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Modular-Invariance of Trace Functions in Orbifold Theory and Generalized Moonshine

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Abstract: The goal of the present paper is to provide a mathematically rigorous foundation to certain aspects of the theory of rational orbifold models in conformal field theory, in other words the theory of rational vertex operator algebras and their automorphisms.

Under a certain finiteness condition on a rational vertex operator algebra V which holds in all known examples, we determine the precise number of g-twisted sectors for any automorphism g of V of finite order. We prove that the trace functions and correlation functions associated with such twisted sectors are holomorphic functions in the upper half-plane and, under suitable conditions, afford a representation of the modular group of the type prescribed in string theory. We establish the rationality of conformal weights and central charge.

In addition to conformal field theory itself, where our conclusions are required on physical grounds, there are applications to the generalized Moonshine conjectures of Conway–Norton–Queen and to equivariant elliptic cohomology.

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1. Introduction

The goals of the present paper are to give a mathematically rigorous study of rational orbifold models, more precisely we study the questions of the *existence* and *modular-invariance of twisted sectors of rational vertex operator algebras.* The idea of orbifolding a vertex operator algebra with respect to an automorphism, and in particular the introduction of twisted sectors, goes back to some of the earliest papers in the subject [FLM1, Le, FLM2, FLM3, DHVW, DHVV, FFR, D3], while the question of modular-invariance underlies the whole enterprise. Apart from a few exceptions such as [DGM], the physical literature tends to treat the existence and modular-invariance of twisted sectors as axioms, while mathematical work has been mainly limited to studying special cases such as affine algebras and lattice theories [KP, Le, FLM2] and fermionic orbifolds [DM1]. Under some mild finiteness conditions on a rational vertex operator algebra V we will, among other things, establish the following:

(A) The precise number of inequivalent, simple g-twisted sectors that V possesses.
(B) Modular-invariance (in a suitable sense) of the characters of twisted sectors.

In order to facilitate the following discussion we assume that the reader has a knowledge of the basic definitions concerning vertex operator algebras as given, for example, in [FLM3, FHL, DM1] and below in Sects. 2 and 3.

Let us suppose that V is a vertex operator algebra. There are several approaches to what it means for V to be *rational*, each of them referring to finiteness properties of V of various kinds (cf. [MS, HMV, AM] for example). Our own approach is as follows (cf. [DLM2, DLM3]): following [Z], an *admissible* V-module is a certain linear space

$$M=\bigoplus_{n\in\mathbb{Z}}^{\infty} M(n) \tag{1.1}$$

which admits an action of V by vertex operators which satisfy certain axioms, the most important of which is the Jacobi identity. However, the homogeneous subspaces $M(n)$ are not assumed to be finite-dimensional. As explained in [DLM2], this definition includes as a special case the idea of an (*ordinary*) V-module, which is a graded linear space of the shape

$$M=\bigoplus_{n\in\mathbb{C}} M_n \tag{1.2}$$

such that M admits an appropriate action of V by vertex operators and such that each M_n has finite dimension, $M_{\lambda+n}=0$ for fixed $\lambda\in\mathbb{C}$ and sufficiently small integer n, and each M_n is the n-eigenspace of the $L(0)$-operator. Although one ultimately wants to establish the rationality of the grading of such modules, it turns out to be convenient to allow the gradings to be *a priori* more general. One then has to *prove* that the grading is rational after all. By the way, the terms "simple module" and "sector" are synonymous in this context.

We call V a *rational vertex operator algebra* in case each admissible V-module is completely reducible, i.e., a direct sum of simple admissible modules. We have proved in [DLM3] that this assumption implies that V has only finitely many inequivalent simple admissible modules, moreover each such module is in fact an ordinary simple V-module. These results are special cases of results proved in (loc.cit.) in which one considers the same set-up, but relative to an automorphism g of V of finite order. Thus one has the notions of *g-twisted V-module*, *admissible g-twisted V-module* and *g-rational vertex operator algebra V*, the latter being a vertex operator algebra all of whose admissible g-twisted modules are completely reducible. We do not give the precise definitions here (cf. Sect. 3), noting only that if g has order T then a simple g-twisted V-module has a grading of the shape

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n/T} \tag{1.3}$$

with $M_\lambda \neq 0$ for some complex number λ which is called the *conformal weight* of M. This is an important invariant of M which plays a rôle in the theory of the Verlinde algebra, for example.

The basic result (loc.cit.) is that a g-rational vertex operator algebra has only finitely many inequivalent, simple, admissible g-twisted modules, and each of them is an ordinary simple g-twisted module. Although the theory of twisted modules includes that of ordinary modules, which corresponds to the case $g = 1$, it is nevertheless common and convenient to refer to the *untwisted theory* if $g = 1$, and to the *twisted theory* otherwise.

It seems likely that if V is rational then in fact it is g-rational for all automorphisms g of finite order, but this is not known. Nevertheless, our first main result shows that there is a close relation between the untwisted and twisted theories. To explain this, we first recall standard notation for the vertex operator determined by v:

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n)z^{-n-1} \tag{1.4}$$

so that each $v(n)$ is a linear operator on V. Define $C_2(V)$ to be the subspace of V spanned by the elements $u(-2)v$ for u, v in V. We say that V satisfies *Condition C_2* if $C_2(V)$ has finite codimension in V. This is closely related to Zhu's condition C [Z], which also includes the hypothesis that V is a sum of highest weight modules for the Virasoro algebra. Zhu asserted in [Z], and we verify in Sect. 12, that condition C_2 holds for a number of the most familiar rational vertex operator algebras. Next, any automorphism h of V induces in a natural way (cf. Sect. 4 of [DM2] and Sect. 3 below) a bijection from the set of isomorphism classes of (simple, admissible) g-twisted modules to the corresponding set of hgh^{-1}-twisted modules, so that if g and h commute then one may consider the set of h-stable g-twisted modules. In particular we have the set of h-stable ordinary (or untwisted) modules, which includes the vertex operator algebra V itself. Our first set of results may now be stated as follows:

Theorem 1.1. *Suppose that V is a rational vertex operator algebra which satisfies Condition C_2. Then the following hold:*

(i) *The central charge of V and the conformal weight of each simple V-module are rational numbers.*

(ii) *If g is an automorphism of V of finite order then the number of inequivalent, simple g-twisted V-modules is at most equal to the number of g-stable simple untwisted V-modules, and is at least 1 if V is simple.*
(iii) *If V is g-rational, the number of inequivalent, simple g-twisted V-modules is precisely the number of g-stable simple untwisted V-modules.*

We actually prove an extension of this result in which V is also assumed to be g^i-rational for all integers i (g as in (ii)), where the conclusion is that each simple g^i-twisted V-module has rational conformal weight. There is a second variation of this theme involving the important class of *holomorphic* vertex operator algebras. This means that V is assumed to be both simple and rational, moreover V is assumed to have a *unique* simple module - which is necessarily the *adjoint* module V itself. Familiar examples include the Moonshine Module [B2, FLM3] and vertex operator algebras associated to positive-definite, even, unimodular lattices [B2], [FLM3]. Proof of rationality and holomorphy of these particular vertex operator algebras can be found in [D1, D2] and [DLM2]. We establish

Theorem 1.2. *Suppose that V is a holomorphic vertex operator algebra which satisfies Condition C_2, g is an automorphism of V of finite order. Then the following hold:*

(i) *V possesses a* unique *simple g-twisted V-module up to isomorphism, call it $V(g)$.*
(ii) *The conformal weight of $V(g)$ is a rational number.*

We turn now to a discussion of the general question of modular-invariance. One is concerned with various trace functions, the most basic of which is the formal character of a (simple) g-twisted sector M. If M has grading (1.3) we define the formal character as

$$\mathrm{char}_q M = q^{\lambda-c/24}\sum_{n=0}^{\infty}\dim M_{\lambda+n/T}q^{n/T}, \tag{1.5}$$

where c is the central charge and q a formal variable.

More generally, if M is an h-stable g-twisted sector as before, then h induces a linear map on M which we denote by $\phi(h)$, and one may consider the corresponding graded trace

$$Z_M(g,h) = q^{\lambda-c/24}\sum_{n=0}^{\infty}\mathrm{tr}_{M_{\lambda+n/T}}\phi(h)q^{n/T}, \tag{1.6}$$

The linear map $\phi(h)$ is only determined up to a nonzero scalar and therefore $Z_M(g,h)$ is also only defined up to a nonzero scalar. Similarly, up to a nonzero scalar $Z_M(g,h)$ is independent of the choice g-twisted sector in the isomorphism class of M. The choice of such a scalar does not interfere with any of the proofs and results in the present paper. As is well-known, it is important to consider these trace functions as special cases of so-called (g,h) *correlation functions*. These may be defined for homogeneous elements $v \in V$ of weight k and any pair of commuting elements (g,h) via

$$T_M(v,g,h) = q^{\lambda-c/24}\sum_{n=0}^{\infty}\mathrm{tr}_{M_{\lambda+n/T}}(v(k-1)\phi(h))q^{n/T}. \tag{1.7}$$

Note that $v(k-1)$ induces a linear map on each homogeneous subspace of M. If we take v to be the vacuum vector then (1.7) reduces to (1.6).

In the special case when V is holomorphic and the twisted sector $V(g)$ is unique up to equivalence, we set

$$Z_{V(g)}(g,h) = Z(g,h), \quad T_{V(g)}(v,g,h) = T(v,g,h). \tag{1.8}$$

Note that in this situation, the uniqueness of $V(g)$ shows that $V(g)$ is h-stable whenever g and h commute. As discussed in [DM1–DM2], this allows us to consider ϕ as a *projective* representation of the *centralizer* $C(g)$ on $V(g)$, in particular (1.7) is defined for all commuting pairs (g,h). Linearizing this projective representation to an ordinary representation of a covering group of $C(g)$ involves the choice of a 2-cocycle with coefficients in $\mathbb{C}^*$. Such a choice corresponds to a choice of the scalar in defining $\phi(h)$, as discussed above.

One can also consider these trace functions less formally. Taking q to be the usual local parameter at infinity in the upper half-plane

$$\mathfrak{h} = \{\tau \in \mathbb{C} | \mathrm{im}\tau > 0\} \tag{1.9}$$

i.e., $q = q_\tau = e^{2\pi i\tau}$, we will see that the trace functions $T_M(v,g,h,\tau)$ converge to holomorphic functions in $\mathfrak{h}$ under suitable conditions on V. By extending T_M linearly to the whole of V one obtains a function

$$T_M : V \times P(G) \times \mathfrak{h} \to P^1(\mathbb{C}), \tag{1.10}$$

where $P(G)$ is the set of commuting pairs of elements in G.

We take $\Gamma = SL(2,\mathbb{Z})$ to operate on $\mathfrak{h}$ in the usual way via Möbius transformations, that is

$$\gamma : \tau \mapsto \frac{a\tau + b}{c\tau + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \tag{1.11}$$

and we let it act on the right of $P(G)$ via

$$(g,h)\gamma = (g^a h^c, g^b h^d). \tag{1.12}$$

Zhu has introduced in [Z] a second vertex operator algebra associated in a certain way to V; it has the same underlying space, however the grading is different. We are concerned with elements v in V which are homogeneous of weight k, say, with regard to the second vertex operator algebra. We write this as $\mathrm{wt}[v] = k$. For such v we define an action of the modular group Γ on T_M in a familiar way, namely

$$T_M|\gamma(v,g,h,\tau) = (c\tau + d)^{-k} T_M(v,g,h,\gamma\tau). \tag{1.13}$$

We state some of our main results concerning modular-invariance.

Theorem 1.3. *Suppose that V is a vertex operator algebra which satisfies Condition C_2, and let G be a finite group of automorphisms of V.*

(i) *For each triple $(v,g,h) \in V \times P(G)$ and for each h-stable g-twisted sector M, the trace function $T_M(v,g,h,\tau)$ converges to a holomorphic function in $\mathfrak{h}$.*
(ii) *Suppose in addition that V is g-rational for each $g \in G$. Let $v \in V$ satisfy $\mathrm{wt}[v] = k$. Then the space of (holomorphic) functions in $\mathfrak{h}$ spanned by the trace functions $T_M(v,g,h,\tau)$ for all choices of g, h and M is a (finite-dimensional) Γ-module with respect to the action (1.13).*

More precisely, if $\gamma \in \Gamma$ *then we have an equality*

$$T_M|\gamma(v, g, h, \tau) = \sum_W \sigma_W T_W(v, (g, h)\gamma, \tau), \tag{1.14}$$

where $(g, h)\gamma$ *is as in (1.12) and* W *ranges over the* $g^a h^c$*-twisted sectors which are* $g^b h^d$*-stable. The constants* σ_W *depend only on* g, h, γ *and* W.

Theorem 1.4. *Suppose that* V *is a holomorphic vertex operator algebra which satisfies Condition* C_2*, and let* G *be a cyclic group of automorphisms of* V*. If* $(g, h) \in P(G)$*,* $v \in V$ *satisfies* $\mathrm{wt}[v] = k$*, and* $\gamma \in \Gamma$*, then* $T(v, g, h, \tau)$ *is a holomorphic function in* $\mathfrak{h}$ *which satisfies*

$$T|\gamma(v, g, h, \tau) = \sigma(g, h, \gamma)T(v, (g, h)\gamma, \tau) \tag{1.15}$$

for some constant $\sigma(g, h, \gamma)$.

Note that if V is as in Theorem 1.4, then $char_q V$ is a modular function on $SL(2, \mathbb{Z})$, possibly with character. It follows from this that the central charge of V is an integer divisible by 8.

We can summarize some of the results above by saying that the functions $T_M(v, g, h, \tau)$ and $T(v, g, h, \tau)$ are *generalized modular forms* of weight k in the sense of [KM]. This means essentially that each of these functions and each of their transforms under the modular group have q-expansions in *rational* powers of q with bounded denominators, and that up to scalar multiples there are only a finite number of such transforms. One would like to show that each of these functions is, in fact, a modular form of weight k and some level N in the usual sense of being invariant under the principal congruence subgroup $\Gamma(N)$. This will require further argument, as it is shown in (loc.cit.) that there are generalized modular forms which are *not* modular forms in the usual sense. All we can say in general at the moment is that $T_M(v, g, h, \tau)$ and $T(v, g, h, \tau)$ have finite level in the sense of Wohlfahrt [Wo]. This is true of all generalized modular forms, and means that $T_M(v, g, h, \tau)$ and $T(v, g, h, \tau)$ and each of their Γ -transforms are invariant under the group $\Delta(N)$ for some N, where we define $\Delta(N)$ to be the normal closure of $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}$ in Γ.

Let us emphasize the differences between Theorem 1.3 (ii) and Theorem 1.4. In the former we assume g-rationality, which in practice is hard to verify, even for known vertex operator algebras. In Theorem 1.4 there is no such assumption, however we have to limit ourselves to *cyclic* pairs (g, h) in $P(G)$. One expects Theorem 1.4 to hold for all commuting pairs (g, h). In [N], Simon Norton conjectured that (1.15) holds in the special case that v is the vacuum vector and V is the Moonshine Module [FLM3] whose automorphism group is the Monster (loc.cit.). His argument was based on extensive numerical evidence in Conway-Norton's famous paper Monstrous Moonshine [CN] which was significantly expanded in the thesis of Larrissa Queen [Q]. A little later it was given a string-theoretic interpretation in [DGH]. We will see in Sect. 12 that the Moonshine Module satisfies Condition C_2 (a proof is also outlined in [Z]), so that Theorem 1.4 applies. Norton also conjectured that each $Z(g, h, \tau)$ is either constant or a *hauptmodul*, the latter being a modular function (weight zero) on some discrete subgroup Γ_0 of $SL(2, \mathbb{R})$ of genus zero such that the modular function in question generates the full field of meromorphic modular functions on Γ_0. By utilizing the results of Borcherds [B2] which establish the original Moonshine conjectures in [CN] for the Moonshine Module we obtain

Theorem 1.5 (*Generalized Moonshine***).** *Let $V^\natural$ be the Moonshine Module, and let g be an element of the Monster simple group M. The following hold:*

(i) *$V^\natural$ has a unique g-twisted sector $V^\natural(g)$.*
(ii) *The formal character* $\mathrm{char}_q V^\natural(g)$ *is a hauptmodul.*
(iii) *More generally, if g and h in M generate a cyclic subgroup then the graded trace $Z(g, h, \tau)$ of $\phi(h)$ on $V^\natural(g)$ is a hauptmodul.*

These results essentially establish the Conway–Norton–Queen conjectures about the Monster for cyclic pairs (g, h). As we have said, the spaces $V^\natural(g)$ support faithful projective representations of the corresponding centralizer $C_M(g)$ of g in M. It was these representations that were conjectured to exist in [N] and [Q], and they are of considerable interest in their own right. See [DLM1] for more information in some special cases. There also appears to be some remarkable connections with the work of Borcherds and Ryba [Ry, BR, B3] on *modular moonshine* which we hope to consider elsewhere. Finally, we mention that Theorem 1.4 can be considerably strengthened, indeed the best possible results can be established, if we assume that g has small order. For simplicity we state only a special case:

Theorem 1.6. *Let V, v, k be as in Theorem 1.4, and assume that the central charge of V is divisible by 24. Suppose that g has order $p = 2$ or 3. Then the following hold:*

(i) *The conformal weight λ of the (unique) g-twisted sector $V(g)$ is in $\frac{1}{p^2}\mathbb{Z}$.*
(ii) *The graded trace $T(v, 1, g, \tau)$ is a modular form of weight k and level p^2.*
(iii) *We have $\lambda \in \frac{1}{p}\mathbb{Z}$ if, and only if, $T(v, g, h, \tau)$ is a modular form on the congruence subgroup $\Gamma_0(p)$.*

These results follow from the previous theorems, and will be proved elsewhere. They can be used to make rigorous some of the assumptions commonly made by physicists (cf. [M, S, T1, T2, T3, V] for example) in the theory of $\mathbb{Z}_p$-orbifolds, and we hope that Theorem 1.6 may provide the basis for a more complete theory of such orbifolds.

We have already mentioned the work of Zhu [Z] on several occasions, and it is indeed this paper to which we are mainly indebted intellectually. In essence, we are going to prove an equivariant version of the theory laid down by Zhu (loc.cit.), though even in the special case that he was studying our work yields improvements on his results. In particular, Zhu's hypothesis that V is a sum of highest weight modules for the Virasoro algebra is eliminated in the present paper, and our notion of rationality, developed in [DLM3], is qualitatively weaker than that of Zhu [Z]. Nevertheless, the broad outline of our proof follows that of [Z]. The equivariant refinement of Zhu's theory began with our paper [DLM3] which also plays a basic rôle in the present paper. In this paper we constructed so-called twisted Zhu algebras $A_g(V)$ which are associative algebras associated to a vertex operator algebra V and automorphism g of finite order. They have the property that, at least for suitable classes of vertex operator algebras, the module category for $A_g(V)$ and the category of g-twisted modules for V are *equivalent*. This reduces the construction of g-twisted sectors to the corresponding problem for $A_g(V)$ (not a priori known to be non-zero!) As in [Z], the rôle of the finiteness condition C_2 is to show that the (g, h) correlation functions that we have considered above satisfy certain differential equations of regular - singular type. These differential equations have coefficients which are essentially modular forms on a congruence subgroup, a fact which is ultimately attributable to the Jacobi identity satisfied by the vertex operators. One attempts to characterize the space of correlation functions as those solutions of the differential equation which possess other technical features related to properties of V and $A_g(V)$. This space is essentially what is sometimes referred to as the (g, h)-*conformal block*, and our results follow from the technical assertion that, under suitable circumstances, the (g, h)-conformal block is indeed spanned by the (g, h)-correlation functions. This whole approach to conformal blocks is inspired by [Z], but is more complicated when twisted sectors are involved.

We point out that the holomorphy of trace functions follows from the fact that they are solutions of suitable differential equations, moreover the Frobenius–Fuchs theory of differential equations with regular - singular points leads to the representation of elements of the conformal block as q-expansions. One attempts to identify coefficients of such q-expansions with the trace function defined by some $A_g(V)$-module, and at the same time show that elements of the conformal blocks are free of logarithmic singularities. The point is that the Frobenius–Fuchs theory plays a critical rôle, as it does also in [Z].

The paper is organized as follows: after some preliminaries in Sects. 2 and 3, we take up in Sect. 4 the study of certain modular and Jacobi-type forms, the main goal being to write down the transformation laws which they satisfy. Our methods (and probably the results too) will be well-known to experts in elliptic functions, but it is fascinating to see how such classical topics such as Eisenstein series and Bernoulli distributions play a rôle in an abstract theory of vertex operator algebras. In Sect. 5 we introduce the space of abstract (g, h) 1-point functions associated to a vertex operator algebra and establish that it affords an action by the modular group (Theorem 5.4). In Sections 6 and 7 we continue the study of such functions and in particular write down the differential equation that they satisfy and the general shape of the solutions in terms of q-expansions and logarithmic singularities. Next we prove in Sect. 8 (Theorems 8.1 and 8.7) that if g and h commute then distinct h-stable g-twisted sectors give rise to linearly independent trace functions which lie in the (g, h)-conformal block, and in Sect. 9 we give as an application of the ideas developed so far a general existence theorem for twisted sectors. Section 10 contains the main theorems which give sufficient conditions under which the (g, h) conformal block is spanned by trace functions. Having reached Sect. 11, we have enough information to be able to apply the methods and results of Anderson and Moore [AM], and this leads to the rationality results stated above in Theorems 1.1 and 1.2 as well as the applications to modular-invariance and to the generalized Moonshine Conjectures, which are discussed in Sect. 13. We also point out how one can use Theorem 1.4 to describe not only other correlation functions but also “Monstrous Moonshine of weight k.” Sect. 12 establishes condition C_2 for a number of well-known rational vertex operator algebras, so that our theory applies to all of these examples.

