Homology.

- What does homology tell you?

  Namely the number of holes in a topological space.

- Advantage: Easier to compute compared with homotopy group.

  Disadvantage: Capture few information, but sometimes being insensitive to some topological information is not necessarily a drawback.

- Chain Complexes.

  K: a simplicial complex. A n-chain is a formal sum of some n-simplices. that is $C = \sum a_i \sigma_i$, where σ_i is a n-simplex, a_i are called coefficients. Let C_n denote the set of all such c, then C_n becomes a R-module. If $a_i \in R$. When $R = \mathbb{Z}$, then C_n is an abelian group; $R = F$ a field, C_n is a vector space!

(In computational topology, we mostly work with $a_i \in \mathbb{F}_2$) C_n is a free R-module obviously. For $p<0$ or $p > \dim K$, $C_p = 0$. When R is a principal ideal domain, (for instance $\mathbb{Z}$, any field, $\cdots$), every submodule of a finitely generated free R-mod is still free.

- [Definition of free R-module] $E \subseteq M$ be a set. If (1) E generates M over R

  (2) E is linearly independent, i.e. for any $\{e_1, \cdots, e_n\} \subseteq E$, $\sum r_i e_i = 0 \Rightarrow r_i = 0$.

  Then E is called a basis of M. If M has a basis, we call it free.

- An Counter Example

  V is a infinite dimensional vector field, such as $\mathbb{R}^\infty$, then $V \cong V \oplus V$.

$R = \mathrm{End}_K(V) = \mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_K(V, V \oplus V) = R \oplus R$ as R module.

when R is a commutative ring, R has IS BN.

— Boundary map ∂_n

$\partial_n : C_n \to C_{n-1}$, is a homomorphism between R-modules. And writing $\sigma_i = [u_0, \cdots, u_n]$, $\partial \sigma_i = \sum_i (-1)^i [u_0, \cdots, \hat{u}_i, \cdots u_n]$. In mod 2 homology, $(-1)^i = 1$.

$\partial_n \partial_{n+1} \sigma = \partial_n \left(\sum_i [u_0, \cdots, \hat{u}_i, \cdots u_{n+1}] (-1)^i \right) = \sum_i (-1)^i \partial_n [u_0, \cdots, \hat{u}_i, u_{n+1}]$

$= \sum_i (-1)^i \left[\sum_{j<i} (-1)^j [u_0, \cdots, \hat{u}_j, \cdots, \hat{u}_i, \cdots u_{n+1}] + \sum_{i<j} (-1)^{j-1} [u_0, \cdots, \hat{u}_i, \cdots \hat{u}_j, \cdots, u_{n+1}] \right]$

$= \sum_{j<i} (-1)^{i+j} [u_0, \cdots, \hat{u}_j, \cdots, \hat{u}_i, \cdots, u_{n+1}] + \sum_{i<j} (-1)^{i+j-1} [u_0, \cdots, \hat{u}_i, \cdots, \hat{u}_j, \cdots u_{n+1}] = 0.$

That is $\partial\partial = 0$, which implies $B_n = \mathrm{Im}\, \partial_{n+1} \subseteq Z_n = \ker \partial_n$, they both free.

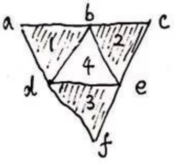
— The chain complex is the sequence $\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to$ with $\partial\partial = 0$.

— Homology. The p-th homology of K, $H_p(K)$ is defined to be Z_n / B_n

The p-th Betti number is the rank of $H_p(K)$, $\beta_p = \mathrm{rank}\, H_p(K)$. In mod 2 homology Z_n, B_n are both subspace of C_n, thus $H_n(K)$ is still a vector space,

$\beta_n = \mathrm{rank}\, H_n(K) = \dim H_n(K) = \dim Z_n - \dim B_n = \mathrm{rank}\, Z_n - \mathrm{rank}\, B_n$.

— Example



$C_0 = \mathbb{Z}_2^6$, $C_1 = \mathbb{Z}_2^9$, $C_2 = \mathbb{Z}_2^3$, $H_2 = Z_2 / B_2$, $B_2 = \partial_3 C_3 = 0$.

$\therefore H_n = Z_2 = \ker \partial_2 = 0$. $H_1 = Z_1 / B_1$, $Z_1 = \ker \partial_1 = \mathbb{Z}_2^4$ generated by C_1, C_2, C_3, C_4, $B_1 = \mathbb{Z}_2^3$ generated by C_1, C_2, C_3,

Hence $H_1 = \mathbb{Z}_2^4 / \mathbb{Z}_2^3 = \mathbb{Z}_2^1$, $\beta_1 = 1$, (one holes)

$H_0 = Z_0 / B_0$, $Z_0 = \ker \partial_0 = C_0 = \mathbb{Z}_2^6$, $B_0 = C_1 / \ker \partial_1 = \mathbb{Z}_2^9 / \mathbb{Z}_2^4$

$= \mathbb{Z}_2^5$, Thus $H_0 = \mathbb{Z}_2^6 / \mathbb{Z}_2^5 = \mathbb{Z}_2$.

