TALK ABOUT A "WILKERSON CRITERION" FOR MORAVA E-THEORY

CHARLES REZK

1. Λ-rings

A Λ-ring is a commutative ring A, together with functions $\lambda^k\colon A \to A$ for $k \geq 0$, such that

(1) $\lambda^0(a) = 1$, $\lambda^1(a) = a$,
(2) $\lambda^k(0) = 0$,
(3) $\lambda^k(1) = 0$ for $k \geq 2$,
(4) $\lambda^k(a+b) = \sum_{0\leq i\leq k} \lambda^i(a)\lambda^{k-i}(b)$,
(5) $\lambda^k(ab) = P(\lambda^i(a), \lambda^j(b))$,
(6) $\lambda^k\lambda^\ell(a) = Q(\lambda^i(a))$.

The P and Q are certain polynomials which are deduced from the "splitting principle". A "line bundle" is an element $a \in A$ such that $\lambda^k(a) = 0$ for $k \geq 2$. The formulae for (5) and (6) are those valid for arbitrary sums of line bundles, where we stipulate that product of two line bundles is a line bundle.

A Λ-ring with involution is a Λ-ring A, together with a homomorphism $\psi^{-1}\colon A \to A$ of Λ-rings such that $\psi^{-1}\psi^{-1} = \mathrm{id}_A$. I write Λ for the category of Λ-rings with involution.

Complex equivariant K-theory $X \mapsto K^0_G(X)$ naturally takes values in Λ; the operations λ^k are defined by kth exterior power, and ψ^{-1} is defined by complex conjugation.

We are going to give another description of Λ, in terms of the multiplicative group scheme $\mathbb{G}_m$. Note that $K^0_{S^1}(\mathrm{pt}) \approx RS^1 \approx \mathbb{Z}[t, t^{-1}]$, and that this carries a coproduct associated to the group homomorphism $S^1 \to S^1 \times S^1$. We identify $\mathbb{G}_m = \mathrm{Spec}(RS^1)$.

The endomorphism ring of $\mathbb{G}_m$ is $\mathbb{Z}$, where $n \in \mathbb{Z}$ corresponds to the map $[n]\colon \mathbb{G}_m \to \mathbb{G}_m$ defined by $[n]^*(f(t)) = f(t^n)$. Let $\mathcal{G} \approx \mathbb{Z} - \{0\}$ denote the monoid of non-zero endomorphisms.

If we consider the base-change $(\mathbb{G}_m)_{\mathbb{F}_p}$ to the finite field $\mathbb{F}_p$, then $[p]$ can be identified with the Frobenius map, since mod p we have $[p]^*(f(t)) = f(t^p) = f(t)^p$.

We define a category $\mathcal{A}$ as follows. The objects are commutative rings A, together with a homomorphism $\psi\colon \mathcal{G} \to \hom_{\mathrm{rings}}(A, A)$, satisfying the following congruence condition.

(1) *Congruence.* For each prime p, we have $\psi^p(a) \equiv a^p \mod p$.

Informally, this means that "ψ takes Frobenius to Frobenius", taking into account that Frobenius is not defined over $\mathbb{Z}$, but rather after base change to a ring of charateristic p. That is, when $f\colon (\mathbb{G}_m)_R \to (\mathbb{G}_m)_R$ is the Frobenius isogeny, then $\psi^f\colon A \otimes R \to A \otimes R$ is the Frobenius map.

Date: September 25, 2006.

There is a functor $\Psi\colon \Lambda \to \mathcal{A}$, which was constructed by Adams, using the formula

$$\sum_{m\geq 1} \psi^m(a)\cdot X^m = -\frac{d}{dX}\log\left[\sum_{k\geq 0}\lambda^k(a)\cdot(-X)^k\right]$$

to define the homomorphisms ψ^m.

Let Λ^{tf} and $\mathcal{A}^{\mathrm{tf}}$ denote the subcategories of torsion free objects.

Theorem 1.1 (Wilkerson)**.** *The functor $\Psi\colon \Lambda^{\mathrm{tf}} \to \mathcal{A}^{\mathrm{tf}}$ is an equivalence of categories.*

The proof uses the "Dwork lemma". That is, for $A \in \mathcal{A}^{\mathrm{tf}}$ there exist functions $\theta_d\colon A \to A$ which are uniquely determined by the expressions

$$\psi^m(a) = \sum_{d|m} d\,\theta_d(a)^{m/d}.$$

That this is so is a consequence of the congruence condition. Then the formal identities

$$\sum_{k\geq 0}\lambda^k(a)\cdot(-X)^k = \exp\left[-\sum_{m\geq 1}\frac{\psi^m(a)}{m}X^m\right] = \prod_{d\geq 1}(1-\theta_d(a)\cdot X^d)$$

show how to obtain the λ^k's from the θ_d's.

