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ON THE ADEM RELATIONS

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§1.

THE PURPOSE of this note is to remark that the Adem relations for Steenrod squares ($p = 2$) and reduced powers ($p > 2$) can be given a much simpler formulation than that generally received (see, e.g. [3]) and that this formulation leads to a simple proof.

First, when $p = 2$ let $P(t)$ denote the formal power series

$$P(t) = \sum_{i \geq 0} t^i Sq^i$$

where t is an indeterminate. Then the Adem relations for Steenrod squares are equivalent to the power-series identity

$$P(s^2 + st)P(t^2) = P(t^2 + st)P(s^2) \tag{1}$$

where s, t are indeterminates. In other words, $P(s^2 + st)P(t^2)$ is symmetrical in s and t.

Next, when $p > 2$, let

$$P(t) = \sum_{i \geq 0} t^i P^i; \tag{2}$$

then the Adem relations for the reduced powers P^i are equivalent to the statement that the formal power series

$$(1 + s\, ad\, \beta)P(t^p + st^{p-1} + \ldots + s^{p-1}t).\ P(s^p)$$

is symmetrical in s and t. Here β denotes the Bockstein homomorphism and $(ad\, \beta)P = \beta P - P\beta$.

In practice, it is simpler (and entails no loss of information) to set $s = 1$, so that if we put $u = 1 + t + \ldots + t^{p-1} = (1-t)^{p-1}$ and $\tau = tu$, our version of the Adem relations becomes

$$P(\tau)P(1) = P(u)P(t^p), \qquad (p \geq 2) \tag{3}$$

$$[\beta, P(\tau)]P(1) = t[\beta, P(u)]P(t^p). \qquad (p > 2) \tag{4}$$

We shall first prove (3) and (4), and then show that they are indeed equivalent to the Adem relations as usually stated.

§2.

We recall that

$$H^*((B\mathbf{Z}/2)^n;\ \mathbf{Z}/2) \simeq \mathbf{Z}/2[x_1, \ldots, x_n]$$

where each x_i has dimension 1, and that for primes $p > 2$

$$H^*((B\mathbf{Z}/p)^n;\ \mathbf{Z}/p) \simeq \mathbf{Z}/p[x_1, \dots, x_n] \otimes E[y_1, \dots, y_n]$$

where each y_i has dimension 1 and each $x_i = \beta y_i$ has dimension 2.

THEOREM (Serre). *The cohomology class $x_1 \dots x_n$ induces an injection of $H^*(K(\mathbf{Z}/2, n);\ \mathbf{Z}/2)$ into $H^*((B\mathbf{Z}/2)^n;\ \mathbf{Z}/2)$ in dimensions $\leq 2n$.*

The class $y_1 \dots y_n x_{n+1} \dots x_{2n}$ induces an injection of $H^*(K(\mathbf{Z}/p, 3n);\ \mathbf{Z}/p)$ into $H^*((B\mathbf{Z}/p)^{2n};\ \mathbf{Z}/p)$ in dimensions $\leq 4n$.

The proof of this result does not involve the Adem relations. For $p = 2$ it is outlined by Serre in [2]: he proves $H^*(K(\mathbf{Z}/2, n);\ \mathbf{Z}/2)$ to be the polynomial algebra on $\{Sq^I \iota_n;\ I$ admissible, excess $(I) < n\}$ and then shows that $\{Sq^I x_1 \dots x_n;\ I$ admissible, degree $(I) \leq n\}$ are linearly independent in $H^*((BZ/2)^n;\ Z/2)$. Details of the two corresponding steps for $p > 2$ can be found in [1, 3].

It follows that to verify relations among Steenrod operations it suffices to evaluate them on $x_1 \dots x_n (p = 2)$ or on $y_1 \dots y_n x_{n+1} \dots x_{2n}$ $(p > 2)$.

Consider the relation (3). By the Cartan formula, $P(1)$, $P(\tau)$, $P(u)$ and $P(t^p)$ are all multiplicative and so we are reduced to verifying (3) for $x \in H^1(B\mathbf{Z}/2;\ \mathbf{Z}/2)(p = 2)$, or for $y \in H^1(B\mathbf{Z}/p;\ \mathbf{Z}/p)$ and $x \in H^2(B\mathbf{Z}/p;\ \mathbf{Z}/p)(p > 2)$. For this we need only the elementary facts that $P(t)x = x + tx^p$ and $P(t)y = y$.

These give $(p > 2)$

$$P(\tau)P(1)y = y = P(u)P(t^p)y$$

and (all p)

$$P(\tau)P(1)x = x + (1 + \tau)x^p + \tau^p x^{p^2},$$

$$P(u)P(t^p)x = x + (u + t^p)x^p + u^p t^p x^{p^2}.$$

The last two expressions are equal if (and only if) $\tau = tu$ and $u = 1 + t + \dots + t^{p-1}$. Thus (3) holds: indeed it is the unique relation of the form $P(a)P(1) = P(b)P(c)$.

