COCYCLE SCHEMES AND $MU[2k,\infty)$–ORIENTATIONS

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Abstract. We recall the study of $MU[2k,\infty)$–orientations as elucidated by Ando, Hopkins, and Strickland. Their work prompts us to investigate a particular algebraic moduli which, after 2–localization, we (together with Adam Hughes and JohnMark Lau) fully describe for all values of k. It gives a strikingly good (but imperfect) approximation of our topological motivator.

1. Integration for oriented cohomology theories

Something homotopy theorists seem to like to do is understand "orientations", or maps of ring spectra $\varphi\colon MX \to E$, where MX is the bordism theory for X–structured manifolds and E is just about anything. For one, evaluating such a map on a point gives rise to an E_*–valued genus $MX_* \to E_*$, but there are tricks around to get a lot more out of it. For instance, writing $\underline{E}_n$ for the n^{th} space in the Ω–spectrum for E, one can consider $MX_n\underline{E}_n$. Using the model for MX as maps in from X–structured manifolds considered up to bordism, one arrives at

$$MX_n\underline{E}_n = \left\{ M^n \xrightarrow{\omega} \underline{E}_n \;\middle|\; \begin{array}{c} M \text{ a closed } n\text{–dimensional } X\text{–manifold,} \\ \omega \in E^n(M) \end{array} \right\} \Big/ \sim .$$

That is, the spectrum $MX \wedge \Sigma^{-n}\Sigma^\infty_+ \underline{E}_n$ contains information in π_0 about X–structured bordism classes which are equipped with top-dimensional E–cohomology classes. Together with φ, one can then build the composite

$$\mathbb{S}^0 \xrightarrow{(M,\omega)} MX \wedge \Sigma^{-n}\Sigma^\infty_+ \underline{E}_n \xrightarrow{\text{colim}} MX \wedge E \xrightarrow{\varphi\wedge 1} E \wedge E \xrightarrow{\mu} E,$$

which gives a recipe for an element of $\pi_0 E$. In familiar examples, this element is identifiable:

- For $\varphi : MO \to H\mathbb{F}_2$, this assignment is $(M^n, \omega \in H^n(M;\mathbb{F}_2)) \mapsto \int_M \omega \in \mathbb{F}_2 = \pi_0 H\mathbb{F}_2$.
- For $\varphi : MSO \to H\mathbb{Z}$, this assignment is $(M^n, \omega \in H^n(M;\mathbb{Z})) \mapsto \int_M \omega \in \mathbb{Z} = \pi_0 H\mathbb{Z}$.

In general, we think of this as a way to extract an integral for E–cohomology classes on X–structured manifolds.

Remembering the Pontryagin–Thom equivalence $\mathbb{S} \simeq M\text{Framed}$, you can see that this construction appears all over the place: to *any* ring spectrum E, the unit map $\eta_E : \mathbb{S} \to E$ associates a theory of integration for E–cohomology classes on framed bordism classes. The above ring map

$$\mathbb{S} \to MSO \to H\mathbb{Z}$$

can then be thought of as a *factorization* which witnesses the "overdeterminacy" of the framed integral — yes, given a framed manifold one can integrate an integral cohomology class, but actually an orientation of the tangent bundle was all that was required to get off the ground.[1] Recognizing that O and SO form the beginning of the Postnikov tower for O, one can then ask what stage of the Postnikov filtration a given ring spectrum's unit map factors. Here's a somewhat longer list of known orientations of cohomology theories:

$$\begin{tikzcd}
M\text{Framed} \arrow[r] \arrow[drr] \arrow[drrr] \arrow[drrrr] \arrow[drrrrr] & \cdots \arrow[r] & M\text{String} \arrow[r, "\sigma"] \arrow[d] & M\text{Spin} \arrow[r] \arrow[d] & MSO \arrow[r] \arrow[d] & MO \arrow[d] \\
 & & \text{tmf} & kO & H\mathbb{Z} & H\mathbb{F}_2.
\end{tikzcd}$$

The first two vertical maps are classical; the orientation of kO is due to Atiyah, Bott, and Shapiro; and the orientation of tmf is due to Ando, Hopkins, and Strickland. One can see an interesting correspondence beginning to take form: the level of Postnikov filtration on the top is matched (somewhat imperfectly) by the chromatic height of the spectra on the bottom. The nature of this correspondence is pretty widely open, and this talk is meant to shed computational

[1]Of course, this is also a claim that even oriented manifolds which *cannot* be framed still support integrals.

