

Homotopy of the $K(2)$ -local Goodwillie derivatives of spheres

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1 Goodwillie tower and symmetric products

We are going to investigate the Goodwillie towers of the odd spheres. Recall that from Goodwillie calculus, there is a tower of spaces $P_0 S^k \leftarrow P_1 S^k \leftarrow \dots$, with inverse limit S^k . Moreover, the fibers of each map is an infinite loop map, (but the maps are not infinite loop maps). From [2], we know that for odd spheres, only the p -power layers are nontrivial. We will call the p^n -layer $L(n)_k$, and there is an induce sequence of spaces $\Omega^\infty L(0)_k \rightarrow \Omega^\infty L(1)_k \rightarrow \dots$ giving rise to the E_1 -term of the Goodwillie spectral sequence. We will denote the Goodwillie differentials (on the E_1 -term) by $d_n : \Omega^\infty L(n)_k \rightarrow \Omega^\infty L(n+1)_k$.

There is another tower of spectra coming from symmetric powers. Let $SP^n(S)$ be the n^{th} symmetric power of the sphere spectrum. Then by Dold-Thom theorem, $SP^\infty(S)$ is a model for $H\mathbb{Z}$. There is the filtration $SP^1(S) \rightarrow SP^p(S) \rightarrow SP^{p^2}(S) \rightarrow \dots$. One finds $L(n)_1 = \Sigma^{-n} SP^{p^n}(S) / SP^{p^{n-1}}(S)$. (We will denote $L(n)_1$ by $L(n)$.) And the filtration induces a sequence of spectra $L(0) \leftarrow L(1) \leftarrow \dots$. We will denote these differentials by $s_n : L(n+1) \rightarrow L(n)$, and call them the transfer map.

From [4] [7], we know that on the infinite loop space level, the two sequences above (when d_n 's apply to the Goodwillie layers for S^1) split each other, i.e. $d_{n-1}\Omega^\infty s_{n-1} + \Omega^\infty s_n d_n$ is a self equivalence. In particular, when applied to any homology theory, each gives rise to a long split exact sequence.

We will analyze the maps s_n more carefully. First we have the map $s_0 : L(1) \rightarrow L(0)$. $L(1)$ can be identified with $\Sigma^\infty B\Sigma_p$. It is well-known that $\Sigma^\infty B\Sigma_p$ has a filtration with the subquotients $\Sigma^{i|v_1|-1} V(0)$ with $i = 1, 2, \dots$, where $V(0)$ denotes the Moore spectrum. There is a similar filtration on $L(0) = S$. We have the map $\Sigma^{-1} S/p^\infty \rightarrow S$, which is an equivalence after p -completion. If p is odd, then we find that, since the Moore spectrum is annihilated by p , the map s_0 factors through $\Sigma^{-1} S/p^\infty$. Moreover, we have the natural filtration $\Sigma^{-1} S/p \rightarrow \Sigma^{-1} S/p^2 \rightarrow \dots$ on $\Sigma^{-1} S/p^\infty$ with subquotients $\Sigma^{-1} S/p$, and the map s_0 can be made to respect the filtrations. Now we have the induced map on the subquotients $\Sigma^{i|v_1|-1} V(0) \rightarrow \Sigma^{-1} V(0)$. From [2], the map s_0 should be an equivalence if we invert v_1 . This indicates that on the subquotients s_0 should be the v_1^i map on $V(0)$.

Now the fiber of s_0 should have an induced filtration. The subquotients are just $\Sigma^{-2} V(0)/v_1^i$ for $i = 1, 2, \dots$. We have the filtration $\Sigma^{-2+(i-1)|v_1|} V(0)/v_1 \rightarrow \Sigma^{-2+(i-2)|v_1|} V(0)/v_1^2 \rightarrow \dots \rightarrow \Sigma^{-2} V(0)/v_1^i$. So we can think of the fiber of s_0 has a double filtration, with the subquotients $\Sigma^{-2+j|v_1|} V(1)$ for $j = 0, \dots, i-1$,

and $i = 1, 2, \dots$. Here $V(1) = V(0)/v_1$ is the Smith-Toda complex. From [2], we know that s_1 factors through the fiber of s_0 , and induces an equivalence of $L(2)$ with the fiber when v_2 is inverted. We can also read off the Poincaré series of $L(2)$ from [2]. All these indicates that we should expect something similar to the previous situation happens for s_1 . This means we might have a double filtration on $L(2)$, with subquotients $\Sigma^{-2+j|v_1|+(i-j)|v_2|}V(1)$, and the map s_1 sends this to the subquotient $\Sigma^{-2+j|v_1|}V(1)$ under the v_2^{i-j} map on $V(1)$, for $j = 0, \dots, i-1$, and $i = 1, 2, \dots$.

Of course this cannot happen for $p = 2, 3$, simply because there does not exist Smith-Toda complexes. But we will show that this is the case for $p \geq 5$, and a variant is true for $p = 2, 3$.

There is an obvious generalization for general $L(n)$. We will show that, on the BP -homology level, we have the expected filtration on the $L(n)$'s.

At the end of this section, we remark that the unstable filtration $S^0 \rightarrow \Omega S^1 \rightarrow \Omega^2 S^2 \rightarrow \dots$ induces a filtration $L(n)_1 \rightarrow L(n)_2 \rightarrow \dots$. We will also refer to it as the unstable filtration on $L(n)$. We will describe how this filtration is related to the filtration above.

