1 The Lagrangian

The multipendulum is described by two static parameters $\vec{m}, \vec{l}$, two dynamic parameters $\vec{\theta}, \dot{\vec{\theta}}$ (which define the phase space), the number of segments N and gravitational acceleration $\vec{g} = -g\hat{y}$.

The vector displacement between the $(n-1)$-th and n-th segment is

$$\vec{r}_n - \vec{r}_{n-1} = l_n \begin{pmatrix} \sin\theta_n \\ -\cos\theta_n \\ 0 \end{pmatrix}$$

The position of the n-th segment is then

$$\vec{r}_n = \sum_{i=1}^{n} l_i \begin{pmatrix} \sin\theta_i \\ -\cos\theta_i \\ 0 \end{pmatrix} = \hat{x} \sum_{i=1}^{n} l_i \sin\theta_i - \hat{y} \sum_{i=1}^{n} l_i \cos\theta_i$$

Similarly, the velocity of the n-th segment relative to the $(n-1)$-th segment is

$$\vec{v}_n - \vec{v}_{n-1} = (\dot{\theta}_n \hat{z}) \times (\vec{r}_n - \vec{r}_{n-1}) = \dot{\theta}_n l_n \begin{pmatrix} \cos\theta_n \\ \sin\theta_n \\ 0 \end{pmatrix}$$

Hence the velocity of the n-th segment is then

$$\vec{v}_n = \sum_{i=1}^{n} \dot{\theta}_i l_i \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \\ 0 \end{pmatrix} = \hat{x} \sum_{i=1}^{n} \dot{\theta}_i l_i \cos\theta_i + \hat{y} \sum_{i=1}^{n} \dot{\theta}_i l_i \sin\theta_i$$

And the square magnitude of that is

$$v_n^2 = \left(\sum_{i=1}^{n} \dot{\theta}_i l_i \cos\theta_i \right)^2 + \left(\sum_{i=1}^{n} \dot{\theta}_i l_i \sin\theta_i \right)^2$$

The potential energy of the n-th segment due to the uniform gravitational field is

$$U_n = -m_n \vec{r}_n \cdot \vec{g} = -m_n g \sum_{i=1}^{n} l_i \cos\theta_i$$

Hence the total gravitational potential energy is

$$U_g = \sum_{i=1}^{N} U_i = -\sum_{i=1}^{N} m_i g \sum_{j=1}^{i} l_j \cos\theta_j$$

And the potential energy due to the torque $\tau(t)$ driving the first segment is

$$U_\tau = -\tau(t)\theta_1$$

Since this force-inducing potential is not a function of $\dot{\vec{\theta}}$, we can absorb it into the Lagrangian. Hence the total potential energy is

$$U = -\tau(t)\theta_1 - \sum_{i=1}^{N} m_i g \sum_{j=1}^{i} l_j \cos\theta_j$$

The kinetic energy of the n-th segment is

$$T_n = \frac{1}{2}m_n v_n^2 = \frac{1}{2}m_n \left(\left(\sum_{i=1}^{n} \dot{\theta}_i l_i \cos\theta_i \right)^2 + \left(\sum_{i=1}^{n} \dot{\theta}_i l_i \sin\theta_i \right)^2 \right)$$

Hence the total kinetic energy is

$$T = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \frac{1}{2} m_i \left(\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right)$$

This yields the Lagrangian of the system:

$$L = T - U = \tau(t)\theta_1 + \sum_{i=1}^{N} m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right) + g \sum_{j=1}^{i} l_j \cos\theta_j \right]$$

It is meaningful to define a term L_n as

$$L_n = T_n - U_n = m_n \left[\frac{1}{2} \left(\left(\sum_{i=1}^{n} \dot{\theta}_i l_i \cos\theta_i \right)^2 + \left(\sum_{i=1}^{n} \dot{\theta}_i l_i \sin\theta_i \right)^2 \right) + g \sum_{i=1}^{n} l_i \cos\theta_i \right]$$

so that

$$L = \tau(t)\theta_1 + \sum_{i=1}^{N} L_i$$

2 The Euler-Lagrange equation

2.1 The $\frac{\partial L}{\partial \theta_n}$ term

First, we wish to find the partial derivative $\frac{\partial L}{\partial \theta_n}$.

