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Power Operations for Morava E - Theory of Height Two at the Prime Three

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ABSTRACT

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Using the universal deformation for the supersingular elliptic curve at the prime 3, we produce a model for a Morava E-theory E associated to the Lubin-Tate ring of deformations of a height 2 formal group; that is, we define an elliptic curve whose associated formal group $\mathbb{G}_E$ is the formal group of the Morava E-theory. We find the 3-isogeny structure of this and related elliptic curves and translate the results to $\mathbb{G}_E$. This allows us to explicitly calculate the Dyer-Lashof algebra of power operations for E using the work of Strickland [**19**]. Rezk [**12**], extending results of Ando, Hopkins and Strickland, [**1, 2**], shows that the homotopy of $K(2)$-local commutative algebras over this Morava E-theory are unstable algebras over the Dyer-Lashof algebra for E satisfying a Cartan formula. With our formulas, we have completely determined the structure of the homotopy of these E-algebras.

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CHAPTER 1

Introduction

Since Steenrod [**15, 16, 17**] introduced his reduced power operations on the cohomology of a space with $\mathbb{F}_p$ coefficients, the algebra of natural operations has enjoyed a fundamental role in stable homotopy theory. Adams introduced operations in K-theory, which Atiyah [**3**] then put into the framework of power operations, using knowledge of the K-theory of the space $DS^0 = \bigvee_m B\Sigma_{m\,+}$ in terms of representations of the symmetric groups. Kudo and Araki [**8**] defined an analogous algebra of homology operations dual to the Steenrod operations, and used it calculate the homology $H_*(QX; \mathbb{F}_2)$ of $QX = \Omega^\infty \Sigma^\infty X$. Dyer and Lashof [**5**], and then May and McClure [**4**], generalized this work defining an algebra of operations for $H\mathbb{F}_p$ and K referred to as the Dyer-Lashof algebra Γ. They constructed a monad C on graded $\mathbb{F}_p$ vector spaces so that the homotopy of the free E_∞-object $\mathbb{P}M$ associated to any $H\mathbb{F}_p$-module M is $C(\pi_* M)$. Further, they showed a C-algebra is a graded commutative $\mathbb{F}_p$-algebra that is also an unstable module over Γ which satisfies a Cartan formula.

Rezk [**12**], extending results of Ando, Hopkins and Strickland, [**1**][**2**], builds an analogous theory for $K(n)$-local commutative algebras over the Morava E-theory E associated to the Lubin-Tate ring of deformations of a height n formal group over a perfect field of characteristic p. In this case a monad $\mathbb{T}$ is defined so that the homotopy of the $K(n)$-localization of the free E_∞-algebra $\mathbb{P}M$ is the completion of $\mathbb{T}(\pi_* M)$ at the the unique maximal ideal $\mathfrak{m} \subset E_*$ for an E-module M:

$$\mathbb{T}(\pi_* M)^{\wedge}_{\mathfrak{m}} \cong \pi_* L_{K(n)} \mathbb{P} M.$$

Rezk also gives an algebraic description of algebras over $\mathbb{T}$ as E_*-algebras that are also unstable algebras over the Dyer-Lashof algebra Γ for E satisfying a Cartan formula. In fact, the goal of [**12**] is to give a congruence criterion for a p-torsion free Γ-algebra to be a $\mathbb{T}$-algebra. Thus, understanding the theory of power operations for Morava E-theory is, in the p-torsion free case, equivalent to calculating the structure of Γ, and finding the Cartan formula it satisfies.

The Dyer-Lashof algebra Γ of power operations is a twisted bialgebra so that there is a homomorphism $\eta : E_0 \to \Gamma$ and a symmetric monoidal structure on Mod^*_Γ. The elements of Γ correspond to the indecomposable natural additive operations arising from the power operations on the homotopy of a $K(n)$-local commutative E-algebra F. Taking $F = E$ and using properties of power operations we deduce the commutation and Adem relations that Γ satisfies. The indecomposable additive operations are parameterized by the indecomposables with respect to a certain product in $E^0 DS^0 = \prod E^0 B\Sigma^+_m$, which is provided by Strickland [**19**]. In that work it is shown that

$$Ind(E^0 DS^0) = \prod_k E^0 B\Sigma_{p^k}/I_{p^k} \cong \prod_k \mathcal{O}_{Sub_k(\mathbb{G}_E)},$$

where $I_{p^k} = \mathrm{Im}[tr : E^0 B\Sigma^p_{p^{k-1}} \to E^0 B\Sigma_{p^k}]$ is the transfer ideal, and $Sub_k(\mathbb{G}_E)$ is the scheme of subgroups of order p^k in the formal group $\mathbb{G}_E = \mathrm{Spf}\ E^0\mathbb{C}P^\infty$ over E_0 associated to E. Thus, we can understand power operations in E by computing the structure of isogenies of rank p^k in $\mathbb{G}_E$. In [**11**] Rezk gives a brief account of these computations for a Morava E-theory of height 2 at the prime 2.

Our work is analogous to Rezk's at the prime 3. We develop an elliptic model for our theory E, a Morava E-theory of height 2 at the prime $p = 3$; i.e. we define an elliptic curve C over E_0 such that the formal group obtained by completing C at the identity is isomorphic to $\mathbb{G}_E$. We may then work with this curve, finding its 3-isogeny structure, and translate the results to $\mathbb{G}_E$. The main result of chapter 4 is the computation of this structure for the curve C. In fact, we compute the 3-isogeny structure for many standard families of curves. Points and subgroups of order N in an elliptic curve are well-behaved when working over schemes with N inverted. The computations of chapter 4 are challenging precisely because we must understand points and subgroups of order p *at the prime* p. Our calculations were aided at almost every step with the open-source mathematics software Sage (http://www.sagemath.org/).

The remainder of chapter 1 introduces the Morava E-theories and the elliptic model used in chapter 4. In chapter 2 we give a brief account of power operations, and describe the structure of Γ using properties of power operations and the results of chapter 4. Chapter 3 is a standard account of classical notions of the moduli of elliptic curves, very closely following the exposition of [**18**], that lays the foundation for our work in chapter 4.

1.1. Deformations of Formal Groups and Morava E-Theory

Generalized cohomology theories are the tools and objects of study in stable homotopy theory. In this section we introduce the theory under study in this work: Morava E-theory of height 2 over the field $\mathbb{F}_3$. We follow Rezk's exposition [**10**] on Lubin and Tate's work [**9**] on deformations of one parameter formal groups of finite height over a perfect field k of characteristic p. Morava showed that the functor associating the ring $A(k, \mathbb{G}_0)$ of all such deformations of the formal group $\mathbb{G}_0$ to complete local rings with residue field k corresponds to a homology theory: the Morava E-theory $E(k, \mathbb{G}_0)$. Goerss, Hopkins and Miller [**6**] show $E(k, \mathbb{G}_0)$ is an E_∞-ring spectrum, and that this structure is essentially unique.

Working with an associative commutative ring spectrum E we get a corresponding cohomology theory with a product. The most valuable of these theories are complex oriented, i.e. there exists a class $x \in \tilde{E}^2\mathbb{C}P^\infty$ that restricts to a unit in $\tilde{E}^2\mathbb{C}P^1 \cong E_0$. Such a spectrum will have a theory of Chern classes. Given a line bundle $\mathcal{L}$ over X classified by a map $f : X \to \mathbb{C}P^1$ we may obtain a E-cohomology class $c_1(\mathcal{L}) = f^*x$. To every such spectrum E we may associate a formal group $\mathbb{G}_E = \operatorname{Spf} E^*\mathbb{C}P^\infty$ over E_* corresponding to the formal group law for $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2)$ in terms of $c_1(\mathcal{L}_1), c_1(\mathcal{L}_2)$. Amongst all complex oriented theories the initial is complex cobordism MU since Quillen showed that MU_* is the Lazaard ring L and that $\mathbb{G}_{MU}$ is the universal formal group classifying formal group laws.

Looking at a prime p complex cobordism MU splits as a wedge of copies of the Brown-Peterson spectrum BP. Formal groups over a prime p are partitioned by height, n, where $[p](x) = v_n x^{p^n} \, mod \, x^{p^n+1}$. This leads to the chromatic filtration of formal groups over fields of characteristic p, and yields a corresponding filtration of cohomology theories. The theory $K(n)$ associated to $\mathbb{F}_p[v_n^{\pm 1}]$ and p-typical formal group F_n with $[p](x) = v_n x^{p^n}$ is called Morava K-theory of height n. The Morava theory $K(0)$ is rational homology

$H\mathbb{Q}$, while $K(1)$ is essentially mod p K-theory, both studied systematically in the last several decades.

A cohomology theory E is even periodic if there exists a unit $u \in \tilde{E}^{-2}$ such that $E_* \cong E_0[u^{\pm 1}]$. We say E is homogeneous if it is both complex oriented and even periodic. Such a theory has a formal group $\mathbb{G}_E = \mathrm{Spf}\, E^0\mathbb{C}P^\infty$ over E_0. Multiplication in $\mathbb{G}_E$ arises from the co-multiplication in $E^0\mathbb{C}P^\infty$. We now consider the homogeneous theory associated to deformations of one parameter formal groups of finite height over a field k of characteristic p.

A deformation of $(k, \mathbb{G}_0)$ to a complete local ring B (with maximal ideal $\mathfrak{m}$ and projection $\pi : B \to B/\mathfrak{m}$) is a pair $(\mathbb{G}, i)$ consisting of a formal group $\mathbb{G}$ over B and a homomorphism $i : k \to B/\mathfrak{m}$, such that $i^*\mathbb{G}_0 = \pi^*\mathbb{G}$. A morphism of deformations $(\mathbb{G}_1, i_1) \to (\mathbb{G}_2, i_2)$ is defined only when $i_1 = i_2$, in which case it consists of an isomorphism $f : \mathbb{G}_1 \to \mathbb{G}_2$ of formal groups over B such that $\pi^* f$ is the identity map of $\pi^*\mathbb{G}_1 = \pi^*\mathbb{G}_2 = i_1^*\mathbb{G}_0 = i_2^*\mathbb{G}_0$. That is, $f(x) \equiv x \mod \mathfrak{m}$. Such an isomorphism f is sometimes called a $\star$-isomorphism. Write $\mathrm{Def}_{\mathbb{G}_0}(B)$ for the category of deformations of $\mathbb{G}_0$ to B and $\star$-isomorphisms.

Lubin and Tate [**9**] show that there exists a complete local ring $A(k, \mathbb{G}_0)$, an isomorphism $i : k \to A(k, \mathbb{G}_0)/\mathfrak{m}$, and a formal group law $\mathbb{G}$ on $A(k, \mathbb{G}_0)$ such that the pair $(\mathbb{G}, id)$ is a universal deformation, in the sense that the functor $B \mapsto \pi_0 \mathrm{Def}_{\mathbb{G}_0}(B)$ is co-represented by the ring $A(k, \mathbb{G}_0)$, so that a map $\phi : A(k, \mathbb{G}_0) \to B$ corresponds to the isomorphism class of $\phi^*\mathbb{G}$ in $\mathrm{Def}_{\mathbb{G}_0}(B)$. Further, Lubin and Tate show that if k is perfect and $\mathbb{G}_0$ has finite height n, then there is a non-canonical isomorphism

$$A(k, \mathbb{G}_0) \cong \mathbb{W}k[[u_1, \ldots, u_{n-1}]].$$

Here we have $\mathbb{W}k$ is the ring of Witt vectors of k, and $A(k, \mathbb{G}_0)$ is a complete local ring with respect to the ideal $\mathfrak{m} = (p, u_1, \ldots, u_{n-1})$.

