# MOORE CONJECTURES

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Simply connected finite CW complexes play a distinguished role in homotopy theory. The problems associated with the fundamental group are significantly different than those dealing with higher homotopy groups. Finite complexes whose universal covering spaces are not finite, such as $S^1 \vee S^2$ are, from this point of view, infinite complexes in disguise. From now on I will use the term "space" or "complex" to mean a simply connected CW complex. If one allows the use of infinite complexes, any collection of abelian groups can be realized as the homotopy groups of a space. But if one restricts one's attention to finite complexes, it becomes possible to prove theorems stating that the homotopy groups of a finite complex have specific properties. An early example of such a theorem would be Serre's Theorem [Se1] stating that the homotopy groups of a finite complex are finitely generated. A more recent example would be the "Serre Conjecture" theorem of McGibbon and Neisendorfer [McN] which shows that for a simply connected finite complex X and prime p, if $X_{(p)}$ is not contractible, then $\pi_n(X)$ has p-torsion for infinitely many n. This strengthens Serre's Theorem [Se2] for the prime 2, later extended by Umeda [U] to odd primes, which claimed that $\pi_n(X)\otimes\mathbb{Z}_{(p)}\neq 0$ for infinitely many n. The question of which sequences of abelian groups can occur as the homotopy groups of a finite complex thus becomes an interesting one. Homotopy theory is of course not restricted to the homotopy groups themselves. Haynes Miller's "Sullivan Conjecture" theorem [M] which shows that for a finite complex X the space of pointed maps $\mathrm{map}_*(BG,X)$ is weakly contractible for any locally finite group G gives an example of a property possessed by finite complexes. Lannes and Schwartz [LS] have defined a quasi-bounded complex X to be one having the property that for each n there exists an integer $\alpha(n)$ such that for all $x\in H^n(X)$, $\Phi(x)=0$ for any Steenrod operation $\Phi$ having degree at least $\alpha(n)$. The fact that Lannes and Schwartz (and others) have found it useful, both for Miller's Theorem and for its use in proving the McGibbon-Neisendorfer Theorem, to enlarge the collection of spaces under consideration to include "quasi-bounded complexes" suggests that it may be interesting and useful to consider the Moore Conjecture and similar questions in a more general context. Even if one is interested only in finite complexes, the con-

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jecture could turn out to be easier to prove when generalized, and considering a class of spaces which is closed under the looping operation (such as quasi-bounded spaces) definitely has its advantages.

One of the aims of this conference was to emphasize the relations between rational homotopy theory and torsion (irrational?) homotopy theory. The Moore Conjecture would provide a striking example of this relationship. Rational homotopy theorists have drawn a distinction between two types of finite complexes. There are "small" complexes, typified by $S^n$, and "big" complexes, typified by $S^n \vee S^k$. More precisely, Felix, Halperin, and Thomas have defined a finite complex to be <u>elliptic</u> (the small case) if there exists N such that $\pi_n(X)\otimes\mathbb{Q}=0$ for $n>N$ and <u>hyperbolic</u> (the large case) if $\pi_n(X)\otimes\mathbb{Q}\neq 0$ for infinitely many n. From their definitions, elliptic spaces are homotopically small and hyperbolic ones are large. But in fact as the following theorem demonstrates, elliptic spaces are very very small and hyperbolic ones very very large in ways not apparent from the definition.

<u>Theorem</u> (Felix-Halperin-Thomas): For a finite complex X, either there exists N such that $\pi_n(X)\otimes\mathbb{Q}=0$ for $n>N$, or there exists N and a constant C such that $\sum_{k=1}^{n} \dim(\pi_k(X)\otimes\mathbb{Q}) > e^{Cn}$ for $n>N$. •

