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THE K-THEORY LOCALIZATIONS AND v_1-PERIODIC HOMOTOPY GROUPS OF H-SPACES

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We determine the mod p K-theory localizations and v_1-periodic homotopy groups of finite H-spaces and of other spaces with torsion-free exterior p-adic K-cohomology algebras at an odd prime p. Our localization results generalize those of Mahowald and Thompson (Topology 1992, **31**, 133–141) for odd-dimensional spheres. We construct our mod p K-theory localizations as homotopy fibers of unstable maps between infinite loop spaces, and similarly construct a wide array of new spaces having torsion-free exterior p-adic K-cohomology algebras with prescribed Adams operations. This leads, for example, to a classification of the odd mod p K-homology spheres.

1. INTRODUCTION

In this paper, we study the localization $X_{K/p}$ of a space X with respect to mod p complex K-homology theory or p-adic K-cohomology theory at an odd prime p (see [4] and 2.5). This localization has previously been determined when X is an infinite loop space [6] or odd sphere [27], but in few other cases. We now let X be a 1-connected space whose p-adic K-cohomology $K^*(X;\hat{Z}_p)$ is isomorphic, as a $Z/2$-graded p-adic λ-ring, to an exterior algebra $\hat{\Lambda}(M)$ generated by a regular torsion-free p-adic Adams module $M \subset K^1(X;\hat{Z}_p)$ (see 2.10, 3.2, and 4.4). For example, by Theorem 6.3, X might be any 1-connected finite H-space such that the multiplication in $H_*(X;Q)$ is associative. In Theorem 4.8, we construct the localization $X_{K/p}$ as a homotopy fiber of a certain map $\Omega^\infty \mathscr{M}(M,1) \to \Omega^\infty \mathscr{M}(M,1)$ with low-dimensional adjustments, where $\mathscr{M}(G,1)$ is a $K\hat{Z}_p^*$-Moore spectrum obtained as follows: by Theorem 3.4, for each stable p-adic Adams module G, there exists a homotopically unique K/p_*-local spectrum $\mathscr{M}(G,1)$ such that $K^0(\mathscr{M}(G,1);\hat{Z}_p)=0$ and $K^1(\mathscr{M}(G,1);\hat{Z}_p)\cong G$.

We also study the v_1-periodic homotopy groups $v_1^{-1}\pi_*(X;V)$ of a pointed space X with coefficients in a finite p-torsion spectrum V (see 7.1). Roughly speaking, these are obtained by choosing a v_1-map $\omega:\Sigma^{\mathrm{d}}V \to V$ and then inverting the action of ω on the homotopy groups of X with coefficients in a desuspension space of V. By [8, 12, Section 6, 21], or [23], there exists a functor $\Phi: Ho_* \to \mathscr{S}$ and natural equivalences

$$v_1^{-1}\pi_*(X;V) \cong [V,\Phi X]_* \cong [V,\tau_p\Phi X]_*$$

where Ho_* is the homotopy category of pointed CW-complexes, where $\mathscr{S}$ is the stable homotopy category, and where $\tau_p\Phi X$ is the p-torsion part of ΦX. From this perspective, the absolute v_1-periodic homotopy groups $v_1^{-1}\pi_*X$, introduced by Davis and Mahowald [20], may be interpreted as stable homotopy groups $v_1^{-1}\pi_*X \cong \pi_*\tau_p\Phi X$ (see [21] and Theorem 7.5). Important examples of v_1-periodic homotopy groups have been computed

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with considerable effort by Bendersky, Davis, Mahowald, Mimura, Thompson, and others as explained in [19].

In this paper, we develop a K-theoretic approach to v_1-periodic homotopy groups, generalizing that used by Langsetmo and Thompson in their calculation of $v_1^{-1}\pi_* S^{2n+1}$ [24, 19]. This approach is based on results of Thompson [33] and the author [10, 14] showing that v_1-periodic homotopy equivalences of spaces are very closely related to K/p_*-equivalences. Here, we deduce that $\Phi X \simeq \Phi(X_{K/p})$ when X is an H-space or an odd sphere or any other K/p_*-durable space (see Theorem 7.9). For a 1-connected finite H-space X with $H_*(X;Q)$ associative, we use our knowledge of $X_{K/p}$ to prove in Theorem 9.2 that $\Phi X \simeq \mathscr{M}(M/\psi^p, 1)$ where $M = \hat{Q}K^1(X;\hat{Z}_p) \cong PK^1(X;\hat{Z}_p)$ is the p-adic Adams module of indecomposables or primitives. We may now use the $K\hat{Z}_p^*$-Adams spectral sequence (Theorems 8.2 and 10.4) to calculate the v_1-periodic homotopy groups $v_1^{-1}\pi_*(X;V) \cong [V, \Phi X]_*$, and we obtain the strikingly simple expressions

$$v_1^{-1}\pi_{2m}X \cong [W^m(M/\psi^p)]^\#$$

$$v_1^{-1}\pi_{2m-1}X \cong [W_1^m(M/\psi^p)]^\#$$

in Theorem 9.2 for the absolute v_1-periodic homotopy groups of X, where $[-]^\#$ denotes the Pontrjagin dual, and where $W^m(M/\psi^p)$ and $W_1^m(M/\psi^p)$, respectively, denote the cokernel and kernel of $\psi^r - r^m \colon M/\psi^p \to M/\psi^p$ for an integer r generating the group of units $(Z/p^2)^\times$. In particular, $v_1^{-1}\pi_{2m}X$ and $v_1^{-1}\pi_{2m-1}X$ are finite p-groups of the same order. To illustrate our approach, we recover the main result of [18] on the v_1-periodic homotopy groups of $SU(n)$.

In the process of constructing our K/p_*-localizations, we also construct a large new family of K/p_*-local spaces. For each regular torsion-free p-adic Adams module M, we obtain a K/p_*-local space X with $M \subset K^1(X;\hat{Z}_p)$ and $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(M)$ in Theorem 4.7. Moreover, we show that X is homotopically unique when $M \cong \hat{Z}_p$. This leads to an almost complete homotopy classification of the K/p_*-local spaces X with $\tilde{K}_0(X;Z/p) = 0$ and $\tilde{K}_1(X;Z/p) \cong Z/p$ in Theorem 5.3. We call these spaces *odd K/p-homology spheres*, and we determine their v_1-periodic homotopy groups in 9.11. The odd K/p-homology spheres X with $\psi^p = 0$ in $K^1(X;\hat{Z}_p) \cong \hat{Z}_p$ are particularly interesting since they are infinite suspension spaces in the K/p_*-local homotopy category of spaces (see 5.8). We remark that they are precisely the spaces of the form $\Theta(E)$ for an odd K/p-homology sphere spectrum E, where Θ is a functor (which we hope to discuss elsewhere) from spectra to K/p_*-local spaces which is left adjoint to the restriction of Φ, and which satisfies the inversion formula

$$(\Sigma^\infty \Theta E)_{K/p} \simeq E_{K/p}$$

for each spectrum E.

The present work sets the stage for a p-adic K-theoretic unstable Adams spectral sequence, analogous to the classical mod p unstable Adams spectral sequence of Massey and Peterson [28] or Bousfield and Kan [15, 16, p. 22]. For a space X, our assumption that $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(M)$, for a regular p-adic Adams module M, is analogous to the Massey–Peterson assumption that $H^*(X;Z/p)$ is a free unstable algebra over the Steenrod algebra in the sense of Steenrod and Epstein [31]. The p-adic K-theoretic unstable E_2-term for X will be given by an unstable Ext for M, and the spectral sequence will converge to $\pi_* X_{K/p}$ above the bottom dimensions by arguments using our results on $X_{K/p}$.

Throughout this paper, we let p denote a fixed odd prime, except in Section 2 where we also allow $p = 2$. We let $\hat{Z}_p$ denote the p-adic integers and let Z_{p^∞} denote the p-torsion subgroup of Q/Z. We use the symbols Z_{p^n} and Z/p^n interchangeably.

2. THE p-ADIC K-COHOMOLOGY OF SPACES AND SPECTRA

In this section, we develop some needed preliminaries on the p-adic K-cohomology of spaces and spectra where p is an arbitrary prime. We start more generally by considering

2.1. Brown–Comenetz duality and p-adic cohomology theories. For a locally compact Hausdorff abelian group G, the *Pontrjagin dual* $G^{\#}$ is given by $\mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{R}/Z)$ with the compact-open topology. This restricts to a duality between the categories of discrete abelian groups and compact Hausdorff abelian groups, and restricts further to a duality between the categories of discrete p-torsion abelian groups and p-profinite abelian groups. Following [17], for a spectrum E, we let cE denote the function spectrum $F(E, cS)$ where cS is determined by the natural equivalence $[X, cS] \cong (\pi_0 X)^{\#}$ for spectra $X \in S$. The spectrum cE is called the *Brown–Comenetz dual* of E, and the associated cohomology theory has a natural *universal coefficient isomorphism*

$$(cE)^n X \cong (E_n X)^{\#}$$

for a space or spectrum X and $n \in Z$. In particular, $\pi_n(cE) \cong (\pi_{-n}E)^{\#}$. Following Anderson [2] or Yosimura [35], we have

2.2. PROPOSITION. *There are canonical equivalences $c(KZ_{p^\infty}) \simeq K\hat{Z}_p$ and $c(KZ/p^i) \simeq KZ/p^i$ for $i \geqslant 1$.*

Proof. We let $e: K\hat{Z}_p \to c(KZ_{p^\infty})$ be the adjoint of the map $K\hat{Z}_p \wedge KZ_{p^\infty} \to cS$ corresponding to the homomorphism

$$\pi_0(K\hat{Z}_p \wedge KZ_{p^\infty}) \to \pi_0 KZ_{p^\infty} = Z_{p^\infty} \subset \mathbb{R}/Z$$

induced by the multiplication map $K\hat{Z}_p \wedge KZ_{p^\infty} \to KZ_{p^\infty}$. Since the multiplication map induces an isomorphism

$$\pi_n K\hat{Z}_p \cong \mathrm{Hom}(\pi_{-n}KZ_{p^\infty}, \pi_0 KZ_{p^\infty}) \cong (\pi_{-n}KZ_{p^\infty})^{\#}$$

for each n, we conclude that $e_*: \pi_* K\hat{Z}_p \cong \pi_* c(KZ_{p^\infty})$ and hence e: $K\hat{Z}_p \simeq c(KZ_{p^\infty})$. The proof for KZ/p^i is similar. □

2.3. COROLLARY. *For a space or spectrum X and $i, n \in Z$ with $i \geqslant 1$, there are natural universal coefficient isomorphisms*

$$K^n(X; \hat{Z}_p) \cong K_n(X; Z_{p^\infty})^{\#}$$

$$K^n(X; Z/p^i) \cong K_n(X; Z/p^i)^{\#}.$$

Thus, each of the cohomology groups $K^n(X; \hat{Z}_p)$ and $K^n(X; Z/p^i)$ has a natural p-profinite abelian group structure. This structure agrees with the usual inverse limit topology [32] since there are natural topological isomorphisms $K^n(X; \hat{Z}_p) \cong \lim_\alpha K^n(X_\alpha; \hat{Z}_p)$ and $K^n(X; Z/p^i) \cong \lim_\alpha K^n(X_\alpha; Z/p^i)$, dual to the corresponding discrete isomorphisms for K_n, where $\{X_\alpha\}$ are the finite CW-subobjects of X.

Using Corollary 2.3 and Bockstein arguments, we see

2.4. PROPOSITION. *For a map $f: X \to Y$ of spaces or spectra and an integer $i \geqslant 1$, the following are equivalent*:

(i) $f^*: K^*(Y; \hat{Z}_p) \cong K^*(X; \hat{Z}_p)$;

(ii) f^*: $K^*(Y;Z/p^i) \cong K^*(X;Z/p^i)$;
(iii) f_*: $K_*(X;Z/p^i) \cong K_*(Y;Z/p^i)$;
(iv) f_*: $K_*(X;Z_{p^\infty}) \cong K_*(Y;Z_{p^\infty})$.

2.5. The K/p_*-localization. We let $X \to X_{K/p}$ denote the localization of a space or spectrum X with respect to the homology $K/p_* = K_*(-;Z/p)$ as in [4] or [5]. By Proposition 2.4, this is the same as the localization with respect to any one of the (co) homologies $K^*(-;\hat{Z}_p)$, $K^*(-;Z/p^i)$, $K_*(-;Z/p^i)$, and $K_*(-;Z_{p^\infty})$. Although each of these (co)homologies may be used to capture the K/p_*-local properties of spaces and spectra, we shall rely primarily on $K^*(-;\hat{Z}_p)$ because of its rich operational structure. To describe this structure for spectra, we introduce

2.6. The stable p-adic Adams modules. By a *finite stable p-adic Adams module*, we mean a finite abelian p-group G with endomorphisms ψ^k: $G \to G$ for $k \in Z - pZ$ such that:

(i) $\psi^1 = \mathrm{Id}$ and $\psi^j\psi^k = \psi^{jk}$ for all $j, k \in Z - pZ$;
(ii) there exists an integer $n \geqslant 1$ such that $\psi^k = \psi^{k+p^nj}$ on G for all $k \in Z - pZ$ and $j \in Z$.

These conditions ensure that the monoidal action of $Z - pZ = \{\psi^k\}_{k \in Z - pZ}$ on G factors through the group of units $(Z/p^n)^\times$ for sufficiently large n and thus extends to a continuous action of the p-adic units $\hat{Z}_p^\times$. By a *stable p-adic Adams module*, we mean the topological inverse limit of an inverse system of finite stable p-adic Adams modules. Let A denote the abelian category of stable p-adic Adams modules or (depending on the context) of $Z/2$-graded stable p-adic Adams modules.

