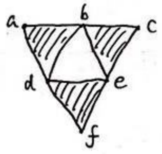
Chapter V Duality.

§ 1.1 Cohomology.

In previous section we have defined chain complexes $\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$ where C_n is a vector space over $\mathbb{Z}_2$ generated by all n-simplices in a simplicial complex K, now dualize this chain complex, i.e. apply the functor $\mathrm{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)$, and let C_n^* denote $\mathrm{Hom}_{\mathbb{Z}_2}(C_n, \mathbb{Z}_2)$. we get another chain complex $\leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1} \leftarrow \cdots$ where $\delta = \partial^*$, that is for $f \in C_n^*$, $\delta f = f\partial : C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f} \mathbb{Z}_2$, $\delta\delta f = \delta(f\partial) = f\partial\partial = 0$

Hence we define the p-th cohomology group $H^{p}(K) = \ker \delta_{p} / \mathrm{Im}\, \delta_{p-1}$. $Z^p = \ker \delta_p$ $B^p = \mathrm{Im}\, \delta_{p-1}$, they all contained in C^p.

Example:



$C_0 = \mathbb{Z}_2^6$ generates by $a^*, b^*, \cdots, f^*$

$Z^0 = \ker \delta^0$, If $c^* \in Z^0$, then $c^*\partial : C_1 \to \mathbb{Z}_2$ is a zero map $\Rightarrow$ For any edge, c^* take the same value on its endpoint. thus the only nontrivial 0-cocycle is $a^* + \cdots + f^*$. $H^0(K) = \mathbb{Z}_2$, $Z^0 = \mathbb{Z}_2$, hence $B^1 = C^0/Z^0 = \mathbb{Z}_2^5$

what about $H^1(K)$?. $Z^1 = \ker \delta^1$. If $c^* \in Z^1$, then $c^*\partial : C_2 \to \mathbb{Z}_2$ is zero. thus 0 on each 2-simplex, which means c^* takes 1 on exactly two edges of each triangle or 0 on all edges. Hence for $[abd]$, $[ab]^* + [bd]^*$ and $[ab]^* + [ad]^*$ are cocycles send $[abd]$ to zero. Therefore $Z^1 = \mathbb{Z}_2^6$, $H^1 = Z^1/B^1 = \mathbb{Z}_2$. $B^2 = C^1/Z^1 = \mathbb{Z}_2^3$, $H^2 = C^2/B^2 = 0$. In this example, we find $H_n = H^n$, this is not a coincidences.

§ 1.2 Coboundary map Matrix.

Recall the representation of a linear map $\phi : V \to W$,

$\phi(v_1, \cdots, v_m) = (w_1, \cdots, w_n) A_{n\times m}$, v_i are basis for V and w_i for W.

Then we take dual, $\mathcal{A}^*: W^* \to V^*$, and take the dual basis w_i^* and v_i^*, we have $\mathcal{A}^*(w_1^*, \cdots w_n^*) = (v_1^*, \cdots, v_m^*) A^T$.

Hence the coboundary matrix $C_n^* \to C_{n+1}^* [\delta_n]$ is the transpose of $[\partial_{n+1}]: C_{n+1} \to C_n$.

i.e. let $r_{n+1} = \operatorname{rank} [\partial_{n+1}] = \operatorname{rank} [\delta_n] = r^n$, $\beta_n = \operatorname{rank} H_n(k) = z_n - b_n = c_n - r_n - r_{n+1}$

$\beta^n = \operatorname{rank} H^n = z^n - b^n = c_n - r^n - r^{n-1} = c_n - r_{n+1} - r_n = \beta_n$. Therefore $H^n(k; \mathbb{Z}_2) = H_n(k; \mathbb{Z}_2)$

§ 1.3 Universal Coefficient Theorem.