This theorem [FHT] which was published in 1982 says, roughly speaking, that if the rational homotopy of X is not finite then it grows exponentially. This contrast between the properties of elliptic and hyperbolic spaces appears elsewhere in rational homotopy theory as well. A natural question to ask them is whether a corresponding contrast exists in the torsion homotopy properties of these two types of finite complexes. For rational vector spaces there is only one invariant of size - the dimension of the vector space. But if instead we fix a prime p and consider modules over $\mathbb{Z}_{(p)}$, we can consider two invariants - the rank and the best exponent for its torsion. For an abelian group G, by the <u>p-rank</u> of M we mean $\dim(G\otimes\mathbb{Z}/p\mathbb{Z})$. An integer M is called an <u>exponent</u> for G if $Mx=0$ for all x in G and the best exponent is the least such integer. In other words, if G has an exponent the best exponent is the order of the element of the group having largest order. An exponent for the subgroup ${}_p\mathrm{Tors}(G)$ of G consisting of these elements whose orders are powers of p will be called a <u>p-exponent</u> for G. For an abelian group G, we will denote the rank and best p-exponent of $G\otimes\mathbb{Z}_{(p)}$ by ${}_p\mathrm{rank}(G)$ and ${}_p\exp(G)$ respectively.

Suppose then that X is a finite complex and p a prime. The most direct analogy with the Felix-Halperin-Thomas Theorem would be the following question.

<u>Question 1</u>: a) What can be said about the rate of growth of ${}_p\mathrm{rank}(\pi_n(X))$?

b) Is the behaviour in the hyperbolic case different from that in the elliptic case? •

Not much is known about this, even in the case where X is a sphere. An upper bound on ${}_p\mathrm{rank}(\pi_n(S^q))$ was given by Selick [S2] but it grows exponentially with n. This bound was improved by Bödigheimer and Henn [BH], but is still exponential. In fact, all evidence to this point (e.g. McGibbon-Neisendorfer Theorem) suggests that ${}_p\mathrm{rank}(\pi_n(X))$ is large for finite complexes X, *whether or not* X is hyperbolic. In addition, stably there is the following theorem [O].

<u>Theorem</u> (Oka): For $p \geq 5$, $\limsup {}_p\mathrm{rank}(\pi_n{}^S)=\infty$, where $\pi_*{}^S$ denotes the stable homotopy groups of spheres. •

Whether or not they grow exponentially is not known.

The Moore Conjecture proposes not that the ranks but rather that the exponents of the homotopy groups of a simply connected finite complex are "large" or "small" depending on whether X is hyperbolic or elliptic. For a fixed prime p and space X there are several possible notions of exponent worth mentioning before stating the Moore conjecture precisely.

<u>Definition</u>: $p^r$ is called a <u>homology exponent</u> for X (at p) if it is a p-exponent for $H_*(X)$. $p^r$ is called a <u>homotopy exponent</u> for X (at p) if it is a p-exponent for $\pi_*(X)$. If X is a co-H-space, $p^r$ is called a <u>co-H-exponent</u> for X (at p) if $p^r(1)=0$ in $[X_{(p)},X_{(p)}]$, where 1 denotes the identity map. If X is an H-space, $p^r$ is called an <u>H-exponent</u> for X if $p^r(1)=0$ in $[X_{(p)},X_{(p)}]$. $p^r$ is called an <u>eventual co-H-space exponent</u> for X (at p) if it is an exponent for $\Sigma^N X$ for sufficiently large N. $p^r$ is called an <u>eventual H-space exponent</u> for X (at p) if it is an exponent for $\Omega^N X$ for sufficiently large N. •

<u>Conjecture</u> (Moore): Let p be a prime and X a simply connected finite complex. Then the following are equivalent:

P1) X is elliptic.

P2) X has a homotopy exponent (at p).

P3) X has an eventual H-space exponent (at p). •

It is trivial that P3 implies P2. It is almost equally obvious that P3 implies P1, for if X has an eventual H-space exponent then $\Omega^N X_{(0)}$ is contractible for sufficiently large N, and this is equivalent to X being elliptic. Thus for X and p as above we wish to show the following two statements.

