Mayer-Vietoris Sequence of de Rham Cohomology

Jacky Wang

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1 A Cheatsheet on Differential Forms

Learn differential forms in 5 minutes!

1. A smooth k-form is a smooth section of $\Lambda^k T^*M$.
2. $\Lambda^k T^*M$ is the smooth vector bundle obtained by gluing together $\Lambda^k T_p^*M$ at each p using a really cool trivialization that encodes the global information of the manifold.
3. $\Lambda^k T_p^*M \cong (\Lambda^k T_pM)^* := \mathrm{Hom}_{\mathsf{Vec}_{\mathbb{R}}}(\Lambda^k T_pM, \mathbb{R})$ which is the space of k-anti-symmetric linear form on T_pM.
4. $T_pM := \{$ the linear space of all directional derivatives at point $p\}$ is the tangent space at p.

2 Mayer-Vietories Sequence of De Rham Cohomology

De Rham cohomology is supposed to be some kind of cohomology theory, so the operator $\Omega^*(\bullet)$ we have met before should also be a contravariant functor. Let's make sense of this.

Definition 2.1 (Pullback of differential forms)**.** For smooth map $F : M \to N$ and differential form ω on N, we define $\Omega^*(F)$ to be the *pullback* $F^*\omega$ of ω, which is a differential form on M defined by

$$(F^*\omega)_p(v_1, \dots, v_k) := \omega_{F(p)}(dF_p(v_1), \dots, dF_p(v_k)).$$

Wait for a minute, what is the type of the functor $\Omega^*(\bullet)$? From the category of smooth manifold to the category of *chain complexes* (to be precise, differential graded algebra). To fit beautifully into the framework of category theory, we need to check that F^* is qualified as a morphism of chain complexes, that is, it commutes with differential operator

$$\begin{tikzcd} \Omega^p(M) \arrow[r, "d"] & \Omega^{p+1}(M) \\ \Omega^p(N) \arrow[r, "d"] \arrow[u, "F^*"] & \Omega^{p+1}(N) \arrow[u, "F^*"'] \end{tikzcd}$$

Let M be a smooth manifold which can be decomposed into open subsets U and V such that $U \cap V \neq \varnothing$. Consider the following diagram

$$\begin{tikzcd} U \cap V \arrow[r, "j_U"] \arrow[d, "j_V"'] & U \arrow[d, "i_U"] \\ V \arrow[r, "i_V"] & U \cup V \end{tikzcd}$$

which turns into

$$\begin{tikzcd} \Omega^*(U \cup V) \arrow[r, "i_U^*"] \arrow[d, "i_V^*"'] & \Omega^*(U) \arrow[d, "j_U^*"] \\ \Omega^*(V) \arrow[r, "j_V^*"] & \Omega^*(U \cap V) \end{tikzcd} \tag{1}$$

after applying the functor $\Omega^*(\bullet)$. It may tempt us into considering the following sequence

$$\Omega^*(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V)\ .$$

We have the following claim:

Proposition 2.1. *The sequence of chain complexes*

$$0 \longrightarrow \Omega^*(U \cup V) \xrightarrow{(i_U^*, i_V^*)} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j_U^* - j_V^*} \Omega^*(U \cap V) \longrightarrow 0$$

is exact.

Proof. First we check the exactness at $\Omega^*(U \cup V)$. For any $\omega \in \Omega^*(M)$, suppose $i_U^*(\omega) = 0$. The intuition is, $i_U^*(\omega)$ may be understood to be some kind of "restriction" $\omega|_U$ of ω as smooth function. Concretely, we have

$$0 = i_U^*(\omega)_p = \omega_p \circ \Lambda^k(\mathrm{d}i_U)_p. \tag{2}$$

for any $p \in U$ by expansion of notations. By the local nature of tangent spaces, the inclusion $i_U : U \hookrightarrow M$ induces isomorphism $(\mathrm{d}i_U)_p : T_pU \to T_pM$ for all p, and taking their wedge product will yield another isomorphism $\Lambda^k T_pU \to \Lambda^k T_pM$. Therefore we have $\omega|_U = 0$. Similarly, $\omega|_V = 0$. Since $M = U \cup V$, ω is zero everywhere.

