MULTIPLICATIVE ORIENTATIONS OF K-THEORY AND P-ADIC ANALYSIS

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Abstract

Recent work of Ando, Blumberg, Gepner, Hopkins and Rezk characterize E_∞ orientations of K-Theory. Their description involves producing measures on the p-adic units with certain moments. In his thesis, Ando produced a criterion for an orientation to be H_∞. His theorem is a condition on the induced formal group law. We connect these two classifications by producing measures of the type above whenever Ando's criterion is satisfied.

To Father and Mother.

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Chapter 1

Introduction

In this article we use techniques from local class field theory concerning formal group laws to study questions in algebraic topology. We compare the classifications of two different types of complex orientations for topological K-Theory.

Let F be a Lubin-Tate group law of height one over the p-adic integers. Recall that F is constructed around a given endomorphism that acts as a lift of the Frobenius. Given any p-adic unit we construct a measure on $\mathbb{Z}_p$ whose moments (which characterize it) involve the Bernoulli numbers of F and certain Euler factors that depend on the unit. There is a natural restriction homomorphism from measures on the p-adic units to measures on all of $\mathbb{Z}_p$. We show that when a certain condition on the endomorphism attached to F is satisfied, all of the measures we construct are in the image of the natural restriction map.

As an application we show that the refinement of an H_∞ complex orientation of K-Theory to special unitary cobordism is an E_∞ map. The condition on the special endomorphism of F mentioned above is equivalent to Ando's criterion [And95] for a complex orientation of K-Theory to be an H_∞ map.

The authors in [ABG+] describe the E_∞ special unitary orientations of K-Theory in terms of sequences of p-adic rational numbers. A refinement of a complex orientation is an E_∞ map if we can produce measures on the p-adic units of the type above whose moments involve the Bernoulli numbers of the induced formal group law. In conclusion, when a complex orientation satisfies the Ando criterion we use the measures described above to prove that refinement is an E_∞ map.

1.1 More Detailed Results

Fix a prime p. Let $\mathbb{Z}_p$ be the p-adic integers.

Definition 1.1.1. *If A is either $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$ let*

$$M(A)$$

be the ring of $\mathbb{Z}_p$ linear maps

$$\mu : \mathrm{Cont}(A, \mathbb{Z}_p) \longrightarrow \mathbb{Z}_p.$$

Given such a $\mu \in M(A)$ and an $f \in \mathrm{Cont}(A, \mathbb{Z}_p)$ we write

$$\int_A f(x)d\mu(x)$$

for the effect of the linear functional μ on the continuous function f.

Definition 1.1.2. *We call a power series $\phi(t) \in \mathbb{Z}_p[[t]]$ a lift of the Frobenius over $\mathbb{F}_p$ to the ring $\mathbb{Z}_p$ if*

1. $\phi(t) \equiv t^p \mod p$
2. $\phi(t) = pt +$ *terms of higher order*

The power series $\phi(t)$ is going to play the role of an endomorphism lifting the Frobenius endomorphism of a height one formal group law over $\mathbb{F}_p$ to the ring $\mathbb{Z}_p$. We are not fixing a formal group law over $\mathbb{F}_p$ in this situation. The results of Lubin and Tate [LT65] construct a formal group law given any $\phi(t)$.

Definition 1.1.3. *Let $F = F_\phi$ be the Lubin-Tate formal group law over $\mathbb{Z}_p$ associated to $\phi(t)$.*

Recall that ϕ is an endomorphism of the formal group law F_ϕ. It is often called the special endomorphism of F.

For a positive integer k, let us agree to write x^k for the continuous function

$$\mathbb{Z}_p \longrightarrow \mathbb{Z}_p$$
$$a \longmapsto a^k$$

Since $\phi(t)$ is congruent to t^p module the ideal (p, t^2) a generalized form of the Weierstrass Preparation Theorem permits us to write

$$\phi(t) = g(t)h(t)$$

where $g(t)$ is a monomial of degree p and $h(t)$ is an invertible power series. We agree that $F[p]$ is the set of zeros of $g(t)$.

Definition 1.1.4. *We say that a formal group law of the type $F = F_\phi$ satisfies the Ando condition if*

$$\prod_{c \in F[p]} t +_F c = \phi(t)$$

as power series over $\mathbb{Z}_p$.

Write G_a for the formal additive group law. Let

$$\exp_F : G_a \to F$$

be the exponential for F. It is a strict isomorphism of formal group laws over $\mathbb{Q}_p$. Define a sequence $B_k(F) \in \mathbb{Q}_p$ for $k \geq 0$ by

$$\frac{x \exp_F' x}{\exp_F x} = \sum_{k=0}^{\infty} B_k(F) \frac{x^k}{k!}.$$

Here the power series $\exp_F' x$ is the derivative of the power series $\exp_F x$ with respect to the x variable.

Our main result concerning measures is

Theorem 1.1.5. *If the formal group law $F = F_\phi$ satisfies the Ando Condition 1.1.4 and $a \neq 1 \in \mathbb{Z}_p^\times$ then there is a $\mu \in M(Z_p^\times)$ such that*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu(x) = -\frac{B_k(F)}{k} \left(1 - p^{k-1}\right)\left(1 - a^k\right)$$

for $k > 0$. For $k = 0$ the value can be interpreted as

$$\frac{1}{p} \log a^{p-1}$$

We prove this statement in Chapter 6.

1.2 An Application

Our calculation has a direct application to algebraic topology. Let MU be complex cobordism. Recall that there is an isomorphism

$$\pi_* MU \cong L$$

where L is Lazard's universal ring for formal group laws. The Ando condition described above is so named because

Theorem 1.2.1 (Ando). *A homotopy ring map*

$$\alpha : MU \to K_p$$

is a map of H_∞ ring spectra if and only if the formal group law

$$G = \alpha_* F_{MU}$$

satisfies the Ando condition with $\phi(t) = [p]_G(t)$.

Let MSU be the Thom spectrum associated to the space BSU. The recent work of Ando, Blumberg, Gepner, Hopkins and Rezk [ABG+] describe the components of the space of E_∞ maps

$$MSU \to K_p.$$

Theorem 1.2.2 (ABGHR). *There is a bijection between the set of $\alpha \in \pi_0 E_\infty(MSU, K_p)$ and the set of sequences*

$$\{t_k\} \in \prod_{k \geq 2} \mathbb{Q}_p$$

that satisfy the following two conditions:

1. *For any $a \neq 1 \in \mathbb{Z}_p^\times$ there exists a $\mu \in M(\mathbb{Z}_p^\times)$ such that*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu(x) = t_k \left(1 - a^k\right)\left(1 - p^{k-1}\right)$$

   for $k \geq 2$ and

2. *there are congruences*

$$t_k \equiv -\frac{B_k}{k} \mod \mathbb{Z}_p.$$

For example, the existence of the Mazur measure 6.0.10 shows that the refinement of the Todd genus is E_∞.

Before we continue we discuss the connection between p-adic K-Theory and Lubin-Tate groups of height one. See Section 6 of [Rez98] for more details.

Let $\mathbb{F}_p$ be the field with p elements. We can consider the formal multiplicative group law

$$x + y + xy$$

over $\mathbb{F}_p$. It has height one. Moreover, we can use the the formal multiplicative group as its own universal deformation to the ring $\mathbb{Z}_p$. This induces a degree -2 group law

$$x +_F y = x + y + uxy$$

over $\mathbb{Z}_p[u^{\pm}]$. The cohomology theory associated to this universal deformation is K_p. Thus any degree zero formal group law constructed from an orientation for K_p will be of the form F_ϕ for some lift of Frobenius ϕ.

We can now use our calculation to construct a non trivial map of sets

$$\mathcal{D} : H_\infty(MU, K_p) \to \pi_0 E_\infty(MSU, K_p).$$

The map $\mathcal{D}$ sends a complex orientation α to the sequence

$$\left\{ -\frac{B_k(\alpha_* F_{MU})}{k} \right\}$$

for $k \geq 2$. Condition two of 1.2.2 follows easily from work of H. Miller [Mil82] on universal Bernoulli numbers. We will address this issue in Section 8.2.

We can consider some related natural maps. For example the map

$$H_\infty(MU, K_p) \to H_\infty(MSU, K_p)$$

induced by restriction on the source. Likewise, there is a function

$$\pi_0 E_\infty(MSU, K_p) \to H_\infty(MSU, K_p)$$

that forgets structure.

The map $\mathcal{D}$ is natural in the sense that

Theorem 1.2.3. *The diagram*

$$\begin{tikzcd} H_\infty(MU, K_p) \arrow[r, "\mathcal{D}"] \arrow[d] & \pi_0 E_\infty(MSU, K_p) \arrow[dl] \\ H_\infty(MSU, K_p) & \end{tikzcd}$$

where the two unlabeled arrows are the natural ones discussed above, commutes.

The document is set up as follows. Chapter two contains the basics that we require on measure theory.

Our calculations occur in chapter four. Chapters five and six are needed to produce a measure on the units of the form in Theorem 1.2.2. The latter two chapters connect our results to topology. The first of these is used to show the equivalence of one of the main theorems we need concerning measures with a classic result of Madsen, Snaith, and Tornehave. The final chapter contains our discussion on orientations.

Chapter 2

Measure Theory

We set up the basic theory of integration over $\mathbb{Z}_p$ in this chapter.

2.1 Setup and Notation

Fix a prime p. In what follows, $\mathbb{Z}_p$ will always be the p-adic integers and $\mathbb{Q}_p$ will be it's fraction field. Recall that the space $\mathbb{Z}_p$ is compact and totally disconnected in the p-adic topology. If $F(x, y)$, or simply F is a formal group law we will write $x +_F y$ for the formal sum. Recall that over a $\mathbb{Q}$-algebra, all formal group laws are isomorphic to the additive group law. We let $\exp_F(x)$ and $\log_F(x)$ be the strict exponential and logarithm of F.

2.1.1 Mahler's Theorem

First we must study the source of our linear functionals. The set of continuous functions from $\mathbb{Z}_p$ to itself have been completely described in terms of infinite sums. To begin we need a

Definition 2.1.2. *The generalized binomial coefficients, written* $\binom{x}{n}$ *, are defined by the formula*

$$\binom{x}{n} = \frac{x(x-1)(x-2)...(x-n-1)}{n!}.$$

for $n \geq 1$*. We set*

$$\binom{x}{0} = 1.$$

These gadgets are of great interest to both topologists and number theorists. For example, they form an *integral* basis for the ring of numerical polynomials. That is polynomials $f(x)$ with rational coefficients such that $f(k) \in \mathbb{Z}$ for any integer k. As Adams, Clarke, Harris and Switzer discovered, this ring arises in topology as the stable degree zero operations in complex K-theory.

Here is a description of continuous functions on $\mathbb{Z}_p$.

Theorem 2.1.3. *[Mahler] If $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a continuous function then*

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

where the a_n are p-adic integers which tend to zero p-adically as n goes to infinity.

A proof can be found in [Mah58].

2.2 Definitions and Notations

We define measures in terms of linear functionals. Later on, we will see that this is equivalent to a Riemann sum definition.

Definition 2.2.1. *Suppose A is a topological abelian group. An $\mathbb{Z}_p$-valued measure on A is a $\mathbb{Z}_p$-module map from* $\mathrm{Cont}(A, \mathbb{Z}_p)$ *to* $\mathbb{Z}_p$.

The group A will usually be $\mathbb{Z}_p$ or $\mathbb{Z}_p^\times$. If μ is such a linear functional, we will often write

$$\int_A f(t) d\mu(t)$$

for the value of μ on the function f. We use

$$M(A)$$

for the $\mathbb{Z}_p$-module of all such gadgets.

