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MATHS IN FOCUS

MATHEMATICS ADVANCED

YEAR 12

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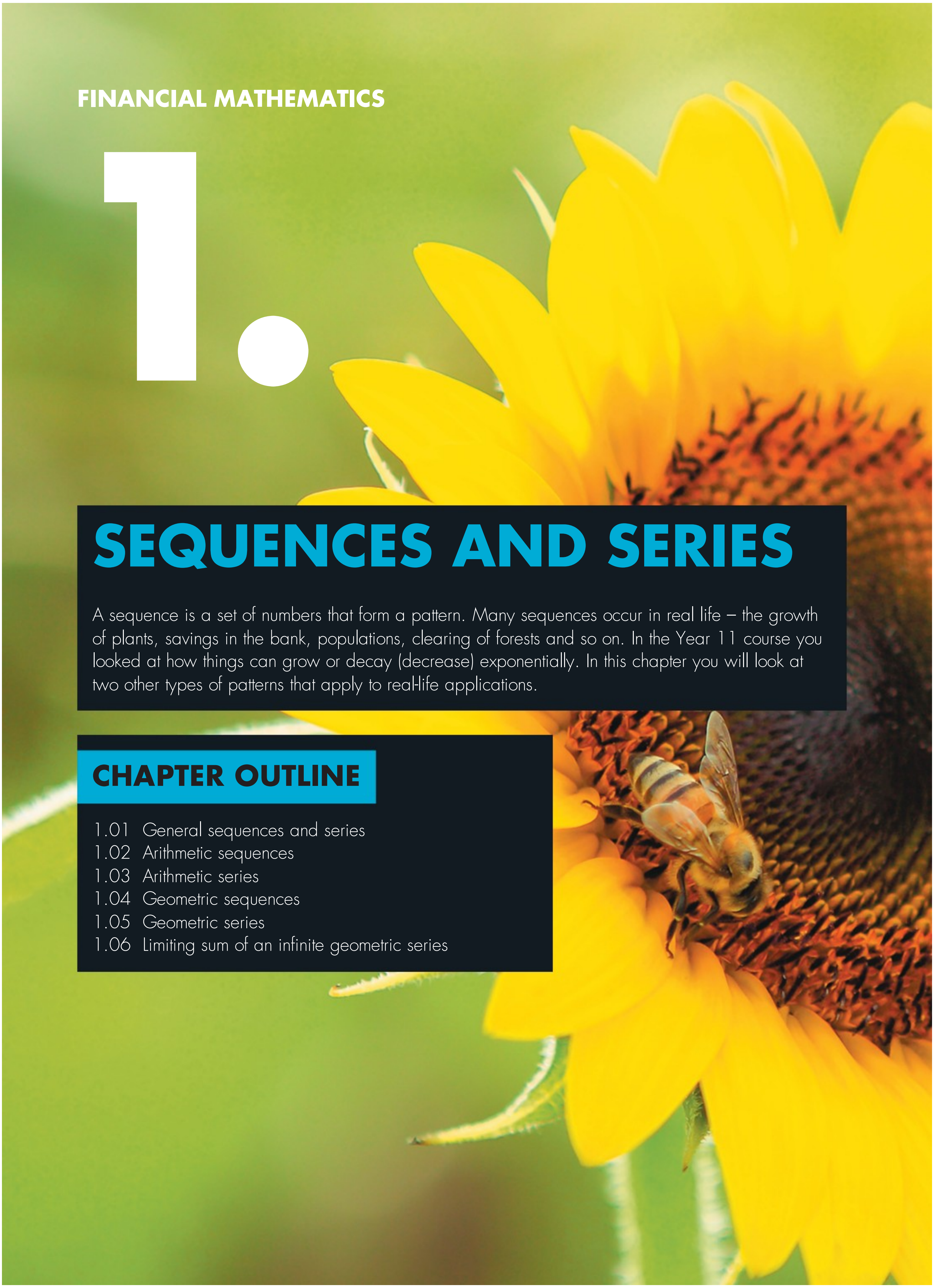
3RD EDITION



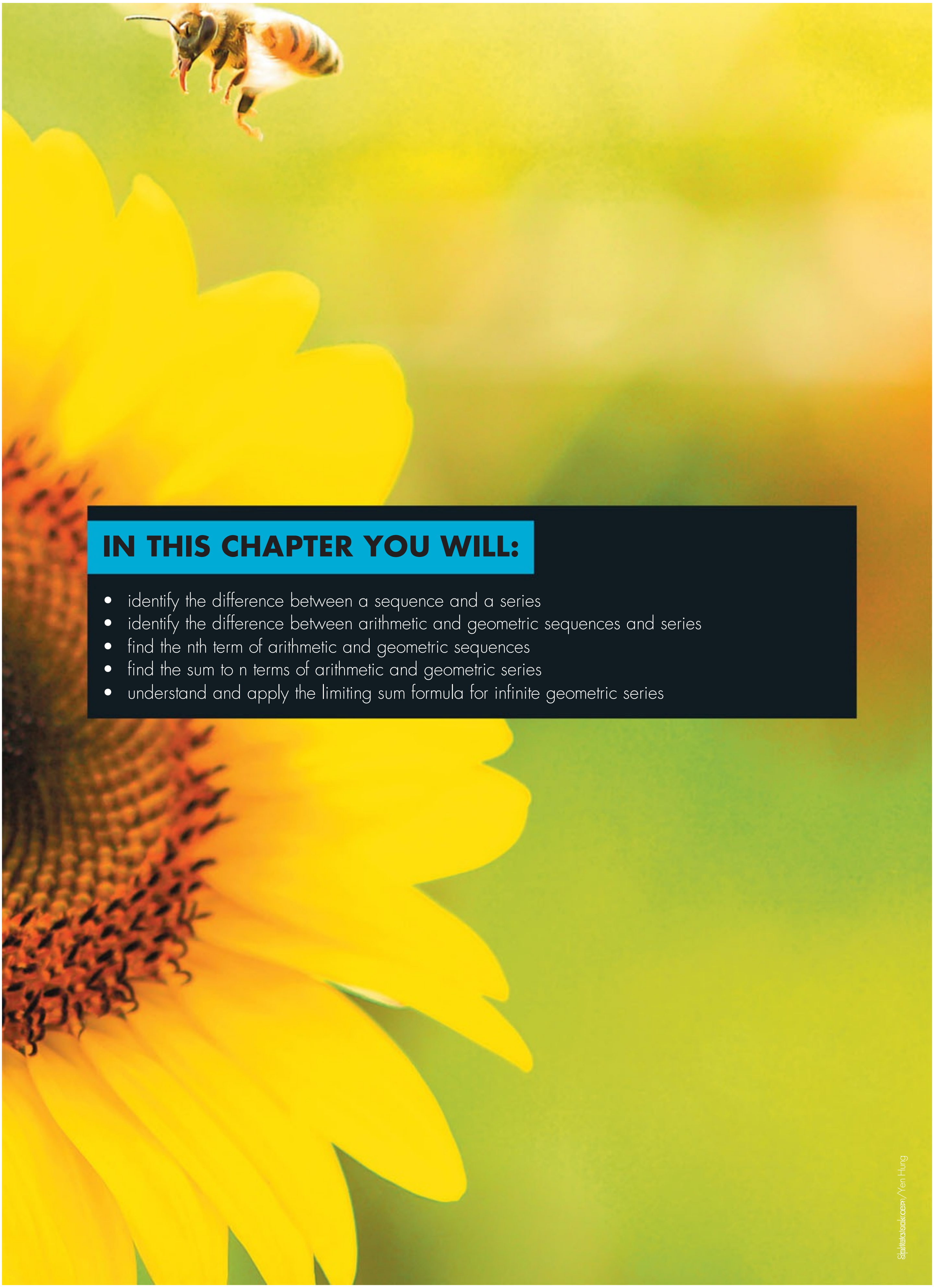
FINANCIAL MATHEMATICS

1.

SEQUENCES AND SERIES

A sequence is a set of numbers that form a pattern. Many sequences occur in real life – the growth of plants, savings in the bank, populations, clearing of forests and so on. In the Year 11 course you looked at how things can grow or decay (decrease) exponentially. In this chapter you will look at two other types of patterns that apply to real-life applications.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- identify the difference between a sequence and a series
- identify the difference between arithmetic and geometric sequences and series
- find the nth term of arithmetic and geometric sequences
- find the sum to n terms of arithmetic and geometric series
- understand and apply the limiting sum formula for infinite geometric series

TERMINOLOGY

arithmetic sequence: A list of numbers where the difference between successive terms is a constant (called the common difference)

arithmetic series: A sum of the terms forming an arithmetic sequence

common difference: The constant difference between successive terms of an arithmetic sequence

common ratio: The constant multiplier of successive terms in a geometric sequence

geometric sequence: A list of numbers where the ratio of successive terms is a constant (called the common ratio)

geometric series: A sum of the terms forming a geometric sequence

limiting sum: The limit, where it exists, of a geometric series as $n \to \infty$

recurrence relation: An equation that defines a term of a sequence or series by referring to its previous term(s)

sequence: A list of numbers where each term of the sequence is related to the previous term by a particular pattern

series: The sum of terms of a sequence of numbers

term: A value of a sequence

WS Sequences and series

WS Classifying sequences

1.01 General sequences and series

Sequences

A sequence is an ordered list of numbers, called terms of the sequence, which follow a pattern. Some patterns are easy to see and some are more difficult to find.

EXAMPLE 1

Find the next 3 terms in the sequence:

a 14, 17, 20, … b 5, 10, 20, 40, … c 5, 1, −3, …

Solution

a For the sequence 14, 17, 20, … we add 3 to each term for the next term.

$14 + 3 = 17$ and $17 + 3 = 20$.

Following this pattern, the next 3 terms are 23, 26 and 29.

b For 5, 10, 20, 40, … we multiply each term by 2 for the next term.

$5 \times 2 = 10$, $10 \times 2 = 20$, $20 \times 2 = 40$.

So the next 3 terms are 80, 160 and 320.

c For the sequence 5, 1, −3, … we subtract 4 (or add −4) to each term for the next term.

$5 - 4 = 1$ and $1 - 4 = -3$.

So the next 3 terms are −7, −11 and −15.

Series

A series is a sum of terms that form a sequence.

EXAMPLE 2

Find the sum of the series with 5 terms:

a $8 + 15 + 22 + \ldots$

b $4 + 8 + 16 + \ldots$

Solution

a We add 7 to each term in the series $8 + 15 + 22 + \ldots$ to find the next term.

So the series with 5 terms is $8 + 15 + 22 + 29 + 36$.

$$\begin{aligned} \text{Sum} &= 8 + 15 + 22 + 29 + 36 \\ &= 110 \end{aligned}$$

b We multiply each term in the series $4 + 8 + 16 + \ldots$ by 2 to find the next term.

So the series with 5 terms is $4 + 8 + 16 + 32 + 64$.

$$\begin{aligned} \text{Sum} &= 4 + 8 + 16 + 32 + 64 \\ &= 124 \end{aligned}$$

DID YOU KNOW?

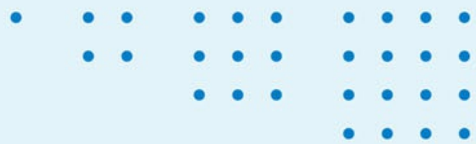
Polygonal numbers

Around 500 BC the Pythagoreans explored different polygonal numbers:

Triangular numbers: $1 + 2 + 3 + 4 + \ldots$

1 $\quad$ 1 + 2 $\quad$ 1 + 2 + 3 $\quad$ 1 + 2 + 3 + 4

Square numbers: $1 + 3 + 5 + 7 + \ldots$



1 $\quad$ 1 + 3 $\quad$ 1 + 3 + 5 $\quad$ 1 + 3 + 5 + 7

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Exercise 1.01 General sequences and series

1 Find the next 3 terms in each sequence.

a $5, 8, 11, \ldots$

b $8, 13, 18, \ldots$

c $11, 22, 33, \ldots$

d $100, 95, 90, \ldots$

e $7, 5, 3, \ldots$

f $12, 3, -6, \ldots$

g $\frac{1}{2}, 1, 1\frac{1}{2}, \ldots$

h $1.3, 1.9, 2.5, \ldots$

i $2, -4, 8, -16, \ldots$

j $\frac{1}{5}, \frac{3}{20}, \frac{9}{80}, \ldots$

2 Find the sum of each series if it has 6 terms.

a $4 + 12 + 36 + \ldots$

b $1 + 2 + 4 + \ldots$

c $3 + 7 + 11 + \ldots$

d $-6 + 12 - 24 + \ldots$

e $1 + 4 + 9 + 16 + \ldots$

f $1 + 8 + 27 + 64 + \ldots$

3 Find the next 3 terms of the sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

4 Find the next 4 terms in the series $3 + 6 + 11 + 18 + 27 + \ldots$

5 What are the next 5 terms in the sequence $1, 1, 2, 3, 5, 8, 13, \ldots$?

6 Complete the next 3 rows in Pascal's triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

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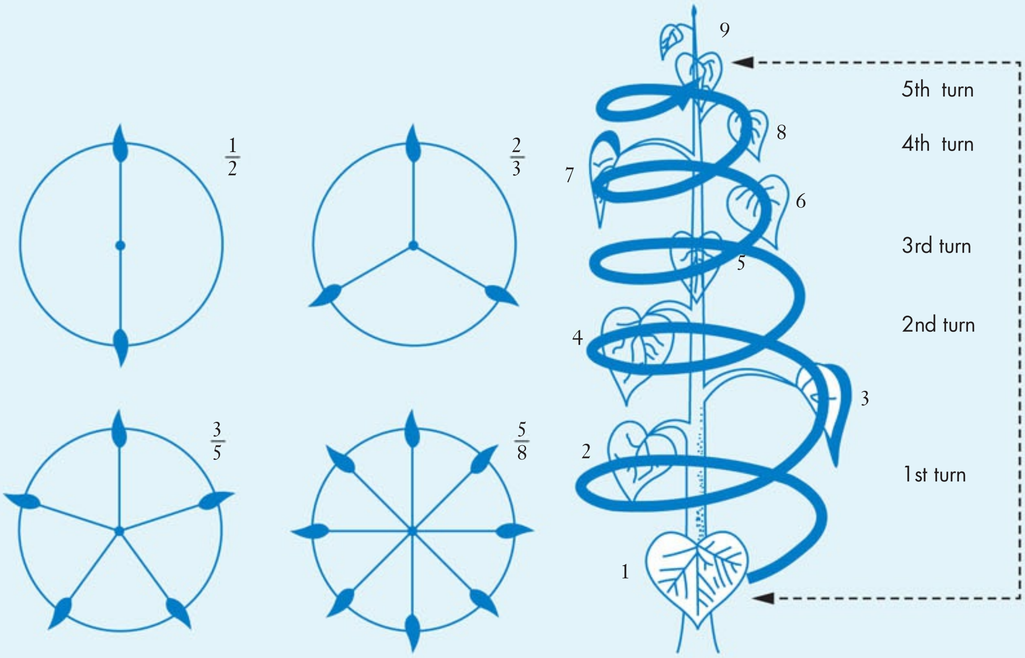
DID YOU KNOW?

Fibonacci numbers

The numbers 1, 1, 2, 3, 5, 8, … are called Fibonacci numbers after Leonardo Fibonacci (1170–1250). These numbers occur in many natural situations.

For example, when new leaves grow on a plant's stem, they spiral around the stem. The ratio of the number of turns to the number of spaces between successive leaves gives the sequence of fractions $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \frac{34}{55}, \ldots$



The Fibonacci ratio is the number of turns divided by the number of spaces.

Research Fibonacci numbers and find out where else they appear in nature.

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WS Arithmetic sequences

WS Arithmetic progressions

1.02 Arithmetic sequences

In an arithmetic sequence, each term is a constant amount more than the previous term. The constant is called the common difference, d.

EXAMPLE 3

Find the common difference of the arithmetic sequence:

a 5, 9, 13, 17, …

b 85, 80, 75, …

Solution

a For this sequence, $9 - 5 = 4$, $13 - 9 = 4$ and $17 - 13 = 4$.

So common difference $d = 4$.

b For this sequence, $80 - 85 = -5$ and $75 - 80 = -5$.

So common difference $d = -5$.

A recurrence relation is an equation that defines a term of a sequence by referring to its previous term. In any arithmetic sequence, a term is d more than the previous term. We can write this as a recurrence relation:

$$T_n = T_{n-1} + d \text{ where } T_n \text{ is the nth term of the sequence}$$

or $T_n - T_{n-1} = d$

EXAMPLE 4

a If 5, x, 31, … is an arithmetic sequence, find x.

b i Evaluate k if $k + 2$, $3k + 2$, $6k - 1$, … is an arithmetic sequence.

ii Write down the first 3 terms of the sequence.

iii Find the common difference d.

Solution

a For an arithmetic sequence,

$T_2 - T_1 = d$ and $T_3 - T_2 = d$

So $T_2 - T_1 = T_3 - T_2$

$$\begin{aligned} x - 5 &= 31 - x \\ 2x - 5 &= 31 \\ 2x &= 36 \\ x &= 18 \end{aligned}$$

Note: x is called the arithmetic mean because $x = \dfrac{5 + 31}{2}$.

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b i For an arithmetic sequence,

$$T_2 - T_1 = T_3 - T_2$$

$$(3k+2)-(k+2)=(6k-1)-(3k+2)$$

$$3k+2-k-2=6k-1-3k-2$$

$$2k=3k-3$$

$$2k+3=3k$$

$$3=k$$

ii Substituting $k = 3$ into the terms of the sequence:

$$T_1 = k+2 = 3+2 = 5$$

$$T_2 = 3k+2 = 3(3)+2 = 11$$

$$T_3 = 6k-1 = 6(3)-1 = 17$$

iii The sequence is 5, 11, 17, …

$$d = 11-5 \text{ or } 17-11 = 6$$

So $d = 6$.

The general term of an arithmetic sequence

Given an arithmetic sequence with 1st term $T_1 = a$ and common difference d:

$$T_1 = a$$

$$T_2 = T_1 + d = a + d$$

$$T_3 = T_2 + d = (a+d)+d = a+2d$$

$$T_4 = T_3 + d = (a+2d)+d = a+3d$$

Notice that the multiple of d is one less than the number of the term. So the multiple of d for the nth term T_n is $n - 1$.

nth term of an arithmetic sequence

$$T_n = a + (n-1)d$$

EXAMPLE 5

a Find the 20th term of the sequence 3, 10, 17, …

b Find a formula for the nth term of the sequence 2, 8, 14, …

c Find the first positive term of the sequence −50, −47, −44, …

Solution

a $a = 3, d = 7, n = 20$

$T_n = a + (n - 1)d$

$T_{20} = 3 + (20 - 1) \times 7$

$= 3 + 19 \times 7$

$= 136$

b $a = 2, d = 6$

$T_n = a + (n - 1)d$

$= 2 + (n - 1) \times 6$

$= 2 + 6n - 6$

$= 6n - 4$

c $a = -50, d = 3$

For the first positive term,

$T_n > 0$

$a + (n - 1)d > 0$

$-50 + (n - 1) \times 3 > 0$

$-50 + 3n - 3 > 0$

$3n - 53 > 0$

$3n > 53$

$n > 17.66...$

So $n = 18$ gives the first positive term.

$T_{18} = -50 + (18 - 1) \times 3$

$= 1$

So the first positive term is 1.

EXAMPLE 6

The 5th term of an arithmetic sequence is 37 and the 8th term is 55. Find the common difference and the first term of the sequence.

Solution

$T_n = a + (n - 1)d$

Given $T_5 = 37$:

$a + (5 - 1)d = 37$

$a + 4d = 37$ [1]

Given $T_8 = 55$:

$a + (8 - 1)d = 55$

$a + 7d = 55$ [2]

Solve [1] and [2] simultaneously:

$3d = 18$ [2] − [1]

$d = 6$

Substitute $d = 6$ into [1]:

$a + 4(6) = 37$

$a + 24 = 37$

$a = 13$

So the common difference is 6 and the first term is 13.

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Exercise 1.02 Arithmetic sequences

1 Find the value of the pronumeral in each arithmetic sequence.

a $5, 9, y, \ldots$ b $8, 2, x \ldots$ c $45, x, 99, \ldots$

d $16, b, 6, \ldots$ e $x, 14, 21, \ldots$ f $32, x-1, 50, \ldots$

g $3, 5k+2, 21, \ldots$ h $x, x+3, 2x+5, \ldots$ i $t-5, 3t, 3t+1, \ldots$

j $2t-3, 3t+1, 5t+2, \ldots$

2 Find the 15th term of each sequence.

a $4, 7, 10, \ldots$ b $8, 13, 18, \ldots$ c $10, 16, 22, \ldots$

d $120, 111, 102, \ldots$ e $-3, 2, 7, \ldots$

3 Find the 100th term of each sequence.

a $-4, 2, 8, \ldots$ b $41, 32, 23, \ldots$ c $18, 22, 26, \ldots$

d $125, 140, 155, \ldots$ e $-1, -5, -9, \ldots$

4 What is the 25th term of each sequence?

a $-14, -18, -22, \ldots$ b $0.4, 0.9, 1.4, \ldots$ c $1.3, 0.9, 0.5, \ldots$

d $1, 2\frac{1}{2}, 4, \ldots$ e $1\frac{2}{5}, 2, 2\frac{3}{5}, \ldots$

5 Find the formula for the nth term of the sequence $3, 5, 7, \ldots$

6 Find the formula for the nth term of each sequence.

a $9, 17, 25, \ldots$ b $100, 102, 104, \ldots$ c $6, 9, 12, \ldots$

d $80, 86, 92, \ldots$ e $-21, -17, -13, \ldots$ f $15, 10, 5, \ldots$

g $\frac{7}{8}, 1, 1\frac{1}{8}, \ldots$ h $-30, -32, -34, \ldots$ i $3.2, 4.4, 5.6, \ldots$

j $\frac{1}{2}, 1\frac{1}{4}, 2, \ldots$

7 Find which term of $3, 7, 11, \ldots$ is equal to 111.

8 Which term of the sequence $1, 5, 9, \ldots$ is 213?

9 Which term of the sequence $15, 24, 33, \ldots$ is 276?

10 Which term of the sequence $25, 18, 11, \ldots$ is equal to -73?

11 Is 0 a term of the sequence $48, 45, 42, \ldots$?

12 Is 270 a term of the sequence $3, 11, 19, \ldots$?

13 Is 405 a term of the sequence $0, 3, 6, \ldots$?

14 Find the first value of n for which the terms of the sequence $100, 93, 86, \ldots$ is less than 20.

15 Find the values of n for which the terms of the sequence $-86, -83, -80, \ldots$ are positive.

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16 Find the first negative term of the sequence 54, 50, 46, ...

17 Find the first term that is greater than 100 in the sequence 3, 7, 11, ...

18 The first term of an arithmetic sequence is −7 and the common difference is 8. Find the 100th term.

19 The first term of an arithmetic sequence is 15 and the 3rd term is 31.

a Find the common difference.

b Find the 10th term of the sequence.

20 The first term of an arithmetic sequence is 3 and the 5th term is 39. Find the common difference.

21 The 2nd term of an arithmetic sequence is 19 and the 7th term is 54. Find the first term and common difference.

22 Find the 20th term in an arithmetic sequence with 4th term 29 and 10th term 83.

23 The common difference of an arithmetic sequence is 6 and the 5th term is 29. Find the first term of the sequence.

24 If the 3rd term of an arithmetic sequence is 45 and the 9th term is 75, find the 50th term of the sequence.

25 The 7th term of an arithmetic sequence is 17 and the 10th term is 53. Find the 100th term of the sequence.

26 a Show that $\log_5 x, \log_5 x^2, \log_5 x^3, \ldots$ is an arithmetic sequence.

b Find the 80th term.

c If $x = 4$, evaluate the 10th term correct to 1 decimal place.

27 a Show that $\sqrt{3}, \sqrt{12}, \sqrt{27}, \ldots$ is an arithmetic sequence.

b Find the 50th term in simplest form.

28 Find the 25th term of $\log_2 4, \log_2 8, \log_2 16, \ldots$

29 Find the 40th term of $5b, 8b, 11b, \ldots$

30 Which term is $213y$ of the sequence $28y, 33y, 38y, \ldots$?

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1.03 Arithmetic series

The sum of an arithmetic series with n terms is given by the formula:

Sum of an arithmetic series with n terms

$S_n = \frac{n}{2}(a + l)$ where a = 1st term and l = last (nth) term

Arithmetic series

Arithmetic series

Arithmetic series

Arithmetic series assignment

Arithmetic series extension

Proof

Let the last or nth term be l.

$S_n = a + (a + d) + (a + 2d) + \ldots + l$ [1]

Writing this around the other way:

$S_n = l + (l - d) + (l - 2d) + \ldots + a$ [2]

[1] + [2]

$$2S_n = (a + l) + (a + l) + (a + l) + \ldots + (a + l) \text{ n times}$$
$$= n(a + l)$$
$$S_n = \frac{n}{2}(a + l)$$

We can find a more general formula if we substitute $T_n = a + (n - 1)d$ for l:

Sum of an arithmetic series with n terms

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Proof

$$S_n = \frac{n}{2}(a + l)$$
$$= \frac{n}{2}[a + a + (n - 1)d]$$
$$= \frac{n}{2}[2a + (n - 1)d]$$

We can also use these formulas to find the sum of the first n terms of an arithmetic sequence (also called the nth partial sum).

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EXAMPLE 7

a Evaluate $9 + 14 + 19 + \ldots + 224$.

b For what value of n is the sum of the series $2 + 11 + 20 + \ldots$ equal to 618?

c The 6th term of an arithmetic sequence is 23 and the sum of the first 10 terms is 210. Find the sum of the first 20 terms of the sequence.

Solution

a
$$a = 9,\ d = 5,\ T_n = 224$$
$$\begin{aligned}
T_n &= a + (n-1)d \\
224 &= 9 + (n-1) \times 5 \\
&= 9 + 5n - 5 \\
&= 5n + 4 \\
220 &= 5n \\
44 &= n
\end{aligned}$$
$$\begin{aligned}
S_n &= \frac{n}{2}(a + 1) \\
&= \frac{44}{2}(9 + 224) \\
&= 5126
\end{aligned}$$

b
$$a = 2,\ d = 9,\ S_n = 618$$
$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
618 &= \frac{n}{2}[2 \times 2 + (n-1) \times 9] \\
1236 &= n(4 + 9n - 9) \\
&= n(9n - 5) \\
&= 9n^2 - 5n
\end{aligned}$$
$$\begin{aligned}
0 &= 9n^2 - 5n - 1236 \\
&= (n - 12)(9n + 103)
\end{aligned}$$
(or use the quadratic formula)

$n - 12 = 0,\ 9n + 103 = 0$

$n = 12$

($9n + 103 = 0$ gives a negative value of n)

c
$$\begin{aligned}
T_n &= a + (n-1)d \\
T^6 &= a + (6-1)d = 23
\end{aligned}$$
$$a + 5d = 23 \qquad [1]$$
$$\begin{aligned}
S_n &= \frac{n}{2}[2a + (n-1)d] \\
S_{10} &= \frac{10}{2}[2a + (10-1)d] = 210
\end{aligned}$$
$$5(2a + 9d) = 210$$
$$2a + 9d = 42 \qquad [2]$$

[1] × 2: $\quad 2a + 10d = 46 \qquad$ [3]

[3] − [2]: $\quad d = 4$

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Substitute $d = 4$ in [1]

$a + 5(4) = 23$

$a + 20 = 23$

$a = 3$

Substitute $a = 3$, $d = 2$, $n = 20$ into the formula for S_n.

$$S_{20} = \frac{20}{2}[2(3) + (20 - 1)4]$$

$$= 10(6 + 19 \times 4)$$

$$= 820$$

Exercise 1.03 Arithmetic series

1 Find the sum of 15 terms of each series.

a $4 + 7 + 10 + \ldots$ b $2 + 7 + 12 + \ldots$ c $60 + 56 + 52 + \ldots$

2 Find the sum of 30 terms of each series.

a $1 + 7 + 13 + \ldots$ b $15 + 24 + 33 + \ldots$ c $95 + 89 + 83 + \ldots$

3 Find the sum of 25 terms of each series.

a $-2 + 5 + 12 + \ldots$ b $5 - 4 - 13 - \ldots$

4 Find the sum of 50 terms of each series.

a $50 + 44 + 38 + \ldots$ b $11 + 14 + 17 + \ldots$

5 Evaluate each arithmetic series.

a $15 + 20 + 25 + \ldots + 535$ b $9 + 17 + 25 + \ldots + 225$

c $5 + 2 - 1 - \ldots - 91$ d $81 + 92 + 103 + \ldots + 378$

e $229 + 225 + 221 + \ldots + 25$ f $-2 + 6 + 14 + \ldots + 94$

g $0 - 9 - 18 - \ldots - 216$ h $79 + 81 + 83 + \ldots + 229$

i $14 + 11 + 8 + \ldots - 43$ j $1\frac{1}{2} + 1\frac{3}{4} + 2 + \ldots + 25\frac{1}{4}$

6 How many terms of the series $45 + 47 + 49 + \ldots$ are needed to give a sum of 1365?

7 For what value of n is the sum of the arithmetic series $5 + 9 + 13 + \ldots$ equal to 152?

8 How many terms of the series $80 + 73 + 66 + \ldots$ are needed to give a sum of 495?

9 How many terms of the series $20 + 18 + 16 + \ldots$ are needed to give a sum of 104?

10 The sum of the first 5 terms of an arithmetic sequence is 110 and the sum of the first 10 terms is 320. Find the first term and the common difference.

11 The sum of the first 5 terms of an arithmetic sequence is 35 and the sum of the next 5 terms is 160. Find the first term and the common difference.

12 Find S_{25}, given an arithmetic series with 8th term 16 and 13th term 81.

13 The sum of 12 terms of an arithmetic series is 186 and the 20th term is 83. Find the sum of 40 terms of the series.

14 The sum of the first 4 terms of an arithmetic series is 42 and the sum of the 3rd and 7th term is 46. Find the sum of the first 20 terms.

15 a Show that $x + 1$, $2x + 4$, $3x + 7$, … are the first 3 terms in an arithmetic sequence.

b Find the sum of the first 50 terms of the sequence.

16 The 20th term of an arithmetic series is 131 and the sum of the 6th to 10th terms inclusive is 235. Find the sum of the first 20 terms.

17 The sum of 50 terms of an arithmetic series is 249 and the sum of 49 terms of the series is 233. Find the 50th term of the series.

18 Prove that $T_n = S_n - S_{n-1}$ for any arithmetic sequence.

19 a Find the sum of all integers from 1 to 100 that are multiples of 6.

b Find the sum of all integers from 1 to 100 that are not multiples of 6.

1.04 Geometric sequences

WS Geometric sequences

WS Geometric progressions 1

In a geometric sequence, each term is formed by multiplying the previous term by a constant. The constant is called the common ratio r.

EXAMPLE 8

Find the common ratio of the geometric sequence:

a $3, 6, 12, \ldots$

b $-2, 10, -50, \ldots$

c $\frac{1}{2}, \frac{1}{5}, \frac{2}{25}, \ldots$

Solution

a For this sequence, $6 \div 3 = 2$, $12 \div 6 = 2$

So common ratio $r = 2$.

b For this sequence, $10 \div -2 = -5$, $-50 \div 10 = -5$

So common ratio $r = -5$.

c $\frac{1}{5} \div \frac{1}{2} = \frac{2}{5}$, $\frac{2}{25} \div \frac{1}{5} = \frac{2}{5}$

So common ratio $r = \frac{2}{5}$.

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In any geometric sequence, a term is r times more than the previous term. We can write this as a recurrence relation:

$$T_n = rT_{n-1}$$

or $$\frac{T_n}{T_{n-1}} = r$$

EXAMPLE 9

a i Find x if 5, x, 45, … is a geometric sequence.

ii Find the sequence.

b Is $\frac{1}{4}, \frac{1}{6}, \frac{1}{18}, \ldots$ a geometric sequence?

Solution

a i For a geometric sequence:

$$\frac{T_2}{T_1} = r \text{ and } \frac{T_3}{T_2} = r$$

$$\text{So } \frac{T_2}{T_1} = \frac{T_3}{T_2}$$

$$\frac{x}{5} = \frac{45}{x}$$

$$x^2 = 225$$

$$x = \pm\sqrt{225}$$

$$= \pm 15$$

Note: x is called the geometric mean because $x = \sqrt{5 \times 45}$.

ii If $x = 15$ the sequence is 5, 15, 45, … $(r = 3)$

If $x = -15$ the sequence is $5, -15, 45, \ldots$ $(r = -3)$

b $$\frac{T_2}{T_1} = \frac{1}{6} \div \frac{1}{4} = \frac{2}{3} \qquad \frac{T_3}{T_2} = \frac{1}{18} \div \frac{1}{6} = \frac{1}{3}$$

$\frac{T_2}{T_1} \neq \frac{T_3}{T_2}$ so the series is not geometric.

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General term of a geometric sequence

Given a geometric sequence with 1st term $T_1 = a$ and common ratio r:

$$T_1 = a \qquad T_3 = T_2 \times r \qquad T_4 = T_3 \times r$$

$$T_2 = T_1 \times r \qquad = (ar) \times r \qquad = (ar^2) \times r$$

$$= ar \qquad = ar^2 \qquad = ar^3$$

Notice that the power of r is one less than the number of the term. So the power of r for T_n is $n - 1$.

nth term of a geometric sequence

$$T_n = ar^{n-1}$$

EXAMPLE 10

a i Find the 10th term of the sequence 3, 6, 12, …

ii Find the formula for the nth term of the sequence.

b Find the 10th term of the sequence – 5, 10, – 20, …

c Which term of the sequence 4, 12, 36, … is equal to 78 732?

d The 3rd term of a geometric sequence is 18 and the 7th term is 1458. Find the first term and the common ratio.

Geometric sequences

Solution

a i This is a geometric sequence with $a = 3$, $r = 2$ and $n = 10$.

$$\begin{aligned} T_n &= ar^{n-1} \\ T_{10} &= 3(2)^{10-1} \\ &= 3 \times 2^9 \\ &= 1536 \end{aligned}$$

ii
$$\begin{aligned} T_n &= ar^{n-1} \\ &= 3(2)^{n-1} \end{aligned}$$

b $a = -5$, $r = -2$, $n = 10$

$$\begin{aligned} T_n &= ar^{n-1} \\ T_{10} &= -5(-2)^{10-1} \\ &= -5 \times (-2)^9 \\ &= 2560 \end{aligned}$$

c This is a geometric sequence with $a = 4$, $r = 3$ and $T_n = 78\,732$.

$$\begin{aligned} T_n &= ar^{n-1} \\ 78\,732 &= 4(3)^{n-1} \\ 19\,683 &= 3^{n-1} \\ \log 19\,683 &= \log 3^{n-1} \\ &= (n-1)\log 3 \end{aligned}$$

$$\begin{aligned} \frac{\log 19\,683}{\log 3} &= n - 1 \\ \frac{\log 19\,683}{\log 3} + 1 &= n \\ 10 &= n \end{aligned}$$

So the 10th term is 78 732.

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d Given $T_3 = 18$

$ar^{3-1} = 18$

$ar^2 = 18$ [1]

Given $T_7 = 1458$

$ar^{7-1} = 1458$

$ar^6 = 1458$ [2]

[2] ÷ [1]:

$$\frac{ar^6}{ar^2} = \frac{1458}{18}$$

$$r^4 = 81$$

$$r = \pm\sqrt[4]{81}$$

$$= \pm 3$$

Substitute $r = 3$ into [1]

$a(3)^2 = 18$

$9a = 18$

$a = 2$

Substitute $r = -3$ into [1]

$a(-3)^2 = 18$

$9a = 18$

$a = 2$

The first term is 2 and the common ratio is ±3.

Here is an example of a geometric sequence involving fractions.

EXAMPLE 11

a Find the 8th term of $\frac{2}{3}, \frac{4}{15}, \frac{8}{75}, \ldots$ in index form.

b Find the first value of n for which the terms of the sequence $\frac{1}{5}, 1, 5, \ldots$ exceed 3000.

Solution

a
$$\frac{T_2}{T_1} = \frac{4}{15} \div \frac{2}{3}$$

$$= \frac{2}{5}$$

$$\frac{T_3}{T_2} = \frac{8}{75} \div \frac{4}{15}$$

$$= \frac{2}{5}$$

So $r = \frac{2}{5}$

$$T_n = ar^{n-1}$$

$$= \frac{2}{3}\left(\frac{2}{5}\right)^{8-1}$$

$$= \frac{2}{3}\left(\frac{2}{5}\right)^{7}$$

$$= \frac{2^8}{3(5^7)}$$

b $a = \frac{1}{5}, r = 5$

$$T_n > 3000$$

$$ar^{n-1} > 3000$$

$$\frac{1}{5}(5)^{n-1} > 3000$$

$$5^{n-1} > 15\ 000$$

$$\log 5^{n-1} > \log 15\ 000$$

$$(n-1)\log 5 > \log 15\ 000$$

$$n - 1 > \frac{\log 15000}{\log 5}$$

$$n > \frac{\log 15000}{\log 5} + 1$$

$$> 6.974\ldots$$

So $n = 7$

The 7th term is the first term to exceed 3000.

Exercise 1.04 Geometric sequences

1 Is each sequence geometric? If so, find the common ratio.

a $5, 20, 60, \ldots$

b $-4, 3, -2\frac{1}{4}, \ldots$

c $\frac{3}{4}, \frac{3}{14}, \frac{3}{49}, \ldots$

d $7, 5\frac{5}{6}, 3\frac{1}{3}, \ldots$

e $-14, 42, -168, \ldots$

f $1\frac{1}{3}, \frac{8}{9}, \frac{8}{27}, \ldots$

g $5.7, 1.71, 0.513, \ldots$

h $2\frac{1}{4}, -1\frac{7}{20}, \frac{81}{100}, \ldots$

i $63, 9, 1\frac{7}{8}, \ldots$

j $-1\frac{7}{8}, 15, -120, \ldots$

2 Find the pronumeral in each geometric sequence.

a $4, 28, x, \ldots$

b $-3, 12, y, \ldots$

c $2, a, 72, \ldots$

d $y, 2, 6, \ldots$

e $x, 8, 32, \ldots$

f $5, p, 20, \ldots$

g $7, y, 63, \ldots$

h $-3, m, -12, \ldots$

i $3, x-4, 15, \ldots$

j $3, k-1, 21, \ldots$

k $\frac{1}{4}, t, \frac{1}{9}, \ldots$

l $\frac{1}{3}, t, \frac{4}{3}, \ldots$

3 Find the formula for the nth term of each sequence.

a $1, 5, 25, \ldots$

b $1, 1.02, 1.0404, \ldots$

c $1, 9, 81, \ldots$

d $2, 10, 50, \ldots$

e $6, 18, 54, \ldots$

f $8, 16, 32, \ldots$

g $\frac{1}{4}, 1, 4, \ldots$

h $1000, -100, 10, \ldots$

i $-3, 9, -27, \ldots$

j $\frac{1}{3}, \frac{2}{15}, \frac{4}{75}, \ldots$

4 Find the 6th term of each sequence.

a $8, 24, 72, \ldots$

b $9, 36, 144, \ldots$

c $8, -32, 128, \ldots$

d $-1, 5, -25, \ldots$

e $\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \ldots$

5 What is the 9th term of each sequence?

a $1, 2, 4, \ldots$

b $4, 12, 36, \ldots$

c $1, 1.04, 1.0816, \ldots$

d $-3, 6, -12, \ldots$

e $\frac{3}{4}, -\frac{3}{8}, \frac{3}{16}, \ldots$

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6 Find the 8th term of each sequence.

a $3, 15, 75, \ldots$

b $2.1, 4.2, 8.4, \ldots$

c $5, -20, 80, \ldots$

d $-\frac{1}{2}, \frac{3}{10}, -\frac{9}{50}, \ldots$

e $1\frac{47}{81}, 2\frac{10}{27}, 3\frac{5}{9}, \ldots$

7 Find the 20th term of each sequence, leaving the answer in index form.

a $3, 6, 12, \ldots$

b $1, 7, 49, \ldots$

c $1.04, 1.04^2, 1.04^3, \ldots$

d $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

e $\frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \ldots$

8 Find the 50th term of 1, 11, 121, … in index form.

9 Which term of the sequence 4, 20, 100, … is equal to 12 500?

10 Which term of 6, 36, 216, … is equal to 7776?

11 Is 1200 a term of the sequence 2, 16, 128, …?

12 Which term of 3, 21, 147, … is equal to 352 947?

13 Which term of the sequence $8, -4, 2, \ldots$ is $\frac{1}{128}$?

14 Which term of 54, 18, 6, … is $\frac{2}{243}$?

15 Find the value of n if the nth term of the sequence $-2, 1\frac{1}{2}, -1\frac{1}{8}, \ldots$ is $-\frac{81}{128}$.

16 The first term of a geometric sequence is 7 and the 6th term is 1701.
Find the common ratio.

17 The 4th term of a geometric sequence is –648 and the 5th term is 3888.

a Find the common ratio.

b Find the 2nd term.

18 The 3rd term of a geometric sequence is $\frac{2}{5}$ and the 5th term is $1\frac{3}{5}$.
Find the first term and common ratio.

19 Find the value of n for the first term of the sequence 5000, 1000, 200, … that is less than 1.

20 Find the first term of the sequence $\frac{2}{7}, \frac{6}{7}, 2\frac{4}{7}, \ldots$ that is greater than 100.

1.05 Geometric series

WS Geometric series

WS Geometric series

WS Geometric series assignment

WS Geometric series problems

The sum of a geometric series with n terms is given by the formulas:

Sum of a geometric series with n terms

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

This formula can also be written as:

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

to be used if r is a fraction, that is, $-1 < r < 1$, also written as $|r| < 1$.

Proof

The sum of a geometric series can be written

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \qquad [1]$$

Multiplying both sides by r:

$$\begin{aligned} rS_n &= r(a + ar + ar^2 + \ldots + ar^{n-1}) \\ &= ar + ar^2 + ar^3 + \ldots + ar^n \qquad [2] \end{aligned}$$

[2] – [1]:

$$rS_n - S_n = ar^n - a$$

$$S_n(r - 1) = a(r^n - 1)$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

[1] – [2] gives the formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

We can also use these formulas to find the sum of the first n terms of a geometric sequence (also called the nth partial sum).

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EXAMPLE 12

a Find the sum of the first 10 terms of the series $3 + 12 + 48 + \ldots$

b Evaluate $60 + 20 + 6\frac{2}{3} + \ldots + \frac{20}{81}$.

c The sum of n terms of $1 + 4 + 16 + \ldots$ is 21 845. Find the value of n.

Solution

a This is a geometric series with $a = 3, r = 4, n = 10$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{3(4^{10} - 1)}{4 - 1}$$

$$= \frac{3(4^{10} - 1)}{3}$$

$$= 4^{10} - 1$$

$$= 1\ 048\ 575$$

b $a = 60,\ r = \frac{1}{3},\ T_n = \frac{20}{81}$

$$T_n = ar^{n-1}$$

$$ar^{n-1} = \frac{20}{81}$$

$$60\left(\frac{1}{3}\right)^{n-1} = \frac{20}{81}$$

$$\left(\frac{1}{3}\right)^{n-1} = \frac{1}{243}$$

$$\frac{1}{3^{n-1}} = \frac{1}{243}$$

So $3^{n-1} = 243$

$$= 3^5$$

$$n - 1 = 5$$

$$n = 6$$

Since $|r| < 1$, we use the second formula.

$$S_n = \frac{a(1 - r^n)}{r - 1}$$

$$S_6 = \frac{60\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}}$$

$$= \frac{60\left(1 - \frac{1}{729}\right)}{\frac{2}{3}}$$

$$= 60 \times \frac{728}{729} \times \frac{3}{2}$$

$$= 89\frac{71}{81}$$

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c $a = 1, r = 4, S_n = 21\ 845$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$21\ 845 = \frac{1(4^n - 1)}{4 - 1}$$

$$= \frac{4^n - 1}{3}$$

$$65\ 535 = 4^n - 1$$

$$65\ 536 = 4^n$$

$$\log 65\ 536 = \log 4^n$$

$$= n \log 4$$

$$\frac{\log 65\ 536}{\log 4} = n$$

$$8 = n$$

So 8 terms give a sum of 21 845.

Exercise 1.05 Geometric series

1 Find the sum of 10 terms of each geometric series.

a $6 + 24 + 96 + \ldots$

b $3 + 15 + 75 + \ldots$

2 Find the sum of 8 terms of each series.

a $-1 + 7 - 49 + \ldots$

b $8 + 24 + 72 + \ldots$

3 Find the sum of 15 terms of each series.

a $4 + 8 + 16 + \ldots$

b $\frac{3}{4} - \frac{3}{8} + \frac{3}{16} - \ldots$ (to 1 decimal place)

4 Evaluate:

a $2 + 10 + 50 + \ldots + 6250$

b $18 + 9 + 4\frac{1}{2} + \ldots + \frac{9}{64}$

c $3 + 21 + 147 + \ldots + 7203$

d $\frac{3}{4} + 2\frac{1}{4} + 6\frac{3}{4} + \ldots + 182\frac{1}{4}$

e $-3 + 6 - 12 + \ldots + 384$

5 For the series $7 + 14 + 28 + \ldots$ find:

a the 9th term

b the sum of the first 9 terms.

6 Find the sum of 30 terms of the series $1.09 + 1.09^2 + 1.09^3 + \ldots$ correct to 2 decimal places.

7 Find the sum of 25 terms of the series $1 + 1.12 + 1.12^2 + \ldots$ correct to 2 decimal places.

8 Find the value of n if the sum of n terms of the series $11 + 33 + 99 + \ldots$ is equal to 108 251.

9 How many terms of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ give a sum of $\frac{1023}{1024}$?

10 The common ratio of a geometric series is 4 and the sum of the first 5 terms is 3069. Find the first term.

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11 Find the number of terms needed to be added for the sum to exceed 1 000 000 in the series $4 + 16 + 64 + \ldots$

12 a Find the sum of 10 terms of the series $2 + 4 + 8 + \ldots$

b Find the sum of 10 terms of the series $1 + 3 + 5 + \ldots$

c Find the sum of the first 10 terms of the series $3 + 7 + 13 + \ldots$

PUZZLES

1 A poor girl saved a rich king from drowning one day. The king offered the girl a reward of a sum of money in 30 daily payments. He gave her a choice of payments:

Choice 1: \$1 the first day, \$2 the second day, \$3 the third day and so on.

Choice 2: 1 cent the first day, 2 cents the second day, 4 cents the third day and so on, the payment doubling each day.

How much money would the girl receive for each choice? Which plan would give the girl more money?

2 Can you solve Fibonacci's problem?

A man entered an orchard through 7 guarded gates and gathered a certain number of apples. As he left the orchard he gave the guard at the first gate half the apples he had and 1 apple more. He repeated this process for each of the remaining 6 guards and eventually left the orchard with 1 apple. How many apples did he gather? (He did not give away any half-apples.)

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Limiting sum of an infinite geometric series

Geometric series crossword

1.06 Limiting sum of an infinite geometric series

In some geometric sequences the sum becomes very large as n increases, for example, the series $2 + 4 + 8 + 16 + 32 + \ldots$ We say these series diverge (their sum is infinite).

In other geometric sequences, however, such as $8 + 4 + 2 + 1 + \ldots$, the sum does not increase greatly after a few terms, but approaches some constant value. We say these series converge (they have a limiting sum that is a specific value, sometimes called the sum to infinity).

EXAMPLE 13

a Find the sum of 15 terms of $2 + 6 + 18 + \ldots$

b By evaluating the sum of 10 terms and 20 terms correct to 4 decimal places for the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, estimate its limiting sum.

Solution

a $a = 2, r = 3, n = 15$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{15} = \frac{2(3^{15} - 1)}{3 - 1}$$

$$= \frac{2(3^{15} - 1)}{2}$$

$$= 3^{15} - 1$$

$$= 14\,348\,906$$

b $a = 2, r = \frac{1}{2}$: $S_n = \frac{a(1 - r^n)}{1 - r}$

Sum to 10 terms:

$$S_{10} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{10}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{10}}\right]}{\frac{1}{2}}$$

$$= 3.9961$$

Sum to 20 terms:

$$S_{20} = \frac{2\left[1 - \left(\frac{1}{2}\right)^{20}\right]}{1 - \frac{1}{2}}$$

$$= \frac{2\left[1 - \frac{1}{2^{20}}\right]}{\frac{1}{2}}$$

$$= 4.0000$$

The limiting sum is 4.

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Can you see why the series $2 + 6 + 18 + \ldots$ does not have a limiting sum and the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ has a limiting sum?

It's because of the common ratio. Only geometric series with common ratios that are fractions, $|r| < 1$, will have a limiting sum.

For $S_n = \frac{a(1-r^n)}{1-r}$:

As $n \to \infty$, $r^n \to 0$ when $-1 < r < 1$

We write $\lim\limits_{n \to \infty} r^n = 0$

Limiting sum of a geometric series

$$S = \frac{a}{1-r} \text{ when } |r| < 1.$$

Proof

$S_n = \frac{a(1-r^n)}{1-r}$

For $|r| < 1$, $\lim\limits_{n \to \infty} r^n = 0$

$S_\infty = \frac{a(1-0)}{1-r}$

$= \frac{a}{1-r}$

EXAMPLE 14

a Find the limiting sum of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$

b Find the sum to infinity of the series $6 + 2 + \frac{2}{3} + \ldots$

c Does the series $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} + \ldots$ have a limiting sum?

Solution

a $a = 2$, $r = \frac{1}{2}$

Since $|r| < 1$, the series has a limiting sum.

$S = \frac{a}{1-r}$

$= \frac{2}{1-\frac{1}{2}}$

$= \frac{2}{\frac{1}{2}}$

$= 2 \times \frac{2}{1}$

$= 4$

So the limiting sum is 4.

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b $a = 6$

$2 \div 6 = \frac{1}{3}$ and $\frac{2}{3} \div 2 = \frac{1}{3}$ so $r = \frac{1}{3}$

Since $|r| < 1$, the series has a limiting sum.

$$\begin{aligned} S &= \frac{a}{1-r} \\ &= \frac{6}{1-\frac{1}{3}} \\ &= \frac{6}{\frac{2}{3}} \\ &= 6 \times \frac{3}{2} \\ &= 9 \end{aligned}$$

So the limiting sum is 9.

c For $\frac{3}{4} + \frac{15}{16} + 1\frac{11}{64} + \ldots$

$r = \frac{15}{16} \div \frac{3}{4}$ or $1\frac{11}{64} \div \frac{15}{16}$

$= 1\frac{1}{4}$

Since $|r| > 1$, this series does not have a limiting sum.

DID YOU KNOW?

A series involving π and e

Here is an interesting series involving π:

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots$$

Gottfried Wilhelm Leibniz (1646–1716) discovered this result. It is interesting that while π is an irrational number, it can be written as the sum of rational numbers.

Here is another interesting series involving e:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Remember: $2! = 2 \times 1$, $3! = 3 \times 2 \times 1$, $4! = 4 \times 3 \times 2 \times 1$ and so on.

Research these and other series.

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Exercise 1.06 Limiting sum of an infinite geometric series

1 Which series has a limiting sum? Find the limiting sum where it exists.

a $9 + 3 + 1 + \ldots$ b $\frac{1}{4} + \frac{1}{2} + 1 + \ldots$ c $16 - 4 + 1 - \ldots$

d $\frac{2}{3} + \frac{7}{9} + \frac{49}{54} + \ldots$ e $1 + \frac{2}{3} + \frac{4}{9} + \ldots$ f $\frac{5}{8} + \frac{1}{8} + \frac{1}{40} + \ldots$

g $-6 + 36 - 216 + \ldots$ h $-2\frac{1}{4} + 1\frac{7}{8} - 1\frac{27}{48} + \ldots$ i $\frac{1}{9} + \frac{1}{6} + \frac{1}{4} + \ldots$

j $2 - \frac{4}{5} + \frac{8}{25} - \ldots$

2 Find the limiting sum of each series.

a $40 + 20 + 10 + \ldots$ b $320 + 80 + 20 + \ldots$ c $100 - 50 + 25 - \ldots$

d $6 + 3 + 1\frac{1}{2} + \ldots$ e $\frac{2}{5} + \frac{6}{35} + \frac{18}{245} + \ldots$ f $72 - 24 + 8 - \ldots$

g $-12 + 2 - \frac{1}{3} + \ldots$ h $\frac{3}{4} - \frac{1}{2} + \frac{1}{3} - \ldots$ i $12 + 9 + 6\frac{3}{4} + \ldots$

j $-\frac{2}{3} + \frac{5}{12} - \frac{25}{96} + \ldots$

3 Find the difference between the limiting sum and the sum of 6 terms of each series, correct to 2 significant figures.

a $56 - 28 + 14 - \ldots$ b $72 + 24 + 8 + \ldots$ c $1 + \frac{1}{5} + \frac{1}{25} + \ldots$

d $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ e $1\frac{1}{4} + \frac{15}{16} + \frac{45}{64} + \ldots$

4 A geometric series has limiting sum 6 and common ratio $\frac{1}{3}$. Evaluate the first term of the series.

5 A geometric series has a limiting sum of 5 and first term 3. Find the common ratio.

6 The limiting sum of a geometric series is $9\frac{1}{3}$ and the common ratio is $\frac{2}{5}$. Find the first term of the series.

7 A geometric series has limiting sum 40 and its first term is 5. Find the common ratio of the series.

8 A geometric series has limiting sum $-6\frac{2}{5}$ and first term -8. Find its common ratio.

9 The limiting sum of a geometric series is $-\frac{3}{10}$ and its first term is $-\frac{1}{2}$. Find the common ratio of the series.

10 The second term of a geometric series is 2 and its limiting sum is 9. Find the values of first term a and common ratio r.

11 A geometric series has 3rd term 12 and 4th term -3. Find a, r and the limiting sum.

12 A geometric series has 2nd term $\frac{2}{3}$ and 4th term $\frac{8}{27}$. Find a, r and its limiting sum.

13 The 3rd term of a geometric series is 54 and the 6th term is $11\frac{83}{125}$. Evaluate a, r and the limiting sum.

14 The 2nd term of a geometric series is $\frac{4}{15}$ and the 5th term is $\frac{32}{405}$. Find the values of a and r and its limiting sum.

15 The limiting sum of a geometric series is 5 and the 2nd term is $1\frac{1}{5}$. Find the first term and the common ratio.

16 The series $x + \frac{x}{4} + \frac{x}{16} + \ldots$ has a limiting sum of $\frac{7}{8}$. Evaluate x.

17 a For what values of k does the limiting sum exist for the series $k + k^2 + k^3 + \ldots$?

b Find the limiting sum of the series when $k = -\frac{2}{3}$.

c Evaluate k if the limiting sum of the series is 3.

18 Show that in any geometric series the difference between the limiting sum and the sum of n terms is $\frac{ar^n}{1-r}$.

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1. TEST YOURSELF

For Questions 1 to 3, select the correct answer A, B, C or D.

WS
Formula sheet: Measurement, Sequences and series

Qz
Practice quiz

1 The sum of n terms of a geometric sequence is:

A $S_n = \frac{n}{2}[2a + (n-1)d]$

B $S_n = \frac{a}{1-r}$

C $S_n = \frac{a(1-r^n)}{1-r}$

D $S_n = \frac{n}{2}(a+1)$

2 The limiting sum of an infinite geometric series exists when:

A $r > 1$ B $|r| > 1$ C $|r| < 1$ D $r < 1$

3 The nth term of the statement that can be proved 12, 9, 6, … is:

A $9 + 3n$ B $15 - 3n$ C $9 - 3n$ D $12n - 15$

4 Find a formula for the nth term of each sequence.

a 9, 13, 17, … b 7, 0, −7, … c 2, 6, 18, …

d $200, 50, 12\frac{1}{2}, \ldots$ e −2, 4, −8, …

5 For the series 156 + 145 + 134 + …:

a Find the 15th term.

b Find the sum of 15 terms.

c Find the sum of 14 terms.

d Write a relationship between T_{15}, S_{15} and S_{14}.

e Find the value of n for the first negative term of the series.

6 Find whether each sequence is:

i arithmetic ii geometric iii neither.

a 97, 93, 89, … b $\frac{2}{3}, \frac{1}{2}, \frac{3}{8}, \ldots$

c $\sqrt{5}, \sqrt{20}, \sqrt{45}, \ldots$ d −1.6, −0.4, 0.6, …

e 3.4, 7.5, 11.6, … f 48, 24, 12, …

g $-\frac{1}{5}, 1, -5, \ldots$ h 105, 100, 95, …

i $1\frac{1}{2}, 1\frac{1}{4}, 1, \ldots$ j $\log x, \log x^2, \log x^3, \ldots$

7 The nth term of the sequence 8, 13, 18, … is 543. Evaluate n.

8 The 11th term of an arithmetic sequence is 97 and the 6th term is 32. Find the first term and common difference.

9 A sequence has nth term given by $T_n = n^3 - 5$. Find:

a the 4th term b the sum of 4 terms c which term is 5827.

10 A sequence has terms 5, x, 45, … Evaluate x if the sequence is:

a arithmetic b geometric.

11 If x, $2x + 3$ and $5x$ are the first 3 terms of an arithmetic series, calculate the value of x.

12 Find the 20th term of:

a 3, 10, 17, … b 101, 98, 95, … c 0.3, 0.6, 0.9, …

13 Find the limiting sum of the series 81 + 27 + 9 + …

14 For each series, find the formula for the sum of n terms.

a $5 + 9 + 13 + \ldots$ b $1 + 1.07 + 1.07^2 + \ldots$

15 a For what values of x does the geometric series $1 + x + x^2 + \ldots$ have a limiting sum?

b Find the limiting sum when $x = \frac{3}{5}$.

c Evaluate x when the limiting sum is $1\frac{1}{2}$.

16 The first term of an arithmetic series is 4 and the sum of 10 terms is 265. Find the common difference.

17 If $x + 2$, $7x - 2$ and $15x + 6$ are consecutive terms in a geometric sequence, evaluate x.

18 Evaluate $8 + 14 + 20 + \ldots + 122$.

19 a Calculate the sum of all the multiples of 7 from 1 to 100.

b Calculate the sum of all numbers from 1 to 100 that are not multiples of 7.

20 The sum of n terms of the series $214 + 206 + 198 + \ldots$ is 2760. Evaluate n.

21 Evaluate n if the nth term of the sequence 4, 12, 36, … is 236 196.

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1. CHALLENGE EXERCISE

1 The nth term of a sequence is given by $T_n = \dfrac{n^2}{n+1}$.

a What is the 9th term of the sequence?

b Which term is equal to $18\frac{1}{20}$?

2 For the series $\dfrac{3\pi}{4}, \pi, \dfrac{5\pi}{4}, \ldots$ find the exact value of:

a the common difference

b the 7th term

c the sum of 6 terms.

3 Evaluate the sum of the first 20 terms of the series:

a $3 + 5 + 9 + 17 + 33 + 65 + \ldots$

b $5 - 2 + 10 - 8 + 15 - 32 + \ldots$

4 Which term of the sequence $\dfrac{7}{9}, \dfrac{14}{45}, \dfrac{28}{225}, \ldots$ is equal to $\dfrac{224}{28125}$?

5 Find the sum of all integers between 1 and 200 that are not multiples of 9.

6 Find the values of n for which $S_n > 24.99$ for the series $20 + 4 + \dfrac{4}{5} + \ldots$

7 The sum of the first 5 terms of a geometric series is 77 and the sum of the next 5 terms is -2464.

a Find the first term and common ratio of the series.

b Find the 4th term of the series.

8 a Find the limiting sum of the series $1 + \cos^2 x + \cos^4 x + \ldots$ where $\cos^2 x \neq 0, 1$.

b Why does this series have a limiting sum?

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FUNCTIONS

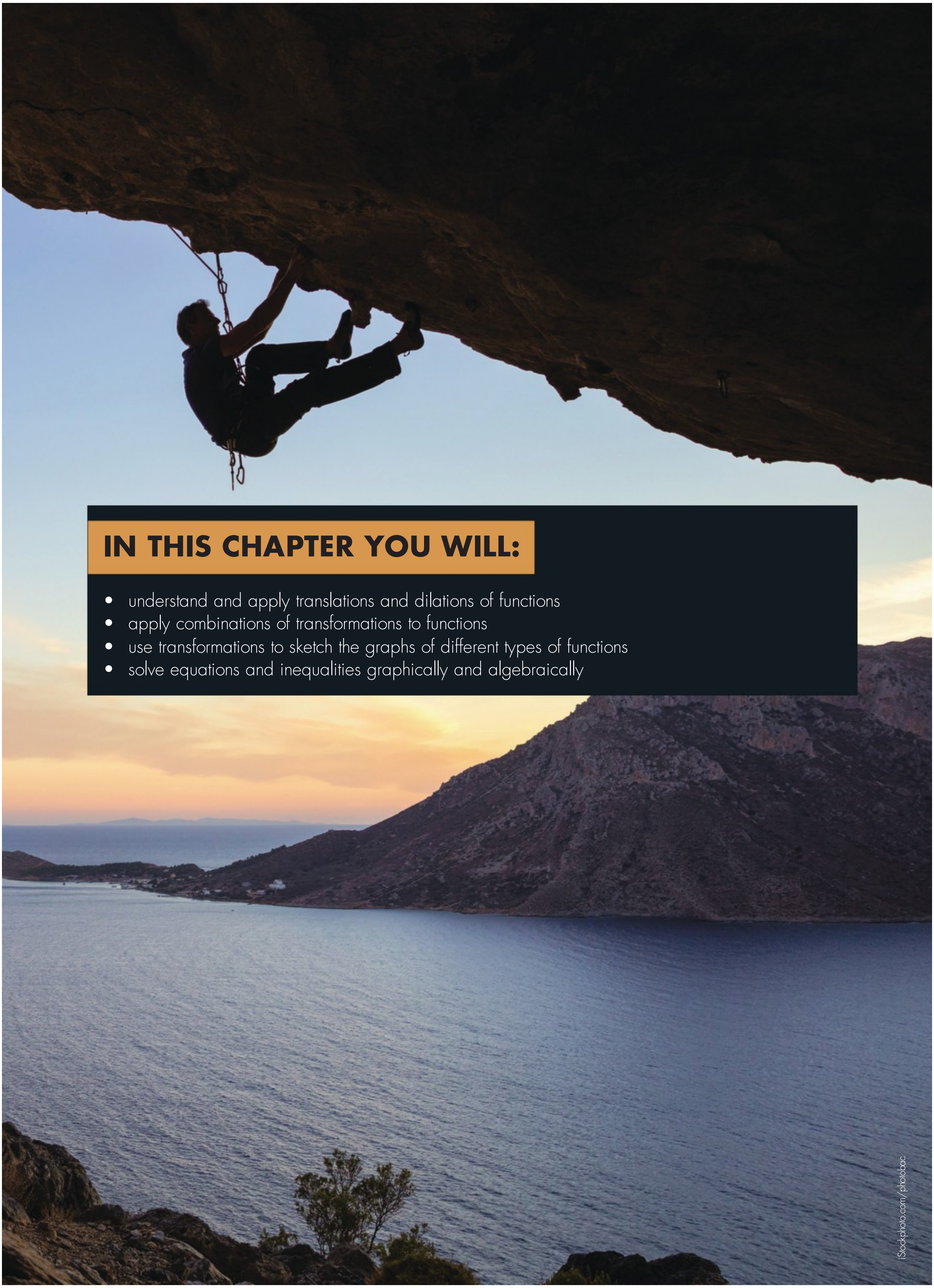
2.

TRANSFORMATIONS OF FUNCTIONS

In this chapter you will explore transformations on the graph of the function $y = f(x)$ that move or stretch the function. We have already met some transformations of functions in Year 11. For example, we learned that the graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis, $y = k \sin x$ is the graph of $y = \sin x$ but stretched vertically to give an amplitude of k, and $y = \cos (x + b)$ is the graph of $y = \cos x$ shifted b units to the right.

You will also look at both graphical and algebraic solutions of equations using the transformations of functions.

CHAPTER OUTLINE



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IN THIS CHAPTER YOU WILL:

- understand and apply translations and dilations of functions
- apply combinations of transformations to functions
- use transformations to sketch the graphs of different types of functions
- solve equations and inequalities graphically and algebraically

TERMINOLOGY

dilation: The process of stretching or compressing the graph of a function horizontally or vertically.
parameter: a constant in the equation of a function that determines the properties of that function and its graph; for example, the parameters for $y = mx + c$ are m (gradient) and c (y-intercept).
scale factor: The value of k by which the graph of a function is dilated.
transformation: A general name for the process of changing the graph of a function by moving, reflecting or stretching it.
translation: The process of shifting the graph of a function horizontally and/or vertically without changing its size or shape.

2.01 Vertical translations of functions

INVESTIGATION

VERTICAL TRANSLATIONS

Some graphics calculators or graphing software use a dynamic feature to show how a constant c (a parameter) changes the graph of a function.
Use dynamic geometry software to explore the effect of c on each graph below. If you don't have dynamic software, substitute different values for c into the equation. Use positive and negative values, integers and fractions.

1 $f(x) = x + c$
2 $f(x) = x^2 + c$
3 $f(x) = x^3 + c$
4 $f(x) = x^4 + c$
5 $f(x) = e^x + c$
6 $f(x) = \ln x + c$
7 $f(x) = \frac{1}{x} + c$
8 $f(x) = |x| + c$

How does the value of c transform the graph? What is the difference between positive and negative values of c?

Notice that c shifts the graph up and down without changing its size or shape. We call this a vertical translation (a shift along the y-axis).

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Vertical translation

For the function $y = f(x)$:

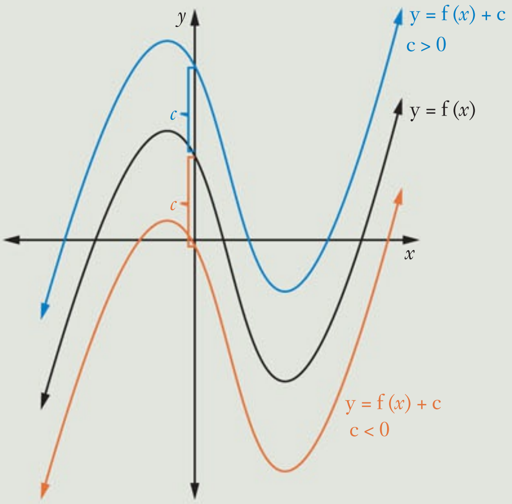
$y = f(x) + c$ translates the graph vertically (along the y-axis).

If $c > 0$, the graph is translated upwards by c units.

If $c < 0$, the graph is translated downwards.

A vertical translation changes the y values of the function.

In Year 11, we learned that $y = \sin x + c$ is the graph of $y = \sin x$ shifted up c units.



EXAMPLE 1

a Explain how the graph of $y = x^2 + 2$ is related to the graph of $y = x^2$.

b If the graph of the function $y = x^2 + 7x + 1$ is translated 4 units down, find the equation of the transformed function.

c The point P(3, –2) lies on the function $y = f(x)$. Find the transformed point (the image of P) if the function is translated:

i 6 units down

ii 8 units up

Solution

a The graph of $y = x^2 + 2$ is a vertical translation 2 units up from the original (parent) function $y = x^2$.

b For a vertical translation 4 units down:

$y = f(x) + c$ where $c = -4$

$y = x^2 + 7x + 1 - 4$

$= x^2 + 7x - 3$

The equation of the transformed function is $y = x^2 + 7x - 3$

c i P(3, –2) is translated 6 units down, so subtract 6 from the y value.

The transformed point is $(3, -2 - 6) \equiv (3, -8)$.

ii P(3, –2) is translated 8 units up, so add 8 to the y value.

The transformed point is $(3, -2 + 8) \equiv (3, 6)$.

For points, we use '≡' (identical to) rather than '='.

EXAMPLE 2

a Sketch the graph of $y = x^3 - 3$.

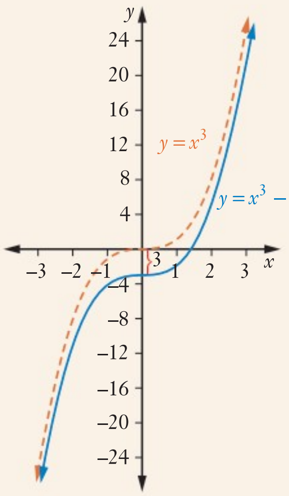
b i State the relationship of $y = \frac{1}{x} - 2$ to $y = \frac{1}{x}$.

ii State the domain and range of $y = \frac{1}{x} - 2$

iii Sketch the graph of $y = \frac{1}{x} - 2$.

Solution

a A vertical translation of -3 units shifts the function $y = x^3$ down to the graph of $y = x^3 - 3$.
If you need to find some points on the graph of $y = x^3 - 3$ you could subtract 3 from y values of $y = x^3$.



b i $y = \frac{1}{x} - 2$ is a vertical translation 2 units down of $y = \frac{1}{x}$.

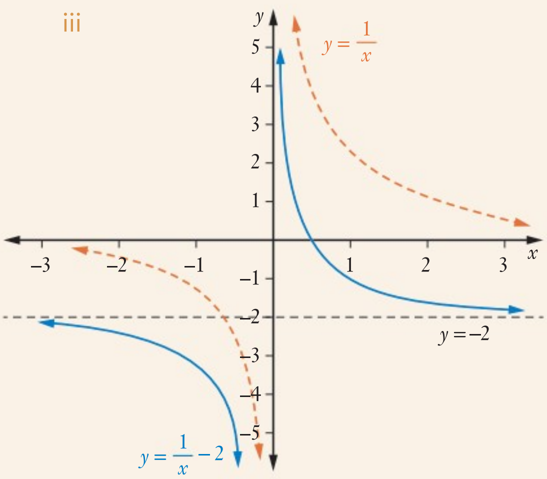
ii Since $x \neq 0$, domain is $(-\infty, 0) \cup (0, \infty)$.

Since $\frac{1}{x} \neq 0$, $\frac{1}{x} - 2 \neq -2$

So range is $(-\infty, -2) \cup (-2, \infty)$.

Since the horizontal asymptote is at $y = -2$, we sketch it as a dotted line.

iii



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Exercise 2.01 Vertical translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = x^2 + 3$ b $y = x^2 - 7$

c $y = x^2 - 1$ d $y = x^2 + 5$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = x^3 + 1$ b $y = x^3 - 4$ c $y = x^3 + 8$

3 Describe how the graph of $y = \frac{1}{x}$ transforms to the graph of $y = \frac{1}{x} + 9$.

4 Find the equation of each translated function.

a $y = x^2$ is translated 3 units downwards

b $f(x) = 2^x$ is translated 8 units upwards

c $y = |x|$ is translated 1 unit upwards

d $y = x^3$ is translated 4 units downwards

e $f(x) = \log x$ is translated 3 units upwards

f $y = \frac{2}{x}$ is translated 7 units downwards

5 Describe the relationship between the graph of $f(x) = x^4$ and:

a $f(x) = x^4 - 1$ b $f(x) = x^4 + 6$

6 Find the equation of the transformed function if:

a $y = 2x^3 + 3$ is translated:

i 5 units down ii 3 units up

b $y = |x| - 4$ is translated:

i 1 unit up ii 2 units down

c $y = e^x + 2$ is translated:

i 1 unit down ii 3 units up

d $y = \log_e x - 1$ is translated:

i 11 units up ii 7 units down

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the transformed (image) point of P if the function is translated:

a 2 units up b 6 units down c m units up

8 Find the original point P on the function $y = f(x)$ if the coordinates of its transformed image are $(-1, 2)$ when the function is translated:

a 1 unit up b 3 units down

9 Sketch each set of functions on the same number plane.

a $y = x^2$, $y = x^2 + 2$ and $y = x^2 - 3$ b $y = 3^x$ and $y = 3^x - 4$

c $y = |x|$ and $y = |x| - 3$

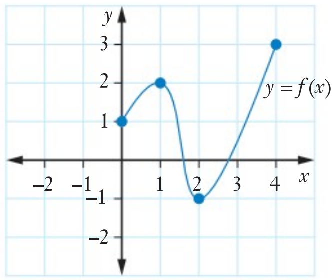
10 a Describe the transformation of $y = \frac{1}{x}$ into $y = \frac{1}{x} + 1$.

b Sketch the graph of $y = \frac{1}{x} + 1$.

11 The graph shows $y = f(x)$. Sketch the graph of:

a $y = f(x) - 1$

b $y = f(x) + 2$



12 a Show that $\frac{3x+1}{x} = \frac{1}{x} + 3$.

b Hence or otherwise, sketch the graph of $y = \frac{3x+1}{x}$.

2.02 Horizontal translations of functions

WS Translations of functions

Translations of functions

WS Graphing translations of functions

INVESTIGATION

HORIZONTAL TRANSLATIONS

Use a graphics calculator or graphing software to explore the affect of parameter b on each graph below. If you don't have dynamic software, substitute different values for b into the equation. Use positive and negative values, integers and fractions for b.

1 $f(x) = (x + b)^2$ 2 $f(x) = (x + b)^3$ 3 $f(x) = (x + b)^4$

4 $f(x) = e^{(x + b)}$ 5 $f(x) = \ln(x + b)$ 6 $f(x) = \frac{1}{x + b}$

7 $f(x) = |x + b|$

How does the graph change as the value of b changes?

What is the difference between positive and negative values of b?

Notice that the parameter shifts the graph to the left or right without changing its size or shape. We call this a horizontal translation (it shifts the function along the x-axis).

For a horizontal translation the shift is in the opposite direction from the sign of b.

To understand why this happens, we change the subject of the equation to x since the translation is a shift along the x-axis. For example:

$$y = (x + 5)^3$$

$$\sqrt[3]{y} = x + 5$$

$$\sqrt[3]{y} - 5 = x$$

This is a shift of 5 units to the left.

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Horizontal translations

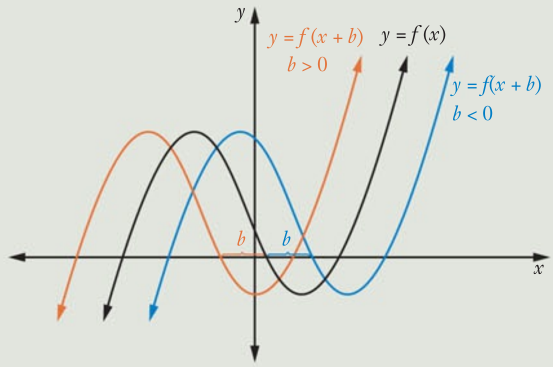
For the function $y = f(x)$:

$y = f(x + b)$ translates the graph horizontally (along the x-axis).

If $b > 0$, the graph is translated to the left by b units.

If $b < 0$, the graph is translated to the right.

A horizontal translation changes the x values of the function.



In Year 11, we learned that $y = \tan(x + b)$ is the graph of $y = \tan x$ shifted left b units.

EXAMPLE 3

a What is the relationship of $f(x) = \log_2 (x + 3)$ to $f(x) = \log_2 x$?

b If the graph $y = (x - 4)^3$ is translated 7 units to the right, find the equation of the transformed function.

c The point P(2, 5) lies on the function $y = f(x)$. Find the corresponding (image) point of P given a horizontal translation with $b = 1$.

d The point Q (3, –4) on the graph of $y = f(x - 2)$ is the image of point P(x, y) on $y = f(x)$. Find the coordinates of P.

Solution

a $f(x) = \log_2 (x + 3)$ is a horizontal translation 3 units to the left from the parent function $f(x) = \log_2 x$.

b If $y = (x - 4)^3$ is translated 7 units to the right:

$y = f(x + b)$ where $b = -7$

$y = (x - 4 - 7)^3 = (x - 11)^3$

So the equation of the transformed function is $y = (x - 11)^3$

c $y = f(x + b)$ describes a horizontal translation (along the x-axis).

When $b = 1$, x values shift 1 unit to the left.

Image of $P \equiv (2 - 1, 5) \equiv (1, 5)$

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d $y = f(x - 2)$ is a horizontal translation 2 units to the right of $y = f(x)$.

So (x, y) becomes $(x + 2, y)$

But $Q(3, -4)$ is the image of $P(x, y)$

So $(x + 2, y) \equiv (3, -4)$

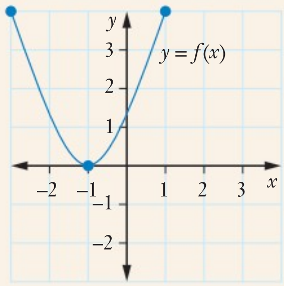
So $x + 2 = 3, y = -4$

$x = 1, y = -4$

So $P \equiv (1, -4)$

EXAMPLE 4

a The graph of $y = f(x)$ shown is transformed into $y = f(x + b)$. Sketch the transformed graph if $b = -3$.



b Sketch the graph of:

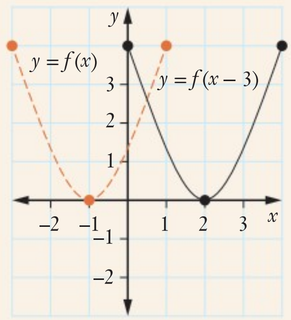
i $y = |x + 3|$

ii $y = \dfrac{1}{x - 2}$

Solution

a The graph $y = f(x + b)$ where $b = -3$ describes a horizontal translation of 3 units to the right.

The transformed graph is 3 units to the right of the original function.



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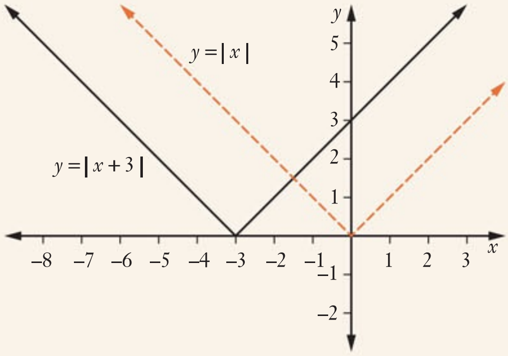
b i The function $y = |x + 3|$ is in the form $y = f(x + b)$ where $b = 3$.

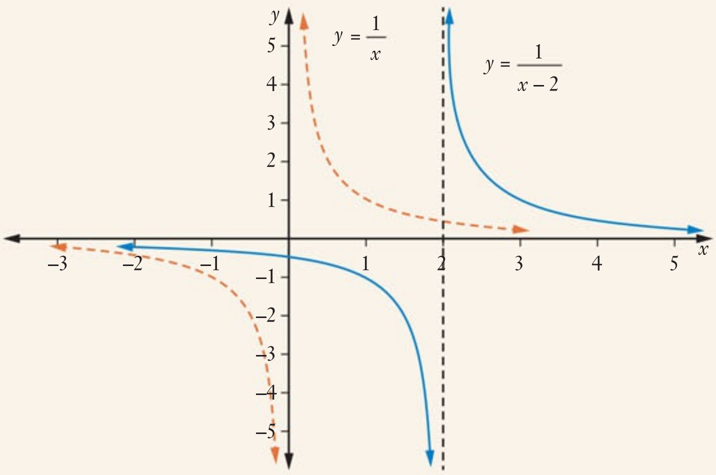
Since $b > 0$, $y = |x|$ is shifted 3 units to the left.



If you need to find some points on the graph of $y = |x + 3|$ you could subtract 3 from x values of $y = |x|$.

ii $y = \dfrac{1}{x - 2}$ is in the form $y = f(x + b)$ where $b = -2$.

Since $b < 0$, $y = \dfrac{1}{x}$ is shifted 2 units to the right.



Exercise 2.02 Horizontal translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = (x - 4)^2$ b $y = (x + 2)^2$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = (x - 5)^3$ b $y = (x + 3)^3$

3 Find the equation of each translated graph.

a $y = x^2$ translated 3 units to the left
b $f(x) = 2^x$ translated 8 units to the right
c $y = |x|$ translated 1 unit to the left
d $y = x^3$ translated 4 units to the right
e $f(x) = \log x$ translated 3 units left

4 Describe how $y = \frac{1}{x}$ transforms to $y = \frac{1}{x-3}$.

5 Describe the relationship between $f(x) = x^4$ and:

a $f(x) = (x + 2)^4$ b $f(x) = (x - 5)^4$

6 Find the equation if:

a $y = -x^2$ is translated

i 4 units to the left ii 8 units to the right

b $y = |x|$ is translated

i 3 units to the right ii 4 units to the left

c $y = e^{x+2}$ is translated

i 4 units to the left ii 7 units to the right

d $y = \log_2(x - 3)$ is translated

i 2 units to the right ii 3 units to the left

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the image point of P if the function is transformed to $y = f(x + b)$ where:

a $b = -4$ b $b = 9$ c $b = t$

8 Find the original point on the function $y = f(x)$ if the coordinates of its image are $(-1, 2)$ when the function is translated:

a 4 units to the left

b 8 units to the right

9 Sketch on the same number plane:

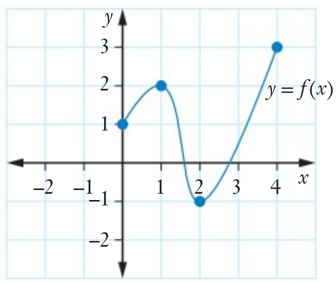
a $y = x^3$ and $y = (x + 1)^3$

b $f(x) = \ln x$ and $f(x) = \ln(x + 2)$

10 The graph shown is $y = f(x)$. Sketch the graph of:

a $y = f(x - 1)$

b $y = f(x + 3)$



11 Find the equation of the transformed function if $f(x) = x^5$ is translated:

a 5 units down b 3 units to the right

c 2 units up d 7 units to the left

12 The point $P(3, -2)$ is the image of a point on $y = f(x)$ after it has been translated 4 units to the left. Find the original point.

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2.03 Vertical dilations of functions

A dilation stretches or compresses a function, changing its size and shape.

INVESTIGATION

VERTICAL DILATION

Explore the effect of parameter k on each graph below. If you don't have dynamic software, substitute different values for k into the equation. Use positive and negative values, integers and fractions for k.

1 $f(x) = kx$

2 $f(x) = kx^2$

3 $f(x) = kx^3$

4 $f(x) = kx^4$

5 $f(x) = ke^x$

6 $f(x) = k \ln x$

7 $f(x) = k\left(\frac{1}{x}\right)$

8 $f(x) = k|x|$

How does the graph change as the value of k changes?

What is the difference between positive and negative values of k?

Notice that k stretches the graph up and down along the y-axis and changes its shape. We call this vertical dilation. The value of the parameter k controls the amount of stretching (expanding) or shrinking (compressing).

We call k the scale factor.

Vertical dilations

For the curve $y = f(x)$:

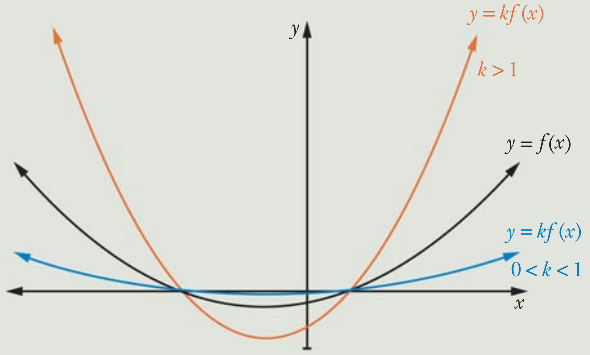
$y = kf(x)$ dilates the curve vertically (along the y-axis) by a scale factor of k.

If $k > 1$, the graph is stretched, or expanded.

If $0 < k < 1$, the graph is shrunk, or compressed.

A vertical dilation changes the y values of the function.

In Year 11, we learned that $y = k \cos x$ is the graph of $y = \cos x$ stretched vertically to give an amplitude of k.



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EXAMPLE 5

a The function $y = x^7$ is dilated vertically by a factor of 3. Find the equation of the transformed function.

b Describe how the function $f(x) = \frac{\log_2 x}{2}$ is related to the function $f(x) = \log_2 x$.

c Find the scale factor of each dilation of a function and state whether the dilation stretches or compresses the graph.

i $y = 7x^2$

ii $y = \frac{e^x}{5}$

Solution

a If a function $y = f(x)$ has a vertical dilation with factor k, the equation of its transformed function is $y = kf(x)$.

So if the function $y = x^7$ has a vertical dilation with factor 3, the equation of the transformed function is $y = 3x^7$.

Since $k > 1$, the function is stretched vertically.

b
$$
\begin{aligned}
f(x) &= \frac{\log_2 x}{2} \\
&= \frac{1}{2}\log_2 x
\end{aligned}
$$

So the function is in the form $y = kf(x)$ where $k = \frac{1}{2}$.

Since $0 < k < 1$, the function is compressed vertically.

So $f(x) = \frac{\log_2 x}{2}$ is the result of $f(x) = \log_2 x$ being dilated (compressed) vertically by a scale factor of $\frac{1}{2}$.

c The function $y = kf(x)$ has scale factor k.

i $y = 7x^2$ has scale factor 7 (stretched)

ii
$$
\begin{aligned}
y &= \frac{e^x}{5} \\
&= \frac{1}{5}e^x
\end{aligned}
$$

Scale factor is $\frac{1}{5}$ (compressed).

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EXAMPLE 6

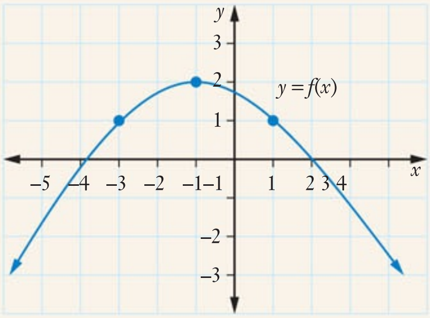
a The point $N = (-1, 8)$ lies on the function $y = f(x)$. Find the image of N on the function $y = kf(x)$ when:

i $k = 5$

ii $k = \frac{1}{2}$

b A function $y = f(x)$ is transformed to $y = kf(x)$. If the image of point A on the transformed function is $(-6, 12)$, find the coordinates of A when $k = 3$.

c The graph shown is $y = f(x)$. Sketch the graph of $y = 2f(x)$.



d Sketch the graphs of $y = x^2$ and $y = \frac{x^2}{2}$ on the same set of axes.

Solution

a $y = kf(x)$ describes a vertical dilation (along the y-axis).

So the y values of the parent function will change.

i When $k = 5$: y values are multiplied by a factor of 5.

Image of $N \equiv (-1, 8 \times 5) \equiv (-1, 40)$

ii When $k = \frac{1}{2}$: y values will be multiplied by a factor of $\frac{1}{2}$ (or divided by 2).

Image of $N \equiv \left(-1, 8 \times \frac{1}{2}\right) \equiv (-1, 4)$

b When $k = 3$, (x, y) becomes $(x, 3y)$.

$(x, 3y) \equiv (-6, 12)$

$x = -6$

$3y = 12$

$y = 4$

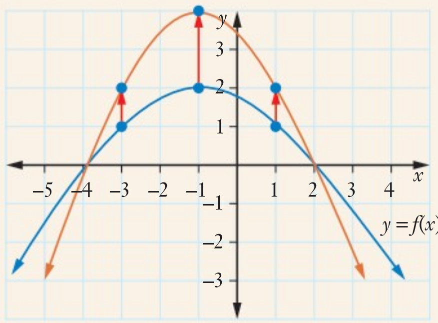
So $A \equiv (-6, 4)$

c The graph of $y = 2f(x)$ is a vertical dilation of $y = f(x)$ with factor 2.

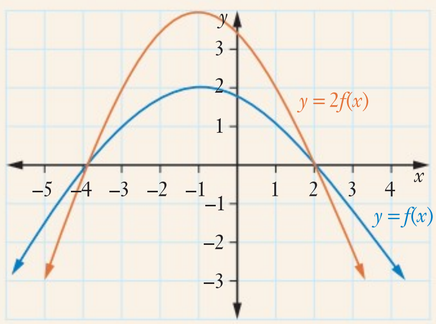
So each y value is doubled and the graph is twice as high as the original graph. For example:

$y = 1$ becomes $y = 2$

$y = 2$ becomes $y = 4$



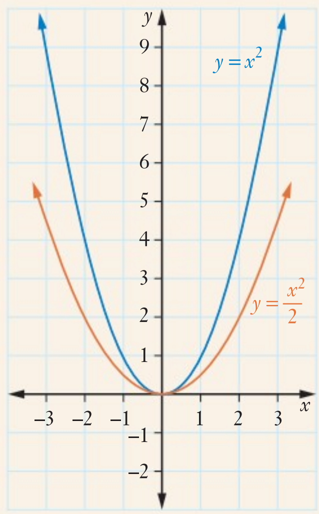
The transformed graph is still a parabola. However it is higher (stretched) and narrower than the original graph.



d $y = \dfrac{x^2}{2}$ is a vertical dilation of $y = x^2$ with scale factor $\dfrac{1}{2}$.

This halves the y values.

$(-3, 9)$	becomes	$\left(-3, 4\frac{1}{2}\right)$
$(-2, 4)$	becomes	$(-2, 2)$
$(-1, 1)$	becomes	$\left(-1, \frac{1}{2}\right)$
$(0, 0)$	becomes	$(0, 0)$
$(1, 1)$	becomes	$\left(-1, \frac{1}{2}\right)$
$(2, 4)$	becomes	$(2, 2)$
$(3, 9)$	becomes	$\left(3, 4\frac{1}{2}\right)$



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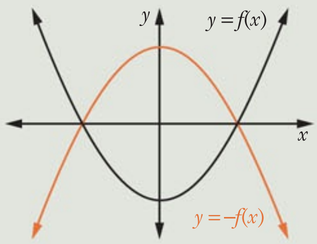
Reflections in the x-axis

You studied reflections in Year 11 in Chapter 5, Fur*th*er functions.

Reflections in the x-axis

$y = -f(x)$ is a reflection of the curve $y = f(x)$ in the x-axis.

This is also a vertical dilation with scale factor $k = -1$.



EXAMPLE 7

a Point P(2, 4) is on the function $y = f(x)$. Find the image of P on the function $y = -f(x)$.

b Sketch the vertical dilation of $f(x) = \frac{1}{x}$ with scale factor –1.

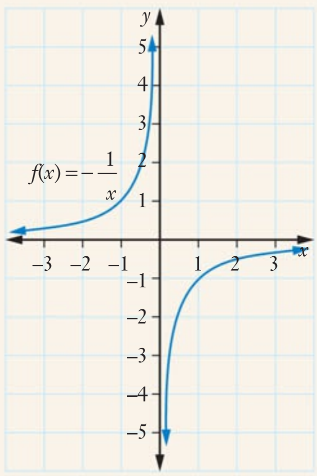
Solution

a The function $y = -f(x)$ is a reflection in the x-axis.

The y values are multiplied by –1.

Image of $P \equiv (2, 4 \times [-1]) \equiv (2, -4)$

b A vertical stretch with scale factor –1 is a reflection of $f(x) = \frac{1}{x}$ in the x-axis.



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Exercise 2.03 Vertical dilations of functions

1 Describe how the constant affects each transformed graph, given the parent function, and state the scale factor.

a $y = x$

i $y = 6x$ ii $y = \frac{x}{2}$ iii $y = -x$

b $y = x^2$

i $y = 2x^2$ ii $y = \frac{x^2}{6}$ iii $y = -x^2$

c $y = x^3$

i $y = 4x^3$ ii $y = \frac{x^3}{7}$ iii $y = \frac{4x^3}{3}$

d $y = x^4$

i $y = 9x^4$ ii $y = \frac{x^4}{3}$ iii $y = \frac{3x^4}{8}$

e $y = |x|$

i $y = 5|x|$ ii $y = \frac{|x|}{8}$ iii $y = -|x|$

f $f(x) = \log x$

i $f(x) = 9 \log x$ ii $f(x) = -\log x$ iii $f(x) = \frac{2\log x}{5}$

2 Find the equation of each transformed graph and state its domain and range.

a $y = x^2$ dilated vertically with a scale factor of 6

b $y = \ln x$ dilated vertically with a scale factor of $\frac{1}{4}$

c $f(x) = |x|$ reflected in the x-axis

d $f(x) = e^x$ dilated vertically with a scale factor of 4

e $y = \frac{1}{x}$ dilated vertically with a scale factor of 7

3 Find the equation of each transformed function after the vertical dilation given.

a $y = 3^x$ with scale factor 5

b $f(x) = x^2$ with scale factor $\frac{1}{3}$

c $y = x^3$ with scale factor -1

d $y = \frac{1}{x}$ with scale factor $\frac{1}{2}$

e $y = |x|$ with scale factor $\frac{2}{3}$

4 Point $M = (3, 6)$ lies on the graph of $y = f(x)$. Find the coordinates of the image of M when $f(x)$ is:

a dilated vertically with a factor of 4

b reflected in the x-axis

c dilated vertically with a factor of 12

d dilated vertically with a factor of $\frac{5}{6}$

5 The coordinates of the image of $X(x, y)$ are $(4, 12)$ when $y = f(x)$ is vertically dilated. Find the coordinates of X if the scale factor is:

a 3 b 2 c $\frac{1}{3}$ d $\frac{3}{4}$ e -1

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6 Sketch each pair of functions on the same set of axes.

a $f(x) = \log_2 x$ and $f(x) = 2\log_2 x$

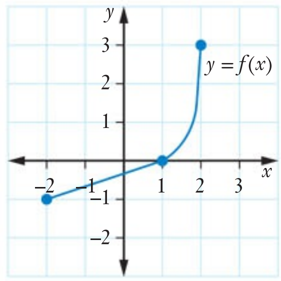
b $y = 3^x$ and $y = 2 \cdot 3^x$

c $y = \frac{1}{x}$ and $y = \frac{3}{x}$

d $y = |x|$ and $y = 2|x|$

e $y = x^3$ and $y = -x^3$

7 Points on a function $y = f(x)$ are shown on the graph. Sketch the graph of the transformed function showing the image points, given a vertical stretch with factor:



a 3

b $\frac{1}{2}$

c -1

8 Sketch the graph of $y = 2\sqrt{1 - x^2}$.

2.04 Horizontal dilations of functions

WS Dilations of functions

WS Advanced graphs

INVESTIGATION

HORIZONTAL DILATIONS

Use dynamic geometry software to explore the affect of parameter a on each graph below. If you don't have dynamic software, substitute different values for a into the equation. Use positive and negative values, integers and fractions for a.

1 $f(x) = ax$

2 $f(x) = (ax)^2$

3 $f(x) = (ax)^3$

4 $f(x) = (ax)^4$

5 $f(x) = e^{ax}$

6 $f(x) = \ln ax$

7 $f(x) = \frac{1}{ax}$

8 $f(x) = |ax|$

How does a transform the graph as the value of a changes?

