PICARD GROUPOIDS AND SPECTRA

The goal of this lecture is to explain in detail the statement that a Picard groupoid is "the same thing as" an Ω-spectrum E with $\pi_i(E) = 0$ for $i \neq 0, 1$. Along the way we introduce the notion of a (very special) Γ-space, which provides one of the possible ways of formalising the concept of an "abelian group structure defined up to all the higher homotopies", and we present an approach to K-theory based on the notion of a permutative category.

1. Picard groupoids. Let $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ be a symmetric monoidal category (α is the associativity constraint, λ and ρ are the left and right unit constraints, and β is the symmetry constraint). We write $\pi_0(\mathcal{C})$ for the set of isomorphism classes of objects of $\mathcal{C}$; it is an abelian monoid, where the operation in $\pi_0(\mathcal{C})$ is induced by $\otimes$. Further, we write $\pi_1(\mathcal{C}) = \mathrm{Aut}_{\mathcal{C}}(\mathbb{1})$; it is an abelian group. The commutativity of $\pi_1(\mathcal{C})$ is a completely general fact about monoidal categories: in fact, the monoid $\mathrm{End}_{\mathcal{C}}(\mathbb{1})$ of endomorphisms of $\mathbb{1}$ is commutative, because it is a submonoid of the Bernstein center of $\mathcal{C}$, defined as the monoid of endomorphisms of the identity functor of $\mathcal{C}$. [To see that $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \hookrightarrow \mathrm{End}(\mathrm{Id}_{\mathcal{C}})$, recall that the identity functor of $\mathcal{C}$ is isomorphic to the functor $X \mapsto \mathbb{1} \otimes X$. The commutativity of $\mathrm{End}(\mathrm{Id}_{\mathcal{C}})$ is a simple exercise.]

Definition: We say that $\mathcal{C}$ is a **Picard groupoid** if the underlying category of $\mathcal{C}$ is a groupoid, and $\pi_0(\mathcal{C})$ is a group (i.e., every object of $\mathcal{C}$ is invertible).

Note that if $\mathcal{C}$ is a Picard groupoid, then for every object $X \in \mathcal{O}b(\mathcal{C})$ we have a *canonical* isomorphism $\mathrm{Aut}_{\mathcal{C}}(\mathbb{1}) \to \mathrm{Aut}_{\mathcal{C}}(X)$ arising from the autoequivalence $\mathcal{C} \to \mathcal{C}$, $Y \mapsto X \otimes Y$. In particular, this allows us to define an important invariant of $\mathcal{C}$, namely, a group homomorphism $\pi_0(\mathcal{C}) \to \pi_1(\mathcal{C})$, $[X] \mapsto c_X$, where $c_X \in \mathrm{Aut}_{\mathcal{C}}(\mathbb{1})$ is the element corresponding to $\beta_{X,X} \in \mathrm{Aut}_{\mathcal{C}}(X \otimes X)$. Note that $c_X^2 = 1$ for all $X \in \mathcal{O}b(\mathcal{C})$.

Definition: We say that a Picard groupoid $\mathcal{C}$ is **strictly commutative**, or simply **strict**, if $c_X = 1$ for all $X \in \mathcal{O}b(\mathcal{C})$.

Remark: The first comment that helps understand this notion is that strict commutativity makes sense for general symmetric monoidal categories, not just for Picard groupoids. Namely, a symmetric monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \beta)$ is said to be **strictly commutative** if $\beta_{X,X} = \mathrm{id}_{X \otimes X}$ for every $X \in \mathcal{O}b(\mathcal{C})$. Consider the category of finite dimensional vector spaces over a given field $\Bbbk$, with the usual symmetric monoidal structure. It is NOT strictly commutative. The point is that if V is a vector space of dimension > 1 and $v, w \in V$ are nonproportional vectors, then $v \otimes w \neq w \otimes v$ in $V \otimes V$, and hence the symmetry automorphism of $V \otimes V$ is not the identity.[1)]

On the other hand, the category of one-dimensional vector spaces **is** strictly commutative. We now present two examples of Picard groupoids, the first of which generalizes the groupoid of one-dimensional vector spaces over a field.

Examples: Let us fix a commutative ring R.

(1) We define Pic_R to be the groupoid of invertible R-modules, with the usual symmetric monoidal structure. In more detail, the objects of Pic_R are the (finitely generated) R-modules M such that there exists another R-module N with $M \underset{R}{\otimes} N \cong R$ (such an M is automatically projective). The morphisms in Pic_R are isomorphisms of R-modules, the monoidal structure is given by $\underset{R}{\otimes}$, and the associativity and commutativity constraints are the obvious ones. This Picard groupoid is STRICTLY commutative. Note that

$$\pi_0(\mathrm{Pic}_R) \cong \mathrm{Pic}(\mathrm{Spec}\, R) \qquad \text{(the usual Picard group of the scheme } \mathrm{Spec}\, R)$$

$$\text{and} \quad \pi_1(\mathrm{Pic}_R) \cong R^\times.$$

(2) We define $\mathrm{Pic}_R^{\mathbb{Z}}$ to be the Picard groupoid of invertible $\mathbb{Z}$-graded R-modules equipped with the "super" commutativity constraint. [Note that in Beilinson's work this groupoid is denoted by $\mathcal{L}_R$.] In more detail, the objects of $\mathrm{Pic}_R^{\mathbb{Z}}$ are pairs of the form $(\mathcal{L}, p_{\mathcal{L}})$, where $\mathcal{L}$ is an invertible R-module and $p_{\mathcal{L}} : \mathrm{Spec}\, R \to \mathbb{Z}$ is a locally constant function. (Here, "p" stands for "parity".) The monoidal structure is given by

$$(\mathcal{L}, p_{\mathcal{L}}) \otimes (\mathcal{M}, p_{\mathcal{M}}) = (\mathcal{L} \otimes \mathcal{M}, p_{\mathcal{L}} + p_{\mathcal{M}}),$$

the associativity constraint is the obvious one, but the commutativity constraint is modified by a sign, namely, it is given by

$$a \otimes b \mapsto (-1)^{p_{\mathcal{L}} p_{\mathcal{M}}} b \otimes a.$$

Because of this, $\mathrm{Pic}_R^{\mathbb{Z}}$ is NOT strictly commutative. However, it is more useful for us than Pic_R. [I should have mentioned above that if $(\mathcal{L}, p_{\mathcal{L}})$ and $(\mathcal{M}, p_{\mathcal{M}})$ are objects of $\mathrm{Pic}_R^{\mathbb{Z}}$, then there are no morphisms between them unless $p_{\mathcal{L}} \equiv p_{\mathcal{M}}$, in which case the morphisms are isomorphisms of R-modules. Thus $\pi_0(\mathrm{Pic}_R^{\mathbb{Z}}) \cong \mathrm{Pic}(\mathrm{Spec}\, R) \oplus \mathbb{Z}$ and $\pi_1(\mathrm{Pic}_R^{\mathbb{Z}}) \cong R^\times$.]