We may also consider a discrete group for A. For example, we will use the notation

$$M(\mathbb{Z}/p^k)$$

to mean the $\mathbb{Z}_p$ module of $\mathbb{Z}_p$ linear maps

$$\mathrm{Maps}(\mathbb{Z}/p^k, \mathbb{Z}_p) \to \mathbb{Z}_p.$$

Example 2.2.2. For any $l \in A$, define $d\psi_l \in M(A)$ by

$$\int_A f(x) d\psi_l(x) = f(l).$$

The measure $d\psi_l$ is called the Dirac measure at l.

We wish to obtain structure theorems for $M(A)$ to help with computation. The case when $A = \mathbb{Z}_p$ has been studied in some detail, and we reproduce some of it here. We will then study $M(\mathbb{Z}_p^\times)$ by understanding its image inside $M(\mathbb{Z}_p)$.

There are two interesting sequences attached to a measure μ. First we have.

Definition 2.2.3. *For $k \geq 0$, the sequence given by*

$$\int_A t^k d\mu(t)$$

will be called the moments of the measure μ.

In light of Mahler's theorem 2.1.3, the second sequence is

Definition 2.2.4. *If $\mu \in M(\mathbb{Z}_p)$ then define*

$$M_i(\mu) = \int_{\mathbb{Z}_p} \binom{x}{i} d\mu(x).$$

We will call these the Mahler moments of the measure μ.

A measure is completely determined by it's Mahler moments. Similarly since we can write

$$x^m = \sum_i c_{i,m} \binom{x}{i}$$

with $c_{i,m} \in \mathbb{Z}$ thus a measure on $\mathbb{Z}_p$ is also determined uniquely by its moments.

2.3 The First Structure Theorem

There are two binary operations on $M(A)$. The first is obvious, we can add two different measures. That is if μ , $\nu \in M(\mathbb{Z}_p)$ we can define a new measure $\mu + \nu$ by

$$\int_A f(t) d(\mu + \nu)(t) = \int_A f(t) d\mu(t) + \int_A f(t) \nu(t).$$

Such a gadget is clearly an $\mathbb{Z}_p$-valued linear functional.

A less obvious thing to do is convolution. Again, given measures μ and ν, we can "compose" the two to form the convolution $\mu \star \nu$. This is the double integral. We need to use the group structure on A to define this however, and we will write this group operation multiplicatively.

Definition 2.3.1. *The convolution of μ and ν, written $\mu \star \nu$ is defined by the formula*

$$\int_A f(t) d\mu \star \nu(t) = \int_A \int_A f(xy) d\mu(x) d\nu(y)$$

It is clear that $\mu \star \nu = \nu \star \mu \in M(A)$ as we've assumed that A is abelian.

2.3.2 Computing the Moments of a Measure

In this section we wish to describe a process to compute the moments of a measure. We can use Mahler's theorem to obtain an isomorphism of groups

$$M(\mathbb{Z}_p) \cong \mathbb{Z}_p[[t]]$$

The map sends a measure μ to the formal power series

$$f_\mu(t) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} d\mu(x) \right) t^i.$$

This construction is clearly additive.

The generalized binomial coefficients form an integral basis for arithmetic polynomials, so in particular we have $c_{m,k} \in \mathbb{Z}$ defined by

$$x^k = \sum_{m=0}^{k} c_{m,k} \binom{x}{m}.$$

For $m > k$ we set $c_{m,k} = 0$.

The difference operator Δ is defined by the formula

$$\Delta^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(j)$$

The difference operator gives rise to the $c_{m,k}$, that is

$$c_{m,k} = \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} i^k \tag{2.3.3}$$

Set $D = (1+x)\frac{d}{dx}$. Later on, we will see that this is a translation invariant derivation for the formal multiplicative group law $x + y + xy$. The main result of the section will follow from

Lemma 2.3.4. *For all $k \geq 0$ and $m \geq 0$ we have*

$$D^k x^m|_{x=0} = c_{m,k}$$

Here, and in what follows, the exponent k on the operator D means to apply the operator k times to the power series.

Proof. We use the change of variables $x \mapsto e^t - 1$ and the formula, see Lemma 4.1.2

$$D^k f(x)|_{x=0} = \frac{d^k}{dt^k} f(e^t - 1)|_{t=0}.$$

Writing

$$x^m = \sum_{i=0}^{m} c_{i,m} \binom{x}{i}$$

and using the binomial theorem after changing variables produces

$$(e^t - 1)^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} e^{jt} \tag{2.3.5}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^n \frac{t^n}{n!} \tag{2.3.6}$$

whose coefficient of $\frac{t^k}{k!}$ is

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} j^k.$$

and we see by formula 2.3.3 that this is precisely $c_{m,k}$. □

Since everything is linear we have

Corollary 2.3.7. *If $f(x) = \sum_{m=0} a_m x^m \in \mathbb{Z}_p[[x]]$ then*

$$D^k f(x)|_{x=0} = \sum_{m=0} a_m c_{m,k}$$

Note that the sum is finite since for $m > k$, $c_{m,k}$ is zero.

Proof. This follows easily from Lemma 2.3.4 and the fact that D is a linear operator. □

Given $f(t) = \sum_{m=0} a_m x^m \in \mathbb{Z}_p[[t]]$ we now have two ways to construct an element of $M(\mathbb{Z}_p)$. The first is

$$\int_{\mathbb{Z}_p} \binom{x}{i} d\mu_f(x) = a_i$$

and Corollary 2.3.7 shows that this measure has moments

$$\int_{\mathbb{Z}_p} x^k d\mu_f(x) = D^k f(t)|_{t=0}.$$

where $D = (1+t)\frac{d}{dt}$. We collect this idea in a

Theorem 2.3.8. *There is an isomorphism of abelian groups*

$$\mathbb{Z}_p[[t]] \to M(\mathbb{Z}_p)$$

where the map sends $f(t)$ to the measure whose moments are

$$D^k f(t)|_{t=0}$$

This is the same map as that constructed using Mahler's theorem, expect we now have an explicit formula for the moments.

This result is something about the formal multiplicative group law $x + y + xy$. We will generalize this identification for any Lubin-Tate group law of height one over $\mathbb{Z}_p$ in chapter four.

Chapter 3

Generalized Bernoulli Numbers

In this chapter we define the sequence of p-adic rationals we are going to study.

Definition 3.0.1. *Let F be a formal group law over a complete local torsion free $\mathbb{Z}_p$ algebra R. Define $B_k(F) \in R \otimes \mathbb{Q}$ for $k \geq 0$ by the expansion*

$$\frac{x \exp'_F(x)}{\exp_F(x)} = \sum_{k=0} B_k(F) \frac{x^k}{k!}$$

We call the sequence $\{B_k(F)\}$ the Bernoulli numbers of the formal group law F. We can do a similar thing with any $h(x) \in R((x))^\times$. Given such an $h(x)$ we can define $B_k(F,h) \in R \otimes \mathbb{Q}$ as

$$\frac{xh'(\exp_F(x)}{\log'_F(\exp_F(x))h(\exp_F(x))} = \sum_{i=0} B_i(F,h) \frac{x^k}{k!}.$$

The definition above is not the standard one in algebraic topology. We prove a lemma that connects our definition of generalized Bernoulli numbers with those studied by H. Miller in [Mil82].

Lemma 3.0.2. *For a formal group law F we have the expansion*

$$\frac{x\exp'_F(x)}{\exp_F(x)} = \sum_{k=0}^{\infty} t_k \frac{x^k}{k!}$$

if and only if we have the expansion.

$$\frac{x}{\exp_F(x)} = exp\left(\sum_{k=1}^{\infty} -\frac{t_k}{k} \frac{x^k}{k!}\right)$$

Proof. Note that the assumption that $\exp'_F(0) = 1$ implies $t_0 = 1$. We start with

$$\frac{1}{x} - \frac{\exp'_F(x)}{\exp_F(x)} = \frac{1}{x} - \sum_{k=0}^{\infty} t_k \frac{x^{k-1}}{k!} \tag{3.0.3}$$

$$= \sum_{k=1}^{\infty} -t_k \frac{x^{k-1}}{k!}, \tag{3.0.4}$$

and integrating both sides yields

$$\ln x - \ln \exp_F(x) = \sum_{k=1}^{\infty} -\frac{t_k}{k}\frac{x^k}{k!} + C \tag{3.0.5}$$
$$= \ln \frac{x}{\exp_F(x)} + C. \tag{3.0.6}$$

Exponentiating both sides above yields the required formula while a comparison of constant terms shows that C must be zero.

The other direction is a similar calculation. □

Definition 3.0.7. *We call the* $-\frac{B_k(F)}{k}$ *the logarithmic Bernoulli numbers.*

Example 3.0.8. Let's look at the formal group law

$$G(x, y) = x + y - xy$$

over $\mathbb{Z}_p$. It's exponential is

$$\exp_G = 1 - e^{-x}.$$

Using the definition to compute $B_k(G)$, we need to expand

$$\frac{xe^{-x}}{1 - e^{-x}}.$$

It follows that

$$B_k(G) = B_k$$

where B_k is the k-th Bernoulli number. Recall these are defined by

$$\frac{x}{e^x - 1} = \sum_{k=0} B_k \frac{x^k}{k!}.$$

We end this section with a classification theorem.

Proposition 3.0.9. *Suppose F and G are isomorphic formal group laws over a torsion free ring R. If*

$$B_k(F) = B_k(G)$$

for all $k \geq 0$, then

$$F = G.$$

Proof. To begin, assume we have a strict isomorphism

$$\Theta : G \to F$$

over formal group laws over R. The hypothesis imply

$$\frac{x}{\exp_F(x)} = \frac{x}{\exp_G(x)}$$

from which it follows that

$$\exp_F(x) = \exp_G(x).$$

However, we also know

$$\exp_F(x) = \Theta(\exp_G(x))$$

and so

$$\exp_G(x) = \Theta(\exp_G(x)).$$

To finish, if we write

$$\Theta = \sum_{j=1} b_j x^j$$

and

$$\exp_G(x) = \sum_{i=1} a_i x^i$$

then since $a_1 = 1$, if follows that

$$\sum_{j=2} b_j \left(\sum_{i=1} a_i x^i \right)^j = 0.$$

A recursive argument, again using that $a_1 = 1$, shows that $b_j = 0$ for all $j \geq 2$. Thus $\Theta = x$ and the we are finished. $\square$

3.1 Formulas

This is a technical section; we require a few formulas for the calculation ahead. These are standard results and can be found in many places. We prove them here for completeness.

Let F be a formal group law over a torsion free ring.. We use the notation $F_2(x, y)$ for the partial derivative of $x +_F y$ at y.

Lemma 3.1.1. *We have*

$$F_2(\exp_F(x), 0) = \exp'_F(x) \tag{3.1.2}$$

and for any $a \in \mathbb{R}$

$$[a]'(\exp_F(x)) = \frac{a \exp'_F(ax)}{\exp'_F(x)} \tag{3.1.3}$$

$$\tag{3.1.4}$$

Proof. Express

$$x +_F y = x + y + \sum_{i,j\geq 1} c_{i,j} x^i y^j.$$

Note here that the $c_{i,j}$ live in R. Recall that $\log'_F(0) = 1$ by definition.