If k is connected, every two vetices can be joint by some 1-simplices. Thus $a-b$ in C_0 is a boundary, actually, generalize this fact If $c=\sum n_i \sigma_i \in C_0$ with $\sum n_i = 0$ in coefficients Ring R. Then $c \in B_0$. Which Implies $C_0/B_0 \cong R$ That is $H_0 = R$ when k is connected.

— If $k = k' \sqcup k''$, $C_n = C_n(k') \oplus C_n(k'')$, $Z_n = Z_n(k') \oplus Z_n(k'')$, $B_n = B_n(k') \oplus B_n(k'')$. Thus complex $C = C' \oplus C''$, and $H_n(C) = H_n(C') \oplus H_n(C'')$

— Reduced homology

$$\to C_2 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} R \to 0$$, augemented by R

$\varepsilon : C_0 \to R$, $\varepsilon(\sum n_i \sigma_i) = \sum n_i$. ε is surjective. And $\varepsilon \partial_1 = 0$.

Since for $c = \sum n_i [a_i, b_i] \in C_1$, $\partial_1 c = \sum n_i (a_i - b_i)$, $\varepsilon \partial_1 c = 0$. So it is still a complex. The homology of this complex is denoted by $\tilde{H}_n$. Clearly we have $\tilde{H}_n = H_n$ for $n > 0$. For $n=0$, ε factor through H_0, $p : c \to \bar{c}$

$$\begin{tikzcd} C_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "P"'] & R \\ H_0 = C_0/\mathrm{Im}\,\partial_1 \arrow[ur, "\tilde{\varepsilon}"'] & \end{tikzcd}$$

and $\bar{\varepsilon} p = \varepsilon$ ~~$\bar{\varepsilon}(c) = \varepsilon c$~~. $\ker \bar{\varepsilon}$ ~~is~~ $= \{ \bar{c} \in H_0 \mid \varepsilon c = 0 \}$ $= \tilde{H}_0$. Hence $H_0 / \tilde{H}_0 \cong R$, which implies
free R-module.
$H_0 = \tilde{H}_0 \oplus R$.

— Induced Maps. $f : X \to Y$ continuous. Generally speaking, in singular homology, f_* takes C_n to C_n and $f_* \partial = \partial f_*$. Thus f_* takes cycles to cycles and boundary to boundarys, ~~these~~ hence induced a map $f_* : H_n(X) \to H_n(Y)$.

— Singular homology. $C_n^S(X) = \sum n_i \sigma_i$, $n_i \in R$, $\sigma_i : \Delta^n \to X$ a continuous map. $\partial : C_n^S(X) \to C_{n-1}^S(X)$. $\partial \sigma = \sum (-1)^i \sigma|[v_0 \cdots, \hat{v}_i, v_n]$. Again $\partial\partial = 0$. Hence we have $H_n^S(X)$. There is a theorem saying that. $H_n^S = H_n$ for simplicial complex.

— Example. If X is a point. Then $H_n(X)=0$, $n>0$, $H_n(X)=R$, $n=0$.

By simplicial homology or singular homology.

— Example. If f and g are homotopic, then $f^*=g^*: H_n(X)\to H_n(Y)$

and $(fg)^*_* = f^*_* \circ g^*_*$. Thus if $X\sim Y$, i.e. $\exists\, f: X\to Y$, $g: Y\to X$ with $fg\sim 1_Y$

$gf\sim 1_X$. Then $f^*_* g^*_* = 1^*_{*H_\cdot(Y)}$ and $g^*_* f^*_* = 1^*_{*H_\cdot(X)}$, which implies $H_\cdot(X)=H_\cdot(Y)$

$H_\cdot(\mathbb{R}^n)=H_\cdot(D^n)=H_\cdot(\{x\})$ since they both are contractible. $\mathbb{R}^n\sim x$ and $D^n\sim x$

— Degree of a Map. $H_n(S^n)=\mathbb{Z}_2$, $f: S^n\to S^n$, then $f_*: H_n(S^n)\to H_n(S^n)$

$f_*: \mathbb{Z}_2\to\mathbb{Z}_2$ has only two elements, $f_*=0$ or $f_*=1$. Generally speaking,

$H_n(S^n)=\underset{(\mathbb{Z})}{R}$, $f_*(\alpha)=t\alpha\in\underset{(\mathbb{Z})}{R}$, where α is the generator of $H_n(S^n)$, then we call

t the degree of map f.

(1) $\deg 1 = 1$; (2) $f\sim g$, $\deg f=\deg g$; (3) $\deg fg=\deg f\deg g$; (4) If f is

not surjective then f factor through $S^n-\{p\}$, i.e. $S^n \xrightarrow{\tilde f} S^n-\{p\} \xrightarrow{i} S^n$,

$f_* = i_*\tilde f_* = 0$, thus $\deg f=0$.