2. Vertex Operator Algebras

The definition of vertex operator algebra [FLM3] entails a $\mathbb{Z}$-graded complex vector space:

$$V = \bigoplus_{n\in\mathbb{Z}} V_n \tag{2.1}$$

satisfying $\dim V_n < \infty$ for all n and $V_n = 0$ for $n << 0$. If $v \in V_n$ we write $\mathrm{wt} v = n$ and say that v is *homogeneous* and has (conformal) *weight* n. For each $v \in V$ there are linear operators $v_n \in \mathrm{End} V$, $n \in \mathbb{Z}$ which are assembled into a *vertex operator*

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v(n) z^{-n-1} \in (\mathrm{End}\, V)[[z, z^{-1}]]$$

which is linear in v. Various axioms are imposed. For $u, v \in V$,

(i)

$$u(n)v = 0 \quad \text{for sufficiently large.} \tag{2.2}$$

(ii) There is a distinguished *vacuum element* $\mathbf{1} \in V_0$ satisfying

$$Y(\mathbf{1}, z) = 1 \tag{2.3}$$

and

$$Y(v, z)\mathbf{1} = v + \sum_{n \geq 2} v(-n)\mathbf{1}z^{n-1}. \tag{2.4}$$

(iii) There is a distinguished *conformal vector* $\omega \in V_2$ with generating function

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$$

such that the component operators generate a copy of the Virasoro algebra represented on V with *central charge* $c \in \mathbb{C}$. That is

$$[L(m), L(n)] = (m-n)L(m+n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c. \tag{2.5}$$

Moreover we have

$$V_n = \{v \in V | L(0)v = nv\}, \tag{2.6}$$

$$\frac{d}{dz}Y(v, z) = Y(L(-1)v, z). \tag{2.7}$$

(iv) The *Jacobi identity* holds, that is

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right)Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right)Y(v, z_2)Y(u, z_1) = z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right)Y\left(Y(u, z_0)v, z_2\right) \tag{2.8}$$

where $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$ and $(z_i - z_j)^n$ is expanded as a formal power series in z_j. Throughout the paper, z_0, z_1, z_2, etc. are independent commuting formal variables.

Such a vertex operator algebra may be denoted by the 4-tuple $(V, Y, \mathbf{1}, \omega)$ or, more usually, by V.

Zhu has introduced a second vertex operator algebra $(V, Y[\,], \mathbf{1}, \tilde{\omega})$ associated to V in Theorem 4.2.1 of [Z]. It plays a crucial rôle in Zhu's theory and also in the present paper, so we give some details of the construction[1].

[1] Our formulae differ from that of Zhu by a factor of $2\pi i$ in the exponent $e^{z\mathrm{wt}v}$.

The conformal vector $\tilde{\omega}$ is defined to be $\omega - \frac{c}{24}$. The vertex operators $Y[v, z]$ are defined for homogeneous v via the equality

$$Y[v, z] = Y(v, e^z - 1)e^{z\mathrm{wt}v} = \sum_{n\in\mathbb{Z}} v[n]z^{-n-1}. \tag{2.9}$$

For integers i, m, p with $i, m \geq 0$ we may define $c(p, i, m)$ in either of two equivalent ways:

$$\binom{p-1+z}{i} = \sum_{m=0}^{i} c(p, i, m)z^m, \tag{2.10}$$

$$m!\sum_{i=m}^{\infty} c(p, i, m)z^i = (\log(1+z))^m(1+z)^{p-1}, \tag{2.11}$$

where, as usual

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n. \tag{2.12}$$

Using a change of variable we calculate that

$$\begin{aligned} v[m] &= \mathrm{Res}_z Y[v, z]z^m && (2.13)\\ &= \mathrm{Res}_z Y[v, \log(1+z)](\log(1+z))^m(1+z)^{-1}, && (2.14)\end{aligned}$$

i.e.,

$$v[m] = \mathrm{Res}_z Y(v, z)(\log(1+z))^m(1+z)^{\mathrm{wt}v-1}. \tag{2.15}$$

So if $m \geq 0$ then

$$v[m] = m!\sum_{i=m}^{\infty} c(\mathrm{wt}v, i, m)v(i). \tag{2.16}$$

For example we have

$$v[0] = \sum_{i=0}^{\infty} \binom{\mathrm{wt}v-1}{i} v(i). \tag{2.17}$$

We also write

$$Y[\tilde{\omega}, z] = \sum_{n\in\mathbb{Z}} L[n]z^{-n-2}. \tag{2.18}$$

For example, one has

$$\begin{aligned} L[-2] &= \omega[-1] - \frac{c}{24}, && (2.19)\\ L[-1] &= L(-1) + L(0), && (2.20)\\ L[0] &= L(0) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+1)} L(n). && (2.21)\end{aligned}$$

Care must be taken to distinguish between the notion of conformal weight in the *original* vertex operator algebra and in $(V, Y[\ ], \mathbf{1}, \tilde{\omega})$. If $v \in V$ is homogeneous in the *latter* vertex operator algebra we denote its conformal weight by $\mathrm{wt}[v]$, and set

$$V_{[n]} = \{v \in V | L[0]v = nv\}. \tag{2.22}$$

In general, V_n and $V_{[n]}$ are distinct, though it follows from (2.21) that for each N we have

$$\bigoplus_{n \leq N} V_n = \bigoplus_{n \leq N} V_{[n]}. \tag{2.23}$$

3. Twisted Modules

Let V be a vertex operator algebra. An *automorphism* g of the vertex operator algebra V is a linear automorphism of V preserving $\mathbf{1}$ and ω such that the actions of g and $Y(v, z)$ on V are compatible in the sense that

$$gY(v, z)g^{-1} = Y(gv, z)$$

for $v \in V$. Let $\mathrm{Aut}V$ be the group of automorphisms of V. If $g \in \mathrm{Aut}V$ has finite order T, say, there are various classes of g-twisted V-modules of concern to us (cf. [Le, FLM2, DLM2, DLM3]). A *weak* g-twisted V-module is a $\mathbb{C}$-linear space M equipped with a linear map $V \to (\mathrm{End}M)[[z^{1/T}, z^{-1/T}]]$, $v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Q}} v(n)z^{-n-1}$ satisfying the following:

For $v \in V$, $w \in M$,

$$v(m)w = 0 \tag{3.1}$$

if m is sufficiently large;

$$Y_M(\mathbf{1}, z) = 1; \tag{3.2}$$

set

$$V^r = \{v \in V | gv = e^{-2\pi ir/T} v\} \tag{3.3}$$

for $0 \leq r \leq T - 1$. Then

$$Y_M(v, z) = \sum_{n \in r/T + \mathbb{Z}} v(n)z^{-n-1} \quad \text{for} \;\; v \in V^r; \tag{3.4}$$

and the twisted Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(u, z_1)Y_M(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(v, z_2)Y_M(u, z_1)$$
$$= z_2^{-1}\left(\frac{z_1 - z_0}{z_2}\right)^{-r/T} \delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(u, z_0)v, z_2) \tag{3.5}$$

if $u \in V^r$. We often write (M, Y_M) for this module. It can be shown (cf. Lemma 2.2 of [DLM2, DLM3]) that $Y_M(\omega, z)$ has component operators which still satisfy (2.5) and (2.7). If we take $g = 1$, we get a weak V-module.

For $m \in \mathbb{C}$ we use the notation $\iota_{z_1,z_2}(z_1-z_2)^m$ to denote the expansion of $(z_1-z_2)^m$ in terms of the nonnegative integral powers of z_2 :

$$\iota_{z_1,z_2}((z_1-z_2)^m) = z_1^m \sum_{i\geq 0} \binom{m}{i} (-1)^i z_2^i z_1^{-i}, \tag{3.6}$$

where $\binom{m}{i} = \frac{m(m-1)\cdots(m-i+1)}{i!}$. It follows from the twisted Jacobi identity (3.5) that

$$\begin{aligned} &\mathrm{Res}_{z_1} Y_M(u,z_1)Y_M(v,z_2)z_1^{r/T}\iota_{z_1,z_2}((z_1-z_2)^m)z_2^n \\ &\quad - \mathrm{Res}_{z_1} Y_M(v,z_2)Y_M(u,z_1)z_1^{r/T}\iota_{z_2,z_1}((z_1-z_2)^m)z_2^n \\ &= \mathrm{Res}_{z_1-z_2} Y_M(Y(u,z_1-z_2)v,z_2)\iota_{z_2,z_1-z_2}z_1^{r/T}(z_1-z_2)^m z_2^n \end{aligned} \tag{3.7}$$

for $m \in \mathbb{Z}$, $n \in \mathbb{C}$. Here, $\iota_{z_2,z_1-z_2}z_1^{r/T} = \sum_{m\geq 0}\binom{r/T}{m} z_2^{r/T-m}(z_1-z_2)^m$.

An *admissible* g-twisted V-module is a weak g-twisted V-module M which carries a $\frac{1}{T}\mathbb{Z}_+$-grading

$$M = \bigoplus_{n\in\frac{1}{T}\mathbb{Z}_+} M(n) \tag{3.8}$$

which satisfies the following

$$v(m)M(n) \subseteq M(n + \mathrm{wt}v - m - 1) \tag{3.9}$$

for homogeneous $v \in V$. We may and do assume that $M(0) \neq 0$ if $M \neq 0$. Again if $g = 1$ we have an admissible V-module.

An (ordinary) g-twisted V-module is a weak g-twisted V-module M which carries a $\mathbb{C}$-grading induced by the spectrum of $L(0)$. That is, we have

$$M = \coprod_{\lambda\in\mathbb{C}} M_\lambda, \tag{3.10}$$

where $M_\lambda = \{w \in M | L(0)w = \lambda w\}$. Moreover we require that dim M_λ is finite and for fixed λ, $M_{\frac{n}{T}+\lambda} = 0$ for all small enough integers n. If $g = 1$ we have an ordinary V-module. If M is a simple g-twisted V-module, then

$$M = \bigoplus_{n=0}^{\infty} M_{\lambda+n/T} \tag{3.11}$$

for some $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$. We define λ as the *conformal weight* of M.

The vertex operator algebra V is called *g-rational* in case every admissible g-twisted V-module is completely reducible, i.e., a direct sum of simple admissible g-twisted modules.

Remark 3.1. There is a subtlety in the definition of these twisted modules. Namely, the definition of V^r given in (3.3) is not quite the same as that which we have used elsewhere ([DLM3, DM1], etc.). Previously we set $V^r = \{v \in V | gv = e^{2\pi ir/T}v\}$, so in effect we have interchanged the notions of g- and g^{-1}-twisted modules. The reason for doing this is that it makes the theorem of modular-invariance (Theorem 5.4 below) have the expected form.

Theorem 3.2 ([DLM3]). *Suppose that V is a g-rational vertex operator algebra, where $g \in \mathrm{Aut} V$ has finite order. Then the following hold:*

(a) *Every simple admissible g-twisted V-module is an ordinary g-twisted V-module.*
(b) *V has only finitely many isomorphism classes of simple admissible g-twisted modules.*

Let V be a vertex operator algebra with g an automorphism of V of finite order T. In [DLM3] a certain associative algebra $A_g(V)$ was introduced which plays a basic rôle in the present work. In case $g = 1$, $A_1(V) = A(V)$ was introduced and used extensively in [Z]. We need to review certain aspects of these constructions here.

Let V^r be as in (3.3). For $u \in V^r$ and $v \in V$ we define

$$u \circ_g v = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt} u - 1 + \delta_r + \frac{r}{T}}}{z^{1+\delta_r}} Y(u, z)v, \tag{3.12}$$

$$u *_g v = \begin{cases} \mathrm{Res}_z (Y(u, z) \frac{(1+z)^{\mathrm{wt}\, u}}{z} v) & \text{if } r = 0 \\ 0 & \text{if } r > 0. \end{cases} \tag{3.13}$$

where $\delta_r = 1$ if $r = 0$ and $\delta_r = 0$ if $r \neq 0$. Extend $\circ_g$ and $*_g$ to bilinear products on V. We let $O_g(V)$ be the linear span of all $u \circ_g v$.

Theorem 3.3 ([DLM3, Z]). *The quotient $A_g(V) = V/O_g(V)$ is an associative algebra with respect to $*_g$.*

Note that if $g = 1$ then $A(V)$ is nonzero, whereas if $g \neq 1$ the analogous assertion may not be true. But if $A_g(V) \neq 0$ then the vacuum element maps to the identity of $A_g(V)$, and the conformal vector maps into the center of $A_g(V)$.

Let M be a weak g-twisted V-module. Define $\Omega(M) = \{w \in V | u(\mathrm{wt} u - 1 + n)w = 0, u \in V, n > 0\}$. For homogeneous $u \in V$ define

$$o(u) = u(\mathrm{wt} u - 1) \tag{3.14}$$

(sometimes called the *zero mode* of u).

Theorem 3.4 ([DLM3]). *Let M be a weak g-twisted V-module. The following hold:*

(a) *The map $v \mapsto o(v)$ induces a representation of the associative algebra $A_g(V)$ on $\Omega(M)$.*
(b) *If M is a simple admissible g-twisted V-module then $\Omega(M) = M(0)$ is a simple $A_g(V)$-module. Moreover, $M \mapsto M(0)$ induces a bijection between (isomorphism classes of) simple admissible g-twisted V-modules and simple $A_g(V)$-modules.*

When combined with Theorem 3.2 one finds

Theorem 3.5 ([DLM3]). *Suppose that V is a vertex operator algebra with an automorphism g of finite order, and that V is g-rational (possibly $g = 1$). Then the following hold:*

(a) *$A_g(V)$ is a finite-dimensional, semi-simple associative algebra (possibly 0).*
(b) *The map $M \mapsto \Omega(M)$ induces an equivalence between the category of ordinary g-twisted V-modules and the category of finite-dimensional $A_g(V)$-modules.*

There are various group actions that we need to explain. Let g, h be automorphisms of V with g of finite order. If (M, Y_g) is a weak g-twisted module for V there is a weak hgh^{-1}-twisted V-module $(M, Y_{hgh^{-1}})$, where for $v \in V$ we define

$$Y_{hgh^{-1}}(v, z) = Y_g(h^{-1}v, z). \tag{3.15}$$

This defines a left action of $\mathrm{Aut}(V)$ on weak twisted modules and on isomorphism classes of weak twisted modules. Symbolically, we write

$$h \circ (M, Y_g) = (M, Y_{hgh^{-1}}) = h \circ M, \tag{3.16}$$

where we sometimes abuse notation slightly by identifying (M, Y_g) with the isomorphism class that it defines. The action (3.16) induces an action

$$h \circ \Omega(M) = \Omega(h \circ M). \tag{3.17}$$

Next, it follows easily from definitions (3.12) and (3.13) that the action of h on V induces an isomorphism of associative algebras

$$\begin{aligned} h : A_g(V) &\to A_{hgh^{-1}}(V) \\ v &\mapsto hv \end{aligned} \tag{3.18}$$

which then induces a functor

$$h : A_{hgh^{-1}}(V) - \mathrm{mod} \to A_g(V) - \mathrm{mod}. \tag{3.19}$$

To describe (3.19), let $(N, *_{hgh^{-1}})$ be a left $A_{hgh^{-1}}(V)$-module (extending the notation of (3.13)). Then $h \circ (N, *_{hgh^{-1}}) = (N, *_g)$, where, for $n \in N$, $v \in V$,

$$v *_{hgh^{-1}} n = h^{-1}v *_g n. \tag{3.20}$$

Now if (M, Y_g) is as before then (3.17) and (3.19) both define actions of h on $\Omega(M)$; they are the same. For if $v \in V$ and we consider the image of v in $A_{hgh^{-1}}(V)$, it acts on $\Omega(h \circ M)$ via the zero mode $o_{hgh^{-1}}(v)$ of v in the vertex operator $Y_{hgh^{-1}}(v, z) = Y_g(h^{-1}v, z)$. In other words, if $m \in \Omega(h \circ M) = \Omega(M)$, then $o_{hgh^{-1}}(v)m = o_g(h^{-1}v)m$, which is precisely what (3.20) says.

We say that the g-twisted V-module M is h-invariant if $h \circ M \cong M$. The set of all such automorphisms, the *stabilizer* of M, is a subgroup C of $\mathrm{Aut}\,V$. There is a *projective* representation of C on M induced by the action (3.16). See [DM1] or [DM2] for more information on this point. Via (3.17) this induces a projective representation of C on $\Omega(M)$.

Next we discuss the C_2*-condition* in more detail. Let V be a vertex operator algebra and M a V-module. We define

$$C_2(M) = \{v(-2)m | v \in V, m \in M\}. \tag{3.21}$$

We say that M *satisfies condition* C_2 in case $C_2(M)$ has finite codimension in M. The most important case is that in which M is taken to be V itself.

Proposition 3.6. *Suppose that V satisfies condition C_2, and let g be an automorphism of V of finite order. Then the algebra $A_g(V)$ has finite dimension.*

Proof. Note from (3.21) that $C_2(V)$ is a $\mathbb{Z}$-graded subspace of V. Since the codimension of $C_2(V)$ is finite, there is an integer k such that $V = C_2(V) + W$, where W is the sum of the first k homogeneous subspaces of V.

We will show that $V_m \subset W + O_g(V)$ for each $m \in \mathbb{Z}$, in which case $V = W + O_g(V)$ and $\dim A_g(V) \leq \dim W$. We proceed by induction on m.

Recall that V^r is the eigenspace of g with eigenvalue $e^{-2\pi i r/T}$ for $0 \leq r \leq T-1$, where g has order T. Since $C_2(V)$ is a homogeneous and g-invariant subspace of V then we may write any $c \in C_2(V) \cap V_m$ in the form

$$c = \sum_{i=1}^{n} u_i(-2)v_i \tag{3.22}$$

for homogeneous elements $u_i, v_i \in V$ satisfying $u_i \in V^r$ for some $r = r(i)$ and $\mathrm{wt}u_i + \mathrm{wt}v_i + 1 = m$.

Suppose first that $u_i \in V^r$ with $r = 0$. According to (3.12), $O_g(V)$ contains

$$\mathrm{Res}_z Y(u_i, z)\frac{(1+z)^{\mathrm{wt}u_i}}{z^2}v_i = \sum_{j \geq 0}\binom{\mathrm{wt}u_i}{j}u_i(j-2)v_i. \tag{3.23}$$

Now $\mathrm{wt}u_i(j-2)v_i = \mathrm{wt}u_i + \mathrm{wt}v_i - j + 1 = m - j$, so if $j \geq 1$ then $u_i(j-2)v_i \in W + O_g(V)$ by induction. But then $W + O_g(V)$ also contains the remaining summand $u_i(-2)v_i$ of (3.23).

Now suppose that $u_i \in V^r$ with $r \geq 1$. By Lemma 2.2 (i) of [DLM3] we have that $O_g(V)$ contains the element $\mathrm{Res}_z Y(u_i, z)\frac{(1+z)^{\mathrm{wt}u_i - 1 + r/T}}{z^2}v_i$, and we conclude once again that $u_i(-2)v_i$ lies in $W + O_g(V)$. So we have shown that all summands of (3.22) lie in $W + O_g(V)$. The proposition follows. $\square$

Proposition 3.7. *Suppose that V satisfies condition C_2, and let g be an automorphism of V of finite order. If $A_g(V) \neq 0$ then V has a simple g-twisted V-module.*

Proof. $A_g(V)$ has finite dimension by Proposition 3.6. Now the result follows from Theorem 9.1 of [DLM3]. $\square$

The following lemma will be used in Sect. 12.

Lemma 3.8. *Let M be a V-module. Then $C_2(M)$ is invariant under the operators $v(0)$ and $v(-1)$ for any $v \in V$.*

Proof. Consider $u(-2)w \in C_2(M)$ for $u \in V$ and $w \in M$. Then for $k = 0, -1$ $v(k)u(-2)w = u(-2)v(k)w + \sum_{i=0}^{\infty}\binom{k}{i}(v(i)u)(-2+k-i)w \in C_2(M)$ as required. $\square$

4. P-Functions and Q-Functions

We study certain functions, which we denote by P and Q, which play a rôle in later sections. The P-functions are essentially Jacobi forms [EZ] and the Q-functions are certain modular forms. The main goal is to write down the transformation laws of these functions under the action of the modular group $\Gamma = SL(2, \mathbb{Z})$.

Let $\mathfrak{h}$ denote the upper half plane $\mathfrak{h} = \{z \in \mathbb{C} | \mathrm{im}\, z > 0\}$ with the usual left action of Γ via Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau = \frac{a\tau + b}{c\tau + d}. \tag{4.1}$$

Γ also acts on the right of $S^1 \times S^1$ via

$$(\mu, \lambda) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mu^a \lambda^c, \mu^b \lambda^d). \tag{4.2}$$

Let t be a torsion point of $S^1 \times S^1$. Thus $t = (\mu, \lambda)$ with $\mu = e^{2\pi i j/M}$ and $\lambda = e^{2\pi i l/N}$ for integers j, l, M, N with $M, N > 0$. For each integer $k = 1, 2, \cdots$ and each t we define a function P_k on $\mathbb{C} \times \mathfrak{h}$ as follows:

$$P_k(\mu, \lambda, z, q_\tau) = P_k(\mu, \lambda, z, \tau) = \frac{1}{(k-1)!} \sideset{}{'}\sum_{n \in \frac{j}{M} + \mathbb{Z}} \frac{n^{k-1} q_z^n}{1 - \lambda q_\tau^n}, \tag{4.3}$$

where the sign $\sum'$ means omit the term $n = 0$ if $(\mu, \lambda) = (1, 1)$. Here and below we write $q_x = e^{2\pi i x}$.

Remark 4.1. (i) (4.3) converges uniformly and absolutely on compact subsets of the region $|q_\tau| < |q_z| < 1$.
(ii) Theorem 4.2 holds also for $(\mu, \lambda) = (1, 1)$ in case $k \geq 3$ but *not* if $k = 1, 2$ (cf. [Z]).

