Branching Neuronal Morphologies.

Classification of neuronal tree.

Distance between trees : edit distance , sequence representon
blastneuron distance
(traditional) functional ~~distortion~~ distance.

d_{Bar} (not stable)

$\uparrow$

Tree $\xrightarrow[\text{TMD.}]{\text{algorithm.}}$ Barcode $\longrightarrow$ PD (bottleneck distance) (stable for $\|f-g\|_\infty$)

Topological Morphology Descriptor.

Or $\longrightarrow$ PD $\xrightarrow{\text{Gaussian kernel.}}$ PI $\longrightarrow$ Machine learning
SVM. RF
CNN. ...

PI $\downarrow$ vector

image (a matrix of pixel values)

d_{Bar} : For each Barcode, we generate a density profile as follow: $\forall x \in \mathbb{R}$, the value of the histogram is number of intervals that contain x.

$$\int_{\mathbb{R}} |\text{Barcode TMD}(T_1, f) - \text{barcode TMD}(T_2, f)| \, dx = d_{Bar}.$$

algorithm. Note that a component may die before it is born.

T: tree R: root N: nodes set B: set of branch points

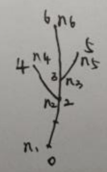
L: set of leaves. ($N = B \cup L$)

f: a real-valued function. $N \to \mathbb{R}$

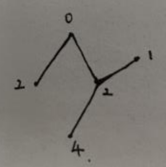
radial distance, path distance, branch length, branch order

for each $n \in N$, T_n denote the subtree with root at the node n.

L_n denote the set of leaves of T_n.



	brith time	death time
[2,4]	2	4
[3,5]	3	5
[0,6]	0	6



brith time	death time.
0	2
0	4
~~4~~	~~4~~
2	1

Let. $v(n) = \max\{f(x) \mid x \in L_n\}$

TMD algorithm.

TMD(T,f) : empty ~~set~~ list to contain pairs of real numbers.

$A \leftarrow L$. ▷ A: set of active node.

for every $l \in L$

  $v(l) = f(l)$

While $R \bar{\in} A$

  for l in A

    p: parent of l

    C: children of p

    if $\forall n \in C, n \in A$

      C_m : randomly choose one of $\{c \mid v(c) = \max_{c'} (v(c'))$ for $c' \in C\}$

      Add. p to A

      for c_i in C

        Remove c_i from A

        if $c_i \neq c_m$

          Add $(v(c_i), f(p))$ to TMD(T,f)

      $v(p) \longleftarrow v(c_m)$

Add $(v(R), f(R))$ to TMD(T,f)

Return TMD(T,f).

Bottleneck distance. $d_B(PD, PD')$

matching between two persistence diagram PD and PD' is a bijection μ between $PD \cup D$ and $PD' \cup D$.

$\mu : PD \cup D \rightarrow PD' \cup D$ (there exists).

D: $\{(x,x) \mid x \geq 0\}$ with infinite multiplicity.

$$d_B(PD, PD') = \inf_\mu \sup_{x \in PD} \| \mu(x) - x \|_\infty$$

$\| \cdot \|_\infty \Rightarrow L_\infty$.

stability. of TMD. $O(\varepsilon)$ for bottleneck distance.

stability.

$$W_\infty(X, Y) \leq \|f - g\|_\infty$$

$X = dgm_p(f)$, $Y = dgm_p(g)$ $\forall p$.

Persistence Image.

Let B be a PD. in brith-death coordinates

$T: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $T(B)$ be the multiset in brith-persistence

$(x,y) \mapsto (x, y-x)$

Let $\phi_u: \mathbb{R}^2 \to \mathbb{R}$ differentiable probability distribution.

mean $u = (u_x, u_y) \in \mathbb{R}^2$

We choose this distribution to be the normal symmetric Gaussian

$\phi_u = g_u$ $\quad$ mean : u $\quad$ variance : σ^2

$$g_u(x,y) = \frac{1}{2\pi\sigma^2} e^{-[(x-u_x)^2 + (y-u_y)^2]/2\sigma^2}$$

Let $f: \mathbb{R}^2 \to \mathbb{R}$ is nonnegative weighting function.

that ~~and~~ $f(x,0) = 0$ $\quad$ (when persistence is zero)

and f is piecewise differentiable. and continuous.

Def. For B, a PD, the corresponding persistence surface

$\rho_B: \mathbb{R}^2 \to \mathbb{R}$ is the function.

$$\rho_B(z) = \sum_{u \in T(B)} f(u)\, \phi_u(z)$$

called persistence surface.

~~is critical.~~

Def. 2. For B, a PD, its persistence image is the collection pixels $I(\rho_B)_p = \iint_p \rho_B \, dy dy$.

$$PD \xrightarrow[\text{distribution}]{\text{probability}} PS \xrightarrow[\text{integral.}]{\text{grid.}} PI$$

$H_0 \quad H_1 \quad \cdots \quad H_k$.

persistence diagram of $H_0, H_1, \cdots, H_k$

$\Downarrow$

persistence image of $H_0, H_1, \cdots, H_k$

$\Downarrow$

a singular vector representing all homological dimensions simultaneously.

Stability.

p-Wasserstein distance is generalization of bottleneck distance

$$W_p(B, B') = \inf_{\gamma: B \to B'} \left(\sum_{u \in B} \| u - \gamma(u) \|_\infty^p \right)^{\frac{1}{p}}$$

For $h: \mathbb{R}^2 \to \mathbb{R}$ differentiable, let $|\nabla h| = \sup_{z \in \mathbb{R}^2} \| \nabla h(z) \|_2$

denote $|\nabla \phi_u|$ by $|\nabla \phi|$, and $\|\phi_u\|_\infty$ by $\|\phi\|_\infty$

Thm 1. let A is maximum area of any pixel in the image

A' is the total area of the image,

n is the number of pixel in the image.

then.

(i) $\| I(\rho_B) - I(\rho_{B'}) \|_\infty \leq \sqrt{10}\, A \left(\|f\|_\infty |\nabla \phi| + \|\phi\|_\infty |\nabla f| \right) W_1(B, B')$

(ii) $\| I(\rho_B) - I(\rho_{B'}) \|_1 \leq \sqrt{10}\, A' \left(\|f\|_\infty |\nabla \phi| + \|\phi\|_\infty |\nabla f| \right) W_1(B, B')$

(iii) $\| I(\rho_B) - I(\rho_{B'}) \|_2 \leq \sqrt{10n}\, A \left(\|f\|_\infty |\nabla \phi| + \|\phi\|_\infty |\nabla f| \right) W_1(B, B')$

Thm 2. The persistence image $I(\rho_B)$ with Gaussian distribution

(i) $\| I(\rho_B) - I(\rho_{B'}) \|_1 \leq \left(\sqrt{5} |\nabla f| + \sqrt{\frac{10}{\pi}} \frac{\|f\|_\infty}{\sigma} \right) W_1(B, B')$

(ii) $\| I(\rho_B) - I(\rho_{B'}) \|_2 \leq \left(\sqrt{5} |\nabla f| + \sqrt{\frac{10}{\pi}} \frac{\|f\|_\infty}{\sigma} \right) W_1(B, B')$

(iii) $\| I(\rho_B) - I(\rho_{B'}) \|_\infty \leq \left(\sqrt{5} |\nabla f| + \sqrt{\frac{10}{\pi}} \frac{\|f\|_\infty}{\sigma} \right) W_1(B, B')$