— Example. BROUWER's Fixed Point Theorem. $f: B^{n+1}\to B^{n+1}$ has at least

one fixed point. If f has no fixed point, then $f(x)\neq x$, let $\tilde f(x)=[x-f(x)]/\|x-f(x)\|$

$\tilde f: B^{n+1}\to S^n$. $\tilde f|_{S^n}\sim 1_{S^n}$ via $\tilde f|_{S^n}(x,t)=[x-tf(x)]/\|x-tf(x)\|$. Hence $\deg \tilde f|_{S^n}=1$

But $\tilde f|_{S^n}=\tilde f\circ i$. $(\tilde f|_{S^n})_* = \tilde f_*\circ i_* = 0$. A contradiction.

$$S^n \xrightarrow{i} B^{n+1} \xrightarrow{\tilde f} S^n$$

$$\mathbb{Z} \xrightarrow{i_*} 0 \xrightarrow{\tilde f_*} \mathbb{Z}$$

Matrix Reduction

Euler-Poincaré Formula. $\chi = \sum (-1)^p \text{rank}\, C_p$, $\text{rank}\, C_p = Z_p + b_{p-1}$, substitute in the formula, $\chi = \sum(-1)^p (Z_p + b_{p-1}) = \sum (-1)^p (Z_p - b_p) = \sum (-1)^p \beta_p$. So how to compute β_p or equivalent how to compute the homology is significant.

— Boundary matrices. $\partial_n : C_n \to C_{n-1}$ is a linear map, thus we can express the ∂_n as a matrix multiplication. If we arrange all the basis of C_n as $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, $e_{c_p} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, ~~then ∂_n should be~~, and the basis of C_{p-1} as f_i. then ∂_n should be.

$$\partial_n = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{c_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{c_p} \\ \vdots & \vdots & & \vdots \\ a_{c_{p-1}}^1 & a_{c_{p-1}}^2 & \cdots & a_{c_{p-1}}^{c_p} \end{bmatrix}$$

, where $a_i^j = 1$ if f_i is a face of e_j, otherwise 0.

— The column space is just the ~~basis~~ B_{n-1}, ~~each column represents a base element of B_{n-1}~~. Every linear independent subset of B columns represent a basis of B_{n-1}

— The null space of $\partial_n \subseteq \mathbb{Z}_2^{c_p}$ is just the Z_n.

— Every column has exactly $p+1$ "1"s.

— If we let r_n denote the rank of ∂_n. Then $Z_n = C_p - r_n$, $r_n = b_{n-1}$. $\beta_n = Z_n - b_n = C_p - r_n - r_{n+1}$. So the key ingredient in this section is to compute the rank of ∂_n. How to do that? Gaussian elimination! Exchanging rows (columns) or adding one row to another does not change the rank of a matrix.

— pseudo code.

```
void REDUCE(x, M)
    if there exist k ≥ x, l ≥ x with M[k, l] = 1, then
    exchange rows x and k; exchange columns x and l;
```

```
for i = x+1 to c_{p-1} do
    if M[i,x] = 1 then add row x to row i end if
end for ;
for j = x+1 to c_p do
    if M[x,j] = 1 then add column x to column j end if
end for ;
x = x+1
REDUCE(x, M)
endif.
Return x-1
```

— Note that $Ax=0$. ~~$A=(I|0)$~~ $A=(\;|0)V$, V invertible.

$Ax=(\;|0)VX=(I|0)Y$, $Y\in \mathrm{span}\{e_{r+1},\cdots e_n\}$, $r=\mathrm{rank}\,A$. $\therefore X=V^{-1}Y$

that is $V^{-1}\begin{pmatrix}0\\I\end{pmatrix}=X$, that is the basis of null(A) is the last $n-r$ columns of V^{-1}.

Relative Homology and Excision and Exact sequence:

— Map between chain complexes.

$f_n: C_n\to C_n'$ and $f_n\partial=\partial f_{n+1}$, such (f) are called morphism between chain complexes. An exact sequence of chain complexes $0\to C'_\cdot\xrightarrow{i}C_\cdot\xrightarrow{j}C''_\cdot\to 0$.

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_{n+1} & \xrightarrow{i} & C_{n+1} & \xrightarrow{j} & C''_{n+1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_n & \xrightarrow{i} & C_n & \xrightarrow{j} & C''_n & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & C'_{n-1} & \xrightarrow{i} & C_{n-1} & \xrightarrow{j} & C''_{n-1} & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

Each row is an exact sequence, actually split since C''_n is free. And the diagram commutes.

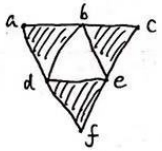
Chapter V Duality.

§ 1.1 Cohomology.

