# Comparison of power operations in Morava $E$-theories

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# Introduction

We are going to discuss power operations of Morava $E$-theories of different heights.

The $E_n$-cohomology $E_n^0(X)$ of a space $X$ naturally has a $\Gamma_n$-module structure,

where $\Gamma_n$ is the algebra of power operations of the $n$th Morava $E$-theory $E_n$.

# Introduction

The $E_n$-cohomology $E_n^0(X)$ also has a natural $G_n$-action, where $G_n$ is the extended Morava stabilizer group.

The $\Gamma_n$-module structure and the $G_n$-module structure on $E_n^0(X)$ are compatible in some sense.

We say that such a module is a $\Gamma_n$-module with compatible $G_n$-action.

# Introduction

We'd like to compare $\Gamma_n$-modules with compatible $G_n$-action and $\Gamma_{n+1}$-modules with compatible $G_{n+1}$-action.

For that purpose, we set

$$B_n = L_{K(n)}(E_n \wedge E_{n+1}).$$

We have two ring homomorphisms

$$\begin{aligned} \mathbf{in} : \quad & E_n^0 \longrightarrow B_n^0, \\ \mathbf{ch} : \quad & E_{n+1}^0 \longrightarrow B_n^0. \end{aligned}$$

# Main Theorem 1

## Theorem 1

*If $M$ is a $\Gamma_{n+1}$-module with compatible $G_{n+1}$-action, then*

$$(B_n^0 \otimes_{E_{n+1}^0} M)^{G_{n+1}}$$

*is a $\Gamma_n$-module with compatible $G_n$-action.*

# Main Theorem 2

## Theorem 2

*For any finite complex $X$, there is a natural isomorphism*

$$(B_n^0 \otimes_{E_{n+1}^0} E_{n+1}^0(X))^{G_{n+1}} \cong E_n^0(X)$$

*of $\Gamma_n$-modules with compatible $G_n$-action.*

# $H_\infty$-ring spectrum

Power operations are defined on an $H_\infty$-ring spectrum.

A spectrum $R$ is an $H_\infty$-ring spectrum if $R$ is equipped with maps

$$\xi_r : E\Sigma_{r+} \wedge_{\Sigma_r} R^{\wedge r} \longrightarrow R$$

for $r \geq 0$ satisfying the associativity and unit conditions up to homotopy.

# Examples of $H_\infty$-ring spectra

## Example 1

- $R = H\mathbb{Z}/p$ the mod $p$ cohomology,
- $R = E_n$ the $n$th Morava $E$-theory.
- If $X$ is a space and $R$ is an $H_\infty$-ring spectrum, then the function spectrum $R^X$ is an $H_\infty$-ring spectrum.

Note that the function spectrum $R^X$ is defined to satisfy

$$\pi_{-q}(R^X) \cong R^q(X).$$

# Power operations

Let $R$ be an $H_\infty$-ring spectrum.

The power operation

$$P_r : \pi_0(R) = R^0 \longrightarrow R^0(B\Sigma_r)$$

is defined as follows.

Suppose $x \in \pi_0(R)$ is represented by

$$x : S^0 \longrightarrow R.$$

# Power operations

This induces a map of Borel constructions

$$1 \wedge x^{\wedge r} : E\Sigma_{r+} \wedge_{\Sigma_r} (S^0)^{\wedge r} \longrightarrow E\Sigma_{r+} \wedge_{\Sigma_r} R^{\wedge r}.$$

The element $P_r(x) \in R^0(B\Sigma_r)$ is given by the composition

$$\begin{aligned} B\Sigma_{r+} = E\Sigma_{r+} \wedge_{\Sigma_r} (S^0)^{\wedge r} &\xrightarrow{1 \wedge x^r} E\Sigma_{r+} \wedge_{\Sigma_r} R^{\wedge r} \\ &\xrightarrow{\xi_r} R \end{aligned}$$

# Power operations

## Remark 2

The inclusion $* \to B\Sigma_r$ induces the restriction map

$$\mathbf{res} : R^0(B\Sigma_r) \longrightarrow R^0(*) = \pi_0(R).$$

We have

$$\mathbf{res} \circ P_r(x) = x^r.$$

# Power operations

The operation $P_r$ is multiplicative

$$P_r(xy) = P_r(x) \cdot P_r(y),$$

but not additive $P_r(x+y) \neq P_r(x) + P_r(y)$.

