Derivation of the closed-form expressions for the small-angle resonant frequencies of the double-pendulum as perturbations of the two principal edge cases

Skippy McDippy

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1 Conversion to normal modes - motivation

By plotting the time evolution of $\frac{\theta_1}{\theta_2}$, we observe that both resonances roughly correspond to the two normal modes: one when $\theta_2 = |n|\theta_1$, other when $\theta_2 = -|n|\theta_1$.

2 The constrained Lagrangian

Suppose the general Lagrangian of the double-pendulum $\mathscr{L}$ allows a stable-state normal mode behaviour, characterised by some constraint equation $f(\vec{q}, \dot{\vec{q}}) = 0$. Then its functional solution $\vec{q}(t)$ must also be a solution to the E-L equation emerging from the constrained Lagrangian $\mathscr{L}^*$, which reduces its coordinates $\vec{q}, \dot{\vec{q}} \to \vec{q}^*, \dot{\vec{q}}^*$ by applying the constraint equation. Hence, to find the behaviour of the two normal modes, we may do this with the constraint function specified for the normal modes, which takes one parameter: n, the *mode coefficient*.

This will hopefully give us a **spectral relation** $\omega_0 = \omega_0(n)$, which relates the resonant frequency ω_0 with the mode coefficient n. Then, we will try to find a second spectral relation by considering an equivalent system with a different Lagrangian. The two spectral relations form a system of two equations of two unknowns, which we will solve for ω_0.

2.1 Formulation of the Lagrangian

For the antiphase normal mode, we may write the constraint equation as

$$\theta_2 = -n\theta_1; n \geq 0, n \in \mathbb{R}$$

We substitute this into the expressions for T and U of the double pendulum:

$$\begin{aligned} T &= \frac{1}{2}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right) \\ &= \frac{1}{2}\left[l_1^2(m_1+m_2) + n^2 l_2^2 m_2 - 2n l_1 l_2 m_2 \cos(\theta_1(1+n))\right]\dot{\theta}_1^2 \end{aligned}$$

$$
\begin{aligned}
U &= -gl_1(m_1+m_2)\cos\theta_1 - gl_2m_2\cos\theta_2 - \theta_1 l_1 F\sin(\omega_F t)\\
&= -gl_1(m_1+m_2)\cos\theta_1 - gl_2m_2\cos(n\theta_1) - \theta_1 l_1 F\sin(\omega_F t)
\end{aligned}
$$

Hence the constrained Lagrangian is

$$
\begin{aligned}
\mathscr{L}^* &= \frac{1}{2}\left[l_1^2(m_1+m_2)+n^2l_2^2m_2-2nl_1l_2m_2\cos(\theta_1(1+n))\right]\dot{\theta}_1^2\\
&\quad +gl_1(m_1+m_2)\cos\theta_1+gl_2m_2\cos(n\theta_1)+\theta_1 l_1 F\sin(\omega_F t)
\end{aligned}
$$

2.2 The equation of motion

Since we only have one coordinate θ_1 (and its velocity counterpart), we will only have one equation of motion:

$$
\frac{\partial\mathscr{L}^*}{\partial\theta_1}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}^*}{\partial\dot{\theta}_1}=0
$$

By direct differentiation, we obtain the two terms:

$$
\begin{aligned}
\frac{\partial\mathscr{L}^*}{\partial\theta_1} &= n(n+1)l_1l_2m_2\dot{\theta}_1^2\sin(\theta_1(1+n))\\
&\quad -gl_1(m_1+m_2)\sin\theta_1-ngl_2m_2\sin(n\theta_1)+Fl_1\sin(\omega_F t)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathscr{L}^*}{\partial\dot{\theta}_1} &= \left[l_1^2(m_1+m_2)+n^2l_2^2m_2-2nl_1l_2m_2\cos(\theta_1(1+n))\right]\ddot{\theta}_1\\
&\quad +2n(n+1)l_1l_2m_2\dot{\theta}_1^2\sin(\theta_1(1+n))
\end{aligned}
$$

From these two expressions, we may directly formulate the equation of motion. Since we're interested in the small-angle case, we shall first apply the following first-order small angle approximations, stemming from $\theta_1=\vartheta\cdot\mathrm{Re}\,p(t)$, where $|p(t)|\approx 1$ is the rotating phasor function and $\vartheta\ll 1$ is the amplitude:

$$
\begin{aligned}
\sin(k\theta_1) &= k\theta_1+O(\theta_1^3)\approx k\theta_1\\
\dot{\theta}_1^2 &= \vartheta^2\left(\mathrm{Re}p(t)\right)^2\approx 0\\
\cos(k\theta_1) &= 1+O(\theta_1^2)\approx 1
\end{aligned}
$$