What is the difference between positive and negative values of a?

Notice that with horizontal dilations, the higher the value of a, the more the graph is compressed along the x-axis from left and right. This is inverse variation and the scale factor for horizontal dilations is $\frac{1}{a}$.

This is because horizontal dilation affects the x values of the function. To see this, we change the subject of the function to x. For example:

$$y = (3x)^3$$

$$\sqrt[3]{y} = 3x$$

$$\frac{\sqrt[3]{y}}{3} = x$$

$$\text{or } x = \frac{1}{3}\sqrt[3]{y}$$

In Year 11, we learned that $y = \sin ax$ is the graph of $y = \sin x$ compressed horizontally to give a period of $\frac{2\pi}{a}$

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This shows a scale factor of $\frac{1}{3}$.

Like horizontal translations, a horizontal stretch works the opposite way to what you would expect, because the equation is in the form $y = f(x)$ rather than $x = f(y)$.

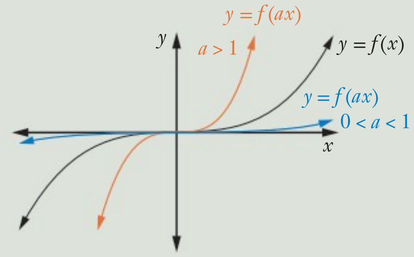
Horizontal dilations

For the curve $y = f(x)$:

$y = f(ax)$ stretches the curve horizontally (along the x-axis) by a scale factor of $\frac{1}{a}$.

If $a > 1$, the graph is compressed.

If $0 < a < 1$, the graph is stretched.



EXAMPLE 8

a Describe how the function $f(x) = x^3$ is related to the function $f(x) = (4x)^3$.

b The function $y = \ln x$ is dilated horizontally by a scale factor of 2. Find the equation of the transformed function.

c Find the scale factor of each function and state whether it stretches or compresses the graph.

i $y = e^{3x}$

ii $f(x) = \left|\frac{x}{4}\right|$

Solution

a The function $y = f(ax)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{a}$.

So the function $f(x) = (4x)^3$ is a horizontal dilation of $f(x) = x^3$ with scale factor $\frac{1}{4}$.

b If $y = \ln x$ is dilated horizontally by a scale factor of 2:

$$\frac{1}{a} = 2$$

$$a = \frac{1}{2}$$

So $y = \ln\left(\frac{1}{2}x\right)$ or $y = \ln\frac{x}{2}$

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c i $y = e^{3x}$ is in the form $y = f(ax)$ where $f(x) = e^x$.

This is a horizontal dilation with $a = 3$.

Scale factor $= \frac{1}{a} = \frac{1}{3}$ (stretched)

ii $f(x) = \left|\frac{x}{4}\right|$ can be written as $f(x) = \left|\frac{1}{4}x\right|$.

The function is in the form $y = f(ax)$ where $f(x) = |x|$.

This is a horizontal dilation with $a = \frac{1}{4}$.

$$\text{Scale factor} = \frac{1}{a}$$

$$= \frac{1}{\frac{1}{4}}$$

$$= 4 \text{ (compressed)}$$

EXAMPLE 9

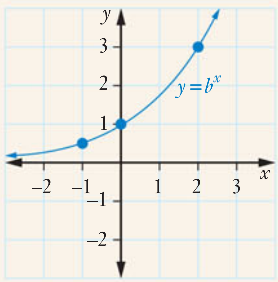
a The points P(–3, 4) and Q(9, 0) lie on the function $y = f(x)$. Find the coordinates of the images of P and Q for the function $y = f(ax)$ when:

i $a = 3$ ii $a = \frac{1}{5}$

b When the function $y = f(x)$ is transformed to $y = f(ax)$, the coordinates of the image of N(x, y) are (16, –5). Find the coordinates of N when:

i $a = 4$ ii $a = \frac{1}{2}$

c The graph of $y = b^x$ shown is transformed to $y = b^{2x}$. Sketch the graph of the transformed function.



d State the scale factor if the graph $y = |x|$ is transformed to $y = \left|\frac{x}{2}\right|$ and sketch both graphs on the same set of axes.

Solution

a The function $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$.

i When $a = 3$, scale factor is $\frac{1}{3}$.

All x values are multiplied by $\frac{1}{3}$ (divided by 3).

Image of $P \equiv \left(-3 \times \frac{1}{3},\ 4\right) \equiv (-1, 4)$

Image of $Q \equiv \left(9 \times \frac{1}{3},\ 0\right) \equiv (3, 0)$

ii When $a = \frac{1}{5}$, scale factor is $\frac{1}{\frac{1}{5}}$ or 5.

All x values are multiplied by 5.

Image of $P \equiv (-3 \times 5, 4) \equiv (-15, 4)$

Image of $Q \equiv (9 \times 5, 0) \equiv (45, 0)$

b We multiply all x values by scale factor $\frac{1}{a}$.

i When $a = 4$, scale factor is $\frac{1}{4}$

So (x, y) becomes $\left(x \times \frac{1}{4},\ y\right) = \left(\frac{x}{4},\ y\right)$

$\left(\frac{x}{4},\ y\right) \equiv (16, -5)$

$\frac{x}{4} = 16$

$x = 64$

$y = -5$

So $N \equiv (64, -5)$

ii When $a = \frac{1}{2}$, scale factor is $\frac{1}{\frac{1}{2}}$ or 2.

So (x, y) becomes $(x \times 2, y) \equiv (2x, y)$

$(2x, y) \equiv (16, -5)$

$2x = 16$

$x = 8$

$y = -5$

So $N \equiv (8, -5)$

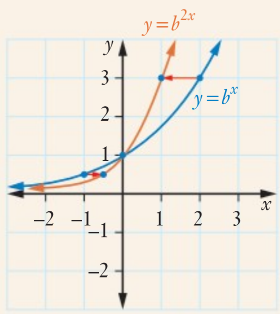
c The graph of $y = b^{2x}$ describes a horizontal dilation of $y = b^x$ with scale factor $\frac{1}{2}$.

So we halve the x values.

$x = -1$ becomes $x = -\frac{1}{2}$

$x = 0$ becomes $x = 0$

$x = 2$ becomes $x = 1$



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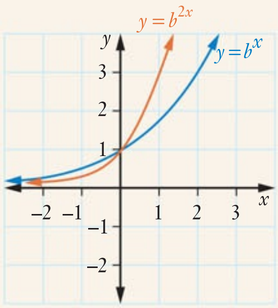
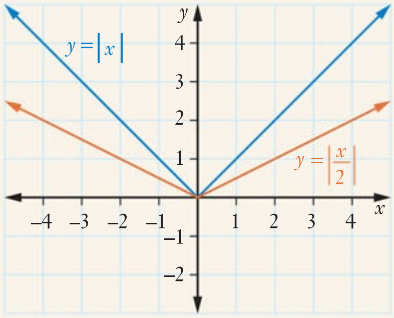
The transformed function is still in the shape of an exponential function, but it has changed shape and size.



d The graph $y = \left|\frac{x}{2}\right|$ is a horizontal dilation of $y = |x|$ with a scale factor $\frac{1}{\frac{1}{2}}$ or 2.

We double the x values.

(–3, 3)	becomes	(–6, 3)
(–2, 2)	becomes	(–4, 2)
(–1, 1)	becomes	(–2, 1)
(0, 0)	becomes	(0, 0)
(1, 1)	becomes	(2, 1)
(2, 2)	becomes	(4, 2)
(3, 3)	becomes	(6, 3)



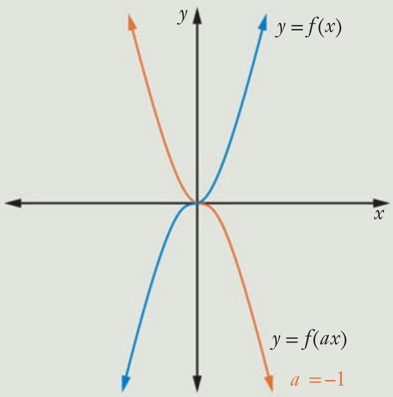
Reflections in the y-axis

You studied reflections in the y-axis in Year 11 in Chapter 5, *Further functions*.

Reflections in the y-axis

$y = f(-x)$ is a reflection of the curve $y = f(x)$ in the y-axis.

This is a horizontal stretch with scale factor $a = \frac{1}{-1} = -1$.



Notice that for even functions $y = f(x) = f(-x)$.

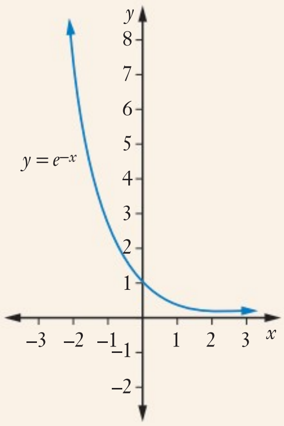
Even functions are already symmetrical about the y-axis so the reflected graph is the same as the original graph.

EXAMPLE 10

Sketch the graph of the horizontal dilation of $y = e^x$ with scale factor -1.

Solution

The horizontal dilation with scale factor -1 is a reflection of $y = e^x$ in the y-axis.



Exercise 2.04 Horizontal dilations of functions

1 Describe the transformation that the constant makes on $f(x) = x^4$ and state the scale factor.

a $f(x) = (8x)^4$

b $f(x) = \left(\frac{x}{5}\right)^4$

c $f(x) = \left(\frac{3x}{7}\right)^4$

d $f(x) = (-x)^4$

2 Describe whether the constant describes a horizontal or vertical dilation and state the scale factor.

a $y = x^2$

i $y = (2x)^2$ ii $y = (5x)^2$ iii $y = \left(\frac{x}{3}\right)^2$

b $y = x^3$

i $y = 4x^3$ ii $y = \left(\frac{x}{2}\right)^3$ iii $y = (-x)^3$

c $y = x^4$

i $y = (7x)^4$ ii $y = \frac{x^4}{8}$ iii $y = \left(\frac{3x}{4}\right)^4$

d $y = |x|$

i $y = |5x|$ ii $y = \left|\frac{x}{2}\right|$ iii $y = \left|\frac{3x}{5}\right|$

e $y = 5^x$

i $y = 5^{3x}$ ii $y = -5^x$ iii $y = 5^{\frac{x}{2}}$

f $f(x) = \log x$

i $f(x) = 8 \log x$ ii $f(x) = \log(-x)$ iii $f(x) = \log \frac{x}{7}$

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3 Find the equation of each transformed graph and state its domain and range.

a $f(x) = |x|$ is dilated horizontally with a scale factor of $\frac{1}{5}$

b $y = x^2$ is dilated horizontally with a scale factor of 3

c $y = x^3$ is reflected in the y-axis.

d $y = e^x$ is dilated vertically with a scale factor $\frac{1}{9}$

e $y = \log_4 x$ is reflected in the x-axis

4 Point X (–2, 7) lies on $y = f(x)$. Find the coordinates of the image of X on $y = f(ax)$ given:

a $a = 2$ b $a = -1$ c $a = \frac{1}{3}$

5 The function $y = f(x)$ is transformed into the function $y = f(ax)$. The coordinates of the image point of (x, y) on the original function are $(-24, 1)$ on the transformed function. Find the values of (x, y) if:

a $a = 3$ b $a = 2$ c $a = \frac{1}{4}$

6 Sketch each pair of functions on the same set of axes:

a $f(x) = \ln x$ and $f(x) = \ln (2x)$

b $y = 2^x$ and $y = 2^{\frac{x}{3}}$

c $y = \frac{1}{x}$ and $y = \frac{1}{3x}$

d $y = |x|$ and $y = |2x|$

e $f(x) = x^2$ and $f(x) = (3x)^2$

f $y = \ln x$ and $y = \ln (-x)$

7 Sketch the graphs of $y = e^x$, $y = e^{2x}$ and $y = 2e^x$ on the same set of axes.

8 Explain why a reflection in the y-axis does not change the graph of:

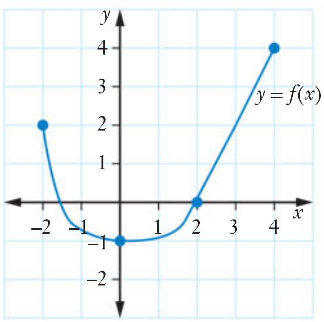
a $y = x^2$

b $f(x) = |x|$

9 Sketch the graph of $y = f(ax)$ given the graph of $y = f(x)$ shown, when:

a $a = \frac{1}{2}$

b $a = 2$



WS
Combinations of transformations

2.05 Combinations of transformations

A function can have any combination of the different types of transformations acting on it.

Transformations

For the curve $y = f(x)$:

$y = f(x) + c$ translates the function vertically:

- up if $c > 0$
- down if $c < 0$

$y = f(x + b)$ translates the function horizontally:

- to the left if $b > 0$
- to the right if $b < 0$

$y = kf(x)$ dilates the function vertically with scale factor k:

- stretches if $k > 1$
- compresses if $0 < k < 1$
- reflects the function in the x-axis if $k = -1$

$y = f(ax)$ dilates the function horizontally with scale factor $\frac{1}{a}$:

- compresses if $a > 1$
- stretches if $0 < a < 1$
- reflects the function in the y-axis if $a = -1$

EXAMPLE 11

a Find the equation of the transformed function if $y = x^4$ is shifted 2 units down and 5 units to the left.

b Find the equation of the transformed function if $y = e^x$ is dilated vertically by a scale factor 3 and translated horizontally 2 units to the right.

Solution

a Starting with $y = x^4$:

A vertical translation 2 units down gives $y = x^4 - 2$.

A horizontal translation 5 units to the left gives $b = 5$.

So the equation becomes $y = (x + 5)^4 - 2$.

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Notice that we could do this the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^4$.

A vertical translation 2 units down gives $y = (x + 5)^4 - 2$.

b Starting with $y = e^x$:

A vertical dilation of scale factor 3 gives $y = 3e^x$.

A horizontal translation 2 units to the right gives $b = -2$.

So the equation becomes $y = 3e^{x-2}$.

Notice that we could do this the other way around:

A horizontal translation 2 units to the right gives $y = e^{x-2}$.

A vertical dilation of scale factor 3 gives $y = 3e^{x-2}$.

When the transformations are both vertical or both horizontal, then the order is important.

EXAMPLE 12

a When the function $y = x^2$ is translated 3 units up (vertically) and vertically dilated by scale factor 4, the equation of the transformed function is $y = 4x^2 + 3$. Find the order in which the transformations were done.

b The equation of the transformed function is $y = (2x + 5)^3$ when the function $y = x^3$ is horizontally dilated by scale factor $\frac{1}{2}$ and translated 5 units (horizontally) to the left. In which order were the transformations done?

c The equation of the transformed function is $y = \ln [3(x - 2)]$ when the function $y = \ln x$ is horizontally dilated by scale factor $\frac{1}{3}$ and translated 2 units (horizontally) to the right. In which order were the transformations done?

Solution

a Starting with $y = x^2$:

A vertical translation 3 units up gives $y = x^2 + 3$.

A vertical dilation by scale factor 4 gives $y = 4(x^2 + 3)$.

This is not the equation of the transformed function.

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Try the other way around:

A vertical dilation by scale factor 4 gives $y = 4x^2$.

A vertical translation 3 units up gives $y = 4x^2 + 3$.

So the correct order is the vertical dilation, then the vertical translation.

b Starting with $y = x^3$:

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x)^3$.

A horizontal translation 5 units to the left gives $b = 5$ so $y = [2(x + 5)]^3$.

This is not the equation of the transformed function.

Try the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^3$.

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x + 5)^3$.

So the correct order is the horizontal translation, then the horizontal dilation.

c Starting with $y = \ln x$:

A horizontal dilation of scale factor $\frac{1}{3}$ gives $y = \ln (3x)$.

A horizontal translation 2 units to the right gives $y = \ln [3(x - 2)]$.

So the correct order is the horizontal dilation, then the horizontal translation.

Doing the horizontal dilation first gives $y = f(a(x + b))$, while doing the horizontal translation first gives $y = f(ax + b)$.

We can state the order we want to perform the transformations.

EXAMPLE 13

Find the equation of the function if $y = x^2$ is first horizontally dilated with scale factor $\frac{1}{2}$, then translated 3 units to the right.

Solution

A horizontal dilation with scale factor $\frac{1}{2}$ gives $a = 2$.

So $y = x^2$ becomes $y = (2x)^2$.

A horizontal translation 3 units to the right gives $b = -3$.

So $y = (2x)^2$ transforms to $y = [2(x - 3)]^2$.

Remember to put brackets around $x - 3$.

We can combine all the transformations into a single expression:

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Equation of a transformed function

$y = kf(a(x + b)) + c$ where a, b, c and k are constants is a transformation of $y = f(x)$:

- a horizontal dilation of scale factor $\frac{1}{a}$
- a horizontal translation of b
- a vertical dilation of k
- a vertical translation of c

Order of transformations

For $y = kf(a(x + b)) + c$:

1. do horizontal dilation (a), then horizontal translation (b)
2. do vertical dilation (k), then vertical translation (c)

It doesn't matter whether you do horizontal or vertical transformations first.

Notice that the horizontal dilation and translation parameters a and b are inside the brackets (they change x values) and the vertical dilation and translation parameters k and c are outside the brackets (they change the y values).

EXAMPLE 14

a Describe the transformations of $y = e^x$ in the correct order to produce the transformed function $y = \frac{1}{2}e^{x+1} - 3$.

b Describe the transformations of $y = x^2$ in order that give the transformed function $y = 3(2x - 6)^2 + 1$.

c Find the equation of the transformed function if $y = f(x)$ undergoes a vertical dilation with factor 5, a horizontal dilation with factor -1, a translation 4 units to the right and 9 units down.

Solution

a For $y = \frac{1}{2}e^{x+1} - 3$:

Horizontal transformations (a and b): No dilation, $b = 1$ gives a translation 1 unit left.

Vertical transformations (k and c): dilation of scale factor $\frac{1}{2}$ and translation 3 units down. Correct order is:

1 Horizontal translation 1 unit left

2 Vertical dilation of scale factor $\frac{1}{2}$

3 Vertical translation 3 units down

Because vertical transformations can be done first, the order 2–3–1 is also possible.

b For $y = 3(2x - 6)^2 + 1$:

First put the equation in the form $y = kf(a(x + b)) + c$.

$y = 3(2x - 6)^2 + 1$

$= 3[2(x - 3)]^2 + 1$

Horizontal transformations: dilation $a = 2$ and translation $b = -3$.

Vertical transformations: dilation $k = 3$ and translation $c = 1$.

1 Horizontal dilation of scale factor $\frac{1}{2}$

2 Horizontal translation 3 units right

3 Vertical dilation of scale factor 3

4 Vertical translation 1 unit up

The order 3–4–1–2 is also possible.

Alternative method: There is another possible order, if you notice that $y = 3(2x - 6)^2 + 1$ is of the form $y = kf(ax + b) + c$, where the $(ax + b)$ is not factorised, so we can do the horizontal translation first, then horizontal dilation.

The horizontal translation is 6 units right $(b = -6)$ followed by a horizontal dilation of scale factor $\frac{1}{2}$, then 3 and 4 as above.

c We require $y = kf(a(x + b)) + c$.

Horizontal transformations: dilation $a = -1$ and translation $b = -4$.

Vertical transformations: dilation $k = 5$ and translation $c = -9$.

Horizontal transformations: $y = kf(-1(x - 4)) + c$

Add vertical transformations: $y = 5f(-(x - 4)) - 9$

This answer can also be written as $y = 5f(-x + 4) - 9$ or $y = 5f(4 - x) - 9$.

Domain and range

We can find the domain and range of functions without drawing their graphs.

Effect of transformations on domain and range

Horizontal transformations change x values so affect the domain.

Vertical transformations change y values so affect the range.

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EXAMPLE 15

Find the domain and range of:

a $f(x) = -3(x-2)^2 + 5$

b $y = 5\sqrt{2x+1}$

Solution

a $y = x^2$ has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

No horizontal dilation.

Horizontal translation 2 units right: domain of $x - 2$ is $(-\infty, \infty)$ so domain of $f(x)$ is unchanged.

Vertical transformations affect the range:

Vertical dilation, scale factor -3: Range of y is $[0, \infty)$, so range of $3y$ is 3 times as much, so no change for $[0, \infty)$.

But the $-$ sign in -3 means the y is reflected in the x-axis, so range of $-3y$ is $(-\infty, 0]$.

Vertical translation 5 units up: Range of $-3y$ is $(-\infty, 0]$ so range of $-3y + 5$ is $(-\infty, 5]$.

So $y = -3(x-2)^2 + 5$ has domain $(-\infty, \infty)$ and range $(-\infty, 5]$.

b $y = \sqrt{x}$ has domain $[0, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

Domain of $2x + 1$ is $[0, \infty)$ so $2x + 1 \geq 0$.

$$2x \geq -1$$

$$x \geq -\frac{1}{2}$$

Vertical transformations affect the range:

Vertical dilation, scale factor 5: Range of $\sqrt{2x+1}$ is $[0, \infty)$, so range of $5\sqrt{2x+1}$ is 5 times as much, so unchanged.

No vertical translation.

So $y = 5\sqrt{2x+1}$ has domain $\left[-\frac{1}{2}, \infty\right)$ and range $[0, \infty)$.

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Exercise 2.05 Combinations of transformations

1 The point $(2, -6)$ lies on the function $y = f(x)$. Find the coordinates of its image if the function is:

a horizontally translated 3 units to the right and vertically translated 5 units down

b translated 4 units up and 3 units to the left

c translated 7 units to the right and 9 units up

d translated 11 units down and 4 units to the left

2 Find the equation of the transformed function where $f(x) = x^5$ is reflected:

a in the x-axis and vertically dilated with scale factor 4

b in the y-axis and horizontally dilated with scale factor 3

3 Find the equation of each transformed function.

a $y = x^3$ is translated 3 units down and 4 units to the left

b $f(x) = |x|$ is translated 9 units up and 1 unit to the right

c $f(x) = x$ is dilated vertically with a scale factor of 3 and translated down 6 units

d $y = e^x$ is reflected in the x-axis and translated up 2 units

e $y = x^3$ is horizontally dilated by a scale factor of $\frac{1}{2}$ and translated down 5 units

f $f(x) = \frac{1}{x}$ is vertically dilated by a factor of 2 and horizontally dilated by a factor of 3

g $f(x) = \sqrt{x}$ is reflected in the y-axis, vertically dilated by a scale factor of 3 and horizontally dilated by a scale factor of $\frac{1}{2}$

h $y = \ln x$ is horizontally dilated by a scale factor of 3 and translated upwards by 2 units

i $f(x) = \log_2 x$ is horizontally dilated by a scale factor of $\frac{1}{4}$ and vertically dilated by a scale factor of 3

j $y = x^2$ is horizontally dilated by a scale factor of 2 and translated down 3 units

4 Describe the transformations to $y = x^3$ in the correct order if the transformed function has equation:

a $y = (x - 1)^3 + 7$

b $y = 4x^3 - 1$

c $y = -5x^3 - 3$

d $y = 2(x + 7)^3$

e $y = 6(2x - 4)^3 + 5$

f $y = 2(3x + 9)^3 - 10$

5 Describe the transformations in their correct order for each of the functions from:

a $y = \log x$ to $y = 2\log(x + 3) - 1$

b $f(x) = x^2$ to $f(x) = -(3x)^2 + 9$

c $y = e^x$ to $y = 2e^{5x} - 3$

d $f(x) = \sqrt{x}$ to $f(x) = 4\sqrt{x - 7} + 1$

e $y = |x|$ to $y = |-2(x + 1)| - 1$

f $y = \frac{1}{x}$ to $y = -\frac{1}{2x} + 8$

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6 The point $(8, -12)$ lies on the function $y = f(x)$. Find the coordinates of the image point when the function is transformed into:

a $y = 3f(x-1)+5$ b $y = -f(2x)-7$ c $y = 2f(x+3)-1$

d $y = 6f(-x)+5$ e $y = -2f(2x-4)-3$

7 Given the function $y = f(x)$, find the coordinates of the image of (x, y) if the function is:

a translated 6 units down and 3 units to the right

b reflected in the y-axis and translated 6 units up

c vertically dilated with scale factor 2 and translated 5 units to the left

d horizontally dilated with scale factor 3 and translated 5 units up

e reflected in the x-axis, vertically dilated with scale factor 8, translated 6 units to the left, horizontally dilated with scale factor 5 and translated 1 unit down

8 Find the equation of the transformed function if $y = f(x)$ is:

a translated 2 units down and 1 unit to the left

b translated 5 units to the right and 3 units up

c reflected in the x-axis and translated 4 units to the right

d reflected in the y-axis and translated up 2 units

e reflected in the x-axis and horizontally dilated with a factor of 4

f vertically dilated by a scale factor of 2 and translated 2 units down

9 Find the equation of the transformed function using the correct order of transformations for $y = kf(a(x+b))+c$.

a $f(x) = \frac{1}{x}$ is reflected in the y-axis, translated up 3 units and dilated vertically by a scale factor of 9

b $y = x^2$ is translated down by 6 units and by 2 units to the left and is horizontally dilated with scale factor $\frac{1}{5}$

c $f(x) = \ln x$ has a vertical dilation with factor 8, a vertical translation of 3 down, a horizontal dilation with factor 2 and a horizontal translation of 5 to the right

d $y = \sqrt{x}$ has a vertical translation of 4 up, a horizontal translation of 4 to the left, a reflection in the y-axis and a vertical dilation with factor 9

e $f(x) = |x|$ is translated up by 7 units, dilated horizontally by a factor of $\frac{1}{6}$ and reflected in the x-axis

f $y = x^3$ is translated 4 units to the left then dilated horizontally with scale factor $\frac{1}{4}$

g $y = 2^x$ is translated up by 5 units, translated 2 units to the right, then is vertically dilated with scale factor 6

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10 Find the domain and range of each function.

a $f(x) = (x + 3)^2 + 5$ b $y = 5|-2x| - 2$ c $f(x) = \frac{1}{2x - 4} + 1$

d $y = 4^{3x} + 2$ e $f(x) = 3 \log (3x - 6) - 5$

11 a By completing the square, write the equation for the parabola $y = x^2 + 2x - 7$ in the form $y = (x + a)^2 + b$.

b Describe the transformations on $y = x^2$ that result in the function $y = x^2 + 2x - 7$.

12 Describe the transformations that change $y = x^2$ into the function $y = x^2 - 10x - 3$.

13 The function $y = f(x)$ is transformed to the function $y = kf(a(x + b)) + c$.
Find the coordinates of the image point of (x, y) when:

a $c = 5, b = -3, k = 2$ and $a = \frac{1}{2}$

b $c = -2, b = 6, k = -1$ and $a = 3$

14 a Find the equation of the transformed graph if $x^2 + y^2 = 9$ is translated 3 units to the right and 4 units up.

b The circle $x^2 + y^2 = 1$ is transformed into the circle $x^2 - 4x + y^2 + 6y + 12 = 0$. Describe how the circle is transformed.

WS

Graphing transformed functions

2.06 Graphs of functions with combined transformations

We can find points and sketch the graphs of functions that are changed by a combination of transformations. Translations are the easiest transformations to use since they shift the graph while keeping it the same size and shape.

EXAMPLE 16

Sketch the graph of $y = (x - 2)^2 - 5$.

Solution

$y = (x - 2)^2 - 5$ is transformed from $y = x^2$ by a horizontal translation of 2 units to the right and a vertical translation of 5 units down.

The vertex (turning point) of parabola $y = x^2$ is $(0, 0)$.

So the vertex of $y = (x - 2)^2 - 5$ is $(0 + 2, 0 - 5) \equiv (2, -5)$.

Sketching the graph, we keep the shape of $y = x^2$ and shift it to the new vertex.

We can find the intercepts for a more accurate graph.

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For x-intercepts, $y = 0$:

$$0 = (x - 2)^2 - 5$$
$$5 = (x - 2)^2$$
$$\pm\sqrt{5} = x - 2$$
$$2 \pm \sqrt{5} = x$$

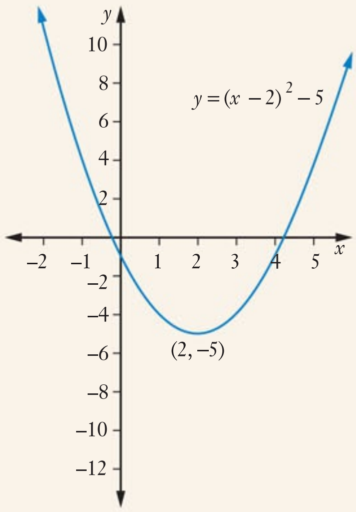
For y-intercepts, $x = 0$:

$$y = (0 - 2)^2 - 5$$
$$= 4 - 5$$
$$= -1$$

So the x-intercepts are approximately 4.2, – 0.2.

To find other points on the graph, you can transform points on $y = x^2$ the same way as for the vertex.

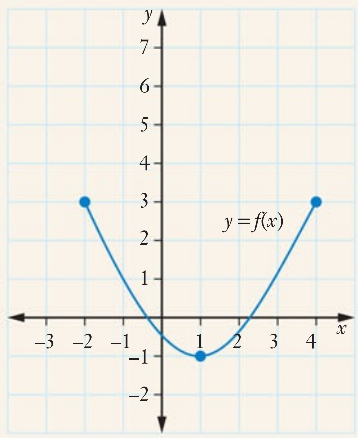
Sketch the graph using a scale on each axis that will show the information. For example, the vertex is at $(2, -5)$ so the y values must go down as far as $y = -5$.



EXAMPLE 17

The graph $y = f(x)$ shown is reflected in the y-axis, dilated vertically with a scale factor of 2 and translated 1 unit up.

Sketch the graph of the transformed function.



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Solution

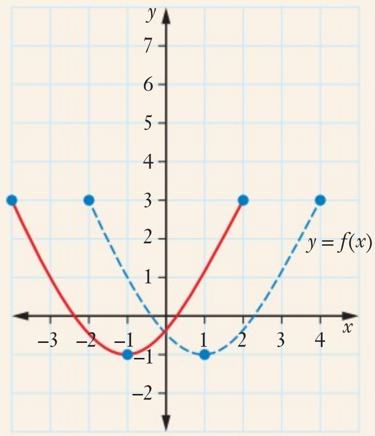
A reflection in the y-axis is a horizontal dilation with scale factor -1.

Multiply each x value by -1.

$x = 1$ becomes $x = -1$

$x = 4$ becomes $x = -4$

$x = -2$ becomes $x = 2$

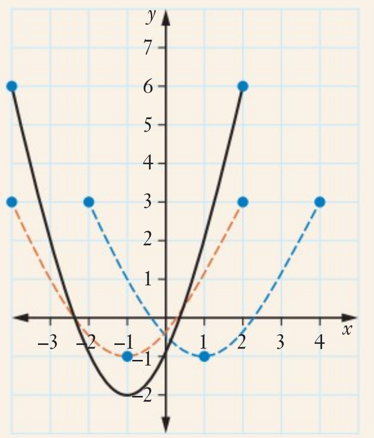


For a vertical dilation with scale factor 2:

Multiply each y value by 2.

$y = -1$ becomes $y = -2$

$y = 3$ becomes $y = 6$

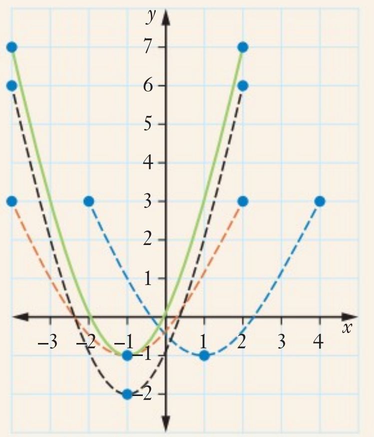


For a vertical translation 1 unit up:

Add 1 to y values.

$y = 6$ becomes $y = 7$

$y = -2$ becomes $y = -1$



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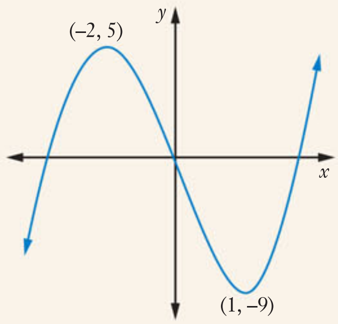
In the previous example, we took one transformation at a time. In the next example, we take transformations together (in the correct order) and plot images of key points on the original (parent) curve.

EXAMPLE 18

a The function $y = f(x)$ is sketched below with stationary (turning) points as shown.



i Describe the transformations if $y = f(x)$ is transformed to $y = 3f(x + 1) - 2$ and how they change the coordinates (x, y) of the parent function.

ii Find the coordinates of the image of each stationary point when the function is transformed.

iii Sketch the graph of $y = 3f(x + 1) - 2$.

b i Describe the transformations if $y = |x|$ is transformed to $y = -\left|\frac{x}{2}\right| + 3$ and the image of point (x, y) on the parent function.

ii Sketch the transformed function.

Solution

a i Transformations (in order) are:

A horizontal translation 1 unit to the left:

So (x, y) becomes $(x - 1, y)$.

A vertical dilation, scale factor 3:

So $(x - 1, y)$ becomes $(x - 1, 3y)$.

A vertical translation 2 units down:

So $(x - 1, 3y)$ becomes $(x - 1, 3y - 2)$.

ii For $(-2, 5)$:

Image becomes $(-2 - 1, 3 \times 5 - 2) \equiv (-3, 13)$.

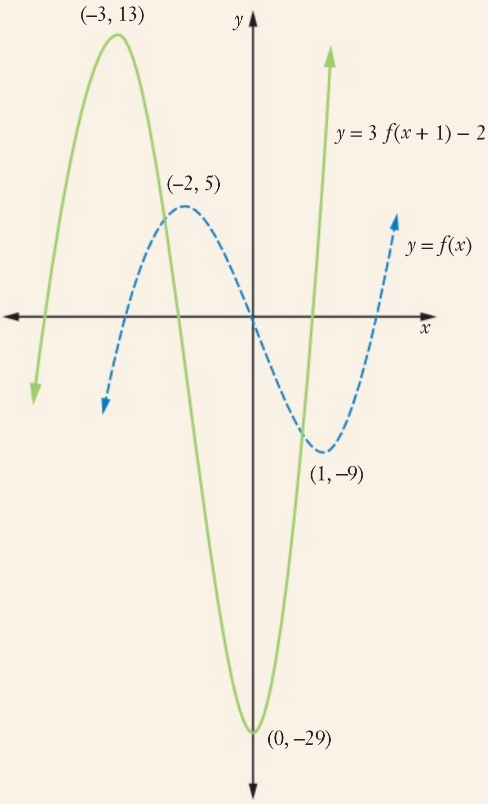
For $(1, -9)$:

Image becomes $(1 - 1, 3 \times [-9] - 2) \equiv (0, -29)$.

iii $y = f(x)$ passes through $(0, 0)$.

Image becomes $(0 - 1, 3 \times 0 - 2)$
$\equiv (-1, -2)$

Sketch the graph showing this information using a suitable scale on each axis. For example, the y values must go up to 13 and down to -29.



b i Transformations (in order) are:

A horizontal dilation, scale factor 2:

So (x, y) becomes $(2x, y)$.

A vertical dilation, scale factor -1 (reflection in the x-axis):

So $(2x, y)$ becomes $(2x, -y)$.

A vertical translation 3 units up:

So $(2x, -y)$ becomes $(2x, -y + 3)$.

ii The intercepts of $y = |x|$ are at $(0, 0)$.

Image of $(0, 0)$ is $(2 \times 0, -0 + 3) \equiv (0, 3)$.

We can find the intercepts on $y = -\left|\frac{x}{2}\right| + 3$

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For x-intercept, $y = 0$:

$$0 = -\left|\frac{x}{2}\right| + 3$$

$$-3 = -\left|\frac{x}{2}\right|$$

$$3 = \left|\frac{x}{2}\right|$$

$$\pm 3 = \frac{x}{2}$$

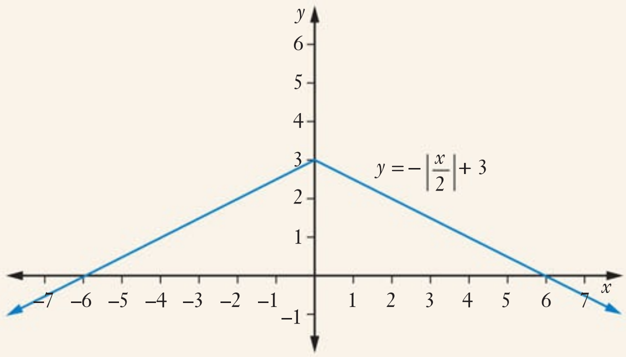
$$\pm 6 = x$$

For y-intercept, $x = 0$:

$$y = -\left|\frac{0}{2}\right| + 3$$

$$= 3$$

Sketching this information using an appropriate scale gives the graph.



Exercise 2.06 Graphs of functions with combined transformations

1 Given $f(x) = x^2$, sketch the graph of:

a $f(x) = x^2 + c$ when

i $c > 0$ ii $c < 0$

b $f(x) = (x + b)^2$ when

i $b > 0$ ii $b < 0$

c $f(x) = kx^2$ when

i $k > 1$ ii $0 < k < 1$ iii $k = -1$

d $f(x) = (ax)^2$ when

i $a > 1$ ii $0 < a < 1$ iii $a = -1$

2 Sketch the graph of the transformed function if the parabola $y = x^2$ is transformed into:

a $y = (x + 2)^2 + 4$ b $y = (x - 3)^2 - 1$ c $y = (x - 1)^2 + 3$

d $y = -(x + 1)^2 - 2$ e $y = 2(x - 1)^2 - 4$

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3 Sketch the graph of the transformed function if the cubic function $y = x^3$ is transformed into:

a $y = (x - 1)^3 + 2$ b $y = (x - 2)^3 - 3$ c $y = -(x + 1)^3 + 4$

d $y = 2(x + 3)^3 - 5$ e $y = 3(x - 1)^3 - 2$

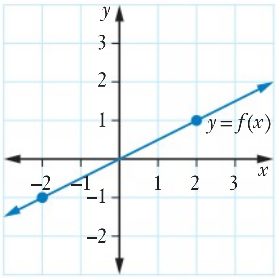
4 A cubic function has stationary points at (6, 1) and (–3, –2).

a Find the images of these points if the function is transformed to $y = -2f(3x) + 1$.

b Sketch the graph of the transformed function.

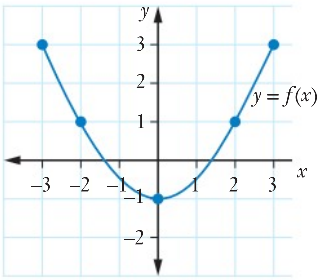
5 Given each function $y = f(x)$, sketch the graph of the transformed function.

a



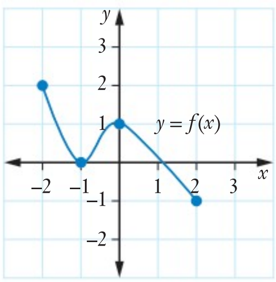
i $y = 3f(x - 1)$ ii $y = -f(2x) + 3$

b



i $y = 3f(x + 3) - 2$ ii $y = -2f\left(\frac{x}{4}\right) + 3$

c



i $y = 2f(-x) - 1$ ii $y = -3f(2x + 4) + 2$

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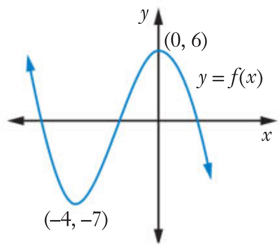
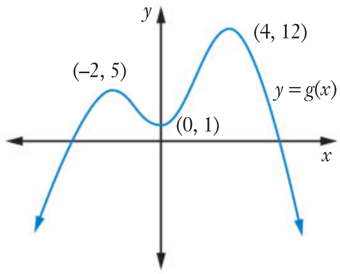
6 For the function $y = f(x)$ with turning points as shown, sketch the transformed function if it is vertically dilated with scale factor 3, translated 4 units down, and horizontally translated 2 units to the left.



7 For the function $y = g(x)$ with turning points as shown, sketch the graph of the transformed function $y = -g[2(x - 1)] - 5$.



8 Sketch the graph of:

a $y = -3(x - 2)^3 + 1$

b $y = 2e^{x + 1} - 4$

c $f(x) = 3\sqrt{x - 2} - 1$

d $y = 2|3x| + 4$

e $y = -(3x)^2 + 1$

9 Sketch the graph of:

a $y = 3 - 2\ln x$

b $f(x) = -2e^x + 1$

c $y = 1 - (x + 1)^3$

d $y = \dfrac{2}{x - 1} + 3$

e $y = -2(x - 3) + 1$

10 a The coordinates of the image of (x, y) when $y = f(x)$ is transformed to $y = 3f(x - 2) + 1$ are $(-3, 2)$. Find the original point (x, y).

b Sketch the graph of the original function $y = f(x)$ if $y = 3f(x - 2) + 1$ is a cubic function with turning points $(-3, 2)$ and $(2, -4)$.

11 The coordinates of the image point of the vertex (x, y) of a parabola are $(-24, 18)$ when $y = f(x)$ is transformed as shown below. Find the coordinates of the original point (x, y) and sketch the graph of the original quadratic function.

a $y = 3f(x - 2) - 5$

b $y = -5f[3(x + 1)]$

c $y = 2f(2x - 6) - 3$

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Solving inequalities graphically

2.07 Equations and inequalities

We can use the graphs of transformed functions to solve equations.

EXAMPLE 19

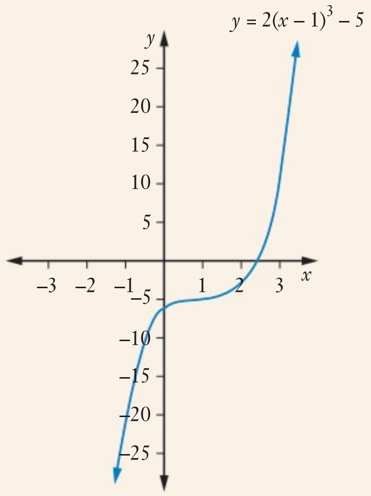
The graph of the cubic function $y = 2(x - 1)^3 - 5$ is shown.

a Solve graphically:

i $2(x - 1)^3 - 5 = 0$

ii $2(x - 1)^3 - 5 = 10$

b Solve each of the equations in part a algebraically.



Solution

a i The solution of $2(x - 1)^3 - 5 = 0$ is where $y = 0$ (x-intercepts).

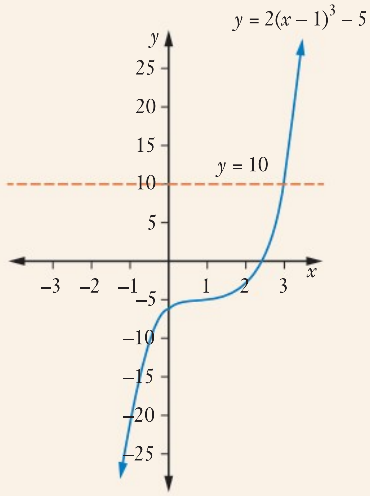
From the graph, the x-intercept is 2.4.

The solution is $x = 2.4$.

ii Draw the line $y = 10$ on the graph.

The solution of $2(x - 1)^3 - 5 = 10$ is where the line intersects the graph.

The solution is $x = 2.9$.



b i
$$\begin{aligned} 2(x - 1)^3 - 5 &= 0 \\ 2(x - 1)^3 &= 5 \\ (x - 1)^3 &= 2.5 \\ x - 1 &= \sqrt[3]{2.5} \\ x &= \sqrt[3]{2.5} + 1 \\ &= 2.36 \end{aligned}$$

ii
$$\begin{aligned} 2(x - 1)^3 - 5 &= 10 \\ 2(x - 1)^3 &= 15 \\ (x - 1)^3 &= 7.5 \\ x - 1 &= \sqrt[3]{7.5} \\ x &= \sqrt[3]{7.5} + 1 \\ &= 2.96 \end{aligned}$$

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We can use transformed functions to find solutions to practical questions.

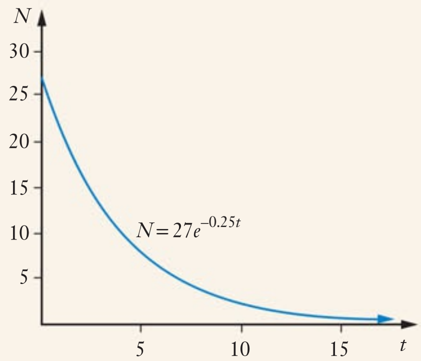
EXAMPLE 20

The graph of $N = 27e^{-0.25t}$ shows the number N of cases of measles over t weeks in a country region.

a Use the graph to find the solution to $27e^{-0.25t} = 10$.

b State the meaning of this solution.

c Solve the equation algebraically.

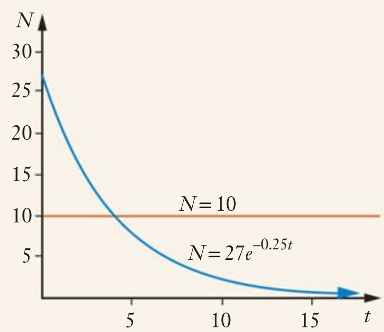


Solution

a Draw the line N = 10 on the graph.

The solution will be where the line intersects the graph.

The solution is t = 4.



b This solution means that after 4 weeks there will be 10 cases of measles.

c
$$27e^{-0.25t} = 10$$
$$e^{-0.25t} = \frac{10}{27}$$
$$\ln\ e^{-0.25t} = \ln\frac{10}{27}$$
$$-0.25t = -0.99325...$$
$$t = \frac{-0.99325...}{-0.25}$$
$$= 3.97300...$$
$$\approx 3.97$$

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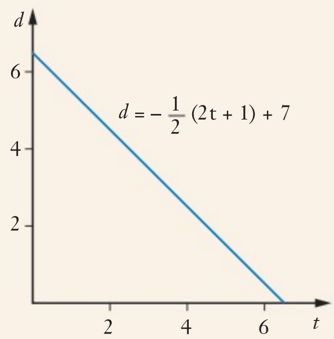
We can solve inequalities graphically.

EXAMPLE 21

a The graph is of the function $d = -\frac{1}{2}(2t + 1) + 7$ where d is the distance (in cm) of a marble at t seconds as it rolls towards a barrier.



Solve graphically and explain the solutions:

i $-\frac{1}{2}(2t + 1) + 7 = 4$

ii $-\frac{1}{2}(2t + 1) + 7 \geq 4$

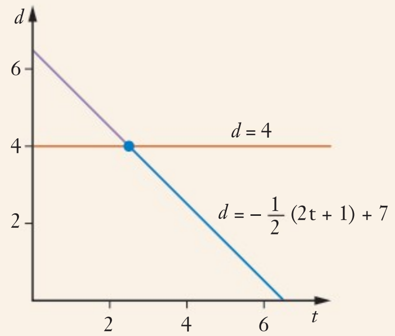
b Sketch the graph of $y = 2(x + 3)^2 - 5$ and solve graphically:

i $2(x + 3)^2 - 5 = 3$

ii $2(x + 3)^2 - 5 < 3$

Solution

a Draw the line $d = 4$ across the graph.



i From the graph, the solution of $-\frac{1}{2}(2t + 1) + 7 = 4$ is $x = 2.5$.

This means that at 2.5 seconds, the marble is 4 cm from the barrier.

ii The solution of $-\frac{1}{2}(2t + 1) + 7 \geq 4$ is all the t values on and above the line $d = 4$, shown in purple.

For this part of the graph, $t \leq 2.5$.

Because $t \geq 0$ (time is never negative), $0 \leq t \leq 2.5$ is the solution.

This means that for the first 2.5 seconds the marble is 4 cm or more from the barrier.

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b The function $y = 2(x + 3)^2 - 5$ is a transformation of $y = x^2$.

The vertex of $y = x^2$ is (0, 0).

The image of (0, 0) is $(0 - 3, 0 \times 2 - 5) \equiv (-3, -5)$

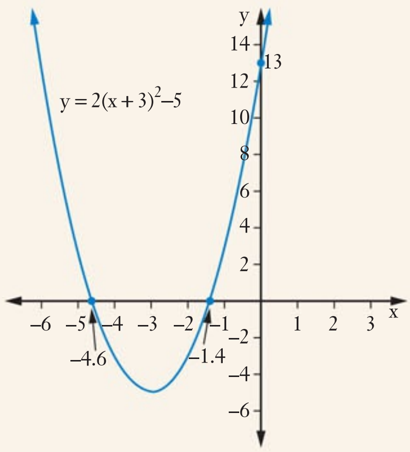
For x-intercept, $y = 0$:

$$\begin{aligned} 0 &= 2(x + 3)^2 - 5 \\ 5 &= 2(x + 3)^2 \\ 2.5 &= (x + 3)^2 \\ \pm\sqrt{2.5} &= x + 3 \\ \pm\sqrt{2.5} - 3 &= x \\ -1.4, -4.6 &= x \end{aligned}$$

For y-intercept, $x = 0$:

$$\begin{aligned} y &= 2(0 + 3)^2 - 5 \\ &= 2(9) - 5 \\ &= 13 \end{aligned}$$

Sketch the graph using a suitable scale on the axes.

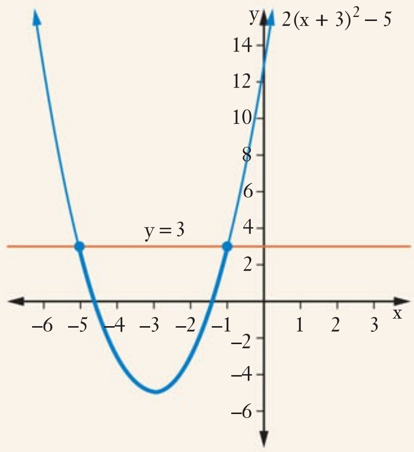


i Draw the line $y = 3$.

From the graph, the solution of $2(x + 3)^2 - 5 = 3$ is $x = -5, -1$.

ii The solution of $2(x + 3)^2 - 5 < 3$ is all x values below the line $y = 3$.

From the graph, the solution of $2(x + 3)^2 - 5 < 3$ is $-5 < x < -1$.



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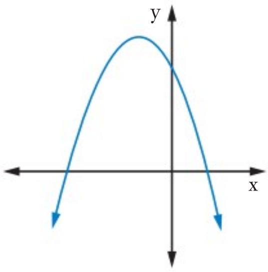
Exercise 2.07 Equations and inequalities

1 For each function $y = f(x)$, state how many solutions there are for the equation $f(x) = 0$.

a



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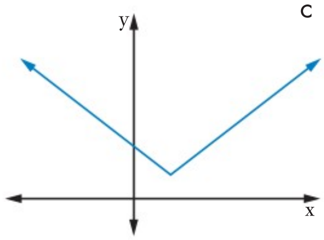

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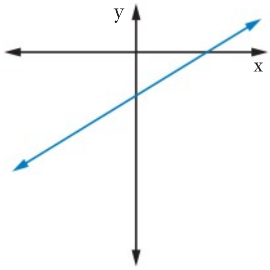

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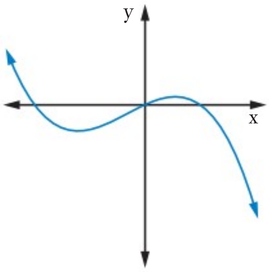

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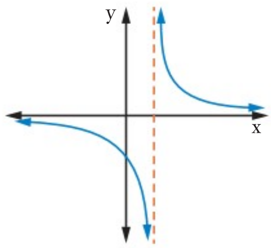

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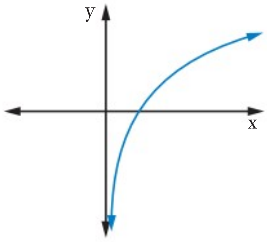

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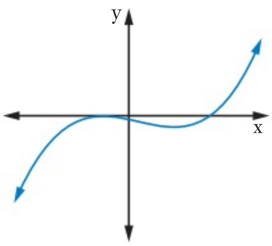

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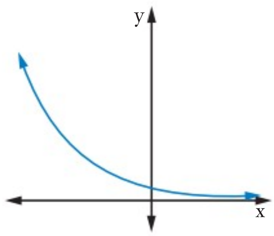

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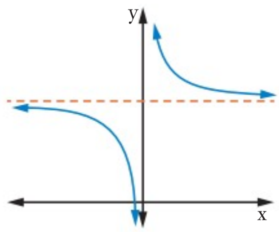

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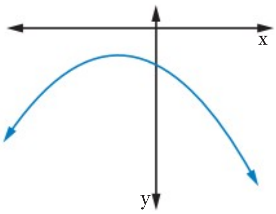

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2 The graph of the quadratic function $f(x) = -2(x + 1)^2 + 3$ is shown.

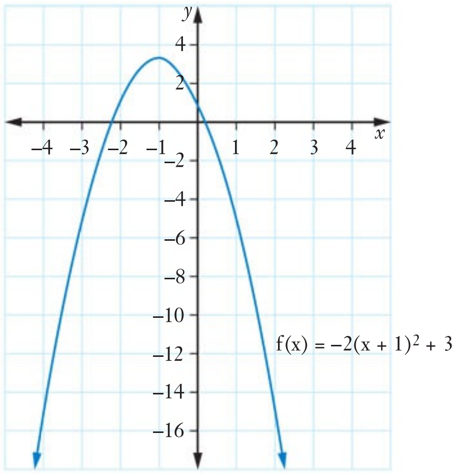
a Solve graphically:

i $-2(x + 1)^2 + 3 = 1$

ii $-2(x + 1)^2 + 3 = -2$

iii $-2(x + 1)^2 + 3 = 0$

b Solve $-2(x + 1)^2 + 3 = 0$ algebraically.



3 The graph of the linear function $f(x) = 3(4x - 5) - 2$ is shown.
Use the graph to solve:

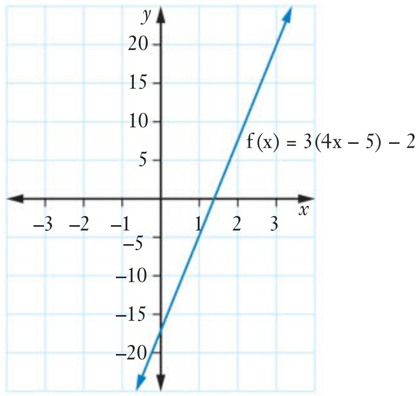
a $3(4x - 5) - 2 = 0$

b $3(4x - 5) - 2 = 5$

c $3(4x - 5) - 2 = -15$

d $3(4x - 5) - 2 > 10$

e $3(4x - 5) - 2 \leq 20$



4 a Sketch the graph of the cubic function $y = -(x + 3)^3 + 1$.

b Solve graphically:

i $-(x + 3)^3 + 1 = 0$

ii $-(x + 3)^3 + 1 = -10$

iii $-(x + 3)^3 + 1 = -20$

c Solve $-(x + 3)^3 + 1 = 0$ algebraically.

5 a Sketch the graph of $y = 3|x - 2| + 4$.

b How many solutions does the equation $3|x - 2| + 4 = 1$ have?

c Solve $3|x - 2| + 4 = 10$ graphically and check your solutions algebraically.

6 a Sketch the graph of the function $f(x) = \frac{2}{x - 3} - 4$, showing asymptotes.

b Solve the equation $\frac{2}{x - 3} - 4 = -5$.

c Solve $\frac{2}{x - 3} - 4 = -2$.

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7 The formula for the area of a garden with side x metres is given by $A = -3(x - 2)^2 + 18$.

a Draw the graph of the area of the garden.

b From the graph, solve the equation $-3(x - 2)^2 + 18 = 10$.

8 A factory has costs according to the formula $C = 2(x + 1)^2 + 3$, where C stands for costs in \$1000s and x is the number of products made.

a Draw the graph of the costs.

b Find the factory overhead (cost when no products are made).

c Solve $2(x + 1)^2 + 3 = 20$ from the graph and explain your answer.

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9 Loudness in decibels (dB) is given by $dB = 10\log\left(\frac{x}{I}\right)$ where I is a constant.

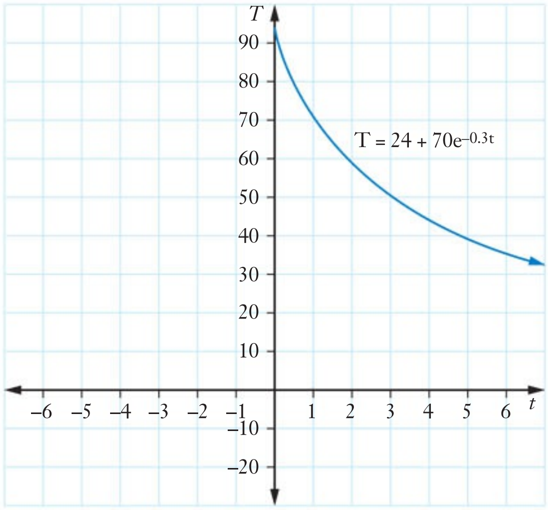
a Sketch the graph of the function given $I = 2$.

b From the graph solve the equation:

i $10\log\left(\frac{x}{I}\right) = 5$

ii $10\log\left(\frac{x}{I}\right) = 2$

10 According to Newton's law of cooling, the temperature T of an object as it cools over time t minutes is given by the formula $T = A + Be^{-kt}$. The graph shown is for the formula $T = 24 + 70e^{-0.3t}$ for a metal ball that has been heated and is now cooling down.



a From the graph, solve these equations and explain what the solutions mean.

i $24 + 70e^{-0.3t} = 50$ ii $24 + 70e^{-0.3t} = 30$

b Solve these equations algebraically:

i $24 + 70e^{-0.3t} = 80$ ii $24 + 70e^{-0.3t} = 26$

c What temperature will the object approach as t becomes large?

Can you give a reason for this?

11 a Sketch the graph of $y = (x - 1)^2 - 2$.

b From the graph, solve:

i $(x - 1)^2 - 2 = 2$ ii $(x - 1)^2 - 2 \geq 2$ iii $(x - 1)^2 - 2 < 2$

12 a Sketch the graph of $f(x) = -(2x + 4)^2 + 1$.

b From the graph, solve:

i $-(2x + 4)^2 + 1 = -3$ ii $-(2x + 4)^2 + 1 > -3$ iii $-(2x + 4)^2 + 1 \leq -3$

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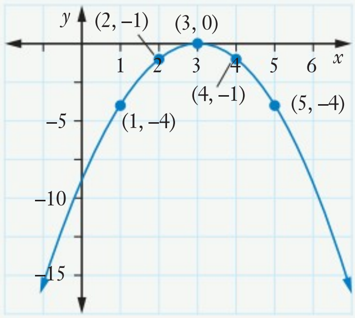
2. TEST YOURSELF

Qz
Practice quiz

For Questions 1 to 3, choose the correct answer A, B, C or D.

1 The function $y = f(x)$ transformed to $y = f(x - 8)$ is:
- A a vertical translation 8 units up
- B a horizontal translation 8 units to the right
- C a vertical translation 8 units down
- D a horizontal translation 8 units to the left

2 The graph below is a transformation of $y = x^2$.
Find its equation.
- A $y = (-x + 3)^2$
- B $y = (-x - 3)^2$
- C $y = -(x + 3)^2$
- D $y = -(x - 3)^2$



3 Find the coordinates of the image of (x, y) when the function $y = f(x)$ is transformed to $y = -2f(x + 1) + 4$.
- A $(x + 1, -2y - 4)$
- B $(x + 1, -2y + 4)$
- C $(x - 1, -2y + 4)$
- D $(-x + 1, 2y + 4)$

4 a Draw the graph of $y = e^{x-1} - 2$.
b Use the graph to solve $e^{x-1} - 2 = 8$.
c Solve $e^{x-1} - 2 = 20$ algebraically.

5 The point (24, 36) lies on the graph of $y = f(x)$. Find the coordinates of its image point if the function is transformed to:
- a $y = 3f(4x) - 1$
- b $y = f[3(x + 2)] + 4$
- c $y = 5f(-x) - 3$
- d $y = -2f(x + 7) - 3$
- e $y = -f(2x - 8) + 5$

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6 Find the equation of each transformed function.

a $y = x^3$ is translated:

i 3 units up   ii 7 units to the left

b $y = |x|$ is dilated:

i vertically with scale factor 3

ii horizontally with scale factor 2

c $f(x) = \ln x$ is dilated vertically with factor 5 and reflected in the y-axis.

d $f(x) = \frac{1}{x}$ is reflected in the x-axis and translated 4 units to the right.

e $f(x) = 3^x$ is dilated vertically with scale factor 9, dilated horizontally with scale factor $\frac{1}{3}$ and translated 6 units down and 2 units to the right.

7 a State the meaning of the constants a, b, c and k in the function $y = kf(a(x + b)) + c$ and the effect they have on the graph of the function $y = f(x)$.

b Describe the effect on the graph of the function if:

i $k = -1$ ii $a = -1$

8 Show that if $y = x^2$ is dilated vertically with scale factor 3, reflected in the x-axis and translated 1 unit up, the transformed function is even.

9 a Draw the graph of $y = 2(x - 3) + 5$.

b From the graph, solve:

i $2(x - 3) + 5 \leq 7$

ii $2(x - 3) + 5 > 9$

10 The population of a city over time t years is given by $P = 2e^{0.4(t+1)}$ where P is population in 10 000s.

a Sketch the graph of the population.

b Use the graph to solve $2e^{0.4(t+1)} = 5$, and explain the meaning of the solution.

11 Find the equation of the transformed function if $f(x) = x^4$ is horizontally translated 4 units to the left.

12 If (8, 2) lies on the graph of $y = f(x)$, find the coordinates of the image of this point when the function is transformed to $y = -4f[2(x + 1)] - 3$.

13 Solve graphically (and also algebraically for part a):

a $2(3x - 6)^2 - 5 = 9$ b $2(3x - 6)^2 - 5 > 9$ c $2(3x - 6)^2 - 5 \leq 9$

14 The function $y = f(x)$ is transformed to $y = -7f(x - 3) - 4$.

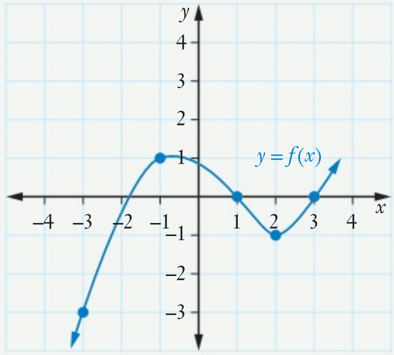
a Find the coordinates of the image of (x, y).

b If the image point is (–3, 3), find the value of x and y.

15 From the graph of $y = f(x)$ shown, draw the graph of:

a $y = 2f(x - 1)$

b $y = -f(x) - 2$



16 By drawing the graph of $y = 2(x + 1)^2 - 8$, solve:

a $2(x + 1)^2 - 8 \le 0$

b $2(x + 1)^2 - 8 > 0$

17 Sketch on the same set of axes:

a $y = x^2$ and $y = -4x^2 + 3$

b $y = |x|$ and $y = -|x - 1| + 2$

c $f(x) = e^x$ and $f(x) = \dfrac{e^{x+2}}{2} - 1$

d $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+2} + 1$

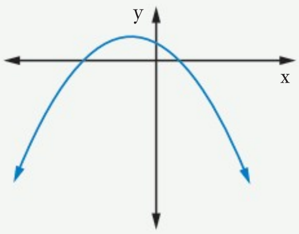
e $y = x^3$ and $y = 2(x - 3)^3 + 1$

f $f(x) = \ln x$ and $f(x) = \ln(-x) + 5$

g $y = \sqrt{x}$ and $y = 2\sqrt{x + 4} - 1$

18 Find the number of solutions of $f(x) = 0$ given the graph of each function $y = f(x)$.

a



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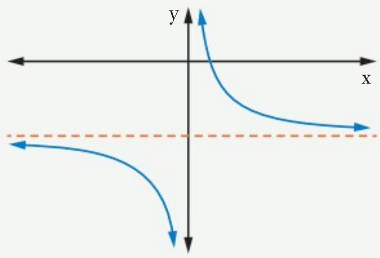

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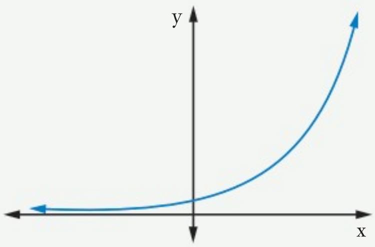

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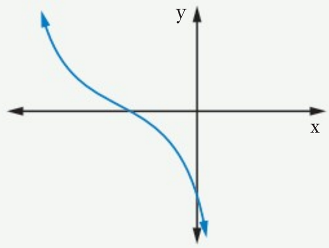

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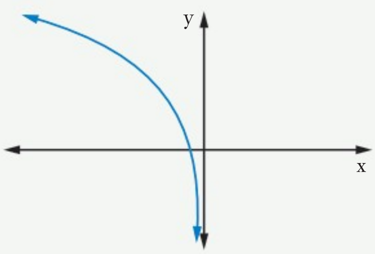

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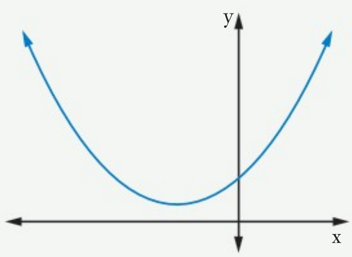

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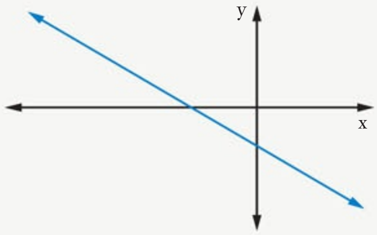

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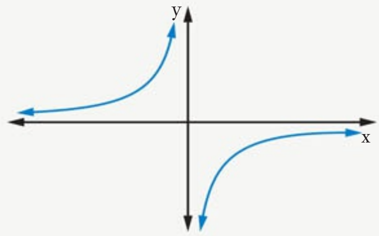

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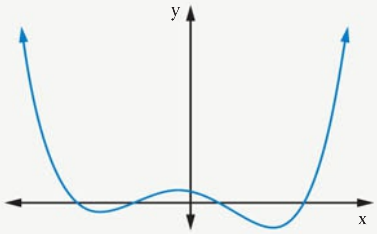

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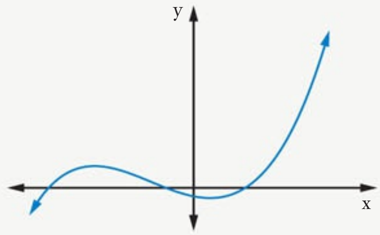

19 a Show that $x^2 + y^2 = r^2$ is not a function and describe its graph.

b Find 2 functions that together form $x^2 + y^2 = r^2$.

c By applying a vertical dilation with scale factor a to both these functions, what shape does the combination of these stretched functions make?

20 The point (x, y) lies on the function $y = f(x)$. The image of (x, y) is the point $(12, 6)$ when the function is transformed to $y = -6f(2x + 8)$. Find the coordinates of (x, y).

21 a Draw the graph of $y = (x - 2)^2 + 1$.

b From the graph, solve:

i $(x - 2)^2 + 1 = 10$ ii $(x - 2)^2 + 1 > 10$ iii $(x - 2)^2 + 1 \leq 10$

22 Point (x, y) lies on $y = f(x)$. Find the image of (x, y) if the function is transformed to:

a $y = 3f(x + 1) - 5$ b $y = -2f[2(x - 6)] + 4$

c $y = 5f(-x) - 3$ d $y = -3f(-3x + 9) - 1$

23 State whether the function $y = f(x)$ is stretched or compressed if it is dilated:

a vertically with scale factor 7

b horizontally with scale factor $\frac{1}{6}$

c horizontally with scale factor 3

d vertically with scale factor $\frac{1}{4}$

e horizontally with scale factor $\frac{7}{6}$

24 Find the domain and range of:

a $y = 3(x - 7)^2 - 10$ b $y = -|x + 1| + 2$ c $y = -\frac{2}{x - 3} - 5$

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2. CHALLENGE EXERCISE

1 A ball is thrown into the air from a height of 1 m, reaches its maximum height of 3 m after 1 second and after 2 seconds it is 1 m high.

a The path of the ball follows the shape of a parabola. Find the equation of the height h of the ball over time t seconds.

b After how long does the ball fall to the ground?

c Put the function in the form $h = kf[a(t + b)] + c$ and describe the transformations to change $h = t^2$ into this equation.

2 a If $(4, -3)$ lies on the function $y = f(x)$, find the coordinates of its image point.

i P on $y = 3f(x + 3) + 1$

ii Q on $y = -f(2x) - 3$

iii R on $y = f(2x - 2) + 1$

b Find the equation of the linear function passing through P that is perpendicular to QR.

c If $y = x$ is transformed into this linear function, describe the transformations.

3 a Show that $\dfrac{2x-7}{x-3} = -\dfrac{1}{x-3} + 2$.

b Sketch the graph of $y = \dfrac{2x-7}{x-3}$ and state its domain and range.

c Solve:

i $\dfrac{2x-7}{x-3} \geq 0$

ii $\dfrac{2x-7}{x-3} < 2$

4 a If $y = \dfrac{1}{x}$ is dilated horizontally with scale factor 2, explain why the equation of the transformed function is the same as if it was dilated vertically with scale factor 2.

b Is this the same result for the function $y = \dfrac{1}{x^2}$? Why?

5 a What is the equation of the axis of symmetry of the quadratic function $f(x) = ax^2 + bx + c$?

b What types of transformations on this function will change the axis of symmetry?

c Find the equation of the axis of symmetry of the quadratic function:

i $f(x) = 2(x + 1)^2 - 2$

ii $y = -(x - 3)^2 + 7$

iii $y = k(x + b)^2 + c$

iv $y = k(ax + b)^2 + c$

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6 The function $y = \sin x$ in the domain $[0, 2\pi]$ is transformed by a reflection in the x-axis, a vertical dilation scale factor 3, a horizontal dilation scale factor 2 and a vertical translation 1 unit down.

a Find the equation of the transformed function.

b State the amplitude, period and centre of the transformed function.

7 The circle $x^2 + 4x + y^2 - 6y + 12 = 0$ is transformed by a vertical translation 3 units down and a horizontal translation 5 units right. Find the equation of the transformed circle.

8 The function $y = 2^x$ is transformed to $y = 3(2^{-3x - 6}) - 5$. Describe the transformations applied to the function.

9 The polynomial $P(x) = x^3 - 3x - 2$ is translated up 2 units and then reflected in the y-axis. Find the equation of the transposed polynomial.

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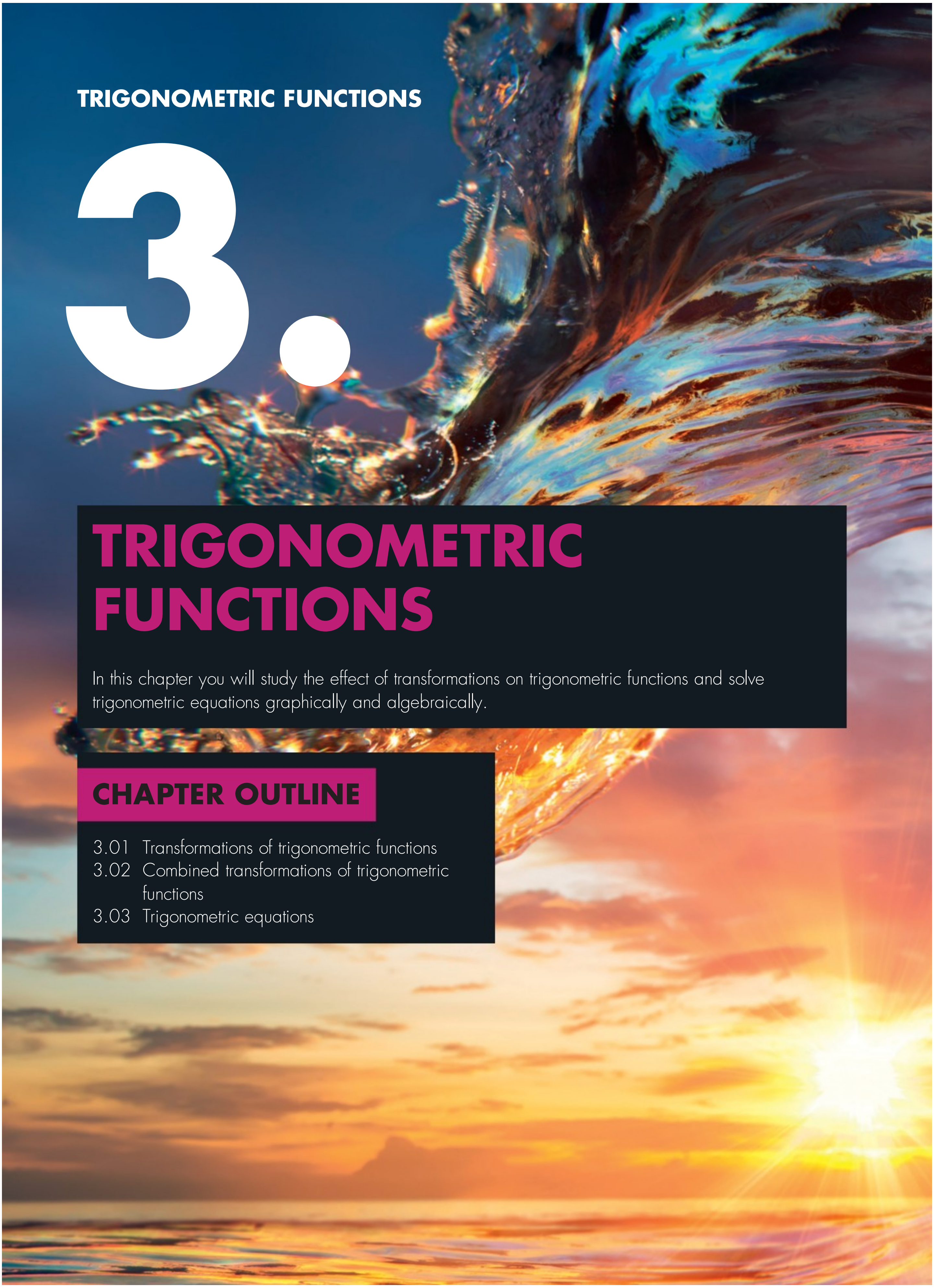


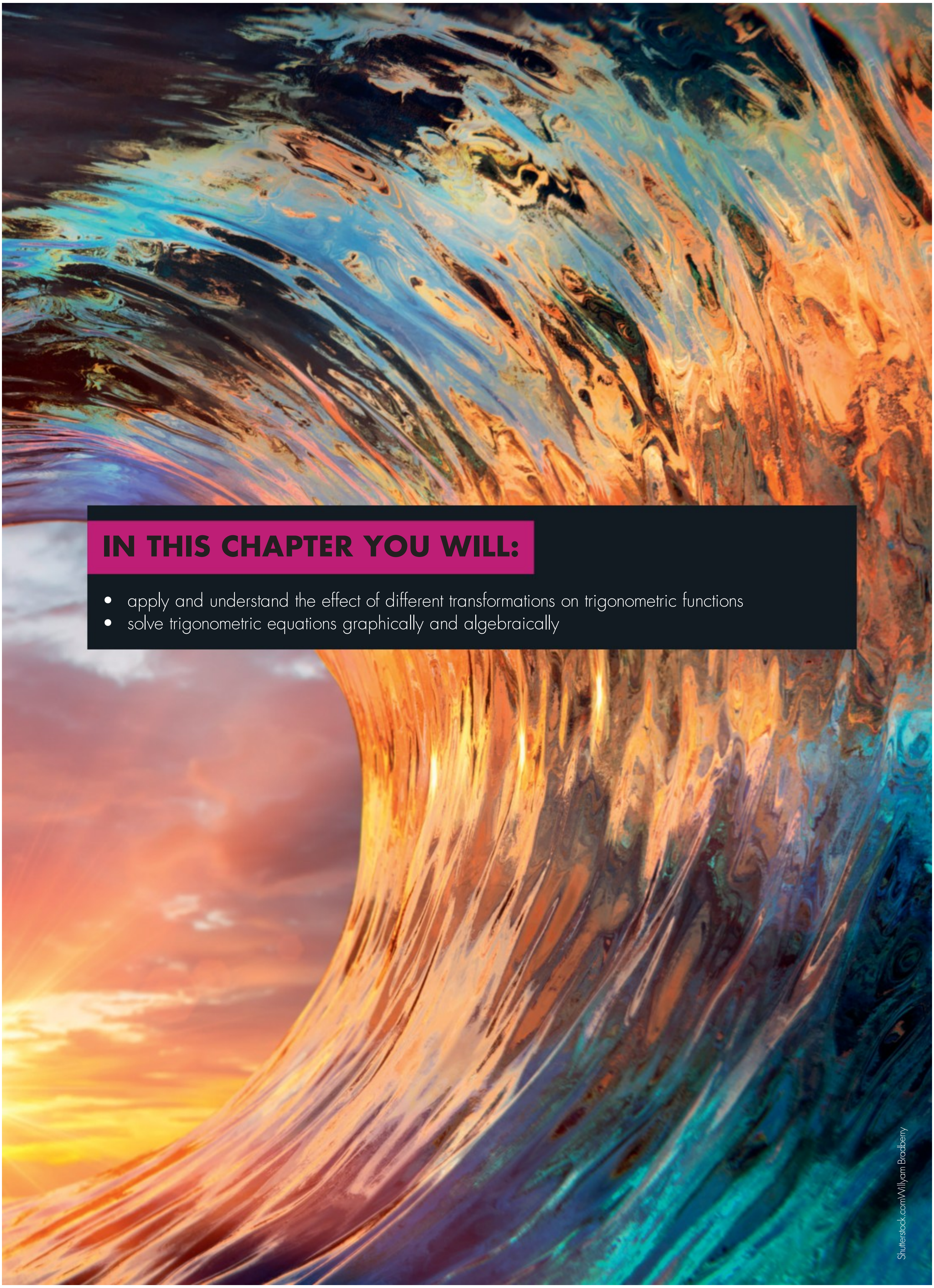
TRIGONOMETRIC FUNCTIONS

3.

TRIGONOMETRIC FUNCTIONS

In this chapter you will study the effect of transformations on trigonometric functions and solve trigonometric equations graphically and algebraically.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- apply and understand the effect of different transformations on trigonometric functions
- solve trigonometric equations graphically and algebraically

TERMINOLOGY

amplitude: The height from the centre of a sine or cosine function to the maximum or minimum values (peaks and troughs of its graph respectively). For $y = k \sin ax$ and $y = k \cos ax$, the amplitude is k.

centre: The mean value of a sine or cosine function that is equidistant from the maximum and minimum values. For $y = k \sin ax + c$ and $y = k \cos ax + c$, the centre is c.

period: The length of one cycle of a periodic function on the x-axis, before the function repeats itself. For $y = k \sin ax$ and $y = k \cos ax$, the period is $\frac{2\pi}{a}$.

phase: A horizontal shift (translation). For $y = k \sin [a(x + b)]$ and $y = k \cos [a(x + b)]$, the phase is b; that is, the graphs of $y = k \sin ax$ and $y = k \cos ax$ respectively are shifted b units to the left.

WS Transforming trigonometric functions

WS Sketching periodic functions – amplitude and period

WS Sketching periodic functions – phase and vertical shift

3.01 Transformations of trigonometric functions

The transformations you studied in Chapter 2, Transformations of functions, can be applied to the trigonometric functions.

Vertical dilations

A vertical dilation of $y = f(x)$ is $y = kf(x)$ with scale factor k.

EXAMPLE 1

a Describe the transformation if $y = \cos x$ is transformed to $y = 3 \cos x$.

b Sketch the graph of $y = 3 \cos x$ in the domain $[0, 2\pi]$ and state its range.

Solution

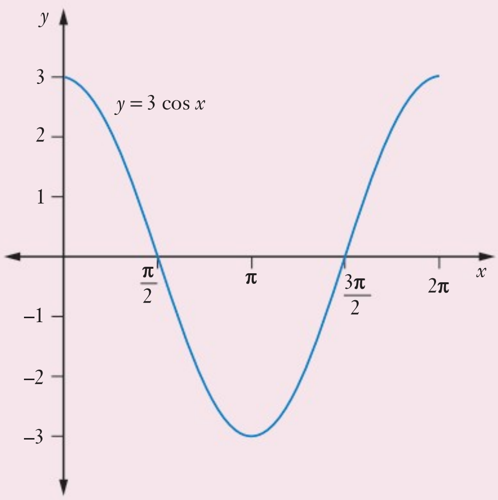
a $y = 3 \cos x$ is a vertical dilation of $y = \cos x$ with scale factor 3.

b A vertical dilation multiplies the y values by 3.

$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$(\pi, -1)$	becomes	$(\pi, -3)$
$\left(\frac{3\pi}{2}, 0\right)$	becomes	$\left(\frac{3\pi}{2}, 0\right)$
$(2\pi, 1)$	becomes	$(2\pi, 3)$

The range is $[-3, 3]$.

Notice that $y = 3 \cos x$ has amplitude 3.



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Amplitude as a vertical dilation

$y = k \sin x$ or $y = k \cos x$ has amplitude k (a vertical dilation with scale factor k).

- If $k > 1$, the function is stretched.
- If $0 < k < 1$, the function is compressed.
- If $k = -1$, the function is reflected in the x-axis.

Note: $y = k \tan x$ does not have an amplitude.

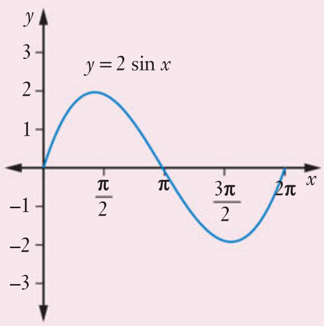
EXAMPLE 2

a Sketch the graph of $f(x) = 2 \sin x$ in the domain $[0, 2\pi]$.

b Find an equation for a cosine function reflected in the x-axis with amplitude 7.

Solution

a This is a vertical dilation of $f(x) = \sin x$ with scale factor 2 (it has amplitude 2).



b $y = \cos x$ is reflected in the x-axis with scale factor $k = -1$.

$y = -\cos x$

Amplitude: $k = 7$

So $y = -7 \cos x$

Horizontal dilations

A horizontal dilation of $y = f(x)$ is $y = f(ax)$ with scale factor $\frac{1}{a}$.

EXAMPLE 3

a Describe the transformation if $f(x) = \sin x$ is transformed to $y = \sin 2x$.

b Draw the graph of $f(x) = \sin 2x$ in the domain $[0, 2\pi]$.

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Solution

a $f(x) = \sin 2x$ is a horizontal dilation of $f(x) = \sin x$ with scale factor $\frac{1}{2}$.

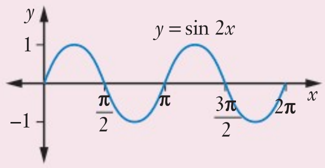
b A horizontal dilation multiplies the x values by $\frac{1}{2}$ (or divides them by 2).

$(0, 0)$	becomes	$(0, 0)$
$\left(\frac{\pi}{2}, 1\right)$	becomes	$\left(\frac{\pi}{4}, 1\right)$
$(\pi, 0)$	becomes	$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, -1\right)$	becomes	$\left(\frac{3\pi}{4}, -1\right)$
$(2\pi, 0)$	becomes	$(\pi, 0)$

Notice that these image points lie in the domain $[0, \pi]$ and not $[0, 2\pi]$.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$.

To sketch the function in the domain $[0, 2\pi]$ we repeat the sine curve from $x = \pi$ to 2π.



Notice that a horizontal dilation compresses the graph of $y = \sin x$, which changes its period. The function $y = \sin 2x$ has 2 complete sine function cycles in the domain $[0, 2\pi]$.

Period as a horizontal dilation

$y = \sin ax$ has period $\frac{2\pi}{a}$.

$y = \cos ax$ has period $\frac{2\pi}{a}$.

$y = \tan ax$ has period $\frac{\pi}{a}$.

- If $a > 1$, the function is compressed horizontally.
- If $0 < a < 1$, the function is stretched horizontally.
- If $a = -1$, the function is reflected in the y-axis.

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EXAMPLE 4

a Find the period of each function.

i $y = \cos x$ ii $f(x) = \sin 5x$ iii $y = \tan 2x$

b Sketch each graph in the domain $[0, 2\pi]$.

i $y = \tan \frac{x}{2}$ ii $y = \sin(-x)$

Solution

a i $y = \cos x$ has period 2π.

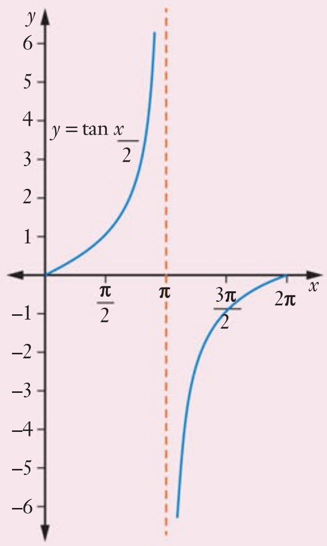
ii $f(x) = \sin 5x$ has period $\frac{2\pi}{5}$.

iii $y = \tan 2x$ has period $\frac{\pi}{2}$.

b i $y = \tan \frac{x}{2}$ is a horizontal dilation of $y = \tan x$.

It has period $\frac{\pi}{\frac{1}{2}}$ or 2π.

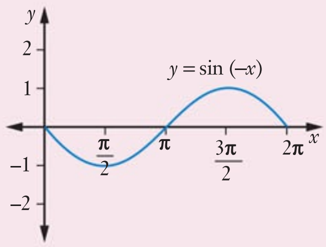
So there will be one cycle of the tan function in the domain $[0, 2\pi]$.



ii $y = \sin(-x)$ is a reflection of $y = \sin x$ in the y-axis, so $a = -1$.

This will change the x values. Transforming points in the domain $[-2\pi, 0]$ will give image points in the domain $[0, 2\pi]$.

$(0, 0)$	becomes	$(0 \times -1, 0) \equiv (0, 0)$
$\left(-\frac{\pi}{2}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(-\pi, 0)$	becomes	$(\pi, 0)$
$\left(-\frac{3\pi}{2}, 1\right)$	becomes	$\left(\frac{3\pi}{2}, 1\right)$
$(-2\pi, 0)$	becomes	$(2\pi, 0)$



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Vertical translations

A vertical translation of $y = f(x)$ is $y = f(x) + c$.

EXAMPLE 5

Sketch the graph of $y = \cos x + 2$ in the domain $[-\pi, \pi]$.

Solution

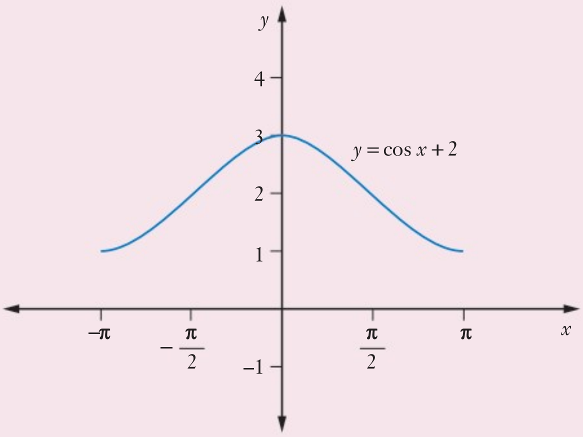
$y = f(x) + c$ is a vertical translation of $y = f(x)$.

So $y = \cos x + 2$ is a vertical translation of $y = \cos x$ up 2 units.

This changes the y values by adding 2 to each.

The domain is $[-\pi, \pi]$.

$(-\pi, -1)$	becomes	$(-\pi, -1 + 2) \equiv (-\pi, 1)$
$\left(-\frac{\pi}{2}, 0\right)$	becomes	$\left(-\frac{\pi}{2}, 2\right)$
$(0, 1)$	becomes	$(0, 3)$
$\left(\frac{\pi}{2}, 0\right)$	becomes	$\left(\frac{\pi}{2}, 2\right)$
$(\pi, -1)$	becomes	$(\pi, 1)$



Notice that the centre of the function is 2.

Centre as a vertical translation

The centre of $y = \sin x + c$ and $y = \cos x + c$ is c.

- If $c > 0$, the centre is translated upwards.
- If $c < 0$, the centre is translated downwards.

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EXAMPLE 6

a Find the centre of the function:

i $f(x) = \sin x - 7$ ii $y = \cos x + 4$

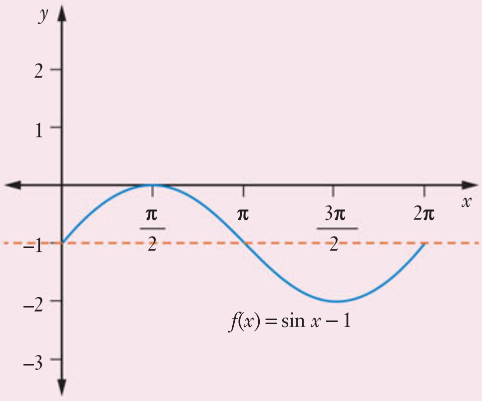
b Sketch the graph of $f(x) = \sin x - 1$.

Solution

a i The centre is –7.

ii The centre is 4.

b This is a vertical translation 1 unit down of $f(x) = \sin x$. The centre is –1.



Horizontal translations

A horizontal translation of $y = f(x)$ is given by $y = f(x + b)$.

EXAMPLE 7

Sketch the graph of $y = \sin\left(x - \frac{\pi}{2}\right)$ in the domain $[0, 2\pi]$.

Solution

$y = \sin\left(x - \frac{\pi}{2}\right)$ is a horizontal translation of $y = \sin x$ by $\frac{\pi}{2}$ units to the right.

We change the x values by adding $\frac{\pi}{2}$.

But since we need the transformed values of x to be in the domain $[0, 2\pi]$, our original values need to be in the domain $\left[0 - \frac{\pi}{2}, 2\pi - \frac{\pi}{2}\right] \equiv \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

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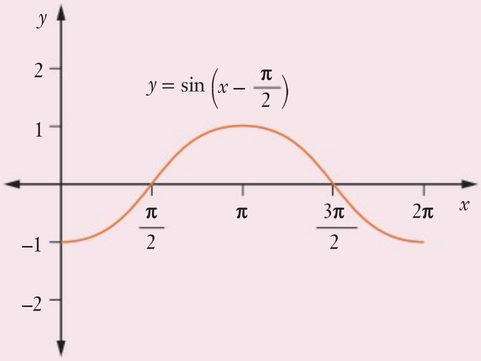
$\left(-\frac{\pi}{2}, -1\right)$ becomes $\left(-\frac{\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (0, -1)$

$(0, 0)$ becomes $\left(0+\frac{\pi}{2}, 0\right) \equiv \left(\frac{\pi}{2}, 0\right)$

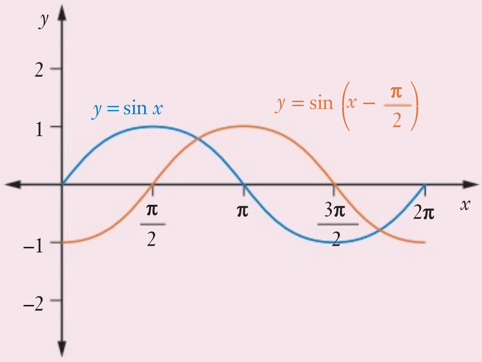
$\left(\frac{\pi}{2}, 1\right)$ becomes $\left(\frac{\pi}{2}+\frac{\pi}{2}, 1\right) \equiv (\pi, 1)$

$(\pi, 0)$ becomes $\left(\pi+\frac{\pi}{2}, 0\right) \equiv \left(\frac{3\pi}{2}, 0\right)$

$\left(\frac{3\pi}{2}, -1\right)$ becomes $\left(\frac{3\pi}{2}+\frac{\pi}{2}, -1\right) \equiv (2\pi, -1)$



Instead of finding points, you could sketch $y = \sin x$ and then shift it $\frac{\pi}{2}$ units to the right.



Phase as a horizontal translation

The phase of $y = \sin(x + b)$, $y = \cos(x + b)$ and $y = \tan(x + b)$ is b.

- If $b > 0$, the phase shift is to the left.
- If $b < 0$, the phase shift is to the right.

Phase shift of trigonometric functions

EXAMPLE 8

a Explain the meaning of $\frac{\pi}{4}$ in the equation $y = \tan\left(x + \frac{\pi}{4}\right)$.

b Sketch the graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ in the domain $[0, 2\pi]$.

Solution

a The function $y = \tan\left(x + \frac{\pi}{4}\right)$ has a phase of $\frac{\pi}{4}$ (to the left).

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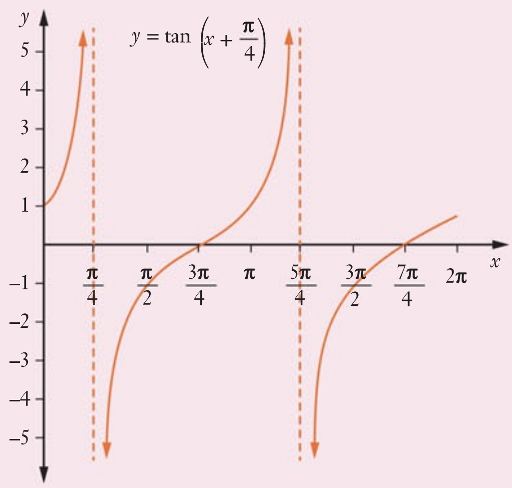
b To find points on the transformed graph, subtract $\frac{\pi}{4}$ from x values.

But since we need the transformed values of x to be in the domain $[0, 2\pi]$,

our original values need to be in the domain $\left[0+\frac{\pi}{4}, 2\pi+\frac{\pi}{4}\right] \equiv \left[\frac{\pi}{4}, \frac{9\pi}{4}\right]$.

