class tautological class e_n to λ_n for all $n \geq 1$, and the class $e_0 \in \pi_0^e(B_{\mathrm{gl}}U(0))$ to 1. The multiplicativity relations $\mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\lambda_{n+m}) = \lambda_n \times \lambda_m$ then mean that the morphism $\hat\lambda$ behaves multiplicatively on all fundamental classes, so λ is in fact a morphism of global ring spectra. We emphasize at this point that the homotopy multiplication on E need not come from an ultra-commutative multiplication, and even if it does, there is no claim that λ can be realized by a morphism of ultra-commutative ring spectra.

Since the classes λ_n were assumed to be units, the morphism λ extends uniquely to a morphism of commutative monoids in the unstable global homotopy category

$$\mathbf{BUP} \;\longrightarrow\; \Omega^\bullet E$$

by the group completion property [...]. Adjoint to this is a morphism of global ring spectra

$$\bar\lambda \;:\; \Sigma^\infty_+\mathbf{BUP} \;\longrightarrow\; E\ .$$

We conclude that the map

$$\mathbb{G}(\underline{\pi}_0(E)) \;\longrightarrow\; \mathcal{GR}(\Sigma^\infty_+\mathbf{BUP}, E) \subset [\![\Sigma^\infty_+\mathbf{BUP}, E]\!]\ , \quad (\lambda_n)_{n\geq 1} \;\longmapsto\; \bar\lambda$$

is a bijection from $\mathbb{G}(\underline{\pi}_0(E))$ to the set of morphisms of global ring spectra from $\Sigma^\infty_+\mathbf{BUP}$ to E.

Example 8.12. We consider a global power functor R that satisfies the *splitting principle*, by which we mean that the restriction map from $U(n)$ to the maximal torus $T^n = U(1)^n$ is an isomorphism

$$\mathrm{res}^{U(n)}_{T^n} \;:\; R(U(n)) \;\longrightarrow\; R(T^n)^{\Sigma_n}$$

to the subring of Σ_n-invariant invariant elements of $R(T^n)$. An example is the complex representation ring functor $\mathbf{R}$, which is isomorphic to $\underline{\pi}_0(\mathbf{KU})$. Other examples are the global power functors $G \mapsto E^0(BG)$, which is isomorphic to $\underline{\pi}_0(bE)$, where E is any ordinary cohomology theory that is complex orientable (in the traditional, non-global sense).

We consider $\{\lambda_n\} \in \mathbb{G}(R)$, then the relation

$$\mathrm{res}^{U(n)}_{U(1)^n}(\lambda_n) \;=\; \lambda_1 \times \cdots \times \lambda_1$$

and the splitting principle show that the entire tuple is already determined by λ_1. Conversely, if $\lambda_1 \in R(U(1))^\times$ is given with $\mathrm{res}^{U(1)}_e(\lambda_1) = 1$, then we can define λ_n as the unique element of $R(U(n))$ that restricts to $\lambda_1^{\times n}$ in $R(T^n)$. This automatically defines an element of $\mathbb{G}(R)$. So if R satisfies the splitting principle, then the map

$$\mathbb{G}(R) \;\longrightarrow\; \{\alpha \in R(U(1))^\times \mid \mathrm{res}^{U(1)}_e(\alpha) = 1\}\ , \quad \{\lambda_n\} \;\longmapsto\; \lambda_1$$

is an isomorphism of groups.

So if E is an orientable global cohomology theory such that the global power functor $\underline{\pi}_0(E)$ satisfies the splitting principle, then the set of global orientations of E is a torsor for the multiplicative group

$$\{\alpha \in (E^0_{U(1)})^\times \mid \mathrm{res}^{U(1)}_e(\alpha) = 1\} \ .$$

Another example is global K-theory. In this case $\pi^T_0(\mathbf{KU}) = \mathbf{R}(T) \cong \mathbb{Z}[s, s^{-1}]$, with s the class of the tautological T-representation. So here the group

$$\{\alpha \in R(U(1))^\times \mid \mathrm{res}^{U(1)}_e(\alpha) = 1\}$$

consists of the powers of s. The Atiyah-Bott-Shapiro orientation is a preferred global orientation of $\mathbf{KU}$; so this argument shows that the set of global orientation of $\mathbf{KU}$ is a torsor over the multiplicative group $\{s^k\}_{k\in\mathbb{Z}}$. [question: what are the λ_n's corresponding to s^k ?]

If we change an orientation $\{\sigma_n\}$ to a new one using $\lambda_1 = s^k$, then the power operations change to

$$P^m(\sigma'_1) \ = \ P^m(\sigma_1 \cdot s^k) \ = \ P^m(\sigma_1) \cdot (P^m(s))^k \ .$$

Now suppose that the original orientation is G_∞; we want to know when the new orientation is G_∞, i.e., when

$$P^m(\sigma'_1) \ = \ \mathrm{res}^{U(m)}_{\Sigma_m \wr T}(\sigma'_m) \ .$$

then

$$P^m(\sigma_1) \cdot (P^m(s))^k \ = \ \mathrm{res}^{U(m)}_{\Sigma_m \wr T}(\sigma_m) \cdot (P^m(s))^k \ .$$

On the other hand,

$$\mathrm{res}^{\Sigma_m \wr T}_{T^m}(\mathrm{res}^{U(m)}_{\Sigma_m \wr T}(\sigma'_m)) \ = \ \mathrm{res}^{U(m)}_{T^m}(\sigma'_n) \ = \ \mathrm{res}^{U(m)}_{T^m}(\sigma_n) \cdot \lambda_1^{\times n} \ .$$

Question: is the splitting principle equivalent to the projective bundle theorem, i.e., that for every equivariant rank k complex vector bundle $\xi : E \longrightarrow X$ the equivariant cohomology of the projectivized bundle $P(E)$ is free over the equivariant cohomology of X with basis the powers $1, e, e^2, \dots e^{k-1}$ of the Euler class of the tautological line bundle over $P(E)$. The projective bundle theorem always holds for products of line bundles, and it holds over the tom Dieck localized theory. This theory is obtained from $MU(-)$ by inverting a multiplicative subset of MU^G_0 (the subset of Euler classes of G-representations without fixed points?) tom Dieck calculates the localization $S^{-1}MU$ in Theorem 3.1 of [**94**]. The localization is made so that $S^{-1}MU$ has a projective bundle theorem (for abelian groups only?) for trivial vector bundles of the form $X \times V$, V a G-representation. Okonek shows in Theorem 2.6 how to get the projective bundle theorem in general.

Example 8.13. We have seen in Theorem IV.5.2 that the forgetful functor $U : \mathcal{GH} \longrightarrow \mathcal{SH}$ from the global to the non-equivariant stable homotopy category has a right adjoint $R : \mathcal{SH} \longrightarrow \mathcal{GH}$ that comes with a preferred lax symmetric monoidal structure. We let F be a homotopy commutative homotopy ring spectrum in the non-equivariant sense, i.e., a commutative monoid in the (non-equivariant) stable homotopy category $\mathcal{SH}$, with respect to the derived smash product. Then RF becomes a global ring spectrum via the lax monoidal structure. The underlying non-equivariant ring spectrum of RF is F again, so Construction 8.9 provides a map

$$\mathrm{Or}(RF) \ \longrightarrow \ \mathrm{Or}(F) \ , \quad \sigma \ \longmapsto \ b(i^*(\sigma_{T,\mathbb{C}})) \cdot \sigma^{-1}_{e,\mathbb{C}} \tag{8.14}$$

from the set of global orientations of the global ring spectrum RF to the set of complex orientations of F.

We claim that with a slight modification, the map (8.14) is a bijection between global orientations of RF and complex orientations of F. The necessary modification comes from the fact that global orientations imply 2-fold periodicity, so we can only expect a bijection for 2-periodic ring spectra F. The 2-periodicity is no real loss of generality, because every homotopy commutative homotopy ring spectrum can be 'periodized' by endowing the wedge

$$FP \ = \ \bigvee_{k\in\mathbb{Z}} \Sigma^{2k} F$$

with a homotopy commutative multiplication that makes the image of $i \in \pi_0(F)$ under inclusion of the summand for $k = 1$ into a unit in $\pi_2(FP)$.

So we consider a '2-periodic complex oriented homotopy ring spectrum', i.e., a triple (F, t, z) consisting of a homotopy commutative homotopy ring spectrum F, a complex orientation $t \in F^2(P(\mathbb{C}^\infty))$ and a unit $z \in \pi_2(F)$. The theory of complex oriented cohomology theories provides Thom classes

$$u(\xi) \ \in \ F^{2n}(M\xi)$$

for all complex vector bundles $\xi : E \longrightarrow X$ of rank n over a CW-complex X. The reduced F-cohomology $F^*(M\xi)$ of the Thom space is then a free module, generated by the Thom class $u(\xi)$, over the unreduced F-cohomology $F^*(X_+)$ of the base space.

Given a complex G-representation V, we can form the associated complex vector $a(V)$ over BG with total space $EG \times_G V$ and Thom space $EG \ltimes_G S^V$. If V has dimension n, then the Thom class of $a(V)$ is an element

$$u(a(V)) \ \in \ F^{2n}(EG \ltimes_G S^V) \ \cong \ (RF)^{2n}_G(S^V) \ ,$$

where the isomorphism is from [...] This is almost where the class $\sigma_{G,V}$ of a global orientation should live, and we can fix up the dimension by multiplying with the unit z^n, or rather its image under the restriction homomorphism $p_G^* : \pi_*(F))\pi_*^e(RF) \longrightarrow \pi_*^G(RF)$. We can thus define

$$\sigma_{G,V} \ = \ u(a(V)) \cdot p_G^*(z^n) \ \in \ F(EG \ltimes_G S^V) \ = \ (RF)_G(S^V) \ .$$

Our next aim is to show that the global bordism spectrum $\mathbf{MP}$ is the universal oriented global ring spectrum. In technical terms, this means that for every global ring spectrum E the map

$$\text{global ring}(\mathbf{MP}, E) \ \longrightarrow \ \mathrm{Or}(E) \ , \quad \psi \ \longmapsto \ \psi(\sigma)$$

that takes a morphism of global ring spectrum $\psi : \mathbf{MP} \longrightarrow E$ to the orientation $\psi(\sigma) = \{\psi_*(\sigma_{G,V})\}$ is a bijection. The proof breaks up into two main steps.

- We consider a commutative unitary ring spectrum $\mathbf{MP}^{\mathrm{re}}$, defined as the Thom spectrum of the restriction of the universal bundle over $\mathbf{BUP}$ along the morphism
$$i \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \mathbf{BUP}$$
defined in Example II.2.26 We then show that $\mathbf{MP}^{\mathrm{re}}$ is a universal *preoriented* global ring spectrum.
- The fact that $\mathbf{BUP}$ is a global group completion of $\mathbf{Gr}^{\mathbb{C}}$ (the unitary analog of Theorem II.4.10) allows us to express $\mathbf{MP}$ by a homotopy pushout square of commutative unitary ring spectra:
$$\begin{tikzcd} \Sigma^\infty_+ \mathbf{Gr}^{\mathbb{C}} \arrow[r, "MD"] \arrow[d] & \mathbf{MP}^{\mathrm{re}} \wedge \overline{\mathbf{MP}}^{\mathrm{re}} \arrow[d, "Mi \vee M\bar{i}"] \\ \mathbb{S} \arrow[r] & \mathbf{MP} \end{tikzcd}$$
Here $\overline{\mathbf{MP}}^{\mathrm{re}}$ is a 'negative' version of the universal preoriented ring spectrum $\mathbf{MP}^{\mathrm{re}}$, defined as the Thom spectrum of the restriction of the universal bundle along the morphism $\bar{i} : \mathbf{BUP}^{\mathrm{re}} \longrightarrow \mathbf{BUP}$ that realizes the *negatives* of representation on π_0.

The relations between the relevant multiplicative global objects are summarized scematically in the following diagram:

$$\begin{tikzcd} \mathbf{MP}^{\mathrm{re}} \arrow[r, "\text{thomified group compl}"] \arrow[d, dotted, "\text{Thom spectrum}"'] & \mathbf{MP} \arrow[d, dotted, "\text{Thom spectrum}"] \\ \mathbf{Gr}^{\mathbb{C}} \arrow[r, "\text{group completion}"'] & \mathbf{BUP} \end{tikzcd}$$

Construction 8.15 (Global preoriented bordism spectrum)**.** We start with the first step, recognizing a certain Thom spectrum $\mathbf{MP}^{\mathrm{re}}$ as a universal preoriented global ring spectrum. We recall from Example II.2.26 the morphism of commutative orthogonal monoid spaces

$$i \ : \ \mathbf{Gr}^{\mathbb{C}} \ \longrightarrow \ \mathbf{BUP} \ .$$

By definition, i is given at a complex inner product space V by

$$\mathbf{Gr}^{\mathbb{C}}(V) \ = \ \coprod_{n\geq 0} Gr_n(V) \ \longrightarrow \ \coprod_{m\geq 0} Gr_m(V^2) = \mathbf{BUP}(V) \ , \quad L \ \longmapsto \ L \oplus V \ .$$

We can restrict the tautological vector bundle over $\mathbf{BUP}(V)$ along $i(V)$ to $\mathbf{Gr}^{\mathbb{C}}(V)$. This is a complex vector bundle (of non-constant rank) over $\mathbf{Gr}^{\mathbb{C}}(V)$ with total space the space of pairs $(L, v, w) \in \mathbf{Gr}^{\mathbb{C}}(V) \times V^2$ such that $v \in L$. We let $\mathbf{MP}^{\mathrm{re}}$ denote the Thom space of this bundle, which is then a wedge, over $n \geq 0$, of the Thom spaces of the individual bundles over $\mathbf{Gr}^{\mathbb{C},[n]}(V)$. As V varies, the maps $i(V) : \mathbf{Gr}^{\mathbb{C}}(V) \longrightarrow \mathbf{BUP}(V)$ are closed embeddings and form a morphism of commutative unitary monoid spaces. So the induced based maps of Thom spaces

$$(Mi)(V) \ : \ \mathbf{MP}^{\mathrm{re}}(V) \ \longrightarrow \ \mathbf{MP}(V)$$

are closed embeddings, and the space $\mathbf{MP}^{\mathrm{re}}(V)$ inherit a unique structure of commutative unitary ring spectrum from $\mathbf{MP}$ such that the maps $(Mi)(V)$ become a morphism of unitary ring spectra

$$Mi \ : \ \mathbf{MP}^{\mathrm{re}} \ \longrightarrow \ \mathbf{MP} \ .$$

Just as the unitary monoid space $\mathbf{Gr}^{\mathbb{C}}$ is a disjoint union of free unitary spaces, the Thom spectrum $\mathbf{MP}^{\mathrm{re}}$ is isomorphic to a wedge of free unitary spectra. For every $n \geq 0$, the $U(n)$-equivariant based map

$$S^{\mathbb{C}^n \oplus \mathbb{C}^n} \ \longrightarrow \ \mathbf{MP}^{\mathrm{re}}(\mathbb{C}^n) \ , \quad (x, y) \ \longmapsto \ (\mathbb{C}^n, x, y)$$

is represented by a morphism of unitary spectra

$$\kappa_n \ : \ F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n \oplus \mathbb{C}^n} \ \longrightarrow \ \mathbf{MP}^{\mathrm{re}} \ .$$

Proposition 8.16. *The morphism*

$$\kappa \ = \ \sum_{n\geq 0} \kappa_n \ : \ \bigvee_{n\geq 0} F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n \oplus \mathbb{C}^n} \ \longrightarrow \ \mathbf{MP}^{\mathrm{re}}$$

is an isomorphism of unitary spectra.