2.7. The stable p-adic cohomology $K^*(E;\hat{Z}_p)$. For a spectrum E, the groups $K^*(E;\hat{Z}_p)$ are stable p-adic Adams modules as in [11, 8.1] with a Bott isomorphism $B: K^*(E;\hat{Z}_p) \cong K^{*-2}(E;\hat{Z}_p)$ such that $\psi^k B = kB\psi^k$ for $k \in Z - pZ$. We shall treat $K^*(E;\hat{Z}_p)$ as a $Z/2$-graded stable p-adic Adams module $\{K^0(E;\hat{Z}_p), K^1(E;\hat{Z}_p)\}$. To similarly describe the p-adic K-theory of spaces, we introduce

2.8. The p-adic Adams modules. By a *finite p-adic Adams module*, we mean a finite abelian p-group M with endomorphisms ψ^k: $M \to M$ for $k \in Z$ such that

(i) $\psi^1 = \mathrm{Id}$ and $\psi^j\psi^k = \psi^{jk}$ for all $j, k \in Z$;
(ii) there exists an integer $n \geqslant 1$ such that $\psi^k = \psi^{k+p^nj}$ on M for all $j, k \in Z$.

These conditions ensure that the monoidal action of $Z \cong \{\psi^k\}_{k \in Z}$ on M factors through Z/p^n for sufficiently large n, and thus extends to a continuous monoidal action of $\hat{Z}_p$. By a p-adic Adams module, we mean the topological inverse limit of an inverse system of finite p-adic Adams modules. Since ψ^0 acts idempotently on a p-adic Adams module M, there is a natural decomposition, $M = M_{\mathrm{red}} \oplus M_{\mathrm{fix}}$, where $M_{\mathrm{red}} = \{x \in M | \psi^0 x = 0\}$ and $M_{\mathrm{fix}} = \{x \in M | \psi^0 x = x\}$. Moreover, since $\psi^k\psi^0 = \psi^0$, we have $\psi^k x = x$ for each $x \in M_{\mathrm{fix}}$ and $k \in Z$. We say that M is *reduced* when $M = M_{\mathrm{red}}$ or equivalently when $\psi^0 = 0$ on M. We let $\mathscr{U}$ denote the abelian category of p-adic Adams modules or (depending on the context) of $Z/2$-graded p-adic Adams modules.

2.9. The p-adic cohomology $K^*(X;\hat{Z}_p)$. For a space X, we may treat $K^*(X;\hat{Z}_p)$ as a $Z/2$-graded p-adic Adams module $\{K^0(X;\hat{Z}_p), K^1(X;\hat{Z}_p)\}$, and we note that there are natural isomorphisms $K^0(X;\hat{Z}_p)_{\mathrm{fix}} \cong H^0(X;\hat{Z}_p)$ and $K^1(X;\hat{Z}_p)_{\mathrm{fix}} \cong H^1(X;\hat{Z}_p)$ by [11, 4.5]. In addition, $K^*(X;\hat{Z}_p)$ is a $Z/2$-graded commutative algebra with $w^2 = 0$ for each

$w \in K^1(X; \hat{Z}_p)$, and the Adams operations ψ^k respect multiplication as follows: for elements $a, b \in K^0(X; \hat{Z}_p)$ and $x, y \in K^1(X; \hat{Z}_p)$, there are identities $\psi^k(ab) = \psi^k(a)\psi^k(b)$, $\psi^k(ax) = \psi^k(a)\psi^k(x)$, $\psi^k(xy) = k\psi^k(x)\psi^k(y)$, and $\psi^k(1) = 1$. Finally, we briefly recall

2.10. The p-adic λ-ring structure of $K^*(X; \hat{Z}_p)$. In [11], we formulated the notion of a $Z/2$-graded p-adic λ-ring, extending the similar ungraded notion of [3], and we showed that $K^*(X; \hat{Z}_p)$ is a $Z/2$-graded p-adic λ-ring for each connected space X. As a part of its structure, a $Z/2$-graded p-adic λ-ring A has a $Z/2$-graded commutative multiplication and has canonical Adams operations $\psi^k : A \to A$, for $k \in Z$ with the properties given above in 2.9. In [11], we showed that the remaining parts of its structure are completely captured by a single non-additive operation $\theta^p : A^0 \to A^0$ which satisfies $\psi^p(x) = x^p + p\theta^p(x)$ (and other conditions) for each $x \in A^0$. In fact, we showed that a $Z/2$-graded p-adic λ-ring is precisely equivalent to a "$Z/2$-graded p-adic θ^p-ring equipped with Adams operations." We refer the reader to [11] for the full details.

3. THE p-ADIC K-COHOMOLOGY OF INFINITE LOOP SPACES

In this section, we recall some results of [11] on the p-adic K-cohomology of infinite loop spaces, where p is a fixed odd prime, and we introduce the fundamental infinite loop spaces which will be our building blocks for K/p_*-localizations of spaces. We shall need

3.1. Free p-adic Adams modules. For a stable p-adic Adams module G, let $\tilde{F}(G)$ denote the reduced p-adic Adams module generated freely by G. Thus, $\tilde{F}(G) = G \times G \times G \times \dots$ with Adams operations

$$\psi^p(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots)$$

$$\psi^k(x_1, x_2, x_3, \dots) = (\psi^k x_1, \psi^k x_2, \psi^k x_3, \dots)$$

for $k \in Z - pZ$, where G is embedded in $\tilde{F}(G)$ by identifying each $x \in G$ with $(x, 0, 0, \dots) \in \tilde{F}(G)$. The inclusion $G \subset \tilde{F}(G)$ is the universal homomorphism from G to a reduced p-adic Adams module.

3.2. Exterior algebras on p-adic Adams modules. For a p-adic Adams module M placed in degree 1, let $\hat{\Lambda}(M)$ denote the $Z/2$-graded p-adic exterior algebra on M given by $\hat{\Lambda}(M) = \lim_\alpha \Lambda(M_\alpha)$ where $\{M_\alpha\}_\alpha$ are the quotient finite p-adic Adams modules of M and Λ is the exterior algebra functor for abelian groups. Then $\hat{\Lambda}(M)$ has a canonical $Z/2$-graded p-adic λ-ring structure by [11, Theorem 6.3], and the inclusion $M \subset \hat{\Lambda}(M)$ is the universal homomorphism from M to a $Z/2$-graded p-adic λ-ring.

For a spectrum E with $K^*(E; \hat{Z}_p)$ torsion-free, we determined the p-adic K-cohomology $K^*(\Omega^\infty E; \hat{Z}_p)$ in [11, Theorem 8.3], and our result specializes to

3.3. THEOREM. *If E is a* 0*-connected spectrum with $H^1(E; \hat{Z}_p) = 0 = H^2(E; \hat{Z}_p)$, with $K^0(E; \hat{Z}_p) = 0$, and with $K^1(E; \hat{Z}_p)$ torsion-free, then there is a natural isomorphism of $Z/2$-graded p-adic λ-rings*

$$\hat{\Lambda}\tilde{F}K^1(E; \hat{Z}_p) \cong K^*(\Omega^\infty E; \hat{Z}_p).$$

We shall apply this theorem to certain spectra $E = \tilde{\mathscr{M}}(G, 1)$ which will be constructed from the following $K\hat{Z}_p^*$-Moore spectra $\mathscr{M}(G, 1)$.

3.4. THEOREM. *For a stable p-adic Adams module G, there exists a K/p_*-local spectrum $\mathscr{M}(G,1)$ with $K^1(\mathscr{M}(G,1);\hat{Z}_p)\cong G$ and $K^0(\mathscr{M}(G,1);\hat{Z}_p)=0$, and such a spectrum is homotopically unique.*

This will be proved later in 10.3. We also let $\mathscr{M}(G,0)$ denote $\Sigma^{-1}\mathscr{M}(G,1)$. To construct $\tilde{M}(G,1)$, we need

3.5. The p-complete spectra. For an abelian group A, *let* SA denote the Moore spectrum with $\pi_i SA=0$ for $i<0$, $H_0SA\cong A$, and $H_iSA=0$ for $i\neq 0$. A spectrum X is called p-complete when it is SZ/p_*-local, or equivalently when $F(SZ[1/p],X)=0$, or equivalently when $\mathrm{Ext}(Z[1/p],\pi_*X)=0=\mathrm{Hom}(Z[1/p],\pi_*X)$ as in [5]. Thus, a spectrum X is p-complete if and only if the groups π_*X are Ext-p-complete in the sense of [16]. Recall that each spectrum X has a natural *p-completion* $X\to\hat{X}_p$ given by the SZ/p_*-localization, where the groups $\pi_*\hat{X}_p$ are given by a splittable short exact sequence

$$0\to\mathrm{Ext}(Z_{p^\infty},\pi_*X)\to\pi_*\hat{X}_p\to\mathrm{Hom}(Z_{p^\infty},\pi_{*-1}X)\to 0$$

as in [5, Proposition 2.5]. *If* X is an E_*-local spectrum for a homology theory E_* with p-torsion coefficient groups π_*E, then X is p-complete since $SZ[1/p]$ is E-acyclic. In particular, the K/p_*-local spectra $\mathscr{M}(G,1)$ of Theorem 3.4 are p-complete. As in [16], an Ext-p-complete abelian group B is called *adjusted* when $\mathrm{Hom}(B,C)=0$ for every torsion-free Ext-p-complete abelian group C, and each Ext-p-complete abelian group A belongs to a splittable short exact sequence $0\to\hat{t}_pA\to A\to A/\hat{t}_pA\to 0$ where $\hat{t}_pA$ is the greatest adjusted Ext-p-complete subgroup of A and where $A/\hat{t}_pA$ is the greatest torsion-free Ext-p-complete quotient group of A. For a p-complete spectrum X and $n\in Z$, we let $X\to\bar{P}^nX$ denote the modified nth Postnikov section with

$$\pi_i\bar{P}^nX=\begin{cases}\pi_iX & \text{if}\quad i<n\\ \pi_nX/\hat{t}_p\pi_nX & \text{if}\quad i=n\\ 0 & \text{if}\quad i>n.\end{cases}$$

For a stable p-adic Adams module G, we now let $\tilde{\mathscr{M}}(G,1)$ denote the homotopy fiber of $\mathscr{M}(G,1)\to\bar{P}^2\mathscr{M}(G,1)$. The map $\tilde{\mathscr{M}}(G,1)\to\mathscr{M}(G,1)$ is a KZ/p_*-localization, or equivalently a $K\hat{Z}_p^*$-localization, by

3.6. LEMMA. *If $f\colon X\to Y$ is a map of spectra with $f_*\colon\pi_iX\cong\pi_iY$ for all sufficiently large i, then $f_*\colon K_*(X;Z/p)\cong K_*(Y;Z/p)$ and $f^*\colon K^*(Y;\hat{Z}_p)\cong K^*(X;\hat{Z}_p)$.*

Proof. This follows by Proposition 2.4 since the Eilenberg–MacLane spectra, and thus the Postnikov spectra, are K/p_*-acyclic. □

We now consider the infinite loop spaces $\Omega^\infty\tilde{\mathscr{M}}(G,1)$ which will serve as our building blocks for K/p_*-localizations of spaces.

3.7. THEOREM. *For a stable p-adic Adams module G, the space $\Omega^\infty\tilde{\mathscr{M}}(G,1)$ is K/p_*-local, and there is a natural isomorphism of $Z/2$-graded p-adic λ-rings $K^*(\Omega^\infty\tilde{\mathscr{M}}(G,1);\hat{Z}_p)\cong\hat{\Lambda}\tilde{F}G$ when G is torsion-free.*

Proof. The first statement follows from Theorem 3.8 below, and the last from Theorem 3.3. □

We have used the following result of [6].

3.8. THEOREM. *For a* 0*-connected p-complete spectrum E, there is a natural equivalence* $(\Omega^\infty E)_{K/p} \simeq \Omega^\infty(E^c_{K/p})$ *where* $E^c_{K/p}$ *is given by the homotopy fiber square*

$$\begin{tikzcd} E^c_{K/p} \arrow[r] \arrow[d] & E_{K/p} \arrow[d] \\ \bar{P}^2 E \arrow[r] & \bar{P}^2 E_{K/p}. \end{tikzcd}$$

To classify maps into the space $\Omega^\infty \tilde{\mathscr{M}}(G,1)$, we use

3.9. THEOREM. *For spectra X and Y such that* $K^*(Y;\hat{Z}_p)$ *is torsion-free, there is a natural short exact sequence*

$$0 \to \operatorname{Ext}^1_{\mathscr{A}}(K^*(Y;\hat{Z}_p), K^*(\Sigma X;\hat{Z}_p)) \to [X, Y_{K/p}] \to \operatorname{Hom}_{\mathscr{A}}(K^*(Y;\hat{Z}_p), K^*(X;\hat{Z}_p)) \to 0.$$

This will be proved later in 10.5. Note that for stable p-adic Adams modules G and G' with G torsion-free, this theorem implies

$$[\mathscr{M}(G',1), \mathscr{M}(G,1)] \cong \operatorname{Hom}_{\mathscr{A}}(G, G').$$

Also, for a spectrum E with $K^*(E;\hat{Z}_p)$ torsion-free, it implies

$$E_{K/p} \simeq \mathscr{M}(K^1(E;\hat{Z}_p),1) \times \mathscr{M}(K^0(E;\hat{Z}_p),0).$$

We use Theorem 3.9 to deduce

3.10. THEOREM. *Let X be a connected space with* $H^1(X;\hat{Z}_p) = 0 = H^2(X;\hat{Z}_p)$, *and let G be a torsion-free stable p-adic Adams module. If* $\phi: K^*(\Omega^\infty \tilde{\mathscr{M}}(G,1);\hat{Z}_p) \to K^*(X;\hat{Z}_p)$ *is a homomorphism of* $Z/2$*-graded p-adic* λ*-rings, then there exists a map* $f: X \to \Omega^\infty \tilde{\mathscr{M}}(G,1)$ *such that* $f^* = \phi$. *Moreover, f is homotopically unique when* $\tilde{K}^0(X;\hat{Z}_p) = 0$.

