Spectral Sequences II

— Some formal definitions

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Ref. J. McCleary, A user's guide to spectral sequences. (2nd edition).

1. Some algebraic Concepts

Let R be a comm. ring.

- graded R-modules $A_* = \bigoplus_n A_n$.
  $x \in A_n$ is called homogeneous of deg. n $(|x| = n)$ } Category.

  $\mathrm{Hom}(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_n)$

  $(A_* \otimes B_*)_n = \bigoplus_{i+j=n} A_i \otimes B_j$.

  $\mathrm{Hom}^k(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_{n+k})$ — deg k homomorphism.

- graded algebra/ring $=$ $\begin{cases} \text{graded module } A_* = \bigoplus_n A_n \\ \text{a degree 0 } R\text{-linear map (multiplication) } A_* \otimes A_* \xrightarrow{\cdot} A_*. \Rightarrow A_n \cdot A_m \subset A_{n+m}, \\ \text{which is associative \& unital} \end{cases}$

  eg. $H^*(X;R)$ — cohomological ring.

- differential graded algebra (dga) $=$ graded ring A_* + a differential $d: A_* \to A_*$

  degree $\begin{cases} +1 \text{ --- cohomology} \\ -1 \text{ --- homology} \end{cases}$ s.t.

  i.) $d^2 = d \circ d = 0$

  ii) Leibniz rule: $d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy$.

eg. $\underset{+1}{C^*(X;R)}$, $\underset{-1}{C_*(X;R)}$ chain complex.

- A filtration of an R-module M is a sequence of submodules:

increasing F_*M : $\cdots \subseteq F_{n-1}M \subseteq F_nM \subseteq F_{n+1}M \subseteq \cdots \subseteq M = \bigcup_n F_nM$

descreasing F^*M : $\cdots \subseteq F^{n+1}M \subseteq F^nM \subseteq F^{n-1}M \subseteq \cdots \subseteq M$

(usually bounded: $M = F_0M \supseteq F_1M \supseteq \cdots \supseteq F_nM \supseteq 0$.)

The associated graded module of a filtered module M is

the graded module $Gr(M) = \bigoplus_n Gr(M)_n$, $Gr(M)_n = \begin{cases} F_nM/F_{n-1}M & \text{for increasing filtration.} \\ F^nM/F^{n+1}M & \text{.. decreasing .. .} \end{cases}$

- filtered differential graded module = dgm (A_*, d) with a "coherent" filtration F^*M :

(A_*, d, F_*)

$d: F^nM \longrightarrow F^nM$; that is, (F^*M, d) is a sub-dga.

"coherent" $\Longrightarrow$ $H(A_*, d) = \operatorname{Ker} d / \operatorname{Im} d$ inherits a filtration:

$$F_nH(A_*, d) = \operatorname{Im}\left(H(F^nA_*, d) \xrightarrow{H(\text{incl})} H(A_*, d) \right).$$

2. Spectral Sequences. (s.s)

- A cohomology s.s. is a sequence of bigraded module $E_r^{p,q}$, $r \geq 1$, $p,q \in \mathbb{Z}$ together with differentials $d_r : E_r^{p,q} \to E_r^{p+r,\, q-r+1}$ s.t. $d_r^2 = 0$

and $E_{r+1}^{p,q} = H(E_r^{p,q}, d_r) = \dfrac{\operatorname{Ker}(E_r^{p,q} \to E_r^{p+r,\, q-r+1})}{\operatorname{Im}(E_r^{p-r,\, q+r-1} \to E_r^{p,q})}$, set $E_\infty^{*,*} = \operatorname{colim}_{\overrightarrow{r}} E_r^{*,*}$.

- We say d_r has bidegree $(r, -r+1)$, total degree 1.

- The s.s. $\{E_r, d_r\}$ converges to the graded R-module M^* ($E_r^{p,q} \Longrightarrow M^{p+q}$) if there exists a filtration F^* of M^* such that $E_\infty^{p,q} \cong F^p M^{p+q} / F^{p+1} M^{p+q}$.

- The s.s. $\{E_r, d_r\}$ collapses at E_{r_0} if $d_r = 0$ for any $r \geq r_0$: $E_\infty^{*,*} = E_{r_0}^{*,*}$.