light on one aspect of it, motivated by studying how Ando, Hopkins, and Strickland prove the existence of their vertical arrow. Their first move is to instead consider maps

$$\begin{tikzcd}
M\mathrm{String} \arrow[d, dashed] \arrow[dr, dashed] & MU[6,\infty) \arrow[l] \arrow[r] \arrow[d] \arrow[dl, dashed] & MUP \arrow[dl, dashed, red] \\
\mathrm{tmf} \arrow[r] & E. &
\end{tikzcd}$$

approximating σ on both sides, and then later try to fill in the dashed maps. In a second step, the solid map is accessed by assuming E to be complex-orientable, so that red maps $MUP \to E$ are well-understood, then trying to use this assumption as a foothold on the Postnikov tower for the unitary group.

2. THE ALGEBRAIC GEOMETRY OF THE THOM CONSTRUCTION

Chromatic homotopy theory, the study of complex-oriented cohomology theories, is at its core a marriage of arithmetic geometry and algebraic topology. According to the taste of the author, the exposition can vary widely in ABV[2], and the tastes of Ando, Hopkins, and Strickland were well over 120 proof. Their basic object is:

Definition. For a space X and ring spectrum E, we write X_E for the formal affine scheme

$$X_E = \operatorname{Spf} E^0 X.$$

Of course, this requires some niceness — for instance, $E^0 X$ has to be a commutative ring, and E has to be sufficiently self-entwined so that "E^0" carries all of the interesting information. Suffice it to say that they arrange their situation so that this is the case. Here's the basic sort of theorem you can expect to see in their language:

Theorem. *Isomorphisms between $\mathbb{C}\mathrm{P}^\infty_E$ and the affine line $\widehat{\mathbb{A}}^1 = \operatorname{Spf} E^0[\![x]\!]$ biject with ring maps $MUP \to E$.* □

Before discussing bordism orientations further, I want to translate some other theorems about complex-orientable cohomology theories into their language. Complex-oriented theories are intrinsically good for one thing: they have Thom isomorphisms for complex vector bundles. Accordingly, the basic theorems about them center on constructions on vector bundles. For instance, the E–cohomology of the projectivization $\mathbb{P}(V)$ of a rank n vector bundle V on a space X has a presentation as

$$E^*\mathbb{P}(V) = E^*X[\![x]\!]/c_V(x),$$

where $c_V(x)$ is the total Chern class of V. In their language, one writes:

Lemma. *The formal scheme $\mathbb{P}(V)_E$ is finite (of rank n) and free over X_E, and it forms a closed subscheme — i.e., a* divisor *— of the curve $\mathbb{C}\mathrm{P}^\infty_E \times X_E$ (thought of as an X_E–scheme).* □

Corollary (Universal example). *The induced map $BU(n)_E \to \operatorname{Div}_n^+ \mathbb{C}\mathrm{P}^\infty_E$ is an isomorphism.* □

Projectivization leads to another essential theorem in vector bundle geometry: the splitting principle. Fixing V, there is a map $f\colon Y \to X$ such that f^*V splits naturally as a sum of line bundles and the induced map E^*f is injective.

Lemma. *There is a map $f\colon Y \to X$ such that $Y_E \to X_E$ is finite and faithfully flat, and $f_E^*\mathbb{P}(V)_E = \mathbb{P}(f^*V)_E$ splits as a sum of points.* □

We can also directly interpret the cohomology of the Thom spectrum X^V, although this takes a little more vocabulary. In the classical situation, a Thom isomorphism is an E^*X–module isomorphism $\widetilde{E}^*X^V \cong E^*X$.

Lemma. *The quasicoherent sheaf $\mathbb{L}(V)$ over X_E determined by E^*X^V is a trivializable line bundle.* □

A remarkable property of this geometric language is that many things we first think of as structures in topology — here is a collection of Thom isomorphisms, for instance — can be stated merely as properties of the associated geometric objects — here are *trivializable invertible sheaves*, without a chosen generating element. In fact, the theorem above can be stated without such a reference object:

Theorem. *Let $\mathscr{L}$ denote the tautological line bundle on $\mathbb{C}\mathrm{P}^\infty$. The sheaf $\mathbb{L}(\mathscr{L})$ on $\mathbb{C}\mathrm{P}^\infty_E$ is invertible exactly when E is complex-orientable, in which case it is trivializable. Trivializations then biject with ring maps $MUP \to E$.*

[2] ABV: Arithmetic By Volume.