We will prove

Theorem 4.2. *Suppose that* $(\mu, \lambda) \neq (1, 1)$. *Then*

$$P_k\left(\mu, \lambda, \frac{z}{c\tau + d}, \gamma\tau\right) = (c\tau + d)^k P_k((\mu, \lambda)\gamma, z, \tau)$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

We can reformulate Theorem 4.2 as follows: for suitable functions $F(\mu, \lambda, z, \tau)$ on $(\mathbb{Q}/\mathbb{Z})^2 \times \mathbb{C} \times \mathfrak{h}$, and for an integer k, we set

$$F|_k\gamma(\mu, \lambda, z, \tau) = (c\tau + d)^{-k} F\left((\mu, \lambda)\gamma^{-1}, \frac{z}{c\tau + d}, \gamma\tau\right). \tag{4.4}$$

As is well-known, this defines a right action of Γ on such functions F. Theorem 4.2 says precisely that P_k is an *invariant* of this action. So it is enough to prove the theorem for the two standard generators $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of Γ. If $\gamma = T$ then Theorem 4.2 reduces to the assertion $P_k(\mu, \lambda, z, \tau + 1) = P_k(\mu, \mu\lambda, z, \tau)$, which follows immediately from definition (4.3).

We also note the equality

$$\frac{d}{dz} P_k(\mu, \lambda, z, \tau) = 2\pi i k P_{k+1}(\mu, \lambda, z, \tau). \tag{4.5}$$

So if Theorem 4.2 holds for k then it holds for $k + 1$ by (4.5) and the chain rule. These observations reduce us to proving Theorem 4.2 in the case that $k = 1$ and $\gamma = S$, when it can be restated in the form

Theorem 4.3. *If* $(\mu, \lambda) \neq (1, 1)$ *then*

$$P_1(\mu, \lambda, \frac{z}{\tau}, \frac{-1}{\tau}) = \tau P_1(\lambda, \mu^{-1}, z, \tau).$$

We will need to make use of several other functions in the proof of Theorem 4.3. First there is the usual Eisenstein series $G_2(\tau)$ with q-expansion

$$G_2(\tau) = \frac{\pi^2}{3} + 2(2\pi i)^2 \sum_{n=1}^{\infty} \frac{nq_\tau^n}{1 - q_\tau^n} \tag{4.6}$$

and well-known transformation law

$$G_2(\gamma\tau) = (c\tau + d)^2 G_2(\tau) - 2\pi i c(c\tau + d), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{4.7}$$

(For these and other facts about elliptic functions, the reader may consult [La].) Let

$$\wp_1(z, \tau) = G_2(\tau)z + \pi i \frac{q_z + 1}{q_z - 1} + 2\pi i \sum_{n=1}^{\infty} \left(\frac{q_\tau^n q_z^{-1}}{1 - q_\tau^n q_z^{-1}} - \frac{q_z q_\tau^n}{1 - q_z q_\tau^n} \right). \tag{4.8}$$

The function $\wp_1(z, \tau)$ is not elliptic, but satisfies

$$\wp_1(z + 1, \tau) = \wp_1(z, \tau) + G_2(\tau), \tag{4.9}$$

$$\wp_1(z + \tau, \tau) = \wp_1(z, \tau) + G_2(\tau)\tau - 2\pi i, \tag{4.10}$$

$$\wp_1(\frac{z}{c\tau + d}, \gamma\tau) = (c\tau + d)\wp_1(z, \tau), \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{4.11}$$

Now introduce a further P-type function

$$P_\lambda(z, \tau) = 2\pi i \sum_{n \in \mathbb{Z}, n \neq 0} \frac{q_z^n}{1 - \lambda q_\tau^n}, \tag{4.12}$$

where λ is a root of unity.

Lemma 4.4. *Suppose that* $|q_\tau| < |q_z| < 1$ *and that* $\lambda^N = 1$. *Then*

$$P_\lambda(z, \tau) = \sum_{k=0}^{N-1} \lambda^k \left(G_2(N\tau)(z + k\tau) - \wp_1(z + k\tau, N\tau) - \pi i \right).$$

Proof. As $|\lambda q_\tau^n| < 1$ for $n \geq 1$ then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{q_z^n}{1 - \lambda q_\tau^n} &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} q_z^n \lambda^m q_\tau^{mn} \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \lambda^m q_z q_\tau^m (q_z q_\tau^m)^n \\
&= \sum_{m=0}^{\infty} \frac{\lambda^m q_z q_\tau^m}{1 - q_z q_\tau^m} \quad (\text{as } |q_z q_\tau^m| < 1 \text{ for } m \geq 0) \\
&= \frac{q_z}{1 - q_z} + \sum_{m=1}^{\infty} \frac{\lambda^m q_z q_\tau^m}{1 - q_z q_\tau^m}.
\end{aligned}$$

Using $|q_z^{-1}q_\tau^m| < 1$ for $m \geq 1$, a similar calculation yields

$$\sum_{n=1}^{\infty} \frac{\lambda^{-1}q_z^{-n}q_\tau^n}{1-\lambda^{-1}q_\tau^n} = \sum_{m=1}^{\infty} \frac{\lambda^{-m}q_z^{-1}q_\tau^m}{1-q_z^{-1}q_\tau^m}.$$

From this and (4.12) we get

$$(2\pi i)^{-1}P_\lambda(z,\tau) = \frac{q_z}{1-q_z} + \sum_{m=1}^{\infty}\left(\frac{\lambda^m q_z q_\tau^m}{1-q_z q_t^m} - \frac{\lambda^{-m}q_z^{-1}q_\tau^m}{1-q_z^{-1}q_\tau^m}\right). \tag{4.13}$$

Next, use the expansion

$$\sum_{m=0}^{\infty} \frac{\lambda^m q_z q_\tau^m}{1-q_z q_\tau^m} = \sum_{n=0}^{\infty}\sum_{k=0}^{N-1} \frac{\lambda^k q_z q_\tau^{nN+k}}{1-q_z q_\tau^{nN+k}} = \sum_{k=0}^{N-1}\lambda^k \sum_{n=0}^{\infty} \frac{q_{z+k\tau}q_{N\tau}^n}{1-q_{z+k\tau}q_{N\tau}^n}$$

and a similar expression for the second term under the summation sign in (4.13) to see that

$$P_\lambda(z,\tau) = 2\pi i \sum_{k=0}^{N-1}\lambda^k \left(\sum_{n=0}^{\infty}\frac{q_{z+k\tau}q_{N\tau}^n}{1-q_{z+k\tau}q_{N\tau}^n} - \sum_{n=1}^{\infty}\frac{q_{z+k\tau}^{-1}q_{N\tau}^n}{1-q_{z+k\tau}^{-1}q_{N\tau}^n}\right). \tag{4.14}$$

Using the formula (4.8) for $\wp_1(z+k\tau, N\tau)$, the lemma follows readily from (4.14). $\square$

For a root of unity λ, set

$$\epsilon(\lambda) = \begin{cases} \frac{1}{1-\lambda}, & \lambda \neq 1 \\ 0, & \lambda = 1. \end{cases} \tag{4.15}$$

Now we are ready for the proof of Theorem 4.3. Let $\nu = e^{2\pi i/M}$ with μ and λ as before. For $t \in \mathbb{Z}$ we then have

$$\sum_{j=1}^{M} \nu^{jt}P_1(\nu^j,\lambda,z,\tau) = \sum_{j=1}^{M}\nu^{jt}\sum_{n\in\frac{j}{M}+\mathbb{Z}}{}' \frac{q_z^n}{1-\lambda q_\tau^n} = \sum_{n\in\mathbb{Z}}{}' \frac{q_{\frac{z+t}{M}}^n}{1-\lambda q_{\frac{\tau}{M}}^n}.$$

From (4.12) and (4.15) we conclude that

$$\sum_{j=1}^{M}\nu^{jt}P_1(\nu^j,\lambda,z,\tau) = \frac{1}{2\pi i}P_\lambda\left(\frac{z+t}{M},\frac{\tau}{M}\right) + \epsilon(\lambda). \tag{4.16}$$

Regarding this as a system of linear equations in $P_1(\nu^j,\lambda,z,\tau)$ for $t = 0, 1, \ldots, M-1$, we may invert to find that (with $\mu = \nu^j$)

$$P_1(\mu,\lambda,z,\tau) = \frac{1}{2\pi i M}\sum_{t=0}^{M-1}\mu^{-t}\left(P_\lambda\left(\frac{z+t}{M},\frac{\tau}{M}\right) + 2\pi i\epsilon(\lambda)\right).$$

Now use Lemma 4.4 to obtain

$$P_1(\mu,\lambda,z,\tau)=\frac{1}{2\pi iM}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\mu^{-t}\lambda^k \tag{4.17}$$
$$\left(G_2(\frac{N\tau}{M})\frac{(t+k\tau)}{M}-\wp_1(\frac{z+t+k\tau}{M},\frac{N\tau}{M})\right)+\frac{\epsilon(\lambda)}{M}\sum_{t=0}^{M-1}\mu^{-t}.$$

(We used $(\mu,\lambda)\neq(1,1)$ to eliminate some terms in (4.17).)

So now we get, using (4.7), (4.11) and (4.17):

$$P_1(\mu,\lambda,\frac{z}{\tau},\frac{-1}{\tau})=\frac{1}{2\pi iM}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\mu^{-t}\lambda^k$$
$$\left(G_2(\frac{-N}{M\tau})\frac{(-k+t\tau)}{M\tau}-\wp_1(\frac{z-k+t\tau}{M\tau},\frac{-N}{M\tau})\right)+\frac{\epsilon(\lambda)}{M}\sum_{t=0}^{M-1}\mu^{-t}$$
$$=\frac{1}{2\pi iM}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\mu^{-t}\lambda^k\left(\left(\left(\frac{M\tau}{N}\right)^2G_2(\frac{M\tau}{N})-2\partial ii\frac{M\tau}{N}\right)\frac{(-k+t\tau)}{M\tau}\right.$$
$$\left.-\,-\frac{M\tau}{N}\wp_1(\frac{z-k+t\tau}{N},\frac{M\tau}{N})\right)+\frac{\epsilon(\lambda)}{M}\sum_{t=0}^{M-1}\mu^{-t}. \tag{4.18}$$

Case 1. $\mu\neq 1\neq\lambda$. Here (4.18) simplifies to

$$P_1(\mu,\lambda,\frac{z}{\tau},\frac{-1}{\tau})=-\frac{\tau}{2\pi iN}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\mu^{-t}\lambda^k\wp_1(\frac{z-k+t\tau}{N},\frac{M\tau}{N})$$
$$=-\frac{\tau}{2\pi iN}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\mu^{-t}\lambda^k\wp_1(\frac{z+N-k+t\tau}{N},\frac{M\tau}{N})$$

using (4.9). From (4.17) this is indeed equal to $\tau P_1(\lambda,\mu^{-1},z,\tau)$, as required.

Case 2. $\lambda\neq 1=\mu$. This time (4.18) reads

$$P_1(\mu,\lambda,\frac{z}{\tau},\frac{-1}{\tau})=\frac{1}{2\pi iM}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\lambda^k\left(\left(\left(\frac{M\tau}{N}\right)^2G_2(\frac{M\tau}{N})-\frac{2\pi iM\tau}{N}\right)\left(\frac{-k}{M\tau}\right)\right. \tag{4.19}$$
$$\left.-\frac{M\tau}{N}\wp_1(\frac{z-k+t\tau}{N},\frac{M\tau}{N})\right)+\epsilon(la).$$

It is easy to check that $\frac{1}{N}\sum_{k=0}^{N-1} k\lambda^k = -\epsilon(\lambda)$, so that (4.19) simplifies to read

$$\begin{aligned} P_1\left(\mu, \lambda, \frac{z}{\tau}, \frac{-1}{\tau}\right) &= -\frac{\tau}{2\pi i N}\sum_{t=0}^{M-1}\sum_{k=0}^{N-1}\lambda^k\left(G_2(\frac{M\tau}{N})\frac{k}{N} + \wp_1(\frac{z-k+t\tau}{N}, \frac{M\tau}{N})\right) \\ &= \frac{\tau}{2\pi i N}\sum_{l=0}^{N-1}\sum_{t=0}^{M-1}\lambda^{-l}\left(G_2(\frac{M\tau}{N})\frac{l}{N} - \wp_1(\frac{z+l+t\tau}{N}, \frac{M\tau}{N})\right) \\ &= \tau P_1(\lambda, \mu^{-1}, z, \tau). \end{aligned}$$

This completes the discussion of Case 2. The final case $\lambda = 1 \neq \mu$ is completely analogous, and we accordingly omit details.This completes the proof of Theorem 4.3, hence also that of Theorem 4.2. $\square$

We discuss some aspects of *Bernoulli polynomials.* Recall [La, Ra] that these polynomials $B_k(x)$ are defined by the generating function

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}. \tag{4.20}$$

For example

$$B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}. \tag{4.21}$$

We will need the following identities (loc.cit.)

$$\sum_{a=0}^{N-1}(a+x)^{k-1} = \frac{1}{k}(B_k(x+N) - B_k(x)), \tag{4.22}$$

$$B_k(1-x) = (-1)^k B_k(x). \tag{4.23}$$

Proposition 4.5. *If $\mu = e^{2\pi i j/M}$ with $1 \leq j \leq M$ and $k \geq 2$ then*

$$\frac{1}{(2\pi i)^k}\sum_{0\neq m\in\mathbb{Z}}\frac{\mu^m}{m^k} = \frac{-B_k(j/M)}{k!}.$$

Proof. This is a typical sort of calculation which we give using results from [La]. Now

$$\begin{aligned} \sum_{0\neq m\in\mathbb{Z}}\mu^m/m^k &= \sum_{m=1}^{\infty}\left\{\frac{\mu^m}{m^k} + (-1)^k\frac{\mu^{-m}}{m^k}\right\} \\ &= \sum_{t=1}^{M}\sum_{n=0}^{\infty}\frac{\mu^t + (-1)^k\mu^{-t}}{(Mn+t)^k} \\ &= M^{-k}\sum_{t=1}^{M}\zeta(k, t/M)(\mu^t + (-1)^k\mu^{-t}), \end{aligned}$$

where $\zeta(k,x)$ is the Hurwitz zeta-function

$$\zeta(k,x)=\sum_{n=0}^{\infty}\frac{1}{(n+x)^k}.$$

For an M^{th} root of unity α define $f_\alpha:\mathbb{Z}/M\mathbb{Z}\to\mathbb{C}$ by $f_\alpha(n)=\alpha^n$. Set

$$\xi(k,f_\alpha)=M^{-k}\sum_{n=1}^{M}f_\alpha(n)\zeta(k,n/M).$$

In fact, one can use this definition for any function $f:\mathbb{Z}/M\mathbb{Z}\to\mathbb{C}$. Thus

$$\sum_{0\neq m\in\mathbb{Z}}\frac{\mu^m}{m^k}=\xi(k,f_\mu)+(-1)^k\xi(k,f_{\mu^{-1}}).$$

Define

$$f_j(n)=\begin{cases}1 & n\equiv j \operatorname{mod} M\\ 0 & n\not\equiv j \operatorname{mod} M\end{cases}$$

and

$$\hat{f}_j(n)=\sum_{a=1}^{M}f_j(a)e^{-2\pi ian/M}.$$

Then we get $\hat{f}_j(n)=\mu^{-n}$.

Use (loc.cit. Theorem 2.1, p. 245) to get (remember $k\geq 2$)

$$\begin{aligned}\xi(1-k,f_j)&=(2\pi i)^{-1}\left(\frac{2\pi}{M}\right)^{1-k}\Gamma(k)\left(\xi(k,f_\mu)e^{\pi i(1-k)/2}-\xi(k,f_{\mu^{-1}})e^{-\pi i(1-k)/2}\right)\\&=(2\pi i)^{-k}M^{k-1}(k-1)!\left(\xi(k,f_\mu)+(-1)^k\xi(k,f_{\mu^{-1}})\right).\end{aligned}$$

(Γ is the gamma-function here!) So now we have

$$\sum_{0\neq m\in\mathbb{Z}}\frac{\mu^m}{m^k}=\frac{(2\pi i)^kM^{1-k}}{(k-1)!}\xi(1-k,f_j).$$

On the other hand by definition,

$$\xi(1-k,f_j)=M^{k-1}\sum_{n=1}^{M}f_j(n)\zeta(1-k,n/M)=M^{k-1}\zeta(1-k,j/M)$$

so

$$\sum_{0\neq m\in\mathbb{Z}}\frac{\mu^m}{m^k}=\frac{(2\pi i)^k}{(k-1)!}\zeta(1-k,j/M).$$

Moreover (loc.cit. Corollary on p, 243)

$$\zeta(1-k,j/M)=-\Gamma(k)\operatorname{Res}_z\frac{z^{-k}e^{zj/M}}{e^z-1}=-(k-1)!B_k(j/M)/k!=-B_k(j/M)/k.$$

So finally

$$\sum_{0\neq m\in\mathbb{Z}} \frac{\mu^m}{m^k} = -\frac{(2\pi i)^k}{k!} B_k(j/M). \qquad \square$$

We next introduce the Q-functions. With $\mu = e^{2\pi ij/M}$ and $\lambda = e^{2\pi il/N}$, we define for $k = 1, 2, \cdots$ and $(\mu, \lambda) \neq (1, 1)$,

$$\begin{aligned} Q_k(\mu, \lambda, q_\tau) &= Q_k(\mu, \lambda, \tau) \\ &= \frac{1}{(k-1)!} \sum_{n\geq 0} \frac{\lambda (n + j/M)^{k-1} q_\tau^{n+j/M}}{1 - \lambda q_\tau^{n+j/M}} \\ &\quad + \frac{(-1)^k}{(k-1)!} \sum_{n\geq 1} \frac{\lambda^{-1} (n - j/M)^{k-1} q_\tau^{n-j/M}}{1 - \lambda^{-1} q_\tau^{n-j/M}} - \frac{B_k(j/M)}{k!}. \end{aligned} \tag{4.24}$$

Here $(n + j/M)^{k-1} = 1$ if $n = 0$, $j = 0$ and $k = 1$. Similarly, $(n - j/M)^{k-1} = 1$ if $n = 1$, $j = M$ and $k = 1$. For good measure we also set

$$Q_0(\mu, \lambda, \tau) = -1. \tag{4.25}$$

We need to justify the notation, which suggests that $Q_k(\mu, \lambda, \tau)$ depends only on τ and the residue classes of j and l modulo M and N respectively. To see this, note that if we provisionally denote by $Q_k'(\mu, \lambda, \tau)$ the value of (4.24) in which j is replaced by $j + M$, then we find that

$$Q_k'(\mu, \lambda, \tau) - Q_k(\mu, \lambda, \tau) = \frac{1}{(k-1)!}(j/M)^{k-1} - \frac{B_k(j/M + 1)}{k!} + \frac{B_k(j/M)}{k!} = 0,$$

the last equality following from (4.22).

We are going to prove

Theorem 4.6. *If $k \geq 0$ then $Q_k(\mu, \lambda, \tau)$ is a holomorphic modular form of weight k. If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ it satisfies*

$$Q_k(\mu, \lambda, \gamma\tau) = (c\tau + d)^k Q_k((\mu, \lambda)\gamma, \tau).$$

As usual one needs to deal with the cases $k = 1, 2$ of Theorem 4.6 separately. To this end, for each element $\boldsymbol{a} = (a_1, a_2) \in \mathbb{Q}^2/\mathbb{Z}^2$, we recall the Klein and Hecke forms (loc.cit.) defined as follows:

$$\begin{aligned} g_{\boldsymbol{a}}(\tau) &= -q_\tau^{B_2(a_1)/2} e^{2\pi i a_2 (a_1 - 1)/2} (1 - q_{a_1\tau + a_2}) \prod_{n=1}^{\infty} (1 - q_\tau^n q_{a_1\tau + a_2})(1 - q_{\tau au}^n q_{a_1\tau + a_2}^{-1}), \\ h_{\boldsymbol{a}}(\tau) &= 2\pi i \left\{ a_1 - \frac{1}{2} - \frac{q_{a_1\tau + a_2}}{1 - q_{a_1\tau + a_2}} - \sum_{m=1}^{\infty} \left\{ \frac{q_\tau^m q_{a_1\tau + a_2}}{1 - q_\tau^m q_{a_1\tau + a_2}} - \frac{q_\tau^m q_{a_1\tau + a_2}^{-1}}{1 - q_\tau^m q_{a_1\tau + a_2}^{-1}} \right\} \right\}. \end{aligned} \tag{4.26}$$

.

Using (4.21) we easily find

Proposition 4.7. *Let $a = (j/M, l/N) \notin \mathbb{Z}$. Then*

(i) $h_{\boldsymbol{a}}(\tau) = -2\pi i Q_1(\mu, \lambda, \tau)$
(ii) $\frac{d}{d\tau}(log g_{\boldsymbol{a}}(\tau)) = -2\pi i Q_2(\mu, \lambda, \tau)$.

Now we can prove Theorem 4.6 in the case $k = 1, 2$. For $k = 1$ we use (i) above together with Theorem 2 (i) and H3 of (loc.cit. p. 248). Similarly if $k = 2$ we use the calculation of (loc.cit. p. 251 et. seq.)

We now consider the case $k \geq 3$. In this case the result is a consequence of the following:

Theorem 4.8. *If* $k \geq 3$ *then*

$$Q_k(\mu, \lambda, \tau) = \frac{1}{(2\pi i)^k} \sideset{}{'}\sum_{m_1, m_2 \in \mathbb{Z}} \frac{\lambda^{-m_1}\mu^{m_2}}{(m_1\tau + m_2)^k},$$

where $\sum'$ *indicates that* $(m_1, m_2) \neq (0, 0)$.