We build an even periodic cohomology theory out of the above functor using Landweber's exact functor theorem. Define the graded ring $E(k, \mathbb{G}_0)_* = A(k, \mathbb{G}_0)[u^{\pm 1}]$ with $|u| = -2$. We then have a graded formal group $\bar{\mathbb{G}}$ corresponding to conjugating $\mathbb{G}$ by u. Thus we have a map $MU_* \to E(k, \mathbb{G}_0)_*$ classifying $\bar{\mathbb{G}}$, and we may define for a space X

$$E_*(X) = E(k, \mathbb{G}_0)_* \otimes_{MU_*} MU_*(X).$$

Morava showed the map $MU_* \to E(k, \mathbb{G}_0)_*$ satisfies the conditions of the Landweber exact functor theorem, and so the functor above is a homology theory. Note that $E_* = E(k, \mathbb{G}_0)_*$ and $E_0 = A(k, \mathbb{G}_0) \cong \mathbb{W}k[[u_1, \ldots, u_{n-1}]]$. We call this the Morava E-theory associated to the pair $(k, \mathbb{G}_0)$. Let $E_n = E(\mathbb{F}_{p^n}, F_n)$ where F_n is the standard Honda formal group. All Morava E-theories agree up to isomorphism in the algebraic closure, i.e. $E(k, \mathbb{G}_0) \otimes \overline{\mathbb{F}}_p \cong E_n \otimes \overline{\mathbb{F}}_p$.

The Morava E-theories E_n play a central role in chromatic stable homotopy theory. The Bousfield localization functor L_{E_n} is essentially "restriction to the closed substack $\mathcal{M}_{FG}^{\geq n}$" of formal groups of height at least n in the moduli stack of formal groups over p-local rings $\mathcal{M}_{FG} \times \operatorname{Spec} \mathbb{Z}_{(p)}$. For a p-local spectrum X we have the chromatic tower which converges in the limit to X:

$$\cdots \to L_{E_n}X \to L_{E_{n-1}}X \to \cdots \to L_{E_1}X \to L_{E_0}X.$$

So knowledge of the structure and symmetries of the theories helps in using them to understand p-local spectra in general.

Further, Hopkins, Miller, and Goerss have shown that $E(k, \mathbb{G}_0)$ has a unique E_∞ structure. At this time there are precious few concrete examples of E_∞-ring spectra. Thus, understanding operations and other features in any examples of these theories is a valuable exercise in building theoretical intuition.

1.2. An Explicit Elliptic Model

We have seen that homogeneous theories have associated formal groups. Another way to obtain a formal group is from any 1-dimensional group scheme that is smooth near the identity, such as $\mathbb{G}_a, \mathbb{G}_m$, or the formal group arising from an elliptic curve. If C is such a group scheme over Spec R, then completion at the identity of the group law in C gives a formal group law $\widehat{C}$ over R. It is classical that formal group laws arising from elliptic curves have height 1 or 2. We say a homogeneous theory E is an elliptic cohomology theory if there is an elliptic curve C over Spec E_0 such that $\mathbb{G}_E \cong \widehat{C}$. It turns out that the Morava E-theories of interest to us are elliptic cohomology theories.

Let $A = \mathbb{Z}[1/2][u_1, (u_1^2 - 1)^{-1}]$ and let C be the Legendre elliptic curve over Spec A given in homogeneous coordinates by

$$C : Y^2 Z = X(X - Z)(X - \lambda Z)$$

where $\lambda = (u_1+1)/(u_1-1)$. At the prime 3 the supersingular locus of C, where the Hasse invariant is zero, is $(3, u_1)$ since the Hasse invariant in this case is $\lambda + 1$. The reduction modulo $(3, u_1)$ of C over Spec A is C_0 over Spec $\mathbb{F}_3$, where

$$C_0 : y^2 = x^3 - x.$$

Since this curve is supersingular, the associated formal group $\hat{C}_0$ has height 2. Let $\hat{A} = A^{\wedge}_{(3,u_1)} \cong \mathbb{Z}_3[[u_1]] \cong A(\mathbb{F}_3, \hat{C}_0)$. It is a classical computation, see for instance [**14**] IV.1, that the formal group law for $\hat{C}$ is

$$\begin{aligned} \hat{C}(z_1, z_2) &= z_1 + z_2 + \frac{2u_1}{u_1 - 1}(z_1^2 z_2 + z_1 z_2^2) \mod (z_1, z_2)^4 \\ &\equiv z_1 + z_2 + u_1(z_1^2 z_2 + z_1 z_2^2) \mod (3, u_1^2, (z_1, z_2)^4). \end{aligned}$$

Following Lubin and Tate [**9**] this shows that $(\hat{A}, \hat{C})$ is isomorphic to the universal deformation of $(\mathbb{F}_3, \hat{C}_0)$. Thus, the elliptic cohomology theory E with $E_0 = \hat{A}$ and formal group $\mathbb{G}_E \cong \hat{C}$ is a Morava E-theory of height 2 at the prime 3 associated to the pair $(\mathbb{F}_3, \hat{C}_0)$.

It must be noted that the "standard" Morava E-theory of height 2 at the prime 3 is $E_2 = E(\mathbb{F}_9, F_2)$ where F_2 is the Honda formal group. The main qualitative difference between this theory and ours is that we do not get all automorphisms of our formal group $\hat{C}_0$ until we extend to $\mathbb{F}_9$ (we need a fourth root of unity). Thus,

$$(E_2)_0 \cong \mathbb{W}\mathbb{F}_9[[u_1]] \cong \mathbb{Z}_3[i][[u_1]] \cong \hat{A}[i].$$

This étale extension doesn't affect our operations because the fourth root of unity i commutes with all operations. So we choose to work over $\mathbb{F}_3$ knowing that our formulas hold for E_2 with only trivial modification.

The subgroup inclusions $\Sigma_i \times \Sigma_j \to \Sigma_m$ for $i + j = m$ induce the transfer maps $tr^{\Sigma_m}_{\Sigma_i \times \Sigma_j} : E^0B(\Sigma_j \times \Sigma_j) \to E^0B\Sigma_m$. Let I_m be the transfer ideal in $E^0B\Sigma_m$ given by

$$I_m = \mathrm{Im}\left[tr : \bigoplus_j E^0B(\Sigma_j \times \Sigma_{m-j}) \to E^0B\Sigma_m\right].$$

Strickland [**19**] shows that $E^0B\Sigma_m/I_m = 0$ unless $m = p^k$, that

$$I_{p^k} = \mathrm{Im}[tr : E^0B\Sigma^p_{p^{k-1}} \to E^0B\Sigma_{p^k}],$$

and

$$E^0B\Sigma_{p^k}/I_{p^k} \cong \mathcal{O}_{Sub_k(\hat{C})} \tag{1.1}$$

represents subgroup schemes of order 3^k in the formal group $\hat{C}$ over $\hat{A}$. This is a crucial ingredient in our method. It allows us to connect the theory of power operations in our Morava E-theory to that of subgroup schemes in the corresponding formal group (and in the end to the corresponding elliptic curve).

1.3. Modular 3-Isogeny Category

The structure that we must compute in our elliptic model to exploit for the calculation of the algebra of power operations for the Morava E-theory is what Rezk [**13**] calls the 3-isogeny module structure for our elliptic curve. In this section we provide the details of that structure, which is essentially determined by the universal example of a degree 3 isogeny from our curve C and its dual.

Write Ell for the category with objects C/S, an elliptic scheme C over a base scheme S, and whose morphisms $C/S \to C'/S'$ are commutative diagrams

$$\begin{tikzcd} C \arrow[r] \arrow[d] & C' \arrow[d] \\ S \arrow[r, "f"] & S' \end{tikzcd}$$

where the induced map $C \to f^*C'$ is an isomorphism of elliptic curves over S.

For any elliptic curve C/S, Katz and Mazur [**7**] let $[3^r\text{-Isog}](C/S)$ be the set of locally free finite commutative S-subgroup schemes $H \leq E$ which are rank 3^r over S. They show that the moduli problem $[3^r\text{-Isog}]$ is finite and relatively representable over Ell. This moduli problem is equivalent to the set of all isogenies of rank 3^r on C/S, since every subgroup $H \leq C$ of $[3^r\text{-Isog}](C/S)$ corresponds to a 3^r-isogeny $\phi : C \to C'$ over S with kernel H. This correspondence is unique up to isomorphisms from C/S. In fact we can consider, also following [**7**], the more general moduli problem $[3^{r_1}, \ldots, 3^{r_n}\text{-Isog}](C/S)$ of chains of subgroups $H_1 \leq H_2 \leq \cdots \leq H_n \leq C$ such that rank $H_k/H_{k-1} = 3^{r_k}$. This is also finite and relatively representable. It corresponds to chains of isogenies $C \to C_1 \to \cdots \to C_n$ of the appropriate rank.

Given an elliptic curve C/S for each S-scheme T we have an associated set $[3^{r_1}, \ldots, 3^{r_n}\text{-Isog}](C_T/T)$. This association is a representable functor $Sch_S^{op} \to Set$. If $S = \operatorname{Spec} A$ is an affine scheme, then the representing scheme $[3^{r_1}, \ldots, 3^{r_n}\text{-Isog}]_{C/S}$ is affine, with ring of functions $A_{r_1,..,r_n}$. These rings are finite and locally free as an A-module. Taken in total this is a category $[\text{3-Isog}]_{C/S}(T)$ in which the set of degree r morphisms is exactly $\mathrm{Hom}_{Sch}(T,\mathrm{Spec}\, A_{r_1,..,r_n})$. Suppose $\mathrm{Spec}\, A$ represents the functor sending T to the set of curves C with a trivialization α of the invertible sheaf of invariant one-forms ω over C. The following maps encode the categorical structure. We have that s, t are the source and target maps respectively, while μ gives composition of morphisms. The map w encodes taking the dual of an isogeny, while ψ encodes the action of the multiplication map $[3] : C \to C'$ on the invariant differential.

$$
\begin{array}{rr}
s = s_r : A \to A_r & (\phi : C \to C', \alpha) \to (C, \alpha) \\
t = t_r : A \to A_r & (\phi : C \to C', \alpha) \to (C', \phi^* \alpha) \\
\mu_{r_1,r_2} : A_{r_1+r_2} \to A_{r_1,r_2} & (C \xrightarrow{\phi} C' \xrightarrow{\phi'} C'', \alpha) \to (\phi'\phi : C \to C'', \alpha) \\
w = w_r : A_r \to A_r & (\phi : C \to C', \alpha) \to (\hat{\phi} : C' \to C, \phi^* \alpha) \\
\pi = \pi_r : A_{2r} \to A & (C, \alpha) \to ([3^r] : C \to C', \alpha) \\
\psi = \psi_r : A_r \to A_r & (\phi : C \to C', \alpha) \to (\phi : C \to C', [3^r]^* \alpha)
\end{array}
$$

Note that $A_{r_1,..,r_n} \cong A_{r_1}\, {}^t\otimes^s_A A_{r_2}\, {}^t\otimes^s_A \cdots {}^t\otimes^s_A A_{r_n}$. There are many relations satisfied by these structure maps. For instance, the basic notion that the dual of isogeny has reversed source and target gives $w_r s_r = t_r$. The fact that the composition of a rank r isogeny and its dual is $\hat{\phi}\phi = [3^r]$ means that $w_r t_r = s_r \psi_r$ and $s_r \pi_r = (1 \otimes w_r)\mu_{r,r}$. All of the structural data of the affine 3-isogeny category may be derived from knowledge of the rings $A = A_0$ and A_1 [**13**], and the ring homomorphisms

$$s, t : A \to A_1, \quad \psi : A \to A, \quad w : A_1 \to A_1.$$

There is still some redundancy in the basic data, since $t = ws$ and $s\psi = wt$. As an example of building the rest of the category, the ring A_2 may be written as the following pullback (identifying $A_{1,1} = A_1{}^t \otimes^s_A A_1$).

$$\begin{tikzcd} A_2 \arrow[r, "\mu_{1,1}"] \arrow[d, "\pi_2"'] & A_{1,1} \arrow[d, "1\otimes w"] \\ A \arrow[r, "s"] & A_1 \end{tikzcd} \tag{1.2}$$

In chapter 4 we have calculated the basic data in the cases of Weierstrass elliptic curves over $\mathbb{Z}[\frac{1}{2}]$, as well as curves with various level structures (including the Legendre curve). We summarize our result, Theorem 4.4.1, of the basic data for the Legendre curves $C_{u_1} : y^2 = x(x-1)(x - \frac{u_1+1}{u_1-1})$ here, with $\psi_1(d) = d^4 - 6d^2 - 8u_1 d - 3$.

$$\begin{aligned}
A_0 &= \mathbb{Z}[1/2][u_1, (u_1^2-1)^{-1}] & A_1 &= A[d]/\psi_1(d) \\
s &: A \to A_1 & s(u_1) &= u_1 \\
t &: A \to A_1 & t(u_1) &= (-8u_1^2+3)d^3 + 3u_1 d^2 + \\
& & &\quad + (48u_1^2 - 19)d + 64u_1^3 - 42u_1 \\
\psi &: A \to A & \psi(u_1) &= u_1 \\
w &: A_1 \to A_1 & w(u_1) &= t(u_1),\ w(d) = -\frac{3}{d} = -d^3 + 6d + 8u_1
\end{aligned} \tag{1.3}$$

These calculations will figure prominently in the development of a presentation of the algebra of power operations for our Morava E-theory. We shall detail in chapter 2 how the extension A_1 determines the generators, while the maps t and w determine the commutation and Adem relations respectively of our algebra.