In previous section we have defined chain complexes $\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$ where C_n is a vector space over $\mathbb{Z}_2$ generated by all n-simplices in a simplicial complex K, now dualize this chain complex, i.e. apply the functor $\mathrm{Hom}_{\mathbb{Z}_2}(-, \mathbb{Z}_2)$, and let C_n^* denote $\mathrm{Hom}_{\mathbb{Z}_2}(C_n, \mathbb{Z}_2)$. we get another chain complex $\leftarrow C_{n+1}^* \xleftarrow{\delta} C_n^* \xleftarrow{\delta} C_{n-1} \leftarrow \cdots$ where $\delta = \partial^*$, that is for $f \in C_n^*$, $\delta f = f\partial : C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{f} \mathbb{Z}_2$, $\delta\delta f = \delta(f\partial) = f\partial\partial = 0$

Hence we define the p-th cohomology group $H^P(K) = \ker \delta_P / \mathrm{Im}\, \delta_{P-1}$. $Z^P = \ker \delta_P$ $B^P = \mathrm{Im}\, \delta_{P-1}$, they all contained in C^P.

Example :



$C_0 = \mathbb{Z}_2^6$ generates by $a^*, b^*, \cdots, f^*$

$Z^0 = \ker \delta^0$, If $c^* \in Z^0$, then $c^*\partial : C_1 \to \mathbb{Z}_2$ is a zero map $\Rightarrow$ For any edge, c^* take the same value on its endpoint. thus the only nontrivial 0-cocycle is $a^* + \cdots + f^*$. $H^0(K) = \mathbb{Z}_2$, $Z^0 = \mathbb{Z}_2$, hence $B^1 = C^0 / Z^0 = \mathbb{Z}_2^5$

what about $H^1(K)$? . $Z^1 = \ker \delta^1$. If $c^* \in Z^1$, then $c^*\partial : C_2 \to \mathbb{Z}_2$ is zero. thus 0 on each 2-simplex, which means c^* takes 1 on exactly two edges of each triangle or 0 on all edges. Hence for $[abd]$, $[ab]^* + [bd]^*$ and $[ab]^* + [ad]^*$ are cocycles send $[abd]$ to zero. Therefore $Z^1 = \mathbb{Z}_2^6$, $H^1 = Z^1 / B^1 = \mathbb{Z}_2$. $B^2 = C^1 / Z^1 = \mathbb{Z}_2^3$, $H^2 = C^2 / B^2 = 0$. In this example, we find $H_n = H^n$, this is not a coincidences.

§ 1.2 Coboundary map Matrix.

Recall the representation of a linear map $\phi : V \to W$,

$\phi(v_1, \cdots, v_m) = (w_1, \cdots, w_n) A_{n \times m}$, v_i are basis for V and w_i for W.

Then we take dual, $\mathcal{A}^*: W^* \to V^*$, and take the dual basis w_i^* and v_i^*, we have $\mathcal{A}^*(w_1^*, \cdots w_n^*) = (v_1^*, \cdots, v_m^*) A^T$.

Hence the coboundary matrix $C_n^* \to C_{n+1}^*$ $[\delta_n]$ is the transpose of $[\partial_{n+1}]: C_{n+1} \to C_n$.

i.e. let $r_{n+1} = \text{rank}\,[\partial_{n+1}] = \text{rank}\,[\delta_n] = r^n$, $\beta_n = \text{rank}\, H_n(k) = z_n - b_n = c_n - r_n - r_{n+1}$

$\beta^n = \text{rank}\, H^n = z^n - b^n = c_n - r^n - r^{n-1} = c_n - r_{n+1} - r_n = \beta_n$. Therefore $H^n(k; \mathbb{Z}_2) = H_n(k; \mathbb{Z}_2)$

§ 1.3 Universal Coefficient Theorem.

~~X is a topological space, $H_n(X)$ denotes the homology group of X in coefficient R, R is a PID. G is a R-module.~~

C is a chain complex of free R-modules with boundary maps R-module homomorphisms, and G is also an R-module. There is a natural split short exact sequences:

$$0 \to \text{Ext}_R(H_{n-1}(C), G) \to H^n(C; G) \to \text{Hom}_R(H_n(C), G) \to 0.$$

where $\text{Ext}_R(A, B)$, A, B R-modules. Take free (projective) resolution for A

$0 \to A_1 \to A_0 \to A \to 0$, apply $\text{Hom}_R(-, B)$ get

$0 \leftarrow A_1^* \leftarrow A_0^* \leftarrow A^* \leftarrow 0$. $\text{Ext}_R(A, B) = H^1(\text{Hom}_R(P_A, B))$. When $G = \mathbb{Z}_2$

H_n is a vector space, free, therefore $\text{Ext} = 0 \Rightarrow H^n(C; \mathbb{Z}_2) = \text{Hom}_{\mathbb{Z}_2}(H_n(C), \mathbb{Z}_2)$.