We notice that L_i is a function of $\theta_1, \theta_2, \dots \theta_i$ and $\dot{\theta}_1, \dot{\theta}_2, \dots \dot{\theta}_i$, in other words, it is invariant to θ_n iff $n > i$. Hence

$$\frac{\partial L_i}{\partial \theta_n} = 0 \quad \text{if} \quad n > i$$

Now we can use this to simplify the expression

$$\frac{\partial L}{\partial \theta_n} = \frac{\partial}{\partial \theta_n}\tau(t)\theta_1 + \frac{\partial}{\partial \theta_n}\sum_{i=1}^{N} L_i = \tau(t)\delta_{1n} + \sum_{i=1}^{N} \frac{\partial L_i}{\partial \theta_n} = \tau(t)\delta_{1n} + \sum_{i=n}^{N} \frac{\partial L_i}{\partial \theta_n}$$

where for each sum term in the last expression $i \geq n$, hence all the terms should be nonzero. For these terms we have

$$\begin{aligned}
\frac{\partial L_i}{\partial \theta_n} &= \frac{\partial}{\partial \theta_n} m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right) + g \sum_{j=1}^{i} l_j \cos\theta_j \right] \\
&= m_i \left[\frac{1}{2} \left(\frac{\partial A^2}{\partial \theta_n} + \frac{\partial B^2}{\partial \theta_n} \right) + g \frac{\partial C}{\partial \theta_n} \right] = m_i \left[A \frac{\partial A}{\partial \theta_n} + B \frac{\partial B}{\partial \theta_n} + g \frac{\partial C}{\partial \theta_n} \right]
\end{aligned}$$

where A, B, C are the sums

$$A = \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j; \quad B = \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j; \quad C = \sum_{j=1}^{i} l_j \cos\theta_j$$

Since $i \geq n$, these sums will have exactly one term for which $j = n$, for which the partial derivative $\frac{\partial}{\partial \theta_n}$ is trivial; for all other terms, it is zero. Hence

$$\frac{\partial A}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} \dot{\theta}_n l_n \cos\theta_n = -\dot{\theta}_n l_n \sin\theta_n; \frac{\partial B}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} \dot{\theta}_n l_n \sin\theta_n = \dot{\theta}_n l_n \cos\theta_n; \frac{\partial C}{\partial \theta_n} = \frac{\partial}{\partial \theta_n} l_n \cos\theta_n = -l_n \sin\theta_n$$

This yields

$$\frac{\partial L_i}{\partial \theta_n} = -m_i l_n \left[\dot{\theta}_n \left(\sin\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j - \cos\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right) + g \sin\theta_n \right]$$

From which we have

$$\begin{aligned}
\frac{\partial L}{\partial \theta_n} &= \tau(t)\delta_{1n} - l_n \sum_{i=n}^{N} m_i \left[\dot{\theta}_n \left(\sin\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j - \cos\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right) + g \sin\theta_n \right] \\
&= \tau(t)\delta_{1n} - l_n \sum_{i=n}^{N} m_i \left[g \sin\theta_n + \dot{\theta}_n \sum_{j=1}^{i} \dot{\theta}_j l_j \left(\sin\theta_n \cos\theta_j - \cos\theta_n \sin\theta_j \right) \right]
\end{aligned}$$

Using the angle subtraction identity $\sin\theta_n \cos\theta_j - \cos\theta_n \sin\theta_j = \sin(\theta_n - \theta_j)$ yields

$$\frac{\partial L}{\partial \theta_n} = \tau(t)\delta_{1n} - l_n \sum_{i=n}^{N} m_i \left[g \sin\theta_n + \dot{\theta}_n \sum_{j=1}^{i} \dot{\theta}_j l_j \sin(\theta_n - \theta_j) \right]$$