~~X is a topological space, $H_n(X)$ denotes the homology group of X in coefficient R, R is a PID. G is a R-module.~~

C is a chain complex of free R-modules with boundary maps R-module homomorphisms, and G is also an R-module. There is a natural split short exact sequences:

$$0 \to \operatorname{Ext}_R(H_{n-1}(C), G) \to H^n(C; G) \to \operatorname{Hom}_R(H_n(C), G) \to 0.$$

where $\operatorname{Ext}_R(A, B)$, A, B R-modules. Take free (projective) resolution for A

$$0 \to A_1 \to A_0 \to A \to 0, \text{ apply } \operatorname{Hom}_R(-, B) \text{ get}$$

$$0 \leftarrow A_1^* \leftarrow A_0^* \leftarrow A^* \leftarrow 0. \quad \operatorname{Ext}_R(A, B) = H^1(\operatorname{Hom}_R(P_A, B)).$$ When $G = \mathbb{Z}_2$

H_n is a vector space, free, therefore $\operatorname{Ext} = 0 \Rightarrow H^n(C; \mathbb{Z}_2) = \operatorname{Hom}_{\mathbb{Z}_2}(H_n(C), \mathbb{Z}_2)$.

By the way. C_n' is the G coefficient chain group, $C_n' = C_n \otimes_{\mathbb{Z}} G$, C_n is $\mathbb{Z}$ coefficient

Then. $0 \to H_n(C) \otimes G \to H_n(C \otimes G) \to \operatorname{Tor}(H_{n-1}(C), G) \to 0$. is a split SES.

where $\operatorname{Tor}(A, B) = H_1(P_A \otimes B)$.

§2. Poincaré Duality.

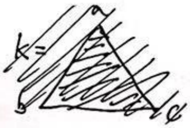
$\mathrm{Star}\,\tau = \{ \sigma \in K \mid \tau \text{ is a face of } \sigma \}$, $\overline{St}\,\tau$ the minimal subcomplex containing $St\,\tau$. link, $Lk\,\tau = \{ v \in \overline{St}\,\tau \mid v \cap \tau = \phi \}$.

- Combinatorial manifolds.

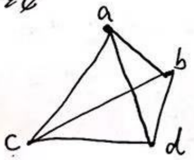
A combinatorial manifold of dimension d, satisfying there is a triangulation s.t. $Lk\,\sigma$ triangulates the sphere of dimension $d-i-1$. This implies $\overline{St}\,\sigma \cong B^d$ (D^d)

$\sigma \in C_i$ means $Lk\,\sigma \cong S^{d-i-1}$

$a \in C_0$, $\overline{St}\,a = K$, $Lk\,a = \{\{b\},\{c\},\{bc\}\}$



An Example : A tetrahedron with empty interior.



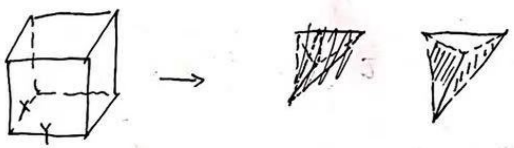
$\overline{St}\,a$ = the union of three standing 2-simplices. $\cong B^2$

$Lk\,a = \{ \{b\},\{c\},\{d\},\{bc\},\{bd\},\{cd\} \} \cong S^1$

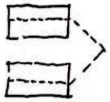
$\overline{St}\,[ac]$ = the union of two 2-simplices intersecting on $[ac] \cong B^2$, $Lk[ac] = \{\{b\},\{d\}\} \cong S^0$

In fact each simplex in $St\,\sigma$ is the join of σ with a simplex in $Lk\,\sigma$, $\overline{St}\,\sigma = \sigma * Lk\,\sigma$. $Lk\,\sigma = S^{d-i-1} = \underset{d-i}{*} S^0$, $\sigma \underset{d-i}{*} S^0 = D^i \underset{d-i}{*} S^0 = S^{d-i} D^i = D^d$.