2 Homology of $L(2)$

2.1 Ordinary homology of $L(2)$

The ordinary homology of $L(2)$ was calculated in [10] and [2]:

Theorem 1. *If we identify the mod p cohomology of $SP^\infty(S) = H\mathbb{Z}$ as the vector space generated by the Steenrod operators $\beta^{\epsilon_1} P^{i_1} \beta^{\epsilon_2} P^{i_2} \beta^{\epsilon_3} \dots P^{i_k}$ with admissible sequences $(\epsilon_1, i_1, \dots, i_k)$, then the filtration defined by the $SP^n(S)$ induces the filtration defined by the length k of the operator. consequently, we find the cohomology of $L(n)$ has a set of generators $\beta^{\epsilon_1} P^{i_1} \beta^{\epsilon_2} P^{i_2} \dots P^{i_n}$ with admissible sequences $(\epsilon_1, i_1, \dots, i_n)$.*

For example, the cohomology of $L(1) \cong \Sigma^\infty B\Sigma_p$, has a set of basis of the form $P^1, \beta P^1, P_2, \beta P^2, \dots$. This corresponds to the usual way of constructing the spaces by attaching cells. Note that we have a desuspension in relating $L(n)$ with the symmetric products, so the lowest cell lies in dimension $2(p-1)-1$, as everyone knows.

Also note that the cohomology of $L(2)$ has a set of basis of the form: $P^p P^1, \beta P^p P^1, P^{p+1} \beta P^1, \beta P^{p+1} \beta P^1, P^{p+1} P^1, \beta P^{p+1} P^1, P^{p+2} \beta P^1, \beta P^{p+2} \beta P^1, \dots, P^{2p} P^2, \beta P^{2p} P^2, P^{2p+1} \beta P^2, \beta P^{2p+1} \beta P^2, \dots$. So in particular, the dimensions of the cells of $L(2)$ lie in $|v_2| - 2, |v_2| - 1, |v_2| + |v_1| - 1, |v_2| + |v_1|, |v_2| + |v_1| - 2, |v_2| + |v_1| - 1, |v_2| + 2|v_1| - 1, |v_2| + 2|v_1|, \dots, 2|v_2| - 2, 2|v_2| - 2, 2|v_2| - 1, 2|v_2| + |v_1| - 1, 2|v_2| + |v_1|, \dots$, which is consistent of the conjectural cell structure of the last section.

We have the unstable filtration on $L(n)$, and by dualizing the result in [2], we find that it induces the filtration by the last component of the admissible sequences on cohomology.

2.2 BP-homology of $L(2)$

2.2.1 Adams spectral sequence approach

The BP (co)homology of $L(2)$ can be computed using the Adams spectral sequence.

Recall that the Milnor elements of the Steenrod algebra are defined inductive by : $Q_0 = \beta$ and $Q_{k+1} = [P^{p^k}, Q_k]$. For example, $Q_1 = P^1\beta + \beta P^1$.

By [5], the Steenrod algebra is Koszul with generators the Steenrod operators P^i and the Milnor elements Q_i , and relations: the usual Adem relations for the generators P_i , and $Q_k Q_l = -Q_l Q_k$, $P^n Q_k = Q_k P^n + Q_{k+1} P^{n-p^k}$ if $n > p^k$, $P^{p^k} Q_k = Q_k P^{p^k} + Q_{k+1}$, $P^n Q_k = Q_k P^n$ if $n < p^k$.

With the map $BP \rightarrow H\mathbb{Z}$, we can identify the ordinary mod p cohomology of BP with the quotient of the Steenrod algebra by the right ideal generated by the Q_i 's. The Adams spectral sequence computing $BP^*L(2)$ has E_2 -term $Ext_A(H\mathbb{F}_p^*BP, H\mathbb{F}_p^*L(2))$. The Q_i 's generates a sub-exterior-algebra $E[Q_i]$ in the Steenrod algebra. By the change of rings theorem, we can identify the E_2 -term with $Ext_{E[Q_i]}(\mathbb{F}_p, H\mathbb{F}_p^*L(2))$.

In $H\mathbb{F}_p^*L(2)$, we have the elements with names $P^i P^j$. These elements generates a \mathbb{F}_p -sub-vector space U . We find $Q_1 P^i P^j = -\beta P^{i+1} P^j + P^{i+1} \beta P^j$. Note the first term has excess one more. Thus we find that the map $E[Q_0, Q_1] \otimes U \rightarrow H\mathbb{F}_p^*L(2)$ is an isomorphism.

We also note that $Q_0 Q_1 P^i P^j$ are annihilated by any Q_i , since the composition has three Q_i 's, hence the expansion using the Adem relations has at least three β 's in each term, in particular with length at least three.

So we conclude that the E_2 -term has the form $\mathbb{F}_p[v_2, v_3, \dots] \otimes U$. Here v_i corresponds to $[Q_i]$ in the bar (or Koszul) complex, and the generators should be the form $v_2^{i_2} v_3^{i_3} \dots v_k^{i_k} [Q_0 Q_1 P^i P^j]$ in the Koszul complex. The reason why v_0 and v_1 do not appear comes from the fact that anything with v_1 , such as $v_1 [P^i P^j]$, is contained as one term of the Koszul differential $d([Q_0 P^i P^j]) = \sum \pm v_i [Q_i Q_0 P^i P^j]$. So we conclude that there is a filtration such that the graded pieces are the $\mathbb{F}_p[v_2, v_3, \dots]$, with generators $x_{2,i,j}$ corresponding to $[Q_0 Q_1 P^i P^j]$. Here we will abbreviate $x_{2,i,j}$ as $x_{i,j}$ if no confusion arises.

Using the commutation relations, one finds the following relation: $Q_2 P^i P^j + Q_1 (P^{i+p-1} P^{j+1} + P^{i+p} P^j) + Q_0 P^{i+p} P^{j+1} = P^{i+p} P^{j+1} \beta$. Note that the right hand side has length 3, hence zero in $H\mathbb{F}_p^*L(2)$. Also note that every term is admissible except $P^{i+p-1} P^{j+1}$ when $i = pj$. But in the latter case we find $P^{jp+p-1} P^{j+1} = 0$ from the Adem relations.