2.2 The $\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\theta}_n}$ term

First, we need to find $\frac{\partial L}{\partial \dot{\theta}_n}$. We can use approach analogous to subsection 2.1:

$$\frac{\partial L}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \tau(t)\theta_1 + \frac{\partial}{\partial \dot{\theta}_n} \sum_{i=1}^{N} L_i = \sum_{i=1}^{N} \frac{\partial L_i}{\partial \dot{\theta}_n} = \sum_{i=n}^{N} \frac{\partial L_i}{\partial \dot{\theta}_n}$$

where

$$
\begin{aligned}
\frac{\partial L_i}{\partial \dot{\theta}_n} &= \frac{\partial}{\partial \dot{\theta}_n} m_i \left[\frac{1}{2} \left(\left(\sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j \right)^2 + \left(\sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right)^2 \right) + g \sum_{j=1}^{i} l_j \cos\theta_j \right] \\
&= m_i \left[\frac{1}{2} \left(\frac{\partial A^2}{\partial \dot{\theta}_n} + \frac{\partial B^2}{\partial \dot{\theta}_n} \right) + g \frac{\partial C}{\partial \dot{\theta}_n} \right] = m_i \left[A \frac{\partial A}{\partial \dot{\theta}_n} + B \frac{\partial B}{\partial \dot{\theta}_n} + g \frac{\partial C}{\partial \dot{\theta}_n} \right]
\end{aligned}
$$

where once again

$$
A = \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j; \quad B = \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j; \quad C = \sum_{j=1}^{i} l_j \cos\theta_j
$$

and the new derivatives are

$$
\frac{\partial A}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \dot{\theta}_n l_n \cos\theta_n = l_n \cos\theta_n; \quad \frac{\partial B}{\partial \dot{\theta}_n} = \frac{\partial}{\partial \dot{\theta}_n} \dot{\theta}_n l_n \sin\theta_n = l_n \sin\theta_n; \quad \frac{\partial C}{\partial \dot{\theta}_n} = 0
$$

Hence

$$
\frac{\partial L_i}{\partial \dot{\theta}_n} = m_i l_n \left[\cos\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j + \sin\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right]
$$

and then

$$
\frac{\partial L}{\partial \dot{\theta}_n} = l_n \sum_{i=n}^{N} m_i \left[\cos\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \cos\theta_j + \sin\theta_n \sum_{j=1}^{i} \dot{\theta}_j l_j \sin\theta_j \right]
$$