With a little bit of work, the category Λ is determined entirely by $\mathcal{A}$. One observes that the functor $U\colon \mathcal{A}^{\mathrm{tf}} \to \mathrm{Ab}^{\mathrm{tf}}$ admits a left adjoint $F\colon \mathrm{Ab}^{\mathrm{tf}} \to \mathcal{A}^{\mathrm{tf}}$, and that $\mathcal{A}^{\mathrm{tf}}$ is equivalent to the category of algebras over the monad $T' = UF$ on $\mathrm{Ab}^{\mathrm{tf}}$. We can prolong this to a monad T on Ab.

Theorem 1.2. *There is an equivalence of categories (T-algebras) $\approx \Lambda$.*

The key point here is that $T'(\mathbb{Z}^{\oplus k})$ can be shown to be isomorphic to the free Λ-ring with involution on k-generators.

2. Sheaves on formal groups

Fix a prime p and an integer $n \geq 1$.

Let $\mathcal{R}$ denote the category of Artinian local rings R such that the residue field $R/\mathfrak{m}$ is perfect of characteristic p. Morphisms are local homomorphisms.

Let $\mathcal{M}_R$ denote the category of R-modules, and $\mathcal{A}_R$ denote the category of R-algebras.

For R in $\mathcal{R}$, let $\mathcal{G}_R$ denote the category whose objects are formal groups G over R such that the restriction G_0 over $R/\mathfrak{m}$ has height n. The morphisms of $\mathcal{G}_R$ are isogenies, which is to say, non-zero homomorphisms of formal groups.

The set of morphisms $\mathcal{G}_R(G, G')$ has a topology. A basic open neighborhood of a map $\alpha\colon G \to G'$ consists of all homomorphisms $\beta\colon G \to G'$ such that α and β agree to order N, for some $N \geq 1$.

Let $\mathcal{G}_R^{\mathrm{iso}}$ denote the maximal subgroupoid of $\mathcal{G}_R$, i.e., the subcategory consisting of all the objects and all the isomorphisms.

For each $f\colon R \to R'$ in $\mathcal{R}$, there are base-change functors

$$f^*\colon \mathcal{M}_R \to \mathcal{M}_{R'}, \qquad f^*\colon \mathcal{A}_R \to \mathcal{A}_{R'}, \qquad f^*\colon \mathcal{G}_R \to \mathcal{G}_{R'}.$$

If $R \xrightarrow{f} R' \xrightarrow{g} R''$, there are natural isomorphisms $(gf)^* \approx g^* f^*$ satisfying the evident coherence property.

We define a category $\mathcal{M}$ as follows. The objects are data $\{M_R, M_f\}$, consisting of, for each R in $\mathcal{R}$, a functor

$$M_R \colon \mathcal{G}_R^{\mathrm{iso}^{\mathrm{op}}} \to \mathcal{M}_R,$$

and for each $f \colon R \to R'$ in $\mathcal{R}$, a natural isomorphism

$$M_f \colon f^* M_R \xrightarrow{\sim} M_{R'} f^*,$$

satisfying the following two conditions.

(1) *Coherence.* For $R \xrightarrow{f} R' \xrightarrow{g} R''$, both ways of constructing a natural map $g^* f^* M_R \to M_{R''} g^* f^*$ are identical.

(2) *Continuity.* The function

$$\mathcal{G}_R^{\mathrm{iso}}(G, G') \to \mathcal{M}_R(M_R(G'), M_R(G))$$

is continuous, where the right-hand side is given the compact-open topology, where the modules are given the discrete topology. Explicitly, this means that for all $m' \in M_R(G')$ and $m \in M_R(G)$, the set of $\alpha \in \mathcal{G}_R^{\mathrm{iso}}(G, G')$ such that $M_R(\alpha)(m') = m$ is open.

Morphisms $M \to N$ in $\mathcal{M}$ are natural transformations $M_R \to N_R$ which commute with the structure.

Given a complete local ring R with perfect characteristic p residue field, and a formal group G over R, we define the notation

$$M_R(G) \overset{\mathrm{def}}{=} \lim_k M_{R/\mathfrak{m}^k}(G \otimes_R R/\mathfrak{m}^k).$$

Let G be the universal deformation of a height n formal group G_0 defined over $\bar{\mathbb{F}}_p = R/\mathfrak{m}$, with $R = \mathbb{W}\bar{\mathbb{F}}_p[\![u_1, \ldots, u_{n-1}]\!]$. Let $\mathbb{G}_n = \mathcal{G}_{\bar{\mathbb{F}}_p}^{\mathrm{iso}}(G_0, G_0) \rtimes \mathrm{Gal}(\bar{\mathbb{F}}_p, \mathbb{F}_p)$, a profinite group. Then $\mathbb{G}_n$ acts on the inverse systems defining G and R, and thus acts on $M_R(G)$.