Proof. A trivialization of $\mathbb{L}(\mathscr{L})$ gives rise to a trivialization of $\mathbb{L}(\mathscr{L}^{\times n})$. Using the splitting principle, $\mathbb{L}(f^*V)$ pulls back from $\mathbb{L}(\mathscr{L}^{\times n})$, hence receives a trivialization. Finally, trivializations descend along finite flat maps. To see that these trivializations are suitably compatible, write $\zeta\colon X_E \to \mathbb{C}\mathrm{P}^\infty_E \times X_E$ for the zero-section and $\mathscr{I}(\mathbb{P}(V)_E)$ for the ideal sheaf of functions vanishing on $\mathbb{P}(V)_E$. There is then is an equivalence $\zeta^*\mathscr{I}(\mathbb{P}(V)_E) \cong \mathbb{L}(V)$. The projectivization construction converts sums of bundles to sums of divisors, and hence the trivialization of $\mathbb{L}(V\oplus W)$ is the product of the trivializations for $\mathbb{L}(V)$ and $\mathbb{L}(W)$. This data constructs a ring map $MUP \to E$. □

3. Orientations for MSU and $MU[6,\infty)$

Let's get back on task: our real goal was to study ring maps $MU[2k,\infty) \to E$ for nonzero k. Because we went through the analysis above, we can see where the crucial pieces of input lie: first we should develop a version of the splitting principle, and then we should tease apart the resulting mess of formal schemes.

Consider first the case of $k=1$, where a vector bundle $V\colon X \to BU\times\mathbb{Z}$ has been lifted as in

$$\begin{tikzcd}
\cdots \arrow[r] & BU[6,\infty) \arrow[r] & BSU \arrow[r] & BU \arrow[r, "+n\cdot 1"] & BU\times\mathbb{Z} \\
 & & & & X. \arrow[u, "V"'] \arrow[ul] \arrow[ull, dotted] \arrow[ulll, dotted] \arrow[ullll, dotted, "\cdots"]
\end{tikzcd}$$

The analog of the splitting principle for BU–structured bundles then says that over Y,

$$f^*(V - n\cdot 1) \cong \bigoplus_{j=1}^n (\mathscr{L}_j - 1),$$

where now both sides have a natural interpretation as vector bundles of virtual rank 0. For the case $k=2$, we record the following Lemma and proof:

Lemma. *Select a factorization $\widetilde{V}$ of $V - n\cdot 1$ through BSU. Then, $f^*\widetilde{V}$ is equivalent as BSU–classes to a sum of bundles of the form $(\mathscr{H}-1)(\mathscr{H}'-1)$ for $\mathscr{H}$, $\mathscr{H}'$ line bundles.*

Proof sketch. Over Y, we may split two lines off of $V = \mathscr{L}_1 + \mathscr{L}_2 + V'$. The difference bundle $\widetilde{V} - (\mathscr{L}_1 - \varepsilon)(\mathscr{L}_2 - \varepsilon)$ has a natural SU–structure, because $(\mathscr{L}_1 - \varepsilon)$ and $(\mathscr{L}_2 - \varepsilon)$ are elements of $kU^2(X)$ and their cup product lies in $kU^4(X) = [X, BSU]$. Forgetting the SU–structure, the underlying bundle has the form

$$\begin{aligned}
\widetilde{V} - (\mathscr{L}_1 - \varepsilon)(\mathscr{L}_2 - \varepsilon) &= (V' + \mathscr{L}_1 + \mathscr{L}_2 - r\varepsilon) - (\mathscr{L}_1 - \varepsilon) - (\mathscr{L}_2 - \varepsilon) + (\mathscr{L}_1\mathscr{L}_2 - \varepsilon) \\
&= (V' + \mathscr{L}_1\mathscr{L}_2) - (r-1)\varepsilon.
\end{aligned}$$

This has rank one fewer, so we can induct. Finally, you need to know that all rank 1 bundles with SU–structure are trivial, which grounds the induction. □

Lemma (Hopkins[3])**.** *In the case $k=3$, $BU[6,\infty)$–bundles decompose into sums of $(\mathscr{H}-1)(\mathscr{H}'-1)(\mathscr{H}''-1)$.* □

We're thus moved to study the universal maps

$$\widehat{\mathbb{G}}_E^{\times 2} \xrightarrow{(\mathscr{L}_1-1)(\mathscr{L}_2-1)} BSU_E, \quad \widehat{\mathbb{G}}_E^{\times 3} \xrightarrow{(\mathscr{L}_1-1)(\mathscr{L}_2-1)(\mathscr{L}_3-1)} BU[6,\infty)_E.$$

