# Spectral Sequences II

## — Some formal definitions

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Ref. J. McCleary, A user's guide to spectral sequences. (2nd edition).

# 1. Some algebraic Concepts

Let $R$ be a comm. ring.

- graded $R$-modules $A_* = \bigoplus_n A_n$.
  $x \in A_n$ is called homogeneous of deg. $n$ $(|x| = n)$ } Category.

  $\mathrm{Hom}(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_n)$

  $(A_* \otimes B_*)_n = \bigoplus_{i+j=n} A_i \otimes B_j$.

  $\mathrm{Hom}^k(A_*, B_*) = \prod_n \mathrm{Hom}(A_n, B_{n+k})$

- graded algebra/ring $= \begin{cases} \text{graded module } A_* = \bigoplus_n A_n \\ \text{a degree 0 } R\text{-linear map (multiplication) } A_* \otimes A_* \xrightarrow{\cdot} A_*. \Rightarrow A_n \cdot A_m \subset A_{n+m}, \\ \text{which is associative \& unital} \end{cases}$

  eg. $H^*(X;R)$ — cohomological ring.

- differential graded algebra (dga) = graded ring $A_*$ + a differential $d: A_* \to A_*$
  degree $\begin{cases} +1 \text{ --- cohomology} \\ -1 \text{ --- homology} \end{cases}$ s.t.

  i.) $d^2 = d \circ d = 0$

  ii) Leibniz rule: $d(xy) = dx \cdot y + (-1)^{|x|} x \cdot dy$.

eg. $\underset{+1}{C^*(X;R)}$, $\underset{-1}{C_*(X;R)}$ chain complex.

- A <u>filtration</u> of an R-module $M$ is a sequence of submodules:

  increasing $F_*M$ : $\cdots \subseteq F_{n-1}M \subseteq F_nM \subseteq F_{n+1}M \subseteq \cdots \subseteq M = \bigcup_n F_nM$

  descreasing $F^*M$ : $\cdots \subseteq F^{n+1}M \subseteq F^nM \subseteq F^{n-1}M \subseteq \cdots \subseteq M$

The <u>associated graded module</u> of a filtered module $M$ is

the graded module $Gr(M) = \bigoplus_n Gr(M)_n$ , $Gr(M)_n = \begin{cases} F_nM/F_{n-1}M & \text{for increasing filtration.} \\ F^nM/F^{n+1}M & \text{.. decreasing -- .} \end{cases}$

- <u>filtered differential graded module</u> $=$ dgm $(A_*, d)$ with a "coherent" filtration $F^*M$ :

  $(A_*, d, F_*)$

  $d : F^nM \longrightarrow F^nM$; that is, $(F^*M, d)$ is a sub-dga.

"coherent" $\Longrightarrow$ $H(A_*, d) = {}^{\ker d}/_{\operatorname{Im} d}$ inherits a filtration:

$$F_nH(A_*, d) = \operatorname{Im}\left( H(F^nA_*, d) \xrightarrow{H(\text{incl})} H(A_*, d) \right).$$

# 2. Spectral Sequences. (s.s.)

- A cohomology s.s. is a sequence of bigraded module $E_r^{p,q}$, $r \geq 1$, $p,q \in \mathbb{Z}$ together with differentials $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ s.t. $d_r^2 = 0$

  and $$E_{r+1}^{p,q} = H(E_r^{p,q}, d_r) = \frac{\mathrm{Ker}(E_r^{p,q} \to E_r^{p+r,q-r+1})}{\mathrm{Im}(E_r^{p-r,q+r-1} \to E_r^{p,q})},$$ set $E_\infty^{*,*} = \underset{\overrightarrow{r}}{\mathrm{colim}}\, E_r^{*,*}$.

  - We say $d_r$ has bidegree $(r, -r+1)$, total degree 1.

- The s.s. $\{E_r, d_r\}$ converges to the graded R-module $M^*$ ($E_r^{p,q} \Longrightarrow M^{p+q}$) if there exists a filtration $F^*$ of $M^*$ such that

$$\begin{array}{c} Gr(M^n)_p \cong E_\infty^{p,n-p}, \ \forall p,n. \\ \wr\wr \\ F^p M^n / F^{p+1} M^n \end{array}$$

- The s.s. $\{E_r, d_r\}$ collapses at $E_{r_0}$ if $d_r = 0$ for any $r \geq r_0$ : $E_\infty^{*,*} = E_{r_0}^{*,*}$.

