CHARACTERS AND COHOMOLOGY OF FINITE GROUPS

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INTRODUCTION

For any finite group G one can consider the integral cohomology ring $H^*(G, \mathbf{Z})$ and the ring $R(G)$ of unitary characters (cf. § 6). For 1-dimensional characters there is a well-known isomorphism

$$H^2(G, \mathbf{Z}) \cong \mathrm{Hom}\,(G, U(1)).$$

The purpose of this paper is to establish a connection between the rings $H^*(G, \mathbf{Z})$ and $R(G)$ which, in a certain sense, provides a generalization of the above isomorphism. We shall prove that there is a spectral sequence $\{E_r^p\}$ with

$$E_2^p = H^p(G, \mathbf{Z})$$
$$E_\infty^p = R_p(G)/R_{p+1}(G),$$

where $R(G) = R_0(G) \supset \ldots \supset R_p(G) \supset R_{p+1}(G) \supset \ldots$ is a certain filtration of $R(G)$. This spectral sequence has the following additional properties.

a) A homomorphism $G \to G'$ induces a homomorphism of spectral sequences $E_r' \to E_r$.

b) A monomorphism $G \to G'$ induces a homomorphism of spectral sequences $E_r \to E_r'$ (compatible with the transfer and induced representations).

c) There is a product structure compatible with the products in $H^*(G, \mathbf{Z})$ and $R(G)$.

d) All the even operators d_{2r} are zero.

e) The filtration of $R(G)$ is even, i.e. $R_{2k-1}(G) = R_{2k}(G)$.

It follows from *d)* and *c)* that, if $H^q(G, \mathbf{Z}) = 0$ *for all odd* q, $H^*(G, \mathbf{Z})$ is isomorphic to the graded ring of $R(G)$. This applies notably to the Artin-Tate groups (with periodic cohomology).

The filtration on $R(G)$ has one further property, which we proceed to describe. Let $\varepsilon\colon R(G) \to \mathbf{Z}$ be the homomorphism obtained by assigning to each character its value at the identity of G, and let $I(G)$ be the kernel of ε. Then we have:

f) The filtration topology of $R(G)$ coincides with its $I(G)$-adic topology.

In view of *f)* the $I(G)$-adic completion $\widehat{R(G)}$ of $R(G)$ plays an important role throughout.

The filtration on $R(G)$ and the spectral sequence are defined topologically, and the whole paper rests heavily on the fundamental results of Bott [3, 4] on the homotopy

of the unitary group. The basic idea may be described as follows. Associated with the group G there is its classifying space B_G, i.e. a space with fundamental group G and contractible universal covering space. The homotopy type of B_G is uniquely determined by G, so that homotopy invariants of B_G give rise to algebraic invariants [1] of G. The most obvious example is provided by the cohomology ring. Now in [1] a homotopy invariant ring $K^*(X)$ was introduced for a finite CW-complex X. This was constructed from complex vector bundles over X, the addition and multiplication in $K^*(X)$ being induced by the direct sum and tensor product of vector bundles. Although B_G is not a finite complex, it is a limit of finite complexes and so [2] $\mathscr{K}^*(B_G)$ can be defined as an inverse limit. Because a representation of G induces a vector bundle over B_G we have a natural ring homomorphism

$$\alpha : R(G) \to \mathscr{K}^*(B_G).$$

The essential content of this paper is the study of the homomorphism α. The main result is that α induces an isomorphism

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G),$$

where $\widehat{R(G)}$ is the $I(G)$-adic completion of $R(G)$ as above. This identifies the *algebraic invariant* $\widehat{R(G)}$ with the *homotopy invariant* $\mathscr{K}^*(B_G)$.

The spectral sequence relating $H^*(G, \mathbf{Z})$ and $\widehat{R(G)}$ now follows from a spectral sequence $H^*(X, \mathbf{Z}) \Rightarrow K^*(X)$ for any finite CW-complex X [1]. Actually this step involves an inverse limit process and has to be treated with care.

The spectral sequence which we have been discussing for the group G is a special case of a more general "Hochschild-Serre" spectral sequence for a normal subgroup V of a group G. This has

$$E_2^p = H^p(S, \widehat{R(V)}),$$
$$E_\infty^p = R_p(G)_S/R_{p+1}(G)_S,$$

where $S = G/V$ operates on $\widehat{R(V)}$ by conjugation, and

$$R(G) = R_0(G)_S \supset \ldots \supset R_p(G)_S \supset \ldots$$

is a filtration on $R(G)$ defined relative to S. This reduces to the previous spectral sequence on taking V to be the identity.

The layout of the paper is as follows. In § 1 we discuss vector bundles and representations. In § 2 we summarize the theory of the ring $K^*(X)$. In § 3 we collect together a number of results on inverse limits and completions which will be needed later. Then in § 4 we extend the theory of § 2, with suitable restrictions, to infinite dimensional complexes. The results of § 4 are applied in § 5 to the classifying space of a finite group. The main result (5.1) asserts the existence of a strongly convergent spectral sequence

$$H^*(S, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_V)_S$$

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[1] These are necessarily invariant under conjugation, since this just corresponds to a change of base point in B_G.
[2] As in [1] we use $\mathscr{K}$ for the inverse limit K.

where V is normal in G and $S = G/V$. At this point the topological side of the problem is essentially completed, and we turn in § 6 to a study of the ring $R(G)$ with its $I(G)$-adic topology. The main result of this section (6.1) asserts that the $I(H)$-adic topology of $R(H)$ is the same as its $I(G)$-adic topology (for $H \subset G$). This is a most important property of this topology and it leads to a number of basic results for the completion $\widehat{R(G)}$. We also identify the kernel of the homomorphism $R(G) \to \widehat{R(G)}$, showing that it is not in general zero, i.e. that the topology of $R(G)$ is not Hausdorff, but that it *is* zero if G is a p-group.

In § 7 we enunciate the main theorems in a precise form. The next four sections are devoted to the proof of the isomorphism of $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$. The case of a cyclic group G is dealt with explicitly in § 8, and the fact that $\hat{\alpha}$ has zero kernel for general G is shown to follow. In § 9 we digress to establish a few simple lemmas on representations. In § 10 we show that $\hat{\alpha}$ is an isomorphism for solvable groups by using an induction argument based on the spectral sequence of (5.1). The results of § 9 are needed at this stage of the proof. Finally in § 11 the main theorem is extended from solvable groups to general groups by using the "completion" of Brauer's theorem [5] on the characters of finite groups.

An important problem which is left outstanding is that of giving an algebraic description of the filtration on $R(G)$. For cyclic groups this is solved by (8.1), and the case of a general group can be reduced to that of p-groups by (4.9). In § 12 we consider a certain algebraic filtration which has been introduced by Grothendieck. One may conjecture that this coincides with our filtration on $R(G)$. In § 13 we compute some illustrative examples.

This paper seems the appropriate place to point out that a representation of a finite group has certain cohomological invariants called Chern classes ([1]). In an appendix we summarize their formal properties and discuss their relation with our spectral sequence.

This paper is based on the joint work of F. Hirzebruch and the author, and much of its content was in fact worked out jointly. The corresponding theory for compact connected Lie groups will be found in [1]. It seems likely that the results of this paper and those of [1] are extreme cases of a theorem valid for arbitrary compact Lie groups ([2]).

On the algebraic side I am greatly indebted to J. Tate and J.-P. Serre for their generous help, without which this paper would not have materialized. This applies in particular to the important § 6.

§ 1. Vector bundles and representations.

For general definitions and properties of fibre bundles we refer to [2], [9] and [12]. We recall that if ξ is a principal bundle over a space X with group G, and if $\rho : G \to H$

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(1) This is of course well-known to topologists.
(2) (Added in proof). This is in fact the case. It will be dealt with in a separate publication.

is a homomorphism, then we have an induced principal bundle over X with group H, which is denoted by $\rho(\xi)$ [2, § 6]. We shall be concerned with the case when ξ is the universal covering space of X, so that $G = \pi_1(X)$ is the fundamental group of X. Moreover we shall suppose that G is finite. For H we take the general linear group $GL(n, \mathbf{C})$, and we shall consider this *a)* with the discrete topology, and *b)* with its ordinary topology. The corresponding principal bundles will be called *discrete* $GL(n, \mathbf{C})$-bundles or *ordinary* $GL(n, \mathbf{C})$-bundles according as we use topology *a)* or topology *b)*. For the discrete case we have ([1]) [12, § 13.9]:

Proposition (**1.1**). — *The mapping* $\rho \to \rho(\xi)$ *sets up a* $(1 - 1)$ *correspondence between the equivalence classes of unitary representations of* G *of degree n and the isomorphism classes of discrete* $GL(n, \mathbf{C})$-*bundles over* X.

The purpose of (1.1) is simply to translate representations into a geometrical form.

From any $GL(n, \mathbf{C})$-bundle (discrete or ordinary) one can form the associated *n*-dimensional complex vector bundle over X, and conversely given the vector bundle the principal bundle may be recovered as the bundle of *n*-frames. We proceed to translate (1.1) into terms of vector bundles. Let E be a complex representation space of G (or G-module). Then we may form the vector bundle $E(\xi)$ over X associated to ξ. $E(\xi)$ may be considered either as an ordinary vector bundle or as a discrete vector bundle according as E is taken with the ordinary or the discrete topology. Then from (1.1) we have.

Proposition (**1.2**). — *The mapping* $E \to E(\xi)$ *sets up a* $(1 - 1)$ *correspondence between the isomorphism classes of complex* G-*modules and the isomorphism classes of discrete complex vector bundles over* X.

For G-modules, discrete vector bundles and ordinary vector bundles one has the following operations and maps.

1) *Direct sum* $E \oplus F$;

2) *Tensor product* $E \otimes F$;

3) *Exterior powers* $\lambda^i(E)$;

4) *Inverse image* $f^* E$;

5) *Direct image* $f_* E$.

1), 2) and 3) need no explanation (for vector bundles see [9, § 3.6]). 4) is to be understood as follows. If $f: H \to G$ is homomorphism of groups, and E is a G-module, then E is also an H-module and as such is denoted by $f^* E$. If $f: Y \to X$ is a continuous map of spaces, and E is a (discrete or ordinary) vector bundle over X, then $f^* E$ is the induced vector bundle over Y. 5) is defined when f is a monomorphism in the group case or a finite covering in the space case. For groups $f_* E$ is the induced representation module, and for coverings $f_* E$ is the direct image bundle, i.e. the fibre $(f_* E)_x$ is defined as the direct sum $\oplus_y E_y$ where $y \in f^{-1}(x)$.

([1]) We suppose X satisfies the requirements of [12, § 13.9]. For example we could take X a finite CW-complex.

It is not difficult to check that 1)-5) are compatible with the mapping of (1.2) and also with the passage from discrete vector bundles to ordinary vector bundles. We observe only that if $\pi_1(X) = G$, $\pi_1(Y) = H$, a map f induces a homomorphism $H \to G$ and that this is a monomorphism if f is a finite covering.

1)-5) have the following properties:

a) $\oplus$ is commutative and associative;

b) $\otimes$ is associative and distributive over $\oplus$;

c) f^* commutes with $\oplus$, $\otimes$ and λ^i;

d) f_* commutes with $\oplus$;

e) $f_*(E \otimes f^*F) \cong f_*(E) \otimes F$.

These follow trivially from the definitions. For representations e) is the so called reciprocity formula.

Our main problem is to study the passage from representations ρ of G to the ordinary $GL(n, \mathbf{C})$-bundle $\rho(\xi)$. In view of (1.1) and using the notation of [9, § 3.1], this means we have to study the map

$$H^1(X, GL(n, \mathbf{C})) \to H^1(X, GL(n, \mathbf{C})_c)$$

where $GL(n, \mathbf{C})$ denotes the constant sheaf and $GL(n, \mathbf{C})_c$ denotes the sheaf of germs of continuous maps $X \to GL(n, \mathbf{C})$. In general this cohomology formulation of the problem is of no help, but when $n = 1$, the sheaves are sheaves of *abelian* groups and the problem can be dealt with as follows.

We have two exact sequences of sheaves, related by homomorphisms:

$$
\begin{array}{ccccccccc}
0 \to & \mathbf{Z} & \longrightarrow & \mathbf{C} & \xrightarrow{\exp 2\pi i} & \mathbf{C}^* & \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & \mathbf{Z} & \longrightarrow & \mathbf{C}_c & \longrightarrow & \mathbf{C}_c^* & \to 0
\end{array}
$$

where $\mathbf{C}^* = GL(1, \mathbf{C})$. These give cohomology exact sequences (cf. [9, § 3.8])

$$
\begin{array}{ccccccccc}
\to H^1(X, \mathbf{C}) & \to & H^1(X, \mathbf{C}^*) & \to H^2(X, \mathbf{Z}) & \to H^2(X, \mathbf{C}) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to H^1(X, \mathbf{C}_c) & \to & H^1(X, \mathbf{C}_c^*) & \to H^2(X, \mathbf{Z}) & \to H^2(X, \mathbf{C}_c) & \to
\end{array}
$$

Now the sheaf $\mathbf{C}_c$ is fine and so $H^q(X, \mathbf{C}_c) = 0$ $(q > 0)$ [9, § 2.11]. Hence we deduce:

Proposition (**1.3**). — *Let* X *have zero Betti numbers in dimensions* 1 *and* 2. *Then we have canonical isomorphisms*:

$$\operatorname{Hom}(\pi_1(X), \mathbf{C}^*) \cong H^1(X, \mathbf{C}^*) \cong H^1(X, \mathbf{C}_c^*) \cong H^2(X, \mathbf{Z}).$$

In (1.3) we may take X to be the 3-skeleton of the classifying space of B_G, where G is finite. We obtain the isomorphism

$$\operatorname{Hom}(G, \mathbf{C}^*) \cong H^2(X, \mathbf{Z})$$

referred to in the introduction (with $\mathbf{C}^*$ replacing $U(1)$), and which it is our purpose to generalize.

The notion of discrete vector bundles was introduced in this section simply as a bridge between representations and ordinary vector bundles. From now on all vector bundles will be ordinary.

§ 2. $K^*(X)$ for finite-dimensional X.

We propose here to recall briefly the definition and basic properties of the functor K^* introduced in [1]. All spaces considered in this section will be finite CW-complexes (cf. [13]).

Let $F(X)$ be the free abelian group generated by the set of all isomorphism classes of complex vector bundles over X. To every triple $t = (\xi, \xi', \xi'')$ of vector bundles with $\xi \cong \xi' \oplus \xi''$ we assign the element $[t] = [\xi] - [\xi'] - [\xi'']$ of $F(X)$, where $[\xi]$ denotes the isomorphism class of ξ. The group $K^0(X)$ is defined as the quotient of $F(X)$ by the subgroup generated by all the elements of the form $[t]$. In this definition we allow a vector bundle to have different dimensions over the different connectedness components of X.

The tensor product of vector bundles defines a commutative ring structure in $K^0(X)$; the unit 1 is given by the trivial bundle of dimension one. $K^0(X)$ is a contravariant functor of X.

Let S^1 denote the circle and let $X \to X \times S^1$ be the embedding given by a base point of S^1. We define $K^1(X)$ to be the kernel of the induced homomorphism

$$K^0(X \times S^1) \to K^0(X),$$

and we put $K^*(X) = K^0(X) \oplus K^1(X)$. The ring structure on $K^0(X)$ extends to give a ring structure on $K^*(X)$, and $K^*(X)$ is again a contravariant functor. Moreover it is an invariant of homotopy type. A map $f: Y \to X$ induces a homomorphism $K^*(X) \to K^*(Y)$ which will be denoted by $f^!$. For a point we have:

(2.1) $\qquad\qquad K^0(\text{point}) \cong \mathbf{Z}, \quad K^1(\text{point}) = 0.$

For a connected space X the fibre dimension defines an "augmentation" $\varepsilon: K^0(X) \to \mathbf{Z}$. In view of (2.1) this is the restriction to $K^0(X)$ of the homomorphism $i^!: K^*(X) \to K^*(\text{point})$ induced by the inclusion of a point in X. Using $i^!$ we extend ε to $K^*(X)$. If we denote the kernel of ε by $\widetilde{K}^*(X)$ there is a canonical decomposition $K^*(X) \cong \widetilde{K}^*(X) \oplus \mathbf{Z}$.

We define a filtration on $K^*(X)$ by putting $K^*_p(X) = \mathrm{Ker}\{K^*(X) \to K^*(X^{p-1})\}$, where X^{p-1} is the $(p-1)$-skeleton of X. If X is connected $K^*_1(X) = \widetilde{K}^*(X)$. This filtration is a homotopy invariant and turns $K^*(X)$ into a filtered ring, i.e.

$$K^*_p(X) . K^*_q(X) \subset K^*_{p+q}(X).$$

It has moreover the following property:

(2.2) $\qquad\qquad K^0_{2k-1}(X) = K^0_{2k}(X), \quad K^1_{2k}(X) = K^1_{2k+1}(X).$

If $f: Y \to X$ is a finite covering the direct image of vector bundles (cf. § 1) induces a group homomorphism $f_!: K^0(Y) \to K^0(X)$. Replacing X, Y by $X \times S^1, Y \times S^1$ this extends to a homomorphism $f_!: K^*(Y) \to K^*(X)$ which preserves filtration (since we may take $Y^{p-1} = f^{-1}(X^{p-1})$ [13, § 4, 5]), multiplies the augmentation by the degree of f and satisfies the formula

$$(2.3) \qquad f_!(y \cdot f^!(x)) = f_!(y) \cdot x, \quad y \in K^*(Y), \quad x \in K^*(X).$$

For elements of K^0 this formula follows at once from $e)$ of § 1. The general case can then be shown to follow (1).