As seen in section 1.2 the completion of $A_0 = A$ at the supersingular locus $(3, u_1)$ is $\hat{A} \cong E_0$ the coefficient ring for our Morava E-theory. Define $\hat{A}_{r_1,\ldots,r_n} = \hat{A} \otimes_A A_{r_1,\ldots,r_n}$ so that $\hat{A}_{r_1,\ldots,r_n} \cong \mathcal{O}_{Sub_{r_1,\ldots,r_n}(\hat{C})}$ represents chains of subgroups of the appropriate rank in the associated formal group $\hat{C}$. Then by (1.1) we have $\hat{A}_k \cong E^0 B\Sigma_{p^k}/I_{p^k}$. Further,

$$\hat{A}_{r_1,\ldots,r_n} \cong E^0 B\Sigma_{p^{r_1}}/I_{p^{r_1}} \otimes_{E_0} \cdots \otimes_{E_0} E^0 B\Sigma_{p^{r_n}}/I_{p^{r_n}}.$$

As we shall see in chapter 2, these rings will be the targets of our power operations.

CHAPTER 2

Power Operations

In chapter 2 we give some background exposition on definitions and basic notions concerning power operations. We introduce the Dyer-Lashof algebra, and discuss Rezk's work [**11, 12**] in the case of Morava E-theory. In section 2.2 we summarize the classical work of Atiyah [**3**] and McClure [**4**] on the Dyer-Lashof algebra for K-theory. In section 2.3 we state the main result, the calculation of the Dyer-Lashof algebra for Morava E-theory of height 2 at the prime 3. We then use calculations detailed in chapter 4 of the 3-isogeny module structure for an elliptic model of our theory to prove the structure of our Dyer-lashof algebra.

2.1. Background

We now give the necessary background in power operations for an E_∞-ring spectrum E and Rezk's work [**12**] defining an algebra of power operations for the height n Morava E-theory of a $K(n)$-local commutative E-algebra.

Power operations arise naturally from structured ring spectra. For any E_∞-ring spectrum E we have a coherent set of maps $\xi_m : (E^{\wedge m})_{h\Sigma_m} \to E$ from the extended powers $E^{\wedge m}_{h\Sigma_m} = \mathrm{E}\Sigma_{m+} \wedge_{\Sigma_m} E^{\wedge m}$ of E, where $\mathrm{E}\Sigma_m$ is the standard contractible free Σ_m-space. Following [**12**] we write $\mathbb{P}^m(F) = \mathrm{E}\Sigma_{m+} \wedge_{\Sigma_m} F^{\wedge_E m}$ for the m**th extended power** of F in the category of E-modules, and $\mathbb{P} = \bigvee_m \mathbb{P}^m : h\mathrm{Mod}_E \to h\mathrm{Alg}_E$ for the free E-algebra functor.

If F is a commutative E-algebra, then we have maps $\mu_m : \mathbb{P}^m(F) \longrightarrow F$ extending the multiplication $\mu : F \wedge_E F \to F$.

Let X be any CW complex. Then $E \wedge \Sigma^\infty_+ X$ is an E-module, and we have that $F^0 X \cong \mathrm{Hom}_{Mod_E}(E \wedge \Sigma^\infty_+ X, F)$. Notice that $\mathbb{P}^m(E \wedge \Sigma^\infty_+ X) \cong E \wedge \Sigma^\infty_+(X^m_{h\Sigma_m})$. Thus, given a map $f : E \wedge \Sigma^\infty_+ X \to F$ we can construct the m**th exterior power** of f,

$$E \wedge \Sigma^\infty_+(X^m_{h\Sigma_m}) \cong \mathbb{P}^m(E \wedge \Sigma^\infty_+ X) \xrightarrow{\mathbb{P}^m(f)} \mathbb{P}^m(F) \xrightarrow{\mu_m} F.$$

Definition 2.1.1. *Let F be a commutative E-algebra spectrum. For any space X and any $m \geq 0$, we obtain an operation, called the m***th exterior power operation***,*

$$\mathcal{P}_m : F^0 X \to F^0(E\Sigma_m \times_{\Sigma_m} X^m),$$

defined so that an E-module map $f : E \wedge \Sigma^\infty_+ X \to F$ is sent to the composite above. It has the property that $\mathcal{P}_m(xy) = \mathcal{P}_m(x)\mathcal{P}_m(y)$.

For a space X we have the following commutative diagram of Σ_m-equivariant maps (we give X the trival action).

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "i"'] & X^m \arrow[d, "i"] \\ X \times B\Sigma_m \arrow[r, "\Delta"] & E\Sigma_m \times_{\Sigma_m} X^m \end{tikzcd}$$

The bottom map induces a map on F-cohomology. Composing this map with the external power operation produces the m**th total power operation**

$$P_m : F^0(X) \to F^0(X \times B\Sigma_m) \cong F^0 X \otimes_{E_0} E^0 B\Sigma_m.$$

We get an isomorphism $F^0(X \times B\Sigma_m) \cong F^0X \otimes_{E_0} E^0B\Sigma_m$ since $E^0B\Sigma_m$ is a finite flat E_0-module [**19**]. If $i : X \hookrightarrow X \times B\Sigma_m$ is induced by basepoint inclusion, then $i^*P_m(x) = x^m$.

The subgroup inclusions $\Sigma_i \times \Sigma_j \to \Sigma_m$ for $i + j = m$ induce the transfer maps $tr^{\Sigma_m}_{\Sigma_i \times \Sigma_j} : E^0B(\Sigma_i \times \Sigma_j) \to E^0B\Sigma_m$. Let I_m be the transfer ideal in $E^0B\Sigma_m$ given by

$$I_m = \mathrm{Im}\left[tr : \bigoplus_j E^0B(\Sigma_j \times \Sigma_{m-j}) \to E^0B\Sigma_m\right].$$

It is well known that

$$P_m(x + y) = \sum_{i+j=m} tr^{\Sigma_m}_{\Sigma_i \times \Sigma_j}(P_i(x) \times P_j(y)).$$

Thus, the internal operations are additive up to transfer,

$$P_m(x + y) = P_m x + P_m y + I_m.$$

So we quotient out the transfer ideal in order to get additive operations.

Definition 2.1.2. *The* m**th reduced total power operation** $\bar{P}_m$ *is the* m*th total power operation* P_m *followed with the quotient of* $E^0B\Sigma_m$ *by the transfer ideal* I_m.

$$\bar{P}_m : F^0(X) \longrightarrow F^0X \otimes_{E_0} E^0B\Sigma_m/I_m.$$

For each $d \in E^0\mathrm{B}\Sigma_m/I_m$ we get an additive natural operation Q_d in F^0 by pairing the result of $\bar{P}_m$ with d.

$$\begin{tikzcd} F^0X \arrow[r, "\bar{P}_m"] \arrow[dr, "Q_d"'] & F^0X \otimes_{E_0} E^0B\Sigma_m/I_m \arrow[d, "\langle\ ,d\rangle"] \\ & F^0X \end{tikzcd}$$

Definition 2.1.3. *We define* Γ *the* **Dyer-Lashof algebra of power operations in** E *to be the set of all additive natural operations on* E^0 *arising from the total power operations above.*

In the case that E is a Morava E-theory, Rezk defines an "algebraic theory of power operations" by constructing a monad $\mathbb{T}$, called the algebraic approximation functor, on the category of E_*-modules so that taking homotopy gives a map from commutative E-algebras $\pi_* : h\mathrm{Alg}^*_E \to \mathrm{Alg}^*_{\mathbb{T}}$. This is analogous to the monad McClure [**4**] defined for $H\mathbb{F}_p$-algebras. Let K be the module spectrum with $K_* = E_*/\mathfrak{m}$, then the localization $L = L_K \simeq L_{K(n)}$. Then if M is a finitely generated and free E-module, define $\mathbb{T}^m(\pi_*M) = \pi_*L\mathbb{P}^m(M)$. More generally, we define $\mathbb{T} = \bigoplus_m \mathbb{T}^m$ to be the left Kan extension along the inclusion of categories $h\mathrm{Mod}^{ff}_E \cong \mathrm{Mod}^{ff}_{E_*} \hookrightarrow \mathrm{Mod}_{E_*}$. If M is simply flat, then the isomorphism above holds only after completion, i.e.

$$\mathbb{T}(\pi_*M)^\wedge_\mathfrak{m} \cong \pi_*L\mathbb{P}(M).$$

The algebra of power operations Γ is analogous to the May-Dyer-Lashof algebra for $H\overline{\mathbb{F}}_p$ so that every algebra over $\mathbb{T}$ is also an algebra over Γ. This forgetful functor $U : Alg^*_{\mathbb{T}} \to Alg^*_\Gamma$ is a rational isomorphism. The bulk of [**12**] is dedicated to giving a congruence criterion for a p-torsion free Γ-algebra to be an algebra over $\mathbb{T}$. In that work *right* modules are used so that what is called Γ there would be called Γ^{op} here.

2.2. Power Operations in K-Theory

The case of E_1, (p-adic) K-theory, provides a simple but enlightening example of the concepts. In this case the coefficient ring is $\pi_0 E \cong \mathbb{W}\mathbb{F}_p \cong \mathbb{Z}_p$. The formal group is the multiplicative group $\mathbb{G}_m$. Choosing coordinates x and y so that $E^0(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong \mathbb{Z}_p[[x,y]]$, the corresponding formal group law is $x +_{\mathbb{G}_m} y = x + y + xy$. Then $[p](x) = (1+x)^p - 1 \equiv x^p \mod p$ so the multiplicative group has height 1.

The power operations on K-theory (with $\mathbb{Z}$ coefficients) may be parameterized by the symmetric powers σ^m or the exterior powers λ^m. With rational coefficients they are equivalent to the Adams operations ψ^m. Working at a prime p the additive operations are determined by ψ^p. The ring $\Gamma \cong \mathbb{Z}_p[\psi^p]$ is the free algebra generated by the p-th Adams operation. There is also the operation θ, which is not additive, exhibiting the congruence modulo p of ψ^p and the pth power map, i.e. $\psi^p x = x^p + p\theta x$. This is the result of Atiyah's classical work [**3**] calculating $KB\Sigma_m$ in terms of representations of the symmetric group.

An object in Alg^*_Γ is a ψ-ring, a $\mathbb{Z}_p$-algebra A with a ring homomorphism $\psi : A \to A$. Using the results of McClure [**4**] we can show that a $\mathbb{T}$-algebra is a $\mathbb{Z}/2$-graded θ-ring, a strictly commutative graded $\mathbb{Z}_p$-algebra B_* equipped with a map $\theta : B_* \to B_*$ satisfying certain properties which allow the map $\psi(x) = x^p + p\theta(x)$ to be a homomorphism of B_0. Wilkerson [**21**] shows that any torsion free algebra A with a map ψ which reduces to the pth power map modulo pA is actually a θ-ring. Rezk [**12**] proves an analogous congruence criterion in order for a torsion free Γ-algebra to be an algebra over $\mathbb{T}$ in the general case of height n Morava E-theory.