- 1st quadrant s.s. : $E_r^{p,q} = 0$ if $p<0$ or $q<0$.

Rmk: A 1st quadrant s.s. $\{E_r^{*,*}, d_r\}$ converges in a strong way: $\forall (p,q), \exists r_0$ s.t. $d_r^{p,q} = 0$ for any $r \geq r_0$.

eg. Let $r > \max(p, q+1)$. $\underset{0}{\underset{\|}{E_r^{p-r,q+r-1}}} \xrightarrow{d_r} E_r^{p,q} \xrightarrow{d_r} \underset{0}{\underset{\|}{E_r^{p+r,q-r+1}}} \Rightarrow E_r^{p,q} = E_{r+1}^{p,q} \rightsquigarrow E_r^{p,q} = E_{r+k}^{p,q} = E_\infty^{p,q}$

▷ Everything above is almost the same for a homological s.s. $E^r_{p,q} \Rightarrow M_{p+q}$ except that $d_r: E^r_{p,q} \to E^r_{p-r,q+r-1}$ has bidegree $(-r, r-1)$, total degree $-1$.

Ex. 1. Suppose $E_\infty^{p,q}=0$ unless $p=k$. Then $H^{k+q}\cong E_\infty^{k,q}$.

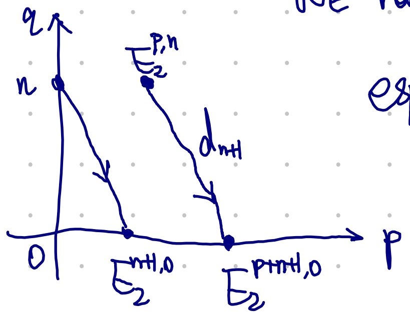
2. Suppose $E_2^{p,q}=0$ unless $q=0,n$ $(n\geq 2)$, then there is a LES:

(Example 1.D)

$$\to H^{p+n} \to E_2^{p,n} \xrightarrow{d_{n+1}} E_2^{p+n+1,0} \to H^{p+n+1} \to E_2^{p+1,n} \xrightarrow{d_{n+1}} E_2^{p+n+2,0} \to \cdots$$

Pf. The only possible nontrivial differential is $d_{n+1}: E_2^{p,n}\to E_2^{p+n+1,0}$, $p\geq 0$.

We have $E_2\cong\cdots\cong E_{n+1}$, $E_{n+2}\cong E_\infty$;




especially, $E_\infty^{p+n+1,0}\cong E_2^{p+n+1,0}/\mathrm{im}\, d_{n+1}$, $E_\infty^{p,n}\cong \ker d_{n+1}$.

$$\Leftrightarrow \quad 0\to E_\infty^{p,n}\to E_2^{p,n}\xrightarrow{d_{n+1}} E_2^{p+n+1,0}\to E_\infty^{p+n+1,0}\to 0 \qquad (1)$$

$E_2^{p,n}\Rightarrow H^{p+n}$, $F^* = F^*H^{p+n}$, $F^0=H^{p+n}\supseteq F^1\supseteq\cdots\supseteq 0$.

$E_\infty^{p-1,n+1} = F^{p-1}/F^p = 0 \Rightarrow F^0=F^1=\cdots=F^p = H^{p+n}$

$E_\infty^{p+1,n-1} = F^{p+1}/F^{p+2}=0 \therefore\Rightarrow F^{p+1}=\cdots=F^{p+n}=E_\infty^{p+n,0}$

$\therefore E_\infty^{p,n}\cong F^p/F^{p+1}\cong H^{p+n}/E_\infty^{p+n,0} \Leftrightarrow 0\to E_\infty^{p+n,0}\to H^{p+n}\to E_\infty^{p,n}\to 0 \qquad (2)$

