Lecture 8 Essential Example: the K3 Surface

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The purpose of this lecture is to introduce the K3 surface through different viepoints and dicuss some properties of the K3 surface.

1 Construction of the Kummer Surface

1.1 Kummer's construction

Consider the 4-torus $\mathbb{T}^4 = S^1 \times S^1 \times S^1 \times S^1$. By considering each S^1 as the unit disc in the complex plane, we can define an involution[1]

$$\sigma : \mathbb{T}^4 \to \mathbb{T}^4, \ \ (x_1, x_2, x_3, x_4) \to (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4),$$

which has $16 = 2^4$ fixed points. Then, the natural quotient map

$$\pi : \mathbb{T}^4 \to \mathbb{T}^4/\sigma$$

sends these fixed points to 16 singular points, which make $\mathbb{T}^4/\sigma$ fail to ba a manifold. Hence, a method of desingularization is inevitable. Kummer's methods consists of following main steps:

(1) Small neighborhoods of singular points in $\mathbb{T}^4/\sigma$ are the cone $\mathcal{C}_{\mathbb{RP}^3}$ of $\mathbb{RP}^3$;

(2) Considering the unit disc subbundle $\mathbb{D}T^*_{S^2}$ of the 2-sphere's complex cotangent bundle $T^*_{S^2}$, which is also bounded by $\mathbb{RP}^3$;

(3) Replacing $\mathcal{C}_{\mathbb{RP}^3}$ by $\mathbb{D}T^*_{S^2}$.

[1] Involution means a function which is the inverse of itself.

The result of above precedures is the Kummer surface. We now show detail of this construction.

Lying in a 4-manifold, a small neighborhood of a fixed point in $\mathbb{T}^4$ is a 4-dick $\mathbb{D}^4$ which can also be regarded as the a cone $\mathcal{C}_{S^3}$. Then, some a small neighborhood of a singular point in $\mathbb{T}/\alpha$ is a cone $\mathcal{C}_{\mathbb{RP}^3}$, which has boundary $\mathbb{RP}^3$. One can visulize $\mathcal{C}_{\mathbb{RP}^3} = \pi(\mathcal{C}_{S^3})$ in the following way: condisering the Hopf bundle $S^3 \to S^2$, which factors through to $\mathbb{RP}^3 \to S^2$ with fiber $\mathbb{RP}^1$.

$$\begin{tikzcd} S^1 \arrow[r] \arrow[d] & S^3 \arrow[r] \arrow[d] & S^2 \arrow[d, equal] \\ \mathbb{RP}^1 \arrow[r] & \mathbb{RP}^3 \arrow[r] & S^2 \end{tikzcd}$$

Since $\mathbb{RP}^1 \cong S^1$, $\mathbb{RP}^3$ can be considered as an S^1-bundle over S^2. Thus, one can visulize $\mathcal{C}_{\mathbb{RP}^3}$ by attaching a disc to each circle-fiber of $\mathbb{RP}^3$, and then identifying all their centers. In this case, previous singular points are just the identified centers.

Similarly, one can visulize the unit disc bundle $\mathbb{D}T^*_{S^2}$ by attaching discs to circle-fibers without identifying centers. This is because $\mathbb{D}T^*_{S^2}$ can just be considered as the disc-bundle over S^2. Since $\partial(\mathbb{D}T^*_{S^2}) = \partial\mathcal{C}_{\mathbb{RP}^3} = \mathbb{RP}^3$, we are able to replace the cone by a disc bundle. As a byproduct, singular points are replaced by some 2-spheres.

Furthermore, through some calculation, once can show that the Euler characteristic of the complex cotangent bundle $\chi(T^*_{S^2}) = -2$. Whence, the induced surface has self-intersection -2. Therefore, above singularization change singular points $\{p_1, \ldots, p_{16}\}$ to some 2-spheres $\{S_1, \ldots, S_{16}\}$ with self-intersection -2.

Definition 1.1. *With notations above, the 4-manifold*

$$X = (\mathbb{T}^4/\sigma - \cup_{i=1}^{16}\mathcal{C}_{\mathbb{RP}^3}) \cup_{16\mathbb{RP}^3} (\bigcup_{i=1}^{16} \mathbb{D}T^*_{S^2})$$

is called the ***Kummer surface****.*

1.2 Holomorphic construction

In fact, the method of desingularization in Kummer's construction is valid because our geometric structure is not that complicated. There is also a canonical approach to deal with singularities.