The first important idea of this story is that $\mathrm{Pic}_R^{\mathbb{Z}}$ is the correct target[2)] for the "determinant functor" on the category of perfect complexes of R-modules. This is explained below.

2. Determinants.

We begin with the following

Definition: Let $\mathcal{C}$ be a Waldhausen category and $\mathcal{P} = (\mathcal{P}, \otimes, \mathbb{1}, \lambda, \rho, \beta)$ is a strictly associative Picard groupoid.

(The assumption that $\mathcal{P}$ is strictly associative, i.e., that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\alpha_{X,Y,Z} = \mathrm{id}$ for all $X, Y, Z \in \mathcal{P}$ is inessential and is only imposed for the purpose of keeping the size of the diagrams that appear below under control.)

A **determinant functor** from $\mathcal{C}$ to $\mathcal{P}$ is a pair $(\det, \psi)$ consisting of a functor

$$\det : w\mathcal{C} \to \mathcal{P}$$

and a collection of isomorphisms

$$\psi(i, p) : \det(A) \otimes \det(C) \xrightarrow{\simeq} \det(B)$$

for every cofibration sequence

$$A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C$$

in $\mathcal{C}$, satisfying the following axioms:

(i) Given a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} A \arrow[r, tail, "i"] \arrow[d, "\lambda_1"', "\wr"] & B \arrow[r, two heads, "p"] \arrow[d, "\lambda_2"', "\wr"] & C \arrow[d, "\lambda_3"', "\wr"] \\ A' \arrow[r, tail, "i'"] & B' \arrow[r, two heads, "p'"] & C' \end{tikzcd}$$

the induced diagram commutes:

$$\begin{tikzcd} \det(A) \otimes \det(C) \arrow[r, "\psi(i,p)"] \arrow[d, "\det(\lambda_1)\otimes\det(\lambda_3)"', "\wr"] & \det(B) \arrow[d, "\wr"', "\det(\lambda_2)"] \\ \det(A') \otimes \det(C') \arrow[r, "\psi(i',p')"] & \det(B') \end{tikzcd}$$

(ii) Given a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d, tail] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d, two heads] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\ A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'' \end{tikzcd}$$

the induced diagram commutes:

$$\begin{tikzcd} \det(A') \otimes \det(A'') \otimes \det(C') \otimes \det(C'') \arrow[r] \arrow[d, "\text{symmetry}"] & \det(A) \otimes \det(C) \arrow[dd] \\ \det(A') \otimes \det(C') \otimes \det(A'') \otimes \det(C'') \arrow[d] & \\ \det(B') \otimes \det(B'') \arrow[r] & \det(B) \end{tikzcd}$$

(the unlabelled maps are induced by ψ).

(iii) We have $\det(0) = \mathbb{1}$. In addition, for each $X \in \mathcal{C}$, consider the two cofibration sequences

$$X \overset{\mathrm{id}_X}{\rightarrowtail} X \overset{0}{\twoheadrightarrow} 0 \quad \text{and}$$

$$0 \overset{0}{\rightarrowtail} X \overset{\mathrm{id}_X}{\twoheadrightarrow} X.$$

Then

$$\psi(\mathrm{id}_X, 0) = \rho_{\det(X)} : \det(X) \otimes \mathbb{1} \to \det(X)$$

and

$$\psi(0, \mathrm{id}_X) = \lambda_{\det(X)} : \mathbb{1} \otimes \det(X) \to \det(X).$$

This is a sort of "normalization" axiom.[3)]

Remark: This definition is purely ad hoc, and may be the wrong one to use for a general Waldhausen category. Note, however, that it does formally imply a "multiplicativity" property of determinants that one wants. Namely, recall that $\mathcal{C}$ has coproducts, namely, if $A, B \in \mathcal{O}b(\mathcal{C})$, then $A \oplus B$ exists and can be constructed as the pushout

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & B \arrow[d, tail] \\ A \arrow[r, tail] & A \oplus B \end{tikzcd}$$

In turn, the arrow $A \rightarrowtail A \oplus B$ fits into a cofibration sequence $A \rightarrowtail A \oplus B \twoheadrightarrow C$, where C can be defined as the pushout

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] & A \oplus B \arrow[d] \\ 0 \arrow[r, tail] & C \end{tikzcd} \tag{$*$}$$

Observe, however, that one can take $C = B$. Indeed, there is a unique map $A \oplus B \to B$ which is the identity on B and which is the composition $A \to 0 \to B$ on A. It is then trivial to verify that this arrow $A \oplus B \to B$ satisfies the universal property of the pushout $(*)$. In particular, the axioms for det imply the existence of natural isomorphisms

$$\det(A) \otimes \det(B) \xrightarrow{\simeq} \det(A \oplus B),$$

functorial with respect to weak equivalences of A's and B's.

(Incidentally, this argument also answers a question that came up during the first lecture. Namely, when one is defining K_0 of a Waldhausen category, one does not need to impose the additivity requirement $[A \oplus B] = [A] + [B]$: it follows automatically from the condition $[X] = [Y] + [X/Y]$ for every cofibration sequence $Y \rightarrowtail X \twoheadrightarrow X/Y$.)

The definition of a determinant functor is taken from the paper

[KM] F. Knudsen and D. Mumford, "The projectivity of the module space of stable curves I: preliminaries on 'det' and 'div' ".