Proof. The non-constant part of $\sum' \frac{\lambda^{-m_1}\mu^{m_2}}{(m_1\tau+m_2)^k}$ is equal to

$$\begin{aligned}
&\sum_{m_2\in\mathbb{Z}} \left(\sum_{m_1=1}^{\infty} \frac{\lambda^{-m_1}\mu^{m_2}}{(m_1\tau + m_2)^k} + (-1)^k \sum_{m_1=1}^{\infty} \frac{\lambda^{m_1}\mu^{m_2}}{(m_1\tau - m_2)^k} \right) \\
&\quad = \sum_{m_1=1}^{\infty} \sum_{t=1}^{M} \sum_{n\in\mathbb{Z}} \mu^t \left(\frac{\lambda^{-m_1}}{(m_1\tau + Mn + t)^k} + (-1)^k \frac{\lambda^{m_1}}{(m_1\tau - Mn - t)^k} \right) \\
&\quad = M^{-k} \sum_{m_1=1}^{\infty} \sum_{t=1}^{M} \sum_{n\in\mathbb{Z}} \mu^t \\
&\qquad \left(\frac{\lambda^{-m_1}}{(m_1\tau/M + n + t/M)^k} + (-1)^k \frac{\lambda^{m_1}}{(m_1\tau/M - n - t/M)^k} \right).
\end{aligned}$$

Use (loc.cit. p. 155) to get this equal to

$$\begin{aligned}
&M^{-k} \sum_{m_1=1}^{\infty} \sum_{t=1}^{M} \frac{(-1)^k \mu^t (2\pi i)^k}{(k-1)!} \\
&\qquad \sum_{n=1}^{\infty} \left(n^{k-1}\lambda^{-m_1} q^n_{m_1\tau/M+n+t/M} + (-1)^k n^{k-1}\lambda^{m_1} q^n_{m_1\tau/M-n-t/M} \right) \\
&\quad = (-1)^k M^{-k} \frac{(2\pi i)^k}{(k-1)!} \sum_{m_1=1}^{\infty} \sum_{t=1}^{M} \mu^t \sum_{n=1}^{\infty} \\
&\qquad \left(n^{k-1}\lambda^{-m_1}\nu^{tn} q^{m_1 n}_{\tau/M} + (-1)^k n^{k-1}\lambda^{m_1}\nu^{-tn} q^{m_1 n}_{\tau/M} \right) \\
&\quad = (-1)^k M^{-k} \frac{(2\pi i)^k}{(k-1)!} \sum_{t=1}^{M} \mu^t \sum_{n=1}^{\infty} \\
&\qquad \left(\sum_{d|n} d^{k-1} (\lambda^{-n/d}\nu^{td} + (-1)^k \lambda^{n/d}\nu^{-td} \right) q^n_{\tau/M}
\end{aligned}$$

(where $\nu = e^{2\pi i/M}$). Using orthogonality relations for roots of unity, this is equal to

$$\begin{aligned}
&\frac{M^{1-k}(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\left(\sum_{\substack{d|n\\ d\equiv -j\ (\mathrm{mod} M)}} d^{k-1}\lambda^{-n/d}+(-1)^k\sum_{\substack{d|n\\ d\equiv j\ (\mathrm{mod} M)}} d^{k-1}\lambda^{n/d}\right)q_{\tau/M}^{n}\\
&=\frac{M^{1-k}(2\pi i)^k}{(k-1)!}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\Big(\lambda^m(Mn+j)^{k-1}q_{\tau/M}^{m(Mn+j)}\\
&\quad+(-1)^k\lambda^{-m}(Mn+M-j)^{k-1}q_{\tau/M}^{m(Mn+M-j)}\Big)\\
&=\frac{M^{1-k}(2\pi i)^k}{(k-1)!}\\
&\quad\sum_{n=0}^{\infty}\left(\frac{\lambda(Mn+j)^{k-1}q_{\tau/M}^{Mn+j}}{1-\lambda q_{\tau/M}^{Mn+j}}+(-1)^k\frac{\lambda^{-1}(Mn+M-j)^{k-1}q_{\tau/M}^{Mn+M-j}}{1-\lambda^{-1}q_{\tau/M}^{Mn+M-j}}\right),
\end{aligned}$$

where $1 \le j \le M$. Together with Proposition 4.5 we now get

$$\begin{aligned}
&\frac{1}{(2\pi i)^k}\sum{}' \frac{\lambda^{-m_1}\mu^{m_2}}{(m_1\tau+m_2)^k}=-\frac{B_k(j/M)}{k!}+\frac{M^{1-k}}{(k-1)!}\\
&\quad\sum_{n=0}^{\infty}\left(\frac{\lambda(Mn+j)^{k-1}q_{\tau/M}^{Mn+j}}{1-\lambda q_{\tau/M}^{Mn+j}}+(-1)^k\frac{\lambda^{-1}(Mn+M-j)^{k-1}q_{\tau/M}^{Mn+M-j}}{1-\lambda^{-1}q_{\tau/M}^{Mn+M-j}}\right)\\
&=Q_k(\mu,\lambda,\tau).
\end{aligned}$$

This completes the proof of Theorem 4.8 and hence also Theorem 4.6. □

We remark the well-known fact that if we take $(\mu, \lambda) = (1, 1)$ in Theorem 4.8 we obtain the Eisenstein series of weight k as long as $k \ge 3$ is even. Thus for $k \ge 4$ even,

$$\begin{aligned}
G_k(\tau) &= \sum_{m_1,m_2\in\mathbb{Z}}{}' \frac{1}{(m_1\tau+m_2)^k}\\
&=(2\pi i)^k\left(\frac{-B_k(0)}{k!}+\frac{2}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right),
\end{aligned}\tag{4.27}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. For this range of values of k, $G_k(\tau)$ is a modular form on $SL(2,\mathbb{Z})$ of weight k.

We make use of the normalized Eisenstein series

$$E_k(\tau)=\frac{1}{(2\pi i)^k}G_k(\tau)=\frac{-B_k(0)}{k!}+\frac{2}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \tag{4.28}$$

for even $k \ge 2$. (Warning: this is not the same notation as used in (loc.cit.), for example.)

We also utilize the differential operator ∂_k which acts on holomorphic functions on $\mathfrak{h}$ via

$$\partial_k f(\tau) = \frac{1}{2\pi i}\frac{df(\tau)}{d\tau} + kE_2(\tau)f(\tau). \tag{4.29}$$

One checks using (4.7) that

$$(\partial_k f)|_{k+2}\gamma = \partial_k(f|_k\gamma). \tag{4.30}$$

We complete this section with a discussion of some further functions related to P and Q which occur later. Again we take $\mu = e^{2\pi ij/M}$ and $\lambda = e^{2\pi il/N}$. Set for $k \geq 1$,

$$\bar{P}_k(\mu,\lambda,z,q_\tau) = \bar{P}_k(\mu,\lambda,z,\tau) = \frac{1}{(k-1)!}\sideset{}{'}\sum_{n\in j/M+\mathbb{Z}}\frac{n^{k-1}z^n}{1-\lambda q_\tau^n} \tag{4.31}$$

with

$$\bar{P}_0 = 0. \tag{4.32}$$

Recall (3.6). We shall need the following result in Sect. 8.

Proposition 4.9. *If* $m \in \mathbb{Z}$, $\mu = e^{2\pi ij/M}$, $k \geq 0$ *and* $(\mu,\lambda) \neq (1,1)$, *then*

$$\begin{aligned}
&Q_k(\mu,\lambda,\tau) + \frac{1}{k}B_k(1-m+j/M)\\
&\quad = \operatorname{Res}_z\left(\iota_{z,z_1}((z-z_1)^{-1})z_1^{m-j/M}z^{-m+j/M}\bar{P}_k\left(\mu,\lambda,\frac{z_1}{z},\tau\right)\right)\\
&\qquad - \operatorname{Res}_z\left(\lambda\iota_{z_1,z}((z-z_1)^{-1})z_1^{m-j/M}z^{-m+j/M}\bar{P}_k\left(\mu,\lambda,\frac{z_1q_\tau}{z},\tau\right)\right).
\end{aligned}$$

Proof. The result is clear if $k = 0$, so take $k \geq 1$. The first of the two residues we must evaluate is equal to

$$\begin{aligned}
&\operatorname{Res}_z\frac{1}{(k-1)!z}\sum_{n=0}^{\infty}\sum_{r\in\mathbb{Z}}\left(\frac{(r+j/M)^{k-1}\left(\frac{z_1}{z}\right)^{r+j/M}}{1-\lambda q_\tau^{r+j/M}}\right)\\
&\qquad = \frac{1}{(k-1)!}\sum_{n=0}^{\infty}\frac{(-n-m+j/M)^{k-1}}{1-\lambda q_\tau^{-n-m+j/M}}\\
&\qquad = \frac{(-1)^k}{(k-1)!}\sum_{n=m}^{\infty}\frac{\lambda^{-1}(n-j/M)^{k-1}q_\tau^{n-j/M}}{1-\lambda^{-1}q_\tau^{n-j/M}}.
\end{aligned}$$

Similarly the second residue is equal to

$$\frac{-1}{(k-1)!}\sum_{n=1-m}^{\infty}\frac{\lambda(n+j/M)^{k-1}q_\tau^{n+j/M}}{1-\lambda q_\tau^{n+j/M}}.$$

Comparing with the definition of $Q_k(\mu, \lambda, \tau)$, we see that we are reduced to establishing that

$$\frac{1}{k!}(B_k(1-m+j/M)-B_k(j/M))$$
$$= \begin{cases} \dfrac{-1}{(k-1)!}\displaystyle\sum_{n=0}^{-m}\frac{\lambda(n+j/M)^{k-1}q_\tau^{n+j/M}}{1-\lambda q_\tau^{n+j/M}} \\ +\frac{(-1)^k}{(k-1)!}\sum_{n=m}^{0}\frac{\lambda^{-1}(n-j/M)^{k-1}q_\tau^{n-j/M}}{1-\lambda^{-1}q_\tau^{n-j/M}}, & m\le 0 \\ \frac{1}{(k-1)!}\sum_{n=1-m}^{-1}\frac{\lambda(n+j/M)^{k-1}q_\tau^{n+j/M}}{1-\lambda q_\tau^{n+j/M}} \\ -\frac{(-1)^k}{(k-1)!}\sum_{n=1}^{m-1}\frac{\lambda^{-1}(n-j/M)^{k-1}q_\tau^{n-j/M}}{1-\lambda^{-1}q_\tau^{n-j/M}}, & m\ge 2 \\ 0, & m=1 \end{cases}. \tag{4.33}$$

The case $m=1$ is trivial. Assume next that $m\le 0$. Then the two summations in (4.33) are equal to

$$\frac{-1}{(k-1)!}\sum_{n=0}^{-m}\frac{\lambda(n+j/M)^{k-1}q_\tau^{n+j/M}}{1-\lambda q_\tau^{n+j/M}}-\frac{1}{(k-1)!}\sum_{n=m}^{0}\frac{(-n+j/M)^{k-1}}{\lambda q_\tau^{-n+j/M}-1}$$
$$=\frac{1}{(k-1)!}\sum_{n=0}^{-m}(n+j/M)^{k-1}.$$

Now the desired result follows from (4.22). Similarly, if $m\ge 2$ the two summands in (4.33) sum to

$$\frac{(-1)^k}{(k-1)!}\sum_{n=1}^{m-1}(n-j/M)^{k-1}.$$

Two applications of (4.22) then yield the desired result. □

5. The Space of 1-Point Functions on the Torus

The following notation will be in force for some time:

(a) V is a vertex operator algebra .
(b) $g, h\in \mathrm{Aut}(V)$ have finite order and satisfy $gh=hg$.
(c) $A=\langle g, h\rangle$.
(d) g has order T, h has order T_1 and A has exponent $N=lcm(T, T_1)$.
(e) $\Gamma(T, T_1)$ is the subgroup of matrices $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ in $SL(2,\mathbb{Z})$ satisfying $a\equiv d\equiv 1$ $(\mathrm{mod} N)$, $b\equiv 0$ $(\mathrm{mod} T)$, $c\equiv 0$ $(\mathrm{mod} T_1)$.
(f) $M(T, T_1)$ is the ring of holomorphic modular forms on $\Gamma(T, T_1)$; it is naturally graded $M(T, T_1)=\oplus_{k\ge 0}M_k(T, T_1)$, where $M_k(T, T_1)$ is the space of forms of weight k. We also set $M(1)=M(1,1)$.
(g) $V(T, T_1)=M(T, T_1)\otimes_{\mathbb{C}} V$.

(h) $O(g,h)$ is the $M(T,T_1)$-submodule of $V(T,T_1)$ generated by the following elements, where $v \in V$ satisfies $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$:

$$v[0]w,\ w \in V,\ (\mu,\lambda) = (1,1), \tag{5.1}$$

$$v[-2]w + \sum_{k=2}^{\infty}(2k-1)E_{2k}(\tau) \otimes v[2k-2]w,\ (\mu,\lambda) = (1,1), \tag{5.2}$$

$$v,\ (\mu,\lambda) \neq (1,1), \tag{5.3}$$

$$\sum_{k=0}^{\infty} Q_k(\mu,\lambda,\tau) \otimes v[k-1]w,\ (\mu,\lambda) \neq (1,1). \tag{5.4}$$

Here, notation for modular forms is as in Sect. 4. These definitions are sensible because of the following:

Lemma 5.1. *$M(T,T_1)$ is a Noetherian ring which contains each $E_{2k}(\tau)$, $k \geq 2$, and each $Q_k(\mu,\lambda,\tau)$, $k \geq 0$, for μ,λ a T^{th}., resp. T_1^{th}. root of unity.*

Proof. It is well-known that the ring of holomorphic modular forms on any congruence subgroup of $SL(2,\mathbb{Z})$ is Noetherian. So the first statement holds.

Each E_{2k} is a modular form on $SL(2,\mathbb{Z})$, whereas the containment $Q_k(\mu,\lambda,\tau) \in M_k(T,T_1)$ follows from Theorem 4.6. $\square$

Lemma 5.2. *Suppose that V satisfies Condition C_2. Then $V(T,T_1)/O(g,h)$ is a finitely-generated $M(T,T_1)$-module.*

Proof. Since $C_2(V)$ is a graded subspace of V of finite codimension, there is an integer m such that $V_n \subset C_2(V)$ whenever $n > m$. Let M be the $M(T,T_1)$-submodule of $V(T,T_1)$ generated by $W = \oplus_{n\leq m}V_n$. The lemma will follow from the assertion that $V(T,T_1) = M + O(g,h)$. This will be established by proving that if $v \in V_{[k]}$ (cf. (2.22)) then $v \in M + O(g,h)$.

If $k \leq m$ then $v \in W$ by (2.23) and we are done, so we may take $k > m$. Since $V_{[k]} \subset C_2(V) + W$ then we may write v in the form

$$v = w + \sum_{i=1}^{p} a_i(-2)b_i$$

with $a_i, b_i \in V$ homogeneous in the vertex operator algebra (V, [ ]) such that $\mathrm{wt}[a_i] + \mathrm{wt}[b_i] + 1 = k$. Clearly, it suffices to show that $a_i(-2)b_i \in M + O(g,h)$. We may also assume that $ga_i = \mu^{-1}a_i$, $ha_i = \lambda^{-1}a_i$ for suitable scalars μ,λ.

Suppose first that $(\mu,\lambda) = (1,1)$. From (5.2) we see that $O(g,h)$ contains

$$a_i[-2]b_i + \sum_{l=2}^{\infty}(2l-1)E_{2l}(\tau) \otimes a_i[2l-2]b_i.$$

Since $\mathrm{wt}[a_i[2l-2]b_i] = k - 2l$, then the sum

$$\sum_{l=2}^{\infty}(2l-1)E_{2l}(\tau) \otimes a_i[2l-2]b_i$$

lies in $M + O(g,h)$ by the inductive hypothesis, whence so too does $a_i[-2]b_i$.

On the other hand, it follows from (2.9) that we have

$$v(n) = v[n] + \sum_{j>n} \alpha_j v[j]$$

for $v \in V$, $j \in \mathbb{Z}$ and scalars α_j. In particular we get

$$a_i(-2)b_i = a_i[-2]b_i + \sum_{j>-2} \alpha_j a_i[j]b_i.$$

Having already shown that each of the summands $a_i[j]b_i$ lies in $M + O(g,h)$, $j \geq -2$, we get $a_i(-2)b_i \in M + O(g,h)$ as desired.

Now suppose that $(\mu, \lambda) \neq (1,1)$. In this case (5.4) tells us that $O(g,h)$ contains the element

$$-a_i[-1]b_i + \sum_{l=1}^{\infty} Q_l(\mu, \lambda, \tau) \otimes a_i[l-1]b_i$$

(cf. (4.25)). More to the point, $O(g,h)$ also contains the same expression with a_i replaced by $L[-1]a_i$. Since $(L[-1]a_i)[t] = -ta_i[t-1]$ by (2.7), we see that $O(g,h)$ contains the element

$$a_i[-2]b_i + \sum_{l=1}^{\infty}(l-1)Q_l(\mu, \lambda, \tau) \otimes a_i[l-2]b_i.$$

Now we proceed as before to conclude that $a_i(-2)b_i \in M + O(g,h)$. □

There is a natural grading on $V(T, T_1)$. Namely, the subspace of elements of degree n is defined to be

$$V(T, T_1)_n = \oplus_{k+l=n} M_k(T, T_1) \otimes V_{[l]}. \tag{5.5}$$

Observe that $O(g,h)$ is a graded subspace of $V(T, T_1)$.

Lemma 5.3. *Suppose V satisfies Condition C_2. If $v \in V$ then there is $m \in \mathbb{N}$ and elements $r_i(\tau) \in M(T, T_1)$, $0 \leq i \leq m-1$, such that*

$$L[-2]^m v + \sum_{i=0}^{m-1} r_i(\tau) \otimes L[-2]^i v \in O(g,h). \tag{5.6}$$

Proof. Let I be the $M(T, T_1)$-submodule of $V(T, T_1)/O(g,h)$ generated by $\{L[-2]^i v, i \geq 0\}$. Since $M(T, T_1)$ is a Noetherian ring, Lemma 5.2 tells us that I is finitely generated and so some relation of the form (5.6) must hold. □

We now define the *space of* (g,h) *1-point functions* $\mathcal{C}_1(g,h)$ to be the $\mathbb{C}$-linear space consisting of functions

$$S : V(T, T_1) \times \mathfrak{h} \to \mathbb{C}$$

which satisfy

(C1) $S(v, \tau)$ is holomorphic in τ for $v \in V(T, T_1)$.
(C2) $S(v, \tau)$ is $M(T, T_1)$-linear in the sense that S is $\mathbb{C}$-linear in v and satisfies

$$S(f \otimes v, \tau) = f(\tau)S(v, \tau) \tag{5.7}$$

for $f \in M(T, T_1)$ and $v \in V$.

(C3) $S(v,\tau)=0$ if $v\in O(g,h)$.
(C4) If $v\in V$ satisfies $gv=hv=v$ then

$$S(L[-2]v,\tau)=\partial S(v,\tau)+\sum_{l=2}^{\infty}E_{2l}(\tau)S(L[2l-2]v,\tau). \tag{5.8}$$

In (5.8), ∂S is the operator which is linear in v and satisfies

$$\partial S(v,\tau)=\partial_k S(v,\tau)=\frac{1}{2\pi i}\frac{d}{d\tau}S(v,\tau)+kE_2(\tau)S(v,\tau) \tag{5.9}$$

for $v\in V_{[k]}$ (cf. (4.29)).

Theorem 5.4 (Modular-Invariance**).** *For $S\in\mathcal{C}_1(g,h)$ and $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z})$ define*

$$S|\gamma(v,\tau)=S|_k\gamma(v,\tau)=(c\tau+d)^{-k}S(v,\gamma\tau) \tag{5.10}$$

for $v\in V_{[k]}$, and extend linearly. Then $S|\gamma\in\mathcal{C}_1((g,h)\gamma)$.

Proof. We need to verify that $S|\gamma$ satisfies (C3)-(C4) with $(g,h)\gamma=(g^ah^c,g^bh^d)$ in place of (g,h).

Step 1. $S|\gamma$ vanishes on $O((g,h)\gamma)$. Pick $v,w\in V$ homogeneous in $(V,Y[\,])$ and with $g^ah^cv=\mu^{-1}v$, $g^bh^dv=\lambda^{-1}v$. Suppose to begin with that $(\mu,\lambda)=(1,1)$. We must show that $S|\gamma(u,\tau)=0$ when u is one of the elements (5.1) and (5.2). This follows easily from (5.10), the equality $S(u,\tau)=0$, and the fact that E_{2k} is modular of weight $2k$.

Now assume that $(\mu,\lambda)\neq(1,1)$. If $gv=\alpha^{-1}v$ and $hv=\beta^{-1}v$ then $(\alpha,\beta)\neq(1,1)$, so that certainly $S|\gamma(v,\tau)=(c\tau+d)^{-\mathrm{wt}[v]}S(v,\gamma\tau)=0$. So it remains to show that $S|\gamma(u,\tau)=0$ for

$$u=\sum_{k=0}^{\infty}Q_k(\mu,\lambda,\tau)\otimes v[k-1]w.$$

First note that we have $(\alpha,\beta)=(\mu,\lambda)\gamma^{-1}$. Then with Lemma 5.1 and Theorem 4.6 we calculate that

$$\begin{aligned}S|\gamma(u,\tau)&=\sum_{k=0}^{\infty}Q_k(\mu,\lambda,\tau)S|\gamma(v[k-1]w,\tau)\\&=\sum_{k=0}^{\infty}Q_k(\mu,\lambda,\tau)(c\tau+d)^{-\mathrm{wt}[v]-\mathrm{wt}[w]+k}S(v[k-1]w,\gamma\tau)\\&=(c\tau+d)^{-\mathrm{wt}[v]-\mathrm{wt}[w]}\sum_{k=0}^{\infty}Q_k(\alpha,\beta,\gamma\tau)S(v[k-1]w,\gamma\tau)\\&=(c\tau+d)^{-\mathrm{wt}[v]-\mathrm{wt}[w]}\sum_{k=0}^{\infty}S(Q_k(\alpha,\beta,\gamma\tau)\otimes v[k-1]w,\gamma\tau)\end{aligned}$$

which is indeed 0 since $S\in\mathcal{C}_1(g,h)$.

