Morse Theory

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1 Introduction

We can get information about a topological space by considering the functions on it. A natural question is, what kind of functions can tell us information as much as possible and meanwhile such functions are general enough. There is another natural question in algebraic topology. What topological space can be equipped a CW structure? A good news is that the smooth manifolds actually have a CW homotopy type by considering the so-called Morse functions on it.

We will discuss about the properties of Morse functions and their generalization–Morse-Smale functions which can be used to construct Morse homology.

2 An example

Before discussing the Morse theory formally, let's see an example.

Here is a torus M standing vertically(see figure 1), let f be the "height function" i.e. the projection to the axis. Let $M^a = \{x \in M \mid f(x) \leq a\}$. Let's figure out what M^a looks like.

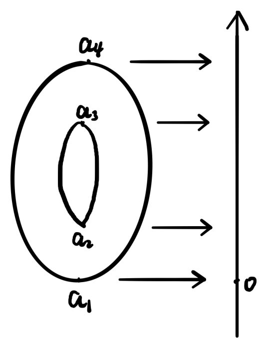

Figure 1: The height function of a torus

(1) If $a < 0 = f(a_1)$, then M^a is empty.

(2) If $f(a_1) < a < f(a_2)$, then M^a looks like Figure 2

Figure 2: $f(a_1) < a < f(a_2)$

(3) If $f(a_2) < a < f(a_3)$, then M^a looks like Figure 3

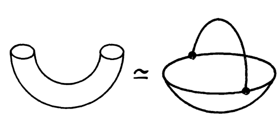

Figure 3: $f(a_2) < a < f(a_3)$

(4) If $f(a_3) < a < f(a_4)$, then M^a looks like Figure 4

(5) If $a > f(a_4)$, then M^a is the whole torus.

Observe that the homotopy type of M^a changes when a passes the values of a_1, a_2, a_3, a_4 respectively. Moreover, it changes by attaching cell of certain dimension. We will see what causes this phenomenon later.

Now, let's start Morse theory formally.

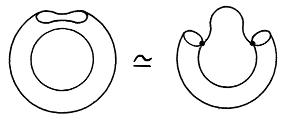

Figure 4: $f(a_3) < a < f(a_4)$

3 Morse functions

Def 3.1 *Let $f \in C^\infty(M)$. p is called critical if* $\mathrm{d}f\,|_p = 0$

Surprisingly, a_1, a_2, a_3, a_4 in the previous example are all critical points!

Def 3.2 Hess$f$ *is a tensor of type (0,2) defined by* $\mathrm{Hess}f_p(V,W) = W(\tilde{V}(f))(p)$, *where* $V, W \in T_pM$ *and* $\tilde{V}$ *is a smooth extension of V. In a local coordinate,* $\mathrm{Hess}f_p = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{i,j}$

It is easy to show that this definition is independent of the extension of V. Hence it is well-defined.

Def 3.3 *A critical point p is called non-degenerate if* $\mathrm{Hess}f_p$ *is non-singular.*

This definition of Hess is different from that defined in Riemannian Geometry in which Hess$f$ is defined as $\nabla^2 f$ so that $\mathrm{Hess}f(V,W) = W(V(f)) - (\nabla_W V)(f)$

Lemma 3.4 (Morse) *Let p be a non-degenerated critical point of f. Then $\exists$ a local coordinate $(U, x_1, \cdots, x_n)$ with $x_i(p) = 0$ s.t. $f(x) = f(p) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_n^2$*

Sketch of the proof. In an arbitrary local coordinate $(\mathrm{U}, y_1, \cdots, y_n)$, $f(y) = f(p) + \sum y_i y_j \int_0^1 (1-t)\frac{\partial^2 f}{\partial y_i \partial y_j}\,\mathrm{d}t$(see Warner, *GTM*94, 13). Note that $\int_0^1 (1-t)\frac{\partial^2 f}{\partial y_i \partial y_j}\,\mathrm{d}t$ is symmetric so that it can be normalized. The complete proof can be found in Milnor, *Morse Theory*, 6

Def 3.5 *$f \in C^\infty(M)$ is called a Morse function if*
(1) Each critical point is non-degenerated
(2) The critical points have distinct values.

In some books, the definition of the Morse functions does not require the condition(2). We will see what role does this condition play in the Morse theory later.

