# Algebraic geometry of ring spectra and multiplicative invariants for families of manifolds

#### Yifei Zhu

Southern University of Science and Technology

Algebraic and geometric topology workshop 2017

#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M)\in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M)\in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $\bullet \ g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M) \in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $\bullet \ g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M) \in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M) \in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $\bullet \ g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M) \in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



#### Definition

A genus is a function which assigns to each closed manifold M of some type an element  $g(M) \in R$  of a commutative ring R, satisfying

- $g(M_1 \coprod M_2) = g(M_1) + g(M_2)$
- $g(M_1 \times M_2) = g(M_1) \cdot g(M_2)$
- $g(\partial N) = 0$



The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$ the cobordism ring of *string manifolds* 

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

 $MO\langle 8 \rangle^{-*}(X) \leadsto \text{genus of a } \textit{family} \text{ parametrized by } X \text{ (orientation)}$ 

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\downarrow \\ MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

#### The Witten genus (Witten '87): $MString_* \rightarrow MF_*$

$$MString_* = MO\langle 8 \rangle_* \coloneqq \text{the cobordism ring of } string \ \text{manifolds}$$
 
$$0 = w_1(M) = w_2(M) = p_1(M)/2$$
 
$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3] \qquad MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

 $MO\langle 8\rangle^{-*}(X) \leadsto \text{genus of a } \textit{family} \text{ parametrized by } X \text{ (orientation)}$ 



The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO\langle 8 \rangle_* :=$ the cobordism ring of *string manifolds* 

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

 $MO\langle 8 \rangle^{-*}(X) \leadsto \text{genus of a } \textit{family} \text{ parametrized by } X \text{ (orientation)}$ 

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \\ MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \\ MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \\ MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO\langle 8 \rangle_* :=$ the cobordism ring of *string manifolds* 

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

 $MO\langle 8\rangle^{-*}(X) \leadsto \text{genus of a } \textit{family} \text{ parametrized by } X \text{ (orientation)}$ 

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO\langle 8 \rangle_* :=$ the cobordism ring of *string manifolds* 

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

 $MO\langle 8\rangle^{-*}(X) \leadsto \text{genus of a } \textit{family} \text{ parametrized by } X \text{ (orientation)}$ 

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$

$$MO\langle 8\rangle \xrightarrow[]{\text{Ando-Hopkins-Rezk '10}} TMF \quad \text{topological modular forms}$$
 
$$\uparrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$
 
$$MU\langle 6\rangle \xrightarrow[]{\text{Ando-Hopkins-Strickland '04}} E \quad \text{an elliptic cohomology theory}$$

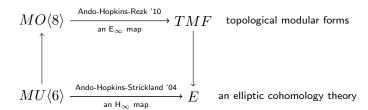
The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$



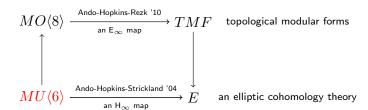
The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$



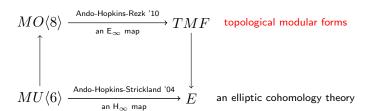
The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$



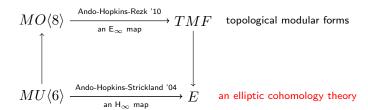
The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{Ell}}, \omega^{\otimes k})$ 

$$MF_k := H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$



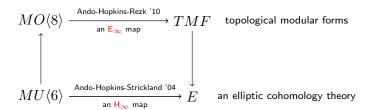
The Witten genus (Witten '87):  $MString_* \rightarrow MF_*$ 

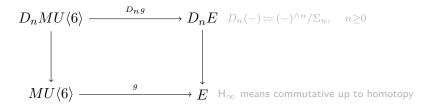
 $MString_* = MO(8)_* :=$  the cobordism ring of string manifolds

$$0 = w_1(M) = w_2(M) = p_1(M)/2$$

$$MF_* \cong \mathbb{Z}[c_4, c_6, (c_4^3 - c_6^2)/(12)^3]$$
  $MF_k := H^0(\overline{\mathcal{M}_{\text{Ell}}}, \omega^{\otimes k})$ 

$$MF_k \coloneqq H^0(\overline{\mathcal{M}_{\mathrm{Ell}}}, \omega^{\otimes k})$$





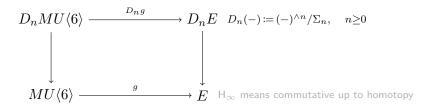
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





