Homology.

- What ~~is~~ does homology tell you?

  Namely the number of holes in a topological space.

- Advantage : Easier to compute ~~com~~ compared with homotopy group.

  Disadvantage: Capture few information, but sometimes being insensitive to some topological information is not necessarily a drawback.

- Chain Complexes.

  K: a simplicial complex. A n-chain is a formal sum of some n-simplices. that is $C = \sum a_i \sigma_i$, where σ_i is a n-simplex, a_i are called coefficients. Let C_n denote the set of all such c, then C_n becomes a R-module. If $a_i \in \mathbb{R}$. When $R = \mathbb{Z}$, then C_n is an abelian group; ~~If~~ $R = F$ a field, C_n is a vector space! (In computational topology, we mostly work with $a_i \in \mathbb{F}_2$) C_n is a free R-module obviously. For $p<0$ or $p > \dim K$, $C_p = 0$. When R is a principal ideal domain, (for instance $\mathbb{Z}$, any field, $\cdots$), every submodule of a finitely generated free R-mod is still free.

- [Definition of free R-module] $E \subseteq M$ be a set. If (1) E generates M over R (2) E is linearly independent, i.e. for any $\{e_1, \cdots e_n\} \subseteq E$, $\sum r_i e_i = 0 \Rightarrow r_i = 0$. Then E is called a basis of M. If M has a basis, we call it free.

- An Counter Example

  V is a infinite dimensional vector field, such as $\mathbb{R}^\infty$, then $V \cong V \oplus V$.

$R = \mathrm{End}_K(V) = \mathrm{Hom}_K(V, V) \cong \mathrm{Hom}_K(V, V \oplus V) = R \oplus R$ as R module.

when R is a commutative ring, R has IBN.

- Boundary map ∂_n

$\partial_n : C_n \to C_{n-1}$, is a homomorphism between R-modules. And writing $\sigma_i = [u_0, \cdots, u_n]$, $\partial \sigma_i = \sum_i (-1)^i [u_0, \cdots, \hat{u}_i, \cdots u_n]$. In mod 2 homology, $(-1)^i = 1$.

$$\partial_n \partial_{n+1} \sigma = \partial_n\Big(\sum_i [u_0, \cdots, \hat{u}_i, \cdots u_{n+1}](-1)^i \Big) = \sum_i (-1)^i \partial_n [u_0, \cdots, \hat{u}_i, u_{n+1}]$$

$$= \sum_i (-1)^i \Big[\sum_{j<i} (-1)^j [u_0, \cdots, \hat{u}_j, \cdots, \hat{u}_i, \cdots u_{n+1}] + \sum_{i<j} (-1)^{j-1} [u_0, \cdots, \hat{u}_i, \cdots \hat{u}_j, \cdots, u_{n+1}] \Big]$$

$$= \sum_{j<i} (-1)^{i+j} [u_0, \cdots, \hat{u}_j, \cdots, \hat{u}_i, \cdots, u_{n+1}] + \sum_{i<j} (-1)^{i+j-1} [u_0, \cdots, \hat{u}_i, \cdots, \hat{u}_j, \cdots u_{n+1}] = 0.$$

That is $\partial\partial = 0$, which implies $B_n = Im\, \partial_{n+1} \subseteq Z_n = \ker \partial_n$, they both free.

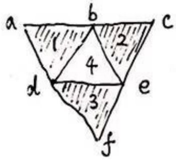
- The chain complex is the sequence $\to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to$ with $\partial\partial = 0$.

- Homology. ~~The p-th homology of k, $H_p(k)$ is defined to be~~ Z_n / B_n

The p-th Betti number is the rank of $H_p(k)$, $\beta_p = \text{rank } H_p(k)$. In mod 2 homology Z_n, B_n are both subspace of C_n, thus $H_n(k)$ is still a vector space,

$\beta_n = \text{rank } H_n(k) = \dim H_n(k) = \dim Z_n - \dim B_n = \text{rank } Z_n - \text{rank } B_n$.

- Example



$C_0 = \mathbb{Z}_2^6$, $C_1 = \mathbb{Z}_2^9$, $C_2 = \mathbb{Z}_2^3$, $H_2 = Z_2 / B_2$, $B_2 = \partial_3 C_3 = 0$.

$\therefore H_n = Z_2 = \ker \partial_2 = 0$. $H_1 = Z_1 / B_1$, $Z_1 = \ker \partial_1 = \mathbb{Z}_2^4$ generated by C_1, C_2, C_3, C_4, $B_1 = \mathbb{Z}_2^3$ generated by C_1, C_2, C_3,

Hence $H_1 = \mathbb{Z}_2^4 / \mathbb{Z}_2^3 = \mathbb{Z}_2^1$, $\beta_1 = 1$, (one holes)

$H_0 = Z_0 / B_0$, $Z_0 = \ker \partial_0 = C_0 = \mathbb{Z}_2^6$, $B_0 = C_1 / \ker \partial_1 = \mathbb{Z}_2^9 / \mathbb{Z}_2^4 = \mathbb{Z}_2^5$, Thus $H_0 = \mathbb{Z}_2^6 / \mathbb{Z}_2^5 = \mathbb{Z}_2$.