$X * Y = X \times Y \times [0,1] / (x_0,0,y), (x,1,y_0)$



$X * S^0 = SX$,

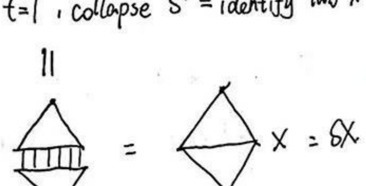


$\to X \times S^0 \times \{t\} \longrightarrow$ $t=0$, collapse X

$t=1$, collapse S^0 = identify two "X"

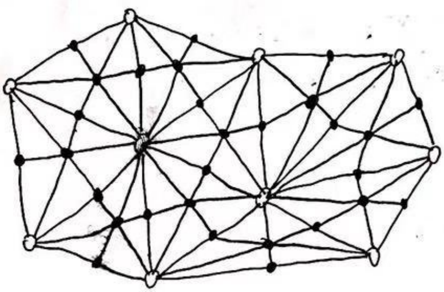
$X \times S^0 \times [0,1]$

||



- Dual Block.

Let M be a compact, combinatorial d-manifold triangulated by K. Recall the barycenter subdivision, SdK. If K has the link property, so does SdK. Label each vertex u in SdK, ~~by its~~ as follow: u is a barycenter of a simplex σ in K, label u by the dimension of σ. Note that each simplex in SdK has distinct labels on its vertices. The vertex with smallest label is therefore unique. Letting u ~~is~~ be the barycenter of σ in K, the dual block denoted by $\hat{\sigma}$, is the union of the simplices in the barycentric subdivision for which u is the vertex with minimum label. Let B be the set of dual blocks and call it the ~~dual~~ block decomposition of M



vertex $\longleftrightarrow$ D^2

line $\longleftrightarrow$ D^1

triangle $\longleftrightarrow$ D^0

If p-simplex σ is a face of $p+1$-simplex τ, then $\hat{\tau}$ ~~is~~ ~~is~~ contained in the boundary of $\hat{\sigma}$. In fact, the boundary $\hat{\sigma}$ is the union of dual blocks $\hat{\tau}$ over all proper cofaces τ of σ. We denote this boundary by $bd\,\hat{\sigma}$, and $\hat{\sigma} = bd\,\hat{\sigma} * u$, u is the barycenter of σ. $\sigma \leq \tau$ then $\hat{\tau}^* \leq \hat{\sigma}^*$. Since SdK is a combinatorial manifold, $bd\,\hat{\sigma} \cong S^{q-1}$, where $p+q=d$.

we now construct a new chain complex. Since σ a p-simplex. $\hat{\sigma}$ is a q-dimensional B^q, Let D_q denotes the vector space over $\mathbb{Z}_2$ generated by all $\hat{\sigma_i}$, where σ_i is a p-simplex. The boundary map $\partial_q : D_q \to D_{q-1}$ is defined by $\partial_q(\hat{\sigma_i}) = \sum \hat{\tau_i}$, where τ_i is a $(p+1)$-simplex and $\sigma_i \leq \tau_i$ is a proper face. $\partial_{q-1} \circ \partial_q = 0$ since there exactly two identical elements in $\partial_{q-1}(\hat{\tau_i})$ and $\partial_{q-1}(\hat{\tau_j})$ containing them as faces.

The next step is to show $H_q(D) = H_q(C)$, where C is the simplicial chain complex. Mapping each ~~p-simplex~~ p-dimensional dual block to the sum of p-simplices it contains, we get a homomorphism $b_p : D_p \to C_p$, b_p commutes with the boundary map., thus forms a chain map $b : D \to C$.
(no formal proof).

? – Block Complex Lemma : The chain map $b : D \to C$ induces $b_* : H_p(D) \to H_p(C)$ being an isomorphism.