Proof. Since $K^*(\Omega^\infty \tilde{\mathscr{M}}(G,1);\hat{Z}_p) \cong \hat{\Lambda}\tilde{F}G$ by Theorem 3.7, ϕ corresponds to a stable p-adic Adams module homomorphism $\bar{\phi}: G \to K^1(X;\hat{Z}_p) \cong K^1(\Sigma^\infty X;\hat{Z}_p)$, and there exists a map $\bar{f}: \Sigma^\infty X \to \mathscr{M}(G,1)$ with $\bar{f}^* = \bar{\phi}$ by Theorem 3.9. Since $\mathscr{M}(G,1)$ is p-complete and $(\Sigma^\infty X)^\wedge_p$ is 1-connected with $\pi_2(\Sigma^\infty X)^\wedge_p$ adjusted, $\bar{f}$ corresponds to a map f': $\Sigma^\infty X \to \tilde{M}(G,1)$, and the adjoint map $f: X \to \Omega^\infty \tilde{\mathscr{M}}(G,1)$ has the desired properties. When $\tilde{K}^0(X;\hat{Z}_p) = 0$, f is homotopically unique by Theorem 3.9, and hence $\bar{f}$ is homotopically unique. □

4. CONSTRUCTIONS OF K/p_*-LOCALIZATIONS

We now give our main results on the K/p_*-localizations of spaces X with $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(M)$ for a reduced p-adic Adams module M, and on the existence of such spaces X, where p is a fixed odd prime. Our constructions are based on

4.1. Lemma. *For a reduced p-adic Adams module M, there is a short exact sequence*

$$0 \to \tilde{F}M \xrightarrow{\partial} \tilde{F}M \xrightarrow{\alpha} M \to 0$$

where α *is the adjunction map with* $\alpha(x_1, x_2, x_3, \dots) = x_1 + \psi^p x_2 + (\psi^p)^2 x_3 + \cdots$ *and* $\partial = \tilde{F}\psi^p - \psi^p$.

This is easily verified and suggests that $\hat{\Lambda}(M)$ might be realized as the fiber of a map $\Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ which realizes $\hat{\Lambda}(\partial)$. To actually do this, we need a weak technical condition on M which will be introduced in 4.4 using

4.2. Linearity, quasilinearity, and strict nonlinearity conditions. As in [13], a p-adic Adams module H is called *linear* when $\psi^k x = kx$ for all $k \in Z$ and $x \in H$, and H is called *quasilinear* when $pH \subset \psi^p H$. The quasilinear subobjects of a p-adic Adams module M are all contained in a largest quasilinear subobject $M_{ql} \subset M$, which includes, for instance, all $x \in M$ with $px = 0$, or $\psi^p x = x$, or $\psi^p x = cpx$ for a p-adic unit c. A p-adic Adams module M is called *strictly nonlinear* when $M_{q\ell} = 0$. This is equivalent to saying that $\mathrm{Hom}(H, M) = 0$ for each quasilinear p-adic Adams module H, and implies that M is reduced (see 2.8) and torsion-free. Note that strict nonlinearity is preserved by inverse limits, extensions, and subobjects. A torsion-free p-adic Adams module with $\psi^p = p^k$ for some $k \geqslant 2$ is strictly nonlinear, and many other examples follow from

4.3. Proposition. *If M is a torsion-free p-adic Adams module with* $(\psi^p)^n M \subset p^{n+1} M$ *for some* $n \geqslant 1$, *then M is strictly nonlinear.*

This is proved in [13, 2.5].

4.4. Regularity. A p-adic Adams module M will be called *regular* when the kernel of $M \to \mathrm{Lin}\, M$ is strictly nonlinear where

$$\mathrm{Lin}\, M = M/((\psi^r - r)M + (\psi^p - p)M)$$

is the largest linear quotient of M, constructed using an integer r generating $(Z/p^2)^\times$. Thus, M is regular whenever it is an extension of a strictly nonlinear submodule by a linear quotient module. If a p-adic Adams module M is regular, then so are all of its submodules. For any connected space X with $K^*(X; \hat{Z}_p)$ torsion-free, we know that $\tilde{K}^0(X;\hat{Z}_p)$ is regular by [11, Theorem 6.3] and [13, Theorem 2.6], and in every case that we have examined with $H^1(X;\hat{Z}_p) = 0$, we have found that $K^1(X; \hat{Z}_p)$ is also regular (see Proposition 5.4 and Lemma 6.1).

4.6. The main construction. By Theorem 3.10, for a torsion-free p-adic Adams module M, there exists a map $f: \Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ with

$$f^* = \hat{\Lambda}(\partial): \hat{\Lambda}(\tilde{F}M) \to \hat{\Lambda}(\tilde{F}M)$$

where $\partial = \tilde{F}\psi^p - \psi^p$. Any such f will be called a *companion map* of M, and $\mathrm{Fib}\, f$ will denote its homotopy fiber. Since $\Omega^\infty \tilde{\mathscr{M}}(M, 1)$ is K/p_*-local, so are $\mathrm{Fib}\, f$ and $\bar{P}^2\, \mathrm{Fib}\, f$, where $\bar{P}^2\, \mathrm{Fib}\, f$

denotes the modified 2nd Postnikov section of $\mathrm{Fib}\, f$ as in 3.5. We let $\widetilde{\mathrm{Fib}}\, f$ denote the homotopy fiber of the map $\mathrm{Fib}\, f \to \bar{P}^2\, \mathrm{Fib}\, f$, and we conclude that $\widetilde{\mathrm{Fib}}\, f$ is K/p_*-local with

$$\pi_i \widetilde{\mathrm{Fib}}\, f = \begin{cases} 0 & \text{if} \quad i < 2 \\ \hat{t}_p(\pi_2 \mathrm{Fib}\, f) & \text{if} \quad i = 2 \\ \pi_i \mathrm{Fib}\, f & \text{if} \quad i > 2. \end{cases}$$

4.7. THEOREM. *For a regular torsion-free p-adic Adams module M and any companion map* $f: \Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$, *there is a canonical isomorphism* $K^*(\widetilde{\mathrm{Fib}}\, f; \hat{Z}_p) \cong \hat{\Lambda}(M)$ *of* $Z/2$*-graded p-adic* λ*-rings with M in degree* 1.

This will be proved later in 11.8 and leads immediately to our main result on K/p_*-localizations.

4.8. THEOREM. *If* X *is a connected space with* $K^*(X; \hat{Z}_p) \cong \hat{\Lambda}(M)$ *for a regular torsion-free p-adic Adams module* $M \subset K^1(X; \hat{Z}_p)$, *then* $X_{K/p} \simeq \widetilde{\mathrm{Fib}}\, f$ *for some companion map* f: $\Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ *of* M. *Moreover,* $H^1(X; \hat{Z}_p) = 0 = H^2(X; \hat{Z}_p)$.

Proof. The last statement follows since $\{Z_p \oplus H^2(X; \hat{Z}_p), H^1(X; \hat{Z}_p)\}$ is a quotient ring of $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(M)$ as shown in [11, 5.4], and since the homomorphism $M \to H^1(X; \hat{Z}_p)$ must be trivial because it factors through $M_{\mathrm{fix}} = 0$. Applying Theorem 3.10 twice, we obtain a map $h: X \to \Omega^\infty \tilde{M}(M, 1)$ with $h^* = \hat{\Lambda}(\alpha): \hat{\Lambda}(\tilde{F}M) \twoheadrightarrow \hat{\Lambda}(M)$ and then obtain a map k: $\mathrm{Cof}\, h \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ with

$$k^* = \hat{\Lambda}(\beta): \hat{\Lambda}(\tilde{F}M) \to K^*(\mathrm{Cof}\, h; \hat{Z}_p) \subset \hat{\Lambda}(\tilde{F}M).$$

Composing the canonical map $\Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \mathrm{Cof}\, h$ with k, we obtain a companion map f: $\Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ of M such that h lifts to a map $w: X \to \mathrm{Fib}\, f$. Since $[X, P^2\, \mathrm{Fib}\, f] = 0$, w lifts to a map $u: X \to \widetilde{\mathrm{Fib}}\, f$ which is a $K^*(-; \hat{Z}_p)$-equivalence by Theorem 4.7. Hence, $u: X \to \widetilde{\mathrm{Fib}}\, f$ is a K/p_*-equivalence to the K/p_*-local space $\widetilde{\mathrm{Fib}}\, f$. $\square$

Similarly, using the homotopical uniqueness statement of Theorem 3.10, we obtain

4.9. THEOREM. *Let* f: $\Omega^\infty \tilde{M}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ *be a companion map of a regular torsion-free p-adic Adams module* M, *and let* X *be a connected space with* $H^1(X; \hat{Z}_p) = 0 = H^2(X; \hat{Z}_p)$ *and* $\tilde{K}^0(X; \hat{Z}_p) = 0$. *Then each homomorphism* $\hat{\Lambda}(M) \to K^*(X; \hat{Z}_p)$ *of* $Z/2$*-graded* λ*-rings is induced by a map* $h: X \to \widetilde{\mathrm{Fib}}\, f$.

We shall apply this theorem below when X is an odd K/p-homology sphere.

5. APPLICATIONS TO K/p_*-HOMOLOGY SPHERES

By an *odd* (resp. *even*) K/p*-homology sphere* we mean a space X such that $\tilde{K}_i(X; Z/p)$ is Z/p for i odd (resp. even) and is trivial otherwise. We shall apply the results of Section 4 to obtain almost complete results on the K/p_*-localizations of odd K/p-homology spheres and on the classification of the resulting K/p_*-local homotopy types, where p is a fixed odd prime. For this purpose, we use

5.1. LEMMA *A space X is an odd K/p-homology sphere if and only if $K^1(X;\hat{Z}_p) \cong \hat{Z}_p$ and $\tilde{K}^0(X;\hat{Z}_p) = 0$.*

Proof. Using the coefficient sequence $0 \to Z/p \to Z_{p^\infty} \to Z_{p^\infty} \to 0$, we see that a space X is a K/p-homology sphere if and only if $\tilde{K}_0(X;Z_{p^\infty}) = 0$ and $K_1(X;Z_{p^\infty}) \cong D$ for a p-torsion group D with $D/p = 0$ and $D\backslash p \cong Z/p$, i.e. for $D \cong Z_{p^\infty}$. The lemma now follows by Corollary 2.3. □

5.2. The spherical p-adic Adams modules. A p-adic Adams module M will be called *spherical* when it is isomorphic to $\hat{Z}_p$ as a p-profinite group. In general, the Adams operations on a p-adic Adams module M are all determined by ψ^p and ψ^r where r is a fixed integer generating the group of units $(Z/p^2)^\times$. It is easy to see that a spherical p-adic Adams module M must satisfy one of the following conditions:

(i) M is spherical of *class* 0 when it has $\psi^k = 1$ for each $k \in Z$;
(ii) M is spherical of *class* n for $1 \leqslant n < \infty$ when it has $\psi^p = u\,p^n$ and $\psi^r = v$ for p-adic units $u, v \in \hat{Z}_p^\times$;
(iii) M is spherical of *class* ∞ when it has $\psi^p = 0$ and $\psi^r = v$ for a p-adic unit $v \in \hat{Z}_p^\times$.

Moreover, these conditions (with no further restrictions on n, u, v) completely classify the spherical p-adic Adams modules up to isomorphism. Note that, for $0 \leqslant n < \infty$, $K^1(S^{2n+1};\hat{Z}_p)$ is spherical of class n with $\psi^k = k^n$ for each $k \in Z$. These spherical p-adic Adams modules will be called *standard*. Our main theorem on odd K/p-homology spheres is

5.3. THEOREM. *Let M be a spherical p-adic Adams module of class n for $0 \leqslant n \leqslant \infty$ which is standard when $n = 1$. Then there exists a homotopically unique odd K/p-homology sphere $S(M, 1)$ which is K/p_*-local with $K^1(S(M,1);\hat{Z}_p) \cong M$. Moreover, if X is any odd K/p-homology sphere with $K^1(X;\hat{Z}_p) \cong M$, then $X_{K/p} \cong S(M, 1)$.*

Proof. For $n = 0$ we may assume that $M = \hat{Z}_p$ with $\psi^k = 1$ for all $k \in Z$, and we may let $S(M, 1) = K(\hat{Z}_p, 1)$. Since

$$[X, K(\hat{Z}_p, 1)] \cong H^1(X;\hat{Z}_p) \cong \mathrm{K}^1(X;\hat{Z}_p)_{\mathrm{fix}} \cong \mathrm{Hom}_{\mathscr{U}}(M, K^1(X;\hat{Z}_p))$$

there is a K/p_*-localization $X \to K(\hat{Z}_p, 1)$ corresponding to an isomorphism $M \cong K^1(X;\hat{Z}_p)$. For $n \geqslant 1$, M is regular since it is linear or strictly nonlinear, and we let $S(M, 1)$ be $\mathrm{Fib}\, f$ for a companion map $f\colon \Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ of M. Then $S(M, 1)$ is K/p_*-local with $K^1(S(M, 1);\hat{Z}_p) \cong M$ and $\tilde{K}^0(S(M, 1);\hat{Z}_p) \cong 0$ by Theorem 4.7. There is also a K/p_*-localization map $X \to S(M, 1)$ by Theorems 4.7 and 4.8. The homotopical uniqueness of $S(M, 1)$ follows since our version of $S(M, 1)$ is the K/p_*-localization of any other version. □

We do not know which, if any, of the nonstandard spherical p-adic Adams modules of class 1 can be realized as $K^1(X;\hat{Z}_p)$ for an odd K/p-homology sphere X. However, since these modules are irregular, such an X could not be a 1-connected H-space by Lemma 6.1 below, and could not be finite dimensional by

5.4. Proposition. *If X is a connected finite dimensional CW-complex with $H^1(X;\hat{Z}_p)=0$ and with $K^1(X;\hat{Z}_p)$ torsion-free, then $K^1(X;\hat{Z}_p)$ is regular.*

The proof will depend on two lemmas.