$\left(\frac{\pi}{4}, 1\right)$	becomes	$(0, 1)$
Undefined at $x=\frac{\pi}{2}$	becomes	Undefined at $x=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$
$\left(\frac{3\pi}{4}, -1\right)$	becomes	$\left(\frac{\pi}{2}, -1\right)$
$(\pi, 0)$	becomes	$\left(\frac{3\pi}{4}, 0\right)$
$\left(\frac{5\pi}{4}, 1\right)$	becomes	$(\pi, 1)$
Undefined at $x=\frac{3\pi}{2}$	becomes	Undefined at $x=\frac{3\pi}{2}-\frac{\pi}{4}=\frac{5\pi}{4}$
$\left(\frac{7\pi}{4}, -1\right)$	becomes	$\left(\frac{3\pi}{2}, -1\right)$
$(2\pi, 0)$	becomes	$\left(\frac{7\pi}{4}, 0\right)$
$\left(\frac{9\pi}{4}, -1\right)$	becomes	$(2\pi, 1)$



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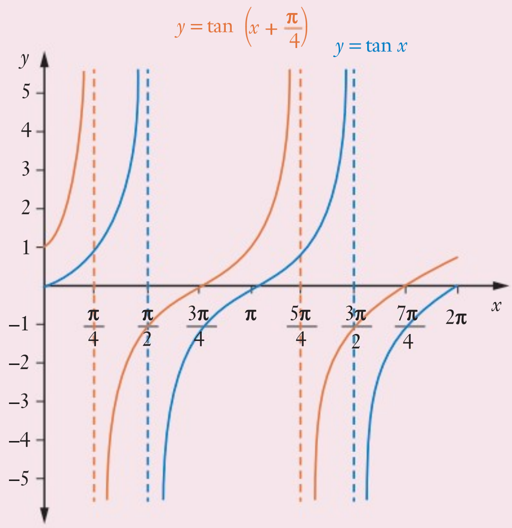
Instead of finding points, you could sketch $y = \tan x$ and then shift it $\frac{\pi}{4}$ units to the left.



Exercise 3.01 Transformations of trigonometric functions

1 Describe whether each transformation of a trigonometric function changes its amplitude, period, centre or phase.

a horizontal translation

b vertical dilation

c horizontal dilation

d vertical translation

2 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 5 \sin x$

b $f(x) = 2 \tan x$

c $y = -\cos x$

d $f(x) = -2 \sin x$

e $y = -\tan x$

3 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \sin x + 1$

b $y = \tan x - 2$

c $f(x) = \cos x - 3$

4 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos 4x$

b $y = \sin \frac{x}{2}$

c $f(x) = \tan 2x$

d $y = \tan \frac{x}{4}$

5 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = \cos (x + \pi)$

b $y = \tan\left(x - \frac{\pi}{2}\right)$

c $y = \sin\left(x - \frac{\pi}{4}\right)$

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6 Find the equation of the transformation of $y = \sin x$ if the transformed function has:

a amplitude 9

b a reflection in the x-axis

c centre -4

d period π

e a phase shift of π units to the right

7 Find the equation of the transformation of $y = \cos x$ if the transformed function has:

a amplitude 4

b a phase of $\frac{\pi}{3}$ units

c centre 8

d period $\frac{\pi}{2}$

e a vertical dilation with scale factor 7

8 Find the equation of the transformation of $y = \tan x$ if the transformed function has:

a period 2π

b a shift of $\frac{\pi}{6}$ units to the right

c a reflection in the y-axis

9 Sketch each graph in the domain $[-\pi, \pi]$.

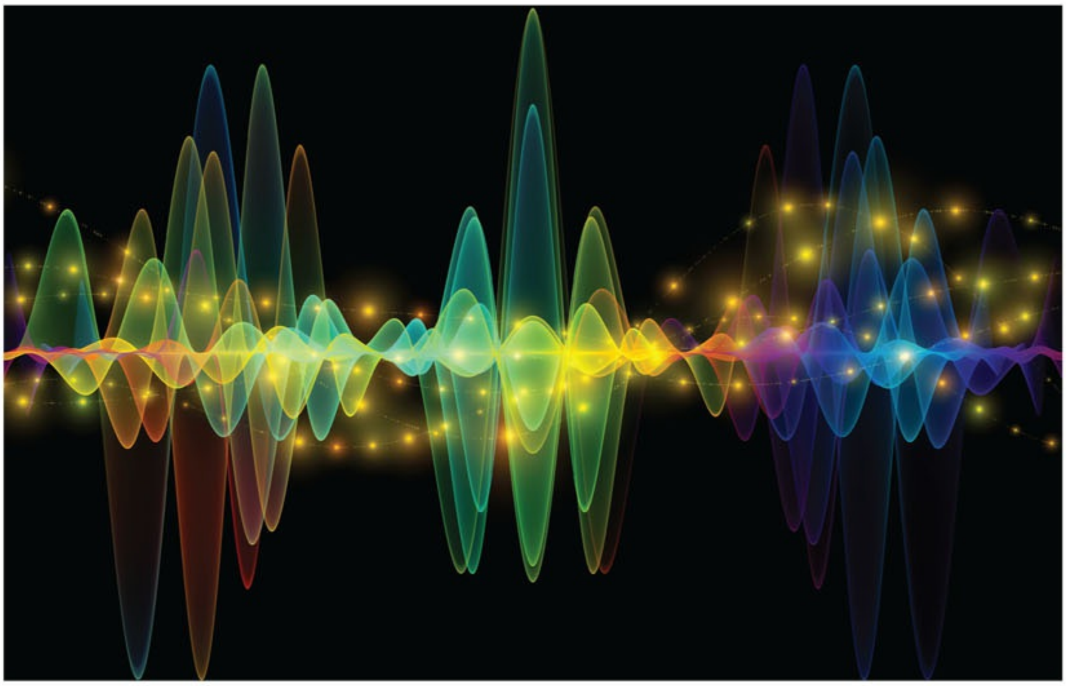
a $y = 3 \sin x$

b $y = \tan(-x)$

c $f(x) = \cos 2x$

d $y = \sin(x - \pi)$

e $f(x) = -\cos x$



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WS Graphing trigonometric functions

WS Sketching periodic functions

3.02 Combined transformations of trigonometric functions

We can put all the information about trigonometric functions together.

General equation of trigonometric functions

Function	Amplitude	Period	Phase	Centre
$y = k\sin[a(x+b)] + c$	k	$\frac{2\pi}{a}$	b Shift left if $b > 0$ Shift right if $b < 0$	c Shift up if $c > 0$ Shift down if $c < 0$
$y = k\cos[a(x+b)] + c$	k	$\frac{2\pi}{a}$		
$y = k\tan[a(x+b)] + c$	No amplitude	$\frac{\pi}{a}$		

EXAMPLE 9

a Sketch each function in the domain $[0, 2\pi]$.

i $y = 4\sin\frac{x}{2} + 1$ ii $y = 3\cos\left(x - \frac{\pi}{4}\right)$

b Find the equation of a cosine function that has amplitude 5, period 4π, centre -2 and a phase of 2 units to the left.

Solution

a i $y = 4\sin\frac{x}{2} + 1$ has amplitude 4,

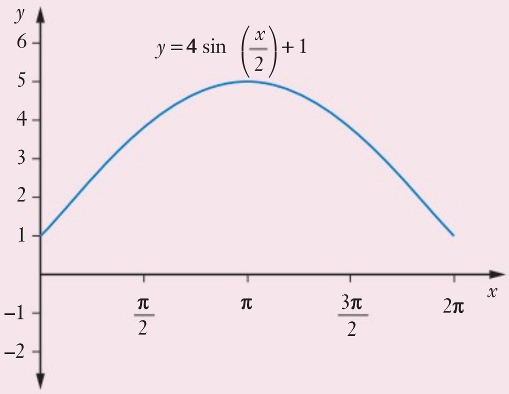
period $\frac{2\pi}{\frac{1}{2}} = 4\pi$ and centre 1.

Period 4π means only half the sine function curve will be in the domain $[0, 2\pi]$.

Centre 1 and amplitude 4 means:

Minimum $1 - 4 = -3$

Maximum $1 + 4 = 5$



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ii $y = 3\cos\left(x - \frac{4}{\pi}\right)$ has amplitude 3, period 2π, phase $\frac{\pi}{4}$ to the right and centre 0.

A phase is a horizontal translation, so it changes the x values and the domain.

Add $\frac{\pi}{4}$ to each x value.

But since we need the transformed values of x to be in the domain $[0, 2\pi]$,

our originalvalues need to be in the domain $\left[0-\frac{\pi}{4}, 2\pi-\frac{\pi}{4}\right] \equiv \left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$.

$\left(-\frac{\pi}{4}, 2.1\right)$ becomes $(0, 2.1)$

$(0, 3)$ becomes $\left(\frac{\pi}{4}, 3\right)$

$\left(\frac{\pi}{4}, 2.1\right)$ becomes $\left(\frac{\pi}{2}, 2.1\right)$

$\left(\frac{\pi}{2}, 0\right)$ becomes $\left(\frac{3\pi}{4}, 0\right)$

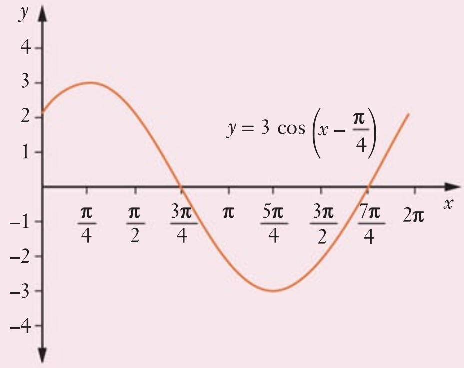
$\left(\frac{3\pi}{4}, -2.1\right)$ becomes $(\pi, -2.1)$

$(\pi, -3)$ becomes $\left(\frac{5\pi}{4}, -3\right)$

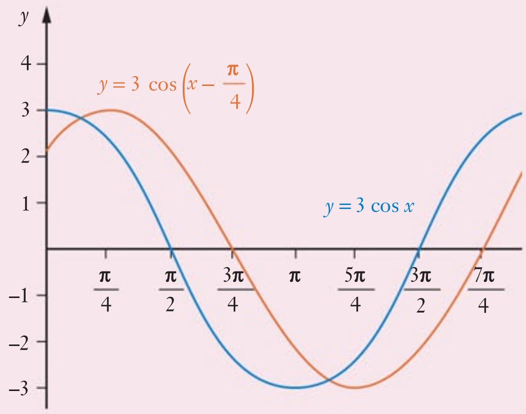
$\left(\frac{5\pi}{4}, -2.1\right)$ becomes $\left(\frac{3\pi}{2}, -2.1\right)$

$\left(\frac{3\pi}{2}, 0\right)$ becomes $\left(\frac{7\pi}{4}, 0\right)$

$\left(\frac{7\pi}{4}, 2.1\right)$ becomes $(2\pi, 2.1)$



Alternatively, sketch $y = 3\cos x$ and then shift it $\frac{\pi}{4}$ units to the right.



b $y = k\cos[a(x + b)] + c$.

Amplitude: $k = 5$

Period: $\frac{2\pi}{a} = 4\pi$

$2\pi = 4\pi a$

$\frac{1}{2} = a$

Phase: $b = 2$

Centre: $c = -2$

The equation is $y = 5\cos\left[\frac{1}{2}(x + 2)\right] - 2$

Exercise 3.02 Combined transformations of trigonometric functions

1 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 2\sin x - 3$

b $y = -\tan 2x$

c $f(x) = \cos\left(x + \frac{\pi}{2}\right) + 1$

d $y = \sin\left(-\frac{x}{2}\right) + 2$

e $f(x) = 3\cos 2x - 2$

2 a Find the equation of the transformed function if $y = \sin x$ is vertically dilated with scale factor 5, horizontally dilated with scale factor $\frac{1}{3}$, vertically translated 6 units down and horizontally translated 5 units to the left.

b Describe each transformation as a change in period, amplitude, centre or phase of the function.

3 Find the equation of the transformed function of $y = \cos x$ if it is:

a vertically dilated with scale factor 4, horizontally dilated with scale factor $\frac{1}{6}$, vertically translated 2 units up and horizontally translated $\frac{\pi}{3}$ units to the right

b reflected in the x-axis, reflected in the y-axis, translated 5 units down and π units to the left.

4 Sketch each graph in the domain $[-\pi, \pi]$.

a $y = 3\sin 2x$

b $y = 2\tan\frac{x}{2} + 1$

c $f(x) = -2\cos 3x$

d $y = 5\sin x - 3$

e $y = \cos(-2x) + 1$

5 Describe the features of each function in terms of amplitude, period, centre, and phase.

a $y = 3\tan 4x - 5$

b $y = 8\cos(x + \pi) - 3$

c $y = 5\sin[2(x - 3)] + 1$

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6 Find the equation of each function.

a a sine function with amplitude 7, period π, phase of 1 unit to the right and centre -3

b a cosine function with amplitude 1, a reflection in the x-axis, period $\frac{2\pi}{5}$ and centre 2

c a tangent function with period 2π, a reflection in the x-axis and a phase of 2 units to the left

d $y = \sin x$ with a vertical dilation scale factor 4, a reflection in the y-axis, a horizontal dilation scale factor 3, a vertical translation 2 units up and a horizontal translation 5 units to the left

7 Describe the features of $y = k \operatorname{cosec}\,[a(x + b)] + c$.

8 Find the equation of the transformed function if $y = \tan x$ is translated 3 units to the right and then dilated horizontally with scale factor $\frac{1}{4}$.

9 The water depth at a harbour entrance is 5 m at low tide and 25 m at high tide. The time between each low tide is around 12 hours.

a Find the centre of the tidal motion.

b What is the amplitude and period?

c Write an equation for the water depth D metres in terms of time t hours as a cosine function.

10 Find an equation for blood pressure, B, as a sine function of time, t minutes, if the maximum blood pressure is 120 and the minimum is 80, with a heart rate of 60 beats per minute.

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Solving trigonometric equations graphically

Trigonometric equations

3.03 Trigonometric equations

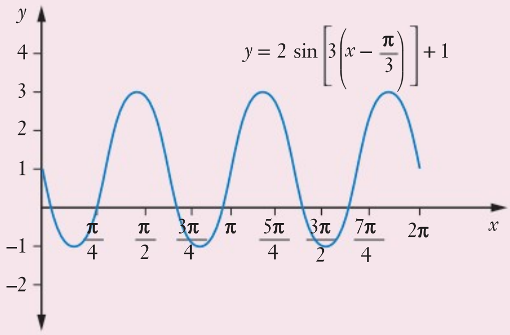
Graphical solutions

We can use the work on transformations to help solve trigonometric equations graphically.

EXAMPLE 10

a The graph of the trigonometric function $y = 2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1$ is shown for $[0, 2\pi]$.

Find the number of solutions to the trigonometric equation $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ for $[0, 2\pi]$.



b i Sketch the graphs of $y = \frac{x}{4} - 1$ and $y = 3\cos x - 2$ for $[0, 2\pi]$.

ii Find the number of solutions to the equation $3\cos x - 2 = \frac{x}{4} - 1$ for $[0, 2\pi]$.

iii Solve the equation graphically.

Solution

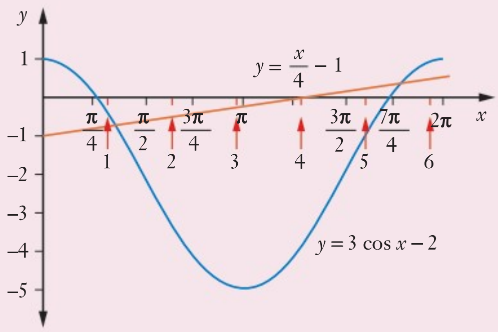
a To solve $2\sin\left[3\left(x - \frac{\pi}{3}\right)\right] + 1 = 0$ graphically, we find the x-intercepts.

The function has 6 x-intercepts in the domain $[0, 2\pi]$.

So the equation has 6 solutions.

b i $y = 3\cos x - 2$ has amplitude 3 and centre –2.

$y = \frac{x}{4} - 1$ is a linear function with x-intercept 4 and y-intercept –1.



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ii The solutions to $3 \cos x - 2 = \frac{x}{4} - 1$ are shown by where the graphs $y = 3 \cos x - 2$ and $y = \frac{x}{4} - 1$ intersect.

The graphs intersect in 2 places in $[0, 2\pi]$ so the equation has 2 solutions.

iii The graphs intersect just after $x = 1$ and $x = \frac{7\pi}{4} \approx 5.5$.

A precise graph drawn on graph paper or using technology would show that the solutions are $x \approx 1.1$ and 5.6.

Algebraic solutions

EXAMPLE 11

Solve for $[0°, 360°]$:

a $\sin 3x = \frac{1}{2}$

b $\cos (2x - 60°) = \frac{1}{\sqrt{2}}$

Solution

a For the domain $[0°, 360°]$

$0° \leq x \leq 360°$

$0° \leq 3x \leq 1080°$

So the new domain is $[0°, 1080°]$ (3 revolutions of the circle)

$\sin 3x = \frac{1}{2}$

sin x > 0 in 1st and 2nd quadrants

$$3x = 30°, 180° - 30°, 360° + 30°, 360° + (180° - 30°), 720° + 30°, 720° + (180° - 30°)$$
$$= 30°, 150°, 390°, 510°, 750°, 870°$$
$$x = 10°, 50°, 130°, 170°, 250°, 290°$$

b $0° \leq x \leq 360°$

$0° \leq 2x \leq 720°$

$-60° \leq 2x - 60° \leq 660°$

So the new domain is $[-60°, 660°]$ (2 revolutions of the circle starting at $-60°$)

$\cos (2x - 60°) = \frac{1}{\sqrt{2}}$

cos x > 0 in 1st and 4th quadrants

$$2x - 60° = -45°, 45°, 360° - 45°, 360° + 45°$$

($720° - 45°$ is outside the domain)

$$= -45°, 45°, 315°, 405°$$
$$2x = 15°, 105°, 375°, 465°$$
$$x = 7.5°, 52.5°, 187.5°, 232.5°$$

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EXAMPLE 12

Solve each equation for $[0, 2\pi]$.

a $6\cos 2x - 3 = 0$

b $\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

Solution

a For the domain $[0, 2\pi]$

$0 \le x \le 2\pi$

$0 \le 2x \le 4\pi$, so when solving for $2x$ we need to go around the circle twice.

$6\cos 2x - 3 = 0$

$6\cos 2x = 3$

$\cos 2x = \frac{1}{2}$

cos x > 0 in 1st and 4th quadrants

$2x = \frac{\pi}{3}, 2\pi - \frac{\pi}{3}, 2\pi + \frac{\pi}{3}, 2\pi + 2\pi - \frac{\pi}{3}$ as $0 \le 2x \le 4\pi$

$= \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b $0 \le x \le 2\pi$

$-\frac{\pi}{4} \le x - \frac{\pi}{4} \le \frac{7\pi}{4}$

So the new domain is $\left[-\frac{\pi}{4}, \frac{7\pi}{4}\right]$ $\left(\text{1 revolution of the circle starting at } -\frac{\pi}{4}\right)$

$\tan\left(x - \frac{\pi}{4}\right) = \sqrt{3}$

tan x > 0 in 1st and 3rd quadrants

$x - \frac{\pi}{4} = \frac{\pi}{3}, \pi + \frac{\pi}{3}$

$= \frac{\pi}{3}, \frac{4\pi}{3}$

$x = \frac{\pi}{3} + \frac{\pi}{4}, \frac{4\pi}{3} + \frac{\pi}{4}$

$= \frac{7\pi}{12}, \frac{19\pi}{12}$

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Exercise 3.03 Trigonometric equations

1 By drawing the graph of $y = 2\sin 3x$ in the domain $[0, 2\pi]$:

a find the number of solutions of $2\sin 3x = 1$

b solve $2\sin 3x = 1$ graphically

2 a Sketch the graphs of $y = -\cos x + 3$ and $y = x - 1$ for $[0, 2\pi]$.

b Solve:

i $-\cos x + 3 = x - 1$ ii $-\cos x + 3 = 2$

3 Solve for $[0°, 360°]$:

a $2\sin 2x = 1$ b $\tan 3x = -1$ c $\cos(x + 90°) = \dfrac{\sqrt{3}}{2}$

d $\tan(x - 45°) = \sqrt{3}$ e $\sin(x + 60°) = 0$

4 Solve for $[0, 2\pi]$:

a $\tan 2x = \sqrt{3}$ b $2\cos 3x + 1 = 0$ c $4\sin^2\left(x - \dfrac{\pi}{3}\right) = 3$

d $2\cos^2 2x - 1 = 0$ e $\cos(x + \pi) = 1$

5 Solve for $[-\pi, \pi]$:

a $\tan 3x = 1$ b $\cos\left(x + \dfrac{\pi}{4}\right) = \dfrac{1}{\sqrt{2}}$ c $\sin 2x = -1$

d $\cos\left(x - \dfrac{\pi}{2}\right) = 0$ e $\tan^2 4x = 0$

6 Solve for $[0, 2\pi]$:

a $\cos 2\left(x - \dfrac{\pi}{2}\right) = \dfrac{1}{2}$ b $2\sin\left(3x + \dfrac{3\pi}{2}\right) = 1$

7 The function $T = 15\cos\dfrac{\pi t}{6} + 20$ models the average monthly temperatures in Nelson Springs, starting in January.

a Find the amplitude, period and centre of the function.

b Solve $15\cos\dfrac{\pi t}{6} + 20 = 35$ and explain the meaning of the solutions.

8 A set of tidal waves has a maximum height of 20 m and a minimum height of 6 m. The waves break every 10 seconds.

a Find the equation of a sine function that describes the motion of the waves.

b Find the first 4 times that the waves reach their maximum height.

c Find the first time that the waves reach their minimum height.

d When will the height of the waves be in the centre?

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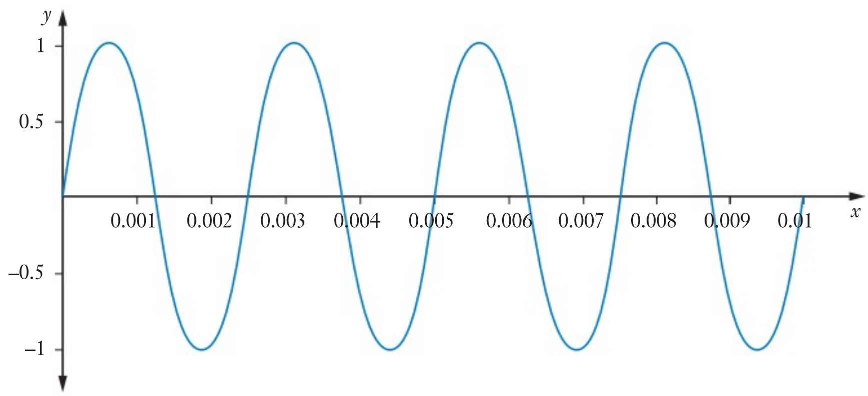
9 Sound waves have the shape of sine functions. The graph below shows the sound wave that occurs when playing the note A above middle C on a piano. Its equation is $y = \sin(880\pi x)$ where x is time in seconds.



a Find the amplitude and period of this sound wave.

b Use the graph to solve for [0, 0.01]:

i $\sin(880\pi x) = 0.5$ ii $\sin(880\pi x) = 0$

c Solve algebraically for [0, 0.01] (to 2 significant figures):

i $\sin(880\pi x) = 0.5$ ii $\sin(880\pi x) = 0$

d The higher the amplitude, the more volume the sound has (it is louder). (The word 'amplifier' comes from this property.) Find the equation of the note A that is 3 times as loud as the one drawn.

e A note that is higher in pitch has a higher frequency (more cycles) than a lower note. Draw a rough sketch of a middle C note with the same volume as the A note above C.

f The unit of measurement for frequency is hertz (Hz), the number of wave cycles of a sound in 1 second. What is the frequency in hertz of note A?

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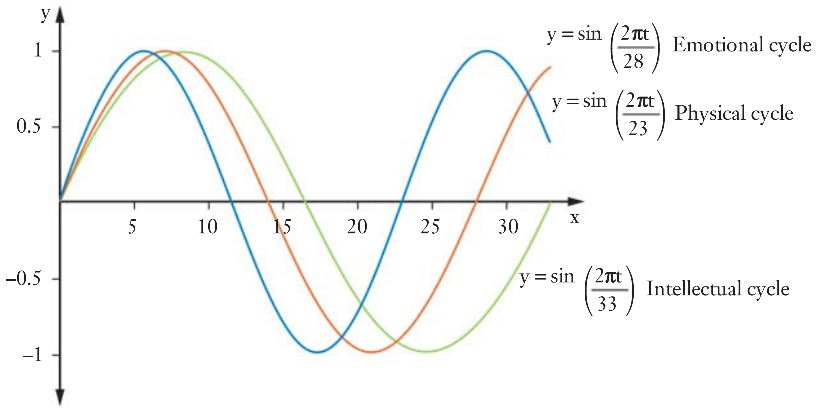
10 Biorhythms is a theory that emotional, physical and mental activity in humans can be modelled by 3 sine functions: physical $y = \sin \frac{2\pi}{23}$, emotional $y = \sin \frac{2\pi t}{28}$ and intellectual $y = \sin \frac{2\pi t}{33}$ where t is time in days, starting from your date of birth. It was first developed by German doctor Wilhelm Fliess in 1878 but became popular in the 1970s when computers were able to chart the 3 biorhythms. Their graphs are sketched below.



a What is the period of each function? What does this mean?

b When do the physical and intellectual graphs intersect?

c When do the emotional and physical graphs intersect?

d Biorhythms are supposed to be at optimal levels when $y = 1$ (maximum points). Estimate the range of times when all 3 biorhythms are near optimal levels together.

11 A vertical spring is pulled down and then let go. It bounces back up and down again according to the equation $h = 12 \cos t + 15$ where h is the height of the spring in cm and t is time in seconds.

a Describe the significance of the 15 in the equation.

b What are the maximum and minimum heights of the spring?

c What is the height of the spring after π seconds?

d At what times will the spring be at its minimum height?

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3. TEST YOURSELF

WS Formula sheet: Functions, Trigonometry and Vectors

Qz Practice quiz

For Questions 1 to 3, choose the correct answer A, B, C or D.

1 The function $y = 2\cos 3x - 7$ has:

A amplitude 2, period 3 and centre -7

B amplitude 7, period $\frac{1}{3}$ and centre 2

C amplitude 2, period $\frac{2\pi}{3}$ and centre 7

D amplitude 2, period $\frac{2\pi}{3}$ and centre -7

2 The equation of a function with phase π units to the left is:

A $y = \tan(x + \pi)$

B $y = \tan(\pi x)$

C $y = \tan x + \pi$

D $y = \tan(x - \pi)$

3 The solution of $\cos 2x = 1$ in the domain $[0, 2\pi]$ is:

A $x = 0, 2\pi$

B $x = 0, \pi, 2\pi$

C $x = \frac{\pi}{2}, \frac{3\pi}{2}$

D $x = \pi$

4 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = 3\cos x$

b $y = \tan\frac{x}{2}$

c $y = \sin x - 2$

d $y = -\sin x$

e $y = 3\cos 2x - 1$

5 The function $h = 3\cos\left(\frac{2\pi t}{3}\right) + 10$ shows the water level in a lock in metres over time t hours.

a Find the maximum and minimum levels of water and when they occur.

b Solve $3\cos\left(\frac{2\pi t}{3}\right) + 10 = 11$ and explain what the solution means.

Photo courtesy Margaret Grove

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6 Simplify:

a $2\cot^2 x + 2$

b $\tan A \operatorname{cosec} A$

c $(\sec A + \tan A)(\sec A - \tan A)$

d $\sin(180° - x)$

7 Solve for $[0, 2\pi]$:

a $4\cos^2 x = 3$

b $2\sin 2x = 1$

c $\cos\left(x - \frac{\pi}{2}\right) = -1$

d $\tan^2\left(x + \frac{\pi}{6}\right) = 3$

8 A person's blood pressure has a maximum pressure of 135 and a minimum pressure of 85 and the heartbeat is 70 beats per minute.

a Write an equation showing this blood pressure y as a sine function of time t minutes.

b Draw a graph showing this function.

9 Solve for $[-\pi, \pi]$:

a $2\cos 2x = 1$

b $\tan\left(x - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{3}}$

c $\sin\left(x + \frac{\pi}{2}\right) = \frac{1}{\sqrt{2}}$

10 The high tide mark on a cliff above a river is 80 m and the low tide mark is 50 m. The average time between high tides in the river is approximately 13 hours.

a Write the equation for the height of the tides as a sine function.

b Find the times when the river is halfway between high and low tides.

11 Find the equation of each transformed function of $y = \cos x$ that has:

a period π

b amplitude 5

c a reflection in the x-axis

d a phase of $\frac{\pi}{6}$ units to the right

e centre 4

12 a Sketch the graph of $y = 2\sin\frac{x}{2} - 1$ for $[0, 2\pi]$.

b From the graph, solve $2\sin\frac{x}{2} - 1 = 0$.

c Solve $2\sin\frac{x}{2} - 1 = 0$ algebraically.

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13 State the amplitude, period, centre and phase of each function.

a $y = 2 \sin 3x - 1$

b $y = \cos\left(\frac{x}{2} + \pi\right)$

c $y = -3 \tan\left(5x - \frac{\pi}{4}\right)$

14 Solve for $[0°, 360°]$:

a $\tan (x + 45°) = 1$

b $\sqrt{2} \cos (x - 20°) + 1 = 0$

c $3 \sin [2(x + 10°)] - 2 = 0$

15 Solve for $[-180°, 180°]$:

a $5 \tan 2x = -5$

b $\cos [3(x - 30°)] + 1 = 0$

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3. CHALLENGE EXERCISE

1 a Find the amplitude, period and phase of the function $y = 2\cos\left(2x - \frac{\pi}{2}\right)$.

b Solve $2\cos\left(2x - \frac{\pi}{2}\right) = \sqrt{3}$ for $[0, 2\pi]$.

2 Find the equation of:

a a cosine function with amplitude 8, period 2π and centre 4

b a sine function with amplitude 2, period $\frac{\pi}{4}$, phase $\frac{\pi}{3}$ units to the right and centre 3

c a tangent function with period 2π and phase $\frac{\pi}{2}$ units to the left

3 Sketch the graph of $y = 3\sec 2x$ for $[0, 2\pi]$.

4 Solve $\operatorname{cosec} 2x = \sqrt{2}$ for $[0, 2\pi]$.

5 a Find the equation of the transformation of $y = \cos x$ if it has a vertical translation 5 units down, a horizontal translation $\frac{\pi}{6}$ units to the right, then a vertical stretch, scale factor 4 and a horizontal stretch, scale factor 3.

b Find the amplitude, period, centre and phase of this transformed function.

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Practice set 1

In Questions 1 to 7, select the correct answer A, B, C or D.

1 For what values of r does the limiting sum of a geometric series exist?

A $|r| > 1$ B $|r| < 1$ C $|r| \geq 1$ D $|r| \leq 1$

2 The transformation of $y = f(x)$ to $y = 3f(2x)$ is:

A Vertical dilation scale factor 3, horizontal dilation scale factor 2

B Horizontal dilation scale factor 3, vertical dilation scale factor 2

C Vertical dilation scale factor 3, horizontal dilation scale factor $\frac{1}{2}$

D Horizontal dilation scale factor 3, vertical dilation scale factor $\frac{1}{2}$

3 Simplify $\frac{\sin\theta}{\cos^2\theta\sec\theta}$.

A $\cot\theta$ B $\tan^2\theta$ C $\tan\theta$ D $\cot^2\theta$

4 The nth term of the sequence 7, 49, 343, … is:

A $7n$ B 7^{n-1} C $7n - 1$ D 7^n

5 $y = \cos(x + \pi) + 3$ has a phase shift of:

A π units to the right B 3 units up

C 3 units down D π units to the left

6 Find the limiting sum of $\frac{3}{5} + \frac{2}{5} + \frac{4}{15} + \ldots$

A $\frac{1}{5}$ B $\frac{9}{10}$ C $1\frac{4}{5}$ D $\frac{6}{15}$

7 The formula for the sum $1 + 1.03 + 1.03^2 + \ldots + 1.03^{n-1}$ is:

A $S = \frac{1.03(1.03^{n-1} - 1)}{1.03 - 1}$ B $S = \frac{1.03(1.03^n - 1)}{1.03 - 1}$

C $S = \frac{1.03^{n-1} - 1}{1.03 - 1}$ D $S = \frac{1.03^n - 1}{1.03 - 1}$

8 a Sketch the graphs of $y = x^2$ and $y = -(x + 2)^2$ on the same set of axes.

b Describe the transformations that changed $y = x^2$ into the transformed function.

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9 Solve each equation for [0°, 360°].

a $\tan 2x + 1 = 0$

b $2 \cos 3x = 1$

c $2 \sin (x - 90°) = \sqrt{3}$

d $\tan (x - 180°) = \sqrt{3}$

e $2 \cos^2(x + 45°) = 1$

10 Describe the amplitude, period, centre and phase shift of each function.

a $y = 4 \cos 5x$

b $y = -2 \sin \left(x - \frac{\pi}{6} \right) + 1$

c $y = \tan \left(\frac{x}{4} + 2 \right)$

11 Find in index form the 10th term of $\frac{3}{4}, \frac{3}{16}, \frac{3}{64}, \ldots$

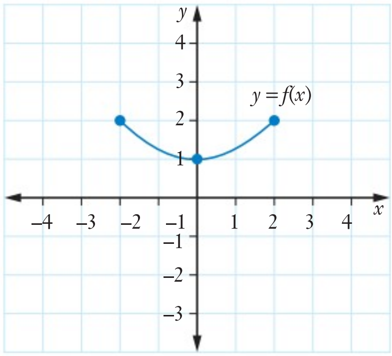
12 Copy the graph $y = f(x)$ and sketch:

a $y = 2f(x)$

b $y = f(x) + 1$

c $y = f(x - 2)$

d $y = f(2x)$



13 The 2nd term of a geometric sequence is 52 and the 4th term is 13.
Find 2 sequences that satisfy these requirements.

14 Find which term $-370$is in the series $17 + 8 - 1 - \ldots$

15 Describe the transformations on $y = x^3$ if the equation of the transformed function is $y = 4(x - 1)^3 - 3$, and state whether any dilations stretch or compress the graph of the function.

16 Solve each equation for [0°, 360°].

a $6\sin^2 x - 7 \sin x + 2 = 0$

b $2\sin[2(x - 30)] = 1$

17 A moving sculpture has a ball on the end of a wire that oscillates backwards and forwards between 2 points. The equation of the distance d cm of the ball from the centre of the sculpture at time t seconds is given by $d = 6 \cos (2\pi t) + 10$.

a Find the centre of motion and the maximum distance of the ball from this centre in both directions.

b How long does it take the ball to complete one complete cycle between the 2 points?

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18 Find the equation of the function if $y = \sqrt{x}$ is dilated vertically with scale factor 4, dilated horizontally with scale factor $\frac{1}{3}$, translated vertically 1 unit down and translated horizontally 7 units to the left.

19 a Find the 50th term of 3, 7, 11, …

b Calculate the sum of 50 terms.

20 The nth term of a series is given by $7n - 3$.

a Find the first 3 terms and the 12th term.

b Evaluate the sum of the first 20 terms.

c Which term is equal to 200?

21 a Sketch the graph of $f(x) = 5|x - 2| - 3$.

b From the graph, solve each equation.

i $5|x - 2| - 3 = 2$

ii $5|x - 2| - 3 = 7$

iii $5|x - 2| - 3 = -3$

c State the domain and range of $f(x)$.

22 Find the exact value of:

a $\cos 120°$ b $\sin 300°$ c $\tan 225°$

d $\cos(-135°)$ e $\tan 690°$

23 The 4th term of an arithmetic sequence is 18 and the 8th term is 62.
Find the formula for the general term of the sequence.

24 Evaluate x if $\sec x = \operatorname{cosec}(2x - 30°)$.

25 Sketch the graph of each function in the domain $[0, 2\pi]$.

a $y = -7\cos x$ b $y = 2\sin x$

c $y = \cos x + 1$ d $y = \tan\left(x + \frac{\pi}{2}\right)$

e $y = 3\cos 2x$ f $y = -4\sin\frac{x}{2} + 3$

26 Prove each identity.

a $\cot x \sec x = \operatorname{cosec} x$

b $\sin^2 x \operatorname{cosec}^2 x - \sin^2 x = \cos^2 x$

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27 Find the equation of the transformed function of $y = \sin x$ if the function has:

a amplitude 2

b period 4π

c centre -3

d a reflection in the x-axis and amplitude 5

e a phase shift $\frac{\pi}{2}$ units to the left

f amplitude 5, period 6π, centre 1 and a phase shift of π units to the right

28 The geometric series $x + x^2 + x^3 + \ldots$ has a sum to infinity of 5. Find the value of x.

29 Solve each equation for $[0, 2\pi]$.

a $\cos x = 0.62$

b $\tan^2 x = 1$

c $2 \sin x - 1 = 0$

d $\cos x = 0$

e $4 \sin^2 x = 3$

f $2 \cos 2x + 1 = 0$

30 Find the first value of n for which the sum of the sequence 20, 4, 0.8, … is greater than 24.85.

31 Evaluate $\frac{2}{25} + \frac{2}{125} + \frac{2}{625} + \ldots$

32 The average temperature T over t months is given by $T = 20 \cos \frac{\pi t}{6} + 18$ where January is $t = 0$.

a Find the amplitude, period and centre of the function. What do these features mean in terms of maximum and minimum temperatures and cycles?

b Find the month with an average temperature of -2°C.

c What is the month with the highest average temperature?

d What is the average temperature in September?

e When is the average temperature 18°C?

33 Solve graphically $(x - 1)^2 - 4 \le 0$.

34 Show that $f(x) = 3x^2 - 2$ is an even function.

35 a Show that log 3, log 9, log 27 … are terms of an arithmetic sequence.

b Find the exact sum of 20 terms of the sequence.

36 a Sketch the graph of $y = 3(x - 2)^2 - 4$.

b From the graph, solve each inequality.

i $3(x - 2)^2 - 4 \ge 8$

ii $3(x - 2)^2 - 4 < 8$

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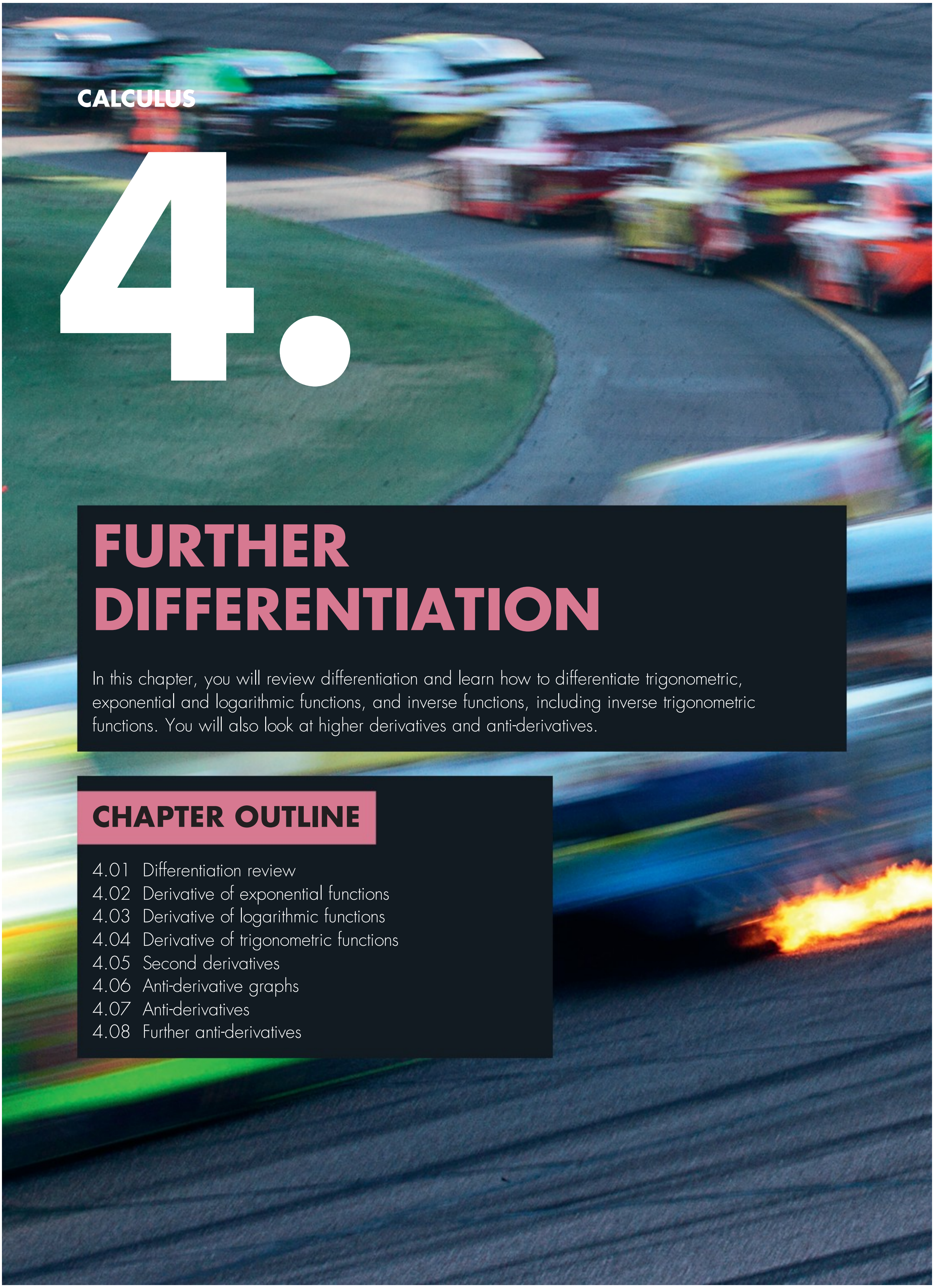


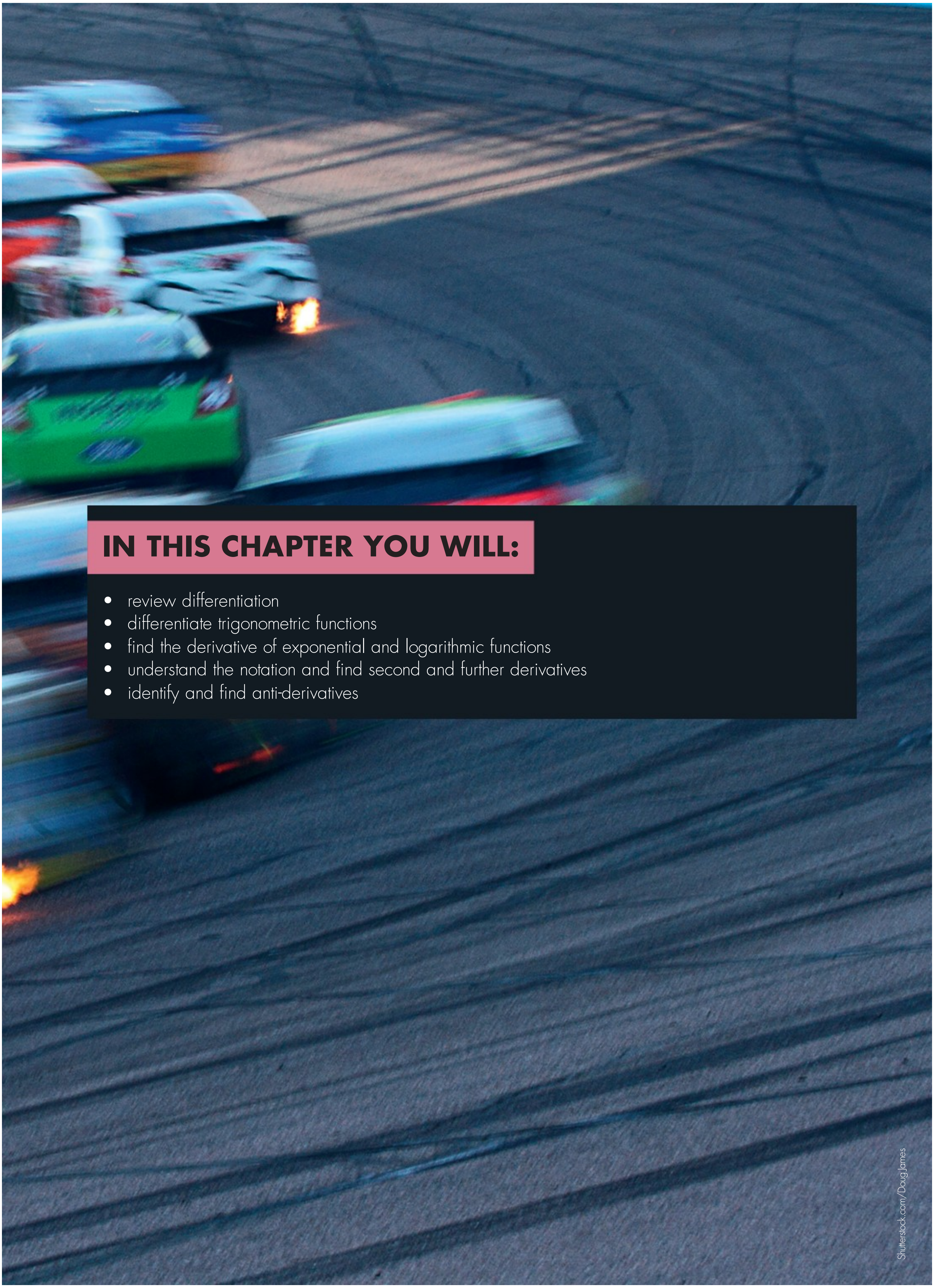
CALCULUS

4.

FURTHER DIFFERENTIATION

In this chapter, you will review differentiation and learn how to differentiate trigonometric, exponential and logarithmic functions, and inverse functions, including inverse trigonometric functions. You will also look at higher derivatives and anti-derivatives.

CHAPTER OUTLINE



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IN THIS CHAPTER YOU WILL:

- review differentiation
- differentiate trigonometric functions
- find the derivative of exponential and logarithmic functions
- understand the notation and find second and further derivatives
- identify and find anti-derivatives

TERMINOLOGY

anti-derivative: A function $F(x)$ whose derivative is $f(x)$, that is, $F'(x) = f(x)$. Also called the primitive or integral function.

anti-differentiation: The process of finding the original function given its derivative.

second derivative: The derivative $f''(x)$ or $\frac{d^2y}{dx^2}$; the derivative of the derivative $f'(x)$ or $\frac{dy}{dx}$.

4.01 Differentiation review

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

Product rule

If $y = uv$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ or $y' = u'v + v'u$.

Quotient rule

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $y' = \frac{u'v - v'u}{v^2}$.

Rates of change

The average rate of change between 2 points (x_1, y_1) and (x_2, y_2) is the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The instantaneous rate of change at point (x, y) is the derivative $f'(x)$ or $\frac{dy}{dx}$.

EXAMPLE 1

Water is pumped into a dam according to the formula $Q = 3t^3 + 2t^2 + 270$ where Q is the amount of water in kL and t is time in hours. Find:

a the amount of water in the dam after 6 hours

b the average rate at which the water is pumped into the dam between 3 and 6 hours

c the rate of change after 6 hours

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Solution

a $Q = 3t^3 + 2t^2 + 270$

When $t = 6$

$Q = 3(6)^3 + 2(6)^2 + 270$

$= 990$

So there is 990 kL of water in the dam after 6 hours.

b When $t = 3$

$Q = 3(3)^3 + 2(3)^2 + 270$

$= 369$

$$\text{Average rate of change} = \frac{Q_2 - Q_1}{t_2 - t_1} = \frac{990 - 369}{6 - 3} = \frac{621}{3} = 207$$

So average rate of change is 207 kL h^{-1}.

c $\frac{dQ}{dt} = 9t^2 + 4t$

When $t = 6$

$\frac{dQ}{dt} = 9(6)^2 + 4(6)$

$= 348$

So the rate of increase after 6 hours is 348 kL h^{-1}.

Exercise 4.01 Differentiation review

1 Differentiate each function.

a $3x^4 - 2x^3 + 7x - 4$

b $2x + 5$

c $6x^2 - 3x - 2$

2 Find the derivative $f'(x)$ given $f(x) = 4x^5 + 9x^2$.

3 Find $\frac{dx}{dt}$ if $x = 2\pi t^3 - 3t^2 + 1$.

4 Find $f'(-2)$ when $f(x) = 8x^3 + 5x - 2$.

5 Differentiate:

a x^{-5}

b $x^{\frac{2}{3}}$

c $\frac{1}{x^2}$

d $\sqrt[4]{x}$

e $-\frac{5}{x^4}$

6 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 8$.

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7 Differentiate:

a $(3x-1)^7$ b $(x^2-x+2)^3$ c $\sqrt{7x-2}$

d $\dfrac{1}{3x-2}$ e $\sqrt[3]{x^2-3}$

8 Find the derivative of:

a $x^2(x+4)$ b $(2x-1)(6x+5)$ c $4x(x^2+1)$

d $(4x+3)(x^2-1)^2$ e $2x^3\sqrt{x+1}$

9 Differentiate:

a $\dfrac{2x+3}{x-5}$ b $\dfrac{x^3}{4x-7}$ c $\dfrac{x^2+3}{2x-3}$

d $\dfrac{3x+1}{(2x+9)^2}$ e $\dfrac{3x+4}{\sqrt{2x-1}}$

10 Find the gradient of the tangent to the curve:

a $y=x^2-2x+5$ at the point where $x=-2$

b $f(x)=x^3-3$ at the point $(-1,-4)$

11 Find the gradient of the normal to the curve:

a $f(x)=3x^4+x^2-2$ at the point where $x=-1$

b $y=x^2+x-3$ at the point $(-3, 3)$

12 Find the equation of the tangent to the curve:

a $y=2x^2-5x-6$ at the point $(3,-3)$

b $y=5x^3-2x^2-x$ at the point where $x=2$

13 Find the equation of the normal to the curve:

a $f(x)=x^3+2x^2-3x-5$ at the point $(-1,-1)$

b $y=x^2-3x+1$ at the point where $x=3$

14 For the curve $y=x^2-8x+15$, find any values of x for which $\dfrac{dy}{dx}=0$.

15 Find the coordinates of the points at which the curve $y=x^3-2$ has a tangent with gradient 12.

16 Function $f(x)=x^2+x-4$ has a tangent parallel to the line $3x+y-4=0$ at point P. Find the equation of the tangent at P.

17 Find the coordinates of P if the gradient of the tangent to $y=\sqrt{x}$ is $\dfrac{1}{4}$ at point P.

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18 For the curve $y = \frac{5x - 3}{4x + 1}$ at the point where $x = 0$, find the equation of:

a the tangent b the normal

19 Find a formula for the rate of change $\frac{dQ}{dt}$ given:

a $Q = 3t^2 + 8$ b $Q = \frac{2}{t - 3}$ c $Q = \sqrt[3]{2x + 3}$

20 The mass M in kg of a snowball as it rolls down a hill over time t seconds is given by $M = t^2 + 3t + 4$.

a Find the average rate at which the mass changes between:

i 2 and 5 seconds ii 6 and 8 seconds

b Find the rate at which the mass is changing after:

i 5 seconds ii a minute

21 According to Boyle's Law, the pressure of a gas in pascals (Pa) is given by the formula $P = \frac{k}{V}$, where k is a constant and V is the volume of the gas in m^3. If $k = 250$ for a certain gas, find the rate of change in the pressure when $V = 10.7$.

22 The height of a ball in metres is given by $h = 4t - 2t^2$ where t is time in seconds.

a Find the height after:

i 1 s ii 1.5 s

b How long does it take for the ball to reach the ground?

c Find the velocity of the ball after:

i 0.5 s ii 1 s iii 2 s

4.02 Derivative of exponential functions

WS Derivatives of exponential functions

You learned how to differentiate $y = e^x$ in Year 11, in Chapter 8, Exponential and logarithmic functions.

Differentiation rules for e^x

$$\frac{d}{dx}e^x = e^x$$

$$\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = f'(x)\,e^{f(x)}$$

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EXAMPLE 2

a If $f(x) = 3e^x$, find the equation of the tangent to the curve at $(2, 3e^2)$.

b Differentiate :

i x^2e^x ii e^{8x} iii e^{5x-2}

Solution

a
$$\begin{aligned} f(x) &= 3e^x \\ f'(x) &= 3e^x \end{aligned}$$
At $(2, 3e^2)$
$$f'(2) = 3e^2$$
So $m = 3e^2$

Equation:
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3e^2 &= 3e^2(x - 2) \\ &= 3e^2x - 6e^2 \\ y &= 3e^2x - 3e^2 \end{aligned}$$
(or $3e^2x - y - 3e^2 = 0$)

b i $y' = u'v + v'u$

where $u = x^2$ and $v = e^x$

$u' = 2x \qquad v' = e^x$

$$\begin{aligned} y' &= 2xe^x + e^xx^2 \\ &= xe^x(2 + x) \end{aligned}$$

ii
$$\begin{aligned} \frac{dy}{dx} &= ae^{ax} \\ &= 8e^{8x} \end{aligned}$$

iii
$$\begin{aligned} \frac{dy}{dx} &= f'(x)e^{f(x)} \\ &= 5e^{5x-2} \end{aligned}$$

We can differentiate other exponential functions.

EXAMPLE 3

Differentiate 2^x.

Solution

$$\begin{aligned} 2 &= e^{\ln 2} \\ 2^x &= (e^{\ln 2})^x \\ &= e^{x \ln 2} \\ \frac{dy}{dx} &= \ln 2\ e^{x \ln 2} \quad \leftarrow \text{ln 2 is a constant} \\ &= \ln 2 \times 2^x \\ &= 2^x \ln 2 \end{aligned}$$

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Derivative of a^x

$$\text{If } y = a^x, \text{ then } \frac{dy}{dx} = a^x \ln a$$

The proof of this has the same steps as in the previous example.

Exercise 4.02 Derivative of exponential functions

1 Differentiate:

a e^{7x}	b e^{-x}	c e^{6x-2}	d e^{x^2+1}
e e^{x^3+5x+7}	f e^{5x}	g e^{-2x}	h e^{10x}
i $e^{2x} + x$	j $x^2 + 2x + e^{1-x}$	k $(x + e^{4x})^5$	l xe^{2x}
m $\frac{e^{3x}}{x^2}$	n x^3e^{5x}	o $\frac{e^{2x+1}}{2x+5}$	

2 If $f(x) = e^{3x-2}$ find the exact value of $f'(1)$.

3 Find the derivative of:

a 3^x b 10^x c 2^{3x-4}

4 Find the gradient of the tangent to the curve $y = e^{5x}$ at the point where $x = 0$.

5 Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point (0, 1).

6 For the curve $y = e^{3x}$ at the point where $x = 1$, find the exact gradient of:

a the tangent b the normal

7 For the curve $y = e^{x^2}$ at the point (1, e), find the equation of:

a the tangent b the normal

8 Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point (0, 4).

9 The population of a city is given by $P = 24\ 500e^{0.038t}$ where t is time in years.

a Find the population after:

i 5 years ii 10 years

b Find the average rate of change in population between:

i the 1st and 5th years ii the 5th and 10th years

c Find the rate of change in population after:

i 5 years ii 10 years

10 The displacement of a particle is given by $s = 10e^{2t} - 5t$ cm after t minutes.

a Find the average rate of change in displacement between 1 and 5 minutes.

b Find the rate of change in displacement after:

i 1 minute ii 2 minutes iii 8 minutes

11 A radioactive substance has a mass of $M = 20e^{-0.021t}$ in grams over time t years.

a Find the initial mass.

b Find the mass after 50 years.

c Find the average rate of change in mass between 50 and 100 years.

d Find the rate of change in mass after:

i 50 years   ii 100 years   iii 200 years

12 An object moves according to the formula $x = 3e^{2t}$ where x is displacement in cm and t is time in s.

a Find the displacement at 5 s.

b Find the velocity at 5 s.

INVESTIGATION

DERIVATIVE OF A LOGARITHMIC FUNCTION

Draw the derivative (gradient) function of a logarithm function.

What is the shape of the derivative function?

WS Derivatives of logarithmic functions

WS Exponential and logarithmic functions

4.03 Derivative of logarithmic functions

Logarithm rules

If $y = a^x$ then $\log_a y = x$

$\log_a xy = \log_a x + \log_a y$

$\log_a \frac{x}{y} = \log_a x - \log_a y$

$\log_a x^n = n \log_a x$

$\log_a x = \frac{\log_b x}{\log_b a}$

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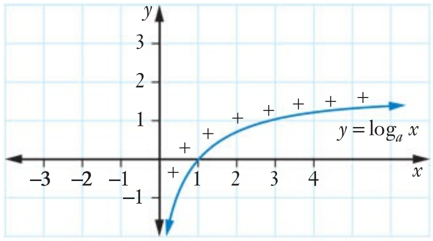
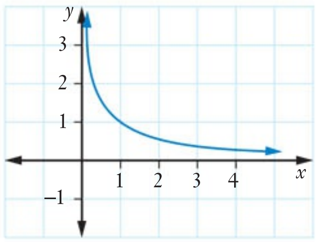
To find the derivative of a logarithmic function, notice that the gradient of the function is always positive but is decreasing.



The derivative function of a logarithmic function is a hyperbola.



There is a special rule for $y = \ln x$.

Derivative of y = ln x

If $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$ where $x > 0$.

Proof

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Given $y = \ln x = \log_e x$

Then $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{e^y}$$

$$= \frac{1}{x}$$

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EXAMPLE 4

a Differentiate $(\ln x + 1)^3$.

b Find the equation of the tangent to the curve $y = \ln x$ at the point $(3, \ln 3)$.

Solution

a $(\ln x + 1)^3$ is a composite function in the form $y = [f(x)]^n$.

$$\begin{aligned}\frac{dy}{dx} &= f'(x)\, nf(x)^{n-1}\\ &= \frac{1}{x} \times 3(\ln x + 1)^2\\ &= \frac{3(\ln x + 1)^2}{x}\end{aligned}$$

b $\dfrac{dy}{dx} = \dfrac{1}{x}$

At $(3, \ln 3)$

$$\frac{dy}{dx} = \frac{1}{3}$$

So $m = \dfrac{1}{3}$

Equation:

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - \ln 3 &= \frac{1}{3}(x - 3)\\ 3y - 3\ln 3 &= x - 3\\ 0 &= x - 3y - 3 + 3\ln 3\end{aligned}$$

Chain rule

If $y = \ln f(x)$, then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$ where $f(x) > 0$

Proof

$y = \ln f(x)$ is a composite function.

Let $y = \ln u$ and $u = f(x)$

$\dfrac{dy}{du} = \dfrac{1}{u}$ and $\dfrac{du}{dx} = f'(x)$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{u} \times f'(x)\\ &= \frac{1}{f(x)} \times f'(x)\\ &= \frac{f'(x)}{f(x)}\end{aligned}$$

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EXAMPLE 5

a Differentiate:

i $\ln(x^2 - 3x + 1)$

ii $\ln\left(\dfrac{x+1}{3x-4}\right)$

b Find the gradient of the normal to the curve $y = \ln(x^3 - 5)$ at the point where $x = 2$.

Solution

a i
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{2x-3}{x^2-3x+1}$$

ii It is easier to simplify first using log laws.

$$y = \ln\left(\frac{x+1}{3x-4}\right)$$
$$= \ln(x+1) - \ln(3x-4)$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{1}{x+1} - \frac{3}{3x-4}$$
$$= \frac{1(3x-4)}{(x+1)(3x-4)} - \frac{3(x+1)}{(3x-4)(x+1)}$$
$$= \frac{3x-4-3(x+1)}{(x+1)(3x-4)}$$
$$= \frac{3x-4-3x-3}{(x+1)(3x-4)}$$
$$= \frac{-7}{(x+1)(3x-4)}$$

b
$$\frac{dy}{dx} = \frac{3x^2}{x^3-5}$$

When $x = 2$

$$\frac{dy}{dx} = \frac{3(2)^2}{2^3-5}$$
$$= 4$$

$m_1 = 4$

The normal is perpendicular to the tangent:

$$m_1 m_2 = -1$$
$$4m_2 = -1$$
$$m_2 = -\frac{1}{4}$$

We can differentiate logarithmic functions with a different base, a.

EXAMPLE 6

Differentiate $y = \log_2 x$.

Solution

$$y = \log_2 x$$

$$= \frac{\ln x}{\ln 2} \text{ using the change of base law}$$

$$= \frac{1}{\ln 2} \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln 2} \times \frac{1}{x}$$

ln 2 is a constant

$$= \frac{1}{x \ln 2}$$

Derivative of $\log_a x$

If $y = \log_a x$, then $\frac{dy}{dx} = \frac{1}{x \ln a}$

The proof of this has the same steps as in the above example.

Exercise 4.03 Derivative of logarithmic functions

1 Differentiate:

a	$x + \ln x$	b	$1 - \ln 3x$	c	$\ln (3x + 1)$
d	$\ln (x^2 - 4)$	e	$\ln (5x^3 + 3x - 9)$	f	$\ln (5x + 1) + x^2$
g	$3x^2 + 5x - 5 + \ln 4x$	h	$\ln (8x - 9) + 2$	i	$\ln (2x + 4)(3x - 1)$
j	$\ln \left(\frac{4x+1}{2x-7}\right)$	k	$(1 + \ln x)^5$	l	$(\ln x - x)^9$
m	$(\ln x)^4$	n	$(x^2 + \ln x)^6$	o	$x \ln x$
p	$\frac{\ln x}{x}$	q	$(2x + 1)\ln x$	r	$x^3 \ln (x + 1)$
s	$\ln (\ln x)$	t	$\frac{\ln x}{x - 2}$	u	$\frac{e^{2x}}{\ln x}$
v	$e^x \ln x$	w	$5(\ln x)^2$		

2 Find $f'(1)$ if $f(x) = \ln \sqrt{2 - x}$.

3 Find the derivative of $\log_{10} x$.

4 Find the equation of the tangent to the curve $y = \ln x$ at the point $(2, \ln 2)$.

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5 Find the equation of the tangent to the curve $y = \ln(x - 1)$ at the point where $x = 2$.

6 Find the gradient of the normal to the curve $y = \ln(x^4 + x)$ at the point $(1, \ln 2)$.

7 Find the exact equation of the normal to the curve $y = \ln x$ at the point where $x = 5$.

8 Find the equation of the tangent to the curve $y = \ln(5x + 4)$ at the point where $x = 3$.

9 Find the derivative of $\log_3(2x + 5)$.

10 Find the equation of the normal to the curve $y = \log_2 x$ at the point where $x = 2$.

11 The formula for the time t in years for kangaroo population growth on Kangaroo Island is given by $t = \dfrac{\ln\left(\dfrac{P}{20\,000}\right)}{0.021}$.

a What is the initial population?

b Find correct to one decimal place the time it takes for the population to grow to:

i 25 000

ii 50 000

c Change the subject of the equation to P.

d Find correct to the nearest whole number the average rate of change in population between 2 and 5 years.

e Find correct to the nearest whole number the rate at which the population is growing after:

i 3 years

ii 5 years

iii 10 years

CLASS INVESTIGATION

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

1 Draw the derivative (gradient) function of sine, cosine and tangent functions.

What is the shape of the derivative function of each graph?

2 By substituting values of x in radians close to 0, find approximations to $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$ and $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$.

3 Differentiate by first principles to find the derivative of each trigonometric function using the above limits. The sine function will use the EXT1 trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

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Derivatives of trigonometric functions

Trigonometric functions and gradient

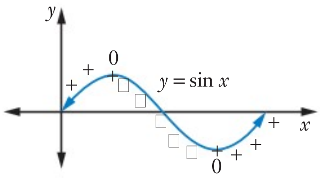
Further trigonometric equations

Differentiating trigonometric functions

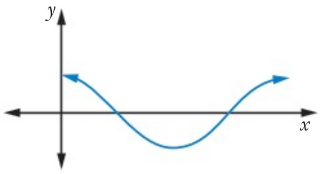
4.04 Derivative of trigonometric functions

Derivative of sin x

We can sketch the derivative (gradient) function of y = sin x.



The sketch of the gradient function is y = cos x.



Derivative of sin x

If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$

Proof

This proof uses trigonometric results from the investigation on the previous page.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \lim_{h \to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \sin x \times 0 + \cos x \times 1$$

$$= \cos x$$

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EXAMPLE 7

a Differentiate $y = x \sin x$.

b Find the equation of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

Solution

a $y = x \sin x$ is in the form $y = uv$

where $u = x$ and $v = \sin x$

$u' = 1$ and $v' = \cos x$

$$\begin{aligned} y' &= u'v + v'u \\ &= 1 \times \sin x + \cos x \times x \\ &= \sin x + x \cos x \end{aligned}$$

b $\dfrac{dy}{dx} = \cos x$

At $(\pi, 0)$

$$\begin{aligned} \frac{dy}{dx} &= \cos \pi \\ &= -1 \end{aligned}$$

So $m = -1$

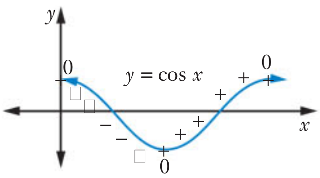
Equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1(x - \pi) \\ y &= -x + \pi \end{aligned}$$

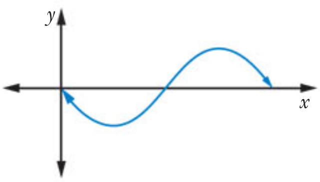
or $x + y - \pi = 0$

Derivative of cos x

We can sketch the derivative (gradient) function of $y = \cos x$.



The sketch of the gradient function below is $y = -\sin x$.



Derivative of cos x

If $y = \cos x$, then $\dfrac{dy}{dx} = -\sin x$

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You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\cos x$ into $\sin\left(\frac{\pi}{2} - x\right)$ and using the derivative of $y = \sin x$.

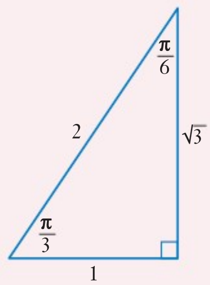
EXAMPLE 8

a Find the derivative of $y = \cos x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = \cos x$ at this point.

Solution

a $\frac{dy}{dx} = -\sin x$



When $x = \frac{\pi}{3}$

$$\frac{dy}{dx} = -\sin\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

b When $x = \frac{\pi}{3}$

$$y = \cos\frac{\pi}{3}$$

$$= \frac{1}{2}$$

Equation:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$2y - 1 = -\sqrt{3}\left(x - \frac{\pi}{3}\right) \quad \text{(multiplying both sides by 2)}$$

$$= -\sqrt{3}x + \frac{\pi\sqrt{3}}{3}$$

$$6y - 3 = -3\sqrt{3}x + \pi\sqrt{3} \quad \text{(multiplying both sides by 3)}$$

$$3\sqrt{3}x + 6y - 3 - \pi\sqrt{3} = 0$$

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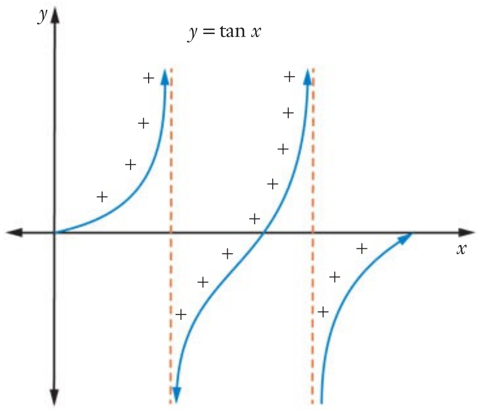
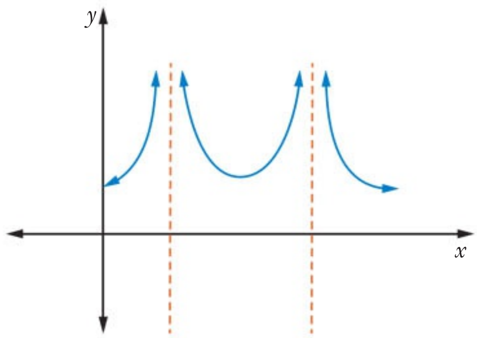
Derivative of tan x

We can sketch the derivative (gradient) function of $y = \tan x$. Notice that the gradient function will have asymptotes in the same place as the original graph, because this is where the tangent is vertical and the gradient is undefined.



The gradient function is $y = \sec^2 x$, where $\sec x = \dfrac{1}{\cos x}$.



Derivative of tan x

$$\text{If } y = \tan x, \text{ then } \frac{dy}{dx} = \sec^2 x$$

You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\tan x$ into $\dfrac{\sin x}{\cos x}$ and using the quotient rule.

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EXAMPLE 9

a Differentiate $y = \dfrac{\tan x}{3x^2}$.

b Find the gradient of the tangent to the curve $f(x) = \tan x$ at the point where $x = \dfrac{\pi}{4}$.

Solution

a $y = \dfrac{\tan x}{3x^2}$ is in the form $y = \dfrac{u}{v}$.

$u = \tan x$ and $v = 3x^2$

$u' = \sec^2 x$ $\quad v' = 6x$

$$y' = \frac{u'v - v'u}{u^2}$$

$$= \frac{\sec^2 x \times 3x^2 - 6x \times \tan x}{(3x^2)^2}$$

$$= \frac{3x(x \sec^2 x - 2\tan x)}{9x^4}$$

$$= \frac{x \sec^2 x - 2\tan x}{3x^3}$$

b $\dfrac{dy}{dx} = \sec^2 x$

At $x = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \sec^2 \frac{\pi}{4}$$

$$= (\sqrt{2})^2$$

$$= 2$$

Chain rule

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x) \cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x) \sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x) \sec^2 f(x)$

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Here is the proof for $y = \sin f(x)$. The others are similar.

Proof

$y = \sin f(x)$ is a composite function

where $y = \sin u$ and $u = f(x)$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times f'(x) \\ &= f'(x) \cos u \\ &= f'(x) \cos f(x)\end{aligned}$$

EXAMPLE 10

a Differentiate each function.

i $y = \sin 7x$

ii $y = \cos\left(4x^3 + \frac{\pi}{3}\right)$

iii $y = \tan(5x - \pi)$

b Find the gradient of the normal to the curve $f(x) = \cos \frac{x}{2}$ at the point where $x = \pi$.

Solution

a i $\frac{dy}{dx} = f'(x) \cos f(x)$

$= 7 \cos 7x$

ii $\frac{dy}{dx} = -f'(x) \sin f(x)$

$= -12x^2 \sin\left(4x^3 + \frac{\pi}{3}\right)$

iii $\frac{dy}{dx} = f'(x) \sec^2 f(x)$

$= 5 \sec^2(5x - \pi)$

b $\frac{dy}{dx} = -f'(x) \sin f(x)$

$= -\frac{1}{2} \sin \frac{x}{2}$

At $x = \pi$:

$\frac{dy}{dx} = -\frac{1}{2} \sin \frac{\pi}{2}$

$= -\frac{1}{2} \times 1$

$= -\frac{1}{2}$

So $m_1 = -\frac{1}{2}$

Normal is perpendicular to the tangent:

$m_1 m_2 = -1$

$-\frac{1}{2} m_2 = -1$

$m_2 = 2$

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While trigonometric functions are usually expressed in radians, we can differentiate angles in degrees by using the conversion $\pi = 180°$.

EXAMPLE 11

Differentiate $y = \sin x°$.

Solution

$180° = \pi$ radians

$$1° = \frac{\pi}{180}$$

$$x° = \frac{\pi x}{180}$$

So $y = \sin x°$ becomes

$$y = \sin \frac{\pi x}{180}$$

$$\frac{dy}{dx} = f'(x) \cos f(x)$$

$$= \frac{\pi}{180} \cos \frac{\pi x}{180}$$

$$= \frac{\pi}{180} \cos x°$$

We can also differentiate composite functions involving trigonometric functions.

EXAMPLE 12

Differentiate:

a $\tan (e^x)$

b $\ln (\cos x)$

Solution

a
$$\frac{dy}{dx} = f'(x) \sec^2 f(x)$$
$$= e^x \sec^2 (e^x)$$

b
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{-\sin x}{\cos x}$$
$$= -\tan x$$

Exercise 4.04 Derivative of trigonometric functions

1 Differentiate:

a	$\sin 4x$	b	$\cos 3x$	c	$\tan 5x$
d	$\tan (3x + 1)$	e	$\cos (-x)$	f	$3 \sin x$
g	$4 \cos (5x - 3)$	h	$2 \cos (x^3)$	i	$7 \tan (x^2 + 5)$
j	$\sin 3x + \cos 8x$	k	$\tan (\pi + x) + x^2$	l	$x \tan x$
m	$\sin 2x \tan 3x$	n	$\frac{\sin x}{2x}$	o	$\frac{3x+4}{\sin 5x}$

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p	$(2x + \tan 7x)^9$	q	$\sin^2 x$	r	$3 \cos^3 5x$
s	$e^x - \cos 2x$	t	$\sin (1 - \ln x)$	u	$\sin (e^x + x)$
v	$\ln (\sin x)$	w	$e^{3x} \cos 2x$	x	$\dfrac{e^{2x}}{\tan 7x}$

2 Find the gradient of the tangent to the curve $y = \tan 3x$ at the point where $x = \dfrac{\pi}{9}$.

3 Find the equation of the tangent to the curve $y = \sin (\pi - x)$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$ in exact form.

4 Differentiate $\ln (\cos x)$.

5 Find the exact gradient of the normal to $y = \sin 3x$ at the point where $x = \dfrac{\pi}{18}$.

6 Differentiate $e^{\tan x}$.

7 Find the equation of the normal to the curve $y = 3 \sin 2x$ at the point where $x = \dfrac{\pi}{8}$ in exact form.

8 Show that $\dfrac{d}{dx} [\ln (\tan x)] = \tan x + \cot x$.

9 Differentiate each function.

a	$y = \tan x°$	b	$y = 3 \cos x°$	c	$y = \dfrac{\sin x°}{5}$

10 Find the derivative of $\cos x \sin^4 x$.

11 The population of salmon in a salmon farm grows and reduces as fish are born and sold. The population is given by $P = 225 \cos \dfrac{2\pi t}{9} + 750$ where t is time in days.

- a What is the centre of the population?
- b What is the minimum number of salmon in the farm at any one time?
- c What is the maximum population?
- d At what times is the population 700?
- e At what rate is the population changing after:
  - i 3 days? ii a week? iii 10 days? iv 18 days?
- f At what times is the population growing at the rate of 25 fish per day?

12 The tide was measured over time at a beach at Merimbula and given the formula $D = 8 \sin \dfrac{\pi t}{6} + 9$ where D is depth of water in metres and t is time in hours.

- a How deep was the water:
  - i initially? ii after 5 hours?
- b When was the water 10 m deep?
- c At what rate was the depth changing after:
  - i 3 hours? ii 11 hours? iii 12 hours?
- d At what times was the depth of water decreasing by 3 m h^{-1}?

The second derivative

First and second derivatives

4.05 Second derivatives

Second derivative

Differentiating $f(x)$ gives $f'(x)$, the first derivative.

Differentiating $f'(x)$ gives $f''(x)$, the **second derivative**.

It is also possible to differentiate further.

Using function notation, differentiating several times gives $f'(x)$, $f''(x)$, $f'''(x)$ and so on.

Using $\frac{dy}{dx}$ notation, differentiating several times gives $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and so on.

The notation $\frac{d^2y}{dx^2}$ comes from $\frac{d^2}{dx^2}(y)$.

EXAMPLE 13

a Find the first 4 derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$.

b Find the second derivative of $y = (2x + 5)^7$.

c If $f(x) = 4 \cos 3x$, show that $f''(x) = -9\, f(x)$

Solution

a
$$f'(x) = 3x^2 - 8x + 3$$
$$f''(x) = 6x - 8$$
$$f'''(x) = 6$$
$$f''''(x) = 0$$

b
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times nf(x)^{n-1}\\ &= 2 \times 7(2x + 5)^6\\ &= 14(2x + 5)^6\end{aligned}$$
$$\begin{aligned}\frac{d^2y}{dx^2} &= f'(x) \times nf(x)^{n-1}\\ &= 2 \times 6 \times 14(2x + 5)^5\\ &= 168(2x + 5)^5\end{aligned}$$

c
$$\begin{aligned}f'(x) &= -f'(x) \times \sin f(x)\\ &= -3 \times 4 \sin 3x\\ &= -12 \sin 3x\end{aligned}$$
$$\begin{aligned}f''(x) &= f'(x) \times \cos f(x)\\ &= 3 \times (-12 \cos 3x)\\ &= -36 \cos 3x\\ &= -9(4 \cos 3x)\\ &= -9f(x) && \text{since } f(x) = 4 \cos 3x\end{aligned}$$

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Exercise 4.05 Second derivatives

1 Find the first 4 derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$, find $f''(x)$.

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find $f'(1)$ and $f''(-2)$, given $f(t) = 3t^4 - 2t^3 + 5t - 4$.

5 Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6 Find the first and second derivatives of $y = 2x^2 - 3x + 3$.

7 If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

8 Find the first and second derivatives of x^{-4}.

9 If $g(x) = \sqrt{x}$, find $g''(4)$.

10 Given $h = 5t^3 - 2t^2 + t + 5$, find $\frac{d^2h}{dt^2}$ when $t = 1$.

11 Find any values of x for which $\frac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

12 Find all values of x for which $f''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$.

13 Find the first and second derivatives of $(4x - 3)^5$.

14 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2 - x}$.

15 Find the first and second derivatives of $f(x) = \frac{x+5}{3x-1}$.

16 Find $\frac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

17 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\frac{d^2y}{dx^2} = -2$ when $x = \frac{1}{2}$.

18 Find $f''(1)$ if $f(t) = t(2t - 1)^7$.

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

20 If $y = e^{4x} + e^{-4x}$, show that $\frac{d^2y}{dx^2} = 16y$.

21 Prove that $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ given $y = 3e^{2x}$.

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22 Show that $\frac{d^2y}{dx^2} = b^2y$ for $y = ae^{bx}$.

23 Find the value of n if $y = e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ny = 0$.

24 Show that $\frac{d^2y}{dx^2} = -25y$ if $y = 2\cos 5x$.

25 Given $f(x) = -2\sin x$, show that $f''(x) = -f(x)$.

26 If $y = 2\sin 3x - 5\cos 3x$, show that $\frac{d^2y}{dx^2} = -9y$.

27 Find values of a and b if $\frac{d^2y}{dx^2} = ae^{3x}\cos 4x + be^{3x}\sin 4x$, given $y = e^{3x}\cos 4x$.

28 Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x-4}$.

29 The displacement of a particle moving in a straight line is given by $x = 2t^3 - 5t^2 + 7t + 8$, where x is in metres and t is in seconds.

a Find the initial displacement.

b Find the displacement after 3 seconds.

c Find the velocity after 3 seconds.

d Find the acceleration after 3 seconds.

30 The height in cm of a pendulum as it swings is given by $h = 8\cos \pi t + 12$ where t is time in seconds.

a What is the height of the pendulum after 3 s?

b What is the maximum and minimum height of the pendulum?

c What is the velocity of the pendulum after:

i 1 s?

ii 1.5 s?

d What is the acceleration of the pendulum:

i initially?

ii after 1 s?

iii after1.5 s?

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4.06 Anti-derivative graphs

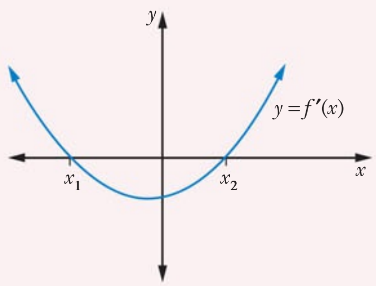
WS
Shapes of curves

The process of finding the original function y = f(x) given the derivative y = f ′(x) is called anti-differentiation, and the original function is called the anti-derivative function, also called the primitive or integral function.

Anti-derivative graphs

EXAMPLE 14

Sketch the graph of the anti-derivative (primitive function) given the graph of the derivative function below and an initial condition, or starting point, of (0, 2).



Solution

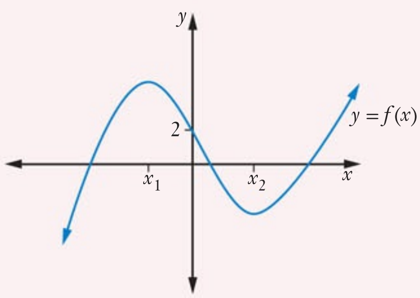
Remember that when you sketch a derivative function, the x-intercepts are where the original function has zero gradient, or stationary (turning) points.

On this graph the stationary points are at $x = x_1$ and $x = x_2$.

Above the x-axis shows where the original function has a positive gradient (it is increasing). On this graph, this is where $x < x_1$ and $x > x_2$.

Below the x-axis shows where the original function has a negative gradient (it is decreasing). On this graph, this is where $x_1 < x < x_2$.

We can sketch this information together with the point (0, 2):



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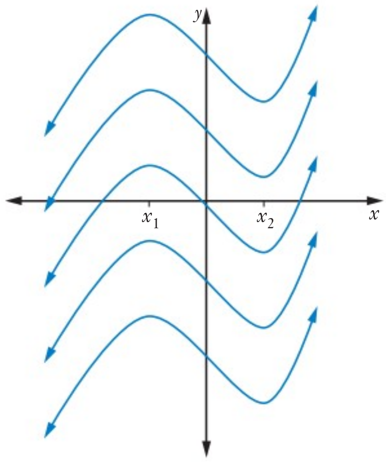
We are not given enough information to sketch a unique graph. There is no way of knowing what the y values of the stationary points are or the stretch or compression of the graph. Also, if we are not given a fixed point on the function, we could sketch many graphs that satisfy the information from the derivative function.

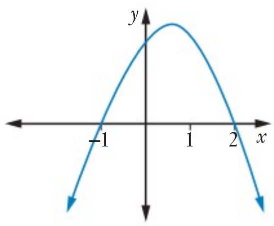
The anti-derivative gives a family of curves.



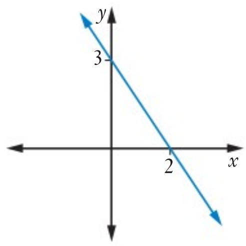
Exercise 4.06 Anti-derivative graphs

1 For each function graphed, sketch the graph of the anti-derivative function given it passes through:

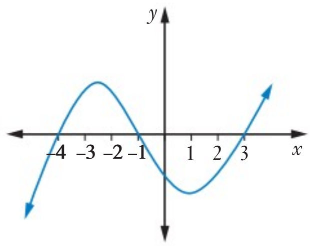
a (0, –1)



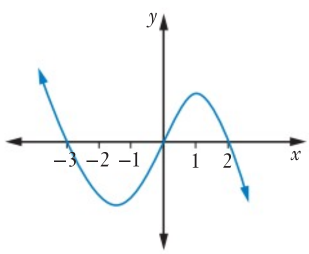
b (1, 2)



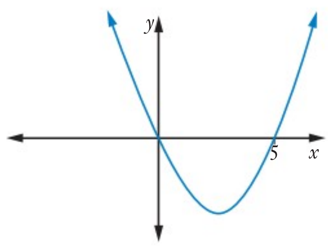
c (0, 3)



d (–1, –1)



e (0, 1)



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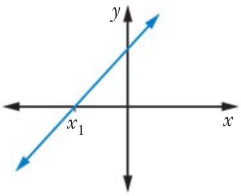
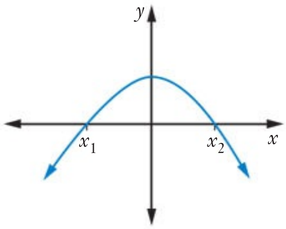
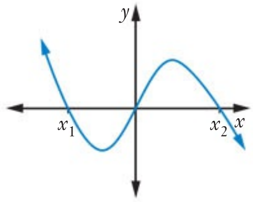
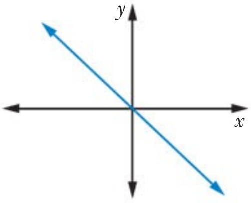
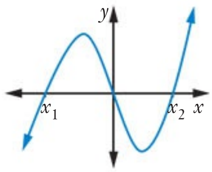
2 Sketch a family of graphs that could represent the anti-derivative function of each graph.

a


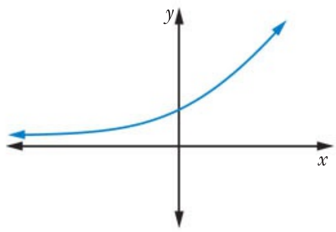
b


c


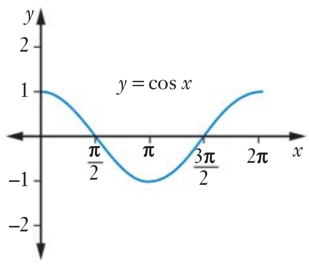
d


e


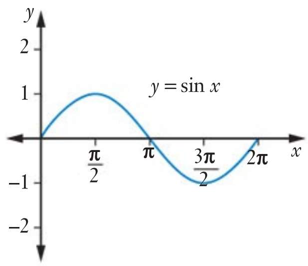
3 The anti-derivative function of the graph below passes through (0, −1). Sketch its graph.



4 Sketch the graph of the anti-derivative function of y = cos x given that it passes through (0, 0).



5 Sketch a family of anti-derivative functions for the graph.



INVESTIGATION

ANTI-DERIVATIVE OF $y = x^n$

1 Differentiate:

a x^2 b $x^2 + 5$ c $x^2 - 7$ d $x^2 + 4$ e $x^2 - 75$

What would be the anti-derivative of $2x$?

2 Differentiate:

a x^3 b $x^3 + 1$ c $x^3 + 6$ d $x^3 - 2$ e $x^3 - 14$

What would be the anti-derivative of $3x^2$?

3 Differentiate:

a x^4 b $x^4 - 3$ c $x^4 + 2$ d $x^4 + 10$ e $x^4 - 1$

What would be the anti-derivative of $4x^3$?

4 Differentiate:

a x^n b $x^n + 7$ c $x^n + 9$ d $x^n - 5$ e $x^n - 2$

What would be the anti-derivative of nx^{n-1}?

Can you find a general rule for anti-derivatives that would work for these examples?

4.07 Anti-derivatives

WS Anti-derivatives

WS Antiderivatives

WS Antidifferentiation

Since anti-differentiation is the reverse of differentiation, we can find the equation of an anti-derivative function.

Anti-derivative of x^n

$$\text{If } \frac{dy}{dx} = x^n, \text{ then } y = \frac{1}{n+1}x^{n+1} + C \text{ where } C \text{ is a constant.}$$

Proof

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) &= \frac{(n+1)x^n}{n+1}\\ &= x^n\end{aligned}$$

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We can apply the same rules to anti-derivatives as we use for derivatives. Here are some of the main ones we use.

Anti-derivative rules

If $\dfrac{dy}{dx} = k$ then $y = kx$.

If $\dfrac{dy}{dx} = kx^n$ then $y = \dfrac{1}{n+1}kx^{n+1} + C$.

If $\dfrac{dy}{dx} = f(x) + g(x)$ then $y = F(x) + G(x) + C$ where $F(x)$ and $G(x)$ are the anti-derivatives of $f(x)$ and $g(x)$ respectively.

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EXAMPLE 15

Find the anti-derivative of $x^4 - 4x^3 + 9x^2 - 6x + 5$.

Solution

If $f(x) = x^4 - 4x^3 + 9x^2 - 6x + 5$

$$F(x) = \frac{1}{5}x^5 - 4 \times \frac{1}{4}x^4 + 9 \times \frac{1}{3}x^3 - 6 \times \frac{1}{2}x^2 + 5x + C$$

Anti-derivative of 5 is 5x.

$$= \frac{x^5}{5} - x^4 + 3x^3 - 3x^2 + 5x + C$$

If we have some information about the anti-derivative function, we can use this to evaluate the constant C.

The anti-derivative function

EXAMPLE 16

a The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$. If the curve passes through the point $(1, -3)$, find its equation.

b If $f''(x) = 6x + 2$ and $f'(1) = f(-2) = 0$, find $f(3)$.

Solution

a $\frac{dy}{dx} = 6x^2 + 8x$

So $y = 6 \times \frac{1}{3}x^3 + 8 \times \frac{1}{2}x^2 + C$

$= 2x^3 + 4x^2 + C$

Substitute $(1, -3)$:

$-3 = 2(1)^3 + 4(1)^2 + C$

$= 6 + C$

$-9 = C$

Equation is $y = 2x^3 + 4x^2 - 9$.

b $f''(x) = 6x + 2$

$f'(x) = 6 \times \frac{1}{2}x^2 + 2 \times \frac{1}{1}x^1 + C$

$= 3x^2 + 2x + C$

Since $f'(1) = 0$:

$0 = 3(1)^2 + 2(1) + C$

$= 5 + C$

$-5 = C$

So $f'(x) = 3x^2 + 2x - 5$

$f(x) = 3 \times \frac{1}{3}x^3 + 2 \times \frac{1}{2}x^2 - 5 \times \frac{1}{1}x^1 + D$

$= x^3 + x^2 - 5x + D$

Since $f(-2) = 0$:

$0 = (-2)^3 + (-2)^2 - 5(-2) + D$

$= -8 + 4 + 10 + D$

$= 6 + D$

$-6 = D$

Equation is $f(x) = x^3 + x^2 - 5x - 6$

$f(3) = 3^3 + 3^2 - 5(3) - 6$

$= 27 + 9 - 15 - 6$

$= 15$

Chain rule

If $\frac{dy}{dx} = (ax + b)^n$, then $y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$ where C is a constant, $a \neq 0$ and $n \neq -1$.

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Proof

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}+C\right)=\frac{a(n+1)(ax+b)^n}{a(n+1)}$$

$$=(ax+b)^n$$

EXAMPLE 17

a Find the anti-derivative of $(3x+7)^8$.

b The gradient of a curve is given by $\frac{dy}{dx}=(2x-3)^4$. If the curve passes through the point $(2,-7)$, find its equation.

Solution

a $\frac{dy}{dx}=(3x+7)^8$

$$y=\frac{1}{a(n+1)}(ax+b)^{n+1}+C$$

$$=\frac{1}{3(8+1)}(3x+7)^{8+1}+C$$

$$=\frac{1}{27}(3x+7)^9+C$$

$$=\frac{(3x+7)^9}{27}+C$$

b $\frac{dy}{dx}=(2x-3)^4$

$$y=\frac{1}{a(n+1)}(ax+b)^{n+1}+C$$

$$=\frac{1}{2(4+1)}(2x-3)^{4+1}+C$$

$$=\frac{1}{10}(2x-3)^5+C$$

Substitute $(2,-7)$:

$$-7=\frac{1}{10}(2\times 2-3)^5+C$$

$$=\frac{1}{10}(1)^5+C$$

$$=\frac{1}{10}+C$$

$$-7\frac{1}{10}=C$$

So the equation is $y=\frac{1}{10}(2x-3)^5-7\frac{1}{10}$

$$=\frac{(2x-3)^5-71}{10}$$

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General chain rule

If $\dfrac{dy}{dx} = f'(x)[f(x)]^n$ then $y = \dfrac{1}{n+1}[f(x)]^{n+1} + C$ where C is a constant and $n \neq -1$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}[f(x)]^{n+1} + C\right) = \frac{1}{n+1} f'(x)(n+1)[f(x)]^{n+1-1}$$
$$= f'(x)[f(x)]^n$$

EXAMPLE 18

Find the anti-derivative of:

a $8x^3(2x^4 - 1)^5$

b $x^2(x^3 + 2)^7$

Solution

a Given $f(x) = 2x^4 - 1$

$$f'(x) = 8x^3$$
$$\frac{dy}{dx} = 8x^3(2x^4 - 1)^5$$
$$= f'(x)[f(x)]^n$$
$$y = \frac{1}{n+1} f(x)^{n+1} + C$$
$$= \frac{1}{5+1}(2x^4 - 1)^{5+1} + C$$
$$= \frac{1}{6}(2x^4 - 1)^6 + C$$
$$= \frac{(2x^4 - 1)^6}{6} + C$$

b Given $f(x) = x^3 + 2$

$$f'(x) = 3x^2$$
$$\frac{dy}{dx} = x^2(x^3 + 2)^7$$
$$= \frac{1}{3} \times 3x^2(x^3 + 2)^7$$
$$= \frac{1}{3} f'(x)[f(x)]^n$$
$$y = \frac{1}{3} \times \frac{1}{n+1} f(x)^{n+1} + C$$
$$= \frac{1}{3} \times \frac{1}{7+1}(x^3 + 2)^{7+1} + C$$
$$= \frac{1}{24}(x^3 + 2)^8 + C$$
$$= \frac{(x^3 + 2)^8}{24} + C$$

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Exercise 4.07 Anti-derivatives

1 Find the anti-derivative of:

a $2x - 3$ b $x^2 + 8x + 1$ c $x^5 - 4x^3$

d $(x - 1)^2$ e 6 f $(3x + 2)^5$

g $8(2x - 7)^4$

2 Find f(x) if:

a $f'(x) = 6x^2 - x$ b $f'(x) = x^4 - 3x^2 + 7$ c $f'(x) = x - 2$

d $f'(x) = (x + 1)(x - 3)$ e $f'(x) = x^{\frac{1}{2}}$

3 Express y in terms of x if:

a $\frac{dy}{dx} = 5x^4 - 9$ b $\frac{dy}{dx} = x^{-4} - 2x^{-2}$ c $\frac{dy}{dx} = \frac{x^3}{5} - x^2$

d $\frac{dy}{dx} = \frac{2}{x^2}$ e $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4 Find the anti-derivative of:

a $\sqrt{x}$ b x^{-3} c $\frac{1}{x^8}$

d $x^{-\frac{1}{2}} + 2x^{-\frac{2}{3}}$ e $x^{-7} - 2x^{-2}$

5 Find the anti-derivative of:

a $2x(x^2 + 5)^4$ b $3x^2(x^3 - 1)^9$ c $8x(2x^2 + 3)^3$

d $15x^4(x^5 + 1)^6$ e $x(x^2 - 4)^7$ f $x^5(2x^6 - 7)^8$

g $(2x - 1)(x^2 - x + 3)^4$ h $(3x^2 + 4x - 7)(x^3 + 2x^2 - 7x)^{10}$

i $(x - 3)(x^2 - 6x - 1)^5$

6 If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$, find an equation for y in terms of x.

7 If $f'(x) = 4x - 7$ and $f(2) = 5$, find an equation for $y = f(x)$.

8 Given $f'(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$, find the value of $f(1)$.

9 Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve.

10 If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$, find x when $t = 4$.

11 Given $\frac{d^2y}{dx^2} = 8$, and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$, find the equation of y in terms of x.

12 If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$, find the equation of the curve.

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13 If $f''(x) = 6x - 2$ and $f'(2) = f(2) = 7$, find the equation of the function $y = f(x)$.

14 Given $f''(x) = 5x^4$, $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$.

15 A curve has $\frac{d^2y}{dx^2} = 8x$ and the tangent at $(-2, 5)$ has an angle of inclination of 45° with the x-axis. Find the equation of the curve.

16 The tangent to a curve with $\frac{d^2y}{dx^2} = 2x - 4$ makes an angle of inclination of 135° with the x-axis at the point $(2, -4)$. Find its equation.

17 A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$, and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function.

18 A curve has $\frac{d^2y}{dx^2} = 6$ and the tangent at $(-1, 3)$ is perpendicular to the line $2x + 4y - 3 = 0$. Find the equation of the curve.

19 A function has $f'(1) = 3$ and $f(1) = 5$. Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20 The velocity of an object is given by $\frac{dx}{dt} = 6t - 5$. If the object has initial displacement of -2, find the equation for the displacement.