Now we want to calculate the time derivative of this expression. For this, we will use the product rule:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_n} &= l_n \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=n}^{N} m_i \left[\cos\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j + \sin\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j\right] \\
&= l_n \sum_{i=n}^{N} m_i \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\sin\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j\right)\right] \\
&= l_n \sum_{i=n}^{N} m_i \left[\frac{\mathrm{d}\cos\theta_n}{\mathrm{d}t} \sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j + \cos\theta_n \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j\right. \\
&\qquad \left. + \frac{\mathrm{d}\sin\theta_n}{\mathrm{d}t} \sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j + \sin\theta_n \frac{\mathrm{d}}{\mathrm{d}t}\sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j\right] \\
&= l_n \sum_{i=n}^{N} m_i \left[\frac{\mathrm{d}\cos\theta_n}{\mathrm{d}t} \sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j + \cos\theta_n \sum_{j=1}^{i} l_j \left(\frac{\mathrm{d}\dot\theta_j}{\mathrm{d}t}\cos\theta_j + \dot\theta_j \frac{\mathrm{d}\cos\theta_j}{\mathrm{d}t}\right)\right. \\
&\qquad \left. + \frac{\mathrm{d}\sin\theta_n}{\mathrm{d}t} \sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j + \sin\theta_n \sum_{j=1}^{i} l_j \left(\frac{\mathrm{d}\dot\theta_j}{\mathrm{d}t}\sin\theta_j + \dot\theta_j \frac{\mathrm{d}\sin\theta_j}{\mathrm{d}t}\right)\right] \\
&= l_n \sum_{i=n}^{N} m_i \left[-\dot\theta_n \sin\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \cos\theta_j + \cos\theta_n \sum_{j=1}^{i} l_j \left(\ddot\theta_j \cos\theta_j - \dot\theta_j^2 \sin\theta_j\right)\right. \\
&\qquad \left. + \dot\theta_n \cos\theta_n \sum_{j=1}^{i} \dot\theta_j l_j \sin\theta_j + \sin\theta_n \sum_{j=1}^{i} l_j \left(\ddot\theta_j \sin\theta_j + \dot\theta_j^2 \cos\theta_j\right)\right] \\
&= l_n \sum_{i=n}^{N} m_i \sum_{j=1}^{i} l_j \left[\cos\theta_n \left(\ddot\theta_j \cos\theta_j + \dot\theta_j \left(\dot\theta_n - \dot\theta_j\right)\sin\theta_j\right) + \sin\theta_n \left(\ddot\theta_j \sin\theta_j - \dot\theta_j\left(\dot\theta_n - \dot\theta_j\right)\cos\theta_j\right)\right] \\
&= l_n \sum_{i=n}^{N} m_i \sum_{j=1}^{i} l_j \left[\ddot\theta_j (\cos\theta_n\cos\theta_j + \sin\theta_n\sin\theta_j) + \dot\theta_j(\dot\theta_n - \dot\theta_j)(\cos\theta_n \sin\theta_j - \sin\theta_n\cos\theta_j)\right] \\
&= l_n \sum_{i=n}^{N} m_i \sum_{j=1}^{i} l_j \left[\ddot\theta_j \cos\left(\theta_n - \theta_j\right) - \dot\theta_j(\dot\theta_n - \dot\theta_j)\sin\left(\theta_n - \theta_j\right)\right]
\end{aligned}$$

2.3 The resulting Euler-Lagrange equation

The E-L equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot\theta_n} - \frac{\partial L}{\partial \theta_n} = 0$$

Plugging our obtained expressions for these terms yields a set of N second-order differential equations:

$$l_n \sum_{i=n}^{N} m_i \left[g\sin\theta_n + \sum_{j=1}^{i} l_j \left[\ddot\theta_j \cos\left(\theta_n - \theta_j\right) - \dot\theta_j(\dot\theta_n - \dot\theta_j)\sin\left(\theta_n - \theta_j\right) + \dot\theta_n\dot\theta_j l_j \sin\left(\theta_n - \theta_j\right)\right]\right] = \tau(t)\delta_{1n} \quad \text{for} \quad n = 1, 2, \dots N$$

which simplifies to

$$l_n \sum_{i=n}^{N} m_i \left[g \sin\theta_n + \sum_{j=1}^{i} l_j \left[\ddot{\theta}_j \cos\big(\theta_n - \theta_j\big) + \dot{\theta}_j^2 \sin\big(\theta_n - \theta_j\big) \right] \right] = \tau(t)\delta_{1n} \quad \text{for} \quad n = 1, 2, \ldots N$$

If $l_i = 0$, the system is identical to a multipendulum of order $N-1$ with one segment being of mass $m_{i-1} + m_i$. As this produces a degeneracy, we may safely ignore it. Since $l_n \neq 0 \forall n$, we can divide each equation by l_n. Let $\tau(t) = l_1 F(t)$. The system of equations becomes

$$\sum_{i=n}^{N} m_i \left[g \sin\theta_n + \sum_{j=1}^{i} l_j \left[\ddot{\theta}_j \cos\big(\theta_n - \theta_j\big) + \dot{\theta}_j^2 \sin\big(\theta_n - \theta_j\big) \right] \right] = F(t)\delta_{1n} \quad \text{for} \quad n = 1, 2, \ldots N$$

This has been checked manually for $N = 1$ and $N = 2$.