To begin, we know that

$$\log_F(F(x,y)) = \log_F(x) + \log_F(y)$$

and applying the partial differential operator $\frac{d}{dy}$ to both sides yields the formula

$$\log'_F(F(x,y))\frac{d}{dy}F(x,y) = 0 + \log'_F(y) \tag{3.1.5}$$

Setting $y = 0$ we are left with

$$\log'_F(x)F_2(x,0) = 1 \tag{3.1.6}$$

$$\log'_F(x) = \frac{1}{F_2(x,0)}. \tag{3.1.7}$$

We need to show that $\frac{1}{\left(\frac{d}{dy}F(x,y)\right)_{y=0}}$ is well defined. To do that, we'll just compute its inverse:

$$\frac{d}{dy}F(x,y)_{y=0} = \left(\frac{d}{dy}\left(x + y\sum_{i,j\geq 1} c_{i,j}x^i y^j\right)\right)_{y=0} \tag{3.1.8}$$

$$= \left(1 + \sum_{i,j\geq 1} c_{i,j} j x^i y^{j-1}\right)_{y=0} \tag{3.1.9}$$

$$= 1 + \sum_{i\geq 1} c_{i,1} x^i. \tag{3.1.10}$$

The last line in 3.1.8 shows that $F_2(x,0)$ is invertible in $R[[x]]$. Thus there are coefficients $c_i \in R$ with

$$\log_F'(x) = \sum_{i=0}^{\infty} c_i x^i \tag{3.1.11}$$

and we see that the derivative of the logarithm has integral coefficients when viewed as a power series. Continuing on, differentiating both sides of $\exp_F(\log_F(x)) = x$ shows that

$$\exp_F'(x) = \frac{1}{\log_F'(\exp_F(x))} \tag{3.1.12}$$

and now we can see that $\exp_F'(x)$ is in $R[[\exp_F(x)]]$ and the first formula in the lemma follows easily.

To see the second formula hit the identity

$$[a]_F\,(\exp_F(x)) = \exp_F(ax)$$

with $\frac{d}{dx}$. □

Chapter 4

The Main Calculation

Let $\hat{\mathbb{G}}_m$ be the formal multiplicative formal group law

$$x + y + xy$$

over $\mathbb{Z}_p$ If we choose

$$\phi_m(t) = (1+t)^p - 1$$

then $\hat{\mathbb{G}}_m = F_{\phi_m}$

Let $\phi(t)$ be another lift of the Frobenius to $\mathbb{Z}_p$. It follows from the Lubin-Tate theory that there exists a strict isomorphism

$$\Theta : \hat{\mathbb{G}}_m \to F = F_\phi.$$

Remark 4.0.1. The isomorphism produced by the theory of Lubin and Tate is generally not a strict isomorphism. However it is an easy calculation to produce a strict isomorphism given any isomorphism of formal group laws over the same ring.

We will be using the isomorphism Θ to change coordinates. First we relate the exponentials of $\hat{\mathbb{G}}_m$ and F.

Lemma 4.0.2. *If* $\exp_F x$ *is the normalized exponential for the formal group law* F *then*

$$\exp_F x = \Theta(\exp_G(x)).$$

Substituting this formula in the definition for $B_k(F)$*, it follows that*

$$B_k(F) = B_k(G, \Theta)). \tag{4.0.3}$$

Proof. Just write out the definitions. $\square$

4.1 Invariant Derivations

A derivation D (over $\mathbb{Z}_p$) of the formal group law F is translation invariant if

$$(Df)(x +_F y) = D\left(f(x +_F y)\right)$$

where the D on the right hand side treats y as a constant. The left hand side applies the operator D to f first and then evaluates at $x +_F y$.

Definition 4.1.1. *Suppose ϕ is a lift of the Frobenius to the complete local torsion free ring R. Define*

$$D_F = \frac{1}{\log'_F(y)} \frac{d}{dy} = F_2(y, 0) \frac{d}{dy}$$

and as before

$$D = (1 + x) \frac{d}{dx}.$$

It is straightforward to check that these are translation invariant derivations on F and $\hat{\mathbb{G}}_m$ respectively. The next lemma is helpful in making calculations.

Lemma 4.1.2. *We have the change of variables formula*

$$D^k f(x)|_{x=0} = \frac{d^k}{dt^k} f(e^t - 1)|_{t=0} = \frac{d^k}{dt^k} f(\exp_G(t))|_{t=0}$$

Proof. We are making the change of variables

$$x \mapsto e^t - 1 = \exp_{G_a} \tag{4.1.3}$$

where G_a is the formal additive group law over $\mathbb{Q}_p$. The invariant derivation for G_a is the operator $\frac{d}{dt}$. A straightforward calculation shows that the mapping 4.1.3 sends the operator D to the invariant derivation for G_a. □

The next lemma is a technical tool. It compares the effects of D_F and D via the strict isomorphism $\Theta : \hat{\mathbb{G}}_m \to F$.

Lemma 4.1.4. *If $f(y) \in \mathbb{Z}_p[[y]]$ then*

$$D_F^k f(y)|_{y=0} = D^k f(\Theta(x))|_{x=0}$$

Proof. This argument is the same as the previous change of variables formula. As operators, D and D_F are related by the isomorphism Θ. For the moment, let G be the group $\hat{\mathbb{G}}_m$. We start with a simply calculation:

$$\Theta(x +_G y) = \Theta(x) +_F \Theta(y) = \Theta(x) + \Theta(y) + \sum_{i,j\geq 2} a_{ij}\Theta(x)^i\Theta(y)^j \tag{4.1.5}$$

$$\Theta'(x +_G y)G_2(x, y) = \Theta'(y) + \sum_{i,j} a_{ij} j\Theta(x)^i\Theta(y)^{j-1}\Theta'(y) \tag{4.1.6}$$

$$\Theta'(x +_G y)G_2(x, y) = \Theta'(y)\left(F_2(\Theta(x), \Theta(y))\right) \tag{4.1.7}$$

and so we see that

$$G_2(x, 0) = \frac{1}{\Theta'(x)}F_2(\Theta(x), 0)$$

and

$$\Theta'(x)dx = dy$$

Thus, under the change of variables, $\Theta(x) = y$, the operator D_F is sent to the operator D. □

Example 4.1.8. Let's look at the group law $x + y + xy$. The Dirac measure at l corresponds to the power series $(1 + t)^l$ under the isomorphism in 2.3.8

For any other group law F that is isomorphic to $\hat{\mathbb{G}}_m$ by a map Θ, the Dirac measure at l is associated to the power series

$$\left(\frac{F_2(y, 0)\Theta'(0)}{\Theta'(\Theta^{-1}x)}\right)^l.$$

Using Lemma 4.1.4, we see that when we evaluate D_F^k of this power series at zero we obtain l^k.

This can all be verified by computing the derivative of

$$F(\Theta(x), \Theta(y)) = \Theta(\hat{\mathbb{G}}_m(x, y))$$

with respect to y and then setting $y = 0$.

We can now state the main calculational tool.

Corollary 4.1.9. *There is an isomorphism of abelian groups*

$$\mathcal{D}_F : \mathbb{Z}_p[[t]] \to M(\mathbb{Z}_p)$$

such that a power series f is sent to a measure whose moments are computed via

$$\int_{\mathbb{Z}_p} x^k d\mu(x) = D_F^k f(t)|_{t=0}$$

for $k \geq 0$.

Proof. Define $\mathcal{D}_F$ so that the diagram

$$\begin{tikzcd} \mathbb{Z}_p[[s]] \arrow[r, "\mathcal{D}"] & M(\mathbb{Z}_p) \\ \mathbb{Z}_p[[t]] \arrow[u, "\Theta"] \arrow[ur, "\mathcal{D}_F"'] & \end{tikzcd}$$

commutes. Here $\mathcal{D}$ is the map in Theorem 2.3.8 Both Θ and $\mathcal{D}$ are isomorphisms. and so $\mathcal{D}_F$ is as well. Finally, Lemma 4.1.4 produces the formula form the moments. $\square$

This statement allows us to change the way we think about measures on $\mathbb{Z}_p$ depending upon the isomorphism

$$\Theta : \hat{\mathbb{G}}_m \to F.$$

This turns out to be the right thing to do for topology. See for example Theorem 7.1.6.

4.2 Producing the Generalized Bernoulli Numbers

In this section we produce a power series whose values at D_F^k are of the desired form. However we do not know if the coefficients of the power series are integers or if we can lift the measure to the units. We address these two problems separately.

See Chapter 5 for a discussion on integrality. Producing a measure on the units is a more difficult, but related to topology in a non-trivial way. See Chapter 6 for the details.

For any unit a in the ring of integers of the completion of the algebraic closure of $\mathbb{Q}_p$ congruent to 1 modulo p the power series

$$\sum_{i=1} \left(-1^{i+1}\right) \frac{(1-a)^i}{i}$$

converges. In other words, we can use the power series expansion of the classical logarithm around 1 to define

$$\log_p a = \sum_{i=1} \left(-1^{i+1}\right) \frac{(1-a)^i}{i}$$

for any unit congruent to one modulo p. We can use a similar definition for a power series $f(t)$ whose constant term is one modulo p. We will often drop the dependency on p in our notation.

Definition 4.2.1. *For $a \in \mathbb{Z}_p$ we let $[a]_F(t)$ be the multiplication on F_ϕ by a given by the formal A-module structure.*

The power series $[a]_F(t)$ has the properties:

1. $[a]_F(t) \equiv aT +$ higher degree terms

2. $[a]_F(x +_F y) = [a]_F(x) +_F [a]_F(y)$

3. $[a+b]_F(t) = [a]_F t +_F [b]_F(t)$.

The formal A module structure is such that

$$[a]_F(t) = \exp_F(a \log_F t)$$

where $\log_F$ and $\exp_F$ are the logarithm and exponential for the formal group law F. See the first few paragraphs of [Haz79] for a short discussion on why this is the case.

Define the divided a-series by

$$\langle a \rangle_F(t) = \frac{[a]_F(t)}{t}.$$

It has leading coefficient a.

The next proposition is the first step in constructing elements of $M(\mathbb{Z}_p)$ whose moments are of the form from Theorem 1.2.2

Proposition 4.2.2. *Let $a \neq 1 \in \mathbb{Z}_p^\times$ then*

$$D_F^k \frac{1}{p} \log \left(\frac{\langle a \rangle_F^p(y)}{\langle a \rangle_F \circ \phi(y)} \right) |_{y=0} = -\frac{B_k(F)}{k}(1-a^k)(1-p^{k-1})$$

for all $k > 0$ and for $k = 0$ the value is

$$\frac{1}{p} \log a^{p-1}$$

Proof. To compute the values

$$D_F^k \frac{1}{p} \log f(y)|_{y=0}$$

we will make a change of variables as suggested by the previous section. We replace y with $\Theta(e^t - 1) = \Theta(\exp_G(t)$ and compute

$$\frac{d^k}{dt^k} \log f(\Theta(\exp_G(t))|_{t=0}.$$

Observing that $\Theta(\exp_G t)) = \exp_F(t)$ we can study t power series

$$\log \frac{\langle a\rangle_F^p(\exp_F)(t)}{\langle a\rangle_F(\exp_F(pt))}$$

instead. The formula

$$\phi \circ \Theta = \Theta \circ [p]_{\hat{\mathbb{G}}_m}$$

is needed for the denominator.