If K is connected, every two vetices can be joint by some 1-simplices. Thus $a-b$ in C_0 is a boundary, actually, generalize this fact If $c=\sum n_i\sigma_i \in C_0$ with $\sum n_i = 0$ in coefficients Ring R. Then $c \in B_0$. Which Implies $C_0/B_0 \cong R$. That is $H_0 = R$ when K is connected.

— If $K = K' \sqcup K''$, $C_n = C_n(K') \oplus C_n(K'')$, $Z_n = Z_n(K') \oplus Z_n(K'')$, $B_n = B_n(K') \oplus B_n(K'')$. Thus complex $C = C' \oplus C''$, and $H_n(C) = H_n(C') \oplus H_n(C'')$

— Reduced homology

$$\to C_2 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} R \to 0$$

, augemented by R

$\varepsilon : C_0 \to R$, $\varepsilon(\sum n_i\sigma_i) = \sum n_i$. ε is surjective. And $\varepsilon\partial_1 = 0$.

Since for $c = \sum n_i[a_i, b_i] \in C_1$, $\partial_1 c = \sum n_i(a_i - b_i)$, $\varepsilon\partial_1 c = 0$. So it is still a complex. The homology of this complex is denoted by $\tilde{H}_n$. Clearly we have $\tilde{H}_n = H_n$ for $n>0$. For $n=0$, ε factor through H_0, $p : c \to \bar{c}$

$$\begin{tikzcd} C_0 \arrow[r, "\varepsilon"] \arrow[d, dashed, "p"'] & R \\ H_0 = C_0/\mathrm{Im}\,\partial_1 \arrow[ur, "\tilde{\varepsilon}"'] & \end{tikzcd}$$

and $\bar{\varepsilon} p = \varepsilon$. $\ker\bar{\varepsilon} = \{\bar{c} \in H_0 \mid \varepsilon c = 0\}$ $= \tilde{H}_0$. Hence $H_0/\tilde{H}_0 \cong R$, which implies

free R-module.

$H_0 = \tilde{H}_0 \oplus R$.

— Induced Maps. $f: X \to Y$ continuous. Generally speaking, in singular homology, $f_*^\#$ takes C_n to C_n and $f_*^\# \partial = \partial f_*^\#$. Thus $f_*^\#$ takes cycles to cycles and boundary to boundarys, hence induced a map $\bar{f}_*^\# : H_n(X) \to H_n(Y)$.

— Singular homology. $C_n^S(X) = \sum n_i\sigma_i$, $n_i \in R$, $\sigma_i : \Delta^n \to X$ a continuous map. $\partial : C_n^S(X) \to C_{n-1}^S(X)$. $\partial\sigma = \sum (-1)^i \sigma|[v_0 \cdots, \hat{v}_i, v_n]$. Again $\partial\partial = 0$. Hence we have $H_n^S(X)$. There is a theorem saying that, $H_n^S = H_n$ for simplicial complex.

— Example. If X is a point. Then $H_n(X)=0$, $n>0$, $H_n(X)=R$, $n=0$.

By simplicial homology or singular homology.

— Example. If f and g are homotopic, then $f^*=g^*: H_n(X)\to H_n(Y)$

and $(fg)^*_* = f^*_*\circ g^*_*$. Thus if $X\sim Y$, i.e. $\exists f:X\to Y$, $g:Y\to X$ with $fg\sim 1_Y$

$gf\sim 1_X$. Then $f^*_*g^*_* = 1^*_{*H_\cdot(Y)}$ and $g^*_*f^*_* = 1^*_{*H_\cdot(X)}$, which implies $H_\cdot(X)=H_\cdot(Y)$

$H_\cdot(\mathbb{R}^n)=H_\cdot(D^n)=H_\cdot(\{x\})$ since they both are contractible. $\mathbb{R}^n\sim x$ and $D^n\sim x$

— Degree of a Map. $H_n(S^n)=\mathbb{Z}_2$, $f: S^n\to S^n$, then $f_*: H_n(S^n)\to H_n(S^n)$

$f_*:\mathbb{Z}_2\to\mathbb{Z}_2$ has only two elements, $f_*=0$ or $f_*=1$. Generally speaking,

$H_n(S^n)=\underset{(\mathbb{Z})}{R}$, $f_*(\alpha)=t\alpha\in\underset{(\mathbb{Z})}{R}$, where α is the generator of $H_n(S^n)$, then we call

t the degree of map f.

(1) $\deg \cancel{f}\, 1 = 1$; (2) $f\sim g$, $\deg f=\deg g$; (3) $\deg fg=\deg f\deg g$; (4) If f is

not surjective then f factor through $S^n-\{p\}$, i.e. $S^n\xrightarrow{\tilde f} S^n-\{p\}\xrightarrow{i} S^n$, ~~degf~~

$f_*=i_*\tilde f_*=0$, thus $\deg f=0$.