Actually, we have

$$P_r(x+y) = \sum_{i+j=r} \mathbf{tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_r}(P_i(x) \times P_j(y)),$$

where $\mathbf{tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_r} : R^0(B\Sigma_i \times \Sigma_j) \to R^0(B\Sigma_r)$ is the transfer map.

# The algebra $\Gamma_n$ of power operations

Rezk considered algebraic structures any $K(n)$-local commutative $E_n$-algebra has.

Let $R$ be a $K(n)$-local commutative $E_n$-algebra.

In particular, this implies $R$ is an $H_\infty$-ring spectrum.

So we can consider power operations on $\pi_0(R)$.

There exists a graded algebra

$$\Gamma_n = \oplus_{k\geq 0}\Gamma_n[k],$$

which is called the algebra of power operations.

For a $K(n)$-local commutative $E_n$-algebra $R$, $\pi_0(R)$ naturally has a $\Gamma_n$-algebra structure.

Namely, $\pi_0(R)$ is a $\Gamma_n$-module and a commutative ring such that the multiplication

$$\pi_0(R) \otimes_{E_n^0} \pi_0(R) \longrightarrow \pi_0(R)$$

is a map of $\Gamma_n$-modules.

The degree $k$-component of $\Gamma_n$ is given by

$$\Gamma_n[k] = \mathrm{Hom}_{E_n^0}(D_n(p^k), E_n^0),$$

where $D_n(p^k)$ is the cokernel

$$\mathrm{coker}\left(\bigoplus_{i=1}^{p^k-1} E_n^0(B\Sigma_i \times \Sigma_{p^k-i}) \xrightarrow{\text{transfer}} E_n^0(B\Sigma_{p^k})\right).$$

Note that $D_n(p^k)$ is a complete local ring since the transfer image is an ideal.

# $\Gamma_n$-action on $\pi_0(R)$

The action of $\Gamma_n$ on $\pi_0(R)$ is given as follows.

If $\theta \in \Gamma_n[k]$ and $x \in \pi_0(R)$, then $\theta(x) \in \pi_0(R) = R^0$ is the image of $x$ under the map

$$\begin{aligned}
\pi_0(R) &\xrightarrow{P_{p^k}} R^0(B\Sigma_{p^k}) \cong E_n^0(B\Sigma_{p^k}) \otimes_{E_n^0} R^0 \\
&\longrightarrow D_n(p^k) \otimes_{E_n^0} R^0 \\
&\xrightarrow{\theta \otimes 1} E_n^0 \otimes_{E_n^0} R^0 \cong \pi_0(R).
\end{aligned}$$

Hence we obtain an operation

$$\theta : \pi_0(R) \longrightarrow \pi_0(R).$$

for $\theta \in \Gamma_n$.

## Remark 3

The operation $\theta : \pi_0(R) \longrightarrow \pi_0(R)$ is additive since it factors through

$$D_n(p^k) \otimes_{E_n^0} R^0 = E_n^0(B\Sigma_{p^k})/(\text{transfer}) \otimes_{E_n^0} R^0.$$

▸ non-additive

# The moduli interpretation of $D_n(p^k)$

Let $\mathbb{F}_n$ be the formal group scheme over $E_n^0$ associated to the Morava $E$-theory $E_n$.

## Theorem 3 (Strickland)

*For any complete local $E_n^0$-algebra $T$,*

$$\mathbf{Hom}^c_{E_n^0\text{-alg}}(D_n(p^k), T) \cong \left\{ \begin{array}{l} \text{finite subgroup schemes } H \text{ of } \mathbb{F}_n \\ \text{over } T \text{ of rank } p^k \end{array} \right\}.$$

# Formal affine graded category scheme

The structure of $D_n = \{D_n(p^k)\}_k$ is described as follows.

The pair $(E_n^0, D_n)$ forms a formal affine graded category scheme.

Namely, for any complete local ring $T$, the pair

$$(\mathrm{Hom}^c_{\mathrm{ring}}(E_n^0, T), \{\mathrm{Hom}^c_{\mathrm{ring}}(D_n(p^k), T)\}_k)$$

naturally has the structure of a graded category.

In particular, we have maps

$$\begin{aligned} s &: & E_n^0 &\longrightarrow & D_n(p^k) \\ t &: & E_n^0 &\longrightarrow & D_n(p^k) \\ u &: & D_n(p^0) &\longrightarrow & E_n^0 \end{aligned}$$

and

$$c : D_n(p^{k+l}) \longrightarrow D_n(p^k) \underset{s,E_n^0,t}{\otimes} D_n(p^l)$$

for $k, l \geq 0$ corresponding to the structure of a graded category.