By the way. C_n' is the G coefficient chain group, $C_n' = C_n \otimes_{\mathbb{Z}} G$, C_n is $\mathbb{Z}$ coefficient.

Then. $0 \to H_n(C) \otimes G \to H_n(C \otimes G) \to \text{Tor}(H_{n-1}(C), G) \to 0$. is a split SES.

where $\text{Tor}(A, B) = H_1(P_A \otimes B)$.

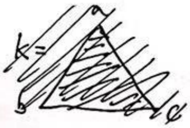
§2. Poincaré Duality.

$\mathrm{Star}\,\tau = \{\sigma \in K \mid \tau \text{ is a face of } \sigma\}$, $\overline{St}\,\tau$ the minimal subcomplex containing $St\,\tau$. link, $Lk\,\tau = \{v \in \overline{St}\,\tau \mid v \cap \tau = \phi\}$.

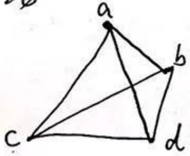
- Combinatorial manifolds.

A combinatorial manifold of dimension d, satisfying there is a triangulation s.t. $\underset{\sigma \in C_i}{Lk\,\sigma}$ $\underbrace{\text{triangulates the sphere of dimension } d-i-1}_{\text{means } Lk\sigma \cong S^{d-i-1}}$. This implies $\overline{St}\,\sigma \cong \underset{(D^d)}{B^d}$

$a \in C_0$, $\overline{St}\,a = K$, $Lk\,a = \{\{b\},\{c\},\{bc\}\}$



An Example : A tetrahedron with empty interior.



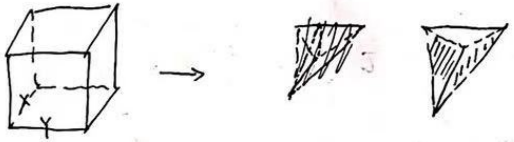
$\overline{St}\,a =$ the union of three standing 2-simplices. $\cong B^2$

$Lk\,a = \{\{b\},\{c\},\{d\},\{bc\},\{bd\},\{cd\}\} \cong S^1$

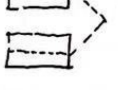
$\overline{St}\,[ac] =$ the union of two 2-simplices intersecting on $[ac] \cong B^2$, $Lk[ac] = \{\{b\},\{d\}\} \cong S^0$

In fact each simplex in $St\,\sigma$ is the join of σ with a simplex in $Lk\,\sigma$, $\overline{St}\,\sigma = \sigma * Lk\,\sigma$. $Lk\,\sigma = S^{d-i-1} = \underset{d-i}{*} S^0$, $\sigma \underset{d-i}{*} S^0 = D^i \underset{d-i}{*} S^0 = S^{d-i} D^i = D^d$.

$X * Y = X \times Y \times [0,1] \,/\, (x_0, 0, y), (x, 1, y_0)$



$X * S^0 = SX$,



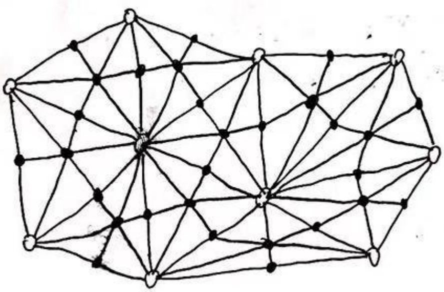
$X \times S^0 \times [0,1]$

$X \times S^0 \times \{t\} \longrightarrow$ $t=0$, collapse X
$t=1$, collapse S^0 = identify two "X"

||

- Dual Block.

Let M be a compact, combinatorial d-manifold triangulated by K. Recall the barycenter subdivision, $\mathrm{Sd}K$. If K has the link property, so does $\mathrm{Sd}K$. Label each vertex u in $\mathrm{Sd}K$, ~~by its~~ as follow: u is a barycenter of a simplex σ in K, label u by the dimension of σ. Note that each simplex in $\mathrm{Sd}K$ has distinct labels on its vertices. The vertex with smallest label is therefore unique. Letting u ~~is~~ be the barycenter of σ in K, the dual block denoted by $\hat{\sigma}$, is the union of the simplices in the barycentric subdivision for which u is the vertex with minimum label. Let B be the set of dual blocks and call it the ~~dual~~ block decomposition of M



vertex $\longleftrightarrow$ D^2

line $\longleftrightarrow$ D^1

triangle $\longleftrightarrow$ D^0

If p-simplex σ is a face of $p+1$-simplex τ, then $\hat{\tau}$ ~~is~~ contained in the boundary of $\hat{\sigma}$. In fact, the boundary $\hat{\sigma}$ is the union of dual blocks $\hat{\tau}$ over all proper cofaces τ of σ. We denote this boundary by $\mathrm{bd}\,\hat{\sigma}$, and $\hat{\sigma} = \mathrm{bd}\,\hat{\sigma} * u$, u is the barycenter of σ. $\sigma \leq \tau$ then $\hat{\tau}^* \leq \hat{\sigma}^*$. Since $\mathrm{Sd}K$ is a combinatorial manifold, $\mathrm{bd}\,\hat{\sigma} \cong S^{q-1}$, where $p+q=d$.