We come now to the most important property for our present purposes, the existence of the spectral sequence. We state this as a proposition.

Proposition (2.4). — There is a spectral sequence $\{E^p_r(X)\}$ with $E^p_2(X) = H^p(X, \mathbf{Z})$, $E^p_\infty(X) = K^*_p(X)/K^*_{p+1}(X)$, and with the following further properties.

a) A map $f: Y \to X$ induces a homomorphism of spectral sequences $E^p_r(X) \to E^p_r(Y)$ which depends only on the homotopy class of f.

b) A finite covering $f: Y \to X$ induces a homomorphism of spectral sequences $E^p_r(Y) \to E^p_r(X)$.

c) The cup-product in $H^*(X, \mathbf{Z})$ induces products in each E_r $(2 \leqslant r \leqslant \infty)$ which for $r = \infty$ coincide with the products induced by the ring structure of $K^*(X)$.

d) The even differentials d_{2r} are all zero, d_3 is the Steenrod operation Sq^3, and $d_r(x) = 0$ for $\dim x \leqslant 2$ and all r.

Remark. — It is understood of course that the homomorphism of $a)$ is compatible with f^* and $f^!$ while that of $b)$ is compatible with f_* (the direct image or trace for cohomology) and $f_!$.

In view of the last part of $d)$ we have an isomorphism:

$$(2.5) \qquad K^*_2(X)/K^*_3(X) \cong H^2(X, \mathbf{Z}).$$

This isomorphism can be described directly as follows. First we observe, using (2.2), that $\qquad K^*_2(X)/K^*_3(X) \cong K^0_1(X)/K^0_3(X)$.

Now by assigning to each vector bundle E over X the 1-dimensional bundle $\det(E)$ (i.e. $\lambda^n(E)$ if E has dimension n), and then using (1.3) we obtain a homomorphism (the first Chern class)

$$c_1: K^0(X) \to H^2(X, \mathbf{Z}).$$

Restricting to $K^0_1(X)$ we obtain the homomorphism which induces (2.5).

Next we turn to the more general spectral sequence for a fibre bundle $\pi: Y \to X$ with fibre F. First we define a filtration on $K^*(Y)$ relative to X by putting

$$K^*_p(Y)_X = \text{Ker}\{K^*(Y) \to K^*(Y^{p-1})\}$$

where $Y^{p-1} = \pi^{-1}(X^{p-1})$. Then we have:

Proposition (2.6). — There is a spectral sequence $\{E^p_r\}$ with $E^p_2 = H^p(X, \mathbf{K}^*(F))$, $E^p_\infty = K^*_p(Y)_X/K^*_{p+1}(Y)_X$, and with the following further properties.

(1) Statements given without proof here or in [1] will be proved in a future publication with F. HIRZEBRUCH.

a) *A commutative diagram*

$$\begin{array}{ccc} Y & \to & Y' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ X & \to & X' \end{array}$$

gives rise to a homomorphism of spectral sequences $E'_r \to E_r$;

b) *The cup-product in* $H^*(X, K^*(F))$, *using the ring structure of* $K^*(F)$, *induces products in each* E_r $(2 \leqslant r \leqslant \infty)$ *which for* $r = \infty$ *coincide with the products induced by the ring structure of* $K^*(Y)$.

c) *If* $K^1(F) = 0$, *all* $d_{2r} = 0$.

Remarks. — 1) Taking F to be a point and $Y = X$, (2.6) reduces to (2.4).

2) $K^*(F)$ denotes the local coefficient system whose group at x is $K^*(\pi^{-1}(x))$.

3) In *a)* we do not insist that $F = F'$.

Taking $Y' = X' = X$ in *a)*, with $Y \to Y'$ being π, we get a homomorphism of the spectral sequence of (2.4) into that of (2.6). Hence from (2.6) *b)* we deduce:

(**2.7**) *The* E_r *of* (2.6) *are modules over the* E_r *of* (2.4).

Applying (2.4) with $X = $ point we get the trivial spectral sequence $\mathbf{Z}$, i.e. $E^0_r = \mathbf{Z}$, $E^p_r = 0$ for $p > 0$ (all r). Since the spectral sequence of $\pi: y_0 \to x_0$ with $y_0 \in Y$ and $x_0 = \pi(y_0) \in X$ is a direct factor of that of (2.6) we deduce:

(**2.8**) *Let* $Y \to X$ *be a fibre bundle with* Y *and* F *(the fibre) connected. Then the spectral sequence* E_r *of* (2.6) *decomposes*: $E_r = \widetilde{E}_r \oplus \mathbf{Z}$, *where* $\mathbf{Z}$ *denotes the trivial spectral sequence*, $\widetilde{E}^p_r = E^p_r$ *for* $p > 0$ *and* $\widetilde{E}^0_2 = H^0(X, \widetilde{K}^*(F))$, $\widetilde{E}^0_\infty = \widetilde{K}^*_0(Y)_X / \widetilde{K}^*_1(Y)_X$.

Easy consequences of (2.4) are the following:

(**2.9**) $K^*(X)$ *is a finitely-generated group.*

(**2.10**) *Let* $f: Y \to X$ *be such that* $f^*H^q(X, \mathbf{Z})$ *is finite for all* $q > 0$ *and suppose* X *connected. Then* $f^!\widetilde{K}^*(X)$ *is finite.*

We now make one formal application of the properties of $K^*(X)$.

Proposition (**2.11**). — *Let* $f: Y \to X$ *be a finite covering of degree* d. *Then* d *annihilates the kernel of* ([1]) $GK^*(X) \to GK^*(Y)$. *If* p *is a prime not dividing* d, *then the* p-*primary component of* $GK^*(X)$ *is a direct factor of* $GK^*(Y)$.

Proof. — Since $f_!$ and $f^!$ preserve filtration they induce homomorphisms $\varphi_!$ and $\varphi^!$ of GK^*, and from (2.3) we deduce

$$\varphi_!(y \cdot \varphi^!(x)) = \varphi_!(y) \cdot x \qquad x \in GK^*(X), y \in GK^*(Y).$$

Taking $y = 1$ we obtain

$$\varphi_! \varphi^!(x) = \varphi_!(1) \cdot x = dx.$$

The proposition follows at once from this formula.

Remark. — This is quite analogous to the corresponding result for cohomology.

([1]) If $A = A_0 \supset A_1 \supset A_2 \supset \ldots$ is a filtered group we denote by GA the graded group $\Sigma_p A_p/A_{p+1}$. The component A_p/A_{p+1} will be denoted by $G^p A$.

§ 3. Inverse limits and completions.

Let M be a filtered abelian group, i.e. we have a sequence of subgroups:

$$M = M_0 \supset M_1 \supset \ldots \supset M_n \supset \ldots$$

This filtration gives M the structure of a topological group, the subgroups M_n being taken as a fundamental system of neighbourhoods of o in M. We denote by $\hat{M}$ (or $M^\wedge$) the completion of M for this topology, i.e.

(3.1) $\hat{M} = \varprojlim M/M_n$ (inverse limit).

We remark that the topology of M is not necessarily Hausdorff so that the natural map $M \to \hat{M}$ may have non-zero kernel. In fact we have:

(3.2) $\mathrm{Ker}\,(M \to \hat{M}) = \bigcap_{n=1}^{\infty} M_n$.

If $\{_n A\}$ is an inverse system of abelian groups (indexed by the non-negative integers), the inverse limit $A = \varprojlim\ {}_n A$ has a natural filtration defined by ([1])

$$A_n = \mathrm{Ker}\{A \to {}_{n-1} A\}$$

Moreover A is complete for the topology defined by this filtration, i.e. $A \cong \hat{A}$. Thus an inverse limit is in a natural way a *complete filtered group*. This applies in particular to the group $\hat{M}$ given by (3.1). It is easy to see that the subgroups of the filtration may be identified with the completions $\hat{M}_n$ of the subgroups M_n (for the induced topology).

If M is a finite group then the filtration necessarily terminates, i.e. $M_n = M_{n+1}$ for all $n \geqslant n_0$, and so $\hat{M} \cong M/M_{n_0}$. We record this for future reference.

Lemma (3.3). — *If* M *is a finite filtered group* $M \to \hat{M}$ *is an epimorphism.*

We also state the following elementary properties of inverse limits, the verifications being trivial.

Lemma (3.4). — *Let* $\{_{\alpha, \beta} A\}$ *be an inverse system indexed by pairs* $(\alpha, \beta) \in I \times J$, *where* I, J *are two directed sets. Then*

$$\varprojlim_{\alpha} \varprojlim_{\beta}\ {}_{\alpha, \beta} A \cong \varprojlim_{(\alpha, \beta)}\ {}_{\alpha, \beta} A \cong \varprojlim_{\beta} \varprojlim_{\alpha}\ {}_{\alpha, \beta} A$$

Lemma (3.5). — *If* $0 \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to 0$ *is an exact sequence of inverse systems* (α *belonging to some directed set), then*

$$0 \to \varprojlim {}_\alpha A \to \varprojlim {}_\alpha B \to \varprojlim {}_\alpha C$$

is exact.

In order for lim to be right exact we need a condition. Following Dieudonné-Grothendieck [8] we adopt the following definition. An inverse system $\{_\alpha A\}$ is said to satisfy the *Mittag-Leffler* condition (ML) if, for each α, there exists $\beta \geqslant \alpha$ such that

$$\mathrm{Im}(_\beta A \to {}_\alpha A) = \mathrm{Im}(_\gamma A \to {}_\alpha A)$$

([1]) We put $_{-1} A = o$ so that $A_0 = A$.

for all $\gamma \geqslant \beta$. Moreover we shall assume from now on that all inverse systems are over *countable* directed sets. The following properties of (ML) are proved in [8, chapter o (complements)].

(**3.6**) *If* $\{_\alpha A\} \to \{_\alpha B\} \to o$ *is exact and* $\{_\alpha A\}$ *satisfies* (ML), *so does* $\{_\alpha B\}$.

(**3.7**) *If* $o \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to o$ *is exact, and if* $\{_\alpha A\}$ *and* $\{_\alpha C\}$ *each satisfy* (ML), *so does* $\{_\alpha B\}$.

(**3.8**) *If* $o \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to o$ *is exact, and if* $\{_\alpha A\}$ *satisfies* (ML), *then*

$$o \to \varprojlim {}_\alpha A \to \varprojlim {}_\alpha B \to \varprojlim {}_\alpha C \to o$$

is exact.

(**3.9**) *Let* $\{_\alpha C^*\}$ *be an inverse system of complexes, with differentials of degree* r. *Suppose that, for each* p, $\{_\alpha C^p\}$ *and* $\{H^p(_\alpha C^*)\}$ *satisfy* (ML), *then* $\varprojlim H^p(_\alpha C^*) \cong H^p(\varprojlim {}_\alpha C^*)$.

Remark. — In [8] the differentials in (3.9) are supposed to have degree 1, but this does not affect the argument.

Using (3.8) we now prove two further lemmas on completions $(^1)$.

Lemma (**3.10**). — *Let* $f : M \to N$ *be a homomorphism of filtered groups. Then the following two statements are equivalent*:

(i) $\hat{f} : \hat{M} \to \hat{N}$ *is an isomorphism of filtered groups*.

(ii) $Gf : GM \to GN$ *is an isomorphism (where* GM *denotes the graded group of* M).

Proof. — Suppose first that (ii) holds. Then $M/M_0 = N/N_0 = o$ and from the diagram:

$$
\begin{array}{ccccccccc}
o \to & M_n/M_{n+1} & \to & M/M_{n+1} & \to & M/M_n & \to o \\
 & \downarrow & & \downarrow & & \downarrow & \\
o \to & N_n/N_{n+1} & \to & N/N_{n+1} & \to & N/N_n & \to o
\end{array}
$$

we deduce, by induction on n, that $M/M_n \to N/N_n$ is an isomorphism for all n. Taking inverse limits we deduce (i).

Conversely let (i) hold, then $G\hat{M} \to G\hat{N}$ is an isomorphism. To prove (ii) it will be sufficient therefore to prove that $GM \cong G\hat{M}$, i.e. to prove the special case where $N = \hat{M}$ and f is the natural map. Now we have an exact sequence:

$$o \to {}_1 M_{n+1}/M_{n+k} \to M_n/M_{n+k} \to M_n/M_{n+1} \to o,$$

and the inverse system $\{M_{n+1}/M_{n+k}\}$ (for n fixed and $k \to \infty$) satisfies (ML) trivially since all maps are epimorphisms. Hence by (3.8) we deduce the exact sequence:

$$o \to \hat{M}_{n+1} \to \hat{M}_n \to M_n/M_{n+1} \to o,$$

which proves that $GM \cong G\hat{M}$ as required.

Lemma (**3.11**). — *Let* $o \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to o$ *be an exact sequence of abelian groups. Let* M_n *be a filtration of* M *and define filtrations of* M', M'' *by* $M'_n = \alpha^{-1}(M_n)$, $M''_n = \beta(M_n)$. *Then* $o \to \hat{M}' \to \hat{M} \to \hat{M}'' \to o$ *is exact.*

$(^1)$ Direct proofs are also possible.

Proof. — From the definitions of M'_n, M''_n it follows that

$$0 \to M'/M'_n \to M/M_n \to M''/M''_n \to 0$$

is exact. Since the inverse system $\{M'/M'_n\}$ satisfies (ML) the lemma follows from (3.8).

In addition to the condition (ML) we shall require later the following:

(F) *For each α there exists $\beta \geqslant \alpha$ so that* $\text{Im}(_\beta A \to _\alpha A)$ *is finite.*

Clearly (F) implies (ML).

Lemma (3.12). — *Let* $0 \to \{_\alpha A\} \to \{_\alpha B\} \to \{_\alpha C\} \to 0$ *be an exact sequence of inverse systems and suppose* $\{_\alpha B\}$ *satisfies* (F). *Then* $\{_\alpha A\}$ *and* $\{_\alpha C\}$ *also satisfy* (F).

Proof. — By hypothesis there exists $\beta \geqslant \alpha$ so that $_{\beta,\alpha}B = \text{Im}(_\beta B \to _\alpha B)$ is finite. Define $_{\beta,\alpha}A$, $_{\beta,\alpha}C$ similarly, then we have exact sequences

$$_{\beta,\alpha}B \to _{\beta,\alpha}C \to 0, \qquad 0 \to _{\beta,\alpha}A \to _{\beta,\alpha}B.$$

These imply that $_{\beta,\alpha}C$ and $_{\beta,\alpha}A$ are finite. Q.E.D.

Lemma (3.13). — *Let* $\{_\alpha A\}$ *satisfy* (F), *then* $\varprojlim {_\alpha A}$ *is a compact Hausdorff group.*

Proof. — Let $_\alpha B = \bigcap_{\beta \geqslant \alpha} \text{Im}(_\beta A \to _\alpha A)$. Then (F) implies that $_\alpha B$ is finite. But, from the definition of inverse limits, $\varprojlim {_\alpha A} \cong \varprojlim {_\alpha B}$. Thus $\varprojlim {_\alpha A}$ is an inverse limit of finite groups and so compact and Hausdorff (for the inverse limit topology, i.e. the topology induced from the direct product $\Pi_\alpha A$).

Lemma (3.14). — *Let* $\{_\alpha A\}$ *be an inverse system indexed by* I, *and let* J *be a cofinal subset of* I. *Then* $\varprojlim_{\alpha \in I} {_\alpha A} \cong \varprojlim_{\alpha \in J} {_\alpha A}$, *and* $\{_\alpha A\}_{\alpha \in I}$ *satisfies* (ML) *or* (F) $\Leftrightarrow \{_\alpha A\}_{\alpha \in J}$ *satisfies* (ML) *or* (F).

Proof. — The isomorphism of the inverse limits is well-known and the implication $\Rightarrow$ is trivial. Suppose $\{_\alpha A\}_{\alpha \in J}$ satisfies (ML), and let $\lambda \in I$. Since J is cofinal there exists $\alpha \in J$, $\alpha \geqslant \lambda$. Since $\{_\alpha A\}_{\alpha \in J}$ satisfies (ML) there exists $\beta \in J$, $\beta \geqslant \alpha$ so that $\text{Im}(_\beta A \to _\alpha A) = \text{Im}(_\gamma A \to _\alpha A)$ for all $\gamma \in J$, $\gamma \geqslant \beta$. Now let $\mu \in I$, $\mu \geqslant \beta$ and (J being cofinal) choose $\gamma \in J$, $\gamma \geqslant \mu$. Then it follows that $\text{Im}(_\beta A \to _\lambda A) = \text{Im}(_\mu A \to _\lambda A)$ which shows that $\{_\alpha A\}_{\alpha \in I}$ satisfies (ML). If $\{_\alpha A\}_{\alpha \in J}$ satisfies (F) then, with the same notation, $\text{Im}(_\beta A \to _\alpha A)$ is finite. This implies that $\text{Im}(_\beta A \to _\lambda A)$ is finite, showing that $\{_\alpha A\}_{\alpha \in I}$ satisfies (F).

Noetherian Completions.

Let A be a Noetherian ring (commutative and with identity), let $\mathfrak{a}$ be an ideal in A and let M be a finitely-generated A-module. We define a filtration on M by $M_n = \mathfrak{a}^n M$. The topology defined by this filtration is called the "$\mathfrak{a}$-adic" topology of M or simply the $\mathfrak{a}$-topology of M. This topology has a number of important properties which we proceed to recall (cf. [6, exp. 18]).

Proposition (3.15). — *Let* M *be a finitely generated A-module and let* N *be a sub-module of* M. *Then the topology of* N *induced by the $\mathfrak{a}$-topology of* M *coincides with the $\mathfrak{a}$-topology of* N.