5.5. Lemma. *If X is a connected CW-complex with $H^1(X;\hat{Z}_p)=0$, then the kernel of the canonical map $K^1(X;\hat{Z}_p)\to H^3(X;\hat{Z}_p)$ is isomorphic to $K^1(X/X^3;\hat{Z}_p)$ where X^3 denotes the 3-skeleton of X.*

Proof. The map $K\hat{Z}_p\to P^2K\hat{Z}_p$ induces a ladder of exact sequences

$$\begin{tikzcd}
\tilde{K}^0(X;\hat{Z}_p) \arrow[r] \arrow[d] & \tilde{K}^0(X^3;\hat{Z}_p) \arrow[r] \arrow[d] & K^1(X/X^3;\hat{Z}_p) \arrow[r] \arrow[d] & K^1(X;\hat{Z}_p) \arrow[r] \arrow[d] & K^1(X^3;\hat{Z}_p) \arrow[d] \\
H^2(X;\hat{Z}_p) \arrow[r] & H^2(X^3;\hat{Z}_p) \arrow[r] & H^3(X/X^3;\hat{Z}_p) \arrow[r] & H^3(X;\hat{Z}_p) \arrow[r] & H^3(X^3;\hat{Z}_p)
\end{tikzcd}$$

in which first vertical map is onto by [11, 5.4], while the second and fifth are isomorphisms. Since $H^3(X/X^3;\hat{Z}_p)=0$, the map $\tilde{K}^0(X;\hat{Z}_p)\to\tilde{K}^0(X^3;\hat{Z}_p)$ is onto, and the lemma follows by a diagram chase. □

A p-adic Adams module M will be called *weakly nonlinear* when $p^iM_{q\ell}=0$ for some $i\geqslant 0$, where $M_{q\ell}\subset M$ is the largest quasilinear submodule of M (see 4.2).

5.6. Lemma. *If X is a finite dimensional CW-complex with $X^3=*$, then $K^1(X;\hat{Z}_p)$ is weakly nonlinear, and is strictly nonlinear when it is torsion-free.*

Proof. It will suffice to show that $K^1(X^n;\hat{Z}_p)$ is weakly nonlinear for $n\geqslant 4$. We assume inductively that $K^1(X^{n-1};\hat{Z}_p)$ is weakly nonlinear and consider the exact sequence $K^1(X^n/X^{n-1};\hat{Z}_p)\to K^1(X^n;\hat{Z}_p)\to K^1(X^{n-1};\hat{Z}_p)$. The image I of the first map has $pI_{q\ell}=0$ since I has $\psi^p=p^j$ for some $j\geqslant 2$. Since $(\,-\,)_{q\ell}$ is left exact, we deduce that $K^1(X^n;\hat{Z}_p)$ is weakly nonlinear. □

5.7. *Proof of Proposition* 5.4. Using the exact sequence

$$0\to K^1(X/X^3;\hat{Z}_p)\to K^1(X;\hat{Z}_p)\to H^3(X;\hat{Z}_p)$$

of Lemma 5.5, we see that $K^1(X;\hat{Z}_p)$ is regular since $H^3(X;\hat{Z}_p)$ is linear and $K^1(X/X^3;\hat{Z}_p)$ is strictly nonlinear by Lemma 5.6.

5.8. Desuspensions of K/p-homology spheres. A p-adic Adams module N "suspends" as in [7, 1.6] to give a p-adic Adams module σN such that $\psi^k:\sigma N\to\sigma N$ equals $k\psi^k:N\to N$ for $k\in Z$. If M is a spherical p-adic Adams module of class $n\geqslant 1$, assumed standard when $n\leqslant 2$, then there is a unique p-adic Adams module N of class $n-1$ with $\sigma N=M$, and there is an equivalence

$$(\Sigma^2 S(N,1))_{K/p}\simeq S(M,1)$$

by Theorem 5.3. Thus if X is an odd K/p-homology sphere with $K^1(X;\hat{Z}_p)$ of class $n\geqslant 1$, assumed standard when $n\leqslant 2$, then X has a unique double desuspension in K/p_*-local homotopy theory. Such desuspensions may be iterated until the lowest possible class is reached. When $K^1(X;\hat{Z}_p)$ is of class ∞, then X desuspends infinitely in K/p_*-local homotopy theory.

6. ON THE K/p_*-LOCALIZATIONS OF H-SPACES

Working at an odd prime p, we shall show that our main K/p_*-localization result, Theorem 4.8, applies to a wide range of H-spaces. Recall that this theorem gives the K/p_*-localization of any connected space Y such that $K^*(Y;\hat{Z}_p) \cong \hat{\Lambda}(M)$ for a regular torsion-free p-adic Adams module $M \subset K^1(Y;\hat{Z}_p)$. This condition implies that $K_*(Y;Z/p)$ is an exterior coalgebra $\Lambda(M_p^\#)$, where $M_p^\#$ is the discrete Pontrjagin dual of M/p. Thus, $K_*(Y;Z/p)$ has trivial even primitives $PK_0(Y;Z/p)=0$ and odd primitives $PK_1(Y;Z/p) \cong M_p^\#$. We shall determine the K/p_*-localizations of most H-spaces X with $PK_0(X;Z/p)=0$, and hence of most finite H-spaces X. By the results 1.8, 2.6, 10.2, and 10.5 of [13], we know

6.1. LEMMA. *If X is a 1-connected H-space with $PK_0(X;Z/p)=0$, then*:

(i) $K^*(X;\hat{Z}_p)$ *and* $K^*(\Omega X;\hat{Z}_p)$ *are torsion-free with* $K^1(\Omega X;\hat{Z}_p)=0$;
(ii) *there is a suspension isomorphism* $\hat{Q}K^1(X;\hat{Z}_p) \cong PK^0(\Omega X;\hat{Z}_p)$ *and both sides are regular torsion-free p-adic Adams modules*;
(iii) *there is a suspension isomorphism* $QK_0(\Omega X;Z/p) \cong PK_1(X;Z/p)$ *and* $K_*(X;Z/p)$ *is an exterior coalgebra which is generated by* $PK_1(X;Z/p)$ *as a* (*possibly non-associative*) *algebra.*

For an H-space X with $K^*(X;\hat{Z}_p)$ torsion-free as above, the multiplication map $X \times X \to X$ induces a comultiplication

$$K^*(X;\hat{Z}_p) \to K^*(X;\hat{Z}_p) \hat{\otimes} K^*(X;\hat{Z}_p).$$

6.2. THEOREM. *Let X be a 1-connected H-space with $PK_0(X;Z/p)=0$. If X is homotopy associative* (*or more generally if $K^*(X;\hat{Z}_p)$ is coassociative*), *then $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(PK^1(X;\hat{Z}_p))$ and the p-adic Adams module $PK^1(X;\hat{Z}_p)$ is regular and torsion-free. Hence, $X_{K/p} \simeq \widetilde{\mathrm{Fib}}\, f$ for some companion map $f\colon \Omega^\infty \tilde{\mathscr{M}}(M,1) \to \Omega^\infty \tilde{\mathscr{M}}(M,1)$ of $M = PK^1(X;\hat{Z}_p)$.*

Proof. Since $K^*(X;\hat{Z}_p)$ is coassociative and torsion-free by Lemma 6.1, $K_*(X;Z/p) \cong K^*(X;\hat{Z}_p)_p^\#$ is associative. Since $PK_0(X;Z/p)=0$, the elements of $PK_1(X;Z/p)$ have trivial Lie brackets, and the induced homomorphism $\Lambda(PK_1(X;Z/p)) \to K_*(X;Z/p)$ is an isomorphism by 6.1(iii) and Proposition 10.4 of [11]. Since $K^*(X;\hat{Z}_p)$ is a torsion-free p-profinite Hopf algebra, $K_*(X;Z/p^n)$ is a Hopf algebra of free Z/p^n-modules with $K_*(X;Z/p^n)\otimes Z/p \cong K_*(X;Z/p)$ for $n \geqslant 1$. Thus, $PK_0(X;Z/p^n)=0$ since $PK_0(X;Z/p)=0$, and the elements of $PK_1(X;Z/p^n)$ must have trivial Lie brackets. Since the algebra $K_*(X;Z/p)$ is generated by elements in the image of the suspension $K_0(\Omega X;Z/p) \to PK_1(X;Z/p)$, and since $K_0(\Omega X;Z/p) \cong K_0(\Omega X;Z/p^n)\otimes Z/p$ by Lemma 6.1, we deduce that the algebra $K_*(X;Z/p^n)$ is likewise generated by elements in the image of the suspension $K_0(\Omega X;Z/p^n) \to PK_1(X;Z/p^n)$. Thus, $K_*(X;Z/p^n)$ is commutative for $n \geqslant 1$, and consequently $K^*(X;\hat{Z}_p)$ is cocommutative. Now [13, Theorem 4.8] gives the desired isomorphism $\hat{\Lambda}(PK^1(X;\hat{Z}_p)) \cong K^*(X;\hat{Z}_p)$, and the p-adic Adams module $PK^1(X;\hat{Z}_p) \cong \hat{Q}K^1(X;\hat{Z}_p)$ is regular by Lemma 6.1. □

The above result applies to most 1-connected finite H-spaces

6.3. THEOREM. *If X is a 1-connected H-space with $H_*(X;Q)$ associative as an algebra and with $H_*(X;Z_{(p)})$ finitely generated over $Z_{(p)}$ (and thus vanishing above some dimension), then $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(PK^1(X;\hat{Z}_p))$ where $PK^1(X;\hat{Z}_p)$ is regular and torsion-free. Thus Theorem 6.2 applies to give $X_{K/p}$.*

Proof. As in [13, Corollary 10.4], this follows from work of Lin [25] which shows that $K_*(X;Z_{(p)})$ is $Z_{(p)}$-free and $PK_0(X;Z/p)=0$. □

7. ON v_1-PERIODIC HOMOTOPY GROUPS AND K/p_*-LOCALIZATIONS OF SPACES

In this section, we study the v_1-periodic homotopy groups of spaces and explain how they may be captured using spectra as in the work of Davis and Mahowald [21], Kuhn [23], and the author [8, 12]. This will set the stage for the next section where we shall determine v_1-periodic homotopy groups of finite H-spaces using our knowledge of their K/p_*-localizations. We first recall

7.1. The v_1-periodic homotopy groups of spaces. By a *finite p-torsion* spectrum $W \in \mathscr{S}$, we mean a finite CW-spectrum with finite p-torsion integral homology. For such a W, a *v_1-map* is a K/p_*-equivalence (or $K(1)_*$-equivalence) $\omega:\Sigma^d W \to W$ with $d>0$ such that $K(n)_*\omega = 0$ for $n>1$, where $K(n)_*$ is the n^{th} Morava K-theory. The Hopkins-Smith periodicity theorem (see [22] or [29]) ensures that each finite p-torsion spectrum W has a v_1-map $\omega:\Sigma^d W \to W$ with $d = 2(p-1)\,p^e$ for some $e \geqslant 0$, and that any two v_1-maps for W become equivalent after sufficient iteration. Since the sequence $W \overset{\omega}{\leftarrow} \Sigma^d W \overset{\omega}{\leftarrow} \Sigma^{2d} W \overset{\omega}{\leftarrow} \cdots$ in $\mathscr{S}$ eventually desuspends uniquely to Ho_*, we may define the *v_1-periodic homotopy groups* of a space $Y \in Ho_*$ with coefficients in a finite p-torsion spectrum W by

$$v_1^{-1}\pi_*(Y;W) = \operatorname*{colim}_m\,[\Sigma^{dm}W,\,Y]_*.$$

By [22] or [23], the groups $v_1^{-1}\pi_*(Y;W)$ do not depend on the choice of ω and are natural in W as well as Y. Following Davis and Mahowald [20], we may also define the absolute *v_1-periodic homotopy groups* of a space $Y \in Ho_*$ by

$$v_1^{-1}\pi_*Y = \operatorname*{colim}_k v_1^{-1}\pi_{*+1}(Y;Z/p^k) = \operatorname*{colim}_k v_1^{-1}\pi_{*+1}(Y;S^{-1}Z/p^k)$$

using the Moore spectra $S^{-1}Z/p^k = S^{-1}\cup_{p^k} e^0$ and the canonical maps $S^{-1}Z/p^{k+1} \to S^{-1}Z/p^k$ which have degree p on the top cell and degree 1 on the bottom cell. The v_1-periodic homotopy groups of spaces are completely captured by

7.2. The functor $\Phi: Ho_* \to \mathscr{S}$. By [8, 12, Section 6, 21], or [23], there is a functor Φ: $Ho_* \to \mathscr{S}$ such that:

(i) for a space $Y \in Ho_*$ and finite p-torsion spectrum $W \in \mathscr{S}$, there is a natural isomorphism $v_1^{-1}\pi_*(Y;W) \cong [W,\Phi Y]_*$;
(ii) ΦY is K/p_*-local for each $Y \in Ho_*$;
(iii) for a spectrum E, there is a natural equivalence $\Phi(\Omega^\infty E) \simeq E_{K/p}$;
(iv) Φ preserves homotopy fiber squares.