# $\Gamma_n$-module and $D_n$-comodule

In other words, $(E_n^0, D_n)$ is a graded Hopf algebroid without antipode.

So we can consider a $D_n$-comodule,

which is an $E_n^0$-module $M$ with structure maps

$$M \longrightarrow D_n(p^k) \underset{s, E_n^0}{\otimes} M$$

for $k \geq 0$ satisfying the associativity and unit conditions.

## Remark 4

The category of $\boldsymbol{\Gamma_n}$-modules is equivalent to the category of $\boldsymbol{D_n}$-comodules.

In the following of this talk we use the words $\boldsymbol{\Gamma_n}$-module and $\boldsymbol{D_n}$-comodule interchangeably.

# The extension $B_n$

We are going to consider the relationship between $\Gamma_n$-modules and $\Gamma_{n+1}$-modules.

For that purpose, we define

$$B_n = L_{K(n)}(E_n \wedge E_{n+1}).$$

Note that $B_n$ is a $K(n)$-local commutative $E_n$-algebra.

Hence $\pi_0(B_n) = B_n^0$ is a $\Gamma_n$-algebra.

# The extension $B_n$

We have maps of $H_\infty$-ring spectra

$$\begin{aligned}\mathbf{in}: \quad & E_n \longrightarrow B_n,\\ \mathbf{ch}: \quad & E_{n+1} \longrightarrow B_n.\end{aligned}$$

In particular, we have ring homomorphisms

$$\begin{aligned}\mathbf{in}: \quad & E_n^0 \longrightarrow B_n^0,\\ \mathbf{ch}: \quad & E_{n+1}^0 \longrightarrow B_n^0.\end{aligned}$$

The following is the key proposition.

**Proposition 5**

*$\boldsymbol{B_n^0}$ is a $\boldsymbol{\Gamma_{n+1}}$-algebra such that $\mathbf{ch} : \boldsymbol{E_{n+1}^0} \to \boldsymbol{B_n^0}$ is a map of $\boldsymbol{\Gamma_{n+1}}$-algebras.*

The proposition follows from the following lemma.

## Lemma 4

*There is an isomorphism*

$$D_{n+1}(p^k) \underset{s,E^0_{n+1}}{\otimes} B^0_n \cong \prod_{r=0}^{k} D_{n+1}(p^k, p^r),$$

*where* $D_{n+1}(p^k, p^r)$ *is a complete local* $B^0_n$*-algebra such that*

$$\mathbf{Hom}^c_{B^0_n\text{-alg}}(D_{n+1}(p^k, p^r), T)$$

$$\cong \left\{ \begin{array}{l} \text{finite subgroup schemes } H \text{ of } \mathbb{F}_{n+1}[p^\infty] \\ \text{over } T \text{ of rank } p^k \text{ such that the rank of} \\ \pi_0(H) \text{ is } p^r \end{array} \right\}.$$

Using the proposition, we obtain

## Proposition 6

*If $M$ is a $B_n^0$-module and a $\Gamma_{n+1}$-module such that*

$$B_n^0 \otimes_{E_{n+1}^0} M \longrightarrow M$$

*is a map of $\Gamma_{n+1}$-modules, then $M$ has a $\Gamma_n$-module structure.*

*Proof.* $\mathbf{ch}$ induces a ring homomorphism

$$\begin{aligned} D_{n+1}(p^k) &= E^0_{n+1}(B\Sigma_{p^k})/(\text{transfer}) \\ &\longrightarrow B^0_n(B\Sigma_{p^k})/(\text{transfer}) \\ &\cong D_n(p^k) \underset{s,E^0_n}{\otimes} B^0_n. \end{aligned}$$

We obtain a $B^0_n$-algebra homomorphism

$$D_{n+1}(p^k) \underset{s,E^0_{n+1}}{\otimes} B^0_n \longrightarrow D_n(p^k) \underset{s,E^0_n}{\otimes} B^0_n.$$

We have a map

$$
\begin{aligned}
M \longrightarrow\ & D_{n+1}(p^k) \underset{s,E^0_{n+1}}{\otimes} M \\
\cong\ & (D_{n+1}(p^k) \underset{s,E^0_{n+1}}{\otimes} B^0_n) \underset{B^0_n}{\otimes} M \\
\longrightarrow\ & (D_n(p^k) \underset{s,E^0_n}{\otimes} B^0_n) \otimes_{B^0_n} M \\
\cong\ & D_n(p^k) \underset{s,E^0_n}{\otimes} M.
\end{aligned}
$$

This gives $M$ a $\Gamma_n$-module structure. □

## Remark 7

This proposition gives $B_n^0$ a $\Gamma_n$-module structure.