Splice ES (1) and (2) ~~together~~:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & & & \\
 & E_\infty^{p+n,0} \arrow[d] & & & 0 \arrow[d] & & & \\
 & H^{p+n} \arrow[d] \arrow[dr] & & & & & & \\
0 \arrow[r] & E_\infty^{p,n} \arrow[r] \arrow[d] & E_2^{p,n} \arrow[r, "d_{n+1}"] & E^{p+n+1,0} \arrow[r] \arrow[dr] & E_\infty^{p+n+1,0} \arrow[r] \arrow[d] & 0 & & \\
 & 0 & & & H^{p+n+1} \arrow[d] \arrow[dr] & & & \\
 & & & 0 \arrow[r] & E_\infty^{p+1,n} \arrow[r] \arrow[d] & E_2^{p+1,n} \arrow[r] & \cdots & \\
 & & & & 0 & & &
\end{tikzcd}$$

Q.E.D.

- Cohomological S.S. from a filtration.

<u>Thm.</u> Each f. dga $(A, d, F^*)$, $d: A^n \to A^{n+1}$, $F^*$ decreasing filtration, determines a cohomological s.s.

$$E_1^{p,q} = H^{p+q}(F^pA/F^{p+1}A)$$

Suppose further the filtration $F^*$ is <u>bounded</u>, then

$$E_1^{p,q} \Rightarrow H^{p+q}(A,d) \text{ ; that is, } E_\infty^{p,q} \cong F^pH^{p+q}(A,d)/F^{p+1}H^{p+q}(A,d).$$

Each SES $0 \to F^{p+1}A \to F^pA \to F^pA/F^{p+1}A \to 0$ induces a LES in homology:

$$\cdots \to H^n(F^{p+1}A) \xrightarrow{i} H^n(F^pA) \xrightarrow{j} H^n(F^pA/F^{p+1}A) \xrightarrow{k} H^{n+1}(F^{p+1}A) \to \cdots$$

Exact couple:

$$\begin{tikzcd} \bigoplus_{p,q} H^{p+q}(F^pA) \arrow[rr, "i"] & & \bigoplus_{p,q} H^{p+q}(F^pA) \arrow[dl, "j"] \\ & \bigoplus_{p,q} H^{p+q}(F^pA/F^{p+1}A) \arrow[ul, "k"] & \end{tikzcd}$$

## • Exact Couple

Let $X$ be a CW-cpx and let $\phi = X_{-1} \subset X_0 \subseteq X_1 \subseteq \cdots \subseteq X$ be a filtration.

e.g. $X_n = n$-skeleton of $X$, cellular filtration.

Each inclusion $X_{p-1} \hookrightarrow X_p$ induces a LES of homology groups

$$\cdots \xrightarrow{k} H_n X_{p-1} \xrightarrow{i} H_n X_p \xrightarrow{j} H_n(X_p, X_{p-1}) \xrightarrow{\partial = k} H_{n-1} X_{p-1} \to \cdots \qquad ji = kj = ik = 0$$

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_{p-2}) \arrow[d, "i"] & & & & \\
\cdots \arrow[r] & H_n(X_{p-1}) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_{p-1}, X_{p-2}) \arrow[r, "k"] & b_3 \in H_{n-1}(X_{p-2}) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(X_{p-2}, X_{p-3}) & \\
\cdots \arrow[r] & H_n(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_n(X_p, X_{p-1}) \arrow[r, "k"] & b_2 \in H_{n-1}(X_{p-1}) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(X_{p-1}, X_{p-2}) \arrow[r] & \cdots \\
& H_n(X_{p+1}) \arrow[r, "j"] \arrow[dd, dotted] & a \in H_n(X_{p+1}, X_p) \arrow[r, "k"] \arrow[rr, bend right, "d_1"] \arrow[urr, bend right, "d_2"] \arrow[uurr, bend left, "d_3"] & H_{n-1}(X_p) \arrow[r, "j"] \arrow[d, "i"] & H_{n-1}(X_p, X_{p-1}) \arrow[r] & \cdots \\
& & & H_{n-1}(X_{p+1}) \arrow[r, "j"] \arrow[d, dotted] & H_{n-1}(X_{p+1}, X_p) \arrow[r] & \cdots \\
& H_n X & & H_{n-1} X & &
\end{tikzcd}$$

Let $E^1_{pq} = H_{p+q}(X_p, X_{p-1})$, $d_1 = jk$ $(n = p+q)$

check $d_1 : E^1_{pq} \to E^1_{p-1,q}$ & $d_1^2 = jkjk = 0$.

