HOMEWORK 4

21-241: Matrices and LinearTransformations, Fall 2018

1. Use Gaussian elimination to solve the system of equations. Clearly indicate the elementary row operations used:

$$\begin{aligned} x + 2y - z &= 1 \\ 2x + 4y - 2z - w &= -1 \\ -3x - 5y + 6z + w &= 3 \\ -x + 2y + 8z - 2w &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 2 & 4 & -2 & -1 & -1 \\ -3 & -5 & 6 & 1 & 3 \\ -1 & 2 & 8 & -2 & 0 \end{array}\right] \xrightarrow[\substack{3R_1 + R_3 \\ R_1 + R_4}]{-2R_1 + R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 1 & 3 & 1 & 6 \\ 0 & 4 & 7 & -2 & 1 \end{array}\right] \quad \begin{array}{c} R_2 \leftrightarrow R_3 \\ \longrightarrow \\ \text{followed by} \\ \text{new } R_3 \leftrightarrow R_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 6 \\ 0 & 4 & 7 & -2 & 1 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right] \xrightarrow[-4R_2 + R_3]{} \left[\begin{array}{cccc|c} \boxed{1} & 2 & -1 & 0 & 1 \\ 0 & \boxed{1} & 3 & 1 & 6 \\ 0 & 0 & \boxed{-5} & -6 & -23 \\ 0 & 0 & 0 & \boxed{-1} & -3 \end{array}\right]$$

$$\begin{cases} -w = -3 \Rightarrow w = 3 \\ -5z - 6(3) = -23 \Rightarrow 5z = 23 - 18 \Rightarrow z = 1 \\ y + 3(1) + 3 = 6 \Rightarrow y = 0 \\ x + 2(0) - 1 + 0 = 1 \Rightarrow x = 2 \end{cases}$$

Solution: $\boxed{x = 2,\ y = 0,\ z = 1,\ w = 3}$

2. Write the system from problem #1 as $A\mathbf{x} = \mathbf{b}$, and factor A as LU or P^TLU, whichever is appropriate. Use this factorization to solve the system.

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & -2 & -1 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \\ 0 \end{bmatrix}}_{\vec{b}}$$

. From #1, we see that we need to use

$P = P_{34} P_{23}$ (in this order)

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}; \quad PA = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \\ 2 & 4 & -2 & -1 \end{bmatrix}$$

find the LU factorization of PA:

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & -5 & 6 & 1 \\ -1 & 2 & 8 & -2 \\ 2 & 4 & -2 & -1 \end{bmatrix}}_{PA} \xrightarrow[\substack{3R_1 + R_2 \\ R_1 + R_3 \\ -2R_1 + R_4}]{} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 4 & 7 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} \xrightarrow[-4R_2 + R_3]{} \underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U \text{ (same as in \#1)}}$$

$\ell_{21} = -3,\ \ell_{31} = -1,\ \ell_{41} = 2,\ \ell_{32} = 4$. L must be 4×4.

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$PA = LU \Rightarrow A = P^{-1}LU = P^TLU$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{P^T} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix}}_{U}$$

Solve $A\vec{x} = \vec{b}$: $\quad P^T L \underbrace{U\vec{x}}_{\vec{c}} = \vec{b} \quad \Rightarrow \quad \underbrace{P^T}_{P^{-1}} L\vec{c} = \vec{b} \quad \Rightarrow \begin{cases} L\vec{c} = P\vec{b} \\ U\vec{x} = \vec{c} \end{cases}$

$L\vec{c} = P\vec{b}$:
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ -1 & 4 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} & c_1 = 1 \\ & -3\cdot 1 + c_2 = 3 \Rightarrow c_2 = 6 \\ & -1 + 4\cdot 6 + c_3 = 0 \Rightarrow c_3 = -23 \\ & 2\cdot 1 + c_4 = -1 \Rightarrow c_4 = -3. \end{aligned}$$

$U\vec{x} = \vec{c}$:
$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -5 & -6 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ -23 \\ -3 \end{bmatrix}$$
(same as in #1)