To extract the v_1-periodic homotopy groups $v_1^{-1}\pi_* Y$ from ΦY, we use

7.3. The p-torsion part of a spectrum. A spectrum $A \in \mathcal{S}$ is called *p-torsion* when $\pi_* A$ is p-torsion. For each spectrum $E \in \mathcal{S}$, there is a universal map $\tau_p E \to E$ from a p-torsion spectrum to E in $\mathcal{S}$, given by the homotopy fiber of the localization $E \to E[1/p]$ away from p. We note that $\tau_p E \simeq E \wedge \tau_p S$, where $\tau_p S$ is the Moore spectrum $S^{-1}Z_{p^\infty}$, and we call $\tau_p E$ the *p-torsion part* of E. The functor τ_p: $\mathcal{S} \to \mathcal{S}$ is left adjoint to the p-completion functor $\widehat{(-)}_p : \mathcal{S} \to \mathcal{S}$ of 3.5, since the p-completion of a spectrum may be constructed as the map of function spectra $E \simeq F(S, E) \to F(\tau_p S, E) \simeq \hat{E}_p$ induced by $\tau_p S \to S$ (see [5, 2.5]). From another standpoint, the maps $\tau_p E \to E$ and $E \to \hat{E}_p$ are the universal examples of SZ/p_*-equivalences into and out of E in $\mathcal{S}$. As in [12, 6.7], we easily deduce

7.4. Proposition. *The adjoint functors $\tau_p : \mathcal{S} \to \mathcal{S}$ and $\widehat{(-)}_p : \mathcal{S} \to \mathcal{S}$ restrict to adjoint equivalences*: (i) *between the full subcategories of p-complete spectra and p-torsion spectra*; and (ii) *between the full subcategories of K/p_*-local spectra and p-torsion K_*-local spectra.*

Thus, the K/p_*-local spectrum ΦY corresponds to the p-torsion K_*-local spectrum $\tau_p \Phi Y$, and we have the following reinterpretation of the v_1-periodic homotopy groups $v_1^{-1}\pi_* Y$ in the spirit Davis and Mahowald [21].

7.5. Theorem. *For a space $Y \in Ho_*$ and a finite p-torsion spectrum $W \in \mathcal{S}$, there are natural isomorphisms*

$$v_1^{-1}\pi_*(Y; W) \cong [W, \Phi Y]_* \cong [W, \tau_p \Phi Y]_*$$

$$v_1^{-1}\pi_* Y \cong \pi_* \tau_p \Phi Y.$$

Proof. The first isomorphisms follow from 7.2 and 7.3, and the last follows by

$$v_1^{-1}\pi_* Y \cong \operatorname*{colim}_k v_1^{-1}\pi_{*+1}(Y; Z/p^k) \cong \operatorname*{colim}_k [S^{-1}Z/p^k, \Phi Y]_{*+1}$$

$$\cong \operatorname*{colim}_k \pi_{*+1}(DS^{-1}Z/p^k \wedge \Phi Y) \cong \pi_*(S^{-1}Z_{p^\infty} \wedge \Phi Y) \cong \pi_* \tau_p \Phi Y.$$

where D is the Spanier–Whitehead duality functor. □

7.6. Corollary. *For a map $f: X \to Y$ in Ho_* and a finite p-torsion spectrum W with $K_*(W;Z/p) \neq 0$, the following are equivalent*:

(i) $f_* : v_1^{-1}\pi_*(X; W) \cong v_1^{-1}\pi_*(Y; W)$;
(ii) $f_* : v_1^{-1}\pi_*(X; Z/p) \cong v_1^{-1}\pi_*(Y; Z/p)$;
(iii) $f_* : v_1^{-1}\pi_* X \cong v_1^{-1}\pi_* Y$;
(iv) $\Phi f : \Phi X \simeq \Phi Y$.

Proof. If W is a finite p-torsion spectrum, then $\langle D(SZ/p)\rangle = \langle DW\rangle$ by Hopkins and Smith (see [22] or [29, Theorem 7.2.7]), and hence the condition $(\Phi f)_*$: $[SZ/p, \Phi X]_* \cong [SZ/p, \Phi Y]_*$ is equivalent to $(\Phi f)_*$: $[W, \Phi X]_* \cong [W, \Phi Y]_*$. Hence (i) $\Leftrightarrow$ (ii) by Theorem 7.5, and the corollary follows easily since ΦX and ΦY are p-complete. □

A map $f: X \to Y$ in Ho_* will be called a *v_1-periodic equivalence* when it satisfies the conditions of Corollary 7.6. The v_1-periodic equivalences of spaces are very closely related to the K/p_*-equivalences by [10, 34]. In [14, 11.12], we proved

7.7. THEOREM. *If $f: X \to Y$ is a K/p_*-equivalence of H-spaces, then f is a v_1-periodic equivalence.*

To generalize this theorem beyond H-spaces, we say that a space $X \in Ho_*$ is *K/p_*-durable* when its K/p_*-localization map $X \to X_{K/p}$ is a v_1-periodic equivalence. By [14, Theorem 11.11], this is equivalent to saying that the natural map $\pi_i(\Omega X)_{K/p} \to \pi_i\Omega(X_{K/p})$ is an isomorphism for sufficiently large i. Each K/p_*-local space is obviously K/p_*-durable, and using the p-completion of [16] or [5, Section 4] for nilpotent spaces, we have

7.8. COROLLARY. *If X is an H-space, or more generally if X is a pointed nilpotent space whose p-completion $\hat{X}_p$ is an H-space, then X is K/p_*-durable.*

Proof. This follows by Theorem 7.7 since the p-completion map $X \to \hat{X}_p$ is a v_1-periodic equivalence as well as a K/p_*-equivalence, and since $X_{K/p}$ is an H-space. □

Note that the odd spheres are K/p_*-durable since their p-completions are H-spaces. Now, Theorem 7.7 immediately extends to

7.9. THEOREM. *If $f: X \to Y$ is a K/p_*-equivalence of K/p_*-durable spaces, then f is a v_1-periodic equivalence.*

We may now approach the v_1-periodic homotopy groups of a K/p_*-durable space by applying Theorem 7.5 to the spectrum $\Phi X \simeq \Phi(X_{K/p})$. To determine this spectrum using our knowledge of $X_{K/p}$, we shall need

7.10. LEMMA. *For a pointed space X, the natural homomorphism $\Phi: K^*(X;\hat{Z}_p) \to K^*(\Phi X;\hat{Z}_p)$ factors through the indecomposable quotient $\hat{Q}K^*(X;\hat{Z}_p)/\psi^p$.*

Proof. Φ factors through the indecomposables $\hat{Q}K^*(X;\hat{Z}_p)$ since it factors through the suspension homomorphism $K^*(X;\hat{Z}_p) \to K^{*-1}(\Omega X;\hat{Z}_p)$ by 7.2(iv). To show that Φ factors through $K^*(X;\hat{Z}_p)/\psi^p$, it suffices to show that it carries $\psi^p: \Omega^\infty\Sigma^n K\hat{Z}_p \to \Omega^\infty\Sigma^n K\hat{Z}_p$ to a trivial map $\Phi\psi^p: \Sigma^n K\hat{Z}_p \to \Sigma^n K\hat{Z}_p$ for $n = 0, 1$. Thus, by Corollary 6.4.8 of [1], it suffices to show that $(\Phi\psi^p)_* = 0: \pi_*\Sigma^n K\hat{Z}_p \to \pi_*\Sigma^n K\hat{Z}_p$. This follows since $(\Phi\psi^p)_*$ is infinitely divisible by p, which in turn follows since $\Phi\psi^p \simeq p\Sigma^2(\Phi\psi^p)$ because $\Omega^2(\Phi\psi^p) \simeq \Phi(\Omega^2\psi^p) \simeq \Phi(p\psi^p) \simeq p\Phi\psi^p$. □

8. ON THE SPECTRA ΦX

For a K/p_*-durable space X such that $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(M)$ for a regular torsion-free p-adic Adams module M, we now determine the spectrum ΦX and show that it is often a $K\hat{Z}_p^*$-Moore spectrum. We also develop results on the stable homotopy theory of $K\hat{Z}_p^*$-Moore spectra which may be used to determine the v_1-periodic homotopy groups $v_1^{-1}\pi_*(X;W) \cong [W, \Phi X]_*$ and $v_1^{-1}\pi_*X \cong \pi_*\tau_p\Phi X$. This work will be applied in Section 9 to derive more explicit results on the v_1-periodic homotopy groups of finite H-spaces and K/p-homology spheres.

Recall from Theorem 3.4 that for each stable p-adic Adams module G, there exists a homotopically unique K/p_*-local spectrum $\mathscr{M}(G,1)$ with $K^1(\mathscr{M}(G,1);\hat{Z}_p)=G$ and $K^0(\mathscr{M}(G,1));\hat{Z}_p)=0$. Also, recall from Theorem 3.9 that for stable p-adic Adams modules G and G' with G torsion-free, there is a natural isomorphism

$$K\hat{Z}_p^*\colon [\mathscr{M}(G',1),\mathscr{M}(G,1)] \cong \mathrm{Hom}_{\mathscr{A}}(G,G').$$

Thus, a homomorphism $h\colon G\to G'$ induces a map $\mathscr{M}(h,1)\colon \mathscr{M}(G',1)\to\mathscr{M}(G,1)$, and we can state

8.1. THEOREM. *If X is a connected K/p_*-durable space (e.g. a connected H-space) with $K^*(X;\hat{Z}_p)\cong\hat{\Lambda}(M)$ for a regular torsion-free p-adic Adams module $M\subset K^1(X;\hat{Z}_p)$, then ΦX is the homotopy fibre of the map $\mathscr{M}(\psi^p,1)\colon\mathscr{M}(M,1)\to\mathscr{M}(M,1)$. In particular, if ψ^p: $M\to M$ is monic, then $\Phi X\simeq\mathscr{M}(M/\psi^p,1)$.*

Proof. By Theorem 4.8, $X_{K/p}\simeq\widetilde{\mathrm{Fib}}\, f$ for some companion map $f\colon\Omega^\infty\tilde{\mathscr{M}}(M,1)\to\Omega^\infty\tilde{\mathscr{M}}(M,1)$ of M. Thus, since X is K/p_*-durable, ΦX is the homotopy fiber of $\Phi f\colon\Phi\Omega^\infty\tilde{\mathscr{M}}(M,1)\to\Phi\Omega^\infty\tilde{\mathscr{M}}(M,1)$. There is a natural equivalence

$$\Phi\Omega^\infty\tilde{\mathscr{M}}(M,1)\simeq\tilde{\mathscr{M}}(M,1)_{K/p}\simeq\mathscr{M}(M,1)$$

by 7.2(iii), and the homomorphism

$$\Phi\colon K^*(\Omega^\infty\tilde{\mathscr{M}}(M,1);\hat{Z}_p)\to K^*(\mathscr{M}(M,1);\hat{Z}_p)$$

corresponds to the natural retraction

$$\hat{\Lambda}(\tilde{F}M)\to\hat{Q}\hat{\Lambda}(\tilde{F}M)/\psi^p\cong M$$

by Theorem 3.7 and Lemma 7.10. Since $f^*=\hat{\Lambda}(\tilde{F}\psi^p-\psi^p)$ on $\hat{\Lambda}(\tilde{F}M)$, we deduce that $\Phi F\simeq\mathscr{M}(\psi^p,1)$ on $\mathscr{M}(M,1)$ and hence ΦX is the homotopy fiber of $\mathscr{M}(\psi^p,1)$. $\square$

The groups $v_1^{-1}\pi_*(X;W)\cong[W,\mathscr{M}(M/\psi^p,1)]_*$ may be calculated in principle using the $K\hat{Z}_p^*$-Adams spectral sequence (see Theorem 10.4), and the following special case (proved in 10.6) will often suffice.

8.2. THEOREM. *For a stable p-adic Adams module G and a spectrum E with $K^0(E;\hat{Z}_p)=0$, there is a splittable short exact sequence*

$$0\to\mathrm{Ext}^2_{\mathscr{A}}(G,K^1(\Sigma^2E;\hat{Z}_p))\to[E,\mathscr{M}(G,1)]\to\mathrm{Hom}_{\mathscr{A}}(G,K^1(E;\hat{Z}_p))\to 0$$

and an isomorphism

$$[\Sigma E,\mathscr{M}(G,1)]\cong\mathrm{Ext}^1_{\mathscr{A}}(G,K^1(\Sigma^2E;\hat{Z}_p)).$$

We now turn to the problem of calculating the groups $\mathrm{Ext}^s_{\mathscr{A}}(G,N)$. Let $\mathscr{G}$ be the abelian category of p-profinite abelian groups. Since $\mathscr{G}$ is Pontrjagin dual to the category of p-torsion abelian groups, it has enough projectives, which are precisely the torsion-free objects, and each object of $\mathscr{G}$ has projective dimension $\leqslant 1$. The forgetful functor $\mathscr{A}\to\mathscr{G}$ has a left adjoint $V\colon\mathscr{G}\to\mathscr{A}$ which is exact by [7, 3.8 and 6.1]. Hence, $\mathscr{A}$ has enough projectives, and there are natural isomorphisms

$$\mathrm{Ext}^s_{\mathscr{A}}(VH,N)\cong\mathrm{Ext}^s_{\mathscr{G}}(H,N)\cong\mathrm{Ext}^s(N^\#,H^\#)$$

for $H \in \mathscr{G}$, $N \in \mathscr{A}$, and $s \geqslant 0$. More generally, to determine $\mathrm{Ext}^s_{\mathscr{A}}(G, N)$ for $G, N \in \mathscr{A}$, we let r be a fixed integer generating the group of units $(Z/p^2)^\times$, and we use the "fundamental exact sequence"

$$0 \to V(G) \xrightarrow{V\psi^r - \psi^r} V(G) \xrightarrow{\alpha} G \to 0$$

where α is the adjunction counit (see [7, 7.5] or [9, 6.10]). By taking the long exact Ext_A-sequence, we obtain

8.3. THEOREM. *For stable p-adic Adams modules G, $N \in \mathscr{A}$, there is a natural exact sequence*

$$0 \to \mathrm{Hom}_{\mathscr{A}}(G, N) \to \mathrm{Hom}_{\mathscr{G}}(G, N) \xrightarrow{\psi^r_G - \psi^r_N} \mathrm{Hom}_{\mathscr{G}}(G, N)$$

$$\to \mathrm{Ext}^1_{\mathscr{A}}(G, N) \to \mathrm{Ext}^1_{\mathscr{G}}(G, N) \xrightarrow{\psi^r_G - \psi^r_N} \mathrm{Ext}^1_{\mathscr{G}}(G, N)$$

$$\to \mathrm{Ext}^2_{\mathscr{A}}(G, N) \to 0$$

and $\mathrm{Ext}^s_{\mathscr{A}}(G, N) = 0$ *for* $s > 2$.