PROOF. We denote by $\mathbf{MP}^{\mathrm{re},[n]}(V)$ the Thom space of the vector bundle over $Gr_n(V)$ with total space consisting of the pairs $(L, v, w) \in Gr_n(V) \times V^2$ such that $v \in L$. As V varies, these spaces form a unitary subspectrum $\mathbf{MP}^{\mathrm{re},[n]}$ of $\mathbf{MP}^{\mathrm{re}}$ with $\mathbf{MP}^{\mathrm{re}} = \bigvee_{n\geq 0} \mathbf{MP}^{\mathrm{re},[n]}$. So we need to show that the morphism $\kappa_n : F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n \oplus \mathbb{C}^n} \longrightarrow \mathbf{MP}^{\mathrm{re},[n]}$ is an isomorphism for every $n \geq 0$. The value of the source of κ_n at an inner product space is given by

$$(F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n \oplus \mathbb{C}^n})(V) \ = \ \mathbf{U}(\mathbb{C}^n, V) \wedge_{U(n)} S^{\mathbb{C}^n \oplus \mathbb{C}^n} \ ;$$

this is the Thom space of the vector bundle over $\mathbf{L}^{\mathbb{C}}(\mathbb{C}^n, V)/U(n) = \mathbf{L}^{\mathbb{C}}_{U(n),\mathbb{C}^n}$ with total space

$$\xi(\mathbb{C}^n, V) \times_{U(n)} (\mathbb{C}^n \oplus \mathbb{C}^n) \ ,$$

where $\xi(\mathbb{C}^n, V)$ is the complex version of the orthgonal complement bundle [ref]. Sending the point

$$[(\varphi : \mathbb{C}^n \longrightarrow V, v), x, y] \ \in \ \xi(\mathbb{C}^n, V) \times_{U(n)} (\mathbb{C}^n \oplus \mathbb{C}^n)$$

to the point

$$(\varphi(\mathbb{C}^n), \, \varphi(x), \, v + \varphi(y)) \in \ Gr_n(V) \times V^2$$

defines a vector bundle isomorphism to the vector bundle over $Gr_n(V)$ previously considered. The map $\kappa_n(V)$ is precisely the induced map on Thom spaces, hence a homeomorphism. $\square$

We let G be a compact Lie group and V a complex G-representation and observe that the periodicity classes

$$\sigma_{G,V} \;\in\; \mathbf{MP}_G^{[n]}(S^V)$$

defined in (7.20) have preferred lifts to equivariant cohomology classes in $\mathbf{MP}^{\mathrm{re}}$ under the thomified group completion morphism $Mi : \mathbf{MP}^{\mathrm{re}} \longrightarrow \mathbf{MP}$. Indeed, the G-equivariant based map

$$S^{V^2} \;\longrightarrow\; \mathbf{MP}^{\mathrm{re}}(V) \ , \quad (v,w) \;\longmapsto\; (V,v,w)$$

represents a class

$$\sigma_{G,V}^{\mathrm{pre}} \quad\in\quad \mathbf{MP}_G^{\mathrm{re}}(S^V) \ .$$

When composed with the map $(Mi)(V) : \mathbf{MP}^{\mathrm{re}}(V) \longrightarrow \mathbf{MP}(V)$, the representative for $\sigma_{G,V}^{\mathrm{pre}}$ equals the representative $\bar\sigma_{G,V}$ for $\sigma_{G,V}$, so the homotopy classes satisfy

$$(Mi)_*(\sigma_{G,V}^{\mathrm{pre}}) \;=\; \sigma_{G,V} \text{ in } \mathbf{MP}_G(S^V) \ .$$

As the next proposition shows, the lifts $\sigma_{G,V}^{\mathrm{pre}}$ form a preorientation of $\mathbf{MP}^{\mathrm{re}}$. We emphasize, though, that the classes $\sigma_{G,V}^{\mathrm{pre}}$ are *not* invertible in the $RO(G)$-graded sense, i.e., they do *not* form an orientation. The next proposition say that $\mathbf{MP}^{re}$ is the universal preoriented global ring spectrum.

Proposition 8.17.
(i) *The classes $\sigma_{G,V}^{\mathrm{pre}}$ form a preorientation of the commutative unitary ring spectrum $\mathbf{MP}^{\mathrm{re}}$.*
(ii) *For every global ring spectrum E the map*

$$\text{global ring}(\mathbf{MP}^{\mathrm{re}}, E) \;\longrightarrow\; \mathrm{PreOr}(E) \ , \quad \psi \;\longmapsto\; \psi(\sigma^{\mathrm{pre}}) = \{\psi_*(\sigma_{G,V}^{\mathrm{pre}})\}$$

that evaluates at the preferred preorientation of $\mathbf{MP}^{\mathrm{re}}$ is bijective.

Proof. Part (i) can be show by the same explicit verification as for the periodicity classes $\sigma_{G,V}$ in Proposition 7.21.

(ii) The isomorphism κ of Proposition 8.16 allows us to identify morphism in the global homotopy category out of $\mathbf{MP}^{\mathrm{re}}$: for every unitary spectrum E (not necessarily a global ring spectrum), the map

$$[\![\mathbf{MP}\mathrm{re}, E]\!] \;\longrightarrow\; \prod_{n\geq 0} [\![F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n\oplus\mathbb{C}^n}, E]\!] \ , \quad \psi \;\longmapsto\; (\psi\circ\kappa_n)_n$$

given by precomposition with the morphism $\kappa_n : F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n\oplus\mathbb{C}^n} \longrightarrow E$ is a bijection; this uses that wedges of unitary spectra are coproducts in the global homotopy category. On the other hand, by [...] the unitary spectrum $F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n\oplus\mathbb{C}^n}$ represents the functor

$$\mathcal{GH} \;\longrightarrow\; \mathcal{A}b \ , \quad E \;\longmapsto\; E_{U(n)}(S^{\mathbb{C}^n}) \ .$$

The composite bijection is given by

$$[\![\mathbf{MP}^{\mathrm{re}}, E]\!] \;\cong\; \prod_{n\geq 0} E_{U(n)}(S^{\mathbb{C}^n}) \ , \quad \psi \;\longmapsto\; \big(\psi_*(\sigma_n^{\mathrm{pre}})\big)_n \ ,$$

where $\sigma_n^{\mathrm{pre}} = \sigma_{U(n),\mathbb{C}^n}^{\mathrm{pre}} \in \mathbf{MP}_{U(n)}^{\mathrm{re}}(S^{\mathbb{C}^n})$. In particular, morphisms in the global homotopy category out of $\mathbf{MP}$re are detemined by their effect on the classes σ_n^{pre} for all $n \geq 0$, so the map of the proposition is injective.

Now we suppose that conversely we are given an preorientation $\tau = \{\tau_{G,V}\}$ of E. By the previous paragraph there is unique morphism $\psi : \mathbf{MP}^{\mathrm{re}} \longrightarrow E$ in the global homotopy category such that $\psi_*(\sigma_n^{\mathrm{pre}}) = \tau_{U(n),\mathbb{C}^n}$ for all $n \geq 0$. In particular, $\psi_*(1) = \psi_*(\sigma_0^{\mathrm{pre}}) = \tau_{e,0} = 1$, so ψ preserves the unit morphisms.

The derived smash product in $\mathcal{GH}$ distributes over the product, so $\mathbf{MP}^{\mathrm{re}} \wedge^L \mathbf{MP}^{\mathrm{re}}$ is isomorphic in $\mathcal{GH}$ to

$$\bigvee_{n,m\geq 0} (F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n\oplus\mathbb{C}^n} \wedge^L F_{U(n),\mathbb{C}^n} S^{\mathbb{C}^n\oplus\mathbb{C}^n}) \;\cong\; \bigvee_{n,m\geq 0} F_{U(n)\times U(m),\mathbb{C}^n\times\mathbb{C}^m} S^{\mathbb{C}^{n+m}\oplus\mathbb{C}^{n+m}} \ .$$

So the commutativity, in the global homotopy category, of the multiplicativity diagram

$$\begin{tikzcd} \mathbf{MP}^{\mathrm{re}} \wedge^L \mathbf{MP}^{\mathrm{re}} \arrow[r, "\mu_{\mathbf{MP}^{\mathrm{re}}}"] \arrow[d, "\psi\wedge^L\psi"'] & \mathbf{MP}^{\mathrm{re}} \arrow[d, "\psi"] \\ E \wedge^L E \arrow[r, "\mu_E"'] & E \end{tikzcd}$$

can be checked by evaluation on the classes $\sigma_n^{\mathrm{pre}} \times \sigma_m^{\mathrm{pre}}$ in $(\mathbf{MP}^{\mathrm{re}} \wedge^L \mathbf{MP}^{\mathrm{re}})_{U(n)\times U(m)}(S^{\mathbb{C}^{n+m}})$ for all $n, m \geq 0$. This verfication is straighforward from the properties of ψ and the preorientation properties of τ and σ^{pre}:

$$\begin{aligned} (\mu_E \circ (\psi \wedge^L \psi))_*(\sigma_n^{\mathrm{pre}} \times \sigma_m^{\mathrm{pre}}) \ &= \ (\mu_E)_*(\psi_*(\sigma_n^{\mathrm{pre}}) \times \psi_*(\sigma_m^{\mathrm{pre}})) \\ &= \ (\mu_E)_*(\tau_{U(n),\mathbb{C}^n} \times \tau_{U(n),\mathbb{C}^n}) \\ &= \ \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\tau_{U(n+m),\mathbb{C}^{n+m}}) \\ &= \ \mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\psi_*(\sigma_{n+m}^{\mathrm{pre}})) \\ &= \ \psi_*(\mathrm{res}^{U(n+m)}_{U(n)\times U(m)}(\sigma_{n+m}^{\mathrm{pre}})) \\ &= \ \psi_*(\sigma_n^{\mathrm{pre}} \times \sigma_m^{\mathrm{pre}}) \ = \ (\psi \circ \mu_{\mathbf{MP}^{\mathrm{re}}})_*(\sigma_n^{\mathrm{pre}} \times \sigma_m^{\mathrm{pre}}) \ . \end{aligned}$$

□

The next theorem is the main result of this section, and makes precise that the global bordism spectrum $\mathbf{MP}$ is the universal complex oriented global ring spectrum.

Theorem 8.18. *For every global ring spectrum E the map*

$$\text{global ring}(\mathbf{MP}, E) \ \longrightarrow \ \mathrm{Or}(E) \ , \quad \psi \ \longmapsto \ \psi(\sigma) = \{\psi_*(\sigma_{G,V})\}$$

that evaluates at the preferred orientation of $\mathbf{MP}$ *is bijective.*

Construction 8.19 (Localization theorem)**.** We review tom Dieck's *localization theorem* that expresses the geometric fixed point homotopy groups of a globally oriented ring spectrum as the localization of the equivariant homotopy groups at the multiplicative subset of all non-trivial Euler classes. Tom Dieck originally proved this for equivariant bordism as Theorem 3.1 in [**94**], and this amounts to considering the universal example $\mathbf{MP}$ of a globally oriented ring spectrum.

We let E be a globally oriented ring spectrum with periodicity classes $\sigma_{G,V} \in E_G(S^V)$. For a compact Lie group G we consider the set

$$S_G \ = \ \{e(V) \mid V \text{ is a complex } G\text{-representation, } V^G = 0 \ \} \ \subset \ \pi_0^G(E)$$

of Euler classes of complex G-representations with trivial fixed points. This is a multiplicative subset of the commutative ring $\pi_0^G(E)$ by the multiplicativity property of Euler classes.

If $V^G = 0$, then

$$\Phi^G(e(V)) \ = \ \Phi^G(\sigma_{G,V}) \ \in \ \Phi_0^G(E) \ ,$$

and so

$$\Phi^G(e(V)) \cdot \Phi^G(\sigma_{G,V}^{-1}) \ = \ 1 \ ;$$

we conclude that the geometric fixed point homomorphism

$$\Phi^G \ : \ \pi_0^G(E) \ \longrightarrow \ \Phi_0^G(E)$$

(see (6.3) of Chapter III) takes all Euler classes in the set S_G to units in the commutative ring $\Phi_0^G(E)$. Since Φ^G is a ring homomorphism, there is a unique extension to a ring homomorphism

$$\hat{\Phi}^G \ : \ \pi_0^G(E)[S_G^{-1}] \ \longrightarrow \ \Phi_0^G(E)$$

from the localization of $\pi_0^G(E)$ at the multiplicative subset S_G.

Theorem 8.20. *Let E be a globally oriented ring spectrum. Then for every compact Lie group G the extension*

$$\hat{\Phi}^G \ : \ \pi_0^G(E)[S_G^{-1}] \ \longrightarrow \ \Phi_0^G(E)$$

of the geometric fixed point homomorphism is an isomorphism of rings.

Remark 8.21. We offer some comments before we prove Theorem 8.20. Every G-representation is isomorphic to a finite sum of irreducible representations; the multiplicativity property of Euler classes then shows that every element of the set S_G is a finite product of Euler classes of non-trivial irreducible G-representations. So the localization of $\pi_0^G(E)$ at the multiplicative subset S_G is effectively inverting the Euler classes of all non-trivial irreducible G-representations.

If G is finite, then there are only finitely many isomorphism classes of irreducible G-representations λ, so the localization $\pi_0^G(E)[S_G^{-1}]$ can be obtained by inverting a single element of $\pi_0^G(E)$, for example the product

$$\prod_{\lambda \neq 1} e(\lambda) \ .$$

Every non-trivial irreducible representation is a summand of the reduced regular representation $\bar{\rho}_G$, so for finite G, the localization $\pi_0^G(E)[S_G^{-1}]$ can be obtained by inverting the Euler class $e(\bar{\rho}_G)$.

The geometric fixed point homomorphism Φ^G annihilates transfers from all proper subgroups of G (see Proposition III.6.5). So Theorem 8.20 implies that such a transfer is annihilated by the Euler class of some G-representation with trivial fixed points. This consequence can also be seen directly as follows. For every proper subgroup H of G there is G-representation V with $V^G = 0$ but $V^H \neq 0$. So $e(V) \in S_G$, but $\mathrm{res}^G_H(e(V)) = e(\mathrm{res}^G_H(V)) = 0$. For all $x \in \pi_0^H(E)$ we then have

$$\mathrm{tr}_H^G(x) \cdot e(V) \ = \ \mathrm{tr}_H^G(x \cdot \mathrm{res}^G_H(e(V))) \ = \ 0$$

by reciprocity.

Proof of Theorem 8.20. The localization theorem is a consequence of the interpretation of the geometric fixed point homotopy groups as the equivariant homotopy groups of the G-spectrum $\tilde{E}\mathcal{P}_G \wedge E\langle G\rangle$ (see Proposition III.6.14) and the complex periodicity. Here $\tilde{E}\mathcal{P}_G$ is the unreduced suspension of a universal space for the family $\mathcal{P}_G$ of proper subgroups of G.

