9TH GRADUATE STUDENT TOPOLOGY AND GEOMETRY CONFERENCE.

Michigan State University

"Puncture Stability"

for the pure mapping class group

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THE UNIVERSITY OF CHICAGO

The setting

Sequence $\{X_n\}$ of groups or spaces
with maps $\phi_n : X_n \to X_{n+1}$.

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Homological Stability:

The map ϕ_n induces isomorphism

$$H_i(X_n) \approx H_i(X_{n+1}),$$

when the parameter *n* is large with respect to *i*.

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Definition (Pure Mapping Class Group)

$$\mathrm{PMod}_{g,r}^n := \pi_o(\mathrm{PDiff}^+(\Sigma_{g,r}^n, rel\,\partial))$$

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$$\mathrm{PMod}_{g,r}^0 = \mathrm{Mod}_{g,r}^0 = \text{ Mapping class group of } \Sigma_{g,r}.$$

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- The pure braid group: $\mathrm{PMod}_{0,1}^n = P_n$

Remarks:

The group $PMod^n_{g,r}$:

- Is a finite index subgroup of the mapping class group $Mod^n_{g,r}$:

$$1 \to \mathrm{PMod}^n_{g,r} \to \mathrm{Mod}^n_{g,r} \to S_n \to 1$$

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- Is related with the topology of

$\mathcal{M}_{g,n} :=$ *the moduli space of n-pointed genus g projective curves,*

since

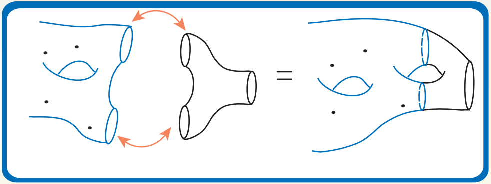
$$H^*(\mathrm{PMod}_g^n;\mathbb{Q}) \approx H^*(\mathcal{M}_{g,n};\mathbb{Q})$$

Parameters g and r: The maps.

Surfaces with boundary:

(1) *Increasing the Genus g.*

The inclusion $\Sigma^n_{g,r} \hookrightarrow \Sigma^n_{g+1,r-1}$



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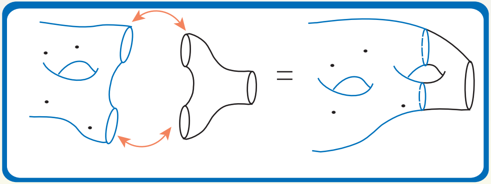

induces

$$\alpha : \mathrm{PMod}^n_{g,r} \to \mathrm{PMod}^n_{g+1,r-1}$$

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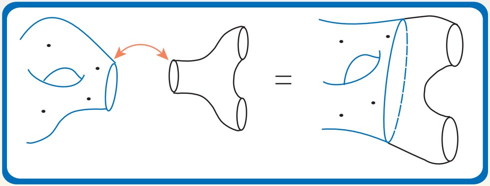

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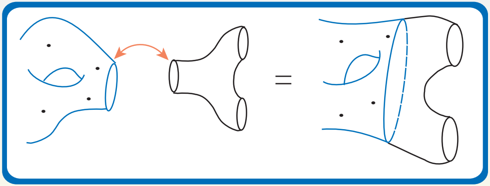

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“Genus” homological stability.

Theorem (Harer 1985)

The groups

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- Stable ranges have been improved by Ivanov and others.

Parameter n: Surfaces with boundary.

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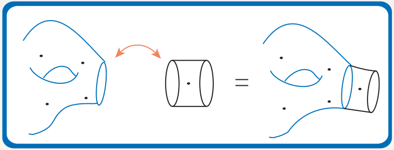

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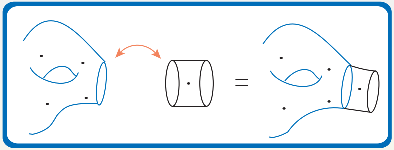

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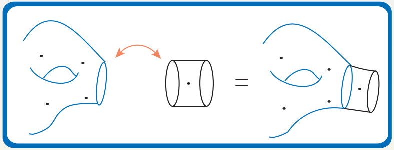

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Pure braid group case:

$$\mu_n : P_n \to P_{n+1}$$

“adding a strand”

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In general, the pure mapping class group fails to satisfy "puncture" homological stability.