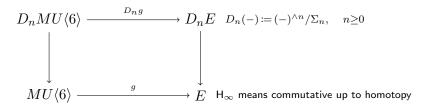
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\Sigma_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





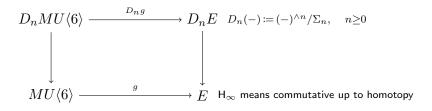
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





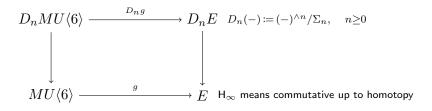
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\Sigma_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





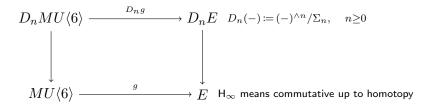
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





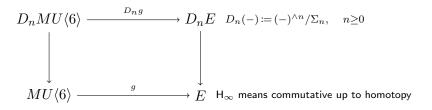
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\Sigma_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





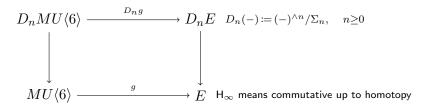
The vertical maps correspond to power operations; e.g., given

$$E^0X \cong \pi_0 E^{\Sigma_+^{\infty} X} =: \pi_0 A = [S, A]_S \cong [E, A]_E \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





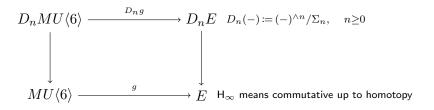
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\Sigma_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





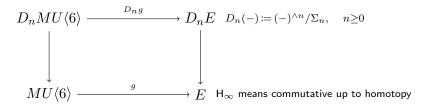
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





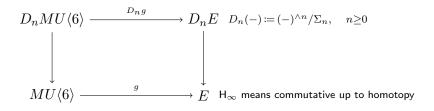
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





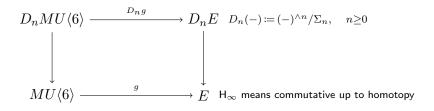
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





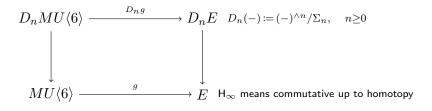
The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$





The vertical maps correspond to power operations; e.g., given

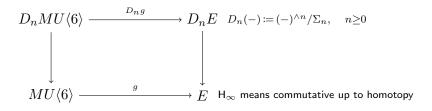
$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation  $Q_{\alpha} \colon E^{0}X \to E^{0}X$ . When E = KU, this is the n-fold tensor product of  $\mathbb{C}$ -vector bundles over X.





The vertical maps correspond to power operations; e.g., given

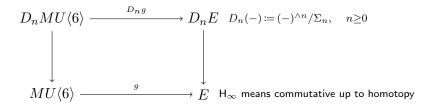
$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation  $Q_{\alpha} \colon E^{0}X \to E^{0}X$ . When E = KU, this is the n-fold tensor product of  $\mathbb{C}$ -vector bundles over X.





The vertical maps correspond to power operations; e.g., given

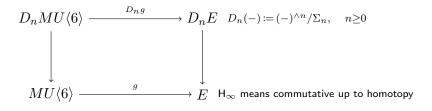
$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation  $Q_{\alpha} \colon E^0 X \to E^0 X$ . When E = KU, this is the n-fold tensor product of  $\mathbb{C}$ -vector bundles over X.





The vertical maps correspond to power operations; e.g., given

$$E^{0}X \cong \pi_{0}E^{\sum_{+}^{\infty}X} =: \pi_{0}A = [S, A]_{S} \cong [E, A]_{E} \ni f$$

we have

$$E \xrightarrow{\alpha} D_n E \xrightarrow{D_n f} D_n A \to A$$

and thus a power operation  $Q_{\alpha} \colon E^{0}X \to E^{0}X$ . When E = KU, this is the n-fold tensor product of  $\mathbb{C}$ -vector bundles over X.