The authors of the paper consider the special case where $\mathcal{C} = \mathcal{C}_R^\bullet = \mathrm{Ch}^{\mathrm{b}}(\mathcal{P}(R))$ (the Waldhausen category of bounded complexes of finitely generated projective R-modules, where the weak equivalences are the quasi-isomorphisms and the cofibrations are the termwise split monomorphisms), and where $\mathcal{P} = \mathrm{Pic}_R^{\mathbb{Z}}$ is the Picard groupoid of "$\mathbb{Z}$-graded superlines over R" introduced above. They prove the following

Theorem: There exists a unique up to canonical isomorphism pair $(\det, \psi)$ which to every commutative ring R associates a determinant functor

$$(\det_R, \psi_R) : \mathcal{C}_R^\bullet \to \mathrm{Pic}_R^{\mathbb{Z}}$$

in a way which is compatible with base change with respect to R, and satisfies the following properties:

(i) If $M \in \mathcal{P}(R)$, viewed as a complex concentrated in degree 0, then

$$\det_R(M) = \left(\bigwedge^{\mathrm{rk}(M)} M, \mathrm{rk}(M) \right)$$

(recall that, in general, $\mathrm{rk}(M)$ is not a number, but a locally constant function $\operatorname{Spec} R \to \mathbb{Z}$).

(ii) If $0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$ is a short exact sequence of R-modules, where $M', M, M'' \in \mathcal{P}(R)$, then the isomorphism

$$\psi_R(i,p) : \bigwedge^{\operatorname{rk} M'}(M') \bigotimes_R \bigwedge^{\operatorname{rk} M''}(M'') \to \bigwedge^{\operatorname{rk} M}(M)$$

is equal to the standard one, defined as follows: on a sufficiently small affine open in $\operatorname{Spec} R$, we may assume that M', M'' (and hence also M) are free, and let $(e_1, \dots, e_r, f_1, \dots, f_s)$ be a basis of M such that $e_1, \dots, e_r$ is a basis of $i(M') \cong M'$, and $p(f_1), \dots, p(f_s)$ is a basis of M''; then the isomorphism $\bigwedge^r M' \otimes \bigwedge^s M'' \xrightarrow{\simeq} \bigwedge^{r+s} M$ is given by

$$(e_1 \wedge \ldots \wedge e_r) \otimes (p(f_1) \wedge \ldots \wedge p(f_s)) \mapsto e_1 \wedge \ldots \wedge f_s.$$

(In fact, what is really proved in [KM] is a natural extension of the theorem above to arbitrary schemes instead of just the affine ones. The category $\mathcal{P}(R)$ is replaced by the exact category $\mathcal{C}_X$ of locally free sheaves of finite rank on a scheme X. Furthermore, Knudsen and Mumford also show that det extends in a canonical way to the category of perfect complexes and quasi-isomorphisms on X. However, it is somewhat nontrivial to formulate the axioms of such an extension, so for the sake of time and space we refer the reader to [KM] for more information.)

3. Strict Picard groupoids. A substantial part of this lecture will be devoted to a description of Picard groupoids from the homotopy-theoretic point of view. We begin by describing the strictly commutative Picard groupoids, since their study requires fewer technical tools and for them we can formulate a precise statement, following

[Del] P. Deligne, "La formule de dualité globale", SGA 4, tome 3.

Let $A^{-1} \xrightarrow{d} A^0$ be a two-step complex of abelian groups. It defines a strictly commutative Picard groupoid $\mathcal{P}(A^0)$ as follows:

Note that $\pi_i(\mathcal{P}(A^0)) = H^{-i}(A^0)$ for i=0,1

- $\mathcal{O}b\,\mathcal{P}(A^0) = A^0$;
- $\mathcal{P}(A^0)(a,b) = \{f \in A^{-1} \,|\, d(f) = b - a\}$ for $a, b \in A^0$;
- the composition of morphisms is induced by the addition in A^{-1};
- the monoidal structure is given by the addition in A^0;
- the associativity and commutativity constraints are given by $0 \in A^{-1}$.

Conversely, every strictly commutative Picard groupoid $\mathcal{P}$ is equivalent to one which arises this way. The idea of the proof is as follows: choose a set of objects $\{X_i\}_{i \in I}$ of $\mathcal{P}$ that is a complete set of representatives for the isomorphism classes of objects of $\mathcal{P}$ (all our categories are assumed to be essentially small). Let $A^0 = \mathbb{Z}^I$, and let $\mathcal{P}(A^0)$ denote the discrete Picard groupoid associated to the complex $0 \to A^0$. Then one can show that there exists a Picard functor (i.e., a monoidal functor which is compatible with the commutativity constraints)

$$F : \mathcal{P}(A^0) \to \mathcal{P}$$

such that $F(e_i) = X_i$, where $e_i \in A^0 = \mathbb{Z}^I$ is the i-th basis vector. [Of course, the assumption that $\mathcal{P}$ is strictly commutative plays a crucial role here: otherwise F could not possibly be a Picard functor.] Then we define*) A^{-1} to be the set of pairs (x, t), where $x \in A^0$ and $t : F(x) \xrightarrow{\sim} F(0)$ is an isomorphism in $\mathcal{P}$. We leave it to the reader to define the addition on A^{-1} and check that the natural functor $\mathcal{P}(A^0) \to \mathcal{P}$ is an equivalence; see [Del], § 1.4 for more details. Deligne also proves the following

Theorem: Let $\mathcal{PLC}^{\text{str}}$ denote the category of small *strict* Picard groupoids, where the morphisms are the isomorphism classes of Picard functors, and let $D^{[-1,0]}(\mathcal{A}b)$ denote the derived category of complexes of abelian groups supported in degrees 0 and -1. Then the construction $A^0 \mapsto \mathcal{P}(A^0)$ extends to an equivalence of categories[4] $D^{[-1,0]}(\mathcal{A}b) \xrightarrow{\sim} \mathcal{PLC}^{\text{str}}$.

In fact, Deligne proves[5] an extension of this theorem to the context of strict Picard stacks (i.e., sheaves of strictly commutative Picard groupoids).

4. Some actual fairytales. Let $\mathcal{PLC}$ denote the category of all small Picard groupoids, not just the strictly commutative ones. On the other hand, let

$$\mathrm{Ho}^{[0,1]}(\mathcal{S}p^{\mathbb{N}}) \subset \mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$$

denote the full subcategory of the stable homotopy category (say, that of spectra of simplicial sets, as discussed in the previous lectures) formed by the spectra E such that $\pi_n(E) = 0$ for $n \neq 0, 1$. A natural extension of the previous result to this context is the following

"Theorem": There is a natural equivalence of categories[6]

$$\mathrm{Ho}^{[0,1]}(\mathcal{S}p^{\mathbb{N}}) \xrightarrow{\sim} \mathcal{PLC}$$

which preserves π_0 and π_1.