Proof. — We have the inclusion

$$\mathfrak{a}^n N \subset \mathfrak{a}^n M \cap N.$$

To prove the proposition therefore it is sufficient to show that, for each n, there exist m so that

1) $$\mathfrak{a}^m M \cap N \subset \mathfrak{a}^n N.$$

But by the lemma of Artin-Rees [6, exp. 2] there exists m_0 such that, for $m \geqslant m_0$,

2) $$\mathfrak{a}^m M \cap N = \mathfrak{a}^{m-m_0}(\mathfrak{a}^{m_0} M \cap N).$$

Taking $m = n + m_0$, 1) follows at once from 2).

Proposition (3.16). — *For finitely-generated* A-*modules,* $\mathfrak{a}$-*adic completion is an exact functor.*

Proof. — Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely-generated A-modules. By (3.15) the $\mathfrak{a}$-topology of M' is induced from the $\mathfrak{a}$-topology of M. Also the $\mathfrak{a}$-topology of M'' is induced from the $\mathfrak{a}$-topology of M. Hence we can apply (3.11) and we deduce the exact sequence $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$, where each completion is the $\mathfrak{a}$-adic completion.

Let G be a finite group. M will be called an A-G-module if it is both an A-module (finitely-generated) and a G-module, and if the operations of A and G on M commute. Then the cohomology groups $H^q(G, M)$ will be (finitely-generated) A-modules and so we can form their $\mathfrak{a}$-adic completions $H^q(G, M)^{\wedge}$. On the other hand, since $g(\alpha(m)) = \alpha(g(m))$ for all $\alpha \in A$, $g \in G$, $m \in M$, it follows that the sub-modules $\mathfrak{a}^n M$ are stable under G. Hence G operates on $M/\mathfrak{a}^n M$ and so on $\hat{M}$. Then we have:

Proposition (3.17). — *Let* M *be an* A-G-*module. Then we have a canonical isomorphism*:

$$H^q(G, M)^{\wedge} \cong H^q(G, \hat{M}).$$

Proof. — Let $\Lambda = \mathbf{Z}[G]$ be the group ring of G and let $\{X_n\}$ be the standard Λ-free resolution of $\mathbf{Z}$ [7, chapter X]. By definition

1) $$H^q(G, M) = H^q(\operatorname{Hom}_\Lambda(X_*, M)).$$

Now $\operatorname{Hom}_\Lambda(X_q, M)$ is, for each q, a finitely-generated A-module. Hence by (3.16) :

2) $$H^q(\operatorname{Hom}_\Lambda(X_*, M))^{\wedge} \cong H^q(\operatorname{Hom}_\Lambda(X_*, M)^{\wedge}).$$

Since X_q is, for each q, a free Λ-module it follows that

3) $$\operatorname{Hom}_\Lambda(X_*, M)^{\wedge} \cong \operatorname{Hom}_\Lambda(X_*, \hat{M}).$$

From 1), 2) and 3) the proposition follows.

Spectral sequences.

We propose next to consider inverse limits of spectral sequences. By a spectral sequence we shall understand a sequence of complexes ([1]) $\{E_r\}$, $2 \leqslant r < \infty$, with given isomorphisms $E_{r+1} \cong H(E_r)$. We suppose that $E_r^p = 0$ for $p < 0$ and that the

---

([1]) I.e. graded abelian groups with a differential (endomorphism d satisfying $d^2 = 0$).

differential d_r of E_r has degree r. Thus each E_r^p, for $2 \leqslant r < \infty$, may be identified with a quotient group Z_r^p/B_r^p, where ([1]) Z_r^p, B_r^p are subgroups of E_2^p. These subgroups are arranged as follows:

$$(\mathscr{S}) : 0 = B_2^p \subset \ldots \subset B_r^p \subset \ldots \subset \ldots \subset Z_r^p \ldots \subset Z_2^p = E_2^p.$$

We then define
$$B_\infty^p = \bigcup_r B_r^p = \varinjlim_r B_r^p,$$

$$Z_\infty^p = \bigcap_r Z_r^p = \varprojlim_r Z_r^p,$$

$$E_\infty^p = Z_\infty^p/B_\infty^p.$$

By a *strongly convergent spectral sequence* we shall mean a spectral sequence $\{E_r\}$ together with a *complete filtered group* M and isomorphisms $E_\infty^p \cong M_p/M_{p+1}$. We shall write $E_2 \Rightarrow M$, and say that the spectral sequence $\{E_r\}$ converges strongly to M. This is in agreement with the terminology of [7, chapter XV, § 2]. If all the differentials d_r are zero we shall say that the spectral sequence collapses. The following is then an immediate consequence of the definitions.

Proposition (3.18). — *Suppose* $E_2 \Rightarrow M$ *and the spectral sequence collapses. Then* $E_2 \cong GM$.

We now prove a result on inverse limits of spectral sequences.

Proposition (3.19). — *Let* $\{_nE_r, {}_nM\}$ *be an inverse system of strongly convergent spectral sequences. Suppose further that*

a) *For each* p, $\{_nE_2^p\}$ *satisfies* (F);

b) $\{_nM\}$ *satisfies* (F).

Then $\{\varprojlim {}_nE_r, \varprojlim {}_nM\}$ *is a strongly convergent spectral sequence.*

Proof. — For each n we have the sequence of inclusions:

$$(_n\mathscr{S}) : 0 = {}_nB_2^p \subset \ldots \subset {}_nB_r^p \subset \ldots \subset {}_nZ_r^p \subset \ldots \subset {}_nZ_2^p = {}_nE_2^p.$$

Now hypothesis $a)$ and (3.12) imply that $\{_nE_r^p\} = \{_nZ_r^p/_nB_r^p\}$ satisfies (F), and so (ML). Hence writing $E_r^p = \varprojlim {}_nE_r^p$ and using (3.9) we see that $E_{r+1}^p \cong H^p(E_r)$, showing that $\{E_r\}$ ($2 \leqslant r < \infty$) is indeed a spectral sequence (the operators are of course defined as $d_r = \varprojlim {}_nd^r$).

Taking the inverse limit of $(_n\mathscr{S})$ we get (by (3.5)) a sequence of inclusions:

$$(\mathscr{S}) \qquad 0 = B_2^p \subset \ldots \subset B_r^p \subset \ldots \subset Z_r^p \subset \ldots \subset Z_2^p = E_2^p,$$

where $B_r^p = \varprojlim {}_nB_r^p$, $Z_r^p = \varprojlim {}_nZ_r^p$. By $a)$ and (3.12) again $\{_nB_r^p\}$ satisfies (F) and so (ML). Hence by (3.8) $E_r^p \cong Z_r^p/B_r^p$, showing that $(\mathscr{S})$ has the same significance as before. Now the fact that $_nE_r^p = 0$ for $p < 0$ and that d_r has degree r imply that, for all n, $_nB_\infty^p \cong {}_nB_r^p$ for $r \geqslant p + 1$.

([1]) This conflicts slightly with the usual notation, but should cause no confusion.

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Hence $\qquad \lim {}_n\mathrm{B}^p_\infty \cong \mathrm{B}^p_r$ for $r \geqslant p + 1$,

$$\cong \mathrm{B}^p_\infty.$$

Also $\qquad \underleftarrow{\lim}{}_n\mathrm{Z}^p_\infty = \lim_{\substack{\longleftarrow \\ n}} \lim_{\substack{\longleftarrow \\ r}} {}_n\mathrm{Z}^p_r$

$$\cong \lim_{\substack{\longleftarrow \\ r}} \lim_{\substack{\longleftarrow \\ n}} {}_n\mathrm{Z}^p_r \qquad\qquad\qquad \text{by } (3.4),$$

$$= \lim_{\substack{\longleftarrow \\ r}} \mathrm{Z}^p_r$$

$$= \mathrm{Z}^p_\infty.$$

We now consider $\mathrm{M} = \underleftarrow{\lim}{}_n\mathrm{M}$. In view of (3.5) this is filtered by subgroups $\mathrm{M}_p = \underleftarrow{\lim}{}_n\mathrm{M}_p$. To complete the proof we have to show

(i) $\mathrm{Z}^p_\infty / \mathrm{B}^p_\infty \cong \mathrm{M}_p / \mathrm{M}_{p+1}$;

(ii) $\mathrm{M} \cong \underleftarrow{\lim}\, \mathrm{M}/\mathrm{M}_p$.

Now $\{{}_n\mathrm{B}^p_\infty\}$ satisfies (F) ((a) and (3.12)), and $\{{}_n\mathrm{M}_{p+1}\}$ satisfies (F)) (b) and (3.12)). Hence, by (3.8), (i) follows from the corresponding isomorphisms ${}_n(\mathrm{i})$ and the isomorphisms $\mathrm{B}^p_\infty \cong \underleftarrow{\lim}{}_n\mathrm{B}^p_\infty$, $\mathrm{Z}^p_\infty \cong \underleftarrow{\lim}{}_n\mathrm{Z}^p_\infty$ established above. Similarly (ii) follows from ${}_n(\mathrm{ii})$, using (3.8) and (3.4).

§ 4. $\mathscr{K}^*(\mathrm{X})$ for infinite-dimensional X.

In this section we shall extend the definition and properties of $\mathrm{K}^*(\mathrm{X})$, as given in § 2, to CW-complexes X all of whose skeletons are finite $(^1)$. Throughout this section a CW-complex will always mean one with this property. We define

$$\mathscr{K}^*(\mathrm{X}) = \underleftarrow{\lim}\, \mathrm{K}^*(\mathrm{X}^n),$$

where X^n is the n-skeleton of X. Then, as remarked in § 3, $\mathscr{K}^*(\mathrm{X})$ is in a natural way a complete filtered group, the filtration being defined by

$$\mathscr{K}^*_p(\mathrm{X}) = \mathrm{Ker}\{\mathscr{K}^*(\mathrm{X}) \to \mathrm{K}^*(\mathrm{X}^{p-1})\}$$

$$\cong \underleftarrow{\lim_n}\, \mathrm{K}^*_p(\mathrm{X}^n) \qquad\qquad\qquad \text{(by (3.5))}.$$

The products in $\mathrm{K}^*(\mathrm{X}^n)$ induce products in $\mathscr{K}^*(\mathrm{X})$, so that $\mathscr{K}^*(\mathrm{X})$ becomes a filtered ring. Also, for connected X, $\mathscr{K}^*(\mathrm{X})$ has an augmentation ε and a direct (group) decomposition $\mathscr{K}^*(\mathrm{X}) = \widetilde{\mathscr{K}}^*(\mathrm{X}) \oplus \mathrm{Z}$.

Lemma (4.1). — *Let* $f : \mathrm{Y} \to \mathrm{X}$ *be a continuous map,* X, Y *being* CW-*complexes. Let* ${}_n\mathrm{A} = \mathrm{K}^*(\mathrm{X}^n)$, ${}_n\mathrm{B} = \mathrm{K}^*(\mathrm{Y}^n)$, ${}_n\mathrm{A}' = \mathrm{Im}({}_{n+1}\mathrm{A} \to {}_n\mathrm{A})$, ${}_n\mathrm{B}' = \mathrm{Im}({}_{n+1}\mathrm{B} \to {}_n\mathrm{B})$. *Then* f *induces a homomorphism* ${}_n f^! : {}_n\mathrm{A}' \to {}_n\mathrm{B}'$ *which depends only on the homotopy class of* f.

Proof. — Let g, h be any two cellular maps homotopic to f. Then there exists a cellular homotopy Φ between g and h. Hence we have maps $g_n : \mathrm{Y}^n \to \mathrm{X}^n$, $h_n : \mathrm{Y}^n \to \mathrm{X}^n$ and $i_n \circ g_n \simeq i_n \circ h_n$ where $i_n : \mathrm{X}^n \to \mathrm{X}^{n+1}$ is the inclusion. This implies that $g^!_n \circ i^!_n = h^!_n \circ i^!_n$

$(^1)$ This restriction is not essential, but it covers the cases we are interested in.

these being homomorphisms $_{n+1}A \to {}_nB$. Thus g_n and h_n induce the same homomorphism $_nA' \to {}_nB'$ and this is the required $_nf^1$ depending only on the homotopy class of f.

Lemma (4.2). — — *In the notation of* (4.1) *suppose f is a homotopy equivalence. Then $_nf^1$ is an isomorphism.*

Proof. — This follows at once from (4.1).

Lemma (4.3). — $\mathscr{K}^*(X)$ *is an invariant of homotopy type (as a filtered ring).*

Proof. — This follows from (4.2) and the fact that $\varprojlim_n A \cong \varprojlim_n A'$ (as filtered rings).

Note. — The above results apply equally to $\widetilde{\mathscr{K}}^*(X)$.

Lemma (4.4). — *Let X and Y be connected CW-complexes of the same homotopy type, and put* $_nA = K^*(X^n)$, $_nB = K^*(Y^n)$, $_n\widetilde{A} = \widetilde{K}^*(X^n)$ $_n\widetilde{B} = \widetilde{K}^*(Y^n)$. *Then*

(i) $\{_nA\}$ *satisfies* (ML) $\Leftrightarrow \{_nB\}$ *satisfies* (ML),

(ii) $\{_n\widetilde{A}\}$ *satisfies* (F) $\Leftrightarrow \{_n\widetilde{B}\}$ *satisfies* (F).

Proof. — This follows from (4.2) and the following facts:

$$\{_nA\} \text{ satisfies } (ML) \Leftrightarrow \{_nA'\} \text{ satisfies } (ML)$$
$$\{_n\widetilde{A}\} \text{ satisfies } (F) \Leftrightarrow \{_n\widetilde{A'}\} \text{ satisfies } (F),$$

where we adopt the notation of (4.1).

In view of (4.4) we may say that $\mathscr{K}^*(X)$ satisfies (ML) or that $\widetilde{\mathscr{K}}^*(X)$ satisfies (F), meaning that, for some cellular structure $\{K^*(X^n)\}$ satisfies (ML) or that $\{\widetilde{K}^*(X^n)\}$ satisfies (F).

Lemma (4.5). — — *Let X be a connected CW-complex, and let $\{T^n\}$ be an increasing sequence of finite connected sub-complexes of X with* $\bigcup_n T^n = X$. *Then.*

(i) $\varprojlim K^*(T^n) \cong \mathscr{K}^*(X)$,

(ii) $\{K^*(T^n)\}$ *satisfies* (ML) $\Leftrightarrow \mathscr{K}^*(X)$ *satisfies* (ML),

(iii) $\{\widetilde{K}^*(T^n)\}$ *satisfies* (F) $\Leftrightarrow \widetilde{\mathscr{K}}^*(X)$ *satisfies* (F).

Proof. — Let I be the directed set of all finite connected sub-complexes of X. Then the sets $\{X^n\}$, $\{T^n\}$ $(n \geqslant 1)$ (X^n being as before the n-skeleton) are confinal in I. The lemma now follows from (3.14).

Lemma (4.6). — *Let X be a connected CW-complex with $H^q(X, \mathbf{Z})$ finite for all $q > 0$. Then $\widetilde{\mathscr{K}}^*(X)$ satisfies* (F).

Proof. — The hypotheses on X imply that $\mathrm{Im}\{H^q(X^{n+1}, \mathbf{Z}) \to H^q(X^n, \mathbf{Z})\}$ is finite for all $q > 0$. Hence, by (2.10), $\mathrm{Im}\{\widetilde{K}^*(X^{n+1}) \to \widetilde{K}^*(X^n)\}$ is finite, and so $\widetilde{\mathscr{K}}^*(X)$ satisfies (F).

Let G be a finite group. Then its classifying space B_G may be taken as a (connected) CW-complex (with finite skeletons) [10]. The homotopy type of B_G is

uniquely determined by G and hence, by (4.3), $\mathscr{K}^*(B_G)$ is a filtered ring depending only on G. Moreover, since an inner automorphism of G induces a map $B_G \to B_G$ homotopic to the identity on each finite skeleton [12, § 13.9], it follows that $\mathscr{K}^*(B_G)$ is invariant under inner automorphisms of G. For a direct definition of $\mathscr{K}^*(B_G)$ by means of universal operations on G-bundles, and for a direct proof of the invariance under inner automorphisms, see [1, § 4.6]. Since the cohomology groups $H^q(B_G, \mathbf{Z})$ are finite for $q > 0$ we deduce, from (4.6).

Corollary (**4.7**). — *Let* G *be a finite group. Then* $\widetilde{\mathscr{K}}^*(B_G)$ *satisfies* (F).

Suppose $f : Y \to X$ is a finite covering, X being a CW-complex. Then Y is a CW-complex and the n-skeletons Y^n of Y are simply the inverse images $f^{-1}(X^n)$ of the n-skeletons of X [13, § 4.5]. Thus f is the limit of maps $f_n : Y^n \to X^n$ of finite CW-complexes. It follows that the homomorphism $f_!$ of § 2 extends to the present infinite-dimensional complexes. Moreover the extended $f_!$ will have all the formal properties described in § 2. In particular (2.11) applies. Taking $X = B_G$ and Y the universal covering of X we deduce:

Proposition (**4.8**). — $G.\widetilde{\mathscr{K}}^*(B_G)$ *is annihilated by the order of* G.

Taking $X = B_G$ and $Y = B_H$, where H is a p-Sylow subgroup of G, we deduce:

Proposition (**4.9**). — *The p-primary component of* $G\mathscr{K}^*(B_G)$ *is a direct factor of* $G\mathscr{K}^*(B_H)$, *where* H *is a p-Sylow subgroup of* G.

Remark. — Both these propositions are analogous to the corresponding results for cohomology.