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{\pmb{C}} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_{G} \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{\text{univ}}), c_1(\mathcal{L}_{\text{univ}})) = c_1(\mathcal{L}_{\text{univ}} \otimes \mathcal{L}_{\text{univ}})$$



#### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_{G} \cong R[\![t]\!] \rightsquigarrow R[\![t_{1}, t_{2}]\!] \ni F(t(P_{1}), t(P_{2})) = t(P_{1} + P_{2})$$
$$F(c_{1}(\mathcal{L}_{\text{univ}}), c_{1}(\mathcal{L}_{\text{univ}})) = c_{1}(\mathcal{L}_{\text{univ}} \otimes \mathcal{L}_{\text{univ}})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(\underline{P_1}), t(\underline{P_2})) = t(\underline{P_1} + \underline{P_2})$$
$$F(c_1(\mathcal{L}_{\text{univ}}), c_1(\mathcal{L}_{\text{univ}})) = c_1(\mathcal{L}_{\text{univ}} \otimes \mathcal{L}_{\text{univ}})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{\text{univ}}), c_1(\mathcal{L}_{\text{univ}})) = c_1(\mathcal{L}_{\text{univ}} \otimes \mathcal{L}_{\text{univ}})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{\text{univ}}), c_1(\mathcal{L}_{\text{univ}})) = c_1(\mathcal{L}_{\text{univ}} \otimes \mathcal{L}_{\text{univ}})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



### Definition (Ando-Hopkins-Strickland '01, Lurie '09)

$$\text{elliptic cohomology theory} = \left\{ \begin{array}{ll} R, & C/R, & E, \\ E^0(*) \cong R, & \operatorname{Spf} E^0(\mathbb{CP}^\infty) \cong \widehat{C} \end{array} \right\}$$

- ullet A formal group G is a group object in formal schemes.
- A formal group law F is G with a chosen coordinate t:

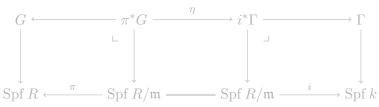
$$\mathcal{O}_G \cong R[\![t]\!] \rightsquigarrow R[\![t_1, t_2]\!] \ni F(t(P_1), t(P_2)) = t(P_1 + P_2)$$
$$F(c_1(\mathcal{L}_{univ}), c_1(\mathcal{L}_{univ})) = c_1(\mathcal{L}_{univ} \otimes \mathcal{L}_{univ})$$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty}\text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$

$$G \longleftarrow \pi^*G \stackrel{\eta}{\longrightarrow} i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spf} R \longleftarrow^{\pi} \operatorname{Spf} R/\mathfrak{m} = \operatorname{Spf} R/\mathfrak{m} \stackrel{i}{\longrightarrow} \operatorname{Spf} k$$

#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} \mathbf{E}^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$

$$G \longleftarrow \pi^*G \stackrel{\eta}{\longrightarrow} i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spf} R \longleftarrow^{\pi} - \operatorname{Spf} R/\mathfrak{m} = \operatorname{Spf} R/\mathfrak{m} \stackrel{i}{\longrightarrow} \operatorname{Spf} k$$

#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty}\text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$

$$G \longleftarrow \pi^*G \stackrel{\eta}{\longrightarrow} i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spf} R \longleftarrow^{\pi} - \operatorname{Spf} R/\mathfrak{m} \longrightarrow \operatorname{Spf} R/\mathfrak{m} \stackrel{i}{\longrightarrow} \operatorname{Spf} k$$

#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty) = \text{universal deformation of a fg } \Gamma \text{ of height } n$  over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$

$$G \longleftarrow \pi^*G \stackrel{\eta}{\longrightarrow} i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spf} R \longleftarrow^{\pi} - \operatorname{Spf} R/\mathfrak{m} \longrightarrow \operatorname{Spf} R/\mathfrak{m} \stackrel{i}{\longrightarrow} \operatorname{Spf} k$$

#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

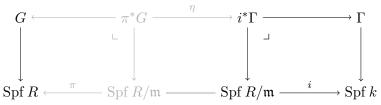
- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[u_1, \dots, u_{n-1}][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

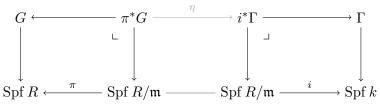
- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$



#### Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$



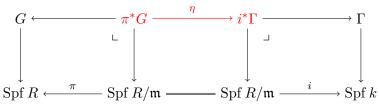
# Elliptic cohomology and Morava E-theory

# Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty} \text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$

A deformation  $(G, i, \eta)$  of  $\Gamma/k$  to R (Lubin-Tate '66):



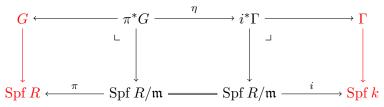
# Elliptic cohomology and Morava E-theory