[proof] : Let X_p be the subcomplex of $Sd\,K$, consisting of all simplicies that lie in blocks of dimension at most p. Clearly $H_p(X_q, X_{q-1}) = \begin{cases} D_p & \text{if } q = p \\ 0, & \text{otherwise.} \end{cases}$ (1) ?

$$\to H_{p+1}(X_q, X_{q-1}) \to H_p(X_{q-1}) \to H_p(X_q) \to H_p(X_q, X_{q-1}) \to \cdots$$

let $q = p+2$. $H_p(X_{q+1}) \cong H_p(X_{q+2}) \cong H_p(Sd\,k)$.

If $p+1 < q$, $H_p(X_{q-1}) \cong H_p(X_q)$

$$\begin{tikzcd}
 & D_{p+1} = H_{p+1}(X_{p+1}, X_p) \arrow[d, "e"] \arrow[dr, dashed, "\partial_{q+1}"] & & 0 = H_{p-1}(X_{p-2}) \arrow[d] \\
0 = H_p(X_{p-1}) \arrow[r] & H_p(X_p) \arrow[r, "f"] \arrow[d, "l"] & D_p = H_p(X_p, X_{p-1}) \arrow[r, "g"] \arrow[dr, dashed, "\partial_q"] & H_{p-1}(X_{p-1}) \arrow[d, "h"] \\
 & H_p(X_{p+1}) \arrow[d] & & D_{p-1} = H_{p-1}(X_{p-1}, X_{p-2}) \\
 & 0 = H_p(X_{p+1}, X_p) & &
\end{tikzcd}$$

② The diagramm commutes.?

③ $\ker \partial_q = \ker g$. $\operatorname{Im} \partial_{q+1} = \operatorname{Im} f \circ e = f(\operatorname{Im} e)$

$= \operatorname{Im} f \cong H_p(X_p)$

$$H_p(D) = \ker \partial_q / \operatorname{Im} \partial_{q+1} = \frac{f(H_p(X_p))}{f(\operatorname{Im} e)} \cong \frac{H_p(X_p)}{\operatorname{Im} e} = \frac{H_p(X_p)}{\ker l} \cong H_p(X_{p+1}).$$

$$= H_p(\operatorname{sd} K).$$

— Poincaré Duality (First form). Let M be a compact, combinatorial d-manifold. Then there is a isomorphism between $H_p(M)$ and $H^q(M)$ for $p+q=d$.

— $\forall$ p-simplex σ in K, let $\sigma^* \in C_p^*$, let $\varphi_q : D_q \longrightarrow C^p$ (bijective) by $\varphi_q(\hat{\sigma}) = \sigma^*$

extending linearly gives a isomorphism between D_q and C^p.

$$\begin{tikzcd}
D_q \arrow[r, "\partial_q"] \arrow[d, "\varphi_q"'] & D_{q-1} \arrow[d, "\varphi_{q-1}"] \\
C^p \arrow[r, "\delta_p"] & C^{p+1}
\end{tikzcd}$$

If this diagram commutes, then we have $H_q(\mathbb{M}) \cong H^p(\mathbb{M})$

$\varphi_{q-1} \circ \partial_q(\hat{\sigma}) = \varphi_{q-1}(\Sigma \hat{\tau}) = \Sigma \tau^*$, where τ is a $p+1$ simplex with σ being its face. $\delta^p \varphi_q(\hat{\sigma}) = \delta^p(\sigma^*) = \sigma^* \partial$; since they ~~evaluate~~ take 1 on each $p+1$ simplex $\tau \geq \sigma$ and 0 otherwise, therefore they are equal, $\Rightarrow$ the diagram commutes.

$$H_p(\mathbb{M}) \cong H^q(\mathbb{M}) = H_q(\mathbb{M}) \quad \text{for } p+q=d$$

In general, If M is R-orientable, then there is an isomorphism $H^k(M;R) = H_{n-k}(M;R)$. Every manifold is $\mathbb{Z}_2$-orientable.