Step 2. $S|\gamma$ satisfies (5.8). First note that if $g^a h^c v = g^b h^d v = v$ then also $gv = hv = v$. Then we calculate using (4.30) that

$$\begin{aligned} S|\gamma(L[-2]v,\tau) &= (c\tau+d)^{-\mathrm{wt}[v]-2} S(L[-2]v,\gamma\tau) \\ &= (c\tau+d)^{-\mathrm{wt}[v]-2}(\partial S(v,\gamma\tau) + \sum_{k=2}^{\infty} E_{2k}(\gamma\tau) S(L[2k-2]v,\gamma\tau)) \\ &= (\partial_{\mathrm{wt}[v]} S)|_{\mathrm{wt}[v]+2}\gamma(v,\tau) \\ &\quad + \sum_{k=2}^{\infty} (c\tau+d)^{2k-\mathrm{wt}[v]-2} E_{2k}(\tau) S(L[2k-2]v,\gamma\tau) \\ &= \partial_{\mathrm{wt}[v]}(S|_{\mathrm{wt}[v]}\gamma)(v,\tau) + \sum_{k=2}^{\infty} E_{2k}(\tau) S|\gamma(L[2k-2]v,\tau). \end{aligned}$$

This completes the proof of Step 2, and with it that of the theorem. □

6. The Differential Equations

In this section we study certain differential equations which are satisfied by functions $S(v,\tau)$ in the space of (g,h) 1-point functions. The idea is to exploit Lemma 5.3 together with (5.8).

We fix an element $S \in \mathcal{C}_1(g,h)$.

Lemma 6.1. *Let $v \in V$ and suppose that V satisfies Condition C_2. There are $m \in \mathbb{N}$ and $r_i(\tau) \in M(T,T_1)$, $0 \le i \le m-1$, such that*

$$S(L[-2]^m v,\tau) + \sum_{i=0}^{m-1} r_i(\tau) S(L[-2]^i v,\tau) = 0. \tag{6.1}$$

Proof. Combine Lemma 5.3 together with (C2) and (C3). □

In the following we extend (5.9) by setting for $f \in M_l(T,T_1)$, $v \in V_{[k]}$,

$$\partial S(f\otimes v,\tau) = \partial_{k+l}(S(f\otimes v,\tau)) = \partial_{k+l}(f(\tau)S(v,\tau)) \tag{6.2}$$

(cf. (4.29). Then define

$$\partial^i S(f\otimes v,\tau) = \partial_{k+l+2(i-1)}(\partial^{i-1} S(f\otimes v,\tau)) \tag{6.3}$$

for $i \ge 1$. Note that

$$\partial S(f\otimes v,\tau) = (\partial_l f(\tau))S(v,\tau) + f(\tau)\partial S(v,\tau). \tag{6.4}$$

Moreover $\partial_l f(\tau) \in M_{l+2}(T,T_1)$ as we see from (4.30).

The simplest case to study is that corresponding to a *primary field*, i.e. a vector v which is a highest weight vector for the Virasoro algebra in $(V, Y[\ ])$. Thus v satisfies $L[n]v = 0$ for $n > 0$. We assume that this holds until further notice.

First note that we have

$$S(L[-2]v,\tau) = \partial S(v,\tau). \tag{6.5}$$

This follows from (5.8) if $gv = hv = v$. In general, it is a consequence of this special case, the linearity of $S(v, \tau)$ in v, and the identity $S(w, \tau) = 0$ if $gw = \mu^{-1}w$, $hw = \lambda^{-1}w$ and $(\mu, \lambda) \neq (1, 1)$. This latter equality follows from (5.3) and (C3). In the same way, we find from (5.8) that

$$S(L[-2]^{i+1}v, \tau) = \partial S(L[-2]^i v, \tau) + \sum_{k=2}^{\infty} E_{2k}(\tau) S(L[2k-2]L[-2]^i v, \tau). \quad (6.6)$$

Using the Virasoro algebra relation we easily find that for $i \in \mathbb{N}$ and $k \geq 2$ there are scalars α_{ijk}, $0 \leq j \leq i-1$ such that

$$L[2k-2]L[-2]^i v = \sum_{j=0}^{i-1} \alpha_{ijk} L[-2]^j v, \quad (6.7)$$

so that (6.6) becomes

$$S(L[-2]^{i+1}v, \tau) = \partial S(L[-2]^i v, \tau) + \sum_{j=0}^{i-1}\sum_{k=2}^{\infty} \alpha_{ijk} E_{2k}(\tau) S(L[-2]^j v, \tau). \quad (6.8)$$

Now proceeding by induction on i, the case $i = 1$ being (6.5), one proves

Lemma 6.2. *Suppose that $L[n]v = 0$ for $n > 0$. Then for $i \geq 1$ there are elements $f_j(\tau) \in M(1)$, $0 \leq j \leq i-1$, such that*

$$S(L[-2]^i v, \tau) = \partial^i S(v, \tau) + \sum_{j=0}^{i-1} f_j(\tau) \partial^j S(v, \tau). \quad (6.9)$$

Combine Lemmas 6.2 and 6.1 to obtain

Lemma 6.3. *Suppose that V satisfies Condition C_2, and that $v \in V$ satisfies $L[n]v = 0$ for $n > 0$. Then there are $m \in \mathbb{N}$ and $g_i(\tau) \in M(T, T_1)$, $0 \leq i \leq m-1$, such that*

$$\partial^m S(v, \tau) + \sum_{i=0}^{m-1} g_i(\tau) \partial^i S(v, \tau) = 0. \quad (6.10)$$

Bearing in mind the definition of ∂ (cf. (5.9), (6.3)), (6.10) may be reformulated as follows:

Proposition 6.4. *Let $R = R(T, T_1)$ be the ring of holomorphic functions generated by $E_2(\tau)$ and $M(T, T_1)$. Suppose that V satisfies Condition C_2, and that $v \in V$ satisfies $L[n]v = 0$ for $n > 0$. Then there are $m \in \mathbb{N}$ and $r_i(\tau) \in R(T, T_1)$, $0 \leq i \leq m-1$, such that*

$$(q_{\frac{1}{T}} \frac{d}{dq_{\frac{1}{T}}})^m S(v, \tau) + \sum_{i=0}^{m-1} r_i(\tau) (q_{\frac{1}{T}} \frac{d}{dq_{\frac{1}{T}}})^i S(v, \tau) = 0, \quad (6.11)$$

where $q_{\frac{1}{T}} = e^{2\pi i \tau / T}$.

We observe here only that $q\dfrac{d}{dq} = \dfrac{1}{T} q_{\frac{1}{T}} \dfrac{d}{dq_{\frac{1}{T}}} = \dfrac{1}{2\pi i}\dfrac{d}{d\tau}$.

Now (6.11) is a homogeneous linear differential equation with holomorphic coefficients $r_i(\tau) \in R$, and such that 0 is a regular singular point. The forms in $R(T, T_1)$ have Fourier expansions at ∞ which are power series in $q_{\frac{1}{T}}$ because they are invariant under $\begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}$. We are therefore in a position to apply the theory of Frobenius–Fuchs concerning the nature of the solutions to such equations. A good reference for the elementary aspects of this theory is [I], but the reader may also consult [AM] where they arise in a context related to that of the present paper. We will also need some results from (loc. cit.) in Sect. 11. Frobenius–Fuchs theory tells us that $S(v, \tau)$ may be expressed in the following form: for some $p \geq 0$,

$$S(v, \tau) = \sum_{i=0}^{p} (\log q_{\frac{1}{T}})^i S_i(v, \tau), \tag{6.12}$$

where

$$S_i(v, \tau) = \sum_{j=1}^{b(i)} q^{\lambda_{ij}} S_{i,j}(v, \tau), \tag{6.13}$$

$$S_{i,j}(v, \tau) = \sum_{n=0}^{\infty} a_{i,j,n}(v) q^{n/T} \tag{6.14}$$

are holomorphic on the upper half-plane, and

$$\lambda_{i,j_1} \not\equiv \lambda_{i,j_2} \; (\mathrm{mod} \frac{1}{T}\mathbb{Z}) \tag{6.15}$$

for $j_1 \neq j_2$.

We are going to prove

Theorem 6.5. *Suppose that V satisfies Condition C_2. For* every *$v \in V$, the function $S(v, \tau) \in \mathcal{C}_1(g, h)$ can be expressed in the form (6.12)-(6.15). Moreover, p is bounded independently of v.*

We begin by proving by induction on k that if $v \in V_{[k]}$ then $S(v, \tau)$ has an expression of the type (6.12). We have already shown this if v is a highest weight vector for the Virasoro algebra and in particular if v is in the top level $V_{[t]}$ of $(V, Y[\,])$, i.e., if $V_{[t]} \neq 0$ and $V_{[s]} = 0$ for $s < t$. This begins the induction. The proof is an elaboration of the previous case. We may assume that $gv = hv = v$.

Lemma 6.6. *Suppose that $l \geq 2$ and $i \geq 0$. Then there are scalars α_{ijl} and $w_{ijl} \in V_{[2i+2-2l-2j+k]}$, $0 \leq j \leq i-1$, such that*

$$L[2l-2]L[-2]^i v = L[-2]^i L[2l-2]v + \sum_{j=0}^{i-1} \alpha_{ijl} L[-2]^j w_{ijl}. \tag{6.16}$$

Moreover $\mathrm{wt}[w_{ijl}] \leq \mathrm{wt}[v]$ *with equality only if $w_{ijl} = v$.*

Proof. By induction on $i + l$, the case $i = 0$ being trivial. Now we calculate

$$L[2l-2]L[-2]^{i+1}v = (L[-2]L[2l-2] + 2lL[2l-4] + \delta_{l,2}\frac{c}{2})L[-2]^i v$$
$$= L[-2]^{i+1}L[2l-2]v+\sum_{j=0}^{i-1}\alpha_{ijl}L[-2]^{j+1}w_{ijl}+2lL[2l-4]L[-2]^i v+\delta_{l,2}\frac{c}{2}L[-2]^i v.$$

Either $l = 2$ or the inductive hypothesis applies to $L[2l-4]L[-2]^i v$, and in either case the lemma follows. □

Now use (5.8) and Lemma 6.6 to see that

$$S(L[-2]^{i+1}v, \tau) = \partial S(L[-2]^i v, \tau)$$
$$+ \sum_{l=2}^{\infty} E_{2l}(\tau)(S(L[-2]^i L[2l-2]v, \tau) + \sum_{j=0}^{i} \alpha_{ijl} S(L[-2]^j w_{ijl}, \tau)). \quad (6.17)$$

Note that (6.17) is the appropriate generalization of (6.8). By induction based on (6.17) we find

Lemma 6.7. *For $i \geq 1$ we have*

$$S(L[-2]^i v, \tau) = \partial^i S(v, \tau) + \sum_{j=0}^{i-1} f_{ij}(\tau)\partial^j S(v, \tau) + \sum_{j=0}^{i-1}\sum_{l} g_{ijl}(\tau)\partial^j S(w_{ijl}, \tau), \quad (6.18)$$

where $f_{ij}(\tau), g_{ijl}(\tau) \in M(1)$ and $\mathrm{wt}[w_{ijl}] < \mathrm{wt}[v]$. □

The analogue of Lemma 6.3 is now

Lemma 6.8. *There is $m \in \mathbb{N}$ such that*

$$\partial^m S(v, \tau) + \sum_{i=0}^{m-1} g_i(\tau)\partial^i S(v, \tau) + \sum_{j=0}^{m}\sum_{l} h_{jl}(\tau)\partial^j S(w_{jl}, \tau) = 0 \quad (6.19)$$

for $g_i(\tau), h_{jl}(\tau) \in M(T, T_1)$, and $\mathrm{wt}[w_{jl}] < \mathrm{wt}[v]$. □

We are now in a position to complete the proof that $S(v, \tau)$ has an expression of the form (6.12)-(6.15). By induction this is true of the terms $S(w_{jl}, \tau)$ in (6.19), and hence the third summand on the l.h.s of (6.19) also has such an expression. Thus as before we may view (6.19) as a differential equation of regular singular type, this time inhomogeneous, namely,

$$(q_{\frac{1}{T}}\frac{d}{dq_{\frac{1}{T}}})^m S(v, \tau) + \sum_{i=0}^{m-1} r_i(\tau)(q_{\frac{1}{T}}\frac{d}{dq_{\frac{1}{T}}})^i S(v, \tau) + \sum_{i=0}^{p}(\log q_{\frac{1}{T}})^i u_i(v, \tau) = 0, \quad (6.20)$$

where $r_i(\tau) \in M(T, T_1)$ (cf. Proposition 6.4) and $u_i(v, \tau)$ satisfies (6.13)-(6.15).

One easily sees (cf. [I, AM]) that the functions $(\log q_{\frac{1}{T}})^i u_i(v, \tau)$, $0 \leq i \leq p$, are themselves solutions of a differential equation of regular singular type (6.11) with coefficients analytic in the upper half plane. Let us formally state this by saying that they are solutions of the differential equation $L_1 f = 0$, where L_1 is a suitable linear differential operator with 0 as regular singular point and coefficients analytic in the upper half plane. Now (6.20) takes the form $L_2 S + f = 0$ for the corresponding linear differential operator L_2, so that we get $L_1 L_2 S = 0$. But $L_1 L_2$ is once again a linear differential operator of the appropriate type, so again the Frobenius–Fuchs theory allows us to conclude that $S(v, \tau)$ indeed satisfies (6.12)-(6.15).

It remains to prove that the integer p in (6.12) can be bounded independently of v. Indeed, we showed in Lemma 5.2 that if $W = \oplus_{n \le m} V_n$ and $V_n \subset C_2(V)$ for $n > m$ then $V(T, T_1)/O(g, h)$ is generated as $M(T, T_1)$-module by W. So for $v \in V$ we have $v \equiv \sum_i f_i(\tau) \otimes w_i$ (mod $O(g, h)$), where $\{w_i\}$ is a basis for W, whence $S(v, \tau) = \sum_i f_i(\tau) S(w_i, \tau)$ since S vanishes on $O(g, h)$. Clearly then, we may take p to be the maximum of the corresponding integers determined by $S(w_i, \tau)$. This completes the proof of Theorem 6.5.

7. Formal 1-Point Functions

Although we dealt with holomorphic functions in Sect. 6, the arguments were all formal in nature. In this short section we record a consequence of this observation.

We identify elements of $M(T, T_1)$ with their Fourier expansions at ∞, which lie in the ring of formal power series $\mathbb{C}[[q_{\frac{1}{T}}]]$. Similarly, the functions $E_{2k}(\tau)$, $k \ge 1$, are considered to lie in $\mathbb{C}[[q]]$. The operator ∂ (cf. (5.9), (6.2)) operates on these and other power series via the identification $\frac{1}{2\pi i} \frac{d}{dq} = \frac{q_{\frac{1}{T}}}{T} \frac{d}{dq_{\frac{1}{T}}}$.

A *formal* (g, h) 1-point function is a map

$$S : V(T, T_1) \to P,$$

where P is the space of formal power series of the form

$$q^{\lambda} \sum_{n=0}^{\infty} a_n q^{n/T} \tag{7.1}$$

for some $\lambda \in \mathbb{C}$, and which satisfies the formal analogues of (C2)-(C4) in Sect. 5. We will establish

Theorem 7.1. *Suppose that S is a formal (g, h) 1-point function. Then S defines an element of $\mathcal{C}_1(g, h)$, also denoted by S, via the identification*

$$S(v, \tau) = S(v, q), q = q_\tau = e^{2\pi i \tau}. \tag{7.2}$$

The main point is to show that if S is a formal (g, h) 1-point function, and if $v \in V$ is such that

$$S(v, q) = q^{\lambda} \sum_{n=0}^{\infty} a_n q^{n/T}$$

then $q_\tau^{\lambda} \sum_{n=0}^{\infty} a_n q_\tau^{n/T}$ is holomorphic in τ. We prove this as in Sect. 6. Namely, by first showing that if v is a highest weight vector for the Virasoro algebra then $S(v, q)$ satisfies a differential equation of type (6.11). Since the coefficients are holomorphic in $\mathfrak{h}$, the Frobenius–Fuchs theory tells us that $S(v, q)$ has the desired convergence.

Proceeding by induction on $\mathrm{wt}[v]$, in the general case we arrive at an inhomogeneous differential equation of type (6.20). Again convergence of $S(v, q)$ follows from the Frobenius–Fuchs theory. Since the proofs of these assertions are *precisely* the same as those of Sect. 6, we omit further discussion.

8. Correlation Functions

In this section we start to relate the theory of 1-point functions to that of twisted V-modules. We keep the notation (a)-(h) introduced at the beginning of Sect. 5, and introduce now a simple g-twisted V-module $M = M(g) = \oplus_{n=0}^{\infty} M_{\lambda+n/T}$ (cf. (3.11)). We further assume that h leaves M stable, that is $h \circ M \simeq M$. As remarked in Sect. 3, there is a projective representation on M of the stabilizer (in $\mathrm{Aut}V$) of M, and we let $\phi(h)$ be a linearized action on M of the element corresponding to h. This all means (cf. (3.15), (3.16)) that if $v \in V$ operates on M via the vertex operator $Y_M(v, z)$ then we have (as operators on M)

$$\phi(h)Y_M(v, z)\phi(h)^{-1} = Y_M(hv, z). \tag{8.1}$$

We define $M' = \oplus_{n=0}^{\infty} M'_{\lambda+n/T}$ to be the *restricted dual* of M, so that $M'_n = \mathrm{Hom}_{\mathbb{C}}(M_n, \mathbb{C})$ and there is a pairing $\langle , \rangle : M' \times M \to \mathbb{C}$ such that $\langle M'_n, M_m\rangle = 0$ if $m \neq n$.

With this notation, a (g, h) 1-*point correlation function* is essentially a trace function, namely

$$\mathrm{tr}Y_M(v, z)q^{L(0)} = \sum_{w,w'} \langle w', Y_M(v, z)q^{L(0)}w\rangle, \tag{8.2}$$

where w ranges over a homogeneous basis of M, w' ranges over the dual basis of M', and q is indeterminate. As Laurent series we have

$$\mathrm{tr}Y_M(v, z)q^{L(0)} = \sum_{w,w'} \sum_{n\in\frac{1}{T}\mathbb{Z}} \langle w', v(n)q^{L(0)}w\rangle z^{-n-1}. \tag{8.3}$$

It is easy to see that the trace function is independent of the choice of basis.

Now we introduce the function T which is linear in $v \in V$, and defined for homogeneous $v \in V$ as follows:

$$T(v) = T_M(v, (g, h), q) = z^{\mathrm{wt}v}\mathrm{tr}Y_M(v, z)\phi(h)q^{L(0)-c/24}. \tag{8.4}$$

Here c is the central charge of V.

Next observe that for $m \in \frac{1}{T}\mathbb{Z}$, $v(m)$ maps M_n to $M_{n+\mathrm{wt}v-m-1}$. So unless $m = \mathrm{wt}v - 1$, we have $\sum\langle w', v(m)\phi(h)w\rangle = 0$. So only the zero mode $o(v) = v(\mathrm{wt}v - 1)$ contributes to the sum in (8.3). Thus $T(v)$ is independent of z, and

$$T(v) = q^{\lambda-c/24} \sum_{n=0}^{\infty} \mathrm{tr}_{M_{\lambda+n/T}} o(v)\phi(h)q^{n/T}. \tag{8.5}$$

We could equally write

$$T_M(v) = \mathrm{tr}_M o(v)\phi(h)q^{L(0)-c/24}. \tag{8.6}$$

We are going to prove

Theorem 8.1. $T(v) \in \mathcal{C}_1(g, h)$.

The strategy is to prove that T is a formal (g, h) 1-point function, then invoke Theorem 7.1. Certainly $T(v)$ has the correct shape as a power series in q (cf. (7.1). So we must establish that $T(v)$ satisfies the formal analogues of (C2)-(C4).

We can impose $M(T, T_1)$-linearity (C2) by extension of scalars. As we shall explain, the proof of (C4) is contained in Zhu's paper [Z]. So it remains to discuss (C3), that is we must show that $T(v)$ vanishes on $O(g, h)$, i.e., on the elements of type (5.1)-(5.4). Again we shall later explain that (5.1) and (5.2) may be deduced from results in [Z], so we concentrate on (5.3) and (5.4).

To this end, let us now fix a homogeneous $v \in V$ such that $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$ and $(\mu, \lambda) \neq (1, 1)$. We need to establish

Lemma 8.2. $T(v) = 0$.

Theorem 8.3. $\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau)T(v[k-1]w) = 0$ *for any* $w \in V$.

The proof of Lemma 8.2 is easy. We have already seen that only the zero mode $o(v)$ of v contributes a possible non-zero term in the calculation of $T(v)$. On the other hand, if $\mu \neq 1$ then from (3.4) we see that $o(v) = 0$. So Lemma 8.2 certainly holds if $\mu \neq 1$.

Suppose that $\lambda \neq 1$. We have

$$T(v) = \mathrm{tr}_M o(v)\phi(h)q^{L(0)-c/24} = \mathrm{tr}_M \phi(h)o(v)q^{L(0)-c/24},$$

i.e.,

$$\mathrm{tr} Y_M(v, z)\phi(h)q^{L(0)} = \mathrm{tr}\phi(h)Y_M(v, z)q^{L(0)}. \tag{8.7}$$

But (8.1) yields

$$\phi(h)Y_M(v, z) = \lambda^{-1}Y_M(v, z)\phi(h). \tag{8.8}$$

As $\lambda \neq 1$, (8.7) and (8.8) yield $\mathrm{tr}Y_M(v, z)\phi(h)q^{L(0)} = 0$. This completes the proof of Lemma 8.2.