21 The acceleration of a particle is given by $\frac{d^2x}{dt^2} = 24t^2 - 12t + 6 \text{ m s}^{-2}$. Its velocity $\frac{dx}{dt} = 0$ when $t = 1$ and its displacement $x = -3$ when $t = 0$. Find the equation for its displacement.

4.08 Further anti-derivatives

Anti-derivative of exponential functions

$$\text{If } \frac{dy}{dx} = e^x, \text{ then } y = e^x + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = e^{ax+b}, \text{ then } y = \frac{1}{a}e^{ax+b} + C$$

$$\text{If } \frac{dy}{dx} = f'(x)e^{f(x)}, \text{ then } y = e^{f(x)} + C$$

Proof (by differentiation)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a}e^{ax+b} + C\right) &= \frac{1}{a} \times ae^{ax+b}\\ &= e^{ax+b}\end{aligned}$$

$$\frac{d}{dx}[e^{f(x)} + C] = f'(x)e^{f(x)}$$

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EXAMPLE 19

a Find the anti-derivative of $e^{4x} + 1$.

b Find the equation of the function $y = f(x)$ given $f'(x) = 6e^{3x}$ and $f(2) = 2e^6$.

Solution

a $\frac{1}{a}e^{ax+b} + C = \frac{1}{4}e^{4x} + C$

b
$$f'(x) = 6e^{3x}$$
$$f(x) = 6 \times \frac{1}{3}e^{3x} + C$$
$$= 2e^{3x} + C$$

If $f(2) = 2e^6$:
$$2e^6 = 2e^{3 \times 2} + C$$
$$= 2e^6 + C$$
$$0 = C$$
So $f(x) = 2e^{3x}$

Anti-derivative of $\frac{1}{x}$

$$\text{If } \frac{dy}{dx} = \frac{1}{x}, \text{ then } y = \ln|x| + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = \frac{f'(x)}{f(x)}, \text{ then } y = \ln|f(x)| + C$$

Proof

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$, because $\ln x$ is defined only for $x > 0$.

So the anti-derivative of $\frac{1}{x}$ when $x > 0$ is $\ln x$.

Suppose $x < 0$.

Then $\ln(-x)$ is defined because $-x$ is positive.

$$\frac{d}{dx}[\ln(-x)] = \frac{f'(x)}{f(x)}$$
$$= \frac{-1}{-x}$$
$$= \frac{1}{x}, \qquad x < 0$$

So if $\frac{dy}{dx} = \frac{1}{x}$, then $y = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases}$,

or more simply, $\quad y = \ln|x| + C$

EXAMPLE 20

a Find the anti-derivative of $\frac{3}{x}$.

b Find the equation of the function that has $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$ and passes through $(3, 3 \ln 4)$.

Solution

a
$$\frac{dy}{dx} = \frac{3}{x}$$
$$= 3 \times \frac{1}{x}$$
$$y = 3 \ln|x|$$

b
$$\frac{dy}{dx} = \frac{6x}{x^2 - 5}$$
$$= 3 \times \frac{2x}{x^2 - 5}$$
$$= 3 \times \frac{f'(x)}{f(x)} \text{ where } f(x) = x^2 - 5$$
$$y = 3 \ln f|x| + C$$
$$= 3 \ln|x^2 - 5| + C$$

Substitute $(3, 3 \ln 4)$:
$$3 \ln 4 = 3 \ln|3^2 - 5| + C$$
$$= 3 \ln 4 + C$$
$$0 = C$$
$$\text{So } y = 3 \ln|x^2 - 5|$$

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Anti-derivatives of trigonometric functions

If $\dfrac{dy}{dx} = \cos x$, then $y = \sin x + C$ since $\dfrac{d}{dx}(\sin x) = \cos x$

If $\dfrac{dy}{dx} = \sin x$, then $y = -\cos x + C$ since $\dfrac{d}{dx}(\cos x) = -\sin x$ so $\dfrac{d}{dx}(-\cos x) = \sin x$

If $\dfrac{dy}{dx} = \sec^2 x$, then $y = \tan x + C$ since $\dfrac{d}{dx}(\tan x) = \sec^2 x$

Chain rule

If $\dfrac{dy}{dx} = \cos(ax + b)$, then $y = \dfrac{1}{a}\sin(ax + b) + C$

If $\dfrac{dy}{dx} = \sin(ax + b)$, then $y = -\dfrac{1}{a}\cos(ax + b) + C$

If $\dfrac{dy}{dx} = \sec^2(ax + b)$, then $y = \dfrac{1}{a}\tan(ax + b) + C$

If $\dfrac{dy}{dx} = f'(x)\cos f(x)$, then $y = \sin f(x) + C$

If $\dfrac{dy}{dx} = f'(x)\sin f(x)$, then $y = -\cos f(x) + C$

If $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$, then $y = \tan f(x) + C$

Proof

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)+C\right] &= \frac{1}{a}\times a\cos(ax+b)\\ &= \cos(ax+b)\end{aligned}$$

The other results can be proved similarly.

EXAMPLE 21

a Find the anti-derivative of $\cos 3x$.

b Find the equation of the curve that passes through $\left(\dfrac{\pi}{4}, 3\right)$ and has $\dfrac{dy}{dx} = \sec^2 x$.

Solution

a $$\begin{aligned}y &= \frac{1}{a}\sin(ax+b)+C\\ &= \frac{1}{3}\sin 3x + C\end{aligned}$$

b $y = \tan x + C$

Substitute $\left(\frac{\pi}{4}, 3\right)$:

$3 = \tan \frac{\pi}{4} + C$

$= 1 + C$

$2 = C$

So $y = \tan x + 2$

Exercise 4.08 Further anti-derivatives

1 Find the anti-derivative of:

a $\sin x$ b $\sec^2 x$ c $\cos x$

d $\sec^2 7x$ e $\sin (2x - \pi)$

2 Anti-differentiate:

a e^x b e^{6x} c $\frac{1}{x}$

d $\frac{3}{3x-1}$ e $\frac{x}{x^2+5}$

3 Find the anti-derivative of:

a $e^x + 5$ b $\cos x + 4x$ c $x + \frac{1}{x}$

d $8x^3 - 3x^2 + 6x - 3 + x^{-1}$ e $\sin 5x - \sec^2 9x$

4 Find the equation of a function with $\frac{dy}{dx} = \cos x$ and passing through $\left(\frac{\pi}{2}, -4\right)$.

5 Find the equation of the function that has $f'(x) = \frac{5}{x}$ and $f(1) = 3$.

6 A function has $\frac{dy}{dx} = 4 \cos 2x$ and passes through the point $\left(\frac{\pi}{6}, 2\sqrt{3}\right)$.
Find the exact equation of the function.

7 A curve has $f''(x) = 27e^{3x}$ and has $f(2) = f'(2) = e^6$. Find the equation of the curve.

8 The rate of change of a population over time t years is given by $\frac{dP}{dt} = 1350e^{0.054t}$.
If the initial population is 35 000, find:

a the equation for population b the population after 10 years

9 The velocity of a particle is given by $\frac{dx}{dt} = 3e^{3t}$ and the particle has an initial displacement of 5 metres. Find the equation for displacement of the particle.

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10 A pendulum has acceleration given by $\frac{d^2x}{dt^2} = -9 \sin 3t$, initial displacement 0 cm and initial velocity 3 cm s^{-1}.

a Find the equation for its velocity.

b Find the displacement after 2 seconds.

c Find the times when the pendulum has displacement 0 cm.

Summary of differentiation rules

Rule	Chain rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x)$

Product rule: $\frac{d}{dx}(uv) = u'v + v'u$

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

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x

4. TEST YOURSELF

Qz Practice quiz

For Questions 1 to 4, choose the correct answer A, B, C or D.

1 The anti-derivative of sin 6x is:

A $\frac{1}{6}\cos 6x$ B $6\cos 6x$ C $-6\cos 6x$ D $-\frac{1}{6}\cos 6x$

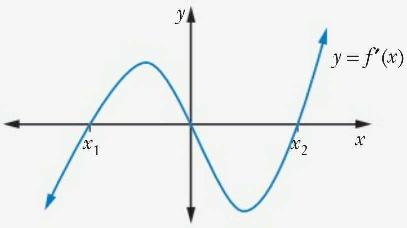
2 The gradient of the tangent to the curve $y = e^{2x} + x$ at (0, 1) is:

A 2 B 3 C e D 1

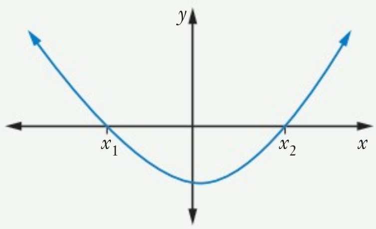
3 If $y = \cos 2x$, then $\frac{d^2y}{dx^2}$ is :

A $-4y$ B $-2y$ C $4y$ D y

4 The graph of the derivative $y = f'(x)$ is shown. Which of the following graphs could be the graph of $y = f(x)$?



A



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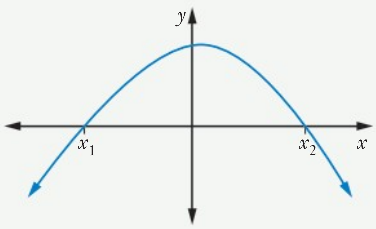
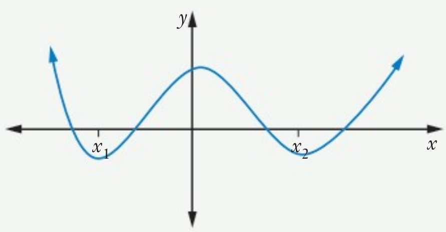
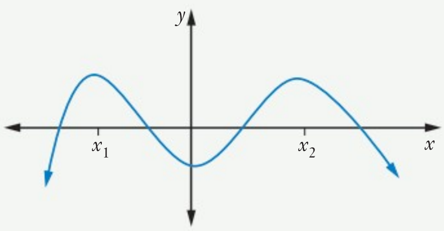


C



D



5 Differentiate:

a e^{5x} b $2e^{1-x}$ c $\ln 4x$ d $\ln(4x + 5)$

e xe^x f $\frac{\ln x}{x}$ g $(e^x + 1)^{10}$

6 Differentiate:

a $\cos x$ b $2\sin x$ c $\tan x + 1$ d $x\sin x$

e $\frac{\tan x}{x}$ f $\cos 3x$ g $\tan 5x$

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7 Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$.

8 Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

9 If $x = \cos 2t$, show that $\frac{d^2x}{dt^2} = -4x$.

10 Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2$.

11 Find the anti-derivative of:

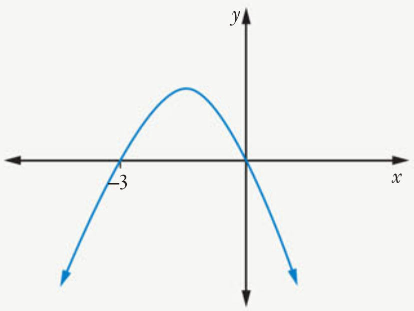
a $10x^4 - 4x^3 + 6x - 3$ b e^{5x} c $\sec^2 9x$

d $\frac{1}{x+5}$ e $\cos 2x$ f $\sin\left(\frac{x}{4}\right)$

12 Find the gradient of the tangent to the curve $y = 3\cos 2x$ at the point where $x = \frac{\pi}{6}$.

13 A curve has $\frac{dy}{dx} = 6x^2 + 12x - 5$. If the curve passes through the point $(2, -3)$, find the equation of the curve.

14 Sketch the graph of the anti-derivative of the following function, given that the anti-derivative passes through $(0, 4)$.



15 Find the equation of the normal to the curve $y = \ln x$ at the point $(2, \ln 2)$.

16 Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

17 Differentiate:

a $(5x^2 + 7)^4$ b $4x(2x - 3)^7$ c $\frac{5x-1}{3x+4}$

d $2x^3e^x$ e $\frac{\tan 3x}{x+1}$

18 If $f''(x) = 15x + 12$ and $f(2) = f'(2) = 5$, find the equation of $y = f(x)$.

19 If $f(x) = 3x^5 - 2x^4 + x^3 - 2$, find:

a $f(-1)$ b $f'(-1)$ c $f''(-1)$

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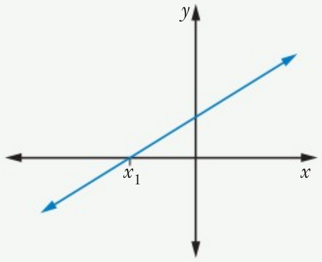
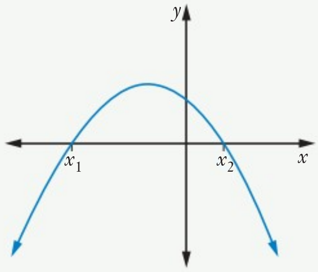
20 Sketch an example of the graph of an anti-derivative function for each graph.

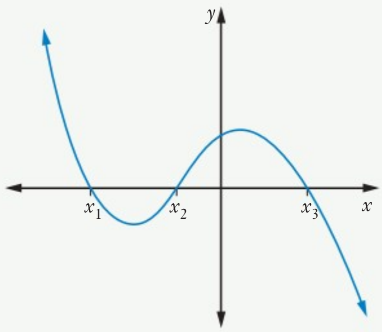
a



b



c



21 A function has $f'(3) = 5$ and $f(3) = 2$. If $f''(x) = 12x - 6$, find the equation of the function.

22 Find the anti-derivative of:

a $x^3(3x^4 - 5)^6$ b $3x(x^2 + 1)^9$

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4. CHALLENGE EXERCISE

1 Find the exact gradient of the tangent to the curve $y = e^{x + \ln x}$ at the point where $x = 1$.

2 Find the first and second derivatives of $\dfrac{5 - x}{(4x^2 + 1)^3}$.

3 Find the anti-derivative of:

a $2xe^{x^2}$ b $x^2 \sin (x^3)$

4 Differentiate $e^{x \sin 2x}$.

5 A curve passes through the point $(0, -1)$ and the gradient at any point is given by $(x + 3)(x - 5)$. Find the equation of the curve.

6 The rate of change of V with respect to t is given by $\dfrac{dV}{dt} = (2t - 1)^2$.
If $V = 5$ when $t = \dfrac{1}{2}$, find V when $t = 3$.

7 Find the derivative of $y = \dfrac{x \log_e x}{e^x}$.

8 a Differentiate $\ln (\tan x)$.

b Find the anti-derivative of $\tan x$.

9 Find the anti-derivative of:

a $x^2 \sin (x^3 - \pi)$ b xe^{x^2}

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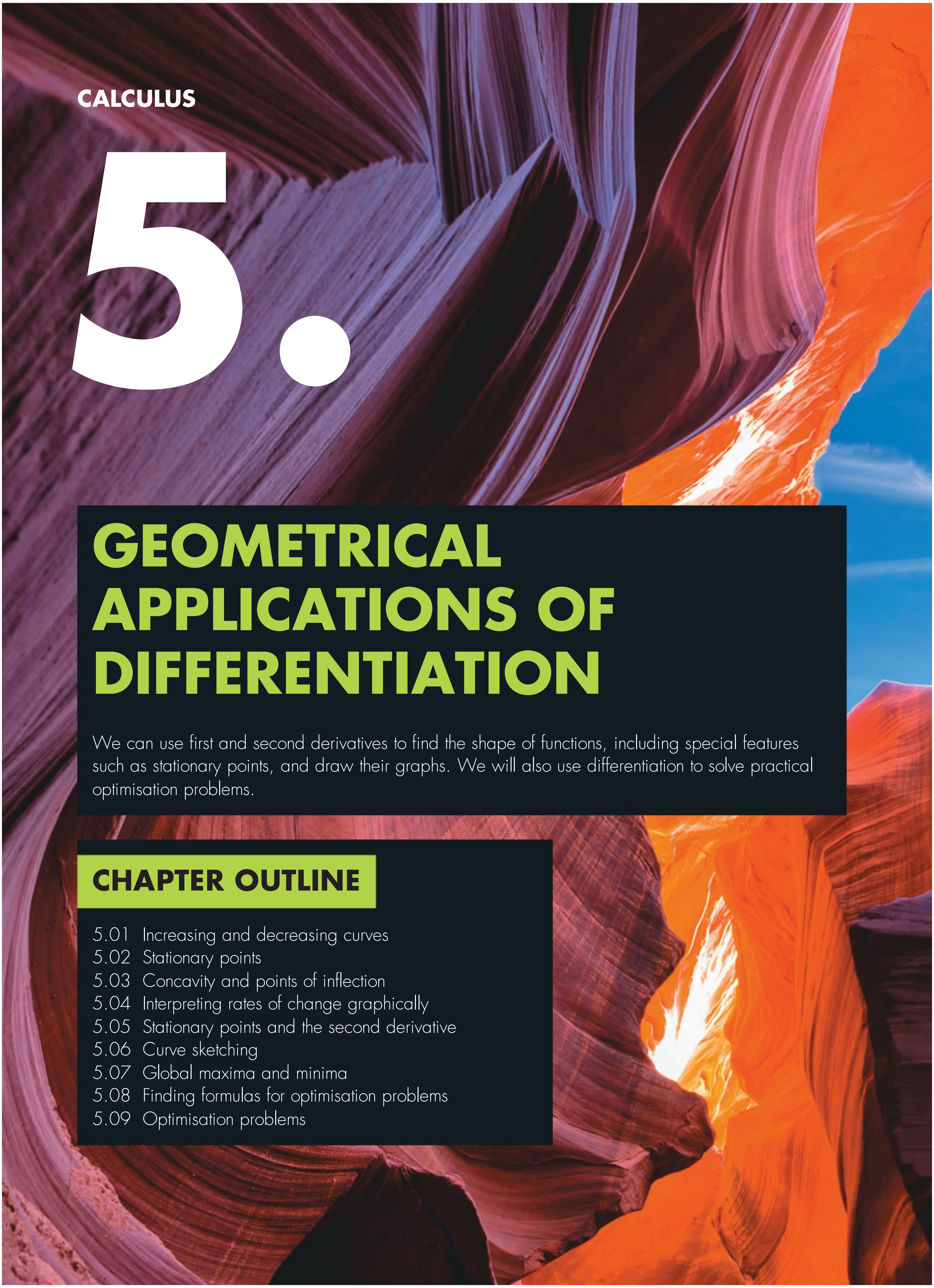


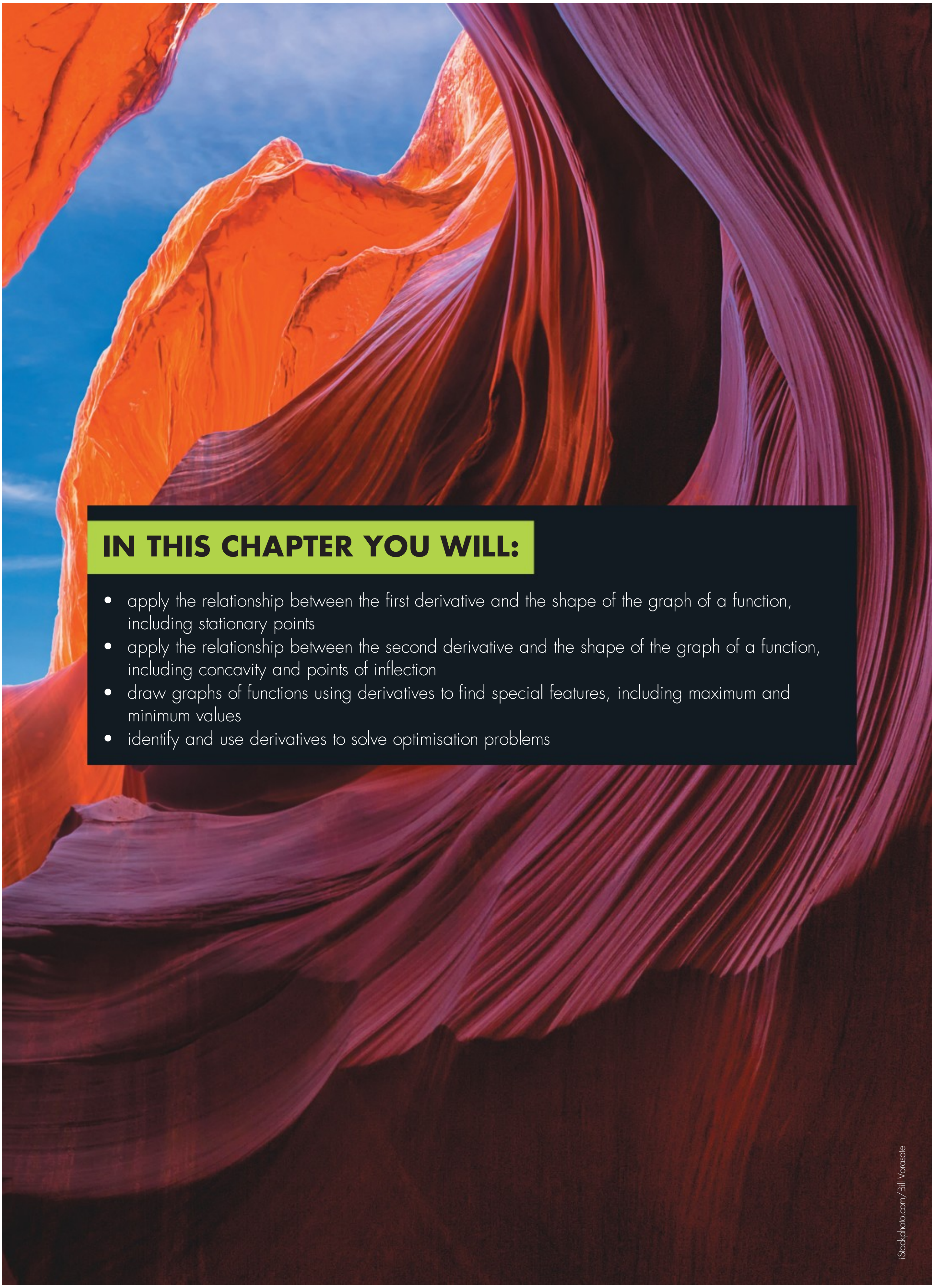
CALCULUS

5.

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

We can use first and second derivatives to find the shape of functions, including special features such as stationary points, and draw their graphs. We will also use differentiation to solve practical optimisation problems.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- apply the relationship between the first derivative and the shape of the graph of a function, including stationary points
- apply the relationship between the second derivative and the shape of the graph of a function, including concavity and points of inflection
- draw graphs of functions using derivatives to find special features, including maximum and minimum values
- identify and use derivatives to solve optimisation problems

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TERMINOLOGY

concavity: The shape of a curve as it bends; it can be concave up or concave down.

global maximum or minimum: The absolute highest or lowest value of a function over a given domain.

horizontal point of inflection: A stationary point where the concavity of the curve changes.

local maximum or minimum: a relatively high or low value of a function shown graphically as a turning point.

maximum point: A stationary point where the curve reaches a peak.

minimum point: A stationary point where the curve reaches a trough.

monotonic increasing or decreasing: A function that is always increasing or decreasing.

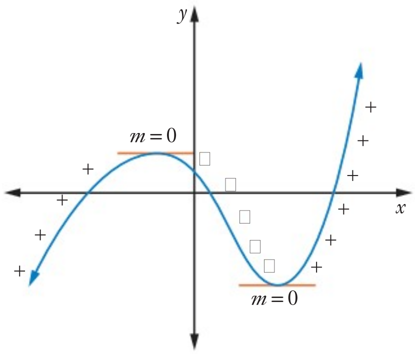
point of inflection: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes.

stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection.

5.01 Increasing and decreasing curves

WS
The sign of the derivative

You have already seen how the derivative describes the shape of a curve.



Sign of the first derivative

If $f'(x) > 0$, the graph of $y = f(x)$ is increasing.

If $f'(x) < 0$, the graph of $y = f(x)$ is decreasing.

If $f'(x) = 0$, the graph of $y = f(x)$ has a stationary point.

Sometimes a curve is monotonic increasing or decreasing (always increasing or decreasing).

Monotonic increasing or decreasing functions

A curve is monotonic increasing if $f'(x) > 0$ for all x.

A curve is monotonic decreasing if $f'(x) < 0$ for all x.

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EXAMPLE 1

a Find all x values for which the curve $f(x) = x^2 - 4x + 1$ is increasing.

b Find any stationary points on the curve $y = x^3 - 48x - 7$.

Solution

a $f'(x) = 2x - 4$

For increasing curve:

$f'(x) > 0$

$2x - 4 > 0$

$2x > 4$

$x > 2$

So the curve is increasing for $x > 2$.

b $y' = 3x^2 - 48$

For stationary points:

$y' = 0$

$3x^2 - 48 = 0$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

When $x = 4$:

$y = 4^3 - 48(4) - 7$

$= -135$

When $x = -4$:

$y = (-4)^3 - 48(-4) - 7$

$= 121$

So the stationary points are $(4, -135)$ and $(-4, 121)$.

Exercise 5.01 Increasing and decreasing curves

1 For what x values is the function $f(x) = -2x^2 + 8x - 1$ increasing?

2 Find all values of x for which the curve $y = 2x^2 - x$ is decreasing.

3 Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

4 Find values of x for which the curve $y = x^2 - 3x - 4$ is:

   a decreasing

   b increasing

   c stationary

5 Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing.

6 Prove that $y = x^3$ is monotonic increasing for all $x \neq 0$.

7 Find the stationary point on the curve $f(x) = x^3$.

8 Find all x values for which the curve $y = 2x^3 + 3x^2 - 36x + 9$ is stationary.

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9 Find all stationary points on the curve of each function.

a $y = x^2 - 2x - 3$

b $f(x) = 9 - x^2$

c $y = 2x^3 - 9x^2 + 12x - 4$

d $y = x^4 - 2x^2 + 1$

10 Find any stationary points on the curve $y = (x - 2)^4$.

11 Find any values of x for which the curve $y = 2x^3 - 21x^2 + 60x - 3$ is stationary.

12 The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Evaluate p.

13 Evaluate a and b if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

14 a Find the derivative of $y = x^3 - 3x^2 + 27x - 3$.

b Show that the curve is monotonic increasing for all values of x.

15 Sketch a function with $f'(x) > 0$ for $x < 2$, $f'(2) = 0$ and $f'(x) < 0$ when $x > 2$.

16 Sketch a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$.

17 Sketch a curve with $\frac{dy}{dx} > 0$ for all $x \neq 1$ and $\frac{dy}{dx} = 0$ when $x = 1$.

18 Sketch a function that has $f'(x) > 0$ in the domain $(-\infty, -2) \cup (5, \infty)$, $f'(x) = 0$ for $x = -2$ and $x = 5$, and $f'(x) < 0$ in the domain $(-2, 5)$.

19 A function has $f(3) = 2$ and $f'(3) < 0$. Show this information on a sketch.

20 The derivative of a function is positive at the point $(-2, -1)$. Show this information on a graph.

21 Find the stationary points on the curve $y = (3x - 1)(x - 2)^4$.

22 Differentiate $y = x\sqrt{x+1}$. Hence find the stationary point on the curve, giving the exact coordinates.

23 The curve $f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$ has a stationary point at $x = 1$.
Find the value of a.

24 Show that $f(x) = \sqrt{x}$ has no stationary points.

25 Show that $f(x) = \frac{1}{x^3}$ has no stationary points.

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5.02 Stationary points

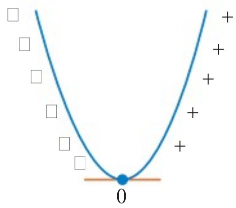
In Year 11, Chapter 6, Introduction to calculus, you learned about 3 types of stationary points: minimum point, maximum point and horizontal point of inflection.

WS
Stationary points

Minimum and maximum turning points

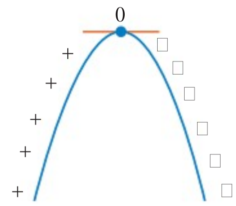
At a local minimum point, the curve is decreasing on the LHS and increasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	< 0	0	> 0

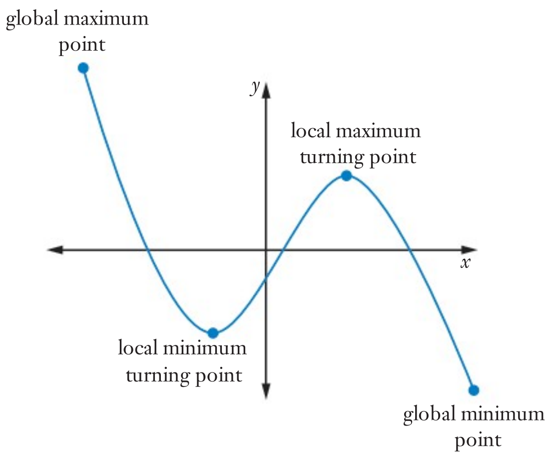


At a local maximum point, the curve is increasing on the LHS and decreasing on the RHS.

x	LHS	Minimum	RHS
$f'(x)$	> 0	0	< 0



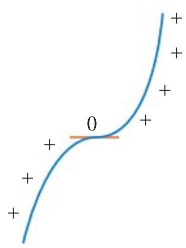
These stationary points are called local maximum or minimum points because they are not necessarily the global maximum or minimum points on the curve.

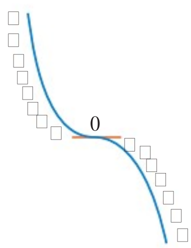


Horizontal point of inflection

These curves are increasing or decreasing on both sides of the horizontal point of inflection.
It is not a turning point since the curve does not turn around at this point.

We will learn more about points of inflection in the next section.





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EXAMPLE 2

Find any stationary points on the curve $f(x) = 2x^3 - 15x^2 + 24x - 7$ and determine their nature.

'determine their nature' means find what type of stationary point they are

Solution

$f'(x) = 6x^2 - 30x + 24$

For stationary points:

$$f'(x) = 0$$
$$6x^2 - 30x + 24 = 0$$
$$6(x^2 - 5x + 4) = 0$$
$$x^2 - 5x + 4 = 0$$
$$(x-1)(x-4) = 0$$
$$x = 1, \qquad x = 4$$

So there are 2 stationary points, where $x = 1$ and $x = 4$.

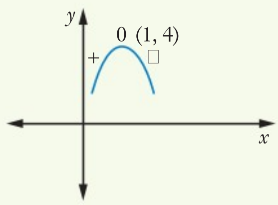
$f(1) = 2(1)^3 - 15(1)^2 + 24(1) - 7$

$\quad = 4$

So (1, 4) is a stationary point.

To determine its nature, choose a point close to (1, 4) on the LHS and RHS, for example, $x = 0$ and $x = 2$, and test the sign of $f'(x)$.

x	0	1	2
$f'(x)$	24	0	−12
	+		−



Positive to negative, so (1, 4) is a maximum point.

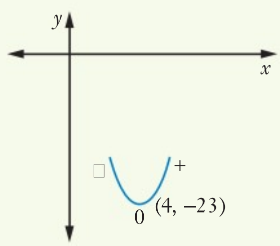
$f(4) = 2(4)^3 - 15(4)^2 + 24(4) - 7$

$\quad = -23$

So (4, −23) is a stationary point.

To determine its nature, choose, for example, $x = 2$ and $x = 5$.

x	2	4	5
$f'(x)$	−12	0	24
	−		+



Negative to positive, so (4, −23) is a minimum point.

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Exercise 5.02 Stationary points

1 Find the stationary point on the curve $y = x^2 - 1$ and show that it is a minimum point.

2 Find the stationary point on the curve $y = x^4$ and determine its type.

3 The function $f(x) = 7 - 4x - x^2$ has one stationary point. Find its coordinates and show that it is a maximum turning point.

4 Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature.

5 For the curve $y = (4 - x)^2$, find the turning point and determine its nature.

6 The curve $y = x^3 - 6x^2 + 5$ has 2 turning points. Find them and use the derivative to determine their nature.

7 Find the turning points on the curve $y = x^3 - 3x^2 + 5$ and determine their nature.

8 Find any stationary points on the curve $f(x) = x^4 - 2x^2 - 3$. What type of stationary points are they?

9 The curve $y = x^3 - 3x + 2$ has 2 stationary points. Find their coordinates and determine their type.

10 The curve $y = x^5 + mx^3 - 2x^2 + 5$ has a stationary point at $x = -1$. Find the value of m.

11 For a certain function, $f'(x) = 3 + x$. For what value of x does the function have a stationary point? What type of stationary point is it?

12 A curve has $f'(x) = x(x + 1)$. For what x values does the curve have stationary points? What type are they?

13 a Differentiate $P = 2x + \dfrac{50}{x}$ with respect to x.

b Find any stationary points on the curve and determine their nature.

14 For the function $A = \dfrac{h^2 - 2h + 5}{8}$, find any stationary points and determine their nature.

15 Find any stationary points on the function $V = 40r - \pi r^3$ correct to 2 decimal places, and determine their nature.

16 Find any stationary points on the curve $S = 2\pi r + \dfrac{120}{r}$ correct to 2 decimal places, and determine their nature.

17 a Differentiate $A = x\sqrt{3600 - x^2}$.

b Find any stationary points on $A = x\sqrt{3600 - x^2}$ (to 1 decimal place) and determine their nature.

WS
Concavity

WS
Shapes of curves

5.03 Concavity and points of inflection

The first derivative $f'(x)$ is the rate of change of the function $y = f(x)$.

Similarly, the second derivative $f''(x)$ is the rate of change of the first derivative $f'(x)$.

This means the relationship between $f''(x)$ and $f'(x)$ is the same as the relationship between $f'(x)$ and $f(x)$.

Relationship between 1st and 2nd derivatives

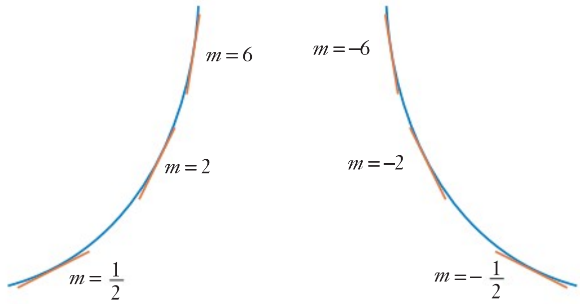
If $f''(x) > 0$ then $f'(x)$ is increasing.

If $f''(x) < 0$ then $f'(x)$ is decreasing.

If $f''(x) = 0$ then $f'(x)$ is stationary.

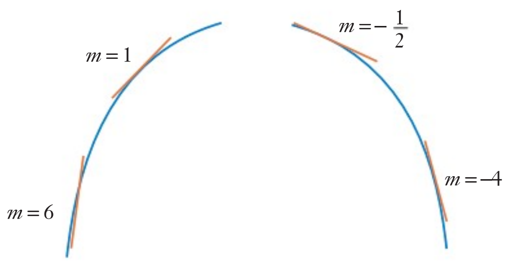
The sign of the second derivative shows the shape of the graph.

If $f''(x) > 0$ then $f'(x)$ is increasing. This means that the gradient of the tangent is increasing.



Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is concave upwards.

If $f''(x) < 0$ then $f'(x)$ is decreasing. This means that the gradient of the tangent is decreasing.



Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is concave downwards.

Sign of 2nd derivative

If $f''(x) > 0$, the curve is concave upwards.

If $f''(x) < 0$, the curve is concave downwards.

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EXAMPLE 3

Find the domain over which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

$f'(x) = 6x^2 - 14x - 5$

$f''(x) = 12x - 14$

For concave downwards:

$f''(x) < 0$

$12x - 14 < 0$

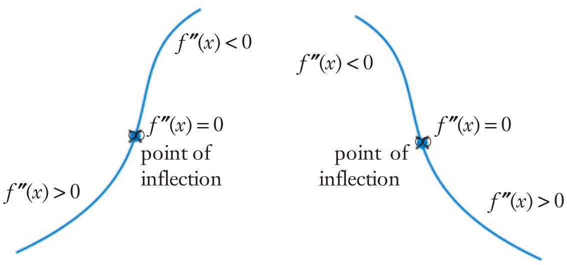
$12x < 14$

$x < \frac{14}{12}$

$x < 1\frac{1}{6}$

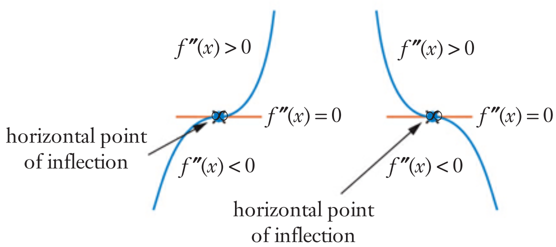
So the domain over which the curve is concave downwards is $(-\infty, 1\frac{1}{6})$

Points of inflection



At the point where $f''(x) = 0$, $f'(x)$ is constant. This means that the gradient of the tangent is neither increasing nor decreasing. This happens when the curve goes from being concave upwards to concave downwards, or concave downwards to concave upwards. We say that the curve is changing concavity at a point of inflection. The diagrams above show a point of inflection and the change in concavity as the curve changes shape.

The diagrams on the right show a horizontal point of inflection that occurs at a stationary point. Notice that the tangent is horizontal at the point of inflection.



Points of inflection

If $f''(x) = 0$, and concavity changes, it is a point of inflection.

If $f'(x) = 0$ also, it is a horizontal point of inflection.

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EXAMPLE 4

a Find the point of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$.

b Does the function $y = x^4$ have a point of inflection?

Solution

a $y' = 3x^2 - 12x + 5$

$y'' = 6x - 12$

For point of inflection, $y'' = 0$ and concavity changes.

$6x - 12 = 0$

$x = 2$

Check that concavity changes by choosing values on the LHS and RHS, for example $x = 1$ and $x = 3$, and testing the sign of the second derivative y''.

x	1	2	3
y''	–6	0	6
	–		+

Since concavity changes (negative to positive), there is a point of inflection at $x = 2$.

When $x = 2$:

$y = 2^3 - 6(2)^2 + 5(2) + 9$

$= 3$

So (2, 3) is a point of inflection.

b $\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = 12x^2$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and concavity changes.

$12x^2 = 0$

$x^2 = 0$

$x = 0$

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Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$, and test the sign of $\frac{d^2y}{dx^2}$.

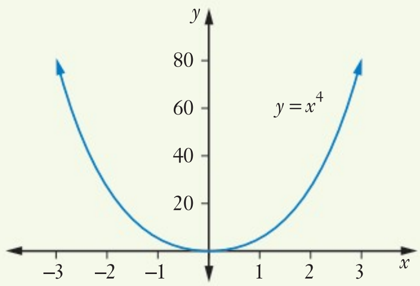
x	-1	0	1
$\frac{d^2y}{dx^2}$	12	0	12
	+		+

Since concavity doesn't change (both sides are positive), (0, 0) is not a point of inflection.

So $y = x^4$ does not have a point of inflection.

We can see this by drawing the graph of $y = x^4$.

This graph has a turning point at (0, 0).



Exercise 5.03 Concavity and points of inflection

1. For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?
2. Find all values of x for which the function $f(x) = (x - 3)^3$ is concave downwards.
3. Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.
4. Show that the curve $y = x^2$ is always concave upwards.
5. Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.
6. Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.
7. Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$.
8. Find the stationary point on the curve $y = x^3 - 2$ and show that it is a point of inflection.
9. Determine whether there are any points of inflection on the curve:
   a $y = x^6$ b $y = x^7$ c $y = x^5$
   d $y = x^9$ e $y = x^{12}$
10. Sketch a curve that is always concave up.
11. Sketch a curve where $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.
12. Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$.
13. Show that $f(x) = \frac{2}{x^2}$ is concave upwards for all $x \neq 0$.

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14 For the function $f(x) = 3x^5 - 10x^3 + 7$:

a Find any points of inflection.

b Find which of these points are horizontal points of inflection (stationary points).

15 a Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection.

b Describe the concavity of the curve.

16 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$, evaluate a.

17 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$.

18 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b.

19 The curve $y = x^6 - 3x^5 + 21x - 8$ has 2 points of inflection.

a Find these points of inflection.

b Show that they are not stationary points.

WS

The shape of a curve

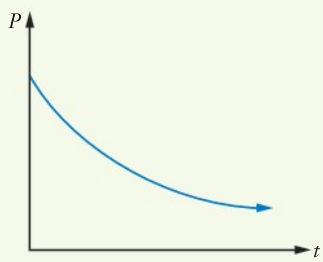
5.04 Interpreting rates of change graphically

We can find out more about the shape of a graph if we combine the results from the first and second derivatives.

EXAMPLE 5

a For a particular curve, $f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Draw the shape of the curve at this point.

b The curve below shows the population (P) of unemployed people over time t months.



i Describe the sign of $\frac{dP}{dt}$ and $\frac{d^2p}{dt^2}$.

ii How is the population of unemployed people changing over time?

iii Is the rate of change of unemployment increasing or decreasing?

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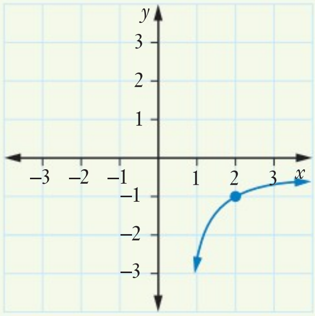
Solution

a $f(2) = -1$ means that the point $(2, -1)$ lies on the curve.

If $f'(2) > 0$, the curve is increasing at this point.

If $f''(2) < 0$, the curve is concave downwards at this point.



b i The curve is decreasing, so $\frac{dP}{dt} < 0$.

The curve is concave upwards, so $\frac{d^2P}{dt^2} > 0$.

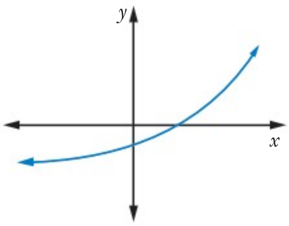
ii Since the curve is decreasing, the number of unemployed people is decreasing.

iii Since the curve is concave upwards, the (negative) gradient is increasing. This means that the rate of change of unemployment is increasing.

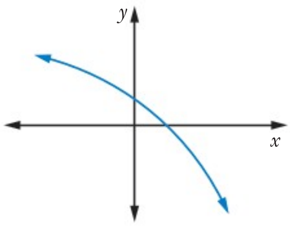
Exercise 5.04 Interpreting rates of change graphically

1 For each curve, describe the sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

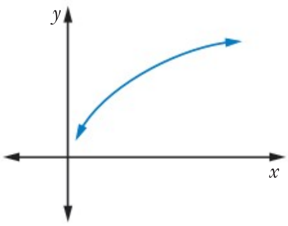
a



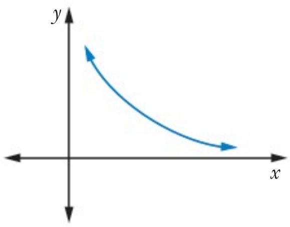
b



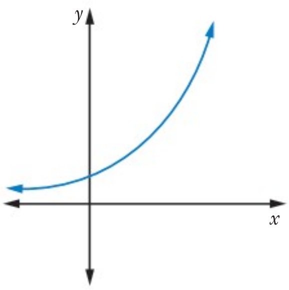
c



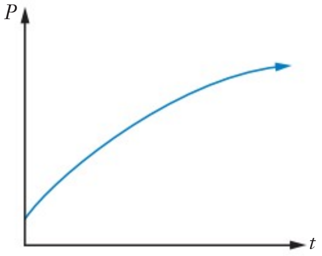
d



e



2 The curve below shows the population of a colony of sea lions.



a Describe the sign of the first and second derivatives.

b Is the rate of change of the sea lion population increasing?

Photo courtesy Margaret Grove

3 Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

4 Draw a sketch to show the shape of each curve.

a $f'(x) < 0$ and $f''(x) < 0$

b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

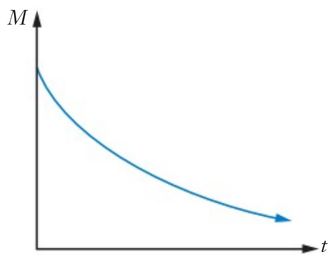
d $f'(x) > 0$ and $f''(x) > 0$

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6 As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

7 The graph shows the decay of a radioactive substance.

Describe the sign of $\frac{dM}{dt}$ and $\frac{d^2M}{dt^2}$.



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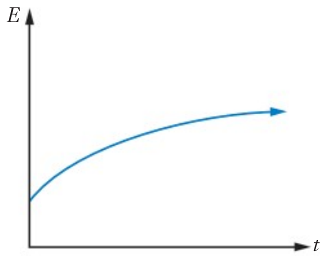
8 The population P of fish in a certain lake was studied over time. At the start of the study the number of fish was 2500.

a During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

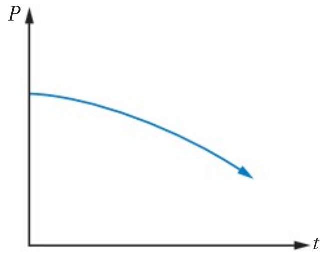
b If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate of change?

c Sketch the graph of the population P against t.

9 The graph shows the level of education of youths in a certain rural area over the past 100 years. Describe how the level of education has changed over this period of time. Include mention of the rate of change.



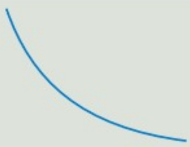
10 The graph shows the number of students in a high school over several years. Describe how the school population is changing over time, including the rate of change.



5.05 Stationary points and the second derivative

Putting the first and second derivatives together gives this summary of the shape of a curve.

Shape of a curve and the derivatives

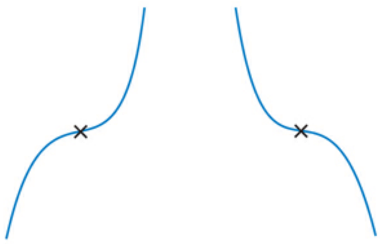
	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

We can use the table to find the requirements for stationary points.

If $f'(x) = 0$ and $f''(x) > 0$, there is a minimum turning point (concave upwards).

If $f'(x) = 0$ and $f''(x) < 0$, there is a maximum turning point (concave downwards).

If $f'(x) = 0$ and $f''(x) = 0$ and concavity changes, then there is a horizontal point of inflection.



Stationary points and the derivatives

Minimum turning point: $f'(x) = 0$ and $f''(x) > 0$

Maximum turning point: $f'(x) = 0$ and $f''(x) < 0$

Horizontal point of inflection: $f'(x) = 0$, $f''(x) = 0$ and concavity changes

Now we can use the second derivative to determine the nature of stationary points.

EXAMPLE 6

a Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and distinguish between them.

b Find the stationary point on the curve $y = 2x^5 - 3$ and determine its nature.

Solution

a $f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

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For stationary points:

$$f'(x) = 0$$

$$6x^2 - 6x - 12 = 0$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1, x = 2$$

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7$$

$$= 14$$

$$f''(-1) = 12(-1) - 6$$

$$= -18$$

$$< 0 \text{ (concave downwards)}$$

So $(-1, 14)$ is a maximum turning point.

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7$$

$$= -13$$

$$f''(2) = 12(2) - 6$$

$$= 18$$

$$> 0 \text{ (concave upwards)}$$

So $(2, -13)$ is a minimum turning point.

b $y' = 10x^4$

$$y'' = 40x^3$$

For stationary points:

$$y' = 0$$

$$10x^4 = 0$$

$$x^4 = 0$$

$$x = 0$$

When $x = 0$:

$$y = 2(0)^5 - 3$$

$$= -3$$

$$y'' = 40(0)^3$$

$$= 0$$

Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$.

x	-1	0	1
y''	-40	0	40
	$-$		$+$

Since concavity changes, $(0, -3)$ is a horizontal point of inflection.

The table also tells us that the curve changes from concave downwards to concave upwards.

Exercise 5.05 Stationary points and the second derivative

1 Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature.

2 Find the stationary point on the curve $f(x) = 3x^4 + 1$ and determine what type of point it is.

3 Find the stationary point on the curve $y = 3x^2 - 12x + 7$ and show that it is a minimum turning point.

4 Determine the stationary point on $y = x - x^2$ and show that it is a maximum point.

5 Show that $f(x) = 2x^3 - 5$ has a horizontal point of inflection and find its coordinates.

6 Does the function $f(x) = 2x^5 + 3$ have a stationary point? If it does, determine its nature.

7 Find any stationary points on $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8 Find the stationary points on the curve $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ and determine whether they are maximum or minimum points.

9 Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10 a Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and distinguish between them.

b Find any points of inflection on the curve.

11 Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature.

12 Find any stationary points on the curve $f(x) = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = \frac{1}{2}$.

a Find the value of a.

b Hence, or otherwise, find the stationary point and determine its nature.

14 The curve $y = x^3 - mx^2 + 5x - 7$ has a stationary point where $x = -1$. Find the value of m.

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1, -2)$. Find the values of a and b.

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5.06 Curve sketching

We can sketch the graph of a function by using special features such as intercepts, stationary points and points of inflection. Here is a summary of strategies for sketching a curve.

WS Curve sketching

WS Further curve sketching

WS Curve sketching

WS Curve sketching with derivatives

WS Further graphs

Curve sketching

Curve sketching

Sketching curves

- Find stationary points $\left(\frac{dy}{dx} = 0\right)$, and determine their nature.
- Find points of inflection $\left(\frac{d^2y}{dx^2} = 0\right)$, and check that concavity changes.
- Find any x-intercepts($y = 0$), and y-intercepts($x = 0$).
- Find domain and range.
- Find any asymptotes or other discontinuities.
- Find limiting behaviour of the function.
- Use the symmetry of the function where possible:
  - check if the function is even: $f(-x) = f(x)$
  - check if the function is odd: $f(-x) = -f(x)$

EXAMPLE 7

a Find any stationary points and points of inflection on the curve $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

b Sketch the curve of the composite function $y = f(g(x))$ where $f(x) = 2x + 1$ and $g(x) = x^3$, showing any important features.

Solution

a $f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

For stationary points:

$$f'(x) = 0$$

$$3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3, \qquad x = -1$$

$$f(3) = 3^3 - 3(3)^2 - 9(3) + 1$$

$$= -26$$

$$f''(3) = 6(3) - 6$$

$$= 12$$

$$> 0 \text{ (concave upwards)}$$

So $(3, -26)$ is a minimum turning point.

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$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1$
$= 6$
$f''(-1) = 6(-1) - 6$
$= -12$
< 0 (concave downwards)

So $(-1, 6)$ is a maximum turning point.

For points of inflection:

$f''(x) = 0$
$6x - 6 = 0$
$6x = 6$
$x = 1$

Check concavity changes by choosing values on LHS and RHS e.g. $x = 0$ and $x = 2$.

x	0	1	2
$f''(x)$	−6	0	6

Since concavity changes, $x = 1$ is at a point of inflection.

$f(1) = 1^3 - 3(1)^2 - 9(1) + 1$
$= -10$

So $(1, -10)$ is a point of inflection.

For x-intercept, $y = 0$:

$0 = x^3 - 3x^2 - 9x + 1$

This has no factors so we can't find the x-intercepts.

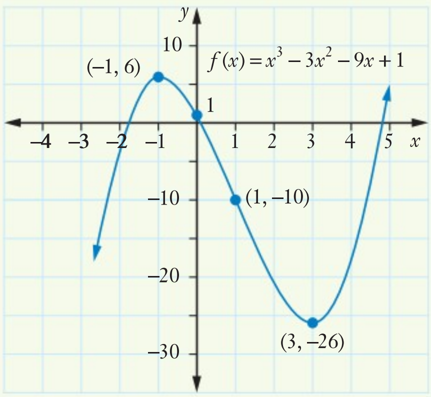
For y-intercept, $x = 0$:

$f(0) = 0^3 - 3(0)^2 - 9(0) + 1$
$= 1$

$f(x) = x^3 - 3x^2 - 9x + 1$ is a cubic function with no symmetry or discontinuities.

It is not an even or odd function.

Notice that the point of inflection at $(1, -10)$ is not a stationary point. It is the point where the graph naturally changes concavity.



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b $y = f(g(x))$

$= 2x^3 + 1$

$\frac{dy}{dx} = 6x^2$

$\frac{d^2y}{dx^2} = 12x$

For stationary points:

$\frac{dy}{dx} = 0$

$6x^2 = 0$

$x^2 = 0$

$x = 0$

When $x = 0$:

$y = 2(0)^3 + 1$

$= 1$

$\frac{d^2y}{dx^2} = 12(0)$

$= 0$

Check concavity either side:

x	−1	0	1
$\frac{d^2y}{dx^2}$	−12	0	12

Since concavity changes, (0, 1) is a horizontal point of inflection.

For x-intercepts, $y = 0$

$0 = 2x^3 + 1$

$-1 = 2x^3$

$-0.5 = x^3$

$\sqrt[3]{-0.5} = x$

$-0.8 \approx x$

For y-intercept, $x = 0$

$y = 2(0)^3 + 1$

$= 1$

This is (0, 1), the point of inflection.

This is a cubic function. We can make the graph more accurate by finding some extra points.

When $x = -1$:

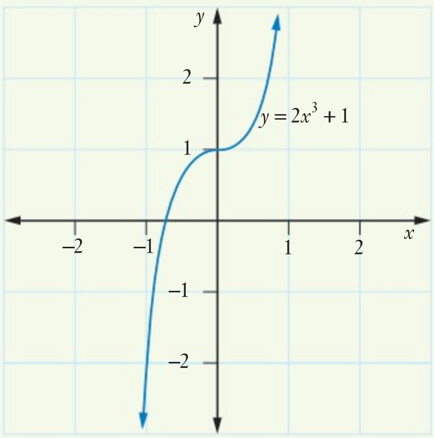
$y = 2(-1)^3 + 1$

$= -1$

When $x = 1$:

$y = 2(1)^3 + 1$

$= 3$



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You can use derivatives to help sketch other functions, for example trigonometric, exponential and logarithmic graphs.

EXAMPLE 8

Sketch the curve $y = xe^x$, showing any important features.

Solution

$y = xe^x$

$y' = u'v + v'u$ where $u = x$ and $v = e^x$

$u' = 1 \qquad v' = e^x$

$y' = 1 \times e^x + e^x \times x$

$\quad = e^x(1 + x)$

$y'' = u'v + v'u$ where $u = e^x$ and $v = 1 + x$

$\qquad\qquad u' = e^x \qquad v' = 1$

$y'' = e^x \times (1 + x) + 1 \times e^x$

$\quad = e^x(2 + x)$

For stationary points:

$$y' = 0$$

$$e^x(1 + x) = 0$$

$$1 + x = 0 \qquad (e^x \neq 0)$$

$$x = -1$$

When $x = -1$:

$$y = -1e^{-1}$$

$$= -\frac{1}{e}$$

$$y'' = e^{-1}(2 + -1)$$

$$= \frac{1}{e}$$

$$> 0 \qquad \text{(concave upwards)}$$

So $\left(-1, -\frac{1}{e}\right)$ is a minimum turning point.

For x-intercepts, $y = 0$:

$0 = xe^x$

$x = 0 \qquad (e^x \neq 0)$

For y-intercepts, $x = 0$:

$y = 0e^0$

$\quad = 0$

The general exponential function $y = a^x$ has an asymptote at the x-axis.

Limiting behaviour as $x \to \pm\infty$:

As $x \to \infty$, $xe^x \to \infty$ since x and e^x are both becoming large as x becomes large.

As $x \to -\infty$, $x \to -\infty$ but $e^x \to 0$ when x is negative $\left(\text{since } e^{-x} = \frac{1}{e^x}\right)$.

So $xe^x \to 0^-$ (it approaches zero from the negative side).

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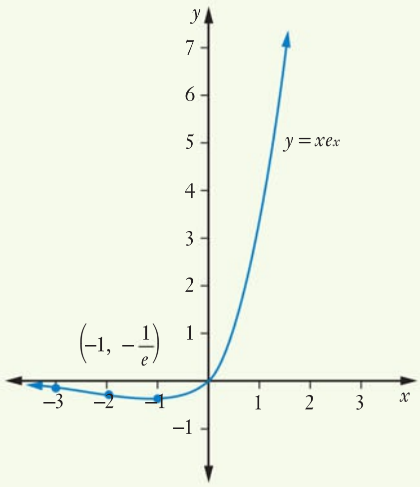
We can sketch this information on a graph.



When the second derivative is hard to find, we can use the first derivative to check the type of stationary points.

EXAMPLE 9

Find any stationary points and sketch the function $y = x\sqrt{16 - x^2}$.

Solution

$$y = x\sqrt{16 - x^2}$$

$$y' = u'v + v'u \qquad u = x \qquad v = \sqrt{16 - x^2} = (16 - x^2)^{\frac{1}{2}}$$

$$u' = 1 \qquad v' = -2x \times \frac{1}{2}(16 - x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{16 - x^2}}$$

$$y' = 1 \times \sqrt{16 - x^2} + \left(-\frac{x}{\sqrt{16 - x^2}}\right) \times x$$

$$= \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}}$$

For stationary points:

$$y' = 0$$

$$\sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}} = 0$$

$$16 - x^2 - x^2 = 0 \qquad \text{(multiplying both sides by } \sqrt{16 - x^2}\text{)}$$

$$16 - 2x^2 = 0$$

$$16 = 2x^2$$

$$8 = x^2$$

$$\pm\sqrt{8} = x$$

When $x = \sqrt{8}$:

$y = \sqrt{8} \times \sqrt{16 - (\sqrt{8})^2}$

$= \sqrt{8} \times \sqrt{8}$

$= 8$

So $(\sqrt{8}, 8)$ is a stationary point.

When $x = -\sqrt{8}$:

$y = -\sqrt{8} \times \sqrt{16 - (-\sqrt{8})^2}$

$= -\sqrt{8} \times \sqrt{8}$

$= -8$

So $(-\sqrt{8}, -8)$ is a stationary point.

Since the second derivative is hard to find, we can check the first derivative on LHS and RHS of $\pm\sqrt{8} \approx 2.8$, to see where the curve is increasing and decreasing.

x	2	2.8	3
y'	+2.3	0	–0.8

Positive to negative, so $(\sqrt{8}, 8)$ is a maximum turning point.

x	–3	–2.8	–2
y'	–0.8	0	+2.3

Negative to positive, so $(-\sqrt{8}, -8)$ is a minimum turning point.

For x-intercepts, $y = 0$:

$0 = x\sqrt{16 - x^2}$

$x = 0, \quad \sqrt{16 - x^2} = 0$

$$16 - x^2 = 0$$

$$16 = x^2$$

$$\pm 4 = x$$

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For y-intercept, $x = 0$:

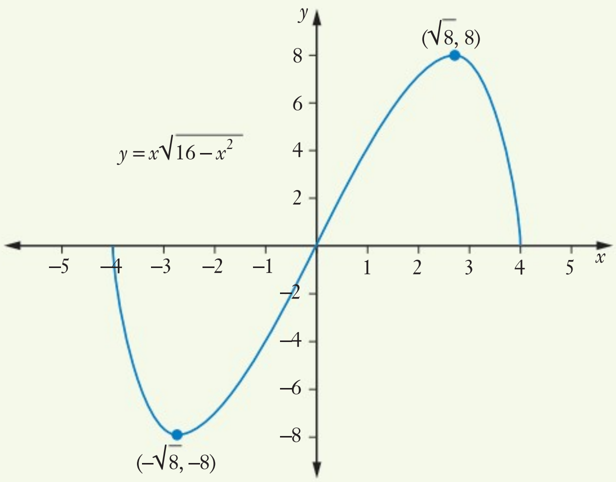
$y = 0\sqrt{16 - 0^2}$

$= 0$

Domain: $\sqrt{16 - x^2} \geq 0$

This simplifies to $-4 \leq x \leq 4$ or $[-4, 4]$ by solving the inequality or by noticing that the graph of $y = \sqrt{16 - x^2}$ is a semicircle with radius 4.

We can sketch this information on a graph.



Exercise 5.06 Curve sketching

1 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the x– and y-intercepts of the graph of $f(x)$ and sketch the curve.

2 Sketch the graph of $y = 6 - 2x - x^2$, showing the stationary point.

3 Find the stationary point on the curve of the composite function $y = f(g(x))$ where $f(x) = x^3$ and $g(x) = x - 1$ and determine its nature. Hence sketch the curve.

4 Sketch the graph of $y = x^4 + 3$, showing any stationary points.

5 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection. Hence sketch the curve.

6 Sketch the graph of $f(x) = x^7$.

7 Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

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8 a Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.

b Find any points of inflection on the curve.

c Sketch the curve.

9 Find any stationary points and points of inflection on the curve $y = f(x) + g(x)$ where $f(x) = x^3 - 7x^2 - 1$ and $g(x) = x^2 + 4$ and hence sketch the curve.

10 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

11 Sketch the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

12 Find all stationary points and points of inflection on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

13 Show that the curve $y = \frac{2}{1+x}$ has no stationary points. By considering the domain and range of the function, sketch the curve.

14 Sketch in the domain $[0, 2\pi]$, showing all stationary points:

a $y = \cos 2x$ b $y = 5 \sin 4x$

15 Draw the graph of each function, showing stationary points, points of inflection and other features.

a $y = x^2 \ln x$ b $y = \frac{x}{e^x}$ c $y = \frac{1}{x^2 - 1}$

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5.07 Global maxima and minima

WS

Greatest and least values

A curve may have local maximum and minimum turning points, but the absolute highest and lowest values of a function over a given domain are called the global maximum or minimum values of the function.

EXAMPLE 10

Find the global maximum and minimum values of y for the function $f(x) = x^4 - 2x^2 + 1$ in the domain $[-2, 3]$.

Solution

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

For stationary points:

$$f'(x) = 0$$

$$4x^3 - 4x = 0$$

$$4x(x^2 - 1) = 0$$

$$4x(x + 1)(x - 1) = 0$$

$x = 0, \quad x = -1, \quad x = 1$

$f(0) = 0^4 - 2(0)^2 + 1$

$= 1$

$f''(0) = 12(0)^2 - 4$

$= -4$

< 0 (concave downward)

So $(0, 1)$ is a maximum turning point.

$f(-1) = (-1)^4 - 2(-1)^2 + 1$

$= 0$

$f''(-1) = 12(-1)^2 - 4$

$= 8$

> 0 (concave upward)

So $(-1, 0)$ is a minimum turning point.

$f(1) = 1^4 - 2(1)^2 + 1$

$= 0$

$f''(1) = 12(1)^2 - 4$

$= 8$

> 0 (concave upward)

So $(1, 0)$ is a minimum turning point.

At the endpoints of the domain:

$f(-2) = (-2)^4 - 2(-2)^2 + 1$

$= 9$

$f(3) = 3^4 - 2(3)^2 + 1$

$= 64$

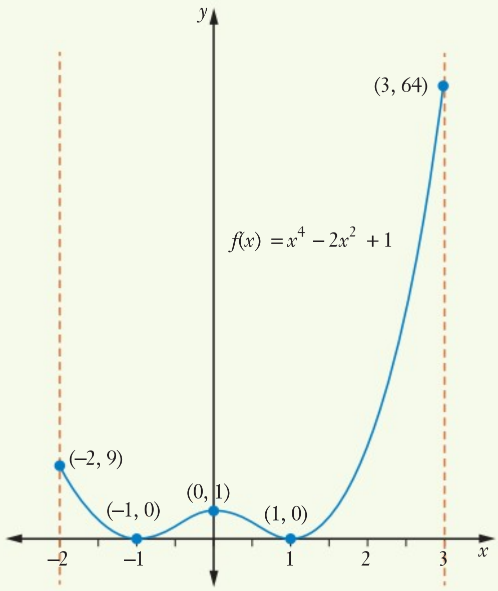
Checking, we also notice that $f(x) = x^4 - 2x^2 + 1$ is an even function.

$f(-x) = (-x)^4 - 2(-x)^2 + 1$

$= x^4 - 2x^2 + 1$

$= f(x)$

Drawing this information:



In the domain [–2, 3], the global maximum value is 64 and the global minimum value is 0.

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Exercise 5.07 Global maxima and minima

1 Sketch the graph of $y = x^2 + x - 2$ in the domain $[-2, 2]$ and find the maximum value of y in this domain.

2 Sketch the graph of $f(x) = 9 - x^2$ over the domain $[-4, 2]$. Hence find the maximum and minimum values of the function over this domain.

3 Find the maximum value of $y = x^2 - 4x + 4$ in the domain $[-3, 3]$.

4 Sketch the graph of $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \le x \le 3$, showing any stationary points. Find the global maximum and minimum values of the function.

5 Find the global maximum for $y = x^5 - 3$ in the domain $[-2, 1]$.

6 Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \le x \le 4$ and find its global maximum and minimum.

7 Find the local and global maximum and minimum of $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ in the domain $[-2, 2]$.

8 Sketch $y = x^3 + 2$ over the domain $[-3, 3]$ and find its global minimum and maximum.

9 Sketch $y = \sqrt{x + 5}$ for $-4 \le x \le 4$ and find its maximum and minimum values.

10 Show that $y = \dfrac{1}{x - 2}$ has no stationary points. Find its maximum and minimum values in the domain $[-3, 3]$.

INVESTIGATION

THE LARGEST DISC

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

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5.08 Finding formulas for optimisation problems

Optimisation problems involve finding maximum or minimum values. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise paddock size.

To solve an optimisation problem, we must first find a formula for the quantity that we are trying to maximise or minimise.

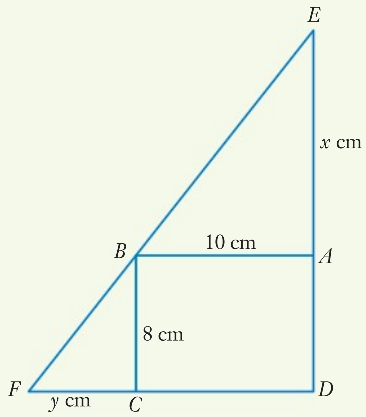
EXAMPLE 11

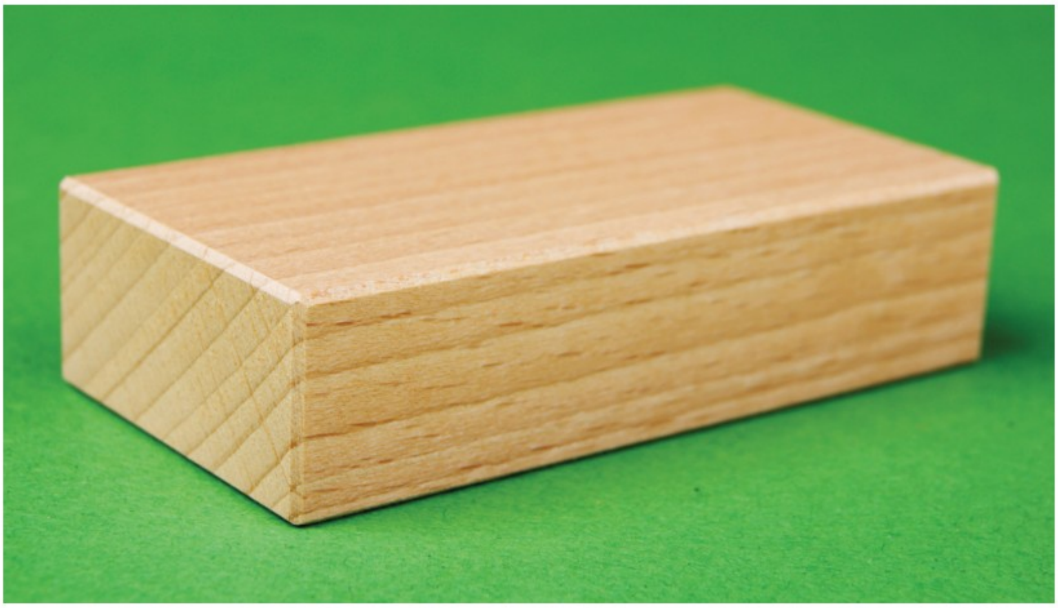
a A rectangular prism has a base with length twice its width. Its volume is 300 cm^3. Show that the surface area is given by $S = 4x^2 + \dfrac{900}{x}$.

b ABCD is a rectangle with AB = 10 cm and BC = 8 cm. Length AE = x cm and CF = y cm.

i Show that $xy = 80$.

ii Show that triangle EDF has area given by $A = 80 + 5x + \dfrac{320}{x}$.





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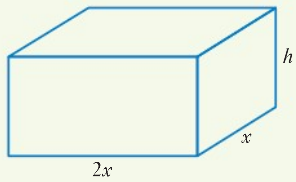
Solution

a Volume:

$$V = lwh$$
$$= 2x \times x \times h$$
$$= 2x^2h$$

$V = 300$:

$$300 = 2x^2h$$
$$\frac{300}{2x^2} = h \qquad [1]$$



Surface area:

$$S = 2(lw + wh + lh)$$
$$= 2(2x^2 + xh + 2xh)$$
$$= 2(2x^2 + 3xh)$$
$$= 4x^2 + 6xh$$

Substitute [1]:

$$S = 4x^2 + 6x \times \frac{300}{2x^2}$$
$$= 4x^2 + \frac{900}{x}$$

b i Triangles AEB and CBF are similar.

So $\frac{10}{y} = \frac{x}{8}$

$$xy = 80 \qquad [1]$$

ii Side FD = $y + 10$ and side ED = $x + 8$

Since $xy = 80$

$$y = \frac{80}{x} \qquad [2]$$

Area:

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(y + 10)(x + 8)$$
$$= \frac{1}{2}(xy + 8y + 10x + 80)$$
$$= \frac{1}{2}\left(80 + 8 \times \frac{80}{x} + 10x + 80\right) \qquad \text{substituting [1] and [2]}$$
$$= \frac{1}{2}\left(160 + \frac{640}{x} + 10x\right)$$
$$= 80 + \frac{320}{x} + 5x$$

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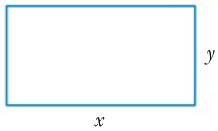
Exercise 5.08 Finding formulas for optimisation problems

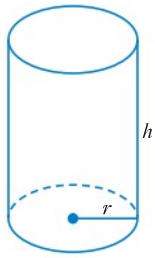
1 The area of a rectangle is to be 50 m^2. Show that its perimeter is given by the equation $P = 2x + \frac{100}{x}$.



2 A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$.

3 The product of 2 numbers is 20. Show that the sum of the numbers is $S = x + \frac{20}{x}$.

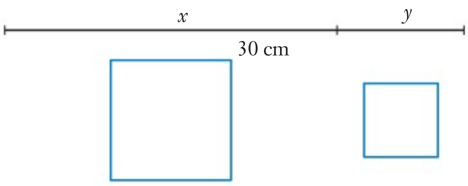
4 A closed cylinder is to have a volume of 400 cm^3. Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$.



5 A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square as shown.

a Show that $y = 30 - x$.

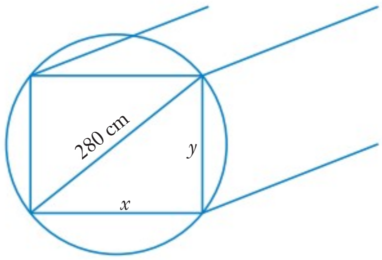
b Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$.



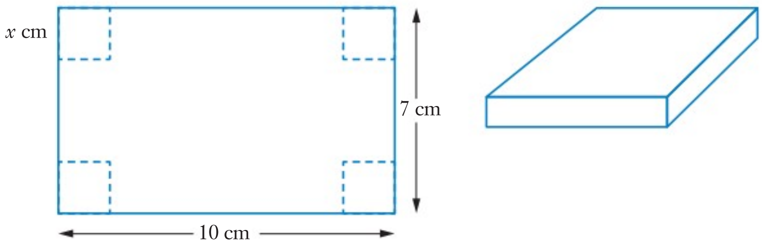
6 A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown.

a Show that $y = \sqrt{78400 - x^2}$.

b Show that the cross-sectional area is given by $A = x\sqrt{78400 - x^2}$.



7 A 10 cm by 7 cm rectangular piece of cardboard has equal square corners with side x cm cut out. The sides are folded up to make an open box as shown. Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$.



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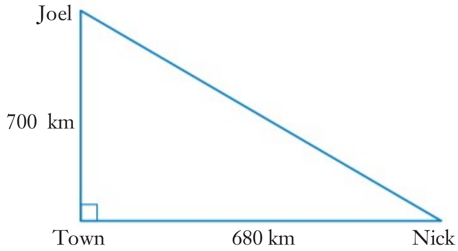
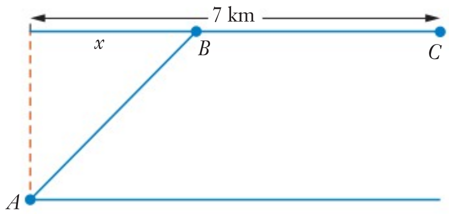
8 A travel agency calculates the expense E per person of organising a holiday in a group of x people as $E = 200 + 400x$. The cost C for each person taking a holiday is $C = 900 - 100x$. Show that the profit to the travel agency on a holiday with a group of x people is given by $P = 700x - 500x^2$.

9 Joel is 700 km north of a town, travelling towards it at an average speed of 75 km h^{-1}. Nick is 680 km east of the town, travelling towards it at 80 km h^{-1}.
Show that, after t hours, the distance between Joel and Nick is given by $d = \sqrt{952400 - 213800t + 12025t^2}$.



10 Taylor swims from point A to point B across a 500 m wide river, then walks along the river bank to point C. The distance along the river bank is 7 km. If she swims at 5 km h^{-1} and walks at 4 km h^{-1}, show that the time taken to reach point C is given by $t = \dfrac{\sqrt{x^2 + 0.25}}{5} + \dfrac{7 - x}{4}$.



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WS Optimisation problems
WS Further optimisation problems
WS Starting maxima and minima problems
WS Applications of optimisation
Maximum volume
WS Applications of derivatives problems
WS Applications of derivatives assignment
WS Second derivative assignment
WS Second derivative problems

5.09 Optimisation problems

You can use derivatives to find the maximum or minimum value of a formula.

Always check that an answer gives a maximum or minimum value.

EXAMPLE 12

The equation for the expense per year, E (in units of \$10 000), of running a certain business is given by $E = x^2 - 6x + 12$, where x is the number (in 100s) of items manufactured.

a Find the expense of running the business if no items are manufactured.

b Find the number of items needed to minimise the expense of the business.

c Find the minimum expense of the business.

Solution

a When $x = 0$:

$$E = 0^2 - 6(0) + 12$$

$$= 12 \qquad \text{(expense is in units of \$10 000)}$$

So the expense of running the business when no items are manufactured is $12 \times \$10\,000 = \$120\,000$ per year.

b For stationary points:

$$\frac{dE}{dx} = 0$$

$$\frac{dE}{dx} = 2x - 6 = 0$$

$$2x = 6$$

$$x = 3 \qquad \text{(x is in 100s of items)}$$

$$\frac{d^2E}{dx^2} = 2$$

$$> 0 \qquad \text{(concave upwards)}$$

So $x = 3$ gives a minimum value.

$3 \times 100 = 300$

So 300 items manufactured each year will give the minimum expense.

c When $x = 3$:

$$E = 3^2 - 6(3) + 12$$

$$= 3$$

So the minimum expense per year is $3 \times \$10\,000 = \$30\,000$.

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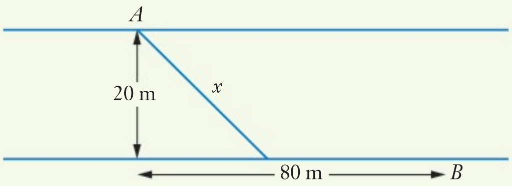
EXAMPLE 13

a The council wants to make a rectangular swimming area at the beach using the seashore on one side and a length of 300 m of shark-proof netting for the other 3 sides. What are the dimensions of the rectangle that encloses the greatest area?

b Kristyn is at point A on one side of a 20 m wide river and needs to get to point B on the other side 80 m along the bank as shown. Kristyn swims to any point on the other bank and then runs along the side of the river to point B.

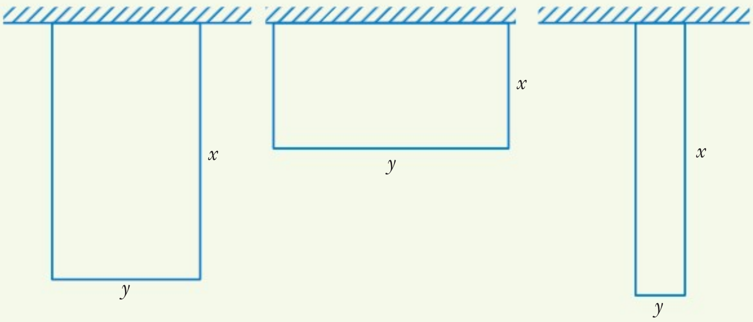


If she can swim at 7 km h^{-1} and run at 11 km h^{-1}, find x, the distance she swims to the nearest metre, to minimise her total travel time.

Optimisation problems

Solution

a Many different rectangles could have a perimeter of 300 m. Let the length of the rectangle be y and the width be x.



Perimeter: $2x + y = 300$ m

$$y = 300 - 2x \qquad [1]$$

Area:

$$\begin{aligned} A &= xy \\ &= x(300 - 2x) \qquad \text{substituting [1]} \\ &= 300x - 2x^2 \end{aligned}$$

$$\frac{dA}{dx} = 300 - 4x$$

For stationary points:

$$\frac{dA}{dx} = 0$$

$$\begin{aligned} 300 - 4x &= 0 \\ 300 &= 4x \\ 75 &= x \end{aligned}$$

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$$\frac{d^2A}{dx^2} = -4$$

< 0 Concave downwards

So $x = 75$ gives maximum area.

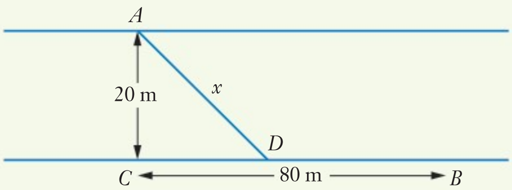
When $x = 75$: Substituting into [1]

$y = 300 - 2(75)$

$= 150$

So the dimensions that give the maximum area are 150 m × 75 m.

b First, we need to find a formula for the time Kristyn takes to run the distance AD + DB.



AD = x so find DB:

$DB = 80 - CD$

By Pythagoras' theorem, $x^2 = 20^2 + CD^2$

$x^2 - 20^2 = CD^2$

$CD = \sqrt{x^2 - 400}$

$DB = 80 - \sqrt{x^2 - 400}$

$\text{Speed} = \frac{\text{distance}}{\text{time}}$

$s = \frac{d}{t}$

$st = d$

$t = \frac{d}{s}$

Time taken to swim AD:

$t_1 = \frac{x}{7}$

Time taken to run DB:

$t_2 = \frac{80 - \sqrt{x^2 - 400}}{11}$

Total time:

$t = t_1 + t_2$

$= \frac{x}{7} + \frac{80 - \sqrt{x^2 - 400}}{11}$

$= \frac{11x}{77} + \frac{7(80 - \sqrt{x^2 - 400})}{77}$

$= \frac{11x + 560 - 7(x^2 - 400)^{\frac{1}{2}}}{77}$

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$$\frac{dt}{dx} = \frac{11 - 7 \times 2x \times \frac{1}{2}(x^2 - 400)^{-\frac{1}{2}}}{77}$$

$$= \frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77}$$

For minimum time: $\frac{dt}{dx} = 0$

$$\frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77} = 0$$

$$11 - 7x(x^2 - 400)^{-\frac{1}{2}} = 0$$

$$11 = 7x(x^2 - 400)^{-\frac{1}{2}}$$

$$11 = \frac{7x}{\sqrt{x^2 - 400}}$$

$$11\sqrt{x^2 - 400} = 7x$$

$$121(x^2 - 400) = 49x^2 \qquad \text{squaring both sides}$$

$$121x^2 - 48\,400 = 49x^2$$

$$72x^2 = 48\,400$$

$$x^2 = 672.222\ldots$$

$$x = \sqrt{672.222\ldots}$$

$$\approx 25.9$$

To check that t is a minimum:

x	25	25.9	26
$\frac{dt}{dx}$	−0.009	0	0.0006

Since the function is decreasing on LHS and increasing on RHS, t is a minimum at $x = 25.9$.

So Kristyn should swim a distance of 25.9 m to minimise her total travel time.

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Exercise 5.09 Optimisation problems

1 The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where t is time in seconds. Find when the ball will reach its maximum height, and what the maximum height will be.

2 The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$, where x is the distance travelled in km. Find the distance that gives the minimum cost.

3 The perimeter of a rectangle is 60 m and its length is x m.

a Show that the area of the rectangle is given by the equation $A = 30x - x^2$.

b Hence find the maximum area of the rectangle.

4 A farmer wants to make a rectangular paddock with an area of 4000 m^2. To minimise fencing costs she wants the paddock to have a minimum perimeter.

a Show that the perimeter is given by the equation $P = 2x + \dfrac{8000}{x}$.

b Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place.

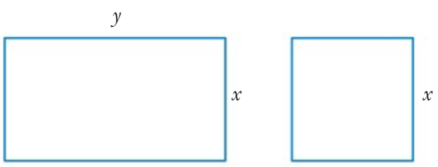
c Calculate the cost of fencing the paddock, at $48.75 per metre.

5 Bill wants to put a small rectangular vegetable garden in his backyard using 2 existing walls as part of its border. He has 8 m of garden edging for the border on the other 2 sides. Find the dimensions of the garden bed that will give the greatest area.

6 Find 2 numbers whose sum is 28 and whose product is a maximum.

7 The difference of 2 numbers is 5. Find these numbers if their product is to be minimum.

8 A piece of wire 10 m long is broken into 2 parts, which are bent into the shape of a rectangle and a square as shown. Find the dimensions x and y that make the total area a maximum.



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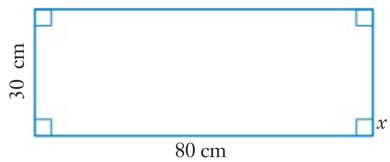
9 A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side x cm from each corner. The edges are turned up to make an open box.



a Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$.

b Find the value of x that gives the box its greatest volume.

c Find the maximum volume of the box.

10 The formula for the surface area of a cylinder is given by $S = 2\pi r(r + h)$ where r is the radius of its base and h is its height.

a Show that if the cylinder holds a volume of 54π m^3, the surface area is given by the equation $S = 2\pi r^2 + \frac{108\pi}{r}$.

b Hence find the radius that gives the minimum surface area.

11 A silo in the shape of a cylinder is required to hold 8600 m^3 of wheat.

a Find an equation for the surface area of the silo in terms of the base radius.

b Find the minimum surface area required to hold this amount of wheat, to the nearest square metre.

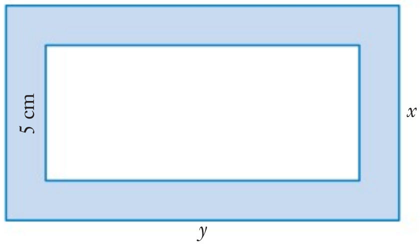
Photo courtesy Margaret Grove

12 A rectangle is cut from a circular disc of radius 6 cm.

a Show that the formula for the area of the rectangle is $A = x\sqrt{144 - x^2}$.

b Find the area of the largest rectangle that can be produced.

13 A poster consists of a photograph bordered by a 5 cm margin. The area of the poster is to be 400 cm^2.

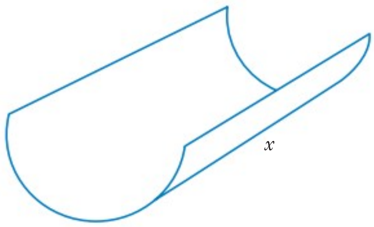


a Show that the area of the photograph is given by the equation $A = 500 - 10x - \frac{4000}{x}$.

b Find the maximum area possible for the photograph.

14 A surfboard is in the shape of a rectangle and semicircle, as shown. The perimeter is to be 4 m. Find the maximum area of the surfboard, correct to 2 decimal places.

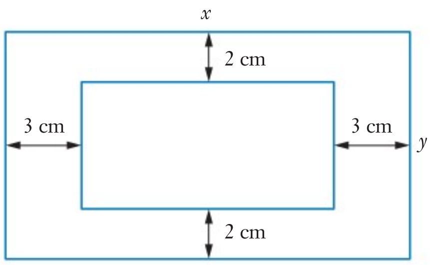
15 A half-pipe is to be made from a rectangular piece of metal of length x m. The perimeter of the rectangle is 30 m.



a Find the dimensions of the rectangle that will give the maximum surface area.

b Find the height from the ground up to the top of the half-pipe with this maximum area, correct to 1 decimal place.

16 The picture frame shown has a border of 2 cm at the top and bottom and 3 cm at the sides. If the total area of the border is to be 100 cm^2, find the maximum area of the frame.



17 A 3 m piece of wire is cut into 2 pieces and bent around to form a square and a circle. Find the size of the 2 lengths, correct to 2 decimal places, that will make the total area of the square and circle a minimum.

 ISBN 9780170413220

18 Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 km h^{-1}, while the other starts at 120 km away and travels towards the intersection at 60 km h^{-1}.

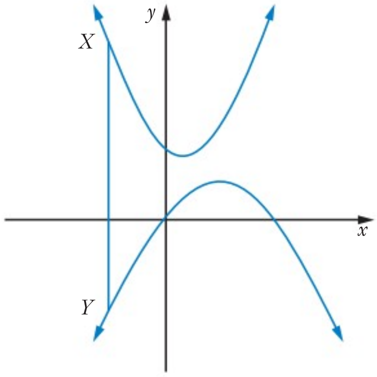
a Show that their distance apart after t hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$.

b Hence find their minimum distance apart.

19 X is a point on the curve $y = x^2 - 2x + 5$. Point Y lies directly below X and is on the curve $y = 4x - x^2$.

a Show that the distance, d, between X and Y is $d = 2x^2 - 6x + 5$.

b Find the minimum distance between X and Y.



20 A truck travels 1500 km at an hourly cost given by $s^2 + 9000$ cents where s is the average speed of the truck.

a Show that the cost for the trip is given by $C = 1500\left(s + \frac{9000}{s}\right)$.

b Find, to the nearest km h^{-1}, the speed that minimises the cost of the trip.

c Find the cost of the trip to the nearest dollar.

Shutterstock.com/Cesar Witkowski

CLASS CHALLENGE

HERON'S PROBLEM

One boundary of a farm is a straight river bank, and on the farm stands a house. Some distance away there is a shed. Each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

LEWIS CARROLL'S PROBLEM

After a battle at least 95% of the combatants had lost a tooth, at least 90% had lost an eye, at least 80% had lost an arm, and at least 75% had lost a leg. At least how many had lost all four?

Dreamstime.com/Olga Rotula

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5. TEST YOURSELF

For Questions 1–4 choose the correct answer A, B, C or D.

Practice quiz

Derivatives find-a-word

Second derivative find-a-word

1 A maximum turning point has:

A $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ B $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ D $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

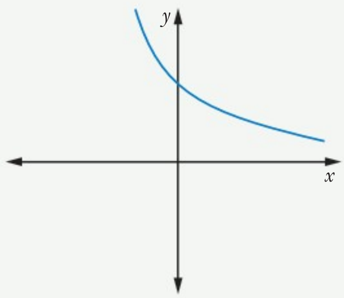
2 For the graph shown:

A $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

B $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$

D $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



3 For a horizontal point of inflection:

A $f''(x) = 0$ B $f'(x) = 0$ and $f''(x) = 0$

C $f''(x) = 0$ and concavity changes D $f'(x) = 0$, $f''(x) = 0$ and concavity changes

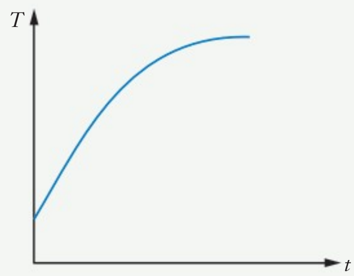
4 The graph below shows temperature T at time t. Which statement describes the shape of the graph?

A The temperature is increasing and the rate of change in temperature is increasing.

B The temperature is decreasing and the rate of change in temperature is increasing.

C The temperature is increasing and the rate of change in temperature is decreasing.

D The temperature is decreasing and the rate of change in temperature is decreasing.



5 Find the stationary points on the curve $y = x^3 + 6x^2 + 9x - 11$ and determine their nature.

6 Find all x values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

7 The height in metres of an object thrown up into the air is given by $h = 20t - 2t^2$, where t is time in seconds. Find the maximum height that the object reaches.

8 Find the domain over which the curve $y = 5 - 6x - 3x^2$ is decreasing.

9 Find the point of inflection on the curve $y = 2x^3 - 3x^2 + 3x - 2$.

10 A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink. Given that 375 mL has a volume of 375 cm^3:

a show that the surface area of a can is given by $S = 2\pi r^2 + \dfrac{750}{r}$

b find the radius of the can that gives the minimum surface area.

11 For the function $y = 3x^4 + 8x^3 + 6x^2$:

a find any stationary points

b determine their nature

c sketch the curve for the domain $[-3, 3]$

d find the maximum and minimum values of the function in this domain.

12 A rectangular prism with a square base is to have a surface area of 250 cm^2.

a Show that the volume is given by $V = \dfrac{125x - x^3}{2}$.

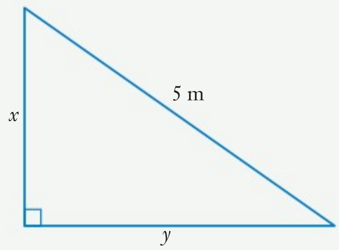
b Find the dimensions that will give the maximum volume.

13 The cost to a business of manufacturing x products a week is given by $C = x^2 - 300x + 9000$. Find the number of products that will give the minimum cost each week.

14 A 5 m length of timber is used to border a triangular garden bed, with the other sides of the garden against the house walls.

a Show that the area of the garden is $A = \dfrac{1}{2}x\sqrt{25 - x^2}$.

b Find the greatest possible area of the garden bed.



15 Find any points of inflection on the curve $f(x) = x^4 - 6x^3 + 2x + 1$.

16 Find the maximum value of the curve $y = x^3 + 3x^2 - 24x - 1$ in the domain $[-5, 6]$.

17 A function has $f'(2) < 0$ and $f''(2) < 0$. Sketch the shape of the function near $x = 2$.

18 Sketch the graph of the function $f(x) = xe^{2x}$ showing all features.

19 Sketch the graph of the function $y = 2\cos 4x$ in the domain $[0, \pi]$.

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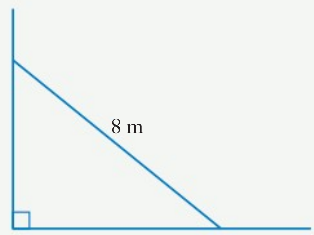
5. CHALLENGE EXERCISE

1 Sketch the curve $y = x(x - 2)^3$ showing any stationary points and points of inflection.

2 Find the maximum possible area if an 8 m length of fencing is placed across a corner to enclose a triangular space.



3 Find the greatest and least values of $f(x) = 4x^3 - 3x^2 - 18x$ in the domain $[-2, 3]$.

4 Show that the function $f(x) = 2(5x - 3)^3$ has a horizontal point of inflection at (0.6, 0).

5 Two circles have radii r and s such that $r + s = 25$.
Show that the sum of areas of the circles is least when $r = s$.

6 Find the equation of a curve that is always concave upwards with a stationary point at (−1, 2) and y-intercept 3.

7 a Show that $y = x^n$ has a stationary point at (0, 0) where n is a positive integer.

b If n is even, show that (0, 0) is a minimum turning point.

c If n is odd, show that (0, 0) is a point of inflection.

8 Find the minimum and maximum values of $y = \dfrac{x+3}{x^2 - 9}$ in the domain $[-2, 2]$.

9 The cost of running a car at an average speed of V km h^{-1} is given by $c = 100 + \dfrac{V^2}{75}$ cents per hour. Find the average speed (to the nearest km h^{-1}) at which the cost of a 1000 km trip is a minimum.