For the rest of the proof we write $\mathcal{U}^\perp = (\mathcal{U}_G^{\mathbb{C}})^\perp = \mathcal{U}_G^{\mathbb{C}} - (\mathcal{U}_G^{\mathbb{C}})^G$ for the orthogonal complement of the G-fixed points of the complexified G-universe $\mathcal{U}_G^{\mathbb{C}}$. As we explained in Example III.6.12, the unit sphere $S(\mathcal{U}^\perp)$ is a model for $\tilde{E}\mathcal{P}_G$, and so we can take

$$\tilde{E}\mathcal{P}_G \ = \ S(\mathcal{U}^\perp)^\diamond \ = \ S^{\mathcal{U}^\perp} \ .$$

We choose an exhausting sequence of finite-dimensional complex G-subrepresentations

$$0 = V_0 \ \subset \ V_1 \ \subset \ \dots \ \subset \ V_m \ \subset \ \dots$$

of $\mathcal{U}^\perp$. Then the inclusions induce G-cofibrations $S^{V_{m-1}} \longrightarrow S^{V_m}$ and hence h-cofibrations of unitary G-spectra

$$S^{V_{m-1}} \wedge E\langle G\rangle \ \longrightarrow \ S^{V_m} \wedge E\langle G\rangle \ .$$

The unitary G-spectrum $\tilde{E}\mathcal{P}_G \wedge E\langle G\rangle = S^{\mathcal{U}^\perp} \wedge E\langle G\rangle$ is a colimit of this sequence of h-cofibrations,

$$\pi_k^G(\tilde{E}\mathcal{P}_G \wedge E\langle G\rangle)$$

is a colimit of the sequence of groups $\pi_k^G(S^{V_m} \wedge E\langle G\rangle)$.

Multiplication by the periodicity element $\sigma_{G,V_m} \in E_G(S^{V_m})$ is an isomorphism

$$- \cdot \sigma_{G,V_m} \ : \ \pi_k^G(S^{V_m} \wedge E\langle G\rangle) \ \cong \ \pi_k^G(E)$$

and we claim that the following diagram commutes:

$$\begin{tikzcd}
\pi_k^G(S^{V_{m-1}} \wedge E\langle G\rangle) \arrow[r] \arrow[d, "\cdot\sigma_{G,V_{m-1}}"', "\cong"] & \pi_k^G(S^{V^{(m)}} \wedge S^{V_{m-1}} \wedge E\langle G\rangle) \arrow[r, "\cong"] \arrow[d, "\cong"', "\cdot\sigma_{G,V_{m-1}}"] & \pi_k^G(S^{V_m} \wedge E\langle G\rangle) \arrow[d, "\cong"', "\cdot\sigma_{G,V_m}"] \\
\pi_k^G(E\langle G\rangle) \arrow[r] \arrow[rr, bend right, "\cdot e(V^{(m)})"'] & \pi_k^G(S^{V^{(m)}} \wedge E\langle G\rangle) \arrow[r, "\cdot\sigma_{G,V^{(m)}}"] & \pi_k^G(E\langle G\rangle)
\end{tikzcd}$$

Here $V^{(m)} = V_m - V_{m-1}$ is the orthogonal complement of V_{m-1} in V_m, and we have used the identification $S^{V^{(m)}} \wedge S^{V_{m-1}} \cong S^{V^{(m)} \oplus V_{m-1}} = S^{V_m}$ and the multiplicativity property

$$\sigma_{G,V_{m-1}} \cdot \sigma_{G,V^{(m)}} \;=\; \sigma_{G,V_{m-1}\oplus V^{(m)}} \;=\; \sigma_{G,V_m} \ .$$

The two left horizontal morphisms are induced by the inclusion $S^0 \longrightarrow S^{V^{(m)}}$ and so the lower horizontal composite is multiplication by the Euler class of $V^{(m)}$. We conclude that the group $\pi_k^G(\tilde{E}\mathcal{P}_G \wedge E)$ is a colimit of the sequence

$$\pi_k^G(E) \xrightarrow{\cdot e(V^{(1)})} \pi_k^G(E) \xrightarrow{\cdot e(V^{(2)})} \dots \xrightarrow{\cdot e(V^{(m-1)})} \pi_k^G(E) \xrightarrow{\cdot e(V^{(m)})} \dots \ .$$

Each of the Euler classes $e(V^{(m)})$ belongs to S_G. Every non-trivial irreducible G-representation λ occurs infinitely often in $\mathcal{U}^{\perp}$, so $e(\lambda)$ is a factor of $e(V^{(m)})$ for infinitely many m, so a colimit of the last sequence is the localization $\pi_k^G(E)[S_G^{-1}]$. □

Example 8.22 (Geometric fixed points of $\mathbf{KU}$)**.** The Bott classes $\beta_V \in \mathbf{KU}_G(S^V)$, for complex G-representations V, form a global orientation of the periodic global K-theory spectrum $\mathbf{KU}$. From the very definition of β_V it follows that the isomorphism $\pi_0^G(\mathbf{KU}) \cong \mathbf{R}(G)$ takes the Euler class

$$e(V) \;=\; i^*(\beta_V) \;\in \pi_0^G(\mathbf{KU})$$

to the element

$$\lambda(V) \;=\; \sum_{k\geq 0}(-1)^k \cdot [\Lambda^k(V)] \;\in \mathbf{R}(G) \ ;$$

here $\Lambda^k(V)$ is the k-th exterior power of the representation V. The localization theorem 8.20 lets us conclude that the geometric G-fixed point homotopy groups of $\mathbf{KU}$ vanish in odd degrees, and in even degrees are isomorphic to the localozation

$$\mathbf{R}(G)[S_G^{-1}]$$

for the multiplicative subset $S_G = \{\lambda(V) \in \mathbf{R}(G) \mid V^G \neq 0\}$.

We make this explicit in the smallest non-trivial example, for $G = C_2$ the cyclic group of order two. Then $\mathbf{R}(C_2) = \mathbb{Z}[x]/(x^2-1)$ where $x = [\sigma]$ is the class of the sign representation. The multiplicative subset S_{C_2} consists of the powers of the Euler class $\lambda(\sigma) = 1 - x$. Because

$$(1-x)^2 \;=\; 2(1-x) \qquad \text{and} \qquad (1-x)(1+x) \;=\; 0$$

in $\mathbf{R}(C_2)$, inverting $1-x$ also inverts 2, and it sets x equal to -1. So

$$\Phi_0^{C_2}(\mathbf{KU}) \;\cong\; \mathbf{R}(C_2)[(1-x)^{-1}] \;\cong\; \mathbb{Z}[1/2] \ .$$

APPENDIX A

Miscellaneous tools

1. Model structures for equivariant spaces

We refer to the standard model structure on the category of compactly generated weak Hausdorff spaces as described for example in [**43**, Thm. 2.4.25]. In this model structure, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of generalized CW-complexes, i.e., cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions.

We let M be a topological monoid, i.e., a compactly generated weak Hausdorff space equipped with an associative and unital multiplication

$$\mu \ : \ M \times M \ \longrightarrow \ M$$

that is continuous with respect to the compactly generated product topology [finer or weaker?] An M-space is then a compactly generated weak Hausdorff space X equipped with an associative and unital action

$$M \times X \ \longrightarrow \ X$$

that is continuous with respect to the compactly generated product topology.

We let N be a submonoid of M and denote by

$$X^N \ = \ \{x \in X \mid nx = x \text{ for all } n \in N\} \tag{1.1}$$

the subspace of N-fixed points. For an individual element $n \in N$ the n-fixed subspace $\{x \in X \mid nx = x\}$ is the preimage of the diagonal under the continuous map $(\mathrm{Id}, n \cdot -) : X \longrightarrow X \times X$, so it is a closed subspace of X by the weak Hausdorff condition. The N-fixed points X^N are then closed in X as an intersection of closed subsets. This means that the subspace topology on X^N is again compactly generated weak Hausdorff and so

$$X^N \xrightarrow{\text{incl}} X \rightrightarrows \mathrm{map}(N, X)$$

is an equalizer diagram in the category of compactly generated weak Hausdorff spaces, where the two maps on the right are adjoint to the projection $N \times X \longrightarrow X$ respectively the composite

$$N \times X \xrightarrow{\text{incl} \times X} M \times X \xrightarrow{\alpha} X \ .$$

Definition 1.2. A submonoid N of a topological monoid M is *biclosed* if the following two conditions hold:

(i) the set N is closed in the topology of M, and
(ii) if $m \in M$ and $n \in N$ satisfy $mn \in N$, then $m \in N$.

Remark 1.3. Eventually we want to define classes of weak equivalences for M-spaces by testing on the fixed point spaces X^N for collections of submonoids. For this purpose it is no loss of generality to restrict to biclosed submonoids, as we now explain. First we observe that any intersection of biclosed submonoids of a topological monoid is again biclosed. So an arbitrary submonoid N of M has a *biclosure* $\bar{N}$, defined as the intersection of all biclosed submonoid of M that contain N, which is the smallest biclosed submonoid of M that contains N.

We will now argue that for every M-space X the N-fixed points agree with the $\bar{N}$-fixed points:

$$X^N = X^{\bar{N}} \ .$$

The *stabilizer* of a point $x \in X$ is the submonoid

$$\mathrm{stab}_M(x) \ = \ \{m \in M \mid mx = x\} \ .$$

The stabilizer is also the preimage of $\{x\}$ under the continuous map $- \cdot x : M \longrightarrow X$; since singletons are closed in compactly generated spaces, the stabilizer is a closed subset of M. Moreover, if $m, n \in M$ are such that n and mn stabilize x, then

$$mx \ = \ m(nx) \ = \ (mn)x \ = \ x \ ,$$

i.e., $m \in \mathrm{stab}_M(x)$. So the point stabilizer of M-spaces are always biclosed submonoid.

More generally, for every subset $S \subseteq X$ the stabilizer

$$\mathrm{stab}_M(S) \ = \ \{m \in M \mid mx = x \text{ for all } x \in S\}$$

is the intersection of the stabilizers of all points in S, so it is another biclosed submonoid of M. If N is a submonoid of M, not necessarily biclosed, then

$$N \ \subseteq \ \mathrm{stab}_M(X^N)$$

and the stabilizer monoid on the right is biclosed. So the biclosure $\bar{N}$ of N is contained in $\mathrm{stab}_M(X^N)$, and hence $X^N = X^{\bar{N}}$.

Example 1.4. For example, every closed subgroup G of a topological monoid M is biclosed, because the assumptions $g \in G$ and $mg \in G$ imply $m = (mg)g^{-1} \in G$.

Another example relevant to global homotopy theory is the topological monoid $\mathcal{L} = \mathbf{L}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ of linear isometric selfembeddings of $\mathbb{R}^\infty$. Here $\mathbb{R}^\infty$ has the colimit topology from the linear topology on its finite dimensional sub-vector spaces, and the topology on $\mathcal{L}$ is as a subspace of the mapping space $\mathrm{map}(\mathbb{R}^\infty, \mathbb{R}^\infty)$, using that every linear self-map of $\mathbb{R}^\infty$ is continuous in the colimit topology. Since $\mathcal{L}$ is closed in $\mathrm{map}(\mathbb{R}^\infty, \mathbb{R}^\infty)$ [justify], the subspace topology is indeed compactly generated (and automatically weakly Hausdorff). If V is a finite dimensional inner product space, then the space $\mathbf{L}(V, \mathbb{R}^\infty)$ of linear isometric embeddings (again with the subspace topology of $\mathrm{map}(V, \mathbb{R}^\infty)$) is an $\mathcal{L}$-space under composition of isometries. For every linear isometric embedding $\alpha : V \longrightarrow \mathbb{R}^\infty$ the stabilizer

$$\mathrm{stab}_{\mathcal{L}}(\alpha) \ = \ \{\varphi \in \mathcal{L} \mid \varphi \circ \alpha = \alpha\}$$

is thus a biclosed submonoid of $\mathcal{L}$, compare the previous remark.

Now we show that the functor sending an M-space X to the set of N-fixed points is representable by an 'orbit space' M/N. We denote by M/N a coequalizer in the category of M-spaces

$$M \times N \ \underset{\mu'}{\overset{\mathrm{proj}}{\rightrightarrows}} \ M \ \xrightarrow{\quad q \quad} \ M/N \ , \tag{1.5}$$

where $\mu = \mu \circ (M \times \mathrm{incl})$. Since the forgetful functor creates colimits, we could equivalently take a coequalizer in the underlying category of compactly generated weak Hausdorff spaces, and that inherits a unique M-action that makes the projection $q : M \longrightarrow M/N$ a homomorphism of M-spaces.

Now we let K be any compactly generated weak Hausdorff space. Since product with K is a left adjoint, the diagram

$$M \times N \times K \ \underset{\mu' \times K}{\overset{\mathrm{proj} \times K}{\rightrightarrows}} \ M \times K \ \xrightarrow{q \times K} \ M/N \times K \ .$$

is another coequalizer of M-spaces, where M acts trivially on K. So for every M-space X, precomposition with $q \times K$ is a bijection from $M\mathbf{U}(M/N \times K, X)$ to the equalizer of the two maps

$$M\mathbf{U}(M \times K, X) \underset{M\mathbf{U}(\mu' \times K, X)}{\overset{M\mathbf{U}(\mathrm{proj} \times K, X)}{\rightrightarrows}} M\mathbf{U}(M \times N \times K, X) \ .$$

The free-forgetful adjunction and the adjunction between $N \times -$ and $\mathrm{map}(N, -)$ identifies this with the set of those continuous maps $f : K \longrightarrow X$ that are equalized by the two right maps in the equalizer diagram (1.1). Since the inclusion X^N is an equalizer, we have shows altogether that evaluation at the class of the identity element is a bijection

$$M\mathbf{U}(M/N \times K, X) \ \longrightarrow \ \mathbf{U}(K, X^N) \tag{1.6}$$

from the set of continuous M-maps from $M/N \times K$ to X to the set of continuous maps from K to the N-fixed points of X.

Remark 1.7. One more time we do not lose any generality by restricting to biclosed submonoids. Indeed, the proof of the adjunction (1.6) did not use any property of the submonoid N. Since $X^N = X^{\bar{N}}$ for every M-space X, then M-spaces M/N and $M/\bar{N}$ represent the same functor, and so they are isomorphic. In other words, the 'orbit space' M/N only depends on the biclosure $\bar{N}$ of the submonoid N.

One should beware that even for biclosed submonoid N, the 'orbit space' M/N may not be what one expects at first sight. To construct M/N, one could start from the equivalence relation $\sim_N$ on M generated by $m \sim_N mn$ for all $m \in M$ and $n \in N$. If N is biclosed, then it is the equivalence class of 1, but the other equivalence classes may still be hard to identify.

Since M is compactly generated, the quotient topology on the set $M/\sim_N$ of equivalence classes will automatically be compactly generated, but not necessarily weakly Hausdorff. So in a second step one has to apply the left adjoint to the inclusion of compactly generated weak Hausdorff spaces into all compactly generated spaces, but this step can change the topology and may even alter the underlying set by identifying different equivalence classes.