The proof of Theorem 8.3 is harder. We first need to define n-point correlation functions. These are multi-linear functions $T(v_1, \ldots, v_n)$, $v_i \in V$, defined for v_i homogeneous via

$$\begin{aligned} &T(v_1, \ldots, v_n) = T((v_1, z_1), \ldots, (v_n, z_n), (g, h), q) \\ &= z_1^{\mathrm{wt} v_1} \cdots z_n^{\mathrm{wt} v_n} \mathrm{tr} Y_M(v_1, z_1) \cdots Y_M(v_n, z_n)\phi(h)q^{L(0)-c/24}. \end{aligned} \tag{8.9}$$

We only need the case $n = 2$. We will prove

Theorem 8.4. *Let* $v, v_1 \in V$ *be homogeneous with* $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$ *and* $(\mu, \lambda) \neq (1, 1)$. *Then*

$$T(v, v_1) = \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z}, q)T(v[k-1]v_1), \tag{8.10}$$

$$T(v_1, v) = \lambda \sum_{k=1}^{\infty} \bar{P}_k(\mu, \lambda, \frac{z_1}{z}q, q)T(v[k-1]v_1), \tag{8.11}$$

where $\bar{P}_k$ *is as in (4.31).*

We start with

Lemma 8.5. *Let $k \in \frac{1}{T}\mathbb{Z}$. Then*

$$\begin{aligned}
&(1-\lambda q^k)\mathrm{tr} v(\mathrm{wt} v-1+k)Y_M(v_1,z_1)\phi(h)q^{L(0)}\\
&\quad=\sum_{i=0}^{\infty}\binom{\mathrm{wt} v-1+k}{i}z_1^{\mathrm{wt} v-1+k-i}\mathrm{tr} Y_M(v(i)v_1,z_1)\phi(h)q^{L(0)} \qquad (8.12)\\
&\quad(1-\lambda q^k)\mathrm{tr} Y_M(v_1,z_1)v(\mathrm{wt} v-1+k)\phi(h)q^{L(0)}\\
&\quad=\lambda q^k\sum_{i=0}^{\infty}\binom{\mathrm{wt} v-1+k}{i}z_1^{\mathrm{wt} v-1+k-i}\mathrm{tr} Y_M(v(i)v_1,z_1)\phi(h)q^{L(0)}. \qquad (8.13)
\end{aligned}$$

Proof. We have

$$\begin{aligned}
&\mathrm{tr} v(\mathrm{wt} v-1+k)Y_M(v_1,z_1)\phi(h)q^{L(0)}\\
&\quad=tr[v(\mathrm{wt} v-1+k),Y_M(v_1,z_1)]\phi(h)q^{L_0} \qquad (8.14)\\
&\quad\quad+\mathrm{tr} Y_M(v_1,z_1)v(\mathrm{wt} v-1+k)\phi(h)q^{L(0)}.
\end{aligned}$$

From (8.8) we get

$$v(\mathrm{wt} v-1+k)\phi(h)=\lambda\phi(h)v(\mathrm{wt} v-1+k),$$

moreover,

$$v(\mathrm{wt} v-1+k)q^{L(0)}=q^kq^{L(0)}v(\mathrm{wt} v-1+k).$$

Hence

$$\mathrm{tr} Y_M(v_1,z_1)v(\mathrm{wt} v-1+k)\phi(h)q^{L(0)}=\lambda q^k\mathrm{tr} v(\mathrm{wt} v-1+k)Y_M(v_1,z_1)\phi(h)q^{L(0)}. \qquad (8.15)$$

Using the relation

$$[v(m),Y_M(v_1,z_1)]=\sum_{i=0}^{\infty}\binom{m}{i}z_1^{m-i}Y_M(v(i)v_1,z_1)$$

which is a consequence of the Jacobi identity (2.3) we get

$$\begin{aligned}
&\mathrm{tr}[v(\mathrm{wt} v-1+k),Y_M(v_1,z_1)]\\
&\quad=\sum_{i=0}^{\infty}\binom{\mathrm{wt} v-1+k}{i}z_1^{\mathrm{wt} v-1+k-i}Y_M(v(i)v_1,z_1). \qquad (8.16)
\end{aligned}$$

Both parts of the lemma follow from (8.15)-(8.16). □

Now we turn to the proof of (8.10) of Theorem 8.4. Using (8.12) in the last lemma and setting $\mu = e^{2\pi i r/T}$ we have

$$
\begin{aligned}
T(v, v_1) &= T((v,z),(v_1,z_1),(g,h),q)\\
&= z^{\mathrm{wt}v} z_1^{\mathrm{wt}v_1} \mathrm{tr} Y_M(v,z) Y_M(v_1,z_1)\phi(h) q^{L(0)-c/24}\\
&= z^{\mathrm{wt}v} z_1^{\mathrm{wt}v_1} \sum_{k\in\mathbb{Z}+\frac{r}{T}} z^{-\mathrm{wt}v-k} \mathrm{tr} v(\mathrm{wt}v-1+k) Y_M(v_1,z_1)\phi(h) q^{L(0)-c/24}\\
&= z_1^{\mathrm{wt}v_1} \sum_{k\in\mathbb{Z}+\frac{r}{T}} z^{-k}(1-\lambda q^k)^{-1}\\
&\qquad \sum_{i=0}^{\infty} \binom{\mathrm{wt}v-1+k}{i} z_1^{\mathrm{wt}v-1+k-i} Y_M(v(i)v_1,z_1)\phi(h)q^{L(0)-c/24}\\
&= \sum_{k\in\mathbb{Z}+\frac{r}{T}} (\frac{z_1}{z})^k (1-\lambda q^k)^{-1} \sum_{i=0}^{\infty} \binom{\mathrm{wt}v-1+k}{i} T(v(i)v_1)\\
&= \sum_{k\in\mathbb{Z}+\frac{r}{T}} (\frac{z_1}{z})^k (1-\lambda q^k)^{-1} \sum_{i=0}^{\infty}\sum_{m=0}^{i} c(\mathrm{wt}v,i,m) k^m T(v(i)v_1)\\
&= \sum_{i=0}^{\infty}\sum_{m=0}^{i} m!\bar{P}_{m+1}(\mu,\lambda,z_1/z,q) c(\mathrm{wt}v,i,m) T(v(i)v_1)\\
&= \sum_{m=0}^{\infty} \bar{P}_{m+1}(\mu,\lambda,z_1/z,q) T(v[m]v_1),
\end{aligned}
$$

where we have used (2.10) and (4.31). This is precisely (8.10) of Theorem 8.4. Equation (8.11) follows in the same way by using (8.13). □

Before proving Theorem 8.3 we still need

Lemma 8.6. *We have*

$$\sum_{k=0}^{\infty} \frac{1}{k!} B_k(1-\mathrm{wt}v+r/T) v[k-1] = \sum_{i=0}^{\infty} \binom{r/T}{i} v(i-1).$$

Proof. The l.h.s. of the equality is equal to

$$
\begin{aligned}
\mathrm{Res}_w Y[v,w] \frac{e^{(1-\mathrm{wt}v+r/T)w}}{e^w-1} &= \mathrm{Res}_w Y(v, e^w-1) \frac{e^{(1+r/T)w}}{e^w-1}\\
&= \mathrm{Res}_z Y(v,z) \frac{(1+z)^{r/T}}{z}\\
&= \sum_{i=0}^{\infty} \binom{r/T}{i} v(i-1)
\end{aligned}
$$

as required. □

Proof of Theorem 8.3. Combine Lemma 8.6 and Proposition 4.9 to get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} Q_k(\mu,\lambda,q)T(v[k-1]w)\\
&=\sum_{k=1}^{\infty}\mathrm{Res}_z\left(\iota_{z,z_1}((z-z_1)^{-1})z_1^{\mathrm{wt}v-r/T}z^{-\mathrm{wt}v+r/T}\bar{P}_k(\mu,\lambda,\frac{z_1}{z},q)\right)T(v[k-1]w)\\
&\quad-\lambda\sum_{k=1}^{\infty}\mathrm{Res}_z\left(\iota_{z_1,z}((z-z_1)^{-1})z_1^{\mathrm{wt}v-r/T}z^{-\mathrm{wt}v+r/M}\bar{P}_k(\mu,\lambda,\frac{z_1q}{z},q)\right)T(v[k-1]w)\\
&\quad-\sum_{i=0}^{\infty}\binom{r/T}{i}T(v(i-1)w).
\end{aligned}
$$

On the other hand, use (3.7) to obtain

$$
\begin{aligned}
\sum_{i=0}^{\infty}\binom{r/T}{i}T(v(i-1)w)&=\sum_{i=0}^{\infty}\binom{r/T}{i}z_1^{\mathrm{wt}v+\mathrm{wt}w-i}\\
&\quad\mathrm{tr}Y_M(v(i-1)w,z_1)\phi(h)q^{L(0)-c/24}\\
&=\sum_{i=0}^{\infty}\binom{r/T}{i}z_1^{\mathrm{wt}v+\mathrm{wt}w-i}\mathrm{Res}_{z-z_1}(z-z_1)^{i-1}\\
&\quad\mathrm{tr}Y_M(Y(v,z-z_1)w,z_1)\phi(h)q^{L(0)-c/24}\\
&=\mathrm{Res}_{z-z_1}\iota_{z_1,z-z_1}\left(\frac{z}{z_1}\right)^{r/T}(z-z_1)^{-1}z_1^{\mathrm{wt}v+\mathrm{wt}w}\\
&\quad\mathrm{tr}Y_M(Y(v,z-z_1)w,z_1)\phi(h)q^{L(0)-c/24}\\
&=\mathrm{Res}_z\iota_{z,z_1}(z-z_1)^{-1}(\frac{z_1}{z})^{\mathrm{wt}v-r/T}T(v,w)\\
&\quad-\mathrm{Res}_z\iota_{z_1,z}(z-z_1)^{-1}(\frac{z_1}{z})^{\mathrm{wt}v-r/T}T(w,v)
\end{aligned}
$$

which by Theorem 8.4 is equal to

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\mathrm{Res}_z\iota_{z,z_1}(z-z_1)^{-1}(\frac{z_1}{z})^{\mathrm{wt}v-r/T}\bar{P}_k(\mu,\lambda,\frac{z_1}{z},q)T(v[k-1]w)\\
&\qquad-\lambda\sum_{k=1}^{\infty}\mathrm{Res}_z\iota_{z_1,z}(z-z_1)^{-1}(\frac{z_1}{z})^{\mathrm{wt}v-r/T}\bar{P}_k(\mu,\lambda,\frac{z_1q}{z},q)T(v[k-1]w).
\end{aligned}
$$

This completes the proof of the theorem. □

In order to complete the proof of Theorem 8.1 we need to explain how (5.8), and the fact that $T(v)$ vanishes on (5.1) and (5.2), follow from [Z]. These results concern the case in which the critical vector v satisfies $gv=hv=v$. Thus v lies in the invariant sub vertex operator algebra V^A. Now Zhu's proof of (5.8), for example, is quite general in the sense that it does not depend on any special properties of V. In particular, his argument applies to V^A, which is what we need. (Note that M is a *module* for V^A.) Similarly, Zhu's argument establishes that T vanishes on (5.1) and (5.2) in the case that g and h both fix v and w. On the other hand we may certainly assume that w is an eigenvector for g and h. If $gw = \alpha w$, $hw = \beta w$ and $(\alpha, \beta) \neq (1, 1)$, we have already seen (cf. the proof of Lemma 8.2) that $T(v[2k-2]w) = 0$, so it is clear in this case that T vanishes on (5.2). This completes the proof of Theorem 8.1.

Theorem 8.7. *Let $M^1, M^2, \ldots$ be inequivalent simple g-twisted V-modules, each of which is h-stable. Let $T_1, T_2, \ldots$ be the corresponding trace functions (8.6). Then $T_1, T_2, \ldots$ are linearly independent elements of $\mathcal{C}_1(g, h)$.*

Proof. Suppose false. Then we may choose notation so that for some $m \in \mathbb{N}$ there are non-zero scalars $c_1, c_2, \ldots, c_m$ such that

$$c_1 T_1 + \cdots + c_m T_m = 0. \tag{8.17}$$

Let Ω_i be the top level of M^i, $1 \leq i \leq m$, and let λ_i be the conformal weight of M^i (cf. Sect. 3). Thus M^i is graded by $\frac{1}{T}\mathbb{Z}$,

$$M^i = \oplus_{n=0}^{\infty} M^i_{\lambda_i + n/T}$$

and $\Omega_i = M^i_{\lambda_i}$.

Define a partial order $<<$ on the λ_i by declaring that

$$\lambda_i << \lambda_j, \text{ if and only if } \lambda_j - \lambda_i \in \frac{1}{T}\mathbb{Z}_+. \tag{8.18}$$

We may, and shall, assume that λ_1 is minimal with respect to $<<$.

By Theorem 3.4 the Ω_i realize inequivalent irreducible representations of the algebra $A_g(V)$. Moreover from the discussion following (3.20) the $A_g(V)$-modules Ω_i are h-invariant, and the corresponding trace functions $\mathrm{tr}|_{\Omega_i} o(v)\phi(h)$ are linearly independent. Thus we may choose v so that $\mathrm{tr}|_{\Omega_1} o(v)\phi(h) = 1$, $\mathrm{tr}|_{\Omega_i} o(v)\phi(h) = 0$ for $i > 1$.

Because of our assumption on λ_1, applying this to (8.17) yields $c_1 = 0$, contradiction. $\square$

9. An Existence Theorem for g-Twisted Modules

We will prove

Theorem 9.1. *Suppose that V is a simple vertex operator algebra which satisfies Condition C_2, and that $g \in \mathrm{Aut}V$ has finite order. Then V has at least one simple g-twisted module.*

The idea is to prove that $A_g(V) \neq 0$. Then the theorem is a consequence of Proposition 3.7. We start with more general considerations that we shall need in Sect. 10. Let (g, h) be a pair of commuting elements in $\mathrm{Aut}(V)$.

Lemma 9.2. *Let $v \in V$ satisfy $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$. Then the following hold:*

(a) *The constant term of $\sum_{k=0}^{\infty} Q_k(\mu, \lambda, q)v[k-1]w$ is equal to $-v \circ_g w$ if $\mu \neq 1$.*
(b) *The constant term of $\sum_{k=0}^{\infty} Q_k(\mu, \lambda, q)(L[-1]v)[k-1]w$ is equal to $-v \circ_g w$ if $\mu = 1, \lambda \neq 1$.*

(c) *The constant term of* $v[-2]w + \sum_{k=2}^{\infty}(2k-1)E_{2k}(q)v[2k-2]w$ *is* $v \circ_g w$ *if* $\mu = \lambda = 1$.

Proof. As usual, (c) follows as in the corresponding proof in [Z]. The proof of (a) is similar to that of Lemma 8.6. For from (4.24) we see that if we take $\mu = e^{2\pi ir/T}$ with $1 \le r \le T$ then the constant term of the expression in (a) is equal to the following (take v homogeneous):

$$\begin{aligned}-\sum_{k=0}^{\infty}\frac{B_k(r/T)}{k!}v[k-1]w &= -\mathrm{Res}_z Y[v,z]w\frac{e^{rz/T}}{e^z-1}\\ &= -\mathrm{Res}_z Y(v,e^z-1)w\frac{e^{(\mathrm{wt}v+r/T)z}}{e^z-1}\\ &= -\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v-1+r/T}}{z}\end{aligned}$$

and by (3.12) this is exactly $-v \circ_g w$ if $r \neq T$.

As for (b), we replace v by $L[-1]v = L(-1)v + L(0)v$ (cf. (2.20)) and set $r = T$ in the foregoing. From (4.24) the constant term of

$$\sum_{k=0}^{\infty} Q_k(1,\lambda,q)v[k-1]w$$

is

$$-\sum_{k=0}^{\infty}\frac{B_k(1)}{k!}v[k-1]w + \frac{1}{1-\lambda}v[0]w = -\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v}}{z} + \frac{1}{1-\lambda}v[0]w.$$

Note that $(L[-1]v)[0]w = 0$. Then the constant term of the expression in (b) is equal to

$$\begin{aligned}&-\mathrm{Res}_z Y(L(-1)v,z)w\frac{(1+z)^{\mathrm{wt}v+1}}{z} - \mathrm{wt}v\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v}}{z}\\ =&-\mathrm{Res}_z\left(\frac{d}{dz}Y(v,z)\right)w\frac{(1+z)^{\mathrm{wt}v+1}}{z} - \mathrm{wt}v\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v}}{z}\\ =&\mathrm{Res}_z Y(v,z)w\frac{d}{dz}\frac{(1+z)^{\mathrm{wt}v+1}}{z} - \mathrm{wt}v\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v}}{z}\\ =&-\mathrm{Res}_z Y(v,z)w\frac{(1+z)^{\mathrm{wt}v}}{z^2}\\ =&-v\circ_g w.\end{aligned}$$

This completes the proof of the lemma. □

Now take $S \in \mathcal{C}_1(g,h)$ and assume that $S \neq 0$. After Theorem 6.5 we may choose p so that (6.12)-(6.15) hold for all $v \in V$, and such that $S_p \neq 0$. We may further choose notation such that $\lambda_{p,1}$ is minimal among all $\lambda_{p,j}$ with respect to the partial order (8.18) and $a_{p,1,0}(v) \neq 0$ for some $v \in V$. Setting

$$S_{p,1}(v,\tau) = \alpha(v) + \sum_{n=1}^{\infty} a_{p,1,n}(v)q^{n/T} \tag{9.1}$$

defines a function $\alpha : V \to \mathbb{C}$ which is not identically zero. Because $S(v, \tau)$ is linear in v, α is a linear functional on V.

Lemma 9.3. *α vanishes on $O_g(V)$.*

Proof. We know that S vanishes on $O(g, h)$, hence on the elements (5.1)-(5.4). Using Lemma 9.2 leads to the required vanishing conditions. For example, if $\mu \neq 1$ in the notation of Lemma 9.2 then

$$\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau) S(v[k-1]w) = 0. \tag{9.2}$$

This identity holds if S is replaced by S_p, and then if S_p is in turn replaced by $S_{p,1}$. Hence

$$\sum_{k=0}^{\infty} Q_k(\mu, \lambda, \tau)(\alpha(v[k-1]w) + \sum_{n=1}^{\infty} a_{p,1,n}(v[k-1]w)q^{n/T}) = 0. \tag{9.3}$$

The constant term in (9.3), necessarily zero, is equal to $\alpha(-v \circ_g w) = 0$ by Lemma 9.2, so that α vanishes on $v \circ_g w$ if $gv \neq v$. The other vanishing conditions follow similarly. □

To complete the proof of Theorem 9.1 we must show that there is some non-zero element S of $\mathcal{C}_1(g, h)$ (for suitable h). For then we know that the function α is non-vanishing on V but vanishes on $O_g(V)$, whence $V \neq O_g(V)$ and $A_g(V) = V/O_g(V)$ is non-zero, as required.

Consider $\mathcal{C}_1(1, g) : V$ is itself a g-stable simple V-module, so that the corresponding trace function $T_V(v, (1, g), q)$ lies in $\mathcal{C}_1(1, g)$ by Theorem 8.1. So $\mathcal{C}_1(1, g) \neq 0$, and since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ induces a linear isomorphism between $\mathcal{C}_1(1, g)$ and $\mathcal{C}_1(g, 1)$ by Theorem 5.4 then also $\mathcal{C}_1(g, 1) \neq 0$. This completes the proof of Theorem 9.1.

10. The Main Theorems

We continue to use the notation introduced in Sect. 5. The next theorem is decisive.

Theorem 10.1. *Suppose that V is g-rational and satisfies Condition C_2, and let $M^1, \dots,$ M^m be all of the inequivalent, simple, h-stable, g-twisted V-modules. Let $T_1, \dots, T_m$ be the corresponding trace functions (8.6). Then $T_1, \dots, T_m$ form a basis of $\mathcal{C}_1(g, h)$.*

There are several important corollaries.

Theorem 10.2. *Suppose that V is rational and satisfies Condition C_2. Suppose further that the group $\langle g, h\rangle$ generated by g and h is cyclic with generator k. Then the dimension of $\mathcal{C}_1(g, h)$ is equal to the number of inequivalent, k-stable, simple V-modules. In particular, the number of inequivalent, simple g-twisted V-modules is at most equal to the number of g-stable, simple V-modules, with equality if V is g-rational.*

Notice that parts (ii) and (iii) of Theorem 1.1 are included in Theorem 10.2 together with Theorem 9.1. Recall next that a simple vertex operator algebra V is called *holomorphic* in case V is rational and if V is the unique simple V-module.

Theorem 10.3. *Suppose that V is holomorphic and satisfies Condition C_2. For each automorphism g of V of finite order, there is a unique simple g-twisted V-module $V(g)$. Moreover if $\langle g, h\rangle$ is cyclic then $\mathcal{C}_1(g, h)$ is spanned by $T_{V(g)}(v, g, h, q)$.*

This establishes Theorem 1.2 (i). First we show how Theorems 10.2 and 10.3 follow from Theorem 10.1. In the situation of Theorem 10.2 we have $\langle g, h\rangle = \langle k\rangle$. Since V is rational, Theorem 10.1 tells us that $\mathcal{C}_1(1, k)$ has a basis consisting of the trace functions $T_M(v, 1, k, q)$ where M ranges over the inequivalent, k-stable, simple V-modules.

We can find $\gamma \in SL(2, \mathbb{Z})$ such that $(g, h)\gamma = (1, k)$. By the theorem of modular-invariance, γ induces a linear isomorphism from $\mathcal{C}_1(g, h)$ to $\mathcal{C}_1(1, k)$. Theorem 10.2 follows from this together with Theorem 8.7.

