Euler characteristic.

$$\chi(M) = 0 \text{ if } \dim M \text{ is odd.}$$

Manifolds with boundary

Lefchetz Duality Theorem, Let M be compact, combinatorial d-manifold with boundary ∂M. Then $H_p(M, \partial M) = H^q(M)$, $H_p(M) = H^q(M, \partial M)$.

Alexander Duality.

Let B be the dual block decomposition, $N \subseteq K$ be a subcomplexes, $X \subseteq B$ a subcomplexes. N and X are complementary : $\sigma \in N \Leftrightarrow \hat{\sigma} \notin X$.

To separate N from X, we subdivide onece more, to get N' and X' and enlarge them to N'' and X''. $\partial N'' = \partial X'' = N'' \cap X''$. and N'' and X'' are deformation retract of N and X.

Let S^d be a C-M with triangulation K, $N < K$, X complement of N.

Then $\tilde{H}_p(N) = \tilde{H}^{d-p-1}(X)$. for $p < d-1$, $\underset{+G}{\tilde{H}^{d-p-1}(X)} \underset{defn}{=} \underset{+G}{\tilde{H}^{d-p-1}(X'')} \underset{d-p-1>0}{=} H^{d-p-1}(X'')$

$$\underset{\text{Lefchetz}}{=} H_{p+1}(X'', \partial X'') \underset{\text{excision}}{=} \tilde{H}_{p+1}(Sd^2K, N'') \underset{\uparrow}{=} \underset{+G}{\tilde{H}_p(N'')} \underset{defn}{=} \underset{+G}{\tilde{H}_p(N)}.$$

$X = A \cup B$, $A \cap B \neq \emptyset$, $(B, A \cap B) \hookrightarrow (X, A)$. induce isomorphism

$\uparrow$: $\tilde{H}_{p+1}(sd^2K) = \tilde{H}_p(sd^2k) = 0$. for $p = d-1$,

$$\overset{0}{\overset{\parallel}{H_d(N'')}} \to H_d(sd^2K) \to H_d(sd^2K, N'') \to H_{d-1}(N'') \to \overset{0}{\overset{\parallel}{H_{d-1}(sd^2K)}}$$

$$\text{since} \quad \begin{array}{ccc} N'' & \xrightarrow{i} & S^d \\ \parallel & & \uparrow \\ Y & \longrightarrow & \mathbb{R}^d \end{array}$$

for the same reason $p = -1, d$,

$H_p(N)$ or $H_p(X) = 0$.

Adding a simplex. $N_{i-1} \subseteq N_i$ subcomplexes of k, $N_i - N_{i-1} = \{\sigma_i\}$.

Consider

$$\to \tilde{H}_p(N_{i-1}) \to \tilde{H}_p(N_i) \to \tilde{H}_p(N_i, N_{i-1}) \xrightarrow{D} \tilde{H}_{p-1}(N_{i-1}) \to \tilde{H}_{p-1}(N_i) \to$$

$H_q(N_i, N_{i-1}) = 0$, if $q \neq p = \dim \sigma_i$, $= G$ if $q = p$.

Hence. $\tilde{H}_q(N_{i-1}) = \tilde{H}_q(N_i)$ for $q < p-1$ or $q > p$.

Case 1 : D is ~~surjective~~ 0, then. $\operatorname{rank} \tilde{H}_p(N_i) = \operatorname{rank} \tilde{H}_p(N_{i-1}) + 1$

and $\operatorname{rank} \tilde{H}_{p-1}(N_{i-1}) = \operatorname{rank} \tilde{H}_{p-1}(N_i)$. , σ_i create a homology class, called poistive simplex.

Case 2 : D is surjective. $\operatorname{rank}_p N_{i-1} = \operatorname{rank}_p (N_i)$.

and. $\operatorname{rank}_{p-1}(N_{i-1}) = \operatorname{rank}_{p-1}(N_i) + 1$, σ_i destroy a homology class called negative simplex.

If we know the simplices are positive or negative then there is an Incremental algorithm to compute the Betti number of Complex N.

$N = \{\sigma_1, \cdots, \sigma_j\}$ be ordered, $N_i = \{\sigma_1, \dots, \sigma_i\}$ is a subcomplexes of N

$\tilde{\beta}_{-1} = 1$; for $p=0$ to d do $\tilde{\beta}_p = 0$ endfor ; $\tilde{\beta}_{-1} = 1$ means $N_0 = \{\phi\}$

for $i=1$ to j do

if σ_i is positive. then $\tilde{\beta}_p = \tilde{\beta}_p + 1$

else $\tilde{\beta}_{p-1} = \tilde{\beta}_{p-1} - 1$

end if

end for.

Assuming we know the classification of the simpliices, the algorithms computes the Betti numbers of all N_i spending only constant time per simplex.