$\Rightarrow \quad x = 2,\ y = 0,\ z = 1,\ w = 3$

3. Use Gaussian elimination to solve the system of equations. Clearly indicate the elementary row operations used:

$$\begin{aligned} -w + 3x - 2y + 4z &= 0 \\ 2w - 6x + y - 2z &= -3 \\ w - 3x + 4y - 8z &= 2 \end{aligned}$$

$$\left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 2 & -6 & 1 & -2 & -3 \\ 1 & -3 & 4 & -8 & 2 \end{array}\right] \xrightarrow[R_1+R_3]{2R_1+R_2} \left[\begin{array}{cccc|c} -1 & 3 & -2 & 4 & 0 \\ 0 & 0 & -3 & 6 & -3 \\ 0 & 0 & 2 & -4 & 2 \end{array}\right] \xrightarrow{\frac{2}{3}R_2+R_3}$$

$$\underbrace{\left[\begin{array}{cccc} -1 & 3 & -2 & 4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right.}_{U} \underbrace{\left|\begin{array}{c} 0 \\ -3 \\ 0 \end{array}\right]}_{\vec{c}} \Rightarrow \begin{cases} 0 = 0 \checkmark \\ -3y + 6z = -3 \Rightarrow y - 2z = 1 \Rightarrow y = 1 + 2z \\ -w + 3x - 2 - \cancel{4z} + \cancel{4z} = 0 \Rightarrow w = 3x - 2 \end{cases}$$

Solution: $\boxed{w = 3x - 2,\ x = x,\ y = 1 + 2z,\ z = z}$

4. Find the LU factorization or the P^TLU factorization of the matrix A of the system in problem #3.

In #3, we didn't use any row exchanges,

$l_{21} = -2,\ l_{31} = -1,\ l_{32} = -\frac{2}{3}$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -\frac{2}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} -1 & 3 & -2 & 4 \\ 0 & 0 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}.$$

5. a) A square matrix is called *strictly lower (upper) triangular* if it is lower (upper) triangular and $a_{ii} = 0$ for all i from 1 to n.

i) Show that any square matrix A can be written as a sum $L + D + U'$, with L strictly lower triangular, D diagonal, and U' strictly upper triangular.

ii) Find matrices L, D, U' to write $A = \begin{bmatrix} 1 & -4 & 2 \\ 3 & 1 & -1 \\ -2 & 0 & 5 \end{bmatrix}$ as $L + D + U'$.

b) Find the LDU' factorization of A. Here, L, D, U' are not the same as the ones from part a). (L is unit lower triangular, D is diagonal, and U' is unit upper triangular.)

a) i) $(L)_{ij} = \begin{cases} 0, & \text{for } i \le j \\ (A)_{ij}, & \text{for } i > j \end{cases}$; $U' = \begin{cases} 0 & \text{for } i \ge j \\ (A)_{ij} & \text{for } i < j \end{cases}$

$(D)_{ij} = \begin{cases} (A)_{ij}, & \text{for } j = i \\ 0, & \text{for } j \ne i \end{cases}$

$\Rightarrow$ $(A)_{ij} = (L)_{ij} + (D)_{ij} + (U')_{ij}$ for i, j taking values from 1 to n.

ii) $$A = \begin{bmatrix} 1 & -4 & 2 \\ 3 & 1 & -1 \\ -2 & 0 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}}_{L} + \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & -4 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{U'}$$

b) $$\begin{bmatrix} 1 & -4 & 2 \\ 3 & 1 & -1 \\ -2 & 0 & 5 \end{bmatrix} \xrightarrow[\substack{-3R_1 + R_2 \\ 2R_1 + R_3}]{} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 13 & -7 \\ 0 & -8 & 9 \end{bmatrix} \xrightarrow[\frac{8}{13} R_2 + R_3]{} \begin{bmatrix} 1 & -4 & 2 \\ 0 & 13 & -7 \\ 0 & 0 & \frac{61}{13} \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{8}{13} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 13 & -7 \\ 0 & 0 & \frac{61}{13} \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -\frac{8}{13} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & \frac{61}{13} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & -7/13 \\ 0 & 0 & 1 \end{bmatrix}}_{U'}$$

6. If A is symmetric and invertible and has an LDU' factorization, show that $U' = L^T$.

A symmetric $\Rightarrow$ $A^T = A$

A invertible $\Rightarrow$ the LU factorization of A is unique (see lecture slides).

$A = LU = LDU'$ where D contains the pivots of U (see #5).