2.3. Dyer-Lashof Algebra for $E(\mathbb{F}_3, \hat{C}_0)$

We now compute the Dyer-Lashof algebra Γ of additive power operations defined in 2.1 for the Morava E-theory $E = E(\mathbb{F}_3, \hat{C}_0)$ defined in 1.2 corresponding to deformations of the height 2 formal group $\hat{C}_0$ obtained from completing the supersingular curve $C_0 : y^2 = x^3 - x$ over $E_0 \cong \hat{A} \cong \mathbb{Z}_3[[u_1]]$. This calculation makes use of our results in chapter 4 calculating the 3-isogeny module structure for the curve

$$C = C_{u_1} : y^2 = x(x-1)(x - \frac{u_1+1}{u_1-1})$$

analogous to Rezk's work [**11**], properties of power operations, and the work of Strickland [**19**] on the Morava theory of symmetric groups.

Recall from section 1.2 that we considered the formal group $\hat{C}$ obtained by completing C at the identity over Spec$\hat{A}$. As discussed in section 2.1, given a $K(2)$-local E-algebra F we have reduced power operations $\bar{P}_m : F^0(X) \longrightarrow F^0(X) \otimes_{E_0} E^0 B\Sigma_m / I_m$. Strickland [**19**] shows that $E^0 B\Sigma_m / I_m = 0$ unless $m = p^k$, and $E^0 B\Sigma_{p^k} / I_{p^k} \cong \mathcal{O}_{Sub_k(\hat{C})} \cong \hat{A}_k$ classifies subgroup schemes of order 3^k in $\hat{C}$. So the only non-trivial reduced power operations are when $m = p^k$,

$$\bar{P}_{3^k} : F^0(X) \longrightarrow F^0(X) \otimes_{\hat{A}} \hat{A}_k.$$

The unique map $B\Sigma_{3^k} \to pt$ induces the map $s : F_0 \to F_0 \otimes_{\hat{A}} \hat{A}_1$, while taking $X = pt$ and $k = 1$ above gives $\bar{P}_3 = t : F_0 \to F_0 \otimes_{\hat{A}} \hat{A}_1$. Taking $F = E$ the maps s and $t = \bar{P}_3$

are the maps classifying the source and target respectively of the universal rank 3 isogeny from $\hat{C}$ over Spec $\hat{A}_1$. These maps are induced by completing the analogous maps for the elliptic curve C. We show in Theorem 4.4.1 that the ring $\hat{A}_1 \cong \hat{A}[d]/\psi_1(d)$ is a free degree 4 extension of $\hat{A}$. So $\bar{P}_3$ is determined by operations $Q_i : F^0(X) \to F^0(X)$ such that

$$P(x) = \bar{P}_3(x) = Q_0(x) + dQ_1(x) + d^2Q_2(x) + d^3Q_3(x).$$

So according to Theorem 4.4.1,

$$P(u_1) = (-8u_1^2 + 3)d^3 + 3u_1d^2 + (48u_1^2 - 19)d + 64u_1^3 - 42u_1.$$

For any F we have that $P(xy) = P(x)P(y)$. Specifically, $P(u_1x) = P(u_1)P(x)$ yields the commutation relations for the Q_i,

$$\begin{aligned}
Q_0u_1 &= (64u_1^3 - 42u_1)Q_0 + (-24u_1^2 + 9)Q_1 + 9u_1Q_2 - 3Q_3,\\
Q_1u_1 &= (48u_1^2 - 19)Q_0 - 18u_1Q_1 + 9Q_2 + u_1Q_3,\\
Q_2u_1 &= 3u_1Q_0 - Q_1 + 3Q_3,\\
Q_3u_1 &= (-8u_1^2 + 3)Q_0 + 3u_1Q_1 - Q_2.
\end{aligned} \tag{2.1}$$

All the reduced power operations $\bar{P}_{3^k}$ are determined by $P = \bar{P}_3$ and knowledge of the subgroups of $\hat{C}$, as can be seen using a transfer argument. For instance, applying the total square map twice on F^0 should have target

$$F^0(X) \otimes^s_{\hat{A}} \hat{A}_1{}^t \otimes^s_{\hat{A}} \hat{A}_1 \cong F^0(X) \otimes \hat{A}_{1,1}.$$

Then $\bar{P}_9 : F^0(X) \to F^0(X) \otimes \hat{A}_2$ is determined by P^2 using the pullback

$$\begin{tikzcd}
A_2 \arrow[r, "\mu_{1,1}"] \arrow[d, "\pi_2"'] & A_{1,1} \arrow[d, "1\otimes w"] \\
A \arrow[r, "s"] & A_1,
\end{tikzcd}$$

and we have that $(1 \otimes w)P^2 = \sum_{i,j} d^i w(d)^j Q_i Q_j$ factors through $F^0 \otimes \hat{A}$.

$$\begin{tikzcd}
F^0 \arrow[drr, bend left, "P^2"] \arrow[dr, dashed] \arrow[ddr, dashed] & & \\
& F^0 \otimes \hat{A}_2 \arrow[r, "\mu_{1,1}"] \arrow[d, "\pi_2"'] & F^0 \otimes \hat{A}_{1,1} \arrow[d, "1\otimes w"] \\
& F^0 \otimes \hat{A} \arrow[r, "s"] & F^0 \otimes \hat{A}_1
\end{tikzcd}$$

The Adem relations come from the fact that the coefficients of the positive powers of d must be zero. So the following Adem relations hold for the Q_i,

$$\begin{aligned}
Q_1Q_0 &= -6Q_0Q_1 + 3Q_2Q_1 - 48u_1Q_0Q_2 + 18Q_1Q_2 - 9Q_3Q_2 \\
&\quad + (-384u_1^2 + 117)Q_0Q_3 + 144u_1Q_1Q_3 - 54Q_2Q_3, \\
Q_2Q_0 &= 3Q_3Q_1 - 3Q_0Q_2 - 24u_1Q_0Q_3 + 9Q_1Q_3, \\
Q_3Q_0 &= Q_0Q_1 + 8u_1Q_0Q_2 - 3Q_1Q_2 \\
&\quad + (64u_1^2 - 18)Q_0Q_3 - 24u_1Q_1Q_3 + 9Q_2Q_3.
\end{aligned} \tag{2.2}$$

Notice that we may use the Adem relations to write an arbitrary product of the Q_i as

$$Q_0^j Q_{i_1} \cdots Q_{i_n} \qquad \text{for} \quad j \geq 0, \quad \text{and} \quad 1 \leq i_k \leq 3. \tag{2.3}$$

This gives us an admissible basis for all operations generated by the Q_i as an $\hat{A}$-module. Using the calculations from Theorem 4.4.1 we have shown that the operations Q_i satisfy the commutation and Adem relations. We now show that these are in fact all the power operations for our Morava E-theory.

Definition 2.3.1. *Let Γ be the associative graded ring generated over $\hat{A} \cong \mathbb{Z}_3[[u_1]]$ by the set of Q_i, where $|Q_i| = 1$ with the commutation relations* (2.1)*, the Adem relations* (2.2)*, and so that the obvious map $\eta : \hat{A} \to \Gamma$ injects $\mathbb{Z}_3$ centrally.*

We now state and prove the main result of this work, namely the structure of the Dyer-Lashof algebra for the Morava E-theory $E(\mathbb{F}_3, \hat{C}_0)$.

Theorem 2.3.2. (Main Theorem) *The Dyer-Lashof algebra of power operations for the Morava E-theory $E(\mathbb{F}_3, \hat{C}_0)$ is the associative ring Γ.*

Proof. For now let Γ' be the Dyer-Lashof algebra for E as described in [**12**]. The degree k part of Γ' corresponds to $E^0 B\Sigma_{3^k}/I_{3^k}$. The above discussion, based on the calculations of Theorem 4.4.1, guarantees that $\Gamma'[1]$ is generated as an $\hat{A}$-module by $\{Q_i\}$, and the operations in Γ' must satisfy both the commutation and Adem relations detailed above. Thus we have a map $\Gamma \to \Gamma'$ which is certainly an isomorphism in degrees 0 and 1. Further, since $\bar{P}_3$ may be used to generate all the higher reduced power operations $\bar{P}_{3^k}$, our map is surjective in all degrees. Strickland [**19**] shows that the rank of $E^0 B\Sigma_{3^k}/I_{3^k}$ over E_0, and so the rank of $\Gamma'[k]$ over $\hat{A}$, is the Gaussian binomial coefficient

$$\binom{k+1}{1}_3 = \frac{3^{k+1}-1}{3-1} = 1 + 3 + \cdots 3^k.$$

Since $|Q_i| = 1$ we have that $\Gamma[0]$ and $\Gamma[1]$ have ranks 1 and 4 respectively. The Adem relations in Γ give us the admissible basis (2.3) of Γ as an $\hat{A}$-module which partitions the basis of $\Gamma[k]$ into elements of the form $Q_0 \cdot \Gamma[k-1]$ and those of the form $Q_{i_1} \cdots Q_{i_k}$ with $1 \leq i_j \leq 3$. Thus we have rank $\Gamma[k] =$ rank $\Gamma[k-1] + 3^k = 1 + 3 + \cdots + 3^k = \Gamma'[k]$. Therefore, the map must be an isomorphism, and Γ is the algebra of power operations. $\square$

Note that although our presentation of Γ was given (necessarily for its context) over the base ring $E_0 \cong \hat{A} \cong \mathbb{Z}_3[[u_1]]$, all of the commutation and Adem relations hold over the subring $\mathbb{Z}[u_1]$. The same thing is true in the following discussion of tensor products for Γ-modules and Cartan formulas.

2.4. Algebras over the Dyer-Lashof Algebra

In this section we completely describe the structure on the homotopy of $K(2)$-local commutative algebras F over our Morava E-theory arising from the E_∞-ring structure on E following Rezk's work [**12**]. We have in the previous section deduced the structure of the Dyer-Lashof algebra Γ for our theory E. We now want an action of Γ on the tensor product $M \otimes_{E_0} N$ of Γ-modules making the category of left Γ-modules symmetric monoidal, i.e. a Cartan formula for Γ. Recall any Γ-module is canonically an $\hat{A}$-module via the map $\eta : \hat{A} \to \Gamma$. The tensor product of two Γ-modules M and N is the $\hat{A}$-module

$M \otimes_{\hat{A}} N$ with the following action of Γ:

(2.4)

$$\begin{aligned}
Q_0(x \otimes y) &= Q_0x \otimes Q_0y + 3Q_3x \otimes Q_1y + 3Q_2x \otimes Q_2y + 3Q_1x \otimes Q_3y + 18Q_3x \otimes Q_3y, \\
Q_1(x \otimes y) &= Q_1x \otimes Q_0y + Q_0x \otimes Q_1y + 8u_1Q_3x \otimes Q_1y + 8u_1Q_2x \otimes Q_2y + 3Q_3x \otimes Q_2y \\
&\quad + 8u_1Q_1x \otimes Q_3y + 3Q_2x \otimes Q_3y + 48u_1Q_3x \otimes Q_3y, \\
Q_2(x \otimes y) &= Q_2x \otimes Q_0y + Q_1x \otimes Q_1y + 6Q_3x \otimes Q_1y + Q_0x \otimes Q_2y + 6Q_2x \otimes Q_2y \\
&\quad + 8u_1Q_3x \otimes Q_2y + 6Q_1x \otimes Q_3y + 8u_1Q_2x \otimes Q_3y + 39Q_3x \otimes Q_3y, \\
Q_3(x \otimes y) &= Q_3x \otimes Q_0y + Q_2x \otimes Q_1y + Q_1x \otimes Q_2y + 6Q_3x \otimes Q_2y \\
&\quad + Q_0x \otimes Q_3y + 6Q_2x \otimes Q_3y + 8u_1Q_3x \otimes Q_3y.
\end{aligned}$$

There is a unique Γ-module structure on $\hat{A}$ compatible with η such that $Q_0 \cdot 1 = 1$, and $Q_i \cdot 1 = 0$ for $i \neq 0$. There is an element $\Psi \in \Gamma$ defined as

$$\begin{aligned}
\Psi &= Q_0Q_0 + 8u_1Q_0Q_1 - 3Q_1Q_1 + (64u_1^2 - 18)Q_0Q_2 - 24u_1Q_1Q_2 + 9Q_2Q_2 \\
&\quad + (512u_1^3 - 288u_1)Q_0Q_3 + (-192u_1^2 + 54)Q_1Q_3 + 72u_1Q_2Q_3 - 27Q_3Q_3.
\end{aligned} \tag{2.5}$$

Then Ψ is central in Γ and for any tensor product of Γ-modules $\Psi \cdot (x \otimes y) = \Psi \cdot x \otimes \Psi \cdot y$.