Hence we find the following differential in the Koszul complex: $d([Q_1 P^i P^j]) = v_0 [Q_0 Q_1 P^i P^j] + v_2 [Q_2 Q_1 P^i P^j] + \dots$. Using the previous relation, we get $Q_2 Q_1 P^i P^j = Q_1 Q_0 P^{i+p} P^{j+1}$. This shows that we have the following relation $p x_{i,j} = \pm v_2 x_{i+p,j+1} + \text{higher terms}$. Similarly we have the relation $v_1 x_{i,j} = \pm v_2 (x_{i+p-1,j+1} + x_{i+p,j}) + \text{higher terms}$. Here we understand that the inadmissible term is zero if occurs.

We can also understand the effect of the transfer maps.

Recall that $BP^*L(0)$ is generated by x_0 . $BP^*L(1)$ is generated by $x_{1,i}$ with relations $p x_{1,i} = v_1 x_{1,i+1} + \text{higher terms}$. The $x_{1,i}$ corresponds to $[Q_0 P^i]$ in the Adams spectral sequence computing $BP^*L(1)$.

Now we have the differential $d(\iota) = v_0 [Q_0] + v_1 [Q_1] + \dots$. But $\beta = \beta(\iota)$ is zero in $H\mathbb{F}_p^*H\mathbb{Z}$. So the leading term of the differential is $v_1 [Q_0 P^1]$. Hence the

transfer map sends x_0 to $v_1 x_{1;1}$ as leading term. Using the relations between $x_{1;i}$'s, we find the leading term for the transfer of $p^k x_0$ is $v_1^{k+1} x_{1;k+1}$.

Now for the $x_{1;i}$, we have the following differentials computing $BP^* H\mathbb{Z}$: $d([Q_0 P^i]) = v_1[Q_1 Q_0 P^i] + v_2[Q_2 Q_0 P^i] + \dots$. But $Q_1 Q_0 P^i = -Q_0 Q_1 P^i = Q_0 Q_0 P^{i+1} - Q_0 P^{i+1} Q_0$, which gives 0 in $H\mathbb{F}_p^* H\mathbb{Z}$. The next term $Q_2 Q_0 P^i = Q_0 Q_1 P^{i+p} - Q_0 P^{i+p} Q_1$. $Q_0 Q_1 P^{i+p} = Q_0 P^{i+p+1} Q_0$ has Q_0 on the right. $P^{i+p} Q_1 = P^{i+p} P^1 Q_0 - P^{i+p} Q_0 P^1$. $P^{i+p} Q_0 = Q_0 P^{i+p} + Q_1 P^{i+p-1}$. So collecting the relevant terms gives $d([Q_0 P^i]) = \pm v_2[Q_1 Q_0 P^{i+p-1} P^1] + \dots$. This shows that the leading term for the transfer of $x_{1;i}$ is $v_2 x_{2;i+p-1;1}$. If we use the filtration on $x_{2;i,j}$ defined by the sum of the last coordinate and the excess, i.e. $j + (i - pj)$, then using the relations among the $x_{2;i,j}$, the leading term for the transfer of $v_1^k x_{1;i}$ is $v_2^{k+1} x_{2;i+pk-k+p-1,k+1}$.

Thus we find that the homological analogue of the picture in the previous section holds.

2.2.2 Invariant theory approach

We know that the $L(n)$ are the Steinberg summand of the Thom spectra over the classifying space of $(\mathbb{Z}/p)^n$ for the vector bundle defined by the multiples of the reduced regular representation. So we can also use invariant theory to compute its BP -cohomology.

Recall that $BP^* B\mathbb{Z}/p$ is the ring of functions of the kernel of the multiplication by p map for the formal group associated to BP . So we have the equation $BP^* \mathbb{Z}/p = BP^*[u]/[p](u)$. One finds that this ring is Landweber exact, since it is free when any v_i is inverted, using the Weierstrass preparation theorem.

So $BP^*(\mathbb{Z}/p)^2 = BP^* \mathbb{Z}/p \otimes_{BP^*} BP^* \mathbb{Z}/p = BP^*[u, v]/([p](u), [p](v))$. Note that is we use the filtration by the powers u, v , and ignore low dimensional irregularities, the graded pieces are $BP^*/(p, v_1)[u, v]$.

From [6], the Steinberg summand is isomorphic to the invariants of the Borel subgroup modulo the sum of the invariants of the minimal parabolic subgroups.

Now we can define the partial Dickson invariants for $GL(2)$. Define $F_1 = \prod_{i \neq 0} [i](v)$, and $F_2 = \prod_{(i,j) \neq (0,0)} ([i](x) +_G [j](y))$. Here G denote the formal group law associated to BP . They are invariant under the Borel subgroup. We note that $F_1 = v^{p-1}$, and F_2 is invariant under $GL_2(\mathbb{F}_p)$. Also note that F_2 is the top Chern class of the complex reduced regular representation.

We can relate the different Thom spectra by inclusion of the zero section. Using the arguments in [1], we find that if we do the direct limit of the cohomology under the tower of negative multiples of the reduced regular representation, we get that the resulting $BP^* GL_2(\mathbb{F}_p)$ -module has Steinberg summand $F_1 BP^*[F_1, F_2][F_2^{-1}]/relations$. And the relations express $pF_1, v_1 F_1, pF_2, v_1 F_2$ into terms of higher filtration. Note that there might be negative powers of F_1 in the invariants, such as F_2/F_1 . But this terms appears in the Dickson invariant D_1 which is the $p^2 - p$ Chern class of the reduced regular representation. Hence in the Steinberg summand it is equivalent to something with positive powers of F_1 .

Now we can include the cohomology of Thom spectra of odd multiples of the real reduced regular representation into the direct limit of the inverse tower. We also know that the Euler class is F_2 , which is the top Chern class. So we conclude that $L(2)$, which is the Steinberg summand of the Thom spectrum of

the real reduced regular representation, is the submodule generated by F_2 . In this way, we get the filtration on $BP^*L(2)$.