— Example. BROUWER's Fixed Point Theorem. $f: B^{n+1}\to B^{n+1}$ has at leas

one fixed point. If f has no fixed point, then $f(x)\neq x$, let $\tilde f(x)=[x-f(x)]/\|x-f$

$\tilde f: B^{n+1}\to S^n$. $\tilde f|_{S^n}\sim 1_{S^n}$ via $\tilde f|_{S^n}(x,t)=[x-tf(x)]/\|x-tf(x)\|$. Hence $\deg \tilde f|_{S^n}$

But $\tilde f|_{S^n}=\tilde f\circ i$. $(\tilde f|_{S^n})_*=\tilde f_*\circ i_*=0$. A contradiction.

$$S^n\xrightarrow{i} B^{n+1}\xrightarrow{\tilde f} S^n$$

$$\mathbb{Z}\xrightarrow{i_*} 0\xrightarrow{\tilde f_*}\mathbb{Z}$$

Matrix Reduction

Euler-Poincaré Formula. $\chi = \sum (-1)^p \operatorname{rank} c_p$, $\operatorname{rank} c_p = z_p + b_{p-1}$, substitute in the formula, $\chi = \sum (-1)^p (z_p + b_{p-1}) = \sum (-1)^p (z_p - b_p) = \sum (-1)^p \beta_p$. So how to compute β_p or equivalent how to compute the homology is significant.

— Boundary matrices. $\partial_n : C_n \to C_{n-1}$ is a linear map, thus we can express the ∂_n as a matrix multiplication. If we arrange all the basis of C_n as $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$ $e_{c_p} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, ~~then ∂_n should be~~, and the basis of C_{p-1} as f_i. then ∂_n should be.

$$\partial_n = \begin{bmatrix} a_1^1 & a_1^2 & \cdots & a_1^{c_p} \\ a_2^1 & a_2^2 & \cdots & a_2^{c_p} \\ \vdots & \vdots & & \vdots \\ a_{c_{p-1}}^1 & a_{c_{p-1}}^2 & \cdots & a_{c_{p-1}}^{c_p} \end{bmatrix}$$

, where $a_i^j = 1$ if f_i is a face of e_j, otherwise 0.

— The column space is just the ~~basis~~ B_{n-1}, ~~each column represents a base element of B_{n-1}~~. Every linear independent subset of ~~B~~ columns represent a basis of B_{n-1}

— The null space of $\partial_n \subseteq \mathbb{Z}_2^{c_p}$ is just the Z_n.

— Every column has exactly $p+1$ "1"s.

— If we let r_n denote the rank of ∂_n. Then $z_n = c_p - r_n$, $r_n = b_{n-1}$. $\beta_n = z_n - b_n = c_p - r_n - r_{n+1}$. So the key ingredient in this section is to compute the rank of ∂_n. How to do that? Gaussian elimination! Exchanging rows (columns) or adding one row to another does not change the rank of a matrix.

— pseudo code.

```
void REDUCE(x, M)
    if there exist k ≥ x, l ≥ x with M[k, l] = 1, then
    exchange rows x and k; exchange columns x and l;
```

for $i = x+1$ to c_{p-1} do

  if $M[i,x] = 1$ then add row x to row i end if

end for ;

for $j = x+1$ to c_p do

  if $M[x,j] = 1$ then add ~~row~~ column x to column j end if

end for ;

~~return x~~ $x = x+1$

~~return M~~

REDUCE (~~x+1~~ x , M)

end if.

Return $x-1$

— Note that $Ax = 0$. ~~$A = (\ | 0)$~~ $A = (\ | 0)V$, V invertible.

$Ax = (\ | 0)VX = (\ | 0)Y$, $Y \in \text{span}\{e_{r+1}, \cdots e_n\}$, $r = \text{rank } A$. $\therefore X = V^{-1}Y$

that is $V^{-1}\begin{pmatrix} 0 \\ I \end{pmatrix} = X$, that is the basis of null (A) is the last $n-r$ columns of V^{-1}.

Relative Homology and Excision and Exact sequence:

— Map between chain complexes.

$f_n: C_n \to C_n'$ and $f_n \partial = \partial f_{n+1}$, such $\{f\}$ are called morphism between chain complexes. An exact sequence of chain complexes $0 \to C'_\cdot \xrightarrow{i} C_\cdot \xrightarrow{j} C''_\cdot \to 0$.

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] & \\
0 \arrow[r] & C_{n+1}' \arrow[r, "i"] \arrow[d] & C_{n+1} \arrow[r, "j"] \arrow[d] & C_{n+1}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_n' \arrow[r, "i"] \arrow[d] & C_n \arrow[r, "j"] \arrow[d] & C_n'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r, "i"] \arrow[d] & C_{n-1} \arrow[r, "j"] \arrow[d] & C_{n-1}'' \arrow[r] \arrow[d] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

Each row is an exact sequence, actually split since C_n'' is free. And the diagram commutes.