Proposition (**4.10**). — *For each prime p dividing the order of* G *let* G_p *be a p-Sylow subgroup of* G. *Then* $\mathscr{K}^*(B_G) \to \sum_p \mathscr{K}^*(B_{G_p})$ *is a monomorphism.*

Proof. — Suppose $x \in \mathrm{Ker}\{\mathscr{K}^*(B_G) \to \mathscr{K}^*(B_{G_p})\}$ for all p, $x \neq 0$. Since $\mathscr{K}^*(B_G)$ is a complete filtered group it is Hausdorff, and so $x \neq 0$ implies that there exists an integer n so that $x \in \mathscr{K}_n^*(B_G)$, $x \notin \mathscr{K}_{n+1}^*(B_G)$. But then x would define an element of $G^n\mathscr{K}^*(B_G)$ giving zero in each $G^n\mathscr{K}^*(B_{G_p})$. In view of (4.8) and (4.9) this is a contradiction, and so the proposition is proved.

The problem of generalizing the spectral sequences (2.4) and (2.6) to infinite-dimensional complexes presents serious difficulties (espectally (2.6)). We shall not attempt this problem in general but in the next section we deal with the case of classifying spaces of finite groups.

§ 5. The spectral sequence of a normal subgroup.

Let G be a finite group, V a normal subgroup and put $S = G/V$. Let B_G, B_S be the classifying spaces of G, S, and let $E_G \to B_G$, $E_S \to B_S$ be the universal bundles (i.e. the universal coverings). Then we have a factorization (cf. [9, Satz 3.44])

$$E_G \to B_V \to B_G$$

where $B_V = E_G/V$ is a classifying space for V, and $B_V \to B_G$ is the bundle associated to the universal bundle with fibre S. Since V is normal $B_V \to B_G$ is moreover a principal S-bundle.

An element $g \in G$ defines a map $E_G \to E_G$ given by $x \to xg$. Since V is normal we have $xgV = xVg$ and so g induces a map $0_g : B_V \to B_V$. On the other hand g induces an automorphism $v \to g^{-1}vg$ of V and hence a weak ([1]) homotopy class of maps $B_V \to B_V$. It is easy to check from the definitions that 0_g is a representative map of this class. Moreover 0_g depends only on the coset $s = gV$ and it describes the way in which s operates on the principal S-bundle B_V.

For simplicity put $X = B_G$, $Y = B_S$, $A = B_V$, $B = E_S$. Let X^n, Y^n be the n-skeletons of X, Y and let A^n, B^n be their inverse images in A, B. As already observed in § 4, A^n, B^n are the n-skeletons of a CW-structure on A, B. Moreover these CW-structures are invariant under the operation of S [13, § 4.5]. Hence $Z = (A \times B)/S$ will have an induced CW-structure and $Z^n = (A^n \times B^n)/S$ will be a finite sub-complex, connected for $n \geq 1$. Now we have two fibrations

$$
\begin{array}{ccc}
& Z & \\
B \swarrow & & \searrow A \\
X & & Y
\end{array}
$$

which are the limits, under inclusion, of the fibrations

$$
\begin{array}{ccc}
& Z^n & \\
B^n \swarrow & & \searrow A^n \\
X^n & & Y^n
\end{array}
$$

Since $B = E_S$ is contractible it follows that $Z \to X$ is a homotopy equivalence. Hence, by (4.3), we have $\mathscr{K}^*(Z) \cong \mathscr{K}^*(B_G)$. Since $Z = \bigcup_n Z^n$ it follows from (4.4), (4.5) and (4.7) that $\{\widetilde{\mathscr{K}}^*(Z^n)\}$ satisfies (F).

From the fibrations $Z^n \to Y^n$ we obtain, by (2.6), a spectral sequence $\{_nE_r\}$ with

$$_nE_2^p = H^p(Y^n, \mathbf{K}^*(A^n))$$
$$_nE_\infty^p = K_p^*(Z^n)/K_{p+1}^*(Z^n).$$

Here $\mathbf{K}^*(A^n)$ is the local coefficient system associated to the operation of

$$\pi_1(Y^n) \cong \pi_1(Y) \cong S \quad (n \geq 2)$$

on A^n defined by the fibration $A^n \to X^n$, and ([2]) $K_p^*(Z^n)$ is the filtration on $K^*(Z^n)$ induced from Y^n. By (2.8) we can decompose $_nE_r = {_n\widetilde{E}_r} \oplus \mathbf{Z}$ where $\mathbf{Z}$ is the trivial spectral sequence, $_n\widetilde{E}_r^p = {_nE_r^p}$ for $p > 0$ and

$$_n\widetilde{E}_2^0 = H^0(Y^n, \widetilde{\mathbf{K}}^*(A^n)),$$
$$_n\widetilde{E}_\infty^0 = \widetilde{K}_0^*(Z^n)/\widetilde{K}_1^*(Z^n).$$

_____

([1]) I.e. defined on each skeleton.

([2]) The suffix of (2.6) is omitted here to simplify notation.

Since Z^n is a finite CW-complex there is no convergence problem for $\{_nE_r\}$ as $r\to\infty$ (n fixed). Hence $\{_n\widetilde{E}_r, \widetilde{K}^*(Z^n)\}$ is an inverse system of strongly convergent spectral sequences in the sense of § 3. We wish to apply (3.19). Now condition $b)$ of (3.19) has already been verified so it remains to consider $a)$.

Since A^n is a finite CW-complex, $K^*(A^n)$ is a finitely-generated group (2.9). Hence $_n\widetilde{E}_2^p$ is finitely-generated. But, for $n>p$, we have:

$$_n\widetilde{E}_2^p \cong H^p(Y, \mathbf{K}^*(\mathbf{A}^n)) \cong H^p(S, K^*(A^n)),$$

since $Y = B_S$. Hence $_n\widetilde{E}_2^p$ for $n>p>0$, is annihilated by the order of S and so is finite. Thus, for $p>0$, $\{_n\widetilde{E}_2^p\}$ satisfies (F). For $p=0$ and $n>0$ we have:

$$_n\widetilde{E}_2^0 = H^0(S, \widetilde{K}^*(A^n)) \cong \widetilde{K}^*(A^n)^S \quad \text{(the invariants)}.$$

By (4.7) we know that $\{\widetilde{K}^*(A^n)\}$ satisfies (F) (since $A = B_V$) and so, by (3.12), $\{\widetilde{K}^*(A_n)^S\}$ also satisfies (F).

Conditions $a)$ and $b)$ of (3.19) therefore hold, and so we obtain a strongly convergent spectral sequence $\{\varprojlim_n \widetilde{E}_r, \mathscr{K}^*(Z)\}$. Adding the trivial spectral sequence $\mathbf{Z}$ does not affect the convergence and gives $\varprojlim H^p(S, K^*(A^n)) \Rightarrow \mathscr{K}^*(Z)$.
Now

$$H^p(S, K^*(A^n)) = H^p(\mathrm{Hom}_\Lambda(L_*, K^*(A^n)))$$

where $\Lambda = \mathbf{Z}[S]$ and L_* is the standard Λ-free resolution of $\mathbf{Z}$ [7, chapter X]. Since $\{\widetilde{K}^*(A^n)\}$ satisfies (F), $\{K^*(A^n)\}$ satisfies (ML) and so for each p $\{\mathrm{Hom}_\Lambda(L_p, K^*(A^n))\}$ satisfies (ML). Moreover as already observed $\{H^p(S, K^*(A^n))\}$ satisfies (F) for $p>0$, and $\{H^0(S, \widetilde{K}^*(A^n))\}$ satisfies (F). Hence for all p $\{H^p(S, K^*(A^n))\}$ satisfies (ML) (adding $\mathbf{Z}$ for $p=0$). Hence $(^1)$, by (3.9),

$$\varprojlim H^p(S, K^*(A^n)) \cong H^p(S, \mathscr{K}^*(A)).$$

Since $\mathscr{K}^*(A) = \mathscr{K}^*(B_V)$ and $\mathscr{K}^*(Z) \cong \mathscr{K}^*(B_G)$ we have established the following theorem.

Theorem (5.1). — *Let* G *be a finite group,* V *a normal subgroup and* $S = G/V$. *Then* $\mathscr{K}^*(B_G)$ *has a filtration defined relative to* S *(denoted by a subscript* S*), and we have a strongly convergent spectral sequence:* $H^*(S, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_G)_S$.

Either by taking $S = G$ in (5.1) or more directly by repeating the proof and using (2.4) instead of (2.6) we obtain

Theorem (5.2). — *Let* G *be a finite group. Then there is a strongly convergent spectral sequence:* $H^*(G, \mathbf{Z}) \Rightarrow \mathscr{K}^*(B_G)$.

All the properties of the spectral sequences (2.6) and (2.4) go over to (5.1) and (5.2). In particular this applies to the product structures and to the conditions under which $d_{2r} = 0$.

$(^1)$ This could have been incorporated in the proof of (3.19) by starting the spectral sequence with E_1.

In (5.1) $\mathscr{H}^*(B_V)$ is an S-module, the operation being induced by conjugation as explained earlier (see [1, § 4.6] for a direct definition of this operation).

The filtrations of (5.1) and (5.2) are such that $\mathscr{H}^*_p(B_G)_S \subset \mathscr{H}^*_p(B_G)$. Thus the "S-topology" of $\mathscr{H}^*(B_G)$ is *finer* than the "G-topology".

A notable case of (5.2) is the following:

Corollary (5.3). — *If* $H^q(G, \mathbf{Z}) = 0$ *for all odd* q, *then* $H^*(G, \mathbf{Z}) \cong G\mathscr{H}^*(B_G)$ *(as graded rings).*

Proof. — The hypothesis and d) of (2.4) imply that $d_r = 0$ for all r. The corollary then follows from (3.18).

A similar result holds for (5.1).

Corollary (5.4). — *If in* (5.1) $H^q(S, \mathscr{H}^*(B_V)) = 0$ *for all odd* q, *and* $\mathscr{H}^1(B_V) = 0$, *then* $H^*(S, \mathscr{H}^*(B_V)) \cong G\mathscr{H}^*(B_G)_S$ *(as graded rings).*

Proof. — This follows from c) of (2.6) and (3.18).

§ 6. The representation ring R(G).

Let G be a finite group. We denote by $R(G)$ the free abelian group generated by the equivalence classes of irreducible complex (or unitary) representations of G. Thus, if $\xi_1, \ldots, \xi_n$ are the (classes of) irreducible representations of G, every element of $R(G)$ can be written uniquely as

$$\rho = \sum_{i=1}^n r_i \xi_i, \qquad r_i \in \mathbf{Z}.$$

The (classes of) representations of G correspond to the "positive" elements of $R(G)$, i.e. those with $r_i \geq 0$ for all i (but not all $r_i = 0$).

The tensor product makes $R(G)$ into a ring. We shall call this the *representation ring* of G — it is isomorphic to the *character ring* of G. In this section we shall identify these two rings.

We define an augmentation $\varepsilon : R(G) \to \mathbf{Z}$ by $\varepsilon(\xi_i) = \dim \xi_i$, and we denote by $I(G)$ the kernel of ε. We shall consider $R(G)$ with the $I(G)$-adic (or augmentation) topology, and its completion $\widehat{R(G)}$ in this topology. The main result of this section will be

Theorem (6.1). — *Let* H *be a subgroup of* G. *Then the* $I(H)$-*adic topology of* $R(H)$ *is the same as its* $I(G)$-*adic topology* ($R(H)$ *being viewed as* $R(G)$-*module via the restriction homomorphism* $R(G) \to R(H)$).

Let G have order g. Let $\chi_1, \ldots, \chi_n$ be the characters of $\xi_1, \ldots, \xi_n$. Then

$$R(G) = \sum_i \mathbf{Z}\chi_i.$$

Let $\theta = \exp(2\pi i/h)$ where h is a multiple of g, and put $A = \mathbf{Z}[\theta]$, so that

$$A = \mathbf{Z} + \mathbf{Z}\theta + \ldots + \mathbf{Z}.\theta^{\varphi(h)-1} \qquad (\varphi \text{ the Euler function}).$$

Define
$$R_A(G) = R(G) \otimes_{\mathbf{Z}} A = \sum A\chi_i.$$

Now a character is a (class) function on G and its values are in A, being sums of eigen-values θ^ν of unitary representations of period dividing g. Hence we may regard $R_A(G)$ as a subring of the ring (¹) A^G of *all* A-valued functions on G.

Lemma (6.2). — (i) *Every prime ideal of* $R_A(G)$ *is the restriction of a prime ideal of* A^G.
(ii) *Every prime ideal of* $R(G)$ *is the restriction of a prime ideal of* $R_A(G)$.

Proof. — A^G is a finitely-generated Z-module. In fact if, for each $S \in G$, we let e_S denote the function taking the value 1 on S and 0 elsewhere, then

$$A^G = \sum_{S \in G} \sum_{\nu = 0}^{\varphi(h) - 1} Z\theta^\nu e_S.$$

Z is imbedded in R(G) by the trivial representation, in A^G by the constant integer valued functions. Since we have inclusions

$$Z \subset R(G) \subset R_A(G) \subset A^G,$$

it follows that

a) A^G is a finitely-generated $R_A(G)$-module;
b) $R_A(G)$ is a finitely-generated $R(G)$-module.

By the theorem of Cohen-Seidenberg [14, p. 257, Th. 3] *a)* and *b)* imply (i) and (ii) respectively.

We may remark, at this point, that all the rings occurring here are finitely-generated Z-modules and so certainly Noetherian.

The prime ideals in A^G are easy to describe, because A^G is just a sum of g copies of A. If $S \in G$ and $\mathfrak{p}$ is a prime ideal of A, then the set of functions $\psi \in A^G$ such that $\psi(S) \in \mathfrak{p}$ is a prime ideal of A^G, and every prime ideal of A^G is of this type. We denote the restriction of this prime ideal to $R_A(G)$ by $P_{\mathfrak{p},S}$. Thus

$$P_{\mathfrak{p},S} = \{\chi \in R_A(G) \mid \chi(S) \in \mathfrak{p}\}.$$

By (6.2) (i) we know that every prime ideal of $R_A(G)$ is of this form for some $\mathfrak{p}$ and some S.

If $\mathfrak{p} \neq (0)$ then $\mathfrak{p} \cap Z = p Z$ for some prime $p \neq 0$ of Z, $\mathfrak{p}$ is a maximal ideal of A and $A/\mathfrak{p}$ is a finite field of characteristic p. We then define S_p by the decomposition $S = S_p.B$, where S_p and B are powers of S, S_p has order *prime to* p and B has order a power of p. If $\mathfrak{p} = (0)$ we define $S_p = S$. S_p is called the p-regular factor of S.

Lemma (6.3). — $P_{\mathfrak{p},S} \supset P_{\mathfrak{p}',S'}$ *if and only if* (i) $\mathfrak{p} \supset \mathfrak{p}'$ *and* (ii) S_p *and* S'_p *are conjugate in* G.

Proof. — Suppose first that (i) and (ii) hold. To prove that $P_{\mathfrak{p},S} \supset P_{\mathfrak{p}',S'}$ it will be sufficient to show that, for any $\mathfrak{p}$, S and $\chi \in R_A(G)$ we have

$$\chi(S) \equiv \chi(S_p) \mod \mathfrak{p}.$$

This is trivial if $\mathfrak{p} = (0)$, so we may suppose $\mathfrak{p} \cap Z = p Z$, $p \neq 0$. Restricting χ to the cyclic subgroup generated by S we see that it is sufficient to deal with the case where G

(¹) Elsewhere this notation is used for the invariants, but there should be no confusion.

is cyclic and generated by S. Also we may suppose χ is irreducible, hence one-dimensional, hence multiplicative: $\chi(S_1 S_2) = \chi(S_1) \cdot \chi(S_2)$.

Let $f_{\mathfrak{p}} : A \to A/\mathfrak{p}$ be the canonical homomorphism. Then from $S = S_{\mathfrak{p}} . B$ we obtain

$$f_{\mathfrak{p}}(\chi(S)) = f_{\mathfrak{p}}(\chi(S_{\mathfrak{p}})) . f_{\mathfrak{p}}(\chi(B)).$$

Since $B^{p^\nu} = 1$ for some ν, we have $(f_{\mathfrak{p}}(\chi(B)))^{p^\nu} = 1$ for some ν. But $A/\mathfrak{p}$ is a finite field of characteristic p, hence $f_{\mathfrak{p}}(\chi(B)) = 1$, and this completes the proof.

Conversely suppose $P_{\mathfrak{p},S} \supset P_{\mathfrak{p}',S'}$. Then

$$\mathfrak{p} = P_{\mathfrak{p},S} \cap A \supset P_{\mathfrak{p}',S'} \cap A = \mathfrak{p}'.$$

Suppose first that $\mathfrak{p} \neq (o)$. Then according to [1] [5, Lemma 3] there exists $\eta \in R_A(G)$ such that

    (i) η has values in $\mathbf{Z}$;

    (ii) $\eta(T) = o$ if $T_{\mathfrak{p}}$ is not conjugate to $S_{\mathfrak{p}}$;

    (iii) $\eta(T) \equiv 1 \pmod{p}$ if $T_{\mathfrak{p}}$ is conjugate to $S_{\mathfrak{p}}$.

If $S_{\mathfrak{p}}'$ and $S_{\mathfrak{p}}$ were not conjugate we would have $\eta \in P_{\mathfrak{p}',S'}$ but $\eta \notin P_{\mathfrak{p},S}$. This gives a contradiction and so $S_{\mathfrak{p}}$ and $S_{\mathfrak{p}}'$ are conjugate. If $\mathfrak{p} = (o)$ and S and S' are not conjugate, then as is well-known there exists $\eta \in R(G)$ with $\eta(S') = o$, $\eta(S) \neq o$, which gives a contradiction also in this case.

This lemma leads at once to the following description of the "scheme" of the prime ideals of $R_A(G)$.