- 1st quadrant s.s. : $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$.

Rmk A 1st quadrant s.s. $\{E_r^{*,*}, d_r\}$ converges in a strong way:

$\forall (p,q)$, $\exists\, r_0$ s.t. $d_r^{p,q} = 0$ for each $r \geq r_0$.

Eg: If $r > \max(p, q+1)$, then $\underset{\overset{\shortparallel}{0}}{E_r^{p-r,\, q+r-1}} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} \underset{\overset{\shortparallel}{0}}{E_r^{p+r,\, q-r+1}}$.

▷ Everything above is almost the same for a homological s.s. $E^r_{p,q} \Rightarrow M_{p+q}$ except that $d_r: E^r_{p,q} \to E^r_{p-r,\,q+r-1}$ has bidegree $(-r, r-1)$, total degree -1.

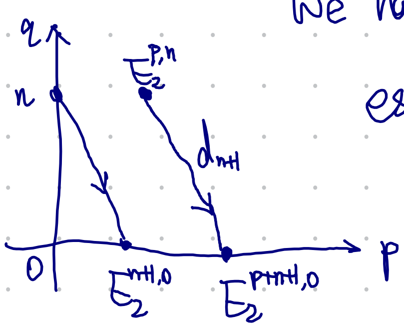
Ex. 1. Suppose $E_\infty^{p,q}=0$ unless $p=k$. Then $H^{k+q}\cong E_\infty^{k,q}$.

2. Suppose $E_2^{p,q}=0$ unless $q=0,n$ $(n\geq 2)$, then there is a LES:

(Example 1.D)

$$\to H^{p+n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to H^{p+n+1} \to E_2^{p+1,n} \xrightarrow{d_{n+1}} E_2^{p+n+2,0} \to \cdots \quad (*)$$

Pf. The only possible nontrivial differential is $d_{n+1}: E_2^{p,n} \to E_2^{p+n+1,0}$, $p\geq 0$.



We have $E_2\cong\cdots\cong E_{n+1}$, $E_{n+1}\cong E_\infty$;

especially, $E_\infty^{p+n+1,0}\cong E_2^{p+n+1,0}/\mathrm{im}\, d_{n+1}$, $E_\infty^{p,n}\cong \operatorname{Ker} d_{n+1}$.

$$\Leftrightarrow \quad 0\to E_\infty^{p,n}\to E_2^{p,n}\xrightarrow{d_{n+1}} E_2^{p+n+1,0}\to E_\infty^{p+n+1,0}\to 0 \qquad (1)$$

$E_2^{p,n}\Rightarrow H^{p+n}$, $F^* = F^*H^{p+n}$, $F^0 = H^{p+n}\supseteq F^1\supseteq\cdots\supseteq 0$.

$E_\infty^{p-1,n+1} = F^{p-1}/F^p = 0 \Rightarrow F^0=F^1=\cdots=F^p = H^{p+n}$

$E_\infty^{p+1,n-1} = F^{p+1}/F^{p+2}=0 \Rightarrow F^{p+1}\supseteq\cdots = F^{p+n} = E_\infty^{p+n,0}$

$\therefore E_\infty^{p,n}\cong F^p/F^{p+1}\cong H^{p+n}/E_\infty^{p+n,0}$. $\Leftrightarrow$ $0\to E_\infty^{p+n,0}\to H^{p+n}\to E_\infty^{p,n}\to 0$ (2)

Splice ES (1) and (2) ~~together~~ $\Longrightarrow$ (*):

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & & \\
 & E_\infty^{p+n,0} \arrow[d] & & & & & \\
 & H^{p+n} \arrow[d] & & & 0 \arrow[d] & & \\
0 \arrow[r] & E_\infty^{p,n} \arrow[r] \arrow[d] & E_2^{p,n} \arrow[r, "d_{n+1}"] & E^{p+n+1,0} \arrow[r] & E_\infty^{p+n+1,0} \arrow[r] \arrow[d] & 0 & \\
 & 0 & & & H^{p+n+1} \arrow[d] & & \\
 & & & 0 \arrow[r] & E_\infty^{p+1,n} \arrow[r] \arrow[d] & E_2^{p+1,n} \arrow[r] & \cdots \\
 & & & & 0 & &
\end{tikzcd}$$

Q.E.D.