In conclusion, $BP^*L(2)$ is generated by terms of the form $F_1^{l_1}F_2^{l_2}$, with $l_1, l_2 \geq 1$. The primary filtration is induced by the unstable filtration, and is defined by the powers of F_2 . The secondary filtration is induced by the powers of F_2 . The associated grade pieces is the ideal generated by F_1F_2 inside $BP^*/(p, v_1)[F_1, F_2]$. Moreover, we note that $F_1^{l_1}F_2^{l_2}$ corresponds to the generator $x_{pl_2+l_1-1, l_2}$ in the previous subsection.

To understand the multiplication by p and v_1 . We know the equations for u and v : $pu + v_1u^p + v_2u^{p^2} + \dots = 0$, $p v + v_1v^p + v_2v^{p^2} + \dots = 0$. Let $t = uv^p - vu^p$, then the leading term for F_2 is t^{p-1} . Combining the equations of u and v to cancel out terms involving p gives $v_1t + v_2(v^{p^2}u - u^{p^2}v) + \dots = 0$. Next we have the Dickson invariant D_1 has leading term $(v^{p^2}u - u^{p^2}v)/(v^pu - u^pv) = (u^{p-1})^p + (u^{p-1})^{p-1}v^{p-1} + \dots + (v^{p-1})^p$. Modulo p , one finds that the right hand side reduces to $(v^{p-1})^p + (\frac{t}{v})^{p-1}$. Plugging in this formula we get the equation $v_1t + v_2t(F_1^p + \frac{F_2^2}{F_1}) + \dots = 0$. Hence we have the equation for multiplication of v_1 on the generators: $v_1F_1^{l_1}F_2^{l_2} = -v_2(F_1^{l_1+p}F_2^{l_2} + F_1^{l_1-1}F_2^{l_2+1}) + \dots$, which is the same relation for the generators $x_{2i,j}$ obtained previously.

Using the equation for v we have the following: $pF_1^{l_1}F_2^{l_2} = (pv)v^{l_1(p-1)-1}F_2^{l_2} = -(v_1v^p + v_2v^{p^2} + \dots)v^{l_1(p-1)-1}F_2^{l_2} = -(v_1F_1^{l_1+1} + v_2F_1^{l_1+(p+1)})F_2^{l_2} + \dots$. With the formula for multiplication by v_1 , we get the following equation: $v_1F_1^{l_1+1}F_2^{l_2} = -v_2(F_1^{l_1+1+p}F_2^{l_2} + F_1^{l_1}F_2^{l_2+1}) + \dots$. The first term cancels the other term in the previous formula, and we arrive at the formula $pF_1^{l_1}F_2^{l_2} = v_2F_1^{l_1}F_2^{l_2+1} + \dots$, as expected.

In this way, at least in principle, we can work out, with the formula of the formal group laws, the meaning of the dots. We can also understand, in principle, the action of the Morava stabilizer group on the cohomology, from the knowledge of the action on protective space.

2.3 BP -homology for general $L(n)$

The method we use also works for general $L(n)$'s. We will briefly state the result.

First we find the existence of a filtration on cohomology, with the associated grade pieces direct sum of $BP^*/(p, \dots, v_{n-1})$'s, and generators $x_{n;i_1, \dots, i_n}$, with (i_1, \dots, i_n) admissible, of the form $[Q_0Q_1 \dots Q_{n-1}P^{i_1}P^{i_2} \dots P^{i_n}]$ in the Adams spectral sequence, in terms of the bar complex.

Using the commutation relations, we have the equation $Q_nP^{i_1}P^{i_2} \dots P^{i_n} = -Q_{n-1}P^{i_1+p^{n-1}}P^{i_2} \dots P^{i_n} + P^{i_1+p^{n-1}}Q_{n-1}P^{i_2} \dots P^{i_n}$. With induction, we can transform the term $Q_{n-1}P^{i_2} \dots P^{i_n}$ into the sum of terms with one Q_i on the left with $i < n-1$, and a term of length n . We also assert that the first superscript on P of each term do not exceed $i_2 + p^{n-2}$. Then we can transform $P^{i_1+p^{n-1}}Q_i$ into $Q_{i-1}P^{i_1+p^{n-1}} + Q_{i+1}P^{i_1+p^{n-1}-p^i}$. In this way, we transform the original one into terms with one Q_i on the left with $i < n$, and a term of length $n+1$. Note the inadmissible term might arise from $P^{i_1+p^{n-1}-p^i}P^{i_2+j}$ with $j \leq p^{n-2}$. But the Adem relations show that this term vanishes.

So we arrive at the conclusion that we can play the same trick as before to obtain the formula for multiplication by p, v_1, \dots, v_{n-1} . In fact the formula

can be obtained inductively using the above procedure. In particular, we have $px_{n;i_1,\dots,i_n} = v_n x_{n;i_1+p^{n-1},i_2+p^{n-2},\dots,i_n+1} + \text{higher terms}$.

The same process can be used to transform $Q_{n+1}P^{i_1}\dots P^{i_n}$ into sum of terms with one Q_i on the left with $i \leq n$ and a term $P^{i_1+p^n}P^{i_2+p^{n-1}}\dots P^{i_n+p}Q_1$. Terms with one Q_n on the left can be transformed into sum of terms with one Q_i on the left with $i \leq n-1$ and a term with Q_0 on the right side. The term $P^{i_1+p^n}P^{i_2+p^{n-1}}\dots P^{i_n+p}Q_1$ equals $P^{i_1+p^n}P^{i_2+p^{n-1}}\dots P^{i_n+p}Q_0P^1$ plus a term with Q_0 on the right. We can transform it back to sum of terms with one Q_i on the left with $i \leq n-1$ and a term $Q_nP^{i_1+p^n-p^{n-1}}P^{i_2+p^{n-1}-p^{n-2}}\dots P^{i_n+p-1}P^1$. So the same trick as before says that the transfer of the generator $x_{n;i_1,\dots,i_n}$ has leading term $x_{n+1;i_1+p^n-p^{n-1},\dots,i_n+p-1,1}$, as expected.