Proposition (**6.4**). — *The prime ideals of* $R_A(G)$ *are all of the form* $P_{\mathfrak{p},S}$. *Two such ideals* $P_{\mathfrak{p},S}$ *and* $P_{\mathfrak{p}',S'}$ *coincide if and only if* $\mathfrak{p} = \mathfrak{p}'$ *and* $S_{\mathfrak{p}}$ *and* $S_{\mathfrak{p}}'$ *are conjugate in* G. *If* $\mathfrak{p} = (o)$, $P_{\mathfrak{p},S}$ *is a minimal prime ideal, while if* $\mathfrak{p} \neq (o)$ $P_{\mathfrak{p},S}$ *is a maximal prime ideal. The maximal prime ideals containing* $P_{0,S}$ *are the ideals* $P_{\mathfrak{p},S}$ *with* $\mathfrak{p} \neq (o)$. *The minimal prime ideals contained in* $P_{\mathfrak{p},S}(\mathfrak{p} \neq (o))$ *are the* $P_{0,S'}$ *with* $S_{\mathfrak{p}}'$ *conjugate to* $S_{\mathfrak{p}}$.

Lemma [2] (**6.5**). — $P_{0,1} = A.I(G)$.

Proof. — Trivially $A.I(G) \subset P_{0,1}$. On the other hand let $\chi \in P_{0,1}$ and write

$$\chi = \sum_{\nu=0}^{\varphi(h) - 1} \chi_\nu \theta^\nu$$

with $\chi_\nu \in R(G)$. Then $\chi(1) = \sum_\nu \chi_\nu(1) . \theta^\nu = o$, with $\chi_\nu(1) \in \mathbf{Z}$. This implies $\chi_\nu(1) = o$ for all ν and $\chi_\nu \in I(G)$, i.e. $\chi \in A.I(G)$.

Now let H be a subgroup of G. To distinguish we shall write $P_{\mathfrak{p},S}(G)$ instead of $P_{\mathfrak{p},S}$. Let $\rho : R_A(G) \to R_A(H)$ be the restriction homomorphism.

Lemma (**6.6**). — *Suppose* $\rho^{-1}(P_{\mathfrak{p},S}(H)) = P_{\mathfrak{p}',1}(G)$. *Then* $P_{\mathfrak{p},S}(H) = P_{\mathfrak{p}',1}(H)$.

Proof. — Since $S \in H$, $\rho^{-1}(P_{\mathfrak{p},S}(H)) = P_{\mathfrak{p},S}(G)$. Hence $P_{\mathfrak{p},S}(G) = P_{\mathfrak{p}',1}(G)$, and so by (6.4) $\mathfrak{p} = \mathfrak{p}'$ and $S_{\mathfrak{p}}$ is conjugate in G to $1_{\mathfrak{p}} = 1$. Hence $S_{\mathfrak{p}} = 1$ and so $S_{\mathfrak{p}}$ is

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[1] This lemma is the main step in the proof of Brauer's theorem given in [5]. Brauer's theorem itself will be needed in § 11.

[2] We denote the identity of G by 1.

conjugate to $\mathrm{I_p}$ in H.   Hence by (6.4) applied to H we conclude that $\mathrm{P_{p,S}(H)} = \mathrm{P_{p',1}(H)}$ as required.

Lemma (**6.7**). — *The prime ideals of* $\mathrm{R(H)}$ *which contain* $\rho\mathrm{I(G)}$ *are the same as those which contain* $\mathrm{I(H)}$.

Proof. — Trivially $\rho\mathrm{I(G)} \subset \mathrm{I(H)}$.   Hence we must show that, if $\mathrm{P_0}$ is a prime ideal of $\mathrm{R(H)}$ which contains $\rho\mathrm{I(G)}$, then $\mathrm{P_0} \supset \mathrm{I(H)}$.   By (6.2) (ii) applied to H there exists p, S so that $\mathrm{P_0} = \mathrm{R(H)} \cap \mathrm{P_{p,S}(H)}$.   Then $\rho\mathrm{I(G)} \subset \mathrm{P_0}$ implies $\rho\mathrm{I(G)} \subset \mathrm{P_{p,S}(H)}$.   Hence by (6.5)

$$\rho\mathrm{P_{0,1}(G)} = \rho(\mathrm{A.I(G)}) \subset \mathrm{A}.\rho\mathrm{I(G)} \subset \mathrm{P_{p,S}(H)} \qquad \text{and so}$$
$$\mathrm{P_{0,1}(G)} \subset \rho^{-1}(\mathrm{P_{p,S}(H)}).$$

By (6.4) this implies $\rho^{-1}(\mathrm{P_{p,S}(H)}) = \mathrm{P_{p',1}(G)}$ and so by (6.6) $\mathrm{P_{p,S}(H)} = \mathrm{P_{p',1}(H)}$. Hence $\mathrm{I(H)} \subset \mathrm{P_{p,S}(H)}$, and therefore $\mathrm{I(H)} \subset \mathrm{R(H)} \cap \mathrm{P_{p,S}(H)} = \mathrm{P_0}$.   Q.E.D.

Theorem (6.1) follows at once from (6.7) and the fact that, in a Noetherian ring, the $\mathfrak{a}$-adic topology is the same as the $\mathfrak{a}'$-adic topology where $\mathfrak{a}'$ is the radical of $\mathfrak{a}$ (i.e. the intersection of all the prime ideals containing $\mathfrak{a}$).   For the proof of this statement see [11, p. 14, Th. 1 and p. 22, Prop. 8].

Lemma (**6.8**). — *Let* B *be a Noetherian ring with no (non-zero) nilpotent elements.   Let* $\mathfrak{b}$ *be a prime ideal of* B, *and let* $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ *be the minimal prime ideals of* B *numbered so that*

$$\begin{aligned} &\textit{for } 1 \leqslant i \leqslant m && \mathfrak{b} + \mathfrak{p}_i \neq \mathrm{B} && \textit{and} \\ &\textit{for } i > m && \mathfrak{b} + \mathfrak{p}_i = \mathrm{B}. \end{aligned}$$

Then
$$\bigcap_{n=1}^{\infty} \mathfrak{b}^n = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \ldots \cap \mathfrak{p}_m.$$

Proof. — In any Noetherian ring we have [11, p. 14, Th. 1]

$$\mathfrak{N} = \bigcap_{i=1}^{k} \mathfrak{p}_i,$$

where $\mathfrak{N}$ is the ideal of nilpotent elements and the $\mathfrak{p}_i$ are the minimal prime ideals. With the hypothesis of the lemma we have $\mathfrak{N} = (\mathrm{o})$.   Thus $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ are the primary components of (o) and the lemma now follows from [14, p. 218 Corollary].

We observe that the condition $\mathfrak{b} + \mathfrak{p}_i \neq \mathrm{B}$ in (6.8) is equivalent to: there exists a maximal prime ideal $\mathfrak{q}$ with $\mathfrak{b} + \mathfrak{p}_i \subset \mathfrak{q}$.   If $\mathfrak{b} + \mathfrak{p}_i \neq \mathrm{B}$ we take $\mathfrak{q}$ to be a maximal ideal containing $\mathfrak{b} + \mathfrak{p}_i$ and recall that a maximal ideal is necessarily prime.

We now apply (6.8) with $\mathrm{R_A(G)}$ for B and $\mathrm{P_{0,1}}$ for $\mathfrak{b}$.   From (6.4) it follows that the maximal prime ideals containing $\mathrm{P_{0,1}}$ are the $\mathrm{P_{p,1}}$ ($\mathrm{p} \neq (\mathrm{o})$), and that the minimal prime ideals contained in $\mathrm{P_{p,1}}$ are the $\mathrm{P_{0,S'}}$ with $\mathrm{S'_p}$ conjugate to $\mathrm{I_p} = \mathrm{I}$, i.e. S' of order a power of p.   Hence (6.8), together with (6.5), gives

Lemma (**6.9**). — $\displaystyle\bigcap_{n=1}^{\infty} (\mathrm{A.I(G)})^n = \{\chi \in \mathrm{R_A(G)} \mid \chi(\mathrm{S}) = \mathrm{o}$ *for all* $\mathrm{S} \in \mathrm{G}$ *having prime power order*$\}$.

Suppose now that J is any ideal of $R(G)$. Then $A.J$ is the ideal of $R_A(G)$ consisting

of elements of the form $\sum\limits_{\nu=0}^{\varphi(h)-1} j_\nu 0^\nu$, $j_\nu \in J$.

Since the 0^ν form a *free* basis for $R_A(G)$ over $R(G)$ it follows that $A.J \cap R(G) = J$. Taking $J = I(G)^n$, and observing that $A.I(G)^n = (A.I(G))^n$, we deduce from (6.9):

Proposition **(6.10)**. — $\bigcap\limits_{n=1}^{\infty} I(G)^n = \{\chi \in R(G) \mid \chi(S) = 0 \ \textit{for all} \ S \in G \ \textit{having prime power order}\}$.

(6.10) and (3.2) together imply

Proposition **(6.11)**. — *If* G *is a p-group then* $R(G) \to \widehat{R(G)}$ *is a monomorphism*.

In general, for each prime p dividing the order of G, let G_p be a p-Sylow subgroup of G. Then we have a restriction homomorphism

$$R(G) \to \sum\limits_{p} R(G_p).$$

Proposition **(6.12)**. — *The kernels of the two homomorphisms*

$$R(G) \to \widehat{R(G)} \quad \textit{and} \quad R(G) \to \sum\limits_{p} R(G_p)$$

coincide.

Proof. — Denote the homomorphisms by α, β respectively. In (6.10) Ker α is explicitly determined and from this it is immediate that Ker $\alpha \subset$ Ker β. Conversely suppose $\chi \in R(G)$ is in Ker β. Then $\chi(S) = 0$ for $S \in G_p$. But every element of G of order p is conjugate to an element S of G_p. Hence $\chi(S) = 0$ for all S of prime power order, i.e. $\chi \in$ Ker α.

Next we shall examine the quotients $I(G)^n/I(G)^{n+1}$.

Proposition **(6.13)**. — *Let* g *be the order of* G, *and let* $n > 0$. *Then* $I(G)^n/I(G)^{n+1}$ *is a finite group annihilated by* g.

Proof. — Since $R(G)$ is a finitely-generated group the same is true of $I(G)^n/I(G)^{n+1}$. Hence it will be sufficient to show that $g.I(G)^n \subset I(G)^{n+1}$ for $n > 0$.

For any subgroup H of G we have the two homomorphisms:

$$i^* : R(G) \to R(H) \qquad \text{(restriction)}$$
$$i_* : R(H) \to R(G) \qquad \text{(induced representation)},$$

and the formula $((e)$ of § 1$)$

$$i_*(i^*(\alpha).\beta) = \alpha.i_*(\beta) \qquad \alpha \in R(G), \ \beta \in R(H).$$

In particular we may take $H = 1$, and apply the formula with $\alpha \in I(G)^n$ $(n > 0)$ and $\beta = 1$. Then $i^*(\alpha) = 0$ and so we deduce $\alpha.i_*(1) = 0$. Now $i_*(1)$ has augmentation g (in fact $i_*(1)$ is the regular representation of G), and so $g - i_*(1) \in I(G)$. Hence

$$g\alpha = (g - i_*(1)).\alpha \in I(G)^{n+1},$$

which completes the proof.

Remark. — This proof is formally similar to that of (4.8) or to the corresponding result for cohomology.

§ 7. Statement of the main theorems.

Let G be a finite group, ξ a principal G-bundle over a finite CW-complex X. Then, as observed in § 1, each complex representation ρ of G defines a complex vector bundle $\rho(\xi)$ over X. Because of properties (1)-(5) of § 1 this extends to a ring homomorphism $R(G) \to K^0(X)$, compatible with inverse images, direct images and exterior powers. If X is an infinite dimensional CW-complex with finite skeletons X^n, then the homomorphisms $R(G) \to K^0(X^n)$ are compatible with each other and so define a homomorphism $R(G) \to \mathscr{K}^0(X) = \varprojlim K^0(X^n)$. In particular, taking $X = B_G$ and ξ the universal G-bundle we obtain a homomorphism

$$\alpha : R(G) \to \mathscr{K}^0(B_G).$$

Following α by the inclusion of $\mathscr{K}^0(B_G)$ in $\mathscr{K}^*(B_G)$ we obtain a homomorphism, which we still denote by α

$$\alpha : R(G) \to \mathscr{K}^*(B_G).$$

α is a ring homomorphism and commutes with inverse images, direct images (for $H \subset G$), exterior powers and augmentation (cf. § 4 and § 6).

If $\rho \in I(G)$ then $\alpha(\rho) \in \widetilde{\mathscr{K}^*}(B_G) = \mathscr{K}_2^*(B_G)$ (by (2.2)), and so

(7.1) $$\alpha(I(G)^n) \subset \mathscr{K}_{2n}^*(B_G).$$

Thus α is *continuous*, $R(G)$ having the $I(G)$-adic topology, and $\mathscr{K}^*(B_G)$ having the filtration (or inverse limit) topology. Hence α induces a homomorphism $\hat{\alpha}$ of the completions. Since $\mathscr{K}^*(B_G)$ is an inverse limit and hence complete (§ 3), it follows that $\hat{\alpha}$ is a homomorphism:

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G).$$

Our main theorem is then:

Theorem **(7.2)**. — $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is a topological isomorphism.*
Obvious corollaries are:

Corollary **(7.3)**. — $\mathscr{K}^1(B_G) = 0$.

Corollary **(7.4)**. — $\mathscr{K}^0(B_G)$ *has no elements of finite order.*

Corollary **(7.5)**. — *The topology on* $R(G)$ *induced by* α *from the filtration on* $\mathscr{K}^*(B_G)$ *coincides with the* $I(G)$*-adic topology.*

Combining (7.2) with (5.2) we obtain

Theorem **(7.6)**. — *Let* G *be a finite group, then* $\widehat{R(G)}$ *has a filtration for which there is a strongly convergent spectral sequence*

$$H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}.$$

This is the spectral sequence referred to in the introduction, bearing in mind that $GR(G) \cong \widehat{GR(G)}$ (3.10). Properties *a)-d)* of the introduction follow from *a)-d)* of (2.4). Property *e)* follows from (2.2).

From (2.5), and the fact that the mapping $\rho \to \rho(\xi)$ commutes with exterior

powers ((3) of § 1) and so in particular with the determinant operation "det", we deduce:

Proposition **(7.7)**. — *The 2-dimensional part of* (7.6) *gives an isomorphism*

$$R_2(G)/R_4(G) \cong H^2(G, \mathbf{Z}).$$

This isomorphism is induced by the mapping $\rho - \varepsilon(\rho) \to \det \rho$, *followed by the isomorphism* $\operatorname{Hom}(G, \mathbf{C}^*) \cong H^2(G, \mathbf{Z})$ *of* (1.3).

Combining (7.2) with (5.1) we obtain the following generalization of (7.6):

Theorem **(7.8)**. — *Let* G *be a finite group,* V *a normal subgroup,* $S = G/V$. *Then* R(G) *has a filtration defined relative to* S *(denoted by a subscript* S*) for which there is a strongly convergent spectral sequence*:

$$H^*(S, \widehat{R(V)}) \Rightarrow \widehat{R(G)}_S.$$

Here $\widehat{R(V)}$ *is an* S-*module, the operation of* S *being induced by conjugation in* G. *This spectral sequence is contravariant in* (V, G, S), *has products and is such that all* $d_{2r} = 0$.

The extra properties of the spectral sequence in (7.8) follow as before from (2.6).

Lemma **(7.9)**. — *To prove* (7.2) *it is sufficient to prove*

(i) $\hat{\alpha}$ *is a monomorphism*;

(ii) $\alpha R(G)$ *is dense in* $\mathscr{K}^*(B_G)$.

Proof. — We decompose

$$R(G) = \mathbf{Z} \oplus I(G), \qquad \mathscr{K}^*(B_G) = \mathbf{Z} \oplus \widetilde{\mathscr{K}}^*(B_G).$$

Then α and $\hat{\alpha}$ decompose accordingly. Consider then

1) $$\hat{\alpha} : \widehat{I(G)} \to \widetilde{\mathscr{K}}^*(B_G).$$

Now $\widehat{I(G)}$ is an inverse limit of finite groups (6.13) and so is a compact Hausdorff group. The same applies to $\widetilde{\mathscr{K}}^*(B_G)$, by (4.7) and (3.13). Hence $\hat{\alpha}(\widehat{I(G)})$ is *closed* in $\widetilde{\mathscr{K}}^*(B_G)$. Now (ii) implies that $\hat{\alpha}(\widehat{I(G)})$ is dense in $\widetilde{\mathscr{K}}^*(B_G)$ and so 1) must be an epimorphism. Together with (i) this proves 1) is an isomorphism. Since $\widehat{I(G)}$ and $\mathscr{K}^*(B_G)$ are compact Hausdorff groups any continuous isomorphism between them must be a homeomorphism. The same is then true for $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$.

§ 8. Cyclic Groups.

In this section we shall prove (7.2) for cyclic groups and then derive (i) of (7.9) for general finite groups.