3. $\{E_r, d_r\}$ a first quadrant S.S. $\Longrightarrow H^*$.

Suppose $E_r^{p,q}=0$ unless $p=0$ or $p=n$, $n\geq 2$. Derive the Wang Sequence:

$$\cdots \longrightarrow H^k \longrightarrow E_2^{0,k} \xrightarrow{d_n} E_2^{n,k-n+1} \longrightarrow H^{k+1} \longrightarrow E_2^{0,k+1} \longrightarrow \cdots$$

- Exact Couple

Let X be a CW-cpx and let $\phi = X_{-1} \subset X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ be a filtration.

e.g. $X_n = n$-skeleton of X, cellular filtration.

Each inclusion $X_{p-1} \to X_p$ induces a LES of homology groups

$$\cdots \xrightarrow{k} H_n X_{p-1} \xrightarrow{i} H_n X_p \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{\partial = k} H_{n-1} X_{p-1} \xrightarrow{i} \cdots \qquad \underline{ji = kj = ik = 0}$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_{p-2}) \arrow[d, "i"] & & & & \\
\cdots \arrow[r] & \underline{H_n(X_{p-1})} \arrow[r, "j"] \arrow[d, "i"] & H_n(X_{p-1}, X_{p-2}) \arrow[r, "k"] & b_3 \in H_{n-1}(X_{p-2}) \arrow[r, "j"] \arrow[d, "i"] & \underline{H_{n-1}(X_{p-2}, X_{p-3})} & \\
\cdots \arrow[r] & \underline{H_n(X_p)} \arrow[r, "j"] \arrow[d, "i"] & H_n(X_p, X_{p-1}) \arrow[r, "k"] & b_2 \in \underline{H_{n-1}(X_{p-1})} \arrow[r, "j"] \arrow[d, "i"] & \underline{H_{n-1}(X_{p-1}, X_{p-2})} \arrow[r] & \cdots \\
& H_n(X_{p+1}) \arrow[r, "j"] \arrow[dd, dotted] & \underline{H_n(X_{p+1}, X_p)} \ni a \arrow[r, "k"] \arrow[rr, bend right, "d_1"] \arrow[ur, bend right, "d_2"] \arrow[uurr, bend left, "d_3"] & \underline{H_{n-1}(X_p)} \arrow[r, "j"] \arrow[d, "i"] & \underline{H_{n-1}(X_p, X_{p-1})} \arrow[r] & \cdots \\
& & & H_{n-1}(X_{p+1}) \arrow[r, "j"] \arrow[d, dotted] & H_{n-1}(X_{p+1}, X_p) \arrow[r] & \cdots \\
& H_n X & & H_{n-1} X & &
\end{tikzcd}$$

Let $E^1_{pq} = H_{p+q}(X_p, X_{p-1})$, $d_1 = jk$ $(n = p+q)$

check $d_1 : E^1_{pq} \to E^1_{p-1,q}$ & $d_1^2 = j\underline{kj}k = 0$.

$E^2_{pq} = \dfrac{\operatorname{Ker} d_1}{\operatorname{Im} d_1}$, $d_2 : E^2_{pq} \to E^2_{p-2,q+1}$

$\forall a \in \operatorname{Ker} d_1$, $jk(a) = 0$, $k(a) \in \operatorname{Ker} j = \operatorname{Im} i$.

$\Rightarrow \exists b$, s.t. $i(b) = k(a)$. Set $d_2(a) = j(b)$.

Check: d_2 is well-defined & $d_2(E^2_{p,q}) \subset E^2_{p-2,q+1}$.

$E^3_{p,q} = \dfrac{\operatorname{Ker} d_2}{\operatorname{Im} d_2}$, define d_3 similarly.

$\forall a \in \operatorname{Ker} d_2$, $j(i^{-1})k(a) = 0 \Rightarrow i^{-1}k(a) = \operatorname{Ker} j = \operatorname{Im} i$

$\Rightarrow i^{-1}k(a) = i(b)$, $k(a) = i^2(b)$

$d_3(a) = j(b)$, $\underline{d_3 = j(i^{-1})^2 k}$

$\vdots$

$d_{r+1} = j(i^{-1})^r k$

- $E^1_{pq} = H_{p+q}(X_p, X_{p-1}) \Longrightarrow H_{p+q}(X)$.