Example 1.8. Here are two examples of particular relevance for us where we can describe an 'orbit space' more explicitly. If G is a compact topological group and H a closed subgroup, then the set G/H of left cosets endowed with the quotient topology is compact, so it is a coequalizer in then sense of (1.5). So in this situation the orbit space notation is unambiguous.

For the monoid $\mathcal{L}$ of linear isometric selfembeddings of $\mathbb{R}^\infty$ and a finite dimensional inner product space V, the $\mathcal{L}$-space $\mathbf{L}(V, \mathbb{R}^\infty)$ ought to be a quotient of $\mathcal{L}$ by the stabilizer of any particular linear isometric embedding $\alpha : V \longrightarrow \mathbb{R}^\infty$. In fact, $\mathbf{L}(V, \mathbb{R}^\infty)$ is transitive as an $\mathcal{L}$-space in the strong sense that any two points are related by the action of an invertible element in $\mathcal{L}$. So the stabilizers of any two points in $\mathbf{L}(V, \mathbb{R}^\infty)$ are conjugate in $\mathcal{L}$.

Lemma 1.9. *For every topological monoid M, every submonoid N and every compact space K the M-space $M/N \times K$ is finite with respect to sequences of closed embeddings of M-spaces.*

PROOF. We let

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$$

be a sequence of morphisms of M-spaces that are closed embeddings of underlying spaces, and

$$f \ : \ M/N \times K \ \longrightarrow \ \mathrm{colim}_{i \geq 0} X_i$$

morphism of M-spaces. The composite

$$K \xrightarrow{(N,-)} M/N \times K \xrightarrow{f} \mathrm{colim}_{i \geq 0} X_i$$

factors through a continuous map $g : K \longrightarrow X_i$ for some $i \geq 0$ by [**43**, Prop. 2.4.2] (this uses that singletons in compactly generated weak Hausdorff spaces are closed). Sequential colimits of compactly generated weak

Hausdorff spaces along closed embeddings are given by the colimits of underlying sequence of sets, so the canonical map $X_i \longrightarrow \mathrm{colim}_{i\geq 0} X_i$ is injective. Since the map $f(N,-) : K \longrightarrow \mathrm{colim}_{i\geq 0} X_i$ lands in the N-fixed points of the colimit, the factorization g lands in the N-fixed points, so it extends uniquely to a morphism of M-spaces $\tilde{g} : M/N \times K \longrightarrow X_i$. Since morphisms out of $M/N \times K$ are determined by their restriction to K, the morphism $\tilde{g}$ is the desired factorization of the original morphism f. □

Now we let $\mathcal{C}$ be a collection of biclosed submonoids of M that is stable under conjugacy by invertible elements of M. We call a morphism $f : X \longrightarrow Y$ of M-spaces a *$\mathcal{C}$-equivalence* (respectively *$\mathcal{C}$-fibration*) if the restriction $f^N : X^N \longrightarrow Y^N$ to N-fixed points is a weak equivalence (respectively Serre fibration) of spaces for all closed submonoids N of M that belong to the collection $\mathcal{C}$. A *$\mathcal{C}$-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously $\mathcal{C}$-equivalences and $\mathcal{C}$-fibrations. The resulting '$\mathcal{C}$-projective model structure' is well known in the case when M is a group and $\mathcal{C}$ is a collection of closed subgroups, and the proof for monoids is not much different and fairly standard. However, we do not know a reference in the monoid case, so we provide the proof.

Proposition 1.10. *Let M be a topological monoid and $\mathcal{C}$ a collection of biclosed submonoids of M. Then the $\mathcal{C}$-equivalences, $\mathcal{C}$-cofibrations and $\mathcal{C}$-fibrations form a model structure, the* $\mathcal{C}$-projective model structure *on the category of M-spaces. This model structure is proper, cofibrantly generated and topological.*

Proof. We refer the reader to [**30**, 3.3] for the numbering of the model category axioms. The forgetful functor from the category of M-spaces to the category of compactly generated weak Hausdorff spaces has a left adjoint free functor $M \times -$ and a right adjoint cofree functor $\mathrm{map}(M,-)$; so the category of M-spaces is complete and cocomplete and all limits and colimits are created in the underlying category of compactly generated weak Hausdorff spaces.

Model category axioms MC2 (saturation) and MC3 (closure properties under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of $\mathcal{C}$-cofibrations. The proof of the remaining axioms uses Quillen's small object argument. We recall the 'standard' set of generating cofibrations and generating acyclic cofibrations. In the category of (non-equivariant) spaces, the set $\{S^{k-1} \longrightarrow D^k\}_{k\geq 0}$ of inclusions of spheres into discs detects Serre fibrations that are simultaneously weak equivalences (where S^{-1} is the empty set). By adjointness (i.e., the bijection (1.6)), the set

$$I_{\mathcal{C}} \;=\; \{M/N \times S^{k-1} \;\longrightarrow\; M/N \times D^k\}_{k\geq 0, N\in\mathcal{C}} \tag{1.11}$$

then detects acyclic fibrations in the $\mathcal{C}$-projective model structure on M-spaces. Similarly, the set of inclusions $\{D^k \times \{0\} \longrightarrow D^k \times [0,1]\}_{k\geq 0}$ detects Serre fibrations; so by adjointness, the set

$$J_{\mathcal{C}} \;=\; \{M/N \times D^k \times \{0\} \;\longrightarrow\; M/N \times D^k \times [0,1]\}_{k\geq 0, N\in\mathcal{C}} \tag{1.12}$$

detects fibrations in the $\mathcal{C}$-projective model structure on M-spaces.

All morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are closed embeddings, and this property is preserved by coproducts, cobase change and sequential colimits in the category of M-spaces. Lemma 1.9 guarantees that sources and targets of all morphisms in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ are finite (sometimes called 'finitely presented') with respect to sequences of closed embeddings of M-spaces. In particular, the sources of all these morphisms are finite with respect to sequences of morphisms in $I_{\mathcal{C}}$-cell and $J_{\mathcal{C}}$-cell.

Now we can prove the factorization axiom MC5. Every morphism in $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$ is a $\mathcal{C}$-cofibration by adjointness. Hence every $I_{\mathcal{C}}$-cofibration or $J_{\mathcal{C}}$-cofibration is a $\mathcal{C}$-cofibration of M-spaces. Hence the small object argument applied to the set $I_{\mathcal{C}}$ gives a (functorial) factorization of any morphism of M-spaces as a $\mathcal{C}$-cofibration followed by a morphism with the right lifting property with respect to $I_{\mathcal{C}}$. Since $I_{\mathcal{C}}$ detects the $\mathcal{C}$-acyclic $\mathcal{C}$-fibrations, this provides the factorizations as cofibrations followed by acyclic fibrations.

For the other half of the factorization axiom MC5 we apply the small object argument to the set $J_{\mathcal{C}}$; we obtain a (functorial) factorization of any morphism of M-spaces as a $J_{\mathcal{C}}$-cell complex followed by a morphism with the right lifting property with respect to $J_{\mathcal{C}}$. Since $J_{\mathcal{C}}$ detects the $\mathcal{C}$-fibrations, it remains to show that every $J_{\mathcal{C}}$-cell complex is a $\mathcal{C}$-equivalence. To this end we observe that the morphisms in $J_{\mathcal{C}}$ are inclusions of deformation retracts internal to the category of M-spaces. This property is inherited by coproducts and cobase changes, so every morphisms obtained by cobase changes of coproducts of morphisms in $J_{\mathcal{C}}$ is a homotopy equivalence of M-spaces, hence also a $\mathcal{C}$-equivalence. We also need to pass to sequential colimits, which is fine because $J_{\mathcal{C}}$-cell complexes are closed embeddings, and taking N-fixed points commutes with sequential colimits over closed embeddings.

It remains to prove the other half of MC4, i.e., that every $\mathcal{C}$-acyclic $\mathcal{C}$-cofibration $j : A \longrightarrow B$ has the left lifting property with respect to $\mathcal{C}$-fibrations. In other words, we need to show that the $\mathcal{C}$-acyclic $\mathcal{C}$-cofibrations are contained in the $J_{\mathcal{C}}$-cofibrations. The small object argument provides a factorization

$$A \xrightarrow{j'} W \xrightarrow{q} B$$

as a $J_{\mathcal{C}}$-cell complex j' followed by a $\mathcal{C}$-fibration q. In addition, q is a $\mathcal{C}$-equivalence since f is. Since f is a $\mathcal{C}$-cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r, "j'"] \arrow[d, "f"'] & W \arrow[d, "q", "\sim"'] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus f is a retract of a morphism q that has the left lifting property for $\mathcal{C}$-fibrations. So f itself has the left lifting for $\mathcal{C}$-fibrations.

[proper] It remains to show that the model structure is topological. The sphere inclusions $i_k : S^{k-1} \longrightarrow D^k$ are closed under pushout product, in the sense that $i_k \Box i_l$ is isomorphic to i_{k+l}. So pushout product with i_k preserves the set $I_{\mathcal{C}}$ of generating $\mathcal{C}$-cofibrations. This implies that the pushout product of a general cofibration of spaces with a $\mathcal{C}$-cofibration is again a $\mathcal{C}$-cofibration. The pushout product of a sphere inclusion i_k with an inclusion $D^n \times \{0\} \longrightarrow D^n \times [0,1]$ is a homotopy equivalence. So the pushout product of i_k with a morphism in $J_{\mathcal{C}}$ is a homotopy equivalence. Similarly, the pushout product of an inclusion $D^n \times \{0\} \longrightarrow D^n \times [0,1]$ with a morphism in $I_{\mathcal{C}}$ is a homotopy equivalence. So pushout product of a generating cofibration with a generating acyclic cofibration is an $\mathcal{C}$-acyclic $\mathcal{C}$-cofibration. This implies that the pushout product axiom in general. $\square$

Definition 1.13. Let M be a topological monoid and $\mathcal{C}$ a collection of biclosed submonoids of M. A *universal space* for the collection $\mathcal{C}$ is an M-space E with the following properties:

(i) for every submonoid $N \in \mathcal{C}$ the fixed point space E^N is weakly contractible;
(ii) E is M-homotopy equivalent to a $\mathcal{C}$-cofibrant M-space.

Any cofibrant replacement of the one-point M-space in the $\mathcal{C}$-projective model structure is a universal space for the family $\mathcal{C}$, so universal spaces exist for any collection. Moreover, the following proposition shows that any two universal spaces for the same collection are M-equivariantly homotopy equivalent.

Proposition 1.14. *Let M be a topological monoid, $\mathcal{C}$ a collection of biclosed submonoids of M, and E a universal M-space for the collection $\mathcal{C}$.*

(i) *Every $\mathcal{C}$-cofibrant M-space admits a homomorphism to E, and any two such morphisms are homotopic as M-maps.*
(ii) *If E' is another universal M-space then every homomorphism from E' to E is an M-equivariant homotopy equivalence.*

Now we are going to formulate a version of Elmendorf's theorem [**31**] for the homotopy theory of M-spaces relative to a collection $\mathcal{C}$ of biclosed submonoids. Again, this is well known for topological groups and the proof for topological monoids is essentially the same. However, we have do not know a reference of the generalization of Elmendorf's theorem to this context, so we provide a proof.

Construction 1.15. Associated to the collection $\mathcal{C}$ of submonoids of M we define the topological *orbit category* $\mathbf{O}_{M,\mathcal{C}}$ as the full topological subcategory of the category of M-spaces with objects the orbits M/N for $N \in \mathcal{C}$. More precisely, we let $\mathcal{C}$ be the object set of $\mathbf{O}_{M,\mathcal{C}}$, and for $N, P \in \mathcal{C}$ the space is

$$\mathbf{O}_{M,\mathcal{C}}(N,P) \;=\; M\mathbf{U}(M/N, M/P) \;\cong\; (M/P)^N \;,$$

where the bijection on the right hand side is by evaluation at the image of the unit element in M/N. The topology of this space is specified by the right hand side, i.e., the subspace topology of the N-fixed points of the orbit space M/P. Composition is given by composition of M-maps.

For every M-space X the various fixed point subspaces assemble into a continuous functor $\Phi : \mathbf{O}_{M,\mathcal{C}}^{\mathrm{op}} \longrightarrow \mathbf{U}$ on the orbit category via

$$\Phi(X)(N) \;=\; X^N \;\cong\; M\mathbf{U}(M/N, X) \;,$$

with subspace topology of X. The functoriality in N as an object of $\mathbf{O}_{M,\mathcal{C}}$ comes from bijection (1.6) and composition of M-maps between the orbit spaces.

For every small topological category J with discrete object set the category $J\mathbf{U}$ of continuous functors from J to spaces has a well-known 'projective' model structure (see for example [**60**, VI Thm. 5.2]) in which the weak equivalence and fibrations are those natural transformations that are weak equivalences respectively Serre fibrations at every object.

In the case of M-space and a collection $\mathcal{C}$ of biclosed submonoid the fixed point functor

$$\Phi \;:\; M\mathbf{U} \;\longrightarrow\; \mathbf{O}_{M,\mathcal{C}}\mathbf{U}$$

has a left adjoint Λ, with value at an $\mathbf{O}_{M,\mathcal{C}}$-space F given by a coend of the functor

$$\Lambda(F) \;=\; \int_{N\in\mathcal{C}} M/N \times F(N) \;,$$

i.e., a coequalizer, in the category of M-spaces, of the two morphisms

$$\textstyle\coprod_{N,N'\in\mathcal{C}} M/N \times \mathbf{O}_{M,\mathcal{C}}(N,N') \times F(N') \rightrightarrows \coprod_{N\in\mathcal{C}} M/N \times F(N) \;\; .$$

All we will need to know about the left adjoint is that for all $N \in \mathcal{C}$ it takes the representable $\mathbf{O}_{M,\mathcal{C}}$-space $\mathbf{O}_{M,\mathcal{C}}(-,N) = \Phi(M/N)$ to M/N. Indeed, the counit $\epsilon_{M/N} : \Lambda(\Phi(M/N)) \longrightarrow M/N$ induces a bijection of morphism sets

$$\begin{aligned} M\mathbf{U}(\Lambda(\Phi(M/N), X) \;&\cong\; \mathbf{O}_{M,\mathcal{C}}\mathbf{U}(\Phi(M/N), \Phi(X)) \;=\; \mathbf{O}_{M,\mathcal{C}}\mathbf{U}(\mathbf{O}_{M,\mathcal{C}}(-,N), \Phi(X)) \\ &\cong\; \Phi(X)(M/N) \;=\; M\mathbf{U}(M/N, X) \;. \end{aligned}$$

So the counit $\epsilon_{M/N} : \Lambda(\Phi(M/N)) \longrightarrow M/N$ is an isomorphism of M-spaces.

The projective $\mathcal{C}$-model structure is defined so that the fixed point functor Φ preserves and detects weak equivalence and fibrations. So (Λ, Φ) is a Quillen adjoint functor pair.