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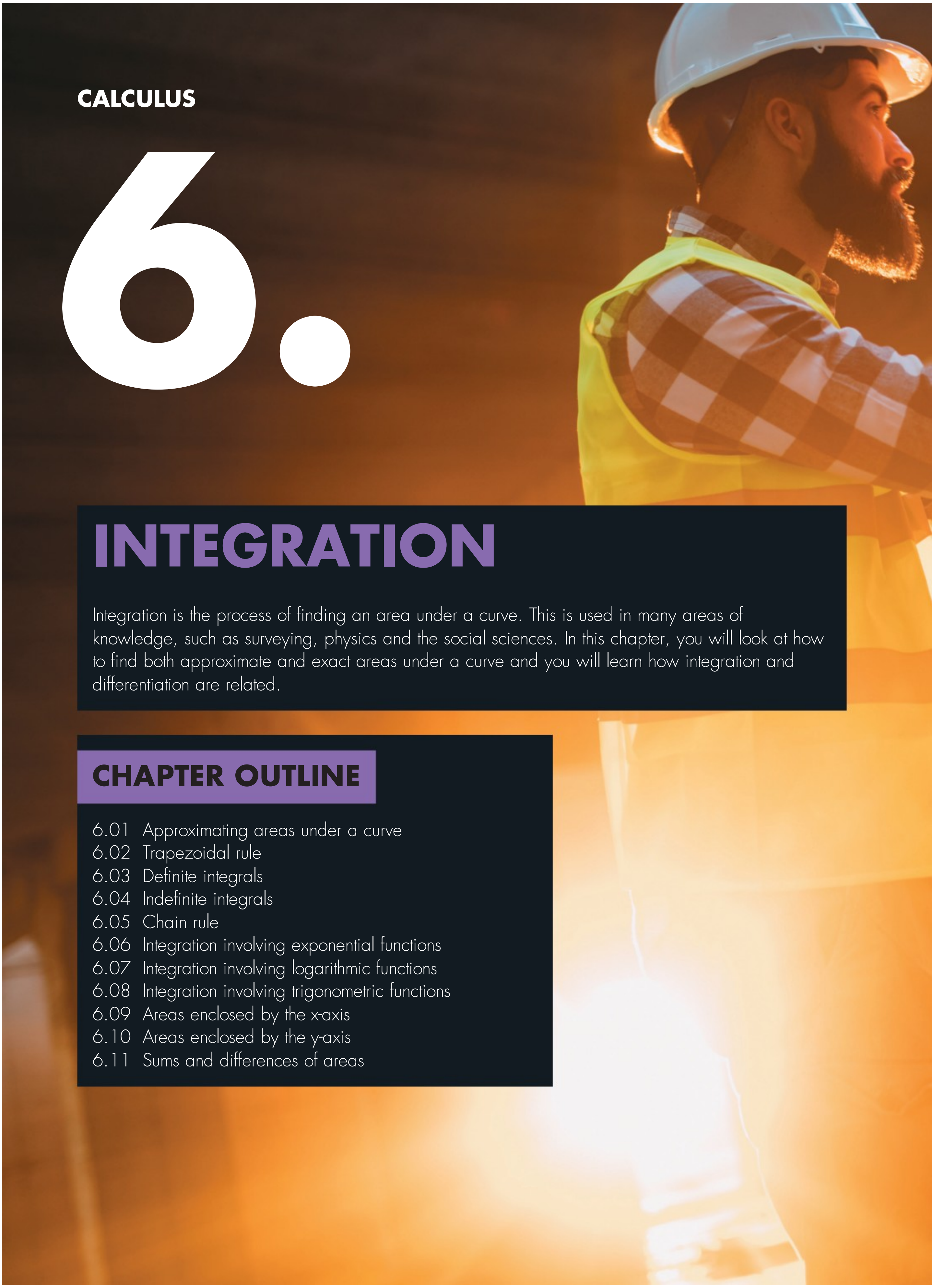
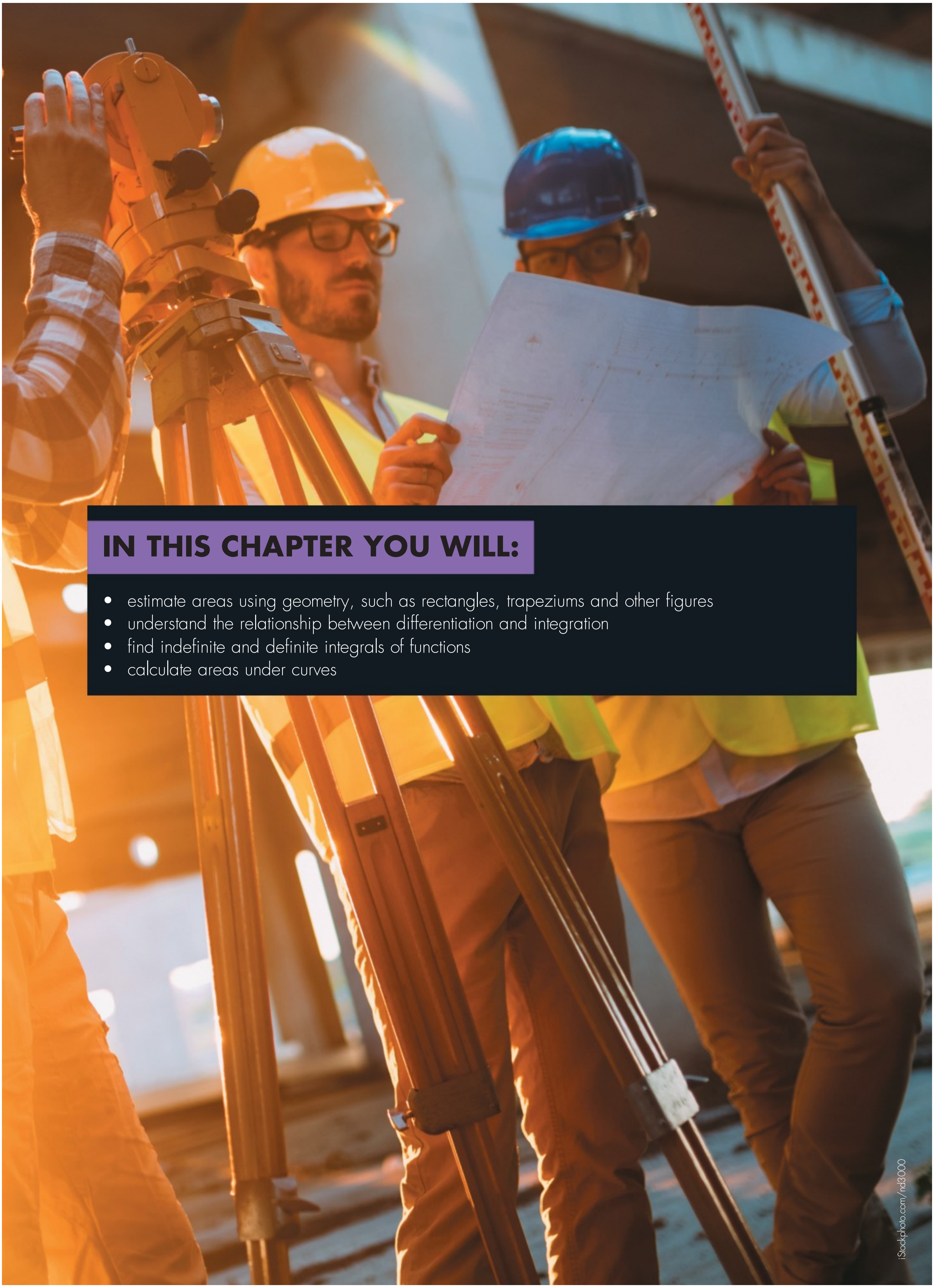


CALCULUS

6. INTEGRATION

Integration is the process of finding an area under a curve. This is used in many areas of knowledge, such as surveying, physics and the social sciences. In this chapter, you will look at how to find both approximate and exact areas under a curve and you will learn how integration and differentiation are related.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- estimate areas using geometry, such as rectangles, trapeziums and other figures
- understand the relationship between differentiation and integration
- find indefinite and definite integrals of functions
- calculate areas under curves

TERMINOLOGY

definite integral: The integral or anti-derivative $y = F(x)$ used to find the area between the curve $y = f(x)$, the x–axis and boundaries $x = a$ and $x = b$, given by $\int_a^b f(x)\,dx = F(b) - F(a)$.

indefinite integral: A general anti-derivative $\int f(x)\,dx$.

integral: An anti-derivative.

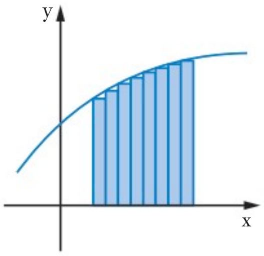
integration: The process of finding an anti-derivative.

trapezoidal rule: A formula for approximating area under a curve by using a trapezium.

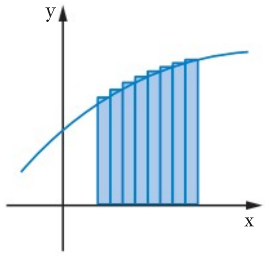
WS
Areas using rectangles

6.01 Approximating areas under a curve

Mathematicians since the time of Archimedes have used rectangles to approximate irregular areas. In more recent times, we use the number plane to find areas enclosed between a curve and the x-axis. We call this the area under the curve.



inner rectangles



outer rectangles

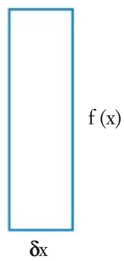
The first diagram has inner or left rectangles that are below the curve, because the top left corners of the rectangles touch the curve.

The second diagram has outer or right rectangles that are above the curve, because the top right corners of the rectangles touch the curve.

The more rectangles we have, the more accurately they approximate the area under the curve.

Integral notation

The diagram at right shows one of the rectangles. The height of each rectangle is $f(x)$ and its width is δx, so its area is $f(x)\delta x$. So the sum of all the rectangles is $\Sigma f(x)\delta x$ for the different values of x.



We can approximate the area under the curve using a large number of rectangles by making the width of each rectangle very small.

Taking an infinite number of rectangles, $\delta x \to 0$.

$$\text{Area} = \lim_{\delta x \to 0}\left(\sum f(x)\delta x\right)$$
$$= \int f(x)\,dx$$

δx is 'delta x' and means a small change in x.
δ is the Greek letter for 'd', for difference.

We use the integral symbol $\int$ to stand for the sum of rectangles (the symbol is an S for sum).

We call $\int f(x)\,dx$ an indefinite integral.

If we are finding the area under the curve $y = f(x)$ between $x = a$ and $x = b$, we can write $\int_a^b f(x)\,dx$.

We call $\int_a^b f(x)\,dx$ a definite integral.

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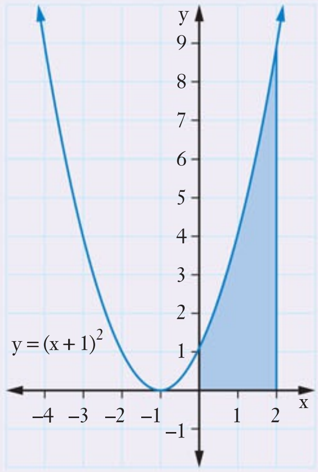
EXAMPLE 1

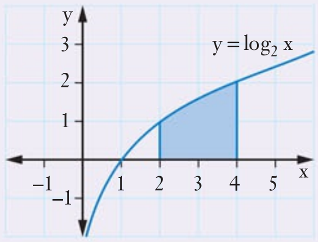
a Find an approximation to the shaded area by using:

i 4 inner rectangles

ii 4 outer rectangles



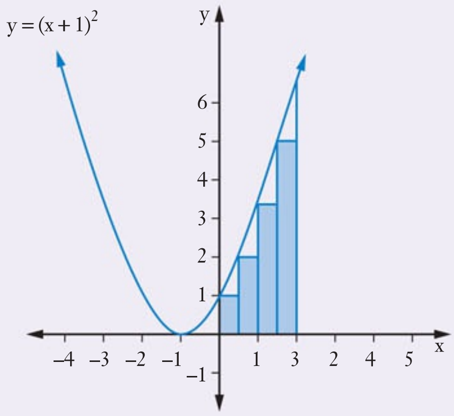
b Find the shaded area below by using a trapezium.



Solution

a i Using inner rectangles, the top left corners touch the curve and they lie below the curve.

Each rectangle has height f (x) and width 0.5 units.



Height of 1st rectangle:

$f(0) = (0 + 1)^2 = 1$

$\text{Area} = 1 \times 0.5 = 0.5$

Height of 2nd rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

$\text{Area} = 2.25 \times 0.5 = 1.125$

Height of 3rd rectangle:

$f(1) = (1 + 1)^2 = 4$

$\text{Area} = 4 \times 0.5 = 2$

Height of 4th rectangle:

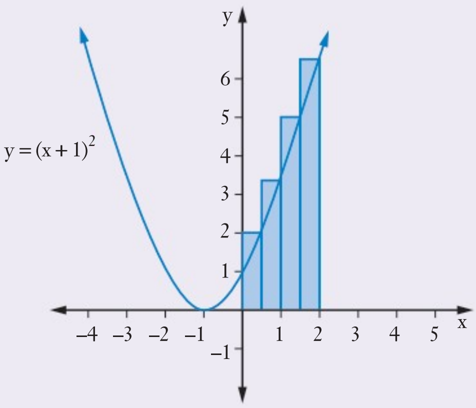
$f(1.5) = (1.5 + 1)^2 = 6.25$

$\text{Area} = 6.25 \times 0.5 = 3.125$

$\text{Total area} = 0.5 + 1.125 + 2 + 3.125 = 6.75$

So area is 6.75 units^2.

ii Using outer rectangles, the top right corners touch the curve and they go above the curve.



Height of 1st rectangle:

$f(0.5) = (0.5 + 1)^2 = 2.25$

Area $= 2.25 \times 0.5 = 1.125$

Height of 2nd rectangle:

$f(1) = (1 + 1)^2 = 4$

Area $= 4 \times 0.5 = 2$

Height of 3rd rectangle:

$f(1.5) = (1.5 + 1)^2 = 6.25$

Area $= 6.25 \times 0.5 = 3.125$

Height of 4th rectangle:

$f(2) = (2 + 1)^2 = 9$

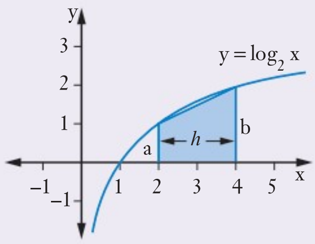
Area $= 9 \times 0.5 = 4.5$

Total area $= 1.125 + 2 + 3.125 + 4.5 = 10.75$

So area is 10.75 units^2

b Area of a trapezium:

$A = \frac{1}{2}h(a + b)$ where h = perpendicular height and a and b are parallel sides.



Here the trapezium has parallel sides at $x = 2$ and $x = 4$.

$$\begin{aligned} a &= f(2) \\ &= \log_2 2 \\ &= 1 \end{aligned}$$

$$\begin{aligned} b &= f(4) \\ &= \log_2 4 \\ &= 2 \end{aligned}$$

$$\begin{aligned} h &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} A &= \frac{1}{2}h(a + b) \\ &= \frac{1}{2}(2)(1 + 2) \\ &= 3 \end{aligned}$$

So area is 3 units^2.

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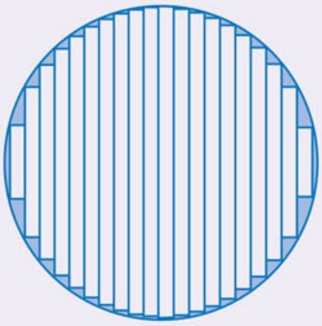
DID YOU KNOW?

Archimedes

Integration has been of interest to mathematicians since very early times. Archimedes (287–212 BCE) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.



Area within curve

Archimedes

TECHNOLOGY

Areas under a curve

We can use a spreadsheet to find approximate areas under a curve using rectangles. Using technology allows us to find sums of large numbers of rectangles without needing to do many calculations. This gives a more accurate approximation to the area under a curve.

For example, we can use a spreadsheet to find the approximate area under the curve $y = (x + 1)^2$ between $x = 0$ and $x = 2$ from Example 1 a i.

We find the y values using the formula =(A2+1)^2 (copy the formula down the column).

The width is =A3–A2 (copy this value down the column).

The area is =B2*C2 (copy the formula down the column).

	A	B	C	D
1	‹	y	Width	Area
2	0	1	0.5	0.5
3	0.5	2.25	0.5	1.125
4	1	4	0.5	2
5	1.5	6.25	0.5	3.125
6				
7			Total area	6.75

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We can use the spreadsheet to find the area using a much larger number of rectangles, for example, 20 or 40.

	A	B	C	D	E	F	G	H	I
1	x	y	Width	Area		x	y	Width	Area
2	0	1	0.1	0.1		0	1	0.05	0.05
3	0.1	1.21	0.1	0.121		0.05	1.1025	0.05	0.055125
4	0.2	1.44	0.1	0.144		0.1	1.21	0.05	0.0605
5	0.3	1.69	0.1	0.169		0.15	1.3225	0.05	0.066125
6	0.4	1.96	0.1	0.196		0.2	1.44	0.05	0.072
7	0.5	2.25	0.1	0.225		0.25	1.5625	0.05	0.078125
8	0.6	2.56	0.1	0.256		0.3	1.69	0.05	0.0845
9	0.7	2.89	0.1	0.289		0.35	1.8225	0.05	0.091125
10	0.8	3.24	0.1	0.324		0.4	1.96	0.05	0.098
11	0.9	3.61	0.1	0.361		0.45	2.1025	0.05	0.105125
12	1	4	0.1	0.4		0.5	2.25	0.05	0.1125
13	1.1	4.41	0.1	0.441		0.55	2.4025	0.05	0.120125
14	1.2	4.84	0.1	0.484		0.6	2.56	0.05	0.128
15	1.3	5.29	0.1	0.529		0.65	2.7225	0.05	0.136125
16	1.4	5.76	0.1	0.576		0.7	2.89	0.05	0.1445
17	1.5	6.25	0.1	0.625		0.75	3.0625	0.05	0.153125
18	1.6	6.76	0.1	0.676		0.8	3.24	0.05	0.162
19	1.7	7.29	0.1	0.729		0.85	3.4225	0.05	0.171125
20	1.8	7.84	0.1	0.784		0.9	3.61	0.05	0.1805
21	1.9	8.41	0.1	0.841		0.95	3.8025	0.05	0.190125
22	2	9	0.1	0.9		1	4	0.05	0.2
23						1.05	4.2025	0.05	0.210125
24						1.1	4.41	0.05	0.2205
25			Total area	9.17		1.15	4.6225	0.05	0.231125
26						1.2	4.84	0.05	0.242
27						1.25	5.0625	0.05	0.253125
28						1.3	5.29	0.05	0.2645
29						1.35	5.5225	0.05	0.276125
30						1.4	5.76	0.05	0.288
31						1.45	6.0025	0.05	0.300125
32						1.5	6.25	0.05	0.3125
33						1.55	6.5025	0.05	0.325125
34						1.6	6.76	0.05	0.338
35						1.65	7.0225	0.05	0.351125
36						1.7	7.29	0.05	0.3645
37						1.75	7.5625	0.05	0.378125
38						1.8	7.84	0.05	0.392
39						1.85	8.1225	0.05	0.406125
40						1.9	8.41	0.05	0.4205
41						1.95	8.7025	0.05	0.435125
42						2	9	0.05	0.45
43									
44								Total area	8.9175
45									

Use technology with a larger number of rectangles for a more accurate area to this question.

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We can find the area under the same curve by using different methods.

We can use other shapes to find areas under a curve.

EXAMPLE 2

Find an approximation to the area under the curve $y = x^2$ between $x = 0$ and $x = 2$ by using:

a squares

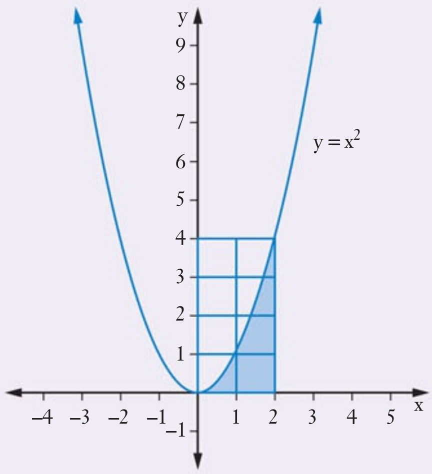
b a triangle

Solution

a On the grid, each square is 1 square unit.
By counting and approximating squares:

$A \approx 3$

So area is 3 units^2.



b Using a triangle:

$b = 2 - 0.5 = 1.5$

$h = f(2)$

$= 2^2$

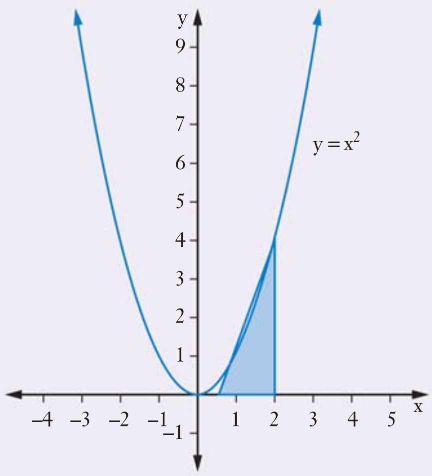
$= 4$

$A = \frac{1}{2}bh$

$= \frac{1}{2} \times 1.5 \times 4$

$= 3$

So area is 3 units^2.



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Exercise 6.01 Approximating areas under a curve

1 Find an approximation to the area under the curve $y = x^2 + 2x$ between $x = 1$ and $x = 2$ by using:

a 2 inner rectangles

b 2 outer rectangles

2 Find an approximation (to 2 decimal places) to the area under the curve $y = \frac{2}{x+1}$ from $x = 1$ to $x = 3$ using:

a 2 inner rectangles

b 2 outer rectangles

c 4 inner rectangles

d 4 outer rectangles

3 Use a trapezium to find an approximate area under the curve:

a $f(x) = x^2$ between $x = 2$ and $x = 3$

b $y = \ln x$ between $x = 4$ and $x = 7$

c $f(x) = x^3 + 1$ between $x = 0$ and $x = 4$

d $f(x) = \sin x$ between $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$ (give answer in exact form)

e $y = 9 - x^2$ between $x = 1$ and $x = 2$

4 Find the approximate area under the curve $f(x) = x^3 + 3$ between $x = 0$ and $x = 4$ by using:

a 2 inner rectangles

b 2 outer rectangles

c a trapezium

5 Use a trapezium to find each area under the curve.

a $y = \frac{1}{x}$ between $x = 1$ and $x = 7$

b $y = x^2 + 5$ between $x = 0$ and $x = 1$

c $f(x) = \cos x$ between $x = 0$ and $x = \frac{\pi}{3}$ (in exact form)

d $y = e^x$ between $x = 1$ and $x = 4$ (in exact form)

e $f(x) = x(x - 4)(x - 9)$ between $x = 2$ and $x = 3$

6 a Sketch the graph of $y = 1 - x^2$ and shade the area under the curve (enclosed between the curve and the x-axis).

b Find this approximate area by using a triangle.

7 Find the approximate area under the curve $y = \sqrt{x-1}$ between $x = 2$ and $x = 5$ by using:

a 6 inner rectangles

b 6 outer rectangles

c a trapezium

d squares

8 Find the exact area under the curve $y = \sqrt{25 - x^2}$.

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9 a Find the exact area under the curve $y = \sqrt{9 - x^2}$.

b Find the approximate area under the curve $y = \sqrt{9 - x^2}$:

i between $x = 1$ and $x = 2$ using a trapezium

ii between $x = 0$ and 3 using 3 outer rectangles

10 Use a triangle to find the approximate area under the curve:

a $y = x^2$ between $x = 0$ and $x = 4$

b $y = \sqrt{x}$ between $x = 0$ and $x = 3$

c $y = \cos x$ between $x = 0$ and $x = \dfrac{\pi}{2}$

11 Find the approximate area under each curve by using:

i 4 inner rectangles   ii 4 outer rectangles

a $y = -x^2 + 4x$

b $y = \sin x$ in the domain $[0, \pi]$ (in exact form)

12 Find the approximate area under the curve $y = x^2 + 5$ between $x = 0$ and $x = 5$ (using technology where available) using:

a 10 inner rectangles

b 10 outer rectangles

6.02 Trapezoidal rule

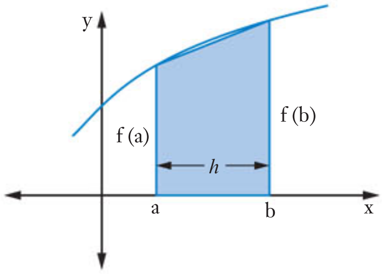
WS Trapezoidal rule

A trapezium usually gives a much closer approximation to the area under a curve than a rectangle does.

The trapezoidal rule is a formula that uses a trapezium to find the area under a curve.

$$A = \frac{1}{2}h[f(a) + f(b)] \text{ where } h = b - a$$

$$= \frac{1}{2}(b - a)[f(a) + f(b)]$$



Trapezoidal rule

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

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EXAMPLE 3

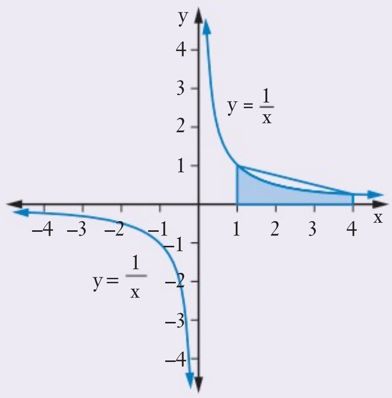
Use the trapezoidal rule to find an approximation for:

a $\int_1^4 \frac{1}{x}\,dx$

b $\int_0^1 x^3\,dx$ using 2 subintervals

Solution

a $\int_1^4 \frac{1}{x}\,dx$ is the area under the curve as shaded in the diagram.

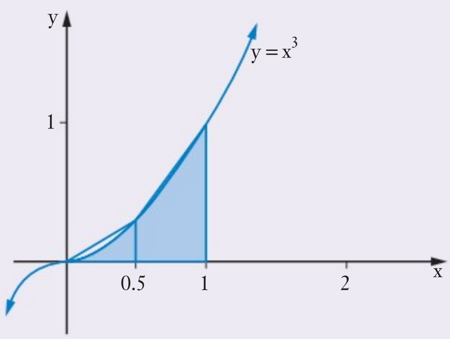


$f(x) = \frac{1}{x}$, a = 1 and b = 4.

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

$$\begin{aligned}\int_1^4 \frac{1}{x}\,dx &\approx \frac{1}{2}(4 - 1)[f(1) + f(4)]\\ &= \frac{1}{2}(3)\left[\frac{1}{1} + \frac{1}{4}\right]\\ &= \frac{15}{8}\\ &= 1\frac{7}{8}\end{aligned}$$

b 2 subintervals means 2 trapezia.
We use the trapezoidal formula twice.



$f(x) = x^3$ and $h = 0.5$

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b - a)[f(a) + f(b)]$$

$$\begin{aligned}\int_0^1 x^3\,dx &= \int_0^{0.5} x^3\,dx + \int_{0.5}^1 x^3\,dx\\ &\approx \frac{1}{2}(0.5 - 0)[f(0) + f(0.5)] + \frac{1}{2}(1 - 0.5)[f(0.5) + f(1)]\\ &= \frac{1}{2}(0.5)[0^3 + 0.5^3] + \frac{1}{2}(0.5)[0.5^3 + 1^3]\\ &= 0.3125\end{aligned}$$

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There is a more general trapezoidal rule when using several subintervals or trapezia.

Trapezoidal rule for n subintervals

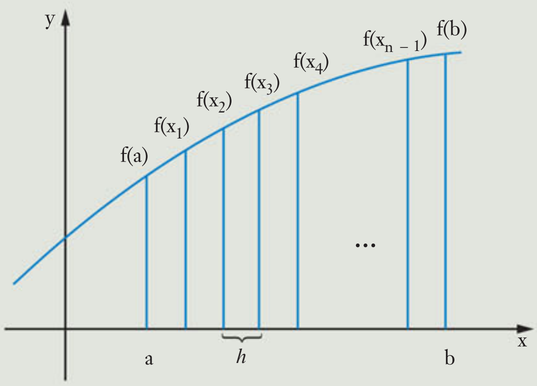
Given n subintervals (trapezia)

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + f(b) + 2\left\{f(x_1) + \ldots + f(x_{n-1})\right\}\right]$$

where $a = x_0$ and $b = x_n$, and the values of $x_0, x_1, x_2, \ldots, x_n$ are found by dividing the interval $a \le x \le b$ into n equal subintervals of width $h = \dfrac{b-a}{n}$.

Since $h = \dfrac{b-a}{n}$, the formula can also be written as

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(a) + f(b) + 2\left\{f(x_1) + \ldots + f(x_{n-1})\right\}\right].$$



Proof

Interval b − a is divided into n trapezia.

So the width of each trapezium is $h = \dfrac{b-a}{n}$.

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{1}{2}h[f(a) + f(x_1)] + \frac{1}{2}h[f(x_1) + f(x_2)] + \frac{1}{2}h[f(x_2) + f(x_3)] + \ldots + \frac{1}{2}h[f(x_{n-1}) + f(b)]\\ &= \frac{h}{2}[f(a) + f(x_1) + f(x_1) + f(x_2) + f(x_2) + f(x_3) + \ldots + f(x_{n-1}) + f(b)]\\ &= \frac{h}{2}(f(a) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_{n-1}) + f(b))\\ &= \frac{h}{2}\left[f(a) + f(b) + 2\left\{f(x_1) + \ldots + f(x_{n-1})\right\}\right]\end{aligned}$$

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Trapezoidal rule

EXAMPLE 4

a Use the trapezoidal rule with 4 subintervals to find an approximation for $\int_2^3 \frac{2}{x-1}\,dx$ correct to 3 decimal places.

b Use the trapezoidal rule with 7 subintervals to find an approximation for $\int_0^{14} (t^2+3)\,dt$.

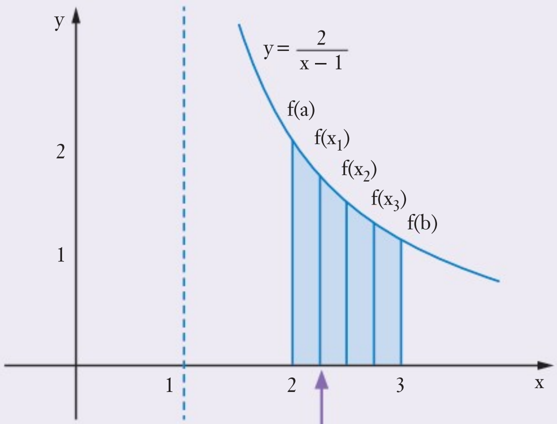
Solution

a From the question and diagram:

$a = 2, b = 3, n = 4.$

The width of each trapezium is

$$h = \frac{b-a}{n} = \frac{3-2}{4} = \frac{1}{4} = 0.25$$



Note that these x values increase by h = 0.25, the width of each trapezium.

Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + f(b) + 2\{f(x_1) + \dots + f(x_{n-1})\}\right]$$

$$\int_2^3 \frac{2}{x-1}\,dx \approx \frac{0.25}{2}\left[f(2) + f(3) + 2\{f(2.25) + f(2.5) + f(2.75)\}\right]$$

$$= 0.125\left[\frac{2}{2-1} + \frac{2}{3-1} + 2\left(\frac{2}{2.25-1} + \frac{2}{2.5-1} + \frac{2}{2.75-1}\right)\right]$$

$$= 0.125[2 + 1 + 2(1.6 + 1.3333\ldots + 1.1428\ldots)]$$

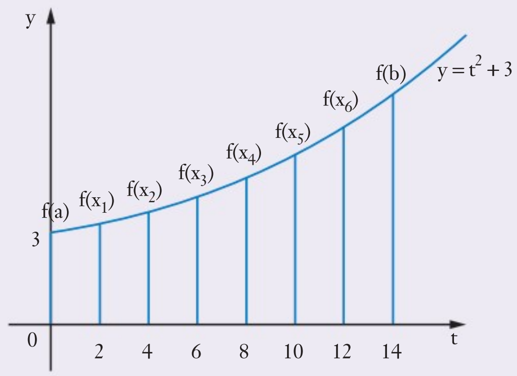
$$= 1.39404\ldots$$

$$\approx 1.394$$

b $a = 0, b = 14, n = 7.$

The width of each trapezium is

$$h = \frac{14-0}{7} = \frac{14}{7} = 2$$



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Substituting into the general trapezoidal rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + f(b) + 2\left\{f(x_1) + \ldots + f(x_{n-1})\right\}\right]$$

$$\int_0^{14} (t^2 + 3)\,dt \approx \frac{2}{2}\left[f(0) + f(14) + 2\left\{f(2) + f(4) + f(6) + f(8) + f(10) + f(12)\right\}\right]$$

$$= [(0^2 + 3) + (14^2 + 3) + 2\{(2^2 + 3) + (4^2 + 3) + (6^2 + 3) + (8^2 + 3)$$
$$+ (10^2 + 3) + (12^2 + 3)\}]$$

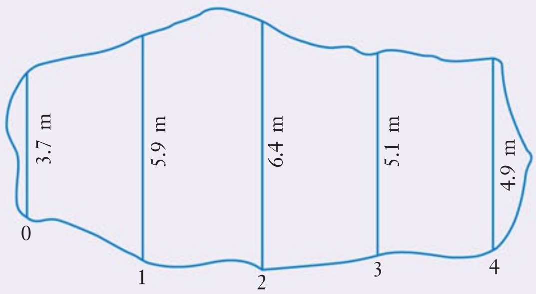
$$= 966$$

We can use the trapezoidal rule to find irregular areas.

EXAMPLE 5

A surveyor needs to find the area of the irregular piece of land shown.

Use the trapezoidal rule to find its approximate area.



Solution

We can use the values in the diagram or put them in the table below.

x	0	1	2	3	4
f(x)	3.7	5.9	6.4	5.1	4.9

$a = 0$, $b = 4$, $n = 4$.

From the diagram or table, the height of each trapezium is 1.

$$\text{Area} \approx \frac{h}{2}\left[f(a) + f(b) + 2\left\{f(x_1) + \ldots + f(x_{n-1})\right\}\right]$$

$$= \frac{1}{2}\left[f(0) + f(4) + 2\left\{f(1) + f(2) + f(3)\right\}\right]$$

$$= \frac{1}{2}\left[3.7 + 4.9 + 2\left\{5.9 + 6.4 + 5.1\right\}\right]$$

$$= 21.7$$

So the area of the land is approximately 21.7 m^2.

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Exercise 6.02 Trapezoidal rule

1 Use the trapezoidal rule to find an approximation for each integral.

a $\int_1^2 x^2\,dx$ b $\int_0^2 (x^3+1)\,dx$ c $\int_1^5 \frac{dx}{x}$ d $\int_1^2 \frac{dx}{x+3}$

2 Find an approximation to $\int_1^3 x^3\,dx$ using the trapezoidal rule with:

a 1 subinterval b 2 subintervals

3 Use the trapezoidal rule with 2 trapezia to find an approximation to:

a $\int_2^3 \log x\,dx$ b $\int_0^2 \frac{dx}{x+4}$

4 Find an approximation to:

a $\int_1^4 \log x\,dx$ using 3 trapezia b $\int_0^2 (x^2 - x)\,dx$ using 4 trapezia

c $\int_0^1 \sqrt{x}\,dx$ using 5 subintervals d $\int_1^5 \frac{dx}{x^2}$ using 4 subintervals

e $\int_3^6 \frac{dx}{x-1}$ using 6 trapezia

5 Given the table of values, find the approximate value of each definite integral.

a $\int_1^9 f(x)\,dx$

x	1	3	5	7	9
f(x)	3.2	5.9	8.4	11.6	20.1

b $\int_1^4 f(t)\,dt$

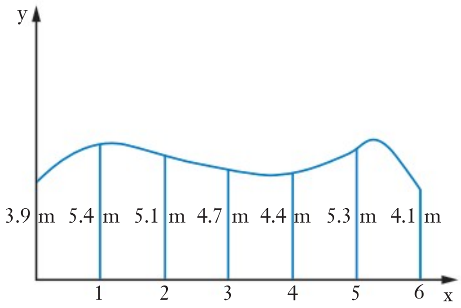
t	1	2	3	4
f (t)	8.9	6.5	4.1	2.9

c $\int_2^{14} f(x)\,dx$

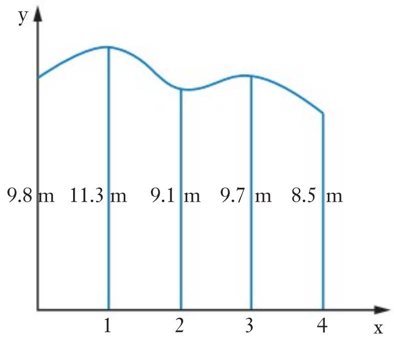
x	2	4	6	8	10	12	14
f(x)	25.1	37.8	52.3	89.3	67.8	45.4	39.9

6 Use the trapezoidal rule to find the approximate area of each irregular figure below.

a



b



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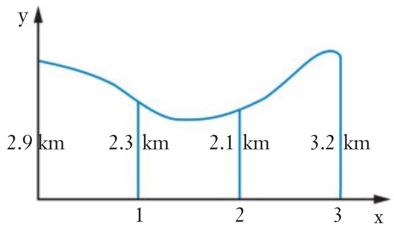
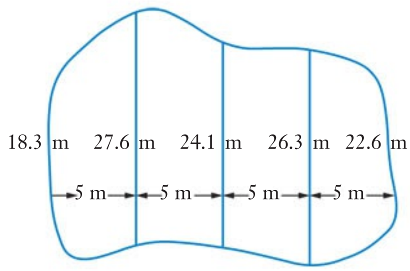
c



d



6.03 Definite integrals

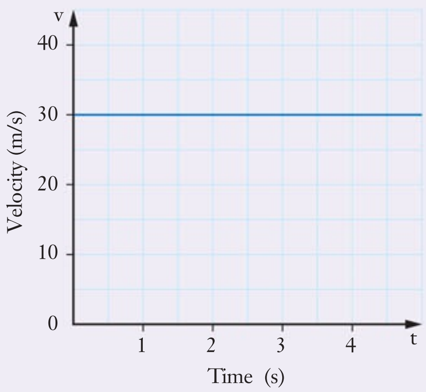
WS Definite integrals

WS Finding definite integrals

We can link the area under a graph to calculus.

EXAMPLE 6

This graph shows the velocity of an object over time as it travels at a constant 30 m s^{-1}.



a Find the distance it travels in:

i 1 s
ii 2 s
iii 3 s

b Find the area under the line between:

i $t = 0$ and $t = 1$
ii $t = 0$ and $t = 2$
iii $t = 0$ and $t = 3$

Solution

a i $s = \frac{d}{t}$, so $d = st$.
In 1 s, the object travels $30 \times 1 = 30$ m.

ii In 2 s, the object travels $30 \times 2 = 60$ m.

iii In 3 s, the object travels $30 \times 3 = 90$ m.

b i The area is $30 \times 1 = 30$ units^2

ii The area is $30 \times 2 = 60$ units^2

iii The area is $30 \times 3 = 90$ units^2

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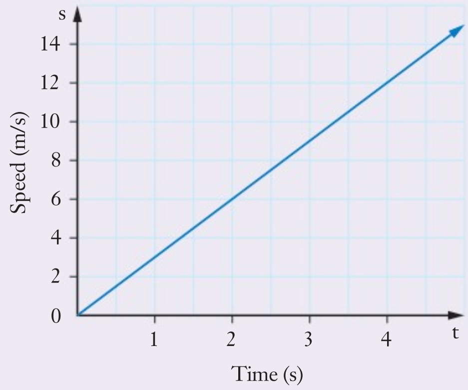
EXAMPLE 7

This graph shows the speed of an object increasing at a steady rate.



a Find the distance travelled in:

i 1 s

ii 4 s

b Find the area under the graph between:

i $t = 0$ and $t = 1$

ii $t = 0$ and $t = 4$

Solution

a i At $t = 0$, $s = 0$, and at $t = 1$, $s = 3$.

$$\text{Average speed} = \frac{0+3}{2} = 1.5 \text{ m s}^{-1}$$

In 1 s, the object travels $1.5 \times 1 = 1.5$ m.

ii At $t = 0$, $s = 0$, and at $t = 4$, $s = 12$.

$$\text{Average speed} = \frac{0+12}{2} = 6 \text{ m s}^{-1}$$

In 4 s, the object travels $6 \times 4 = 24$ m.

b i
$$A = \frac{1}{2}bh = \frac{1}{2} \times 1 \times 3 = 1.5 \text{ units}^2$$

ii
$$A = \frac{1}{2}bh = \frac{1}{2} \times 4 \times 12 = 24 \text{ units}^2$$

The graphs in the last 2 examples show velocity (rate of change of displacement) and speed (rate of change of distance) against time. The area under each curve gave the information about the original variable. This is the anti-derivative.

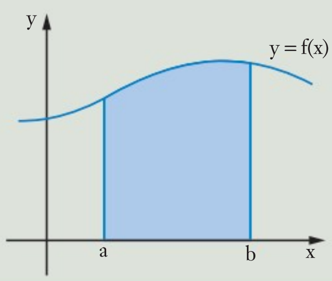
In the same way, the area under any rate of change graph will give the original variable, or the anti-derivative.

Fundamental theorem of calculus

The area enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F(x)$ is the anti-derivative of function $f(x)$.



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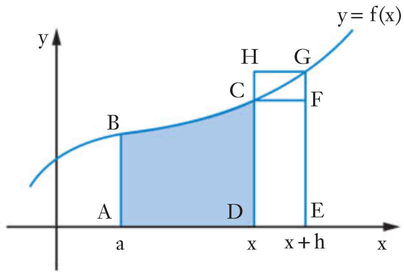
Proof

Consider a continuous curve $y = f(x)$ for all values of $x > a$.



Let area ABCD be $A(x)$

Let area ABGE be $A(x + h)$

Then area DCGE is $A(x + h) - A(x)$

Area DCFE < area DCGE < area DHGE

$$f(x) \times h < A(x + h) - A(x) \quad < f(x + h) \times h$$

$$f(x) < \frac{A(x + h) - A(x)}{h} \quad < f(x + h)$$

$$\lim_{h \to 0} f(x) < \lim_{h \to 0} \frac{A(x + h) - A(x)}{h} < \lim_{h \to 0} f(x + h)$$

This is the formula for the derivative of A(x) from first principles, from Year 11, Chapter 6, Introduction to calculus.

$$f(x) < A'(x) < f(x)$$

So $A'(x) = f(x)$

$A(x)$ is an anti-derivative of $f(x)$.

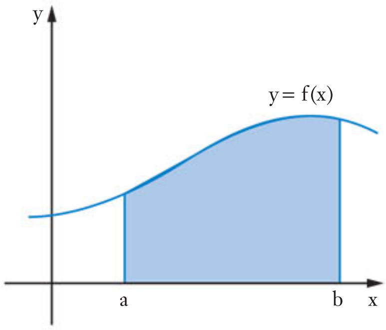
Let $F(x)$ be the anti-derivative of $f(x)$ with a constant term of 0.

Then $A(x) = F(x) + C$ [1]

Now $A(x)$ is the area under $y = f(x)$ between a and x.

$A(a) = 0$

Substitute in [1]:



$A(a) = F(a) + C$

$0 = F(a) + C$

$-F(a) = C$

$A(x) = F(x) - F(a)$

If $x = b$ where $b > a$:

$A(b) = F(b) - F(a)$

So $\int_b^a f(x)\,dx = F(b) - F(a)$

EXAMPLE 8

Evaluate:

a $\int_3^4 (2x + 1)\,dx$ b $\int_0^5 3x^2\,dx$ c $\int_0^2 (-3x^2)\,dx$ d $\int_{-1}^1 x^3\,dx$

Solution

You learned in Chapter 4, Further differentiation, that the anti-derivative of x^n is $\frac{1}{n+1} x^{n+1} + C$. Let $F(x)$ be the anti-derivative where $C = 0$.

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a $\int_3^4 (2x+1)\,dx = \left[x^2 + x\right]_3^4$

The anti-derivative

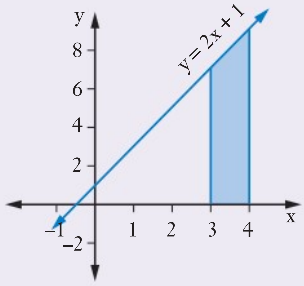
$= (4^2 + 4) - (3^2 + 3)$

F(b) – F(a)

$= 20 - 12$

$= 8$

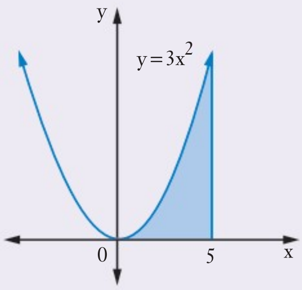
The graph shows the area that this integral calculates.
Can you find its area an easier way?



b $\int_0^5 3x^2\,dx = \left[x^3\right]_0^5$

$= 5^3 - 0^3$

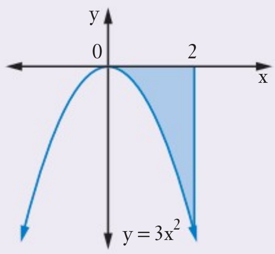
$= 125$



c $\int_0^2 (-3x^2)\,dx = \left[-x^3\right]_0^2$

$= -2^3 - (-0^3)$

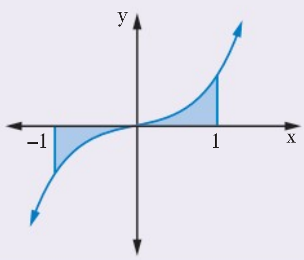
$= -8$



d $\int_{-1}^1 x^3\,dx = \left[\dfrac{x^4}{4}\right]_{-1}^1$

$= \dfrac{1^4}{4} - \dfrac{(-1)^4}{4}$

$= 0$



We can also find the definite integral of x^n when n is a fraction or negative.

EXAMPLE 9

Evaluate:

a $\int_1^2 \dfrac{2x-3}{x^3}\,dx$

b $\int_1^8 \sqrt[3]{x}\,dx$

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Solution

a $\displaystyle\int_1^2 \frac{2x-3}{x^3}\,dx = \int_1^2 \frac{2x}{x^3} - \frac{3}{x^3}\,dx$

$$= \int_1^2 \frac{2}{x^2} - \frac{3}{x^3}\,dx$$

$$= \int_1^2 (2x^{-2} - 3x^{-3})\,dx$$

$$= \left[\frac{2x^{-1}}{-1} - \frac{3x^{-2}}{-2}\right]_1^2$$

$$= \left[-\frac{2}{x} + \frac{3}{2x^2}\right]_1^2$$

$$= \left[-\frac{2}{2} + \frac{3}{2(2)^2}\right] - \left[-\frac{2}{1} + \frac{3}{2(1)^2}\right]$$

$$= -1 + \frac{3}{8} + 2 - \frac{3}{2}$$

$$= -\frac{1}{8}$$

b $\displaystyle\int_1^8 \sqrt[3]{x}\,dx = \int_1^8 x^{\frac{1}{3}}\,dx$

$$= \left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^8$$

$$= \left[\frac{3x^{\frac{4}{3}}}{4}\right]_1^8$$

$$= \left[\frac{3\sqrt[3]{x^4}}{4}\right]_1^8$$

$$= \frac{3\sqrt[3]{8^4}}{4} - \frac{3\sqrt[3]{1^4}}{4}$$

$$= \frac{3\times 16}{4} - \frac{3\times 1}{4}$$

$$= 11\frac{1}{4}$$

We can use the definite integral to find original information given a rate of change.

EXAMPLE 10

The velocity of a particle is given by $v = 8t^3 - 3t^2 + 6t + 1$ cm s^{-1}.
Find the change in displacement in the first 3 seconds.

Solution

$$v = \frac{dx}{dt} = 8t^3 - 3t^2 + 6t + 1$$

$$x = \int_0^3 (8t^3 - 3t^2 + 6t + 1)\,dt$$

$$= \left[8\frac{t^4}{4} - 3\frac{t^3}{3} + 6\frac{t^2}{2} + t\right]_0^3$$

$$= \left[2t^4 - t^3 + 3t^2 + t\right]_0^3$$

$$= [2(3)^4 - 3^3 + 3(3)^2 + 3] - [2(0)^4 - 0^3 + 3(0)^2 + 0]$$

$$= 165$$

So the change in displacement in the first 3 seconds is 165 cm.

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DID YOU KNOW?

Differentiation vs integration

Many mathematicians in the 17th century were interested in the problem of finding areas under a curve. The Englishman Isaac Barrow (1630–77) is said to be the first to discover that differentiation and integration are inverse operations. This discovery is called the fundamental theorem of calculus.

Barrow was an outstanding Greek scholar as well as making contributions in the areas of mathematics, theology, astronomy and physics. However, when he was a schoolboy, he was so often in trouble that his father was overheard saying to God in his prayers that if he decided to take one of his children, he could best spare Isaac.

Another English mathematician named Isaac, Sir Isaac Newton (1643–1727), was also a scientist and astronomer, and helped to discover calculus. He was not interested in his school work, but spent most of his time inventing things, such as a water clock and sundial.

Newton left school at 14 to manage the family estate after his stepfather died. However, he spent so much time reading that he was sent back to school. He went on to university and developed the theories in mathematics and science that have made him famous today.

Exercise 6.03 Definite integrals

1 Evaluate each definite integral.

a $\int_0^2 4x\,dx$ b $\int_1^3 (2x+1)\,dx$ c $\int_{-1}^6 3x^2\,dx$

d $\int_1^2 (4t-7)\,dt$ e $\int_{-1}^1 (6y+5)\,dy$ f $\int_0^3 6x^2\,dx$

g $\int_1^2 (x^2+1)\,dx$ h $\int_0^2 4x^3\,dx$ i $\int_{-1}^4 (3x^2-2x)\,dx$

2 Evaluate:

a $\int_{-1}^1 x^2\,dx$ b $\int_{-2}^3 (x^3+1)\,dx$ c $\int_{-2}^2 x^5\,dx$

d $\int_1^4 \sqrt{x}\,dx$ e $\int_0^1 (x^3-3x^2+4x)\,dx$ f $\int_1^2 (2x-1)^2\,dx$

g $\int_{-1}^1 (y^3+y)\,dy$ h $\int_3^4 (2-x)^2\,dx$ i $\int_{-2}^2 4t^3\,dt$

j $\int_2^4 \frac{x^2}{3}\,dx$ k $\int_1^3 \frac{5x^4}{x}\,dx$ l $\int_2^4 \frac{x^4-3x}{x}\,dx$

m $\int_1^2 \frac{4x^3+x^2+5x}{x}\,dx$ n $\int_3^5 \frac{x^3-2x^2+3x}{x}\,dx$ o $\int_3^4 \frac{x^2+x+3}{3x^5}\,dx$

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3 For each velocity function, find the change in displacement between 2 and 4 seconds.

a $v = 3t^2 + 7$ ms^{-1}

b $v = 8t - 5$ km h^{-1}

c $v = 4t^3 + 2t + 3$ cm s^{-1}

d $v = (t + 3)^2$ ms^{-1}

e $v = 5 - 6t + 9t^2$ cm s^{-1}

4 A high-power hose fills an empty swimming pool at the rate of $r = 25 + 4t^3$ L min^{-1}. Find the volume to the nearest litre after:

a 5 minutes

b 15 minutes

c half an hour

INVESTIGATION

AREAS

Look at the results of definite integrals in the examples and exercises. Sketch the graphs where possible and shade in the areas found.

Can you see why the definite integral sometimes gives a negative answer?

Can you see why it will sometimes be zero?

6.04 Indefinite integrals

To find the indefinite integral $\int f(x)\,dx$, we find the anti-derivative of the function.

WS Indefinite integrals

WS Integrating functions

Integral of x^n

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \text{ where } n \neq -1$$

EXAMPLE 11

Find each indefinite integral.

a $\int 5x^9\,dx$

b $\int \left(\frac{1}{x^3} + \sqrt{x}\right)dx$

Solution

a
$$\begin{aligned}\int 5x^9\,dx &= 5\int x^9\,dx\\ &= 5\times\frac{x^{10}}{10} + C\\ &= \frac{x^{10}}{2} + C\end{aligned}$$

b
$$\begin{aligned}\int \left(\frac{1}{x^3} + \sqrt{x}\right)dx &= \int \left(x^{-3} + x^{\frac{1}{2}}\right)dx\\ &= \frac{x^{-2}}{-2} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C\\ &= -\frac{1}{2x^2} + \frac{2\sqrt{x^3}}{3} + C\end{aligned}$$

DID YOU KNOW?

John Wallis

English clergyman and mathematician John Wallis (1616–1703) found that the area under the curve $y = 1 + x + x^2 + x^3 + \ldots$ is given by:

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots$$

He found this result independently of the fundamental theorem of calculus.

We can use the indefinite integral to find original information given a rate of change.

EXAMPLE 12

The rate of air flow into a container is given by

$$R = 4 + 3t^2 \text{ mm}^3 \text{ s}^{-1}$$

If there is initially no air in the container, find the volume of air in it after 12 seconds.

Solution

$R = \frac{dV}{dt} = 4 + 3t^2$

$V = \int (4 + 3t^2)\, dt$

$= 4t + t^3 + C$

When $t = 0$, $V = 0$

$0 = 4(0) + 0^3 + C$

$= C$

So $V = 4t + t^3$

When $t = 12$:

$V = 4(12) + 12^3$

$= 1776$

So the container will hold 1776 mm^3 of air after 12 seconds.

Exercise 6.04 Indefinite integrals

1 Find each indefinite integral.

a $\int x^2\, dx$ b $\int 3x^5\, dx$ c $\int 2x^4\, dx$

d $\int (m + 1)\, dm$ e $\int (t^2 - 7)\, dt$ f $\int (h^7 + 5)\, dh$

g $\int (y - 3)\, dy$ h $\int (2x + 4)\, dx$ i $\int (b^2 + b)\, db$

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2 Find:

a $\int (x^2+2x+5)\,dx$

b $\int (4x^3-3x^2+8x-1)\,dx$

c $\int (6x^5+x^4+2x^3)\,dx$

d $\int (x^7-3x^6-9)\,dx$

e $\int (2x^3+x^2-x-2)\,dx$

f $\int (x^5+x^3+4)\,dx$

g $\int (4x^2-5x-8)\,dx$

h $\int (3x^4-2x^3+x)\,dx$

i $\int (6x^3+5x^2-4)\,dx$

j $\int (3x^{-4}+x^{-3}+2x^{-2})\,dx$

3 Find each indefinite integral.

a $\int \frac{dx}{x^8}$

b $\int x^{\frac{1}{3}}\,dx$

c $\int \frac{x^6-3x^5+2x^4}{x^3}\,dx$

d $\int (1-2x)^2\,dx$

e $\int (x-2)(x+5)\,dx$

f $\int \frac{3}{x^2}\,dx$

g $\int \frac{dx}{x^3}$

h $\int \frac{4x^3-x^5-3x^2+7}{x^5}\,dx$

i $\int (y^2-y^{-7}+5)\,dy$

j $\int (t^2-4)(t-1)\,dt$

k $\int \sqrt{x}\,dx$

l $\int \frac{2}{t^5}\,dt$

m $\int \sqrt[3]{x}\,dx$

n $\int x\sqrt{x}\,dx$

o $\int \sqrt{x}\left(1+\frac{1}{\sqrt{x}}\right)dx$

4 The rate of change of the angle sum S of a polygon with n sides is a constant 180°. If S = 360° when n = 4, find S when n = 7.

5 For a certain graph, the rate of change of y values with respect to its x values is given by $R = 3x^2 - 2x + 1$. If the graph passes through the point (–1, 3), find its equation.

6 The rate of change in velocity over time is given by $\frac{dx}{dt} = 4t + t^2 - t^3$.
If the initial velocity is 2 cm s^{-1}, find the displacement after 15 s.

7 The rate of flow of water into a dam is given by R = 500 + 20t L h^{-1}. If there is 15 000 L of water initially in the dam, how much water will there be in the dam after 10 hours?

6.05 Chain rule

You found the anti-derivative of $y = (ax + b)^n$ in Chapter 4, Further differentiation. We can write this as an integral.

Chain rule for $(ax + b)^n$

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \text{ where } n \neq -1$$

EXAMPLE 13

Find:

a $\displaystyle\int (5x-9)^3\,dx$ b $\displaystyle\int_2^3 (3-x)^8\,dx$ c $\displaystyle\int_1^3 \sqrt{4x-3}\,dx$

Solution

a
$$\int (5x-9)^3\,dx = \frac{(5x-9)^4}{5\times 4} + C$$
$$= \frac{(5x-9)^4}{20} + C$$

b
$$\int_2^3 (3-x)^8 dx = \left[\frac{(3-x)^9}{-1\times 9}\right]_2^3$$
$$= \left[-\frac{(3-x)^9}{9}\right]_2^3$$
$$= -\frac{(3-3)^9}{9} - \left(-\frac{(3-2)^9}{9}\right)$$
$$= 0 + \frac{1}{9}$$
$$= \frac{1}{9}$$

c
$$\int_1^3 \sqrt{4x-3}\,dx = \int_1^3 (4x-3)^{\frac{1}{2}}\,dx$$
$$= \left[\frac{(4x-3)^{\frac{3}{2}}}{4\times\frac{3}{2}}\right]_1^3$$
$$= \left[\frac{\sqrt{(4x-3)^3}}{6}\right]_1^3$$
$$= \frac{\sqrt{(4\times 3-3)^3}}{6} - \frac{\sqrt{(4\times 1-3)^3}}{6}$$
$$= \frac{\sqrt{9^3}}{6} - \frac{\sqrt{1^3}}{6}$$
$$= \frac{27}{6} - \frac{1}{6}$$
$$= 4\frac{1}{3}$$

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In Chapter 4, *Further differentiation*, you also learned about the general chain rule for $f'(x)[f(x)]^n$.

Chain rule for $f'(x)[f(x)]^n$

$$\int f'(x)[f(x)]^n \, dx = \frac{1}{n+1}[f(x)]^{n+1} + C \quad \text{where } n \neq -1$$

EXAMPLE 14

a Find $\int x(x^2+1)^3 \, dx$.

b Find the exact value of $\int_1^2 x^2\sqrt{x^3-1} \, dx$.

Solution

a $f(x) = x^2 + 1$

$f'(x) = 2x$

$$\int x(x^2+1)^3 \, dx$$

$$= \int \frac{1}{2} \times 2x(x^2+1)^3 \, dx$$

$$= \frac{1}{2}\int 2x(x^2+1)^3 \, dx$$

$$= \frac{1}{2} \times \frac{1}{4}(x^2+1)^4 + C$$

$$= \frac{1}{8}(x^2+1)^4 + C$$

applying the chain rule formula

b $f(x) = x^3 - 1$

$f'(x) = 3x^2$

$$\int_1^2 x^2\sqrt{x^3-1} \, dx$$

$$= \int_1^2 \frac{1}{3} \times 3x^2(x^3-1)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{3}\int_1^2 3x^2(x^3-1)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{3}\left[\frac{1}{\frac{3}{2}}(x^3-1)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{1}{3}\left[\frac{2}{3}\sqrt{(x^3-1)^3}\right]_1^2$$

$$= \frac{2}{9}\left[\sqrt{(x^3-1)^3}\right]_1^2$$

$$= \frac{2}{9}\left(\sqrt{(2^3-1)^3} - \sqrt{(1^3-1)^3}\right)$$

$$= \frac{2}{9}\left(\sqrt{7^3} - \sqrt{0^3}\right)$$

$$= \frac{2}{9}(7\sqrt{7})$$

$$= \frac{14}{9}\sqrt{7}$$

Exercise 6.05 Chain rule

1 Find each indefinite integral.

a $\int (3x-4)^2\, x$ b $\int (x+1)^4\, dx$ c $\int (5x-1)^9\, dx$

d $\int (3y-2)^7\, dy$ e $\int (4+3x)^4\, dx$ f $\int (7x+8)^{12}\, dx$

g $\int (1-x)^6\, dx$ h $\int \sqrt{2x-5}\, dx$ i $2\int (3x+1)^{-4}\, dx$

j $\int 3(x+7)^{-2}\, dx$ k $\int \frac{1}{2(4x-5)^3}\, dx$ l $\int \sqrt[3]{4x+3}\, dx$

m $\int (2-x)^{-\frac{1}{2}}\, dx$ n $\int \sqrt{(t+3)^3}\, dt$ o $\int \sqrt{(5x+2)^5}\, dx$

2 Evaluate:

a $\int_1^2 (2x+1)^4\, dx$ b $\int_0^1 (3y-2)^3\, dy$ c $\int_1^2 (1-x)^7\, dx$

d $\int_0^2 (3-2x)^5\, dx$ e $\int_0^1 \frac{(3x-1)^2}{6}\, dx$ f $\int_4^5 (5-x)^6\, dx$

g $\int_3^6 \sqrt{x-2}\, dx$ h $\int_0^2 \frac{5}{(2n+1)^3}\, dn$ i $\int_1^4 \frac{2}{\sqrt{(5x-4)^3}}\, dx$

3 Find each indefinite integral.

a $\int 4x^3(x^4+5)^2\, dx$ b $\int 2x(x^2-3)^5\, dx$

c $\int 3x^2(x^3+1)^3\, dx$ d $\int (2x+3)(x^2+3x-2)^4\, dx$

e $\int x(3x^2-7)^6\, dx$ f $\int x^2(4-5x^3)^2\, dx$

g $\int 4x^5(2x^6-3)^4\, dx$ h $\int 3x(5x^2+3)^7\, dx$

i $\int (x+2)(x^2+4x)^5\, dx$ j $\int (3x^2-2)(3x^3-6x-2)^3\, dy$

4 Evaluate:

a $\int_0^2 x(2x^2+3)^2\, dx$ b $\int_0^1 x^2(x^3-1)^5\, dx$

c $\int_1^2 x^4(x^5+2)^3\, dx$ d $\int_0^1 x^3(5-x^4)^7\, dx$

e $\int_2^4 3x(x^2+2)^4\, dx$ f $\int_{-1}^1 5x^2(2x^3-7)^3\, dx$

g $\int_{-1}^0 (x-1)(x^2-2x+3)^6\, dx$ h $4\int_0^1 (x^2+2)(x^3+6x-1)^2\, dx$

i $5\int_{-2}^2 x^2(x^3-1)(x^6-2x^3-1)^4\, dx$

5 A function has $\frac{dy}{dx} = x^2(x^3-2)^4$ and passes through the point (1, 4). Find its equation.

6 The velocity of an object is given by $v = x(x^2-3)^4$ m s^{-1}. If the displacement is 0 after 2 s, find:

a the equation for the displacement

b the displacement after 3 s.

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6.06 Integration involving exponential functions

We can write the anti-derivative of e^x as an integral.

Integral of e^x

$$\int e^x \, dx = e^x + C$$

EXAMPLE 15

Evaluate $\int_0^2 4e^x \, dx$.

Solution

$$\begin{aligned}\int_0^2 4e^x \, dx &= 4\left[e^x\right]_0^2 \\ &= 4(e^2 - e^0) \\ &= 4(e^2 - 1)\end{aligned}$$

Integral of a^x

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

EXAMPLE 16

Find $\int 2^x \, dx$.

Solution

$$\int a^x \, dx = \frac{1}{\ln a} a^x + C$$

$$\int 2^x \, dx = \frac{1}{\ln 2} 2^x + C$$

Chain rule for e^{ax+b}

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

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EXAMPLE 17

a Find $\int (e^{2x} - e^{-x})\,dx$.

b Evaluate $\int_1^2 5e^{3x}\,dx$.

Solution

a $\int (e^{2x} - e^{-x})\,dx = \frac{1}{2}e^{2x} - \frac{1}{-1}e^{-1} + C$

$= \frac{1}{2}e^{2x} + e^{-x} + C$

b $\int_1^2 5e^{3x}\,dx = \left[5 \times \frac{1}{3}e^{3x}\right]_1^2$

$= \left[\frac{5e^{3x}}{3}\right]_1^2$

$= \frac{5e^{3\times 2}}{3} - \frac{5e^{3\times 1}}{3}$

$= \frac{5e^6}{3} - \frac{5e^3}{3}$

$= \frac{5e^3}{3}(e^3 - 1)$

Exercise 6.06 Integration involving exponential functions

1 Find each indefinite integral.

a $\int e^{4x}\,dx$
b $\int e^{-x}\,dx$
c $\int e^{5x}\,dx$
d $\int e^{-2x}\,dx$
e $\int e^{4x+1}\,dx$
f $\int -3e^{5x}\,dx$
g $\int e^{2t}\,dt$
h $\int (e^{7x} - 2)\,dx$
i $\int (e^{x-3} + x)\,dx$

2 Evaluate in exact form:

a $\int_0^1 e^{5x}\,dx$
b $\int_0^2 -e^{-x}\,dx$
c $\int_1^4 2e^{3x+4}\,dx$
d $\int_2^3 (3x^2 - e^{2x})\,dx$
e $\int_0^2 (e^{2x} + 1)\,dx$
f $\int_1^2 (e^x - x)\,dx$
g $\int_0^3 (e^{2x} - e^{-x})\,dx$

3 Evaluate correct to 2 decimal places:

a $\int_1^3 e^{-x}\,dx$
b $\int_0^2 2e^{3y}\,dy$
c $\int_5^6 (e^{x+5} + 2x - 3)\,dx$
d $\int_0^1 (e^{3t+4} - t)\,dt$
e $\int_1^2 (e^{4x} + e^{2x})\,dx$

4 Find the indefinite integral of:

a 5^x
b 7^{3x}
c 3^{2x-1}

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5 a Differentiate x^2e^x.

b Hence find $\int x(2+x)e^x\,dx$.

6 A function has $f'(x) = x^2e^{2x^3}$ and passes through the point (0, 0). Find the equation of the function.

7 A particle moves so that its velocity over time t is given by $v = 2e^{t} - 1$ ms^{-1}.
If displacement $x = 10$ when $y = 0$, find x when $t = 3$.

6.07 Integration involving logarithmic functions

We can write the anti-derivatives that involve the logarithmic function as an integral.

WS Integrals involving exponential and logarithmic functions

WS Integration of $\frac{1}{x}$

WS Integration of reciprocals

WS Differentiation and integration involving exponential and logarithmic functions

Integral of $\frac{1}{x}$

$$\int \frac{1}{x}\,dx = \ln|x| + C \quad \text{where } x \neq 0$$

EXAMPLE 18

Evaluate $\int_1^5 \frac{3}{x}\,dx$.

Solution

$$\int_1^5 \frac{3}{x}\,dx = \left[3\ln|x|\right]_1^5$$
$$= 3\ln 5 - 3\ln 1$$
$$= 3\ln 5 - 3\times 0$$
$$= 3\ln 5$$

Integral of $\frac{f'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C \quad \text{where } f(x) \neq 0$$

EXAMPLE 19

a Find $\int \frac{x^2}{x^3+7}\,dx$.

b Find the exact value of $\int_0^3 \frac{x+1}{x^2+2x+4}\,dx$.

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Solution

a $f(x) = x^3 + 7$, $f'(x) = 3x^2$

$$\int \frac{x^2}{x^3+7}\,dx = \int \frac{1}{3} \times \frac{3x^2}{x^3+7}\,dx$$
$$= \frac{1}{3}\int \frac{3x^2}{x^3+7}\,dx$$
$$= \frac{1}{3}\ln|x^3+7| + C$$

applying the formula

b $f(x) = x^2 + 2x + 4$, $f'(x) = 2x + 2$

$$\int_0^3 \frac{x+1}{x^2+2x+4}\,dx = \frac{1}{2}\int_0^3 \frac{2x+2}{x^2+2x+4}\,dx \qquad \text{as } x+1 = \frac{1}{2}(2x+2)$$
$$= \frac{1}{2}\Big[\ln|x^2+2x+4|\Big]_0^3$$
$$= \frac{1}{2}[\ln|3^2+2(3)+4| - \ln|0^2+2(0)+4|]$$
$$= \frac{1}{2}[\ln 19 - \ln 4]$$
$$= \frac{1}{2}\ln\left(\frac{19}{4}\right)$$

Exercise 6.07 Integration involving logarithmic functions

1 Find the integral of each function.

a $\frac{2}{2x+5}$ b $\frac{4x}{2x^2+1}$ c $\frac{5x^4}{x^5-2}$ d $\frac{1}{2x}$ e $\frac{2}{x}$

f $\frac{5}{3x}$ g $\frac{2x-3}{x^2-3x}$ h $\frac{x}{x^2+2}$ i $\frac{3x}{x^2+7}$ j $\frac{x+1}{x^2+2x-5}$

2 Find:

a $\int \frac{4}{4x-1}\,dx$ b $\int \frac{dx}{x+3}$ c $\int \frac{x^2}{2x^3-7}\,dx$

d $\int \frac{x^5}{2x^6+5}\,dx$ e $\int \frac{x+3}{x^2+6x+2}\,dx$

3 Evaluate correct to one decimal place:

a $\int_1^3 \frac{2}{2x+5}\,dx$ b $\int_2^5 \frac{dx}{x+1}$ c $\int_1^7 \frac{x^2}{x^3+2}\,dx$

d $\int_0^3 \frac{4x+1}{2x^2+x+1}\,dx$ e $\int_3^4 \frac{x-1}{x^2-2x}\,dx$

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4 a Show that $\dfrac{3x+3}{x^2-9}=\dfrac{1}{x+3}+\dfrac{2}{x-3}$.

b Hence find $\displaystyle\int \frac{3x+3}{x^2-9}\,dx$.

5 a Show that $\dfrac{x-6}{x-1}=1-\dfrac{5}{x-1}$.

b Hence find $\displaystyle\int \frac{x-6}{x-1}\,dx$.

6 A function has $f'(x)=\dfrac{x^2}{3x^3-1}$ and passes through the point (1, 0). Find the equation of the function.

7 A particle has velocity $v=\dfrac{5t}{t^2+4}$ m s^{-1}. Find its displacement after 5 s if its initial displacement is 4 m.

8 The number of people with measles is increasing at the rate given by $R=\dfrac{x^2}{3x^3+1}$ people/week. If 3 people had measles initially, find the number with measles after 8 weeks.

6.08 Integration involving trigonometric functions

WS Integrals of trigonometric functions

WS Mixed integration questions

WS Finding indefinite integrals 1

WS Finding indefinite integrals 2

WS Finding definite integrals

Integrals of trigonometric functions

$$\int \cos x\,dx=\sin x+C$$

$$\int \sin x\,dx=-\cos x+C$$

$$\int \sec^2 x\,dx=\tan x+C$$

EXAMPLE 20

Find the exact value of $\displaystyle\int_0^{\frac{\pi}{3}} \sec^2 x\,dx$.

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sec^2 x\,dx &= [\tan x]_0^{\frac{\pi}{3}}\\ &= \tan\frac{\pi}{3}-\tan 0\\ &= \sqrt{3}\end{aligned}$$

Chain rule for trigonometric functions

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+C$$

EXAMPLE 21

a Find $\int \sin 3x\,dx$

b Find $\int \cos x°\,dx$.

c Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x\,dx$.

Solution

a $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

b
$$\begin{aligned}\int \cos x°\,dx &= \int \cos\left(\frac{\pi x}{180}\right)dx\\ &= \frac{1}{\frac{\pi}{180}}\sin\left(\frac{\pi x}{180}\right)+C\\ &= \frac{180}{\pi}\sin x° + C\end{aligned}$$

c
$$\begin{aligned}\int_0^{\frac{\pi}{8}} \sin 2x\,dx &= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{8}}\\ &= -\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right) - \left[-\frac{1}{2}\cos(2\times 0)\right]\\ &= -\frac{1}{2}\cos\frac{\pi}{4} + \frac{1}{2}\cos 0\\ &= -\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\\ &= -\frac{1}{2\sqrt{2}}+\frac{1}{2}\\ &= \frac{2-\sqrt{2}}{4}\end{aligned}$$

Exercise 6.08 Integration involving trigonometric functions

1 Find the integral of each function.

a $\cos x$	b $\sin x$	c $\sec^2 x$
d $\frac{\sin x°}{4}$	e $\sin 3x$	f $-\sin 7x$
g $\sec^2 5x$	h $\cos(x+1)$	i $\sin(2x-3)$
j $\cos(2x-1)$	k $\sin(\pi - x)$	l $\cos(x+\pi)$
m $2\sec^2 7x$	n $4\sin\left(\frac{x}{2}\right)$	o $3\sec^2\left(\frac{x}{3}\right)$

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2 Evaluate each definite integral, giving exact answers where appropriate.

a $\int_0^{\frac{\pi}{2}} \cos x \, dx$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$

c $\int_{\frac{\pi}{2}}^{\pi} \sin \frac{x}{2} \, dx$

d $\int_0^{\frac{\pi}{2}} \cos 3x \, dx$

e $\int_0^{\frac{1}{2}} \sin(\pi x) \, dx$

f $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

g $\int_0^{\frac{\pi}{12}} 3 \cos 2x \, dx$

h $\int_0^{\frac{\pi}{10}} -\sin(5x) \, dx$

3 Find:

a $\int \left(\cos x \cos \frac{\pi}{3} - \sin x \sin \frac{\pi}{3} \right) dx$

b $\int (\sin \pi \cos x - \cos \pi \sin x) \, dx$

4 A curve has $\frac{dy}{dx} = \cos 4x$ and passes through the point $(\pi, \frac{\pi}{4})$.

Find the equation of the curve.

5 A pendulum swings at the rate given by $\frac{dx}{dt} = 12\pi \cos \frac{2\pi t}{3}$ cm s^{-1}.

It starts 2 cm to the right of the origin.

a Find the equation of the displacement of the pendulum.

b Find the exact displacement after:

i 1 s

ii 5 s

6 The rate at which the depth of water changes in a bay is given by $R = 4\pi \sin \frac{\pi t}{6}$ m h^{-1}.

a Find the equation of the depth of water d over time t hours if the depth is 2 m initially.

b Find the depth after 2 hours.

c Find the highest, lowest and centre of depth of water.

d What is the period of the depth of water?

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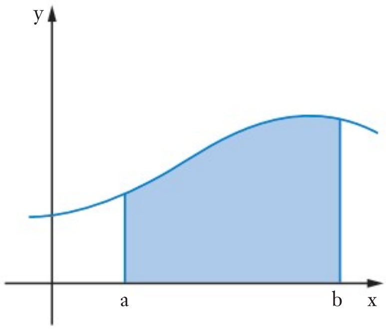
WS
Starting areas

Area under a curve

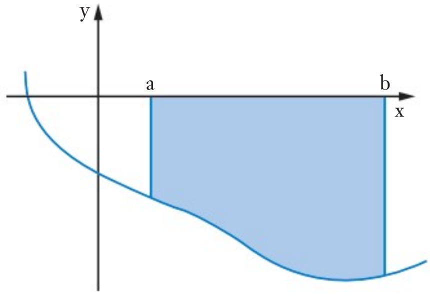
6.09 Areas enclosed by the x-axis

The definite integral gives the signed area under a curve.

Areas above the x-axis give a positive definite integral.



Areas below the x-axis give a negative definite integral.



So, to find areas below the x-axis, we take the absolute value of the definite integral.

Area under a curve

$$\text{Area} = \left|\int_a^b f(x)\,dx\right|$$

It is important to sketch the graph to see where the area is in relation to the x-axis.

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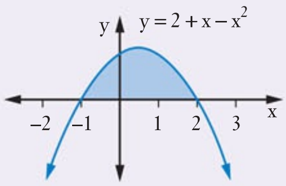
EXAMPLE 22

a Find the area enclosed by the curve $y = 2 + x - x^2$ and the x-axis.

b Find the area bounded by the curve $y = x^2 - 4$ and the x-axis.

Solution

a Sketch the graph of $y = 2 + x - x^2$ and shade the area enclosed between the curve and the x-axis.



The area is above the x-axis, so the definite integral will be positive.

$$\text{Area} = \int_{-1}^{2} 2 + x - x^2\, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$

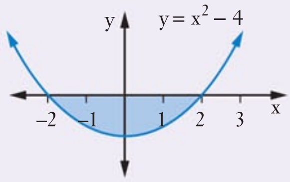
$$= \left(2(2) + \frac{2^2}{2} - \frac{2^3}{3}\right) - \left(2(-1) + \frac{(-1)^2}{2} - \frac{(-1)^3}{3}\right)$$

$$= \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 4\frac{1}{2}$$

So the area is $4\frac{1}{2}$ units^2.

b Sketch the graph of $y = x^2 - 4$.



The definite integral will be negative because the area is below the x-axis.

$$\int_{-2}^{2} (x^2 - 4)\, dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$$

$$= \left(\frac{2^3}{3} - 4(2)\right) - \left(\frac{(-2)^3}{3} - 4(-2)\right)$$

$$= \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right)$$

$$= -10\frac{2}{3}$$

$$\text{Area} = \left|-10\frac{2}{3}\right|$$

$$= 10\frac{2}{3}\text{ units}^2$$

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Area under a curve

EXAMPLE 23

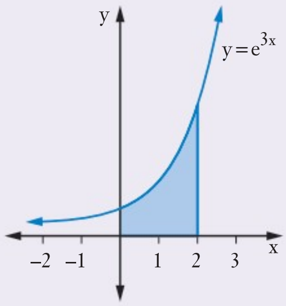
a Find the exact area enclosed between the curve $y = e^{3x}$, the x-axis and the lines $x = 0$ and $x = 2$.

b Find the exact area enclosed between the hyperbola $y = \frac{1}{x}$, the x-axis and the lines $x = 1$ and $x = 2$.

c Find the area enclosed between the curve $y = \cos x$, the x-axis and the lines $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

Solution

a Sketch the graph of $y = e^{3x}$.

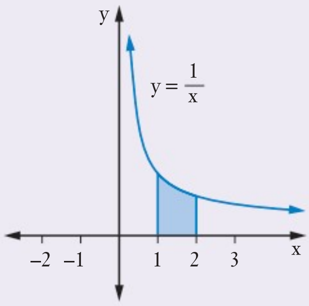


The definite integral will be positive because the area is above the x-axis.

$$\begin{aligned}
\text{Area} &= \int_0^2 e^{3x}\,dx \\
&= \left[\frac{1}{3}e^{3x}\right]_0^2 \\
&= \frac{1}{3}e^{3\times 2} - \frac{1}{3}e^{3\times 0} \\
&= \frac{1}{3}e^{6} - \frac{1}{3}e^{0} \\
&= \frac{1}{3}(e^6 - 1)
\end{aligned}$$

So the area is $\frac{1}{3}(e^6 - 1)$ units^2.

b Sketch the graph of $y = \frac{1}{x}$. For $x = 1$ and $x = 2$, we only need to sketch the graph in the 1st quadrant.



The definite integral will be positive because the area is above the x-axis.

$$\begin{aligned}
\text{Area} &= \int_1^2 \frac{dx}{x} \\
&= \Big[\ln|x|\Big]_1^2 \\
&= \ln 2 - \ln 1 \\
&= \ln 2 - 0 \\
&= \ln 2
\end{aligned}$$

Absolute value not required as 2 and 1 are positive.

So the area is $\ln 2$ units^2.

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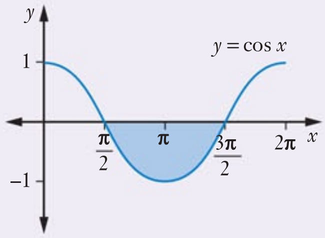
c Sketch the graph of $y = \cos x$.



The definite integral will be negative because the area is below the x-axis.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x \, dx = [\sin x]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \sin\left(\frac{3\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)$$

$$= -1 - 1$$

$$= -2$$

$$\text{Area} = |-2|$$

$$= 2 \text{ units}^2$$

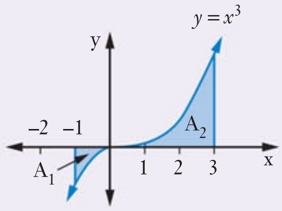
If the area has some parts above the x-axis and some below the x-axis, we need to find these separately.

EXAMPLE 24

Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -1$ and $x = 3$.

Solution

Sketch the graph of $y = x^3$.



There are 2 areas, marked A_1 and A_2 on the diagram.

A_1 is below the x-axis so the integral will be negative.

A_2 is above the x-axis so the integral will be positive.

A_1:

$$\int_{-1}^{0} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{0}$$

$$= \frac{0^4}{4} - \frac{(-1)^4}{4}$$

$$= -\frac{1}{4}$$

$$\text{So } A_1 = \left|-\frac{1}{4}\right|$$

$$= \frac{1}{4} \text{ units}^2$$

$$A_2 = \int_{0}^{3} x^3 \, dx$$

$$= \left[\frac{x^4}{4}\right]_{0}^{3}$$

$$= \frac{3^4}{4} - \frac{0^4}{4}$$

$$= \frac{81}{4} \text{ units}^2$$

$$\text{Total area} = A_1 + A_2$$

$$= \frac{1}{4} + \frac{81}{4}$$

$$= 20\frac{1}{2} \text{ units}^2$$

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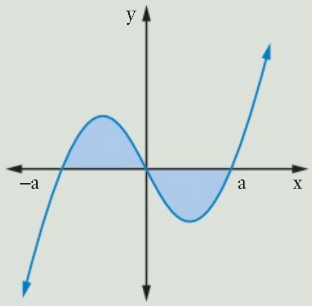
Odd and even functions

Some functions have special properties that we can use to find their areas.

Odd functions

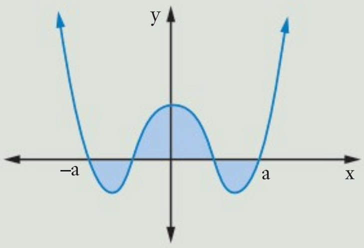
If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\,dx = 0$.

The positive and negative areas cancel each other out.



Even functions

If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.



EXAMPLE 25

Find the area between the curve:

a $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$

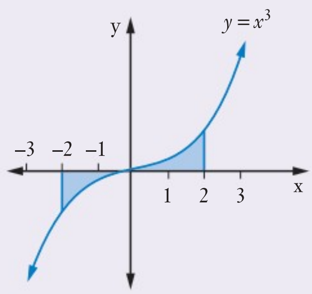
b $y = x^2$, the x-axis and the lines $x = -4$ and $x = 4$

Solution

a Sketch the graph of $y = x^3$ and shade the area bounded by the curve, the x-axis and the boundaries $x = \pm 2$.

$y = x^3$ is an odd function since $f(-x) = -f(x)$.

This means that the shaded areas are symmetrical. We can find the area between $x = 0$ and $x = 2$.



The total area will be twice this area.

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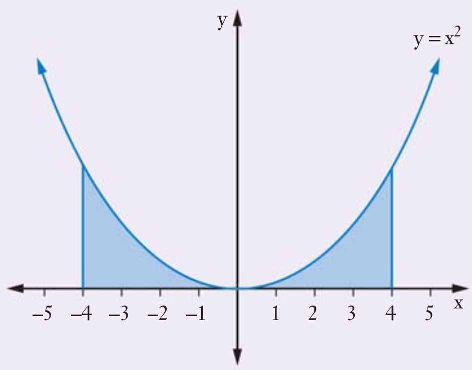
$$\text{Area} = 2\int_0^2 x^3\,dx$$
$$= 2\left[\frac{x^4}{4}\right]_0^2$$
$$= 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right)$$
$$= 8$$

So area is 8 units^2.

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the x-axis and the lines $x = \pm 4$.



$y = x^2$ is an even function since $f(-x) = f(x)$.

$$\text{Area} = \int_{-4}^4 x^2\,dx$$
$$= 2\int_0^4 x^2\,dx$$
$$= 2\left[\frac{x^3}{3}\right]_0^4$$
$$= 2\left(\frac{4^3}{3} - \frac{0^3}{3}\right)$$
$$= 2 \times \frac{64}{3}$$
$$= 42\frac{2}{3}$$

So area is $42\frac{2}{3}$ units^2.

Exercise 6.09 Areas enclosed by the x-axis

1 Find the area enclosed between the curve $y = 1 - x^2$ and the x-axis.

2 Find the area bounded by the curve $y = x^2 - 9$ and the x-axis.

3 Find the area enclosed between the curve $y = x^2 + 5x + 4$ and the x-axis.

4 Find the area enclosed between the curve $y = x^2 - 2x - 3$ and the x-axis.

5 Find the area bounded by the curve $y = -x^2 + 9x - 20$ and the x-axis.

6 Find the area enclosed between the curve $y = -2x^2 - 5x + 3$ and the x-axis.

7 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = 0$ and $x = 2$.

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8 Find the area enclosed between the curve $y = x^4$, the x-axis and the lines $x = -1$ and $x = 1$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -2$ and $x = 2$.

10 Find the area enclosed between the curve $y = x^3$, the x-axis and the lines $x = -3$ and $x = 2$.

11 Find the exact area enclosed by the curve $y = 2e^{2x}$, the x-axis and the lines $x = 1$ and $x = 2$.

12 Find the exact area bounded by the curve $y = e^{4x-3}$, the x-axis and the lines $x = 0$ and $x = 1$.

13 Find the area enclosed by the curve $y = x + e^{-x}$, the x-axis and the lines $x = 0$ and $x = 2$, correct to 2 decimal places.

14 Find the area bounded by the curve $y = e^{5x}$, the x-axis and the lines $x = 0$ and $x = 1$, correct to 3 significant figures.

15 Find the area enclosed between the curve $y = \sin x$ and the x-axis in the domain $[0, 2\pi]$.

16 Find the exact area bounded by the curve $y = \cos 3x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{12}$.

17 Find the area enclosed between the curve $y = \sec^2 \frac{x}{4}$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$, correct to 2 decimal places.

18 Find the area bounded by the curve $y = 3x^2$, the x-axis and the lines $x = -1$ and $x = 1$.

19 Find the area enclosed between the curve $y = x^2 + 1$, the x-axis and the lines $x = -2$ and $x = 2$.

20 Find the area enclosed between the curve $y = x^2$, the x-axis and the lines $x = -3$ and $x = 2$.

21 Find the area enclosed between the curve $y = x^2 + x$, and the x-axis.

22 Find the area enclosed between the curve $y = \frac{1}{x^2}$, the x-axis and the lines $x = 1$ and $x = 3$.

23 Find the area enclosed between the curve $y = \frac{2}{(x-3)^2}$, the x-axis and the lines $x = 0$ and $x = 1$.

24 Find the exact area between the curve $y = \frac{1}{x}$, the x-axis and the lines $x = 2$ and $x = 3$.

25 Find the exact area bounded by the curve $y = \frac{1}{x-1}$, the x-axis and the lines $x = 4$ and $x = 7$.

26 Find the area bounded by the curve $y = \frac{x}{x^2+1}$, the x-axis and the lines $x = 2$ and $x = 4$, correct to 2 decimal places.

27 Find the area bounded by the curve $y = \sqrt{x}$, the x-axis and the line $x = 4$.

28 Find the area bounded by the curve $y = \sqrt{x+2}$, the x-axis and the line $x = 7$.

29 Use the trapezoidal rule with 4 subintervals to find the area bounded by the curve $y = \ln x$, the x-axis and the line $x = 5$, correct to 2 decimal places.

30 Find the area bounded by the x-axis, the curve $y = x^3$ and the lines $x = -a$ and $x = a$.

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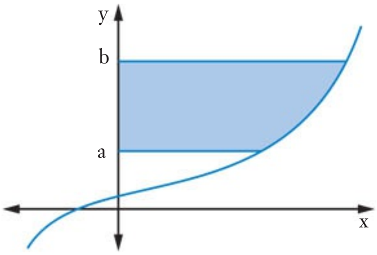
6.10 Areas enclosed by the y-axis

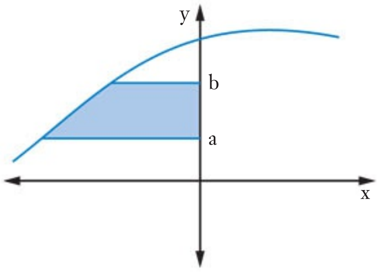
We can find an area bounded by a graph and the y-axis by writing the equation in the form $x = f(y)$.

The definite integral gives the signed area.

Areas to the right of the y-axis give a positive definite integral.

Areas to the left of the y-axis give a negative definite integral.





Area bounded by a curve and the y-axis

For the curve $x = f(y)$:

$$\text{Area} = \left| \int_a^b f(y)\,dy \right|$$

EXAMPLE 26

a Find the area enclosed by the curve $x = y^2$, the y-axis and the lines $y = 1$ and $y = 3$.

b Find the area enclosed by the curve $y = x^2$, the y-axis and the lines $y = 0$ and $y = 4$ in the first quadrant.

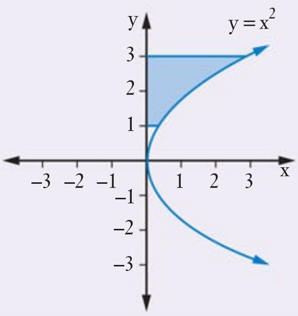
Solution

a Sketch the graph of $x = y^2$ and shade the area bounded by the curve, the y-axis and the lines $y = 1$ and $y = 3$.

This is the same shape as the parabola $y = x^2$ with the x and y values swapped.

For example, when $x = 1$, $y = \pm 1$, when $x = 4$, $y = \pm 2$.

The area is to the right of the y-axis so the integral will be positive.



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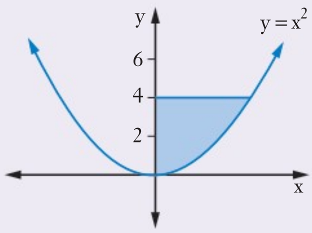
$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_1^3 y^2\,dy$$
$$= \left[\frac{y^3}{3}\right]_1^3$$
$$= \frac{3^3}{3} - \frac{1^3}{3}$$
$$= 8\frac{2}{3}$$

So the area is $8\frac{2}{3}$ units^2

b Sketch the graph of $y = x^2$ and shade the area enclosed between the curve, the y-axis and the lines $y = 0$ and $y = 4$.



The area is to the right of the y-axis so the integral will be positive.

Change the subject of the equation to x.

$$y = x^2$$
$$\pm\sqrt{y} = x$$

In the first quadrant:

$$x = \sqrt{y}$$
$$= y^{\frac{1}{2}}$$

$$\text{Area} = \int_a^b f(y)\,dy$$
$$= \int_0^4 y^{\frac{1}{2}}\,dy$$
$$= \left[\frac{y^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$
$$= \left[\frac{2\sqrt{y^3}}{3}\right]_0^4$$
$$= \frac{2\sqrt{4^3}}{3} - \frac{2\sqrt{0^3}}{3}$$
$$= 5\frac{1}{3}$$

So the area is $5\frac{1}{3}$ units^2.

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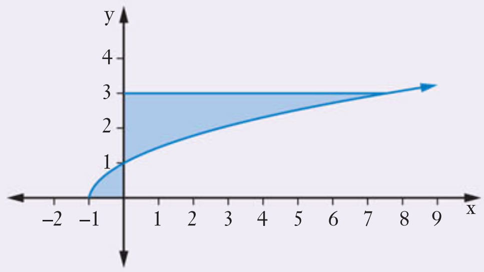
EXAMPLE 27

Find the area enclosed between the curve $y = \sqrt{x+1}$, the y-axis and the lines $y = 0$ and $y = 3$.

Solution

The area between $y = 0$ and $y = 1$ is to the left of the y-axis so the integral will be negative.

The area between $y = 1$ and $y = 3$ is to the right of the y-axis so the integral will be positive.



Changing the subject:

$$y = \sqrt{x+1}$$

$$y^2 = x + 1$$

$$y^2 - 1 = x$$

$$\int_0^1 (y^2 - 1)\,dy = \left[\frac{y^3}{3} - y\right]_0^1$$

$$= \left(\frac{1^3}{3} - 1\right) - \left(\frac{0^3}{3} - 0\right)$$

$$= -\frac{2}{3}$$

$$A_1 = \left|-\frac{2}{3}\right|$$

$$= \frac{2}{3} \text{ units}^2$$

$$A_2 = \int_1^3 (y^2 - 1)\,dy$$

$$= \left[\frac{y^3}{3} - y\right]_1^3$$

$$= \left(\frac{3^3}{3} - 3\right) - \left(\frac{1^3}{3} - 1\right)$$

$$= 6\frac{2}{3} \text{ units}^2$$

$$\text{Total area} = A_1 + A_2$$

$$= \frac{2}{3} + 6\frac{2}{3}$$

$$= 7\frac{1}{3} \text{ units}^2$$

Exercise 6.10 Areas enclosed by the y-axis

1 Find the area bounded by the y-axis, the curve $x = y^2$ and the lines $y = 0$ and $y = 4$.

2 Find the area enclosed between the curve $x = y^3$, the y-axis and the lines $y = 1$ and $y = 3$.

3 Find the area in the first quadrant enclosed between the curve $y = x^2$, the y-axis and the lines $y = 1$ and $y = 4$.

4 Find the area between the lines $y = x - 1$, $y = 0$, $y = 1$ and the y-axis.

5 Find the area bounded by the line $y = 2x + 1$, the y-axis and the lines $y = 3$ and $y = 4$.

6 Find the area bounded by the curve $y = \sqrt{x}$, the y-axis and the lines $y = 1$ and $y = 2$.

7 Find the area bounded by the curve $x = y^2 - 2y - 3$ and the y-axis.

8 Find the area bounded by the curve $x = -y^2 - 5y - 6$ and the y-axis.

9 Find the area enclosed by the curve $y = \sqrt{3x - 5}$, the y-axis and the lines $y = 2$ and $y = 3$.

10 Find the area in the first quadrant enclosed between the curve $y = \frac{1}{x^2}$, the y-axis and the lines $y = 1$ and $y = 4$.

11 Find the area enclosed between the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.

12 Find the area enclosed between the curve $y = x^3 - 2$ and the y-axis between $y = -1$ and $y = 25$.

13 Find the area in the second quadrant enclosed between the lines $y = 4$ and $y = 1 - x$.

14 Find the area enclosed between the y-axis and the curve $x = y(y - 2)$.

15 Find the area in the first quadrant bounded by the curve $y = x^4 + 1$, the y-axis and the lines $y = 1$ and $y = 3$, correct to 2 significant figures.

16 Find the area between the curve $y = \ln x$, the y-axis and the lines $y = 2$ and $y = 4$, correct to 3 significant figures.

6.11 Sums and differences of areas

WS Sums and differences of areas

WS Calculating physical areas

WS Calculating areas between curves

WS Areas between curves

WS Areas between curves 2

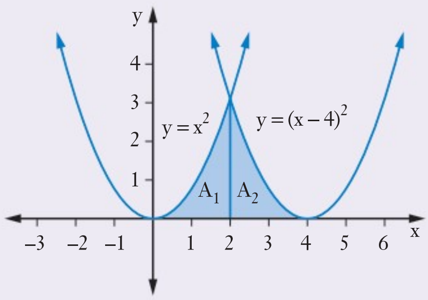
EXAMPLE 28

a Find the area enclosed between the curves $y = x^2$, $y = (x - 4)^2$ and the x-axis.

b Find the area enclosed between the curve $y = x^2$ and the line $y = x + 2$.

Solution

a Sketch the graphs of $y = x^2$ and $y = (x - 4)^2$ (translation of $y = x^2$ by 4 units to the right) and shade the area enclosed between the curves and the x-axis.



Find the x values of their intersection ($x = 2$, from the graph or by solving simultaneous equations).

Shaded area $= A_1 + A_2$

$$= \int_0^2 x^2\,dx + \int_2^4 (x - 4)^2\,dx$$

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Area A_1:

$$\int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{2^3}{3} - \frac{0^3}{3}$$

$$= 2\frac{2}{3}$$

Area A_2:

$$\int_2^4 (x-4)^2\,dx = \left[\frac{(x-4)^3}{3}\right]_2^4$$

$$= \frac{(4-4)^3}{3} - \frac{(2-4)^3}{3}$$

$$= 0 + \frac{8}{3}$$

$$= 2\frac{2}{3}$$

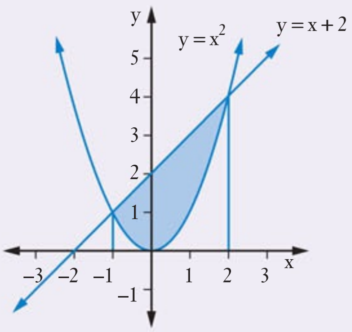
Total area $= A_1 + A_2$

$$= 2\frac{2}{3} + 2\frac{2}{3}$$

$$= 5\frac{1}{3}$$

So area is $5\frac{1}{3}$ units^2.

b Sketch the graphs of $y = x^2$ and $x + 2$ and shade the area enclosed between them.



We can find the x values of the points of intersection of the functions from the graph or by solving simultaneous equations:

$y = x^2$ [1]

$y = x + 2$ [2]

Substituting [1] into [2]:

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2, \qquad x = -1$$

Notice that between $x = -1$ and $x = 2$, the graph of $y = x + 2$ is above the graph of $y = x^2$.

So we can find the area by integrating $(x + 2)$ and x^2 between $x = -1$ and $x = 2$ and then finding their difference.

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$$A = \int_{-1}^{2} (x + 2)\, dx - \int_{-1}^{2} x^2\, dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

$$= \left[\frac{2^2}{2} + 2(2) - \frac{2^3}{3}\right] - \left[\frac{(-1)^2}{2} + 2(-1) - \frac{(-1^3)}{3}\right]$$

$$= \frac{10}{3} - \left(\frac{-7}{6}\right)$$

$$= \frac{9}{2}$$

$$= 4\frac{1}{2}$$

So area is $4\frac{1}{2}$ units^2.