Let G be a cyclic group of order n, and let ρ the representation which maps a generator of G to $\exp \dfrac{2\pi i}{n}$. Then from (1.3) it follows that ρ corresponds to a generator x of $H^2(G, \mathbf{Z})$. Now it is well-known [7, p. 251] that

1) $H^*(G, \mathbf{Z}) = \mathbf{Z}[x]$, where $nx = 0$, $\mathbf{Z}[x]$ denoting the graded ring of polynomials

in x. In particular there is no odd-dimensional cohomology and so, from (5.3) we have an isomorphism

$$2) \qquad H^*(G, \mathbf{Z}) \cong G\mathscr{H}^*(B_G).$$

Now $R(G) = \mathbf{Z}[\rho]$ where $\rho^n = 1$. If we put $\sigma = \rho - 1$, then $R(G) = \mathbf{Z}[\sigma]$ where $(1 + \sigma)^n = 1$, and $I(G)$ is the ideal (σ). Since

$$0 = (1 + \sigma)^n - 1 \equiv n\sigma \bmod \sigma^2,$$

it follows that, for $k > 0$, $I(G)^k/I(G)^{k+1}$ is cyclic of order n generated by the class of σ^k. If we filter $R(G)$ by defining $R_{2k}(G) = R_{2k-1}(G) = I(G)^k$ it follows that we have

3) $GR(G) = \mathbf{Z}[\bar{\sigma}]$ where $n\bar{\sigma} = 0$, and $\bar{\sigma}$ is the residue class of σ mod $I(G)^2$.

Consider now the homomorphism

$$\alpha : R(G) \to \mathscr{H}^*(B_G).$$

It is a homomorphism of filtered rings, and so induces a homomorphism of graded rings:

$$G\alpha : GR(G) \to G\mathscr{H}^*(B_G).$$

If we identify $G\mathscr{H}^*(B_G)$ with $H^*(G, \mathbf{Z})$ by 2) we find, from (2.5), that

$$G\alpha(\bar{\sigma}) = c_1\alpha(\rho) = x.$$

Hence from 1) and 3) $G\alpha$ is an isomorphism. From (3.10) we deduce:

Proposition (**8.1**). — *Let* G *be a cyclic group and filter* $R(G)$ *by putting*

$$R_{2k-1}(G) = R_{2k}(G) = I(G)^k.$$

Then $\widehat{R(G)}$ *has an induced filtration, and* $\hat{\alpha} : \widehat{R(G)} \to \mathscr{H}^*(B_G)$ *is an isomorphism of filtered groups.*

This is, for cyclic groups, a more precise result than (7.2).

We proceed now to prove (i) of (7.9).

Lemma (**8.2**). — *Let* G *be a finite group,* $\{G_\lambda\}$ *the family of all cyclic subgroups of* G. *Then* $R(G) \to \sum_\lambda R(G_\lambda)$ *(given by the restriction) is a monomorphism.*

Proof. — If $\rho \in R(G)$ gives zero in each G_λ, then $\chi_\rho | G_\lambda = 0$, where χ_ρ is the character of ρ. Since $G = \bigcup_\lambda G_\lambda$, this implies $\chi_\rho = 0$ and so $\chi = 0$.

Lemma (**8.3**). — *With the same notation as* (8.2)

$$\widehat{R(G)} \to \sum_\lambda R(G_\lambda)^\wedge$$

is a monomorphism (where each completion is with respect to the augmentation ideal of the corresponding group).

Proof. — By (8.2) we have an exact sequence

$$0 \to R(G) \to \sum_\lambda R(G_\lambda).$$

By (6.1) the $I(G_\lambda)$-topology of $R(G_\lambda)$ is the same as the $I(G)$-topology. Hence regarding $R(G)$ and $\sum_\lambda R(G_\lambda)$ as $R(G)$-modules, and completing with respect to the $I(G)$-topology, we get (by (3.16)) an exact sequence:

$$o \to R(G) \to (\sum_\lambda R(G_\lambda))^\smallfrown.$$

But $(\sum_\lambda R(G_\lambda))^\smallfrown \cong \sum_\lambda R(G_\lambda)^\smallfrown$, and so the lemma is proved.

Proposition (**8.4**). — *For any finite group* G

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{H}^*(B_G)$$

is a monomorphism.

Proof. — Let $\{G_\lambda\}$ be the family of all cyclic subgroups of G. Then we have a commutative diagram

$$\begin{array}{ccc} \widehat{R(G)} & \xrightarrow{\ \theta\ } & \sum_\lambda R(G_\lambda)^\smallfrown \\ \hat{\alpha} \downarrow & & \downarrow \varphi \\ \mathscr{H}^*(B_G) & \to & \sum_\lambda \mathscr{H}^*(B_{G_\lambda}). \end{array}$$

Ker $\theta = o$ (8.3) and φ is an isomorphism (8.1). Hence Ker $\hat{\alpha} = o$ as required.

§ 9. Some lemmas on representations.

Let V be normal in G, $S = G/V$. Let N be an irreducible G-module (complex representation space), $M \subset N$ an irreducible V-module. Then $\sum_{g \in G} gM \subset N$ and is invariant under G, hence $\sum_{g \in G} gM = N$. Now each gM is an irreducible V-module. Hence we can find a subset $g_1, \ldots, g_m$ of elements of G such that

$$N = \sum_{i=1}^m g_i M \quad \text{as } direct\ sum.$$

To see this consider a maximal subspace of N of the form $\sum_{i=1}^m g_i M$ (direct sum). If this is different from N, then some further gM exists which is not contained in $\sum_{i=1}^m g_i M$; since gM is irreducible $gM \cap \sum_{i=1}^m g_i M = o$, and so $\sum_{i=1}^m g_i M$ is not maximal.

Let ρ, σ be respectively the isomorphism classes of N (as G-module) and M (as V-module). Then if $i^* : R(G) \to R(V)$ is the restriction homomorphism, we have

$$(1) \qquad\qquad i^*(\rho) = \sum_{i=1}^m s_i(\sigma),$$

where $s_i = g_i^{-1} V$. We have just to recall that S operates on R(V), and this operation is such that $s(\sigma)$ is the class of gM if $s = g^{-1} V$. In detail, if $s = hV$ a representative V-module for $s(\sigma)$ is given by defining a new V-module structure on M as follows

$$v[x] = hvh^{-1}.x, \qquad x \in M.$$

Hence $x \to h^{-1}.x$ defines an isomorphism of this new V-module structure on M with the original V-module structure on $h^{-1}M$.

Now S operates trivially on R(G) and so $i^*(\rho)$ must be invariant under S. But

by (1) every irreducible component of $i^*(\rho)$ is of the form $s(\sigma)$ for some $s \in S$. Hence we must have $i^*(\rho) = n(\sum_j \sigma_j)$, where $\{\sigma_j\}$ is the complete set of (distinct) conjugates of σ (i.e. the "orbit" $S\sigma$). We state this as a lemma.

Lemma (9.1). — *Let* V *be normal in* G, ρ *an irreducible representation of* G. *Then*

$$i^*(\rho) = n\Sigma\sigma_j,$$

where $\{\sigma_j\}$ *is a complete set of conjugate irreducible representations of* V.

Lemma (9.2). — *Let* V *be normal in* G *with* $G/V = S$. *Let* σ *be an irreducible representation of* V, S_σ *the stabilizer of* σ *in* S. *Let*

$$i_*(\sigma) = \sum_k m_k \rho_k$$

be the decomposition of $i_*(\sigma)$ *(the induced representation) into irreducible representations* ρ_k *of* G. *Then* $\Sigma m_k^2 = (S_\sigma : 1)$.

Proof. — From $i_*(\sigma) = \Sigma m_k \rho_k$ we have

(2) $$(S : 1).\dim \sigma = \Sigma m_k \dim \rho_k.$$

By Frobenius's theorem and (9.1) we have $i^*(\rho_k) = m_k(\Sigma\sigma_j)$, where $\{\sigma_j\}$ is the complete set of conjugates of σ. Hence

(3) $$\dim \rho_k = (S : S_\sigma).m_k.\dim \sigma.$$

From (2) and (3) we deduce

$$(S : 1) = (S : S_\sigma)\,\Sigma m_k^2,$$

and so $$(S_\sigma : 1) = \Sigma m_k^2.$$

Lemma (9.3). — *Let* V *be normal in* G *with* $G/V = S$. *Let* σ *be an irreducible representation of* V *with stabilizer* S_σ, *and let* $\{\sigma_j\}$ *be the complete set of conjugates of* σ. *Suppose that* $(S_\sigma : 1)$ *is square-free. Then*

$$\Sigma\sigma_j \in i^*R(G).$$

Proof. — From (9.2) we have $\Sigma m_k^2 = (S_\sigma : 1)$. Since $(S_\sigma : 1)$ is square-free this implies that the m_k have no common factor. Hence there exist integers a_k such that $\Sigma a_k m_k = 1$. Hence

$$i^*(\Sigma a_k \rho_k) = (\Sigma a_k m_k)(\Sigma\sigma_j) = \Sigma\sigma_j.$$

Lemma (9.4). — *Let* V *be normal in* G *with* $G/V = S$. *Suppose that* $(S : 1)$ *is square-free. Then* $$R(V)^S = i^*R(G),$$

where $R(V)^S$ *denotes the invariants of* S.

Proof. — We have already remarked that $i^*R(G) \subset R(V)^S$. Now a **Z**-basis for $R(V)^S$ is given by the sums of complete sets of conjugates $\Sigma\sigma_j$. But for any σ, since $S_\sigma \subset S$ and $(S : 1)$ is square-free, it follows that $(S_\sigma : 1)$ is square-free. Hence (9.4) follows at once from (9.3).

The special case of (9.4) which we shall need later is explicitly:

Proposition (9.5). — *Let* V *be normal in* G *with* $G/V = Z_q$ *cyclic of prime order* q. *Then*

$$R(V)^{Z_q} = i^*R(G).$$

§ 10. Solvable groups.

In this section we shall prove (7.2) for solvable groups. The main step is the following:

Proposition (**10.1**). — *Let* V *be normal in* G *with* $G/V = Z_q$ *(q prime). Suppose* $\hat{\alpha}_V : \widehat{R(V)} \to \mathscr{K}^*(B_V)$ *is an isomorphism. Then* $\hat{\alpha}_G : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is an isomorphism.*

Proof. — By (5.1) we have a strongly convergent spectral sequence:

$$H^*(Z_q, \mathscr{K}^*(B_V)) \Rightarrow \mathscr{K}^*(B_G)_{Z_q}.$$

By hypothesis $\mathscr{K}^*(B_V) \cong \widehat{R(V)}$, so that

(1) $$E_2^p \cong H^p(Z_q, \widehat{R(V)}) \cong H^p(Z_q, R(V))^{\widehat{}} \qquad \text{by (3.17).}$$

Here we must observe the following: $\widehat{R(V)}$ is the completion in the I(V)-adic topology. By (6.1) this is the same as the I(G)-adic topology, R(V) being viewed as R(G)-module. Since $i^*R(G) \subset R(V)^{Z_q}$ it follows that R(V) is an $R(G)$-Z_q-module as required for (3.17). Moreover $H^p(Z_q, R(V))^{\widehat{}}$ denotes the I(G)-adic completion.

Let $\xi_1, \ldots, \xi_n$ be the (classes of) irreducible representations of V. Suppose ξ_i for $1 \leqslant i \leqslant r$ are invariant under Z_q, and that the ξ_i for $i > r$ fall into sets of q conjugates (these are the only possibilities since q is prime). Then as a Z_q-module

(2) $$R(V) = Z\xi_1 \oplus Z\xi_2 \oplus \ldots \oplus Z\xi_r \oplus M,$$

where M is a free Z_q-module. Hence $H^{2k+1}(Z_q, R(V)) = 0$, and so $E_2^{2k+1} = 0$. Since $\widehat{R(V)} \cong \mathscr{K}^0(B_G)$ and $\mathscr{K}^1(B_G) = 0$ (by assumption) it follows from (5.4), and (1), that

(3) $$G\mathscr{K}^*(B_G)_{Z_q} \cong H^*(Z_q, R(V))^{\widehat{}}.$$

To prove $\hat{\alpha}_G$ an isomorphism it is only necessary, by (7.9), to show that $\alpha(R(G))$ is dense in $\mathscr{K}^*(B_G)$. Since $\alpha(R(G)) \subset \hat{\alpha}\widehat{R(G)}$, and since the Z_q-topology of $\mathscr{K}^*(B_G)$ is finer than its G-topology it will be sufficient to prove that $\hat{\alpha}(\widehat{R(G)})$ is dense in $\mathscr{K}^*(B_G)$ *for the* Z_q-*topology*. This means we have to prove, for each p, that

(4) $$G^p\widehat{R(G)} \to G^p\mathscr{K}^*(B_G)_{Z_q}$$

is an epimorphism, where we give $\widehat{R(G)}$ the induced filtration: $\widehat{R_p(G)} = \hat{\alpha}_G^{-1}\mathscr{K}_p^*(B_G)_{Z_q}$. For $p = 0$ we have to show (using 3)) that

(5) $$\widehat{R(G)} \to (R(V)^{Z_q})^{\widehat{}} \to 0$$

is exact. But this follows from the fact that

(6) $$R(G) \to R(V)^{Z_q} \to 0$$

is exact (9.5), and that I(G)-adic completion is an exact functor (3.16). For $p = 2k+1$ it is trivial. Suppose therefore $p = 2k$, $k > 0$. To prove (4) in this case it will be sufficient, using (5), to prove that

$$\lambda : G^{2k}\widehat{R(Z_q)} \otimes_Z (R(V)^{Z_q})^{\widehat{}} \to G^{2k}\mathscr{K}^*(B_G)_{Z_q}$$

is an epimorphism, where λ is defined using the module multiplication of (2.7). But we have a commutative diagram:

$$H^{2k}(Z_q, \mathbf{Z}) \otimes_{\mathbf{Z}} R(V)^{Z_q} \xrightarrow{\mu} H^{2k}(Z_q, R(V))$$

$$H^{2k}(Z_q, \mathbf{Z}) \otimes_{\mathbf{Z}} (R(V)^{Z_q})^{\wedge} \xrightarrow{\lambda} H^{2k}(Z_q, R(V))^{\wedge},$$

where we have substituted for $G^{2k}\widehat{R(Z_q)}$ and $G^{2k}\mathscr{K}^*(B_G)$ by (8.1) and (3), and where λ, μ are now the obvious maps (as follows from (2.7)). From (2) we see that μ is an epimorphism, and the finiteness of $H^{2k}(Z_q, R(V))$ implies, by (3.3), that τ is an epimorphism. Hence λ is an epimorphism as required. This completes the proof.

Proposition (10.2). — *Let* G *be a solvable group. Then* $\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$ *is an isomorphism.*

Proof. — A solvable group has, by definition, a composition series

$$G = G_0 \supset G_1 \supset G_2 \supset \ldots \supset G_{n-1} \supset G_n = 1.$$

With G_{i-1}/G_i cyclic of prime order. The length n depends only on G. We prove (10.2) by induction on n. For $n = 1$, G is cyclic and so the result follows from (8.1). Suppose it is true for groups of length $n-1$. If G is of length n, then G_1 has length $n-1$. Hence $\hat{\alpha}_{G_1}$ is an isomorphism and $G/G_1 = Z_q$ (q prime). Hence by (10.1) $\hat{\alpha}_G$ is an isomorphism.

§ 11. The "completion" of Brauer's theorem.

We recall that an *elementary group* is a product of a p-group and a cyclic group. In particular an elementary group is solvable.

Let G be a finite group, $\{H_\lambda\}_{\lambda \in \Lambda}$ the family of all elementary subgroups of G. Let Q denote the group of inner automorphisms of G. Then Q operates on Λ, $\sigma(\lambda) \in \Lambda$ being defined for $\sigma \in Q$, $\lambda \in \Lambda$ by $H_{\sigma(\lambda)} = \sigma H_\lambda$. For any (ordered) triple λ, μ, $\nu \in \Lambda$ we define a homomorphism

$$\Theta_{\lambda,\mu,\nu} : R(H_\lambda) \to R(H_\mu \cap H_\nu)$$

as follows:
$$\Theta_{\lambda,\mu,\nu} = \begin{cases} \text{restriction if } \lambda = \mu, \lambda \neq \nu \\ -\text{restriction if } \lambda = \nu, \lambda \neq \mu \\ 0 \text{ otherwise.} \end{cases}$$

For any triple λ, μ, σ with λ, $\mu \in \Lambda$ and $\sigma \in Q$ we define a homomorphism

$$\Phi_{\lambda,\mu,\sigma} : R(H_\lambda) \to R(H_{\sigma(\mu)})$$

as follows
$$\Phi_{\lambda,\mu,\sigma} = \begin{cases} 1 & \text{if } \lambda = \sigma(\mu), \ \lambda \neq \mu \\ -\sigma^* & \text{if } \lambda \neq \sigma(\mu), \ \lambda = \mu \\ 1 - \sigma^* & \text{if } \lambda = \mu = \sigma(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The set of all $\Theta_{\lambda,\mu,\nu}$ and $\Phi_{\lambda,\mu,\sigma}$ defines a homomorphism

$$\Psi : \sum_{\lambda \in \Lambda} R(H_\lambda) \to \sum_{\mu,\nu \in \Lambda} R(H_\mu \cap H_\nu) \oplus \sum_{\substack{\mu \in \Lambda \\ \sigma \in Q}} R(H_{\sigma(\mu)}).$$

We shall now examine the kernel of Ψ. For any pair $\mu, \nu \in \Lambda$ and $\rho = \sum_\lambda \rho_\lambda \in \sum_\lambda R(H_\lambda)$, the component $\Psi(\rho)_{\mu,\nu}$ is by definition

$$\Psi(\rho)_{\mu,\nu} = \sum_\lambda \Theta_{\lambda,\mu,\nu}(\rho_\lambda),$$
$$= r_{\mu,\nu}(\rho_\mu) - r_{\nu,\mu}(\rho_\nu),$$

where $r_{\mu,\nu}$ is the restriction $R(H_\mu) \to R(H_\mu \cap H_\nu)$.