Similarly, suppose $\omega \in \Omega(U)$ and $\eta \in \Omega(V)$ satisfy $j_U(\omega) = j_V(\eta)$. Informally it means $\omega|_{U\cup V} = \eta|_{U\cup V}$, which means ω and η can be glued together to form a global section on M. To fulfill this dream, we need to deal with these tedious definitions. Drawing diagrams will be useful to avoid getting lost. The assumption gives us

$$\omega_p \circ \Lambda^k(\mathrm{d}j_U)_p = \eta_p \circ \Lambda^k(\mathrm{d}j_V)_p \tag{3}$$

for each $p \in U \cap V$.

Now construct $\tau \in \Omega^k(M)$ to be

$$p \in M \mapsto \begin{cases} \omega_p \circ (\Lambda^k \mathrm{d}i_{Up})^{-1} & p \in U \\ \eta_p \circ (\Lambda^k \mathrm{d}i_{Vp})^{-1} & p \in V \end{cases} \tag{4}$$

and clearly such τ satisfy $i_U^*(\tau) = \omega$ and $i_V^*(\tau) = \eta$. To make this well-defined, we need to check that for $p \in U \cap V$,

$$\omega_p \circ (\Lambda^k \mathrm{d}i_{Up})^{-1} = \eta_p \circ (\Lambda^k \mathrm{d}i_{Vp})^{-1}. \tag{5}$$

In fact, we have the diagram

$$\begin{tikzcd}
\Lambda^k T_pU \cap V \arrow[rr, "\Lambda^k \mathrm{d}j_{Up}"] \arrow[dd, "\Lambda^k \mathrm{d}j_{Vp}"'] & & \Lambda^k T_pU \arrow[dd, "\omega_p"] \\
& \Lambda^k T_pM \arrow[ul, "w"] \arrow[ur, "(\Lambda^k \mathrm{d}i_{Up})^{-1}"'] \arrow[dl, "(\Lambda^k \mathrm{d}i_{Vp})^{-1}"] & \\
\Lambda^k T_pU \cap V \arrow[rr, "\eta_p"'] & & \mathbb{R}
\end{tikzcd} \tag{6}$$

in which w is (the lift of) the inverse of witness of commutativity in diagram (1). There is

$$\omega_p \circ (\Lambda^k di_{Up})^{-1} = \omega_p \circ \Lambda^k(\mathrm{d}j_U)_p \circ \text{witness} = \eta_p \circ \Lambda^k(\mathrm{d}j_V)_p \circ \text{witness} = \eta_p \circ (\Lambda^k di_{Vp})^{-1} \tag{7}$$

for $p \in U \cap V$. Note: we are allowed to perform such kind of acrobatics thanks to the fact that inclusions induce isomorphisms between tangent spaces, which is then due to the idea that tangent space is some kind of *local* object.

On the other hand, a form $\omega \in \Omega^*(M)$ traveling all the way through $\Omega^*(U) \oplus \Omega^*(V)$ into $\Omega^*(U \cap V)$ will annihilate since

$$(j_U^* \circ i_U^*)(\omega) = (j_U \circ i_U)^*(\omega) = (j_V \circ i_V)^*(\omega) = (j_V^* \circ i_V^*)(\omega) \tag{8}$$

by the functorality of pullback. So the exactness is also satisfied at $\Omega^*(U) \oplus \Omega^*(V)$.

Remark: the two steps that we have done on $\Omega^*(U) \oplus \Omega^*(V)$ actually checks that the contravariant functor $\Omega^*(\bullet)$ is actually a *sheaf* of differential graded algebra on the space M if we consider these steps on arbitrary open cover instead of just U and V.

Finally we check the same thing at $\Omega^*(U \cap V)$. The seemingly redundant magic mentioned before, partition of unity, is used at this point. We first use the simple case of $M = \mathbb{R}$ to demonstrate the ideal. Pick $f \in \Omega^0(\mathbb{R})$, which is a smooth function on $\mathbb{R}$. We need to find a pair of functions defined on U and V respectively whose difference on $U \cap V$ happens to be f. Since U and V form an open cover of M, there is a partition of unity $\{\rho_U, \rho_V\}$ subordinate to this cover. We can see that $\rho_V \cdot f$ is roughly actually a function on U, since ρ_V is annihilating on $U - U \cap V$ so it doesn't matter whatever f evaluates to over U, even if f is undefined. And here's the trick:

$$(\rho_U f) - (-\rho_V f) = f \tag{9}$$

holds everywhere on $U \cap V$.

