1-parameter families.

straight-line homotopy.

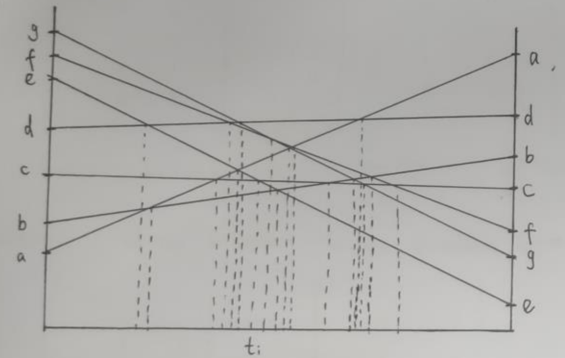
$f, g: K \to \mathbb{R}$ monotonic.

$F: K \times [0,1] \longrightarrow \mathbb{R}$ straight-line homotopy.

$F(\sigma, t) = (1-t) f(\sigma) + t\, g(\sigma)$

obviously, for $\forall t \in [0,1]$ $F(\ ,t) = f_t$ is monotonic.

①



②

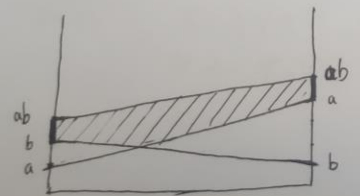
$\{a, b, ab\}$

$f: a, b, ab : 1, 2, 3$

$g: a, b, ab : 4, 1, 5$

$f \Rightarrow [f(b), f(ab)]$

$g \Rightarrow [g(a), g(ab)]$



1

We have the total order of the simplex in K, that is define by f_t. And we know the order is same, when $\forall t \in [t_i, t_{i+1}]$.

$\Rightarrow Dgm(f_t) = Dgm(f_{t'})$, $\forall t \in [t_i, t_{i+1}]$, for all i.

So, we only consider the location of $[t_i - \varepsilon, t_i + \varepsilon]$.

Simplify, we assume that there are at most two simplexes that satisfy the condition of $f_t(\sigma) = f_t(\tau)$. $t \in (0,1)$.

$f_t \Rightarrow$ boundary matrix $\partial_t \Rightarrow$ Reduced R_t

Reduced: $low(j_1) \neq low(j_0)$, $j_0 \neq j_1$)

$f_{t_i-\varepsilon}(\sigma) < f_{t_i-\varepsilon}(\tau) \Rightarrow f_{t_i+\varepsilon}(\tau) < f_{t_i+\varepsilon}(\sigma)$

$\partial_{t_i-\varepsilon} \Rightarrow P \partial_{t_i-\varepsilon} P = \partial_{t_i+\varepsilon}$

$$P = \begin{bmatrix} I_{i-1} & & \\ & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \\ & & I_{n-i-1} \end{bmatrix}$$

Given, $R = \partial V \Rightarrow \partial = RU$ $\quad$ R: reduced, upper triangular, invertible.

U: upper triangular, invertible.

Goal: decomposition $\partial_{t_i+\varepsilon} = P \partial_{t_i-\varepsilon} P = R_{t_i+\varepsilon} \cdot U_{t_i+\varepsilon}$

$R = \partial V \Rightarrow \partial = RU$.

Note $P^2 = I$. $\Rightarrow P\partial P = PRUP = \underbrace{PRP}_{\text{non-reduced}}\underbrace{PUP}_{\text{non-triangular.}}$

~~non-triangular~~

PR may change the ~~non-triangular~~ reduced property of R, so we consider how one fix the deficiency.

Deficiency : 1. 2

$$R \quad \begin{array}{c|c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c|c} & \begin{array}{cc} k & l \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$low(k) = i$, $low(l) = i+1$ $\qquad$ $low(k) = i+1$, $low(l) = \cancel{i+1}\ i$

Fixing : Add the k-column to the l-column, before the transposition.

$$\tilde{R} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

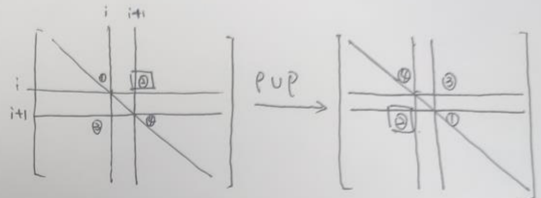
$R = \partial V$ $\qquad \tilde{R} = \partial \cdot V \cdot (I + E_{kl}) = \partial \tilde{V}$

$\Rightarrow \partial = \tilde{R} \cdot (\tilde{V})^{-1} \triangleq \tilde{R} \cdot \hat{U}$

$\hat{U}$ is ~~tri~~ upper-triangular

$\tilde{R}$ isn't reduced.

Deficiency: PUP non-triangular, if $U[i, i+1] = 1$



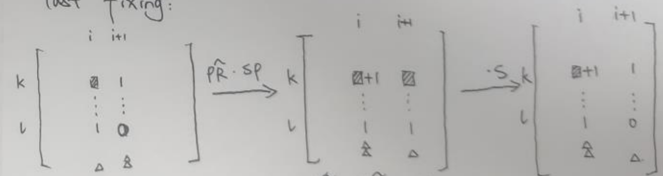
Fixing: Add the $i+1$-row to i-row $\qquad S = I + E_{i,i+1}$ $(S^2 = I)$

$\partial' = P\tilde{R}\hat{U}P = \boxed{P\tilde{R}SP}\boxed{PS\hat{U}P}$

$PS\hat{U}P$ is triangular.

but $P\tilde{R}SP$ may not reduced, because of S.

last fixing:



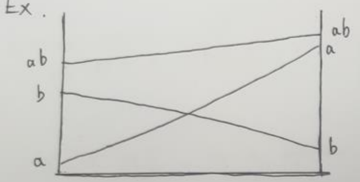
$P\tilde{R}$ $\qquad P\tilde{R}\cdot SP$ $\qquad P\tilde{R}SPS$

$P\partial P = \underbrace{P\tilde{R}SPS}\,\underbrace{SPS\hat{U}P}$ (Encounter ~~a~~ deficiency)

$\partial_{t,t+\varepsilon} = R_{t,t+\varepsilon} \cdot U_{t,t+\varepsilon}$.

3.

Ex.



$(b, ab) \to (a, ab)$

Ex.