We now consider commutative monoid objects in the category of left Γ-modules.

Definition 2.4.1. *A Γ- ring is a commutative $\hat{A}$-algebra equipped with a Γ-module structure compatible with the $\hat{A}$-module structure via $\eta : \hat{A} \to \Gamma$, and which satisfies the Cartan* *formulas*

$$
\begin{aligned}
Q_0(xy) &= Q_0xQ_0y + 3Q_3xQ_1y + 3Q_2xQ_2y + 3Q_1xQ_3y + 18Q_3xQ_3y,\\
Q_1(xy) &= Q_1xQ_0y + Q_0xQ_1y + 8u_1Q_3xQ_1y + 8u_1Q_2xQ_2y + 3Q_3xQ_2y\\
&\quad + 8u_1Q_1xQ_3y + 3Q_2xQ_3y + 48u_1Q_3xQ_3y,\\
Q_2(xy) &= Q_2xQ_0y + Q_1xQ_1y + 6Q_3xQ_1y + Q_0xQ_2y + 6Q_2xQ_2y\\
&\quad + 8u_1Q_3xQ_2y + 6Q_1xQ_3y + 8u_1Q_2xQ_3y + 39Q_3xQ_3y,\\
Q_3(xy) &= Q_3xQ_0y + Q_2xQ_1y + Q_1xQ_2y + 6Q_3xQ_2y\\
&\quad + Q_0xQ_3y + 6Q_2xQ_3y + 8u_1Q_3xQ_3y.
\end{aligned}
\tag{2.6}
$$

Suppose the Γ-ring B is such that the operation Q_0 obeys the "Frobenius congruence," $Q_0x \equiv x^3 \mod 3B$ for all $x \in B$. As in [**12**], we single out with a definition Γ-rings with an operation θ that exhibits this congruence.

Definition 2.4.2. *An* **amplified Γ-ring** *is one that has an operation $\theta : B \to B$ such that*

$$Q_0x = x^3 + 3\theta x. \tag{2.7}$$

They play the analogous role that algebras over the Steenrod algebra play in mod p homology, as the following theorem of Rezk [**12**] demonstrates.

Theorem 2.4.3. (Rezk) *For a $K(2)$-local commutative E-algebra F, π_0F naturally has the structure of an amplified Γ-ring.*

So our explicit computation of Γ and its Cartan formula gives us a complete description of the structure in the homotopy of E-algebras arising from the E_∞ structure in E itself. The operation θ satisfies many identities owing to (2.7) and the commutation (2.1), Adem (2.2), and Cartan relations (2.6) above including:

$$\begin{aligned}
\theta(x+y) &= \theta x + \theta y - (x^2 y + x y^2), \\
\theta(u_1 x) &= (64u_1^3 - 42u_1)\theta x + (21u_1^3 - 14u_1)x^3 + (-8u_1^2 + 3)Q_1 x + 3u_2 Q_2 x - Q_3 x, \\
\theta(xy) &= y^3\theta x + x^3\theta y + 3\theta x\theta y + Q_3 x Q_1 y + Q_2 x Q_2 y + Q_1 x Q_3 y + 6Q_3 x Q_3 y, \\
3Q_k\theta x &= Q_k Q_0 x - Q_k(x^3).
\end{aligned}$$

The free amplified Γ-ring on one generator x is

$$\hat{A}[\theta^j Q_{i_1}\cdots Q_{i_n} x \,\big|\, j, n \geq 0, 1 \leq i_k \leq 3].$$

CHAPTER 3

Moduli of Elliptic Curves and Level 2-Structures

We have written down the generators and relations of the Dyer-Lashof algebra for the Morava E-theory $E(\mathbb{F}_3, \hat{C}_0)$ in chapter 2. We justified these with calculations to be carried out in the elliptic model C_{u_1} in chapter 4. In this chapter we collect the results, all classical, of [**18**] giving an explicit presentation of the moduli stack $\mathcal{M}(2)$ of generalized elliptic curves with level 2-structure in order to lay the groundwork for our calculations in the elliptic curve.

3.1. Weierstrass Curves with 2 Inverted

Let C/S be a generalized elliptic curve over a scheme on which 2 is invertible, and whose geometric fibers are either smooth or have a nodal singularity (i.e. are Néron 1-gons). Then, Zariski locally, C is isomorphic to a Weierstrass curve of a specific and particularly simple form. Explicitly, there is a cover $U \to S$ and functions x, y on U such that the map $U \to \mathbb{P}^2_U$ given by $[x : y : 1]$ is an isomorphism between $C_U = C \times_S U$ and a Weierstrass curve in $\mathbb{P}^2_U$ of the form:

$$C_{\vec{b}} : y^2 = x^3 + b_2x^2 + b_4x + b_6 =: f_{\vec{b}}(x), \tag{3.1}$$

such that the identity for the group structure on C_U is mapped to the point at infinity $[0 : 1 : 0]$ [**14**], [**7**]. Any two Weierstrass equations for C_U are related by a affine change of variables of the form:

$$x \to u^{-2}x + r \tag{3.2}$$
$$y \to u^{-3}y.$$

The object which classifies locally Weierstrass curves of the form (3.1), together with isomorphisms which are given as affine change of variables (3.2), is a stack $\mathcal{M}_{weier}[1/2]$, and the above assignment $C \to C_{\vec{b}}$ of a locally Weierstrass curve to an elliptic curve defines a map $w : \mathcal{M}[1/2] \to \mathcal{M}_{weier}[1/2]$.

Following [**18**], the Weierstrass curve (3.1) associated to a generalized elliptic curve C has the following properties: C is smooth if and only if the discriminant of $f_{\vec{b}}(x)$ has no repeated roots after any base change, and C has a nodal singularity if and only if $f_{\vec{b}}(x)$ has a double root. Moreover, non-isomorphic elliptic curves cannot have isomorphic Weierstrass presentations. Thus the map $w : \mathcal{M}[1/2] \to \mathcal{M}_{weier}[1/2]$ injects $\mathcal{M}[1/2]$ into the open substack $U(\Delta)$ of $\mathcal{M}_{weier}[1/2]$, which is the locus where the discriminant of $f_{\vec{b}}$ has order of vanishing at most one.

Conversely, any Weierstrass curve of the form (3.1) has genus one, is smooth if and only if $f_{\vec{b}}(x)$ has no repeated roots, and has a nodal singularity whenever it has a double root. Thus $w : \mathcal{M}[1/2] \to U(\Delta)$ is also surjective, hence an isomorphism. Using this and the fact that points of order two on an elliptic curve are well understood when 2 is inverted, we will find a fairly simple presentation of $\mathcal{M}(2)$.

The moduli stack of locally Weierstrass curves is represented by the Hopf algebroid

$$(B = Z[1/2][b_2, b_4, b_6], B[u^{\pm 1}, r]).$$

Explicitly, $\mathcal{M}_{weier}[1/2]$ is the homotopy colimit of the diagram:

$$\operatorname{Spec} B[u^{\pm 1}, r] \underset{\eta_R}{\overset{\eta_L}{\longrightarrow}} \operatorname{Spec} B$$

where η_R is Spec of the inclusion of B in $B[u^{\pm 1}, r]$ and η_L is Spec of the map:

$$b_2 \to u^2(b_2 + 12r)$$

$$b_4 \to u^4(b_4 + rb_2 + 6r^2)$$

$$b_6 \to u^6(b_6 + 2rb_4 + r^2 b_2 + 4r^3)$$

which is obtained by plugging in the transformation (3.2) into (3.1). In other words, $\mathcal{M}_{weier}[1/2]$ is obtained from $\operatorname{Spec} B$ by enforcing the isomorphisms that come from the change of variables (3.2).

3.2. Level 2-Structures

The elliptic curve of interest is the Legendre curve, which has well-behaved points of order 2 in that it has a level 2-structure. Thus, we may restrict our attention to the moduli stack of such curves. Following closely the exposition in [**18**], suppose C/S is a smooth elliptic curve which is given locally as a Weierstrass curve (3.1), and let $\phi : (\mathbb{Z}/2)^2 \to C$ be a level-2-structure. For convenience in the notation, define $g_0 = (1,1), g_1 = (1,0), g_2 = (0,1) \in (\mathbb{Z}/2)^2$. Then $\phi(g_i)$ are all points of exact order 2 on C, and thus have y-coordinate equal to zero since $[-1](x,y) = (x,-y)$. Then (3.1) becomes

$$y^2 = (x - e_0)(x - e_1)(x - e_2), \tag{3.3}$$

where $e_i = x(\phi(g_i))$ are all different. If C is a generalized elliptic curve which is singular, let $\tilde{C}$ denote the blow-up of C at the singular point. Then $\tilde{C}$ is a Néron 2-gon, and a choice of level-2- structure makes $\tilde{C}$ locally isomorphic to the blow-up of (3.3), with $e_i = e_j \neq e_k$, for $\{i, j, k\} = \{0, 1, 2\}$.

So let $R = Z[1/2][e_0, e_1, e_2]$, L be the line in $\operatorname{Spec} R$ defined by the ideal $(e_0 - e_1, e_1 - e_2, e_2 - e_0)$, and let $\operatorname{Spec} R - L$ be the open complement. The change of variables (3.2) translates to a $(\mathbb{G}_a \rtimes \mathbb{G}_m)$-action on $\operatorname{Spec} R$ that preserves L and is given by:

$$e_i \to u^2(e_i - r).$$

Consider the isomorphism $\psi : (\operatorname{Spec} R - L) \to (\mathbb{A}^2 - 0) \times \mathbb{A}^1$:

$$(e_0, e_1, e_2) \to ((e_1 - e_0, e_2 - e_0), e_0).$$

We see that $\mathbb{G}_a$ acts trivially on the $(\mathbb{A}^2 - 0)$-factor, and freely by scaling on $\mathbb{A}^1$. Therefore the quotient $(\operatorname{Spec} R - L)//\mathbb{G}_a$ is

$$\tilde{\mathcal{M}}(2) = \mathbb{A}^2 - 0 = \operatorname{Spec}(\mathbb{Z}[1/2][\lambda_1, \lambda_2]) - 0,$$

the quotient map being ψ composed with the projection onto the first factor. This corresponds to choosing coordinates in which C is of the form:

(3.4) $$C_{\vec{\lambda}} : y^2 = x(x - \lambda_1)(x - \lambda_2).$$

The $\mathbb{G}_m$-action is given by grading R as well as $\Lambda = Z[1/2][\lambda_1, \lambda_2]$ so that the degree of each e_i and λ_i is 2. It follows that $\mathcal{M}(2) = \tilde{\mathcal{M}}(2)//\mathbb{G}_m$ is the weighted projective line

$\operatorname{Proj}\Lambda = (\operatorname{Spec}\Lambda - 0)//\mathbb{G}_m$. Note that we are taking homotopy quotient which makes a difference: -1 is a non-trivial automorphism on $\mathcal{M}(2)$ of order 2.

The sheaf of invariant differentials $\omega_{\mathcal{M}(2)}$ is an ample invertible line bundle on $\mathcal{M}(2)$, locally generated by the invariant differential $\eta_{C_{\vec{\lambda}}} = \frac{dx}{2y}$. From (3.2) we see that the $\mathbb{G}_m$ $= \operatorname{Spec}\mathbb{Z}[u^{\pm 1}]$ action changes $\eta_{C_{\vec{\lambda}}}$ to $u \cdot \eta_{C_{\vec{\lambda}}}$. Hence, $\omega_{\mathcal{M}(2)}$ is the line bundle on $\mathcal{M}(2) =$ $\operatorname{Proj}\Lambda$ which corresponds to the shifted module $\Sigma^{-1}\Lambda$, standardly denoted by $\mathcal{O}(1)$. We summarize the above discussion with the following proposition from [**18**].