Cor 3.6 *If f is a Morse function, then* $\mathrm{Crit}(f)$ *is discrete where* $\mathrm{Crit}(f)$ *is the set of all critical points of f.*

Proof. It follows from the Morse Lemma immediately.

Here is a question. Given a smooth manifold M, does M admit a Morse function? How general are the Morse functions?

Prop 3.7 *Let $M \subset \mathbb{R}^N$ be a submanifold of dimension n. Then for almost every $p \in \mathbb{R}^N$, the function $f_p : M \to \mathbb{R},$, $x \mapsto \|x - p\|^2$ is a Morse function*

Proof. The derivative of f_p is given by $\mathrm{d}f_{p,x}(v) = 2\langle x - p, v\rangle$. Therefore the critical points occur exactly when T_xM is normal to $x - p$. Choose a local coordinate $(u_1, \cdots, u_n)$ around x. Then

$$\frac{\partial f_p}{\partial u_i} = 2(x-p)\cdot\frac{\partial x}{\partial u_i}, \quad \frac{\partial^2 f_p}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_i}\cdot\frac{\partial x}{\partial u_j} + (x-p)\cdot\frac{\partial^2 x}{\partial u_i \partial u_j}\right).$$

By definition, x is a non-degenerate critical point iff x-p is normal to T_xM and the matrix on the right is non-degenerate. By Sard's Thm, it suffices to show that the $p \in \mathbb{R}^N$ s.t. $x - p$ is normal to T_xM and the matrix on the right is singular, are exactly the critical points of some smooth map.

To find such a map, consider the normal bundle NM of M in $mathbbR^N$. Define the map $NM \to \mathbb{R}^N$, $(x, v) \mapsto x + v$. It can be verified that $p = x + v$ is a critical point iff $2(\frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + v\frac{\partial^2 x}{\partial u_i \partial u_j})$ is singular. Hence, this function is what we need.

Rem 3.8 *By Whitney embedding theorem, every smooth manifold can be embedded in some Euclidean space. Thus for any smooth manifold, there exists a Morse function on it.*

Cor 3.9 *The Morse functions are open and dense in* $C^\infty(M)$

Although almost every smooth function on a manifold is a Morse function, there are still many functions which are not Morse function. Here are some non-examples:

(1)The "height function" of a horizontal torus. The critical points form two circle which are not discrete, so is not a Morse function. However, it is a so-called Morse-Bott function which is a generalization of Morse functions.

(2)$f(x) = x^3$. Even this usual function is not a Morse function.

(3)$f(x, y) = x^2y^2$. $f^{-1}(0)$ is not a manifold so it is not even a Morse-Bott function.

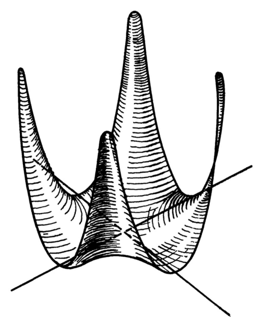

Figure 5: $f(x, y) = x^2y^2$

4 Fundamental theorems of Morse theory

Def 4.1 *Let f be a Morse function. λ in the lemma 2.2.4 is called the index of f at p and it is denoted as* $\mathrm{Ind}_f(p)$*. Equivalently,* $\mathrm{Ind}_f(p)$ *is the dimension of the largest subspace of T_pM on which* Hess_f *is negative define.*

One can easily figure out the indices of the "height function" h at the critical points in the initial example. i.e. $\mathrm{Ind}_h(a_1) = 0$, $\mathrm{Ind}_h(a_2) = 1$, $\mathrm{Ind}_h(a_3) = 1$, $\mathrm{Ind}_h(a_4) = 2$.

Recall that every manifold can be equipped with a Riemannian metric. The torus in the initial example is embedded in $\mathbb{R}^3$ so that we can consider the "height" because it is automatically a Riemannian manifold.

Recall that if f is a smooth function on a Riemannian manifold (M, g), then the gradient $\mathrm{grad}f$ is defined to be $(\mathrm{d}f)_\#$. In a local coordinate, $\mathrm{grad}f = \frac{\partial f}{\partial x_i} g^{ij} \frac{\partial}{\partial x_j}$.

Thm 4.2 (Regular interval theorem) *Suppose $f : M \to [a, b]$ be a smooth map on a compact Riemannian manifold with boundary. Suppose that f has no critical points and that $f(\partial M = \{a, b\}$. Then there is a diffeomorphism*

$$F : f^{-1}(a) \times [a, b] \to M$$

s.t. $\pi = f \circ F$ where π is the projection from $f^{-1}(a) \times [a, b]$ to $[a, b]$.