Note that L is the same matrix in both factorizations; L is unique.

Since U is also unique and $U = DU'$ with D containing the pivots of U $\Rightarrow$ D and U' are unique (determined by the pivots).

$\Rightarrow$ the LDU' factorization is also unique.

Find A^T:

$$A^T = (LDU')^T = (U')^T D^T L^T = (U')^T D L^T$$

(If D is diagonal, D^T is also diagonal: $(D^T)_{ij} = D_{ji} = 0$ for $i \neq j$)

Use that $A^T = A$:

$$\underbrace{(U')^T}_{\text{unit l.t.}} D \underbrace{L^T}_{\text{unit u.t.}} = \underbrace{L}_{\text{unit l.t.}} D \underbrace{U'}_{\text{unit u.t.}}$$

Since the LDU' factorization is also unique $\Rightarrow$ $(U')^T = L$ or $U' = L^T$.

7. Parts a) and b) are not related.

a) Write the following matrices as P_{ij} for specific values of i and j, or as products of such matrices. Remember that in products of permutation matrices, the matrix on the right gets applied first.

$$\text{(i) } \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{(ii) } \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b) Consider the matrix:

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists). (Check your answer, that is, show $AA^{-1} = I$.)

a) i) $P_{13} P_{23} P_{14} = P_{13} P_{23} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = P_{13} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ ✓

ii) $P_{24} P_{13} = P_{24} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ (or $P_{13} P_{24}$)

b) $$\left[\underbrace{\begin{matrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{matrix}}_{A} \;\middle|\; \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I}\right] \xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{matrix} 1 & -2 & -1 \\ 2 & 3 & 0 \\ 2 & 0 & -1 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \xrightarrow[\substack{-2R_1+R_2 \\ -2R_1+R_3}]{}$$

$$\left[\begin{matrix} 1 & -2 & -1 \\ 0 & 7 & 2 \\ 0 & 4 & 1 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & -2 & 1 \end{matrix}\right] \xrightarrow[\frac{1}{7}R_2]{} \left[\begin{matrix} 1 & -2 & -1 \\ 0 & 1 & \frac{2}{7} \\ 0 & 4 & 1 \end{matrix} \;\middle|\; \begin{matrix} 0 & 1 & 0 \\ \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & -2 & 1 \end{matrix}\right] \xrightarrow[-4R_2+R_3]{}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & -\frac{6}{7} & 1 \end{array}\right] \xrightarrow[-7R_3]{} \left[\begin{array}{ccc|ccc} 1 & -2 & -1 & 0 & 1 & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \xrightarrow[2R_2+R_1]{}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{7} & \frac{2}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{2}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & 4 & 6 & -7 \end{array}\right] \xrightarrow[\substack{\frac{3}{7}R_3+R_1 \\ -\frac{2}{7}R_3+R_2}]{} \underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right.}_{I} \underbrace{\left|\begin{array}{ccc} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{array}\right]}_{A^{-1}}$$

Check: $AA^{-1} = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -2 & -1 \\ 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -3 \\ -1 & -2 & 2 \\ 4 & 6 & -7 \end{bmatrix} = I$

8. Determine whether the following sets W are subspaces of the corresponding vector spaces V. (If W is not a subspace, indicate which condition fails. If W is a subspace, prove that all the conditions are satisfied.)

a) $V = \mathbb{R}^3,\ W = \{\langle a, b, |a|\rangle\}$

b) $V = \mathcal{M}_{22},\ W = \left\{\begin{bmatrix} a & b \\ b & 2a \end{bmatrix}\right\}$

c) $V = \mathcal{P}_2,\ W = \{a + bx + cx^2 : abc = 0\}$

a) (i) $\vec{0}_{\mathbb{R}^3} = \langle 0,0,0\rangle \in W$ (for $a=0, b=0$)

(ii) Let $\vec{v} \in W$: $\vec{v} = \langle a, b, |a|\rangle$; $a, b \in \mathbb{R}$

Let $c \in \mathbb{R}$: $c\vec{v} = \langle ca, cb, c|a|\rangle$. But $c|a| \neq |ca|$ if c is negative.