These maps have some evident properties:

1. They're *symmetric*: trading $\mathscr{L}_1$ and $\mathscr{L}_2$, for instance, gives an isomorphic bundle.
2. They're *rigid*: replacing $\mathscr{L}_1$ by the trivial line, you get the zero bundle.
3. They're $\operatorname{Div}\widehat{\mathbb{G}}_E$*–linear*: acting by $\mathscr{H} \in kU^0\mathbb{C}\mathrm{P}^\infty$, there are the decompositions

$$\begin{aligned}
(\mathscr{L}_1 - 1)\mathscr{H}(\mathscr{L}_2 - 1) &= (\mathscr{L}_1 - 1)(\mathscr{H}\mathscr{L}_2 - \mathscr{H}) = (\mathscr{L}_1 - 1)(\mathscr{H}\mathscr{L}_2 - 1) - (\mathscr{L}_1 - 1)(\mathscr{H} - 1) \\
&= (\mathscr{H}\mathscr{L}_1 - \mathscr{H})(\mathscr{L}_2 - 1) = (\mathscr{H}\mathscr{L}_1 - 1)(\mathscr{L}_2 - 1) - (\mathscr{H} - 1)(\mathscr{L}_2 - 1).
\end{aligned}$$

With these observations in hand, one is led to investigate trivializations of the line bundle $\mathbb{L}(\bigotimes_{j=1}^k (\mathscr{L}_j - 1))$ on $(\mathbb{C}\mathrm{P}^\infty)_E^{\times k}$ satisfying a symmetry condition, a rigidity condition, and the linearity condition

$$f(x,y,\ldots)\cdot f(t, x +_{\widehat{\mathbb{G}}_E} y, \ldots) = f(t +_{\widehat{\mathbb{G}}_E} x, y, \ldots)\cdot f(t, x, \ldots),$$

[3] I do not know an elementary proof of this, along the lines of the BSU case. I'd love to learn one!

as these are the sections which one "expects" to come from $BU[2k,\infty)$–structures. The main theorem of Ando, Hopkins, and Strickland is that this is sufficient:

Theorem (Ando–Hopkins–Strickland). *If E is complex-orientable and $k \le 3$, then such a trivializing section determines a ring map $MU[2k,\infty) \to E$.*

Notes on proof. They reduce from general E to $E = MU$, then from there to $E = Hk$ for k a prime field. They then perform the brutal calculation of the available such sections and compare with $\operatorname{Spec} Hk_* MU[2k,\infty)$, made accessible by Singer's calculation below. □

In the case of $k = 0$ and $k = 1$, this degenerates to exactly the analysis above. In the case where E is an *elliptic spectrum*, they further show that the associated elliptic curve begets a canonical section in the $k = 3$ case.[4] This is the main punchline in the construction of the (complex, nonparametrized) σ–orientation. It's also possible, by placing more hypotheses on E, to extend this same mode of analysis to maps $M\mathrm{String} \to E$.

4. Orientations for $MU[2k,\infty)$?

Several things go wrong in trying to extend this analysis to $MU[2k,\infty)$ for $k \ge 4$, which is what I want to spend the remainder of the talk discussing. In fact, something immediately obstructs any attempt to continue their algebro-geometric analysis. The following theorem is key to their result:

Theorem (Singer, Stong). *There are classes $\theta_{2j} \equiv c_j$ (mod decomposables) such that*

$$H\mathbb{F}_2{}^*(BU[2k,\infty)) \cong \frac{H\mathbb{F}_2{}^*(BU)}{\langle \theta_{2j} \mid \sigma_2(j-1) < k-1 \rangle} \otimes \operatorname{Op}[\mathrm{Sq}^3\, \iota_{2k-3}],$$

where $\operatorname{Op}[\mathrm{Sq}^3\, \iota_{2k-3}]$ is the Steenrod–Hopf sub-algebra of $H\mathbb{F}_2{}^*K(\mathbb{Z}, 2k-3)$ generated by $\mathrm{Sq}^3\, \iota_{2k-3}$. □

Corollary. *There are odd classes in $H\mathbb{F}_{2*}BU[2k,\infty)$ for $k \ge 4$.*

Proof. The class $\mathrm{Sq}^7\,\mathrm{Sq}^3\, \iota_{2k-3} \in \operatorname{Op}[\mathrm{Sq}^3\, \iota_{2k-3}]$ is finally non-zero once $k \ge 4$. □