The reason the word theorem appears in quotation marks is that I have not seen the result stated in this form anywhere in the literature, nor have I worked out the details of the proof. However, this is not so much a fairytale, since it is known how to construct functors in both directions, and the proof of the theorem is probably not very difficult. Note that one also expects a similar result for sheaves of spectra and sheaves of Picard groupoids. For more details on this, see § 5.5 in

V. Drinfeld, "Infinite-dimensional vector bundles in algebraic geometry (an introduction)", math.AG/0300155.

The constructions of the two functors between $\mathrm{Ho}^{[0,1]}(\mathcal{S}p^{\mathbb{N}})$ and $\mathcal{PLC}$ are explained below, after some preliminaries on Γ-spaces. Here we explain how the strictly commutative Picard spectra fit into this picture. If $A^0 = (A^{-1} \xrightarrow{d} A^0)$ is as before, let $\widetilde{A}$ denote the simplicial abelian group associated to A^0 (recall that the category of complexes of abelian groups in nonpositive degrees is equivalent to the category of simplicial abelian groups — that is, simplicial objects in $\mathcal{A}b$). Now any simplicial abelian group has a classifying simplicial set $B\widetilde{A}$, which, if constructed correctly, is itself a simplicial abelian group, whence we can iterate this construction, obtaining a sequence $B\widetilde{A},\ BB\widetilde{A}, \ldots$. Now we get a spectrum

$$\widetilde{A} \cong \Omega B\widetilde{A},\ \Omega BB\widetilde{A},\ \Omega BBB\widetilde{A},\ \ldots$$

(the structure maps are induced by the natural homotopy equivalences

$$B^n\widetilde{A} \to \Omega B^{n+1}\widetilde{A}\ ).$$

This is the Ω-spectrum corresponding to the strict Picard groupoid $\mathcal{P}(A^0)$.

Two special cases of this construction deserve explicit mention. If we start with a discrete Picard groupoid $\mathcal{P}$, i.e., one with $\pi_1(\mathcal{P}) = 0$, then $\mathcal{P}$ is just an abelian group, namely, $\pi_0(\mathcal{P})$, and the corresponding Ω-spectrum is the sequence of Eilenberg-MacLane spaces

$$K(\pi_0(\mathcal{P}), 0),\ K(\pi_0(\mathcal{P}), 1),\ K(\pi_0(\mathcal{P}), 2), \dots.$$

Similarly, if $\mathcal{P}$ is a strict Picard groupoid such that $\pi_0(\mathcal{P}) = 0$, then the corresponding Ω-spectrum is the sequence

$$K(\pi_1(\mathcal{P}), 1),\ K(\pi_1(\mathcal{P}), 2),\ K(\pi_1(\mathcal{P}), 3), \dots.$$

Next let $\mathcal{C}$ be a Waldhausen category, let $K(\mathcal{C})$ denote its K-theory spectrum (the construction of $K(\mathcal{C})$ as a symmetric spectrum was explained in the first lecture, but here it is enough to think of it as an ordinary spectrum), and let $\prod(K(\mathcal{C}))$ denote the Picard groupoid corresponding to the $[0,1]$-truncation of $K(\mathcal{C})$, as in the theorem above.

Hope (possibly unreasonable): There exists a determinant functor

$$(\det, \psi) : \mathcal{C} \to \prod(K(\mathcal{C}));$$

moreover, if $\mathcal{C}$ is a "reasonable" Waldhausen category, this functor is "universal" with respect to all determinant functors from $\mathcal{C}$ to Picard groupoids.

At present I do not know how to define the terms "reasonable" and "universal", nor do I know how to prove the result above.[7)] However, the statement is partially motivated by precise results of P. May and R. Thomason. (Reference: a paper of May-Thomason and another paper of May in Topology, vol. 17.) They work with algebraic K-theory in a different context (namely, that of "permutative categories"=strictly associative and strictly unital symmetric monoidal categories), and prove that the entire K-theory spectrum construction $\mathcal{C} \mapsto K(\mathcal{C})$ for such categories can be characterized in a completely axiomatic way.

To the best of my knowledge, no analogues of the results of May and Thomason for Waldhausen categories exist at present, and it is not even clear how to formulate these analogues.

However, in the special case where

$$\mathcal{C} = \mathrm{Ch}^{\mathrm{b}}(\mathcal{P}(R)),$$

one can observe a consequence of the conjecture stated above, namely, the determinant functor $\mathcal{C} \to \mathrm{Pic}_R^{\mathbb{Z}}$ does induce a Picard functor

$$\prod K(R) = \prod K(\mathcal{C}) \to \mathrm{Pic}_R^{\mathbb{Z}},$$

or, equivalently, a morphism of spectra

$$K(R) = K(\mathcal{C}) \to K(\mathrm{Pic}_R^{\mathbb{Z}}),$$

where the RHS is the K-theory spectrum of $\mathrm{Pic}_R^{\mathbb{Z}}$ produced by the constructions of May and Segal, explained below.[8)]

5. Γ-spaces. We now briefly discuss the category of Γ-spaces (also called $\mathcal{F}$-spaces in the works of May and Thomason), following [BF] and the paper

[Seg] G. Segal, “Categories and cohomology theories”, Topology **13** (1974), 293 – 312.

The formalism of Γ-spaces provides one of the precise ways of talking about spaces equipped with an abelian group structure “up to all homotopies”. The precise definition is given below; note that we follow the conventions of [BF] in that what Segal calls a Γ-space we will call a *special topological* Γ-*space*.

Definition: Let Γ be the category whose objects are the finite sets and whose morphisms are defined as follows: if $S, T \in \Gamma$, then a morphism $S \to T$ is a map of sets $\theta : S \to 2^T$ such that $\theta(s_1) \cap \theta(s_2) = \varnothing$ whenever $s_1, s_2 \in S$ and $s_1 \neq s_2$. The composition is defined in the obvious way.[9)]

On the other hand, let $\mathcal{F}$ denote the category of finite pointed sets. It was first observed by Anderson that $\mathcal{F}$ is equivalent to the opposite category Γ^{op}. More precisely, we can define a functor

$$\begin{aligned} \Phi : \mathcal{F} &\to \Gamma^{\mathrm{op}} \\ (S, *) &\mapsto S \setminus \{*\} \end{aligned}$$

which acts on morphisms as follows: if

$$f : (S, *) \to (T, *)$$

is a morphism in $\mathcal{F}$, then

$$\Phi(f) : T \setminus \{*\} \to 2^{S \setminus \{*\}}$$

is given by

$$\Phi(f)(t) = f^{-1}(t).$$

Definition: If $\mathcal{C}$ is a category, a **Γ-object** of $\mathcal{C}$ is a functor $\Gamma^{\mathrm{op}} \to \mathcal{C}$, or, equivalently, a functor $\mathcal{F} \to \mathcal{C}$.