Alternative: The forgetful map

The inclusion $\Sigma_{g,r}^{n+1} \hookrightarrow \Sigma_{g,r}^{n}$ induces

$$\boxed{f_n : \mathrm{PMod}_{g,r}^{n+1} \to \mathrm{PMod}_{g,r}^{n}}$$

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- f_n is defined for closed surfaces.
- For surfaces with boundary: $f_n \circ \mu_n = \mathrm{id}$.

Cohomology groups as S_n-representations

(a) $H^i(\mathrm{PMod}_{g,r}^n;\mathbb{Q})$ is a finite dimensional $\mathbb{Q}$-vector space with an S_n-action from

$$1 \to \mathrm{PMod}_{g,r}^n \to \mathrm{Mod}_{g,r}^n \to S_n \to 1$$

It is a finite dimensional rational S_n-representation.

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(b) The induced maps

$$f^i_n : H^i(PMod^n_{g,r};\mathbb{Q}) \to H^i(PMod^{n+1}_{g,r};\mathbb{Q})$$

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$\{H^i(\mathrm{PMod}^n_{g,r};\mathbb{Q}), f^i_n\}$ is a **consistent sequence** of S_n-representations.

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- Any S_n-representation decomposes over $\mathbb{Q}$ as a direct sum of irreducible representations.
- Irreducible representations of S_n are classified by partitions of n:

Irreducible S_n-representations	**Partitions of n**
$V(\lambda)_n = V(\lambda_1, \dots \lambda_l)_n$	$(n - \sum \lambda_i \geq \lambda_1 \geq \dots \geq \lambda_l)$
$V(0)_n =$ Trivial Representation of S_n	$(n-0,0) = (n)$
$V(1)_n =$ Standard Representation of S_n	$(n-1,1)$
$V(\underbrace{1,\dots,1}_{k})_n = \bigwedge^{k}(\text{standard})$	$(n-k,\underbrace{1,\dots,1}_{k})$

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Main Theorem (informal statement):

The decomposition into irreducibles is eventually independent of n. Roughly speaking this is the notion of

"representation stability".

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Theorem (Church-Farb 2010)

The sequence $\{H^i(P_n;\mathbb{Q}), f_n^i\}$ is uniformly representation stable.

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Corollary (Rational Arnol'd 1969)

The braid groups satisfy homological stability:

$$H_i(B_n;\mathbb{Q}) \approx H_i(B_{n+1};\mathbb{Q})$$

if n *is large with respect to* i.

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The multiplicity of the trivial representation is constant for n large enough and

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Corollary (Rational Hatcher-Wahl 2010)

The groups $\mathrm{Mod}^n_{g,r}$, with $r > 0$, satisfy "puncture" homological stability:

$$H_i(\mathrm{Mod}^n_{g,r};\mathbb{Q}) \approx H_i(\mathrm{Mod}^{n+1}_{g,r};\mathbb{Q})$$

if n is large with respect to i.

Ingredients for the proof:

- The Birman exact sequence for $g \geq 2$:

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- Associated “sequence of Hochschild-Serre spectral sequences”

$$E_2^{p,q}(n) = H^p\big(\mathrm{Mod}_{g,r}; H^q(\pi_1(\mathrm{Conf}_n(\Sigma_g^r)); \mathbb{Q})\big) \Rightarrow H^{p+q}(\mathrm{PMod}_{g,r}^n; \mathbb{Q}).$$

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Theorem (Church)

The sequence $\{H^q(\pi_1(Conf_n(\Sigma_g^r)), f_n^q\}$ is uniformly representation stable and monotone.

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Induction argument on the pages of the spectral sequence.

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- Base of the induction: E_2-page

Theorem (Representation stability with changing coefficients, J.R.)

Let $\{V_n.\phi_n\}$ be a consistent sequence compatible with G-actions. If the sequence is monotone and uniformly representation stable, then so it is the sequence $\{H^p(G; V_n), \phi_n^p\}$.

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- The conclusion of the Theorem is recovered from the E_∞-page.

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  Representation stability for the cohomology of $\overline{\mathcal{M}}_{g,n}$ the Deligne-Mumford compatification of $\mathcal{M}_{g,n}$.

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Thank you