Now for general manifolds, consider $\omega \in \Omega^k(U \cap V)$. We define $\tau \in \Omega^k(U)$ as

$$p \mapsto \begin{cases} \rho_V(p) \cdot \omega_p & p \in U \cap V \\ 0 & p\text{otherwise.} \end{cases}$$

and $\eta \in \Omega^k(V)$ likewise. Then the same trick in equation (9) still holds. $\square$

The short exact sequence of the graded differential algebra mentioned above is actually a short exact sequence of chain complexes since the horizontal induced maps cannot change the degree of differential forms and these squares commutes.

$$
\begin{tikzcd}
 & \cdots & \cdots & \cdots & \\
0 \arrow[r] & \Omega^{q+1}(M) \arrow[r] \arrow[u] & \Omega^{q+1}(U) \oplus \Omega^{q+1}(V) \arrow[r] \arrow[u] & \Omega^{q+1}(U \cap V) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Omega^{q}(M) \arrow[r] \arrow[u, "d"] & \Omega^{q}(U) \oplus \Omega^{q}(V) \arrow[r] \arrow[u, "d"] & \Omega^{q}(U \cap V) \arrow[r] \arrow[u, "d"] & 0 \\
 & \cdots \arrow[u] & \cdots \arrow[u] & \cdots \arrow[u] &
\end{tikzcd}
$$

By snake lemma such short exact sequence induce a long exact sequence of cohomology groups

$$
\begin{tikzcd}
 & H^{q+1}(M) \arrow[r] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[r] & H^{q+1}(U \cap V) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{q}(M) \arrow[r] & H^{q}(U) \oplus H^{q}(V) \arrow[r] & H^{q}(U \cap V) \arrow[ulll, "d^*"] &
\end{tikzcd}
$$

and this is the main result of this lecture.

Example 2.1 (Cohomology of S^1)*.* Decompose the circle S^1 into upper and lower hemisphere called S_+ and S_-. Then we get the following exact sequence

$$
\begin{tikzcd}
 & 0 \arrow[r] & 0 \arrow[r] & \cdots \\
 & H^1(S^1) \arrow[r] & H^1(S_+) \oplus H^1(S_-) = 0 \arrow[r] & H^1(S_+ \cap S_-) = 0 \arrow[ull, "d^*"] \\
0 \arrow[r] & H^0(S^1) = \mathbb{R} \arrow[r] & H^0(S_+) \oplus H^0(S_-) = \mathbb{R} \oplus \mathbb{R} \arrow[r] & H^0(S_+ \cap S_-) = \mathbb{R} \oplus \mathbb{R} \arrow[ull, "d^*"]
\end{tikzcd}
$$

in which $H^0(S^1) \cong H^0(S_+) \cong H^0(S_-) \cong \mathbb{R}$ is easy to see (and this is a common property of a qualified cohomology theory); $H^1(S_+) \cong H^1(S_-) \cong H^1(S_+ \cap S_-) = 0$ is pretty intuitive since these spaces are "flat". Therefore we get the following

$$
0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1(S^1) \longrightarrow 0
$$

and it is quite lucky that we can calculate $H^1(S^1) = \mathbb{R}$ directly from exactness without looking into the details of these connecting arrows. But no, no cheating! We are going to concretely calculate these and find their actual generator for future use, instead of just abstractly calculate its structure.

3 Mayer-Vietories Sequence of De Rham Cohomology with Compact Support

We can build the Mayer-Vietoris sequence of de Rham cohomology with compact supports almost alongside that of ordinary de Rham cohomology but with some subtle points to deal with.

Although also entitled with the name "cohomology", de Rham cohomology with compact support is actually *not* a qualified cohomology theory according to Eilenberg–Steenrod axioms, for it does not satisfy homotopy invariance and it is even a *covariant* functor! However, as we will see soon, this kind of "cohomology" actually fits into the place of homology when we talk about pairings in the context of Poincáre's duality.