we now construct a new chain complex. Since σ a p-simplex. $\hat{\sigma}$ is a q-dimensional B^q, Let D_q denotes the vector space over $\mathbb{Z}_2$ generated by all $\hat{\sigma_i}$, where σ_i is a p-simplex. The boundary map $\partial_q : D_q \to D_{q-1}$ is defined by $\partial_q(\hat{\sigma_i}) = \sum \hat{\tau_i}$, where τ_i is a $(p+1)$-simplex and $\sigma_i \leq \tau_i$ is a proper face. $\partial_{q-1} \circ \partial_q = 0$ since there exactly two identical elements in $\partial_{q-1}(\hat{\tau_i})$ and $\partial_{q-1}(\hat{\tau_j})$ containing them as faces.

The next step is to show $H_q(D) = H_q(C)$, where C is the simplicial chain complex. Mapping each ~~p-simplex~~ p-dimensional dual block to the sum of p-simplices it contains, we get a homomorphism $b_p : D_p \to C_p$, b_p commutes with the boundary map., thus forms a chain map $b: D \to C$.
(no formal proof).

? — Block Complex Lemma : The chain map $b: D \to C$ induces $b_*: H_p(D) \to H_p(C)$ being an isomorphism.

[proof] : Let X_p be the subcomplex of Sdk, consisting of all simplicies that lie in blocks of dimension at most p. Clearly $H_p(X_q, X_{q-1}) = \begin{cases} D_p & \text{if } q = p \\ 0, & \text{otherwise.} \end{cases}$ (1) ?

$$\to H_{p+1}(X_q, X_{q-1}) \to H_p(X_{q-1}) \to H_p(X_q) \to H_p(X_q, X_{q-1}) \to \cdots$$

let $q = p+2$, $H_p(X_{q+1}) \cong H_p(X_{q+2}) \cong H_p(Sdk)$.

If $p+1 < q$, $H_p(X_{q-1}) \cong H_p(X_q)$

$$
\begin{tikzcd}
 & D_{p+1} = H_{p+1}(X_{p+1}, X_p) \arrow[d, "e"] \arrow[dr, dashed, "\partial_{q+1}"] & & 0 = H_{p-1}(X_{p-2}) \arrow[d] \\
0 = H_p(X_{p-1}) \arrow[r] & H_p(X_p) \arrow[r, "f"] \arrow[d, "\iota"] & D_p = H_p(X_p, X_{p-1}) \arrow[r, "g"] \arrow[dr, dashed, "\partial_q"] & H_{p-1}(X_{p-1}) \arrow[d, "h"] \\
 & H_p(X_{p+1}) \arrow[d] & & D_{p-1} = H_{p-1}(X_{p-1}, X_{p-2}) \\
 & 0 = H_p(X_{p+1}, X_p) & &
\end{tikzcd}
$$

② The diagramm commutes.?

③ $\ker \partial_q = \ker g$. $\operatorname{Im} \partial_{q+1} = \operatorname{Im} f \circ e = f(\operatorname{Im} e)$

$= \operatorname{Im} f \cong H_p(X_p)$

$$H_p(D) = \ker \partial_q / \operatorname{Im} \partial_{q+1} = \frac{f(H_p(X_p))}{f(\operatorname{Im} e)} \cong \frac{H_p(X_p)}{\operatorname{Im} e} = \frac{H_p(X_p)}{\ker \iota} \cong H_p(X_{p+1})$$

$$= H_p(\operatorname{sd} K).$$

— Poincaré Duality (First form). Let M be a compact, combinatorial d-manifold. Then there is an isomorphism between $H_p(M)$ and $H^q(M)$ for $p+q=d$.

— $\forall$ p-simplex σ in K, let $\sigma^* \in C_p^*$, let $\varphi_q : D_q \to C^p$ (bijective) by $\varphi_q(\hat{\sigma}) = \sigma^*$ extending linearly gives a isomorphism between D_q and C^p.

$$
\begin{tikzcd}
D_q \arrow[r, "\partial_q"] \arrow[d, "\varphi_q"'] & D_{q-1} \arrow[d, "\varphi_{q-1}"] \\
C^p \arrow[r, "\delta_p"] & C^{p+1}
\end{tikzcd}
$$

If this diagram ~~commutes~~ commutes, then we have $H_q(\mathbb{M}) \cong H^p(\mathbb{M})$

$\varphi_{q-1} \circ \partial_q(\hat{6}) = \varphi_{q-1}(\Sigma \hat{\tau}) = \Sigma \tau^*$, where τ is a $p+1$ simplex with 6 being its face. $\delta^p \varphi_q(\hat{6}) = \delta^p(6^*) = 6^*\partial$; since they ~~evaluate~~ take 1 on each $p+1$ simplex $\tau \geq 6$ and 0 otherwise, therefore they are equal, $\Rightarrow$ the diagram commutes.

