1 The normal modes of a multipendulum

The multipendulum has the following energies:

$$\begin{aligned}
T &= \frac{1}{2}\sum_{i=1}^{N} m_i \left[\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right] \\
V^* &= -g \sum_{i=1}^{N} m_i \sum_{j=1}^{i} l_j \cos\theta_j
\end{aligned}$$

We will add a constant V_0 to V^* where

$$V_0 = g \sum_{i=1}^{N} m_i \sum_{j=1}^{i} l_j$$

so $V = V^* + V_0$ becomes

$$V = -g \sum_{i=1}^{N} m_i \sum_{j=1}^{i} l_j \left(\cos\theta_j - 1 \right)$$

This will prove helpful in later calculations.

1.1 Finding the stable equilibrium positions

Equilibrium positions $\vec{\theta}_m$ are such that

$$\left. \frac{\mathrm{d}V^*}{\mathrm{d}\vec{\theta}} \right|_{\vec{\theta}_m} = 0$$

Since V^* is a linear combination of $\cos\theta_i$ terms, the stable equilibrium points will occur at points in space where

$$\theta_i = 0 \text{ or } \theta_i = \pi \text{ for } i = 1, 2, \ldots N$$

By inspection, we see that the only stable equilibrium is where $\vec{\theta} = \vec{0}$. Hence we can treat $\vec{\theta}$ as the perturbation about the equilibrium.

1.2 Perturbing the Lagrangian to the 2nd order about the equilibrium

We have $L = T - V$. We can perturb T and L separately and bring them together afterwards. In our ansatz:

$$\theta_i(t) = \varepsilon e^{i\omega_i t}, \dot{\theta}_i(t) = \varepsilon \omega e^{i\omega_i t}$$

where $O(\varepsilon^3) = 0$.

1.2.1 Perturbing the T term

Consider the following expansion of $\sin\theta$:

$$\sin\theta = \theta + O(\theta^3) = \theta$$

Then the term

$$\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 = \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \theta_j \right)^2 = O(\theta^4) + O(\theta^2\dot{\theta}^2) + O(\dot{\theta}^4) = O(\varepsilon^4) = 0$$

Using the expansion of $\cos\theta$

$$\cos\theta = 1 - \frac{1}{2}\theta^2 + O(\theta^4) = 1 - \frac{1}{2}\theta^2$$

we can rewrite T as

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \left(1 - \frac{1}{2}\theta_j^2\right)\right)^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} \dot{\theta}_j l_j - \frac{1}{2}\dot{\theta}_j l_j \theta_j^2\right)^2 = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\sum_{j=1}^{i} \dot{\theta}_j l_j\right)^2$$

since $\dot{\theta}_j \theta^2 = O(\varepsilon^3) = 0$.

1.2.2 Perturbing the V term

This is more straightforward; we substitute the expansion of $\cos\theta$ into our expression for the potential:

$$V = -g\sum_{i=1}^{N} m_i \sum_{j=1}^{i} l_j \left(1 - \frac{1}{2}\theta_j^2 - 1\right) = \frac{1}{2}g\sum_{i=1}^{N} m_i \sum_{j=1}^{i} l_j \theta_j^2$$

1.3 Changing to natural coordinates

We wish to make a coordinate transformation $\vec{\theta} \to \vec{q}$ such that the kinetic term becomes

$$T = \frac{1}{2}\dot{\vec{q}} \cdot \dot{\vec{q}}$$

By observing our kinetic term, we can guess the correct answer:

$$q_i = \sqrt{m_i}\sum_{j=1}^{i} l_j \theta_j,\ \dot{q}_i = \sqrt{m_i}\sum_{j=1}^{i} l_j \dot{\theta}_j$$

We now wish to have an expression for V in terms of $\vec{q}$. First, we would like to obtain θi as a function of $\vec{q}$.
First we form a recurrence relation for $\vec{q}$:

$$\begin{aligned} q_1 &= \sqrt{m_1} l_1 \theta_1 \\ q_n &= \sqrt{\frac{m_n}{m_{n-1}}} q_{n-1} + \sqrt{m_m} l_n \theta_n \end{aligned}$$

From this we have

$$\begin{aligned} \theta_1 &= \frac{q_1}{l_1\sqrt{m_1}} \\ \theta_n &= \frac{1}{l_n}\left(\frac{q_n}{\sqrt{m_n}} - \frac{q_{n-1}}{\sqrt{m_{n-1}}}\right) \end{aligned}$$