$E^2_{pq} = \frac{\ker d_1}{\operatorname{Im} d_1}$, $d_2 : E^2_{pq} \to E^2_{p-2,q+1}$

$\forall\, a \in \ker d_1$, $jk(a) = 0$, $k(a) \in \ker j = \operatorname{Im} i$.

$\Rightarrow \exists\, b$, s.t. $i(b) = k(a)$. Set $d_2(a) = j(b)$.

Check: $d_2$ is well-defined & $d_2(E^2_{p,q}) \subset E^2_{p-2,q+1}$.

$E^3_{p,q} = \frac{\ker d_2}{\operatorname{Im} d_2}$, define $d_3$ similarly.

$\forall\, a \in \ker d_2$, $j(i^{-1})k(a) = 0 \Rightarrow i^{-1}k(a) = \ker j = \operatorname{Im} i$

$\Rightarrow i^{-1}k(a) = i(b)$, $k(a) = i^2(b)$

$d_3(a) = j(b)$, $d_3 = j(i^{-1})^2 k$

$\vdots$

$d_{r+1} = j(i^{-1})^r k$

• $E^1_{pq} = H_{p+q}(X_p, X_{p-1}) \Longrightarrow H_{p+q}(X)$.

**Def.** An <u>exact couple</u> consists of a pair of modules $D, E$ and morphisms $i, j, k$ making the following triangle exact at each corner.

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

Define differential $d: E \to E$ by $d = jk$, $d^2 = jkjk = 0$.

<u>derived exact couple</u>:

$$\begin{tikzcd} D_1 \arrow[rr, "i_1"] & & D_1 = i(D) \arrow[dl, "j_1"] \\ & E_1 = H(E, d) \arrow[ul, "k_1"] & \end{tikzcd}$$

$$\left.\begin{array}{l} i_1 = i|_D \\ j_1 = j \circ i^{-1} : i(x) \mapsto [j(x)] \\ k_1 = \bar{k} : [y] \mapsto [ky] \end{array}\right\} \text{well-defined \& exact.}$$

define $d_1: E_1 \to E_1$, $d_1 = j_1 k_1$

....

$r^{th}$ derived exact couple.

$$\begin{tikzcd} D_r \arrow[rr, "i_r"] & & D_r \arrow[dl, "j_r"] \\ & E_r \arrow[ul, "k_r"] & \end{tikzcd}$$

$d_r: E_r \to E_r$, $d_r = j_r k_r$

$E_{r+1} = H(E_r, d_r)$

If modules $D, E$ are graded, then so is $E_r$.

- - - - - bigraded, - - - - - - .

**Thm** Given an exact couple of bigraded modules

$$\begin{tikzcd} D^{*,*} \arrow[rr, "i"] & & D^{*,*} \arrow[dl, "j"] \\ & E^{*,*} \arrow[ul, "k"] & \end{tikzcd}$$

bi-deg$(i) = (-1, 1)$

bi-deg$(j) = (0, 0)$

bi-deg$(k) = (1, 0)$

. Then there exists a cohomology s.s. $\{E_r^{*,*}, d_r\}$.

$E_r^{*,*} = (r-1)^{th}$ derived module from $E^{*,*}$

and $d_r = j_r k_r$.

- Serre Spectral Sequences.

Recall that a map $p: E \to B$ is a (Serre) fibration if it satisfies the homotopy lifting properties (HLP) for finite CW-cpxes $X$.

$$\begin{tikzcd} X\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X\times I \arrow[r, "H"] \arrow[ur, dashed, "\exists \tilde{H}"] & B \end{tikzcd}$$

<u>prop1</u> If $B$ is path-connected, then $p^{-1}(b_1) \simeq p^{-1}(b_2)$ for any $b_1, b_2 \in B$.

$E$: total space, $B$: base space, $F = p^{-1}(b)$ is called the fibre

<u>prop2.</u> Given $F \hookrightarrow E \xrightarrow{p} B$ a fibration, $\pi_1(B)$ acts on $F$.

<u>Cor.</u> $\pi_1(B)$ acts on $\pi_*(F)$, $H_*(F)$, $H^*(F)$.