S1) If X is elliptic then X has an eventual H-space exponent at p.

S2) If X is hyperbolic then X has no homotopy exponent at p.

This conjecture was proposed by John Moore during the academic year 1977-78 at Princeton. Although it was well-known in the area, it was not published until the paper "Moore Spaces have Exponents" written in 1979 by Cohen, Moore, and Neisendorfer [CMN3]. Unfortunately this paper has not yet appeared! The first appearance of the conjecture in the literature may be in a paper by Neisendorfer and Selick [NS], which appeared in 1982. It is interesting that the conjecture predates the introduction of the concepts of "elliptic" and "hyperbolic" into rational homotopy theory, and that the authors of those terms had no knowledge of the conjecture at the time of their introduction. Thus people studying torsion homotopy and rational homotopy were independently led to this division of finite complexes.

Possibly the most fascinating feature of the conjecture is that conditions P2 and P3 are stated in terms of a particular prime p, while P1 refers only to the rational homotopy type of X. Thus the conjecture would imply that existence or non-existence of exponents for a finite complex is determined by its rational homotopy type, and in particular is independent of the prime p.

Corollary of Conjecture: If $\Sigma X$ satisfies P2 or P3, then so does X. •

This of course follows immediately since $\Sigma X$ elliptic implies X elliptic. The reverse implication does not hold, as can be seen by considering the example $S^n \times S^k$. Proof of this corollary would be a useful first step in proving the conjecture. In particular, it would reduce the proof of S1 to the case where X is a suspension, or in fact a k-fold suspension. Along these lines there is a further conjecture by Barratt which for rationally contractible finite complexes proposes exactly what the eventual exponent should be.

Conjecture (Barratt): Let X be a finite simply connected complex such that $\Sigma X$ has co-H-space exponent $p^r$. Then $\Omega^2\Sigma^2 X$ has H-space exponent $p^{r+1}$. •

The list of spaces for which any of these conjectures has been proved is not long. The evidence for Moore's conjecture is seen most clearly in the contrasting behaviour of our canonical examples $S^n$ and $S^n \vee S^k$. The hyperbolic example, $S^n \vee S^k$ (with n, k>1), can be seen not to have a homotopy exponent by applying the Hilton-Milnor Theorem [H], [Mi]. This shows that its homotopy contains the homotopy of $S^{2N+1}$ as a retract for arbitrarily large N. But a theorem of Mahowald [Ma] for p=2 and Gray [G] for p odd shows that $\pi_*(S^{2N+1})$ has elements of order $p^N$. The elliptic example, $S^n$, was shown to have a homotopy exponent at each prime by the following theorem of James [J2] and Toda[T].

Theorem (James - p=2; Toda - p>2): $p^{2n}$ is a homotopy exponent for $S^{2n+1}$ at p. •

Exponents for even spheres follow from this by the fibration $S^{2n-1} \to S^{2n} \to S^{4n-1}$. For odd primes, the theorem above was improved by Cohen, Moore, and Neisendorfer [CMN1], [CMN2], [N1] to the following theorem in which $S^{2n+1}\langle 2n+1\rangle$ denotes the 2n+1 connected cover of $S^{2n+1}$, defined as the homotopy-theoretic fibre of the map $S^{2n+1} \to K(Z,2n+1)$ which represents a generator of $\pi_*(K(Z,2n+1))$.

Theorem (Neisendorfer - p=3; Cohen-Moore-Neisendorfer - p>3): $p^n$ is an H-space exponent for $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$ at p. •

Corollary: $\Omega^N S^{2n+1}$ has an H-space exponent for $N>2n+1$. •

This theorem is most important for giving the best possible homotopy exponent at odd primes (see Gray's Theorem mentioned above), but from the present point of view the most interesting feature is the replacement of a homotopy exponent by an eventual H-space exponent. Moore has also shown [c.f. MN] how to rephrase James' argument to show that $2^{2n}$ is a space exponent for $\Omega^{2n}S^{2n+1}\langle 2n+1\rangle$ at 2. This has been improved to approximately $2^{(3/2)n}$ by Selick [S4], but the published version (inexplicably!) fails to mention the space exponent. The best possible exponent for spheres at the prime 2 is unknown but Barratt and Mahowald have conjectured that it is approximately $2^n$ (c.f. [S4] for the exact statement).