If we hit this power series with the operator $\frac{d}{dt}$ we are left with the mess

$$\frac{p\langle a\rangle_F^{p-1}(\exp_F t)\langle a\rangle'(\exp_F t)\exp'_F(t)\langle a\rangle_F \exp_F pt}{\langle a\rangle_F^2(\exp_F pt)} \cdot \frac{\langle a\rangle(\exp_F(pt))}{\langle a\rangle^p(\exp_F t)} \tag{4.2.3}$$

$$-\frac{\langle a\rangle'_F(\exp_F pt)\exp'_F(pt)p\langle a\rangle^p(\exp_F(t))}{\langle a\rangle_F^2(\exp_F pt)} \cdot \frac{\langle a\rangle(\exp_F(pt))}{\langle a\rangle^p(\exp_F t)} \tag{4.2.4}$$

$$\tag{4.2.5}$$

Thankfully, this monstrosity simplifies quite nicely to the expected form

$$\frac{p\langle a\rangle'(\exp_F(t)\exp'_F(t)}{\langle a\rangle(\exp_F t)} - \tag{4.2.6}$$

$$\frac{\langle a\rangle'(\exp_F pt)\exp'_F(pt)p}{\langle a\rangle(\exp_F pt)}. \tag{4.2.7}$$

Using the formula

$$\exp'_F t\langle a\rangle(\exp_F t) + \exp_F t\langle a\rangle'(\exp_F t)\exp'_F t = \exp'_F(at)a. \tag{4.2.8}$$

we can rewrite the first term of 4.2.6 as

$$\frac{\exp'_F(at)a - \exp_F t\langle a\rangle \exp_F t}{\langle a\rangle(\exp_F t)\exp_F t} \tag{4.2.9}$$

and after expanding the numerator we are left with

$$\frac{a\exp'_F(at)}{\exp_F at)} - \frac{\exp'_F t}{\exp_F t}. \tag{4.2.10}$$

Similarly for the terms involving p we obtain

$$-\frac{1}{p}\frac{pa\exp'_F(pat)}{\exp_F(pat)} + \frac{1}{p}\frac{p\exp'_F(pt)}{\exp_F pt} \tag{4.2.11}$$

Recalling Definition 3.0.1 we can compute the power series expansions of 4.2.10 and 4.2.11;

$$\sum_{k=0} -B_k(F)(1-a^k)(1-p^{k-1})\frac{t^{k-1}}{k!}. \tag{4.2.12}$$

Notice that there is no $\frac{1}{t}$ term so we can rewrite 4.2.12 as

$$\sum_{k=1} -\frac{B_k(F)}{k}(1-a^k)(1-p^{k-1})\frac{t^{k-1}}{(k-1)!}.$$

Integrating term by terms produces

$$\sum_{k=1} -\frac{B_k(F)}{k}(1-a^k)(1-p^{k-1})\frac{t^k}{k!} + C$$

for some constant C. We now see that the claim is true for $k > 0$.

For $k = 0$, observe that none of our substitutions changed the constant term of the power series. Thus we can read this off from the original function $\frac{1}{p}\log\frac{\langle a\rangle^p(y)}{\langle a\rangle(\phi(y))}$, and this value is

$$\frac{1}{p}\log a^{p-1}$$

as claimed. $\square$

Chapter 5

The Coleman Norm Operator

Our investigations are motivated by the similarities of the Lubin isogeny for quotient formal groups, see [And95] for a good explanation of its roll in topology, and the Coleman Norm operator [Col79].

In Definition 1.1.2 we defined a power series $\phi(t)$ over $\mathbb{Z}_p$ to be a lift of the Frobenius over $\mathbb{F}_p$ if

1. $\phi(t) \equiv t^p \mod p$ and

2. $\phi(t) = pt +$ higher degree terms .

The gadget $\phi(t)$ is trying to be a lift of the Frobenius morphism of some height one formal group law over $\mathbb{F}_p$.

As before, for each such $\phi(t)$ there exist a unique formal group law F_ϕ over $\mathbb{Z}_p$ such that ϕ is an endomorphism of F_ϕ. We call ϕ the special endomorphism of F_ϕ. We will often drop the dependence on ϕ when there is no confusion.

The following operator is useful in producing norm coherent sequence in a division tower. See [Col79] for more details. We are going to use this operator to verify that our power series from Proposition 4.2.2 has integral coefficients. Recall the definition of $F[p]$ from 1.1.4.

Proposition 5.0.1 (R. Coleman). *Fix a ϕ as above.. Then there exist a unique multiplicative operator*

$$N_F : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p[[t]]$$

such that for any $g(t) \in \mathbb{Z}_p[[t]]$ we have

$$N_F(g) \circ \phi(t) = \prod_{c \in F[p]} g(t +_F v). \tag{5.0.2}$$

Proof. We follow the proof from [dS87]. The original is due to R. Coleman and can be found in [Col79]

The formula 5.0.2 characterizes the power series $N_F(g)$ uniquely and multiplicity follows easily. Therefore, all that is left to do is show existence. The idea is that the right hand side of 5.0.2 is invariant under translation by elements of $F[p]$, and thus must factor though ϕ.

We proceed by recursion. let g_0 be the right hand side of 5.0.2. Then for any $v \in F[p]$, $g_0(t +_F v) = g_0$ so

$$g_0(t) - g_0(0) = g_1(t)\phi(t)$$

by the Weierstrass preparation theorem.

Now $g_1(t)$ is also $F[p]$ translation invariant, and so we can repeat the division step:

$$g_1(t) - g_1(0) = g_2(t)\phi(t).$$

Continuing in this way we produce a sequence of power series, the $g_i(t)$'s. Substituting for the formula for $g_1(t)$ shows

$$g_0(t) - g_0(0) = g_1(0)\phi(t) + g_2(t)\phi(t)^2.$$

Repeating this process produces

$$g_0(t) = g_0(0) + g_1(0)\phi(t) + g_2(0)\phi(t)^2 + \ldots$$

This infinite series converges in the (π, t) topology of $\mathcal{O}_{\mathbb{C}_p}[[t]]$ since $\phi(t)$ has no linear terms. Thus

$$N_F g(t) = g_0(0)t + g_1(0)t + \ldots$$

$\square$

Definition 5.0.3. *We call N_F the Coleman norm operator. We will often drop the dependence on F.*

5.0.4 Some Examples

The ring $\mathbb{Z}_p$ is contained in the center of the endomorphism ring of the group law F_ϕ. In particular, the element $p \in \mathbb{Z}_p$ corresponds to the special endomorphism ϕ. For more details on this see the very nice article of Lubin and Tate [LT65].

Recall the definition of the a-series from 4.2.1.

Lemma 5.0.5. *If $a \in \mathbb{Z}_p$ and N is the Coleman norm operator attacked to ϕ then the following conditions are equivalent:*

1. $N_F t = t$
2. $N_F[a]_F(t) = [a]_F(t)$

Proof. We have the chain of equalities.

$$Nt \circ [a]_F(t) \circ \phi(t) = Nt \circ \phi(t) \circ [a]_F(t) \tag{5.0.6}$$

$$= \prod_{v \in F[p]} (t +_F v) \circ [a]_F(t) = \prod_{v \in F[p]} [a]_F(t) +_F v \tag{5.0.7}$$

$$= \prod_{v \in F[p]} [a]_F(t) +_F [a]_F(v) = \prod_{v \in F[p]} [a]_F(t +_F v) \tag{5.0.8}$$

$$= N[a](t) \circ \phi(t). \tag{5.0.9}$$

However, since $\phi(0) = 0$, it is an invertible power series under composition we obtain the identity

$$Nt \circ [a]_F(t) = N[a]_F(t).$$

We can now read off the claim. □

Observe that the Ando condition 1.1.4 is equivalent to the Coleman norm operator fixing the identity power series. It follows that if F satisfies the Ando condition then

$$N_F \langle a \rangle_F(t) = \langle a \rangle_F(t). \tag{5.0.10}$$

We will return to this point in the next section.

5.1 Obtaining Integral Coefficients

We saw in Proposition 4.2.2 that we could reproduce sequences of the form in Theorem 1.2.2 using an invariant derivation. The techniques in Corollary 4.1.9 can produce a measure on $\mathbb{Z}_p$ with the sequence

$$-\frac{B_k}{k}(1 - a^k)(1 - p^{k-1})$$

as its moments if

$$\frac{1}{p} \log \frac{\langle a \rangle_F^p(t)}{\langle a \rangle_F \circ \phi(t)}$$

has integral coefficients.

Suppose $g(t)$ is an invertible element of $\mathbb{Z}_p[[t]]$ and $\phi(t)$ is a lift of Frobenius. There is a chain of congruences

$$g(t)^p \equiv g(t^p) \equiv g \circ t^p \equiv g \circ \phi \mod p.$$

This implies

$$\frac{g(t)^p}{g \circ \phi(t)} \equiv 1 \mod p$$

and so the logarithm of such a gadget is congruent to zero modulo p.

However, we are interested in producing measures on the units. In light of Katz' Theorem 6.0.4 and the desire to involve the Coleman norm operator the following definition is useful for our ends. Fix a ϕ and let F be the associated Lubin-Tate group law.

Definition 5.1.1. *If $g(t) \in \mathbb{Z}_p[[t]]^\times$ let*

$$\widetilde{\log} g(t) = \frac{1}{p} \log\left(\frac{(g(t))^p}{N_F g \circ \phi(t)}\right)$$

Power series of this form will always produce measures on the units, as we will see in the next chapter. We can now show integrality of the power series $\widetilde{\log} g(t)$ under the right conditions.

Lemma 5.1.2. *Suppose $g(t) \in \mathbb{Z}_p[[t]]^\times$ and $N_F g(t) = g(t)$. Then*

$$\widetilde{\log} g(t)$$

is an element of $\mathbb{Z}_p[[t]]$.

Proof. By definition

$$\phi(t) \equiv t^p \mod p$$

and so

$$N_F g \circ \phi(t) \equiv N_F g \circ t^p \equiv g \circ t^p \equiv g(t)^p \mod p$$

Since $g(t)$ is invertible we have

$$\frac{(g(t))^p}{N_F g \circ \phi(t)} \equiv 1 \mod p$$

Taking the logarithm of both sides shows that p times $\widetilde{\log} g(t)$ is congruent to zero module p which proves the lemma. $\square$

Combining Lemma 5.1.2, Corollary 4.1.9, and 5.0.10 we have

Corollary 5.1.3. *Suppose F satisfies the Ando condition 1.1.4. If $a \in \mathbb{Z}_p^\times$ then there exists a $\mu \in M(Z_p)$ such that*

$$\int_{\mathbb{Z}_p} x^k d\mu(x) = -\frac{B_k(F)}{k}(1 - a^k)\left(1 - p^{k-1}\right)$$

for $k \geq 1$.

This is not quite enough. Theorem 1.2.2 concerns measures on the units, $\mathbb{Z}_p^\times$. In the next chapter we will compare $M(\mathbb{Z}_p)$ and $M(\mathbb{Z}_p^\times)$

Chapter 6

Katz' Theorem

We are interested in the moments of measures on $\mathbb{Z}_p^\times$ and not on $\mathbb{Z}_p$. The next theorem will help to determine when we can produce the former from the latter.

There is an obvious homomorphism

$$\text{res} : M(\mathbb{Z}_p^\times) \to M(\mathbb{Z}_p) \tag{6.0.1}$$

which restricts a continuous map

$$f : \mathbb{Z}_p \to \mathbb{Z}_p$$

to $\mathbb{Z}_p^\times$ and then integrates. We are interested in the image of this homomorphism.

Let's study this map in a bit more detail. As described in Chapter 2 an element of $M(\mathbb{Z}_p)$ is completely determined by its moments. The same is true for $M(\mathbb{Z}_p^\times)$. The set of polynomials $h(x) \in \mathbb{Q}_p[x]$ such that $h(0) = 0$ and $h(a) \in \mathbb{Z}_p$ whenever $a \in \mathbb{Z}_p^\times$ is dense in the set $\text{Cont}(\mathbb{Z}_p^\times, \mathbb{Z}_p)$. Thus any element of $M(\mathbb{Z}_p^\times)$ is also determined by its sequence of moments for $k \geq 0$. It follows that the restriction homomorphism is injective.