The groups $\mathrm{Ext}^s_{\mathscr{A}}(G, N)$ are particularly accessible when N is *m-powered* for an integer m, that is, when $\psi^k = k^m$ on N for all $k \in Z - pZ$. For a stable p-adic Adams module G and integer m, we let $W^m G$ denote the largest m-powered quotient module of G. The functor W^m and its first left derived functor W^m_1 are given by

$$W^m G = \mathrm{coker}(\psi^r - r^m)$$
$$W^m_1 G = \ker(\psi^r - r^m)$$

for $\psi^r - r^m : G \to G$, and we have

8.4. THEOREM. *For stable p-adic Adams modules G, $N \in \mathscr{A}$ such that N is m-powered, there are natural isomorphisms*

$$\mathrm{Hom}_{\mathscr{A}}(G, N) \cong \mathrm{Hom}_{\mathscr{G}}(W^m G, N)$$
$$\mathrm{Ext}^2_{\mathscr{A}}(G, N) \cong \mathrm{Ext}^1_{\mathscr{G}}(W^m_1 G, N)$$

and a splittable natural short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathscr{G}}(W^m G, N) \to \mathrm{Ext}^1_{\mathscr{A}}(G, N) \to \mathrm{Hom}_{\mathscr{G}}(W^m_1 G, N) \to 0.$$

Proof. There is a natural isomorphism $Hom_{\mathscr{A}}(X, N) \cong \mathrm{Hom}_{\mathscr{G}}(W^m X, N)$ for $X \in \mathscr{A}$, and the functor W^m carries projectives to projectives since it is left adjoint to an exact functor. Hence, the theorem follows by a universal coefficient or Grothendieck spectral sequence argument. □

Finally, if X is a space with $\Phi X \simeq \mathscr{M}(M/\psi^p, 1)$ as in Theorem 8.1, then the v_1-periodic homotopy groups $v_1^{-1}\pi_* X \cong \pi_*(\tau_p \Phi X)$ may be calculated using

8.5. THEOREM. *For a stable p-adic Adams module G, there are natural isomorphisms $\pi_{2m}(\tau_p \mathscr{M}(G, 1)) \cong (W^m G)^\#$ and $\pi_{2m-1}(\tau_p \mathscr{M}(G, 1)) \cong (W^m_1 G)^\#$ for $m \in Z$.*

This will be proved in 10.7.

9. ON THE v_1-PERIODIC HOMOTOPY GROUPS OF FINITE H-SPACES AND K/p-HOMOLOGY SPHERES

We now apply the results of Section 8 to determine the v_1-periodic homotopy groups of finite H-spaces and K/p-homology spheres. We discuss the example of $SU(n)$ in some detail, recovering the main result of Davis [18]. Since the associated spectra will be of the form $\mathscr{M}(G, 1)$, we start by collecting

9.1. Some properties of the spectra $\mathscr{M}(G, 1)$. Let G be a stable p-adic Adams module with $p^e G = 0$ for some $e \geqslant 1$. Then the spectrum $\mathscr{M}(G, 1)$ is periodic with $\Sigma^{2(p-1)p^{e-1}} \mathscr{M}(G, 1) \simeq \mathscr{M}(G, 1)$ by Theorem 3.4, and $p^e \simeq 0 : \mathscr{M}(G, 1) \to \mathscr{M}(G, 1)$ by Theorem 8.2. Hence, $\tau_p \mathscr{M}(G, 1) \simeq \mathscr{M}(G, 1)$. Moreover, by Theorem 8.5, $\pi_{2m} \mathscr{M}(G, 1) \cong (W^m G)^{\#}$ and $\pi_{2m-1} \mathscr{M}(G, 1) \cong (W_1^m G)^{\#}$ for $m \in Z$, where $W^m G$ and $W_1^m G$ are respectively the cokernel and kernel of $\psi^r - r^m : G \to G$. In particular, if G is finite, then $\pi_{2m} \mathscr{M}(G, 1)$ and $\pi_{2m-1} \mathscr{M}(G, 1)$ are finite p-groups of the same order for each $m \in Z$.

Our main result on the v_1-periodic homotopy groups of finite H-spaces is

9.2. THEOREM. *If X is a 1-connected H-space with $H_*(X;Q)$ associative and with $H_*(X;Z_{(p)})$ finitely generated over $Z_{(p)}$, then $\Phi X \simeq \mathscr{M}(M/\psi^p, 1)$ where $M = PK^1(X;\hat{Z}_p) \cong \hat{Q}K^1(X;\hat{Z}_p)$. Moreover, M/ψ^p is finite, and the v_1-periodic homotopy groups of X are given by $v_1^{-1}\pi_{2m} X \cong [W^m(M/\psi^p)]^{\#}$ and $v_1^{-1}\pi_{2m-1} X \cong [W_1^m(M/\psi^p)]^{\#}$, which are of the same order for each $m \in Z$.*

Proof. By Theorem 6.3, $PK^1(X;\hat{Z}_p)$ is a regular torsion-free p-adic Adams module and $K^*(X;\hat{Z}_p) \cong \hat{\Lambda}(PK^*(X;\hat{Z}_p))$. Hence, Theorem 8.1 applies to X, and it suffices to show that ψ^p is monic on $K^1(X;\hat{Z}_p)$. Since $K_*(X;Z_{(p)})$ is finitely generated and $Z_{(p)}$-free by Lin [25], we have $K^1(X;\hat{Z}_p) \cong \hat{Z}_p \otimes K^1(X;Z_{(p)})$ and $K^1(X;Z_{(p)}) \subset K^1(X;Q)$. Hence, since ψ^p is monic on $K^1(X;Q)$, it is also monic on $K^1(X;\hat{Z}_p)$. □

To illustrate the use of this theorem, we shall recover the main result of [18] on

9.3. The v_1-periodic homotopy groups of $SU(n)$. Applying Theorem 9.2 to $SU(n)$ for $n \geqslant 2$, we see that $\Phi SU(n) \simeq \mathscr{M}(M_n/\psi^p, 1)$ where

$$M_n \cong \hat{Q}K^1(SU(n);\hat{Z}_p) \cong K^1(\Sigma CP^{n-1};\hat{Z}_p) \cong \tilde{K}^0(CP^{n-1};\hat{Z}_p).$$

Since $K^0(CP^{n-1};\hat{Z}_p)$ is the truncated polynomial algebra $\hat{Z}_p[x]/(x^n)$ generated by $x = \xi - 1$ where ξ is the canonical line bundle on CP^{n-1}, we have $M_n = \hat{Z}_p\{x, x^2, \ldots, x^{n-1}\}$ with $\psi^k x = \sum_{i=1}^{n-1} \binom{k}{i} x^i$ and $\psi^k x^m = (\psi^k x)^m$ for $k \in Z$ and $1 \leqslant m \leqslant n-1$. The v_1-periodic homotopy groups of $SU(n)$ are now given algebraically by $v_1^{-1}\pi_{2m} SU(n) \cong [W^m(M_n/\psi^p)]^{\#}$ and $v_1^{-1}\pi_{2m-1} SU(n) \cong [W_1^m(M_n/\psi^p)]^{\#}$ for $m \in Z$. Before describing these groups more explicitly in Theorem 9.10, we shall discuss the structure of M_n/ψ^p as an abelian group.

9.4. PROPOSITION. *The stable p-adic Adams module M_n/ψ^p is of order $p^{\binom{n}{2}}$ with a composition series $Z/p, Z/p^2, \ldots, Z/p^{n-1}$.*

Proof. Using the filtration of M_n by its powers $M_n^m = \hat{Z}_p[x^m, x^{m+1}, \ldots, x^{n-1}]$, we see that $\psi^p = p^m$ on $M_n^m/M_n^{m+1} \cong \hat{Z}_p$ for $1 \leqslant m \leqslant n-1$. □

By the *p-exponent* of an object A in an additive category, we mean the smallest integer $e \geqslant 0$ such that $p^e = 0: A \to A$ (when such an integer exists). Also, by the *exponent of p* in a nonzero integer k, we mean the largest integer $e \geqslant 0$ such that $p^e | k$, and we write $v_p(k) = e$. Our computations suggest

9.5. CONJECTURE. *The p-exponent of M_n/ψ^p is $(n-1) + v_p((n-1)!)$ for $n \geqslant 2$.*

The p-exponent of M_n/ψ^p is of interest since it is also the p-exponent of the spectrum $\Phi SU(n) \simeq \mathscr{M}(M_n/\psi^p, 1)$ and determines its periodicity by 9.1. In this conjecture, $v_p((n-1)!)$ may be evaluated using the following theorem of Legendre (see [30, p. 546]). For an integer $m = \sum_{i \geqslant 0} a_i p^i$ with $0 \leqslant a_i < p$ for each i, let $\alpha(m) = \sum_{i \geqslant 0} a_i$ denote the p-adic weight of m.

9.6. THEOREM. (Legendre). *For $m \geqslant 1$, the exponent of p in $m!$ is given by*

$$v_p(m!) = (m - \alpha(m))/(p-1).$$

We can easily prove a weak version of Conjecture 9.5.

9.7. PROPOSITION. *The p-exponent of M_n/ψ^p (and hence of $\Phi SU(n)$) is at least $n-1$ and at most $2n-3$ for $n \geqslant 2$.*

Proof. By induction on $m \geqslant 1$, we have

$$\psi^p(x^m) = p^m x^m + k_1 p^{m-1} x^{m+1} + k_2 p^{m-2} x^{m+2} + \cdots + k_m x^{2m} + (\text{higher terms})$$

in M_n for integers $k_i \geqslant 0$ depending on m. Thus, by another induction, the elements $p^{n-1}x^{n-1}, p^n x^{n-2}, \ldots, p^{2n-3}x$ are all in the image of $\psi^p: M_n \to M_n$. Hence, the p-exponent of M_n/ψ^p is at most $2n-3$, and it is at least $n-1$ by Proposition 9.4. $\square$

We now proceed to evaluate the groups $v_1^{-1}\pi_{2m} SU(n) \cong [W^m(M_n/\psi^p)]^{\#}$.

9.8. LEMMA. *For $n \geqslant 2$ and $m \in Z$, $W^m(M_n/\psi^p)$ is the p-finite quotient of $\hat{Z}_p$ by the relations $T_p(m,j) = 0$ for all $j \geqslant n$ where*

$$T_p(m,j) = \sum_{\substack{i \geqslant 0 \\ (i,p)=1}} (-1)^{i+j} \binom{j}{i} i^m.$$

Proof. We may obtain $W^m(M_n/\psi^p)$ from $K^0(CP^\infty; \hat{Z}_p)$ by taking its largest p-profinite quotient group with: (i) $\psi^p w = 0$ and $\psi^r w = r^m w$ for each $w \in K^0(CP^\infty; \hat{Z}_p)$; and (ii) $(\xi - 1)^j = 0$ for each $j \geqslant n$. Letting $C_k = \{1, \xi, \xi^2, \ldots, \xi^{k-1}\}$ denote the cyclic group of order k on the generator ξ, we have $K^0(CP^\infty; \hat{Z}_p) \cong \lim_s Z_{p^s} C_{p^s}$. The p-profinite quotient of $K^0(CP^\infty; \hat{Z}_p)$ by the relations in (i) is just $\lim_s Z_{p^s}\{\xi\} \cong \hat{Z}_p\{\xi\} \cong \hat{Z}_p$ since $Z_{p^n}\{\xi\} \cong Z_{p^s}$ is the quotient of $Z_{p^s} C_{p^s}$ by the relations: $\xi^k = 0$ when $p|k$, and $\xi^k = k^m \xi$ when $(k,p) = 1$. Thus $W_m(M_n/\psi^p)$ is the p-finite quotient of $\hat{Z}_p\{\xi\}$ by the relations in (ii). $\square$

The numbers $T_p(m,j)$ have been studied by Lundell [26, p. 41] and are related to the Stirling numbers of the second kind, $S(m,j)$, which satisfy

$$j! S(m,j) = \sum_{i \geqslant 0} (-1)^{i+j} \binom{j}{i} i^m$$

for $m, j \geqslant 1$. In fact, following Davis [18], we may replace Lemma 9.8 by

9.9. LEMMA. *For $n \geqslant 2$ and $m \geqslant n$, $W^m(M_n/\psi^p)$ is the p-finite quotient of $\hat{Z}_p$ by the relations $j!S(m,j) = 0$ for $j = n, n+1, \ldots, m$.*

Proof. Since $S(m,m) = 1$, we have $v_p(m!S(m,m)) = (m - \alpha(m))/(p-1) < m$ by Theorem 9.6. Thus since $j!S(m,j) \equiv T_p(m,j) \bmod p^m$ for all j, $W^m(M_n/\psi^p)$ is the p-finite quotient of $\hat{Z}_p$ by the relations $j!S(m,j) = 0$ for all $j \geqslant n$. The lemma now follows since $S(m,j) = 0$ for $j > m$. □

Combining this lemma with 9.3, we recover the following main result of Davis [18].