Theorem-Definition 1.2. *Let X be a complex surface and p be any of its points, then there exists another surface $\widetilde{X}$, containing a complex curve E of* *genus zero and self-intersection $E \cdot E = -1$, together with a map $\pi : \widetilde{X} \to X$ such that $\sigma(E) = p$ and π induces an isomorphism from $\widetilde{X} - E$ to $X - P$.*

*In this case, $\widetilde{X}$ is called the **blow-up** of X in p, E is called the **exceptional curve**, π is called a **monoidal transformation** (or a σ-**process**).*

Example 1.3. *Consider the blow-up of $\mathbb{C}^2$ in the origin. Recall the line bundle $\mathcal{O}(-1) = \{(l, z) \in \mathbb{P}^1 \times \mathbb{C}^2 \mid z \in l\}$. Denote $\mathbb{CP}^1$ simply as $\mathbb{P}^1$. Consider the following commutative diagram*

$$\begin{tikzcd}
\mathbb{P}^1 \arrow[r, hook] \arrow[d] & \mathcal{O}(-1) \arrow[r, hook] \arrow[d, "\pi"'] & \mathbb{P}^1 \times \mathbb{C}^2 \arrow[r] \arrow[d] & \mathbb{P}^1 \\
\{0\} \arrow[r, hook] & \mathbb{C}^2 \arrow[r, equal] & \mathbb{C}^2 &
\end{tikzcd}$$

The fiber of the projection $p : \mathcal{O}(-1) \to \mathbb{P}^1$ over a line $l \in \mathbb{P}^1$ is isomorphic to l itself. Consider the fiber of another projection $\pi : \mathcal{O}(-1) \to \mathbb{C}^2$. For nonzero $z \in \mathbb{C}^2$, $\pi^{-1}(z) = (l_z, z)$, where $z \in l_z$. For $z = 0$, $\pi^{-1}(0) = \mathbb{P}^1 \times \{0\}$ since all lines in $\mathbb{C}^2$ contain the origin. It is easy to verify that the blow-up of $\mathbb{C}^2$ in $\{0\}$ is just the line bundle $\mathcal{O}(-1)$ together with the natural projection $\pi : \mathcal{O}(-1) \to \mathbb{C}^2$.

This example illustrates that the blow-up in a point can be considered as replacing a small affine neighborhood of that point to $\mathcal{O}(-1)$, and that point p becomes the exceptional curve $\mathbb{P}^1 \times \{p\}$.

Consider the 4-torus as a complex torus, i.e. $\mathbb{T}^4 = (\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}) \bigoplus (\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z})$. Then the previous involution becomes

$$\sigma : \mathbb{T}^4 \to \mathbb{T}^4, \ \ (z_1, z_2) \mapsto (-z_1, -z_2).$$

Denote $\widetilde{\mathbb{T}}^4$ as the blow-up of $\mathbb{T}^4$ in 16 fixed points. Lying in a complex surface, one may choose affine neighborhoods of each singular point. According to 1.2 and the paricular case when $n = 1$ of 1.3, singularities are blowed up to the closure of some exceptional curves $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$.[2] Then involution σ then induces a involution $\widetilde{\sigma}$ of $\widetilde{\mathbb{T}}^4$ given by

$$\widetilde{\sigma} : \mathcal{O}(-1) \to \mathcal{O}(-1), \ \ (x, y) \times (u, v) \mapsto (x, y) \times (-u, -v).$$

Now, the quotient $\widetilde{\mathbb{T}}^4/\widetilde{\sigma}$ is smooth, and is exactly Kummer surface.

Definition 1.4. *With notations above, the complex surface $\widetilde{\mathbb{T}}^4/\widetilde{\sigma}$ is called Kummer surface.*

[2]The reason why it is $\overline{\mathbb{CP}^2}$ instead of $\mathbb{CP}^2$ can be found in ([6], p.299)

Intuitively, our two definitions of Kummer surfaces are compatible: The image of exceptional curves corresponds to spheres with self-intersection -2.

Remark 1.5. *Notice that the intersection number of two constructions are different: The first one means the intersetion form, while the other one is the intersection number of complex divisers. However, they are in fact compatible. Detail can be found in ([1], p.83).*

Remark 1.6. *Another viewpoint of Kummer surface is to consider it as the minimal resolution of $\mathbb{T}^4/\sigma$, and hence we have the following commutative diagram.*

$$\begin{tikzcd}
\widetilde{\mathbb{T}}^4 \arrow[r, "blow-up"] \arrow[d, "\widetilde{\pi}"'] & \mathbb{T}^4 \arrow[d, "\pi"] \\
X \arrow[r, "minimal", "resolution"'] & \mathbb{T}^4/\sigma
\end{tikzcd}$$

2 Properties of K3 Surface

Proposition 2.1. *Let X be a Kummer surface, then*

(a) X is simply connected;

(b) $\mathcal{K}_X = \mathcal{O}_X$;

(c) $H^1(X, \mathcal{O}_X) = 1$.