We now reindex teh double summation in V:

$$V = \frac{1}{2}g\sum_{i=1}^{N} l_i \theta_i^2 \sum_{j=1}^{N} m_j$$

To make matters simpler, we define a new vector $\vec{\mu}$ such that

$$\mu_i = \sum_{j=i}^{N} m_j$$

Then we substitute $\theta_n(q_1, q_2, \dots q_N)$ into V:

$$\begin{aligned} V &= \frac{1}{2}g\left[\frac{\mu_1}{l_1}\left(\frac{q_1}{\sqrt{m_1}}\right)^2 + \sum_{i=2}^{N}\frac{\mu_i}{l_i}\left(\frac{q_i}{\sqrt{m_i}} - \frac{q_{i-1}}{\sqrt{m_{i-1}}}\right)^2\right] \\ &= \frac{1}{2}g\left[\frac{q_N^2}{l_N} + \sum_{i=1}^{N-1}\left(\left(\frac{\mu_i}{l_i} + \frac{\mu_{i+1}}{l_{i+1}}\right)\frac{q_i^2}{m_i} - 2\frac{\mu_{i+1}}{l_{i+1}}\frac{q_i q_{i+1}}{\sqrt{m_i m_{i+1}}}\right)\right] \end{aligned}$$

1.4 Determining the k matrix

We can now write our Lagrangian in the form

$$L = \frac{1}{2}\dot{\vec{q}} \cdot \dot{\vec{q}} - \frac{1}{2}\vec{q} \cdot k\vec{q}$$

where k is a matrix defined as

$$k_{ij} = \frac{\partial^2 V}{\partial q^i \partial q^j}$$

Now:

$$\frac{\partial V}{\partial q^j} = \begin{cases} \frac{g}{l_N}\left(q_N - \sqrt{\frac{m_N}{m_{N-1}}}q_{N-1}\right) & j = N \\ g\left[\left(\frac{\mu_1}{l_1} + \frac{\mu_2}{l_2}\right)\frac{q_1}{m_1} - \frac{\mu_2}{l_2\sqrt{m_1 m_2}}q_2\right] & j = 1 \\ g\left[\left(\frac{\mu_j}{l_j} + \frac{\mu_{j+1}}{l_{j+1}}\right)\frac{q_j}{m_j} - \frac{\mu_j}{l_j\sqrt{m_{j-1}m_j}}q_{j-1} - \frac{\mu_{j+1}}{l_{j+1}\sqrt{m_j m_{j+1}}}q_{j+1}\right] & \text{otherwise} \end{cases}$$

Taking the second partial derivative and setting this equal to the ij-th element of k:

$$k_{ij} = \begin{cases} \frac{g}{l_N} & i = j = N \\ -\frac{g}{l_N}\sqrt{\frac{m_N}{m_{N-1}}} & i = N-1, j = N \\ -\frac{g}{l_{j+1}}\frac{\mu_{j+1}}{\sqrt{m_j m_{j+1}}} & i = j+1, j \neq N \\ \frac{g}{m_j}\left(\frac{\mu_j}{l_j} + \frac{\mu_{j+1}}{l_{j+1}}\right) & i = j \neq N \\ -\frac{g}{l_j}\frac{\mu_j}{\sqrt{m_{j-1}m_j}} & i = j-1, j \neq N \\ 0 & \text{otherwise} \end{cases}$$

We see that $k_{ij} = k_{ji} \to k$ is symmetric, as expected.

1.5 Determining the normal modes and associated natural frequencies

Obtaining the set of eigenvectors $\vec{v}_i$ and associated eigenvalues λ_i of matrix k, we know that the normal modes are in the form

$$\vec{q}_n(t) = \vec{v}_n e^{i\omega_n t}, \omega_n = \sqrt{\lambda_n}$$

We now wish to represent this motion in the original coordinates, that is, find $\vec{\theta}_n(t) = f(\vec{v}_n, \omega_n)$. To do this, we just substitute the i-th and $(i-1)$-th elements of $\vec{q}_n$ into our expression for $\theta_i(q_i, q_{q-1})$:

$$\begin{aligned} \theta_{n,1}(t) &= \frac{v_{n,1}}{l_1\sqrt{m_1}} e^{i\omega_n t} \\ \theta_{n,i}(t) &= \frac{1}{l_i}\left(\frac{v_{n,i}}{\sqrt{m_i}} - \frac{v_{n,i-1}}{\sqrt{m_{i-1}}}\right) e^{i\omega_n t} \end{aligned}$$

where n is the index of the normal mode and i is the index of the segment.