9.10. THEOREM. *If $m \geqslant n \geqslant 2$, then the group $v_1^{-1}\pi_{2m}SU(n)$ is cyclic of order p^e where*

$$e = \min\{v_p(j!S(m,j)) \mid n \leqslant j \leqslant m\}$$

and the group $v_1^{-1}\pi_{2m-1}SU(n)$ is of the same order.

Note that this describes all of the groups $v_1^{-1}\pi_* SU(n) \cong \pi_* \tau_p \mathscr{M}(M_n/\psi^p, 1)$ by periodicity (see 9.1). Finally, we determine

9.11. The v_1-periodic homotopy groups of K/p-homology spheres. Let $X \in Ho_*$ be an odd K/p-homology sphere with $\tilde{K}^0(X;\hat{Z}_p) = 0$ and $K^1(X;\hat{Z}_p) = N$ for a spherical p-adic Adams module N (see 5.2), and assume that X is K/p_*-durable (as it is when $X = S^{2n+1}$ or $X = S(N,1)$). Suppose that $N \cong \hat{Z}_p$ has Adams operations $\psi^p = up^n$ and $\psi^r = v$ for $1 \leqslant n < \infty$ and $u, v \in \hat{Z}_p^\times$, where $u = 1$ and $v = r$ when $n = 1$. Now choose an integer m such that $v \equiv r^m \bmod p^n$, and observe that $N/\psi^p \cong Z/p^n$ has Adams operations $\psi^k = k^m$ for each $k \in Z - pZ$. Hence, $\Phi X \simeq \mathscr{M}(N/\psi^p, 1)$ is the K/p_*-localization of the mod p^n Moore spectrum $S^{2m} \cup_{p^n} e^{2m+1}$ by Theorems 8.1 and 3.4, and X has v_1-periodic homotopy groups

$$v_1^{-1}\pi_{2i}X \cong v_1^{-1}\pi_{2i-1}X \cong \begin{cases} Z/p^{\min(n, v_p(a)+1)} & \text{if } i = m + (p-1)a \\ 0 & \text{otherwise} \end{cases}$$

by 9.1, since $v_p(a) + 1 = v_p(r^m - r^i)$. This generalizes the result of Thompson (see [33, 24] or [19]) for the ordinary odd spheres S^{2n+1}.

We obtain very different results when we suppose that the spherical p-adic Adams module N is of class ∞ (see 5.2), so that $N = \hat{Z}_p$ has Adams operations $\psi^p = 0$ and $\psi^r = v$ for a p-adic unit $v \in \hat{Z}_p^\times$. In this case, $\Phi X \simeq \mathscr{M}(N,1) \vee \mathscr{M}(N,0)$ by Theorem 8.1. If $v = r^m$ for some integer m, then N has Adams operations $\psi^k = k^m$ for each $k \in Z - pZ$, and $\mathscr{M}(N,1)$ is the K/p_*-localization of the sphere spectrum S^{2m+1}. Moreover,

$$v_1^{-1}\pi_{2i}X \cong \begin{cases} Z/p^{v_p(a)+1} & \text{if } i = m + (p-1)a \text{ for } a \neq 0 \\ Z_{p^\infty} & \text{if } i = m \text{ or } m-1 \\ 0 & \text{otherwise} \end{cases}$$

$$v_1^{-1}\pi_{2i-1}X \cong \begin{cases} Z/p^{v_p(a)+1} & \text{if } i = m + (p-1)a \text{ for } a \neq 0 \\ Z_{p^\infty} \oplus Z_{p^\infty} & \text{if } i = m \\ 0 & \text{otherwise} \end{cases}$$

by Theorems 7.5 and 8.5. Similarly, if v is not an integral power of r, then

$$v_1^{-1}\pi_{2i}X \cong v_1^{-1}\pi_{2i-1}X \cong Z/p^{v_p(v - r^i)}$$

for each integer i.

10. ON $K\hat{Z}_p^*$-MOORE SPECTRA AND THE $K\hat{Z}_p^*$-ADAMS SPECTRAL SEQUENCE

In [7, 9], we obtained detailed results on $KZ_{(p)_*}$-Moore spectra and the $KZ_{(p)_*}$-Adams spectral sequence. We now use that work to derive some previously claimed results (Theorems 3.4, 3.9, 8.2, and 8.5) on $K\hat{Z}_p^*$-Moore spectra and the $K\hat{Z}_p^*$-Adams spectral sequence. In preparation, we show that the $K\hat{Z}_p^*$-cohomologies of p-complete spectra correspond to the $KZ_{(p)_*}$-homologies of p-torsion spectra.

Let $Z_{(p)}^\times$ be the group of units in the p-local integers. For each $k \in Z_{(p)}^\times$, there is a unique map of spectra $\psi^k : K_{(p)} \to K_{(p)}$ with $\psi^k = k^n : \pi_{2n} K_{(p)} \cong \pi_{2n} K_{(p)}$ for each $n \in Z$ as in [7, Section 2]. This induces the Adams operation ψ^k in the associated homology and cohomology theories, and we have

10.1. Proposition. *For a spectrum X, there is a natural isomorphism*

$$K^*(X; \hat{Z}_p) \cong K_{*-1}(\tau_p X; Z_{(p)})^\#$$

such that ψ^k corresponds to $(\psi^{1/k})^\#$ for each $k \in Z_{(p)}^\times$.

Proof. Using Corollary 2.3 and the fiber sequence $KZ_{(p)} \to KQ \to KZ_{p^\infty}$, we obtain natural isomorphisms

$$K^*(X; \hat{Z}_p) \cong K_*(X; Z_{p^\infty})^\# \cong K_*(\tau_p X; Z_{p^\infty})^\# \cong K_{*-1}(\tau_p X; Z_{(p)})^\#$$

and we see that ψ^k corresponds to $(\psi^{1/k})^\#$ because the map $\psi^k : K\hat{Z}_p \to K\hat{Z}_p$ corresponds to $c(\psi^{1/k}) : c(KZ_{p^\infty}) \to c(KZ_{p^\infty})$ by Proposition 2.2 and [7, Section 2]. □

10.2. Pontrjagin duality for stable Adams modules. By a *stable p-torsion Adams module*, we mean a direct limit of a directed system of finite stable p-adic Adams modules (see 2.6), or equivalently we mean a p-torsion object in the category $\mathscr{A}(p)$ of [7, Section 1]. For a stable p-adic Adams module G, the Pontrjagin dual $G^\#$ is now a stable p-torsion Adams module equipped with the operations $\psi^k = (\psi^{1/k})^\#$ for $k \in Z_{(p)}^\times$. Moreover, the Pontrjagin duality functor now gives a contravariant equivalence between the category $\mathscr{A}$ of stable p-adic Adams modules and the category $\mathscr{A}^\#$ of stable p-torsion Adams modules. Propositions 7.4 and 10.1 combine to show that a p-complete spectrum X corresponds to a p-torsion spectrum $\tau_p X$ such that, $K_{*-1}(\tau_p X; Z_{(p)}) \in \mathscr{A}^\#$ is Pontrjagin dual to $K^*(X; \hat{Z}_p) \in \mathscr{A}$.

10.3. Proof of Theorem 3.4. By [7, 3.8 and 8.7], for a stable p-adic Adams module $G \in \mathscr{A}$, there exists a p-torsion K_*-local spectrum Y with $K_0(Y; Z_{(p)}) \cong G^\# \in \mathscr{A}^\#$ and $K_1(Y; Z_{(p)}) = 0$, and this spectrum is unique up to equivalence. Using the above correspondence, we now obtain Theorem 3.4 by taking $\mathscr{M}(G, 1) = \hat{Y}_p$. □

We similarly obtain a $K\hat{Z}_p^*$-Adams spectral sequence. Let $\mathscr{A}$ now denote the category of $Z/2$-graded stable p-adic Adams modules and note that Ext_A^s is trivial for $s > 2$ by Theorem 8.3.

10.4. Theorem. *For spectra X and Y, there is a natural spectral sequence $\{E_r^{s,t}(X, Y)\}$ converging strongly to $[X, Y_{K/p}]_{t-s}$ with*

$$d_r : E_r^{s,t}(X, Y) \to E_r^{s+r, t+r-1}(X, Y)$$

$$E_2^{s,t}(X, Y) = \mathrm{Ext}_{\mathscr{A}}^s(K^*(Y; \hat{Z}_p), K^*(\Sigma^t X; \hat{Z}_p))$$

$$E_3^{s,t}(X, Y) = E_\infty^{s,t}(X, Y) = (F^s/F^{s+1})[X, Y_{K/p}]_{t-s}$$

$$[X, Y_{K/p}]_* = F^0[X, Y_{K/p}]_* \supset \cdots \supset F^3[X, Y_{K/p}]_* = 0.$$

Proof. This follows by letting $\{E_r^{s,t}(X, Y)\}$ be the $KZ_{(p)_*}$-Adams spectral sequence of [7, Section 8] for $\tau_p X$ and $\tau_p Y$, and by using 10.2 to obtain the present statement. □

10.5. *Proof of Theorem* 3.9. If $K^*(Y;\hat{Z}_p)$ is torsion-free, then it has projective dimension $\leqslant 1$ in $\mathscr{A}$ by Theorem 8.3 since it is projective in $\mathscr{G}$. Hence, in Theorem 10.4, we have $E_2^{s,t}(X, Y) = 0$ for $s > 1$, and the spectral sequence collapses to the form given in Theorem 3.9. □

10.6. *Proof of Theorem* 8.2. If E is a spectrum with $K^0(E;\hat{Z}_p) = 0$ and $K^1(E;\hat{Z}_p) = H \in \mathscr{A}$, then $E_{K/p} \simeq \mathscr{M}(H, 1)$ by Theorem 3.4. Hence, by [7, Section 9], the $K\hat{Z}_p^*$-Adams spectral sequence for $[E, \mathscr{M}(G, 1)]_* \cong [\mathscr{M}(H, 1), \mathscr{M}(G, 1)]_*$ collapses to the form given in Theorem 8.2. □

10.7. *Proof of Theorem* 8.5. Since $\tau_p\mathscr{M}(G, 1)$ is a p-torsion K_*-local spectrum with $K_0(\tau_p\mathscr{M}(G, 1); Z_{(p)}) \cong G^\#$ and $K_1(\tau_p\mathscr{M}(G, 1); Z_{(p)}) = 0$, the $KZ_{(p)_*}$-Adams spectral sequence of [7] for $\pi_*\tau_p\mathscr{M}(G, 1)$ collapses to give Theorem 8.5. □

11. PROOF OF THEOREM 4.7

For a regular torsion-free p-adic Adams module M and a companion map $f:\Omega^\infty\tilde{\mathscr{M}}(M, 1) \to \Omega^\infty\tilde{\mathscr{M}}(M, 1)$, we must establish an isomorphism of $Z/2$-graded p-adic λ-rings $K^*(\widetilde{\mathrm{Fib}}\, f;\hat{Z}_p) \cong \hat{\Lambda}(M)$. We shall first determine the p-adic K-cohomology of $\Omega\Omega^\infty\tilde{\mathscr{M}}(M, 1)$ using

11.1. A free p-adic λ-ring functor. Following [11, 13], we say that a (degree 0 or ungraded) p-adic λ-ring R is *linear* when $xy = 0$, $\theta^p x = x$, and $\psi^k x = kx$ for each $x, y \in \tilde{R}$ and $k \in Z$. A p-adic λ-ring A has a universal linear quotient $A/\hat{\Gamma}^2\tilde{A}$, and

$$\tilde{K}^0(X;\hat{Z}_p)/\hat{\Gamma}^2\tilde{K}^0(X;\hat{Z}_p) \cong H^2(X;\hat{Z}_p)$$

for each connected space X by [11, 5.4]. Recall that a (degree 0 or ungraded) p-adic Adams module H is called *linear* when $\psi^k x = kx$ for each $k \in Z$. By an *augmented p-adic Adams module* $M{\downarrow}H$, we mean a p-adic Adams module M with a given map to a linear p-adic Adams module H. As in [13, 3.5], there is a *free p-adic λ-ring functor* $U:\bar{\mathscr{U}} \to \mathscr{K}$ from the category $\bar{\mathscr{U}}$ of augmented p-adic Adams modules to the category $\mathscr{K}$ of p-adic λ-rings, where U is left adjoint to the forgetful functor sending $A \in \mathscr{K}$ to $\tilde{A}{\downarrow}(\tilde{A}/\hat{\Gamma}^2\tilde{A}) \in \bar{\mathscr{U}}$. For a 1-connected space X, there is a natural suspension homomorphism of p-adic λ-rings

$$\sigma: U(\hat{Q}K^1(X;\hat{Z}_p){\downarrow}H^3(X;\hat{Z}_p)) \to K^0(\Omega X;\hat{Z}_p),$$

and Theorems 10.2 and 10.5 of [13] show

11.2. THEOREM. *If X is a 1-connected H-space with $PK_0(X;Z/p) = 0$, then $K^1(\Omega X;\hat{Z}_p) = 0$ and $K^0(\Omega X;\hat{Z}_p)$ is torsion-free with*

$$\sigma: U(\hat{Q}K^1(X;\hat{Z}_p){\downarrow}H^3(X;\hat{Z}_p)) \cong K^0(\Omega X;\hat{Z}_p).$$

By Theorem 3.7, this applies to $X = \Omega^\infty \tilde{\mathscr{M}}(G, 1)$ for a torsion-free stable p-adic Adams module G. For a 1-connected space X, the natural augmentation map $\hat{Q}K^1(X; \hat{Z}_p) \to H^3(X; \hat{Z}_p)$ induces a map

$$\alpha : \mathrm{Lin}(\hat{Q}K^1(X; \hat{Z}_p)) \to H^3(X; \hat{Z}_p)$$

where Lin is the linearization functor for p-adic Adams modules (see 4.4).