Remark: A Γ-object of $\mathcal{C}$ can be thought of as a simplicial object equipped with some extra data. Namely, we have a natural faithful functor $\triangle \to \Gamma$ which induces a bijection on isomorphism classes of objects, defined as follows:

$$\begin{gathered} [m] \mapsto \{1, 2, \ldots, m\} =: \underline{m} \\ (f : [m] \to [n]) \longmapsto (i \mapsto \theta(i) = \{j |\, f(i-1) < j \leq f(i)\}) \end{gathered}$$

By means of this functor, every Γ-object yields a simplicial object of $\mathcal{C}$.

Definition: Suppose $\mathcal{C}$ is a category with finite products and a suitable notion of “homotopy equivalence”. The principal examples to keep in mind are:

(1) $\mathcal{C} = \mathcal{C}at$, the category of small categories, where homotopy equivalence means equivalence of categories;

(2) $\mathcal{C} = \mathcal{T}op_*$, the category of (say, weakly Hausdorff and compactly generated) pointed topological spaces;[10)]

(3) $\mathcal{C} = \mathcal{S}_*$, the category of pointed simplicial sets.

The Γ-objects in these examples will be called, respectively, Γ-categories, topological Γ-spaces, and Γ-spaces, respectively. In a general situation like this, a Γ-object A is said to be **special** if for each $n \geq 0$, the morphism

$$A(\underline{n}) \to A(\underline{1}) \times \ldots \times A(\underline{1}),$$

induced by the n nontrivial morphisms

$$A(\underline{n}) \to A(\underline{1})$$

corresponding to the morphisms $\underline{1} \to \underline{n}$ in Γ given by $1 \mapsto \{k\} \in \underline{n}$ $(1 \leq k \leq n)$, is a homotopy equivalence. Note that the case $n = 0$ of this definition is the statement that the unique map from $A(\underline{0}) = A(\varnothing)$ to the terminal object of $\mathcal{C}$ is a homotopy equivalence, i.e., $A(\underline{0})$ is contractible.

Exercise: A Γ-object of $\mathcal{C}$ satisfying the stronger condition that the morphisms $A(\underline{n}) \to A(\underline{1}) \times \ldots \times A(\underline{1})$ constructed above are *isomorphisms* for all $n \geq 0$ (in particular, this forces $A(\underline{0})$ to be a terminal object of $\mathcal{C}$) is the same thing as a commutative monoid in $\mathcal{C}$.

More precisely, starting from such a Γ-object A, we obtain a binary operation on $A(\underline{1})$ as the composition

$$A(\underline{1}) \times A(\underline{1}) \to A(\underline{2}) \to A(\underline{1})$$

(the first arrow is the inverse of $A(\underline{2}) \to A(\underline{1}) \times A(\underline{1})$, and the second arrow is induced by the morphism $\underline{1} \to \underline{2}$, $1 \mapsto \{1, 2\}$). On the other hand, the unique morphism $\underline{1} \to \underline{0}$ in Γ induces a map $A(\underline{0}) \to A(\underline{1})$, and the object $A(\underline{1})$ of $\mathcal{C}$ together with the two maps $A(\underline{1}) \times A(\underline{1}) \to A(\underline{1})$ and $A(\underline{0}) \to A(\underline{1})$ is an abelian monoid.

In general, a special topological Γ-space A in particular yields an H-space structure on $A(\underline{1})$ obtained by choosing a homotopy inverse $A(\underline{1}) \times A(\underline{1}) \to A(\underline{2})$ for the map $A(\underline{2}) \to A(\underline{1}) \times A(\underline{1})$ and looking at the composition

$$A(\underline{1}) \times A(\underline{1}) \to A(\underline{2}) \to A(\underline{1}).$$

Moreover, this H-space structure is homotopy commutative. However, a special topological Γ-space structure is much more than just a homotopy commutative H-space structure.[11)]

Definition: A topological Γ-space A is said to be **very special** if it is special and, in addition, the H-space structure on $A(1)$ admits a homotopy inverse. A Γ-space $A : \Gamma^{\mathrm{op}} \to \mathcal{S}_*$ is said to be **very special** if its geometric realization

$$|A| : \Gamma^{\mathrm{op}} \to \mathcal{T}op$$

is very special. A Γ-category $A : \Gamma^{\mathrm{op}} \to \mathcal{C}at$ is **very special** if it is special and its classifying space $BA : \Gamma^{\mathrm{op}} \to \mathcal{T}op$ is very special.[12)]

Remark: Probably, in order for a Γ-space $A : \Gamma^{\mathrm{op}} \to \mathcal{S}_*$ to be very special, it is enough to require that A is special and that the abelian monoid $\pi_0 A(\underline{1})$ is a group.[13)]

The category of Γ-spaces $A : \Gamma^{\mathrm{op}} \to \mathcal{S}_*$ is studied in the work [BF] of Bousfield and Friedlander. In particular, they construct a strict model structure and a stable model structure on this category; the second one produces a homotopy category which is equivalent to the stable homotopy category of connective spectra. We do not have enough time to go into the construction of these model structures, but we will explain how to go between Γ-spaces and (connective) spectra.

6. Γ-spaces and connective spectra. Recall the category $\mathcal{S}p^{\mathbb{N}}$ of spectra of simplicial sets; it is complete and cocomplete. In particular, the product of two spectra E, F is denoted by $E \times F$, and coproduct of E and F is denoted by $E \vee F$ and called the **wedge** of E and F. Observe that since $\mathcal{S}p^{\mathbb{N}}$ is pointed (the initial object is the same as the terminal object), we have a natural morphism $E \vee F \to E \times F$. Moreover, we have a closed right action of $\mathcal{S}_*$ on $\mathcal{S}p^{\mathbb{N}}$ (it was discussed in one of the previous lectures for symmetric spectra, but for ordinary spectra the definition is the same). In particular, if $E, F \in \mathcal{S}p^{\mathbb{N}}$, we have the simplicial set $\mathrm{Map}(E, F)$ of maps from E to F.