<u>prop3.</u> A fibration $F \hookrightarrow E \to B$ induces a LES of homotopy groups:

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots$$

- **Thm. (Serre S.S.)**

Let $F \hookrightarrow E \xrightarrow{p} B$ be a (homotopy) fibration with $F$-connected and $B$ path-connected.

If $\pi_1 B = 0$ or $\pi_1(B) \curvearrowright F$ trivially, then there are 1st quadrant S.S. for any abelian group $R$.

cohomology. $E_2^{p,q} = H^p(B; H^q(F;R)) \Longrightarrow H^{p+q}(E;R)$

homology. $E^2_{p,q} = H_p(B; H_q(F;R)) \Longrightarrow H_{p+q}(E;R)$.

- prop. If $R$ is a comm. ring and $H^p(B;R), H^q(F;R)$ are free $R$-module of finite types for all $p,q$.

then $E_2^{*,*} \cong H^*(B;R) \otimes_R H^*(F;R)$.

- $d(x^n) = n x^{n-1} dx$ if $|x|$ is even.

Example 1. Loop path fibration

$\Omega X = \mathrm{map}_*(S^1, X)$ compact-open ~~topology~~

$PX = \mathrm{map}_*(I, X)$, $I = [0,1]$

$$\underset{\simeq *}{\Omega X \longrightarrow PX} \xrightarrow{\pi} X$$

$$\alpha \longmapsto \alpha(1)$$

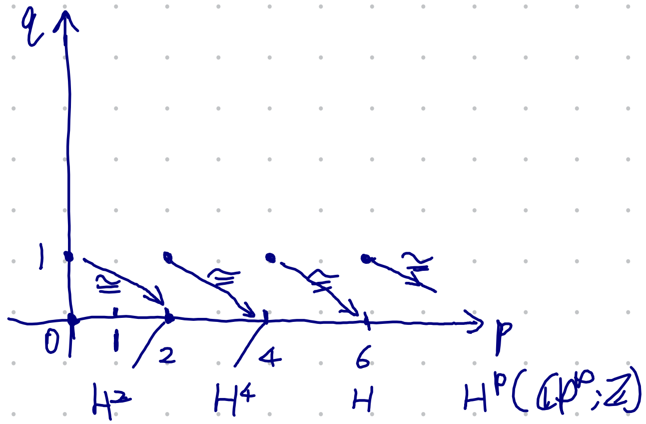
$$E_2^{p,q} = H^p(X; H^q(\Omega X)) \Rightarrow H^{p+q}(PX) = 0 \text{ except } p+q=0.$$

$$E_\infty^{p,q} = 0 \text{ except } p+q=0.$$

$$K(\mathbb{Z},1) \longrightarrow * \longrightarrow K(\mathbb{Z},2) \text{ or } S^1 \hookrightarrow * \longrightarrow \mathbb{C}P^\infty.$$

$$E_2^{p,q} = H^p(\mathbb{C}P^\infty; \mathbb{Z}) \otimes H^q(S^1; \mathbb{Z})$$

$q = 0, 1.$



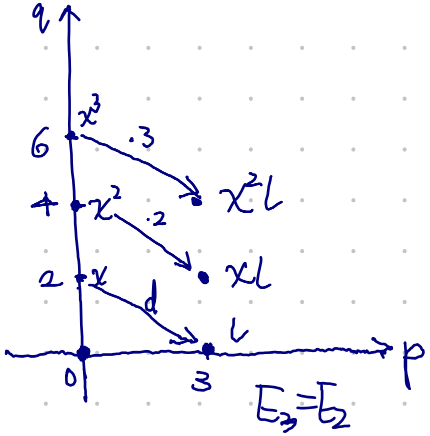

Since $E_\infty^{p,q} = 0$ except $p=q=0$

we get $H^p(\mathbb{C}P^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & p = 2n \\ 0 & p = 2n+1 \end{cases}$..