Next, for any pair $\pi \in \Lambda$, $\sigma \in Q$ the component $\Psi(\rho)_{\pi,\sigma}$ is by definition

$$\Psi(\rho)_{\pi,\sigma} = \sum_\lambda \Phi_{\lambda,\pi,\sigma}(\rho_\lambda),$$
$$= \rho_{\sigma(\pi)} - \sigma^*(\rho_\pi).$$

Hence $\rho \in \mathrm{Ker}\,\Psi$ if and only if

(i) $r_{\mu,\nu}(\rho_\mu) = r_{\nu,\mu}(\rho_\nu)$ for all $\mu, \nu \in \Lambda$,

(ii) $\rho_{\sigma(\pi)} = \sigma^*(\rho_\pi)$ for all $\pi \in \Lambda$, $\sigma \in Q$.

Consider now the character $\chi_\lambda = \chi(\rho_\lambda)$. It is a function on H_λ. (i) and (ii) are equivalent to:

(i') $\chi_\mu = \chi_\nu$ in $H_\mu \cap H_\nu$, for all $\mu, \nu \in \Lambda$,

(ii') $\chi_{\sigma(\pi)} = \sigma^*(\chi_\pi)$ for all $\pi \in \Lambda$, $\sigma \in Q$.

From (i') the set of χ_λ defines a single-valued function (with values in $\mathbf{C}$) on $\bigcup_\lambda H_\lambda$. Since the family of elementary groups includes all cyclic groups it follows that $\bigcup_\lambda H_\lambda = G$. Thus we have a function χ on G. Then (ii') asserts that $\sigma^*(\chi) = \chi$ for all $\sigma \in Q$. Hence $\rho = \sum \rho_\lambda$ belongs to the kernel of Ψ if and only if $\chi_\lambda = \chi(\rho_\lambda)$ is, for all λ, the restriction to H_λ of a class function χ on G. But the theorem of Brauer [5, Theorem B] asserts that such a class function χ is necessarily a character of G. Thus we may reformulate Brauer's theorem as follows:

Lemma (**11.1**). — *We have an exact sequence*

$$0 \to R(G) \overset{r}{\to} \sum_{\lambda \in \Lambda} R(H_\lambda) \overset{\Psi}{\to} \sum_{\mu,\nu \in \Lambda} R(H_\mu \cap H_\nu) \oplus \sum_{\substack{\pi \in \Lambda \\ \sigma \in Q}} R(H_{\sigma(\pi)})$$

where r is the restriction, $\{H_\lambda\}_{\lambda \in \Lambda}$ is the family of all elementary subgroups of G and Ψ is the homomorphism defined above.

Lemma (**11.2**). — *In the notation of* (11.1) *we have an exact sequence.*

$$0 \to \widehat{R(G)} \xrightarrow{\hat{r}} \sum_\lambda \widehat{R(H_\lambda)} \xrightarrow{\hat{\Psi}} \sum_{\mu,\nu} R(H_\mu \cap H_\nu)^\smallfrown \oplus \sum_{\pi,\sigma} R(H_{\sigma(\pi)})^\smallfrown$$

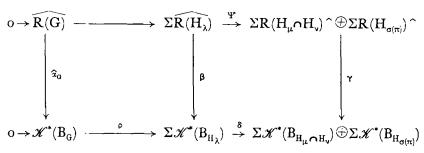*where each completion is with respect to the augmentation topology.*

Proof. — This follows at once from (11.1), (6.1) and (3.16).

Proposition (**11.3**). — *For any finite group*

$$\hat{\alpha} : \widehat{R(G)} \to \mathscr{K}^*(B_G)$$

is an epimorphism.

Proof. — With the notation introduced above we have a commutative diagram:

$$
\begin{array}{ccccc}
0 \to \widehat{R(G)} & \longrightarrow & \Sigma\widehat{R(H_\lambda)} & \xrightarrow{\Psi} & \Sigma R(H_\mu \cap H_\nu)^\smallfrown \oplus \Sigma R(H_{\sigma(\pi)})^\smallfrown \\
\Big\downarrow{\hat{\alpha}_\sigma} & & \Big\downarrow{\beta} & & \Big\downarrow{\gamma} \\
0 \to \mathscr{K}^*(B_G) & \xrightarrow{\rho} & \Sigma\mathscr{K}^*(B_{H_\lambda}) & \xrightarrow{\delta} & \Sigma\mathscr{K}^*(B_{H_\mu \cap H_\nu}) \oplus \Sigma\mathscr{K}^*(B_{H_{\sigma(\pi)}})
\end{array}
$$

The top line is exact by (11.2). In the bottom line we have exactness at $\mathscr{K}^*(B_G)$, i.e. ρ a monomorphism, by (4.10) and the fact that the H_λ include all Sylow subgroups of G. δ is defined in a precisely analogous way to Ψ, and since Q (the group of inner automorphisms of G) operates trivially on $\mathscr{K}^*(B_G)$ (cf. § 4) it follows that $\delta\rho = 0$. Now β is an isomorphism, since the H_λ are solvable (10.2). Also γ is a monomorphism (8.4). Hence $\rho\hat{\alpha}_G : \widehat{R(G)} \to \mathrm{Ker}\,\delta$ is an epimorphism. Since $\rho\mathscr{K}^*(B_G) \subset \mathrm{Ker}\,\delta$, it follows that

$$\rho\mathscr{K}^*(B_G) = \mathrm{Ker}\,\delta = \rho\hat{\alpha}_G(\widehat{R(G)}).$$

Since ρ is a monomorphism this implies $\mathscr{K}^*(B_G) = \hat{\alpha}_G(\widehat{R(G)})$, i.e. $\hat{\alpha}_G$ is an epimorphism.

(11.3) and (8.4) together complete the proof of the main theorem (7.2) (in view of (7.9)).

§ 12. The filtration of $R(G)$.

As remarked in the introduction the filtration on $R(G)$ has been defined topologically, via B_G, and the problem of giving an algebraic definition of the filtration is left unsolved. There is however a good candidate for such an algebraic definition due to Grothendieck, which we shall proceed to describe.

We recall first the notion of a λ-*ring*, introduced by Grothendieck. A λ-ring is a commutative ring R (with identity) with operators

$$\lambda^i : R \to R$$

(i an integer $\geqslant 0$), satisfying the following conditions:

(1) $$\lambda^0 x = 1, \quad \lambda^1 x = x, \quad \lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x).\lambda^{n-i}(y).$$

If we put, for any $x \in R$,

(2) $$\lambda_t(x) = \sum_{n \geqslant 0} \lambda^n(x) t^n \in R[[t]],$$

the relations (1) express the fact that $x \to \lambda_t(x)$ is a homomorphism of the additive group R into the multiplicative group $1 + R[[t]]^+$ (formal power series with constant term 1) which is a right inverse of the homomorphism $(1 + \sum_{i \geqslant 1} x_i t^i) \to x_1$.

The ring $\mathbf{Z}$ of integers has a unique λ-ring structure such that

(3) $$\lambda_t(1) = 1 + t.$$

Then we have

(4) $$\lambda_t(n) = (1+t)^n, \quad \lambda^i(n) = \binom{n}{i}.$$

An *augmented λ-ring* will then mean a λ-ring R together with a homomorphism $\varepsilon : R \to \mathbf{Z}$ of λ-rings, $\mathbf{Z}$ having the λ-structure just described.

If R is any λ-ring, Grothendieck defines operators γ^n by the formula:

(5) $$\gamma^n(x) = \lambda^n(x+n-1),$$

and $\gamma_t(x)$ by

(6) $$\gamma_t(x) = \sum_{n \geqslant 0} \gamma^n(x) t^n.$$

Then γ_t and λ_t are related by the formula

(7) $$\gamma_t(x) = \lambda_{t/1-t}(x),$$

or equivalently

(8) $$\lambda_s(x) = \gamma_{s/1+s}(x).$$

These show, in particular, that the γ^n also satisfy the identities (1).

Now let R be an augmented λ-ring and let $I = \text{Ker } \varepsilon$, where $\varepsilon : R \to \mathbf{Z}$ is the augmentation. Then the filtration on R defined by Grothendieck is as follows [1]: R_{2n} *is the subgroup generated by the monomials*

$$\gamma^{n_1}(x_1).\gamma^{n_2}(x_2), \ldots, \gamma^{n_k}(x_k)$$

with $x_i \in I$ *and* $\sum_{1}^{k} n_i \geqslant n$. We shall refer to this as the *γ-filtration* of the augmented λ-ring.

Since ε commutes with λ_t it also, by (7), commutes with γ_t, and hence if $x \in I$

$$\varepsilon \gamma^n(x) = \gamma^n \varepsilon(x) = \gamma^n(0) = 0 \qquad \text{for } n \geqslant 1.$$

This shows that, in the γ-filtration, we have

(9) $$R_2 = I, \qquad R_0 = R.$$

From the definition it is clear that the γ-filtration makes R a *filtered ring*, i.e.

$$R_{2n}.R_{2m}^{\sharp} \subset R_{2n+2m}.$$

_____

[1] We adopt an "even" notation for the filtration in order to conform with the topological aspect.

As pointed out by Grothendieck the rings $K^0(X)$ (for a connected finite CW-complex X) and R(G) are augmented λ-rings, the λ^i being the exterior powers. Thus if $x \in K^0(X)$ is represented by the formal sum

$$x = \Sigma n_i \xi_i,$$

where the ξ_i are vector bundles on X, $\lambda_t(x)$ is defined by

$$\lambda_t(x) = \Pi \lambda_t(\xi_i)^{n_i},$$

where $\lambda^k(\xi_i)$ is the k-th exterior power of the vector bundle ξ_i. It is not difficult to show that this definition of $\lambda_t(x)$ is unambiguous, and makes $K^0(X)$ an augmented λ-ring. In a similar way if $\rho \in R(G)$ is given by

$$\rho = \Sigma n_i \rho_i,$$

where the ρ_i are the irreducible representations of G, then $\lambda_t(\rho)$ is defined by

$$\lambda_t(\rho) = \Pi \lambda_t(\rho_i)^{n_i}$$

where $\lambda^k(\rho_i)$ is the k-th exterior power of the representation ρ_i. Since R(G) is a free abelian group with the ρ_i as basis, there is in this case nothing to prove. It is perhaps worth remarking that if we identify R(G) with the character ring of G, so that $R(G)[[t]]$ becomes a subring of the ring of all functions $G \to C[[t]]$, then for any representation ρ of G, $\lambda_t(\rho)$ is the function given by

$$g \to \det(1 + t\rho(g)).$$

We shall now consider the γ-filtration of R(G), and to distinguish it from the topological filtration we shall denote the subgroups of the γ-filtration by $R'_{2n}(G)$.

Proposition (**12.1**). — *Let* G *be a finite group*, $\rho_1, \ldots, \rho_k$ *its irreducible representations. Put* $\sigma_{ij} = \lambda^i(\rho_j - \varepsilon(\rho_j) + i - 1)$ *and define the weight of* σ_{ij} *to be* i. *Then* $R'_{2n}(G)$, *the n-th subgroup of the γ-filtration of* R(G), *is the subgroup generated by the monomials of weight* $\geq n$ *in the elements* $\sigma_{ij}(i = 1, 2, \ldots, \varepsilon(\rho_j); j = 1, \ldots, k)$.

Proof. — The elements $\rho_j - \varepsilon(\rho_j)$ form an additive base for I(G). Now by (7) it follows that $\gamma^i(nx + my)$ is expressible as a polynomial of weight i in the $\gamma^k(x), \gamma^k(y)$ (where weight $\gamma^k = k$) for all integers m, n. Hence, from the definition of $R'_{2n}(G)$, we see that it is generated additively by the monomials of weight $\geq n$ in the σ_{ij}. However for $i > \varepsilon(\rho_j)$ we have

$$\sigma_{ij} = \lambda^i(\rho_j + k)$$

where $k \geq 0$ and $i > \varepsilon(\rho_j) + k$, so that $\sigma_{ij} = 0$. This completes the proof.

Corollary (**12.2**). — *The graded ring associated to the γ-filtration of* R(G) *is finitely-generated. The number of generators can be chosen equal to the sum of the dimensions of the irreducible representations of* G.

The γ-filtration of R(G) defines a topology which we shall call the γ-topology.

Corollary (**12.3**). — *The γ-topology of* R(G) *coincides with the* I(G)-*adic topology.*

Proof. — From the definition we have

$$I(G)^n \subset R'_{2n}(G).$$

Conversely, let n be given, and put $m = sn$ where $s = \max \varepsilon(\rho_j)$. Consider any monomial in the σ_{ij} of weight $\geq m$. Since $i \leq s$, the degree of the monomial must be $\geq \frac{m}{s} = n$, where we define the degree of each σ_{ij} to be 1. But by 9) $\sigma_{ij} \in I(G)$. Hence

$$R'_{2m}(G) \subset I(G)^n,$$

completing the proof.

Corollary (12.4). — *For an abelian group* G, *we have*

$$R'_{2n}(G) = I(G)^n$$

Proof. — If G is abelian all the irreducible representations ρ_j have dimension 1. Hence, by (12.1), $R'_{2n}(G)$ is generated by the monomials in the elements $\sigma_{1j} = (\rho_j - 1)$ of weight $\geq n$. Since the σ_{1j} form an additive basis for $I(G)$ we have

$$R'_{2n}(G) = I(G)^n$$

as required.

We turn next to consider the γ-filtration of the augmented ring $K^0(X)$. Again to distinguish this from the topological filtration we shall denote the subgroups by $K^{0'}_{2n}(X)$.

Proposition (12.5). — *For all n we have*

$$K^{0'}_{2n}(X) \subset K^0_{2n}(X).$$

Proof. — Since $K^0(X)$, with the topological filtration, is a filtered ring it will be sufficient to show that if $x \in K^0(X)$ with $\varepsilon(x) = 0$ then

$$\lambda^n(x + n - 1) \in K^0_{2n}(X).$$

In view of (2.2) it will be sufficient to show that, if dim $X \leq 2(n-1)$, then

$$\lambda^n(x + n - 1) = 0.$$

Now since $\varepsilon(x + n - 1) = n - 1 \geq \frac{1}{2} \dim X$ it follows that $x + n - 1$ is in the "stable range" and so (it is easy to show) can be represented by a vector bundle ξ of dimension $n - 1$. Then $\lambda^n(x + n - 1)$ is represented by $\lambda^n(\xi)$ and this is zero since $n > \dim \xi$.

Since $K^0(X)$ is an augmented λ-ring, for all finite connected CW-complexes X, it follows that the inverse limit group $\mathscr{K}^0(B_G)$ is also an augmented λ-ring. Moreover from the definitions it is immediate (cf. § 1) that

$$\alpha : R(G) \to \mathscr{K}^0(B_G)$$

is a homomorphism of augmented λ-rings, and hence $\alpha(R'_{2n}(G)) \subset \mathscr{K}^{0'}_{2n}(B_G)$. From (12.5) therefore we deduce

Proposition (12.6). — *Let* $\{R'_{2n}(G)\}$ *be the γ-filtration of the augmented λ-ring* R(G), *and let* $\{R_{2n}(G)\}$ *be the topological filtration. Then, for all n, we have*

$$R'_{2n}(G) \subset R_{2n}(G).$$

Next we need an elementary lemma.

Lemma (**12.7**). — *In any* λ-*ring, with* γ *being defined by* (5), *we have the identity*

$$\lambda^n(x) = \sum_{i=0}^{n} \gamma^i(x-n).$$

Proof. — By (1) and (7)

$$\lambda_t(x) = \lambda_t(x-n) \cdot \lambda_t(n) = \gamma_{t/1+t}(x-n) \cdot (1+t)^n$$
$$= \sum_{i \geqslant 0} \gamma^i(x-n)t^i(1+t)^{n-i}.$$

Equating coefficients of t^n, the lemma follows.

Proposition (**12.8**). — *For* $n = 0$, 1, 2 *we have*

$$R'_{2n}(G) = R_{2n}(G).$$

Proof. — In view of (9) we need only consider the case $n = 2$. Let $y \in R_4(G)$, then we can write $y = \rho - \tau$, where ρ, τ are representations of G of dimension n and by (7.7), $\lambda^n(\rho) = \lambda^n(\tau)$. Thus

$$y = \{\rho - \lambda^n(\rho) - n + 1\} - \{\tau - \lambda^n(\tau) - n + 1\}$$

Applying (12.7) with ρ, τ instead of x we see that

$$\rho - \lambda^n(\rho) - n + 1 = -\sum_{i=2}^{n} \gamma^i(\rho - n)$$
$$\tau - \lambda^n(\tau) - n + 1 = -\sum_{i=2}^{n} \gamma^i(\tau - n).$$

This shows that $y \in R'_4(G)$ which, in view of (12.6), completes the proof.

The preceding results make it not unreasonable to conjecture that, for all n and G, we have $R'_{2n}(G) = R_{2n}(G)$. We shall in fact verify this conjecture in the next section for a few explicit groups.

In connection with this conjecture, (12.2) should be compared with a recent result of L. Evens, to the effect that $H^*(G, \mathbf{Z})$ is finitely-generated [1].

§ 13. Some examples.

In this section we shall compute a few illustrative examples of the spectral sequence $H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}$.

The symmetric group S_3.

The character table of S_3 is

	(Conjugacy classes)		
	1^3	$2\,1$	3
1	1	1	1
x	1	-1	1
y	2	0	-1

(Irreducible representations)

---

[1] (Added in proof) It can in fact be proved that $G\mathscr{H}^*(B_G)$ is finitely generated.

Now by (4.9) the filtration on $R = R(S_3)$ is determined by its p-Sylow subgroups ($p = 2, 3$). Since these are cyclic (8.1) enables us to determine the filtration completely. Putting $\alpha = 1 - x$, $\beta = 2 - y$ we find

$$R_2 = \{\alpha, \beta\}, \qquad R_4 = \{2\alpha, \alpha + \beta\}$$
$$R_{4k+4} = (\alpha + \beta) R_{4k} \quad (k \geqslant 1),$$

where for R_2, R_4 we have written the generators.