There are similar statements at primes $p > 2$. This means that we're not allowed to write "$\operatorname{Spec} H\mathbb{F}_{2*}BU[2k,\infty)$" for $k \ge 4$, and so the comparison part of the Ando–Hopkins–Strickland proof is immediately stymied. Nonetheless, we can still follow their lead and perform a purely algebraic analysis: write $C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)$ for the scheme parametrizing such $\mathbb{G}_m$–valued functions, and consider the induced ring map

$$\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \mathbb{G}_m)) \to H\mathbb{F}_{2*}BU[2k,\infty).$$

More than this, this map is equivariant for the coaction of $\mathcal{O}(\operatorname{Aut}\widehat{\mathbb{G}}_a)$, i.e., it is a map of comodules for the dual Steenrod algebra. Accordingly, it factors

$$\begin{array}{ccc}
\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_m)) & \longrightarrow & H\mathbb{F}_{2*}BU[2k,\infty) \\
& \searrow & \uparrow \\
& & \text{even part} \\
& \searrow & \uparrow \\
& & \text{even sub-Steenrod-Hopf algebra.}
\end{array}$$

Conjecture (Hughes–Lau–P.). The longest map is an isomorphism.

Evidence. We completely computed a presentation of ring of functions $\mathcal{O}(C^k(\widehat{\mathbb{G}}_a; \widehat{\mathbb{G}}_m)) \times \operatorname{Spec} \mathbb{Z}_{(2)}$, and Singer's method gives partial information about the coaction of the dual Steenrod algebra on the topological side. Base changing to $\operatorname{Spec} \mathbb{F}_2$, as far out as we can check we see that the Poincaré series for the two sides agree.

This means that their algebraic approximation is actually pretty good — much, much closer than you might initially think! This begets a natural further question, if we want to use this calculation in homotopy theory:

[4] Remarkably, there is *not* a canonical section in the $k < 3$ cases!

Question. Is there a space $X(k)$ with $X(k)_E \cong C^k(\widehat{\mathbb{G}}_E; \mathbb{G}_m)$? That is, is there a space whose homology realizes this particular subset of the homology of $BU[2k, \infty)$?

These spaces should probably have some other nice properties if they're to be truly useful. However, because these hypothetical spaces are so close to such already-famous spaces, there are many dangerous theorems to steer around. Here are two that seem especially important:

Theorem (Adams–Priddy)**.** *Any connective spectrum delooping BSU is p–locally equivalent to $kU[4, \infty)$.* □

Theorem (Hovey)**.** *There is no map of ring spectra $MO[k, \infty) \to EO_{p-1}$ for $p \geq 5$ and* any k.

Notes on proof. The homotopy of $MO[k, \infty)$ is not known, but because $MO[k, \infty)$ is defined using the real J–homomorphism, it is at least possible to conclude that the image of J is sent to zero in $\pi_* MO\langle k\rangle$ in degrees at and above $k-1$.[5] On the other hand, calculations of Hopkins and Miller show that $\alpha_{1/1} \in \pi_{2p-3}\mathbb{S}$ is visible in $\pi_* EO_{p-1}$, so any orientation of EO_{p-1} must have $k - 2 \geq 2p - 3$. Finally, because integral Eilenberg–Mac Lane spaces are $K(p-1)$–acyclic in degrees $p + 1$ and higher, it follows that there is a natural equivalence

$$L_{K(p-1)} MO[k > p + 2, \infty) \xrightarrow{\simeq} L_{K(p-1)} MO[p + 2, \infty).$$

When $p \geq 5$, these two bounds contradict each other. □

Finally, the spaces for which the Ando–Hopkins–Strickland analysis was successful actually do fit into an interesting sequence of p–local spaces: they are the first examples of *Wilson spaces*, which are the indecomposable H–spaces with $\mathbb{Z}_{(p)}$–free homotopy and homology. The basic structure theorem about them, due to Wilson, is that the sequence continues through different deloopings of the spectra $BP\langle k\rangle$. Their ordinary homology is known, as computed by Sinkinson after Ravenel and Wilson's analysis of the Hopf ring $H\mathbb{F}_{p*}\underline{MU}_{2*}$, and it doesn't look very much like what we described above. Nonetheless, perhaps some connection would be uncovered if we could answer the following first question:

Question. Is there a splitting principle for $\underline{BP\langle 2\rangle}_8$–classes?

[5] This is true for the complex image of J and $MU[k, \infty)$ too.