A note on H_∞ structures

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Abstract

We give a source of (coconnective) examples of H_∞ structures that do not lift to E_∞ structures, based on Mandell's proof of the equivalence between certain cochain algebras and spaces.

1 Introduction

The diagonal map on a space X gives rise to a structure on its cochains which is not commutative, but associative and commutative up to higher coherences (an E_∞ structure). From this structure, the Steenrod reduced power operations arise; this was axiomatized in [May70]. One perspective would regard this as the structure of an E_∞ algebra on the function spectrum $(H\mathbb{F}_p)^X$, parametrizing maps from X into an Eilenberg-Mac Lane spectrum.

Power operations and the Adem relations between them do not make use of the entirety of an E_∞ structure. In the paper [May78] and in the book [BMMS86], the notion of an H_∞ ring spectrum was developed. This structure gives enough data to produce geometric power operations, but only depends on structure in the homotopy category.

An E_∞ structure seems to contain strictly more information than an H_∞ structure. However, decades passed before Noel produced an example of a

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ring spectrum with an H_∞ structure that does not lift to an E_3 structure [Noe], based on a counterexample to the transfer conjecture due to Kraines and Lauda.

In this paper we observe another source of such examples. In order to give the reader an idea of where we are going, we present the following.

Theorem 1. *Let R_k be a wedge of Eilenberg-Mac Lane spectra such that $\pi_* R_k$ is isomorphic to the graded ring $\mathbb{F}_2[x]/(x^3)$, with $|x| = -2^k$. For $k > 3$, there exists an H_∞ structure on R_k which does not come from an E_∞ structure.*

The H_∞ structures in the theorem can still be defined for $k \leq 3$, but these do come from E_∞ algebras: specifically, the function spectra $(H\mathbb{F}_2)^{\mathbb{RP}^2}$, $(H\mathbb{F}_2)^{\mathbb{HP}^2}$, and $(H\mathbb{F}_2)^{\mathbb{OP}^2}$.

In fact, our construction is essentially equivalent to the Hopf invariant one problem. We observe that an H_∞ structure on such a spectrum can be specified by describing Dyer-Lashof operations on the homotopy groups, and that these Dyer-Lashof operations are determined by the underlying H_∞ structure. However, to have a lift to an E_∞ structure, compatibility with the connective cover and results of Mandell [Man01] imply that we must be able to find a space with this cohomology algebra.

Both this paper and Noel's rely on heavy machinery. (A more direct proof here would lift Adams' proof from secondary Steenrod operations to secondary Dyer-Lashof operations, which would be intrinsically worthwhile.) Folklore has it that an H_∞ structure does not automatically determine an A_4-structure in the sense of Stasheff's associahedra, and so one might suspect that a simpler obstruction exists in terms of Toda brackets.

In addition, it is not clear in either case whether there are situations where there is a ring spectrum which *admits* an H_∞ structure, but not a (possibly unrelated) E_∞ structure, because both results rely on the impossibility of lifting a fixed H_∞ structure.

2 H_∞ structures

We will work in a highly developed category of spectra where smash-powers are homotopically well-behaved, and E_∞ algebras and commutative ring objects have the same homotopy theory; these include the S-modules of [EKMM97] and the category of symmetric spectra with the positive stable model structure (the S-model structure of [Shi04]). We leave the reader to fill in appropriate cofibrancy assumptions.

Let R be a commutative ring object in spectra, with category R-mod of R-modules. There is a monad $\mathbb{P}_R$ whose algebras are precisely commutative R-algebras:

$$\mathbb{P}_R(M) = \bigvee_{m \geq 0} M^{\wedge_R m}/\Sigma_m$$

Write R-alg for the category of commutative R-algebras.

Proposition 2. *There is a natural isomorphism $T \wedge_R \mathbb{P}_R(M) \cong \mathbb{P}_T(T \wedge_R M)$.*

Proof. Both functors are left adjoint to the composite in the commutative diagram of forgetful functors

$$\begin{tikzcd} T\text{-alg} \arrow[r] \arrow[d] & T\text{-mod} \arrow[d] \\ R\text{-alg} \arrow[r] & R\text{-mod}. \end{tikzcd}$$

Therefore, they are naturally isomorphic. □

The monad $\mathbb{P}_R$ is homotopically well-behaved, and descends to a monad on the homotopy category $Ho(R\text{-mod})$. We still denote this monad by $\mathbb{P}_R$, and refer to its algebras as H_∞ R-algebras [BMMS86]. If R is the sphere, we simply refer to these as H_∞ algebras.

For T a commutative R-algebra, there is a map of monads $\mathbb{P}_R \to \mathbb{P}_T$. This represents the forgetful functor from T-algebras to R-algebras, and on the homotopy category it represents the forgetful functor from H_∞ T-algebras to H_∞ R-algebras.

We note that this makes the natural isomorphism adjoint to the composite

$$\mathbb{P}_R(M) \to \mathbb{P}_R(T \wedge_R M) \to \mathbb{P}_T(T \wedge_R M).$$

Lemma 3. *Suppose T is a commutative R-algebra and M is a T-module which is a retract of $T \wedge_R N$ for some R-module N. Then any H_∞ T-algebra structure on M is determined by the underlying H_∞ R-algebra structure.*

In particular, if M is a wedge of suspensions of T, this automatically holds.