As for Theorem 10.3, since V is holomorphic it is certainly rational, so Theorem 10.2 applies. So if $\langle g, h\rangle = \langle k\rangle$ is cyclic then $\dim \mathcal{C}_1(g, h)$ is equal to 1 since V is the only simple V-module and it is certainly k-stable. By Theorem 9.1, V has at least one simple g-twisted V-module, call it $V(g)$, and from Theorem 8.7 there can be no more than one since $\dim \mathcal{C}_1(g, 1) = 1$. So $V(g)$ is unique, hence h-stable whenever $gh = hg$. So $T_{V(g)}(v, g, h, q)$ spans $\mathcal{C}_1(g, h)$ by Theorem 8.1. This establishes Theorem 10.3.

Turning to the proof of Theorem 10.1, we consider first an arbitrary function $S \in \mathcal{C}_1(g, h)$. We have seen in Sect. 9 that S can be represented as

$$S(v, \tau) = \sum_{i=0}^{p} (log q_{\frac{1}{T}})^i S_i(v, \tau) \tag{10.1}$$

for fixed p and all $v \in V$, with each S_i satisfying (6.13)-(6.15). We will prove

Proposition 10.4. *Each S_i is a linear combination of the functions $T_1, \ldots, T_m$.*

Proposition 10.5. $S_i = 0$ *if* $i > 0$.

Theorem 10.1 obviously follows from these two propositions. First we show that Proposition 10.5 follows from Proposition 10.4. To this end, pick $v \in V$ such that $gv = \mu^{-1}v$, $hv = \lambda^{-1}v$. It suffices to show that $pS_p(v, \tau) = 0$. If $(\mu, \lambda) \neq (1, 1)$ this follows from (C3) and (5.3), so we may assume $gv = hv = v$.

Set

$$w = L[-2]v - \sum_{k=1}^{\infty} E_{2k}(\tau)L[2k-2]v. \tag{10.2}$$

From (5.8) and (5.9) we get

$$S(w, \tau) = \frac{q_{\frac{1}{T}}}{T} \frac{d}{dq_{\frac{1}{T}}} S(v, \tau). \tag{10.3}$$

Now Proposition 10.4 combined with Theorem 8.1 tells us that (10.3) is satisfied by each S_i. Then we calculate that

$$S(w, \tau) = \sum_{i=0}^{p} (\frac{i}{T}(\log q_{\frac{1}{T}})^{i-1} S_i(v, \tau) + (\log q_{\frac{1}{T}})^i S_i(w, \tau)). \tag{10.4}$$

We may identify the parts of (10.1) which involve a given power $(\log q_{\frac{1}{T}})^i$. Taking $i = p-1$, we see that

$$S_{p-1}(w,\tau) = \frac{p}{T} S_p(v,\tau) + S_{p-1}(w,\tau),$$

so that $pS_p(v,\tau) = 0$, as desired.

We turn our attention to the proof of Proposition 10.4. We assume without loss that $S_p \neq 0$ and that each $S_{p,j} \neq 0$ (cf. (6.13)). We are then in the situation that was in effect in Sect. 9. We adopt the notation (9.1). It was shown that $\alpha : V \to \mathbb{C}$ vanishes on $O_g(V)$, and thus defines a linear functional

$$\alpha : A_g(V) \to \mathbb{C}.$$

We continue this line of reasoning, and now prove

Lemma 10.6. *Suppose that $u, v \in V$ and satisfy $hu = \rho u$, $hv = \sigma v$, $\rho, \sigma \in \mathbb{C}$. Then*

$$\alpha(u *_g v) = \rho \delta_{\rho\sigma,1} \alpha(v *_g u). \tag{10.5}$$

Proof. We may assume that $gu = \xi u$ and $gv = \nu v$ for scalars ξ, ν. If ξ or ν is not equal to 1 then u (resp. v) lies in $O_g(V)$ (cf. Lemma 2.1 of [DLM3]), whence so too do $u *_g v$ and $v *_g u$ by Theorem 3.3. So in this case both sides of (10.5) are equal to 0. So we may assume $gu = u$, $gv = v$.

Similarly, $u *_g v$ is an eigenvector for h with eigenvalue $\rho\sigma$, so if $\rho\sigma \neq 1$ then $u *_g v$ and $v *_g u$ lie in $O(g,h)$ by (5.3). Then $S(u *_g v) = S(v *_g u) = 0$ by (C3), which again leads to both sides of (10.5) being 0. So we may assume that $\rho\sigma = 1$ and try to prove that $\alpha(u *_g v) = \rho\alpha(v *_g u)$.

Now we know from [Z] (also see Lemma 2.2 (iii) of [DLM3]) that if V^g is the space of g-invariants of V then for u homogeneous

$$u *_g v - v *_g u \equiv \mathrm{Res}_z Y(u,z)v(1+z)^{\mathrm{wt}u-1} \pmod{O(V^g)}.$$

Using (2.17) we get

$$u *_g v - v *_g u \equiv \sum_{i=0}^{\infty} \binom{\mathrm{wt}u - 1}{i} u(i)v \equiv u[0]v \pmod{O(V^g)}. \tag{10.6}$$

Now certainly $O(V^g) \subset O_g(V)$ (loc. cit.), and if $\rho = 1$ then $u[0]v \in O(g,h)$ by (5.1). So in this case (10.6) leads to $\alpha(u *_g v - v *_g u) = 0$ as desired.

So we may take $\rho \neq 1$. In this case we follow the calculation of Lemma 9.2. Bearing in mind that $gu = u$ and $hv = \rho v$ with $\rho \neq 1$, we see from the proof of Lemma 9.2 (b) that the constant term of

$$\sum_{k=0}^{\infty} Q_k(1, \rho^{-1}, q) u[k-1]v$$

is

$$-u *_g v + \frac{1}{1-\rho^{-1}} u[0]v.$$

Since S vanishes on $\sum_{k=0}^{\infty} Q_k(1, \rho^{-1}, q)u[k-1]v \in O(g, h)$, this shows that

$$\alpha(u *_g v) = \frac{1}{1-\rho^{-1}}\alpha(u[0]v).$$

However (10.6) still applies, so that

$$\alpha(u *_g v) = \frac{1}{1-\rho^{-1}}(\alpha(u *_g v) - \alpha(v *_g u)),$$

which is equivalent to the desired result. □

We will need

Lemma 10.7. *Let A be a finite-dimensional semi-simple algebra over $\mathbb{C}$ with decomposition $A = \oplus_{i \in I} A_i$ into simple components.*

Let $h : A \to A$ be an automorphism of A of finite order, and suppose that $F : A \to \mathbb{C}$ is a linear map which satisfies

$$F(ab) = \rho\delta_{\rho\sigma,1}F(ba) \tag{10.7}$$

whenever $ha = \rho a$ and $hb = \sigma b$, $\rho, \sigma \in \mathbb{C}$. Then F can be written as a linear combination with scalars α_j :

$$F(a) = \sum_{j \in J} \alpha_j \mathrm{tr}_{W_j}(a\gamma_j), \tag{10.8}$$

where in (10.8), $\{A_j\}_{j \in J}$ ranges over the h-invariant simple components of A, W_j is the simple A-module such that $A_j W_j = W_j$, and $\gamma_j \in A_j^*$ satisfies $ha = \gamma_j a \gamma_j^{-1}$ for $a \in A_j$.

Remark. The existence of γ_j is the Skolem–Noether Theorem.

Proof. Proceed by induction on the order of h and cardinality of I. The group $\langle h \rangle$ permutes the A_i among themselves, and the conditions of the lemma apply to any h-invariant sum of A_i. So we may assume that $\langle h \rangle$ is transitive in its action on the A_i.

First assume that there are at least two components. Then there are no h-invariant components, so we must show that $F = 0$ in this case. If $\sigma \neq 1$ and $hb = \sigma b$, taking $a = 1$ in (10.7) shows that $F(b) = 0$. So we only need show that F is zero on the algebra A^h of h-invariants of A.

If there is $1 \neq k \in \langle h \rangle$ such that k fixes each A_i then the algebra of k-invariants B is a semi-simple algebra admitting $\langle h \rangle / \langle k \rangle$. By induction we see that $F(B) = 0$, so we are done as $A^h \subset B$. So without loss there is no such k. So h has order $|I|$, the number of components. Thus if A_0 is the first component then we may set $A_i = h^i A_0$, $0 \leq i \leq |I| - 1$. Then

$$A^h = \{\sum_{i=0}^{|I|-1} h^i x | x \in A_0\} \simeq A_0.$$

By (10.7) $F(ab) = F(ba)$ for $a, b \in A^h$, so F is a trace function , i.e., $F(a) = \alpha \mathrm{tr}_W a$ for some $\alpha \in \mathbb{C}$, W the simple A^h-module. So we must show $F(1_A) = 0$. Let λ be an $|I|^{\mathrm{th}}$ root of unity, let $u, v \in A_0$ be units such that $uv = 1_{A_0}$, and let

$$a = \sum_{i=0}^{|I|-1} \lambda^i h^i(u), b = \sum_{i=0}^{|I|-1} \lambda^{-i} h^i(v).$$

Then $ba = ab = \sum_{i=0}^{|I|-1} h^i(u)h^i(v) = \sum_i h^i(uv) = 1_A$. On the other hand $ha = \lambda^{-1}a$, $hb = \lambda b$ and $\lambda \neq 1$. So (10.7) yields $F(1_A) = F(ab) = F(ba) = \lambda F(ab)$. So $F(1_A) = 0$ as desired.

This reduces us to the case that A is itself a simple algebra. Pick $\gamma \in A^*$ such that $h(a) = \gamma a \gamma^{-1}$ and consider $F_1 : A \to \mathbb{C}$ defined by $F_1(a) = F(a\gamma^{-1})$. If $ha = \rho a$ and $\rho \neq 1$ then $F(a\gamma^{-1}) = 0$ by (10.7), so $F_1(a) = 0$ for such a. On the other hand we get for $hb = \sigma b$,

$$\begin{aligned} F_1(ab) &= F(ab\gamma^{-1}) = \rho\delta_{\rho\sigma,1}F(b\gamma^{-1}a) \\ &= \rho\delta_{\rho\sigma,1}F(b\gamma^{-1}a\gamma\gamma^{-1}) = \delta_{\rho\sigma,1}F(ba\gamma^{-1}) = \delta_{\rho\sigma,1}F_1(ba). \end{aligned}$$

From this we conclude that $F_1(ab) = F_1(ba)$ for all $a, b \in A$. So F_1 is a trace function $F_1(a) = \alpha \mathrm{tr}_W a$, so that $F(a) = \alpha \mathrm{tr}_W a\gamma$. This completes the proof of the lemma. □

Now we return to the situation of Lemma 10.6. From (3.18) h induces an automorphism of $A_g(V)$ via $h : v \mapsto hv$, and since V is g-rational, then $A_g(V)$ is semi-simple and Lemma 10.7 applies. From the discussion in Sect. 3, the h-invariant components of $A_g(V)$ correspond precisely to the h-invariant simple $A_g(V)$-modules, and these correspond to the h-invariant simple g-twisted V-modules. For such a simple $A_g(V)$-module Ω we have $\phi(h)o(v)\phi(h)^{-1} = o(hv)$ (cf. (8.1)), $o(v)$ being the corresponding zero mode (3.14). Also $o(hv) = \gamma o(v)\gamma^{-1}$ if γ represents h in the sense of Lemma 10.7. So γ and $\phi(h)$ differ by a scalar when considered as operators on Ω. By Lemmas 10.6 and 10.7 we get

Lemma 10.8. *The linear function $\alpha : A_g(V) \to \mathbb{C}$ can be represented in the form*

$$\alpha(v) = \sum_j \alpha_j \mathrm{tr}_{\Omega(M^j)} o(v)\phi(h), \tag{10.9}$$

where α_j are scalars and the spaces Ω_{M^j} range over the top levels of the h-invariant simple g-twisted V-modules M^j.

Recall that we have

$$S_p(v, \tau) = \sum_{j=1}^{b} q^{\lambda_{p,j}} S_{p,j}(v, \tau) \tag{10.10}$$

with $S_{p,1}$ as in (9.1).

Lemma 10.9. *Suppose that $\alpha_j \neq 0$ in (10.9). Then the conformal weight of the corresponding g-twisted module M^j is equal to $\lambda_{p,1} + c/24$.*

Proof. We use the method of proof of Proposition 10.5 once more. For $v \in V$, let $w = w(v)$ be as in (10.2). Thus (10.3) holds whenever $S \in \mathcal{C}_1(g, h)$.

Applying (10.3) with $S = T_{M^j}$ (cf. (8.4)) and considering leading terms yields

$$\mathrm{tr}_{\Omega(M^j)} o(w)\phi(h) = (\lambda_j - c/24)\mathrm{tr}_{\Omega(M^j)} o(v)\phi(h), \tag{10.11}$$

where λ_j is the conformal weight of M^j. Similarly applying (10.3) to S itself and considering the leading term of S_p yields

$$\alpha(w) = \lambda_{p,1}\alpha(v). \tag{10.12}$$

Using (10.5), we find that for $v \in V$,

$$\lambda_{p,1}\sum_j \alpha_j \mathrm{tr}_{\Omega(M^j)} o(v)\phi(h) = \lambda_{p,1}\alpha(v) = \alpha(w) = \sum_j \alpha_j \mathrm{tr}_{\Omega(M^j)} o(w)\phi(h)$$
$$= \sum_j \alpha_j(\lambda_j - c/24)\mathrm{tr}_{\Omega(M^j)} o(v)\phi(h).$$

The linear independence of characters of $A_g(V)$ implies that $\lambda_{p,1}\alpha_j = \alpha_j(\lambda_j - c/24)$. The lemma follows. $\square$

We are ready for the final argument. We have in the previous notation

$$q^{\lambda_{p,1}} S_{p,1}(v,\tau) = q^{\lambda_{p,1}}(\sum_j \alpha_j \mathrm{tr}_{\Omega(M^j)} o(v)\phi(h) + \sum_{n=1}^{\infty} a_{p,1,n}(v) q^{n/T})$$
$$= \sum_j q^{\lambda_j - c/24}\alpha_j \mathrm{tr}_{\Omega(M^j)} o(v)\phi(h) + q^{\lambda_{p,1}} \sum_{n=1}^{\infty} a_{p,1,n}(v) q^{n/T}).$$

Now also

$$T_{M^j}(v,\tau) = q^{\lambda_j - c/24}(\mathrm{tr}_{\Omega(M^j)} o(v)\phi(h) + \sum_{n=1}^{\infty} \mathrm{tr}_{M^j_{\lambda_j+n/T}} o(v)\phi(h) q^{n/T}).$$

So we see that the function

$$S'(v,\tau) = S(v,\tau) - (\log q_{\frac{1}{T}})^p \sum_j \alpha_j T_{M^j}(v,\tau)$$

again has the form (6.12)-(6.15), but the leading term of the piece corresponding to S_p now has a higher degree than S_p itself. We now continue the argument, replacing S with S' and S_p with S'_p. We find, since each T_{M^j} already lies in $\mathcal{C}_1(g,h)$, and since there are only finitely many M^j, that indeed S_p is a linear combination of T_{M^j}. But our argument applies equally well to each S_i, so each S_i is a linear combination of T_{M^j}. This completes the proof of Proposition 10.4.

11. Rationality of Central Charge and Conformal Weights

Recall from (3.11) that a simple g-twisted V-module M has grading of the form $M = \oplus_{n=0}^{\infty} M_{\lambda+n/T}$ for some $\lambda \in \mathbb{C}$ called the conformal weight of M. We will show that, under suitable rationality conditions on V, the conformal weight λ of M is a rational number. We prove even more, namely

Theorem 11.1. *Suppose that V is a holomorphic vertex operator algebra which satisfies Condition C_2, and let $g \in \mathrm{Aut}V$ have finite order. Let $V(g)$ be the unique simple g-twisted V-module whose existence is guaranteed by Theorem 10.3. Then the conformal weight of $V(g)$ is rational, and the central charge c of V is also rational.*

Theorem 11.2. *Suppose that V is a vertex operator algebra which satisfies Condition C_2, and let $g \in \mathrm{Aut}V$ have finite order. Suppose that V is g^i-rational for all integers i. Then each simple g^i-twisted V-module has rational conformal weight, and the central charge c of V is rational.*

Theorem 11.3. *Suppose that V is a rational vertex operator algebra which satisfies condition C_2. Then each simple V-module has rational conformal weight, and the central charge of V is rational.*

These theorems complete the proofs of Theorems 1.1 and 1.2. Note that Theorem 11.3 is simply a restatement of Theorem 11.2 in the special case that $g = 1$. We will prove Theorems 11.1 and 11.2 simultaneously. Indeed, at this point in the paper the proof follows from ideas in a paper of Anderson and Moore [AM]: we have only to assemble the relevant facts.

First observe that to prove Theorem 11.2 it suffices to show that each simple g-twisted V-module has rational conformal weight, and that c is rational. With this in mind, let $f(q)$ be one of the following q-expansion: $q^{-c/24}\sum_{n\geq n_0}(\dim V_n)q^n$, where $V = \oplus_{n\geq n_0}V_n$; $q^{\lambda-c/24}\sum_{n\geq n_0}(\dim V(g)_{\lambda+n/T})q^n$ with λ the conformal weight of $V(g)$, where $V(g)$ is either the unique simple g-twisted V-module in the situation of Theorem 11.1, or any simple g-twisted V-module in the situation of Theorem 11.2.

Let U be the $SL(2,\mathbb{Z})$-module of holomorphic functions on $\mathfrak{h}$ generated by $f(q)$. In each case U is a finite-dimensional $\mathbb{C}$-linear space, and the elements of U have q-expansions in (not necessarily rational) powers of q. This assertion follows from Theorems 10.1 and 10.3. This puts us in the position of being able to apply methods and results of Anderson and Moore (loc.cit.) . The argument proceeds as follows.

Define by $\lambda = \lambda(\tau)$ the usual Picard function which generates the field of rational functions on the compactification of $\mathfrak{h}/\Gamma(2)$. With $E = \frac{d}{d\lambda}$, there are unique meromorphic functions k_i such that U is precisely the space of solutions of the differential equation

$$E^n y + \sum_{i=0}^{n-1} k_i E^i y = 0. \tag{11.1}$$

The k_i are then in $\mathbb{C}(\lambda)$ (Proposition 1 of (loc.cit.)).

For a given $\phi \in \mathrm{Aut}(\mathbb{C})$, and for $r(q) \in U$, let r^ϕ be as defined in (loc.cit.). By the Frobenius–Fuchs theory, the r^ϕ are then q-expansions of the solutions of the ϕ-transform of (11.1), namely

$$E^n y + \sum_{i=0}^{n-1} k_i^\phi E^i y = 0. \tag{11.2}$$

We claim that the solutions of (11.2) also afford a representation of $SL(2,\mathbb{Z})$. First note that since each k_i lies $\mathbb{C}(\lambda)$ then the actions of ϕ and $\gamma \in SL(2,\mathbb{Z})$ on $\mathbb{C}(\lambda)$ commute: this follows from the well-known formulae for the action of the modular group on λ. Then if $y|\gamma$ is the γ-image of a solution y of (11.1) we find that $(y|\gamma)^\phi|\gamma^{-1}$ is a solution of (11.2). The claim follows from this observation.

Now $f(q) \in U$ has the form

$$f = q^{\lambda-c/24}\sum_{n\geq N} a_n q^{n/T},$$

where $\lambda = 0$ and $T = 1$ in the first case, and where $a_n \in \mathbb{Z}$ in all cases. Then

$$f^{\phi} = q^{\phi(\lambda - c/24)} \sum_{n \geq N} a_n q^{n/T},$$

i.e.,

$$f^{\phi} = q^{\phi(\lambda - c/24) - (\lambda - c/24)} f. \tag{11.3}$$

One now applies S to both sides of (11.3) to obtain

$$f^{\phi}|S = e^{-\alpha/\tau} f|S, \tag{11.4}$$

where $\alpha = 2\pi i(\phi(\lambda - c/24) - (\lambda - c/24))$. On the other hand, we showed above that both $f^{\phi}|S$ and $f|S$ have q-expansions. This leads to a contradiction by using the limit argument of (loc.cit.) unless $\alpha = 0$. As this holds for all ϕ, we conclude that $c/24 \in \mathbb{Q}$ and $\lambda - c/24 \in \mathbb{Q}$, which completes the proofs of the theorems.

Let us formalize the situation which prevails in case V is a holomorphic vertex operator algebra which satisfies Condition C_2 and is equipped with a finite group G of automorphisms. Let $V(g)$ be the unique simple g-twisted V-module (Theorem 10.3) for $g \in G$, and let $C(g) = \{h \in G | gh = hg\}$ be the *centralizer* of g in G.
According to Theorem 11.1 the grading of $V(g)$ has the form $V(g) = \oplus_{n=0}^{\infty} V(g)_{\lambda+n/T}$, where g has order T and $\lambda = \lambda(g) \in \mathbb{Q}$. As $V(g)$ is unique, it admits a (projective) representation of $C(g)$, so for any $h \in C(g)$ we may consider the trace function $T_{V(g)}(v, g, h, q)$. As usual, this is really only defined up to a nonzero scalar. By Theorem 10.3 this trace function spans the 1-dimensional space $\mathcal{C}_1(g, h)$ if $\langle g, h\rangle$ is cyclic. The shape of the trace function is as in (8.5), with $\lambda - c/24 \in \mathbb{Q}$. Thus it has a q-expansion with *rational* powers of q of bounded denominator, and is holomorphic as a function on $\mathfrak{h}$.