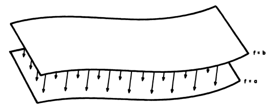

Figure 6: flow along $\frac{\mathrm{grad} f}{|\mathrm{grad} f|^2}$

Sketch of the proof. Let $\eta_x(t)$ be the integral curve of $\frac{\mathrm{grad} f}{|\mathrm{grad} f|^2}$. Then $F(x,t) := \eta_x(t)$ is such a function.

There is also a intuitive explanation. Note that $\mathrm{grad} f_p = 0 \Leftrightarrow p \in \mathrm{Crit}(f)$ and $\mathrm{grad}(f)$ is orthogonal to the level sets. Then each point of each level can flow along $\frac{\mathrm{grad} f}{|\mathrm{grad} f|^2}$ to another level set.

Cor 4.3 (Fundamental theorem 1) *Let M be a compact manifold, and $f : M \to \mathbb{R}$ a Morse function. Let $a < b$ and suppose that $f^{-1}[a,b]$ contains no critical points. Then M^a is diffeomorphic to M^b. Furthermore, M^a is a deformation retract of M^b.*

*Proof.*By the regular interval theorem, there is such F. Since $f^{-1}(a) \times \{a\}$ is a deformation retract of $f^{-1}(a) \times [a,b]$, we see that $f^{-1}(a)$ is a deformation retract of $f^{-1}([a,b])$. We can now paste this deformation retraction with the identity on M^a to obtain the deformation retraction from M^b to M^a.

To prove that M^a is diffeomorphic to M^b we apply the same principle, but we need to be more careful to preserve smoothness during the patching process.

Since $\mathrm{Crit}(f)$ is a closed subset of of the compact M, it is also compact. Therefor there are real numbers c and d with $c < d < a$ so that there are no critical values in $[c,b]$.

By the regular interval theorem there is a natural diffeomorphism F from $f^{-1}([c,b])$ to $f^{-1}c \times [c,b]$, that maps $f^{-1}[c,a]$ diffeomorphically onto $f^{-1}c \times [c,a]$. There is also a diffeomorphism $H : f^{-1}c \times [c,b] \to f^{-1}c \times [c,a]$, and we can insist that it be the identity on $f^{-1}c \times [c,d]$. Thus

$$F^{-1} \circ H \circ F : f^{-1}([c,b]) \to f^{-1}([c,a])$$

is a diffeomorphism that is the identity on $f^{-1}([c,d])$, and thus we can patch it together with the identity on M^d to creat a diffeomorphism from M^b to M^a.

Cor 4.4 (Reeb) *If M is a compact manifold and f is a Morse function on it with only two critical points, then M is homeomorphic to a sphere.*

This is an easy application of the fundamental theorem 1 or the regular interval theorem, so I just give the hint.

(1)The two critical points must be the minimum and maximum points. We may assume the values are 0 and 1 respectively.

(2)Let ϵ be small enough s.t. $f^{-1}(\epsilon)$ and $f^{-1}(1-\epsilon)$ are contained in the Morse charts. Then what can you say about them?

Thm 4.5 (Fundamental theorem 2) *Let $f \in C^\infty(M)$, and let p be a non-degenerate critical point with index λ. Setting $f(p) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact, and contains no critical point of f other than p, for some $\epsilon > 0$. Then $\forall$ sufficiently small ϵ, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a λ-cell attached.*

The proof is omitted. But one can check this theorem by applying it to the previous example. It is benefit for you to think about what submanifold is attached while passing a critical point in the example.

5 Morse inequalities

Recall the Betti numbers are the ranks of the homology groups. Homology groups are related to the CW structure. The fundamental theorem 2 tells us the CW structure of a manifold is related to the critical points of a Morse function. So what is the relation between the Betti numbers and the number of the critical points?

Thm 5.1 (Morse inequalities) *Let f be a Morse function on a manifold, b_k be the k^{th} Betti number, and c_k be the number of critical points of index k. Then*

$$(1) c_k(f) \geq b_k(M) \quad \forall k$$

$$(2) \sum_{k=0}^{n} (-1)^k c_k \geq \sum_{k=0}^{n} (-1)^k b_k$$

(1) is called weak Morse inequality and (2) is called strong Morse inequality.