Exercise 6.11 Sums and differences of areas

1 Find the area bounded by the line $y = 1$ and the curve $y = x^2$.

2 Find the area enclosed between the line $y = 2$ and the curve $y = x^2 + 1$.

3 Find the area enclosed by the curve $y = x^2$ and the line $y = x$.

4 Find the area bounded by the curve $y = 9 - x^2$ and the line $y = 5$.

5 Find the area enclosed between the curve $y = x^2$ and the line $y = x + 6$.

6 Find the area bounded by the curve $y = x^3$ and the line $y = 4x$.

7 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = (x + 1)^2$ and the x-axis.

8 Find the area enclosed between the curve $y = x^2$ and the line $y = -6x + 16$.

9 Find the area enclosed between the curve $y = x^3$, the x-axis and the line $y = -3x + 4$.

10 Find the area enclosed by the curves $y = (x - 2)^2$ and $y = (x - 4)^2$.

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11 Find the area enclosed between the curves $y = x^2$ and $y = x^3$.

12 Find the area enclosed by the curves $y = x^2$ and $x = y^2$.

13 Find the area bounded by the curve $y = x^2 + 2x - 8$ and the line $y = 2x + 1$.

14 Find the area bounded by the curves $y = 1 - x^2$ and $y = x^2 - 1$.

15 Find the exact area enclosed between the curve $y = \sqrt{4 - x^2}$ and the line $x - y + 2 = 0$.

16 Find the exact area in the first quadrant between the curve $y = \frac{1}{x}$, the x–axis and the lines $y = x$ and $x = 2$.

17 Find the exact area bounded by the curves $y = \sin x$ and $y = \cos x$ in the domain $[0, 2\pi]$.

18 Find the exact area enclosed between the curve $y = e^{2x}$ and the lines $y = 1$ and $x = 2$.

19 Find the exact area enclosed by the curve $y = \sin x$ and the line $y = \frac{1}{2}$ for $[0, 2\pi]$.

Summary of integration rules

Rule	Chain rule
$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C$	$\int f'(x)[f(x)]^n \, dx = \frac{1}{n+1}\left[f(x)^{n+1}\right] + C$
$\int e^x \, dx = e^x + C$	$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + C$
$\int a^x \, dx = \frac{1}{\ln a} a^x + C$	
$\int \frac{1}{x} \, dx = \ln\lvert x\rvert + C$	$\int \frac{f'(x)}{f(x)} \, dx = \ln\lvert f(x)\rvert + C$
$\int \cos x \, dx = \sin x + C$	$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$
$\int \sin x \, dx = -\cos x + C$	$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$

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6. TEST YOURSELF

WS Derivatives and Integrals

Qz Practice quiz

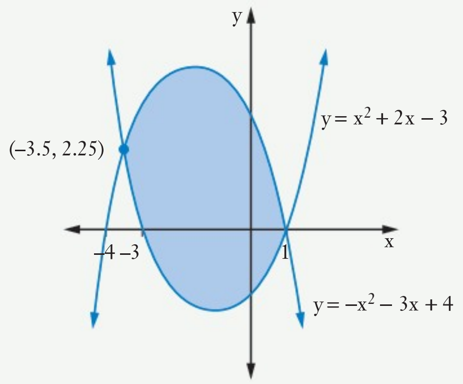
For Questions 1 to 4, choose the correct answer A, B, C or D.

1 Find $\int \sin(6x)\,dx$.

A $\frac{1}{6}\cos(6x) + C$ B $6\cos(6x) + C$

C $-6\cos(6x) + C$ D $-\frac{1}{6}\cos(6x) + C$

2 Find the shaded area below.



A $\int_{-4}^{1}(-x^2-3x+4)\,dx-\int_{-4}^{1}(x^2+2x-3)\,dx$

B $\int_{-3.5}^{1}(-x^2-3x+4)\,dx-\int_{-3.5}^{1}(x^2+2x-3)\,dx$

C $\int_{-3}^{1}(-x^2-3x+4)\,dx+\int_{-3}^{1}(x^2+2x-3)\,dx$

D $\int_{-3.5}^{1}(-x^2-3x+4)\,dx+\int_{-3.5}^{1}(x^2+2x-3)\,dx$

3 Find $\int 4e^{3x}\,dx$.

A $\frac{4}{3}e^{3x}+C$ B $\frac{3}{4}e^{3x}+C$ C $12e^{3x}+C$ D $\frac{1}{12}e^{3x}+C$

4 Find $\int \frac{x}{x^2+3}\,dx$.

A $\frac{2}{(x^2+3)^2}+C$ B $2\ln|x^2+3|+C$

C $\frac{1}{2(x^2+3)^2}$ D $\frac{1}{2}\ln|x^2+3|+C$

5 a Use the trapezoidal rule with 2 subintervals to find an approximation to $\int_1^2 \frac{dx}{x^2}$.

b Use integration to find the exact value of $\int_1^2 \frac{dx}{x^2}$.

6 Find the integral of:

a $3x+1$ b $\frac{5x^2-x}{x}$ c $\sqrt{x}$

d $(2x+5)^7$ e $x^3(3x^4-2)^4$

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7 Find $\int 3^x \, dx$.

8 Find the approximate area under the curve $f(x) = x^3$ between $x = 1$ and $x = 3$ by using:

a 4 inner rectangles b 4 outer rectangles c a trapezium

9 Evaluate:

a $\int_0^2 (x^3 - 1)\, dx$ b $\int_{-1}^1 x^5 \, dx$ c $\int_0^1 (3x - 1)^4 \, dx$

d $\int_0^1 x^2(x^3 - 5)^2 \, dx$ e $\int_{-1}^2 3x\,(x^2 + 1)^3 \, dx$

10 Find the area enclosed between the curve $y = \ln x$, the y-axis and the lines $y = 1$ and $y = 3$.

11 Find the area bounded by the curve $y = x^2$, the x-axis and the lines $x = -1$ and $x = 2$.

12 Find $\int \sin x° \, dx$.

13 Find the area enclosed between the curves $y = x^2$ and $y = 2 - x^2$.

14 Find the indefinite integral of:

a e^{4x} b $\dfrac{x}{x^2 - 9}$ c e^{-x}

d $\dfrac{1}{x + 4}$ e $(x - 3)(x^2 - 6x + 1)^8$

15 Evaluate $\int_1^2 \dfrac{3x^4 - 2x^3 + x^2 - 1}{x^2} \, dx$.

16 Find the exact area in the first quadrant bounded by $x^2 + y^2 = 9$, the y-axis and the lines $y = 0$ and $y = 3$.

17 Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 0$ and $y = 1$.

18 Find the integral of $(7x + 3)^{11}$.

19 Find the area bounded by the curve $y = x^2 - x - 2$, the x-axis and the lines $x = 1$ and $x = 3$.

20 Find the exact area bounded by the curve $y = e^{2x}$, the x-axis and the lines $x = 2$ and $x = 5$.

21 Use the trapezoidal rule with 4 strips to find the area bounded by the curve $y = \ln (x^2 - 1)$, the x-axis and the lines $x = 3$ and $x = 5$.

22 Evaluate $\int_0^4 (3t^2 - 6t + 5)\, dt$.

23 Find the indefinite integral of:

a $\sin 2x$ b $3 \cos x$ c $\sec^2 5x$ d $1 + \sin x$

24 Find the area bounded by the curve $y = x^2 + 2x - 15$ and the x-axis.

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25 The rate at which a metal cools is given by $R = -16\, e^{-0.4t}$ degrees min^{-1}.
If the temperature is initially 215°C, find:

a the equation for the temperature T of the metal

b the temperature, to the nearest degree, of the metal after:

i 5 minutes ii half an hour

26 Evaluate:

a $\int_0^{\frac{\pi}{4}} \cos x\, dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\, dx$

27 Find:

a $\int 5(2x-1)^4 dx$ b $\int \frac{3x^5}{4} dx$

28 Find the exact area bounded by the curve $y = \sin x$, the x-axis and the lines $x = \frac{\pi}{4}$ and $x = \frac{\pi}{2}$.

29 Find:

a $\int x^2(x^3-2)^5\, dx$ b $\int x(5x^2+2)^4\, dx$

c $\int 5x^3(2x^4-1)^2 dx$ d $\int (x+2)(x^2+4x-3)^3\, dx$

30 Find the area bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ and $x = \pi$.

31 Find (in exact form) the approximate area bounded by the curve $y = \sqrt{x-2}$, the x-axis and the line $x = 4$, using:

a a triangle b 2 inner rectangles c 2 outer rectangles

32 Find f (x) given f ′(x) and a point on the graph of f(x).

a $f'(x) = 3x(2x^2-1)^4$ and passing through (1, 3)

b $f'(x) = \sec^2 2x$ and passing through $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

c $f'(x) = e^{5x}$ and passing through $(0, \frac{1}{5})$

d $f'(x) = x^3(x^4-15)^3$ and passing through (2, 0)

e $f'(x) = \frac{3x^3}{x^4+1}$ and passing through (0, 2)

33 The velocity of a particle is given by $v = \frac{t^2}{\sqrt{t^3+9}}$ m s^{-1}.

If the initial displacement is −2 m, find:

a the equation for displacement

b the displacement after 5 s

c when the displacement is 10 m

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6. CHALLENGE EXERCISE

1 a Show that $f(x) = x^3 + x$ is an odd function.

b Hence find the value of $\int_{-2}^{2} f(x)\,dx$.

c Find the total area between $y = f(x)$, the x-axis and the lines $x = -2$ and $x = 2$.

2 a Show that $\sec x \operatorname{cosec} x = \dfrac{\sec^2 x}{\tan x}$.

b Hence, or otherwise, find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec} x \sec x\,dx$.

3 Find the area enclosed between the curves $y = (x - 1)^2$ and $y = 5 - x^2$.

4 Find the exact value of $\int_{\frac{\pi}{12}}^{\frac{\pi}{8}} \sec^2 2x\,dx$.

5 Evaluate $\int_0^1 \dfrac{x}{(3x^2 - 4)^2}\,dx$.

6 Use the trapezoidal rule with 4 subintervals to find the area enclosed between the curve $y = \dfrac{3}{x-2}$, the y-axis and the lines $y = 1$ and $y = 3$.

7 a Sketch the curve $y = x(x - 1)(x + 2)$.

b Find the total area enclosed between the curve and the x-axis.

8 Find the area bounded by the parabola $y = x^2$ and the line $y = 4 - x$ correct to 2 decimal places.

9 a Find the derivative of $x\sqrt{x+3}$.

b Hence find $\int \dfrac{x+2}{\sqrt{x+3}}\,dx$.

10 a Find $\dfrac{d}{dx}(x^2 \ln x)$.

b Hence find the exact value of $\int_1^3 2x(1 + 2\ln x)\,dx$.

11 Find the area enclosed between the curves $y = \sqrt{x}$ and $y = x^3$.

12 a Find the sum of 50 terms of the sequence $2^0, 2^{0.2}, 2^{0.4}, 2^{0.6}, \ldots$

b Hence use 50 inner rectangles to find the approximate area under the curve $y = 2^x$ between $x = 0$ and $x = 10$.

c Find this approximate area by using 100 outer rectangles.

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Practice set 2

In Questions 1 to 6, select the correct answer A, B, C or D.

1 The area of a rectangle with sides x and y is 45. Its perimeter P is given by:

A $P = x + 45x^2$

B $P = x + \frac{45}{x}$

C $P = 2x + \frac{90}{x}$

D $P = 2x + \frac{45}{x}$

2 The area enclosed between the curve $y = x^3 - 1$, the y-axis and the lines $y = 1$ and $y = 2$ is given by:

A $\int_1^2 (x^3 - 1)\, dy$

B $\int_1^2 (y + 1)\, dy$

C $\int_1^2 (\sqrt[3]{y} + 1)\, dy$

D $\int_1^2 (\sqrt[3]{y + 1})\, dy$

3 Find $\int 4x^2(5x^3 + 4)^7\, dx$.

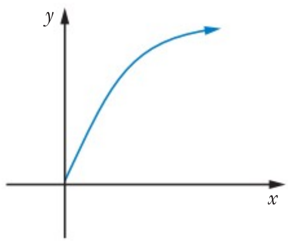
A $\frac{4(5x^3 + 4)^8}{15} + C$

B $\frac{(5x^3 + 4)^8}{30} + C$

C $\frac{(5x^3 + 4)^8}{2} + C$

D $\frac{(5x^3 + 4)^8}{120} + C$

4 For the curve shown, which inequalities are correct?



A $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$

B $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} < 0$

C $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} > 0$

D $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} < 0$

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5 A cone with base radius r and height h has a volume of 300 cm^3.
Its slant height, l, is given by:

A $l = \sqrt{\dfrac{\pi h^3 + 900}{\pi h}}$ B $l = \sqrt{\dfrac{h^2 + 900}{\pi h}}$

C $l = \sqrt{\dfrac{h^2 + 810000}{\pi h}}$ D $l = \sqrt{\dfrac{h^3 + 900}{\pi h}}$

6 The rate at which a waterfall is flowing over a cliff is given by $R = 4t + 3t^2$ m^3 s^{-1}. Find the amount of water flowing after a minute if the amount of water is 10 970 m^3 after 20 seconds.

A 223 220 m^3 B 8800 m^3

C 225 370 m^3 D 226 250 m^3

7 Find all values of x for which the curve $y = (2x - 1)^2$ is decreasing.

8 Find $\int (3x^2 - 2x + 1)\ dx$.

9 Find the maximum value of the curve $y = x^2 + 3x - 4$ in the domain $[-1, 4]$.

10 For the graph of $y = 8 \sin 3x + 5$, find:

a the amplitude b the period c the centre

11 The area of a rectangle is 4 m^2. Find its minimum perimeter.

12 If $y = \sin 7x$, show that $\dfrac{d^2y}{dx^2} = -49y$.

13 Find the anti-derivative of $3x^8 + 4x$.

14 Sketch the curve $y = x^3 - 3x^2 - 9x + 2$, showing all stationary points and points of inflection.

15 Find the area enclosed between the curve $y = x^2 - 1$ and the x-axis.

16 Find $\int \dfrac{3x}{2x^2 - 5}\ dx$.

17 If $f(x) = x^3 - 2x^2 + 5x - 9$, find $f'(3)$ and $f''(-2)$.

18 Evaluate $\int_1^3 (6x^2 + 4x)\ dx$.

19 Find the domain over which the curve $y = 3x^3 + 7x^2 - 3x - 1$ is concave upwards.

20 Evaluate $\int_1^2 x\sqrt{3x^2-3}\,dx$.

21 Find $\int \sec^2 x\,(\tan x+1)^3\,dx$.

22 a If $f(x)=2x^4-x^3-7x+9$, find $f(1)$, $f'(1)$ and $f''(1)$.

b What is the geometrical significance of these results? Illustrate by a sketch of $y=f(x)$ at $x=1$.

23 A piece of wire of length 4 m is cut into 2 parts. One part is bent to form a rectangle with sides x and $3x$, and the other part is bent to form a square with sides y.

a Prove that the total area of the rectangle and square is given by $A=7x^2-4x+1$.

b Find the dimensions of the rectangle and square when the area has the least value.

24 Given the function $f(x)=x^2$, find the equation of the transformed function if $y=f(x)$ is translated 5 units up, 4 units to the left, stretched horizontally by a factor of 2 and stretched vertically by a factor of 3.

25 The gradient function of a curve is given by $f'(x)=4x-3$. If $f(2)=-3$, find $f(-1)$.

26 Evaluate $\int_0^3 (2x+1)\,dx$.

27 The following table gives values for $f(x)=\frac{1}{x^2}$.

x	1	2	3	4	5
f (x)	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$	$\frac{1}{25}$

Use the table together with the trapezoidal rule to evaluate $\int_1^5 \frac{dx}{x^2}$ correct to 3 decimal places.

28 a Find the stationary point on the curve $y=(x-2)^3$ and determine its nature.

b Hence sketch the curve.

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29 Two families travelling on holidays drive along roads that intersect at right angles. One family is initially 230 km from the intersection and drives towards the intersection at an average of 65 km h^{-1}. The other family is initially 125 km from the intersection and travels towards it at an average of 80 km h^{-1}.

a Show that their distance apart after t hours is given by $d^2 = 10\,625t^2 - 49\,900t + 68\,525$.

b Hence find how long it will take them to reach their minimum distance apart.

c Find their minimum distance apart.

30 For the sequence 100, –50, 25, … find:

a the 10th term

b the sum of the first 10 terms

c the limiting sum

31 A rectangle is cut from a circular disc of radius 15 cm. Find the area of the largest rectangle that can be produced.

32 Find $\int (3x+5)^7\,dx$.

33 Evaluate $\int_1^3 \frac{dx}{x}$ correct to 3 decimal places.

34 Find the stationary points on the curve $f(x) = x^4 - 2x^2 + 3$ and distinguish between them.

35 Evaluate $\int_1^2 \sqrt{5x-1}\,dx$ as a fraction.

36 Find the area enclosed between the curve $y = (x-1)^2$ and the line $y = 4$.

37 If a function has a stationary point at (–1, 2) and $f''(x) = 2x - 4$, find $f(2)$.

38 Find the area enclosed between the curves $y = x^2$ and $y = -x^2 + 2x + 4$.

39 Differentiate $x^3 + e^{2x}$.

40 Water is flowing out of a pool at the rate given by $R = -20$ litres per minute. If the volume of water in the pool is initially 8000 L, find:

a the volume after 5 minutes

b how long it will take to empty the pool.

41 Find the exact value of $\int_0^3 3xe^{x^2+1}\,dx$.

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42 The velocity of a particle is given by $v = 12t^2 + 4t + 80$ m s^{-1}. If the particle is initially 3 m to the right of the origin, find its displacement after 5 s.

43 The graph of $y = f(x)$ has a stationary point at $(3, 2)$. If $f''(x) = 6x - 8$, find the equation of $f(x)$.

44 Find the derivative of $\ln (4x + 3)^3$.

45 Find $\int \frac{2x+1}{3x^2 + 3x - 2} dx$.

46 Differentiate:

a $\frac{x}{e^{2x}}$

b $\log_3 x$.

47 Find the equation of the tangent to $y = e^{x+1}$ at the point where $x = -1$.

48 Find the stationary point on the curve $y = xe^{2x}$ and determine its nature.

49 Find the equation of $y = f(x)$ passing through $(\pi, 1)$ and with $f'(x) = -6 \sin 3x$.

50 Find $\int_0^{\frac{\pi}{2}} \sin 2x \, dx$.

51 Differentiate:

a $\ln (\sin x)$

b $\tan (e^{5x} + 1)$

52 Find an approximation to $\int_0^{\frac{\pi}{4}} \tan x \, dx$ correct to 3 decimal places by using a triangle.

53 Find the area under the curve $y = 4 - x^2$ by using:

a 4 inner rectangles

b 4 outer rectangles

54 Differentiate each function.

a $e^x \sin x$

b $\tan^3 x$

c $2 \cos\left(3x - \frac{\pi}{2}\right)$

55 Find the equation of the tangent to the curve $y = \tan 3x$ at the point where $x = \frac{\pi}{4}$.

56 Differentiate:

a $\sin^3 (e^x)$

b $\tan (\ln x + 1)$

57 Find the exact area bounded by the curve $y = \ln (x + 4)$, the y-axis and the lines $y = 0$ and $y = 1$.

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58 Find the anti-derivative of each function.

a e^{3x}

b $\sec^2 \pi x$

c $\dfrac{1}{2x}$

d $\cos\left(\dfrac{x}{5}\right)$

e $\sin 8x$

59 Find $\int \dfrac{3x^2 - 2x + 5}{x^2}\,dx$.

60 Find $\int (e^{5x} - \sin \pi x)\,dx$.

61 Find the exact area enclosed between the curve $y = e^x$, the x-axis, the y-axis and the line $x = 2$.

62 Evaluate $\int_{\frac{\pi}{3}}^{\pi} \cos\left(\dfrac{x}{2} + \pi\right) dx$.

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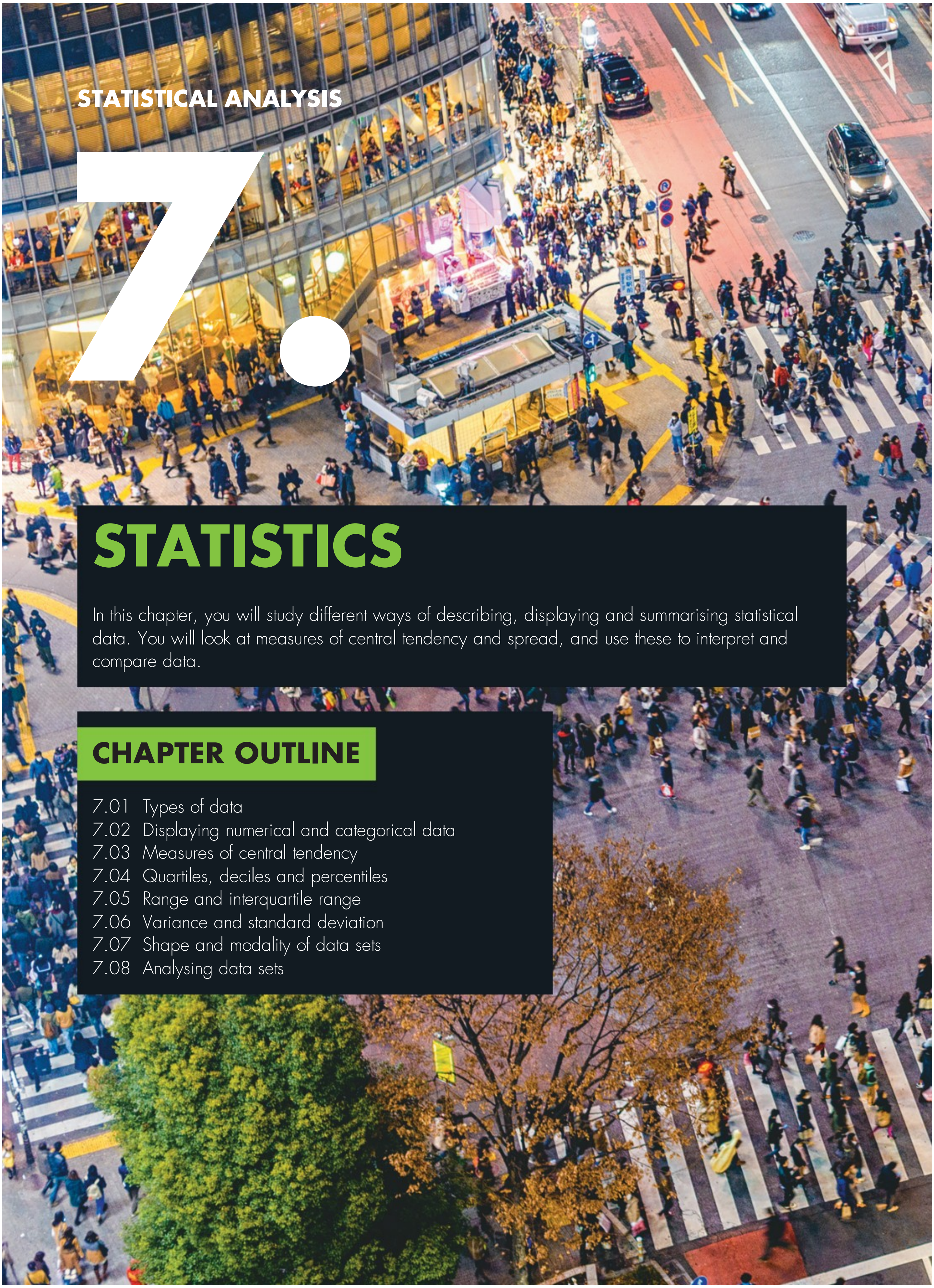

7. STATISTICS

In this chapter, you will study different ways of describing, displaying and summarising statistical data. You will look at measures of central tendency and spread, and use these to interpret and compare data.

CHAPTER OUTLINE

7.01 Types of data
7.02 Displaying numerical and categorical data
7.03 Measures of central tendency
7.04 Quartiles, deciles and percentiles
7.05 Range and interquartile range
7.06 Variance and standard deviation
7.07 Shape and modality of data sets
7.08 Analysing data sets

IN THIS CHAPTER YOU WILL:

- identify different types of data
- display data in tables and graphs
- calculate measures of central tendency: the mean, median and mode
- calculate measures of spread: the range, quantiles, interquartile range, variance and standard deviation
- identify outliers
- recognise different modalities and shapes of data sets
- identify bias in data
- compare 2 sets of data

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TERMINOLOGY

bar chart: Graph with vertical or horizontal columns, also called a column graph.
bimodal: A graph with 2 peaks.
box plot: Graphical display of five-number summary, also called a box-and-whisker plot.
categorical data: Data that are named by categories.
continuous data: Numerical data that can take any value that lies within an interval.
decile: One of the values that divide a data set into 10 equal parts.
discrete data: Numerical data that can only take specific distinct values.
dot plot: A column graph of dots.
five-number summary: The lowest and highest values, median, and lower and upper quartiles of a data set.
frequency polygon: Frequency line graph.
histogram: Bar chart of frequencies with no gap between columns.
interquartile range: Measure of spread, the difference between the upper and lower quartiles.
mean: Average score, calculated by dividing the sum of scores by the total number of scores.
median: The middle score when all scores are placed in order.
modality: The number of peaks in a set of data.
mode: The score with the highest frequency.
multimodal: Having many peaks in a set of data.
nominal data: Categorical data that is listed by name with no order.
numerical data: Data whose values are numbers.
ogive: Cumulative frequency polygon.
ordinal data: Categorical data that can be ordered.
outlier: A score that is clearly apart from other scores – it may be much higher or lower than the other scores.
Pareto chart: A chart containing both a bar chart and a line graph where individual values are represented in descending order by the bars and the cumulative total is represented by the line graph.
percentile: One of the values that divide a data set into 100 equal parts.
pie chart: Circular graph showing categories as sectors.
quantile: One of the values that divide a data set into equal parts.
quartile: One of the values that divide a data set into 4 equal parts.
range: Difference between the highest and lowest scores.
skewness: The shape or asymmetry of a graph to one side.
standard deviation: Measure of the spread of data values from the mean. The square root of variance.
stem-and-leaf plot: Graphical display of tens (stem) and units (leaves).
symmetrical distribution: A distribution where the left and right sides are mirror images of each other.
two-way table: A table that combines the effects of 2 separate variables (usually categorical).
variance: Measure of spread, the square of standard deviation.

7.01 Types of data

WS
Statistical data match-up

There are many different types of data, for example, the type of public transport people use to go to work, the heights of basketball players or the marks students gain in an exam.

There are 2 main types of data:

- Categorical data uses categories described by words or symbols
- Numerical data uses numbers or quantities.

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These types can be divided further:

Categorical and numerical data

Categorical data:

- Nominal data, which cannot be put in order
- Ordinal data, which can be ordered

Numerical data:

- Discrete data, which can be counted as separate values
- Continuous data, which is measured along a smooth scale

For example:

- Public transport – bus, train, tram, ferry – is categorical nominal data since it cannot be put into an order.
- Ratings – strongly disagree, disagree, agree, strongly agree – are categorical ordinal since they can be put in order.
- Shoe sizes – 6, $6\frac{1}{2}$, 7, $7\frac{1}{2}$, … – are numerical discrete since they can be counted.
- Heights of basketball players – 181 cm, 173.64 cm, 192.1 cm … – are numerical continuous since they are along a smooth scale.

EXAMPLE 1

Describe each type of data.

a The breeds of dogs

b Exam marks

c The volume of water in a dam

d Audience size for TV programs

e Makes of cars

f Months of the year

Solution

a Categorical nominal

b Numerical discrete

c Numerical continuous

d Numerical discrete

e Categorical nominal

f Categorical ordinal

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Exercise 7.01 Types of data

1 State whether each type of data is categorical (C) or numerical (N).

a Length of a fence
b Number of koalas in captivity
c Shoe size
d Colour
e Area of land
f Scores on a test
g Number of lollies in a packet
h Gender
i Speed
j Type of swimming strokes
k Attendance at a football match
l Meals on a menu
m Width of a building
n Age
o Weight
p Ranking of quality of a movie
q Surface area of a balloon as it is blown up
r Shirt sizes
s Type of sports offered at a school
t Length of a swimming race

2 State whether each type of data is categorical nominal (N), categorical ordinal (O), numerical discrete (D), numerical continuous (C).

a Survey of radio stations: Excellent, very good, good, poor, very poor
b Weight of truck loads
c Make and model of motorbikes
d Eye colour
e Volume of water in rivers
f Scores on a maths exam
g Number of jellybeans in a packet
h Nationality
i Acceleration
j Olympic sports
k Concert attendance
l Choice of desserts on a menu
m Types of trees in a park

3 Give 3 examples of:

a categorical data
b numerical data
c numerical discrete data
d categorical ordinal data
e numerical continuous data
f categorical nominal data

INVESTIGATION

DATA COLLECTION

Certain organisations are specially set up to collect and analyse data. The Australian Bureau of Statistics (ABS) collects all sorts of data, including the organisation of a regular census.

The census attempts to collect details of every person living in Australia on a particular day. Questions asked include where a person lives, occupation, salary, number of children, religion and marital status. Governments and other organisations use this data to plan future policies in areas such as education, transport, housing. For example, if the number of children in a certain region is increasing, then extra schools could be planned in that area.

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There is evidence that a census was done back in ancient Roman times. Investigate the methods that the Romans or some other ancient civilisation used for collecting data and writing reports.

What information do you think a census should collect? Is there information that you think that is not useful or invades privacy and therefore should not be collected?

Go to the ABS website and find out more about what this organisation does. Other worldwide organisations such as the World Health Organization (WHO) and the United Nations also collect data. Research these and other organisations that collect data, such as universities and the CSIRO.

7.02 Displaying numerical and categorical data

WS Graphing data

WS Two-way tables and Pareto charts

WS Every picture tells a story

WS Cumulative frequency graphs

Data can be displayed in many different ways using tables and graphs.

Numerical data

Frequency tables, histograms, frequency polygons and other graphs can be used to display numerical data.

EXAMPLE 2

a For the following Year 12 English essay marks (out of 10):

8, 4, 5, 4, 8, 6, 7, 8, 9, 5, 6, 7, 7, 5, 4, 6, 7, 9, 3, 5, 5

- i draw a frequency distribution table
- ii draw a histogram for this data
- iii draw a frequency polygon on the same set of axes as the histogram
- iv how many scores are less than 5?
- v what percentage of scores are over 6?

b The assessment scores for a Year 12 mathematics class are below.

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

- i Draw a frequency table that shows the results of the class test using groups of 40–49, 50–59 and so on.
- ii Add a column for class centre and cumulative frequency.
- iii Draw a cumulative frequency histogram and a cumulative frequency polygon (ogive).
- iv What percentage of students scored less than 60?

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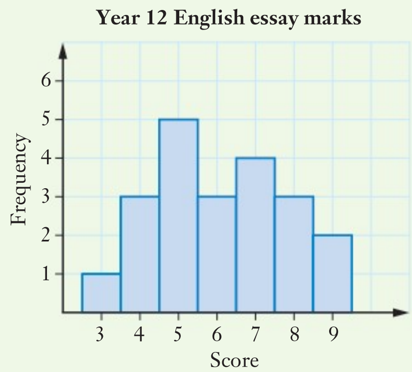
Solution

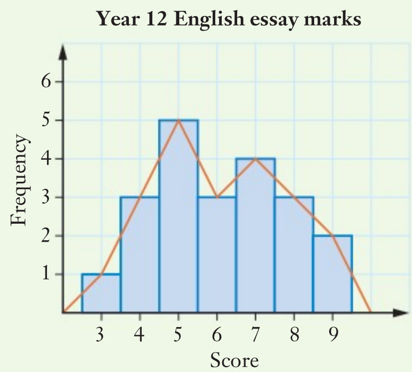
a i The scores range from 3 to 9. We arrange them in a table as shown.

Score	Tally	Frequency
3	|	1
4	|||	3
5	~~||||~~	5
6	|||	3
7	||||	4
8	|||	3
9	||	2

ii The histogram is a bar chart or column graph where the centre of the column is lined up with the score and the columns join together.



iii The frequency polygon is a line graph as shown. It starts and ends on the horizontal axis.



iv Reading from either the table or the graph, scores of 3 and 4 are less than 5. There is one score of 3 and 3 scores of 4.

So there are $1 + 3 = 4$ scores less than 5.

v There are $4 + 3 + 2 = 9$ scores over 6 out of a total of 21 scores.

$$\frac{9}{21} \times 100\% \approx 42.9\%$$

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b i

Scores	Tally	Frequency
40–49	I	1
50–59	IIII	4
60–69	卌 III	8
70–79	卌 I	6
80–89	卌	5
90–99	I	1

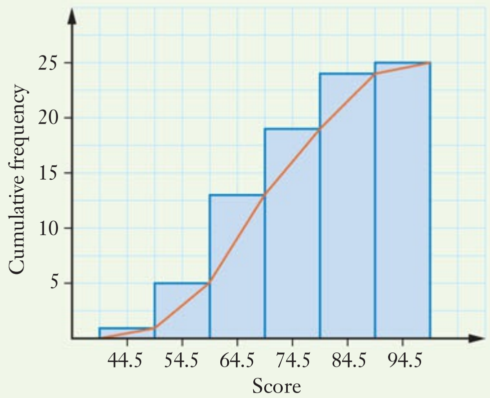
ii The class centre is the average of the highest and lowest possible score in each group. For example, $\frac{40+49}{2} = 44.5$.

Add each score to the previous total for cumulative frequencies.

Scores	Class centre	Frequency	Cumulative frequency
40–49	44.5	1	1
50–59	54.5	4	5
60–69	64.5	8	13
70–79	74.5	6	19
80–89	84.5	5	24
90–99	94.5	1	25

iii Use the class centres for the scores on the graph. The cumulative frequency polygon or ogive starts at the bottom left of the first column and ends at the top right corner of the last column.

Class test results



iv 5 students scored less than 60.

$\frac{5}{25} \times 100\% = 20\%$.

You can also draw stem-and-leaf plots to show discrete data.

EXAMPLE 3

The heartbeat rates in beats per minute of a sample of hospital patients were taken:

75, 53, 58, 71, 68, 51, 60, 87, 62, 62, 89, 65, 69, 47, 70, 72, 75, 68, 76, 83, 62, 88, 94, 53, 85

Draw a stem-and-leaf plot to display these scores.

Solution

On the left of a vertical line, put in the 10s for the scores (the stem). On the right, place the unit for each score in order (the leaf).

For example, for a score of 68 show 6 | 8.

Stem	Leaf
4	7
5	1 3 3 8
6	0 2 2 2 5 8 8 9
7	0 1 2 5 5 6
8	3 5 7 8 9
9	4

Note: The stem-and-leaf plot keeps the actual scores whereas grouping them into a frequency distribution table loses this individual information.

Categorical data

We can display categorical data in different tables and graphs, including two-way tables, bar charts, pie charts and Pareto charts.

EXAMPLE 4

In a survey of Year 12 students, it was found that 47 students had a dog but not a cat, 19 had both a dog and a cat, 32 had a cat but not a dog, and 54 had neither a dog nor a cat.

a Draw a two-way table showing this data.

b Find the percentage of students who have:

   i both a dog and cat

   ii a cat but not a dog

   iii neither a cat nor a dog

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Solution

a A two-way table separates out the students with dogs from those with cats.

	Has a dog	Does not have a dog
Has a cat	19	32
Does not have a cat	47	54

Note that this table could be the other way around, with the cats at the top and the dogs down the side.

b There are $19 + 32 + 47 + 54 = 152$ students altogether.

i 19 students out of 152 have both a dog and a cat: $\frac{19}{152} \times 100\% \approx 12.5\%$

ii 32 students out of 152 have a cat but not a dog: $\frac{32}{152} \times 100\% \approx 21.1\%$

iii 54 students out of 152 have neither a dog nor a cat: $\frac{54}{152} \times 100\% \approx 35.5\%$

EXAMPLE 5

The table shows the eye colour of students.

Represent this data in:

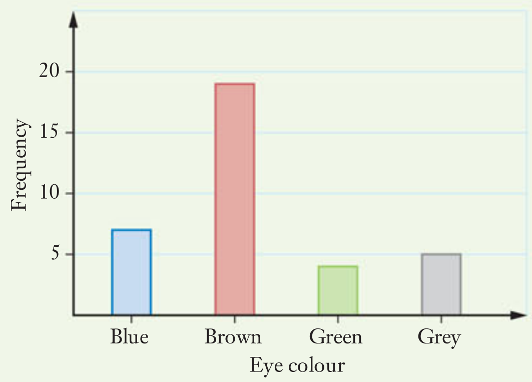
a a bar chart

b a pie chart

Colour	Frequency
Blue	7
Brown	19
Green	4
Grey	5

Solution

a Unlike a histogram, in a bar chart the columns do not need to join up.



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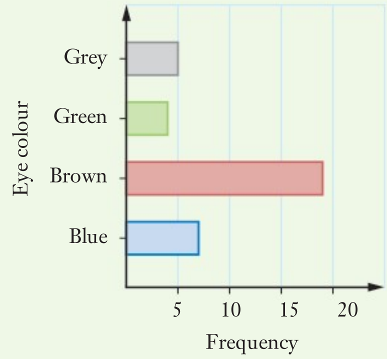
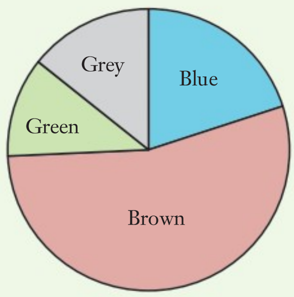
You can also draw the data as a horizontal bar chart like this.



b A pie chart is a circle divided into portions (sectors). Since the angle inside a circle is 360°, each frequency is a proportion of 360°.

There were 35 students surveyed.

Colour	Frequency	Angle
Blue	7	$\frac{7}{35} \times 360° \approx 72°$
Brown	19	$\frac{19}{35} \times 360° \approx 195°$
Green	4	$\frac{4}{35} \times 360° \approx 41°$
Grey	5	$\frac{5}{35} \times 360° \approx 51°$



Note: The number of degrees calculated adds to only 359° because the answers are not exact. This will not greatly affect the pie chart.

A Pareto chart is useful for displaying categorical data from the most to the least important.

WS
Pareto chart template

EXAMPLE 6

The table shows a survey group's preferences for types of TV shows.

a Arrange the table in descending order of frequency and add a percentage frequency column and a cumulative percentage frequency column.

b Draw a Pareto chart to show this data.

Type	Frequency
News	68
Drama	78
Comedy	73
Reality	107
Sport	174
	500

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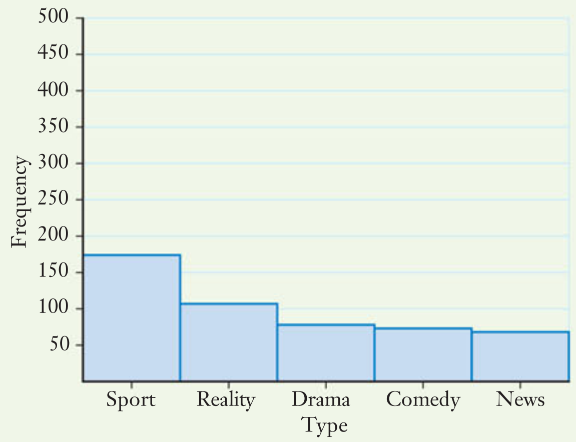
Solution

a

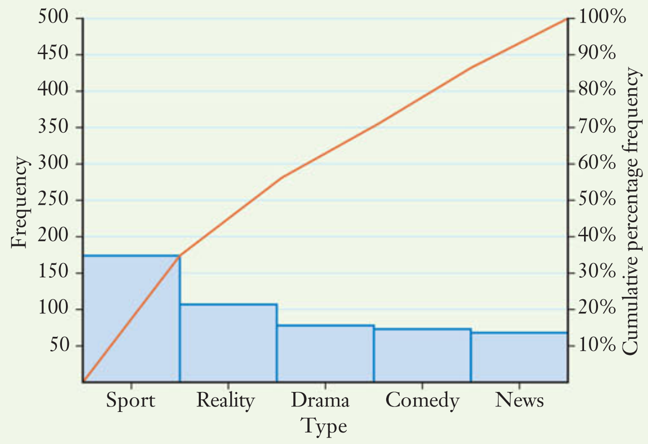
Type	Frequency	Percentage frequency	Cumulative percentage frequency
Sport	174	34.8%	34.8%
Reality	107	21.4%	56.2%
Drama	78	15.6%	71.8%
Comedy	73	14.6%	86.4%
News	68	13.6%	100%
Total	500		

For example, percentage frequency for Sport = $\frac{174}{500} \times 100\% = 34.8\%$

b Step 1: Draw a bar chart of the frequencies using the left axis.



Step 2: Draw a line graph of cumulative percentages using the right axis.



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Statistical graphs

INVESTIGATION

GRAPHS AND SPREADSHEETS

You can draw different types of graphs, including Pareto charts, using a spreadsheet. Enter the data into the spreadsheet, highlight the table and select the chart you want to use. If you are not sure of how to do this, search for online tutorials.

DID YOU KNOW?

Vilfredo Pareto

The Pareto chart is named after Vilfredo Pareto (1848–1923), an economist, sociologist, engineer and philosopher. The chart can be used as a tool for quality control.

Research the Pareto chart, the Pareto principle and the 80/20 rule. Find examples of its uses.

Exercise 7.02 Displaying numerical and categorical data

1 For each set of scores on the next page:

- i draw a frequency distribution table
- ii draw a histogram and frequency polygon
- iii find the highest and lowest scores (groups for parts d and e)
- iv find the most frequent score (group for parts d and e).

a Results of a class quiz:

8, 6, 5, 7, 6, 8, 3, 2, 6, 5, 8, 4, 7, 3, 8, 7, 5, 6, 5, 8, 6, 4, 9, 6, 5

b The number of people ordering pizzas each night:

15, 12, 17, 18, 18 ,15, 16, 13, 15, 17, 18, 12, 17, 14, 16, 15, 17, 18, 19, 15, 15, 12

c The number of people attending a gym:

110, 112, 114, 109, 112, 113, 108, 110, 113, 112, 113, 110, 109, 110, 110, 112, 114, 114, 112, 114, 113

d The results of an assessment task:

45, 79, 65, 48, 69, 50, 62, 74, 38, 69, 88, 96, 90, 58, 52, 68, 63, 61, 79, 74, 50, 65, 77, 91, 56, 77, 63, 81, 90, 59, 67, 50, 61

(Use groups of 30–39, 40–49 and so on.)

e The heights of students (in cm) in a Year 12 class:

159, 173, 182, 166, 172, 179, 181, 163, 178, 169, 183, 158, 162, 167, 174, 175, 180, 174, 176, 159, 161, 171, 174, 179, 180, 159, 157

(Use groups of 155–159, 160–164, 165–169 and so on.)

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2 For each data set:

i add a cumulative frequency column and class centre where necessary

ii sketch a cumulative frequency histogram and ogive (cumulative frequency polygon).

a Number of cars in a school car park

Number of cars	Frequency
10	4
11	8
12	11
13	9
14	5

b Results of a science experiment

Score	Frequency
1	7
2	1
3	3
4	0
5	2
6	5

c Number of sales made in a shoe shop

Sales	Class centre	Frequency
0–4		6
5–9		2
10–14		3
15–19		5
20–24		8
25–29		9
30–34		5

d Results of an assessment task

Scores	Class centre	Frequency
0–19		3
20–39		2
40–59		7
60–79		6
80–99		1

3 The table shows the number of daily rescues at a beach over a period of time.

a Draw a histogram showing this data.

b How many times were more than 6 rescues made?

c What was the most common number of daily rescues during the survey?

Rescues	Frequency
4	1
5	3
6	6
7	4
8	5
9	0
10	2

4 The volume of traffic on a stretch of highway was measured and the results are shown in the table.

a Draw a histogram to show this data.

b What was the percentage volume of traffic between 21 and 40 minutes?

Volume/min	Frequency
1–10	7
11–20	10
21–30	8
31–40	5
41–50	4

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5 a Draw a two-way table for the following data:

- 27 people play soccer but not tennis
- 35 people play tennis but not soccer
- 28 play neither sport
- 12 play both sports

b What percentage of people play:

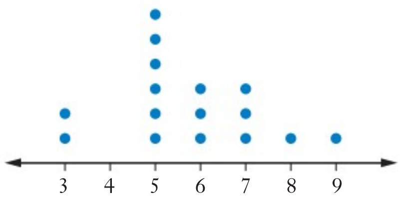
i both sports? ii neither sport?

c What percentage of people who play at least one of these sports play only soccer but not tennis?

d What fraction of people who play soccer play tennis as well?

e What percentage of tennis players do not also play soccer?

6 Lauren surveyed her friends and had them rank a film from 1 to 10. The dot plot shows the results of her survey.



a How many friends did Lauren survey?

b What was the most common ranking?

c Draw the results in a histogram.

d What percentage of rankings were above 4?

e What fraction of Lauren's friends ranked the film below 4?

7 The stem-and-leaf plot shows the weights (in kg) of a group of people surveyed at a local gym.

Stem	Leaf
5	4 6 8 9
6	1 3 7 8
7	0 3 4 5 5 9
8	1 2 2 4 7 8 9
9	3 5
10	2 6 7

a Arrange these weights in a frequency distribution table using groups of 50–59, 60–69 and so on, and include class centre and cumulative frequency columns.

b Draw an ogive of this data.

c How many people weighed:

i 80 kg or more?

ii less than 60 kg?

d What percentage of people surveyed weighed from 70 kg to 89 kg?

e What fraction of people weighed between 50 kg and 80 kg?

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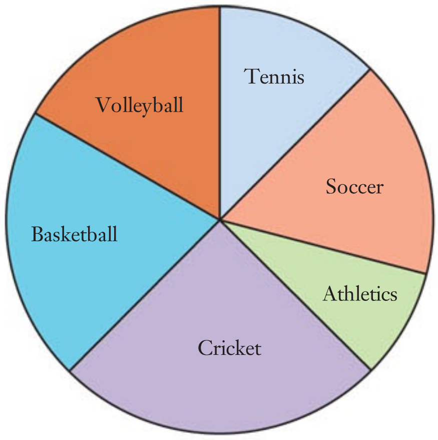
8 The pie chart shows the number of students taking different school sports.

a What percentage of students play cricket?

b There are 720 students at the school who play these sports. By measuring the angles in the pie chart, complete a table showing the frequencies for the different sports.



9 The table shows the results of a survey of university students asking what degree they were doing.

a What percentage of students were studying law?

b What percentage of students were studying medicine or music?

c Draw a pie chart showing this information.

Degree	Frequency
Medicine	104
Arts	87
Music	58
Science	93
Economics	79
Law	101

10 The two-way table shows the results of a survey into the protective effect of vaccination on a new virus.

	Vaccinated	Not vaccinated
Infected	11	76
Not infected	159	58

a How many people in the survey were infected with the virus?

b What percentage of people surveyed were vaccinated?

c What percentage of vaccinated people had the virus?

d How many people with the virus were not vaccinated?

11 The two-way table shows the number of people taking part in a trial of a new medication to prevent asthma.

	Taking medication	Control group
Asthmatic	104	105
Not asthmatic	112	109

a How many people took part in the trial?

b What percentage of people were asthmatic?

c What percentage of asthmatic people were in the control group?

d How many people who were not asthmatic took part in the trial?

e What fraction of the non-asthmatic people took medication?

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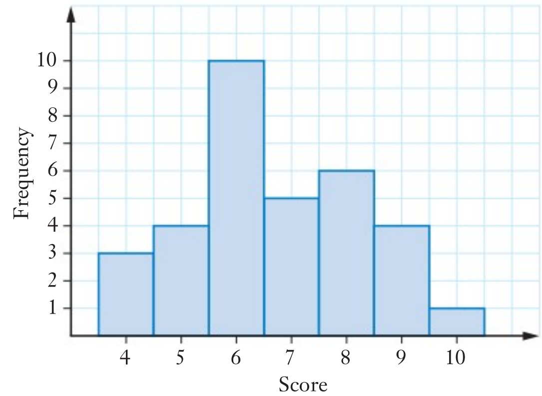
12 The frequency histogram shows the scores on a maths quiz.

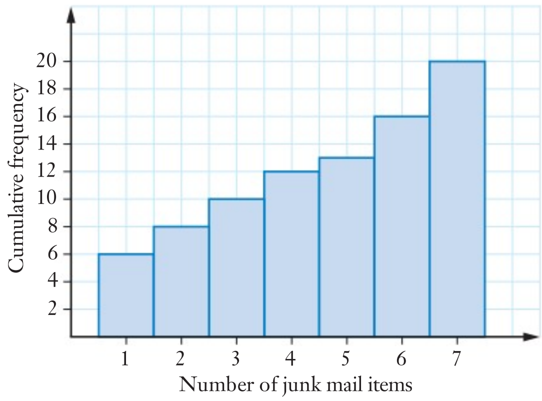
Draw a cumulative frequency histogram and polygon.



13 The cumulative frequency histogram shows data collected in a survey on the number of junk mail items that people received daily in their inbox.

a Draw a frequency distribution table to show the number of junk mail items people receive daily.

b Construct a frequency histogram to show this data.



14 Explain how a stem-and-leaf plot and grouped frequency distribution table can be used for the same data. What are the advantages and disadvantages of each?

15 Draw a Pareto chart for each set of data.

a A survey into why people like a movie:

Reason	Votes
Acting	33
Storyline	29
Music	12
Characters	26

b Customer complaints about an Internet provider:

Complaint	Frequency
Internet speed	34
Cost	61
Data allowance	59
Technical difficulties	46

c Votes for best café in a suburb:

Café	Votes
Coffee Haus	32
Coffee Bean	48
Café Focus	21
Jumping Bean	63
Caffeine Café	36

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7.03 Measures of central tendency

When we analyse data, we try to find a 'typical', 'normal' or 'average' score. For example we might want to know the average crowd size at football matches through the season. You would usually expect to find this score somewhere in the centre of the data.
There are 3 measures of central tendency: the mean, the mode and the median.

WS Measures of central tendency

Statistical measures

Statistics from a frequency table

WS Mean, median and mode

The mean

$$\text{Mean} = \frac{\text{Sum of scores}}{\text{Total number of scores}}$$

$$\bar{x} = \frac{\Sigma x}{n}$$

The mean has symbol $\bar{x}$, n is the number of scores and Σx is the sum of scores. Note: Σ is the Greek letter 'sigma' and is used in mathematics to stand for a sum.

The symbol $\bar{x}$ usually represents the mean of a sample. For the mean of a population, the correct symbol is μ, the Greek letter mu.

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EXAMPLE 7

There are 5 children in a family, aged 13, 19, 11, 17 and 10. Find the mean of their ages.

Solution

$$\bar{x} = \frac{\Sigma x}{n}$$

$$= \frac{13 + 19 + 11 + 17 + 10}{5}$$

There are 5 children so n = 5.

$$= \frac{70}{5}$$

$$= 14$$

So the mean age of the children is 14.

The mean can also be calculated using a calculator's statistics mode.

	Casio scientific	Sharp scientific
Place your calculator in statistical mode.	MODE STAT 1 VAR	MODE STAT =
Clear the statistical memory.	SHIFT 1 EDIT , DEL -A	2ndF DEL
Enter data.	SHIFT 1 Data to get table. 13 = 19 = etc. to enter in column. AC to leave table.	13 M+ 19 M+ etc.
Calculate mean.	SHIFT 1 VAR $\bar{x}$ =	RCL $\bar{x}$
Check the number of scores.	SHIFT 1 VAR n =	RCL n
Change back to normal mode.	MODE COMP	MODE 0

For the mean of larger data sets, it is easier to sort the data into a frequency distribution table to add up the scores.

The mean of data in a frequency table

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$

where Σfx is the sum of each score × its frequency and Σf is the sum of frequencies

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EXAMPLE 8

Find the mean number of hours that members of a class practise piano each week. The number of hours for each student is:

1, 4, 6, 1, 3, 6, 2, 1, 1, 3, 2, 5, 6, 6, 1, 2, 6, 2, 5, 6, 6, 2, 3, 6, 2

Solution

First draw up a frequency distribution table for the hours of practice. Include an fx column for multiplying each score by its frequency.

The table gives us a quick way of finding the sum of scores. For example, we know from the table that there are 6 lots of 2, so we can use $2 \times 6 = 12$. The sum of the fx column gives us the sum of all scores.

Hours (x)	Frequency (f)	Score × frequency (fx)
1	5	5
2	6	12
3	3	9
4	1	4
5	2	10
6	8	48
	$\Sigma f = 25$	$\Sigma fx = 88$

$$\bar{x} = \frac{\Sigma fx}{\Sigma f}$$
$$= \frac{88}{25}$$
$$= 3.52$$

So the mean time students practise each week is 3.52 hours.

On a calculator:

Operation	Casio scientific	Sharp scientific
Clear the statistical memory first (see previous example, previous page).		
Enter data.	SHIFT MODE scroll down to STAT, Frequency? ON SHIFT 1 Data to get table. 1 = 2 = etc. to enter in x column. 5 = 6 = etc. to enter in FREQ column. AC to leave table.	1 2ndF STO 5 M+ 2 2ndF STO 6 M+ etc.
Calculate mean.	SHIFT 1 VAR $\bar{x}$ =	RCL $\bar{x}$

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Although grouped data is not completely accurate because we don't know exactly what scores are in each group, we can still calculate an estimate of the mean.

EXAMPLE 9

From the table, find the mean commuting time that a sample of people take to travel to work.

Minutes	Frequency
0–8	3
9–17	5
18–26	7
27–35	8
36–44	2

Solution

Add class centre and fx columns to the table.

Use the class centres as the scores when calculating fx.

Minutes	Class centre (x)	Frequency (f)	Score × frequency (fx)
0–8	4	3	12
9–17	13	5	65
18–26	22	7	154
27–35	31	8	248
36–44	40	2	80
		$\Sigma f = 25$	$\Sigma fx = 559$

$$\begin{aligned}\bar{x} &= \frac{\Sigma fx}{\Sigma f}\\ &= \frac{559}{25}\\ &\approx 22.36\end{aligned}$$

The mean time taken to travel to work is 22.36 minutes.

The mode

The mode is the most frequent score.

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There is no mode if all the scores are different, or there could be several scores with the same frequency.

EXAMPLE 10

a Find the mode of these scores:

7, 4, 3, 5, 7, 1, 2

b Find the mode for these shoes sold at a shoe store.

Shoe size	5	$5\frac{1}{2}$	6	$6\frac{1}{2}$	7	$7\frac{1}{2}$	8	$8\frac{1}{2}$	9	$9\frac{1}{2}$
Frequency	8	9	15	28	53	61	58	29	12	10

Solution

a There are two 7s and only one of the other scores, so the mode is 7.

b The shoe size with the highest frequency is $7\frac{1}{2}$ (there were 61 of them).

So the mode is $7\frac{1}{2}$.

With grouped data, instead of finding the mode we find the modal class.

EXAMPLE 11

Find the modal class in this data set showing the ages of people at a caravan park.

Scores	Frequency
10–19	2
20–29	0
30–39	1
40–49	5
50–59	7
60–69	3

Solution

The group or class with the highest frequency is 50–59.

While we do not know the individual score with the highest frequency, we say the modal class is 50–59.

The mode is useful when looking at trends such as the most popular types of clothing. It can also be used for categorical data.

The median

The median is the middle score when all scores are in order.

If there are 2 middle scores, the median is the average of those scores.

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EXAMPLE 12

Find the median age of a group of people in a band:

18, 15, 20, 18, 17, 16, 11, 13

Solution

Put the ages in order.

11, 13, 15, 16, 17, 18, 18, 20

There are 2 middle ages, 16 and 17, so we find their average.

$$\text{Median} = \frac{16 + 17}{2} = 16.5$$

So the median age of the band members is 16.5.

The median can also be calculated using a calculator's statistics mode.

Operation	Casio scientific
Clear the statistical memory.	
Enter data.	SHIFT 1 Data to get table. 18 = 15 = etc. to enter in column. AC to leave table.
Calculate the median.	SHIFT 1 MinMax med =
Change back to normal mode.	MODE COMP

You can find the median of data in a frequency table. If there is a large number of scores, you can find the position of the middle score using a cumulative frequency column.

EXAMPLE 13

Find the median of this data set.

Score	Frequency
5	3
6	2
7	4
8	7
9	6
10	3

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Solution

Add a column for cumulative frequencies.

Score	Frequency	Cumulative frequency
5	3	3
6	2	5
7	4	9
8	7	16
9	6	22
10	3	25

There are 25 scores so the position of the middle score is $\frac{25+1}{2} = 13\text{th}$.

The 10th to 16th scores are 8, so the 13th score is 8.

You can check this by writing the scores in order.

The median is 8.

The position of the median

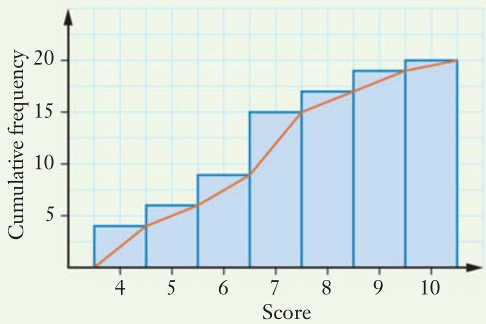
The median of n scores is the $\frac{n+1}{2}$ th score.

If n is even, then the median is the average of the 2 middle scores on both sides of the $\frac{n+1}{2}$ th position.

Another way to find the median is from an ogive (cumulative frequency polygon). We simply use the halfway point on the cumulative frequency axis of the graph.

EXAMPLE 14

Find the median from the cumulative frequency polygon below.



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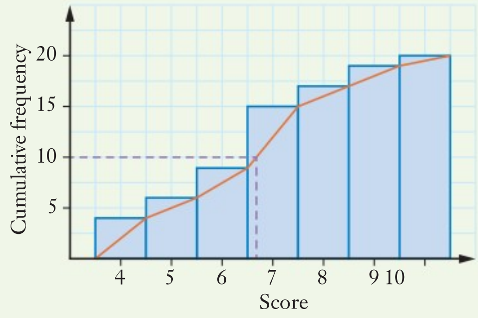
Solution

There are 20 scores in the data set, so the halfway point is at the 10th score as shown on the cumulative frequency axis. The dotted line meets the ogive inside the 7 column.

The median is 7.



Outliers

Sometimes a set of data contains a score that is unusual, compared with the other scores. This unusual or extreme value is called an outlier.

EXAMPLE 15

The prices of houses sold in the town of Greenfield in a particular week are:

\$355 000, \$420 000, \$320 000, \$285 000, \$390 000, \$1 200 000, \$415 000, \$320 000, \$435 000, \$380 000

a Is there an outlier? Why do you think an outlier may be in this data?

b Find the mean house price with and without the outlier.

c Find the median with and without the outlier.

d Find the mode with and without the outlier.

Solution

a The outlier is \$1 200 000 as this is much higher than the other prices. It may be that there is one special house in the area that is much larger than the others, or a certain street with huge houses in it that is unusual for the area.

b Using a calculator:

With the outlier, the mean house price is \$452 000. Notice the big difference.

Without the outlier, the mean house price is \$368 888.89.

c With the outlier, the median house price is \$385 000. Notice the small difference.

Without the outlier, the median house price is \$380 000.

d The mode is \$320 000 in both cases since this is the most frequent price with or without the outlier.

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Note: When real estate agents talk about house prices, they usually use the median price since this is not as affected by outliers as the mean.

Photo courtesy Margaret Grove

INVESTIGATION

OUTLIERS

Which measures of central tendency do outliers tend to affect most?

Find other examples of data that contain outliers and find the mean, mode and median. How do they change if the outlier is removed? Should outliers be looked at closely and discarded or is there a place for them?

Exercise 7.03 Measures of central tendency

1 For each data set find:

i the mean ii the mode iii the median

a Number of people auditioning for parts in a play: 5, 5, 7, 6, 5, 6, 1

b Number of minutes for an ambulance to respond to a call: 1, 4, 6, 8, 7, 4, 6, 4, 5

c Ages of students on a basketball team: 15, 18, 14, 19, 18, 17, 11

d Scores on a class quiz: 4, 6, 5, 4, 7, 8

e Prices of petrol (in dollars): 1.43, 1.66, 1.55, 1.49, 1.27, 1.81, 1.49, 1.38

2 Find the mode of each data set:

a

Hair colour	Frequency
Brown	28
Blond	21
Red	8
Black	12
Grey	17

b

Type of cat	Frequency
Siamese	18
Burmese	12
Russian blue	9
Tabby	32
Ginger	26
Persian	19

3 For each data set find:

i the mean ii the median iii the mode

a Judge's scores on a dance contest

Score	Frequency
3	3
4	4
5	2
6	7
7	6
8	2
9	3

b Number of matches in each match box surveyed

Score	Frequency
50	1
51	6
52	5
53	3
54	4
55	2

c Results in a History assignment

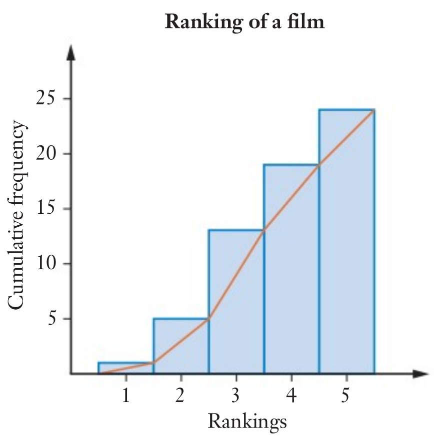
Score	Frequency
14	4
15	2
16	1
17	4
18	3
19	5
20	6

d Attendances at hockey matches

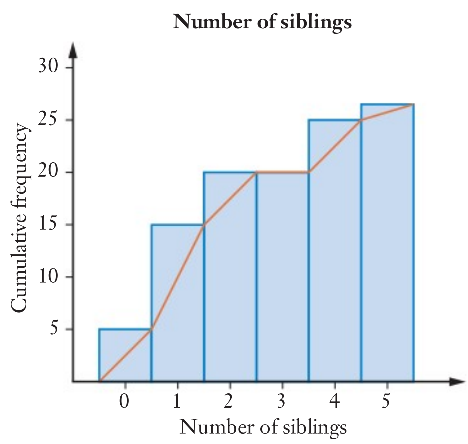
Attendance	Frequency
100	3
101	0
102	2
103	1
104	6
105	5

4 Find the median from each ogive:

a



b



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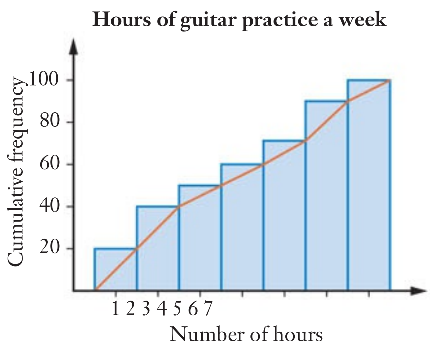
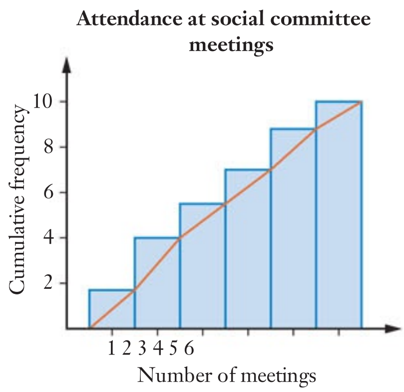
c



d



5 For each data set find:

i the mean

ii the modal class

a Games of chess played each week by members of a chess club:

Score	Frequency
2–4	5
5–7	4
8–10	7
11–13	4
14–16	3
17–19	2

b Hours per week that gymnasts train:

Score	Frequency
0–4	3
5–9	2
10–14	6
15–19	8
20–24	9
25–29	5

c Results in a Legal Studies exam:

Score	Frequency
10–24	4
25–39	0
40–54	1
55–69	5
70–84	9
85–99	8

d Time it takes for computers to boot up:

Time (s)	Frequency
20–24	12
25–29	8
30–34	9
35–39	7
40–44	8
45–49	11
50–54	12
55–59	6

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6 For each data set:

i add a cumulative frequency column

ii draw a cumulative frequency polygon

iii find the median from the graph (estimate for parts c and d)

a Number of athletes representing their school over a 30-year period:

Athletes	Frequency
1	5
2	6
3	4
4	8
5	5
6	2

b Number of lollies in a bag:

Number	Frequency
45	3
46	5
47	1
48	7
49	3
50	1

c Hours a week worked by employees in a cafe:

Hours	Frequency
1–5	7
6–10	5
11–15	3
16–20	6
21–25	7
26–30	2

d Time to complete a race:

Time (min)	Frequency
2.5–2.8	3
2.9–3.2	2
3.3–3.6	0
3.7–4.0	6
4.1–4.4	1
4.5–4.8	4
4.9–5.2	4

7 For each data set:

i draw a frequency distribution table including cumulative frequency

ii find the mean

iii find the mode or modal class

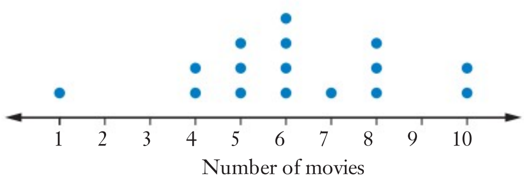
iv draw an ogive

v find the median from the ogive

a Home runs scored over a baseball season

4, 6, 5, 8, 8, 6, 5, 3, 4, 9, 6, 3, 5, 6, 5, 4, 7, 5, 8, 5, 6, 2, 3

b Number of movies seen in a year:



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c Ages of people living in a block of units (use classes of 20–29, 30–39 and so on):

Stem	Leaf
2	1 3 5 5 8 9
3	0 2 4 5 6 6 7 8 8 9
4	2 3 3 6 8
5	0 1 1 1 4 5 5 7 8 8 9
6	3 3 4 5 6 7 7
7	1 5 6
8	1 2 2 4 5 6

8 For each set of data, find:

i the outlier

ii the mean, mode and median

iii the mean, mode and median without the outlier

a Weights (in kg) of people in a lift:

69, 75, 58, 77, 32, 68, 60, 64, 59

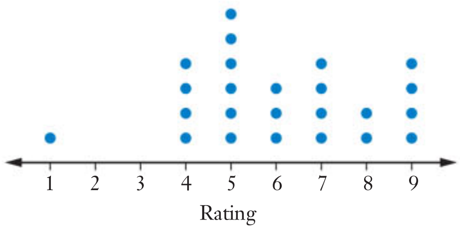
b Number of questions attempted in an exam:

Questions	Frequency
1	1
2	0
3	0
4	1
5	3
6	6
7	5
8	3
9	4

c Ages of people at a family party:

Stem	Leaf
2	3 5 6
3	2 7
4	1 3
5	0 0 3 4
6	3 5
7	
8	
9	7

d Rating of a venue for a dance party:



9 For the set of times (in minutes) students are recorded as late for school shown below, find the outlier. Which measures of central tendency (mean, median, mode) does it change?

5, 3, 6, 4, 7, 1, 6, 8, 7, 9, 6, 5, 8, 6, 7, 4, 5, 7, 4

10 The stem-and-leaf plot below shows the results of a class test:

Stem	Leaf
1	7
2	
3	8 9
4	4 5 6
5	1 3 4 6 7 8
6	4 5 5 7 7 9 9 9 9
7	0 2 3 3 4 5 8
8	3 4 5 7
9	0 1 1 3

a State which score is an outlier and what this means.

b Draw a frequency table including a cumulative frequency column, using groups 10–19, 20–29, 30–39 and so on.

c Use the table to estimate the mean and find the modal class:

i with the outlier included ii without the outlier included.

d Draw an ogive excluding the outlier and find the median.

INVESTIGATION

LIMITATIONS OF CENTRAL TENDENCY

Find the mean, mode and median of each set of data. What do you notice?

Set 1: 5, 6, 7, 7, 8, 9

Set 2: 1, 2, 7, 7, 12, 13

How do the 2 sets of data differ? Can we find out by using the measures of central tendency? How else could we describe how they are different from each other?

WS Quartiles, deciles and percentiles

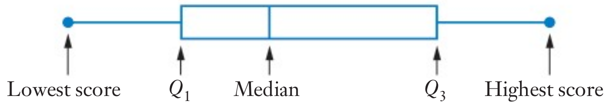
Box-and-whisker plots

7.04 Quartiles, deciles and percentiles

The measures of central tendency give us good information about data sets, but they don't describe the spread of data. As we have seen, the median divides data sets so that half the values lie below the median and half lie above it. A measure that divides a data set into parts of equal size is called a quantile. The median gives only a very rough description of the data set, but with more divisions we can describe the data's spread in more detail.

Quartiles

A quartile divides a data set into quarters.



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The 1st quartile (Q_1) is called the lower quartile and the 3rd quartile (Q_3) is called the upper quartile. The 2nd quartile (Q_2) is the median.

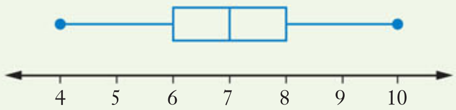
A box plot (also called a box-and-whisker plot) gives a way of showing a five-number summary: the quartiles and highest and lowest scores.

EXAMPLE 16

a Find Q_1, Q_2 and Q_3 for the number of runs in a softball game:

4, 3, 7, 8, 7, 9, 5, 6, 8, 3, 9

b Find Q_1, Q_3, the median, the highest and lowest score for this data.



c Find the quartiles of the data in this frequency table.

Score	Frequency
1	1
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a Put the 11 scores in order.

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Find Q_2 (the median) in the usual way: the 6th score is 7.

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Q_1 is the middle of the scores below the median: the 3rd score, 4.

If a quartile falls between 2 scores, we take their average.

3, 3, 4, 5, 6, 7, 7, 8, 8, 9, 9

Q_3 is the middle of the scores above the median: the 9th score, 8.

So $Q_1 = 4$, $Q_2 = 7$, $Q_3 = 8$

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On a calculator:

Operation	Casio Scientific
Clear the statistical memory.	
Enter data.	SHIFT 1 Data to get table. 4 = 3 = etc. to enter in column. AC to leave table.
Calculate Q_1.	SHIFT 1 MinMax Q_1 =
Calculate Q_2 (median).	SHIFT 1 MinMax med =
Calculate Q_3.	SHIFT 1 MinMax Q_3 =

b From the box plot:

Lowest score = 4, $Q_1 = 6$, Median = 7, $Q_3 = 8$, Highest score = 10.

c Add a cumulative frequency column to the table.

Score	Frequency	Cumulative frequency
1	1	1
5	3	4
6	5	9
7	8	17
8	11	28
9	13	41
10	9	50

There are 50 scores so the median is the $\frac{50+1}{2}$ or 25.5th score (average of 25th and 26th scores).

So $Q_2 = 8$, reading from the cumulative frequency column.

Q_1 is the $\frac{25+1}{2}$ or 13th score (middle of the 1st 25 scores).

So $Q_1 = 7$, reading from the cumulative frequency column.

Q_3 is the $\frac{50+25+1}{2}$ or 38th score (middle of the last 25 scores, halfway between the 26th to 50th scores).

So $Q_3 = 9$, reading from the cumulative frequency column.

So $Q_1 = 7$, $Q_2 = 8$, $Q_3 = 9$.

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EXAMPLE 17

Find the median and upper and lower quartiles from the ogive.

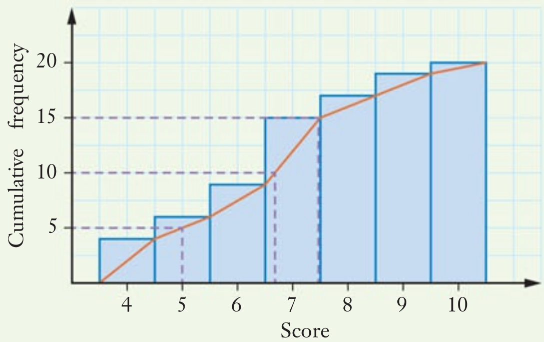
Solution

There are 20 scores.

The median is halfway:

$$\frac{1}{2} \times 20 = 10$$

So the median is 7, reading across from 10 on the cumulative frequency axis.



To find the 1st (lower) quartile:

$$\frac{1}{4} \times 20 = 5$$

So reading across from 5, $Q_1 = 5$.

To find the 3rd (upper) quartile:

$$\frac{3}{4} \times 20 = 15$$

So reading across from 15, $Q_3 = 7.5$.

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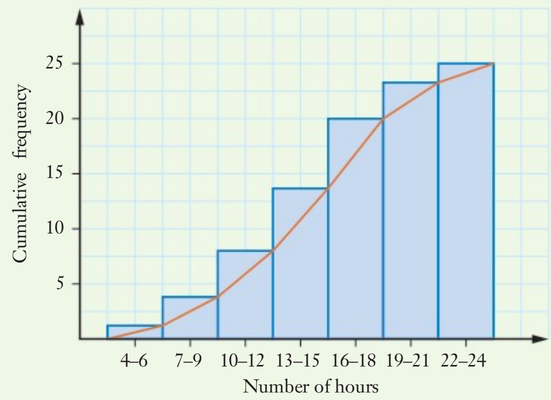
Deciles and percentiles

For a more detailed description of the spread we can divide the data set into smaller parts. Deciles divide the data set into 10 parts and percentiles divide data sets into 100 parts.

Deciles and percentiles

EXAMPLE 18

The ogive shows the number of hours that a rock group rehearses each week over 25 weeks. The scores have been sorted into groups.



Use the ogive to estimate:

a the 35th percentile

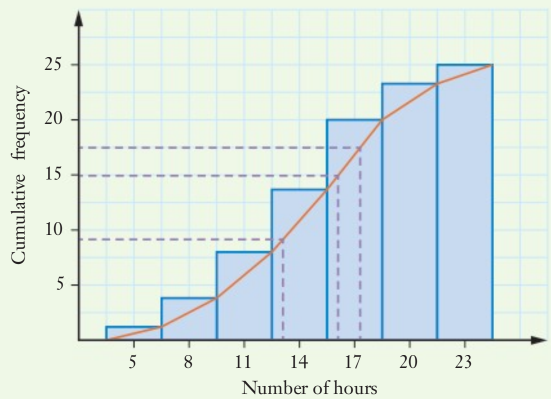
b the 60th percentile

c the 7th decile

We can only estimate because the scores have been grouped into classes.

Solution

Redraw the ogive using class centres for number of hours, and use the cumulative frequency axis to find answers.



a The 35th percentile is the score that separates the lower 35% of the scores.

35% of 25 scores = 8.75th score.

The 35th percentile ≈ 13.

b The 60th percentile separates the lower 60% of the scores.

60% of 25 scores = 15th score.

The 60th percentile ≈ 16.

c The 7th decile separates the lower $\frac{7}{10}$ of the scores.

$\frac{7}{10} \times 25$ scores = 17.5th score.

The 7th decile ≈ 17.

You can use a graphics calculator or software to draw graphs and find quartiles, deciles and percentiles more accurately.

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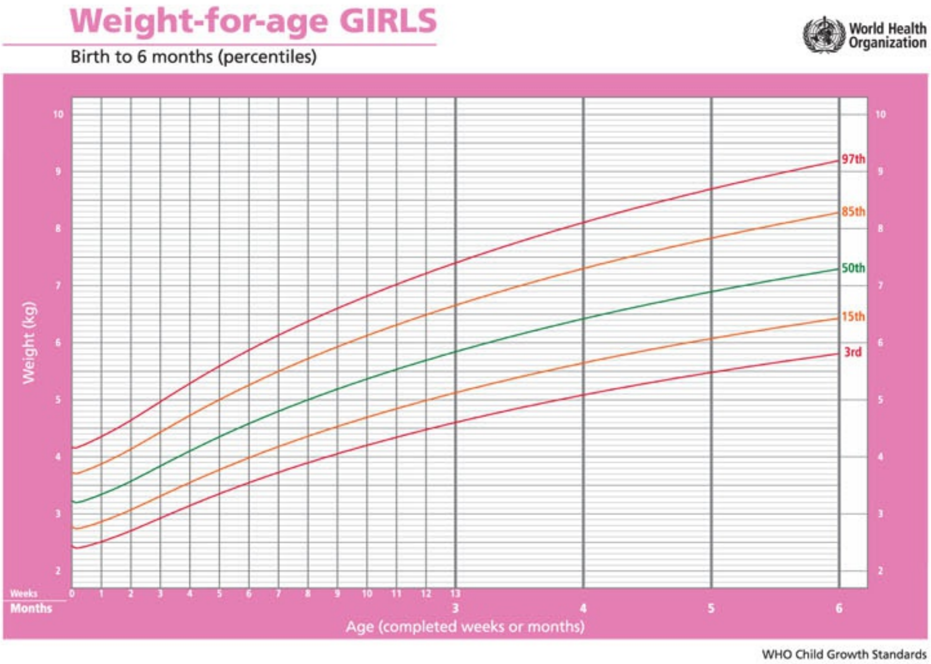
INVESTIGATION

RESEARCHING QUANTILES

Research the words quartile, decile and percentile. When were they first used? Where are they used now? There are other measures such as tercile and quintile. What are they?

Percentiles are used in many applications including graphs of infants' and children's growth rates.



http://www.who.int/childgrowth/standards/cht_wfa_girls_p_0_6.pdf?ua=1

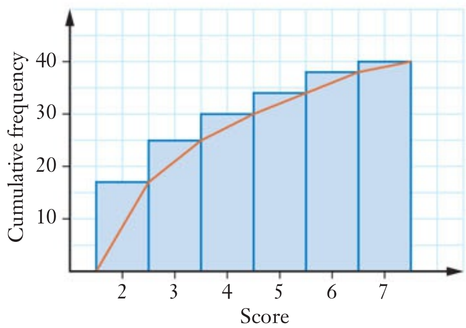
Research and compare different types of growth charts used in Australia and those of the World Health Organization (WHO).

Exercise 7.04 Quartiles, deciles and percentiles

1 For each set of data, find:

i the 1st quartile ii the 2nd quartile iii the 3rd quartile

a



b

Score	Frequency
10	3
11	5
12	4
13	6
14	0
15	2

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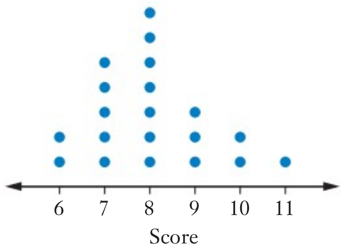
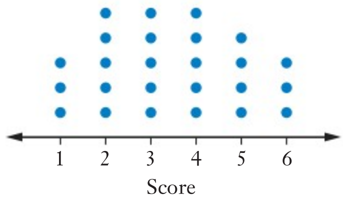
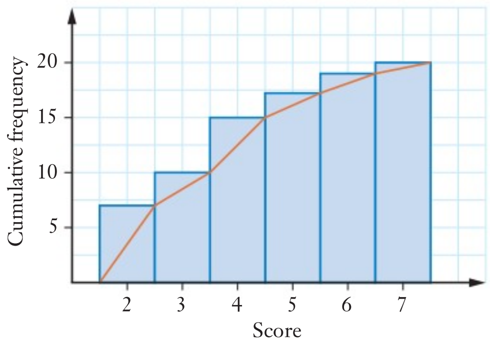
c



d



2 Find the 1st and 3rd quartiles for the following data.



3 For the following data find:

a the 23rd, 55th and 91st percentiles

b the 2nd and 8th decile.

Score	Frequency
23	13
24	19
25	23
26	21
27	9
28	15

4 For the dot plot:

a draw a cumulative frequency polygon

b find:

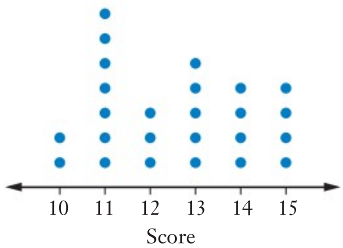
i the 1st quartile

ii the 3rd quartile

iii the 35th percentile

iv the 7th decile

v the 1st decile



5 John measured the weights of children in a particular year at school and organised his findings in a table. Estimate the weight that is:

a the median

b the 1st quartile

c the 3rd quartile

d the 60th percentile

Weight (kg)	Frequency
30–34	1
35–39	9
40–44	8
45–49	5
50–54	2

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6 The number of different dress sizes in Huang's Sportwear shop was counted in a stocktake, and the results set out in a table.

Size	Frequency
8	12
10	23
12	20
14	21
16	13
18	11

a How many dresses were counted?

b What percentage of dresses in the store were size 12?

c Find the median dress size.

d Find the 3rd quartile.

e What percentile is size 14?

7 Antonietta surveyed a number of people to find out how many pets they have. Her results are shown.

Number of pets	Frequency
0	7
1	11
2	3
3	2
4	1

a What percentage of the people surveyed had 2 pets?

b Draw a cumulative frequency polygon for this data.

c From the graph, find:

i the median

ii the 1st quartile

iii the 3rd quartile

8 Abdul measured the reaction times of a group of drivers and placed his results in a table.

Time (s)	Frequency
0.65–0.69	2
0.70–0.74	14
0.75–0.79	19
0.80–0.84	8
0.85–0.89	7

a What was the mean reaction time?

b What percentage of people surveyed reacted within 0.75 and 0.79 seconds?

c Draw an ogive to show this data.

d Use the graph to estimate:

i the 30th percentile

ii the median reaction time

iii reaction times between the 1st and 3rd quartiles

9 For each data set, find:

i the median   ii the lower quartile   iii the upper quartile

a The number of dogs at a pound over several days:

36, 79, 38, 29, 45, 83, 85, 47, 51, 72, 64

b The number of flying hours that Alexis had during a helicopter flying course:

3, 4, 9, 8, 14, 17, 15, 11, 12

c The distance (in km) travelled by a taxi during several shifts:

128.3, 143.2, 103.7, 99.5, 137.5, 203.4, 154.6, 115.3, 192.3, 125.4

d The number of people attending a choir rehearsal over several weeks:

15, 14, 12, 16, 15, 19, 17, 18

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WS
Range and interquartile range

Interquartile range

WS
Interquartile range

Statistical match-up

WS
Cumulative frequency graphs

7.05 Range and interquartile range

The range and interquartile range measure the spread of data.

Range and interquartile range

$$\text{Range} = \text{highest score} - \text{lowest score}$$

$$\text{Interquartile range} = Q_3 - Q_1$$

EXAMPLE 19

Find the range and interquartile range of these scores: 8, 13, 5, 15, 20, 21, 17, 16, 9

Solution

$$\begin{aligned}\text{Range} &= \text{highest score} - \text{lowest score} \\ &= 21 - 5 \\ &= 16\end{aligned}$$

Put the 9 scores in order to find the quartiles.

5, 8, 9, 13, 15, 16, 17, 20, 21

$Q_2 = 15$

5, 8, | 9, 13, 15, 16, 17, | 20, 21

$Q_1 = \dfrac{8+9}{2} = 8.5$ $\qquad Q_3 = \dfrac{17+20}{2} = 18.5$

$$\begin{aligned}\text{Interquartile range} &= Q_3 - Q_1 \\ &= 18.5 - 8.5 \\ &= 10\end{aligned}$$

agefotostock/Martin Barraud

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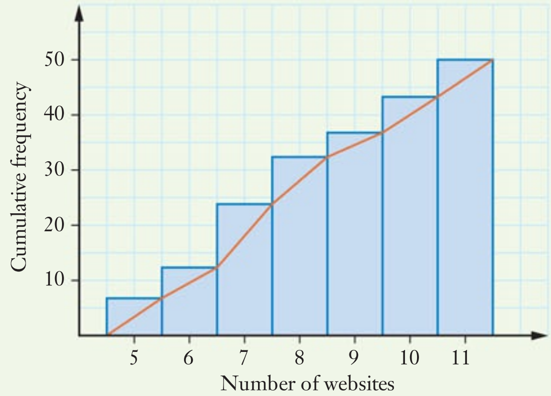
EXAMPLE 20

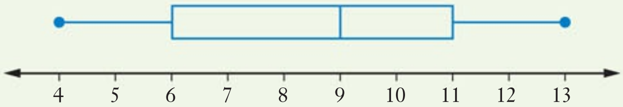
a This set of data shows the results of a survey into the number of travel websites people visit regularly. Find:

i the range ii the interquartile range



b From this box plot, find:

i the median ii the range iii the interquartile range



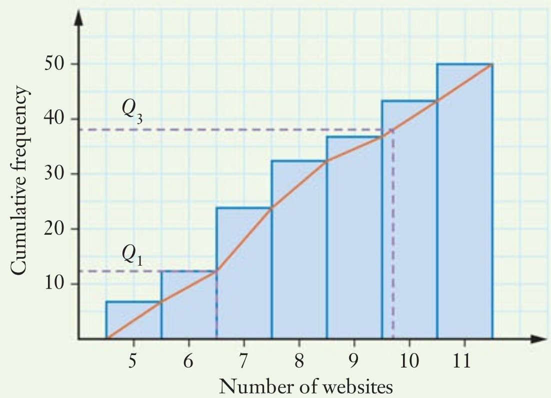
Solution

a i Range = 11 – 5 = 6 highest – lowest

ii There are 50 scores.

$\frac{1}{4} \times 50 = 12.5$ so Q_1 will be the 12.5th score.

$\frac{3}{4} \times 50 = 37.5$ so Q_3 will be the 37.5th score.



$Q_1 = 6.5$ and $Q_3 = 10$

Interquartile range $= Q_3 - Q_1$

$= 10 - 6.5$

$= 3.5$

b i Median = 9 ii Range = 13 – 4 = 9 iii Interquartile range = 11 – 6 = 5

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Outliers

Outliers

Outliers are extreme scores. They affect the range because an outlier will be the highest or lowest score. However, outliers do not affect the interquartile range because the interquartile range does not depend on the highest or lowest scores.

Some outliers are more obvious than others. There is a formal definition of outlier that allows us to test if it's an outlier rather than just deciding by inspection.

Outlier

A score is an outlier if it is more than 1.5 times the interquartile range (IQR) below Q_1 or above Q_3.

An outlier is below $Q_1 - 1.5 \times \text{IQR}$ or above $Q_3 + 1.5 \times \text{IQR}$.

EXAMPLE 21

a For the scores 5, 2, 9, 10, 6, 7, 6, 5, 10, 9, 5, 7, 8, 7, 6, determine if 2 is an outlier.

b For this table of data, find a score that looks like an outlier and use the definition to determine if it is an outlier.

Score	Frequency
1	1
2	0
3	0
4	0
5	3
6	5
7	8
8	11
9	13
10	9

Solution

a $Q_1 = 5$ and $Q_3 = 9$

$$\begin{aligned}\text{IQR} &= Q_3 - Q_1\\ &= 9 - 5\\ &= 4\end{aligned}$$

$$\begin{aligned}1.5 \times \text{IQR} &= 1.5 \times 4\\ &= 6\end{aligned}$$

$$\begin{aligned}Q_1 - 1.5 \times \text{IQR} &= 5 - 6\\ &= -1\end{aligned}$$

$$\begin{aligned}Q_1 + 1.5 \times \text{IQR} &= 9 + 6\\ &= 15\end{aligned}$$

Any outlier would have to be less than −1 or greater than 15. So 2 is not an outlier.

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b A score of 1 looks like an outlier.

From Example 16c,

$Q_1 = 7$ and $Q_3 = 9$

$$IQR = Q_3 - Q_1 \\ = 9 - 7 \\ = 2$$

$$1.5 \times IQR = 1.5 \times 2 \\ = 3$$

$$Q_1 - 1.5 \times IQR = 7 - 3 \\ = 4$$

$$Q_3 + 1.5 \times IQR = 9 + 3 \\ = 12$$

Any outlier would have to be less than 4 or greater than 12. So 1 is an outlier.

Exercise 7.05 Range and interquartile range

1 Find the range of each data set.

a 7, 4, 9, 8, 11, 4, 3, 19, 7, 16

b 56, 89, 43, 99, 45, 28, 37, 78

c 103, 108, 99, 112, 126, 87, 101, 123

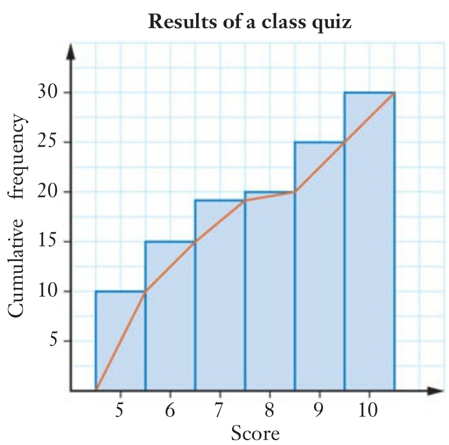
d

Score	Frequency
8	5
9	3
10	7
11	0
12	8
13	7

2 For each set of data, find:

i the median ii the range iii the interquartile range

a



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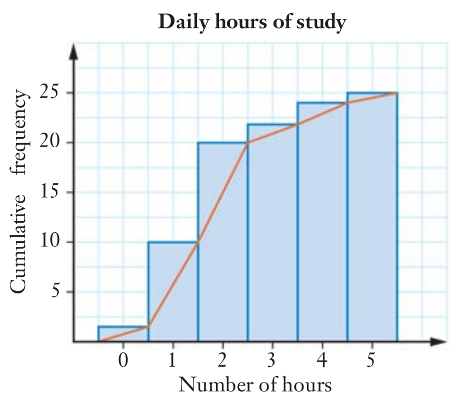

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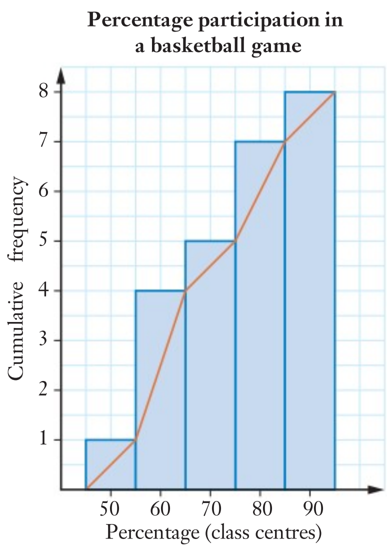
c



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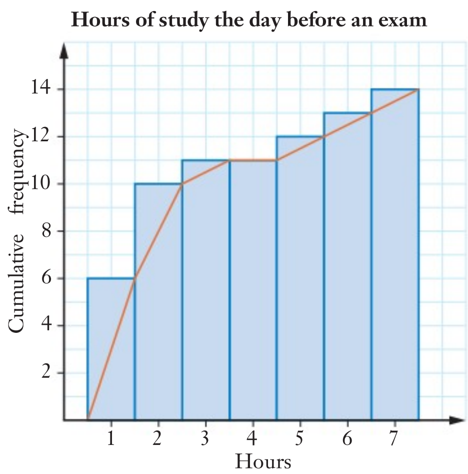


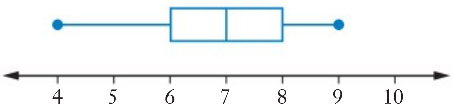
3 For each set of data, find:

i the median

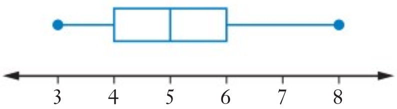
ii the range

iii the interquartile range

a



b



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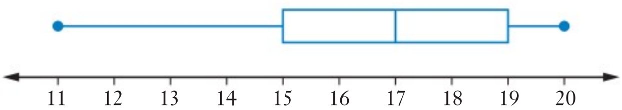

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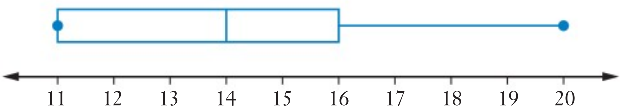
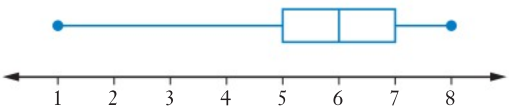


e



4 For this set of data, find:

a the mean

b the mode

c the median

d the range

e the interquartile range

Rain (mm)	Frequency
5	4
6	7
7	8
8	3
9	3

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5 Find a potential outlier in each set of data and use the definition to see if it really is an outlier.

a 100, 76. 93, 54, 32, 66, 53, 97, 51, 80

b 11, 5, 7, 19, 5, 3, 7, 5, 6, 10, 11. 2, 5, 7, 4, 6, 1

c

Score	Frequency
1	1
2	0
3	0
4	0
5	5
6	4

7.06 Variance and standard deviation

WS Variance and standard deviation

WS Statistical calculations

Statistical measures puzzle

Variance is another measure of spread. It measures how far the scores in a data set are from the mean of the data. You studied variance and standard deviation when studying discrete probability distributions in Year 11, Chapter 10, Discrete probability distributions.

The formula for variance, σ^2, is:

$$\sigma^2 = \frac{\Sigma(x - \bar{x})^2}{n}$$

However, you do not have to use it as the calculator's statistical mode can calculate it more easily. The following example will show you what the above formula means, but you don't have to learn it.

EXAMPLE 22

The data below shows the times (in minutes) taken for a fire engine to reach the site of a fire. Find the variance for this data.

4, 7, 3, 9, 4, 5, 1, 3, 5, 9

Solution

First we need to find the mean.

$$\bar{x} = \frac{\Sigma x}{n} = \frac{50}{10} = 5$$

Now we find the difference between each score and the mean. Then we square each difference because we only want positive values. This is shown in the table next page.

x	$x - \bar{x}$	$(x - \bar{x})^2$
1	$1 - 5 = -4$	16
3	$3 - 5 = -2$	4
3	$3 - 5 = -2$	4
4	$4 - 5 = -1$	1
4	$4 - 5 = -1$	1
5	$5 - 5 = 0$	0
5	$5 - 5 = 0$	0
7	$7 - 5 = 2$	4
9	$9 - 5 = 4$	16
9	$9 - 5 = 4$	16
		$\Sigma(x - \bar{x})^2 = 62$

Variance is the mean of these squared differences.

$$\begin{aligned}\sigma &= \frac{\Sigma(x^2 - \bar{x})^2}{n}\\ &= \frac{62}{10}\\ &= 6.2\end{aligned}$$

Standard deviation is another measure of spread, and it is simply the square root of variance. For the data in the previous example, the standard deviation is $\sqrt{6.2} \approx 2.49$.

We use s for standard deviation of a sample and $\boldsymbol{\sigma}$ (the lowercase Greek sigma) for the standard deviation of a population. In this course we will use s most of the time.

The formula for standard deviation, $\boldsymbol{\sigma}$, is:

$$\sigma = \sqrt{\frac{\Sigma(x - \bar{x})^2}{n}}$$

This example shows how to calculate standard deviation and variance using the calculator's statistical mode.