§ 3. Intersection Theory

Let M be a combinatorial d-manifold. $p+q=d$. If σ is a p-simplex, then $\hat{\sigma}$ is q-dimensional. $\sigma \cap \hat{\sigma} = u$ the barycenter of σ. (This is because u has the minimal label in all simplices in ~~Sd~~ $\hat{\sigma}$). If $\sigma \neq \tau$, then $\sigma \cap \hat{\tau} = \phi$, τ is another p-simplex. (How to prove?). Define $\sigma \cdot \hat{\tau} = \begin{cases} 1 & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau \end{cases} : C_p \times D_q \longrightarrow G = \mathbb{Z}_2$

Suppose $c = \sum_i a_i \sigma_i$ is a p-cycle in $\underset{C_p}{K}$ and $d = \sum_j b_j \hat{\tau}_j$ is a q-cycle in D_q. then $c \cdot d = \sum_{i,j} a_i b_j (\sigma_i \cdot \hat{\tau}_j)$ is the intersection number of two cycles in modulo 2. $c \cdot d = 0$ if they are disjoint or meet in an even number, $c \cdot d = 1$ if they meet in an odd number.

In fact, if $c \sim c_0$, then $c \cdot d = c_0 \cdot d$, so does $d \sim d_0$. Let γ be a $p+1$ simplex we want to show $\partial \gamma \cdot d = 0$. For $\hat{\sigma}$ is a summand of d, $\partial \gamma \cdot \hat{\sigma} \neq 0$ iff σ is a face of γ. $\gamma \cap \hat{\sigma} =$ the line segment connecting the barycenters of γ and σ.

? Completing the intersection between γ and d, the edge extends to either a closed curve or a path with two endpoints. Thus $\partial\gamma \cdot d = 0$.

— Parings. $\# : H_p(M) \times H_q(M) \longrightarrow G$ defined by $\#(\gamma, \delta) = c \cdot d$, where c and d are representatives. Call this map the intersection paring of the homology groups, $p+q=d$. Bilinear. U, V be vector spaces, $\# : U \times V \longrightarrow G$ gives a natural homomorphism $\phi_\# : V \longrightarrow Hom(U, G)$, $f_V(u) = \#(u, v)$. A paring is perfect if for every nonzero $u \in U$, $\exists\, V_0 \in V$, s.t. $\#(u, v_0) = 1$. and $\forall\, v \neq 0$, $\exists\, u_0 \in U$, $\#(u_0, v) = 1$.

— Perfect Paring Lemma. The paring $\# : U \times V \to G$ is perfect iff $\phi_\# : V \to Hom(U, G)$ is an isomorphism. If $\phi_\#$ is an iso, then $\forall\, v \neq 0$, $f_V\,{}^\# \in Hom(U, G) \neq 0$. $\therefore \exists\, u_0 \in U$ s.t. $\#(u_0, v) = 1$. Since $\phi\#$ is surjective. for $u \in U$, u^* takes u to 1, $\exists\, v_0 \in V$ s.t. $f_{v_0} = u^*$, $f_{v_0}(u) = u^*(u) = 1$. Conversely. Suppose $\#$ is perfect. $\phi_\#$ is injective. otherwise $\exists\, v_0$, $f_{v_0} = 0$ in $Hom(U, G)$. $rank V \leq rank\, Hom(U, G) = rank\, U$. but $rank U \leq rank V$ in the same way. $\therefore$ $rank V = rank Hom(U, G) \Rightarrow$ $\phi_\#$ is an isomorphism. $\Rightarrow$ $V \cong Hom(U, G) \cong U$.

[— σ a p-simplex of k, $\hat{\sigma}$ its q-dimensional dual block, let $\varphi_q(\hat{\sigma}) = \sigma^*$. and. $\in C_p^*$

$\langle \sigma^*, \tau \rangle = \langle \hat{\sigma}, \tau \rangle$]

— Poincaré Duality (Second Version).

Let M be a compact, combinatorial manifold. Then the paring $\#\ H_p(M) \times H_q(M) \to G$ defined by $\#(\gamma, \delta) = \gamma \cdot \delta$ is perfect for all $p+q=d$.