Proof. We only prove the weak one. We may regard M as $M^{+\infty}$. When a passes a critical point of index λ, then there is a λ cell attached to M^a by the fundamental theorem 2. Since attaching a λ cell may or may not cause that $b_k(M^a)$ plus 1, we get the weak Morse inequality.

Cor 5.2 *If $c_{\lambda+1} = c_{\lambda-1} = 0$, then $c_\lambda = b_\lambda$ and $b_{\lambda+1} = b_{\lambda-1} = 0$.*

Cor 5.3 *Let f be a Morse function on M. Then f has at least as many critical points as the sum of the ranks of the homology groups of M.*

6 Morse-Smale functions

Def 6.1 *Let θ be the flow with respect to* grad$f$ *i.e.* $\frac{\mathrm{d}}{\mathrm{d}t}\theta + \nabla_\theta f = 0$. *Let $a \in \mathrm{Crit}(f)$. We define the two subsets of M:*

$$W^s(a) = \{x \in M | \lim_{t \to +\infty} \theta_x(t) = a\}, W^u(a) = \{x \in M | \lim_{t \to -\infty} \theta_x(t) = a\}$$

They are called the stable manifold of a and the unstable manifold of a respectively.

In other words, $W^s(a)$ is the set of points on M that flow down to a, and $W^u(a)$ is the set of points that would flow up to a if the gradient flow were reversed. They are called "manifolds" is justified by the stable manifold theorem:

Thm 6.2 (Stable manifold theorem) *Let $f : M^n \to \mathbb{R}$ be a Morse function, $a \in \mathrm{Crit}(f)$ of index λ. Then $W^u(a)$ and $W^s(a)$ are smooth submanifolds diffeomorphic to the open disks D^λ and $D^{n-\lambda}$ respectively.*

This proof is very complicated, but you can find it on the internet easily. Try to convince yourself by considering the example.

Prop 6.3 *Let f be a Morse function on a compact Riemannian manifold (M,g). Then*

$$M = \bigcup_{a \in \mathrm{Crit}(f)} W^u(a)$$

is a partition of M into disjoint sets.

Proof. The fact that the union of the the $W^u(a)$ is M comes from the fact that every point of M lies on a flow curve θ, and we can always find $\lim_{t\to-\infty} \theta(t)$

The fact that the $W^u(a)$ and $W^u(b)$ are disjoint is due to the fact that θ is unique.

So far, you may aks whether we can get the CW structure by this proposition. Unfortunately, the answer is No. When applying the proposition to the initial example, you will find the the way that the disks attach don't follow the rules of CW complex. However, we will fix this problem by introduce Morse-Smale functions.

Def 6.4 *Suppose f is a Morse function satisfying the extra condition that $\forall a, b \in \mathrm{Crit}(f)$, $W^u(a)$ intersects $W^s(b)$ transversely. Then f is called a Morse-Smale function.*

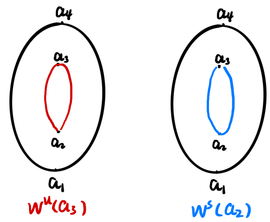

Figure 7: $W^u(a_3)$ and $W^s(a_2)$

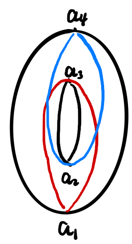

Figure 8: $W^u(a_3)$ and $W^s(a_2)$ after tilting

Now you can see that the initial example is not a Morse-Smale function since $W^u(a_3)$ does not intersect $W^s(a_2)$ transversely.

However if you let the torus tilt a little, the the height function will become a Morse-Smale function. See the figure 8.

Prop 6.5 *Let f be a Morse-Smale function on a compact Riemannian manifold. $p, q \in \mathrm{Crit}(f)$. Then $W(p,q) := W^u(p) \cap W^s(q)$ is an embedded submanifold of dimension $\lambda_p - \lambda_q$*

Proof. It follows from the transversality and Stable manifold theorem immediately.

Cor 6.6 *Keep the conditions above. The index of the critical points is strictly decreasing along gradient flow lines. That is, if $p, q \in \mathrm{Crit}(f)$ with $W(p,q) \neq \varnothing$, then $\lambda_p > \lambda_q$.*

Proof. $W(p,q)$ contains one flow line from p to q which is of dimension 1, thus dim $W(p,q) \geq 1$. Then use the proposition above.