$H_p(\mathbb{M}) \cong H^q(\mathbb{M}) = H_q(\mathbb{M})$ for $p+q=d$

In general, If M is R-orientable, then there is an isomorphism $H^k(M;R) = H_{n-k}(M;R)$. Every manifold is $\mathbb{Z}_2$-orientable.

§ 3. Intersection Theory

Let M be a combinatorial d-manifold. $p+q=d$. If 6 is a p-simplex, then $\hat{6}$ is q-dimensional. $6 \cap \hat{6} = u$ the barycenter of 6. (This is because u has the minimal label in all simplices in ~~Sdk~~ $\hat{6}$). If $6 \neq \tau$, then $6 \cap \hat{\tau} = \phi$, τ is another p-simplex. (How to prove?). Define $6 \cdot \hat{\tau} = \begin{cases} 1 & \text{if } 6=\tau \\ 0 & \text{if } 6 \neq \tau \end{cases} : C_p \times D_q \longrightarrow G = \mathbb{Z}_2$

Suppose $c = \sum_i a_i 6_i$ is a p-cycle in $\underset{C_p}{K}$ and $d = \sum_j b_j \hat{\tau}_j$ is a q-cycle in D_q. then $c \cdot d = \sum_{i,j} a_i b_j (6_i \cdot \hat{\tau}_j)$ is the intersection number of two cycles in modulo 2. $c \cdot d = 0$ if they are disjoint or meet in an even number, $c \cdot d = 1$ if they meet in an odd number.

In fact, if $c \sim c_0$, then $c \cdot d = c_0 \cdot d$, so does $d \sim d_0$. Let γ be a $p+1$ simplex we want to show $\partial\gamma \cdot d = 0$. For $\hat{6}$ is a summand of d, $\partial\gamma \cdot \hat{6} \neq 0$ iff 6 is a face of γ. $\gamma \cap \hat{6}$ = the line segment connecting the barycenters of γ and 6.

? Completing the intersection between γ and d, the edge extends to either a closed curve or a path with two endpoints. Thus $\partial\gamma \cdot d = 0$.

— Parings. $\# : H_p(M) \times H_q(M) \longrightarrow G$ defined by $\#(\gamma, \delta) = c \cdot d$, where c and d are representatives. Call this map the intersection paring of the homology groups, $p+q=d$. Bilinear. U, V be vector spaces, $\# : U \times V \to G$ gives a natural homomorphism $\phi_\# : V \longrightarrow \mathrm{Hom}(U, G)$, $f_V(u) = \#(u, v)$. A paring is perfect if for every nonzero $u \in U$, $\exists\, v_0 \in V$, s.t. $\#(u, v_0) = 1$. and $\forall\, v \neq 0$, $\exists\, u_0 \in U$, $\#(u_0, v) = 1$.

— Perfect Paring Lemma. The paring $\# : U \times V \to G$ is perfect iff $\phi_\# : V \to \mathrm{Hom}(U, G)$ is an isomorphism. If $\phi_\#$ is an iso, then $\forall\, v \neq 0$, $f_V{}^\# \in \mathrm{Hom}(U, G) \neq 0$. $\therefore \exists\, u_0 \in U$ s.t. $\#(u_0, v) = 1$. Since $\phi_\#$ is surjective. for $u \in U$, u^* takes u to 1, $\exists\, v_0 \in V$ s.t. $f_{v_0} = u^*$, $f_{v_0}(u) = u^*(u) = 1$. Conversely. Suppose $\#$ is perfect. $\phi_\#$ is injective. otherwise $\exists\, v_0$, $f_{v_0} = 0$ in $\mathrm{Hom}(U, G)$. $\mathrm{rank}\, V \leq \mathrm{rank}\, \mathrm{Hom}(U, G) = \mathrm{rank}\, U$. but $\mathrm{rank}\, U \leq \mathrm{rank}\, V$ in the same way. $\therefore$ $\mathrm{rank}\, V = \mathrm{rank}\, \mathrm{Hom}(U, G) \Rightarrow$ $\phi_\#$ is an isomorphism. $\Rightarrow V \cong \mathrm{Hom}(U, G) \cong U$.

[— σ a p-simplex of K, $\hat{\sigma}$ its q-dimensional dual block, let $\varphi_q(\hat{\sigma}) = \sigma^* \in C_p^*$. and $\langle \sigma^*, \tau \rangle = \langle \hat{\sigma}, \tau \rangle$]

— Poincaré Duality (Second Version).

Let M be a compact, combinatorial manifold. Then the paring $\# : H_p(M) \times H_q(M) \to G$ defined by $\#(\gamma, \delta) = \gamma \cdot \delta$ is perfect for all $p+q=d$.