Morava E-theory is a functor

$$\mathcal{E} \colon \mathrm{Spectra} \to \mathcal{M}.$$

It is defined by

$$\mathcal{E}(X)_R(G) \overset{\mathrm{def}}{=} (E_{G_0})_0(X) \otimes_{(E_{G_0})_0} R,$$

where $G_0 = G \otimes_R R/\mathfrak{m}$, where E_{G_0} is the Morava E-theory spectrum associated to the universal deformation of G_0, and where $(E_{G_0})_0 \to R$ classifies G, viewed as a deformation of G_0.

For any ring A which contains $\mathbb{F}_p$, let $\phi \colon A \to A$ denote the ring homomorphism $\phi(a) = a^p$. When A is an R-algebra, let $\mathrm{Frob} \colon \phi^* A \to A$ denote the relative Frobenius defined by

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d] & R \arrow[r, equal] \arrow[d] & R \arrow[d] \\
A \arrow[r] \arrow[rr, bend right, "\phi"'] & \phi^* A \arrow[r, "\mathrm{Frob}"] & A
\end{tikzcd}$$

This defines a natural transformation $\mathrm{Frob}\colon \phi^*A \to A$ of R-algebras, whenever $\mathbb{F}_p \subset R$. Applying this to the ring of functions on a formal group gives the Frobenius isogeny

$$\mathrm{Frob}\colon G \to \phi^*G$$

of formal groups over R.

We define a category $\mathcal{A}$ as follows. The objects are data $\{A_R, A_f\}$, consisting of, for each R in $\mathcal{R}$, a functor

$$A_R\colon \mathcal{G}_R^{\mathrm{op}} \to \mathcal{A}_R,$$

and for each $f\colon R \to R'$ in $\mathcal{R}$ a natural isomorphism

$$A_f\colon f^*A_R \xrightarrow{\sim} A_{R'}f^*,$$

satisfying the following three conditions.

(1) *Coherence,* as in the definition of $\mathcal{M}$.
(2) *Continuity,* as in the definition of $\mathcal{M}$.
(3) *Congruence.* For all R in $\mathcal{R}$ containing $\mathbb{F}_p$, and all G in $\mathcal{G}_R$, the diagram

$$\begin{tikzcd} \phi^*A_R(G) \arrow[r, "A_\phi", "\sim"'] \arrow[dr, "\mathrm{Frob}"'] & A_R(\phi^*G) \arrow[d, "A_R(\mathrm{Frob})"] \\ & A_R(G) \end{tikzcd}$$

commutes.

3. Morava E-theory of commutative S-algebras

Let Comm_S denote the category of commutative S-algebra spectra. Then the Morava E-theory homology functor extends to a functor

$$\mathcal{E}\colon \mathrm{Comm}_S \to \mathcal{A}.$$

This is essentially an unpublished result of Ando-Hopkins-Strickland.

More can be said. Let $\mathcal{M}^{\mathrm{tf}} \subset \mathcal{M}$ denote the full subcategory of torsion free objects; that is, the full subcategory of M in $\mathcal{M}$ such that if we evaluate at a universal deformation G over R, we get an R-module $M_R(G)$ which has no p-torsion. Let $\mathcal{A}^{\mathrm{tf}} \subset \mathcal{A}$ denote the analogous full subcategory of torsion free objects. Then the forgetful functor $U\colon \mathcal{A}^{\mathrm{tf}} \to \mathcal{M}^{\mathrm{tf}}$ admits a left adjoint $F\colon \mathcal{M}^{\mathrm{tf}} \to \mathcal{A}^{\mathrm{tf}}$. Let $T' = UF$ denote the monad on $\mathcal{M}^{\mathrm{tf}}$. This can be prolonged to a monad T on $\mathcal{M}$.

Let $\mathbb{P}\colon \mathrm{Spectra} \to \mathrm{Comm}_S$ denote the free commutative S-algebra functor. There is a natural transformation

$$\gamma_X\colon \mathcal{E}(\mathbb{P}X) \to T(\mathcal{E}(X))$$

of functors $\mathrm{Spectra} \to \mathcal{A}$, with the property that if X is a spectrum such that $\mathcal{E}(X)$ is isomorphic (in $\mathcal{M}$) to a direct sum of copies of $\mathcal{E}(S^{2q})$, then γ_X is an isomorphism.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
E-mail address: `rezk@math.uiuc.edu`