Proposition 1.16. *Let M be a topological monoid and $\mathcal{C}$ a collection of biclosed submonoids of M.*

(i) *For every cofibrant $\mathbf{O}_{M,\mathcal{C}}$-space F the adjunction unit $F \longrightarrow \Phi(\Lambda F)$ is an isomorphism.*
(ii) *The adjoint functor pair*

$$M\mathbf{U} \;\; \underset{\Phi}{\overset{\Lambda}{\rightleftarrows}} \;\; \mathbf{O}_{M,\mathcal{C}}$$

is a Quillen equivalence with respect to the $\mathcal{C}$-projective model structure on M-spaces and the projective model structure for $\mathbf{O}_{M,\mathcal{C}}$-spaces.

Proof. (i) We let $\mathcal{G}$ denote the class of $\mathbf{O}_{M,\mathcal{C}}$-spaces for which the adjunction unit is an isomorphism. We show the following property: For every index set I, every I-indexed family N_i of monoids in $\mathcal{C}$, all numbers $n_i \geq 0$ and every pushout square $\mathbf{O}_{M,\mathcal{C}}$-spaces

$$\begin{tikzcd} \coprod_{i\in I} \mathbf{O}_{M,\mathcal{C}}(-,N_i)\times S^{n_i-1} \arrow[r] \arrow[d] & \coprod_{i\in I} \mathbf{O}_{M,\mathcal{C}}(-,N_i)\times D^{n_i} \arrow[d] \\ F \arrow[r] & G \end{tikzcd} \tag{1.17}$$

such that F belongs to $\mathcal{G}$, the $\mathbf{O}_{M,\mathcal{C}}$-space G also belongs to $\mathcal{G}$.

As a left adjoint, Λ preserves pushout and coproducts. For every space K the functor $-\times K$ is a left adjoint, so it commutes with colimits and coends. So Λ also commutes with products with spaces. Thus Λ takes the original square to a pushout square of M-spaces:

$$\begin{tikzcd} \coprod_{i\in I} M/N_i\times S^{n_i-1} \arrow[r] \arrow[d] & \coprod_{i\in I} M/N_i\times D^{n_i} \arrow[d] \\ \Lambda F \arrow[r] & \Lambda G \end{tikzcd}$$

The upper horizontal morphisms in this square is a closed embedding. For every biclosed submonoid P of M the P-fixed point functor $(-)^P$ commutes with disjoint unions, products with spaces and pushouts along closed embeddings. So the square

$$\begin{tikzcd} \coprod_{i\in I} (M/N_i)^P\times S^{n_i-1} \arrow[r] \arrow[d] & \coprod_{i\in I} (M/N_i)^P\times D^{n_i} \arrow[d] \\ (\Lambda F)^P \arrow[r] & (\Lambda G)^P \end{tikzcd}$$

is a pushout in the category of compactly generated weak Hausdorff spaces. Colimits and products with spaces of $\mathbf{O}_{M,\mathcal{C}}$-spaces are formed objectwise, so letting P run through the monoids in the collection $\mathcal{C}$ shows that the square

$$\begin{tikzcd} \coprod_{i\in I} \Phi(M/N_i)\times S^{n_i-1} \arrow[r] \arrow[d] & \coprod_{i\in I} \Phi(M/N_i)\times D^{n_i} \arrow[d] \\ \Phi(\Lambda F) \arrow[r] & \Phi(\Lambda G) \end{tikzcd}$$

is a pushout in the category of $\mathbf{O}_{M,\mathcal{C}}$-spaces. The adjunction units form induce compatible maps from the original pushout square (1.17) to this last square. Since $\Phi(M/N_i) = \mathbf{O}_{M,\mathcal{C}}(-,N_i)$ and the unit $\eta_F : F \longrightarrow \Phi(\Lambda F)$ is an isomorphism, the unit $\eta_G : G \longrightarrow \Phi(\Lambda G)$ is also an isomorphism.

(ii) Since (Λ,Φ) is a Quillen pair and the right adjoint Φ preserves and detects weak equivalences. Moreover, for every cofibrant $\mathbf{O}_{M,\mathcal{C}}$-space F the adjunction unit $\eta_F : F \longrightarrow \Phi(\Lambda(F))$ is an isomorphism by (i), hence a weak equivalence. So (Λ,Φ) is a Quillen equivalence. □

1.1. Compact Lie groups. Now we specialize to equivariant spaces with actions of compact Lie groups. We let G be a compact Lie group and $\mathcal{F}$ be a family of subgroup of G that is stable under conjugacy and passage to closed subgroups. We call a morphism $f : A \longrightarrow B$ of G-spaces an *$\mathcal{F}$-equivalence* if the restriction $f^H : A^H \longrightarrow B^H$ to H-fixed points is a weak equivalence of spaces for all closed subgroups H of G that belong to the family $\mathcal{F}$.

The category of G-spaces then admits two model structures with $\mathcal{F}$-equivalences as the weak equivalences. A morphism $f : X \longrightarrow Y$ of G-space is an *$\mathcal{F}$-fibration* if the restriction $f^H : X^H \longrightarrow Y^H$ to H-fixed points is a Serre fibration for all subgroups H in the family $\mathcal{F}$. A *$\mathcal{F}$-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously $\mathcal{F}$-equivalences and $\mathcal{F}$-fibrations. The resulting '$\mathcal{F}$-projective model structure' is well known (under various names) and just a special case of Proposition 1.10 above.

Proposition 1.18. *Let G be a compact Lie group and $\mathcal{F}$ a family of closed subgroups of G. Then the $\mathcal{F}$-equivalences, $\mathcal{F}$-cofibrations and $\mathcal{F}$-fibrations form a model structure, the* $\mathcal{F}$-projective model structure *on the category of G-spaces. This model structure is proper, cofibrantly generated and topological.*

For later reference we recall the 'standard' set of generating cofibrations and generating acyclic cofibrations for the $\mathcal{F}$-projective model structure on G-spaces. In the category of (non-equivariant) spaces, the set $\{S^{n-1} \longrightarrow D^n\}_{n\geq 0}$ of inclusions of spheres into discs detects Serre fibrations that are simultaneously weak equivalences (where S^{-1} is the empty set). By adjointness, the set

$$\{G/H \times S^{n-1} \ \longrightarrow \ G/H \times D^n\}_{n\geq 0, H\in\mathcal{F}} \tag{1.19}$$

then detects acyclic fibrations in the $\mathcal{F}$-projective model structure on G-spaces (with trivial G-action on the sphere and disc). Similarly, the set of inclusions $\{D^n \times \{0\} \longrightarrow D^n \times [0,1]\}_{n\geq 0}$ detects Serre fibrations; so by adjointness, the set

$$\{G/H \times D^n \times \{0\} \ \longrightarrow \ G/H \times D^n \times [0,1]\}_{n\geq 0, H\in\mathcal{F}} \tag{1.20}$$

detects fibrations in the $\mathcal{F}$-projective model structure on G-spaces (with trivial G-action on the disc and interval). Sources and targets for all these maps are finite (sometimes called 'finitely presented') with respect to closed embeddings, see [**43**, Prop. 2.4.2].

We want to clarify the relationship between $\mathcal{F}$-cofibrations and relative $\mathcal{F}$-CW-complexes. A *relative $\mathcal{F}$-CW-complex* is a relative G-CW-complex where all relative cells are of orbit type G/H for H in the family $\mathcal{F}$. Equivalently, the isotropy group of all points in the complement of the image are in $\mathcal{F}$. Relative $\mathcal{F}$-CW-complexes are built from the generating cofibrations (1.19) by coproducts, cobase change and countable compositions, so every relative $\mathcal{F}$-CW-complex is an $\mathcal{F}$-cofibration. Conversely, every $\mathcal{F}$-cofibration is G-homotopy equivalent to a relative $\mathcal{F}$-CW-complex, in the following precise sense:

Proposition 1.21. *Let $\mathcal{F}$ be a family of closed subgroups of a compact Lie group G and $i : A \longrightarrow B$ an $\mathcal{F}$-cofibration of G-spaces. Then there is a relative $\mathcal{F}$-CW-complex $j : A \longrightarrow B'$ and G-maps $f : B \longrightarrow B'$ and $g : B' \longrightarrow B$ such that both gf and fg are G-homotopic, relative A, to the respective identity maps. In particular, every $\mathcal{F}$-cofibrant G-space is G-homotopy equivalent to a G-CW-complex with all isotropy groups in $\mathcal{F}$.*

Proof. By attaching $\mathcal{F}$-cells we can factor the map $i : A \longrightarrow B$ as a relative $\mathcal{F}$-CW-complex $j : A \longrightarrow B'$ followed by G-map $g : B' \longrightarrow B$ that is an $\mathcal{F}$-equivalence. The category of G-spaces under A inherits a model structure in which a morphism (of G-spaces under A) is a weak equivalence, cofibration or fibration if the underlying G-map (i.e., after forgetting all references to A) is an $\mathcal{F}$-equivalence, $\mathcal{F}$-cofibration respectively $\mathcal{F}$-fibration. The morphisms $i : A \longrightarrow B$ and $j : A \longrightarrow B'$ are then cofibrant-fibrant objects in this under category, and g is a weak equivalence from j to i. By general model category theory, every weak equivalence between cofibrant-fibrant objects is a homotopy equivalence, and in the case of the under category at hand, this means precisely that g has a G-homotopy inverse relative A. □

If H and K are closed subgroups of a compact Lie groups G and the dimension of K is strictly smaller than that of G, then the homogeneous space G/H typically has no preferred K-CW-structure. Hence the underlying K-spaces of G-CW-complexes cannot be made into K-CW-complexes in any natural way. The next proposition is usually sufficient to remedy this, because it says in particular that the restriction functor takes cofibrant G-spaces (for example G-CW-complexes) to cofibrant K-spaces. Also, a *G-cofibration* is a morphism with the right lifting property with respect to all morphisms that are simultaneously weak equivalences and Serre fibrations on the fixed points for all subgroups of G. Equivalently, G-cofibrations are the $\mathcal{F}$-cofibrations for the maximal family $\mathcal{F} = \mathcal{A}ll$ of all subgroups of G. Thus we have the following implications between the various kinds of 'nice equivariant embeddings':

$$\text{relative } G\text{-CW-complex} \implies \text{relative } G\text{-cell complex} \implies G\text{-cofibration} \implies \text{h-cofibration of } G\text{-spaces}$$

All these implications are strict. However, Proposition 1.21 (for the family of all subgroups) makes precise that relative G-CW-complex, relative G-cell complex and G-cofibration are all equally good 'up to G-homotopy'.

Proposition 1.22. *Let G and K be compact Lie groups.*

(i) *For every continuous homomorphism $\alpha : K \longrightarrow G$ the restriction functor $\alpha^* : G\mathbf{U} \longrightarrow K\mathbf{U}$ takes G-cofibrations to K-cofibrations.*
(ii) *For every closed subgroup H of G the induction functor $G \times_H - : H\mathbf{U} \longrightarrow G\mathbf{U}$ takes H-cofibrations to G-cofibrations.*
(iii) *For every closed subgroup H of G the fixed point functor $(-)^H : G\mathbf{U} \longrightarrow WH\mathbf{U}$ takes G-cofibrations to WH-cofibrations, where $WH = N_G H/H$ is the Weyl group of H.*

Proof. (i) The restriction functor α^* preserves all colimits, so it suffices to show that it takes the generating G-cofibrations $S^{n-1} \times G/H \longrightarrow D^n \times G/H$ to K-cofibrations, for any subgroup H of G. We treat two special cases separately.

If α is surjective, then $\alpha^*(G/H)$ is K-equivariantly homeomorphic to K/L for the closed subgroup $L = \alpha^{-1}(H)$ of K. So in this case α^* takes the generating G-cofibration to a K-map isomorphic to $S^{n-1} \times K/L \longrightarrow D^n \times K/L$, which is then a K-cofibration.

If α is the inclusion of a closed subgroup K of G, then the left translation action is a smooth K-action on the smooth compact manifold G/H. Illman's theorem [**45**, Cor. 7.2] provides a finite K-CW-structure on G/H, so in particular G/H is cofibrant as a K-space. Since the projective model structure on K spaces (for the family of all subgroups) is topological, the map $S^{n-1} \times G/H \longrightarrow D^n \times G/H$ is a K-cofibration.

Any continuous homomorphism factors as a continuous epimorphism onto its image, followed by the inclusion of the image, so the two special cases combine to show that α^* takes G-cofibrations to K-cofibration.

(ii) Since the $G \times_H -$ preserves all colimits, so it suffices to show that it takes the generating H-cofibrations $S^{n-1} \times H/J \longrightarrow D^n \times H/J$ to G-cofibrations, for any subgroup J of H. This in turn is clear since $G \times_H (H/J)$ is G-homeomorphic to G/J.

(iii) The fixed point functor $(-)^H$ commutes with disjoint unions and pushout ans sequential composites along G-cofibrations, so it suffices to show that it takes the generating G-cofibrations $S^{n-1} \times G/K \longrightarrow D^n \times G/K$ to WH-cofibrations, for any subgroup K of G. This in turn is true because $(G/K)^H$ is a disjoint union of finitely many WH-orbits [**19**, II Cor. 5.7], hence cofibrant as a WH-spaces. □

Let G and K be compact Lie groups and $\mathcal{F}$ be a family of subgroups of G. We denote by $\mathcal{F} \ltimes K$ the family of those subgroups of K that are epimorphic images of groups in $\mathcal{F}$. So $\mathcal{F} \ltimes K$ consists of all groups of the form $\alpha(H)$ for some $H \in \mathcal{F}$ and some continuous homomorphism $\alpha : H \longrightarrow K$.

Proposition 1.23. *Let G and K be compact Lie groups and A a cofibrant $(G \times K^{\mathrm{op}})$-space such that the right K-action is free. Then the functor*

$$A \times_K - \ : \ K\mathbf{U} \longrightarrow G\mathbf{U}$$

takes K-cofibrations to G-cofibrations For every family $\mathcal{F}$ of subgroups of G, the functor $A \times_K -$ takes $\mathcal{F} \ltimes K$-equivalences to $\mathcal{F}$-equivalences.

Proof. We denote by $\mathcal{F}[G;K]$ the family of those subgroups $\Gamma \leq G \times K^{\mathrm{op}}$ that intersect $\{1\} \times K^{\mathrm{op}}$ only in the identity element. We start with the first claim and show a slightly more general statement, namely that the pushout product, with respect to the functor

$$- \times_K - \ : \ (G \times K^{\mathrm{op}})\mathbf{U} \times K\mathbf{U} \longrightarrow G\mathbf{U} \ ,$$

of an $\mathcal{F}[G;K]$-cofibration with a K-cofibration is a G-cofibration. The claim is a special case because the $\mathcal{F}[G;K]$-cofibrant $(G \times K^{\mathrm{op}})$-spaces are precisely those cofibrant $(G \times K^{\mathrm{op}})$-spaces on which the right K-action is free.