Definition 2.2. *A* ***K3 surface*** *is a compact connected complex manifold X of dimension two such that its canonical bundle is trivial and $H^1(X, \mathcal{O}_X) = 0$.*

Corrolary 2.3. *Kummer surface is a K3 surface.*

Theorem 2.4. *Any two complex K3 surfaces are diffeomorphic.*

Proposition 2.5. *All nonsingular curves on a K3 surface have even intersection number.*

Proof. This is straight from the Riemann-Roch theorem on surface

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D.(K - D) + 1 - p_a = \frac{1}{2}(D)^2 + 1 - p_a.$$

□

Remark 2.6. *This gives the reason why Kummer surface is in actually the minimal resolution of* $\mathbb{T}^4/\sigma$. *Since otherwise, the resolution has to contract some exceptional curves according to the defition of minimal resolution, which is impossible.*

Theorem 2.7 (Hirzebruch-Riemann-Roch)**.** *Let* $\mathcal{E}$ *be a vector bundle of rank* r *on a smooth projective variety* X *of dimension* n*, then*

$$\chi(X, \mathcal{E}) = deg(ch(\mathcal{E}).td(\mathcal{T}))_n.$$

where

$$ch(\mathcal{E}) = r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \dots,$$

and

$$td(\mathcal{E}) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \dots.$$

$(\ )_n$ *denotes the component of degree* n *in the Chow ring* $A(X) \otimes \mathbb{Q}$.

Corrolary 2.8. *In particular, when* $\mathcal{E} = \mathcal{O}_X$, $c_i(\mathcal{E}) = 0$ *for all* $i > 0$. *If furthermore* X *has dimension 2, the formula is reduced to*

$$\chi(X, \mathcal{O}_X) = \frac{1}{12}(c_1^2(X) + c_2(X)).$$

This formula is called the ***Noether formula****.*

Proposition 2.9. *Let* X *be a K3 surface. Then*

(a) $H_2(X; Z) = 22$;

(b) $Q_X = (-E_8)^{\oplus 2} \oplus H^{\oplus 3}$.

Proof. We only prove (a), while the proof of (b) uses some knowledge of lattice and can be found in ([4], p.304).

By Serre duality theorem

$$h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{K}_X) = h^0(X, \mathcal{O}_X) = 1.$$

As $h^2(X, \mathcal{O}_X) = 1$, the Euler characteristic of trivial bundle is

$$\chi(X, \mathcal{O}_X) = \sum_{i=0}^{2} h^i(X, \mathcal{O}_X) = 2.$$

Because $c_1(X) := c_1(\mathcal{T}_X) = -c_1(\Omega_X) = -c_1(\wedge^2\Omega_X) = 0$, by Noether formula, the top Chern class $c_2(X)$ is 24, which coincides with the topological Eular characteristic. Then

$$\chi(X) = \sum_{i=0}^{4} b_i(X) = 24.$$

Because X is simply connected, $H^1(X) = 0$. Thus, by Poincare duality and the isomorphism between sheaf cohomology and singular cohomology, we conclude that

$$b_0(X) = b_4(X) = 1, \;\; b_1(X) = b_3(X) = 0.$$

And finally, $b_2(X) = 22$. □

3 Elliptic Fibration

Definition 3.1. *An* ***elliptic surface*** *is a surface that has an elliptic fibration, in other words a proper morphism with connected fibers to an algebraic curve such that almost all fibers are smooth curves of genus 1. These fibers are called* ***elliptic fibers****, while others are called* ***singular fibers****.*

In fact, the $K3$ surface is an elliptic surface. Consider the projection

$$pr_1 : \mathbb{T}^4 = \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$$

by projecting $\mathbb{T}^4$ to its first factor. This induces a projection

$$\mathbb{T}^4/\sigma \to \mathbb{T}^2/\sigma.$$

The space T^2/σ, called the pillowcase, looks like a cornered sphere, and is diffeomorphic[3] to S^2. For a point away from four corner-points, say $(p, q) \in \mathbb{T}^2/\sigma$, the fiber over it is the elliptic fiber as a torus identified by two tori $(p, q) \times \mathbb{T}^2$ and $(\overline{p}, \overline{q}) \times \mathbb{T}^2$ through σ. While for each corner-points, the fiber over it is the singular fiber $(p, q) \times \mathbb{T}^2/\sigma$, which fails to be a manifold since now it contains four of the original 16 singular points, it .

In this case, we still consider the desingularization of $\mathbb{T}^4/\sigma$, which replace 16 singular points by 16 spheres and induces a map

$$K3 \to \mathbb{T}^2/\sigma \approx \mathbb{CP}^1.$$

The generic fiber is still a torus, while each of the four singular fibers becomes a five transversely-intersecting spheres, which are one of the old singular sphere-fiber of $\mathbb{T}^4/\sigma$ and four desingularizating spheres respectively.

[3] Although it is nonsingular with the induces quotient topology, it can be embeded with a smooth structure.

Remark 3.2. *The fibration structure of Kummer surface provides us an intuitive view to the following:*

(a)The K3 surface is simply connected: Since loops on torus fibers can be pushed to singular fibers and then contract there. Also, desingularization does not creat any new loops.

(b)The main sphere of the singular fiber has self-intersection -2: Denote the main sphere of a singular fiber by S, four desingularization spheres by S_1, S_2, S_3, S_4. Consider an elliptic fiber F approaching S. It covers S twice and covers S_i once, which implies that

$$F = 2S + S_1 + S_2 + S_3 + S_4$$

in homology. By $F^2 = 0$ and $(S_i)^2 = -2$, $S^2 = -2$.

References

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