11.3. Proposition. *If $X = \Omega^\infty \tilde{\mathscr{M}}(G, 1)$ for a torsion-free stable p-adic Adams module G, then α: $\mathrm{Lin}(\hat{Q}K^1(X; \hat{Z}_p)) \cong H^3(X; \hat{Z}_p)$.*

Proof. For the spectrum $E = \tilde{\mathscr{M}}(G, 1)$, there is a suspension isomorphism

$$W^1K^1(E; \hat{Z}_p) \cong W^1G \cong \mathrm{Lin}(\tilde{F}G) \cong \mathrm{Lin}(\hat{Q}K^1(X; \hat{Z}_p))$$

by Theorem 3.7, and there is also a suspension isomorphism $H^3(E; \hat{Z}_p) \cong H^3(X; \hat{Z}_p)$. Thus, it suffices to show that the stable augmentation map $K^1(E; \hat{Z}_p) \to H^3(E; \hat{Z}_p)$ induces an isomorphism $W^1K^1(E; \hat{Z}_p) \cong H^3(E; \hat{Z}_p)$. Hence, by Proposition 10.1 and Lemma 11.4 below, it suffices to show that the Hurewicz homomorphism $\pi_2(\tau_p E) \to K_0(\tau_p E; Z_{(p)})$ induces an isomorphism from $\pi_2(\tau_p E)$ to the kernel of $\psi^r - r : K_0(\tau_p E; Z_{(p)}) \to K_0(\tau_p E; Z_{(p)})$. This follows using the $KZ_{(p)_*}$-Adams spectral sequence [7] for $\pi_2(\tau_p E) \cong \pi_2(\tau_p \mathscr{M}(G, 1))$. $\square$

We have used

11.4. Lemma. *If X is a 1-connected space or spectrum whose p-torsion part $\tau_p X$ (the homotopy fiber of $X \to X[1/p]$) is also 1-connected, then there is a natural isomorphism $H^3(X; \hat{Z}_p) \cong (\pi_2(\tau_p X))^\#$.*

Proof. This follows since there is a natural isomorphism $H^3(X; \hat{Z}_p) \cong H_2(\tau_p X; Z_{(p)})^\#$ obtained using the equivalence $c(HZ_{p^\infty}) \simeq H\hat{Z}_p$ as in Corollary 2.3 and Proposition 10.1. $\square$

For a torsion-free p-adic Adams module M and a companion map f: $\Omega^\infty \tilde{\mathscr{M}}(M, 1) \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$, the adjusted fiber $\widetilde{\mathrm{Fib}}\, f$ of 4.6 belongs to a ladder of p-complete fiber sequences

$$\begin{tikzcd}
\widetilde{\mathrm{Fib}}\, f \arrow[r] \arrow[d] & X \arrow[r, "\tilde{f}"] \arrow[d] & Y \arrow[d] \\
\mathrm{Fib}\, f \arrow[r] & \Omega^\infty \tilde{\mathscr{M}}(M,1) \arrow[r, "f"] & \Omega^\infty \tilde{\mathscr{M}}(M,1)
\end{tikzcd}$$

such that: $\tau_p \widetilde{\mathrm{Fib}}\, f$ is the 1-connected cover of $\tau_p \mathrm{Fib} f$; $\tau_p Y$ is the 2-connected cover of $\tau_p \Omega^\infty \tilde{\mathscr{M}}(M, 1)$; and $\tau_p X$ is 1-connected with $\pi_i(\tau_p X) \cong \pi_i(\tau_p \Omega^\infty \tilde{\mathscr{M}}(M, 1))$ for $i > 2$ and

$$\pi_2(\tau_p X) = \ker(f_* : \pi_2(\tau_p \Omega^\infty \tilde{\mathscr{M}}(M, 1)) \to \pi_2(\tau_p \Omega^\infty \tilde{\mathscr{M}}(M, 1))).$$

Let

$$0 \to (\tilde{F}M \downarrow 0) \xrightarrow{\partial} (\tilde{F}M \downarrow \mathrm{Lin}\, M) \xrightarrow{\alpha} (M \downarrow \mathrm{Lin}\, M) \to 0$$

be the short exact sequence of augmented p-adic Adams modules induced by the short exact sequence of p-adic Adams modules $0 \to \tilde{F}M \xrightarrow{\partial} \tilde{F}M \xrightarrow{\alpha} M \to 0$ in Lemma 4.1.

11.5. PROPOSITION. *The map of p-adic λ-rings* $\Omega\tilde{f}^*: K^0(\Omega Y; \hat{Z}_p) \to K^0(\Omega X; \hat{Z}_p)$ *is equivalent to* $U(\partial): U(\tilde{F}M{\downarrow}0) \to U(\tilde{F}M{\downarrow}\mathrm{Lin}\, M)$. *Moreover,* $K^0(\Omega X; \hat{Z}_p)$ *and* $K^0(\Omega Y; \hat{Z}_p)$ *are torsion-free, while* $K^1(\Omega X; \hat{Z}_p) = 0 = K^1(\Omega Y; \hat{Z}_p)$.

Proof. Theorem 11.2 applies to X and Y since the maps $X \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ and $Y \to \Omega^\infty \tilde{\mathscr{M}}(M, 1)$ are K/p_*-equivalences of infinite loop spaces by [6]. Thus, since $\tilde{f}^*$: $\hat{Q}K^1(Y; \hat{Z}_p) \to \hat{Q}K^1(X; \hat{Z}_p)$ is equivalent to $\partial: \tilde{F}M \to \tilde{F}M$, it suffices to show that $\tilde{f}^*: H^3(Y; \hat{Z}_p) \to H^3(X; \hat{Z}_p)$ is equivalent to 0: $0 \to \mathrm{Lin}\, M$. This follows by Proposition 11.3 and Lemma 11.4 since the short exact sequence $0 \to \tilde{F}M \xrightarrow{\partial} \tilde{F}M \xrightarrow{\alpha} M \to 0$ induces a right exact sequence $\mathrm{Lin}(\tilde{F}M) \to \mathrm{Lin}(\tilde{F}M) \to \mathrm{Lin}(M) \to 0$. □

11.6. PROPOSITION. *If M is a regular torsion-free p-adic Adams module, then* $\Omega\tilde{f}_*: K_0(\Omega X; Z/p) \to K_0(\Omega Y; Z/p)$ *is onto. Moreover, the Frobenius is monic in both* $K_0(\Omega X; Z/p)$ *and* $K_0(\Omega Y; Z/p)$, *while* $K_1(\Omega X; Z/p) = 0 = K_1(\Omega Y; Z/p)$.

Proof. Following [13, 5.1], for an augmented p-adic Adams module $N{\downarrow}H$, let $U(N{\downarrow}H)^\#_p$ be the Pontrjagin dual of $U(N{\downarrow}H)/p$, and recall that $U(N{\downarrow}H)^\#_p$ belongs to the abelian category $\mathscr{H}(p)^{ev}$ of bicommutative irreducible Z/p-Hopf algebras. Since M is regular and torsion-free, $M{\downarrow}\mathrm{Lin}\, M$ is properly torsion-free in the sense of [13, 4.5], and

$$Z/p \to U(M{\downarrow}\mathrm{Lin}\, M)^\#_p \xrightarrow{\alpha^*} U(\tilde{F}M{\downarrow}\mathrm{Lin}\, M)^\#_p \xrightarrow{\partial^*} U(\tilde{F}M{\downarrow}0)^\#_p \to Z/p$$

is a short exact sequence in $\mathscr{H}(p)^{ev}$ by [13, 6.10]. Moreover, the Frobenius is monic in each of these three objects by [13, 8.8]. The stated results now follow since ∂^* is equivalent to $\Omega\tilde{f}_*: K_0(\Omega X; Z/p) \to K_0(\Omega Y; Z/p)$ by Proposition 11.5. □

11.7. PROPOSITION. *If M is a regular torsion-free p-adic Adams module, then the map* $K_*(\widetilde{\mathrm{Fib}}\, f; Z/p) \to K_*(X; Z/p)$ *is an injection onto the kernel of* $\tilde{f}_*: K_*(X; Z/p) \to K_*(Y; Z/k)$ *in the category of* $Z/2$*-graded augmented cocommutative* Z/p*-coalgebras.*

Proof. For the principal fibration $\Omega X \to \Omega Y \to \widetilde{\mathrm{Fib}}\, f$, we consider the K/p_*-bar (or K/p_*-Eilenberg–Moore) spectral sequence of graded coalgebras abutting to $K_*(\widetilde{\mathrm{Fib}}\, f; Z/p)$ with

$$E^2_s \approx \mathrm{Tor}_s^{K_*(\Omega X; Z/p)}(K_*(\Omega Y; Z/p), Z/p)$$

as in [6]. This maps to the K/p_*-bar spectral sequence of graded Hopf algebras abutting to $K_*(X; Z/p)$ with $E^2_s = \mathrm{Tor}_s^{K_*(\Omega Y/Z/p)}(Z/p, Z/p)$, and this in turn maps to the K/p_*-bar spectral sequence of graded Hopf algebras abutting to $K_*(Y; Z/p)$ with $E^2_s = \mathrm{Tor}_s^{K_*(\Omega Y; Z/p)}(Z/p, Z/p)$. Let $A \in \mathscr{H}(p)^{ev}$ denote the kernel of the epimorphism $\Omega\tilde{f}_*: K_0(\Omega X; Z/p) \to K_0(\Omega Y; Z/p)$ in $\mathscr{H}(p)^{ev}$ (see Proposition 11.6). Then the sequence of indecomposables

$$0 \to QA \to QK_0(\Omega X; Z/p) \to QK_0(\Omega Y; Z/p) \to 0$$

is short exact, while the corresponding sequence of derived indecomposables is trivial by [13, B.5]. Thus, there are natural isomorphisms of E^2-terms

$$\mathrm{Tor}_*^{K_*(\Omega X;Z/p)}((K_*(\Omega Y;Z/p),Z/p) \cong \Lambda(QA)$$
$$\mathrm{Tor}_*^{K_*(\Omega X;Z/p)}(Z/p,Z/p) \cong \Lambda(QK_0(\Omega X;Z/p))$$
$$\mathrm{Tor}_*^{K_*(\Omega X;Z/p)}(Z/p,Z/p) \cong \Lambda(QK_0(\Omega Y;Z/p))$$

by [6, 4.6, 13, 8.9], and our three spectral sequences must all collapse with $E^2 = E^\infty$, since the second and third are generated by infinite cycles, while the first injects into the second. Now, the map $K_*(\widetilde{\mathrm{Fib}\, f}; Z/p) \to K_*(X;Z/p)$ is an injection onto the coalgebraic kernel of $\tilde{f}_*: K_*(X;Z/p) \to K_*(Y;Z/p)$, since the associated graded map $\Lambda(QA) \to \Lambda(QK_0(\Omega X;Z/p))$ is an injection onto the coalgebraic kernel of $\Lambda(QK_0(\Omega X;Z/p)) \to \Lambda(QK_0(\Omega Y;Z/p))$. □

11.8. *Proof of Theorem* 4.7. Since the map $\tilde{f}^*: K^*(Y;\hat{Z}_p) \to K^*(X;\hat{Z}_p)$ is equivalent to $\hat{\Lambda}(\partial)$: $\hat{\Lambda}(\tilde{F}M) \to \hat{\Lambda}(\tilde{F}M)$, it has a cokernel $\hat{\Lambda}(M)$ in the category of $Z/2$-graded p-adic λ-rings. Hence, there is a canonical homomorphism of $Z/2$-graded p-adic λ-rings $u: \hat{\Lambda}(M) \to K^*(\widetilde{\mathrm{Fib}\, f}; \hat{Z}_p)$. Since $K_*(\widetilde{\mathrm{Fib}\, f}; Z/p)$ maps injectively to $K_*(X;Z/p)$, it has trivial Bockstein operations, and hence $K^*(\widetilde{\mathrm{Fib}\, f}; \hat{Z}_p)$ is torsion-free (like $\hat{\Lambda}(M)$. Thus, to show that u is an isomorphism, it suffices to show that $u_p^\#$: $K^*(\widetilde{\mathrm{Fib}\, f}; \hat{Z}_p)_p^\# \to \hat{\Lambda}(M)_p^\#$ is an isomorphism. This follows since $K^*(\widetilde{\mathrm{Fib}\, f}; \hat{Z}_p)_p^\# \cong K_*(\widetilde{\mathrm{Fib}\, f}; Z/p)$ and since both $K_*(\widetilde{\mathrm{Fib}\, f}; Z/p)$ and $\hat{\Lambda}(M)_p^\#$ represent the coalgebraic kernel of $K_*(X;Z/p) \to K_*(Y;Z/p)$ by Proposition 11.7. □

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