Since the functor $- \times_K -$ preserves colimits in both variables, it suffices to check the pushout product of a generating $\mathcal{F}[G;K]$-cofibration (1.19) with a generating K-cofibration. The former are of the form

$$(G \times K^{\mathrm{op}})/\Gamma \times S^{n-1} \longrightarrow (G \times K^{\mathrm{op}})/\Gamma \times D^n$$

for some $\Gamma \in \mathcal{F}[G;K]$ and some $n \geq 0$. Similarly, the latter are of the form

$$K/L \times S^{m-1} \longrightarrow K/L \times D^m$$

for some subgroup $L \leq K$ and some $m \geq 0$. Since the pushout product with the sphere inclusion $S^{n-1} \longrightarrow D^n$ preserves cofibrations, it suffices to check that the G-spaces of the form

$$((G \times K^{\mathrm{op}})/\Gamma) \times_K (K/L)$$

are cofibrant, where $\Gamma \in \mathcal{F}[G;K]$ and L is any subgroup of K. The subgroup Γ is the graph of a homomorphism $\alpha : H \longrightarrow K^{\mathrm{op}}$ defined on a subgroup H of G, so the previous balanced product is G-isomorphic to

$$((G \times K^{\mathrm{op}})/\Gamma) \times_K (K/L) \;\cong\; G \times_H (\alpha^*(K/L)) \;,$$

and this is indeed G-cofibrant by parts (i) and (ii) of Proposition 1.22.

Now we prove the claim about equivalences. We must show that for every $\mathcal{F} \ltimes K$-weak equivalence $f : X \longrightarrow Y$ the morphism

$$A \times_K f \;:\; A \times_K X \longrightarrow A \times_K Y$$

is an $\mathcal{F}$-weak equivalence. Must therefor check that for every subgroup $H \leq G$ with $H \in \mathcal{F}$ the H-fixed point map $(A \times_K f)^H$ is a weak equivalence. The underlying $(H \times K^{\mathrm{op}})$-space of A is cofibrant (by Proposition 1.22 (i)), hence equivariantly homotopy equivalent to an $(H \times K^{\mathrm{op}})$-CW-complex; so it is no loss of generality to assume a $(H \times K^{\mathrm{op}})$-CW-structure on A with skeleta A^n. We start by showing the claim for all A^n (instead of A), by induction on n. The induction starts with A^{-1}, which is empty, and there is nothing to show. Then we let $n \geq 0$ and assume the claim for A^{n-1}. By hypotheses there is a pushout square of $(G \times K^{\mathrm{op}})$-spaces

$$\begin{tikzcd}
\coprod_{j \in J} (H \times K^{\mathrm{op}}/\Gamma_j) \times S^{n-1} \arrow[r] \arrow[d] & \coprod_{j \in J} (H \times K^{\mathrm{op}}/\Gamma_j) \times D^n \arrow[d] \\
A^{n-1} \arrow[r] & A^n
\end{tikzcd}$$

Here J is an indexing set of the n-cells of the equivariant CW-structure and Γ_j is a subgroup of $H \times K^{\mathrm{op}}$. Since the right K-action on A is free, each of the subgroups Γ_j is the graph of a continuous homomorphism $\alpha_j : I_j \longrightarrow K^{\mathrm{op}}$ defined on a subgroup I_j of H.

Applying $(- \times_K X)^H$ takes this to a pushout square of spaces; this uses that H-fixed points commutes with pushouts in which one of the legs is a closed embedding [ref]. We rewrite the resulting pushout, using that

$$(((H \times K^{\mathrm{op}}/\Gamma) \times D^n) \times_K X)^H \;\cong\; D^n \times (((H \times_I \alpha^*(X))^H \;\cong\; \begin{cases} D^n \times X^{\alpha(H)} & \text{for } I = H, \text{ and} \\ \emptyset & \text{for } I \neq H, \end{cases}$$

where Γ is the graph of $\alpha : I \longrightarrow K^{\mathrm{op}}$. So we obtain a pushout square of spaces

$$\begin{tikzcd} S^{n-1} \times \coprod_{j\in J'} X^{\alpha_j(H)} \arrow[r] \arrow[d] & D^n \times \coprod_{j\in J'} X^{\alpha_j(H)} \arrow[d] \\ (A^{n-1} \times_K X)^H \arrow[r] & (A^n \times_K X)^H \end{tikzcd}$$

where $J' \subset J$ is the subset of those indices for which $I_j = H$. The same kind of pushout arises from Y instead of X. Since $H \in \mathcal{F}$ the group $\alpha_j(H)$ belongs to $\mathcal{F} \ltimes K$ for all $j \in J'$. So the map

$$f^{\alpha_j(H)} \ : \ X^{\alpha_j(H)} \ \longrightarrow \ Y^{\alpha_j(H)}$$

is a weak equivalence. The disjoint union, over J', of these maps is then also a weak equivalence; by the inductive hypothesis and the gluing lemma [**29**, Lemma A.1], the induced map on pushouts

$$(A^n \times_K f)^H \ : \ (A^n \times_K X)^H \ \longrightarrow \ (A^n \times_K X)^H$$

is then also a weak equivalence.

Now we pass to the colimit over n. Since the skeleton inclusions $A^{n-1} \longrightarrow A^n$ are h-cofibrations of $(H \times K^{\mathrm{op}})$-spaces, the induce maps

$$(A^{n-1} \times_K X)^H \ \longrightarrow \ (A^n \times_K Y)^H$$

are h-cofibrations (of non-equivariant spaces). Since $(A \times_K X)^H$ is a colimit of the sequence of spaces $(A^n \times_K X)^H$ and a colimit of weak equivalences over sequences of h-cofibrations is another weak equivalence, the claim follows. $\square$

Now we discuss when taking cartesian product preserves equivariant cofibrations. There are two related questions, namely 'external products' of a G-space and a K-space and 'internal products' of two G-spaces with diagonal action.

Proposition 1.24. *Let G and K be topological groups.*

(i) *The pushout product of a G-cofibration with a K-cofibration is a $(G \times K)$-cofibration.*
(ii) *If G is a compact Lie group, then the pushout product of two G-cofibrations is a G-cofibration with respect to the diagonal G-action.*

Proof. (i) The product functor

$$\times \ : \ G\mathbf{U} \times K\mathbf{U} \ \longrightarrow \ (G \times K)\mathbf{U}$$

preserves colimits in each variable, so it suffices to check the pushout product of a generating G-cofibration $G/H \times i_m$ with a generating K-cofibration $K/L \times i_n$, where $i_n : S^{n-1} \longrightarrow D^n$ is the inclusion. The pushout product of these is isomorphic to

$$(G \times K)/(H \times L) \times (i_n \Box i_m)$$

and hence a cofibration of $(G \times K)$-spaces.

(ii) By (i) the pushout product of two G-cofibrations is a $(G \times G)$-cofibration. Since G is compact Lie and the diagonal is a closed subgroup, restriction along the diagonal embedding $G \longrightarrow G \times G$ preserves cofibrations by Proposition 1.22. $\square$

To describe the fibrations in a second $\mathcal{F}$-model structure we need a universal G-space for the family $\mathcal{F}$. In Definition 1.13 we introduced universal spaces for collections of submonoids of a topological monoid. We are mostly interested in universal spaces for topological *groups* with respect to a *family* of closed subgroup, i.e., a collection that is also closed under passage to subgroups. In this situation universal spaces have a slightly different characterization that may be more familiar. Traditionally, it is often required that a universal space 'is' (or can be given the structure of) a G-CW-complex. We only require the weaker condition (iii) that still guarantees that a universal space is equivariantly homotopy equivalent to a G-CW-complex.

Proposition 1.25. *Let G be a topological group. Then a G-space E is a universal G-space for a family $\mathcal{F}$ of closed subgroups of G if and only if it satisfies the following three conditions.*

(i) *for every closed subgroup $H \in \mathcal{F}$ the fixed point space E^H is weakly contractible;*
(ii) *for every closed subgroup $H \notin \mathcal{F}$ the fixed point space E^H is empty;*
(iii) *E is G-homotopy equivalent to a G-cofibrant G-space.*

Proof. A universal G-space E for the family $\mathcal{F}$ has property (i) by definition. It also has property (iii) because every $\mathcal{F}$-cofibrant G-space is in particular G-cofibrant. If H is a closed subgroup of G that does not belong to $\mathcal{F}$, then it is not subconjugate to any group in $\mathcal{F}$ (because $\mathcal{F}$ is closed under subconjugates), so the fixed points $(G/K)^H$ are empty for all $K \in \mathcal{F}$. Any $\mathcal{F}$-cofibrant G-space is a retract of an G-space built by attaching cells of the form $G/K \times D^n$ for $K \in \mathcal{F}$. So for every $\mathcal{F}$-cofibrant G-space A and every $H \notin \mathcal{F}$ the fixed points A^H are empty. If E is homotopy equivalent to an $\mathcal{F}$-cofibrant G-space A, then in particular there exists a G-map $E \longrightarrow A$, and hence a map $E^H \longrightarrow A^H = \emptyset$, so E^H is also empty. So every universal G-space for $\mathcal{F}$ satisfies condition (ii).

Now suppose that conversely a G-space E satisfies conditions (i), (ii) and (iii) of this proposition, and let A be a G-homotopy equivalent G-CW-complex. Then there exists a G-map $A \longrightarrow E$, hence a map $A^H \longrightarrow E^H = \emptyset$, for all $H \notin \mathcal{F}$, so A^H is also empty. So the equivariant cells used in any G-CW-structure for A must be of the form $G/K \times D^n$ for $K \in \mathcal{F}$. So A is $\mathcal{F}$-cofibrant, and E satisfies both defining conditions of a universal G-space for the family $\mathcal{F}$. □

In the following we write $E\mathcal{F}$ for any universal G-space for a given family $\mathcal{F}$ of closed subgroups.

Proposition 1.26. *Let G be a compact Lie group, $E\mathcal{F}$ be a universal G-space for a family $\mathcal{F}$ and $E\mathcal{F}'$ be a universal G-space for a family $\mathcal{F}'$. Then $(E\mathcal{F}) \times (E\mathcal{F}')$ is a universal G-space for a family $\mathcal{F} \cap \mathcal{F}'$. with respect to the diagonal G-action.*

Proof. We check the criterion given in Proposition 1.25. The G-space $(E\mathcal{F}) \times (E\mathcal{F}')$ is equivariantly homotopy equivalent to a cofibrant G-space by Proposition 1.24 (ii). Moreover, if $H \leq G$ is a subgroup, then

$$((E\mathcal{F}) \times (E\mathcal{F}'))^H \; = \; (E\mathcal{F})^H \times (E\mathcal{F}')^H \; .$$

If $H \in \mathcal{F} \cap \mathcal{F}'$, then both factors on the right hand side are weakly contractible, hence so is the product. If $H \notin \mathcal{F} \cap \mathcal{F}'$, then at least one of the factors on the right hand side is empty, hence so is the product. □

We let $\mathrm{map}(E\mathcal{F}, A)$ denote the space of continuous maps (not necessarily equivariant) from $E\mathcal{F}$ to A with conjugation action by G. The projection $A \times E\mathcal{F} \longrightarrow A$ is adjoint to a G-map $\eta_A : A \longrightarrow \mathrm{map}(E\mathcal{F}, A)$ that sends a point $a \in A$ to the constant map with value a. For all $H \in \mathcal{F}$ the underlying H-space of $E\mathcal{F}$ is H-equivariantly contractible; so the map $\eta_A^H : A^H \longrightarrow \mathrm{map}^H(E\mathcal{F}, A)$ is homotopy equivalence for all $H \in \mathcal{F}$. Hence η_A is an $\mathcal{F}$-equivalence (but not generally a G-weak equivalence).

We call a G-map $f : X \longrightarrow Y$ a *mixed $\mathcal{F}$-fibration* if the restriction $f^H : X^H \longrightarrow Y^H$ to H-fixed points is a Serre fibration for all closed subgroups H of G (not just for those in $\mathcal{F}$!) and the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & \mathrm{map}(E\mathcal{F}, A) \arrow[d, "\mathrm{map}(E\mathcal{F}, f)"] \\ B \arrow[r, "\eta_B"'] & \mathrm{map}(E\mathcal{F}, B) \end{tikzcd}$$

is G-homotopy cartesian. A commutative square of G-spaces is G-homotopy cartesian if and only if for every subgroup H of G the square of H-fixed point spaces is homotopy cartesian (in the non-equivariant sense). Since η_A and η_B are $\mathcal{F}$-equivalences, a G-map f is a mixed $\mathcal{F}$-fibration if and only if

- for all subgroups H of G the restriction $f^H : X^H \longrightarrow Y^H$ to H-fixed points is a Serre fibration and

- for all subgroups H of G not in the family $\mathcal{F}$ the square of H-fixed point spaces

$$\begin{tikzcd} A^H \arrow[r, "\eta_A^H"] \arrow[d, "f^H"'] & \mathrm{map}^H(E\mathcal{F}, A) \arrow[d, "\mathrm{map}^H(E\mathcal{F}, f)"] \\ B^H \arrow[r, "\eta_B^H"'] & \mathrm{map}^H(E\mathcal{F}, B) \end{tikzcd} \tag{1.27}$$

is homotopy cartesian.

Proposition 1.28. *Let G be a compact Lie group and $\mathcal{F}$ a family of closed subgroups of G. The $\mathcal{F}$-equivalences, G-cofibrations and mixed $\mathcal{F}$-fibrations form a model structure, the* $\mathcal{F}$-flat model structure *on the category of G-spaces. This model structure is proper, cofibrantly generated and topological.*

Proof. I am not aware of a reference for the $\mathcal{F}$-flat model structure, so here is a proof. We apply Bousfield's localization theorem [**17**, Thm. 9.3] to the projective model structure, with respect to the family of all subgroups of G, on G-spaces (which is proper). We need a localization functor on the category of G-spaces, for which we take $QX = \mathrm{map}(E\mathcal{F}, X)$ with natural transformation $\eta_X : X \longrightarrow QX$. The verification of Bousfield's axioms is based on the following observations:

- The map $\eta_X : X \longrightarrow \mathrm{map}(E\mathcal{F}, X)$ is an $\mathcal{F}$-equivalence.
- The product $E\mathcal{F} \times E\mathcal{F}$, with diagonal G-action, is another universal G-space for the family $E\mathcal{F}$, by Proposition 1.26. So the two projections $E\mathcal{F} \times E\mathcal{F} \longrightarrow E\mathcal{F}$ are G-homotopy equivalences (Proposition 1.14 (ii)). So for every G-space X the two maps
$$\begin{aligned} \mathrm{map}(E\mathcal{F}, \eta_X)\ , \eta_{\mathrm{map}(E\mathcal{F},X)}\ :\ & \mathrm{map}(E\mathcal{F}, X) \\ \longrightarrow\ & \mathrm{map}(E\mathcal{F}, \mathrm{map}(E\mathcal{F}, X)) \cong \mathrm{map}(E\mathcal{F} \times E\mathcal{F}, X) \end{aligned}$$
are G-homotopy equivalences.
- A G-map f is an $\mathcal{F}$-equivalence if and only if Qf is a G-weak equivalence. Indeed, for every $\mathcal{F}$-equivalence $f : X \longrightarrow Y$ and every $\mathcal{F}$-CW-complex A, the induced map $\mathrm{map}(A, f) : \mathrm{map}(A, X) \longrightarrow \mathrm{map}(A, Y)$ is a G-weak equivalence, as one can show by induction over a CW-structure; since $E\mathcal{F}$ is G-homotopy equivalence to a G-CW-complex, this shows that every $\mathcal{F}$-equivalence f gives rise to a G-weak equivalence Qf. Conversely, if Qf is a G-weak equivalence, then because η_X and η_Y are $\mathcal{F}$-equivalences, the original map f is an $\mathcal{F}$-equivalence.