Observe that if μ is in the image of the restriction map with moments m_i then its preimage has the same sequence of moments for $i \geq 0$.

Definition 6.0.2. *A measure μ is said to be supported on the units if it is in the image of the map in 6.0.1.*

This definition is equivalent to the following statement. A measure μ is supported on the units if for any continuous function $f : \mathbb{Z}_p \to \mathbb{Z}_p$ that vanishes on all of $\mathbb{Z}_p^\times$ we have

$$\int_{\mathbb{Z}_p} f d\mu = 0.$$

This condition can be a bit more handy for determining whether or not something is in the image of the restriction map.

The following is analogous to the Coleman norm operator.

Definition 6.0.3. *For a formal group law $F = F_\phi$ define*

$$T_F : \mathbb{Z}_p[[t]] \to \mathbb{Z}_p[[t]]$$

by

$$T_F f(t) = \sum_{c \in F[p]} f(t +_F c).$$

Observe that T_F is an additive function. For any power series $f(t)$, $T_F f(t)$ is fixed by the Galois group of the field $Q_p(F[p])$ and so has integral coefficients.

The following can be found in [Kat77] without proof. We will return to the proof later.

Theorem 6.0.4 (Katz). *Under the identification in Corollary 4.1.9, a measure μ is supported on $\mathbb{Z}_p^\times$ if and only if the corresponding power series $f_\mu(t) \in \mathbb{Z}_p[[t]]$ satisfies*

$$T_F f_\mu(t) = 0.$$

The idea of the theorem and the proof concerns Dirac measures at powers of p. The power series

$$T_F f(t)$$

turns out to be a power series in the Dirac measure at p. We will see later on that this theorem is equivalent to one of Madsen, Snaith and Tornehave describing the deloopings of stable self maps of p-adic K-theory.

For now let us finish the construction of our measure on the units.

Example 6.0.5. Recall the definition of $\widetilde{\log}$ 5.1.1 Suppose $g(t) \in \mathbb{Z}_p[[t]]$ and $g(0) \in \mathbb{Z}_p^\times$. Using the definition we have

$$\sum_{c \in F[p]} \widetilde{\log} g(t +_F c) = \frac{1}{p} \sum_{c \in F[p]} \log \frac{g^p(t +_F c)}{g \circ \phi(t +_F c)} = \frac{1}{p} \log \left(\prod_{c \in F[p]} \frac{g^p(t +_F c)}{g \circ \phi(t +_F c)} \right)$$

Since $N_\phi g \circ \phi = g \circ \phi$ substituting produces

$$\frac{1}{p}\log\left(\prod_{c\in F[p]} \frac{g^p(t+_F c)}{\prod\limits_{d\in F[p]} g(t+_F d)\circ(t+_F c)}\right) = \frac{1}{p}\log\left(\prod_{c\in F[p]} \frac{g^p(t+_F c)}{\prod\limits_{d\in F[p]} g(t+_F d +_F c)}\right) \tag{6.0.6}$$

$$= \frac{1}{p}\log\left(\frac{\prod\limits_{c\in F[p]} g^p(t+_F c)}{\prod\limits_{c\in F[p]}\prod\limits_{d\in F[p]} g^p(t+_F c)}\right) \tag{6.0.7}$$

$$\frac{1}{p}\log\left(\frac{\prod\limits_{c\in F[p]} g^p(t+_F c)}{\prod\limits_{c\in F[p]} g^p(t+_F c)}\right) \tag{6.0.8}$$

which vanishes.

Thus whenever $\widetilde{\log}f(y)$ has integral coefficients, its D_F^k's are the moments of a measure on $\mathbb{Z}_p^\times$ by Katz' theorem.

Theorem 6.0.9. *Suppose F satisfies the Ando condition. Then for any $a \in \mathbb{Z}_p^\times$ there exists a $\mu \in M(\mathbb{Z}_p^\times)$ such that*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu(x) = -\frac{B_k(F)}{k}(1-a^k)\left(1-p^{k-1}\right)$$

for $k > 0$.

Proof. This follows easily 5.1.3, 6.0.4 and the previous example. □

There is a classical example of our main theorem. Recall that B_k is the k-th Bernoulli number defined by

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}$$

While investing the existence of a p-adic continuation of the Riemann zeta function, the following powerful theorem arises. It can be found in many places in the literature, including [Kob77] and [Kat75] and in some form [KL64].

Corollary 6.0.10 (Mazur). *If $a \in \mathbb{Z}_p^\times - \{1\}$ then there is a measure μ such that*

$$\int_{\mathbb{Z}_p^\times} x^k d\mu = -\frac{B_k}{k}\left(1-p^{k-1}\right)\left(1-a^k\right)$$

for all $k \geq 1$.

Proof. This is the situation when F is the formal group law

$$x + y - xy.$$

One can easily check that the Ando condition is satisfied for this choice of formal group law. It is the Lubin-Tate group for the lift $\phi(t) = 1 - (1 - t)^p$. □

As we will see later, the existence of such a measure shows that a refinement of the Todd genus

$$MU \to K_p$$

to MSU is an E_∞ map.

6.1 Integrals as Riemann Sums

In order to prove Katz' theorem without topology, it is useful to describe integration in terms of limits of Riemann sums. The computation will turn out to be useful when we consider topology.

6.1.1 A Digression to Complete Group Rings

Many of the results in the section can be found in the section of [Was97] on the Iwasawa algebra.

Definition 6.1.2. *Suppose T is a topological group. An element γ is a topological generator for T if the closure of the group generated by γ is all of T*

For example, 1 is a topological generator for $\mathbb{Z}_p$, since the integers are dense in the p-adic.

Let Γ be a profinte group that is isomorphic to the additive group $\mathbb{Z}_p$. Let Γ_n be the subgroup with index p^n in Γ. For example, $p^n\mathbb{Z}_p$ so that $\Gamma/\Gamma_n \cong \mathbb{Z}/p^n$, the cyclic group of order p^n. Let γ be a topological generator of the group Γ.

Definition 6.1.3. *The completed group ring, $\mathbb{Z}_p[[\Gamma]]$, is defined as the limit*

$$\varprojlim \mathbb{Z}_p[\Gamma/\Gamma_n].$$

This object will turn out to be useful in studying $M(\mathbb{Z}_p)$.

Proposition 6.1.4. *There is an isomorphism*

$$\mathbb{Z}_p[[t]] \cong Z_p[[\Gamma]]$$

This object is know as the Iwasawa algebra. The map inducing the isomorphism sends the topological generator γ to $1 + t \in \mathbb{Z}_p[[t]]$.

Proof. See Theorem 7.1 of [Was97]. □

6.1.5 Measures on Γ/Γ_n

Let Maps$(\Gamma/\Gamma_n, \mathbb{Z}_p)$ denote the set of functions from Γ/Γ_n to $\mathbb{Z}_p$. We agree that

$$M(\Gamma/\Gamma_n)$$

is the $\mathbb{Z}_p$ module of $\mathbb{Z}_p$ linear maps from Maps$(\Gamma/\Gamma_n, \mathbb{Z}_p)$ to $\mathbb{Z}_p$. The goal of this section is to describe any measure $\mu \in M(\Gamma)$ as a limit of $M(\Gamma/\Gamma_n)$. This description will help to produce an analog of the Reisz representation theorem. In other words, a linear functional will lead to a definition of an integral using the standard Riemann sums. This is already worked out in [Kob77] but we need to show the equivalence of these two definitions.

Our first step is to describe the measures on Γ/Γ_n. Recall that this group is isomorphic to $\mathbb{Z}/p^n$.

Proposition 6.1.6. *There is an abelian group isomorphism* $M(\Gamma/\Gamma_n) \cong \mathbb{Z}_p[\Gamma/\Gamma_n]$.

Proof. This is straight forward. We have a good basis for functions whose source is Γ/Γ_n, just the characteristic functions. Thus, define

$$\chi_{\gamma^i}(\gamma^j) = \delta_{i,j}$$

i.e. it's 1 if $i = j$ and zero otherwise. Then all we do is send a measure to the element in the group ring of the form

$$\sum_{i=0}^{p^n-1} \left(\int_{\mathbb{Z}/p^k} \chi_i(\alpha) d\mu(\alpha) \right) \gamma^i.$$

In this expression, we must view γ modulo Γ_n.

The map just described is clearly additive and a bijection, hence the claim is proved. □

6.1.7 Density of Dirac Measures

The ring of $\mathbb{Z}_p$ valued measures can be constructed as the limit of measures on Γ/Γ_n. Explicitly we have

Lemma 6.1.8. *There is an isomorphism*

$$M(\Gamma) \cong \varprojlim \mathbb{Z}_p[\mathbb{Z}/p^k] \tag{6.1.9}$$

Proof. Observe

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Cont}(\Gamma, \mathbb{Z}_p), \mathbb{Z}_p\right) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Cont}(\varprojlim \Gamma/\Gamma_n, \mathbb{Z}_p), \mathbb{Z}_p\right) \cong \\ \mathrm{Hom}_{\mathbb{Z}_p}\left(\varinjlim \mathrm{Maps}(\Gamma/\Gamma_n, \mathbb{Z}_p), \mathbb{Z}_p\right) \cong \varprojlim \mathrm{Hom}_{\mathbb{Z}_p}\left(\mathrm{Maps}(\Gamma/\Gamma_n, \mathbb{Z}_p), \mathbb{Z}_p\right) \cong \\ \varprojlim M(\Gamma/\Gamma_n). \end{aligned}$$

□

We specialize to the case $\Gamma = \mathbb{Z}_p$. The goal of the next few lemmas is to show that to compute the integral over $\mathbb{Z}_p$, we can instead compute the limit of the integrals over $\mathbb{Z}/p^k$.

Suppose μ is a measure on all of $\mathbb{Z}_p$. If $g : \mathbb{Z}/p^k \to \mathbb{Z}_p$ is a function then the composite

$$\mathbb{Z}_p \to \mathbb{Z}/p^k \xrightarrow{g} \mathbb{Z}_p$$

where the first map is the quotient map is continuous. We can than integrate this function against μ.

We have constructed a linear functional $\bar{\mu} \in M(\mathbb{Z}/p^k)$.

Proposition 6.1.10. *Suppose μ is in $M(\mathbb{Z}_p)$. We can represent it uniquely by the sequence*

$$\left\{ s_k = \sum_{i=0}^{p^k-1} a_i(k) d\psi_i \right\}$$

using the isomorphism in 6.1.9. With this notation, if $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is any continuous function then

$$\int_{\mathbb{Z}_p} f(t) d\mu(t) = \lim_{k \to \infty} \sum_{i=0}^{p^k-1} a_i(k) f([\gamma]^i)$$

Proof. First recall that the $d\psi_i$ in the formula above are measures on the group $\mathbb{Z}/p^k$, thus the i means i mod $\mathbb{Z}/p^k$.