## Example 2

$K(\mathbb{Z},2) \longrightarrow S^3\langle 3\rangle \xrightarrow{p} S^3 \xrightarrow{\iota} K(\mathbb{Z},3)$, $\iota \in H^3(S^3;\mathbb{Z})$ a generator

$$E_2^{p,q} = H^p(S^3; H^q(K(\mathbb{Z},2);\mathbb{Z}))$$
$$\cong H^p(S^3) \otimes H^q(K(\mathbb{Z},2))$$




$S^3\langle 3\rangle$ is 3-connected ($\pi_{i\le 3}(S^3\langle 3\rangle)=0$)

$\rightsquigarrow d_3(x) = \iota$

$d_3(x^n) = n\cdot x^{n-1}\iota : E_3^{0,2n} \xrightarrow{\cdot n} E_3^{3,2n-2}$

$E_\infty^{3,2n-2} \cong \mathbb{Z}/n \Rightarrow H^{2n+1}(S^3\langle 3\rangle)$

$\rightsquigarrow \tilde{H}^*(S^3\langle 3\rangle;\mathbb{Z}) = \begin{cases} \mathbb{Z}/n & * = 2n+1 \\ 0 & * = 2n \end{cases}$

$\rightsquigarrow H_*(S^3\langle 3\rangle) \cong \begin{cases} \mathbb{Z}/n & * = 2n \\ 0 & * = 2n+1. \end{cases}$

Example 3. the Gysin Sequences (Example 5.C)

Thm. Let $F \to E \xrightarrow{p} B$ be a fibration with $F$ a homology sphere for some $n \geq 1$ ($H_*(F) \cong H_*(S^n)$)

($S^n \to E \xrightarrow{p} B$)

Suppose $\pi_1 B$ acts trivially or $B$ is 1-connected. Then there is an exact seq:

cohomological: $\to H^k(B;R) \xrightarrow{d_{n+1}} H^{k+n+1}(B;R) \xrightarrow{p^*} H^{k+n+1}(E;R) \to H^{k+1}(B;R) \to \cdots$

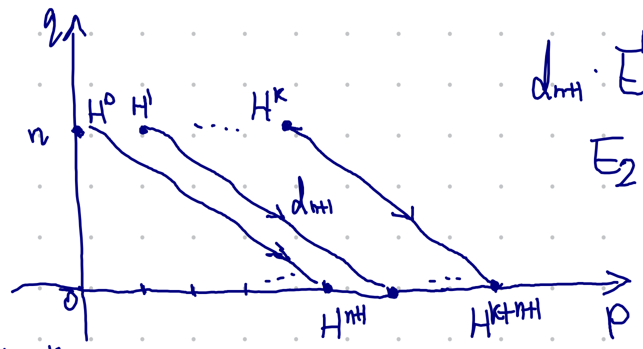
where $d_{n+1} = c \cup -$ for some $c \in H^{n+1}(B;R)$; if $n$ is even and $2 \neq 0$ in $R$, ~~then~~ $2c = 0$.

(homological $\cdots \to H_k(E;R) \xrightarrow{p_*} H_k(B;R) \xrightarrow{d^{n+1}} H_{k-n-1}(B;R) \to H_{k-1}(E;R) \to \cdots$)

Partial proof. $E_2$-page in the S.S.S. with R-coefficients:

$$E_2^{p,q} = H^p(B; H^q(S^n)) \cong H^p(B;R) \otimes H^q(S^n;R) \Rightarrow H^{p+q}(E;R).$$

$(q = 0, n)$




$H^k = H^k(B;R)$

$d_{n+1}: E_2^{k,n} = H^k \to E_2^{k+n+1,0} = H^{k+n+1}$

$E_2 = \cdots = E_{n+1}$, $E_{n+2} = E_\infty$.

Then the exact sequence follows by Ex. 2 on Page 5:

$$\begin{tikzcd}
\cdots \arrow[r] & E_2^{k,n} \arrow[r, "d_{n+1}"] \arrow[d, equal] & E_2^{k+n+1,0} \arrow[r] \arrow[d, equal] & H^{k+n+1}(E;R) \arrow[r] \arrow[dr] & E_2^{k+1,n} \arrow[r, "d_{n+1}"] \arrow[d, equal] & E_2^{k+n+2,0} \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^k(B;R) \arrow[r] & H^{k+n+1}(B;R) \arrow[ur] & & H^{k+1} \arrow[r] & H^{k+n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Q.E.D.

THANKS !