EXAMPLE 23

The table shows the number of hours of karate practice that students at a karate club do each week.
Find correct to one decimal place:

a the standard deviation

b the variance

Practice times (h)	Frequency
2	2
3	4
4	3
5	4
6	1
7	2
8	4

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Solution

a

Operation	Casio scientific	Sharp scientific
Clear the statistical memory.	SHIFT 1 Edit, Del-A	2ndF DEL
Enter data.	SHIFT 1 Data to get table. 2 = 3 = etc. to enter in x column. 2 = 4 = etc. to enter in FREQ column. AC to leave table.	2 2ndF STO 2 M+ 3 2ndF STO 4 M+ etc.
Calculate standard deviation.	SHIFT 1 VAR sx =	RCL sx

$s = 2.0774 \ldots \approx 2.1$

b Variance $= s^2$

$= 2.0774 \ldots^2$

$= 4.3157 \ldots$

≈ 4.3

Exercise 7.06 Variance and standard deviation

1 For each set of data find:

i the mean ii the standard deviation

a Number of minutes kept on hold on the telephone: 7, 4, 9, 8, 4, 6, 2, 4, 5

b Travel time (in minutes) to get into the city: 23, 45, 67, 54, 69, 38, 59, 70, 59, 41

c Height of children (in cm): 101, 112, 131, 122, 130, 143, 152, 107, 112

d Number of repetitions on gym equipment: 8, 6, 9, 5, 5, 7, 6, 4, 8, 9, 6, 3, 6

e Age of performers in a play: 18, 19, 17, 16, 20, 18, 15, 19, 14, 20

2 For each data set, find:

i the standard deviation ii the variance

a Weights (in kg): 51, 67, 64, 53, 60, 48, 58, 49, 61, 71, 67, 58

b Class quiz results: 4, 6, 5, 3, 7, 9, 8, 10, 4, 6, 7, 6, 5, 8, 6, 7, 9, 10, 5, 4, 8

c Time spent waiting in a queue (in mins): 11, 14, 15, 25, 31, 54, 36, 39, 31, 41, 44, 50

d Weight of crates (in kg): 87, 88, 56, 91, 68, 73, 55

e Response time (in mins) for helicopter rescue: 1, 5, 7, 3, 8, 6, 5, 5, 4, 8, 9, 3

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3 For each data set find:

i the mean ii the standard deviation iii the variance

a Number of books read:

Books	Frequency
5	3
6	5
7	6
8	2
9	1
10	3
11	5
12	4

b Piano practice time per week:

Practice (h)	Frequency
1	3
2	0
3	2
4	5
5	7
6	3
7	2

c Weight of luggage:

Weight (kg)	Frequency
31	3
32	0
33	2
34	5
35	7
36	3

d Results of a half-yearly exam:

Score	Frequency
10–19	1
20–29	4
30–39	8
40–49	12
50–59	15
60–69	11
70–79	7

4 In a taste test, the people surveyed had to rank a new biscuit on a scale from 1 to 5. The table shows the results of the survey.

a What is the range?

b Find the interquartile range.

c Find the standard deviation.

Rank	Frequency
1	5
2	11
3	18
4	21
5	9

5 A Year 12 Art class received these results for their major work.

a Calculate:

i the mean ii the standard deviation

b Remove the outlier and calculate:

i the mean ii the standard deviation

Class	Frequency
20–29	1
30–39	0
40–49	2
50–59	4
60–69	7
70–79	9
80–89	8
90–99	7

6 The data shows the ages of students in an MBA course:

20, 25, 31, 34, 17, 27, 29, 53, 20, 31, 19, 23, 30, 29, 18, 25

a Show that there is an outlier.

b Calculate the standard deviation:

i with the outlier ii without the outlier

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7 Jane recorded the number of crocodile sightings each week in a region of the Northern Territory over several weeks.

5, 8, 9, 4, 11, 7, 9, 15, 17, 10, 8, 5, 9, 12

a What was the mean number of crocodiles sighted per week?

b Find the standard deviation.

c Calculate the variance.

7.07 Shape and modality of data sets

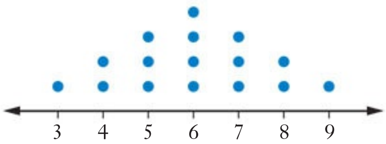
WS Shape of a distribution

The shape of a frequency distribution

WS Shapes of distributions

The measures of central tendency and spread are called summary statistics, and they help us make decisions about data. Other features of data can help with these decisions, and one of these features is the shape of the data. The shape often gives us an idea of the centre and spread even before we measure them, while the modality describes the number of peaks in the distribution of data.

A data set where the mean, mode and median are equal has a symmetrical distribution.



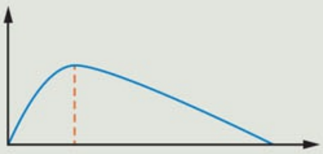
Notice how symmetrical this dot plot is.

Other graphs that are not so symmetrical can be described by their skewness.

The shape of a statistical distribution

This distribution is negatively skewed as most of the area is to the left (or negative direction) of the centre. We can say that the 'tail' points to the low scores in the negative direction.

This distribution is positively skewed as most of the area is to the right (or positive direction) of the centre. We can say that the 'tail' points to the high scores in the positive direction.



This distribution is symmetrical.

All these graphs are also called unimodal since they only have one peak.

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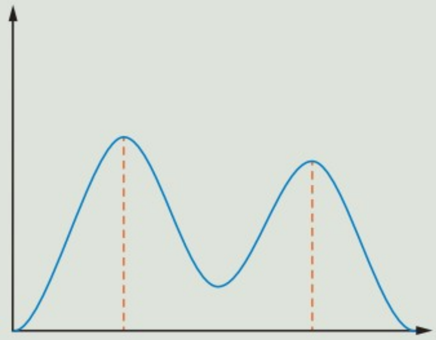
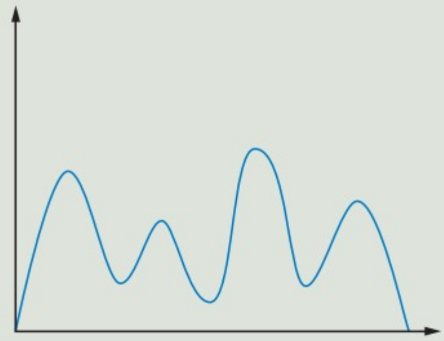
The modality of a statistical distribution

This graph is described as bimodal because it has 2 peaks.



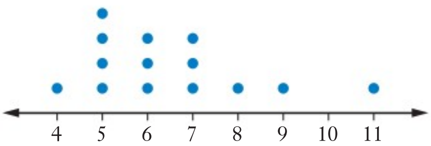
This graph is described as multimodal because it has many peaks.



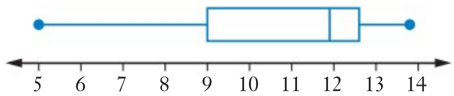
Exercise 7.07 Shape and modality of data sets

1 Describe the shape and modality of each graph.

a



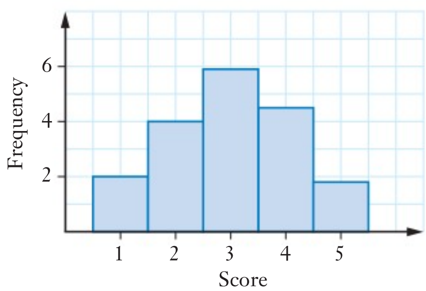
b



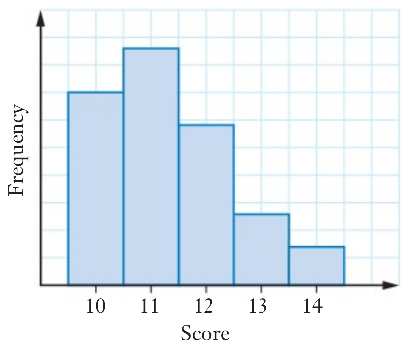
c

Stem	Leaf
1	3 4 4 5 5 7 9
2	0 1 1 7 8 8 9 9
3	2 2 5 6 6 8 9 9
4	1 1 3 5
5	0 0 3
6	3 4 5 8
7	8 9

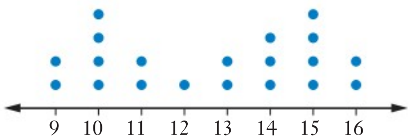
d



e



f



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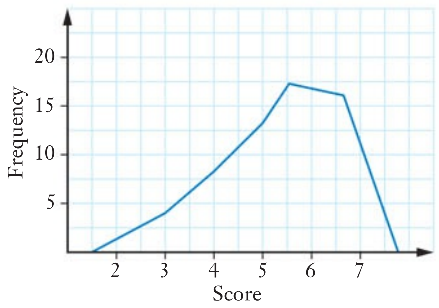

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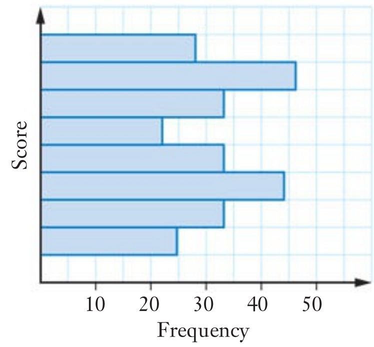
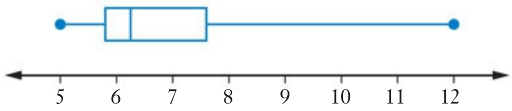


i



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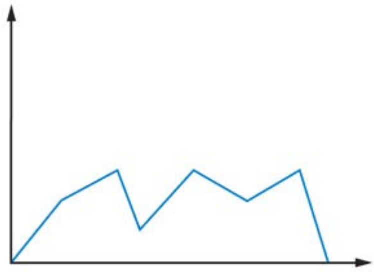

2 a Draw a dot plot for this data.

5, 9, 4, 8, 9, 10, 7, 5, 3, 9, 7, 8, 6, 12, 8, 9

b Describe the shape of the dot plot.

3 Describe the shape and modality of each data set.

a

Score	Frequency
1	7
2	9
3	5
4	3
5	1
6	1

b

Score	Frequency
12	3
13	7
14	11
15	14
16	9
17	2

c

Score	Frequency
10–14	3
15–19	6
20–24	7
25–29	4
30–34	7
35–39	5

d

Score	Frequency
6	1
7	2
8	5
9	8
10	7
11	4

e

Score	Frequency
50–59	2
60–69	4
70–79	6
80–89	3
90–99	1

4 Draw graphs with the following shapes.

a bimodal

b skewed negatively

c symmetrical

d skewed positively

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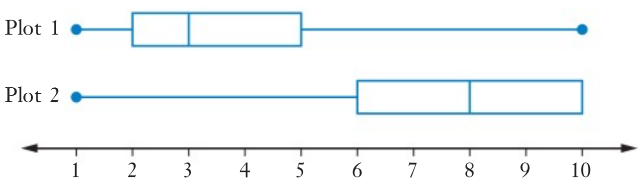
5 Describe the shape and modality of each set of data in the back-to-back stem-and-leaf plot.

Set 1		Set 2
4	1	3 3 4
7 7 6 3	2	1 2 5 6 8 9 9
8 7 5 5 2 0	3	0 2 2 5 6
8 6 6	4	3 6 7
9 8 8 7 1 1	5	0 1
5 4 0	6	3
1 1	7	

6 a Describe the shape of the distributions summarised by the parallel box plots.



b What is the difference between their medians?

c Find the difference in their interquartile ranges.

7 The heights of a number of students were measured and the results are below.

159, 175, 181, 153, 177, 168, 175, 163, 155, 184, 167, 179, 157, 149, 160, 171, 180, 160, 162, 169, 163, 179, 145, 187, 161, 148, 182, 151, 150, 178

a Draw a frequency distribution table for the heights, using groups of 145–149, 150–154, 155–159 and so on.

b Describe the type of distribution for this data.

8 Draw a box plot that describes a symmetrical distribution.

9 This table shows the results of an assessment task.

a Find:

i the mean

ii the median

iii the mode

b Describe the shape of the distribution.

Score	Frequency
4	3
5	5
6	6
7	9
8	6
9	5
10	3

10 Choose a random sample of about 50 people and collect data on the number of siblings (brothers and sisters) each one has. Graph the data and describe the shape and modality of the graph.

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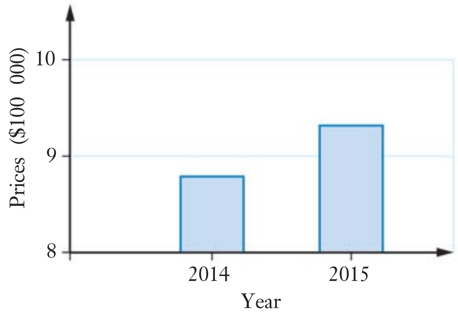
INVESTIGATION

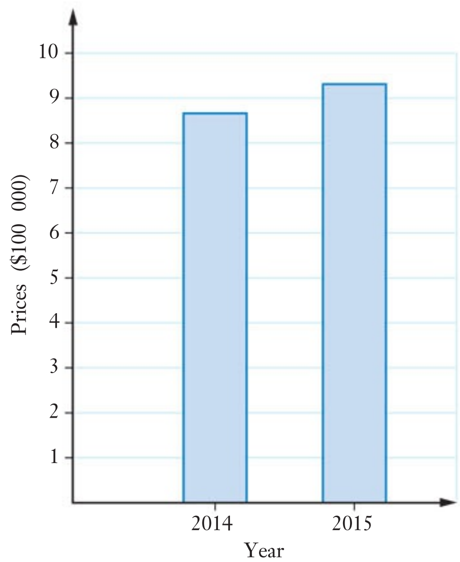
MISLEADING GRAPHS

Sydney's median house price increased from $886 408 in 2014 to $929 842 in 2015.



The column graph shows this information. Looking at the graph, you would think that this was a huge price rise because the second column is almost twice as tall as the first column.

Now look at this graph. What is the difference between the 2 graphs? Which one do you think shows the information better? Is one of the graphs misleading? Why?



Search online for other misleading graphs. Collect them into a portfolio and share with the class. Write an account of why each one is misleading and how you could change it to give a better reading of the information.

Statistics can be misleading in different ways. In the investigation this was caused by the scale on the graph. Sometimes the measures of central tendency or spread can be misleading as well.

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7.08 Analysing data sets

Double box plots

Comparing city temperatures

Comparing word lengths

Comparing sports scores

EXAMPLE 24

The table shows the heights of students in a Year 12 class.

Height (cm)	Class centre	Frequency
150–154	152	3
155–159	157	18
160–164	162	27
165–169	167	31
170–174	172	12
175–179	177	15
180–184	182	25
185–189	187	11
190–194	192	3

a Find the mean and standard deviation of the heights.

b Are the mean and standard deviation misleading for this data? Why?

Solution

a Using a calculator:

$\bar{x} = 170.6$

$s = 10.2$

b Looking at the table, the data looks to be bimodal. This might be because the survey is for both males and females.

If this is the case, the mean of 170.6 is misleading because it doesn't tell us about differences in male and female heights. The spread may be less than 10.2 if we split the data into male and female data.

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Comparing two or more sets of data

Sometimes we need to compare different data sets to see how similar or different they are.

EXAMPLE 25

a Two surveys were made into the number of people attending an outdoor cinema: one in 2019 and one in 2020.

For the 2019 survey, the mean was 112 and the standard deviation was 6.7.

For the 2020 survey, the mean was 95 and the standard deviation was 11.9.

Describe how these results differ.

b This back-to-back stem-and-leaf plot shows the results of tests of the life of 2 brands of batteries batteries (measured in hours).

Buzz		Eternity
9 9 1	4	5
5 2 1 1 1 0	5	1 3 4 9
7 7 4 0	6	1 2 3
4 2 0	7	0 1 2 8
4 0	8	1 2 5 5 7
	9	0

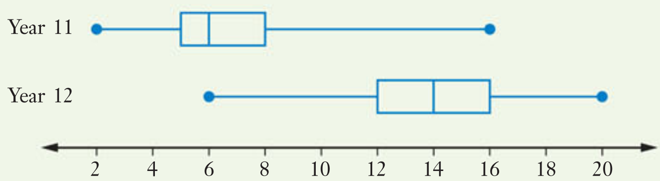
i Describe the shape of the distribution for each brand.

ii Find the mean result for each brand.

iii Find the standard deviation for each brand.

iv Compare the results for the 2 brands of batteries.

c The parallel box plots below show the results of 2 surveys into the number of hours 2 groups of students study each week.



i What is the median number of hours studied for each Year group?

ii Calculate the range for each group.

iii What is the interquartile range for each group?

iv What is the highest number of hours studied in each group surveyed?

v What is the main difference between the 2 groups?

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Solution

a The mean was lower in the second survey, so it looks as if, on average, fewer people were going to the movies in 2020 than in 2019.

The standard deviation was higher in the second survey, so there was a greater variation in the number of people going to the movies.

b i Buzz is slightly positively skewed and Eternity is approximately bimodal.

ii Using a calculator:

The mean for Buzz is 60.4.

The mean for Eternity is 69.4.

iii The standard deviation for Buzz is 12.3 and the standard deviation for Eternity is 14.0.

iv The mean was higher on Eternity so these batteries had longer lives overall.

The standard deviation was slightly higher on Eternity, showing slightly more variability in the life of these batteries. That is, the battery lives were more spread out than for Buzz, but there wasn't a big difference between the 2 brands.

c i Year 11: median is 6.

Year 12: median is 14.

ii Year 11: range = 16 – 2 = 14

Year 12: range = 20 – 6 = 14

iii Year 11: interquartile range = 8 – 5 = 3

Year 12: interquartile range = 16 – 12 = 4

iv Year 11: highest hours = 16

Year 12: highest hours = 20

v Year 12 students generally study for more hours.

CLASS DISCUSSION

ANALYSING DATA SETS

Why do you think the surveys in the example give different results? Are they taken from the same population? How could you tell? What other information could help you decide?

Find other examples online, in newspapers or in magazines that compare 2 or more sets of data. Is the information taken from the same or different populations? Can you tell?

Put these examples in a portfolio and present a report to the class.

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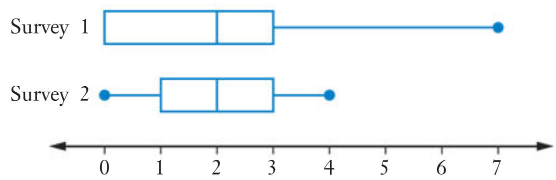
Exercise 7.08 Analysing data sets

1 The parallel box plots show the results of 2 surveys into the number of children in families.



a What was the largest number of children in a family in:

i survey 1? ii survey 2?

b What is the median number of children in the families in:

i survey 1? ii survey 2?

c Find the interquartile range for:

i survey 1 ii survey 2

d Do you think the surveys were taken from the same population or from different populations?

2 The heights (in centimetres) of people were taken from 2 samples.

Sample 1: 158, 169, 177, 147, 160, 153, 167, 164, 171, 159, 162, 166, 172, 148, 157, 176, 155, 170, 149, 150

Sample 2: 181, 166, 179, 181, 164, 173, 168, 165, 176, 181, 157, 162, 178, 182, 157, 172, 180, 173, 159, 151

a Plot this information in a back-to-back stem-and-leaf plot.

b Find the median of each sample.

c Find the range of each sample.

d Do you think the samples were taken from the same population?

3 Surveys were taken at 3 different banks to measure the length of time customers waited before being served. The results are shown in the table.

Time (min)	Bank 1	Bank 2	Bank 3
0–2	29	59	2
3–5	38	26	8
6–8	15	12	11
9–11	9	3	21
12–14	5	0	28
15–17	3	0	20
18–20	1	0	10

a Find i the mean and ii the median waiting times for customers at each bank.

b Find the standard deviation for each bank.

c Do you think there is a significant difference in waiting times at the banks?

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4 Mrs Spell's piano students earned the following marks in their piano examinations:

Class 1: 91, 86, 74, 92, 85, 89, 63, 71, 80, 91, 85, 72, 54, 78

Class 2: 97, 87, 69, 91, 88, 89, 93, 94, 71, 79, 84, 85, 88

a Sketch parallel box plots showing this information.

b For each class, find:

i the median | ii the interquartile range

iii the mean | iv the range

5 Two speed cameras at different locations recorded speeds (in km h^{-1}) of vehicles travelling over the speed limit:

Camera 1: 85, 66, 75, 69, 72, 83, 80, 69, 74, 77, 73, 74, 90, 84, 65, 73, 69, 89, 76, 103

Camera 2: 122, 142, 120, 118, 116, 135, 140, 123, 135, 124, 120, 119, 138, 131, 122, 119, 125, 130, 130, 113

a Draw a back-to-back stem-and-leaf plot to show this data.

b Find the mean speeds recorded by each camera.

c What do you think was the speed limit at the site where each camera was placed?

6 Jon sat for the HSC in one year and scored 56, 48, 61, 53, 41 and 35 for his maths assessments. He resat his HSC the next year and his maths assessment scores were 73, 58, 67, 74, 59 and 68.

a By how much did his mean scores increase the second year?

b What was the difference in the median scores?

c Calculate the difference in the range of scores for each year.

d By how much does the standard deviation differ over the 2 years?

7 These 2 sets of scores have the same median.

Test 1: 1, 2, 4, 6, 7, 9, 10 Test 2: 4, 5, 5, 6, 6, 7, 8

a What is the mean of:

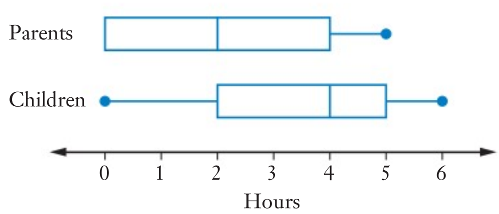
i Test1? | ii Test2?

b Calculate the standard deviation of:

i Test 1 | ii Test 2

c Describe how the 2 sets of scores differ.

8 The parallel box plots show the results of 2 surveys into the number of hours people spend watching TV each day.



a What is the median for each group?

b For each group, find:

i the interquartile range | ii the range

c What is the highest number of hours of TV watched?

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9 For the back-to-back stem-and-leaf plot, find:

a the median of each test

b the mean of each test

c the standard deviation of each test

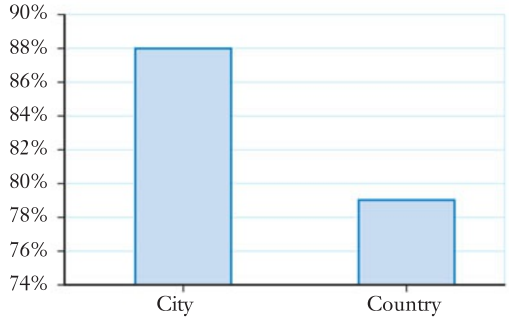
d the difference in range between the 2 tests.

Science		English
	4	3
2 1 1 0	5	8 9
8 6 5 3 3 1	6	0 1 2 2 5
9 9 6 3 2	7	4 6
1 0 0	8	1 2 3
4 2	9	7

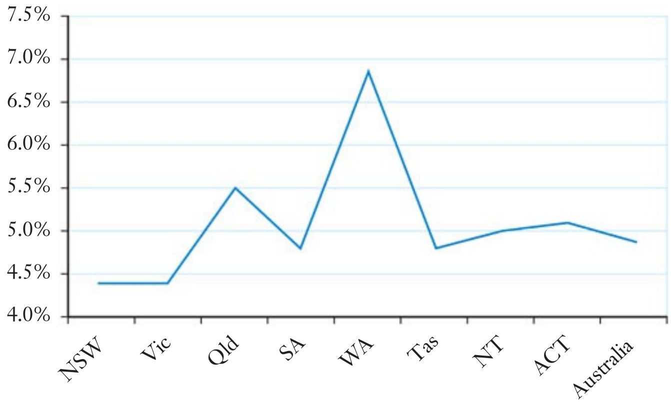
10 The graph compares access to the Internet for city and country households in 2015.

a Describe how this graph is misleading.

b Redraw the graph so it is not misleading.



11 The graph shows the average annual growth in incomes from 2006 to 2011.



Redraw the graph so it is not misleading.

INVESTIGATION

COMPARING SURVEY RESULTS

Conduct a survey among your friends. Make up your own topic, such as what sports they play, subjects they study or their heights. Alternatively, you could carry out an experiment such as counting numbers of people travelling in cars or measuring the time taken for the same journey to school on different days.

To check your results, take another sample and do the same survey or experiment. Are the new results the same? Can you explain why?

7. TEST YOURSELF

Practice quiz

Statistics review

Statistics crossword

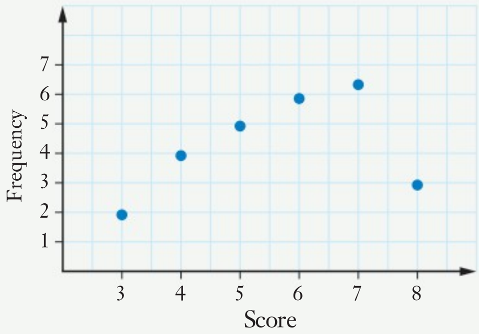
For Questions 1 to 6, select the correct answer A, B, C or D.

1 What type of data is a person's body temperature?

A categorical ordinal
B numerical continuous
C numerical discrete
D categorical nominal

2 This frequency distribution is:

A symmetrical
B positively skewed
C negatively skewed
D multimodal



3 What type of data is a flower's colour?

A categorical ordinal
B numerical continuous
C numerical discrete
D categorical nominal

4 The median of scores 7, 3, 2, 6, 4, 3, 8, 9 is:

A 6 B 4 C 3 D 5

5 Which measure is most affected by an outlier?

A median
B interquartile range
C mode and interquartile range
D mean and range

6 Test 1 has a mean of 4 and standard deviation of 1.2. Test 2 has a mean of 4 and a standard deviation of 2.5. Which statement below is true when comparing the centres and spreads for both tests?

A The centres are the same and test 2 has a larger spread than test 1.
B The spreads are the same and test 2 has a higher centre than test 1.
C The spreads are the same and test 2 has a lower centre than test 1.
D The centres are the same and test 2 has a smaller spread than test 1.

7 Find the mode, median and range of this data set:

8, 6, 8, 4, 5, 6, 8, 5, 7, 4, 7, 8, 6, 8, 9

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8 The table shows the results of a survey to find the distance people must travel to work.

a Find the modal class.

b Find the mean.

c What is the median?

d Find the standard deviation and variance.

Distance (km)	Frequency
0–4	6
5–9	8
10–14	4
15–19	7
20–24	3

9 Find the mean and standard deviation of this data set:

16, 34, 29, 80, 65, 77, 91, 58, 67, 40

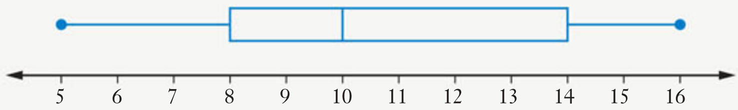
10 From this box plot, find:

a the median

b the range

c the 3rd quartile

d the interquartile range.



11 The table shows the results of a survey into the number of years that people keep a car before selling it.

a Draw a cumulative frequency polygon to show this data.

b Estimate the median age of the cars.

c Estimate:

i the 20th percentile ii the 3rd quartile

iii the 91st percentile iv the 3rd decile v the 9th decile

Years	Frequency
0–2	15
3–5	37
6–8	13
9–11	18
12–14	17

12 The table records the number of times people visited a doctor in the past 6 months.

a Find the mean number of visits.

b Find the median number of visits.

c What is the range?

d Find the interquartile range.

Visits	Frequency
0	9
1	8
2	5
3	1
4	3
5	2

13 'Most families have 2 children.' Is this statement about a mean, median or mode?

14 Nikola conducted a survey to find the number of people in each car that travelled across the Anzac Bridge one morning. These were her results:

a Draw a cumulative frequency polygon to illustrate the results.

b From the polygon, find:

i the median number of people in a car

ii the interquartile range

Occupants	Frequency
1	43
2	32
3	12
4	8
5	5

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15 The table shows the results of Mr Cheung's history class.

Score	Frequency
6	7
7	6
8	3
9	6
10	2

a Find the mean score.

b Find the standard deviation.

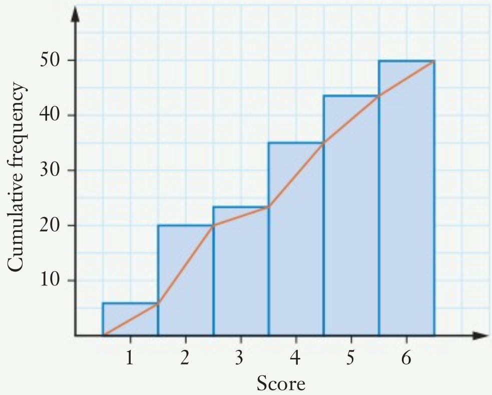
c What is the mode?

16 Find the mean, mode and median of this data set:
45, 49, 49, 48, 43, 45, 41, 40, 49, 48, 44, 40, 42

17 From the graph, find:

a the interquartile range

b the median



18 A class test gave the following scores:
15, 19, 12, 2, 19, 16, 13, 18, 11, 15, 17, 11, 18, 14, 14, 16, 18, 14, 12

a Find:

i the range
ii the mean
iii the mode
iv the median

b Show that one score is an outlier. Which is it?

c Find, without this outlier:

i the range
ii the mean
iii the mode
iv the median

d Does the outlier have much effect on all these measures?

19 The two-way table shows the results of a survey into the number of pets microchipped at a veterinary surgery.

	Male	Female	Total
Cats	369	473	842
Dogs	578	664	1242
Total	947	1137	2084

a What percentage of microchipped animals were:

i female?
ii female cats?
iii male dogs?
iv dogs?

b Draw this information in a Pareto chart.

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20 The table shows the reasons employees gave for leaving their jobs in a large organisation. Draw a Pareto chart of this data.

Work too difficult	185
Boring work	139
Not paid enough	104
Not getting on with co-workers	56
Unsuitable hours	116

21 A back-to-back stem-and-leaf plot shows the number of mushrooms found in 2 regions of a forest in New Zealand.

North		West
	2	2 5 6
5 4 2	3	4 8 9
8 7 6 3 3	4	0 0 1 3 7
6 5 4 4 2 2	5	1 1 2 3 4 4 7 8

a Calculate the median number of mushrooms recorded in each region.

b Find the mean and standard deviation of North region.

c Find the mean and standard deviation of West region.

d Compare and contrast the 2 regions.

22 Below are the results of 2 English assessments.

Term 1: 8, 7, 9, 8, 6, 5, 8, 7, 7, 5, 9, 9

Term 2: 5, 7, 8, 4, 6, 6, 5, 5, 5, 4, 7

a Draw a box plot for each set of data.

b What is the median of each assessment?

c Find the interquartile range of each set.

d Find the mean and standard deviation for each assessment.

e Compare and contrast the 2 assessments.

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7. CHALLENGE EXERCISE

1 The table shows the heights of students in Year 12.

a Describe the modality of the distribution. Can you explain this?

b Find the mean height and the variance.

Height (cm)	Frequency
150–154	7
155–159	5
160–164	15
165–169	9
170–174	8
175–179	15
180–184	6
185–189	2

2 The mean of a data set is 4.5. The table containing the data has a frequency missing. What is it?

Score	Frequency
2	2
3	5
4	
5	3
6	2
7	4

3 a Sketch a box plot for the following scores.

7, 9, 5, 6, 8, 11, 25, 14, 16

b What does this plot show? What is its shape? Can you make a more symmetrical box plot by taking out one of the scores?

4 For the scores 7, 11, 15, 19, 23

a Find:

i the mean

ii the standard deviation

b Copy and complete this table:

Score x	$x - \bar{x}$	$(x - \bar{x})^2$
7		
11		
15		
19		
23		
	$\Sigma(x - \bar{x}) =$	$\Sigma(x - \bar{x})^2 =$

c Find the standard deviation by using the formula $\sigma = \sqrt{\dfrac{\Sigma(x - \bar{x})^2}{n}}$.

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5 The mean of 7 scores is 25. If an extra score of 28 is also included, what will be the new mean?

6 Compare and contrast each pair of data sets.

a Set A has a mean of 54 and a standard deviation of 5.6

Set B has a mean of 76 and a standard deviation of 2.1

b Set A has a mean of 11.6 and a standard deviation of 2.7

Set B has a mean of 21.3 and a standard deviation of 9.2

7 A Year 12 class received the following scores out of 10 on their maths quiz.

a Find the mean and standard deviation of the scores.

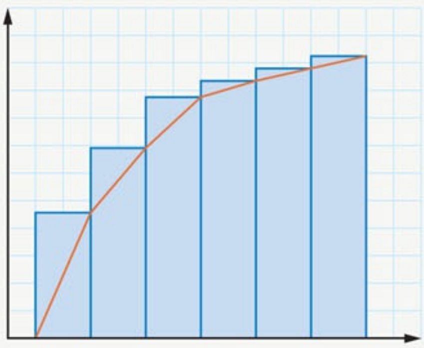
b Draw a box plot for this set of scores.

c Describe the shape of the distribution.

Score	Frequency
4	7
5	11
6	15
7	9
8	4
9	2
10	1

8 Describe the shape of each distribution, given the ogive.

a



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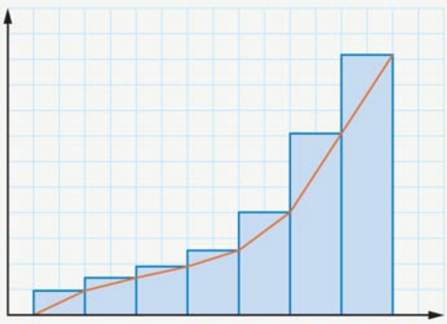
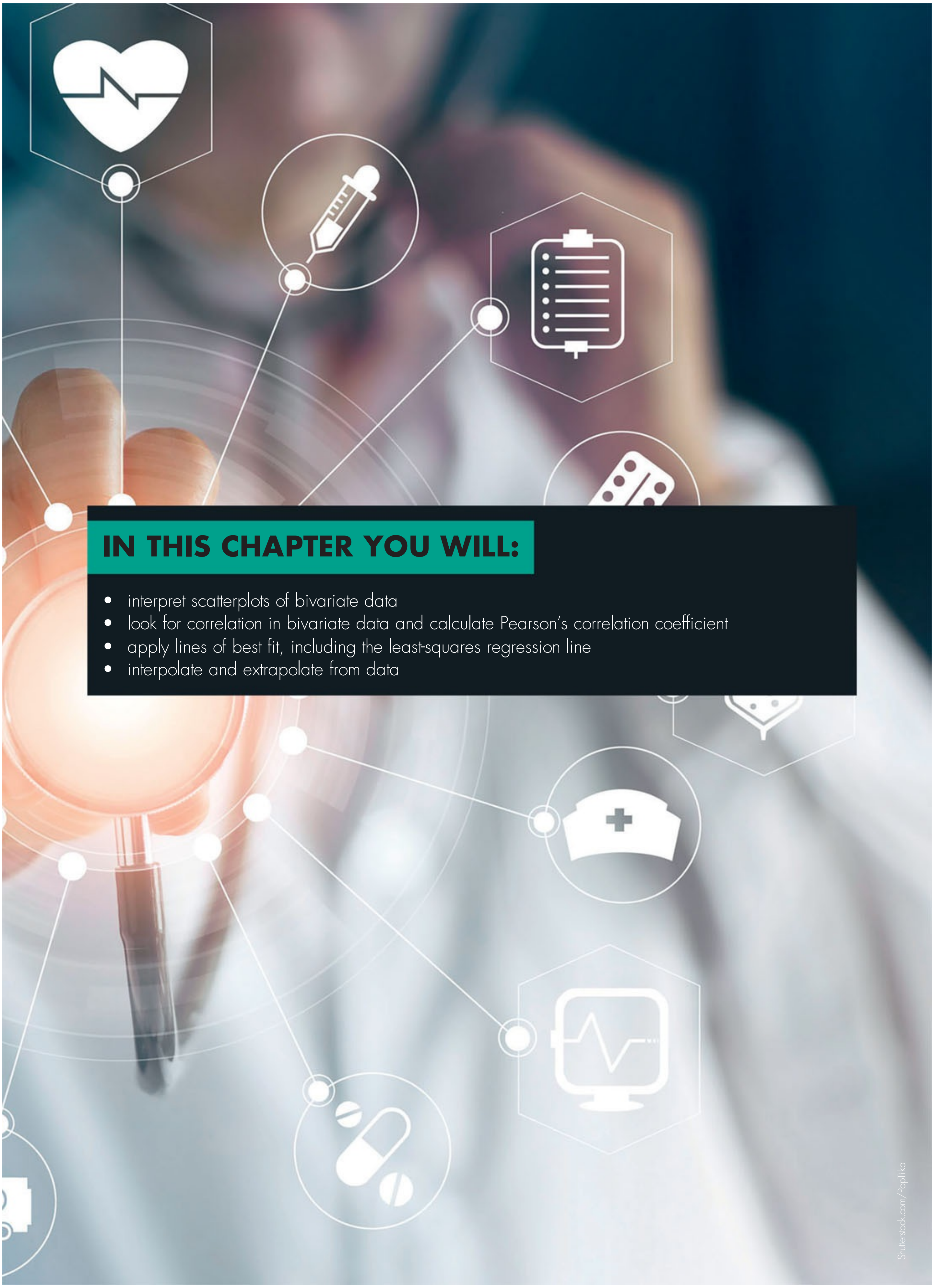


STATISTICAL ANALYSIS

8. CORRELATION AND REGRESSION

This chapter looks at comparisons between variables and uses correlation to measure the strength of these relationships. We also look at regression and lines of best fit to make predictions from this data using interpolation and extrapolation.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- interpret scatterplots of bivariate data
- look for correlation in bivariate data and calculate Pearson's correlation coefficient
- apply lines of best fit, including the least-squares regression line
- interpolate and extrapolate from data

TERMINOLOGY

bivariate data: Data relating to 2 variables that have been measured from the same data set.
extrapolation: Making predictions from a model using values outside the range of the original data set.
interpolation: Making predictions from a model using values lying within the range of the original data set.
least-squares regression line: A line of best fit where the squares of the distances from each point in the scatterplot to the line are minimised.
line of best fit: A line drawn through a scatterplot that best models the relationship between 2 variables in bivariate data.
Pearson's correlation coefficient: A calculated value, r, that measures how closely 2 variables are related in a linear relationship. Its value is always between –1 and 1.
scatterplot: A graph showing the value of 2 variables in a bivariate data set.

8.01 Bivariate data

WS Scatterplots

WS A page of scatterplots

WS Body measurements

In Chapter 7, Statistics, we looked at data with one variable.

Bivariate data measures 2 variables on the same data set to see if they correlate with (are related to) each other. For example, we might want to see if a person's level of education correlates with the amount of money the person earns.

We draw scatterplots to graph bivariate data.

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EXAMPLE 1

This table shows the bivariate data for the number of days each year the temperature in a city was over 30°C and the number of people admitted to the local hospital for heat stroke.

Year	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
Number of days over 30°C	14	17	15	23	21	26	31	33	29	38
Number of patients	64	75	68	72	71	76	83	77	83	92

Draw a scatterplot for this data.

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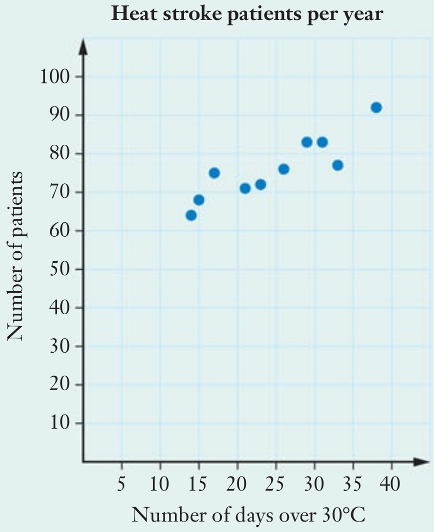
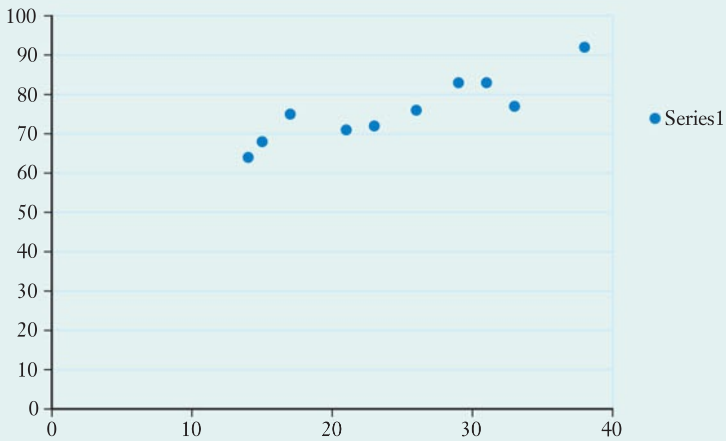
Solution

We draw the graph with number of days over 30°C on the horizontal axis and number of patients on the vertical axis.



You could also put the data from the table into a spreadsheet and choose the scatterplot chart.



We look for patterns in the scatterplot to see if the 2 variables are related. For instance, in the example above there seems to be a linear pattern (it is roughly a straight-line graph).

Not all relationships are linear. We can have scatterplots that have curves or non-linear shapes, or no shape at all.

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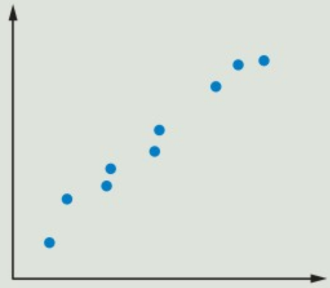
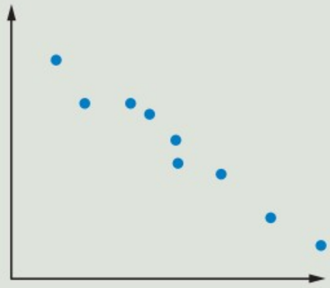
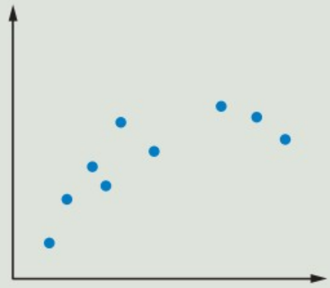
Shape of scatterplots



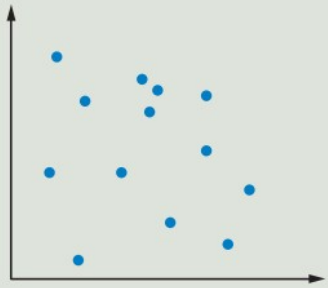
Linear (positive)



Linear (negative)



Non-linear

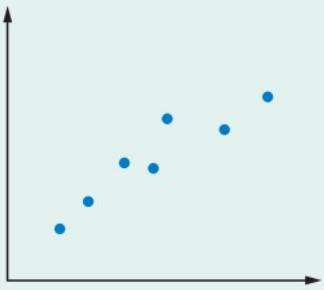


No pattern

EXAMPLE 2

Describe the shape of each scatterplot.

a



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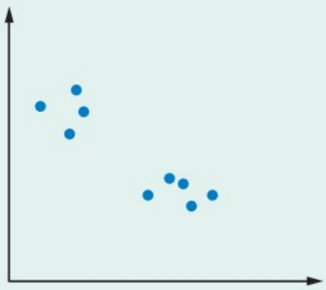

Solution

a This data is linear (positive).

b This data is linear (negative).

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Exercise 8.01 Bivariate data

1 Biometric data is bivariate data drawn from body measurements such as arm length or height. This table shows the heights and weights of 10 people surveyed.

Height (m)	161.8	175.3	159.5	182.3	166.4	167.9	186.4	164.7	154.8	171.2
Weight (kg)	59.4	73.7	55.3	74.8	63.5	68.2	73.5	62.1	49.9	83.6

a Draw a scatterplot to show this data.

b Describe the shape of the scatterplot.

c Describe the relationship between height and weight (if possible).

2 A group of 8 people were surveyed at random about how many people in their family lived at home. They were also asked about the number of bedrooms in their home.

Number of people in family	4	3	6	7	2	1	5	8
Number of bedrooms	3	2	5	4	2	3	4	6

a Draw a scatterplot showing the data.

b Describe the shape of the scatterplot.

c Describe the relationship (if any) between the number of people in the family and the number of bedrooms in the home.

3 A survey of 10 people measured their IQ with the amount they earned each week.

IQ	114	127	95	130	123	141	136	83	109	96
Amount earned per week ($)	689	945	510	874	751	769	553	350	1250	884

a Draw a scatterplot for this data.

b Describe the shape of the scatterplot.

c Describe any relationship between IQ and amount earned.

4 The table shows the amount of sleep students have and their exam results.

Hours of sleep	6.5	9	7.5	6	7	10	11.5	4	8	8.5
Exam results (%)	87	76	43	87	69	55	60	78	94	72

a Draw a scatterplot of this data.

b Describe the shape of the scatterplot.

c What is the relationship between the amount of sleep and exam results?

5 The table shows the amount of study time and exam results for some students.

Hours of study/week	15	26	12	17	5	10	2	8	20	3
Exam results (%)	78	86	70	80	63	67	43	77	92	58

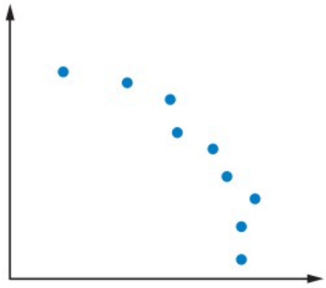
a Draw a scatterplot of the data.

b Describe the form of the scatterplot.

c What is the relationship between the amount of study and the exam results?

6 Describe the pattern of each set of bivariate data.

a



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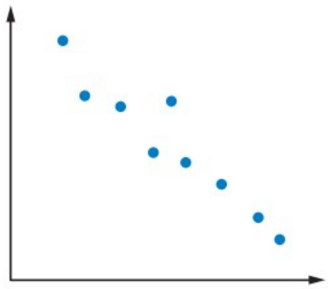

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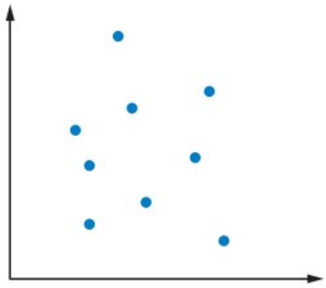

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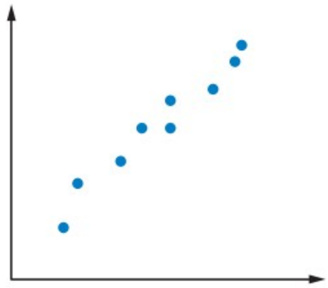

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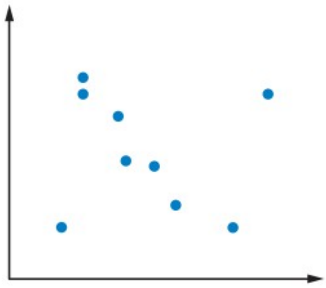

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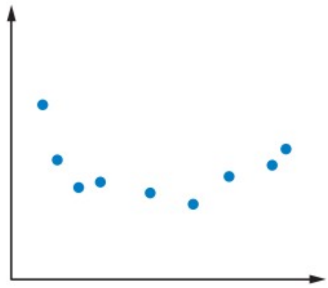

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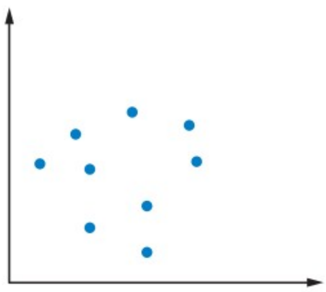

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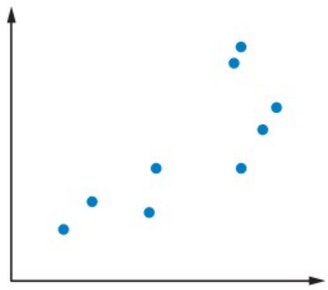

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i

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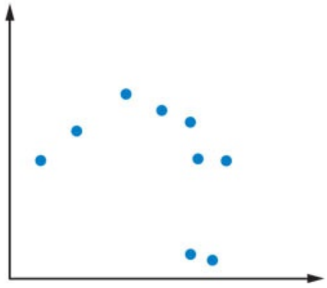


7 a Draw a scatterplot for this set of bivariate data.

x	8	3	7	10	2	9	4	5	9	1
y	2	7	1	2	2	8	4	5 7	3	7

b Describe the shape of the data.

8 State whether each scatterplot has:

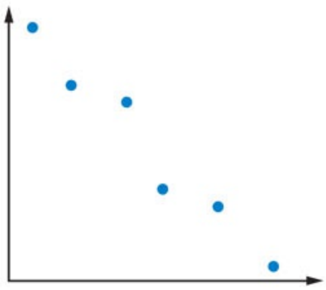
A a positive linear relationship

B a negative linear relationship

C a non-linear relationship

D little or no relationship

a



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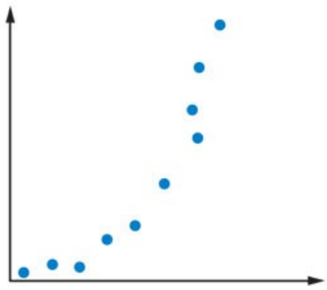

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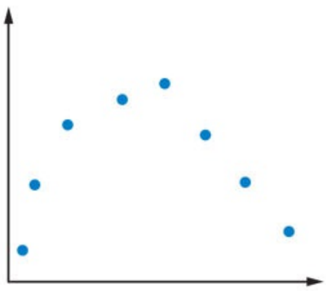

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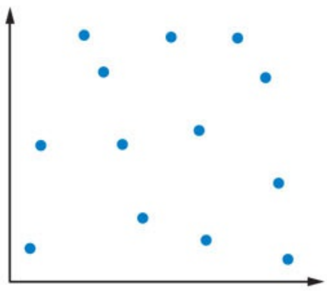

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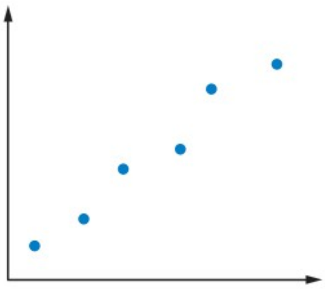
e



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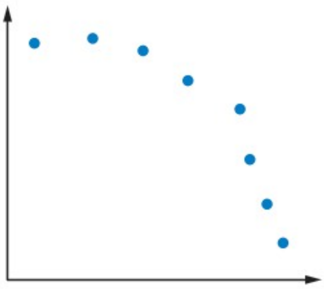
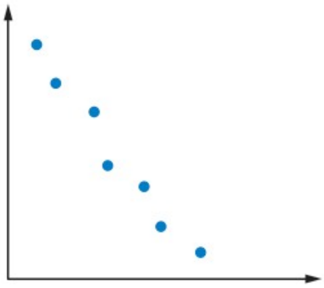


g



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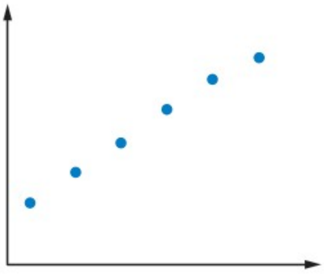

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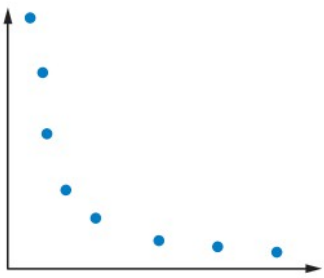

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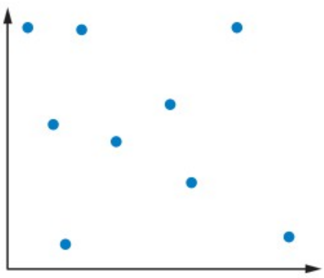


WS Correlation

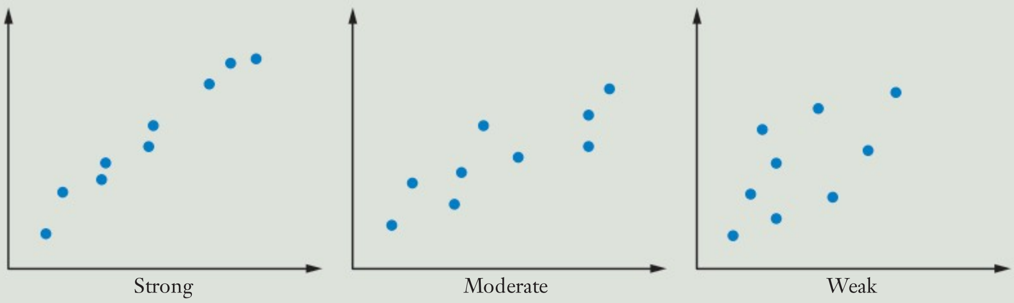
Correlations matching game

Correlation

8.02 Correlation

Correlation measures how well 2 variables are related if there seems to be a linear relationship between them. The relationship could be strong, moderate or weak.

Linear scatterplots



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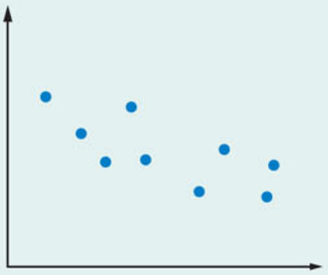
EXAMPLE 3

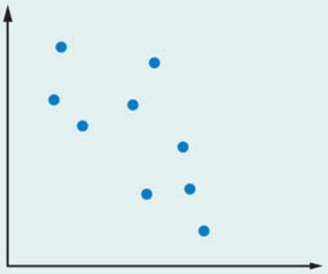
Describe the strength of the linear pattern in each scatterplot as strong, moderate or weak.

a

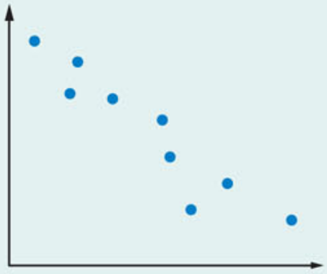


b

c



d



Solution

a The linear pattern is moderate.

b The linear pattern is strong.

c The linear pattern is weak.

d The linear pattern is strong.

Pearson's correlation coefficient

The terms strong, moderate or weak are very general and not very accurate. We use a measurement (r) called the Pearson's correlation coefficient to determine how closely related variables are in a linear relationship.

The formula for r is complex, but you can use a calculator or spreadsheet to calculate it.

The correlation coefficient always lies between -1 and 1.

Correlation coefficient

- $-1 \leq r \leq 1$ for all correlation coefficients
- $0 < r \leq 1$ for a scatterplot with positive direction, where 1 is perfect positive correlation
- $-1 \leq r < 0$ for a scatterplot with negative direction, where -1 is perfect negative correlation
- $r = 0$ means no correlation

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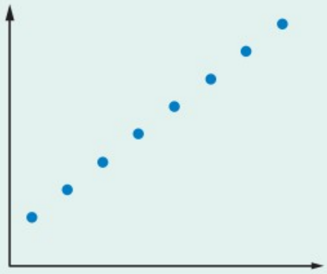
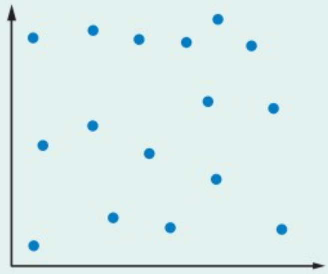
EXAMPLE 4

Match each scatterplot with its correct correlation coefficient.

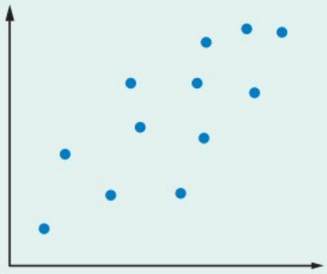
a



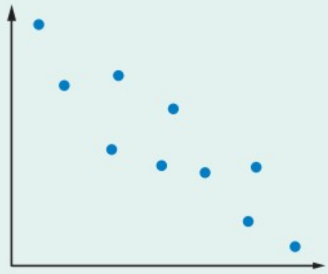
b



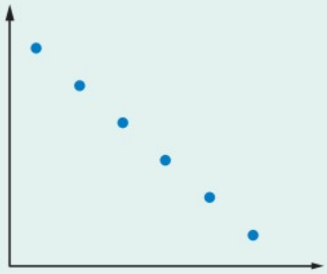
c



d



e



A 0.5 B 1 C −1 D −0.4 E 0

Solution

a B b E c A d D e C

DID YOU KNOW?

Karl Pearson

Karl Pearson (1857–1936), an Englishman, developed the formula for the correlation coefficient r. The coefficient's full name is Pearson's product moment correlation coefficient. Karl Pearson was a mathematician and statistician, but he also studied history, law and German literature.

Research Karl Pearson to find out more about his life and studies.

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EXAMPLE 5

A group of students was surveyed for the number of hours that they studied and their result in a maths exam. Find the correlation coefficient for this bivariate data.

Number of hours studied	6	5	12	8	15	9	14
Exam results (%)	71	46	74	67	76	77	83

Solution

Operation	Casio Scientific	Sharp Scientific
Place your calculator in statistical mode.	MODE 2: STAT 2:A+ BX	MODE 1 STAT 1 LINE
Clear the statistical memory.	SHIFT 1 3 : Edit 2 : Del-A	2ndF DEL
Enter data.	SHIFT 1 2 : Data 6 = 5 = etc. for 1st column 71 = 46 = etc. for 2nd column AC	6 2ndF STO 71 M+ 5 2ndF STO 46 M+ etc.
Calculate r.	SHIFT 1 5 : Reg 3 : r =	ALPHA r =
Change back to normal mode.	MODE 1 : COMP	MODE 0

r = 0.738 (correct to 3 decimal places)

Newpix/Ian Currie

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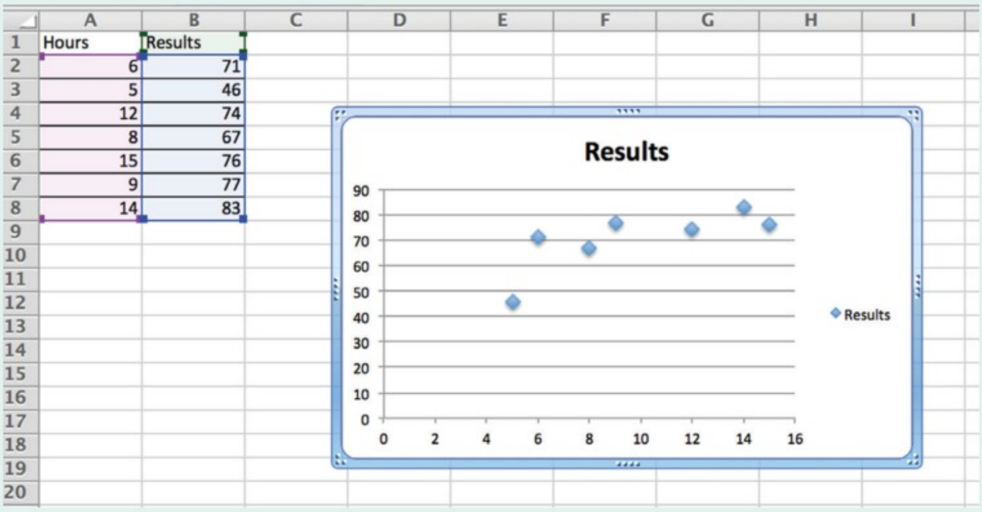
EXAMPLE 6

a Enter the data from Example 5 above into a spreadsheet and draw a scatterplot.

b Use the spreadsheet to find the Pearson's correlation coefficient for the data.

Solution

a Put the values from the table into a spreadsheet and select scatterplot from the charts.



b In an empty cell, type =PEARSON(A2:A8, B2:B8)

$r = 0.738$ (to 3 decimal places)

Causality

Two variables can have a high correlation without one causing the other. For example, does a person's height cause them to weigh more? Possibly, since most tall people would have heavier, longer bones. However, there are other causes for higher weight that aren't related to height.

EXAMPLE 7

For each set of bivariate data, find which ones have a causal relationship.

a Number of people in a family and the number of TVs

b Speed of a boat and time taken to travel across a lake

Solution

a Not causal: the number of people in a family doesn't determine how many TVs there are.

b Causal: The speed of a boat will determine the time taken to travel across a lake (higher speed means less time).

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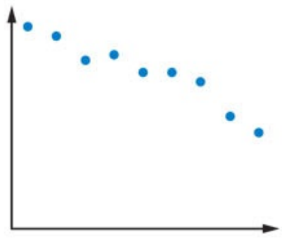
Exercise 8.02 Correlation

1 Match each graph with the correct correlation coefficient.

a


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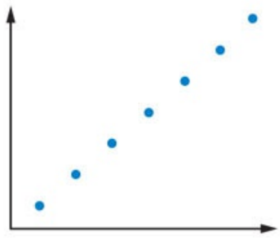
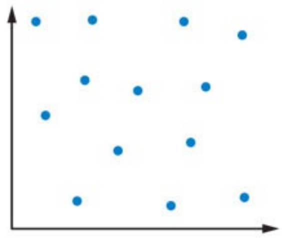
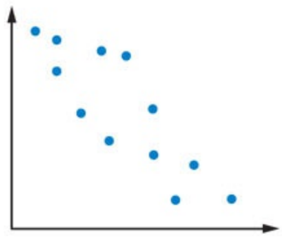
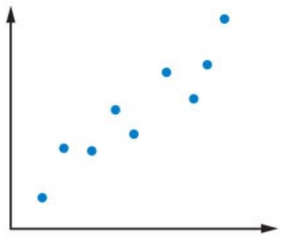
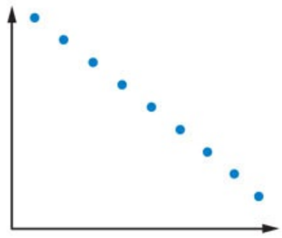


c


d


e


f


A −0.5 B 1 C 0.8 D −1

E −0.9 F 0 G −0.7

2 For each table of values:

i draw a scatterplot ii find the correlation coefficient

a

Height (m)	1.72	1.85	1.61	1.74	1.59	1.79
Weight (kg)	91.3	85.2	58.3	61.9	74.5	102.6

b

Height (cm)	167	180	174	171	154	190
Speed (m s^{-1})	3.7	2.1	3.4	2.2	2.8	4.1

c

Study time (h)	13	21	8	11	18	17
Results (%)	45	89	81	67	74	53

d

Temperature (°C)	15	18	21	24	26	30	35
Attendance at beach	28	19	54	88	190	245	108

e

Forest cleared (ha)	41	58	87	99	132	168
Number of birds	1200	854	530	201	157	92

f

Number of cars in car park	1100	1450	1809	2004	2234	2569	2871	2906
Pollution (ppm)	1.21	1.54	1.78	2.34	2.99	3.35	4.76	5.97

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g

Age	15	19	27	34	49	57
Annual income ($)	2 851	12 600	27 890	38 740	41 834	29 450

h

Exercise (h/week)	14	8	2	10	6	4	32
Weight (kg)	51.8	87.2	74.8	68.4	62.1	63.9	58.9

i

Height (m)	1.59	1.77	1.64	1.78	1.89	1.42
Shoe size	5	7	6	10	9.5	4

j

Exam results (%)	68	92	38	51	77	84
Hours of sleep	7	6	8	6.5	9	7.5

3 For the bivariate data in question 2, which do you think have causality?

4 Find the correlation coefficient of each set of data, correct to 2 decimal places.

a

x	y
3	7
5	9
4	3
11	7
15	12
8	4
9	1

b

x	y
5	67
6	49
3	81
9	23
11	55
8	91
4	61

c

x	y
8	11
4	8
7	11
2	4
9	12
14	16
23	23

d

x	y
5	21
3	28
6	19
5	17
9	21
4	26
11	15
15	18
9	12

5 Determine whether each pair of variables are likely to have a causal relationship.

a Population of a city and pollution
b Head circumference and weight
c Hours training and fitness
d Weight and health
e Size of house and number of pets
f Size of house and selling price

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INVESTIGATION

CAUSALITY

Discuss whether each pair of variables have a high correlation and, if they do, whether one variable causes the other.

1 A person's height and shoe size
2 A person's smoking and lung cancer
3 Amount of study and success in an exam
4 Mathematical and musical ability
5 The number of people at a party and the amount of food and drink consumed
6 The amount of time sunbaking and the incidence of skin cancer
7 The amount of time practising basketball and the number of baskets scored in a game
8 Results in English and Maths exams
9 The length of a person's leg and their walking speed
10 The temperature and the number of people swimming at the beach

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Discuss other relationships between variables. Can you find other examples of highly correlated variables where one causes the other? Can you find examples of highly correlated variables where there is no causality?

Discuss causality in each situation described below.

1 The time a sales representative has been with a company and the number of sales gives a correlation coefficient of –0.6.

2 Height of basketball players and number of baskets scored have a correlation coefficient of 0.87.

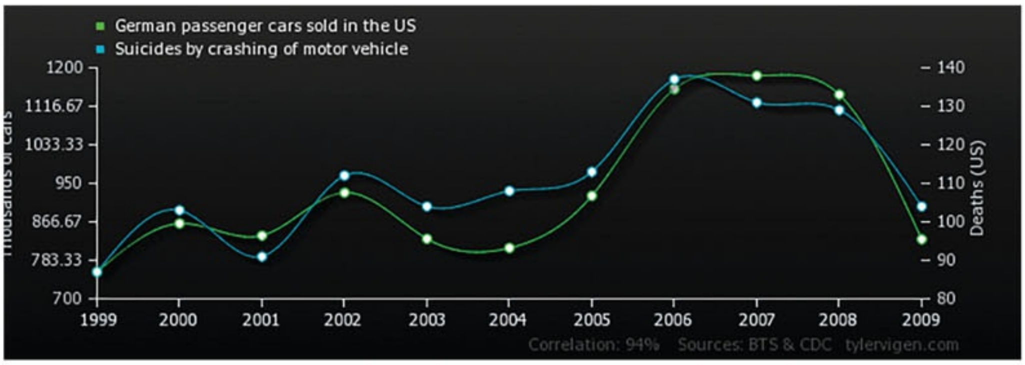
3 Height and self-esteem have a correlation coefficient of 0.32.

4 Temperature and growth of grass have a correlation coefficient of –0.75.

5 Number of hours study and results in the HSC have a correlation coefficient of 0.85.

Collect data from the Internet, newspapers or magazines, or do your own experiments to compare two sets of data. Draw a scatterplot and find the correlation coefficient, then, if there is a high correlation investigate causality. Is the correlation positive or negative? Is it linear?

Here is an example of a high correlation of totally unrelated variables: German cars sold vs suicides by car crashes in the US each year.



http://tylervigen.com/page?page=2

There are many other examples of correlated data that have no causality.

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8.03 Line of best fit

Statistical data is rarely perfect, but we can often see trends in a scatterplot. If there seems to be a linear correlation, we can draw a regression line and find its equation. We can then use this line to make predictions.

The easiest regression line to find is the line of best fit.

Using a ruler, we draw the line that represents as many points as possible. We try to draw a line where about half the points are above the line and half are below it, so that the distance between the line and the points is kept to a minimum.

WS Lines of best fit

WS Lines of fit

WS Height vs shoe size

EXAMPLE 8

A ball is rolled down a ramp and its velocity is measured over time.

t (s)	1	2	3	4	5	6
v (m s^{-1})	0.7	1.5	3.6	6.1	7.8	9.9

a Draw a scatterplot of the data and draw a line of best fit.

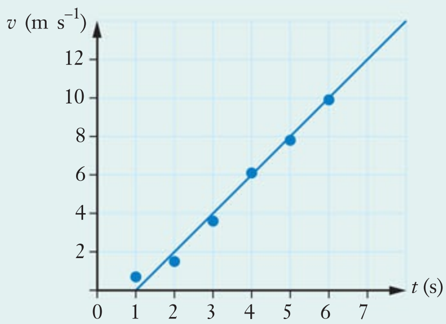
b Use the line of best fit to find the velocity after 3.5 seconds.

c Find the equation of this line.

d Use the equation to find the velocity after 10 seconds.

Solution

a



b After 3.5 seconds, the velocity is 5 ms^{-1}.

c Choose 2 points on the line, say, (2, 2) and (4, 6).

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6-2}{4-2} = \frac{4}{2} = 2$$

Equation: $y - y_1 = m(x - x_1)$

$v - 2 = 2(t - 2)$ using $m = 2$ and (2, 2)

$= 2t - 4$

$v = 2t - 2$

d When $t = 10$:

$v = 2(10) - 2$

$= 18$

So after 10 seconds, the velocity is 18 m s^{-1}.

Interpolation and extrapolation

Interpolation is using a model to make predictions about values lying within the range of the original data set.

Extrapolation is using a model to make predictions about values outside the range of the original data set.

In the above example, finding v when $t = 3.5$ is interpolation while finding v when $t = 10$ is extrapolation.

DID YOU KNOW?

Regression

The word regression comes from the Latin regressio, meaning 'a return'.

Sir Francis Galton (1822–1911) was the first to use this name. He created the concepts of correlation and regression, and used statistics to develop questionnaires and surveys on human differences.

Research correlation, regression and Sir Francis Galton.

CLASS DISCUSSION

EXTRAPOLATION

Extrapolation is not always accurate. For example, a formula may work well at first, then the conditions may change in a way that means the formula is no longer a good model.

Think of examples of bivariate data from the previous examples. Can you always extrapolate answers from a line of best fit? Why? What are the risks?

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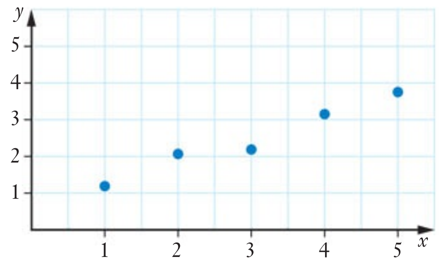
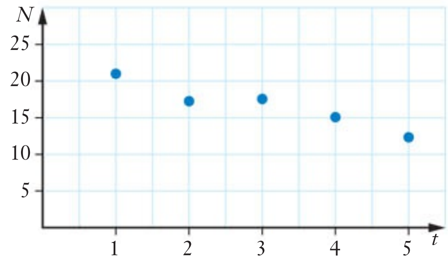
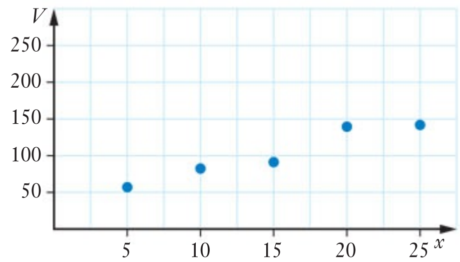
Exercise 8.03 Line of best fit

1 Copy each scatterplot, draw a line of best fit and find its equation.

a


b


c


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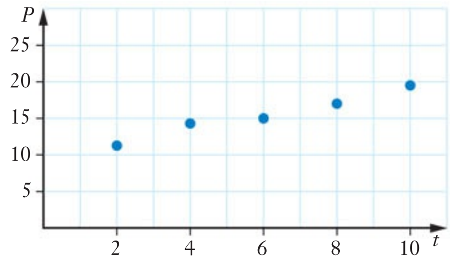

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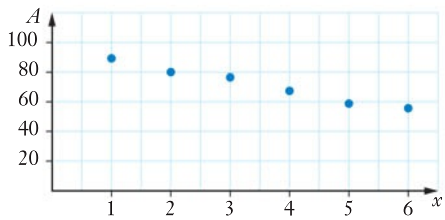

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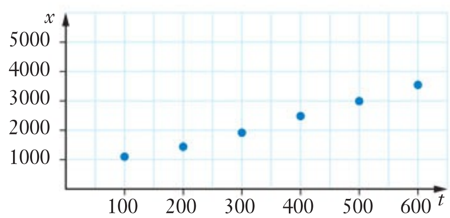

2 The following table shows the results of an experiment testing the temperature of a liquid as it cools down.

t (min)	5	10	15	20	25	30
T (°C)	87	78	69	56	53	41

a Plot this data on a number plane and draw a line of best fit.

b Find the equation of the line.

c Use the equation to find the temperature after:

  i 17 minutes

  ii 35 minutes

d If room temperature is 23°C, is the line of best fit a good model for the cooling of the liquid? Why?

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3 The population of birds in a particular area was sampled over several years.

t (years)	1	2	3	4	5	6
P	1030	983	968	954	915	899

a Plot this data on a number plane and draw a line of best fit.

b Find the equation of the line.

c Use the equation to find the population of birds after 7 years.

d At this rate of decline, after how many years would you expect there to be no more birds in this area?

4 The effect of a dose of medicine on a child's temperature over time was measured from a sample of children, with the following results.

t (min)	2	4	6	8	10
T (°C)	39.5	39.1	38.9	38.2	37.7

a Plot this data on a number plane and draw a line of best fit.

b Find the equation of the line.

c Use the equation to find the child's temperature after 15 minutes.

d Is this equation reliable as a measure of temperature after a longer time, say, 1hour?

5 This table shows the results of a survey into the number of people who attend a new restaurant over a number of weeks.

t (weeks)	1	2	3	4	5	6
No. of people	76	114	163	187	228	274

a Draw a scatterplot and sketch a line of best fit.

b Find the equation of this line.

c Use the equation to find the number of people you would expect to attend the restaurant after 10 weeks.

d Is this equation a good model for the number of people attending the restaurant?

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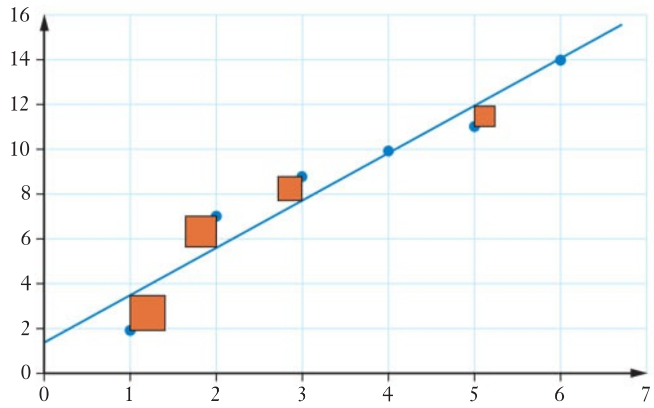
8.04 Least-squares regression line

WS Least-squares regression line

WS Least-squares regression line

The line of best fit relies on our eyes and ruler for its accuracy.

There are several different models of regression lines that try to give a more accurate result. The most popular model is called the least-squares regression line. It uses a line of best fit for which the squares of the distances from each point in the scatterplot to the line are minimised (see the diagram below). Squaring the distances takes away any negative values (a similar technique to finding standard deviation).



The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\dfrac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$, where r = correlation coefficient, $\bar{x}$ and $\bar{y}$ are the sample means and s_x and s_y are the sample standard deviations.

However, you don't need to use these formulas because the regression line can be found using a scientific calculator, graphics calculator, online calculator or spreadsheet.

EXAMPLE 9

The table shows the results of a survey into the number of cigarettes people smoke during the year and the number of days they are absent from work.

Cigarettes/year	2000	3000	4000	5000	6000	7000	8000	9000	10 000
Absences	23	27	54	49	63	81	107	128	147

Find the least-squares regression line by using:

a a calculator

b a spreadsheet

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Solution

a

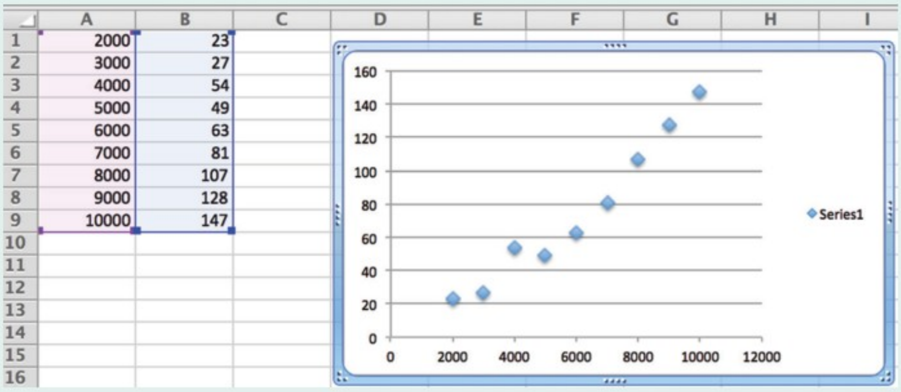
Operation	Casio scientific	Sharp scientific
Enter data.	SHIFT 1 2 : Data 2000 = 3000 = etc. for 1st column 23 = 27 = etc. for 2nd column AC	2000 2ndF STO 23 M+ 3000 2ndF STO 27 M+ etc.
Calculate a.	SHIFT 1 5 : Reg 1 : A =	ALPHA a =
Calculate b.	SHIFT 1 5 : Reg 2 : B =	ALPHA b =

$a = -18.26$, $\quad b = 0.0156$

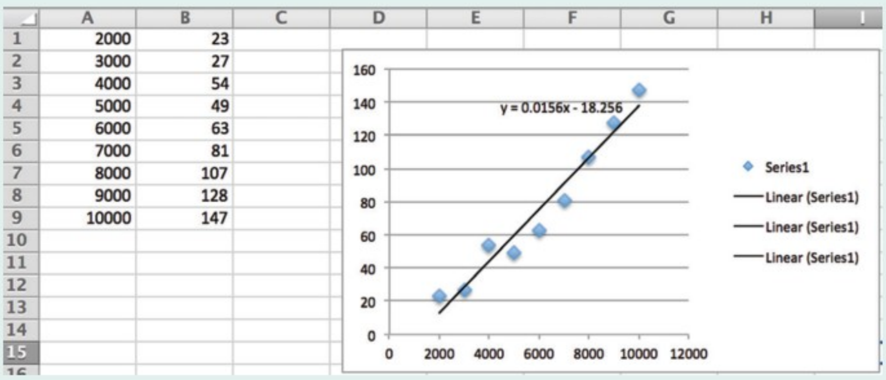
On a scientific calculator, the equation is in the form $y = a + bx$ or $y = bx + a$

So $y = 0.0156x - 18.26$

b Enter the 2 columns in a spreadsheet and draw a scatterplot.



In Chart Layout select Trendline, then Linear trendline. You can display the equation of the line by going to Trendline again, selecting Trendline options and selecting Display equation on chart.



The equation of the least-squares regression line is $y = 0.0156x - 18.256$.

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Exercise 8.04 Least-squares regression line

1 For each data set, find:

i the correlation coefficient

ii the gradient of the least-squares regression line

a

x	1	2	3	4	5	6	7
y	3	4	7	10	11	15	17

b

x	2	4	6	8	10	12	14	16
y	8	11	19	29	34	41	45	67

2 For each data set:

i find the equation of the least-squares regression line

ii sketch the scatterplot and regression line on the same axes

a Age of machine and breakdown rates

Age (years)	1	2	3	4	5
Breakdowns	0	2	5	9	15

b Length of time in office and popularity of a political party

Time (years)	1	2	3	4	5	6
Popularity (%)	52.3	43.8	43.7	42.1	37.9	37.6

c Length of drought and yield of crops

Time (years)	1	2	3	4	5	6
Yield (t)	107.3	101.8	100.2	87.6	63.5	47.1

d Engine size of cars and number of accidents

Size (L)	1.3	1.8	2.0	2.1	2.4	3.8
Accidents	459	447	513	519	506	625

e Age and number wearing glasses

Age	10	20	30	40	50	60	70	80
Glasses	34	28	41	56	87	105	156	209

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3 A block of ice was taken out of a freezer and left to thaw. The results are in the table below.

Time t (min)	5	10	15	20	25	30	35	40
Mass m (kg)	23.7	18.8	11.3	8.7	6.2	5.5	2.3	1.5

a Find the correlation coefficient. Is there a high correlation? Is it positive or negative? What does this mean?

b Find the equation of the least-squares regression line.

c Use the equation to estimate the mass of the ice after:

i 18 minutes ii an hour

d Discuss why extrapolation may not be useful in this situation.

4 This table shows the weight of gemstones sold at an auction and their selling price.

Weight (carat)	0.05	0.8	1.5	1.7	2.5	2.8	3.1	4.0
Price ($)	144	672	1245	1478	2100	2500	2881	3215

a Draw a scatterplot for this data.

b Find the correlation coefficient. Is there a high correlation between the weight of a gemstone and its cost? Is it positive or negative?

c Find the equation of the least-squares regression line.

d How much would you expect to pay for a 2 carat gemstone?

e How much would you expect a gemstone to weigh if it cost $10 000?

5 The table shows the results of a survey into ages and earnings of a group of people.

Age	15	32	19	28	43	67
Earnings/week ($)	689	1205	840	1154	1587	986

a Draw a scatterplot for this data.

b Find the correlation coefficient.

c Find the equation of the least-squares regression line.

d From this equation find the earnings of a 50-year-old.

e Is this equation a good model to extrapolate?

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8. TEST YOURSELF

Practice quiz

Fitting the data find-a-word

For Questions 1 to 5, select the correct answer A, B, C or D.

1 Which variables are not correlated?

A Hand size and height

B Hours of study and exam results

C Height and hours of employment

D Speed of car and fuel economy

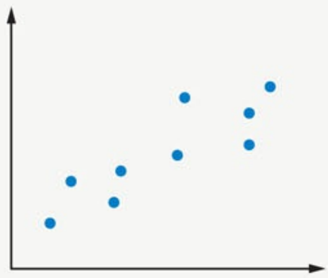
2 Describe the correlation in the scatterplot shown.

A Weak negative correlation

B Moderate positive correlation

C Moderate negative correlation

D Strong positive correlation



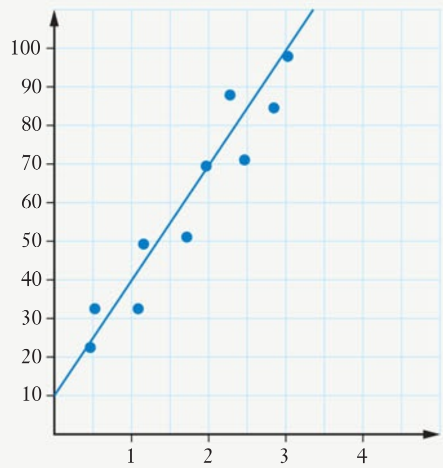
3 Find the equation of the line of best fit.

A $y = 60x + 10$

B $y = 10x + 30$

C $y = 30x - 10$

D $y = 30x + 10$



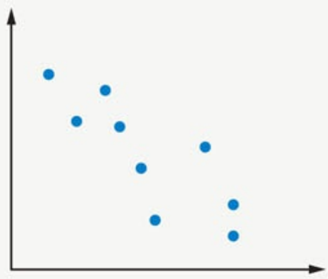
4 Estimate the correlation coefficient of this bivariate data.

A -0.5

B 0.5

C -1

D 1



5 Using the equation of a line of best fit to predict the value of a variable within the domain of the data set is called:

A extrapolation

B causality

C interpolation

D correlation

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6 Make a scatterplot of this table of bivariate data, then:

x	1	2	3	4	5	6	7	8	9
y	17	21	24	29	36	43	44	52	58

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation.

7 For each scatterplot, state whether it has:

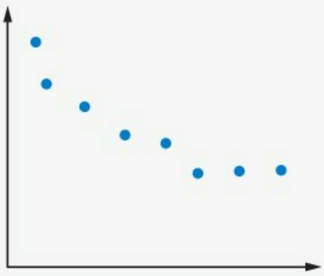
A a positive linear correlation

B a negative linear correlation

C a non-linear correlation

D little or no correlation

a



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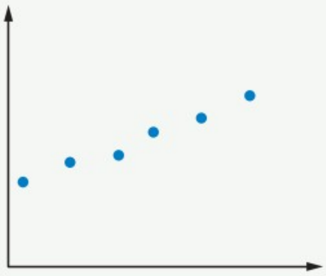

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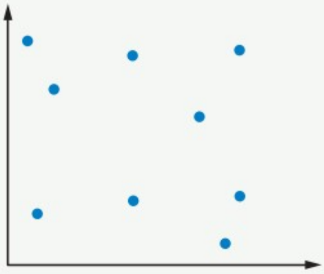

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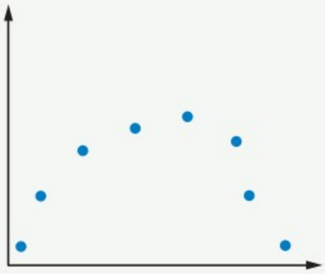

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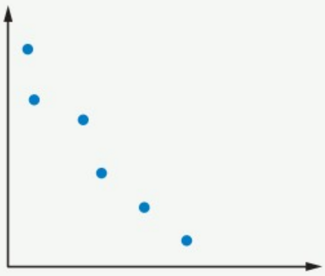

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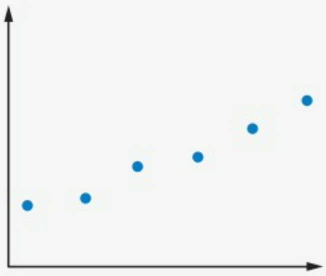

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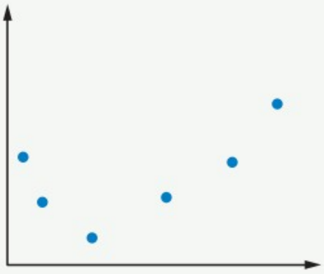

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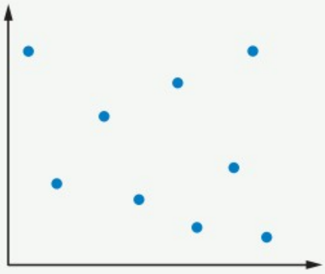

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8 A group of students was surveyed to find whether there was a correlation between a student's music and maths assessment marks. The results are in the table below.

Music	79	58	91	93	65	43	39	64	82	51
Maths	62	63	82	79	73	57	29	52	76	40

a Find the correlation coefficient.

b Find the equation of the least-squares regression line.

c Using the equation, find the maths assessment mark for a student who scores 60 in music.

d Find the music assessment mark for a student who scores 70 in maths.

9 Determine whether each pair of variables are likely to have a causal relationship.

a Height and amount of food eaten

b Number of years playing sport and weight

c Height and arm length

d Number of chickens and number of eggs

e Size of bookshelves and number of books

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8. CHALLENGE EXERCISE

1 The equation of the least-squares regression line is given by $y = mx + c$ where $m = r\dfrac{s_y}{s_x}$ and $c = \bar{y} - m\bar{x}$, where r = correlation coefficient, $\bar{x}$ and $\bar{y}$ are the sample means and s_x and s_y are the sample standard deviations. Evaluate r and $\bar{y}$ given the equation of the least-squares regression line is $y = 2x + 4$, $\bar{x} = 1.2$, $s_x = 1.8$ and $s_y = 4.5$.

2 Sketch a scatterplot that shows a linear correlation of:

a -1 b approximately 0.8 c 0

d approximately -0.2 e 1

3 The table below shows the results of an experiment into the volume of water evaporating from a body of water at different temperatures.

T (°C)	10	15	20	25	30	35	40	45
V (L)	0.5	1.3	2.9	5.8	10.3	15.7	39.8	76.1

a Draw a scatterplot to show this data.

b Why would a least-squares regression line not give a good approximation for this data?

c Use technology or otherwise to find an equation that might approximately model this data.

4 The table shows heights and shoe sizes of several males.

Shoe size	4	5	6	7	8	9	10	11
Height (m)	1.54	1.65	1.68	1.73	1.59	1.82	1.89	1.95

a Draw a scatterplot to show this data.

b Find the correlation coefficient.

c Find the equation of the least-squares regression line.

d The male in the sample with shoe size 8 is found to be an outlier for this data. For the sample without this outlier, find:

i the correlation coefficient

ii the equation of the least-squares regression line

e Is this equation a good model for shoe sizes and height?

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Practice set 3

In Questions 1 to 5, select the correct answer A, B, C or D.

1 Find the median of 3, 10, 1, 4, 9, 6.

A 5 B 1 C 4 D 6

2 Which pair of variables are not correlated?

A Size of house and size of family

B Height and foot size

C Number of babies born and number of nappies used

D Distance travelled and average speed over 2 hours

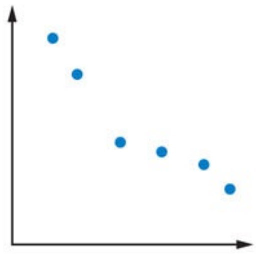
3 Describe the correlation in the scatterplot.

A Strong positive linear correlation

B Moderate positive linear correlation

C Moderate negative linear correlation

D Strong negative linear correlation



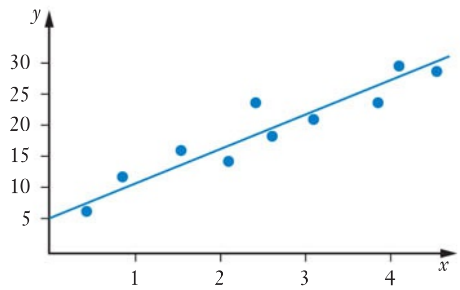
4 The equation of this line of best fit is closest to:

A $y = 10x + 5$

B $y = 5x - 5$

C $y = 10x - 5$

D $y = 5x + 5$



5 The correlation coefficient of the bivariate data shown on this scatterplot is closest to:

A −0.5

B 0.5

C −1

D 1

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6 These scores are the results of a maths quiz:

7, 9, 5, 8, 9, 5, 6, 8, 7, 9, 5, 5, 7, 6, 5

a Complete a frequency distribution table for these scores.

b Draw a frequency polygon and histogram for this data.

c Draw a cumulative frequency histogram and polygon.

d Find the median.

e Find the interquartile range.

f Draw a box plot to show the five-number summary for this data.

7 For this table of bivariate data:

x	1	2	3	4	5	6	7	8
y	18	23	29	38	41	46	52	60

a draw a line of best fit and find its equation

b draw a least-squares regression line and find its equation

8 Shoppers were asked what they most liked about the shopping centre. Draw a Pareto chart for the survey results.

Variety of shops	34
Amenities	11
Child-friendly	28
Parking	27

9 Find the mean, standard deviation and variance of the scores 3, 5, 9, 8, 6, 5, 8, 7.

10 The table shows students' scores on a maths test.

a Find the mean and standard deviation.

b Show that the score of 3 is an outlier.

c Find the mean and standard deviation excluding the outlier.

Score	Frequency
3	1
4	0
5	0
6	2
7	4
8	6
9	8
10	3

11 Find the mean, mode, median and range of the scores 8, 9, 4, 7, 6, 5, 6.

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12 The table below shows the results of a survey into the ratings of a TV show.

Rating	Frequency
1	1
2	7
3	15
4	19
5	10

a How many people were surveyed?

b Find correct to 2 decimal places the mean and standard deviation.

c Find the median.

d Draw a box plot for these results.

13 Find the correlation coefficient for this set of data, correct to 2 decimal places.

x	3	7	4	8	12	2
y	15	11	9	8	7	18

14 Draw an example of statistical data that is:

a positively skewed

b negatively skewed

c symmetrical

d bimodal

e multimodal

15 Describe each data set as categorical nominal, categorical ordinal, quantitative discrete or quantitative continuous.

a Types of trees planned for a park

b Length of road between towns

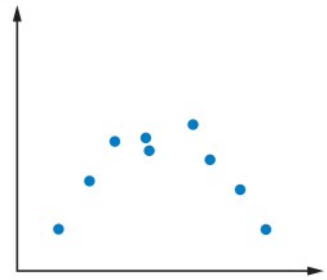
c Survey ratings of Poor, Good, Very good, Excellent

d Dress sizes

e Test scores

16 Match each scatterplot to its description.

a



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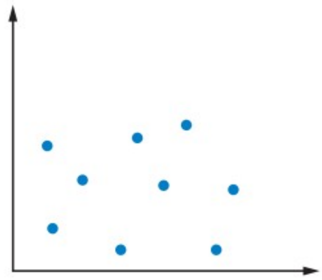

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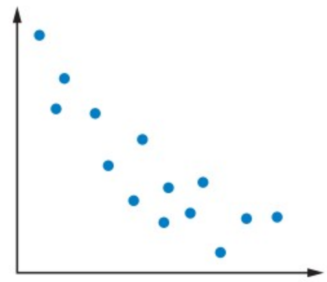

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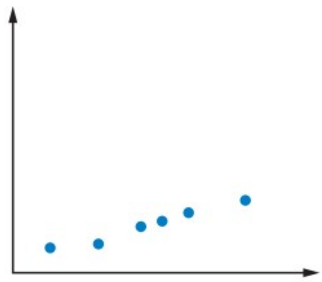

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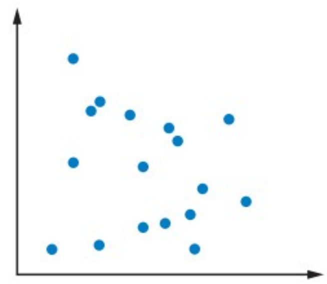

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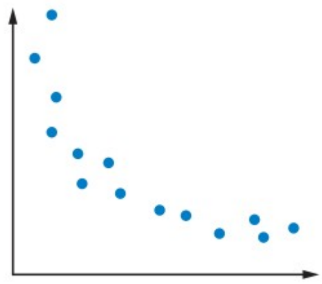


A positive linear correlation

B negative linear correlation

C non-linear correlation

D no correlation

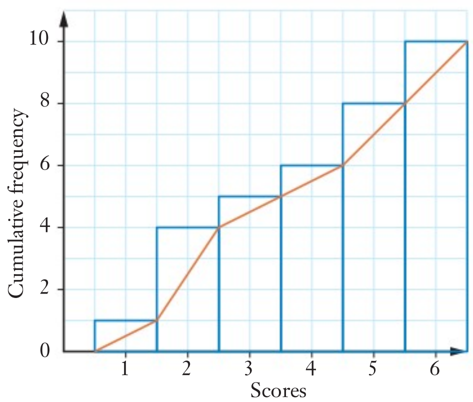
17 For this cumulative frequency polygon, find:

a the median

b the first quartile

c the 60th percentile

d the 4th decile



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18 Find all stationary points and points of inflection on the graph of the function $f(x) = 2x^3 - 6x^2 - 48x + 17$.

19 Solve each equation for $[0, 2\pi]$.

a $2 \sin x = 1$

b $\tan^2 x = 1$

c $2 \cos 2x + 1 = 0$

20 For the sequence 3, 7, 11, … find:

a the 100th term

b the sum of the first 100 terms

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FINANCIAL MATHEMATICS

9. INVESTMENTS, ANNUITIES AND LOANS

Series and sequences have many applications. Financial mathematics is an important part of everyday living as we put money in the bank, pay off credit cards, take out superannuation, buy houses and cars, and many other things. In this chapter you will study the finances of investments, annuities and loans and see how they relate to series.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- identify arithmetic and geometric growth and decay
- solve practical problems of growth and decay
- solve problems involving compound interest investments using repeated calculations, tables and formulas
- solve problems involving annuities using repeated calculations, tables and geometric series
- solve problems involving reducing balance loans using repeated calculations, tables and geometric series

TERMINOLOGY

annuity: An investment for a fixed period of time where payments are made or received regularly.

compound interest: The interest earned on both the principal and previous interest payments of an investment.

future value: The total value at the close of an investment including all payments and interest earned.

present value: A single payment (called the principal) that will produce a future value over a given time.

reducing balance loan: A loan that is repaid by making regular payments with interest calculated on the amount still owing (the reducing balance of the loan) after each payment.

superannuation: A fixed portion of income that is invested regularly to provide a lump sum or pension when a person retires from the paid workforce; an example of an annuity.