Now we have known that the Morse functions are general enough (see Cor 3.0.9) and Morse-Smale functions are more strict than Morse functions. So are Morse-Smale still general enough? The answer is Yes.

Def 6.7 *Let X be a topological space. $A \subset X$ is called residual if it is a contable intersection of open dense subsets of X. $B \subset X$ is called generic if it contains a residual set.*

Thm 6.8 (Kupka-Smale) *If (M,g) is compact, then the set of Morse-Smale functions is generic. Moreover, If f is a Morse fucntion, then for an open dense set of metrics, f is a Morse-Smale function.*

Def 6.9 $\mathrm{grad} f$ *is said to be in standard form near $p \in \mathrm{Crit} f(f)$ if $\exists$ a chart near p s.t.* $\mathrm{grad} f$ *is of the form* $\sum c_i x_i \frac{\partial}{\partial x_i}$

In the Morse chart is $\mathrm{grad} f = -2x_1 \frac{\partial}{\partial x_1} - \cdots - 2x_\lambda \frac{\partial}{\partial x_\lambda} + 2x_{\lambda+1} \frac{\partial}{\partial x_{\lambda+1}} + \cdots + 2x_n \frac{\partial}{\partial x_n}$? No. Because $\mathrm{grad} f$ is relied on the metric, we cannot take the derivative like in the Euclidean spaces.

However, the good news is that every metric can be modified arbitrarily small s.t. $\mathrm{grad} f$ at each critical point p is of the standard form. We say such a metric is compatible with the Morse chart of f.

To see this, it easy to find such a metric locally, and then glue them together by partition of unity to get this metric.

Def 6.10 *Two finite CW complexes X and Y will be called cell equivalent providing there is a homotopy equivalence $h : X \to Y$ with the property that there is a one-to-one correspondence between cells of X and cells of Y s.t. if $e \subset X$ corresponds to $e' \subset Y$ then h maps X(e) to Y(e') and is a homotopy equivalence of these subcomplexes, where X(e) denotes the subcomplex generated by e.*

Thm 6.11 (Cell equivalence theorem) *Let f be a Morse-Smale function on a finite dimensional compact Riemannian manifold (M,g), and g is compatible with the Morse chart. Then ∃! CW complex, up to cell equivalence, and a homotopy equivalence $h : M \to X$ s.t. $\forall p \in \mathrm{Crit}(f)$ of index λ, the image $h(W^u(p))$ is contained in the base $X(e^\lambda)$ of a unique λ-cell $e^\lambda \subset X$.*

This theorem tells us we can get the CW structure of a manifold by considering the Morse-Smale functions on it. Specificly, we can get the cells of the CW structure are unstable manifols. (Check it for the tilt torus.)

7 Morse homology

Def 7.1 *Let f be a Morse-Smale function, $p, q \in \mathrm{Crit}(f)$. $M(p,q)$:=the moduli space of flow lines from p to q. We can identify M(p,q) with $W(p,q)/\mathbb{R}$ where $\mathbb{R}$ acts on W(p,q) by the flow of* grad*f.*

By the Morse-Smale condition and quotient manifold theorem, $M(p,q)$ has the structure of a $(\mathrm{Ind}_p - \mathrm{Ind}_q - 1)$-dimensional manifold.

Thm 7.2 *If M(p,q) can be compactified by adding in broken flow lines from p to q, i.e. paths from p to q which are the concatenation of flow lines through intermediate critical points. The compactification $\overline{M(p,q)}$ is a smooth manifold with corners, and the k-times broken flowlines form the codimension k parts of the boundary.*

In particular, if $\mathrm{Ind}_p = \mathrm{Ind}_q + 1$, there are no possible intermediate critical points so $M(p,q) = \overline{M(p,q)}$ is compact and therefore finite.