Proof. The free-forgetful adjunction between T-modules and R-modules allows us to factor the retraction into a sequence of T-module maps

$$M \to T \wedge_R N \to T \wedge_R M \to M,$$

with composite the identity. Applying $\mathbb{P}_T$ and then making use of Proposition 2, we get a retraction

$$\mathbb{P}_T(M) \to T \wedge_R \mathbb{P}_R(M) \to \mathbb{P}_T(M).$$

We then calculate homotopy classes of maps, finding that

$$[\mathbb{P}_T(M), M]_{T\text{-mod}} \to [\mathbb{P}_R(M), M]_{R\text{-mod}}$$

is the inclusion of a retract. Therefore, any H_∞ structure map $\mathbb{P}_T(M) \to M$ is determined by the induced map $\mathbb{P}_R(M) \to M$. $\square$

3 H_∞ algebras over a field

Fix a prime $p > 0$ and let $H = H\mathbb{F}_p$ be an Eilenberg-Mac Lane spectrum, equipped with a commutative ring structure. We recall the following.

Theorem 4. *The map π_*, from $Ho(H\text{-}mod)$ to the category of graded $\mathbb{F}_p$-vector spaces, is an equivalence of categories.*

Under this equivalence, the monad $\mathbb{P}_H$ becomes the monad sending a graded vector space V to the free object on V in the category of graded-commutative algebras over $\mathbb{F}_p$ equipped with Dyer-Lashof operations.

(The statement about the monad follows from [BMMS86, IX.2.2.1]; we have also found the references [Rez, Lur07] beneficial.)

In particular, this makes the category of H_∞ H-algebras equivalent to the category of graded-commutative algebras over $\mathbb{F}_p$ equipped with Dyer-Lashof operations.

As H-modules are all wedges of suspensions of H, Lemma 3 implies the following.

Proposition 5. *For any H_∞ H-algebra A, the Dyer-Lashof operations on $\pi_* A$ are uniquely determined by the underlying H_∞ structure on A.*

We now recall the following consequence of the Adem relations.

Proposition 6. *The functor $A \mapsto \pi_* A$, when restricted to the category of H_∞ H-algebras where the operation P^0 acts as the identity, is an equivalence to the category of unstable algebras over the Steenrod algebra.*

The standard calculations with the Adem relations in the mod-2 Steenrod algebra provide the following.

Proposition 7. *A graded ring isomorphic to $\mathbb{F}_2[x]/(x^3)$ is the homotopy of an H_∞ H-algebra, with the operation Q^0 acting as the identity, if and only if $|x|$ is a power of 2.*

4 Coconnective E_∞ algebras

We begin by pointing out the following consequence of the fact that connective covers are compatible with E_∞ algebra structures [BR08].

Proposition 8. *Suppose that A is an E_∞ algebra which is coconnective, in the sense that $\pi_* A = 0$ for $* > 0$. Then A canonically admits the structure of an E_∞ $H\pi_0(A)$-algebra.*

Corollary 9. *Suppose that A is a H_∞ H-algebra which is coconnective, in the sense that $\pi_* A = 0$ for $* > 0$. Then the H_∞ structure on A lifts to an E_∞ structure if and only if the H_∞ H-algebra structure lifts to an E_∞ H-algebra structure.*

Proof. One direction is clear. The previous proposition shows that a lifted E_∞ structure automatically comes from an E_∞ H-algebra structure, and Lemma 3 then implies that the two H_∞ H-algebra structures must agree, because they have the same underlying H_∞ structure by assumption. □

Therefore, to prove nonexistence results it suffices to work entirely within H-modules, which is equivalent to working with E_∞ algebras in the category of differential graded modules over $\mathbb{F}_p$.

Proposition 10. *Suppose that an E_∞ $\mathbb{F}_p$-chain complex A satisfies $H^i(A) = 0$ for $i < 0$, $H^0(A) = \mathbb{F}_p$, $H^1(A) = 0$, and $H^i(A)$ finite dimensional over $\mathbb{F}_p$. Then $A \otimes \overline{\mathbb{F}}_p$ is equivalent, as an E_∞ differential graded algebra over $\overline{\mathbb{F}}_p$, to the singular cochain complex of a space.*

Proof. The chain complex $A \otimes \overline{\mathbb{F}}_p$ is an E_∞ algebra, and the homology groups of $A \otimes \overline{\mathbb{F}}_p$ are isomorphic to $H^i(A) \otimes \overline{\mathbb{F}}_p$ as algebras over the Dyer-Lashof algebra. As a result, $A \otimes \overline{\mathbb{F}}_p$ satisfies the hypotheses of Mandell's Characterization Theorem [Man01]. □

This application of Mandell's theorem provides us a large library of nonexistence results merely by requiring that there exist a space realizing a given ring with its Dyer-Lashof operations.

Proposition 11. *A graded ring isomorphic to $\mathbb{F}_2[x]/(x^3)$, with Dyer-Lashof operations uniquely determined by the requirement that Q^0 acts by the identity, can be the homotopy ring of an E_∞ algebra A if and only if $|x|$ is 1, 2, 4, or 8.*

Proof. The nonexistence of maps of Hopf invariant one, due to Adams, implies that there are no spaces with cohomology ring $H^*(X; \mathbb{F}_2) \cong \mathbb{F}_2[x]/(x^3)$ unless x has one of the given degrees. As $H^*(X; \mathbb{F}_2)$ is the fixed subring of $H^*(X; \overline{\mathbb{F}}_2)$ under the action of Q^0, we then find that the ring $\overline{\mathbb{F}}_2[x]/(x^3)$, where $Q^0 x = x$, cannot be the cohomology ring of a space X. □

Combining Propositions 7 and 11 yields Theorem 1.

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