This coincides with the $\Gamma_n$-module structure given by the $K(n)$-local commutative $E_n$-algebra structure on $B_n$.

## $\Gamma_n$-module with compatible $G_n$-action

We consider the action of $G_n$ on a $\Gamma_n$-module.

For a space $X$, $\pi_0(E_n^X) = E_n^0(X)$ is a $\Gamma_n$-module and a twisted $E_n^0$-$G_n$-module.

We introduce a $\Gamma_n$-module with compatible $G_n$-action.

Note that the $G_n$-action on $E_n^0(B\Sigma_{p^k})$ induces a $G_n$-action on

$$D_n(p^k) = E_n^0(B\Sigma_{p^k})/(\text{transfer}).$$

# $\Gamma_n$-module with compatible $G_n$-action

## Definition 5

Let $M$ be a $\Gamma_n$-module.We say $M$ is a $\Gamma_n$-module with compatible $G_n$-action if $M$ is a $\Gamma_n$-module and a twisted $E_n^0$-$G_n$-module such that the structure map

$$M \longrightarrow D_n(p^k) \underset{s,E_n^0}{\otimes} M$$

is $G_n$-equivariant for any $k \geq 0$.

# $\Gamma_n$-module with compatible $G_n$-action

Namely, the diagram

$$\begin{array}{ccc} M & \longrightarrow & D_n(p^k) \underset{s,E_n^0}{\otimes} M \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle g \otimes g} \\ M & \longrightarrow & D_n(p^k) \underset{s,E_n^0}{\otimes} M \end{array}$$

commutes for any $g \in G_n$ and any $k \geq 0$.

Suppose $M$ is a $\Gamma_{n+1}$-module with compatible $G_{n+1}$-action.

The tensor product

$$B_n^0 \otimes_{E_{n+1}^0} M$$

is a $\Gamma_{n+1}$-module with compatible $G_{n+1}$-action.

Furthermore, it has a $\Gamma_n$-module structure (by Proposition 6) with compatible $G_n$-action.

From this, we obtain the following theorem.

## Theorem 6 (Main Theorem 1)

*If $M$ is a $\Gamma_{n+1}$-module with compatible $G_{n+1}$-action, then*

$$(B_n^0 \otimes_{E_{n+1}^0} M)^{G_{n+1}}$$

*is a $\Gamma_n$-module with compatible $G_n$-action.*

# $\Gamma_n$-module $E_n^0(X)$

Now we consider the relationship between the $\Gamma_n$-module $E_n^0(X)$ and the $\Gamma_{n+1}$-module $E_{n+1}^0(X)$ for a space $X$.

The map $\mathbf{ch} : E_{n+1} \to B_n$ induces a map

$$E_{n+1}^0(X) \longrightarrow B_n^0(X)$$

of $\Gamma_{n+1}$-modules with compatible $G_{n+1}$-action.

This implies a map

$$(B_n^0 \otimes_{E_{n+1}^0} E_{n+1}^0(X))^{G_{n+1}} \longrightarrow B_n^0(X)^{G_{n+1}}$$

of $\Gamma_n$-modules with compatible $G_n$-action.

We have a natural isomorphism

$$B_n^0(Y)^{G_{n+1}} \cong E_n^0(Y)$$

for any spectrum $Y$, and a natural isomorphism

$$B_n^0 \otimes_{E_{n+1}^0} E_{n+1}^0(Z) \xrightarrow{\cong} B_n^0(Z)$$

for any finite spectrum $Z$.

Hence we obtain the following theorem.

## Theorem 7 (Main Theorem 2)

*For any finite complex $X$, there is a natural isomorphism*

$$E_n^0(X) \cong (B_n^0 \otimes_{E_{n+1}^0} E_{n+1}^0(X))^{G_{n+1}}$$

*of $\Gamma_n$-modules with compatible $G_n$-action.*

In particular, we can recover the $\Gamma_n$-module $E_n^0(X)$ with compatible $G_n$-action from the $\Gamma_{n+1}$-module $E_{n+1}^0(X)$ with compatible $G_{n+1}$-action if $X$ is a finite complex.

**Thank you for your attention**