The invariant theory method can also be applied. Here we have the space $B(\mathbb{Z}/p)^n$. The group $GL_n(\mathbb{F}_p)$ acts on it, with the reduced regular representation as an equivariant bundle. In BP -cohomology, we have the Dickson invariants as the $p^n - p^k$ Chern classes of the complex reduced regular representation.

We can also define the partial Dickson invariants, which are invariants for the Bore subgroup. We know $BP^*(B(\mathbb{Z}/p)^n) = BP^*[u_n, \dots, u_1]/([p](u_n), \dots, [p](u_1))$. We define $F_k = \prod_{(a_k, \dots, a_1) \neq 0} ([a_k](u_k) +_G \dots +_G [a_1](u_1))$.

Then as before, the same argument show that $F_1^{l_1} \dots F_n^{l_n}, l_i \geq 1$ generate the BP -cohomology of $L(1)$. If we define a filtration using the lexicographic order, so that the primary filtration is defined by the powers of F_n , then the associated graded pieces give the principle ideal generated by $F_1 \dots F_n$ inside $BP^*/(p, \dots, v_{n-1})[F_1, \dots, F_n]$. Since F_n is the top Chern class of the complex reduced regular representation, the unstable filtration is the defined by the powers of F_n .

The two descriptions can be compared by associating the generator $x_{n;i_1,\dots,i_n}$ with $F_1^{i_1-p^{i_2+1}} \dots F_{n-1}^{i_{n-1}-p^{i_n+1}} F_n^{i_n}$.

3 Filtration on $L(2)$

In this section, we will discuss how much the filtration on cohomology can be realized homotopically, so that the $L(n)$'s decompose into extensions of Smith-Toda complexes.

The primary filtration is clear, which is the unstable filtration, defined by the sequences $L(n)_1 \rightarrow L(n)_3 \rightarrow L(n)_5 \rightarrow \dots$. By [3] and its odd primary analogue, the fiber of $L(n)_{2k-1} \rightarrow L(n)_{2k+1}$ is the cofiber of a certain map $L(n-1)_{2pk+1} \rightarrow L(n-1)_{2pk-1}$. Cohomologically, the fiber has generators terms of the form $F_1^{l_1} \dots F_{n-1}^{l_{n-1}} F_n^k$. It is isomorphic to the cokernel of the map sending $F_1^{l_1} \dots F_{n-1}^{l_{n-1}+p}$ to $v_{n-1} F_1^{l_1+p^{n-2}} \dots F_{n-1}^{l_{n-1}+(p+1)}$.

So we will concentrate on filtrations on the fiber of the double suspension map. For $L(2)$, it is enough to have a secondary suspension on these fibers.

3.1 Odd prime case

The odd primary case for $L(2)$ follows directly from [8]. In this case, first note that, by [11], the composition of the map $L(n-1)_{2pk+1} \rightarrow L(n-1)_{2pk-1}$ with the double suspension is the multiplication by p map. It follows that the map $L(1)_{2pk+1} \rightarrow L(1)_{2pk-1}$ is the unique map factoring the multiplication by p map

on the stunted p -primary protective space. Then we find that, as described in [8], the cofiber has a filtration, with the subquotients suspensions of $V(1)$'s.

So for odd primes, there is a secondary filtration on the fibers of the double suspension $L(2)_{2k-1} \rightarrow L(2)_{2k+1}$, which is compatible with the filtration on cohomology described in the previous section.

3.2 Cross effect and secondary suspension

3.2.1 $p = 2$ case

We cannot have an analogous filtration on $L(2)$ at prime 2. In fact, at prime 2, the complex $V(1)$ does not even exist, so the analogue does not make sense.

In stead, we will expect a courser filtration. In fact, from [9], we have a filtration on the fiber $L(2)_{2k-1}^{2k+1}$ induced by the secondary suspension, and the subquotients are copies of $A(1)$'s. So we find that in this case the subquotients are two copies of $V(1)$'s homologically. So this is two time courser a filtration than the odd prime case.

The secondary suspension can be understood using the notion of cross effects. In this case, by sending a complex vector space V to the Thom space of $\rho_{\mathbb{R}} \oplus \rho \otimes V$, where ρ is the reduce regular representation for $(\mathbb{Z}/2)^2$, defines an (complex) orthogonal functor, and we can apply the orthogonal calculus to get the cross effects. In particular, we have the secondary suspension map $\Sigma^2 L(2)_{2k-1}^{2k+1} \rightarrow L(2)_{2k+1}^{2k+3}$, whose cofiber is the second cross effect. This secondary suspension map defines a filtration on $L(2)_1^3$.

To compare this with the homological filtration discussed previously, we need a more concrete description of the secondary suspension. In fact, in [12], the cross effect of Thom functors are computed. In our case, we have the Thom spectra $(B(\mathbb{Z}/2)^2)^{\rho_{\mathbb{R}} \oplus k\rho_{\mathbb{C}}}$. The double suspension map is simply induced by inclusion of the zero section. Then we find that the fiber of the double suspension are the Thom spectra of $\rho_{\mathbb{R}} \oplus k\rho_{\mathbb{C}}$ restricted to the sphere bundle $S_{\rho_{\mathbb{C}}}$. Now when restricted to the sphere bundle, we have a tautological section, so the vector bundle splits off a trivial summand. So we find $\rho_{\mathbb{C}}|_{S_{\rho_{\mathbb{C}}}} = \mathbf{1}_{\mathbb{C}} \oplus \bar{\rho}_{\mathbb{C}}$. Then the secondary suspension is defined by inclusion of the zero section of $\bar{\rho}_{\mathbb{C}}$.