Proposition 3.2.1. *The moduli stack of generalized elliptic curves with a choice of a level-2-structure $\mathcal{M}(2)$ is isomorphic to $\operatorname{Proj}\Lambda = (\operatorname{Spec}\Lambda - 0)//\mathbb{G}_m$, via the map $\mathcal{M}(2) \to \operatorname{Proj}\Lambda$ which classifies the sheaf of invariant differentials $\omega_{\mathcal{M}(2)}$ on $\mathcal{M}(2)$. The universal curve over the locus of smooth curves $\mathcal{M}(2)_0 = \operatorname{Proj}\Lambda - \{0, 1, \infty\}$ is* (3.4)*. The fibers at 0, 1, and ∞, are Néron 2-gons obtained by blowing up* (3.4) *at the singularity.*

We now proceed to understand the action of the group $GL_2(\mathbb{Z}/2)$ on the global sections of ω, $H^0(\mathcal{M}(2), \omega^{\otimes *}) = \Lambda$. By definition, the action comes from the natural action of $GL_2(\mathbb{Z}/2)$ on $(\mathbb{Z}/2)^2$ and hence on the level structure maps $\phi : (\mathbb{Z}/2)^2 \to C[2]$. If we think of $GL_2(\mathbb{Z}/2)$ as the symmetric group Σ_3, then this action is the permutation action on the non-zero elements $\{g_0, g_1, g_2\}$ of $(\mathbb{Z}/2)^2$, which translates to the permutation action on $\{e_i = x(\phi(g_i))\}$. So, Σ_3 acts by permuting the coordinates on

$$H^0(\operatorname{Spec} R - L, \mathcal{O}_{\operatorname{Spec} R - L}) = \mathbb{Z}[e_0, e_1, e_2],$$

i.e. setting $\Sigma_3 = \mathrm{Perm}\{0,1,2\}$, we have that for $\sigma \in \Sigma_3$, $\sigma \cdot e_i = e_{\sigma i}$. The map on H^0 induced by the projection $(\mathrm{Spec}\, R - L) \to \tilde{\mathcal{M}}(2)$ is

$$Z[\lambda_1, \lambda_2] \to Z[e_0, e_1, e_2]$$

$$\lambda_i \to e_i - e_0.$$

Therefore, we obtain that $\sigma \cdot \lambda_i$ is the inverse image of $e_{\sigma i} - e_{\sigma 0}$. That is, $\sigma \cdot \lambda_i = \lambda_{\sigma i} - \lambda_{\sigma 0}$, where we implicitly understand that $\lambda_0 = 0$.

Choose, for example, the generators of $\Sigma_3 = \mathrm{Perm}\{0,1,2\}, \sigma = (012)$ and $\tau = (12)$. Then the above gives:

$$\begin{aligned} \tau :&\lambda_1 \to \lambda_2 & \sigma :&\lambda_1 \to \lambda_2 - \lambda_1 \\ &\lambda_2 \to \lambda_1 & &\lambda_2 \to -\lambda_1. \end{aligned} \tag{3.5}$$

This fully describes the global sections $H^0(\mathcal{M}(2), \omega^{\otimes *})$ as an Σ_3-module.

3.3. Legendre Curves

We have seen above in section 3.2 that the moduli object of smooth curves with a full level 2-structure is the open substack

$$\mathcal{M}(2)_0 = \mathrm{Proj}\ (S) \subset \mathcal{M}(2)$$

where $S = \mathbb{Z}[1/2, \lambda_1, \lambda_2, 1/\lambda_1\lambda_2(\lambda_2 - \lambda_1)]$. This is a weighted version of $\mathbb{P}^1$ minus three points and almost an affine scheme, in the usual way. If we let $A = S^0$ be the elements of degree 0 in S, then

$$\mathcal{M}(2)_0 = (\mathrm{Spec}\ A)//C_2$$

where C_2 acts on $\operatorname{Spec} A$ as the kernel of the map

$$\mathbb{G}_m \xrightarrow{\mu \to \mu^2} \mathbb{G}_m.$$

There is a presentation of A as follows. We privilege $\gamma = \lambda_2 - \lambda_1$ (although we could choose λ_1 or λ_2) and set

$$u_1 = \frac{\lambda_1 + \lambda_2}{\gamma}.$$

Then

$$A \cong \mathbb{Z}[1/2, u_1, (u_1^2 - 1)^{-1}],$$

and we have that $S = A[\gamma^{\pm 1}]$.

We have seen that the sheaf of invariant differentials ω defines an invertible sheaf on $\mathcal{M}(2)$. Further, $\omega^{\otimes k} = \mathcal{O}(k)$, the sheaf defined by the graded Λ-module $\Sigma^{-k}\Lambda$. Thus, we have

$$H^0(\mathcal{M}(2), \omega^{\otimes *}) = \Lambda$$

as graded rings. Modulo 3, the Hasse invariant is

$$v_1 = \lambda_1 + \lambda_2 \in H^0(\mathcal{M}(2), \omega^{\otimes 2}/3)$$

and, hence, the supersingular locus in $\mathcal{M}(2)$ is defined by $I = (3, u_1)$. Then the completion $\hat{A}_I \cong \mathbb{Z}_3[[u_1]]$, and the formal neighborhood of the supersingular locus is the formal stack

$$\operatorname{Spf}\,(\mathbb{Z}_3[[u_1]])//C_2.$$

Since $S = A[\gamma^{\pm 1}]$, the graded completion

$$\hat{S}_I \cong Z_3[[u_1]][u^{\pm 1}].$$

Here u is the invariant differential of some choice of deformation of the supersingular curve; then, $u = \gamma = \lambda_2 - \lambda_1$ up to multiplication by an invertible power series in u_1. This is the graded Lubin-Tate ring and realizable by an E_∞-ring spectrum.

The supersingular curve at 3 does not get all its automorphisms until we base change to $\mathbb{F}_9$. This is why the popular choice of supersingular curve is

$$C_0 : y^2 = x(x-1)(x+1)$$

over $\mathbb{F}_9$. If we classify this curve by a map

$$\mathrm{Spec}\ (\mathbb{F}_9) \to \mathcal{M}(2),$$

then the universal deformation is now over $\mathbb{W}\mathbb{F}_9[[u_1]] \cong \mathbb{Z}_3[i][[u_1]]$, which is an étale extension of $\hat{A}_I$. This then is the usual model for the spectrum E_2 at the prime 3. We have chosen to work with the curve C_0 over $\mathbb{F}_3$ since all the formulas and calculations hold in this subring and $\mathbb{W}\mathbb{F}_9$ is central in the algebra Γ of power operations.

CHAPTER 4

3-Isogenies of the Universal Curve over $\mathcal{M}[\frac{1}{2}]$

In chapter 4 we calculate what Rezk calls the 3-isogeny module structure, introduced in section 1.3, for the elliptic curve C_{u_1}, the elliptic model for our Morava E-theory of height 2 at the prime 3. This amounts to calculating the basic data for this structure: formulas for the universal example of a rank 3 isogeny from our curve and its dual isogeny. The formulas, which are the content of Proposition 4.4.1, are best expressed relative to the coordinates $(z, w) = (-x/y, -1/y)$ of the neighborhood $\{Y \neq 0\}$ of the identity $O = [0 : 1 : 0]$ of C (section 4.4). However, these formulas were prohibitively difficult to work with directly. Thus, our method is to compute these formulas for the general curve with 2 inverted with coordinates (x, y) in the neighborhood $\{Z \neq 0\} \subset \mathbb{P}^2$ of $[0 : 0 : 1]$ (sections 4.1 and 4.2), determine the interplay between the 3-isogeny and a level 2-structure of a curve (4.3), and finally convert these calculations to the zw-coordinates.

This final coordinate change is cumbersome, but more tractable than understanding all isogenies in the zw-coordinates. The nature of the difficulty in these computations is that we must work with subgroups of order 3 *at the prime 3*. Looking away from 3, i.e. with 3 inverted, the situation becomes significantly simpler. Our process closely adheres to Rezk's general method in [**11**] in which he calculates the 2-isogeny module structure for the elliptic model of a Morava E-theory of height 2 at the prime 2. But our method of initially avoiding coordinates at the identity is novel. This chapter may be considered a direct extension of Rezk's work to the other topologically interesting prime.

4.1. Subroups of Order 3 in $C_{\vec{b}}$

In this section we describe the subgroups of order 3 in elliptic curves over schemes with 2 inverted. The following is a novel compilation and presentation of classical results. As discussed in section 3.1, the universal example of an elliptic curve over a scheme over $\mathbb{Z}[\frac{1}{2}]$ is

$$C_{\vec{b}} : y^2 = x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}$$

over the ring $B = \mathbb{Z}[\frac{1}{2}, b_2, b_4, b_6, \Delta^{-1}]$, where Δ is usual discriminant. According to Silverman [**14**], III, Thm 2.3, for a point $P = (x, y)$ in the curve $C_{\vec{b}}$

$$x([2]P) = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4y^2} = \frac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6}$$

and $[-1]P = (x, -y)$. Points of order 3 are exactly the points such that $[2](P) = [-1](P)$. Setting $x([2]P) = x([-1]P)$ we get

$$x^4 - b_4x^2 - 2b_6x - b_8 = x(4x^3 + b_2x^2 + 2b_4x + b_6),$$

$$0 = 3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8.$$

From now on let

$$\psi_3(x) = 3x^4 + b_2x^3 + 3b_4x^2 + 3b_6x + b_8$$

$$f_{\vec{b}}(x, y) = y^2 - x^3 + \frac{b_2}{4}x^2 + \frac{b_4}{2}x + \frac{b_6}{4}.$$

We have shown that a point Q of order 3 has coordinates $Q = (h, k)$ if $\psi_3(h) = 0$ and $f_{\vec{b}}(h, k) = 0$. Let $B'_3 = B[\frac{b_2}{3}, h, k]/(\psi_3(h), f_{\vec{b}}(h, k))$. Then $\mathrm{Spec} B'_3$ represents the functor of points of order 3 in $C_{\vec{b}}$. The point Q generates a subgroup of order 3:

$$H = \langle Q \rangle = \{O, (h, k), (h, -k)\}.$$

Notice that the group C_2 acts on the points of this subgroup by inversion, and that the subgroups of order 3 correspond to the points of order 3 modulo this action. The action of C_2 on $B[h,k]$ fixes $B[h]$ and sends $k \to -k$, so that $B[h,k]^{C_2} = B[h,k^2]$. All of this descends to the quotient, B_3'. Thus, the ring of invariants is $B_3 := (B_3')^{C_2} = B[\frac{b_2}{3}, h]/\psi_3(h)$, and Spec$B_3$ classifies subgroups of order 3 in $C_{\vec{b}}$, i.e.

$$\begin{aligned} Sub_3(C_{\vec{b}}) &= \mathrm{Spec}\left(B[\frac{b_2}{3}, h, k]/(\psi_3(h), f_{\vec{b}}(h,k)\right)//C_2 \\ &= \mathrm{Spec}\ ((B_3')^{C_2}) \\ &= \mathrm{Spec}\ (B_3). \end{aligned}$$

4.2. Universal Example of an Isogeny of Rank 3

A subgroup H of rank N of an elliptic curve C corresponds to a degree N isogeny $\phi : C \to C'$ with kernel H. The universal example of a subgroup of order 3 in an elliptic curve over $\mathbb{Z}[\frac{1}{2}]$ was described above; it occurs over the ring B_3. This $H \leq C_{\vec{b}}$ corresponds to the universal example of a rank 3 isogeny $v : C_{\vec{b}} \to C_{\vec{b}'}$.