For left properness we consider a pushout square of G-spaces

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] & D \end{tikzcd}$$

such that i is a G-cofibration and f is an $\mathcal{F}$-equivalence. We let H be a subgroup of G in the family $\mathcal{F}$. Taking H-fixed points preserves pushouts along G-cofibrations, so the square

$$\begin{tikzcd} A^H \arrow[r, "i^H"] \arrow[d, "f^H"'] & B^H \arrow[d, "g^H"] \\ C^H \arrow[r] & D^H \end{tikzcd}$$

is a pushout. Moreover the map i^H is a cofibration of WH-spaces, hence of spaces, by part (iv) respectively (i) of Proposition 1.22. Since f^H is a weak equivalence and the model category of spaces is left proper, the cobase change g^H is a weak equivalence. Since H was any group from the family $\mathcal{F}$, we conclude that g

is an $\mathcal{F}$-equivalence. Since mixed $\mathcal{F}$-fibrations are in particular $\mathcal{F}$-fibrations, right properness follows from right properness of the $\mathcal{F}$-projective model structure.

To show that the model structure is cofibrantly generated we exhibit explicit sets of generating cofibrations and generating acyclic cofibrations. The cofibrations in the $\mathcal{F}$-flat model structure are independent of $\mathcal{F}$ and coincide with $\mathcal{F}$-cofibrations for the family of all subgroups of G. So the set of maps (1.19) for the family of all subgroups detects the acyclic fibrations.

The mixed $\mathcal{F}$-fibrations are detected by the union of the set (1.20), for all subgroups H of G, with the set of pushout product maps

$$(G \times_H E\mathcal{F} \longrightarrow G \times_H C(E\mathcal{F})) \,\square\, (S^{n-1} \longrightarrow D^n) \tag{1.29}$$

for all $n \geq 0$ and all subgroups H of G. Here $C(E\mathcal{F}) = [0,1] \wedge E\mathcal{F}^+$ is the cone of the universal $\mathcal{F}$-space $E\mathcal{F}$. (For $H \in \mathcal{F}$ the space $E\mathcal{F}$ is H-equivariantly contractible, so then the map is an equivariant deformation retract; thus it would suffice to take the maps only for $H \notin \mathcal{F}$). Indeed, a G-map $f : X \longrightarrow Y$ has the right lifting property with respect to all the maps (1.20) (for all subgroups of G) if and only if all the fixed point maps $f^H : X^H \longrightarrow Y^H$ are Serre fibrations. The right lifting property with respect to the maps (1.29) is equivalent to the map

$$\mathrm{map}^H(C(E\mathcal{F}), Y) \;\longrightarrow\; \mathrm{map}^H(E\mathcal{F}, Y) \times_{\mathrm{map}^H(E\mathcal{F},X)} \mathrm{map}^H(C(E\mathcal{F}), X)$$

being a Serre fibration and a weak equivalence for all $H \in \mathcal{F}$. Since the cone $C(E\mathcal{F})$ is G-equivariantly contractible to the cone point, this in turn is equivalent to the square (1.27) being homotopy cartesian for all subgroups H of G. □

A special case of the $\mathcal{F}$-flat model structure is the 'mixed model structure' considered by Shipley in [**82**, Prop. 1.3], namely when G is a finite group and $\mathcal{F} = \langle e \rangle$ consists only of the trivial subgroup of G (however, Shipley works with simplicial sets, and not with topological spaces).

We record a few consequences of the definition:

- The cofibrations in the $\mathcal{F}$-flat model structure do not depend on the family $\mathcal{F}$.
- For the family $\mathcal{F} = \mathcal{A}ll$ of all subgroups of G, the flat and projective $\mathcal{F}$-model structures coincide.
- A G-space X is fibrant in the $\mathcal{F}$-flat model structure if and only if the map η_X is a G-weak equivalence. Since η_X is always an $\mathcal{F}$-equivalence, this is equivalent to the requirement that for every subgroup H of G with $H \notin \mathcal{F}$ the map
$$X^H \;\longrightarrow\; \mathrm{map}^H(E\mathcal{F}, X)$$
that sends an H-fixed point to the constant map with that value is a weak equivalence.

2. Spaces of linear isometries

In this section we recall some basic facts about spaces of linear isometric embeddings. We will consider real inner product spaces of finite or countably infinite dimension, i.e., $\mathbb{R}$-vector spaces equipped with a scalar product whose algebraic dimension is countable. Up to linear isometric isomorphism, any such inner product space is uniquely determined by its dimension, and it is isomorphic to $\mathbb{R}^n$ for some $n \geq 0$, or to $\mathbb{R}^\infty$, with the canonical basis as orthogonal basis.

We discuss the homotopical properties of the space $\mathbf{L}(V, W)$ of $\mathbb{R}$-linear isometric embeddings. We will also consider actions of compact Lie groups on the inner products spaces and discuss the equivariant homotopy types of such spaces of isometries.

We start by recalling the standard topology on the space $\mathbf{L}(V, W)$ that we shall use throughout the book. The *linear topology* on a real vector space V is defined as follows. When V is finite dimensional, the linear topology is the metric topology from any norm on V. If V is an arbitrary $\mathbb{R}$-vector space, then the linear topology is the colimit topology as the colimit of the finite dimensional sub-vector spaces, equipped with the linear topology. Any linear map of $\mathbb{R}$-vector spaces is continuous with respect to the linear topologies. When V is infinite dimensional, then the linear topology has more open sets than the norm topology from the standard scalar product. For example, the linear map $f : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ given by

$$f(x_1, x_2, x_3, \dots) \ = \ f(x_1, 2x_2, 3x_3, \dots)$$

is continuous with respect to the linear topology on source and target, but it is unbounded and not continuous with respect to the norm topology.

The linear topology on any $\mathbb{R}$-vector space is compactly generated (as a colimit of compactly generated spaces); it is also Hausdorff: for every non-zero vector $v \in V$ there is a linear form $\psi : V \longrightarrow \mathbb{R}$ with $\psi(v) \neq 0$. Since linear maps are continuous in the linear topology, v can be separated from 0 by open sets in V.

Next, we let V and W be two $\mathbb{R}$-vector spaces, endowed with the linear topology. Then the space $\mathrm{map}(V, W)$ of all continuous maps (not necessarily $\mathbb{R}$-linear) is given the mapping space topology within the category of compactly generated weak Hausdorff spaces. [how far is this from the compact-open topology?] Being linear is a closed condition: the map

$$\mathrm{map}(V, W) \times \mathbb{R} \times V^2 \ \longrightarrow \ W \ , \quad (f, \lambda, v, v') \ \longmapsto \ f(\lambda v + v') - \lambda f(v) - f(v')$$

is continuous, hence so is its adjoint

$$\mathrm{map}(V, W) \ \longrightarrow \ \mathrm{map}(\mathbb{R} \times V^2, W) \ ;$$

since the set $\mathrm{Hom}_{\mathbb{R}}(V, W)$ of $\mathbb{R}$-linear maps is the inverse image of the zero function under the last map, it is a closed subset of $\mathrm{map}(V, W)$. We endow $\mathrm{Hom}_{\mathbb{R}}(V, W)$ with the subspace topology of $\mathrm{map}(V, W)$, and it is then compactly generated and weakly Hausdorff. If V is finite dimensional, then this subspace topology on $\mathrm{Hom}_{\mathbb{R}}(V, W)$ should be he same as the linear topology from the vector space structure on $\mathrm{Hom}_{\mathbb{R}}(V, W)$; however, when V is infinite dimensional, and W is non-zero, then the subspace topology has fewer open sets than the linear topology. [check this...]

Similarly, preserving the scalar product (or, equivalently, the norm) is another closed condition. For this we assume that V and W are endowed with a scalar product. A linear map from V to W preserves the scalar products if and only if it preserves the induced norm. The map

$$\mathrm{Hom}_{\mathbb{R}}(V, W) \times V \ \longrightarrow \ \mathbb{R} \ , \quad (f, v) \ \longmapsto \ ||f(v)|| - ||v||$$

is continuous, hence so is its adjoint $\mathrm{map}(V, W) \longrightarrow \mathrm{map}(V, \mathbb{R})$; since the set $\mathbf{L}(V, W)$ of linear isometric embeddings is the inverse image of the zero function under the last map, it is a closed subset of $\mathrm{Hom}_{\mathbb{R}}(V, W)$, and hence of $\mathrm{map}(V, W)$. We endow $\mathbf{L}(V, W)$ with the subspace topology of $\mathrm{map}(V, W)$.

There is another way to describe the topology on the space $\mathbf{L}(V, W)$ of linear isometric embeddings; this falls into three different categories, depending on whether the dimension of source or target is finite or infinite.

- First, if both V and W are finite dimensional, then up to linear isomorphism they are of the form $V = \mathbb{R}^m$ and $W = \mathbb{R}^{n+m}$, endowed with the standard scalar products. Then the bijection

  $$O(n + m)/O(n) \ \longrightarrow \ \mathbf{L}(\mathbb{R}^m, \mathbb{R}^{n+m}) \ , \quad A \cdot O(n) \ \longmapsto \ A \circ i$$

  is a homeomorphism, where $i : \mathbb{R}^m \longrightarrow \mathbb{R}^{n+m}$ is the embedding $i(x_1, \dots, x_m) = (0, \dots, 0, x_1, \dots, x_m)$. So whenever V and W are finite dimensional, $\mathbf{L}(V, W)$ is homeomorphic to a homogeneous space $O(n + m)/O(n)$, in particular compact and cofibrant.
- Secondly, when V is finite dimensional and W of countably infinite dimension, then $\mathbf{L}(V, W)$ is a colimit of the sets $\mathbf{L}(V, W')$ as W' runs over the poset of finite dimensional subvector spaces of W. Moreover, the subspace topology on $\mathbf{L}(V, W)$ inside $\mathrm{map}(V, W)$ is the colimit topology with respect to the subspaces $\mathbf{L}(V, W')$ [check...]. In this situation, $\mathbf{L}(V, W)$ is no longer compact, but it is still cofibrant [ref] and also contractible (Proposition 2.1 (ii))

- Finally, V and W are both of countable infinite dimension, then $\mathbf{L}(V, W)$ is an inverse limit of the sets $\mathbf{L}(V', W)$ as V' runs over the poset of finite dimensional subvector spaces of V. Moreover, the subspace topology on $\mathbf{L}(V, W)$ inside $\mathrm{map}(V, W)$ is the inverse limit topology with respect to the system of spaces $\mathbf{L}(V', W)$. [justify this] Now $\mathbf{L}(V, W)$ is neither compact not cofibrant, but it is contractible (Proposition 2.1 (ii)).

The contractiblity of $\mathbf{L}(V, \mathcal{U})$ for infinite dimensional $\mathcal{U}$ goes back, at least, to Boardman and Vogt [**12**].

Proposition 2.1. *Let U, V and W be real inner product space of finite or countably infinite dimensions.*

(i) *The composition map*

$$\mathbf{L}(V, W) \times \mathbf{L}(U, V) \longrightarrow \mathbf{L}(U, W)$$

is continuous.

(ii) *We let G be a compact Lie group, V a G-representation and $\mathcal{U}$ a G-universe such that V embeds into $\mathcal{U}$. Then the space $\mathbf{L}(V, \mathcal{U})$, equipped with the conjugation action by G, is G-equivariantly contractible.*

Proof. Since $\mathcal{U}$ is a G-universe containing V, it is G-equivariantly isomorphic to the direct sum of $\mathcal{U}$ and V. So $\mathbf{L}(V, \mathcal{U})$ is G-homeomorphic to $\mathbf{L}(V, \mathcal{U} \oplus V)$ and it suffices to construct a G-homotopy that contracts the G-space $\mathbf{L}(V, \mathcal{U} \oplus V)$ into the distinguished point $i \in \mathbf{L}(V, \mathcal{U} \oplus V)$ given by $i(v) = (0, v)$.

In a first step we construct a path ω in $\mathbf{L}^G(\mathcal{U} \oplus V, \mathcal{U} \oplus V)$ from the identity of $\mathcal{U} \oplus V$ to a G-equivariant linear isometric self-embedding φ whose image is contained in the first summand of $\mathcal{U} \oplus V$ [...].

Acting by the path ω gives a G-equivariant homotopy

$$[0,1] \times \mathbf{L}(V, \mathcal{U} \oplus V) \longrightarrow \mathbf{L}(V, \mathcal{U} \oplus V) , \quad (t, j) \longmapsto \omega(t) \circ j$$

from the identity of $\mathbf{L}(V, \mathcal{U} \oplus V)$ to the map

$$\varphi \circ - \; : \; \mathbf{L}(V, \mathcal{U} \oplus V) \longrightarrow \mathbf{L}(V, \mathcal{U} \oplus V) .$$

We define a second homotopy

$$H \; : \; [0,1] \times \mathbf{L}(V, \mathcal{U} \oplus \mathcal{U}) \longrightarrow \mathbf{L}(V, \mathcal{U} \oplus \mathcal{U})$$

by

$$H(t, j)(v) \; = \; \sqrt{1 - t^2} \; \cdot \varphi(j(v)) \; + \; t \cdot i(v) .$$

The fact that the image of φ is orthogonal to the second summand of $\mathcal{U} \oplus V$ guarantees that each map $H(t, j)$ is indeed a linear isometric embedding. Moreover, this homotopy starts with the map $\varphi \circ -$ and ends with the constant map with value i. So both homotopies together contract the G-space $\mathbf{L}(V, \mathcal{U} \oplus V)$ equivariantly into the point i. □

Proposition 2.2. *Let G be a compact Lie group and H and K two closed subgroups of G. The map*

$$K/(K \cap H) \longrightarrow G/H , \quad k(K \cap H) \longmapsto kH$$

is a cofibration of K-spaces.

Now we consider two compact Lie groups G and K, a G-representation V and a K-representation W, possibly of infinite dimensions. The space $\mathbf{L}(V, W)$ of linear isometric embeddings inherits a continuous left K-action and a compatible continuous right G-action from the actions on the target and source, respectively. In other words, for $\varphi \in \mathbf{L}(V, W)$, $g \in G$ and $k \in K$ we set

$$(k \cdot \varphi \cdot g)(v) \; = \; k \cdot \varphi(g \cdot v) .$$

Proposition 2.3. *Let G and K be compact Lie groups, V a finite-dimensional faithful G-representation, $\mathcal{U}$ a K-representation of finite or countably infinite dimension, and $U \subset \mathcal{U}$ a finite-dimensional K-subrepresentation.*

(i) *Then the morphism of* $(K \times G^{\mathrm{op}})$*-spaces*

$$\mathbf{L}(V, U) \longrightarrow \mathbf{L}(V, \mathcal{U})$$

induced by the inclusion is a $(K \times G^{\mathrm{op}})$*-cofibration.*

(ii) *The* $(K \times G^{\mathrm{op}})$*-space* $\mathbf{L}(V, \mathcal{U})$ *is* $(K \times G^{\mathrm{op}})$*-cofibrant.*

(iii) *The* K*-space* $\mathbf{L}(V, \mathcal{U})/G$ *is* K*-cofibrant.*

Proof. (i) We start by showing that the map

$$\mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n}) \longrightarrow \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n+m}) \tag{2.4}$$

induced by the inclusion $\mathbb{R}^{k+n} \longrightarrow \mathbb{R}^{k+n+m}$ is a cofibration of $(O(k+n) \times O(m) \times O(k)^{\mathrm{op}})$-spaces, where $O(m)$ acts trivially on the source.