Using this, we find that cohomologically, the secondary suspension is defined by powers of the top Chern class of $\bar{\rho}_{\mathbb{C}}$, which is $c_2(\rho_{\mathbb{C}}$ restricted to the sphere bundle. But we know that $c_2(\rho_{\mathbb{C}}) \equiv v_1 \bmod p$, from the equation for the c_1 of its summands. Thus we find the cohomological filtration induced by the secondary suspension is the same as coarsening the previous filtration by combining two $V(1)$'s into a single $A(1)$.

3.2.2 Secondary suspension for $p = 3$ case

In the 3-primary case, we already know the existence of a filtration of the $L(2)_{2k-1}^{2k+1}$ with subquotients $V(1)$'s. Here we will see we can do the cross effect thing to produce a coarser filtration, which might correspond to the secondary suspension. We suspect this filtration might be a more "natural" one in $p = 3$ case.

In this case, we still have the primary filtration induced by the inclusion of the zero section of $\rho_{\mathbb{C}}$. And the fibers are Thom spectra over the sphere bundle of $\rho_{\mathbb{C}}$.

But now we are not to do the second cross effect, because in 3-primary case, the second derivative is zero. This is reflected by the fact that, (after inverting v_2), the 7th Chern class of $\rho_{\mathbb{C}}$ is zero mod 3. Instead, the 6th Chern class is v_1 mod 3. So if we take the second cross effect, we do not get a filtration, but introduce an exterior generator in cohomology.

What we want to do is to skip the second cross effect, and do the third cross effect directly. Since we find in this situation, the top Chern class lies in c_6 , we expect a filtration induced by inclusion of the zero section of some 6-dimensional bundle. This means that, since the original $\rho_{\mathbb{C}}$ is 8 dimensional, we need to split off two trivial bundles. Over the sphere bundle, there is a tautological section, so we need to construct a second independent section.

The sphere bundle comes from the Borel construction of the equivariant sphere in $\rho_{\mathbb{C}}$. So if we could construct an equivariant map from the unit sphere to 2-frames in $\rho_{\mathbb{C}}$, the Borel construction would provide the desired section.

Now the eight representations of $(\mathbb{Z}/3)^2$ can be grouped into 4 pairs of conjugate representations. Up to an order inside each pair, we can give it an equivariant quaternion structure. Then there is an equivariant tautological section over the sphere as a quaternion bundle. This provides two sections as complex bundle.

In this way, we construct a filtration over the Thom spectrum of $\rho_{\mathbb{R}}$ over the sphere bundle of $\rho_{\mathbb{C}}$. Since the top Chern class of the bundle, after splitting off the two sections, is v_1 , we find the filtration is defined (after inverting v_2) by powers of v_1 . This section is not $GL_2(\mathbb{F}_3)$ -equivariant. However we can brutally project it to the Steinberg summand, to get a filtration on $L(2)$, still defined by powers of v_1 . One finds this one is tree time coarser than the previous defined one. One also finds the the subquotients are copies of $A(1)$'s, as in the $p = 2$ case/

3.2.3 Filtration on $L(3)$ at $p = 2$

The same trick can be played for $L(3)$ at $p = 2$. As before, the first and second derivative goes well, and we find the primary and secondary filtrations are defined by powers of p and v_1 respectively. When trying to define the ternary suspension map, we meet the same problem as in the $p = 3$ case.

In this case, If we want to play the same trick, the same argument leads to the problem of finding a third equivariant section over the complex Stiefel manifold of 2-frame of the complex reduced regular representation of $(\mathbb{Z}/2)^3$.

In this case, we will use octonions \mathbb{O} . By looking at the multiplication table of octonions, one finds that it can be made $(\mathbb{Z}/2)^3$ -equivariant when it is given the real regular representation. So the multiplication of the complexified octonions $\mathbb{O}_{\mathbb{C}}$ is also equivariant.

Now on the imaginary octonions we have the corss product $x \times y = xy - yx$. It has the property that $x \times y$ is perpendicular to both x and y , and non-zero when x and y are linearly independent. So when complexified, we have $\overline{x} \times \overline{y}$ is perpendicular to both x and y with the Hermitian inner product, and non-zero when x and y are linearly independent.

In this way, we get the third section, and as before it gives rise to the ternary filtration defined (after inverting v_3) by powers of v_2 . Also the subquotients are copies of $A(2)$'s.

3.3 Multiplication by p map

From homological calculations, we find that in $L(2)$ the multiplication by p map maps the generator $F_1^{l_1} F_2^{l_2}$ to $v_2 F_1^{l_1} F_2^{l_2} - 1$. We have observed that the filtration on cohomology can be induced from a topological filtration. So we would like to know if multiplication by p map also has a similar description homotopically.

This is the case for $p \geq 5$. In fact, by [8], the fibers $L(2)_{2k-1}^{2k+1}$ have exponent p , which means the multiplication by p map on them are homotopic to zero. So we can use a homotopy to contract the restriction to $L(2)_1^3$ of the multiplication by p map on $L(2)_1$, and get an map $L(2)_3 \rightarrow L(2)_1$. Its composition with the quotient map to $L(3)$ is the multiplication by p map on $L(2)_3$. Then we do the same thing to get a map $L(2)_3 \rightarrow L(2)_5$, and so on. In the end, we get a system of maps $L(2)_{2k+1} \rightarrow L(2)_{2k-1}$, compatible with suspension. So we get maps $L(2)_{2k+1}^{2k+3} \rightarrow L(2)_{2k-1}^{2k+1}$. The composition of this with the secondary suspension map, i.e the quotient map, is a v_1 map from homological computations. Still by [8], the fiber of the quotient map, which is a $A(2)$ type complex, has exponent v_1 , i.e the restriction of any v_1 map is homotopic to zero. In fact we also know the fiber of the finer filtration, which are $V(1)$ type complexes, also have exponent v_1 . So the same techniques applies, and we conclude that the multiplication by p map can be factored through the quotient map, and we get a map $L(2)_3 \rightarrow L(2)_1$, compatible with the double filtration. From the homological computations, we find that the induced map on the subquotients are the v_2 self map on $V(1)$'s.