Def. An <u>exact couple</u> consists of a pair of modules D, E and morphisms i, j, k making the following triangle exact at each corner.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Define differential $d: E \to E$ by $d = jk$, $d^2 = jkjk = 0$.

<u>derived exact couple</u>:

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 = i(D) \arrow[dl, "j_1"] \\ & E_1 = H(E, d) \arrow[ul, "k_1"] & \end{tikzcd}$$

$\left.\begin{array}{l} i_1 = i|_D \\ j_1 = j \circ i^{-1} : i(x) \mapsto [j(x)] \\ k_1 = \bar{k} : [y] \mapsto [ky] \end{array}\right\}$ well-defined & exact.

define $d_1 : E_1 \to E_1$, $d_1 = j_1 k_1$

. . . .

r^{th} derived exact couple.

$$\begin{tikzcd} D_r \arrow[rr, "i_r"] & & D_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$$

$d_r : E_r \to E_r$, $d_r = j_r k_r$

$E_{r+1} = H(E_r, d_r)$

If modules D, E are graded, then so is E_r.

— — — — — bigraded, — — — — —.

Thm Given an exact couple of bigraded modules

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

bi-deg $(i) = (-1, 1)$

bi-deg $(j) = (0, 0)$

bi-deg $(k) = (1, 0)$

. Then there exists a cohomology s.s. $\{E_r^{*,*}, d_r\}$.

$E_r^{*,*} = (r-1)^{th}$ derived module from $E^{*,*}$

and $d_r = j_r k_r$.

- Cohomological S.S. from a filtration.

<u>Thm.</u> Each f. dga (A, d, F^*), $d: A^n \to A^{n+1}$, F^* decreasing filtration, determines a cohomological s.s.

$$E_1^{p,q} = H^{p+q}(F^pA/F^{p+1}A)$$

Suppose further the filtration F^* is <u>bounded</u>, then

$$E_1^{p,q} \Rightarrow H^{p+q}(A,d) \text{ ; that is, } E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d).$$

Each SES $0 \to F^{p+1}A \to F^pA \to F^pA/F^{p+1}A \to 0$ induces a LES in homology:

$$\cdots \to H^n(F^{p+1}A) \xrightarrow{i} H^n(F^pA) \xrightarrow{j} H^n(F^pA/F^{p+1}A) \xrightarrow{k} H^{n+1}(F^{p+1}A) \to \cdots$$

Exact couple:

$$\begin{tikzcd}
\bigoplus_{p,q} H^{p+q}(F^pA) \arrow[rr, "i"] & & \bigoplus_{p,q} H^{p+q}(F^pA) \arrow[dl, "j"] \\
& \bigoplus_{p,q} H^{p+q}(F^pA/F^{p+1}A) \arrow[ul, "k"] &
\end{tikzcd}$$

3. Serre Spectral Seq.

Recall that a map $p: E \to B$ is a (Serre) fibration if it satisfies the homotopy lifting properties (HLP) for finite CW-cpxes X.

$$\begin{tikzcd} X\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"] \arrow[ur, dotted, "\exists \tilde{H}"] & B \end{tikzcd}$$

prop1 If B is path-connected, then $p^{-1}(b_1) \simeq p^{-1}(b_2)$ for any $b_1, b_2 \in B$.

E: total space, B: base space, $F = p^{-1}(b)$ is called the fibre

prop2. Given $F \hookrightarrow E \xrightarrow{p} B$ a fibration, $\pi_1(B)$ acts on F.

Cor. $\pi_1(B)$ acts on $\pi_*(F)$, $H_*(F)$, $H^*(F)$.

prop3. A fibration $F \hookrightarrow E \to B$ induces a LES of homotopy groups.

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

- **Thm. (Serre S.S.)**

  Let $F \hookrightarrow E \xrightarrow{p} B$ be a (homotopy) fibration with F-connected and B path-connected.