We consider the Lie group $G = O(k+n+m) \times O(k)^{\mathrm{op}}$ and its subgroup

$$H = \{(A, B) \in O(k+n+m) \times O(k)^{\mathrm{op}} \mid A|_{\mathbb{R}^k} = B^{-1} \in \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n+m})\} .$$

The group H is abstractly isomorphic to $O(n+m) \times O(k)^{\mathrm{op}}$ by sending (C, B) to $(B^{-1} \oplus C, B)$.

We define

$$K = O(k+n) \times O(m) \times O(k)^{\mathrm{op}} ,$$

considered as a subgroup of G via the block sum embedding $O(k+n) \times O(m) \longrightarrow O(k+n+m)$. The intersection is then given by

$$K \cap H = \{(X, Y, B) \in O(k+n) \times O(m) \times O(k)^{\mathrm{op}} \ : \ X_{\mathbb{R}^k} = B^{-1}\}$$

and it is isomorphic to $O(n) \times O(m) \times O(k)^{\mathrm{op}}$ via

$$O(n) \times O(m) \times O(k)^{\mathrm{op}} \longrightarrow K \cap H , \quad (C, D, B) \longmapsto (B^{-1} \oplus C, D, B) .$$

We let K act on $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n})$ by

$$(X, Y, B) \cdot \varphi = X \cdot \varphi \cdot B^{-1} ;$$

so in particular the middle factor acts trivially. The square

$$\begin{tikzcd} K \arrow[r] \arrow[d] & G \arrow[d] \\ \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n}) \arrow[r] & \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n+m}) \end{tikzcd}$$

commutes where the vertical maps are the actions on the inclusions $\mathbb{R}^k \longrightarrow \mathbb{R}^{k+n}$ respectively $\mathbb{R}^k \longrightarrow \mathbb{R}^{k+n+m}$. Both actions are transitive, $K \cap H$ is the stabilizer in $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n})$ of the inclusion $\mathbb{R}^k \longrightarrow \mathbb{R}^{k+n}$, and H is the stabilizer in $\mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n+m})$ of the inclusion $\mathbb{R}^k \longrightarrow \mathbb{R}^{k+n+m}$. So the square passes to a commutative square of K-spaces

$$\begin{tikzcd} K/(K \cap H) \arrow[r] \arrow[d, "\cong"'] & G/H \arrow[d, "\cong"] \\ \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n}) \arrow[r] & \mathbf{L}(\mathbb{R}^k, \mathbb{R}^{k+n+m}) \end{tikzcd}$$

in which the vertical maps are isomorphisms. Proposition 2.2 shows that the upper horizontal map is a cofibration of K-spaces; so the lower horizontal map above is a cofibration of K-spaces.

Now we can prove the proposition we $\mathcal{U}$ (and hence U) is finite dimensional. We can assume that the underlying inner product spaces of V and $\mathcal{U}$ are $\mathbb{R}^k$ respectively $\mathbb{R}^{k+n++m}$ with the standard scalar product, and that $U = \mathbb{R}^{k+n}$. The G-action on V is then given by a continuous homomorphism $\rho : G \longrightarrow O(k)$ and the K-action on $\mathcal{U}$ is given by a continuous homomorphism $\psi : K \longrightarrow O(k+n+m)$. Since U is a

$\mathcal{U}$-subrepresentation, the image of ψ must be contained in the subgroup $O(k+n)\times O(m)$. The $(K\times G^{\mathrm{op}})$-action on the map (2.4) is then obtained by restriction of the $(O(k+n)\times O(m)\times O(k)^{\mathrm{op}})$-action along the homomorphism

$$\psi\times\rho^{\mathrm{op}} \ : \ K\times G^{\mathrm{op}} \ \longrightarrow \ (O(k+n)\times O(m)\times O(k)^{\mathrm{op}}) \ .$$

Restriction along any continuous homomorphism between compact Lie groups preserves cofibrations by Proposition 1.22 (i), so the map (2.4) is a $(K\times G^{\mathrm{op}})$-cofibration by the first part.

Now we can treat the case when the dimension of $\mathcal{U}$ is infinite. We choose an exhausting nested sequence of K-subrepresentations

$$U = U_0 \subset U_1 \subset U_2 \subset \dots$$

Then all the morphism $\mathbf{L}(V,U_{n-1}) \longrightarrow \mathbf{L}(V,U_n)$ are $(K\times G^{\mathrm{op}})$-cofibrations by the above. Since cofibrations are closed under sequential composite, the morphism

$$\mathbf{L}(V,U_0) \ \longrightarrow \ \mathrm{colim}_n\, \mathbf{L}(V,U_n) \ \cong \ \mathbf{L}(V,\mathcal{U})$$

is also a $(K\times G^{\mathrm{op}})$-cofibration.

(ii)

(iii) Since the G-action on V is faithful, the G-action on $\mathbf{L}(V,\mathcal{U})$ is free. Since $\mathbf{L}(V,\mathcal{U})$ is $(K\times G^{\mathrm{op}})$-cofibrant by part (ii), the functor

$$\mathbf{L}(V,\mathcal{U})\times_G - \ : \ G\mathbf{U} \ \longrightarrow \ K\mathbf{U}$$

takes G-cofibrations to K-cofibrations by Proposition 1.23. Since $\mathbf{L}(V,\mathcal{U})/G = \mathbf{L}(V,\mathcal{U})\times_G *$ and the one-point G-space is cofibrant, the claim follows. □

Proposition 2.5. *Let G and K be compact Lie groups, V a G-representation (possibly infinite dimensional) and $\mathcal{U}_K$ a complete K-universe.*

(i) *If G acts faithfully on V, then the $(K\times G^{\mathrm{op}})$-space $\mathbf{L}(V,\mathcal{U}_K)$ is a universal space for the family $\mathcal{F}(K;G^{\mathrm{op}})$ of graph subgroups.*

(ii) *For every faithful G-subrepresentation U of V the restriction morphism*

$$\rho^V_U \ : \ \mathbf{L}(V,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(U,\mathcal{U}_K)$$

is a $(K\times G^{\mathrm{op}})$-homotopy equivalence.

Proof. (i) We let Γ be any closed subgroup of $K\times G^{\mathrm{op}}$. Since the G-action on V is faithful the induced right G-action on $\mathbf{L}(V,\mathcal{U}_K)$ is free. So if Γ intersects $1\times G^{\mathrm{op}}$ non-trivially, then $\mathbf{L}(V,\mathcal{U}_K)^\Gamma$ is empty. On the other hand, if $\Gamma\cap(1\times G^{\mathrm{op}}) = \{(1,1)\}$, then Γ is the graph of a unique continuous homomorphism $\alpha : L\longrightarrow G$, where L is the projection of Γ to K. Then

$$\mathbf{L}(V,\mathcal{U}_K)^\Gamma \ = \ \mathbf{L}^L(\alpha^*V,\mathcal{U}_K)$$

is the space of L-equivariant linear isometric embeddings from the L-representation α^*V to the underlying L-universe of $\mathcal{U}_L$. Since $\mathcal{U}_K$ is a complete K-universe, the underlying L-universe is also complete, and so the space $\mathbf{L}^L(\alpha^*V,\mathcal{U}_K)$ is contractible by Proposition 2.1 (ii).

It remains to show that $\mathbf{L}(V,\mathcal{U}_K)$ is homotopy equivalent to a cofibrant $(K\times G^{\mathrm{op}})$-space. If V is finite dimensional, then $\mathbf{L}(V,\mathcal{U}_K)$ is $(K\times G^{\mathrm{op}})$-cofibrant itself [...].

If the dimension of V is infinite, then $\mathbf{L}(V,\mathcal{U}_K)$ is *not* cofibrant, and we have to work a little harder. We choose an exhausting nested sequence

$$U_0 \ \subset \ U_1 \ \subset \ U_2 \ \dots$$

of finite-dimensional G-subrepresentation of $\mathcal{U}_G$ such that U_0 is faithful. We claim that the restriction morphism

$$\rho^V_{U_0} \ : \ \mathbf{L}(V,\mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(U_0,\mathcal{U}_K)$$

is a $(K\times G^{\mathrm{op}})$-homotopy equivalence; since the target is $(K\times G^{\mathrm{op}})$-cofibrant, this finishes the proof.

We claim that all the restriction maps

$$\rho_n \ : \ \mathbf{L}(U_n, \mathcal{U}_K) \ \longrightarrow \ \mathbf{L}(U_{n-1}, \mathcal{U}_K)$$

are $(K \times G^{\mathrm{op}})$-acyclic fibrations, i.e., for every subgroup $\Gamma \leq K \times G^{\mathrm{op}}$ the fixed point map

$$(\rho_n)^\Gamma \ : \ \mathbf{L}(U_n, \mathcal{U}_K)^\Gamma \ \longrightarrow \ \mathbf{L}(U_{n-1}, \mathcal{U}_K)^\Gamma$$

is a weak equivalence and Serre fibration. Since G acts faithfully on U_0 it also acts faithfully on U_n for all $n \geq 0$, and so the Γ-fixed points of source and target are empty whenever $\Gamma \cap (1 \times G^{\mathrm{op}})) = \{(1,1)\}$. Otherwise Γ is the graph of continuous homomorphism $\alpha : L \longrightarrow G$ with $L \leq K$. Sp the fixed point map $(\rho_n)^\Gamma$ is the restriction map

$$(\rho_n)^\Gamma \ : \ \mathbf{L}^L(\alpha^*(U_n), \mathcal{U}_K) \ \longrightarrow \ \mathbf{L}^L(\alpha^*(U_{n-1}), \mathcal{U}_K) \ .$$

Source and target of this map are contractible by Proposition 2.1 (ii), so the map $(\rho_n)^\Gamma$ is a weak equivalence. But $(\rho_n)^\Gamma$ is also a locally trivial fiber bundle, so it is also a Serre fibration. The $(K \times G^{\mathrm{op}})$-space $\mathbf{L}(V, \mathcal{U}_K)$ is the inverse limit of the tower of $(K \times G^{\mathrm{op}})$-acyclic fibrations ρ_n between cofibrant $(K \times G^{\mathrm{op}})$-spaces. So Proposition 2.6 below shows that the restriction map from the inverse limit $\mathbf{L}(V, \mathcal{U}_K)$ to $\mathbf{L}(U_0, \mathcal{U}_K)$ is a $(K \times G^{\mathrm{op}})$-equivariant homotopy equivalence.

(ii) Since G acts faithfully on U it also acts faithfully on V, so source and target of the map ρ_U^V are a universal spaces for the family $\mathcal{F}(K; G^{\mathrm{op}})$ of graph subgroups by (i). Any equivariant map between universal spaces for the same family of subgroups is an equivariant homotopy equivalence. □

Proposition 2.6. *Let $\mathcal{C}$ be a topological model category in which every object is fibrant and let*

$$\cdots \xrightarrow{p_{n+1}} X_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0$$

be a tower of acyclic fibrations between cofibrant objects. Then the canonical morphism

$$p_\infty \ : \ X_\infty \ = \ \lim_{n \geq 0} X_n \ \longrightarrow \ X_0$$

is a homotopy equivalence.

Proof. Since $p_n : X_n \longrightarrow X_{n-1}$ is an acyclic cofibration between cofibrant objects we can choose a section $s_n : X_{n-1} \longrightarrow X_n$ to p_n and a homotopy

$$H_n \ : \ [0,1] \times X_n \ \longrightarrow \ X_n$$

from the identity to $s_n \circ p_n$ such that $p_n \circ H_n : [0,1] \times X_n \longrightarrow X_{n-1}$ is the constant homotopy from p_n to itself. The morphisms

$$s_n \circ s_{n-1} \circ \cdots \circ s_1 \ : \ X_0 \ \longrightarrow \ X_n$$

are then compatible, so they assemble into a morphism $s_\infty : X_0 \longrightarrow X_\infty$ to the inverse limit, and s_∞ is a section to p_∞.

To prove the claim we construct compatible homotopies

$$K_n \ : \ [0,1] \times X_\infty \ \longrightarrow \ X_n$$

by induction on n satisfying

(i) $p_n \circ K_n = K_{n-1}$
(ii) $K_n(t, -) = p_\infty^{(n)}$ for all $t \in [0, \frac{1}{n+1}]$
(iii) $K_n(1, -) = s_n \circ s_{n-1} \circ \cdots s_1 \circ p_\infty$

The induction starts by defining K_0 as the constant homotopy from $p_\infty : X_\infty \longrightarrow X_0$ to itself. Now we assume $n \geq 0$ and suppose that the homotopies $K_0, \ldots, K_{n-1}$ have already been constructed. We define K_n symbolically by

$$K_n(t,-) \;=\; \begin{cases} p_\infty^{(n)} & \text{for } t \in [0, \frac{1}{n+1}], \\ H_n(n(n+1)t - n,\, p_\infty^{(n)}(-)) & \text{for } t \in [\frac{1}{n+1}, \frac{1}{n}], \text{ and} \\ s_n(K_{n-1}(t,-)) & \text{for } t \in [\frac{1}{n}, 1]. \end{cases}$$

This is well-defined at the intersections of the intervals because

$$H_n\left(n(n+1)\frac{1}{n+1} - n,\, p_\infty^{(n)}(-)\right) \;=\; H_n(0, p_\infty^{(n)}(-)) \;=\; p_\infty^{(n)}$$

and

$$\begin{aligned} H_n\left(n(n+1)\frac{1}{n} - n,\, p_\infty^{(n)}(-)\right) \;&=\; H_n(1, p_\infty^{(n)}(-)) \;=\; s_n \circ p_n \circ p_\infty^{(n)} \\ &=\; s_n \circ p_\infty^{(n-1)} \;=\; s_n\left(K_{n-1}(1/n, -)\right) \end{aligned}$$

Then condition (i) holds because

$$\begin{aligned} p_n(K_n(t,-)) \;&=\; \begin{cases} p_n \circ p_\infty^{(n)} & \text{for } t \in [0, \frac{1}{n+1}], \\ p_n(H_n(n(n+1)t - n,\, p_\infty^{(n)}(-))) & \text{for } t \in [\frac{1}{n+1}, \frac{1}{n}], \text{ and} \\ p_n(s_n(K_{n-1}(t,-))) & \text{for } t \in [\frac{1}{n}, 1], \end{cases} \\ &=\; \begin{cases} p_\infty^{(n-1)} & \text{for } t \in [0, \frac{1}{n}], \\ K_{n-1}(t,-) & \text{for } t \in [\frac{1}{n}, 1], \end{cases} \\ &=\; K_{n-1}(t,-)\ . \end{aligned}$$

Now we can finish the proof. By condition (i) the homotopies K_n are compatible, they assemble into a morphism $K_\infty : [0,1] \times X_\infty \longrightarrow X_\infty$. Property (ii) shows that K_∞ starts with the identity of X_∞ and property (iii) ensures that K_∞ ends with the morphism $s_\infty \circ p_\infty$. So s_∞ and p_∞ are mutually inverse homotopy equivalences. $\square$

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