We now calculate a Weierstrass equation for the target $C_{\vec{b}'}$ and formulas for the isogeny. This amounts to choosing coordinates x_1 and y_1 such that x_1 has a pole of order 2 and y_1 has a pole of order 3 at infinity subject to the condition that $y^2/x^3(O) = 1$ and the usual valuation v_P on x, y is non-negative for all $P \neq O$. We start with the choice of Vélu [**20**], in which he describes the coordinates for the isogeny with kernel a finite subgroup H.

$$\begin{aligned} x_1(P) &= x(P) + x(P+Q) - x(Q) + x(P-Q) - x(-Q) \\ y_1(P) &= y(P) + y(P+Q) - y(Q) + y(P-Q) - y(-Q) \end{aligned} \tag{4.1}$$

Then we can simplify the above formula for x_1 to

$$x_1 = x + \frac{t}{x-h} + \frac{4k^2}{(x-h)^2}$$

where

$$t = 6h^2 + b_2h + b_4,$$

$$4k^2 = 4h^3 + b_2h^2 + 2b_4h + b_6.$$

It turns out that for our purposes we should adjust Vélu's choice of coordinates by composing with an automorphism of our target curve. Let $v : C_{\vec{b}} \to C_{\vec{b}'}$ be given by

$$\begin{aligned} x' &= x_1 + \frac{b_2}{3} \\ y' &= y_1. \end{aligned} \tag{4.2}$$

Then our quotient curve has Weierstrass equation

$$C_{\vec{b}'} : 4y^2 = 4x^3 + b_2'x^2 + 2b_4'x + b_6'$$

where

$$b_2' = -3b_2,$$

$$b_4' = -9b_4 - 60h^2 - 10b_2h + \frac{b_2^2}{3},$$

$$b_6' = -27b_6 - 280h^3 - 80b_2h^2 - (\frac{8}{3}b_2^2 + 84b_4)h - \frac{b_2^3}{27} - 4b_2b_4.$$

Now we proceed as with $C_{\vec{b}}$ to find the subgroups of order 3 in $C_{\vec{b}'}$. Suppose that $Q' = (h', k')$ is point of order 3 in $C_{\vec{b}'}$. Then

$$0 = \psi_{3,3}(x) = 3x^4 + b_2'x^3 + 3b_4'x^2 + 3b_6'x + b_8'$$

and let $B'_{3,3} = B'_3[h', k']/(\psi_{3,3}(h'), f_{\vec{b}'}(h', k'))$ and $B_{3,3} = B_3[h']/\psi_{3,3}(h')$. Then the ring $B_{3,3}$ represents nested chains of subgroups $H_1 \leq H_2 \leq C_{\vec{b}}$ such that $|H_1| = |H_2/H_1| = 3$. Repeating the process (4.1),(4.2) above but substituting (b'_2, b'_4, b'_6, h') for (b_2, b_4, b_6, h) gives an isogeny v' of rank 3 from $C_{\vec{b}'}$ with

$$x'' = x + \frac{t'}{x - h'} + \frac{4k'^2}{(x - h')^2} + \frac{b'_2}{3}.$$

We find the Weierstrass equation for the target of this second isogeny is given by

$$C_{\vec{b}''} : 4y^2 = 4x^3 + b''_2 x^2 + 2b''_4 x + b''_6,$$

where

$$\begin{aligned}
b''_2 &= -3b'_2 = 3^2 b_2, \\
b''_4 &= -9b'_4 - 60h'^2 - 10b'_2 h' + \frac{b'^2_2}{3} = 3^4 b_4, \\
b''_6 &= -27b'_6 - 280h'^3 - 80b'_2 h'^2 - (\frac{8}{3}b'^2_2 + 84b'_4)h' - \frac{b'^3_2}{27} - 4b'_2 b'_4 = 3^6 b_6.
\end{aligned}$$

This Weierstrass curve is isomorphic to $C_{\vec{b}}$ under the isomorphism

$$x \mapsto 3^2 x''$$

$$y \mapsto 3^3 y''.$$

The composition of this isomorphism with $v'v$ is exactly the multiplication by 3 map, $[3] : C_{\vec{b}} \to C_{\vec{b}}$. Therefore, v' is the dual isogeny $\hat{v}$ up to an isomorphism of our curve. The dual isogeny $\hat{v} : C_{\vec{b}'} \to C_{\vec{b}}$ corresponds to a subgroup of rank 3 in $C_{\vec{b}'}$ over B_3, namely its kernel. This means that one of the roots of $\psi_{3,3}(x)$ must lie in B_3. Let

$P = (x, y) \in ker[3] \setminus ker\, v$, and $Q' = v(P)$. Then the map $B_{3,3} \to B_3$ sending $h \to h$ and $h' \to x(Q') \in B_3$ classifies $ker\, v \leq C_{\vec{b}}[3] \leq C_{\vec{b}'}$. Using the fact that $\psi_3(x)/(x-h) = 0$ it can be shown that

$$\begin{aligned} x(Q') &= \frac{x(x-h)^2 + t(x-h) + 4k^2}{(x-h)^2} + \frac{b_2}{3} \\ &= -3h - \frac{b_2}{3} + \frac{b_2}{3} \\ &= -3h. \end{aligned}$$

To obtain the kernel of both v and its dual, we must adjoin $b_2/3$. We can also see how the adjusted Vélu coordinate (4.2) simplifies this result. The map $B_{3,3} \to B_3$ above allows us to have the universal example of a rank 3 isogeny and its dual over the ring B_3.

In the language of Rezk we have calculated the following maps, which form the basic data for the category associated to the isogeny module for the subgroups of $C_{\vec{b}}$. Here we have s and t classifying the source and target of the universal 3-isogeny, while w classifies its dual. The map ψ classifies the automorphism of the curve $C_{\vec{b}}$ connecting $v'v$ and the multiplication map [3].

$$\begin{aligned} s : B &\to B_3 & s(b_i) &= b_i \\ t : B &\to B_3 & t(b_i) &= b_i' \\ \psi : B &\to B & \psi(b_i) &= 3^i b_i \\ w : B_3 &\to B_3 & w(b_i) &= b_i' \quad w(h) = -3h \end{aligned}$$

4.3. Accounting for Level 2-Structures

We now lift the formulas of the previous section to the moduli of curves with full level 2-structures $\mathcal{M}(2)$=ProjΛ, where $\Lambda = \mathbb{Z}[\frac{1}{2}, \lambda_1, \lambda_2]$ as discussed in section 3.2. The universal such curve is

$$C_{\vec{\lambda}} : y^2 = x(x-\lambda_1)(x-\lambda_2)$$

and is elliptic over $S = \mathbb{Z}[\frac{1}{2}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}, (\lambda_2 - \lambda_1)^{-1}]$. We lift the formulas via a map $\operatorname{Spec} S \to \operatorname{Spec} B$ given by

$$b_2 \to -4(\lambda_1 + \lambda_2)$$

$$b_4 \to 2\lambda_1\lambda_2$$

$$b_6 \to 0.$$

The universal example of a rank 3 isogeny exists over $\Lambda_3 = S[\frac{\lambda_1+\lambda_2}{3}, h]/\psi_3(h)$. Such an isogeny should preserve points of exact order 2. We can indeed see this is the case since these points are exactly the points $P = (x, y)$ with $y = 0$, and if $v(P) = (x', y')$, then y' is divisible by y. Thus our isogeny $v : C_{\vec{\lambda}} \to C_{\vec{e'}}$ maps to a curve $C_{\vec{e'}}$ with Weierstrass equation

$$C_{\vec{e'}} : y^2 = (x - e_0')(x - e_1')(x - e_2')$$

where $e_i' = x'(\lambda_i, 0)$ and we set $\lambda_0 = 0$. Composing v with the automorphism translating by e_0' we have a rank 3 isogeny $\phi : C_{\vec{\lambda}} \to C_{\vec{\lambda'}}$ where

$$C_{\vec{\lambda'}} : y^2 = x(x - \lambda_1')(x - \lambda_2')$$

and $\lambda_i' = e_i' - e_0'$.

More specifically, if $ker\phi = H = \langle Q \rangle$ and $Q = (h, k)$, then

$$4k^2 = 4h(h-\lambda_1)(h-\lambda_2),$$

$$t = 2h(h-\lambda_1) + 2h(h-\lambda_2) + 2(h-\lambda_1)(h-\lambda_2).$$

Combining this with (4.2) we find that

$$\begin{aligned} e_0' &= -2(h-\lambda_1) - 2(h-\lambda_2) + \frac{2(h-\lambda_1)(h-\lambda_2)}{h} - \frac{4(\lambda_1+\lambda_2)}{3}, \\ \lambda_1' &= e_1' - e_0' = -\lambda_1 + \frac{2h(h-\lambda_2)}{h-\lambda_1} - \frac{2(h-\lambda_1)(h-\lambda_2)}{h}, \\ \lambda_2' &= e_2' - e_0' = -\lambda_2 + \frac{2h(h-\lambda_1)}{h-\lambda_2} - \frac{2(h-\lambda_1)(h-\lambda_2)}{h}. \end{aligned} \tag{4.3}$$

Note that $(h-\lambda_i)^{-1} \in \Lambda_3$ since $\psi_3(\lambda_i) = -\Delta/\lambda_{3-i}^2 \in S^*$. As above, we iterate this process on $C_{\vec{\lambda}'}$ to obtain a map $\phi' : C_{\vec{\lambda}'} \to C_{\vec{\lambda}''}$, and can show that $\lambda_i'' = 3^2\lambda_i$. We may define $\Lambda_{3,3} = \Lambda_3[h']/\psi_3'(h')$ which will classify chains of subgroups of length two. The map

$$\Lambda_{3,3} \to \Lambda_3 \quad h \to h,\ h' \to -3h - e_0' \tag{4.4}$$

classifies the chain $H \leq C_{\vec{\lambda}}[3] \leq C_{\vec{\lambda}}$.

4.4. Subgroups and Isogenies in zw-Coordinates

In order to find a presentation of the model in which the ring classifying subgroups of rank 3 is a finite free extension of our base ring, we now consider another choice of coordinates. Let $z = -X/Y$ and $w = -Z/Y$ in the affine neighborhood $\{Y \neq 0\}$ of $O = [0:1:0]$. In these coordinates

$$C_{\vec{\lambda}} : w = z(z - \lambda_1 w)(z - \lambda_2 w)$$

is a smooth elliptic curve over $S = \mathbb{Z}[\frac{1}{2}, \lambda_1^{\pm 1}, \lambda_2^{\pm 1}, (\lambda_2 - \lambda_1)^{-1}]$. These are the preferred coordinates for another reason. The formal group corresponding to the Morava E-theory of interest is obtained from our elliptic curve by completing at the identity. Thus, having coordinates at the identity is helpful. Following the method of Katz and Mazur [**7**] points of order 3 in C are exactly the inflection points of the curve. Such a point $Q = (p, q)$ is on a line $w = az + b$ and is a triple root of

$$0 = f_{\vec{\lambda}}(z, w) = z(z - \lambda_1 w)(z - \lambda_2 w) - w.$$

Combining these conditions we have

$$\begin{aligned} 0 &= (1 - \lambda_1 a)(1 - \lambda_2 a)z^3 + b(2\lambda_1\lambda_2 a - (\lambda_1 + \lambda_2))z^2 + (\lambda_1\lambda_2 b^2 - a)z - b \\ &= \alpha(z - p)^3. \end{aligned} \tag{4.5}$$

Comparing coefficients we see that

$$\begin{aligned} \alpha &= (1 - \lambda_1 a)(1 - \lambda_2 a), \\ 3\alpha p &= b((\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2 a), \\ 3\alpha p^2 &= \lambda_1\lambda_2 b^2 - a, \\ \alpha p^3 &= b. \end{aligned} \tag{4.6}$$

Note that $\alpha \neq 0$ since if it was, then $b = 0$ would imply that $a = 0$ and $\alpha = 1$. So $a \neq \lambda_1^{-1}, \lambda_2^{-1}$ as in this case $\alpha = 0$. Back-solving the second, third and fourth equations in (4.6) give us that

$$b = \alpha p^3 \Rightarrow q = ap + \alpha p^3,$$

$$a = \lambda_1 \lambda_2 \alpha^2 p^6 - 3\alpha p^2,$$

$$3\alpha p = \alpha p^3(\lambda_1 + \lambda_2 - 2\lambda_1\lambda_2(\lambda_1\lambda_2\alpha^2 p^6 - 3\alpha p^2)).$$