Note in passing that it is possible, even when $\Omega^k Y$ is already rationally contractible, for $\Omega^{k+1}Y$ to have an exponent while $\Omega^k Y$ does not as the following theorem [NS] demonstrates. For an example in which Y is a finite complex, see the discussion of Moore spaces below.

Theorem (Neisendorfer-Selick): $\Omega^{2n-2}S^{2n+1}\langle 2n+1\rangle$ has no H-space exponent at any prime. •

Nowadays it is possible to give the following trivial proof of this by means of Miller's Theorem (Sullivan Conjecture). From the fibration $K(\mathbb{Z},2) \to \Omega^{2n-2}S^{2n+1}\langle 2n+1\rangle \to \Omega^{2n-2}S^{2n+1}$ and the fact that $\mathrm{map}_*(BG,\Omega^{2n-2}S^{2n+1})$ is weakly contractible for any locally finite group G, we conclude that $[BG,K(\mathbb{Z},2)] \cong [BG,\Omega^{2n-2}S^{2n+1}\langle 2n+1\rangle]$ for any locally finite G. Thus if $\Omega^{2n-2}S^{2n+1}\langle 2n+1\rangle$ has an H-space exponent at p, then $H^2(BG)=[BG,K(\mathbb{Z},2)]$ has a p-exponent which is independent of the group G, and this is absurd. Whether or not $\Omega^{2n-2}S^{2n+1}\langle 2n+1\rangle$ has an exponent is not known.

Another class of spaces which were important in motivating both Moore's conjecture and Barratt's conjecture are the mod-p Moore spaces. Let $P^n(p^r)$ denote the homotopy-theoretic cofibre of the degree $p^r$ self-map of $S^{n-1}$. According to Barratt's conjecture $p^{r+1}$ should be an H-space exponent for $\Omega^2 P^{n+2}(p^r)$. For odd primes $\Omega^2 P^{n+2}(p^r)$ was shown to have an exponent by Cohen, Moore, and Neisendorfer [CMN3] and the exponent was then improved by Neisendorfer [N2] to $p^{r+1}$ which is known to be the best possible exponent from results in [CMN2].

Theorem (Neisendorfer): $p^{r+1}$ is an H-space exponent for $\Omega^2 P^{n+2}(p^r)$. •

It is necessary to loop twice to get this result since, as is pointed out in [CMN3], $\Omega\Sigma X$ can never have an H-space exponent at p if $\tilde{H}_*(X;\mathbf{Z}/p\mathbf{Z})\neq 0$.

Comparison of the $k^{th}$ power map on $\Omega\Sigma X$ (which will be denoted simply by k) with the loop on the degree p map of co-H-spaces (denoted $\Omega(\underline{k})$) has long interested homotopy theorists. While the maps are of course not equal in general, they are not totally unrelated either. In the 1950's, Hilton demonstrated relations with his "distributivity formulas" involving the Hilton-Hopf invariants. For spheres $S^{2n-1}$ it is known that $4k=\Omega(\underline{4k})$. (This may not have appeared precisely in the literature, although a proof that they induce the same homomorphism of homotopy groups is given by Whitehead [W].) On the other hand for $S^{2n-1}$, $2\neq\Omega(\underline{2})$ unless n=1,2, or 4. If $p^r$ is a co-H-space exponent for $\Sigma X$, Barratt [B] gave bounds on the order of $\pi_n(X)$. However these bounds do not qualify as a homotopy exponent under our definition since they increase with n. The theorem of Neisendorfer mentioned above provides the only examples in which Barratt's conjecture is known to hold and are the only rationally contractible finite complexes (other than a point) known to have any exponent. In fact there is no example (except a point) of a space having co-H-space exponent a power of 2 which is known to have a homotopy exponent.