Euler characteristic.

$\chi(M) = 0$ if $\dim M$ is odd.

Manifolds with boundary

Lefchetz Duality Theorem, Let M be compact, combinatorial d-manifold with boundary ∂M. Then $H_p(M, \partial M) = H^q(M)$, $H_p(M) = H^q(M, \partial M)$.

Alexander Duality.

Let B be the dual block decomposition, $N \subseteq K$ be a subcomplexes, $X \subseteq B$ a subcomplexes. N and X are complementary : $\sigma \in N \Leftrightarrow \hat{\sigma} \notin X$.

To separate N from X, we subdivide onece more, to get N' and X' and enlarge them to N'' and X''. $\partial N'' = \partial X'' = N'' \cap X''$. and N'' and X'' are deformation retract of N and X.

Let S^d be a C-M with triangulation K, $N < K$, X complement of N.

Then $\tilde{H}_p(N) = \tilde{H}^{d-p-1}(X)$. for $p < d-1$, $\underset{+G}{\tilde{H}^{d-p-1}(X)} \underset{\text{defn}}{=} \underset{+G}{\tilde{H}^{d-p-1}(X'')} \underset{d-p-1>0}{=} H^{d-p-1}(X'')$

$\underset{\text{Lefchetz}}{=} H_{p+1}(X'', \partial X'') \underset{\text{excision}}{=} \tilde{H}_{p+1}(\mathrm{Sd}^2 K, N'') \underset{\uparrow}{=} \underset{+G}{\tilde{H}_p(N'')} \underset{\text{defn}}{=} \underset{+G}{\tilde{H}_p(N)}$.

$X = A \cup B$, $A \cap B \neq \emptyset$, $(B, A\cap B) \hookrightarrow (X, A)$. induce isomorphism

$\uparrow$: $\tilde{H}_{p+1}(\mathrm{Sd}^2 K) = \tilde{H}_p(\mathrm{Sd}^2 K) = 0$. for $p = d-1$,

$$\underset{0}{H_d(N'')} \to H_d(\mathrm{Sd}^2 K) \to H_d(\mathrm{Sd}^2 K, N'') \to H_{d-1}(N'') \to \underset{0}{H_{d-1}(\mathrm{Sd}^2 K)}$$

since
$$\begin{tikzcd} N'' \arrow[r, "i"] \arrow[d, equal] & S^d \\ Y \arrow[r] & \mathbb{R}^d \arrow[u] \end{tikzcd}$$

for the same reason $p = -1, d$,

$H_p(N)$ or $H_p(X) = 0$.

Adding a simplex. $N_{i-1} \subseteq N_i$ subcomplexes of K, $N_i - N_{i-1} = \{\sigma_i\}$.

Consider

$$\to \tilde{H}_p(N_{i-1}) \to \tilde{H}_p(N_i) \to \tilde{H}_p(N_i, N_{i-1}) \xrightarrow{D} \tilde{H}_{p-1}(N_{i-1}) \to \tilde{H}_{p-1}(N_i) \to$$

$H_q(N_i, N_{i-1}) = 0$, if $q \neq p = \dim \sigma_i$, $= G$ if $q = p$.

Hence. $\tilde{H}_q(N_{i-1}) = \tilde{H}_q(N_i)$ for $q < p-1$ or $q > p$.

Case 1 : D is ~~surjective~~ 0, then. $\operatorname{rank} \tilde{H}_p(N_i) = \operatorname{rank} \tilde{H}_p(N_{i-1}) + 1$

and $\operatorname{rank} \tilde{H}_{p-1}(N_{i-1}) = \operatorname{rank} \tilde{H}_{p-1}(N_i)$. , σ_i create a homology class, called poistive simplex.

Case 2 : D is surjective. $\operatorname{rank}_p N_{i-1} = \operatorname{rank}_p(N_i)$.

and. $\operatorname{rank}_{p-1}(N_{i-1}) = \operatorname{rank}_{p-1}(N_i) + 1$, σ_i destroy a homology class called negative simplex.

If we know the simplices are positive or negative then there is an Incremental algorithm to compute the Betti number of Complex N.

$N = \{\sigma_1, \cdots, \sigma_j\}$ be ordered, $N_i = \{\sigma_1, \cdots, \sigma_i\}$ is a subcomplexes of N

$\tilde{\beta}_{-1} = 1$; for $p=0$ to d do $\tilde{\beta}_p = 0$ endfor; $\qquad \tilde{\beta}_{-1} = 1$ means $N_0 = \{\phi\}$

for $i=1$ to j do

  if σ_i is positive. then $\tilde{\beta}_p = \tilde{\beta}_p + 1$

    else $\tilde{\beta}_{p-1} = \tilde{\beta}_{p-1} - 1$

  end if

end for.

Assuming we know the classification of the simplices, the algorithms computes the Betti numbers of all N_i spending only constant time per simplex.