The last equation may be rewritten as

$$0 = \alpha p[2\lambda_1^2\lambda_2^2\alpha^2 p^8 - 6\lambda_1\lambda_2\alpha p^4 - (\lambda_1 + \lambda_2)p^2 + 3]. \tag{4.7}$$

But α is not independent of p. From the first and second equations in (4.6) we have

$$2\lambda_1\lambda_2 p^2 a = (\lambda_1 + \lambda_2)p^2 - 3,$$

$$\begin{aligned} 4\lambda_1\lambda_2 p^4 \alpha &= (3 + (\lambda_2 - \lambda_1)p^2)(3 - (\lambda_2 - \lambda_1)p^2) \\ &= 9 - (\lambda_2 - \lambda_1)^2 p^4. \end{aligned}$$

So multiplying (4.7) by 8 and using the above relation for α we get

$$0 = \alpha p((\lambda_2 - \lambda_1)^4 p^8 - 6(\lambda_2 - \lambda_1)^2 p^4 - 8(\lambda_1 + \lambda_2)p^2 - 3).$$

Let $\gamma = \lambda_2 - \lambda_1, u = \lambda_1 + \lambda_2 \in S$. Define $\phi_1(p) = \gamma^4 p^8 - 6\gamma^2 p^4 - 8up^2 - 3$. Then we get nontrivial points of order 3 over the ring

$$S_1' = S[p]/\phi_1(p).$$

Note that $q = ap + \alpha p^3$ is an element of S_1', and that $2\lambda_1\lambda_2 ap^2 - up^2 + 3$ implies

$$2\lambda_1\lambda_2 p^2 a = 7up^2 + 6\gamma^2 p^4 - \gamma^4 p^8 + \phi_1(p),$$

so that

$$a = \frac{7u + 6\gamma^2 p^2 - \gamma^4 p^6}{2\lambda_1\lambda_2}$$

is in $S[p]/\phi_1(p) = S_1'$.

The cyclic group C_2 acts freely on the non-trivial points of order 3 by inversion. Since $[-1](p, q) = (-p, -q)$ the induced action on S_1' takes p and q to $-p$ and $-q$ respectively. The set of subgroups of order 3 is the quotient of the non-trivial points of order 3 by this action, and is represented by $(S_1')^{C_2}$.

Notice that $\phi_1(p)$ is invariant under the action on S_1'. Write

$$d = \gamma p^2 = (\lambda_2 - \lambda_1)p^2, \quad u_1 = \frac{u}{\gamma} = \frac{\lambda_2 + \lambda_1}{\lambda_2 - \lambda_1},$$

and

$$\psi_1(d) = d^4 - 6d^2 - 8u_1 d - 3 \tag{4.8}$$

so that $\psi_1(\gamma p^2) = \phi_1(p)$. Then we have that

$$S_1 = (S_1')^{C_2} = S[d]/\psi_1(d).$$

The action of $\mathbb{G}_m$ on our curves detailed in section 3.2 makes both S and S_1 graded rings in which the degree of λ_i, u, γ is two, while the degree of p is -1. We will want a presentation of S and S_1 as extensions of their respective degree zero subrings. Notice that $|d| = |u_1| = 0$, and let

$$A = \mathbb{Z}[\frac{1}{2}, u_1, (u_1^2 - 1)^{-1}], \quad A_1 = A[d]/\psi_1(d).$$

Then $A = S^0$, $A_1 = S_1^0$, and $S_k = A_k[\gamma^{\pm 1}]$, since

$$2\lambda_1 = \gamma(u_1 - 1), \quad 2\lambda_2 = \gamma(u_1 + 1).$$

We have shown above that the universal example of a rank 3 isogeny of elliptic curves with our normalized level 2-structure is $\phi : C_{\vec{\lambda}} \to C_{\vec{\lambda}'}$ where $C_{\vec{\lambda}'}$ in zw-coordinates is given by

$$C' : w = z(z - \lambda_1' w)(z - \lambda_2' w).$$

This isogeny exists over S_1. Previously we had written the λ_i' in terms of $x(Q) = h$ with the following formulas,

$$\begin{aligned} \lambda_1' &= -\lambda_1 + \frac{2h(h-\lambda_2)}{h-\lambda_1} - \frac{2(h-\lambda_1)(h-\lambda_2)}{h}, \\ \lambda_2' &= -\lambda_2 + \frac{2h(h-\lambda_1)}{h-\lambda_2} - \frac{2(h-\lambda_1)(h-\lambda_2)}{h}. \end{aligned} \tag{4.9}$$

Using the coordinate change $z = -x/y$ and $w = -1/y$, we have $q = p(p - \lambda_1 q)(p - \lambda_2 q) = p(a + \alpha d/\gamma)$ and

$$\begin{aligned} p &= -\frac{h}{k}, \quad h = \frac{p}{q}, \quad d = \frac{\gamma h^2}{k^2}, \\ q &= -\frac{1}{k}, \quad k = -\frac{1}{q}, \quad d = \frac{\gamma h}{(h-\lambda_1)(h-\lambda_2)}. \end{aligned}$$

Using these relations we find that

(4.10)

$$\gamma h^{-1} = \gamma a + \alpha d, \quad \gamma(h-\lambda_1)^{-1} = \frac{\gamma a + \alpha d}{1-\lambda_1(a+\alpha d/\gamma)}, \quad \gamma(h-\lambda_2)^{-1} = \frac{\gamma a + \alpha d}{1-\lambda_2(a+\alpha d/\gamma)}.$$

The key to rewriting these calculations in our current presentation is to notice

$$\frac{3}{d} = \frac{d^4 - 6d^2 - 8u_1 d}{d} = d^3 - 6d - 8u_1 \in A_1.$$

Then using equations (4.6) $(u_1^2 - 1)d(\gamma a + \alpha d) = -d^2 + 2u_1 d + 3$. We may write

$$\begin{aligned} \psi_1(d) &= (-d^2 + 2u_1 d + 3)(d^2 + 2u_1 d + 1) - 4(u_1^2 - 1)d^2 \\ &= d(u_1^2 - 1)[(\gamma a + \alpha d)(d^2 + 2u_1 d + 1) - 4d]. \end{aligned} \tag{4.11}$$

The equation (4.11) as a relation in S_1 gives us

$$12h = \frac{12\gamma}{\gamma a + \alpha d} = \frac{3\gamma}{d}(d^2 + 2u_1 d + 1) = \gamma(d^3 - 3d - 2u_1). \tag{4.12}$$

Similar calculations show that

$$\begin{aligned} \frac{4}{1-\lambda_1(a+\alpha d/\gamma)} &= d^2 - 2d + 1 = (d-1)^2, \\ \frac{4}{1-\lambda_2(a+\alpha d/\gamma)} &= d^2 + 2d + 1 = (d+1)^2. \end{aligned} \tag{4.13}$$

Combining (4.9), (4.11) and (4.13) we can write λ_i' in S_1 as

$$\begin{aligned} 2\lambda_1' &= -2\lambda_1 + \frac{4\gamma}{d(1-\lambda_1(a+\alpha d/\gamma))} - \frac{4\gamma}{d} &= -\gamma(d^3 + d^2 - 9d - 9u_1), \\ 2\lambda_2' &= -2\lambda_2 + \frac{4\gamma}{d(1-\lambda_2(a+\alpha d/\gamma))} - \frac{4\gamma}{d} &= -\gamma(d^3 - d^2 - 9d - 9u_1). \end{aligned} \tag{4.14}$$

Thus $\gamma' = \lambda'_2 - \lambda'_1 = \gamma d^2$ while $u'_1 = (-d^3 + 9d + 9u_1)/d^2$. This procedure may be run again on the curve $C_{\vec{\lambda}'}$. Then the ring $S_{1,1} = S_1[d']/\psi'_1(d')$ where $\psi'_1(d') = d'^4 - 6d'^2 - 8u'_1 d' - 3$ classifies chains of subgroups in $C_{\vec{\lambda}}$. Remember from (4.4) that the map $\Lambda_{3,3} \to \Lambda_3$ sending $h \to h$ and $h' \to w(h) = -3h - e'_0$ classified $ker\phi \leq C_{\vec{\lambda}}[3] \leq C_{\vec{\lambda}}$. Then the map $S_{1,1} \to S_1$ sending $d \to d$ and d' to

$$\frac{\gamma' w(h)}{(w(h) - \lambda'_1)(w(h) - \lambda'_2)} = -\frac{3}{d} = -d^3 + 6d + 8u_1 \tag{4.15}$$

plays the same role in this new presentation. As before, the target of the isogeny $\phi' : C_{\vec{\lambda}'} \to C_{\vec{\lambda}''}$ is the curve with $\lambda''_i = 3^2\lambda_i$.

The Legendre curve we are truly interested in is the one in which we have normalized our level 2-structure even further by requiring that $[0 : 0 : 1]$ and $[1 : 0 : 1]$ are points of exact order 2,

$$C_\lambda : w = z(z - w)(z - \lambda w).$$

This curve is elliptic over $A = \mathbb{Z}[\frac{1}{2}, u_1, (u_1^2 - 1)^{-1}] = S^0$. This is a non-standard presentation of the ring, where

$$\lambda = \frac{u_1 + 1}{u_1 - 1}, \quad \text{and} \quad u_1 = \frac{\lambda + 1}{\lambda - 1}.$$

As shown in (4.8), we get the universal example of an isogeny of rank 3 over $A_1 = A[d]/\psi_1(d) = S_1^0$. This is a map $\phi : C_\lambda \to C_{\lambda'}$ where $\lambda' = (u'_1+1)/(u'_1-1)$, and using (4.14) we have $u'_1 = (-d^3+9d+9u_1)/d^2 \in A_1$. Finally, we have shown $A_{1,1} = A_1[d']/\psi'_1(d') = S_{1,1}^0$ classifies chains of subgroups $H_1 \leq H_2 \leq C_\lambda$ such that rank H_1 = rank $H_2/H_1 = 3$, and as in (4.15) the map form $A_{1,1} \to A_1$ sending $d \to d$ and $d' \to -3/d$ classifies $H \leq C_\lambda[3] \leq C_\lambda$.

We have justified the following calculations.

Theorem 4.4.1. *The basic data for the 3-isogeny structure of the Legendre curve* C_{u_1} : $y^2 = x(x-1)(x - \frac{u_1+1}{u_1-1})$ *is (where* $\psi_1(d) = d^4 - 6d^2 - 8u_1 d - 3$*)*

$$
\begin{aligned}
A_0 &= \mathbb{Z}[1/2][u_1, (u_1^2-1)^{-1}] & A_1 &= A[d]/\psi_1(d) \\
s &: A \to A_1 & s(u_1) &= u_1 \\
t &: A \to A_1 & t(u_1) &= (-8u_1^2+3)d^3 + 3u_1 d^2 + \\
& & &\quad + (48u_1^2 - 19)d + 64u_1^3 - 42u_1 \\
\psi &: A \to A & \psi(u_1) &= u_1 \\
w &: A_1 \to A_1 & w(u_1) &= t(u_1),\ w(d) = -\frac{3}{d} = -d^3 + 6d + 8u_1.
\end{aligned}
$$

Note here that $t(u_1) = (-d^3 + 9d + 9u_1)/d^2$ is written as an element of $A[d]/\psi_1(d)$.

We may build the entire 3-isogeny structure for C_{u_1} using the basic data above. Completing $A = A_0$ at the supersingular locus $(3, u_1)$ gives the coefficient ring for our Morava E-theory. Moreover, using [**19**] we find that $\hat{A}_k = E^0 B\Sigma_{p^k}/I_{p^k}$, and so our power operations are parameterized by $\prod \hat{A}_k$. The maps s, t, w may be used to deduce the various relations that our operations satisfy.

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