3 Solving for second time derivative terms

To be able to numerically integrate these equations of motion, we need to express $\ddot{\theta}_i$ as $\ddot{\theta}_i(\theta_1, \theta_2, \ldots \theta_N, \dot{\theta}_1, \dot{\theta}_2, \ldots \dot{\theta}_N)$. For this, we need to formulate the equations in the form

$$M\vec{\ddot{\theta}} = \vec{S}$$

where M is a square matrix $N \times N$. The a-th equation of motion then becomes

$$\vec{M}_a \cdot \ddot{\theta} = S_a \iff \sum_{b=1}^{N} M_{ab} \ddot{\theta}_b = S_a$$

so we see that M_{ab} is the coefficient of $\ddot{\theta}_b$ in the a-th equation of motion.

Collecting terms in our equations of motion:

$$\sum_{i=a}^{N} m_i \left[\sum_{j=1}^{i} l_j \ddot{\theta}_j \cos\big(\theta_a - \theta_j\big) \right] = F(t)\delta_{1a} - \sum_{i=a}^{N} m_i \left[g \sin\theta_a + \sum_{j=1}^{i} l_j \dot{\theta}_j^2 \sin\big(\theta_a - \theta_j\big) \right] = S_a$$

from which we have

$$S_a = F(t)\delta_{1a} - \sum_{i=a}^{N} m_i \left[g \sin\theta_a + \sum_{j=1}^{i} l_j \dot{\theta}_j^2 \sin\big(\theta_a - \theta_j\big) \right]$$

Let's take a look at the sum

$$\sum_{i=a}^{N} m_i \left[\sum_{j=1}^{i} l_j \ddot{\theta}_j \cos\big(\theta_a - \theta_j\big) \right] = \sum_{i=a}^{N} \sum_{j=1}^{i} X_{ij} \quad \text{where} \quad X_{ij} = m_i l_j \ddot{\theta}_j \cos\big(\theta_a - \theta_j\big)$$

We now want to swap the order of the summation (into j first, i second), but we need to consider which pairs of indices i, j are present in the sum. For $j < a$, i ranges from a to N. For $j \geq a$, i ranges from j to N. This allows us to rewrite the sum as

$$\sum_{i=a}^{N} \sum_{j=1}^{i} X_{ij} = \sum_{j=1}^{N} \sum_{i=\max(a,j)}^{N} X_{ij} = \sum_{j=1}^{N} l_j \ddot{\theta}_j \cos\big(\theta_a - \theta_j\big) \sum_{i=\max(a,j)}^{N} m_i \quad \text{where} \quad \max(a,j) = \begin{cases} a, & \text{if } a > j \\ j, & \text{if } a \leq j \end{cases}$$

From this we see that

$$M_{ab} = l_b \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^{N} m_i$$

And by applying Cramer's rule, we obtain our desired solution:

$$\ddot{\theta}_n = \frac{\det(M_n)}{\det(M)}$$

where M_n is M with the n-th column by $\vec{S}$.

We can simplify the expression further still. Notice that M has a property that $M_{ab} \propto l_b$; in other words, we can write M as

$$M = M^* \begin{pmatrix} l_1 & & & \\ & l_2 & & \\ & & \ddots & \\ & & & l_N \end{pmatrix} \quad \text{where} \quad M^*_{ab} = \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^{N} m_i$$

Then obviously

$$\det(M) = \det(M^*) \prod_{i=1}^{N} l_i$$

You can convince yourself that

$$\det(M_n) = \det(M_n^*) \prod_{i=1, i\neq n}^{N} l_i$$

Then we can rewrite the result in terms of the simpler matrix M^*:

$$\ddot{\theta}_n = \frac{\det(M_n^*)}{l_n \det(M^*)}$$

Now: consider the matrix element M^*_{ba}:

$$M^*_{ba} = \cos(\theta_b - \theta_a) \sum_{i=\max(b,a)}^{N} m_i = \cos(\theta_a - \theta_b) \sum_{i=\max(a,b)}^{N} m_i = M^*_{ab}$$

Hence M^* is symmetric.