# Theorem (Morava '78, Goerss-Hopkins-Miller '90s-'04)

 $\mathcal{E} \colon \{ \text{formal groups over perfect fields, isos} \} \to \{ E_{\infty}\text{-ring spectra} \}$ 

- $\operatorname{Spf} E^0(\mathbb{CP}^\infty)=$  universal deformation of a fg  $\Gamma$  of height n over a perfect field k of char p
- $E_* \cong \mathbb{W}(k)[[u_1, \dots, u_{n-1}]][u^{\pm 1}], \quad |u| = 2$

A deformation  $(G, i, \eta)$  of  $\Gamma/k$  to R (Lubin-Tate '66):



# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_{\Gamma}$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_{\Gamma}$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_{\infty}$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ ? \text{ genus}$

# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate x on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting x such that  $MU\langle 0 \rangle \stackrel{x}{\to} E$  is an  $H_{\infty}$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \implies E_* = \text{completion of } MF_* \implies ? \text{ genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ ? \text{ genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ ? \text{ genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- ullet (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg, E
  eq elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- ullet (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg, E
  eq elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg,  $E\neq$  elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ? \ \text{genus}$

# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

#### Remark

- ullet (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg, E
  eq elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.

 $E = {\rm local} \ TMF \ \leadsto \ E_* = {\rm completion} \ {\rm of} \ MF_* \ \leadsto \ \ ? \ {\rm genus}$ 



### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

### Remark

- ullet (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg, E
  eq elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.

 $E = {\rm local} \ TMF \ \leadsto \ E_* = {\rm completion} \ {\rm of} \ MF_* \ \leadsto \ \ ? \ {\rm genus}$ 



# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- ullet (Ando '95)  $k=\mathbb{F}_p$ ,  $\Gamma=$  Honda fg, E
  eq elliptic cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ ? \text{ genus}$



# Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_\Gamma$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_\Gamma$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_\infty$  map.

- (Ando '95)  $k = \mathbb{F}_p$ ,  $\Gamma = \mathsf{Honda}$  fg,  $E \neq \mathsf{elliptic}$  cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = {\sf local} \; TMF \; \leadsto \; E_* = {\sf completion} \; {\sf of} \; MF_* \; \leadsto \; ? \; {\sf genus}$



### Theorem (Z.)

Let k be an algebraic extension of  $\mathbb{F}_p$ ,  $\Gamma$  be a formal group over k of height n, and E be the Morava E-theory associated to  $\Gamma/k$ . Given any coordinate  $x_{\Gamma}$  on  $\Gamma$ , there exists a unique coordinate x on  $G_E$  lifting  $x_{\Gamma}$  such that  $MU\langle 0 \rangle \xrightarrow{x} E$  is an  $H_{\infty}$  map.

- (Ando '95)  $k = \mathbb{F}_p$ ,  $\Gamma = \mathsf{Honda}$  fg,  $E \neq \mathsf{elliptic}$  cohomology
- When n=2, the composite  $MU\langle 6\rangle \to MU\langle 0\rangle \to E$  does not factor through the Witten genus.
  - $E = \text{local } TMF \ \leadsto \ E_* = \text{completion of } MF_* \ \leadsto \ \ ?$  genus



There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)[\![u_1,\ldots,u_{n-1}]\!]$$

power operations deformations of Frobenius

$$G \longleftarrow \pi^*G \longrightarrow i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad i^*\operatorname{Frob}^r \downarrow \qquad \operatorname{Frob}^r \downarrow$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad i^*\Gamma(p^r) \longrightarrow \Gamma(p^r)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

### There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)[[u_1, \dots, u_{n-1}]]$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)[u_1, \dots, u_{n-1}]$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)[\![u_1,\ldots,u_{n-1}]\!]$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \leftrightsquigarrow \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$G \longleftarrow \pi^*G \longrightarrow i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad i^*\operatorname{Frob}^r \downarrow \qquad \operatorname{Frob}^r \downarrow$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad i^*\Gamma(p^r) \longrightarrow \Gamma(p^r)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$G \longleftarrow \pi^*G \longrightarrow i^*\Gamma \longrightarrow \Gamma$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad i^*\operatorname{Frob}^r \downarrow \qquad \operatorname{Frob}^r \downarrow$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad i^*\Gamma(p^r) \longrightarrow \Gamma(p^r)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$\begin{matrix} G \longleftarrow & \pi^*G & \xrightarrow{\eta} & i^*\Gamma & \longrightarrow \Gamma \\ \downarrow & & \downarrow & & i^*\operatorname{Frob}^r \downarrow & \operatorname{Frob}^r \downarrow \\ \downarrow & & \downarrow & & i^*\Gamma^{(p^r)} & \longrightarrow \Gamma^{(p^r)} \\ \downarrow & & \downarrow & & \downarrow & & i'=i\circ\sigma^r & \downarrow \\ G' \longleftarrow & \pi^*G' & \longrightarrow i'^*\Gamma & \longrightarrow \Gamma \end{matrix}$$

#### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$\begin{matrix} G \longleftarrow & \pi^*G & \xrightarrow{\eta} i^*\Gamma & \longrightarrow \Gamma \\ \downarrow & \downarrow & \downarrow & i^*\operatorname{Frob}^r \downarrow & \operatorname{Frob}^r \downarrow \\ \downarrow & \downarrow & \downarrow & i^*\Gamma(p^r) & \longrightarrow \Gamma(p^r) \\ \downarrow & \downarrow & \downarrow & \downarrow & i'=i\circ\sigma^r & \downarrow \\ G' \longleftarrow & \pi^*G' & \longrightarrow i'^*\Gamma & \longrightarrow \Gamma \end{matrix}$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$\begin{matrix} G \longleftarrow & \pi^*G & \xrightarrow{\eta} i^*\Gamma & \longrightarrow \Gamma \\ \downarrow & \downarrow & \downarrow & i^*\operatorname{Frob}^r \downarrow & \operatorname{Frob}^r \downarrow \\ \downarrow & \downarrow & \downarrow & i^*\Gamma^{(p^r)} & \longrightarrow \Gamma^{(p^r)} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow i' = i \circ \sigma^r & \downarrow \\ G' \longleftarrow & \pi^*G' & \longrightarrow i'^*\Gamma & \longrightarrow \Gamma \end{matrix}$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$\begin{matrix} G \longleftarrow & \pi^*G & \xrightarrow{\eta} i^*\Gamma & \longrightarrow \Gamma \\ \downarrow & \downarrow & \downarrow & \downarrow & \text{Frob}^r \downarrow & \text{Frob}^r \downarrow \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \uparrow^*\Gamma(p^r) & \longrightarrow \Gamma(p^r) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ G' \longleftarrow & \pi^*G' & \xrightarrow{\eta'} & \downarrow^{\prime*}\Gamma & \longrightarrow \Gamma \end{matrix}$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

$$\begin{matrix} G \longleftarrow & \pi^*G & \xrightarrow{\eta} i^*\Gamma & \longrightarrow \Gamma \\ \downarrow & \downarrow & \downarrow & i^*\operatorname{Frob}^r \downarrow & \operatorname{Frob}^r \downarrow \\ \downarrow & \downarrow & \downarrow & i^*\Gamma(p^r) & \longrightarrow \Gamma(p^r) \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow i' = i \circ \sigma^r & \downarrow \\ G' \longleftarrow & \pi^*G' & \longrightarrow i'^*\Gamma & \longrightarrow \Gamma \end{matrix}$$

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

#### Theorem (Ando-Hopkins-Strickland '04, Rezk '09)

There is a correspondence

$$E(\Gamma/k) \iff \Gamma/k \xrightarrow{\text{univ defo}} G_E/\mathbb{W}(k)\llbracket u_1, \dots, u_{n-1} \rrbracket$$

power operations deformations of Frobenius

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

A level structure on G is a choice of finite subgroup. This theorem gives universal examples of "descent data" for level structures:

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G, i, \eta) \to (G', i', \eta')$  are classified by rings  $A_r$ ,  $r \ge 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\operatorname{Spf} A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G, i, \eta) \to (G', i', \eta')$  are classified by rings  $A_r$ ,  $r \ge 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\operatorname{Spf} A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{ ext{univ}} imes_{A_0} A_r = s_r^* G_{ ext{univ}} \xrightarrow{\psi_{ ext{univ}}^{(p^r)}} t_r^* G_{ ext{univ}} = ?$$

$$\operatorname{Spf} A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\mathrm{univ}} \times_{A_0} A_r = s_r^* G_{\mathrm{univ}} \xrightarrow{\psi_{\mathrm{univ}}^{(p^r)}} t_r^* G_{\mathrm{univ}} = ?$$

$$\operatorname{Spf} A_r$$

# Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

$$H = \text{finite subgroup of } G \qquad f_H \colon G \to G/H \qquad x = \text{coord on } G$$
 