For the cohomology of S_3 it is well-known that we have

$$H^{4k+2}(S_3, \mathbf{Z}) = Z_2, \qquad H^{2q+1}(S_3, Z) = 0$$
$$H^{4k+4}(S_3, \mathbf{Z}) = Z_6,$$

and that the generator of $H^4(S_3, \mathbf{Z})$ gives the periodicity (by cup-products).

Since there are only even dimensions the spectral sequence collapses and we have (5.3) a ring isomorphism $H^*(S_3, \mathbf{Z}) \cong GR(S_3)$. This checks with the above formulae, α mod R_4 giving the generator of $H^2(S_3, \mathbf{Z})$ and $(\alpha + \beta)$ mod R_6 giving the generator of $H^4(S_3, \mathbf{Z})$.

Since $\lambda^2 y = x$, the elements σ_{ij} of (12.1) which generate the γ-filtration of $R(G)$ are
$$-\alpha, \ -\beta \in R_2'(G), \ -\alpha + \beta \in R_4'(G).$$

Since $\alpha^2 = 2\alpha$, $\beta^2 = 3\beta - \alpha$, $\alpha\beta = 2\alpha$ it follows that $R_{2n}'(G) = R_{2n}(G)$ for all n, in accordance with the conjecture of § 12.

The Quaternion group.

G is now the group whose elements are ± 1, $\pm i$, $\pm j$, $\pm k$ under quaternion multiplication.

The character table of G is

(Conjugacy classes)

	I	-1	$\pm i$	$\pm j$	$\pm k$
I	I	I	I	I	I
x_i	I	I	I	-1	-1
x_j	I	I	-1	I	-1
x_k	I	I	-1	-1	I
y	2	-2	0	0	0

(Irreducible representations)

The cohomology of G is [7, p. 254]
$$H^{4k+2}(G, \mathbf{Z}) = Z_2 \oplus Z_2,$$
$$H^{4k+4}(G, \mathbf{Z}) = Z_8,$$
$$H^{2q+1}(G, \mathbf{Z}) = 0,$$

and the generator of $H^4(G, \mathbf{Z})$ gives the periodicity.

Thus again the spectral sequence is trivial and so we have a ring isomorphism $H^*(G, \mathbf{Z}) \cong GR(G)$. However, unlike the case of S_3, we have no guaranteed method of determining the filtration on $R(G)$. In fact we *can* determine the filtration as follows.

First put $\alpha = 1 - x_i$, $\beta = 1 - x_i$, $\gamma = 3 - x_i - x_j - x_k$, $\delta = 2 - y$. Then $R_2 = \{\alpha, \beta, \gamma, \delta\}$ and the products are given by:

$$\alpha^2 = 2\,\alpha, \quad \alpha\beta = 2\,\alpha + 2\,\beta + \gamma, \quad \alpha\gamma = 4\,\alpha, \quad \alpha\delta = 2\,\alpha,$$
$$\beta^2 = 2\,\beta \qquad\qquad \beta\gamma = 4\,\beta, \quad \beta\delta = 2\,\beta,$$
$$\gamma^2 = 4\,\gamma, \quad \gamma\delta = 2\,\gamma,$$
$$\delta^2 = 2\,\delta - \gamma.$$

Now to determine R_4 it is sufficient by (7.7) to consider determinants (or the first Chern class). It is easy to see that $c_1(\alpha) = a$, $c_1(\beta) = b$ are generators of $H^2(G, \mathbf{Z})$, and that $c_1(\gamma) = 0$. Since there is an automorphism of G permuting x_i, x_j, x_k cyclically, it follows by symmetry that $\det y = 1$, i.e. $c_1(\delta) = 0$. Thus $R_4 = \{2\,\alpha, 2\,\beta, \gamma, \delta\}$. From the product formulae we find

$$R_2 R_4 = \delta R_2 = \{2\,\alpha, 2\,\beta, 2\,\gamma, 4\,\delta - \gamma\}.$$

This is of index 8 in R_4 and so must be R_6. Moreover δ mod R_6 gives a generator d of $H^4(G, \mathbf{Z})$. The fact that d gives the periodicity of $H^*(G, \mathbf{Z})$ then shows that the filtration of $R(G)$ is given by

$$R_{4k+2} = \delta^k R_2, \quad R_{4k+4} = \delta^k R_4.$$

Since $\lambda^2 y = 1$, the elements σ_{ij} of (12.1) are

$$-\alpha, \quad -\beta, \quad \alpha + \beta - \gamma \in R_2'(G), \quad -\delta \in R_4'(G).$$

Since $x_i x_j x_k = 1$ we deduce

$$(1 - \alpha)\,(1 - \beta)\,(1 + \alpha + \beta - \gamma) = 1$$

and hence $\gamma \in I(G)^2 \subset R_4'(G)$. Also $\alpha^2 = 2\,\alpha$, $\beta^2 = 2\,\beta$, so that we have $R_{2n}'(G) = R_{2n}(G)$ for all n, in accordance with our conjecture.

We can now use the product formulae in $R(G)$ to compute cup-products. We get

$$a^2 = b^2 = 0, \qquad ab = 4\,d.$$

Remark. — Whenever, as in this example, the odd cohomology groups vanish *and* the filtration on $R(G)$ is known the cup-products in $H^*(G, \mathbf{Z})$ can be read off from the character table of G.

A product of cyclic groups of order 2.

Let $G = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ (n factors). Then $H^*(G, \mathbf{Z})$ has non-zero odd-dimensional groups, so that the spectral sequence does not collapse. Now the first operator d_3 of the spectral sequence is the Steenrod operation Sq^3 (2.4), $d)$. A direct

calculation ([1]) shows that $E_4 = H(H^*(G, \mathbf{Z}), d_3)$ is generated multiplicatively over $\mathbf{Z}_2$ by elements x_i (of dimension 2) $i = 1, \ldots, n$ with relations

(1) $$x_i^2 x_j = x_i x_j^2 \quad (i, j = 1, \ldots, n).$$

Thus E_4 has only even-dimensional terms and so (cf. (5.3)) $E_4 \cong E_\infty \cong GR(G)$.

On the other hand $R(G)$ is generated over $\mathbf{Z}$ by elements ρ_i $(i = 1, \ldots, n)$ with $\rho_i^2 = 1$. Putting $\alpha_i = 1 - \rho_i$, we get the relations $\alpha_i^2 = 2\,\alpha_i$. These imply the equations

(2) $$\alpha_i^2 \alpha_j = 2\,\alpha_i \alpha_j = \alpha_i \alpha_j^2$$

α_i mod R_4 gives an element of E_4 which is easily seen to be x_i. The relations (2) then check with the relations (1). Moreover we see that $R_{2n}(G) = I(G)^n$ which, in view of (12.4), agrees with our conjecture.

Remark. — The calculation for a product of cyclic groups $\mathbf{Z}_p$ (p a prime $\neq 2$) is quite similar. One has to use the operator d_{2p-1} of the spectral sequence.

A direct description of the filtration on $R(G)$, for example a proof of the conjecture of § 12, would lead to lower bounds for the cohomology groups of G. In the absence of such a description we can only give a weak qualitative result in this direction.

Theorem (**13.1**). — *Let* G *be a finite group containing more than one element. Then there exist arbitrarily large integers n so that* $H^n(G, \mathbf{Z}) \neq 0$.

Proof. — The hypothesis on G and (6.10) imply that

$$\mathrm{Im}\{I(G) \to \widehat{I(G)}\} = I(G) / \bigcap_{n=1}^{\infty} I(G)^n$$

is a free abelian group of rank > 0. Now if $H^n(G, \mathbf{Z}) = 0$ for all sufficiently large n then by (7.6) $\widehat{I(G)}$ would be finite. This gives a contradiction, and so the theorem is proved.

APPENDIX

Chern Classes.

If ξ is an n-dimensional complex vector bundle over a CW-complex X, then ξ has Chern classes $c_i(\xi) \in H^{2i}(X, \mathbf{Z})$. For the definition and properties of these classes we refer to [9, § 4] or [2, § 9]. Taking $X = B_G$, the classifying space of a finite group G, we deduce

(1) To each complex representation ρ of G there are associated Chern classes $c_i(\rho) \in H^{2i}(G, \mathbf{Z})$, $c_0(\rho) = 1$ and $c_i(\rho) = 0$ for $i > \dim \rho$.

The Chern classes $c_i(\rho)$ are thus defined topologically. It would be highly desirable to have a direct algebraic definition of them, but like the corresponding problem for the spectral sequence $H^*(G, \mathbf{Z}) \Rightarrow \widehat{R(G)}$ this is still unsolved.

We proceed now to give the formal properties of Chern classes.

(2) *If* $f : G' \to G$ *is a homomorphism and ρ is a representation of G, then*

$$c_i(f^*\rho) = f^* c_i(\rho).$$

([1]) I am indebted to C.T.C. Wall for this calculation.

(3) *For 1-dimensional representations*

$$c_1 : \mathrm{Hom}(G, U(1)) \to H^2(G, \mathbf{Z})$$

is an isomorphism.

This is in fact the isomorphism (1.3).

(4) *If ρ^* is the dual (or contragredient) representation of ρ, then*

$$c_i(\rho^*) = (-1)^i c_i(\rho).$$

It is convenient to consider the sum of all the Chern classes

$$c(\rho) = \sum_i c_i(\rho) \in H^*(G, \mathbf{Z}).$$

(5) $c(\rho \oplus \sigma) = c(\rho) . c(\sigma)$.

Suppose now that $\mu : U(n) \to U(m)$ is a representation of the unitary group $U(n)$. Then μ has associated with it m integral linear forms $w_1, \ldots, w_m$ (the weights) in variables $x_1, \ldots, x_n$ (cf. [2, § 10]). Moreover the elementary symmetric functions $\sigma_i(w_1, \ldots, w_m)$ are symmetric in $x_1, \ldots, x_n$ and hence expressible as polynomials in the elementary symmetric function $\sigma_j(x_1, \ldots, x_n)$. Let $P_{\mu,i}$ be this polynomial, thus

$$\sigma_i(w_1, \ldots, w_m) = P_{\mu,i}(\sigma_1(x_1, \ldots, x_n), \ldots, \sigma_n(x_1, \ldots, x_n)).$$

Now if $\rho : G \to U(n)$ is a representation of G then $\mu\rho : G \to U(m)$ is another representation. The relation between their Chern classes is given by [2, § 10].

(6) $c_i(\mu\rho) = P_{\mu,i}(c_1(\rho), \ldots, c_n(\rho))$.

In particular, taking μ to be the n-th exterior power representation $\lambda^n : U(n) \to U(1)$ we deduce

(7) *If $\dim \rho = n$, then*

$$c_1(\rho) = c_1(\lambda^n \rho).$$

In view of (3) this means that $c_1(\rho)$ is effectively known for any ρ.

If $x_1, \ldots, x_n, y_1, \ldots, y_m$ are two sets of indeterminates with elementary symmetric functions a_i, b_i respectively, we can define polynomials Q_k by the formula

$$\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (1 + t(x_i + y_j)) = \sum_k Q_k(a_1, \ldots, a_n, b_1, \ldots, b_m) t^k$$

where t is an indeterminate. The Chern classes of a tensor product are then given by

(8) $c_k(\rho \otimes \sigma) = Q_k(c_1(\rho), \ldots, c_n(\rho), c_1(\sigma), \ldots, c_m(\sigma))$.

Note that if $\dim \rho = \dim \sigma = 1$, (8) gives $c_1(\rho \otimes \sigma) = c_1(\rho) + c_1(\sigma)$ which is part of the assertion of (3).

In view of (5) the "total" Chern class c may be extended to give a homomorphism

$$c : R(G) \to A(G)$$

of the additive group $R(G)$ into the multiplicative group $A(G)$ consisting of elements of $\prod_{k=0}^{\infty} H^{2k}(G, \mathbf{Z})$ with constant term 1. For example suppose G is a cyclic group of order n, and let ρ be the basic 1-dimensional representation and x the corresponding generator of $H^2(G, \mathbf{Z})$ (cf. § 8). Then the elements of $A(G)$ are formal power series

$$1 + \sum_{k=1}^{\infty} a_k x^k, \qquad a_k \in \mathbf{Z}_n.$$

$R(G)$ is a free abelian group generated by $1, \rho, \rho^2, \ldots, \rho^{n-1}$ and c is given by

$$c\left(\sum_{k=0}^{n-1} m_k \rho^k\right) = \prod_{k=0}^{n-1} (1 + kx)^{m_k},$$

where on the right k is regarded as an element of $\mathbf{Z}_n$ and if $m_k < 0$ we expand $(1 + kx)^{m_k}$ as a formal power series. If $n = p$ is a prime then $\mathbf{Z}_p$ is a field and so

$$\prod_{k=0}^{p-1} (1 + kx)^{m_k} = 1 \iff m_k = 0 \quad \text{for} \quad k \neq 0.$$

Thus, for G cyclic of *prime order,* $\qquad c : I(G) \to A(G)$

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is a monomorphism. This certainly cannot hold for general cyclic groups, since $\theta : R(G) \to \widehat{R(G)}$ is not in general a monomorphism (6.10) and $\theta(\sigma) = 0 \Rightarrow c(\sigma) = 0$ (see below). However even in the case of a p-group, when θ is a monomorphism (6.11), it is still possible for $c : I(G) \to A(G)$ to have a non-zero kernel. As an example let G be cyclic of order p^2, then $c(p\rho^p - p) - (1 + px)^p = 1$.

It appears unlikely that the formal properties of Chern classes listed above are sufficient to prove their uniqueness (working only within the category of finite groups). It is probable that one would have to add a formula for $c_i(i_*(\rho))$ where $i_*(\rho)$ is the "induced representation". However it is easy to see that there can be no simple formula involving only $c_i(\rho)$. For example let $i : G' \to G$ be the inclusion with $G = Z_p$ and G' the identity, and take $\rho = 1$. Then $i_*(\rho)$ is the regular representation $\sum_{k=0}^{p-1} \rho^k$ of G and

$$c\left(\sum_{k=0}^{p-1} \rho^k\right) = \prod_{k=0}^{p-1} (1 + kx) = (1 - x^{p-1}).$$

Thus $c_p(i_*(\rho)) \neq 0$ while $c_i(\rho) = 0$ for all $i > 0$.

If a formula for $c_i(i_*(\rho))$ were known, for ρ 1-dimensional, then one could use Brauer's Theorem [5] and (3) to determine all Chern classes.

Relation with the spectral sequence.

We shall now describe the relation between the Chern classes of a representation and our spectral sequence. The statements which follow are given without proof, but they are all elementary consequences of the results in [1].

We recall that $R(G)$ is filtered by subgroups $R_{2n}(G)$ defined topologically. If we filter the group $A(G)$ by defining $A_{2n}(G)$ to be the subgroup of elements $a = \prod_{\nu=0}^{\infty} a_{2\nu}$ with $a_{2\nu} = 0$ for $1 \leq \nu \leq n - 1$, then

(9) $c : R(G) \to A(G)$ *is a homomorphism of filtered groups.*

In particular c induces a homomorphism of completions $(A(G)$ is itself complete)

$$\hat{c} : \widehat{R(G)} \to A(G),$$

and $c(\sigma) = 0$ for σ in the kernel of $R(G) \to \widehat{R(G)}$ as stated above.

Let $H'(G, Z) \subset H^*(G, Z)$ denote the subgroup of "universal cycles" in the spectral sequence $H^*(G, Z) \Rightarrow \widehat{R(G)}$, i.e. $H'(G, Z) = Z_\infty$ in the notation of § 3. From the spectral sequence we obtain an epimorphism

$$\varphi : H'(G, Z) \to GR(G).$$

Then we have:

(10) *For all* $\rho \in R(G)$ *and all* i $\qquad c_i(\rho) \in H'(G, Z)$,

(11) *Let* $\rho \in R_{2n}(G)$, $[\rho]$ *the image of* ρ *in* $G^{2n}R(G)$. *Then*

$$\varphi(c_n(\rho)) = (-1)^{n-1}(n-1)! [\rho].$$

There is also a close relation between Chern classes and the operators γ^n of § 12. If $\rho \in R(G)$ then, by definition of $R_{2n}(G)$, $\qquad \gamma^n(\rho - \varepsilon(\rho)) \in R'_{2n}(G)$.

Since $R'_{2n}(G) \subset R_{2n}(G)$ (12.6) we obtain an element $[\gamma^n(\rho - \varepsilon(\rho))] \in G^{2n}R(G)$. Then

(12) *For any* $\rho \in R(G)$ *we have* $\qquad \varphi(c_n(\rho)) = [\gamma^n(\rho - \varepsilon(\rho))]$.

From this we see that the conjecture that $R'_{2n}(G) = R_{2n}(G)$ which was made in § 12 is equivalent to the following conjecture: the subring of $H'(G, Z)$ generated by all Chern classes is mapped, by φ, *onto* $GR(G)$.

We already know (2.5) that c_1 induces an isomorphism $G^2R(G) \to H^2(G, Z)$. It follows from (11) that c_2 induces a *monomorphism* [1] $\qquad G^4R(G) \to H^4(G, Z)$.

Thus up to this dimension the filtration is determined by the Chern classes. This is no longer true in higher dimensions as is shown by the example above with G cyclic of order p^2.

In conclusion we may add that for *real* representations ρ one can introduce Stiefel-Whitney classes $w_i(\rho) \in H^i(G, Z_2)$ [2, § 10]. Their formal properties are similar to those of Chern classes.

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[1] From the spectral sequence view-point this corresponds to the fact that the first non-zero group B^p_∞ arises for $p = 6$.

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