We now assume that $\langle g, h\rangle$ is cyclic and fix $v \in V_{[k]}$. It follows from Theorem 5.4 that $T_{V(g)}|_k\gamma(v, g, h, \tau) = (c\tau + d)^{-k} T_{V(g)}(v, g, h, \gamma\tau)$ lies in $\mathcal{C}_1((g, h)\gamma, \tau)$ and hence is a scalar multiple of $T_{V(g^a h^c)}(v, (g, h)\gamma, \tau)$. Here $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ lies in $SL(2, \mathbb{Z})$. Thus there are scalars $\sigma(g, h, \gamma)$ such that the following holds:

$$(c\tau + d)^{-k} T_{V(g)}(v, g, h, \gamma\tau) = \sigma(g, h, \gamma) T_{V(g^a h^c)}(v, (g, h)\gamma, \tau). \tag{11.5}$$

Equation (11.5) together with the rationality of the corresponding q-expansions says precisely that each $T_{V(g)}(v, g, h, \tau)$ is a *generalized modular form of weight* k in the language of [KM]. Note that Theorem 1.4 follows from these results. Theorem 1.3 follows in the same way.

A case of particular interest is when we take v to be the vacuum element $\mathbf{1}$, in which case $k = 0$. In this case

$$Z(g, h) = T_{V(g)}(\mathbf{1}, g, h, \tau) \tag{11.6}$$

(see (1.6)-(1.8)). This is essentially the graded trace of $\phi(h)$ on the g-twisted module $V(g)$, sometimes called a *partition function* or *McKay-Thompson series*. In this case, we have proved

Theorem 11.4. *Let V be a holomorphic vertex operator algebra which satisfies Condition C_2, and let G be a finite group of automorphisms of V. For each pair of commuting elements (g, h) which generates a cyclic group, $Z(g, h)$ is a generalized modular function (i.e., of weight zero) which is holomorphic on $\mathfrak{h}$ and satisfies*

$$\gamma : Z(g, h) \mapsto \sigma(g, h, \gamma) Z((g, h)\gamma) \tag{11.7}$$

for $\gamma \in SL(2, \mathbb{Z})$.

12. Condition C_2

In order to be able to apply the preceding results to known vertex operator algebras, we need verify that Condition C_2 is satisfied. We do this for some of the best known rational vertex operator algebras in this section. Refer to Sect. 3 for the definition of Condition C_2.

Lemma 12.1. *If V is a vertex operator algebra and M is V-module, then $C_2(M)$ contains $v(-n)M$ for all $v \in V$ and $n \geq 2$.*

Proof. This follows from definition (3.21) together with the equality

$$(L(-1)^m v)(-2) = (m+1)! v(-m-2). \qquad \square$$

Lemma 12.2. *Let $V_1, \ldots, V_k$ be vertex operator algebras such that for each i, all simple V_i-modules satisfy Condition C_2. Then the same is true for the tensor product vertex operator algebra $V_1 \otimes \cdots \otimes V_k$.*

Proof. See [FHL] for tensor product vertex operator algebras and their modules. We may assume that $k = 2$. One knows (loc.cit.) that the simple $V_1 \otimes V_2$-modules are precisely those of the form $M_1 \otimes M_2$ with M_i a simple V_i-module.

If $v \in V_1$ then $(v \otimes \mathbf{1})(-2) = v(-2) \otimes id$, from which it follows that $C_2(M_1 \otimes M_2)$ contains $C_2(M_1) \otimes M_2$. Similarly it contains $M_1 \otimes C_2(M_2)$. The lemma follows immediately. $\square$

Now we discuss Condition C_2 for the most well-known rational vertex operator algebras, namely,

(i) The vertex operator algebra $L(c_{p,q}, 0)$ associated with the (discrete series) simple Virasoro algebra Vir-module of highest weight 0 and central charge $c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$ ([DMZ, FZ, Wa]).
(ii) The moonshine module $V^\natural$ ([B1, FLM3]).
(iii) The vertex operator algebra V_L associated with a positive definite even lattice L ([B1, D1, FLM3]).
(iv) The vertex operator algebra $L(k, 0)$ associated to a $\hat{\mathfrak{g}}$-module of highest weight 0 and positive integral level k, $\mathfrak{g}$ a simple Lie algebra ([DL, FZ, Li]).

Lemma 12.3. *$L(c_{p,q}, 0)$ satisfies Condition C_2.*

Proof. Set $L = L(c_{p,q}, 0)$. It is a quotient of the corresponding Verma module $M = M(c_{p,q}, 0)$ and we have $M \simeq U(Vir_-) \cdot 1$ (cf. [FZ]) where $Vir_- = \oplus_{n=1}^{\infty} \mathbb{C}L(-n)$ and the $L(n)$ are the usual generators of the Virasoro algebra Vir.

Now $Y(\omega, z) = \sum_{n\in\mathbb{Z}} L(n)z^{-n-2}$, so that $C_2(V)$ contains $L(-n)L$ for all $n \geq 3$ by Lemma 12.1.

We have $L = M/J$ and J contains two singular vectors [FF]. The first is $L(-1) \cdot 1$, which shows that $C_2(L)$ contains $U(Vir_-)L(-1)1$. From this we see that $L = C_2(L) + \sum_{k=0}^{\infty} \mathbb{C}L(-2)^k 1$.

The second singular vector has the form

$$v = L(-2)^{pq}\mathbf{1} + \sum a_{n_1,\cdots,n_r} L(-n_1 - 2) \cdots L_(- n_r - 2)1, \tag{12.1}$$

where the sum ranges over certain $(n_1, \cdots, n_r) \in \mathbb{Z}_+^r$ with $n_1 + \cdots + n_r \neq 0$, $a_{n_1,\ldots,n_r} \in \mathbb{C}$ (cf. Eq. (3.11) of [DLM2]). From the previous paragraph we see that the terms under the summation sign in (12.1) each lie in $C_2(L)$, whence also $L(-2)^{pq}1$ lies in $C_2(L)$. By Lemma 3.8, $C_2(L)$ is invariant under $L(-2)$. We conclude that $L = C_2(L) + \sum_{k=0}^{pq-1} \mathbb{C}L(-2)^k 1$, and the proposition follows. □

The following result was stated without proof in [Z]

Proposition 12.4. *The moonshine module $V^\natural$ satisfies Condition C_2.*

Proof. Let U be the tensor product $L(\frac{1}{2}, 0)^{\otimes 48}$. It is shown in [DMZ] that $V^\natural$ contains a sub vertex operator algebra isomorphic to U. Moreover when considered as a U-module, $V^\natural$ is a direct sum of finitely many simple U-modules.

Suppose that each simple module for $L(\frac{1}{2}, 0)$ satisfies Condition C_2. Then this is true also for U by Lemma 12.2, so that the space spanned by $u(-2)v$ for $u \in U$ and $v \in V^\natural$ already has finite codimension in $V^\natural$. So it suffices to show that the simple $L(\frac{1}{2}, 0)$-modules indeed satisfy Condition C_2.

The proof of this later assertion is similar to that of the last proposition. Apart from $L = L(\frac{1}{2}, 0)$ itself there are just two other simple modules for L, namely $L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{2}, \frac{1}{16})$ [DMZ]. Let $M(\frac{1}{2}, h)$ $(h = \frac{1}{2}, \frac{1}{16})$ be the corresponding Verma module with $L(\frac{1}{2}, h) = M(\frac{1}{2}, h)/J_h$. As in the previous proposition we have $L(-n)L(\frac{1}{2}, h) \subset C_2(L(\frac{1}{2}, h))$ for $n \geq 3$, so that $L(\frac{1}{2}, h) = C_2(L(\frac{1}{2}, h)) + \sum_{a,b\geq 0} \mathbb{C}L(-2)^a L(-1)^b v_h$, where v_h is a highest weight vector. Moreover J_h contains two singular vectors (cf. [FF]). One of them is $(L(-2) - \frac{3}{4}L(-1)^2)v_{\frac{1}{2}}$ or $(L(-2) - \frac{4}{3}L(-1)^2)v_{\frac{1}{16}}$ (cf. [DMZ]), from which we conclude that $L(\frac{1}{2}, h) = C_2(L(\frac{1}{2}, h)) + \sum_{b\geq 0} \mathbb{C}L(-1)^b v_h$. Because of the existence of a second singular vector we see that $C_2(L(\frac{1}{2}, h))$ necessarily contains $L(-1)^b v_h$ for big enough b, whence our claim follows. □

The reader is referred to [FLM3] for the construction of V_L and associated notation, which we use in the next result.

Proposition 12.5. *V_L satisfies Condition C_2.*

Proof. Set $H = L \otimes_{\mathbb{Z}} \mathbb{C}$. Then it is easy to see that

$$V_L = C_2(V_L) + S(H \otimes t^{-1}) \otimes \mathbb{C}\{L\}. \tag{12.2}$$

Let $0 \neq \alpha, \gamma \in L$ and set $\beta = \gamma - \alpha$. Then $C_2(V_L)$ contains $u(-2)v$, where we take $u = L(-1)^k \iota(a)$ $(k \geq 0)$ and $v = \iota(b)$, where $a, b, c \in \hat{L}$ are such that $\bar{a} = \alpha$, $\bar{b} = \beta$ and $c = ab$. Then $\bar{c} = \gamma$. So $C_2(V_L)$ contains

$$\begin{aligned}&\mathrm{Res}_z z^{-k-2} Y(\iota(a), z)\iota(b)\\ &\qquad = \mathrm{Res}_z z^{-k-2} \exp\left(\sum_{n<0} \frac{-\alpha(n)}{n} z^{-n}\right) \exp\left(\sum_{n>0} \frac{-\alpha(n)}{n} z^{-n}\right) a z^{\alpha} \iota(b).\end{aligned}$$

Now $az^{\alpha}\iota(b) = z^{\langle \alpha, \beta \rangle}\iota(c)$. Using (12.2) we now see that $C_2(V_L)$ contains

$$\mathrm{Res}_z z^{-k-2} e^{\alpha(-1)z} z^{\langle \alpha, \beta \rangle} \iota(c),$$

and we conclude that $C_2(V_L)$ contains

$$\alpha(-1)^{1+k-\langle \alpha, \beta \rangle} \iota(c) \tag{12.3}$$

whenever $k \geq 0$ and $k \geq 1 - \langle \alpha, \beta \rangle$.

So if $\langle \alpha, \beta \rangle \geq 1$ we may choose k appropriately to see that $C_2(V_L)$ contains $\iota(c)$. So we have shown that $C_2(V_L)$ contains $\iota(c)$, and hence $S(H \otimes t^{-1}) \otimes \iota(c)$ by Lemma 3.8, unless $\langle \alpha, \gamma \rangle \leq \langle \alpha, \alpha \rangle$ for all $\alpha \in L$. Let Γ denote the set of $\gamma \in L$ with this latter property.

Now Γ is a finite set. Fix a $\mathbb{Z}$-basis B of L and let $M = \max_{\gamma \in \Gamma, \beta \in B}(1 - \langle \beta, \gamma - \beta \rangle)$. From (12.3) we see that $\beta(-1)^M \otimes \iota(c) \in C_2(V_L)$ for all $\beta \in B, \gamma \in \Gamma$. Now from the above calculations, we see that $C_2(V_L)$ contains $S^r(H \otimes t^{-1}) \otimes \mathbb{C}\{L\}$ for all big enough integers r and also $C_2(V_L)$ contains $S(H \otimes t^{-1}) \otimes \mathbb{C}\iota(c)$ for all $c \in \hat{L}$ such that $\bar{c} \in L \backslash \Gamma$. It follows from (12.2) that indeed $C_2(V_L)$ has finite codimension in V_L. □

Proposition 12.6. *Let k be a positive integer and $\mathfrak{g}$ a complex simple Lie algebra. Then the vertex operator algebra $L(k, 0)$ associated to $\mathfrak{g}$ and k satisfies Condition C_2.*

Proof. See [FZ] for the vertex operator algebra structure of $L(k, 0)$ and also the corresponding Verma module $M(k, 0)$. By definition

$$M(k, 0) = U(\hat{\mathfrak{g}}) \otimes_{U(\sum_{n=0}^{\infty} t^n \otimes \mathfrak{g} + \mathbb{C}c)} \mathbb{C} \simeq U(\sum_{n=1}^{\infty} t^{-n} \otimes \mathfrak{g})$$

(linearly). Then $L = L(k, 0)$ is the quotient of $M(k, 0)$ by the maximal $\hat{\mathfrak{g}}$-submodule. For $a \in \mathfrak{g}$, $Y(a, z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1}$, so $C_2(L)$ contains $a(-n)L$ for all $a \in \mathfrak{g}$ and all $n \geq 2$ by Lemma 12.1. Thus $L = C_2(L) + U(t^{-1} \otimes g)1$. It is enough to show that $C_2(L)$ contains $a_1(-1)^{m_1} \cdots a_d(-1)^{m_d} 1$ whenever $m_i \geq 0$ and $m_1 + \cdots + m_d$ is large enough; here $a_1, \ldots, a_d$ is a basis of $\mathfrak{g}$.

By Lemma 3.6 of [DLM2] we may choose the a_i so that

$$[Y(a_i, z_1), Y(a_i, z_2)] = 0, \quad Y(a_i, z)^{3k+1} = 0$$

for each i. Now the constant term in $Y(a_i, z)^{3k+1} 1$ is equal to $a_i(-1)^{3k+1} 1 + r$ where r is a sum of products of the form $a_i(n_1)^{e_1} \cdots a_i(n_{3k+1})^{e_{3k+1}} 1$ with some $n_j \leq -2$. Since the operators $a_i(n_j)$ commute, $r \in C_2(L)$. Hence $a_i(-1)^{3k+1} 1 \in C_2(L)$. We can now conclude that $C_2(L)$ contains $a_1(-1)^{m_1} \cdots a_d(-1)^{m_d} 1$ whenever $m_i \geq 3k+1$ for some i. The proposition follows immediately. □

13. Applications to the Moonshine Module

We now apply our results to the study of the conjectures of Conway–Norton–Queen as discussed in the Introduction. Recall that the moonshine module $V^\natural$ is a vertex operator algebra whose automorphism group is precisely the Monster $\mathbb{M}$. See [B1, FLM3, G] for details. The first author proved that $V^\natural$ has a unique simple module, namely $V^\natural$ itself, in [D2], and in [DLM3] we showed that in fact $V^\natural$ is rational, that is every admissible $V^\natural$-module is completely reducible. Thus $V^\natural$ is holomorphic. It also satisfies Condition C_2 (Prop. 12.4). By Theorem 10.3 we conclude that there is a unique simple g-twisted $V^\natural$-module $V^\natural(g)$ for each $g \in \mathbb{M}$. For each pair of commuting elements (g, h) in $\mathbb{M}$, recall from Sect. 11 that $Z(g, h, \tau) = Z(g, h)$ is the corresponding partition function. The function $Z(1, h)$ is precisely the graded character of $h \in \mathbb{M}$ on $V^\natural$. By the results of Borcherds [B2], which confirm the original Conway-Norton conjecture [CN], each $Z(1, h)$ is a *Hauptmodul* – in fact the Hauptmodul conjectured in [CN]. We use this to prove the next result, which completes the proof of Theorem 1.5.

Theorem 13.1. *The following hold:*

(i) *There is a scalar $\sigma = \sigma(g)$ such that the graded dimension $Z(g, 1, \tau)$ of $V^\natural(g)$ is equal to $\sigma Z(1, g, S\tau)$. In particular, $Z(g, 1, \tau)$ is a Hauptmodul.*
(ii) *More generally, if a commuting pair $g, h \in \mathbb{M}$ generates a cyclic group then $Z(g, h, \tau)$ is Hauptmodul.*

Proof. Suppose that $\langle g, h\rangle = \langle k\rangle$. Then there is $\gamma \in SL(2, \mathbb{Z})$ such that $(g, h)\gamma^{-1} = (1, k)$. By Theorem 11.4 we have

$$Z(g, h, \tau) = \sigma Z(1, k, \gamma\tau)$$

for some scalar σ. Since $Z(1, k, \tau)$ is a Hauptmodul, so too is $Z(g, h, \tau)$. If $h = 1$ then we may take $k = g$ and $\gamma = S$. Both parts of the theorem now follow. □

More is known in special cases. Huang has shown [H] that if g is of type $2B$ (in ATLAS notation) then in fact the constant $\sigma(g)$ in part (i) is equal to 1. This also follows from our results and [FLM3]. Similarly, if g is of type $2A$ it is shown in [DLM1], on the basis of Theorem 13.1, that again $\sigma(g) = 1$. In this case, the precise description of $Z(g, h, \tau)$ for $gh = hg$ and h of odd order is given in [DLM1].

As discussed in [DM1] and [DM2], the uniqueness of $V^\natural(g)$ leads to a projective representation of the centralizer $C_{\mathbb{M}}(g)$ on $V^\natural(g)$. These are discussed in some detail, in the case that g has order 2, in (loc.cit.). The conjectures of Conway–Norton–Queen state that there are (projective) representations of $C_{\mathbb{M}}(g)$ on suitably graded spaces such that all graded traces are either Hauptmoduln or zero. There is no doubt that the twisted modules $V^\natural(g)$ are the desired spaces. One would still like to show that $\sigma(g) = 1$ in part (i) of the theorem for all $g \in \mathbb{M}$, and to compute the Mckay-Thompson series $Z(g, h, \tau)$ for $\langle g, h\rangle$ *not* cyclic.

Finally, we calculate some correlation functions. We fix a holomorphic vertex operator algebra V satisfying Condition C_2. Recall that $T(v, (g, h), \tau)$ is the (g, h) 1-point correlation function associated with v and a pair of commuting elements $g, h \in \mathrm{Aut}(V)$ as defined in (8.4). If $\mathrm{wt}[v] = k$ then we have seen that $T(v, (1, 1), \tau)$ is a generalized modular form, holomorphic in the upper half-plane $\mathfrak{h}$. In fact, Eq. (11.5) tells us that $T(v, (1, 1), \tau)$ spans a 1-dimensional $SL(2, Z)$-module under the action (1.13), that is we have

$$T|\gamma(v, (1, 1), \tau) = \sigma(\gamma)T(v, (1, 1), \tau) \tag{13.1}$$

for some character $\sigma : SL(2, Z) \to \mathbb{C}^*$. We use this to prove

Lemma 13.2. *Let V be a holomorphic vertex operator algebra which satisfies Condition C_2 and let $v \in V$ satisfy* $\mathrm{wt}[v] = k$. *If k is odd, then the correlation function $T(v, (1, 1), \tau)$ is identically zero.*

Proof. We observe that the q-expansion of $T(v, (1, 1), \tau)$ lies in $\mathbb{C}[[q^{1/3}, q^{-1/3}]]$. This is because V is holomorphic and so $8|c$ (cf. the remark following Theorem 1.4). It follows that T acts on $T(v, (1, 1), \tau)$ as multiplication by a cube root of unity, and since T covers the abelianization of $SL(2, Z)$ then the kernel of the character σ has index dividing 3. Since $S^4 = id$ it follows that S, and in particular S^2, lies in the kernel of σ. Now setting $\gamma = S^2$ in (13.1) yields

$$T(v, (1, 1), \tau) = T|S^2(v, (1, 1), \tau) = (-1)^k T(v, (1, 1), \tau).$$

The lemma follows. □

Now let G be a finite group of automorphisms of V. Then $T(v, 1, g, \tau)$ is essentially the graded trace of $o(v)g$ on V for $g \in G$. If we choose v to be the conformal vector $\tilde{\omega}$ of $(V, Y[])$ then $\mathrm{wt}[\tilde{\omega}] = 2$, so $T(\tilde{\omega}, 1, g, \tau)$ is a form of weight 2. It is easy to describe:

Lemma 13.3. $T(\tilde{\omega}, 1, g, \tau) = \frac{1}{2\pi i}\frac{d}{d\tau} Z(1, g, \tau)$.

Proof. One could proceed by setting $\tilde{\omega} = \omega - c/24$ and using $Y(\omega, z) = \sum_n L(n)z^{-n-2}$, but it is simpler to use (5.8) and (5.9), as we may because $g\tilde{\omega} = \tilde{\omega}$. As $\tilde{\omega} = L[-2]\mathbf{1}$, the lemma follows. □

If we write $V = \oplus_n V_n$, then of course we have

$$Z(1, g, \tau) = q^{-c/24} \sum_n (\mathrm{tr}|_{V_n} g)q^n,$$
$$T(\tilde{\omega}, 1, g, \tau) = q^{-c/24} \sum_n (n - c/24)(\mathrm{tr}|_{V_n} g)q^n,$$

and we may think of $T(\tilde{\omega}, 1, g, \tau)$ as arising from a sequence of virtual characters of G. That is, instead of “Moonshine of weight 0,” one now has “Moonshine of weight 2.” This is relevant because of the work of Devoto [De] in which such things are interpreted as being elements of degree 2 in the elliptic cohomology of BG.

Similarly suppose that $v \in V$ satisfies $\mathrm{wt}[v] = k$ with $gv = v$ for all $g \in G$. Then G commutes with $o(v)$ in its action on each V_n, so each eigenspace of the semisimple part $o(v)_s$ of $o(v)$ on V_n is a G-module and gives rise to a “generalized module” for G, i.e., of the form $\sum_i \lambda_n^i V_n^i$ with $\lambda_n^i \in \mathbb{C}$ the distinct eigenvalues of $o(v)_s$ on V_n and V_n^i the corresponding eigenspaces of $o(v)_s$. In this way, the pair (V, v) gives rise to a sequence of generalized modules $\sum_{n,i} \lambda_n^i V_n^i$ for G such that the corresponding trace functions $T(v, 1, g, \tau)$ are modular forms of weight k. This is “Moonshine of weight k,” and together with the analogues for the twisted sectors gives rise to elements of $Ell^k BG$ as in [De]. Actually, this is not quite what we have proved, because in [De] there are additional arithmetic requirements. It seems likely that the appropriate conditions do hold, but that remains to be investigated.

The 1-point correlation functions for the Moonshine Module are completely described in a forthcoming paper [DM3].

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