We begin by showing that the limit converges. To see this recall that the $a_i(k)$ and $a_j(k+1)$ are related in some way. Explicitly we have the formula

$$a_i(k) = \sum_{l=0}^{p-1} a_{i+lp^k}(k+1).$$

If we rewrite the k-th term in the limit we get

$$\sum_{i=0}^{p^k-1} \sum_{l=0}^{p-1} a_{i+lp^k}(k+1) f(i) \tag{6.1.11}$$

We need to compare this sum with

$$\sum_{j=0}^{p^{k+1}-1} a_j(k+1) f(j). \tag{6.1.12}$$

We can rewrite 6.1.12 as

$$\sum_{i=0}^{p^k-1} \sum_{l=0}^{p-1} a_{i+lp^k}(k+1) f(i+lp^k).$$

There are the same number of terms in this new expression and in 6.1.11. Comparing the coefficient of $a_{i+lp^k}(k+1)$ shows that we need to compare $f(i)$ and $f(i+lp^k)$. Since f was uniformly continuous ($\mathbb{Z}_p$ is compact) these two function values are uniformly bounded, thus the difference of the sums 6.1.11 and 6.1.12 is small uniformly and so the limit must exist.

To finish up we need to show that this limit actually computes the value of the linear functional μ. To this end, we investigate the map, call it ρ, that sends the sequence

$$\left\{ s_k = \sum_{i=0}^{p^k-1} a_i(k)(t+1)^i \right\}$$

associated to the measure μ to the linear functional μ' defined by

$$\int_{\mathbb{Z}_p} f(t) d\mu' = \lim_{k\to\infty} \sum_{i=0}^{p^k-1} a_i(k) f(i).$$

We will show that ρ is the identity.

The function ρ is a $\mathbb{Z}_p$ module map. In addition, it is the identity on the Dirac measures. The restriction of ρ to $\mathbb{Z}/p^k$ is also the identity since everything can be computed via the Dirac measures there. Thus the diagram

$$\begin{tikzcd} M(\mathbb{Z}/p^{k+1}) \arrow[r, "\rho"] \arrow[d] & M(\mathbb{Z}/p^{k+1}) \arrow[d] \\ M(\mathbb{Z}/p^{k}) \arrow[r, "\rho"] & M(\mathbb{Z}/p^{k}) \end{tikzcd}$$

commutes and has identity morphisms for horizontal arrows. It follows that

$$\rho : M(\mathbb{Z}_p) \to M(\mathbb{Z}_p)$$

is the identity. □

6.1.13 A Special Case of Katz' Theorem

In this section we will verify Katz' theorem in the case $F = \hat{\mathbb{G}}_m$. We apply the technical results of the previous sections in order to describe the image of the the map in 6.0.1.

To help the flow of the proof of the main theorem, we need to go over a quick calculation. Let $G = \hat{\mathbb{G}}_m$. The power series associated to the Dirac measure ψ^l where l is an integer is given by $(1+x)^l$. We are going to investigate how the restriction map effects this measure. Suppose l is prime to p, then ψ^l is clearly supported on the units. We will write ζ for a primitive p root of unity over $\mathbb{Z}_p$ and $\zeta_j = \zeta^j - 1$ For the power series computation we compute

$$\sum_{j=0}^{p-1}(1+(x+_G\zeta_j))^l = \sum_{j=0}^{p-1}(1+x+\zeta^j-1+x\zeta^j-x)^j \tag{6.1.14}$$

$$= \sum_{j=0}^{p-1}(\zeta^j(x+1))^l \tag{6.1.15}$$

$$= (x+1)^l\sum_{j=0}^{p-1}\zeta^{jl} \tag{6.1.16}$$

$$= 0. \tag{6.1.17}$$

We end up with zero since p is prime to l and we are summing of the p-th roots of unity.

Theorem 6.1.18 (Katz). *A measure μ is supported on $\mathbb{Z}_p^\times$ if and only if the corresponding power series $f_\mu(t) \in \mathbb{Z}_p[[t]]$ satisfies*

$$\sum_{j=0}^{p-1} f_\mu(x+_G\zeta_j) = 0$$

where G is the formal group law $x+y+xy$.

In this statement it is important to remember that the power series $f(t)$ is produced using the operator $D = (1+t)\frac{d}{dt}$.

Proof. First, suppose we have a measure μ that is supported on the units and the corresponding power series

$$f_\mu(t) = \sum_i \left(\int_{\mathbb{Z}_p} \binom{x}{i} d\mu(x) \right) t^i.$$

The coefficient of t^l of the trace of $f_m u(t)$ is

$$\int_{\mathbb{Z}_p} \sum_{j=0}^{p-1} \zeta^{jx} \binom{x}{l} d\mu(x).$$

However, the function inside the integral side vanishes on the units and so by hypothesis the integral must be zero.

Now suppose the trace of $f_\mu(t)$ is zero. This time we know

$$\int_{\mathbb{Z}_p} \sum_{j=0}^{p-1} \zeta^{jx} \binom{x}{l} d\mu(x) = 0 \tag{6.1.19}$$

since it computes the coefficient of t^l in the trace of $f_\mu(t)$.

Now, using our Riemann sum representation we can compute 6.1.19 as

$$\lim_k \sum_{i=0}^{p^k-1} a_i(k) \sum_{j=0}^{p-1} \zeta^{ji} \binom{i}{l} = 0.$$

We can rewrite this expression as

$$\lim_k \sum_{\substack{i=0 \\ (i,p)\neq 1}}^{p^k-1} a_i(k) \binom{i}{l} = 0. \tag{6.1.20}$$

We've dropped the factor of p since $\mathbb{Z}_p$ is torsion free.

Now suppose $f : \mathbb{Z}_p \to \mathbb{Z}_p$ is a continuous function and

$$f(\mathbb{Z}_p^\times) = 0.$$

By Mahler's theorem 2.1.3 we can represent f as

$$f(x) = \sum_l b_l \binom{x}{l}.$$

To compute its integral we need to calculate

$$\lim_k \sum_{i=0}^{p^k-1} a_i(k) f(i) = \tag{6.1.21}$$

$$\lim_k \sum_{i=0}^{p^k-1} a_i(k) \sum_{l=0} b_l \binom{i}{l} = \tag{6.1.22}$$

$$\sum_{l=0} b_l \lim_k \sum_{i=0}^{p^k-1} a_i(k) \binom{i}{l} = \tag{6.1.23}$$

$$\sum_{l=0} b_l \lim_k \sum_{\substack{i=0 \\ (i,p)\neq 1}}^{p^k-1} a_i(k) \binom{i}{l} = \tag{6.1.24}$$

$$0. \tag{6.1.25}$$

The second equality follows since $\mathbb{Z}_p$ is compact. In other words, by uniform continuity

$$\int_{\mathbb{Z}_p} \sum_n b_n \binom{x}{n} d\mu = \sum_n \int_{\mathbb{Z}_p} b_n \binom{x}{n} d\mu.$$

The third equality follows from the hypothesis $f(\mathbb{Z}_p^\times) = 0$. The last equality follows from the calculation in 6.1.20 $\square$

6.2 Proof of the General Case

Let ϕ be a lift of the Frobenius over $\mathbb{F}_p$ to $\mathbb{Z}_p$ and $F = F_\phi$ the associated Lubin-Tate formal group law over $\mathbb{Z}_p$. Recall the strict isomorphism

$$\Theta : \hat{G}_m \to F.$$

Theorem 6.2.1. *Under the identification*

$$\mathcal{D}_F : M(\mathbb{Z}_p) \to \mathbb{Z}_p[[t]]$$

a measure μ is supported on the units if and only if

$$T_F f_\mu(t) = 0.$$

Proof. The map

$$\Theta : \mathbb{Z}_p[[s]] \to \mathbb{Z}_p[[t]]$$

that sends s to $\Theta(t)$ is an isomorphism since $\Theta(t)$ is invertible under composition of power series. Recall that $\Theta'(0) = 1$.

Suppose $f(t)$ is such that $T_F f(t) = 0$. Then under the change of variables $t \mapsto \Theta(s)$, this quantity maps to $T_G f(\Theta(s)) = 0$. We need to observe here that the set of $\Theta(v)$ for $v \in F[p]$ is the same as the set of $u \in \hat{\mathbb{G}}_m[p]$ since Θ is an isomorphism.

The measure associated to $f(t)$ via the isomorphism $\mathcal{D}_F$ is the same as the measure attached to $f(\Theta(s))$; they have the same moments. The claim now follows from our proof of Katz' theorem in the multiplicative group law case; Theorem 6.1.18. One can use a similar argument to prove the "only if" part of the theorem.

$\square$

Chapter 7

The Effect of a Measure on $\mathbb{CP}^\infty$

One way to prove Katz' theorem is to understand the relationship between measures on $\mathbb{Z}_p$ and unstable operations in K-Theory. Fix an odd prime p. Let K_p be completed periodic complex K-Theory. The first observation is

Proposition 7.0.1. *There is an isomorphism*

$$K_p^0 K_p \cong M(\mathbb{Z}_p^\times).$$

such that if $\mu \in M(\mathbb{Z}_p^\times, \mathbb{Z}_p)$ is considered as an operation in K_p than

$$\mu_{2k} : \pi_{2k} K_p \to \pi_{2k} K_p$$

is multiplication by

$$\int_{\mathbb{Z}_p^\times} x^k d\mu(x)$$

for all $k \in \mathbb{Z}$.

Proof. This follows from the computation of Adams, Harris and Switzer [AHS71]. A more general result can be found in [Hov04]. □

It follows from the proposition that the Dirac measure at $l \in \mathbb{Z}_p^\times$ maps to the Adams operation ψ^l.

In the next few sections We are going to describe the effect of the operation/measure on the tautological line bundle over $\mathbb{CP}^\infty$.

7.1 Collection of Statements

The results in the section can be found in section two of [MST77]. Recall that for p-adic K-Theory, the cohomology of $\mathbb{CP}^\infty$ can be computed via the cohomology of the cyclic groups of p-powers. Explicitly we have

Lemma 7.1.1. *There is an isomorphism*

$$K_p^0\mathbb{CP}^\infty \cong \varprojlim K_p^0\left(B\mathbb{Z}/p^i\right). \tag{7.1.2}$$

We produced a similar statement for R-valued measures on $\mathrm{Cont}(\mathbb{Z}_p, \mathbb{Z}_p)$. Lemma 6.1.9 provides us with an isomorphism

$$M(\mathbb{Z}_p) \cong \varprojlim M(\mathbb{Z}/p^i, R) \tag{7.1.3}$$

The following notation can be found in [MST77]

Definition 7.1.4. *Let $\hat{A}_p$ be the set of H-maps inside the set of all continuous maps $[\Omega^\infty K_p, \Omega^\infty K_p]$.*

Notice that $\hat{A}_p$ is just the set of additive unstable operations for p-adic K-Theory. We now quote a lemma of Adams. It is Lemma 6 in [Ada69]

Lemma 7.1.5 (Adams). *There is an isomorphism*

$$\alpha : \hat{A}_p \cong K_p\mathbb{CP}^\infty$$

which sends an operation μ to its effect on the tautological line bundle $\mathbb{L}$.

The above lemma together with 7.1.2 allows us to write any element of $\hat{A}_p$ uniquely as a sequence

$$\left\{ s_k = \sum_{i=0}^{p^k-1} a_i(k)\psi^i \right\}$$

where ψ^i is the Adams operation at i and $a_i(k) \in \mathbb{Z}_p$.

Our goal is the following

Theorem 7.1.6. *Suppose a coordinate has been chosen for K_p so that we have a given isomorphism*

$$K_p^0\mathbb{CP}^\infty \cong \mathbb{Z}_p[[t]].$$

Let F be the induced formal group law over $\mathbb{Z}_p$. Under these conditions, the diagram

$$\begin{tikzcd} M(\mathbb{Z}_p^\times) \arrow[r, "\mathrm{res}"] \arrow[d] & M(\mathbb{Z}_p) \arrow[r, "\mathcal{D}_F"] & \mathbb{Z}_p[[t]] \\ K_p^0 K_p \arrow[r, "\Omega^\infty"] & \hat{A}_p \arrow[r] & K_p^0 \mathbb{CP}^\infty \arrow[u] \end{tikzcd} \tag{7.1.7}$$

commutes.