The analogue for $p = 2, 3$, cannot happen, simply because we don't have a v_2 map on $V(1)$'s in these cases.

However, we conjecture that the multiplication by p map for $p = 3$ case, do factor through the suspension, and the resulting map $L(2)_3 \rightarrow L(2)_1$ respects the coarser filtration defined by the secondary suspension, and on the subquotients we have the v_2 map on the $A(2)$ complex. In this way, we would conclude that the towers of the secondary root invariant of p power maps, defined by multiples of v_2 map on $A(3)$, would give towers of p powers in homotopy groups in $L(2)$.

The case for $p = 2$ has even weaker result. In this case, we know there is a candidate of the factorisation of the multiplication by 2 map, which is induced from the *EHP* differential. From [11], we know that the *EHP* differentials $\Omega^3 S^{4k+1} \rightarrow \Omega S^{4k-1}$, after composing with the double suspension, gives the multiplication by p map. Note that we only have half of them, as opposed to the odd case. Then these maps are compatible with quadruple suspensions, since the composition of any map is still multiplication by p map. Note we cannot use the double suspension here, other wise the other half of *EH* differentials are involved, and the composition of those with double suspension give $1 + \Omega(-1)$.

The conclusion is, when the homotopy in [11] is functorial, we get a map from $L(2)_5 \rightarrow L(2)_3$, respecting the filtration defined by quadruple suspension, which is two times coarser than the usual filtration defined by the double suspension.

In this case, we no longer can compose maps. But we can still construct a tower on any element on homotopy group. We can use the map $L(2)_{4k+1}^{4k+5} \rightarrow L(2)_{4k-1}^{4k+3}$ to construct the " v_2 -multiple" of any element. The only difference is that we view it in a different cell when view it as source instead of target.

We can avoid using unstable homotopy by working directly with Thom spectra. Let ρ be a real vector bundle on M . Then we have cofiber sequence $S(\rho)^{k\rho} \rightarrow M^{k\rho} \rightarrow M^{(k+1)\rho} \rightarrow \Sigma S(\rho)^{k\rho}$. Recall that $S(\rho)$ is the sphere bundle,

and there is a tautological section over it. The stable *EHP* differential is the composite $S(\rho)^{(k+1)\rho} \rightarrow M^{(k+1)\rho} \rightarrow \Sigma S(\rho)^{k\rho}$. Using the tautological section, we find the restriction of ρ to $S(\rho)$ is splitted into $\mathbf{1} \oplus \bar{\rho}$. Then we have the isomorphism $S(\rho)^{k\rho} = \Sigma^k S(\rho)^{k\bar{\rho}}$.

In coordinate terms, we can write the map as follows. If $(m, x, (v_1, \dots, v_{k+1}))$, with $m \in M$, $x \in S(\rho)_m$, $(v_1, \dots, v_{k+1}) \in \rho_m^{k+1}$ represents a point in $S(\rho)^{(k+1)\rho}$. Then it maps to the point $(m, (v_1, \dots, v_{k+1}))$ in $M^{(k+1)\rho}$. Then, if we identify ΣX with the Thom space of the trivial bundle $\mathbb{R} \times X$ for any X , we find that the differential maps that point to $(\log |v_1|, \frac{v_1}{|v_1|}, (v_2, \dots, v_{k+1}))$ in $\Sigma S(\rho)^{k\rho}$. Now the identification of $S(\rho)^{k\rho}$ with $\Sigma^k S(\rho)^{k\bar{\rho}}$ maps $(m, x, (v_1, \dots, v_k))$ to the point $(v_1 \cdot x, \dots, v_k \cdot x, m, x, (\bar{v}_1, \dots, \bar{v}_k))$.

Now if we compose with the inclusion of the zero section of $\bar{\rho}$, then we find that this can be identified with the subspace of $S(\rho)^{(k+1)\rho}$ such that the coordinate we named v_1 is proportional to the direction defined by the point x . Then the composition of this with the previous map, is given by sending $(m, x, (\lambda v_1, v_2, \dots, v_{k+1}))$ to $(\log \lambda, m, x, (v_2, \dots, v_{k+1}))$ if $\lambda > 0$, and the base point if $\lambda = 0$, and $(\log(-\lambda), m, -x, (v_2, \dots, v_{k+1}))$ if $\lambda < 0$.

So we find that the composition is the sum of the identity and the antipodal map. We need to understand the antipodal map. What we really interest in is its Steinberg summand, which is related to the derivative of the sphere via the James-Hopf map. The antipodal map is induced from the fiber of $S^k \rightarrow \Omega \Sigma S^k$ of the map, which is the identity on S^k , and on $\Omega \Sigma S^k$ it inverts the parameters in Ω and Σ . In the Snaith splitting, this corresponds to the map which reverses the order of multiplication. Since the James-Hopf map is defined by $\sum [x_i \wedge x_j]$ where the sum is the loop sum, and over all $i < j$ with lexicographical order. This means that the induce map on $\Omega \Sigma S^{2k}$ is the map which switches the factors in $S^k \wedge S^k$, and reverses the parameters in $\Omega \Sigma$. So on the derivative, the induced map is to reverse the first factor of ρ' (the reduced regular representation for the group defining the previous derivative), and switches the order of $\rho^k \oplus \rho'^k$. In case k is odd, the net effect is reverse $k + 1$ factors, which is even, so we can use a complex structure to continuously transform it into the identity. This shows when k is odd, the attaching map for $L(n + 1)$ gives a factorization for the multiplication by 2 map on $L(n)_{2k+1}$.