The list of spaces for which the statement S1 has been proved is as follows. As mentioned above, S1 holds for spheres. As a consequence it holds for Lie groups since their homotopy types can be constructed inductively by taking fibrations over spheres. More generally Long [Lo] and Wilkerson (unpublished) have shown that S1 holds for finite H-spaces. At odd primes we can add mod-p Moore spaces to the list. In addition for odd primes, Neisendorfer and Selick [NS] showed that it holds for elliptic 2-cell complexes and a few elliptic 3-cell com-complexes.

If unable to prove that S1 holds for all elliptic complexes, instead of restricting attention to special cases, another approach is to restrict the number of primes to be considered. This has been done successfully by McGibbon and Wilkerson [MW] who proved the following theorem.

<u>Theorem</u> (McGibbon-Wilkerson): X elliptic implies that for almost all primes p, $\Omega X \simeq \pi S^{2n_j+1} \times \pi\Omega S^{2n_j+1}$ after localizing at p, where the products are each over finite index sets (and almost all means at most finitely many exceptions). •

<u>Corollary</u>: If X is elliptic then statement S1 holds for almost all primes. •

Given this theorem one is tempted to ask if it can be generalized to hyperbolic spaces if we allow infinite products instead of finite ones.

However Fröberg, Gulliksen, and Löfwall [FGL] have given an example of a Hopf algebra which has primitive torsion elements of any order and independently Anick [An1] and Avramov [Av] have developed this idea further to produce a finite complex X in which for every integer m, the image of the Hurewicz map in $H_*(\Omega X)$ has an element of order m. Thus $\Omega X$ is not homotopy equivalent to a product of (homologically) torsion-free spaces after inverting any proper subset of primes.

Turning now to statement S2, the list of spaces for which it is known is as follows. As noted above, the Hilton-Milnor Theorem implies that S2 holds for $S^n \vee S^k$ and there are other trivial examples of this form. At odd primes Selick [S3] showed that hyperbolic torsion-free suspensions satisfy S2.

<u>Theorem</u> (Selick): Let p be an odd prime, and let $X=\Sigma Y$ be a finite hyperbolic complex such that $H_*(X_{(p)})$ is torsion-free. Then $\pi_*(X)$ has no p-exponent. •

In addition, with one exception, at odd primes Neisendorfer and Selick [NS] were able to handle all hyperbolic 2-cell complexes. The analogue of the McGibbon-Wilkerson Theorem is not known in the hyperbolic case, but the following theorem of Anick [An1] was presented at this conference.

<u>Theorem</u> (Anick): Let X be a hyperbolic complex having (Lusternik-Schnirelmann) category 2. Then for almost all primes p, $\pi_*(X)$ has no p-exponent. •

I will close by giving some other exponent-related Moore conjectures proposed at this conference.

<u>Conjecture</u> (Moore): Let $f : X \to Y$ be any map, where X and Y are simply connected complexes such that $\pi_*(X)$ has an exponent. Then $\pi_*(F)$ has an exponent where F denotes the homotopy-theoretic fibre of the canonical map $Y \to Y \cup_f CX$. •

Note: $\pi_*$ denotes the direct sum the homotopy groups and an exponent is required for the group, not just for its torsion.

<u>Conjecture</u> (Moore): Let $F \to E \to B$ be a fibration, where F, E, and B are simply connected complexes with F finite. Then $H_*(E)$ has an exponent if and only if $H_*(B)$ does. •

In connection with the latter conjecture Moore also asks the following question. Let $F \to E \to B$ be a fibration, where F, E, and B are simply connected complexes. Suppose that $p^r$ is an exponent for $H_*(F)$ and that $p^s$ is an exponent for $H_*(B)$. Must $H_*(E)$ have an exponent, and if so, can the best exponent be more than $p^{r+s}$?

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