$\Rightarrow c\vec{v} \notin W \Rightarrow W$ is not a subspace of V.

b) (i) $\vec{0}_V = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$ (for $a=0, b=0$)

(ii) Let $\vec{u}, \vec{v} \in W$. $\vec{u} = \begin{bmatrix} a_1 & b_1 \\ b_1 & 2a_1 \end{bmatrix}$; $\vec{v} = \begin{bmatrix} a_2 & b_2 \\ b_2 & 2a_2 \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ b_1+b_2 & 2(a_1+a_2) \end{bmatrix} \in W$

$c\vec{u} = \begin{bmatrix} ca_1 & cb_1 \\ cb_1 & 2(ca_1) \end{bmatrix} \in W$

$\therefore$ W is a subspace of V

c) (i) $\vec{0}_V = 0$ (the zero polynomial) $\in W$ for $a=b=c=0$. ($abc = 0$ in this case)

(ii) Let $p_1, p_2 \in W$: $p_1(x) = a_1 + b_1x + c_1x^2$ with $a_1b_1c_1 = 0$

$p_2(x) = a_2 + b_2x + c_2x^2$ with $a_2b_2c_2 = 0$.

$(p_1+p_2)(x) = p_1(x) + p_2(x) = (a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2$. This doesn't belong to W, since $(a_1+a_2)(b_1+b_2)(c_1+c_2)$ doesn't have to be 0.

- take $p_1(x) = x + x^2$; $p_2(x) = 1 + 2x^2$ in W

$$(p_1+p_2)(x) = 1 + x + 3x^2 \; ; \; a = 1, b = 1, c = 3$$

$$abc \neq 0.$$

$\Rightarrow$ W is not a subspace.

9. Determine whether $\mathbb{R}^2$, with the usual scalar multiplication but addition defined by

$$\langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + y_1 + 1, x_2 + y_2 + 1 \rangle$$

is a vector space. Check all the axioms in the definition of a vector space.

Addition : $\vec{V_1} + \vec{V_2} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + y_1 + 1, x_2 + y_2 + 1 \rangle$

Scalar multiplication : $c\vec{v} = \langle cx, cy \rangle$

1) $\vec{V_1} + \vec{V_2} \in \mathbb{R}^2$ since $x_1 + y_1 + 1 \in \mathbb{R}$ and $x_2 + y_2 + 1 \in \mathbb{R}$

2) $\vec{V_1} + \vec{V_2} = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle = \langle x_1 + y_1 + 1, x_2 + y_2 + 1 \rangle$

$\vec{V_2} + \vec{V_1} = \langle x_2, y_2 \rangle + \langle x_1, y_1 \rangle = \langle x_2 + y_2 + 1, x_1 + y_1 + 1 \rangle \neq \vec{V_1} + \vec{V_2}$

$\Rightarrow$ addition is not commutative (not a vector space)

3) $\vec{V_1} = \langle x_1, y_1 \rangle$, $\vec{V_2} = \langle x_2, y_2 \rangle$, $\vec{V_3} = \langle x_3, y_3 \rangle$

$(\vec{V_1} + \vec{V_2}) + \vec{V_3} = \langle x_1 + y_1 + 1, x_2 + y_2 + 1 \rangle + \langle x_3, y_3 \rangle = \langle x_1 + x_2 + y_1 + y_2 + 3,$
$x_3 + y_3 + 1 \rangle$

$\vec{V_1} + (\vec{V_2} + \vec{V_3}) = \langle x_1, y_1 \rangle + \langle x_2 + y_2 + 1, x_3 + y_3 + 1 \rangle$

$= \langle x_1 + y_1 + 1, x_2 + x_3 + y_2 + y_3 + 3 \rangle \neq (\vec{V_1} + \vec{V_2}) + \vec{V_3}$

- not associative.

4) Let $\vec{0} = \langle a, b \rangle$, $\vec{v} = \langle x, y \rangle$

$\vec{v} + \vec{0} = \langle x + y + 1, a + b + 1 \rangle = \langle x, y \rangle$

$\cancel{x} + y + 1 = \cancel{x} \Rightarrow y = -1$

$a + b + 1 = y \Rightarrow \boxed{a + b = -2} \Rightarrow \vec{0} = \langle a, -2 - a \rangle$ not unique

$\vec{0} = \langle 0, -2 \rangle$ or $\vec{0} = \langle -1, -1 \rangle$, etc.