  If $\pi_1 B = 0$ or $\pi_1(B) \curvearrowright F$ trivially, then there are 1st quadrant S.S. for any abelian group R.

  cohomology: $E_2^{p,q} = H^p(B; H^q(F;R)) \Longrightarrow H^{p+q}(E;R)$

  homology: $E^2_{p,q} = H_p(B; H_q(F;R)) \Longrightarrow H_{p+q}(E;R)$.

- prop. If R is a comm. ring and $H^p(B;R), H^q(F;R)$ are free R-module of finite types for all p, q.

  then $E_2^{*,*} \cong H^*(B;R) \otimes_R H^*(F;R)$.

- $d(x^n) = n x^{n-1} dx$ if $|x|$ is even.

Example 1. Loop path fibration

$\Omega X = \mathrm{map}_*(S^1, X)$ compact-open ~~topology~~

$PX = \mathrm{map}_*(I, X)$, $I = [0,1]$

$\simeq *$

$\Omega X \longrightarrow PX \xrightarrow{\pi} X$

$\alpha \longmapsto \alpha(1)$

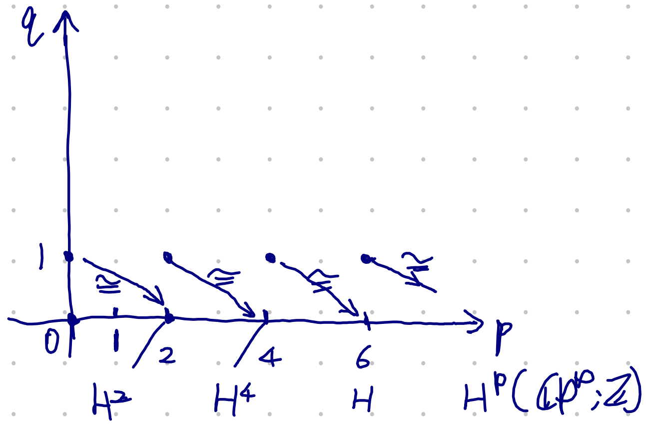
$E_2^{p,q} = H^p(X; H^q(\Omega X)) \Rightarrow H^{p+q}(PX) = 0$ except $p+q=0$.

$E_\infty^{p,q} = 0$ except $p+q=0$.

$K(\mathbb{Z},1) \longrightarrow * \longrightarrow K(\mathbb{Z},2)$ or $S^1 \hookrightarrow * \longrightarrow \mathbb{C}P^\infty$.

$E_2^{p,q} = H^p(\mathbb{C}P^\infty; \mathbb{Z}) \otimes H^q(S^1; \mathbb{Z})$

$q = 0, 1$.



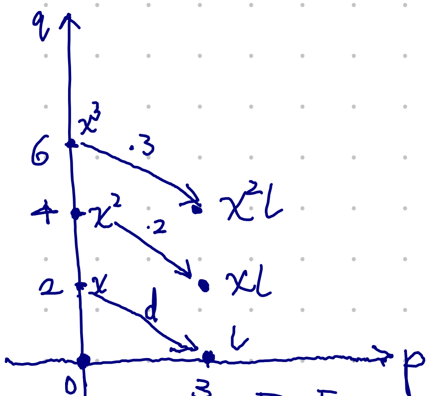
Since $E_\infty^{p,q} = 0$ except $p=q=0$

We get $H^p(\mathbb{C}P^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 2n \\ 0 & p = 2n+1 \end{cases}$ ..

Example 2

$K(\mathbb{Z},2) \longrightarrow S^3\langle 3\rangle \xrightarrow{p} S^3 \xrightarrow{\iota} K(\mathbb{Z},3)$, $\iota \in H^3(S^3;\mathbb{Z})$ a generator

$$E_2^{p,q} = H^p(S^3; H^q(K(\mathbb{Z},2);\mathbb{Z})) \cong H^p(S^3)\otimes H^q(K(\mathbb{Z},2))$$



$E_3 = E_2$

$S^3\langle 3\rangle$ is 3-connected ($\pi_{i\le 3}(S^3\langle 3\rangle)=0$)