So we get the desired map to factorize the multiplication by 2 map, and when this map is compatible with taking cross effects, we find that it preserves those filtrations, giving the $p = 2$ analogue of the claim that the p -power towers in $L(2)$ are given by the secondary root invariant of powers of p .

To address the problem of compatibility with filtrations, we note that it is the attaching map for the fiber of the double suspension, which are Thom spectra over the sphere bundle of the complexified vector bundle $\rho_{\mathbb{C}}$. So we have the secondary suspension induced from the bundle which is obtained from the restriction of $\rho_{\mathbb{C}}$ by subtracting the complex tautological section. It follows that the attaching map is compatible with the secondary suspension. A similar argument would show that the attaching map is also compatible with the ternary suspension in $L(3)$, which give the desired result for the multiplication by 2 map on $L(2)$.

To be precise, We have the inclusion map $S(\rho_{\mathbb{R}}) \rightarrow S(\rho_{\mathbb{C}})$, which induce the attaching map. The filtration is defined by the restriction of multiples of $\rho_{\mathbb{C}}$. Its splitting properties on $S(\rho_{\mathbb{C}})$ pulls back on $S(\rho_{\mathbb{R}})$, which means we have

a compatible system of filtrations. The only thing to be checked is how the induced filtration on Thom spectra on $S(\rho_{\mathbb{R}})$ compares with the filtration on the previous derivative via the James-Hopf map. At least the cohomological behavior is as expected.

4 Some AHSS differentials

In this section, we will show some computations of the AHSS differentials in $L(2)$ associated to the filtration introduced in the previous section. To simplify the situation, we will only work with prime $p \geq 5$.

In that case, we have a double filtration on $L(2)$, with subquotients $V(1)$ complexes. If we localize at $K(2)$, the homotopy groups of these are known, and we have the AHSS with E_1 term the homotopy groups of $V(1)$'s.

We have already computed the BP -cohomology of $L(2)$. To set up the ANSS, we need to understand the action of the Morava stabilizer group. In principle, the generators given by the partial Dickson invariants can be used to compute the action. In practice, we will use a simpler method to understand partially the action.

We have noticed that the graded pieces of $L(2)$ can be labeled by $e_{i,j}$ with $i, j \geq 1$, each represents a $V(1)$ complex, corresponding to the generator $F_1^j F_2^i$ in cohomology. We can introduce the spectrum $L(2)_{-\infty}$, which is the inverse limit of the Steinberg summand of Thom spectra for $-k\rho$. The graded pieces of it are the same, except that now we allow i be any integer. One can deduce from [1] that this is the completed sphere S^{-2} . That means that the -2 cell is invariant in whatever spectral sequence. This cell is the bottom cell for the $V(1)$ labeled by $e_{0,1}$.

Now since we know that the multiplication by p map sends $e_{i,j}$ into the v_2 map on $e_{i-1,j}$, we will identify $e_{i,j}$ with $\frac{v_2}{p} e_{i-1,j}$. This would introduce no ambiguity inside $L(2)_{-\infty}$, since there is no torsion there. We could also use the formula for multiplication by v_1 map to identify $e_{i,j+1}$ with $\frac{v_1}{v_2} e_{i+1,j}$, as long as $j \leq p$. And we should subtract $\frac{v_2}{p} e_{i+1,1}$ for $e_{i,p+2}$, since the v_1 multiple of $e_{i+1,p+1}$ has a summand $\frac{v_2}{p} e_{i+1,1}$ which lies in a larger filtration.

Since the cell $e_{0,1}$ is invariant, it is plausible to label it by 1. Then the other cells can be labeled, so that we would label $e_{i,j}$ with $\frac{v_2^{i-j+1} v_1^{j-1}}{p^{i+j-1}}$, when $j \leq p+1$.

Now we can use the action on the v_i 's to compute the differentials. We will do the example for the v_2 multiples on the $e_{i,1}$ cells.

In this case, the bottom cell is labeled $\frac{v_2^i}{p^i}$. Then the ANSS differential sends it to $\frac{iv_2^{i-1} v_1 t^p}{p^i}$. When i is not divisible by p , we can view this as $[t_1^p] \frac{v_2^{i-1} v_1}{p^i}$. This represents the element β_1 of the bottom cell for the $V(1)$ complex labeled by $e_{i-1,2}$.

When i is divisible by p , there is a further multiplication by p coming from the factor i . When i is exactly divisible by p^k , then the effect is to send the cell $e_{i-1,2}$ into v_2^k multiples on the cell $e_{i-k+1,2}$. Then the element β_1 is sent to β_{k+1} under this map. So we conclude that the AHSS differential send the identity on the bottom cell of the complex $e_{i,1}$ into the bottom β_{k+1} in the cell $e_{i-k+1,2}$.

Using the multiplication by p map, we will further conclude that the AHSS differential would send p multiples of that, i.e. the element v_2^s on the cell $e_{i-s,1}$,

to the p multiples of the bottom β_{k+1} , i.e. β_{k+1+s} on the cell $e_{i-k+1-s,2}$.

Thus we have a system of towers of β family in $L(2)$ related by AHSS differentials, in much the same way of the well known α towers of AHSS differentials in $L(1)$. The difference is that the α towers are seen in the *EHP* sequence directly, while these β towers are hidden behind the Goodwillie tower, so does not appear directly in *EHP* sequence.

A final remark is that direct computations in prime 3 shows that we have the same towers of differentials in AHSS for $L(2)$ in this case. But we need to use the root invariants of the α family to play the role of the β family in $p = 3$ case, when the homological β family does not survive the *ANSS*.

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