To verify the relation (4) we introduce an indeterminate v which we treat as having odd dimension; then the operation $P(t) + v[\beta, P(t)]$ is multiplicative $(v^2 = 0)$ and we can combine the relations (3) and (4) in the single multiplicative formula

$$(P(\tau) + v[\beta, P(\tau)])P(1) = (P(u) + vt[\beta, P(u)])P(t^p). \tag{5}$$

Verification of (5) by evaluation on x and y is now an elementary exercise.

§3.

Finally, we shall derive the Adem relations in their usual form. Consider first the formula (3). It shows that, for any integers $a, b \geq 0$, $P^a P^b$ is equal to the coefficient of τ^a in

$$[P(u)P(t^p)]_{a+b} = \sum_{j \geq 0} u^{a+b-j} t^{pj} P^{a+b-j} P^j,$$

that is to say,

$$P^aP^b = \operatorname*{Res}_{\tau=0}\,[P(u)P(t^p)]_{a+b}\,\frac{d\tau}{\tau^{a+1}}.$$

Now since $\tau = t(1-t)^{p-1}$ we have

$$\frac{d\tau}{dt} = (1-t)^{p-1} - t(p-1)(1-t)^{p-2} = (1-t)^{p-2}$$

since we are working modulo p, and hence

$$P^aP^b = \operatorname*{Res}_{t=0}\,[P(u)P(t^p)]_{a+b}\,\frac{(1-t)^{p-2}\,dt}{(tu)^{a+1}}$$

which is equal to the coefficient of t^a in

$$\sum_{j\geq 0}(1-t)^{(p-1)(b-j)-1}t^{pj}P^{a+b-j}P^j.$$

Consequently

$$P^aP^b = \sum_{j\geq 0}(-1)^{a-pj}\binom{(p-1)(b-j)-1}{a-pj}P^{a+b-j}P^j,$$

which is the first Adem relation, without restriction on a and b.

As to the second relation, we have from (4)

$$\beta P(\tau)P(1) - P(\tau)\beta P(1) = t\beta P(u)P(t^p) - tP(u)\beta P(t^p)$$

and therefore, using (3),

$$P(\tau)\beta P(1) = ((1-t)\beta P(u) + tP(u)\beta)P(t^p).$$

A calculation similar to that just performed now leads to the usual form of the second Adem relations.

4. REMARK

G. Segal has pointed out to us that the Adem relations for Steenrod squares, in the form (1), can be more naturally explained as follows.

For any space X there is a total squaring operation $S: H^n(X) \to H^{2n}(X \times B\Sigma_2)$, where Σ_2 is the symmetric group on two letters. Iterating it gives $S^2: H^n(X) \to H^{4n}(X \times B\Sigma_2 \times B\Sigma_2)$. This is the restriction of a total fourth-power operation $T: H^n(X) \to H^{4n}(X \times B\Sigma_4)$, by the cartesian product embedding of $\Sigma_2 \times \Sigma_2$ in Σ_4. Because inner automorphisms of Σ_4 act on $B\Sigma_4$ by maps which are homotopic to the identity it follows that for any ξ the element $S^2\xi$ is invariant under the action of the normalizer of $\Sigma_2 \times \Sigma_2$ in Σ_4, in particular under the operation of interchanging the factors of $\Sigma_2 \times \Sigma_2$. The Adem relations express this invariance.

If we identify $H^*(X \times B\Sigma_2)$ with $H^*(X)[t]$, with $t \in H^1(B\Sigma_2)$ then $S\xi = \sum_k t^{n-k}Sq^k\xi$

for $\xi \in H^n(X)$. If $H^*(X \times B\Sigma_2 \times B\Sigma_2) = H^*(X)[t, s]$, then

$$S^2\xi = s^{2n} \sum_{m,k} s^{-m} Sq^m(t^{n-k} Sq^k \xi).$$

But $\Sigma s^{-m} Sq^m$ is a ring homomorphism,.and it takes t to $t + s^{-1}t^2$, so

$$S^2\xi = s^{2n} \sum (t + s^{-1}t^2)^{n-k} s^{-m} Sq^m Sq^k \xi$$

$$= s^n t^n (s+t)^n \sum s^{-m}(t + s^{-1}t^2)^{-k} Sq^m Sq^k \xi$$

$$= s^n t^n (s+t)^n P(s^{-1}) P((t + s^{-1}t^2)^{-1}) \xi$$

in the notation of the paper. Hence $P(s^{-1})P((t + s^{-1}t^2)^{-1})$ is symmetric in (s, t).

Write $s^{-1} = u(u+v)$, $t^{-1} = v(u+v)$. Then $(t + s^{-1}t^2)^{-1} = v^2$, and we find that

$$P(u(u+v))P(v^2)$$

is symmetric in (u, v).

An analogous discussion applies to the case of odd primes, but the details are more complicated.

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