I. Appendix

A. Mathematical Proofs

Proof: (Lemma 2) We first assume that $A = 1$ for simplicity, then generalize the proof for any constant value of A. The trigonometric parametrization of the ellipse can be used for proof. The general quadratic equation $ax^2+bxy+cy^2-\rho=0$ represents an ellipse with center at the origin if the following conditions are satisfied:

$$\begin{cases} b^2-4ac<0 \\ \delta = \begin{vmatrix} 2a & b & 0 \\ b & 2a & 0 \\ 0 & 0 & -2\rho \end{vmatrix} \neq 0, \\ \delta.\sigma < 0, \quad \text{where } \sigma = a + c. \end{cases} \tag{1}$$

Moreover, the angle of rotation of the ellipse can be obtained using the following equation

$$\tan(2\theta) = \frac{b}{a-c}. \tag{2}$$

For notational convenience let $C := \cos\theta$ and $S := \sin\theta$. The radii of the ellipse are then

$$\alpha = \sqrt{\frac{\rho}{P}}, \quad \beta = \sqrt{\frac{\rho}{Q}}, \tag{3}$$

where $P = aC^2 + cS^2 + bCS$ and $Q = aS^2 + cC^2 - bCS$. According to lemma 3, the delay coordinate embedding of $A\sin(\frac{2\pi}{T}t)$ with delay τ is equivalent to the delay embedding of $A\sin(t)$ with the delay $\frac{2\pi}{T}\tau$. Therefore, if we prove that the delay-coordinate embedding of $\sin(t)$ constructs an ellipse centered at the origin with the angle of rotation $45°$ provided that $\tau \neq k\pi$ where $k \in \mathbb{Z}$, the first part of the lemma will be proved. Let us construct the quadratic equation using $x(t) = \sin(t)$ and its delayed version $y(t) = \sin(t+\tau)$. Simplifying the obtained equation yields,

$$\begin{aligned} &\sin^2 t\left(a + b\cos\tau + c\cos^2\tau\right) + \cos^2 t\left(c\sin^2\tau\right) \\ &\quad + \cos t\sin t\left(b\sin\tau + 2c\sin\tau\cos\tau\right) - \rho = 0. \end{aligned} \tag{4}$$

In order for this equation to be true for all t, the following conditions should be satisfied

$$\begin{cases} a + b\cos\tau + c\cos^2\tau = c\sin^2\tau, \\ b\sin\tau + 2c\sin\tau\cos\tau = 0. \end{cases} \tag{5}$$

Simplifying Equation (5) yields

$$\begin{cases} a = c, \\ b + 2c\cos\tau = 0. \end{cases} \tag{6}$$

In this case, Equation (4) becomes

$$c\sin^2\tau - \rho = 0. \tag{7}$$

Thus we can check the conditions. We can first calculate

$$b^2 - 4ac = (-2c\cos\tau)^2 - 4c^2 = 4c^2\left(\cos^2\tau - 1\right) \tag{8}$$

First conditions are therefore satisfied since $\tau \neq k\pi,\ k \in \mathbb{Z}$. According to (6) and (7), one can calculate δ as

$$\begin{aligned} \delta = -2\rho\left(4a^2 - b^2\right) &= -8\rho a^2(1-\cos^2\tau) \\ &= -8a^3\left(1-\cos^2\tau\right)^2, \end{aligned} \tag{9}$$

which can never be zero since $\tau \neq k\pi,\ k \in \mathbb{Z}$. Moreover, it is of opposite sign to a and thus of opposite sign of σ, since $\sigma = 2a$. Hence, all three conditions are satisfied and the delay coordinate embedding of $\sin(t)$ with delay $\tau \neq k\pi, k \in \mathbb{Z}$ defines an ellipse centered at the origin. Additionally, according to (2), the angle of rotation is $45°$. The same results are also true with delay $\frac{2\pi}{T}\tau$, where $\tau \neq \frac{kT}{2}$. Therefore, $C = S = \sqrt{2}/2$. Now, the radii of the ellipse can be calculated since

$$\begin{aligned} P &= (a+b+c)/2 = a(1-\cos\tau), \\ Q &= (a-b+c)/2 = a(1+\cos\tau), \end{aligned} \tag{10}$$