$$\Longrightarrow x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H$$
 
$$\mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{H}(x) = f_{H}^{*}(x_{H})$$
 for any finite  $H \subset G$ 



$$H = \text{finite subgroup of } G \qquad f_H \colon G \to G/H \qquad x = \text{coord on } G$$
 
$$\Longrightarrow x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot \, x) \text{ is a coord on } G/H$$
 
$$\mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{H}(x) = f_{H}^{*}(x_{H})$$
 for any finite  $H \subset G$ 



$$H = \text{finite subgroup of } G \qquad f_H \colon G \to G/H \qquad x = \text{coord on } G$$
 
$$\Longrightarrow x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H$$
 
$$\mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$H = \text{finite subgroup of } G \qquad f_H \colon G \to G/H \qquad x = \text{coord on } G$$
 
$$\Longrightarrow \ x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H$$
 
$$\mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$H = \text{finite subgroup of } G \qquad f_H \colon G \to G/H \qquad x = \text{coord on } G$$
 
$$\Longrightarrow \ x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H$$
 
$$\mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H \qquad x = \text{coord on } G \\ & \Longrightarrow x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H & x = \text{coord on } G \\ & \Longrightarrow x_H \coloneqq \mathrm{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\mathrm{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H \qquad x = \text{coord on } G \\ \Longrightarrow & x_H \coloneqq \mathrm{Norm}_{f_H^*}(x) = \det(\,\cdot\,x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\mathrm{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H & x = \text{coord on } G \\ \Longrightarrow & x_H \coloneqq \mathrm{Norm}_{f_H^*}(x) = \det(\,\cdot\,x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\mathrm{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_{\scriptscriptstyle H}(x) = f_{\scriptscriptstyle H}^*(x_{\scriptscriptstyle H}) \qquad \text{for any finite } H \subset G$$



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H & x = \text{coord on } G \\ & \Longrightarrow x_H \coloneqq \operatorname{Norm}_{f_H^*}(x) = \det(\cdot x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\operatorname{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_H(x) = f_H^*(x_H)$$
 for any finite  $H \subset G$ 



$$\begin{split} H = \text{finite subgroup of } G & f_H \colon G \to G/H & x = \text{coord on } G \\ \Longrightarrow & x_H \coloneqq \mathrm{Norm}_{f_H^*}(x) = \det(\,\cdot\,x) \text{ is a coord on } G/H \\ & \mathcal{O}_{G/H} \xrightarrow{f_H^*} \mathcal{O}_G \xrightarrow{\mathrm{Norm}_{f_H^*}} \mathcal{O}_{G/H} \end{split}$$

Explicitly,

$$f_H^*(x_H) = \prod_{Q \in H} \left( x + x(Q) \right)$$

#### Definition

A coordinate x on G is norm-coherent if

$$\psi_H(x) = f_H^*(x_H)$$
 for any finite  $H \subset G$ 



### Theorem (Strickland '97)

Fix  $\Gamma/k$ . Then deformations of Frobenius  $(G,i,\eta) \to (G',i',\eta')$  are classified by rings  $A_r$ ,  $r \geq 0$ , with  $p^r$  the order of the subgroup scheme  $\ker(G \to G') \subset G$ .

#### Remark

$$G_{\text{univ}} \times_{A_0} A_r = s_r^* G_{\text{univ}} \xrightarrow{\psi_{\text{univ}}^{(p^r)}} t_r^* G_{\text{univ}} = ?$$

$$\text{Spf } A_r$$

### Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0\rangle \to E$  is an  ${\rm H}_{\infty}$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0\rangle \to E$  is an  $\mathsf{H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

#### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_{\infty}$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

#### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



### Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_{\infty}$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_{\infty}$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

#### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



# Theorem (Ando '95, Ando-Hopkins-Strickland '04)

Let E be the Morava E-theory associated to  $\Gamma/k$  as before. Then the orientation  $MU\langle 0 \rangle \to E$  is an  ${\rm H}_\infty$  map if and only if its corresponding coordinate on  $G_E$  is norm-coherent.

### Theorem (Z.)

- Any coordinate on  $\Gamma$  over k extends uniquely to a norm-coherent coordinate on  $G_E$  over  $\pi_0 E$ .
- This construction is functorial under base change of  $\Gamma/k$ , under k-isogeny out of  $\Gamma$ , and under k-Galois descent.



Thank you.