Key observation: $\mathrm{Map}(E, F)$ can be extended to a Γ-space $\mathrm{Hom}(E, F)$ such that $\mathrm{Hom}(E, F)(\underline{1}) = \mathrm{Map}(E, F)$. Namely, we define

$$\mathrm{Hom}(E, F)(\underline{n}) = \mathrm{Map}(\underbrace{E \times \ldots \times E}_{n \text{ times}}, F).$$

The structure maps of this Γ-space are defined in the obvious way.[14)]

In particular, the natural maps

$$\mathrm{Hom}(E, F)(\underline{n}) \to \mathrm{Map}(E, F) \times \ldots \times \mathrm{Map}(E, F)$$

are induced by the inclusions

$$E \vee \ldots \vee E \to E \times \ldots \times E.$$

Moreover, assuming that E and F are fibrant and cofibrant with respect to the stable model structure on $\mathcal{S}p^{\mathbb{N}}$ (probably, weaker hypothesis suffice),[15)] the Γ-space $\mathrm{Hom}(E, F)$ is very special. (This has to do with the fact that the inclusion $E \vee E \vee \ldots \vee E \to E \times \ldots \times E$ is a stable homotopy equivalence for any spectrum E.)

Remark: This leads to one way of turning the stable category $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ into an additive category. Namely, recall from the previous lecture that $\mathrm{Ho}(\mathcal{S}p^{\mathbb{N}})$ is equivalent to the more concrete category $\mathrm{ho}(\mathcal{S}p^{\mathbb{N}})$ whose objects are the fibrant and cofibrant spectra, and where the set of morphisms between E and F is given by $\pi_0 \mathrm{Map}(E, F)$. As we have just explained, this set can be made into an abelian group in a canonical way. On the other hand, for the applications to the Beilinson's work it is quite essential to remember that $\mathrm{Map}(E, F)$ is the degree 1 component of a very special Γ-space, i.e., that $\mathrm{Map}(E, F)$ has an "abelian group structure up to all higher homotopies".

Proposition (Bousfield & Friedlander): Let S denote the sphere spectrum. The functor

$$\mathrm{Hom}(S, \cdot) : \mathcal{S}p^{\mathbb{N}} \longrightarrow \Gamma\text{-spaces}$$

has a left adjoint. We will denote it by

$$A \mapsto S \wedge A.$$

It is a special case of a construction which to every Γ-space A associates a functor

$$\begin{aligned} \mathcal{S}p^{\mathbb{N}} &\to \mathcal{S}p^{\mathbb{N}}, \\ E &\mapsto E \wedge A. \end{aligned}$$

Bousfield and Friedlander construct this functor in three steps. First, every Γ-space A, viewed as a functor $\mathcal{F} \to \mathcal{S}_*$, extends uniquely to a colimit-preserving functor $\mathcal{S}ets_* \to \mathcal{S}_*$ (where $\mathcal{S}ets_*$ denotes the category of all pointed sets, not necessarily finite). Next, the functor $\mathcal{S}ets_* \to \mathcal{S}_*$ extends in an obvious way to a functor

$$\mathcal{S}_* \to \mathcal{S}_*^2 = \{\text{bisimplicial sets}\}$$

(just apply the functor $\mathcal{S}ets_* \to \mathcal{S}_*$ to each of the components of a given simplicial set). Composing it with the diagonal, we obtain a functor $\mathcal{S}_* \to \mathcal{S}_*$. Finally, the latter functor extends to a functor $\mathcal{S}p^{\mathbb{N}} \to \mathcal{S}p^{\mathbb{N}}$, again by applying it to each level separately.16)

Proposition (Bousfield & Friedlander): If (v.s. Γ-spaces) denotes the category of very special Γ-spaces, then the functors above restrict to a pair of adjoint functors

$$S \wedge \cdot : (\text{v.s. }\Gamma\text{-spaces}) \rightleftarrows \Omega\mathcal{S}p^{\mathbb{N}} : \mathrm{Hom}(S, \cdot)$$

Moreover, if we define a morphism of Γ-spaces $f : A \to B$ to be a **weak equivalence** if the corresponding morphism of spectra $S \wedge A \to S \wedge B$ is a stable homotopy equivalence, then the functors $S\wedge\cdot$ and $\mathrm{Hom}(S, \cdot)$ induce mutually quasi-inverse equivalences of categories17)

$$\mathrm{Ho}(\Gamma\text{-spaces})^{\text{stable}} \rightleftarrows \mathrm{Ho}\left(\underbrace{\mathcal{S}p^{\mathbb{N}}_{\text{conn}}}_{\text{connective spectra}}\right)^{\text{stable}}$$

Alternatively, one can define a morphism $f : A \to B$ of very special Γ-spaces to be a **strict weak equivalence** if the corresponding map $A(\underline{n}) \to B(\underline{n})$ is a weak equivalence of simplicial sets for all $n \geq 1$, and then we also obtain mutually quasi-inverse equivalences of categories

$$\mathrm{Ho}(\text{v.s. }\Gamma\text{-spaces})^{\text{strict}} \rightleftarrows \mathrm{Ho}\left(\underbrace{\Omega\mathcal{S}p^{\mathbb{N}}_{\text{conn}}}_{\text{connective }\Omega\text{-spectra}}\right)^{\text{strict}}.$$

This completes our discussion of the relationship between Γ-spaces and spectra.

7. Very special Γ-spaces and Picard groupoids. The last topic of the lecture is a description of the relationship between Picard groupoids (or, more generally, permutative categories) and (very) special Γ-spaces. Together with the results discussed in the previous section, this provides in particular a link between Picard groupoids and spectra.

Construction 1. If $\mathcal{P}$ is a permutative18) category, there is a construction defined by P. May (based on [Seg]) that associates to $\mathcal{P}$ a special Γ-category $K(\mathcal{P})$; if $\pi_0(\mathcal{P})$ is an abelian group rather than merely an abelian monoid, then $K(\mathcal{P})$ is very special.