$\rightsquigarrow d_3(x)=\iota$

$d_3(x^n)=n\cdot x^{n-1}\iota : E_3^{0,2n} \xrightarrow{\cdot n} E_3^{3,2n-2}$

$E_\infty^{3,2n-2} \cong \mathbb{Z}/n \Rightarrow H^{2n+1}(S^3\langle 3\rangle)$

$\rightsquigarrow \tilde{H}^*(S^3\langle 3\rangle;\mathbb{Z}) = \begin{cases}\mathbb{Z}/n & *=2n+1\\ 0 & *=2n\end{cases}$

$\rightsquigarrow H_*(S^3\langle 3\rangle) \cong \begin{cases}\mathbb{Z}/n & *=2n\\ 0 & *=2n+1.\end{cases}$

$$E_\infty = E_4^{3,2n-2} \cong H^{2n+1}$$

Example 3. the Gysin Sequences (Example 5.C)

$(S^n \to E \xrightarrow{p} B)$ **Thm.** Let $F \to E \xrightarrow{p} B$ be a fibration with F a homology sphere for some $n \geq 1$ $(H_*(F) \cong H_*(S^n))$

Suppose $\pi_1 B$ acts trivially or B is 1-connected. Then there is an exact seq:

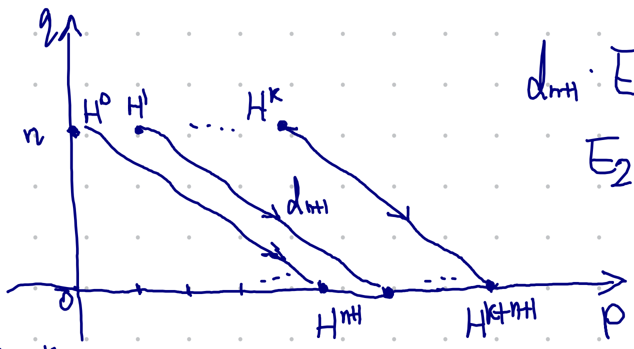
cohomological: $\to H^k(B;R) \xrightarrow{d_{n+1}} H^{k+n+1}(B;R) \xrightarrow{p^*} H^{k+n+1}(E;R) \longrightarrow H^{k+1}(B;R) \longrightarrow \cdots$

where $d_{n+1} = c \cup -$ for some $c \in H^{n+1}(B;R)$; if n is even and $2 \neq 0$ in R, ~~then~~ $2c = 0$.

(homological $\cdots \longrightarrow H_k(E;R) \xrightarrow{p_*} H_k(B;R) \xrightarrow{d^{n+1}} H_{k-n-1}(B;R) \longrightarrow H_{k-1}(E;R) \longrightarrow \cdots$)

Partial proof. E_2-page in the S.S.S. with R-coefficients:

$$E_2^{p,q} = H^p(B; H^q(S^n)) \cong H^p(B;R) \otimes H^q(S^n;R) \Rightarrow H^{p+q}(E;R).$$
$$(q = 0, n)$$



$H^k = H^k(B;R)$

$$d_{n+1} : E_2^{k,n} = H^k \longrightarrow E_2^{k+n+1,0} = H^{k+n+1}$$

$E_2 = \cdots = E_{n+1}$, $E_{n+2} = E_\infty$.

Then the exact sequence follows by Ex. 2 on Page 5:

$$\begin{tikzcd}
\cdots \arrow[r] & E_2^{k,n} \arrow[r, "d_{n+1}"] \arrow[d, equal] & E_2^{k+n+1,0} \arrow[rr] \arrow[d, equal] & & H^{k+n+1}(E;R) \arrow[rr] \arrow[drr] & & E_2^{k+1,n} \arrow[r, "d_{n+1}"] \arrow[d, equal] & E_2^{k+n+2,0} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^k(B;R) \arrow[r] & H^{k+n+1}(B;R) \arrow[urr] & & & & H^{k+1} \arrow[r] & H^{k+n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Q.E.D.

Ex. $S^n \to E \to B$, $n \geq 2$. the Wang Seq. has the form:

$$\cdots \to H^k(E;R) \to H^k(F;R) \xrightarrow{\theta} H^{k-n+1}(F;R) \to H^{k+1}(E;R) \to \cdots$$

THANKS !