$\alpha = \sqrt{1+\cos\tau}$ and $\beta = \sqrt{1-\cos\tau}$. Clearly, $\alpha, \beta \neq 0$ since $\tau \neq k\pi$. So if we change the delay to $\frac{2\pi}{T}\tau$, the radii of the ellipse are going to be $\alpha = \sqrt{1+\cos(\frac{2\pi}{T})\tau}$ and $\beta = \sqrt{1-\cos(\frac{2\pi}{T}\tau)}$. Multiplying by the amplitude does not change the conditions since it just affects ρ, and multiplies it by A^2 in (4). Therefore the radii of the ellipse would be multiplied by A according to (3). Since the delay coordinate embedding of $A\sin(\frac{2\pi}{T}t)$ with delay τ is equivalent to the delay embedding of $A\sin(t)$ with delay $\frac{2\pi}{T}\tau$, the lemma is proved. ■

Proof: (Lemma 3) Using a change of variable $t' = \frac{T_1}{T_2}t$,

$$\begin{aligned} U_2(t') &= \left(A\sin\left(\frac{2\pi}{T_1}t'\right), A\sin\left(\frac{2\pi}{T_1}t' + \frac{2\pi}{T_2}\tau_1\right)\right) \\ &= \left(A\sin\left(\frac{2\pi}{T_1}t'\right), A\sin\left(\frac{2\pi}{T_1}t' + \frac{T_1}{T_2}\tau_1\right)\right) \end{aligned} \tag{11}$$

So, $U_1(t) = U_2(t')$ for $\tau_2 = \frac{T_1}{T_2}\tau_1$. Thus, the sets $U_1(t)$ and $U_2(t)$ contain the same points implying that the two sets are equal. ■

Proof: (Lemma 4) Let W_s be the ellipse with the smallest radii $\alpha_s, \beta_s = A_s\sqrt{1 \pm \cos(\frac{2\pi}{T_s})\tau}$. In the continuous case, the persistent diagram for $\cup_{i=1}^{n} W_i$ would contain a persistent bar with birth time $t_b = 0$ and death time $t_d = \beta_s$ corresponding to this small ellipse. In the discrete case, the birth time of this bar will be close to $t_b = 0$ depending on the T_s, and the death time $t_d \geq \beta_s$. Therefore, the persistent diagram has a bar of length close to β_s. ■

Proof: (Theorem 1) The delay τ is chosen between t_{c1} and t_{c2}. According to our experimental results, selecting the second zero of ACL function gives the best informative delay embedding. Similar to the proof of lemma 4, the smallest ellipse W_s will create a significant 1-dimensional persistent bar of length close to β_s. The points in K do not interfere with this ellipse since they are in between $W_i's$ and close to them (Fig. 2). ■

B. The model construction

We construct a model corresponding to each wheeze signal by generating a signal with similar frequency and amplitude as those of the wheeze signal at each time instance. This objective is achieved in two steps. First, we estimate the frequency of the signals at different time intervals using the zero crossing points. The intervals between successive zero crossing indexes can be considered as a random variable $d_i, i = 1, 2, ..., n_z - 1$ where n_z is the number of zero crossings . We can find the time intervals at which the frequency is constant using the distribution of d_i, and detect the intervals with almost constant frequencies denoted by $T_j, j = 1, 2, .., n_f$, where n_f is the number of different frequencies of the signal. We then find the frequency during each time interval as $f_j = {}^1/_{2\mu_j}$, where $\mu_j, j = 1, 2, ..n_f$ denotes the mean value of d_i during T_j. Next, we find the envelope of the signal using its critical points and multiply it by a sinusoidal function with the frequencies obtained in the first step. Only critical points with positive amplitude are used since the breathing sound signals are to a large extent, almost symmetric about the time axis. Next, we perform an interpolation between them to obtain the envelope of the signal denoted by $a(t)$ and multiply it by the obtained piecewise sinusoidal function.