Explicitly, for every finite set S, we define $K(\mathcal{P})(S)$ to be the category whose objects are collections of sets X_U, one for each subset $U \subseteq S$, together with isomorphisms $X_U \otimes X_V \xrightarrow{\simeq} X_{U\cup V}$ for all pairs of *disjoint* subsets $U, V \subset S$ that are compatible with the symmetry constraint in the sense that the diagrams

$$\begin{CD} X_U \otimes X_V @>>> X_{U\cup V} \\ @V{\beta}VV @| \\ X_V \otimes X_U @>>> X_{V\cup U} \end{CD}$$

commute, and satisfy an obvious compatibility condition for each triple of pairwise disjoint subsets $U, V, W \subset S$. One also requires $X_\varnothing = \mathbb{1}$, and one requires the isomorphisms $X_\varnothing \otimes X_U \to X_U$ to coincide with the unit constraint in $\mathcal{P}$. The structure maps in this Γ-category are defined in the obvious way. By abuse of notation, we write $K(\mathcal{P}) = S \wedge K(\mathcal{P})$ and call it the **K-theory spectrum of** $\mathcal{P}$; this provides one of the possible approaches to higher algebraic K-theory. In general, $K(\mathcal{P})$ is an Ω-spectrum starting with level 1, and it is an Ω-spectrum on the nose whenever $\pi_0(\mathcal{P})$ is a group.[19)]

Construction 2. To a very special Γ-space A one associates the Picard groupoid

$$\Pi(A) := \text{the fundamental groupoid (or Poincaré groupoid) of } |A(1)|.$$

The monoidal structure on $\Pi(A)$ is induced by the H-space structure on $|A(1)|$.

Thus we obtain the following picture:[20)]

$$\begin{tikzcd}
\left\{\begin{matrix}\text{strictly}\\ \text{associative}\\ \text{Picard}\\ \text{groupoids}\end{matrix}\right\} \arrow[r, bend left, "\text{May \& Segal}"] \arrow[d, phantom, "\bigcap"] & \left\{\begin{matrix}\text{very special}\\ \Gamma\text{-spaces}\end{matrix}\right\} \arrow[l, bend left, "\text{Poincaré groupoid}"] \arrow[r, bend left, "S\wedge\cdot"] \arrow[d, phantom, "\bigcap"] & \left\{\begin{matrix}\text{connective}\\ \Omega\text{-spectra}\end{matrix}\right\} \arrow[l, bend left, "\mathrm{Hom}(S,\cdot)"] \\
\left\{\begin{matrix}\text{permutative}\\ \text{categories}\end{matrix}\right\} \arrow[r, "\text{May-Segal}"] & \left\{\begin{matrix}\text{special}\\ \Gamma\text{-spaces}\end{matrix}\right\} &
\end{tikzcd}$$

In particular, remember that the classifying space of a Picard groupoid is not quite an abelian group, but rather a very special Γ-space, which then leads to a connective Ω-spectrum.

Post-lecture footnotes.

1) An even simpler example of a non-strict symmetric monoidal category is the category of vector spaces with *direct sum* as the monoidal functor.

2) In fact, this can already be seen at the level of linear algebra. If $(\mathrm{vect}_{\Bbbk}, \oplus)$ is the symmetric monoidal category of finite dimensional vector spaces over a field $\Bbbk$ with direct sum as the monoidal structure, it is natural to wish for the determinant to be a symmetric monoidal functor from this category to some Picard groupoid. However, if we look at the naive functor $V \mapsto \wedge^{\mathrm{top}} V$, then it is a monoidal functor, but it is not compatible with the symmetry constraint, because the determinant of the automorphism of $V \oplus V$ that switches the two factors is equal to $(-1)^{\dim V}$. This already shows that the target of the determinant functor on $(\mathrm{vect}_{\Bbbk}, \oplus)$ cannot be a strictly commutative Picard groupoid. On the other hand, it is a simple exercise to check that the functor

$$(\mathrm{vect}_{\Bbbk}, \oplus) \to \mathrm{Pic}^{\mathbb{Z}}_{\Bbbk}, \qquad V \mapsto (\wedge^{\mathrm{top}} V, \dim_{\Bbbk} V)$$

can be made into a monoidal functor in the obvious way, and it IS compatible with the symmetry constraints.

3) As Drinfeld pointed out, one should also require a compatibility with the associativity constraint, namely, if we start with a three-step filtration of an object of $\mathcal{C}$,

$$0 = A_0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail A_3 = A,$$

then the following diagram should commute:

$$\begin{tikzcd}
(\det(A_1) \otimes \det(A_2/A_1)) \otimes \det(A_3/A_2) \arrow[dr, "\simeq"] \arrow[dddd, "\alpha"', "\wr"] & \\
& \det(A_2) \otimes \det(A_3/A_2) \arrow[d, "\wr"] \\
& \det(A_3) \\
& \det(A_1) \otimes \det(A_3/A_1) \arrow[u, "\wr"'] \\
\det(A_1) \otimes (\det(A_2/A_1) \otimes \det(A_3/A_2)) \arrow[ur, "\simeq"] &
\end{tikzcd}$$

However, M. Shulman explained that this axiom follows from condition (ii) in the definition of the determinant by taking $A'' = 0$, $A' = A = A_1$, $B' = A_2$, $B = A_3$.

$^{*)}$ Thus, in some sense, A^{-1} is the kernel of F.

$^{4)}$ Actually, it is better to prove a statement about an equivalence between 2-categories. In fact, $\mathcal{PLC}^{\mathrm{str}}$ has a natural 2-category structure, and we have only made it into a 1-category in a brutal way.

$^{5)}$ It appears that the statement is originally due to Grothendieck. However, it was first stated and proved explicitly in [Del].

$^{6)}$ In other words, if one replaces complexes of abelian groups with spectra and removes the words "strictly commutative", the result proved in Deligne's paper remains true. Note that this generalization, as well as its analogue for sheaves, are also (presumably) due to Grothendieck.

$^{7)}$ As Drinfeld remarked during the lecture, perhaps it is not the Waldhausen category but rather the definition of the determinant functor that should be reasonable. In other words, the notion of a determinant functor should be formulated in such a way that the "Hope" we have stated is forced to be true.

This discussion provides a way of how one could have invented algebraic K-theory. Let us briefly explain this for the simpler setup of exact categories. Let $\mathcal{A}$ be any exact category. The first observation is that if A is an abelian group, one has the notion of a **dimension function** on $\mathcal{A}$ with values in A. Namely, it is a map $\dim : \mathcal{O}b(\mathcal{A}) \to A$ which is additive with respect to short exact sequences. (The conditions $\dim(0) = 0$ and $\dim X = \dim Y$ for $X \cong Y$ are automatically forced by this property.) Moreover, it turns out that there exists a universal dimension function on $\mathcal{A}$ (in the obvious sense), and the target for it is the group $K_0(\mathcal{A})$. This is one of the possible ways of discovering the Grothendieck group construction.