In other words, the effect of an operation on the tautological line bundle is given by the power series corresponding to the measure under the map $\mathcal{D}_F$.

7.2 Comparing Measures and K-Theory

We will break down the proof of Theorem 7.1.6 into a few steps. The first is to study the integration theory together with the profinite group structure of $\mathbb{Z}_p$ in a bit more detail.

7.2.1 Measures on $\mathbb{Z}/p^k$

The first step is to return to our study measures on $\mathbb{Z}/p^k$. We have seen in Proposition 6.1.6 that

$$M(\mathbb{Z}/p^k) \cong \mathbb{Z}_p[\mathbb{Z}/p^k] \cong \mathbb{Z}_p[\gamma]/\gamma^{p^k} - 1.$$

Where γ is the reduction to $\mathbb{Z}/p^k$ of the a topological generator for $\mathbb{Z}_p$.

If χ_i is the characteristic function at i then the map sends

$$\mu \mapsto \sum_{i=o}^{p^k-1} \left(\int_{\mathbb{Z}/p^k} \chi_i(x) d\mu(x) \right) \gamma^i$$

There is an inverse system involving these gadgets. Hitting the sequence

$$\ldots \to \mathbb{Z}/p^k \to \mathbb{Z}/p^{k-1} \to \ldots$$

with the covariant functor $M(-)$ we end up with the inverse system

$$\ldots \to M(\mathbb{Z}/p^k) \to M(\mathbb{Z}/p^{k-1}) \to \ldots .$$

Each map sends a measure μ the measure $\bar{\mu}$ defined by

$$\int_{\mathbb{Z}/p^{k-1}} f d\bar{\mu} = \int_{\mathbb{Z}/p^k} \bar{f} d\mu.$$

Here $\bar{f}$ lifts f in the sense that

$$\begin{tikzcd} \mathbb{Z}/p^{k-1} \arrow[r, "f"] & \mathbb{Z}_p \\ \mathbb{Z}/p^k \arrow[u] \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

commutes. For example a Dirac measure, $d\psi_a$ is mapped to $d\psi_{\bar{a}}$ where $\bar{a}$ is the class represented by the reduction of a modulo p^{k-1}.

In general we have

$$\sum_{i=0}^{p^k-1} a_i(k) d\psi_i \mapsto \sum_{i=0}^{p^k-1} a_i(k) d\psi_{\bar{i}}.$$

However, each class modulo p^{k-1} gets hit precisely p times. Collecting terms, we can rewrite this sum as

$$\sum_{j=0}^{p^{k-1}-1} \sum_{l=0}^{p-1} a_{i+lp^{k-1}}(k) d\psi_j.$$

In other words, we have the formula

$$a_i(k-1) = \sum_{l=0}^{p-1} a_{i+lp^{k-1}}(k).$$

7.2.2 The K-Theory of $\mathbb{CP}^\infty$

The next step is to expand on the description of the p-adic K-Theory of $\mathbb{CP}^\infty$. It is the inverse limit of the p-adic K-Theory of the cyclic groups p-th powers as mentioned above. The goal is to compare the inverse limit description of the K-Theory of $\mathbb{CP}^\infty$ with the limit description of measures on $\mathbb{Z}_p$.

If we consider the direct system

$$....\mathbb{Z}/p^{k-1} \xrightarrow{\cdot p} \mathbb{Z}/p^k \to ...$$

and apply K-Theory we can produce an inverse system

$$....KB(\mathbb{Z}/p^k) \to KB(\mathbb{Z}/p^{k-1}) \to ...$$

It is easier to describe the maps in the system of we use Atiyah's calculation to replace these gadgets with the representation rings

$$...R(\mathbb{Z}/p^k) \otimes \mathbb{Z}_p \to R(\mathbb{Z}/p^{k-1}) \otimes \mathbb{Z}_p \to ...$$

These are easily computed as

$$...\mathbb{Z}_p[s]/(s^{p^k} - 1) \to \mathbb{Z}_p[s]/(s^{p^{k-1}} - 1) \to ...$$

Here s is the one dimensional irreducible representation

$$\mathbb{Z}/p^k \to GL_1(\mathbb{C}) \cong \mathbb{C}^\times \cong S^1$$

that sends the generator γ to the appropriate primitive p-th root of unity. These one dimensional representations are compatible under the structure maps. Thus we have no notation to describe the source of the representation.

We can now describe the maps involved. It is very similar to what we saw in the measure theory situation involving the same groups. The map induced on the representation rings by multiplication by p sends the generator s to s^p. Thus we have for a general element

$$\sum_{i=0}^{p^k-1} a_i(k)s^i \to \sum_{i=0}^{p^k-1} a_i(k)s^{ip}.$$

Collecting terms once more, we can rewrite the image as

$$\sum_{j=0}^{p^{k-1}-1} \sum_{l=0}^{p-1} a_{j+lp^{k-1}} s^{jp}.$$

We can connect this to unstable operations via an isomorphism

$$\hat{A}_p \cong \lim_k K_p^0 B(\mathbb{Z}/p^k) \cong \lim_k R(B\mathbb{Z}/p^k) \otimes \mathbb{Z}_p$$

such that any operation corresponds to a sequence

$$\left\{ s_k = \sum_{i=0}^{p^k-1} a_i(k)s^i \right\}$$

where ψ^i matches with

$$s^i$$

inside

$$K_p^0 B(\mathbb{Z}/p^k) \cong \mathbb{Z}_p[[s]]/(s^{p^k} - 1).$$

Lemma 7.2.3. *The diagram*

$$\begin{tikzcd}
M(\mathbb{Z}/p^k) \arrow[r] \arrow[d] & M(\mathbb{Z}/p^{k-1}) \arrow[d] \\
\mathbb{Z}_p[t]/(s^{p^k} - 1) \arrow[r] & \mathbb{Z}_p[t]/(s^{p^{k-1}} - 1)
\end{tikzcd}$$

where the vertical maps send $d\psi_i$ to s^i commutes.

Proof. Obvious. □

Indeed, the vertical arrows are isomorphisms and so the limits are isomorphic:

$$\lim_k M(\mathbb{Z}/p^k, \mathbb{Z}_p) \cong \lim_k \mathbb{Z}_p[s]/(s^{p^k} - 1).$$

The right hand side is known to compute the Iwasawa algebra $\mathbb{Z}_p[[\mathbb{Z}_p]]$. It is isomorphic to the power series ring $\mathbb{Z}_p[[t]]$ where the maps sends s in each $\mathbb{Z}_p[s]/s^{p^k} - 1$ to $1 + t$.

We are now in the situation to compute over $\mathbb{Z}_p$. Suppose $\mu \in M(\mathbb{Z}_p^\times)$. Let $\bar{\mu}$ be the corresponding element of $M(\mathbb{Z}/p^k)$. It is represented by a formal sum

$$\sum_{i=0}^{p^k-1} a_i(k) d\psi_i.$$

Likewise, if we consider μ as an element of $\hat{A}_p$ and then consider its image inside $K_p(B\mathbb{Z}/p^k)$ it is described by

$$\mu(s) = \sum_{i=0}^{p^k-1} b_i(k) s^i.$$

We have described the map between these two gadgets. It sends s^i to $d\psi_i$. It remains to check if

$$a_i(k) = b_i(k)$$

for each i if we consider μ as a measure on $\mathbb{Z}_p^\times$ or as a stable operation.

It's clear that if μ is a sum of Adams operations, these two coefficients agree since they are both counting the number of copies of an Adams operation in the original operation. This remains true when passing to limits.

The main result of this discussion is

Proposition 7.2.4. *The diagram*

$$\begin{tikzcd} M(\mathbb{Z}_p^\times) \arrow[r] \arrow[d] & \lim\limits_k M(\mathbb{Z}/p^k, \mathbb{Z}_p) \arrow[d] \\ K_p^0 K_p \arrow[r] & \lim\limits_k K_p^0(B\mathbb{Z}/p^k) \end{tikzcd}$$

commutes.

7.3 Proof of Theorem 7.1.6

In order to prove 7.1.6 we will first verify the theorem in the case when $F = \hat{\mathbb{G}}_m$. We will then use formal techniques to produce the result for any other coordinate for K-Theory.

We need to study the diagram

$$\begin{tikzcd} M(\mathbb{Z}_p^\times) \arrow[r] \arrow[d] & \lim\limits_k M(\mathbb{Z}/p^k) \arrow[d] & M(\mathbb{Z}_p) \arrow[l] \arrow[r] & \mathbb{Z}_p[[t]] \\ K_p^0 K_p \arrow[r] & \lim\limits_k K_p^0(B/\mathbb{Z}/p^k) & \hat{A}_p \arrow[l] \arrow[r] & K(\mathbb{CP}^\infty) \arrow[u] \end{tikzcd} \tag{7.3.1}$$

Proposition 7.2.4 shows that the left hand rectangle commutes. We will compute both directions around the diagram.

The top horizontal arrow is computed via

$$\mu \mapsto \sum_{j=0}^{\infty} \left(\int\limits_{\mathbb{Z}_p} \binom{x}{j} d\mu \right) t^j \tag{7.3.2}$$

In other words we are computing $f_\mu(t)$, the power series associated to the measure by Mahler's theorem. Recall we are in the case when $F = \hat{\mathbb{G}}_m$. We can use the Riemann sums version of integration in Proposition 6.1.10 to compute the right hand side of 7.3.2. If μ is represented by the sequence

$$\left\{ s_k = \sum_k a_i(k) d\psi_i \right\}$$

then the j-th coefficient of 7.3.2 can be computed via

$$\lim_k \sum_{i=0}^{p^k-1} a_i(k)\binom{i}{j}.$$

The bottom row is similar. The isomorphisms between $\hat{A}_p$ and $K_p^0(\mathbb{CP}^\infty)$ sends an operation to its effect on the tautological line bundle.

The element $\mu(\mathbb{L})$ inside the K-Theory of $\mathbb{Z}/p^k$ is represented by the element

$$\mu(s) = \sum_{i=0}^{p^k-1} a_i(k)s^i.$$

for the same choice of $a_i(k)$ as above, by Proposition 7.2.4. The sequence

$$s_k = \sum_{i=0}^{p^k-1} a_i(k)s^i$$

is mapped to

$$s_k = \sum_{i=0}^{p^k-1} a_i(k)\psi^i$$

inside $\hat{A}_p$. The isomorphism of Adams sends this sequence to

$$s_k = \sum_{i=0}^{p^k-1} a_i(k)\mathbb{L}^i$$

inside $K_p^0\mathbb{CP}^\infty$. Finally, under the isomorphism of this cohomology group with a power series ring the sequence is sent to

$$s_k = \sum_{i=0}^{p^k-1} a_i(k)(1+t)^i.$$

We can compute the power series representing this sequence in a natural way using the fact that $\mathbb{Z}_p[[t]]$ is complete with respect to the (p,t) topology.

Putting all of this together, the bottom row of our diagram sends μ to

$$\lim_k \sum_{i=0}^{p^k-1} a_i(k)\mathbb{L}^i = \lim_k \sum_{i=0}^{p^k-1} a_i(k)(1+t)^i = \lim_k \sum_{i=0}^{p^k-1} a_i(k)\sum_{l=0}^{i}\binom{i}{l}t^l$$

Pulling off the coefficient of t^j and comparing with the calculation of 7.3.2 we see that we obtain the same element in the power series ring.