Example 7.3 *Here is the tilting torus. The height function on it is a Morse-Smale function as mentioned before. There are 8 broken flow lines from a_1 to a_4. $W(a_1, a_4)$ has 4 components. When $W(a_1, a_4)$ is acted by the gradient flow, each component becomes an open interval. Moreover, every two intervals don't have the common boundary. The 8 broken flow lines are exact the boundary of these 4 open intervals. Hence, $M(a_1, a_4)$ consists of 4 open intervals and it can be compactified into 4 closed intervals by adding the broken flow lines.*

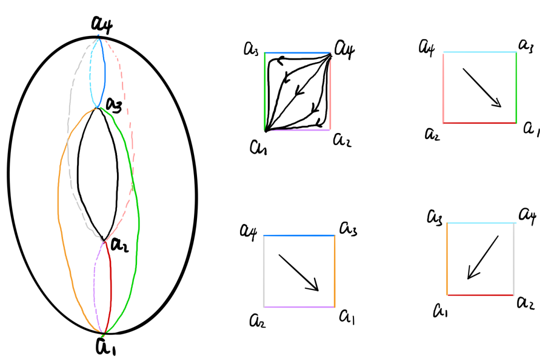

Figure 9: Compactification of $M(a_1, a_4)$

Def 7.4 *The Morse chain complex is defined as follows:*

$$C_k := \mathrm{Z}/2\mathrm{Z}\,\mathrm{Crit}(f), \partial_k(p) = \sum_{q \in \mathrm{Crit}_{k-1}(f)} \#_2 M(p,q) q$$

Where $\#_2$ means the cardinality mod 2

We need to prove this does define a chain complex. To see this, only need to verify that $\partial_{k-1} \circ \partial_k = 0$. By direct computation,

$$\partial_{k-1} \circ \partial_k(p) = \sum_{r \in \mathrm{Crit}_{k-1}(f)} \#_2 M(p,r)(\sum_{q \in \mathrm{Crit}_{k-2}(f)} \#_2 M(r,q)) = \sum_q (\sum_r \#_2 M(p,r) \#_2 M(r,q))q$$
$$= \sum_q \#_2(\bigcup_r M(p,r) \times M(r,q))q = \sum_q \#_2(\partial \overline{M(p,q)})q = 0$$

The last "=" holds because $\partial \overline{M(p,q)}$ is a compact 1-dimensional manifold with corner, thus it has to be a finite disjoint union of closed intervals or circles. Hence, the cardinality of its boundary must be an even number.

Now, we have a complex so that we can define homology groups with respect to this complex. This homology is called Morse homology.

Rem 7.5 *Actually, $\mathbb{Z}/2\mathbb{Z}$ can be replaced by $\mathbb{Z}$. But if we do this, then we have to consider the orientation of the unstable manifolds.*

The Morse homology is isomorphic to the singular homology.

The key to prove it is to find a homotopy between this complex and the complex with respect to the cellular homology. We have known that a Morse-Smale function can give a CW structure, so this relation is natural.

Rem 7.6 *Because of this theorem, it follows immediately that the Morse homology does not depend on the choice of the Morse function and the Riemannian metric. In other word, Morse homology is a topological property.*

Example 7.7 *Here is a sphere M. Let f be the height function. Then f is a Morse-Smale function.* $\mathrm{Ind}(a_1) = 0$, $\mathrm{Ind}(a_2) = 1$, $\mathrm{Ind}(a_3) = \mathrm{Ind}(a_4) = 2$.
$M(a_1, a_2)$ have two elements (blue and purple curves), $M(a_2, a_3)$. Thus, $\partial_1 = 0$.
$M(a_2, a_3)$ and $M(a_2, a_4)$ are both one-element set (red and green respectively). Thus, ∂_2 is surjective.
The chain complex is as follows:

$$0 \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} 0$$

Then $H_0(M) = Ker\partial_0/Im\partial_1 = (\mathbb{Z}/2\mathbb{Z})/0 = \mathbb{Z}/2\mathbb{Z}$, $H_1(M) = Ker\partial_1/Im\partial_2 = (\mathbb{Z}/2\mathbb{Z})/(\mathbb{Z}/2\mathbb{Z}) = 0$, $H_2(M) = Ker\partial_2/Im\partial_3 = (\mathbb{Z}/2\mathbb{Z})/0 = \mathbb{Z}/2\mathbb{Z}$. These are the same as the singular homology of a sphere.

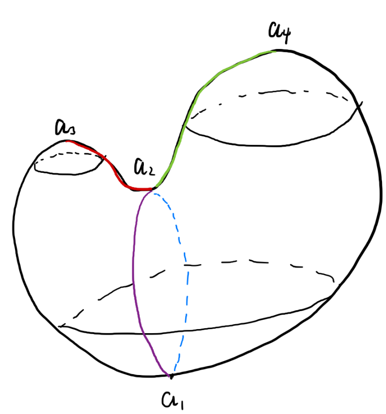

Figure 10: A twisted sphere