However, it is impossible to describe the group $K_1(\mathcal{A})$ just by itself in a similar way. We will not give a precise proof of this statement; however, it is already illustrated well by the example of the category $\mathrm{vect}_{\Bbbk}$. Namely, if we want the determinant functor to be compatible with the symmetry constraint, then, for an object $V \in \mathrm{vect}_{\Bbbk}$, we must remember not only its determinant in the usual sense, i.e., $\wedge^{\mathrm{top}} V$ (the top exterior power of

V), but also the dimension of V, i.e., its class in $K_0(\mathrm{vect}_{\Bbbk})$. Thus what can be described by a universal property is not the group $K_1(\mathcal{A})$ but rather the pair of groups $(K_0(\mathcal{A}), K_1(\mathcal{A}))$. More precisely, they are the π_0 and π_1 of the Picard groupoid which is the target of the universal determinant functor on $\mathcal{A}$. (The definition of a determinant functor on an exact category can be obtained by specializing the definition for Waldhausen categories given above.)

[8)] Let us make once again the following comment, which was mentioned several times in the course of these lectures. The simplest reason for the appearance of spectra in Beilinson's work is essentially the fact from undergraduate linear algebra that if V is a finite dimensional vector space and $\tau \in \mathrm{Aut}(V \oplus V)$ is the automorphism which interchanges the two factors, then $\det(\tau) = (-1)^{\det V}$. This explains why the target for a correct determinant functor is a non-strictly commutative Picard groupoid, and it is a fact of life that in homotopy theory such groupoids correspond to (non-abelian) spectra. O course, one could artificially get rid of spectra by tensoring the stable homotopy category $\mathrm{Ho}\left(\mathcal{S}p^{\mathbb{N}}\right)$ with $\mathbb{Q}$, so that it becomes equivalent to the derived category of complexes of $\mathbb{Q}$-vector spaces, but spectra are just more natural objects to work with.

[9)] If the reader prefers, she/he may define the composition of morphisms in Γ in a way that forces the map $\mathcal{F} \to \Gamma^{\mathrm{op}}$ described below to become a functor.

[10)] As remarked by Peter May, it is better to use weak homotopy equivalences in the topological situation; for example, this leads to the correct model structure on the category of topological Γ-spaces.

[11)] In other words, Segal pointed out that in order to specify a (weak) H-space structure on $A(\underline{1})$ one can specify a morphism $A(\underline{2}) \to A(\underline{1})$, where $A(\underline{2})$ is another space equipped with a weak equivalence $A(\underline{2}) \to A(\underline{1}) \times A(\underline{1})$. Then the question becomes: how to formulate the conditions of associativity and commutativity of this operation "up to higher homotopies". Segal's discovery was that all the required information can be conveniently encoded in a single (special) Γ-space.

Note also that another way of thinking about the operation in $A(\underline{1})$ is as follows: we have the weak equivalence $\pi : A(\underline{2}) \to A(\underline{1}) \times A(\underline{1})$, and then for every pair $(x, y) \in A(\underline{1}) \times A(\underline{1})$ we have not one way of multiplying them, but many different ones, parameterized by the (homotopy) fiber of π over (x, y), which is contractible.

[12)] In fact, since we have decided to replace homotopy equivalence with weak homotopy equivalence in the topological situation, it follows that in each of the three examples the notion of a very special Γ-object can be stated more economically as follows: a Γ-object A is very special if and only if it is special and $\pi_0 A(\underline{1})$ (which is then an abelian monoid) is a group.

[13)] In fact, an arbitrary H-space X admits weak homotopy inverses if and only if $\pi_0(X)$ is a group.

[14)] Namely: consider a morphism $f : \underline{m} \to \underline{n}$ in Γ. The corresponding map

$$\mathrm{Map}(E^n, F) \to \mathrm{Map}(E^m, F)$$

is induced by the map $E^m \to E^n$, defined by taking the i-th component of E^m to the sub-product of E^n corresponding to the subset $f(i) \subseteq \underline{n}$ via the diagonal map. Note that if $f(i) = \varnothing$, then the corresponding factor E in E^m goes to the spectrum $*$ (the unital object in $\mathcal{S}p^{\mathbb{N}}$).

[15] In fact, it seems that the only assumption we need is that F is an Ω-spectrum, and no conditions on E are required. Note in particular that below we apply this construction to $E = S$, the sphere spectrum, which is not an Ω-spectrum.

[16] There is another construction which produces a spectrum from a special Γ-space A, which is an Ω-spectrum starting with level 1, and is even an Ω-spectrum on the nose if A is very special. This construction is due to Segal and is somewhat more intuitive than the construction of Bousfield and Friedlander. Namely, it turns out that any special Γ-space A has a classifying space BA, which is a very special Γ-space. Hence we can iterate this construction and obtain a sequence of spaces $A(\underline{1}), (BA)(\underline{1}), (BBA)(\underline{1}), \ldots$, which is the desired spectrum.

[17] An elementary way of memorizing this is to remember the *incorrect* statement that a connective Ω-spectrum is "the same thing" as an abelian topological group. To get a correct statement one has to work with "topological abelian groups up to all the higher homotopies", and the rest is just an explanation of how to do this formally.

Note once again that the "abelian" (or "additive") version of this situation is simpler: namely, by the Dold-Puppe theorem, a complex of abelian groups concentrated in nonpositive degrees **is** "the same thing as" a simplicial abelian group.

[18] Recall again that a permutative category is the same thing as a strictly associative and strictly unital symmetric monoidal category. The strictness assumption is not really essential here, and it is best not to worry about it. In fact, the whole theory can be formulated for arbitrary symmetric monoidal categories, only the formulation would be much more clumsy.

[19] In fact, the statement that the Γ-category $K(\mathcal{P})$ is very special whenever $\pi_0(\mathcal{P})$ is a group is a tautology, because it is obvious that $\pi_0(\mathcal{P}) \cong \pi_0(K(\mathcal{P})(\underline{1}))$ as monoids. Also, it is not hard to see that whenever $\mathcal{P}$ is a Picard groupoid, we have $\pi_i(K(\mathcal{P})(\underline{1})) = 0$ for all $i \geq 2$.

[20] Note that the construction of Segal (written down explicitly by May) provides one of the possible approaches to higher algebraic K-theory, because if R is any ring and $\mathcal{P}(R)$ is the permutative category of finitely generated projective R-modules with direct sum as the monoidal functor, then $K(\mathcal{P}(R))$ is equivalent to the K-theory spectrum $K(R)$ produced by any of the other known constructions.