5) Let $-\vec{v} = \langle A, B \rangle$ such that $-\vec{v} + \vec{v} = \vec{0}$

$\langle A, B \rangle + \langle x, y \rangle = \langle a, -2 - a \rangle$

$\langle A + B + 1, x + y + 1 \rangle = \langle a, -2 - a \rangle$

$$\begin{cases} A + B + 1 = a \\ x + y + 1 = -2 - a \Rightarrow a = -3 - x - y \end{cases}$$

$A + B + 1 = -3 - x - y \Rightarrow A + B = -4 - x - y$

A and B are not unique for a given x and y.

$x = 1, y = 1 \Rightarrow A + B = -6$

$A = 0, B = -6$ or $A = 3, B = -9$, etc.

6) ÷ 10) The remaining axioms are satisfied since # is the regular scalar multiplication in $\mathbb{R}^2$.

6) $c\vec{v} = \langle cx, cy \rangle \in \mathbb{R}^2$

7) The identity element is the number 1.

$1 \langle x, y \rangle = \langle x, y \rangle$

8) $(cd)\vec{v} = \langle cdx, cdy \rangle$

$c(d\vec{v}) = c\langle dx, dy \rangle = \langle cdx, cdy \rangle = (cd)\vec{v}$

9) $c(\vec{v_1} + \vec{v_2}) = c\langle x_1 + y_1 + 1, x_2 + y_2 + 1 \rangle = \langle cx_1 + cy_1 + c, cx_2 + cy_2 + c \rangle$

$c\vec{v_1} + c\vec{v_2} = \langle cx_1, cy_1 \rangle + \langle cx_2, cy_2 \rangle = \langle cx_1 + cy_1 + 1, cx_2 + cy_2 + 1 \rangle$

$c(\vec{v_1} + \vec{v_2}) \neq c\vec{v_1} + c\vec{v_2}$ for arbitrary c.

10) $(c+d)\vec{v} = \langle (c+d)x, (c+d)y \rangle$

$c\vec{v} + d\vec{v} = \langle cx, cy \rangle + \langle dx, dy \rangle = \langle cx + cy + 1, dx + dy + 1 \rangle \neq (c+d)\vec{v}$.

10. Let U and W be subspaces of V. Define the **sum** of U and W to be:

$$U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$$

Show that $U + W$ is a subspace of V.

i) Let $\vec{0}_V$ be the zero vector of V. Since U and W are subspaces of V,

then $\vec{0}_V \in U$ and $\vec{0}_V \in W$. $\Rightarrow$ $\vec{0}_V \in U+W$ since $\vec{0}_V$ can be written as $\underbrace{\vec{0}_V}_{\in U} + \underbrace{\vec{0}_V}_{\in W}$.

$\Rightarrow$ $U+W$ is nonempty.

ii) Let $\vec{v_1}, \vec{v_2} \in U+W$ $\Rightarrow$ $\vec{v_1} = \vec{u_1} + \vec{w_1}$ for some $\vec{u_1} \in U$, $\vec{w_1} \in W$

$\vec{v_2} = \vec{u_2} + \vec{w_2}$ for some $\vec{u_2} \in U$, $\vec{w_2} \in W$

$$\vec{v_1} + \vec{v_2} = \vec{u_1} + \vec{w_1} + \vec{u_2} + \vec{w_2} = \vec{u_1} + \vec{u_2} + \vec{w_1} + \vec{w_2} = \underbrace{(\vec{u_1} + \vec{u_2})}_{\in U} + \underbrace{(\vec{w_1} + \vec{w_2})}_{\in W}$$

addition in V is commutative

associativity

U, W are subspaces

$\Rightarrow$ $\vec{v_1} + \vec{v_2} \in U+W$

iii) Let $\vec{v} \in U+W$ $\Rightarrow$ $\vec{v} = \vec{u} + \vec{w}$ for some $\vec{u} \in U$ and $\vec{w} \in W$.

Let $c \in \mathbb{F}$.

$c\vec{v} = c(\vec{u} + \vec{w}) = c\vec{u} + c\vec{w} \in U+W$, since $c\vec{u} \in U$, $c\vec{w} \in W$ (U, W are subspaces)

(9) in v.s.

$\Rightarrow$ $U+W$ is a subspace of V.