7.3.3 Changing Coordinates

We have asked for a specific coordinate to be choose in order to make the calculation of the previous section. To finish the theorem we have to understand how the theory changes as we change the coordinate.

Recall that $\hat{\mathbb{G}}_m$ is the multiplicative formal group law $x+y+xy$. Suppose for the moment that we have another coordinate on the formal multiplicative group and thus a new group law for K-Theory called F.

As before, let

$$\Theta : \hat{\mathbb{G}}_m \to F$$

be a strict isomorphism. We will use t for the coordinate on $\hat{\mathbb{G}}_m$ and s for that of F. Thus, we have the change of variables

$$\Theta(s) = t.$$

Recall the invariant derivation $D = (1+x)\frac{d}{dx}$. Set

$$D_F = \frac{1}{\log_F(y)}\frac{d}{dy} = F_2(y,0)\frac{d}{dy}$$

This is a translation invariant derivation for the formal group law F.

We can now state our main result for this section:

Proposition 7.3.4. *Under the bijection in 4.1.9 the effect of μ on $\mathbb{L}$ is given by $g_\mu(y)$.*

Proof. We write

$$\Theta : \mathbb{Z}_p[[y]] \to \mathbb{Z}_p[[x]]$$

for the map that sends y to $\Theta(x)$. It is classical that the diagram

$$\begin{tikzcd} {[K_p, K_p]} \arrow[r, "F"] \arrow[dr, "G"'] & \mathbb{Z}_p[[y]] \arrow[d, "\Theta"] \\ & \mathbb{Z}_p[[x]] \end{tikzcd}$$

commutes where we've used the formal group law as labels for the isomorphism. If follows that if $\mathbb{L} = (1+x) \in \mathbb{Z}_p[[x]]$ then $\mathbb{L} = 1 + \Theta^{-1}(y) \in \mathbb{Z}_p[[y]]$.

Pushing symbols around produces

$$\int_{\mathbb{Z}_p} \left(1+\Theta^{-1}(y)\right)^x d\mu(x) = f_\mu(\Theta^{-1}(y)) = g_\mu(y)$$

and the claim is verified. $\square$

Putting this result together with Proposition 6.0.9 we have

Corollary 7.3.5. *If $B(F)$ is the stable operation in K_p corresponding to the Mazur measure with moments*

$$-\frac{B_k(F)}{k}(1-a^k)(1-p^{k-1})$$

then

$$B(F) : K_p\mathbb{CP}^\infty \to K_p\mathbb{CP}^\infty$$

sends the tautological line bundle $\mathbb{L}$ over $\mathbb{CP}^\infty$ to

$$\widetilde{\log}\langle a\rangle_F(y) \in \mathbb{Z}_p[[y]] \cong K_p\mathbb{CP}^\infty.$$

7.4 A New Proof of Madsen-Snaith-Tornehave

The Madsen-Snaith-Tornehave theorem describes the image of Ω^∞ inside $\hat{A}_p$ using the idea of transfer. As one can see from the diagram in theorem 7.1.6, describing the image of Ω^∞ is equivalent to describing the image of the restriction map res.

Their theorem says that a sequence of the form

$$s_k = \sum_{i=0}^{p^k-1} a_i(k)\psi^i$$

is in the image of Ω^∞ if and only if whenever p divides i we have

$$a_i(k) = 0.$$

Suppose we have a measure $\mu \in M(\mathbb{Z}_p)$ that is supported on the units. Let $\bar{\mu}$ be its image in $M(\mathbb{Z}/p^k)$. Fix some $c \in \mathbb{N}$ such that $(c,p) \neq 1$. Let us compute the integral of the characteristic function at c, χ_c, as a function on $\mathbb{Z}/p^k$. By our previous descriptions

$$\int_{\mathbb{Z}/p^k} \chi_c d\bar{\mu} = \int_{\mathbb{Z}_p} \tilde{\chi}_c d\mu \tag{7.4.1}$$

where

$$\tilde{\chi}_c : \mathbb{Z}_p \to \mathbb{Z}/p^k \xrightarrow{\chi_c} \mathbb{Z}_p.$$

Now the left hand side is computed via

$$\sum_{i=0}^{p^k-1} a_i(k)\chi_c(i).$$

This expression is equal $a_c(k)$. However, the right hand side of 7.4.1 is zero since $\tilde{\chi}_c$ vanishes on $\mathbb{Z}_p^\times$ and μ is supported on the units.

This argument works for all $k \geq 0$ and all $(c, p) \neq 1$. Hence all of the

$$a_c(k) = 0$$

which is the description of Ω^∞ given by Madsen, Snaith, and Tornehave. Similarly if μ is not in the image of the restriction map, it is not in the image of Ω^∞ by Proposition 7.1.6.

The converse is true as well. Suppose we know the operation μ is in the image of Ω^∞. Then we know each $a_c(k)$ vanishes whenever $(c, p) \neq 1$ by MST. The coefficient of t^l of the corresponding power series is computed by

$$\lim_k \sum_{\substack{i=0 \\ (i,p)\neq 1}} a_i(k)\binom{i}{l}.$$

However, by hypothesis this expression dies for each l. It follows from Proposition 7.1.6 that the measure is in the image of the restriction map.

Chapter 8

Orientations

In this chapter we review the construction of the Hirzebruch series attached to an orientation. It is a power series that describes the relationship between a given coordinate and the additive coordinate in singular cohomology. After this discussion, we can move on to describe the bijection in Theorem 1.2.2.

8.1 Hirzebruch Series

Suppose

$$\alpha : MU \to K_p$$

is a homotopy ring map. Adams showed that such a map determines a complex orientation for K_p and thus a theory of Thom classes for complex vector bundles.

Let X be a connected space and V a virtual complex vector bundle of rank zero over X. Let X^V be the associated Thom space. A Thom isomorphism is an identification

$$K_p^0(X) \to \widetilde{K}_p^0 X^V$$

that makes $\widetilde{K}_p^0 X^V$ into a free module of rank one over $K_p^0 X$. A Thom class is a choice of generator for this free module.

There is a canonical orientation of the rationalization of K_p. If E is any spectrum, let $E \otimes \mathbb{Q}$ be its rationalization, i.e. localization with respect to Morava $K(0)$. Let H represent the integral Eilenberg Maclane spectrum. There is a natural map

$$MU \to H \to H \otimes \mathbb{Q}. \tag{8.1.1}$$

However, by Serre's work $H \otimes \mathbb{Q}$ is weakly equivalent to $S \otimes \mathbb{Q}$.

Since K_p is a ring spectrum there is a map

$$S \to K_p$$

where S is the sphere spectrum. Composing with 8.1.1 we obtain a rational orientation

$$\beta : MU \to H \to H \otimes \mathbb{Q} \to S \otimes Q \to K_p \otimes \mathbb{Q}$$

by rationalizing the unit morphism for the spectrum K_p. Let $\mathbb{L}$ be the tautological line bundle over $\mathbb{CP}^\infty$. Denote the Thom class of $\mathbb{L}$ determined by β $Th_\beta(\mathbb{L})$.

We can compare α and β via their rational Thom classes. The Thom class in rational K-theory of $\mathbb{L}$ determined by the map

$$\alpha : MU \to K_p \to K_p \otimes \mathbb{Q}.$$

will be denoted by $Th_\alpha(\mathbb{L})$. Since $Th_\alpha(\mathbb{L})$ and $Th_\beta(\mathbb{L})$ generate the cohomology of the Thom space over the base there must be an element in the base so that

$$Th_\alpha(\mathbb{L}) = \Delta(\alpha, \beta) Th_\beta(\mathbb{L})$$

Let x be the standard Euler class of the tautological bundle $\mathbb{L}$ in ordinary cohomology.

Definition 8.1.2. *The Hirzebruch series of the orientation α is the class*

$$1 + O(x) \in H^0(\mathbb{CP}^\infty, K_p \otimes \mathbb{Q})^\times.$$

If F is the formal group law induced by an orientation

$$\alpha : MU \to K_p$$

then the difference class can be computed as

$$\Delta(\alpha, \beta) = \frac{x}{\exp_F x}.$$

8.2 The ABGHR Theorem

In this section we describe the identification in Theorem 1.2.2 in more detail.

Suppose

$$\alpha' : MSU \to K_p$$

is the refinement of the homotopy ring map

$$\alpha : MU \to K_p.$$

Suppose α is an E_∞ map, and so corresponds to a point of $\pi_0 E_\infty(MSU, K_p)$. The sequence attached to α' under the bijection of 1.2.2 is given by

$$-\frac{B_k(\alpha_* F_{MU})}{k}$$

for $k \geq 2$. These are the logarithmic Bernoulli numbers of F. In particular, than can be computed from the Hirzebruch series of the orientation α.

The converse is also true. Given any map

$$\alpha : MU \to K_p$$

we can ask if the sequence

$$-\frac{B_k(\alpha_* F_{MU})}{k}$$

satisfies the two conditions in Theorem 1.2.2. If it does, then the refinement of α to MSU is an E_∞ map.

It's now clear how to prove Theorem 1.2.3. Given any H_∞ map

$$\alpha : MU \to K_p$$

let F be the associated formal group law. By hypothesis, F satisfies the Ando condition. Moreover, K_p is the cohomology theory attached to the universal deformation $\hat{\mathbb{G}}_m$. Thus F is of the form F_ϕ for some choice of ϕ.

Remark 8.2.1. We are using the two periodicity of K_p to construct a formal group law of degree zero over $\mathbb{Z}_p$ from the orientation α.

We can now use Theorem 6.0.9 to produce a measure on $\mathbb{Z}_p^\times$ satisfying the conditions of Theorem 1.2.2 where t_k is the sequence of logarithmic Bernoulli numbers attached to the group law $\alpha_* F_{MU}$.

We are not quite done. There is a congruence condition in the description of E_∞ maps. In [Mil82] H. Miller showed that for any two maps

$$\alpha, \beta : MU \to K_p$$

there is a congruence

$$-\frac{B_k(\alpha_* F_{MU})}{k} \equiv -\frac{B_k(\beta_*(F_{MU})}{k} \mod \mathbb{Z}_p.$$

Let α to be the orientation

$$x = 1 - \mathbb{L} \in K_p^0 \mathbb{CP}^\infty.$$

This is also know as the Todd orientation. A straight forward calculation shows

$$B_k(\alpha_* F_{MU}) = B_k$$

in this case. Condition two of Theorem 1.2.2 now follows for any choice of complex orientation by the calculations of H. Miller.

Putting all of this together we obtain

Theorem 8.2.2. *The diagram*

$$\begin{tikzcd}
H_\infty(MU, K_p) \arrow[r, "\mathcal{D}"] \arrow[d] & \pi_0 E_\infty(MSU, K_p) \arrow[dl] \\
H_\infty(MSU, K_p) &
\end{tikzcd}$$

where the two unlabeled arrows are the natural maps, commutes.

In addition, the Todd orientation satisfies the Ando condition and is therefore an H_∞ map. It follows from Theorem 1.2.2 and the existence of the Mazur measure 6.0.10 that the refinement of this map to MSU is an E_∞ map.

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Author's Biography

Barry John Walker was born in New Jersery on 25 December 1978. He grew up with his parents and his siblings. Barry has two loving sisters and three brothers. He stayed in Chesterfield, New Jersery until he went of to Rutgers University in the Fall of 1997. After sucesfulling completing a Bachelors degree in Mathematics and graduatin with honors, Barry decieded to go to graduate school to continue his education. He attend the University of Illinois at Urbana-Champaign as a graduate student in the Fall of 2000. Following the completion of the Ph.D. program in mathematics, Barry will begin working at Northwestern University as a Visiting Assistatn Professor.