WS
Arithmetic growth and decay

9.01 Arithmetic growth and decay

We can use arithmetic sequences and series to describe growth (increase) and decay (decrease) in practical problems. This is sometimes called discrete linear growth and decay.

EXAMPLE 1

A stack of cans on a display at a supermarket has 5 cans on the top row. The next row down has 2 more cans and the next one has 2 more cans and so on.

a Calculate the number of cans in the 11th row down.

b If there are 320 cans in the display altogether, how many rows are there?

Solution

a The first row has 5 cans, the 2nd row has 7 cans, the 3rd row 9 cans and so on. This forms an arithmetic sequence with $a = 5$ and $d = 2$.

For the 11th row, we want $n = 11$:

$$T_n = a + (n - 1)d$$
$$T^{11} = 5 + (11 - 1) \times 2$$
$$= 5 + 10 \times 2$$
$$= 25$$

So there are 25 cans in the 11th row.

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b If there are 320 cans altogether, this is the sum of cans in all rows.

$$S_n = 320$$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$
$$320 = \frac{n}{2}[2 \times 5 + (n-1) \times 2]$$
$$= \frac{n}{2}(10 + 2n - 2)$$
$$= \frac{n}{2}(2n + 8)$$
$$= n^2 + 4n$$

$$0 = n^2 + 4n - 320$$
$$= (n - 16)(n + 20)$$
$$n - 16 = 0, \quad n + 20 = 0$$
$$n = 16, \quad n = -20$$

Since n must be a positive integer, then $n = 16$.

There are 16 rows of cans.

We could substitute $n = 16$ into S_n to check this.

Exercise 9.01 Arithmetic growth and decay

1 A market gardener plants daffodil bulbs in rows, starting with a row of 45 bulbs.
Each successive row has 5 more bulbs than the row before.

a Calculate the number of bulbs in the 34th row.

b Which row would be the first to have more than 100 bulbs in it?

c The market gardener plants 10 545 bulbs altogether. How many rows are there?

Photo courtesy Margaret Grove

2 A stack of logs has 1 on the top, then 3 on the next row down, and each successive row has 2 more logs than the one on top of it.

a How many logs are in the 20th row?

b Which row has 57 logs?

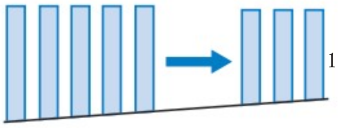
c If there are 1024 logs altogether, how many rows are in the stack?

3 A set of books is stacked in layers, where each layer contains 3 books fewer than the layer below. There are 6 books in the top layer, 9 in the next layer, 12 in the next and so on. There are n layers altogether.

a Write down the number of books in the bottom layer.

b Show that there are $\frac{3}{2}n(n+3)$ books in the stack altogether.

4 A timber fence is to be built on sloping land, with the shortest piece of timber 1.2 m and the longest 1.8 m. There are 61 pieces of timber in the fence.

1.8 m

1.2 m

a What is the difference in height between each piece of timber?

b Assuming no wastage, what length of timber is needed for the fence altogether?

5 A sculpture consists of a set of poles set in a row, with the tallest pole 2.4 m high, the next pole 2.1, the next one 1.8 and so on, down to the last pole which is 0.6 m high.

a How many poles are in the sculpture?

b The poles are made of timber. What length of timber is there altogether in the poles?

6 Johanna has \$2000 in a term deposit that earns simple interest of 2.5% p.a. How much money does she have, including interest, after:

a 1year? b 2years? c 3 years?

d 10 years? e 30 years?

7 Each house in a row of terraced houses is to have a new fence. The houses are on a hill so the first fence will be 1 m high, the second will be 1.05 m high, the third 1.1 m high and so on.

a How high will the fence need to be for the 6th house?

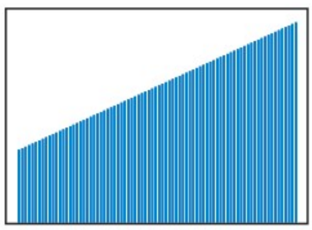
b If the height of the last fence is 1.35 m, how many houses are there?

8 At a courier company, there are different price categories for different weights of parcels. The 1st category is parcels in the range 0–0.5 kg, then 0.5–1 kg, then 1–1.5 kg and so on.

a What is the 10th weight category?

b Which category is 8.5–9 kg?

9 A logo is made with vertical lines equally spaced as shown. The shortest line is 25 mm, the longest is 217 mm, and the sum of the lengths of all the lines is 5929 mm.

25 mm

217 mm

a How many lines are in the logo?

b Find the difference in length between adjacent lines.

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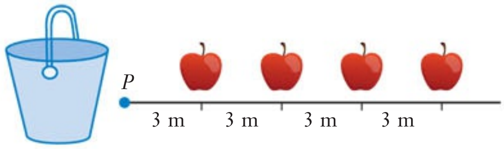
10 In a game, a child starts at point P and runs and picks up an apple 3 m away. She then runs back to P and puts the apple in a bucket. The child then runs to get the next apple 6 m away, and runs back to P to place it in the bucket. This continues until she has all the apples in the bucket.



a How far does the child run from P to pick up the kth apple?

b How far does the child run to fetch all k apples, including return trips to P?

c The child runs 270 m to fetch all the apples and return them to the bucket. How many apples are there?

9.02 Geometric growth and decay

WS
Geometric growth and decay
WS
Practical applications of series

We can use geometric sequences and series to describe growth and decay in practical problems. This is called geometric growth and decay. It is also called exponential growth and decay because $T_n = ar^{n-1}$ is an exponential function. You studied exponential growth and decay involving e^x in the Year 11 course, Chapter 8, Exponential and logarithmic functions.

EXAMPLE 2

A layer of tinting for a car window lets in 95% of light.

a What percentage of light is let in by:

i 2 layers of tinting? ii 3 layers of tinting? iii 10 layers of tinting?

b How many layers will let in 40% of light?

Solution

a i 1 layer lets in 95% of light.

So 2 layers lets in 95% × 95% of light.

$$\begin{aligned} 95\% \times 95\% &= 0.95 \times 0.95 \\ &= 0.9025 \\ &= 90.25\% \end{aligned}$$

So 2 layers lets in 90.25% of light.

ii 1 layer lets in 95% or 0.95 of light.

2 layers lets in $0.95 \times 0.95 = 0.95^2$ of light.

3 layers lets in $0.95^2 \times 0.95 = 0.95^3$ of light.

$$\begin{aligned} 0.95^3 &\approx 0.857 \\ &= 85.7\% \end{aligned}$$

So 3 layers lets in 85.7% of light.

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iii The number of layers forms the geometric sequence $0.95, \; 0.95^2, 0.95^3, \ldots$ with $a = 0.95, r = 0.95$.

For 10 layers, $n = 10$.

$$T_n = ar^{n-1}$$
$$\begin{aligned} T_{10} &= 0.95(0.95)^{10-1} \\ &= 0.95(0.95)^9 \\ &= 0.95^{10} \\ &\approx 0.5987 \\ &= 59.87\% \end{aligned}$$

So 10 layers lets in 59.87% of light.

b We want to find n when the nth term is 40% or 0.4.

$$\begin{aligned} T_n &= ar^{n-1} \\ 0.4 &= 0.95(0.95)^{n-1} \\ &= 0.95^n \\ \log 0.4 &= \log 0.95^n \\ &= n \log 0.95 \\ \frac{\log 0.4}{\log 0.95} &= n \\ 17.9 &\approx n \end{aligned}$$

So around 18 layers of tinting will let in 40% of light.

EXAMPLE 3

A car bought for \$35 000 depreciates (loses value) by 12% p.a.

a Find its value after:

i 1 year ii 2 years iii 3 years

b Write the value of the car as a sequence.

c Find what the car is worth after 10 years.

d When will the value of the car drop below \$15 000? Answer to the nearest year.

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Solution

a i After 1 year the car is worth \$35 000 – 12% of \$35 000.

$$
\begin{aligned}
\$35\,000 - 12\% \text{ of } \$35\,000 &= \$35\,000(1 - 12\%)\\
&= \$35\,000(1 - 0.12)\\
&= \$35\,000(0.88)\\
&= \$30\,800
\end{aligned}
$$

So the car is worth \$30 800 after 1 year.

ii After 2 years the car is worth \$30 800 – 12% of \$30 800.

$$
\begin{aligned}
\$30\,800 - 12\% \text{ of } \$30\,800 &= \$30\,800(1 - 12\%)\\
&= \$30\,800(0.88)\\
&= \$27\,104
\end{aligned}
$$

So the car is worth \$27 104 after 2 years.

iii After 3 years the car is worth \$27 104 – 12% of \$27 104.

$$
\begin{aligned}
\$27\,104 - 12\% \text{ of } \$27\,104 &= \$27\,104(0.88)\\
&= \$23\,851.52
\end{aligned}
$$

So the car is worth \$23 851.52 after 3 years.

b 1st year = \$30 800, 2nd year = \$27 104, 3rd year = \$23 851.52, …

So 30 800, 27 104, 23 851.52, … is a geometric sequence with $a = 30\,800$ and $r = 0.88$.

c When $n = 10$:

$$
\begin{aligned}
T_n &= ar^{n-1}\\
T_{10} &= 30\,800\,(0.88)^{10-1}\\
&= 30\,800\,(0.88)^9\\
&= 9747.53
\end{aligned}
$$

So the car is worth \$9747.53 after 10 years.

d We want $T_n < 15\,000$.

$$30\,800(0.88)^{n-1} < 15\,000$$
$$0.88^{n-1} < 0.487$$
$$\log 0.88^{n-1} < \log 0.487$$
$$(n-1)\log 0.88 < \log 0.487$$
$$(n-1) > \frac{\log 0.487}{\log 0.88} \quad \text{(The inequality reverses because } \log x < 0 \text{ for } 0 < x < 1\text{)}$$
$$n > \frac{\log 0.487}{\log 0.88} + 1$$
$$> 6.63$$

We can substitute $n = 7$ into T_n to check this.

So it will take approximately 7 years for the car to be worth less than \$15 000.

We can also use the limiting sum to model some types of problems.

EXAMPLE 4

a Write $0.\dot{5}$ as a fraction.

b A ball is dropped from a height of 1 metre and bounces up to $\frac{1}{3}$ of its height. It continues bouncing, rising $\frac{1}{3}$ of its height on each bounce.

i Draw a diagram showing the motion.

ii What is the total distance through which the ball travels?

Solution

a $0.\dot{5} = 0.5555555\ldots$

$$= \frac{5}{10} + \frac{5}{100} + \frac{5}{1000} + \ldots$$

This is a geometric series with $a = \frac{5}{10} = \frac{1}{2}$ and $r = \frac{1}{10}$.

$$S = \frac{a}{1-r}$$
$$= \frac{\frac{1}{2}}{1 - \frac{1}{10}}$$
$$= \frac{\frac{1}{2}}{\frac{9}{10}}$$
$$= \frac{5}{9}$$

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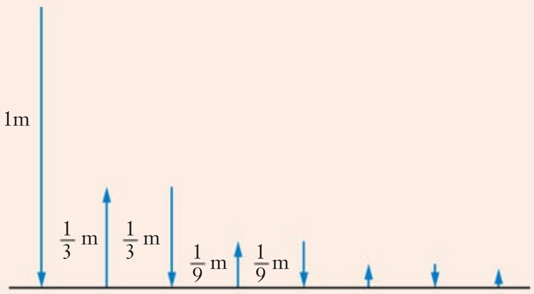
b i



ii Notice that there is a series for the ball coming downwards and another series upwards. There is more than one way of calculating the total distance. Here is one way of solving it.

$$\text{Total distance} = 1 + \frac{1}{3} + \frac{1}{3} + \frac{1}{9} + \frac{1}{9} + \frac{1}{27} + \frac{1}{27} + \ldots$$

$$= 1 + 2\left(\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots\right)$$

$\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots$ is a geometric series with $a = \frac{1}{3}$ and $r = \frac{1}{3}$

$$S = \frac{a}{1-r}$$

$$= \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$= \frac{1}{2}$$

$$\text{Total distance} = 1 + 2\left(\frac{1}{2}\right)$$

$$= 2$$

So the ball travels 2 metres altogether.

INVESTIGATION

LIMITING SUM APPLICATIONS

1 In the above example, in theory will the ball ever stop?

2 Kim owes \$1000 on her credit card. If she pays back 10% of the amount owing each month, she will never finish paying it off. Is this true or false?

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Exercise 9.02 Geometric growth and decay

1 Water evaporates from a pond at an average rate of 7% each week.

a What percentage of water is left in the pond after:

i 1 week? ii 2 weeks? iii 3 weeks?

b What percentage is left after 15 weeks?

c If there was no rain, approximately how long (to the nearest week) would it take for the pond to only have 25% of its water left?

2 The price of shares in a particular company is falling by an average of 2% each day.

a What percentage of their initial value do they have after 2 days?

b Approximately how many days will it take for the shares to halve in value?

c After how many days will the shares be worth 10% of their initial value?

3 A painting appreciates (increases its value) by 16% p.a. It is currently worth \$20 000.

a How much will it be worth in:

i 1 year? ii 2 years? iii 3 years?

b How much will it be worth in 11 years?

c How long will it take for it to be worth \$50 000?

4 A southern brown bandicoot population in Western Australia is decreasing by 5% each year.

a What percentage of the population is left after 5 years?

b After how many years will the population be only 50% of its current level?

c How many years will it take for the population to decrease by 80%?

5 Write each recurring decimal as a fraction.

a $0.\dot{4}$ b $0.\dot{7}$ c $1.\dot{2}$ d $0.\dot{2}\dot{5}$ e $2.\dot{8}\dot{1}$

f $0.2\dot{3}$ g $1.4\dot{7}$ h $1.01\dot{5}$ i $0.1\dot{3}\dot{2}$ j $2.\dot{3}6\dot{1}$

6 A frog jumps 0.5 metres. It then jumps 0.1 m and on each subsequent jump it travels 0.2 of the previous distance. Find the total distance through which the frog jumps.

7 A tree grows by $\frac{4}{5}$ of each previous year's growth.

If it was initially 3 m high, find the ultimate height of the tree.

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8 An 8 cm seedling grows by 4.8 cm in the first week, and then keeps growing by 0.6 of its previous week's growth. How tall will it grow?

9 An object rolls 0.5 m in the first second. Then each second after, it rolls by $\frac{5}{6}$ of its previous roll. Find how far it will roll altogether.

10 A 100 m cliff erodes by $\frac{2}{7}$ of its height each year.

a What will the height of the cliff be after 10 years?

b After how many years will the cliff be less than 50 m high?

11 A lamb grows by $\frac{2}{5}$ of its previous growth each month. If a lamb is 45 cm tall:

a how tall will it be after 6 months?

b what will its final height be?

Photo courtesy Margaret Grove

12 A weight on an elastic string drops down 60 cm and then bounces back to $\frac{2}{3}$ of its initial height. It keeps bouncing, each time rising back to $\frac{2}{3}$ of its previous height. What is the total distance through which the weight travels?

13 Mary bounces a ball, dropping it from 1.5 m on its first bounce. It then rises up to $\frac{2}{5}$ of its height on each bounce. Find the distance through which the ball travels.

14 A roadside wall has a zigzag pattern on it as shown. The two longest lines are each 2 m long, then the next two lines are $1\frac{3}{4}$ m long and the lines in each subsequent pair are $\frac{7}{8}$ of the length of the previous pair. Find the total length of the lines.

15 Frankie receives a text message that she is asked to send to 8 friends. Frankie forwards this text on to 8 friends and each of them sends it on to 8 friends, and so on.

a Describe the number of people receiving the text as a sequence (including Frankie's text).

b How many people would receive the message in the 9th round of texts?

c How many people would have received the text altogether if it is sent 9 times?

9.03 Compound interest

An investment is money that is put into the bank or used to pay for something that will increase in value, or appreciate, in the future. An investment can include real estate, art, jewellery or antiques.

The amount invested is called the present value or principal. The amount the investment is worth after a period of time is called the future value.

Future value of investments (FV)

Money in the bank earns interest. Some investments earn simple interest, but most earn compound interest. For example, a term deposit or investment account gives the option of adding the interest back into the account (compound interest) or taking the interest as cash or another investment (simple interest).

EXAMPLE 5

Mahmoud invests \$7000 into a term deposit account for 3 years, where it earns 5% p.a.

a How much interest does he earn over the 3 years if the interest is paid into the term deposit account at the end of each year?

b What is the future value of the investment?

Solution

a Interest is $5\% = 0.05$

The interest is added to the principal each time it is paid.

$$\begin{aligned}\text{Amount after 1 year} &= \$7000 + 0.05 \times \$7000\\ &= \$7000(1 + 0.05)\\ &= \$7000(1.05)\\ &= \$7350\end{aligned}$$

$$\begin{aligned}\text{Amount after 2 years} &= \$7350 + 0.05 \times \$7350\\ &= \$7350(1.05)\\ &= \$7717.50\end{aligned}$$

$$\begin{aligned}\text{Amount after 3 years} &= \$7717.50 + 0.05 \times \$7717.50\\ &= \$7717.50(1.05)\\ &\approx \$8103.38\end{aligned}$$

$$\begin{aligned}\text{Amount of interest} &= \$8103.38 - \$7000\\ &= \$1103.38\end{aligned}$$

b The future value is \$8103.38.

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We also use compound interest to calculate the future value of other investments.

EXAMPLE 6

Rachel and Wade buy a house in Sydney for \$1 250 000. House prices in that area go up by an average of 11.5% p.a. What is their house worth after 2 years?

Solution

Interest is $11.5\% = 0.115$

$$\begin{aligned}\text{Amount after 1 year} &= \$1\,250\,000 + 0.115 \times \$1\,250\,000\\ &= \$1\,250\,000(1 + 0.115)\\ &= \$1\,250\,000(1.115)\\ &= \$1\,393\,750\end{aligned}$$

$$\begin{aligned}\text{Amount after 2 years} &= \$1\,393\,750(1.115)\\ &= \$1\,554\,031.25\end{aligned}$$

So value after 2 years is \$1 554 031.25

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Compound interest tables simplify calculations. The values in the table next page are called future value interest factors as they give the future values of an investment of \$1 at a certain interest rate and time.

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Future value interest factors on $1							
Periods	1%	2%	5%	8%	10%	15%	20%
1	1.0100	1.0200	1.0500	1.0800	1.1000	1.1500	1.2000
2	1.0201	1.0404	1.1025	1.1664	1.2100	1.3225	1.4400
3	1.0303	1.0612	1.1576	1.2597	1.3310	1.5209	1.7280
4	1.0406	1.0824	1.2155	1.3605	1.4641	1.7490	2.0736
5	1.0510	1.1041	1.2763	1.4693	1.6105	2.0114	2.4883
6	1.0615	1.1262	1.3401	1.5869	1.7716	2.3131	2.9860
7	1.0721	1.1487	1.4071	1.7138	1.9487	2.6600	3.5832
8	1.0829	1.1717	1.4775	1.8509	2.1436	3.0590	4.2998
9	1.0937	1.1951	1.5513	1.9990	2.3579	3.5179	5.1598
10	1.1046	1.2190	1.6289	2.1589	2.5937	4.0456	6.1917
11	1.1157	1.2434	1.7103	2.3316	2.8531	4.6524	7.4301
12	1.1268	1.2682	1.7959	2.5182	3.1384	5.3503	8.9161
13	1.1381	1.2936	1.8856	2.7196	3.4523	6.1528	10.6993
14	1.1495	1.3195	1.9799	2.9372	3.7975	7.0757	12.8392
15	1.1610	1.3459	2.0789	3.1722	4.1772	8.1371	15.4070
16	1.1726	1.3728	2.1829	3.4259	4.5950	9.3576	18.4884
17	1.1843	1.4002	2.2920	3.7000	5.0545	10.7613	22.1861
18	1.1961	1.4282	2.4066	3.9960	5.5599	12.3755	26.6233

To see how the table works, we can use the example of Mahmoud's term deposit.

EXAMPLE 7

Mahmoud invests \$7000 into a term deposit account for 3 years where it earns 5% p.a. Use the table to find the future value of the investment.

Solution

From the table:

For 3 years, $n = 3$ and interest is 5%.
Finding the column for 3 years at 5% gives 1.1576.
1.1576 is the future value on \$1.

$$\begin{aligned}\text{So future value on } \$7000 &= \$7000 \times 1.1576 \\ &= \$8103.20\end{aligned}$$

The table gives a slightly different answer from Example 6 because the future value interest factors are rounded to 4 decimal places. So using a table is quicker but not as accurate!

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We can use the table for investments of less than a year. The value of n stands for time periods, not years.

EXAMPLE 8

Stephanie invests \$2000 into a term deposit account for 5 months where it earns 12% p.a., paid monthly. Use the table to find the future value of the investment.

Solution

Interest is 12% p.a.

p.a. or per annum means each year

So interest per month = 12% ÷ 12 = 1%

The value across from n = 5 months in the 1% column is 1.0510

Future value on \$2000 = \$2000 × 1.0510

= \$2102

Present value of investments (PV)

Sometimes you want to know how much you would need to invest now to end up with a certain amount in the future. For example, you may be saving up for a holiday or a deposit for a house. This value you need to invest now to achieve a future value is called the present value.

EXAMPLE 9

Geordie wants to invest enough money now so that he will have \$5000 in 4 years' time to buy a car. Use the table of future value interest factors to calculate how much present value he would need to invest if the interest rate is 5% p.a.

Solution

We use n = 4 and 5%. We know FV = 5000 and we want to find the present value.

Let PV = x

From the table, the value across from n = 4 in the 5% column is 1.2155.

This is the future value on \$1.

So FV = x × 1.2155 or 1.2155x

But FV = 5000

So 1.2155x = 5000

$$x = \frac{5000}{1.2155}$$

$$= 4113.53$$

So Geordie needs to invest a present value of \$4113.53 to have \$5000 in 4 years' time.

Exercise 9.03 Compound interest

1 Calculate the future value if \$6500 is invested for:

a 2 years at 3% p.a.
b 3 years at 2.5% p.a.
c 4 years at 4.1% p.a.
d 3 years at 1.8% p.a.
e 2 years at 5.3% p.a.

2 Calculate the future value of each investment.

a \$2500 for 3 years at 4.5% p.a.
b \$10 000 for 4 years at 6.2% p.a.
c \$3400 for 5 years at 3.5% p.a.
d \$5000 for 3 years at 6% p.a.
e \$80 000 for 2 years at 4.5% p.a.

3 Christian and Kate buy a house for \$750 000. What is the house worth after 3 years if its value increases by 6% p.a.?

4 Aparna bought a diamond ring for \$3000. How much was it worth 3 years later if it appreciated by 5.8% p.a.?

5 A painting bought for \$15 000 appreciates by 9% p.a. What is its future value after 4 years?

6 The present value of a necklace is \$950. What is its future value after 4 years if it appreciates at 3% p.a.?

7 Hien deposits \$4500 into a tour fund where it earns interest of 2.9% p.a. What will be the future value of the tour fund after 3 years?

8 Use the table of future value interest factors on page 380 to calculate the future value of each investment.

a \$800 for 7 years at 5% p.a.
b \$2000 for 10 years at 1% p.a.
c \$5000 for 6 years at 20% p.a.
d \$60 000 for 5 years at 10% p.a.
e \$100 000 for 8 years at 15% p.a.
f \$673.25 for 6 years at 5% p.a.
g \$1249.53 for 4 years at 1% p.a.
h \$3000 for 3 months at 12% p.a., paid monthly
i \$1000 for 6 months at 12% p.a. paid monthly
j \$3500 for 10 months at 12% p.a. paid monthly

9 Use the table of future value interest factors on page 380 to calculate the present value if the future value is \$10 000 after:

a 7 years at 2% p.a.
b 5 years at 15% p.a.
c 10 years at 8% p.a.
d 3 years at 1% p.a.
e 4 years at 5% p.a.

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9.04 Compound interest formula

The calculations on compound interest follow a pattern called a recurrence relation.

WS Compound interest

EXAMPLE 10

Patrick invests \$2000 at the beginning of the year at 6% p.a. Find a formula for the amount in the bank at the end of n years.

Solution

$6\% = 0.06$

Amount after 1 year:

$$\begin{aligned} A_1 &= \$2000 + 0.06 \text{ of } \$2000 \\ &= \$2000(1 + 0.06) \\ &= \$2000(1.06) \end{aligned}$$

Amount after 2 years:

$$\begin{aligned} A_2 &= A_1 + 0.06A_1 \\ &= [\$2000(1.06)] + 0.06 \times [\$2000(1.06)] \\ &= [\$2000(1.06)](1 + 0.06) \\ &= [\$2000(1.06)](1.06) \\ &= \$2000(1.06)^2 \end{aligned}$$

Amount after 3 years:

$$\begin{aligned} A_3 &= A_2 + 0.06A_2 \\ &= [\$2000(1.06)^2] + 0.06 \times [\$2000(1.06)^2] \\ &= [\$2000(1.06)^2](1 + 0.06) \\ &= [\$2000(1.06)^2](1.06) \\ &= \$2000(1.06)^3 \end{aligned}$$

The recurrence relation is $A_{n+1} = A_n + 0.06A_n$

$A_1, A_2, A_3, \ldots$ is a geometric sequence with $a = 2000(1.06)$ and $r = 1.06$.

$$\begin{aligned} T_n &= ar^{n-1} \\ &= 2000(1.06)(1.06)^{n-1} \\ &= 2000(1.06)^n \end{aligned}$$

So the amount after n years is $\$2000(1.06)^n$.

Compound interest

$A = P(1 + r)^n$

where P = principal (present value)

r = interest rate per period, as a decimal

n = number of periods

A = future value

EXAMPLE 11

Find the amount that will be in the bank after 6 years if \$2000 is invested at 12% p.a. with interest paid:

a yearly   b quarterly   c monthly

Solution

$P = 2000$

a $r = 12\% = 0.12, n = 6$

$$\begin{aligned} A &= P(1 + r)^n \\ &= 2000(1 + 0.12)^6 \\ &= 2000(1.12)^6 \\ &= 3947.65 \end{aligned}$$

So the amount is \$3947.65.

b For quarterly interest, the annual interest rate is divided by 4.

$r = 0.12 \div 4 = 0.03$

Interest is paid 4 times a year.

$n = 6 \times 4 = 24$

$$\begin{aligned} A &= P(1 + r)^n \\ &= 2000(1 + 0.03)^{24} \\ &= 2000(1.03)^{24} \\ &= 4065.59 \end{aligned}$$

So the amount is \$4065.59.

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c For monthly interest, the annual interest rate is divided by 12.

$r = 0.12 \div 12 = 0.01$

Interest is paid 12 times a year.

$n = 6 \times 12 = 72$

$$\begin{aligned}A &= P(1 + r)^n \\ &= 2000(1 + 0.01)^{72} \\ &= 2000(1.01)^{72} \\ &= 4094.20\end{aligned}$$

So the amount is \$4094.20.

We can find the present value using the compound interest formula.

EXAMPLE 12

Geoff wants to invest enough money to pay for a \$10 000 holiday in 7 years' time. If interest is 2.5% p.a., what present value does Geoff need to invest now?

Solution

$A = 10\,000$, $r = 2.5\%$ or 0.025, $n = 7$

We want to find the present value P.

$$\begin{aligned}A &= P(1 + r)^n \\ 10\,000 &= P\,(1 + 0.025)^7 \\ &= P\,(1.025)^7\end{aligned}$$

$$\frac{10000}{1.025^7} = P$$

$$P \approx 8412.65$$

The present value to invest is \$8412.65.

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We can use the compound interest formula to find the interest rate or time period by rearranging the formula.

EXAMPLE 13

a Silvana invested \$1800 at 6% p.a. interest and it grew to \$2722.66.
For how many years was the money invested if interest was paid twice a year?

b Find the interest rate if a \$1500 investment is worth \$1738.91 after 5 years.

Solution

a $P = 1800$ and $A = 2722.66$.

Interest is paid twice a year:

$r = 0.06 \div 2 = 0.03$

$$\begin{aligned} A &= P(1 + r)^n \\ 2722.66 &= 1800(1 + 0.03)^n \\ &= 1800(1.03)^n \\ \frac{2722.66}{1800} &= 1.03^n \\ 1.51259 &= 1.03^n \\ \log(1.51259) &= \log(1.03)^n \\ &= n \log(1.03) \\ \frac{\log(1.51259)}{\log(1.03)} &= n \\ 14 &\approx n \end{aligned}$$

Since interest is paid in twice a year, the number of years will be $14 \div 2 = 7$.

So the money was invested for 7 years.

b $P = 1500$, $A = 1738.91$, $n = 5$

$$\begin{aligned} A &= P(1 + r)^n \\ 1738.91 &= 1500(1 + r)^5 \\ \frac{1738.91}{1500} &= (1 + r)^5 \\ 1.15927 &= (1 + r)^5 \\ \sqrt[5]{1.15927} &= 1 + r \\ 1.03 &= 1 + r \\ 0.03 &= r \\ r &= 3\% \end{aligned}$$

So the interest rate is 3% p.a.

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Exercise 9.04 Compound interest formula

1 Find the amount of money in the bank after 10 years if:

a $500 is invested at 4% p.a.

b $7500 is invested at 7% p.a.

c $8000 is invested at 8% p.a.

d $5000 is invested at 6.5% p.a.

e $2500 is invested at 7.8% p.a.

2 Sam banks $1500 where it earns interest at the rate of 6% p.a. Find the amount after 5 years if interest is paid:

a annually

b twice a year

c quarterly

3 Chantelle banks $3000 in an account that earns 5% p.a. Find the amount in the bank after 10 years if interest is paid:

a quarterly

b monthly

4 Reza put $350 in the bank where it earns interest of 8% p.a. Find the amount there will be in the account after 2 years if interest is paid:

a annually

b monthly

5 How much money will there be in an investment account after 3 years if the present value is $850 and interest of 4.5% p.a. is paid:

a twice ayear?

b quarterly?

6 Find the amount of money there will be in a bank after 8 years if $1000 earns interest of 7% p.a. with interest paid:

a twice a year

b quarterly

c monthly

7 Tanya left $2500 in a credit union account for 4 years, with interest of 5.5% p.a. paid yearly.

a How much money did she have in the account at the end of that time?

b What would be the difference in the future value if interest was paid quarterly?

8 a Find the amount of money there will be after 15 years if Hannah banks $6000 and it earns 9% p.a. interest, paid quarterly.

b How much more money will Hannah have than if interest was paid annually?

9 How much money will be in a bank account after 5 years if $500 earns 6.5% p.a. with interest paid monthly?

10 Find the amount of interest earned over 4 years if $1400 earns 6% p.a. paid quarterly.

11 How much money will be in a credit union account after 8 years if $8000 earns 7.5% p.a. interest paid monthly?

12 Elva wins a lottery and invests $500 000 in an account that earns 8% p.a. with interest paid monthly. How much will be in the account after 12 years?

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13 Calculate the principal invested for 4 years at 5% p.a. to achieve a future value of:

a $5000 b $675 c $12 000 d $289.50 e $12 800

14 What present value is required to accumulate to $5400 in 3 years with interest of 5.8% p.a. paid quarterly?

15 How many years ago was an investment made if $5000 was invested at 6% p.a. paid monthly and it is now worth $6352.45?

16 Find the number of years that $10 000 was invested at 8% p.a. with interest paid twice a year if there is now $18 729.81 in the bank.

17 Jude invested $4500 five years ago at $x\%$ p.a. Evaluate x if the amount in his bank account is now:

a $6311.48 b $5743.27 c $6611.98 d $6165.39 e $6766.46

18 Hamish is given the choice of a bank account in which interest is paid annually or quarterly. If he deposits $1200, find the difference in the amount of interest paid over 3 years if interest is 7% p.a.

19 Kate has $4000 in a bank account that pays 5% p.a. with interest paid annually, and Rachel has $4000 in a different account paying 4% quarterly. Which person will receive more interest over 5 years, and by how much?

20 A bank offers investment account A at 8% p.a. with interest paid twice a year and account B with interest paid at 6% p.a. at monthly intervals. If Georgia invests $5000 over 6 years, which account pays more interest? How much more does it pay?

21 A hairdresser earns $36 400 for the first year of work. His salary increases each year by 2%.

a What is his salary in his:
 i 5th year of work? ii 8th year of work?

b When will his salary reach $60 000?

22 Yuron earns $120 000 in his 1st year, then his salary goes up by 3.5% each year after that.

a How much does Yuron earn in his:
 i 3rd year? ii 12th year? iii 20th year?

b What is the first year in which Yuron earns over $300 000?

23 Masae invests $5000 in a bank account.

a How many years at 2% p.a. interest will it take for her investment to grow to:
 i $5410? ii $7000?

b At what interest rate would the investment grow to $6000 after:
 i 6 years? ii 10 years?

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24 a Use the table of future value interest factors on page 380 to find the interest factor for an investment on $1 over 9 years at 8% p.a.

b Prove this interest factor is correct by using the compound interest formula.

25 Show that the future value interest factor of 1.5209 is true for an investment over 3 years at 15% p.a.

9.05 Annuities

WS Investments and annuities

WS Annuity table problems

WS Annuity: spreadsheet

A better way to build up money faster is to make regular contributions to an investment. This is called an annuity, a name that comes from the same Latin word 'annus' as annual, meaning yearly. However contributions to an annuity could be made more frequently than this. For example, payments into superannuation can be made every week or fortnight when an employee is paid.

Future value of an annuity

EXAMPLE 14

Stevie's grandparents put $100 into a bank account for her on her first birthday. They deposit $100 into the account on each birthday until Stevie is 18 and give her the total amount for her 18th birthday.

How much is in the account at the end of 3 years if interest is 2% p.a.?

Solution

Amount at the end of the 1st year:

Since $100 is deposited at the end of the 1st year on Stevie's 1st birthday, it earns no interest in that year.

$A_1 = \$100$

Amount at the end of the 2nd year:

$$
\begin{aligned}
A_2 &= \$100 + 0.02 \times \$100 \\
&= \$100(1 + 0.02) \\
&= \$100(1.02) \\
&= \$102
\end{aligned}
$$

But another $100 is deposited at the end of the 2nd year on Stevie's 2nd birthday.

$$
\begin{aligned}
\text{So amount} &= \$102 + \$100 \\
&= \$202
\end{aligned}
$$

Amount at the end of the 3rd year:

$$
\begin{aligned}
A_3 &= \$202(1.02) \\
&= \$206.04
\end{aligned}
$$

But another $100 is deposited at the end of the 3rd year on Stevie's 3rd birthday.

$$
\begin{aligned}
\text{So amount} &= \$206.04 + \$100 \\
&= \$306.04
\end{aligned}
$$

So after 3 years the annuity is worth $306.04.

This is the $300 put in by Stevie's grandparents plus interest of $6.04.

WS Future value table for annuities

Future value of an annuity with a contribution of $1 at the end of each period

Periods	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	2.0100	2.0200	2.0300	2.0400	2.0500	2.0600	2.0700	2.0800	2.0900	2.1000	2.1100	2.1200	2.1300	2.1400	2.1500
3	3.0301	3.0604	3.0909	3.1216	3.1525	3.1836	3.2149	3.2464	3.2781	3.3100	3.3421	3.3744	3.4069	3.4396	3.4725
4	4.0604	4.1216	4.1836	4.2465	4.3101	4.3746	4.4399	4.5061	4.5731	4.6410	4.7097	4.7793	4.8498	4.9211	4.9934
5	5.1010	5.2040	5.3091	5.4163	5.5256	5.6371	5.7507	5.8666	5.9847	6.1051	6.2278	6.3528	6.4803	6.6101	6.7424
6	6.1520	6.3081	6.4684	6.6330	6.8019	6.9753	7.1533	7.3359	7.5233	7.7156	7.9129	8.1152	8.3227	8.5355	8.7537
7	7.2135	7.4343	7.6625	7.8983	8.1420	8.3938	8.6540	8.9228	9.2004	9.4872	9.7833	10.0890	10.4047	10.7305	11.0668
8	8.2857	8.5830	8.8923	9.2142	9.5491	9.8975	10.2598	10.6366	11.0285	11.4359	11.8594	12.2997	12.7573	13.2328	13.7268
9	9.3685	9.7546	10.1591	10.5828	11.0266	11.4913	11.9780	12.4876	13.0210	13.5795	14.1640	14.7757	15.4157	16.0853	16.7858
10	10.4622	10.9497	11.4639	12.0061	12.5779	13.1808	13.8164	14.4866	15.1929	15.9374	16.7220	17.5487	18.4197	19.3373	20.3037
11	11.5668	12.1687	12.8078	13.4864	14.2068	14.9716	15.7836	16.6455	17.5603	18.5312	19.5614	20.6546	21.8143	23.0445	24.3493
12	12.6825	13.4121	14.1920	15.0258	15.9171	16.8699	17.8885	18.9771	20.1407	21.3843	22.7132	24.1331	25.6502	27.2707	29.0017
13	13.8093	14.6803	15.6178	16.6268	17.7130	18.8821	20.1406	21.4953	22.9534	24.5227	26.2116	28.0291	29.9847	32.0887	34.3519
14	14.9474	15.9739	17.0863	18.2919	19.5986	21.0151	22.5505	24.2149	26.0192	27.9750	30.0949	32.3926	34.8827	37.5811	40.5047
15	16.0969	17.2934	18.5989	20.0236	21.5786	23.2760	25.1290	27.1521	29.3609	31.7725	34.4054	37.2797	40.4175	43.8424	47.5804
16	17.2579	18.6393	20.1569	21.8245	23.6575	25.6725	27.8881	30.3243	33.0034	35.9497	39.1899	42.7533	46.6717	50.9804	55.7175
17	18.4304	20.0121	21.7616	23.6975	25.8404	28.2129	30.8402	33.7502	36.9737	40.5447	44.5008	48.8837	53.7391	59.1176	65.0751
18	19.6147	21.4123	23.4144	25.6454	28.1324	30.9057	33.9990	37.4502	41.3013	45.5992	50.3959	55.7497	61.7251	68.3941	75.8364
19	20.8109	22.8406	25.1169	27.6712	30.5390	33.7600	37.3790	41.4463	46.0185	51.1591	56.9395	63.4397	70.7494	78.9692	88.2118
20	22.0190	24.2974	26.8704	29.7781	33.0660	36.7856	40.9955	45.7620	51.1601	57.2750	64.2028	72.0524	80.9468	91.0249	102.4436
21	23.2392	25.7833	28.6765	31.9692	35.7193	39.9927	44.8652	50.4229	56.7645	64.0025	72.2651	81.6987	92.4699	104.7684	118.8101
22	24.4716	27.2990	30.5368	34.2480	38.5052	43.3923	49.0057	55.4568	62.8733	71.4027	81.2143	92.5026	105.4910	120.4360	137.6316
23	25.7163	28.8450	32.4529	36.6179	41.4305	46.9958	53.4361	60.8933	69.5319	79.5430	91.1479	104.6029	120.2048	138.2970	159.2764
24	26.9735	30.4219	34.4265	39.0826	44.5020	50.8156	58.1767	66.7648	76.7898	88.4973	102.1742	118.1552	136.8315	158.6586	184.1678
25	28.2432	32.0303	36.4593	41.6459	47.7271	54.8645	63.2490	73.1059	84.7009	98.3471	114.4133	133.3339	155.6196	181.8708	212.7930
26	29.5256	33.6709	38.5530	44.3117	51.1135	59.1564	68.6765	79.9544	93.3240	109.1818	127.9988	150.3339	176.8501	208.3327	245.7120
27	30.8209	35.3443	40.7096	47.0842	54.6691	63.7058	74.4838	87.3508	102.7231	121.0999	143.0786	169.3740	200.8406	238.4993	283.5688
28	32.1291	37.0512	42.9309	49.9676	58.4026	68.5281	80.6977	95.3388	112.9682	134.2099	159.8173	190.6989	227.9499	272.8892	327.1041
29	33.4504	38.7922	45.2189	52.9663	62.3227	73.6398	87.3465	103.9659	124.1354	148.6309	178.3972	214.5828	258.5834	312.0937	377.1697
30	34.7849	40.5681	47.5754	56.0849	66.4388	79.0582	94.4608	113.2832	136.3075	164.4940	199.0209	241.3327	293.1992	356.7868	434.7451

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It is assumed that annuity payments are made at the end of each period, unless stated otherwise. If they are made at the beginning of each period, then the calculations would be different.

You can use the table on the previous page to calculate annuities. You can also download a copy from NelsonNet.

EXAMPLE 15

Stevie's grandparents put \$100 into a bank account for her on each birthday, with the final deposit on her 18th birthday. They give Stevie the total amount of the money for her 18th birthday.

a Use the table of future value of annuities factors to calculate how much is in the account after 3 years if interest is 2% p.a.

b How much will Stevie receive on her 18th birthday?

Solution

a From the table:

The value across from n = 3 years in the 2% column is 3.0604

This is the future value on \$1.

Future value on \$100 = \$100 × 3.0604

= \$306.04

b The value across from n = 18 years in the 2% column is 21.4123.

Future value on \$100 = \$100 × 21.4123

= \$2141.23

So Stevie will receive \$2141.23 on her 18th birthday.

We can use the table of future values of an annuity to calculate how much to contribute regularly to achieve a particular future value.

EXAMPLE 16

a Christopher wants to save a certain amount at the end of each year for 5 years until he has \$20 000 to buy a car. If the interest rate is 3% p.a., find the amount of each annual contribution Christopher needs to make.

b Alexis wants to save up a \$50 000 deposit for a home over 7 years. She wants to make contributions at the end of each quarter. Interest is 8% p.a., paid quarterly. What size contribution would she make?

Solution

a From the future value for annuities table on the previous page, the value across from n = 5 in the 3% column is 5.3091.

If we call the contribution x:

Future value = x × 5.3091

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So $5.3091x = 20\,000$

$$x = \frac{20000}{5.3091}$$
$$= 3767.12$$

So each contribution is \$3767.12

b The contribution is quarterly, or 4 times a year.

Interest rate = 8% ÷ 4 = 2%

Alexis makes 4 contributions each year for 7 years.

Number of periods = $7 \times 4 = 28$

From the table, the value across from $n = 28$ in the 2% column is 37.0512.

If we call the contribution x:

Future value = $x \times 37.0512$

So $37.0512x = 50\,000$

$$x = \frac{50000}{37.0512}$$
$$= 1349.48$$

So each contribution is \$1349.48.

Annuities with regular withdrawals

Another type of annuity is a sum of money earning compound interest that has regular withdrawals or payouts coming out of it.

EXAMPLE 17

Yasmin retires with a lump sum superannuation payment of \$145 000. She puts the money into a financial management company that guarantees 12% p.a. on her annuity, with interest paid monthly. Yasmin withdraws \$1800 at the end of each month as a pension.

Find what Yasmin's annuity is worth after 3 months.

Solution

Monthly interest = 12% ÷ 12 = 1%

Amount at the end of 1st month = $\$145\,000(1 + 0.01)^1$

$= \$145\,000(1.01)$

$= \$146\,450$

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But Yasmin withdraws \$1800

So amount = \$146 450 – \$1800

= \$144 650

Amount at the end of 2nd month = $\$144\,650(1.01)^1$

= \$146 096.50

But Yasmin withdraws \$1800

So amount = \$146 096.50 – \$1800

= \$144 296.50

Amount at the end of 3rd month = $\$144\,296.50(1.01)^1$

= \$145 739.47

But Yasmin withdraws \$1800

So amount = \$144 739.47 – \$1800

= \$143 939.47

So after 3 months Yasmin's annuity is worth \$143 939.47.

Notice that Yasmin's annuity is gradually decreasing. If she took a little less money out each month, she could keep the value of her annuity at around \$145 000 or increase its value a little. Try doing the above example with different values to see if Yasmin could draw a pension while keeping her lump sum the same.

TECHNOLOGY

Annuities and spreadsheets

1 The formula for the future value of an annuity is: $FV = a\left[\frac{(1+r)^n - 1}{r}\right]$ where a = regular contribution, r = interest rate and n = number of periods.

You can use this formula to draw up a spreadsheet for future values of an annuity.

Does this formula look familiar? It comes from the sum of a geometric series.

We can use this formula to write a table of future values in a spreadsheet.

Using rows 1 and 2 for headings, we can put 1 in A3 and the formula =1+A3 in A4. Drag this formula down the column for the periods 1, 2, 3, …

We will use $a = 1$ and $r = 0.05$ (5% interest)

In B3 put the formula =((1+0.05)^A3–1)/0.05 and drag it down the column.

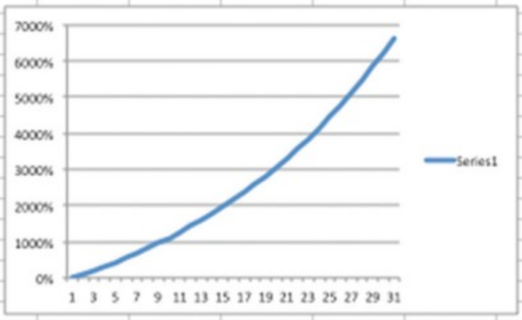
This gives a set of values for the future value of an annuity at 5% p.a.

Now highlight the column of future values and select the line graph from Charts.

Can you find future values from the graph?

Change the formula to a different interest rate. For example, use 0.08 instead of 0.05 in the formula and drag it down the column. How does this change the graph? Try other interest rate changes and look at how the graph changes.

	A	B
1	ture value of annuities	
2	Periods	5%
3	1	1.0000
4	2	2.0500
5	3	3.1525
6	4	4.3101
7	5	5.5256
8	6	6.8019
9	7	8.1420
10	8	9.5491
11	9	11.0266
12	10	12.5779
13	11	14.2068
14	12	15.9171
15	13	17.7130
16	14	19.5986
17	15	21.5786
18	16	23.6575
19	17	25.8404
20	18	28.1324
21	19	30.5390
22	20	33.0660
23	21	35.7193
24	22	38.5052
25	23	41.4305
26	24	44.5020
27	25	47.7271
28	26	51.1135
29	27	54.6691
30	28	58.4026
31	29	62.3227
32	30	66.4388



2 Use the formula $PV = a\left[\frac{(1+r)^n - 1}{r(1+r)^n}\right]$ for the present value of an annuity to draw up a spreadsheet and graph for present value interest factors using similar steps.

For example, for 5% interest, in B3 use the formula =((1.05)^A3–1)/(0.05*(1.05)^A3) for the value of 1 in A3, then drag the formula down the column.

How does the graph change if you change the interest rate?

Exercise 9.05 Annuities

1 Calculate the future value of an annuity with a yearly contribution (at the end of each year) of:

a $5000 for 2 years at 2.5% p.a.

b $1200 for 3 years at 4% p.a.

c $875 for 3 years at 3.6% p.a.

d $10 000 for 2 years at 4.1% p.a.

e $2000 for 3 years at 3.2% p.a.

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Use the future value table for annuities on page 390 to answer Questions 2 to 7.

2 Find the future value of an annuity with annual contributions of:

a $6300 for 7 years at 4% p.a.

b $980 for 5 years at 6% p.a.

c $7500 for 10 years at 5% p.a.

d $495.75 for 4 years at 3% p.a.

e $20 500 for 12 years at 2% p.a.

f $647.12 for 6 years at 1% p.a.

g $800 for 15 years at 7% p.a.

h $598 for 14 years at 9% p.a.

i $15 000 for 11 years at 13% p.a.

j $160 000 for 8 years at 4% p.a.

3 Find the future value of an annuity with contributions of:

a $400 a month for 2 years at 12% p.a. paid monthly

b $940 a quarter for 5 years at 8% p.a. paid quarterly

c $2500 twice a year for 8 years at 14% p.a. paid every 6 months

d $550 three times a year for 5 years at 6% p.a. paid every 4 months

e $587 a month for 18 months at 12% p.a. paid monthly

4 At the end of each year, Alicia puts $3500 into a superannuation fund where it earns 9% p.a. How much will she have in superannuation after 30 years?

5 The future value of an annuity is $35 000 after 12 years. If interest is 6% p.a., find the amount of each yearly contribution.

6 Find the amount of each annual contribution needed to give a future value of:

a $8450 after 5 years at 7% p.a.

b $25 000 after 8 years at 3% p.a.

c $10 000 after 7 years at 4% p.a.

d $3200 after 5 years at 2% p.a.

e $1 000 000 after 20 years at 5% p.a.

7 Emlynn wants to put aside a regular amount of money each month for 2 years at 12% p.a., paid monthly, so she will have $8000 to pay for a film-making course. How much will she need to contribute?

8 Ilona wins $50 000 in a lottery and invests it in a holiday fund annuity where she withdraws $5000 at the end of each year. The annuity pays interest of 4% p.a.

a What is the value of her annuity after:

i 1 year? ii 2 years? iii 3 years?

b What will the annuity be worth after 3 years if Ilona decides to withdraw $4000 each year instead?

c What will the annuity be worth after 3 years if the interest is 2.7% and Ilona withdraws $4000 each year?

9 Dave puts his \$125 000 superannuation payout into an annuity and takes out a pension of \$500 a month. The annuity pays interest of 12% p.a., paid monthly.

What is the value of the annuity after:

a 1 month? b 2 months? c 3 months?

e 3 months if interest is 6% p.a.?

d 3 months if Dave decides to take a pension of \$1000 each Month?

10 Graph A shows an annuity of \$15 000 earning 1% interest per month, with a regular withdrawal of \$500 per month. Graph B shows the same \$15 000 annuity paying interest of 2% per month with a regular withdrawal of \$500 each month.

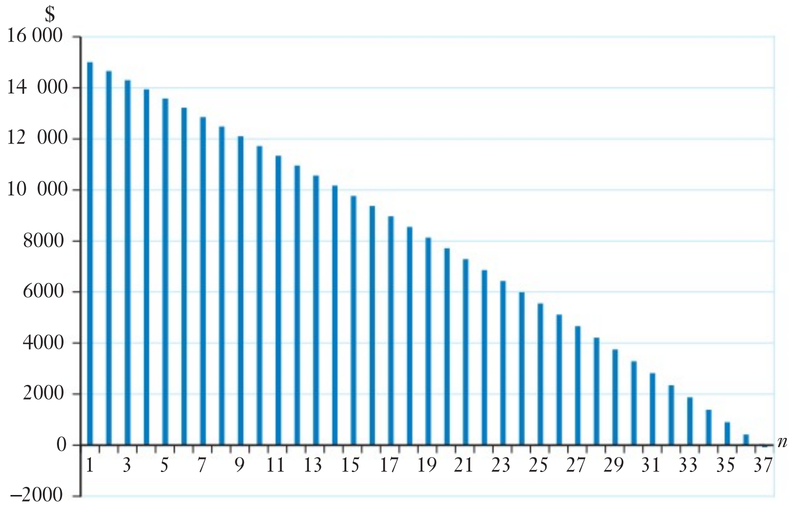
For each graph, determine:

a after how many months the annuity will be worth \$8000

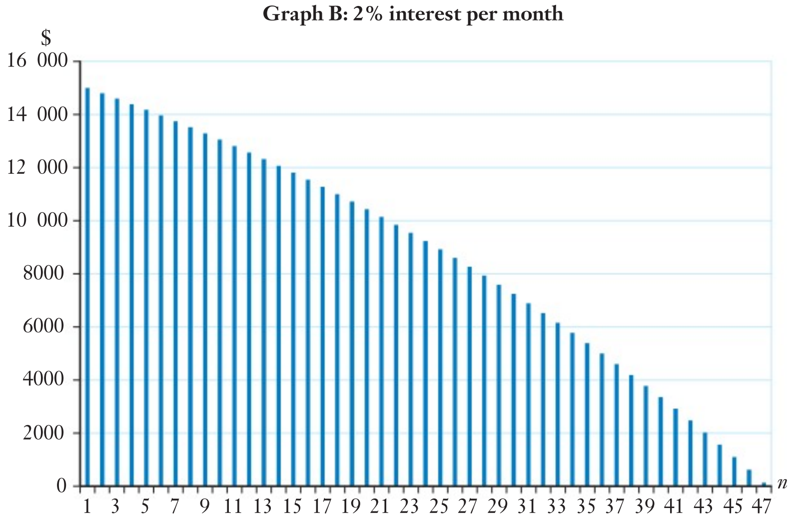
b what the annuity will be worth after a year

c how long it will take for the annuity to run out

Graph A: 1% interest per month



Graph B: 2% interest per month



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Use the future value table on p. 390 to answer Questions 11 and 12.

11 Ryuji pays \$2000 into a superannuation fund at the end of each year at 6% p.a. interest. How many payments would Ryuji make for his superannuation to be greater than:

a \$50 000?

b \$80 000?

12 Find the interest rate if \$1500 is invested at the end of each year grows to:

a \$11 284.95 after 6 years

b \$17 195.85 after 10 years

9.06 Annuities and geometric series

WS Annuities and geometric series

Annuities

Annuities

EXAMPLE 18

A sum of \$1500 is invested at the end of each year in a superannuation fund.
If interest is paid at 6% p.a., how much money will be available at the end of 25 years?

Solution

It is easier to keep track of each annual contribution separately.

Use $A = P(1 + r)^n$ with $P = 1500$ and $r = 0.06$.

The 1st contribution goes in at the end of the 1st year, so it only earns interest for 24 years.

$$A_1 = 1500(1 + 0.06)^{24}$$
$$= 1500(1.06)^{24}$$

The 2nd contribution goes in at the end of the 2nd year, so it earns interest for 23 years.

$$A_2 = 1500(1.06)^{23}$$

Similarly, the 3rd contribution earns interest for 22 years.

$$A_3 = 1500(1.06)^{22}$$

This pattern continues until the final contribution.

The 25th contribution goes in at the end of the 25th year, so it earns interest for 0 years.

$$A_{25} = 1500(1.06)^0$$

The future value is the total of all these contributions together with their interest.

$$FV = A_1 + A_2 + A_3 + \ldots + A_{25}$$
$$= 1500(1.06)^{24} + 1500(1.06)^{23} + 1500(1.06)^{22} + \ldots + 1500(1.06)^0$$
$$= 1500(1.06)^0 + 1500(1.06)^1 + 1500(1.06)^2 + \ldots + 1500(1.06)^{24}$$
$$= 1500(1.06^0 + 1.06^1 + 1.06^2 + \ldots + 1.06^{24}) \qquad \text{(factorising)}$$

$1.06^0 + 1.06^1 + 1.06^2 + \ldots + 1.06^{24}$ is a geometric series with $a = 1.06^0 = 1$, $r = 1.06$ and $n = 25$.

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{25} = \frac{1(1.06^{25} - 1)}{(1.06 - 1)}$$

$$\approx 54.86$$

So FV $\approx 1500(54.86)$

$= 82\ 296.77$

So the total amount of superannuation after 25 years is \$82 296.77.

In the previous example, the contributions were made at the end of each year. If they were made at the beginning of each year, they would all earn an extra year of interest. The 1st contribution would be invested for 25 years, the 2nd for 24 years, and so on until the last contribution for 1 year.

EXAMPLE 19

An amount of \$50 is put into an investment account at the end of each month. If interest is paid at 12% p.a. paid monthly, how much is in the account at the end of 10 years?

Solution

We use the compound interest formula where $P = 50$.

$r = 0.12 \div 12 = 0.01$, $n = 10 \times 12 = 120$

The 1st contribution goes in at the end of the 1st month, so it only earns interest for 119 months.

$A_1 = 50(1 + 0.01)^{119}$

$= 50(1.01)^{119}$

The 2nd contribution goes in at the end of the 2nd month, so it earns interest for 118 months.

$A_2 = 50(1.01)^{118}$

The 3rd contribution earns interest for 117 months.

$A_3 = 50(1.01)^{117}$

This pattern continues until the final contribution.

The 120th contribution earns interest for 0 months.

$A_{120} = 50(1.01)^0$

$FV = A_1 + A_2 + A_3 + \ldots + A_{120}$

$= 50(1.01)^{119} + 50(1.01)^{118} + 50(1.01)^{117} + \ldots + 50(1.01)^0$

$= 50(1.01)^0 + 50(1.01)^1 + 50(1.01)^2 + \ldots + 50(1.01)^{119}$

$= 50(1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119})$ (factorising)

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$1.01^0 + 1.01^1 + 1.01^2 + \ldots + 1.01^{119}$ is a geometric series with $a = 1.01^0$ or 1, $r = 1.01$ and $n = 120$.

$$S_n = \frac{a(r^n - 1)}{(r - 1)}$$

$$S_{120} = \frac{1(1.01^{120} - 1)}{(1.01 - 1)}$$

$$\approx 230.04$$

So $FV \approx 50(230.04)$

$= 11\,501.93$

So the total amount after 10 years is \$11 501.93.

Exercise 9.06 Annuities and geometric series

1 A sum of \$1500 is invested at the end of each year for 15 years at 8% p.a. Find the amount of superannuation available at the end of the 15 years.

2 Liam wants to save up \$15 000 for a car in 5 years' time. He invests \$2000 at the end of each year in an account that pays 7.5% p.a. interest. How much more will Liam have to pay at the end of 5 years to make up the \$15 000?

3 A school invests \$5000 at the end of each year at 6% p.a. to go towards a new library. How much will the school have after 10 years?

4 Jacqueline puts aside \$500 at the end of each year for 5 years. If the money is invested at 6.5% p.a., how much will she have at the end of the 5 years?

5 Miguel's mother invests \$200 for him each birthday up to and including his 18th birthday. The money earns 6% p.a. How much money will Miguel have on his 18th birthday?

6 Xuan is saving up for a holiday. She invests \$800 at the end of each year at 7.5% p.a. How much will she have for her holiday after 5 years' time?

7 A couple saves \$3000 at the beginning of each year towards a deposit on a house. If the interest rate is 5% p.a., how much will the couple have saved after 6 years?

8 Lucia saves up \$2000 each year and at the end of the year she invests it at 6% p.a.

a She does this for 10 years. What is her investment worth?

b Lucia continues investing \$2000 a year for 5 more years. What is the future value of her investment?

9 Jodie starts work in 2019 and puts \$1000 in a superannuation fund at the end of the year. She keeps putting in this same amount at the end of every year until she retires at the end of 2036. If interest is paid at 10% p.a., calculate how much Jodie will have when she retires.

10 Bol invests \$1000 at the beginning of each year. The interest rate is 8% p.a.

a How much will her investment be worth after 6 years?

b How much more would Bol's investment be worth after 6 years if she had invested \$1200 each year?

11 Asam cannot decide whether to invest \$1000 at the end of each year for 15 years or \$500 for 30 years in a superannuation fund. If the interest rate is 5% p.a., which would be the better investment for Asam?

12 Pooja is saving up to go overseas in 8 years' time. She invests \$1000 at the end of each year at 7% p.a. and estimates that the trip will cost her around \$10 000. Will she have enough? If so, how much over will it be? If she doesn't have enough, how much will she need to add to this money to make it up to the \$10 000?

13 Mila puts aside \$20 at the beginning of each month for 3 years. How much will she have then if the investment earns 8.2% p.a., paid monthly?

14 a Find the future value on an investment of \$1 at the end of each year for 19 years at 7% p.a. using the table of future values of an annuity on page 390.

b Prove that this table value is correct.

9.07 Reducing balance loans

Reducing balance loans

Reducing balance loan: spreadsheet

Reducible loans

Loan repayment tables

People take out loans for many reasons – to buy items such as a car, boat or furniture, to consolidate debts, to buy a home, and for home renovations. A home loan is called a mortgage.

An investment or annuity increases in value over time while a loan decreases as the loan is paid off. This is called a reducing balance loan. The amount of time taken to pay off the loan is called the term of the loan. A reducing balance loan is similar to an annuity with regular withdrawals.

EXAMPLE 20

Trang borrows \$8000 over 3 years to buy furniture. Interest on the loan is 1.25% per month and monthly repayments are \$277.32. Find the amount owing after:

a 1 month b 2 months c 3 months

Solution

Use $A = P(1 + r)_n$ with $P = 8000$ and $r = 1.25\% = 0.0125$.

Let A_n be the amount owing after n months.

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a 1st month: Amount owing is $8000 plus interest less the repayment.

$A^1 = 8000(1 + 0.0125)^1 - 277.32$

$= 8000(1.0125) - 277.32$

$= 7822.68$

Amount owing = $7822.68

b 2nd month: Amount owing is $7822.68 plus interest less the repayment.

$A^2 = 7822.68(1.0125)^1 - 277.32$

$= 7643.14$

Amount owing = $7643.14

c 3rd month: Amount owing is $7643.14 plus interest less the repayment.

$A^3 = 7643.14(1.0125)^1 - 277.32$

$= 7461.36$

Amount owing = $7461.36

If you know the term of the loan and the amount of the regular contributions, you can calculate the amount of interest owing.

You can use a loan repayments table to calculate the amount you need to contribute to pay off a loan. Here is a table that gives the monthly loan repayments on a $1000 loan.

	Term (years)					
Interest rate (% p.a.)	5	10	15	20	25	30
2	$17.53	$9.20	$6.44	$5.06	$4.24	$3.70
2.5	$17.75	$9.43	$6.67	$5.30	$4.49	$3.95
3	$17.97	$9.66	$6.91	$5.55	$4.74	$4.22
3.5	$18.19	$9.89	$7.15	$5.80	$5.01	$4.49
4	$18.42	$10.12	$7.40	$6.06	$5.28	$4.77
4.5	$18.64	$10.36	$7.65	$6.33	$5.56	$5.07
5	$18.87	$10.61	$7.91	$6.60	$5.85	$5.37
5.5	$19.10	$10.85	$8.17	$6.88	$6.14	$5.68
6	$19.33	$11.10	$8.44	$7.16	$6.44	$6.00
6.5	$19.57	$11.35	$8.71	$7.46	$6.75	$6.32
7	$19.80	$11.61	$8.99	$7.75	$7.07	$6.65
7.5	$20.04	$11.87	$9.27	$8.06	$7.39	$6.99
8	$20.28	$12.13	$9.56	$8.36	$7.72	$7.34

EXAMPLE 21

a Piri wants to borrow \$350 000 over 30 years to buy a unit, but she is not sure she can afford to pay the monthly repayments. If interest is 4.5% per month, calculate:

i the amount of each monthly repayment

ii the total amount Piri would pay

b Hamish borrows \$25 000 over 5 years to buy a car. Interest is 2% per month. Find:

i the amount of each monthly repayment

ii the total amount Hamish pays

iii the flat rate of interest on the loan

Solution

a i From the table, 30 years at 4.5% p.a. gives \$5.07.

This is on a loan of \$1000, so for \$350 000 we multiply the value by 350.

$\$5.07 \times 350 = \$1774.50$

So Piri would pay \$1774.50 each month.

ii $30 \text{ years} = 30 \times 12 = 360$ months

$$\begin{aligned}\text{Total amount repaid} &= \$1774.50 \times 360 \\ &= \$638\,820\end{aligned}$$

b i From the table, 5 years at 2% p.a. gives \$17.53.

This is on a loan of \$1000, so for \$25 000 we multiply the value by 25.

$\$17.53 \times 25 = \$438.25$

So Hamish pays \$438.25 each month.

ii $5 \text{ years} = 5 \times 12 = 60$ months

$$\begin{aligned}\text{Total amount repaid} &= \$438.25 \times 60 \\ &= \$26\,295\end{aligned}$$

iii $$\begin{aligned}\text{Interest} &= \$26\,295 - \$25\,000 \\ &= \$1295\end{aligned}$$

$$\frac{25000}{1295} \times 100\% = 5.18\%$$

So the flat rate of interest is 5.18%.

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You can use the table to do other calculations.

EXAMPLE 22

a The monthly repayments on a loan of \$70 000 at 5% p.a. are \$553.70. Find the term of the loan.

b A \$150 000 loan with a term of 20 years has monthly instalments of \$1119. Find the interest rate.

Solution

a Let the value in the table be x.

The table is for loans of \$1000.

$\$70\,000 \div \$1000 = 70$

$70 \times x = \$553.70$

$$x = \frac{\$553.70}{70}$$

$$\approx \$7.91$$

Looking at the table in the 5% row, \$7.91 is in the 15 year column.

So the term of the loan is 15 years.

b Let the value in the table be x.

$\$150\,000 \div \$1000 = 150$

$150 \times x = \$1119$

$$x = \frac{\$1119}{150}$$

$$\approx \$7.46$$

Looking at the table in the 20 year column, \$7.46 is in the 6.5% row.

So the interest rate is 6.5% p.a.

INVESTIGATION

FINANCIAL CALCULATORS

Most bank and other financial websites have calculators rather than tables for loan repayments, values of investments and annuities. Search the websites of banks or general websites that have these and try using these calculators.

Exercise 9.07 Reducing balance loans

1 Calculate the amount owing after 3 months on a loan of:

a \$20 000 at 0.9% per month with repayments of \$432.87 per month

b \$3500 at 1.3% per month with repayments of \$151.57 per month

c \$100 000 at 2.2% per month with repayments of \$2203.22 per month

d \$2000 at 2% per month with repayments of \$105.74 per month

e \$45 800 at 12% p.a. with repayments of \$504.30 per month

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2 For each loan below, find:

i the total amount repaid

ii total amount of interest paid

iii the flat interest rate of interest p.a.

a $5000 over 3 years with a monthly payment of $166.07

b $15 900 over 5 years with a monthly payment of $403.76

c $80 000 over 12 years with a monthly payment of $1109.62

d $235 000 over 25 years with a monthly payment of $907.09

e $1348 over 2 years with a monthly payment of $71.27

Use the table of loan repayments on page 401 to answer the rest of the questions.

3 Find the amount of the monthly repayment on a loan of:

a $8 000 over 5 years at 6% p.a.

b $15 000 over 5 years at 8% p.a.

c $72 000 over 10 years at 7.5% p.a.

d $430 000 over 20 years at 4% p.a.

e $312 000 over 15 years at 5.5% p.a.

f $137 000 over 25 years at 3.5% p.a.

g $49 000 over 10 years at 7% p.a.

h $765 000 over 30 years at 2.5% p.a.

i $925 000 over 25 years at 2% p.a.

j $1 000 000 over 30 years at 5.5% p.a.

4 Markus takes out a mortgage of $680 500 over 20 years at 3.5% interest.

a Find his monthly repayment.

b Find the total amount he will pay.

c How much interest does he pay?

d Calculate the flat rate of interest over the whole loan.

5 Find the term of each loan given the monthly payments of:

a $81.12 for a $4 000 loan at 8% p.a.

b $777 for a $75 000 loan at 4.5% p.a.

c $937.95 for a $169 000 loan at 3% p.a.

d $1560.25 for a $395 000 loan at 2.5% p.a.

e $232.20 for a $20 000 loan at 7% p.a.

f $3131.25 for a $625 000 loan at 3.5% p.a.

g $1302 for a $120 000 loan at 5.5% p.a.

h $1809.64 for a $281 000 loan at 2% p.a.

i $72.15 for a $6 500 loan at 6% p.a.

j $474.15 for a $81 750 loan at 3.5% p.a.

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6 Find the interest rate of each loan if the monthly instalment is:

a $57.99 for a $3000 loan for 5 years

b $619.65 for an $81 000 loan for 15 years

c $2307.36 for a $456 000 loan for 20 years

d $2571.56 for a $212 000 loan for 10 years

e $6515.36 for a $947 000 loan for 20 years

f $178.20 for a $9000 loan for 5 years

g $2709.70 for a $686 000 loan for 30 years

h $1422 for a $300 000 loan for 25 years

i $6814.73 for an $845 500 loan for 20 years

j $3127.24 for a $422 600 loan for 15 years

9.08 Loans and geometric series

WS Loans and geometric series

WS Applications of series

We can apply the formulas for compound interest and geometric series to work out the amount of the regular repayments of a reducing balance loan.

EXAMPLE 23

Find the amount of each monthly repayment on a loan of $20 000 at 12% p.a. over 4 years.

Solution

Let M stand for the monthly repayment.

Number of payments is $4 \times 12 = 48$.

Monthly interest is $12\% \div 12 = 1\% = 0.01$

Each month, we add interest and subtract the repayment.

Amount owing after 1 month:

$A^1 = 20\,000(1 + 0.01)^1 - M$

$= 20\,000(1.01)^1 - M$

Amount owing after 2 months:

$A^2 = A^1(1.01)^1 - M$

$= [20\,000(1.01)^1 - M](1.01)^1 - M$

$= 20\,000(1.01)^2 - M(1.01)^1 - M$ (expanding brackets)

$= 20\,000(1.01)^2 - M(1.01^1 + 1)$ (factorising)

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Amount owing after 3 months:

$A_3 = A_2(1.01)^1 - M$

$= [20\ 000(1.01)^2 - M(1.01^1 + 1)](1.01)^1 - M$

$= 20\ 000(1.01)^3 - M(1.01^1 + 1)(1.01)^1 - M$ (expanding brackets)

$= 20\ 000(1.01)^3 - M(1.01^2 + 1.01^1) - M$

$= 20\ 000(1.01)^3 - M(1.01^2 + 1.01^1 + 1)$ (factorising)

Continuing this pattern, after 48 months the amount owing is:

$A_{48} = 20\ 000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1)$

But the loan is paid out after 48 months.

So $A_{48} = 0$

$0 = 20\ 000(1.01)^{48} - M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1)$

$M(1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1) = 20\ 000(1.01)^{48}$

$$M = \frac{20000(1.01)^{48}}{1.01^{47} + 1.01^{46} + 1.01^{45} + \ldots + 1.01^2 + 1.01^1 + 1}$$

$$= \frac{20000(1.01)^{48}}{1 + 1.01^1 + 1.01^2 + 1.01^3 + \ldots + 1.01^{46} + 1.01^{47}}$$

The denominator is a geometric series with $a = 1$, $r = 1.01$ and $n = 48$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{48} = \frac{1(1.01^{48} - 1)}{1.01 - 1}$$

$$= \frac{1.01^{48} - 1}{0.01}$$

≈ 61.223

$$M \approx \frac{20000(1.01)^{48}}{61.223}$$

$= 526.68$

So the monthly repayment is \$526.68.

We can use this method to find loan repayments for more complex questions.

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EXAMPLE 24

A store charges 9% p.a. for loans, and repayments do not have to be made until the 4th month. Ivan buys $8000 worth of furniture and pays it off over 3 years.

a How much does Ivan owe after 3months?

b What are his monthly repayments?

c How much does Ivan pay altogether?

Solution

a Number of payments = 3 × 12 – 3 = 33 (3 months of no repayments)

Monthly interest rate = 0.09 ÷ 12 = 0.0075

Let M stand for the monthly repayment.

The first repayment is made in the 4th month.

After 3 months, the amount owing is

$A = P(1 + r)_n$

$A^3 = 8000(1 + 0.0075)_3$

$= 8000(1.0075)^3$

$= 8181.35$

So the amount owing after 3 months is $8181.35.

b Amount owing after 4 months:

$A_4 = A^3(1.0075)^1 – M$

$= [8000(1.0075)^3](1.0075)^1 – M$

$= 8000(1.0075)^4 – M$

Amount owing after 5 months:

$A^5 = A^4(1.0075)^1 – M$

$= [8000(1.0075)^4 – M](1.0075)^1 – M$

$= 8000(1.0075)^5 – M(1.0075)^1 – M$

$= 8000(1.0075)^5 – M(1.0075 + 1)$ (factorising)

Continuing this pattern, after 36 months the amount owing will be:

$A^{36} = 8000(1.0075)^{36} – M(1.0075^{32} + 1.0075^{31} + 1.0075^{30} + \ldots + 1.0075^1 + 1)$

But the loan is paid out after 36 months.

So $A^{36} = 0$

$0 = 8000(1.00075)^{36} – M(1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^1 + 1)$

$$M(1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^{1} + 1) = 8000(1.0075)^{36}$$

$$M = \frac{8000(1.0075)^{36}}{1.0075^{32} + 1.0075^{31} + \ldots + 1.0075^{1} + 1}$$

$$= \frac{8000(1.0075)^{36}}{1 + 1.0075^{1} + 1.0075^{2} + \ldots + 1.0075^{32}}$$

The denominator is a geometric series with $a = 1$, $r = 1.0075$ and $n = 33$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{33} = \frac{1(1.0075^{33} - 1)}{1.0075 - 1}$$

$$= \frac{1.0075^{33} - 1}{0.0075}$$

$$\approx 37.2849$$

$$M = \frac{8000(1.0075)^{36}}{37.2849\ldots}$$

$$\approx 280.79$$

So the monthly repayment is \$280.79.

c $\$280.79 \times 33 = \$9266.07$

So Ivan pays \$9266.07 altogether.

Exercise 9.08 Loans and geometric series

1 An amount of \$3000 is borrowed at 22% p.a. and paid off over 5 years with yearly repayments. How much is each repayment?

2 The sum of \$20 000 is borrowed at 18% p.a. interest calculated monthly over 8 years. How much are the monthly repayments?

3 David borrows \$5000 from the bank and pays back the loan in monthly instalments over 4 years. If the loan incurs interest of 15% p.a. calculated monthly, find the amount of each instalment.

4 Tri and Mai mortgage their house for \$150 000.

a Find the amount of the monthly repayments they will have to make if the mortgage is over 25 years with interest at 6% p.a. compounded monthly.

b If they want to pay their mortgage out after 15 years, what monthly repayments would they need to make?

5 A loan of \$6000 is paid back in equal annual instalments over 3 years. If the interest is 12.5% p.a., find the amount of each annual instalment.

6 Santi buys a car for \$38 000, paying a 10% deposit and taking out a loan for the balance. If the loan is over 5 years with interest of 1.5% monthly, find:

a the amount of each monthly loan repayment

b the total amount that Santi paid for the car.

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7 A \$2000 loan is offered at 18% p.a. with interest charged monthly, over 3 years.

a If no repayment need be paid for the first 2 months, find the amount of each repayment.

b How much will be paid back altogether?

8 Breanna thinks she can afford a mortgage payment of \$800 each month. How much can she borrow, to the nearest \$100, over 25 years at 11.5% p.a.?

9 Get Rich Bank offers a mortgage at $7\frac{1}{2}$% p.a. over 10 years and Capital Bank offers a mortgage at $5\frac{1}{2}$% p.a. over 25 years, both with interest calculated monthly.

a Find the amount of the monthly repayments for each bank on a loan of \$80 000.

b Find the difference in the total amount paid on each mortgage.

10 Majed buys a \$35 000 car. He puts down a 5% deposit and pays the balance back in monthly instalments over 4 years at 12% p.a.

a Find the amount of the monthly payments.

b Find the total amount that Majed pays for the car.

11 Amy borrowed money over 7 years at 15.5% p.a. and she pays \$1200 a month. How much did she borrow?

12 NSW Bank offers loans at 9% p.a. with no repayments for the first 3 months, while Sydney Bank offers loans at 7% p.a. Compare these loans on an amount of \$5000 over 3 years and state which bank offers the better loan and why.

13 Danny buys a home cinema system for \$10 000. He pays a \$1500 deposit and borrows the balance at 18% p.a. over 4 years.

a Find the amount of each monthly repayment.

b How much did Danny pay altogether?

14 A store offers furniture on hire purchase at 20% p.a. over 5 years, with no repayments for 6 months. Ali buys furniture worth \$12 000.

a How much does Ali owe after 6months?

b What are the monthly repayments?

c How much does Ali pay for the furniture altogether?

15 A loan of \$6000 over 5 years at 15% p.a. interest, charged monthly, is paid back in 5 annual instalments.

a What is the amount of each instalment?

b How much is paid back altogether?

16 a Using the table of loan repayments on page 401, find the amount of the monthly payments on a \$1000 loan over 10 years at 4.5% p.a.

b Show that this table value is correct.

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9. TEST YOURSELF

WS Formula sheet: Measurement, Sequences and series

Qz Practice quiz

In Questions 1 to 3, select the correct answer A, B, C or D.

1 An amount of $2500 invested at 6% p.a. for 5 years with interest paid twice a year has a future value of:

A $\$2500(1.06)^5$ B $\$2500(1.03)^5$

C $\$2500(1.03)^{10}$ D $\$2500(1.06)^{10}$

2 An investment has a present value of $68 000 and a future value of $79 500. Find the flat interest rate on the investment.

A 11.5% B 16.9%

C 85.5% D 14.5%

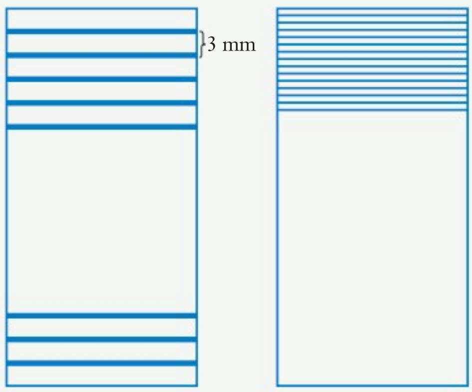
3 A tree is planted when it is 1.2 m tall. Every year its growth is $\frac{3}{8}$ of its previous year's height. Find how tall the tree will grow.

A 1.92 m B 2 m C 2.5 m D 3.2 m

4 A loan of $22 000 at 12% p. a. is paid off in monthly payments of $226.29. Find the amount owing after 3 months.

5 Zac puts $1500 into a savings account that earns 3.7% p.a. How much will Zac have in the account after 3 years?

6 A bamboo blind has 30 slats. It is attached to the window at the top and when the blind is down, the gap between each slat and the next, and between the top slat and the top of the window, is 3 mm. When the blind is up, the slats have no gaps between them.



a Show that when the blind is up, the bottom slat rises 90 mm.

b How far does the next slat rise?

c Explain briefly why the distances the slats rise when the blind is up form an arithmetic sequence.

d Find the distance the 17th slat from the bottom rises.

e What is the sum of the distances that all slats rise?

7 Cristina borrows $62 500 over 5 years with monthly repayments of $2022. Find:

a the total amount Cristina pays

b the amount of interest she pays

c the flat interest rate on the total loan

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8 Use the table of future values of an investment on page 380 to find the future value of:

a $595 over 4 years at 5% p.a.

b $5 000 over 9 years at 8% p.a.

c $1651.20 over 6 years at 10% p.a.

d $13 500 over 11 years at 2% p.a.

e $9 485 over 18 years at 1% p.a.

9 Murat earned $20 000 in one year. At the beginning of the 2nd year he received a salary increase of $450. He now receives the same increase each year.

a What will his salary be after 10 years?

b How much will Murat earn altogether over the 10 years?

10 Ana puts her superannuation payout of $186 900 into an account that earns 3% p.a. paid monthly. She withdraws $2500 at the end of each month as a pension.
Find the amount in the account after:

a 1 month

b 2 months

c 3 months

11 Gerri wants to contribute a certain amount of money at the end of each year into a superannuation fund so that she will have $200 000 at the end of 25 years. If the fund averages 13% p.a., find the amount of the money Gerri would contribute each year.

12 A supermarket stacks boxes with 20 boxes in the bottom stack, 18 boxes in the next stack, 16 in the next and so on.

a How many stacks are there?

b How many boxes are there?

13 Convert each recurring decimal to a fraction.

a $0.\dot{4}$

b $0.7\dot{2}$

c $1.\dot{5}\dot{7}$

14 Find the amount invested in a bank account at 9.5% p.a. if the balance in the account is $5860.91 after 6 years.

15 Haylee borrows $50 000 for farm machinery at 18% p.a. over 5 years and makes equal yearly repayments on the loan at the end of each year.

a How much does she owe at the end of the first year, just before she makes the first repayment?

b How much is each yearly repayment?

16 a If $2000 is invested at 4.5% p.a., how much will it be worth after 4years?

b If interest is paid quarterly, how much would the investment be worth after 4years?

17 Pedro borrows $200 000 to buy a house. If the interest is 6% p.a. compounded monthly and the loan is over 20 years:

a how much is each monthly repayment?

b how much does Pedro pay altogether?

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18 a Find the annual contribution needed for an annuity to have a future value of \$12 000 after 4 years at 5% p.a. if the contribution is made at the beginning of the year.

b Find the single investment that would need to be invested at the same interest rate now to have this future value.

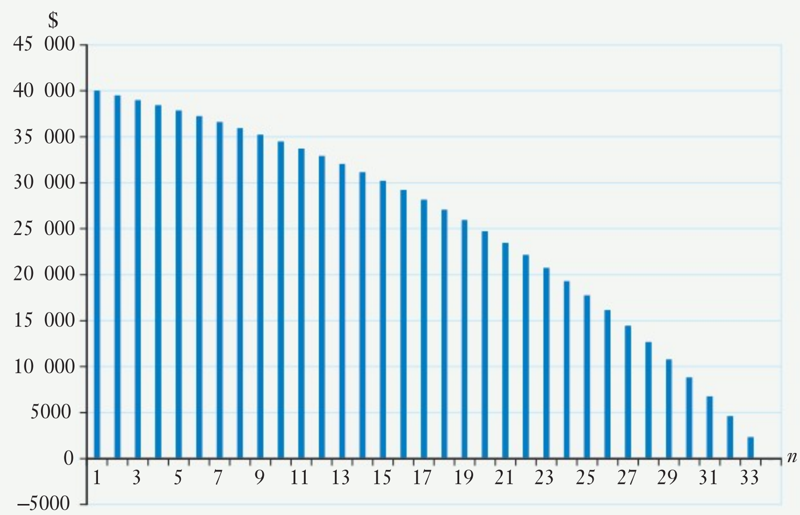
19 Every week during a typing course, Jamal improves his typing speed by 3 words per minute until he reaches 60 words per minute by the end of the course.

a If he can type 18 words per minute in the first week of the course, how many words per minute can he type by week 8?

b How many weeks does the course run for?

20 A ball drops from a height of 1.2 metres then bounces back to $\frac{3}{5}$ of this height. On the next bounce, it bounces up to $\frac{3}{5}$ of this height and so on. Through what distance will the ball travel?

21 This graph shows an annuity of \$40 000 earning 5% p.a. with withdrawals of \$2500 at the end of each year.



a What is the annuity worth after:

i 10 years? ii 20 years?

b When is the annuity worth:

i \$30 000? ii \$10 000?

c How long will it take for the annuity to run out?

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9. CHALLENGE EXERCISE

1 Jane puts $300 into an account at the beginning of each year to pay for her daughter's education in 5 years' time. If 6% p.a. interest is paid quarterly, how much money will Jane have at the end of the 5 years?

2 A factory sells shoes at $60 a pair. For 10 pairs of shoes there is a discount, whereby each pair costs $58. For 20 pairs, the cost is $56 a pair and so on. Find:

a the price of each pair of shoes on an order of 100 pairs of shoes

b the total price on an order of 60 pairs of shoes

3 Find the amount of money in a bank account if $5000 earns 8.5% p.a. for 4 years, then 6.5% p.a. for 3 years, with interest paid monthly for all 7 years.

4 A metal is heated to 500°C.A minute later it cools to 425°C, then a minute later it cools down to 361.25°C. If the metal continues to cool in the same way, find:

a its temperature after:

i 10 minutes ii 15 minutes

b how long it will take to cool down to:

i 200° ii 100°

5 Lukas puts $1000 into a superannuation account at the beginning of each year where it earns 6% p.a. He retires and collects the superannuation at the end of 25 years.

a How much will the first $1000 be worth at the end of 25 years, in index form?

b When Lukas deposits the second $1000 at the end of the 2nd year, how much will it be worth after 25 years?

c How much will the third $1000 be worth after 25 years?

d How much will the final $1000 be worth that Lukas deposits at the beginning of the 25th year?

e Show that the total amount in the account after 25 years is $1000 \times \frac{1.06(1.0625 - 1)}{0.06}$.

f Find the amount that Lukas will have at the end of the 25 years.

6 Kim borrows $10 000 over 3 years at a rate of 1% interest compounded each month. If she pays off the loan in three equal annual instalments, find:

a the amount Kim owes after one month

b the amount she owes after the first year, just before she pays the first instalment

c the amount of each instalment

d the total amount of interest Kim pays.

7 A superannuation fund paid 6% p.a. for the first 10 years and then 10% p.a. after that. If Thanh put $5000 into this fund at the beginning of each year, how much would she have at the end of 25 years?

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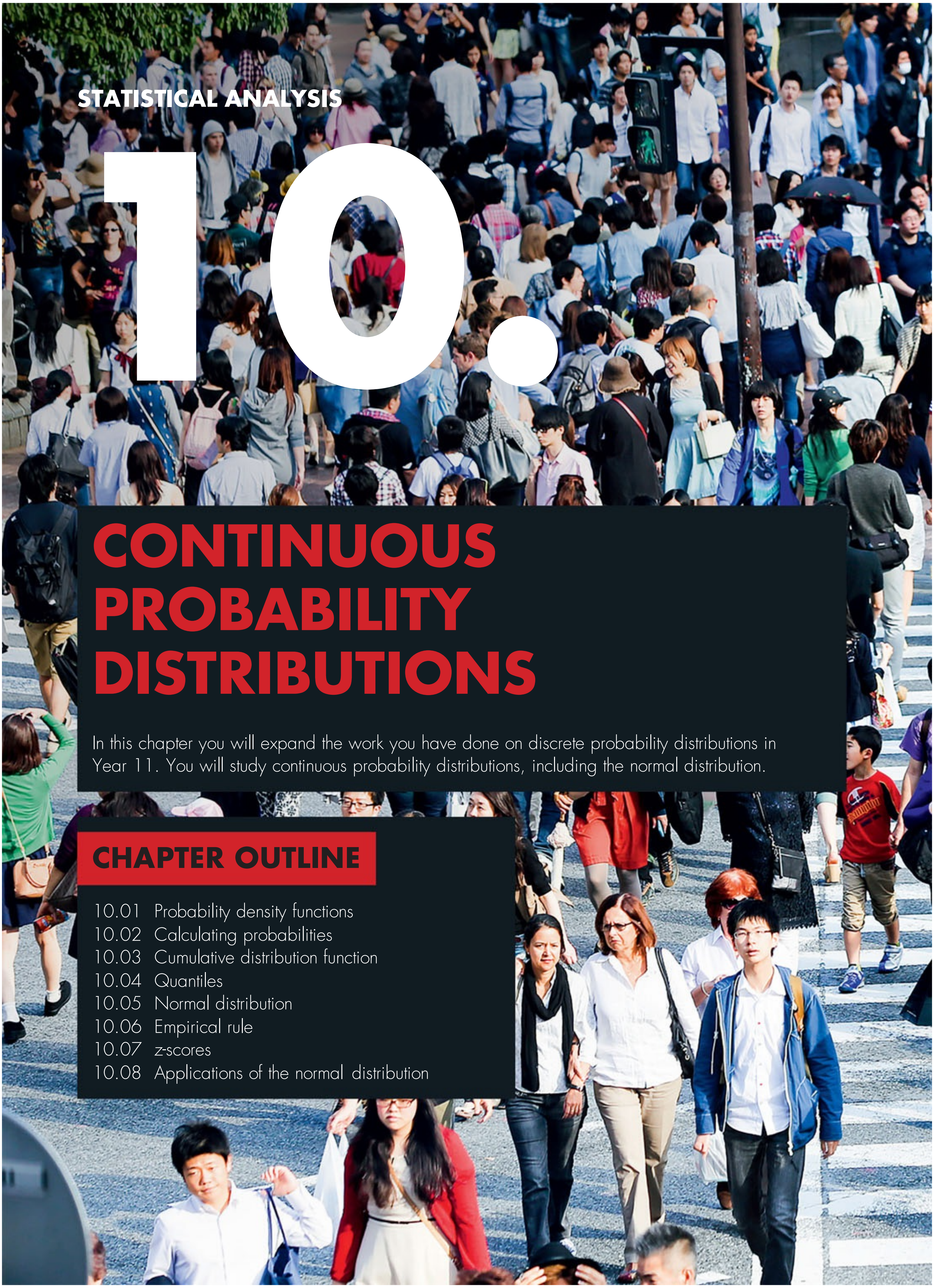

STATISTICAL ANALYSIS

10.

CONTINUOUS PROBABILITY DISTRIBUTIONS

In this chapter you will expand the work you have done on discrete probability distributions in Year 11. You will study continuous probability distributions, including the normal distribution.

CHAPTER OUTLINE

IN THIS CHAPTER YOU WILL:

- recognise continuous random variables
- understand the properties of a probability density function (PDF)
- find cumulative distribution functions (CDF)
- find probabilities of continuous data
- calculate measures of central tendency and spread for continuous probability distributions
- recognise the normal distribution and identify its properties
- calculate probabilities and quantiles for normal distributions
- understand the standard normal distribution and z-scores
- apply the normal distribution to solving practical problems

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TERMINOLOGY

continuous random variable: A random variable that can have any value along a continuum, for example, the height of a basketball player.

cumulative distribution function: A function $F(x)$ for the probability $P(X \leq x)$.

empirical rule: The percentage probabilities (68%, 95%, 99.7%) that normally-distributed scores will lie within 1, 2, and 3 standard deviations, respectively, from the mean.

normal distribution: A continuous probability distribution in which the mean, mode and median are at the centre of a symmetrical bell-shaped graph.

probability density function: A function of a continuous random variable whose integral gives the probability $P(X \leq x)$.

random variable: A variable whose values are based on a chance experiment; for example, the number of road accidents in an hour

uniform probability distribution: A probability distribution in which every outcome has the same probability

z-score: Measures how many standard deviations above or below the mean a score is.

10.01 Probability density functions

Probability density functions

WS Probability density function

EXAMPLE 1

This table gives the results of a survey of different times that runners take to complete a race.

Time (min)	Frequency
0–<4	6
4–<8	8
8–<12	11
12–<16	4
16–<20	2
20–<24	1

a Add a column of relative frequencies.

b Sketch a frequency histogram for the relative frequencies.

c Estimate each probability:

i $P(X < 12)$ ii $P(X \geq 16)$ iii $P(4 \leq X < 8)$

Solution

a

Time (min)	Frequency	Relative frequency
0–<4	6	$\frac{6}{32} = \frac{3}{16}$
4–<8	8	$\frac{8}{32} = \frac{1}{4}$
8–<12	11	$\frac{11}{32}$
12–<16	4	$\frac{4}{32} = \frac{1}{8}$
16–<20	2	$\frac{2}{32} = \frac{1}{16}$
20–<24	1	$\frac{1}{32}$

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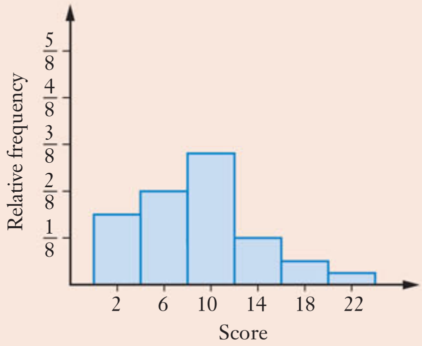
b



c i $P(X < 12) = \frac{6}{32} + \frac{8}{32} + \frac{11}{32}$

$= \frac{25}{32}$

ii $P(X \geq 16) = \frac{2}{32} + \frac{1}{32}$

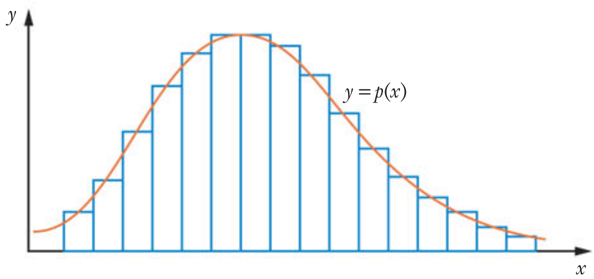
$= \frac{3}{32}$

iii $P(4 \leq X < 8) = \frac{1}{4}$

The times in the above example are values of a continuous random variable, but sorted into groups. While we can estimate probabilities using relative frequency, we use other methods when dealing with continuous data.

Continuous probability distributions

With continuous data we can't really draw a histogram as we did in Example 1 or there would be an 'infinite' number of columns with 'zero' widths. Instead, the probability distribution is a continuous curve.



A continuous probability distribution is represented by a function $P(X = x)$ or $p(x)$ called a probability density function (PDF) where X is the random variable. As with discrete probability distributions, the sum of all probabilities must be 1.

With a continuous probability distribution, we cannot calculate the probability for a single outcome, so $P(X = x) = 0$. Instead, we can only calculate the probability for a range of values such as $P(4 \leq X < 8)$.

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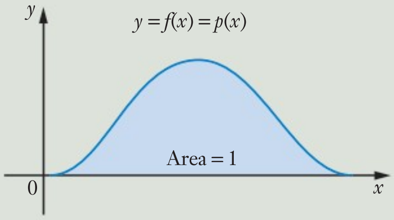
Area under a probability density function

The area under a probability density function is 1.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

where $f(x) \geq 0$ (since $0 \leq p(x) \leq 1$)



EXAMPLE 2

a A function is given by $f(x) = \begin{cases} \dfrac{3x^2}{26} & \text{for } 1 \leq x \leq 3 \\ 0 & \text{for all other } x \end{cases}$.

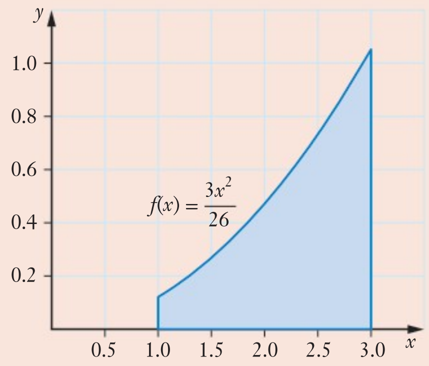
Show that it is a continuous probability distribution.

b A function is given by $f(x) = ax^2$ defined for the domain [0, 5].
Find the value of a for which this is a probability density function.

Solution

a For a continuous probability distribution, the area under the curve must be 1.

Drawing the graph, notice that the area will be 0 for all x values outside $1 \leq x \leq 3$.



$$\begin{aligned}
\text{Area} &= \int_1^3 \frac{3x^2}{26}\,dx \\
&= \frac{1}{26}\int_1^3 3x^2\,dx \\
&= \frac{1}{26}\left[x^3\right]_1^3 \\
&= \frac{1}{26}[3^3 - 1^3] \\
&= \frac{1}{26}[26] \\
&= 1
\end{aligned}$$

So f (x) is a continuous probability distribution.

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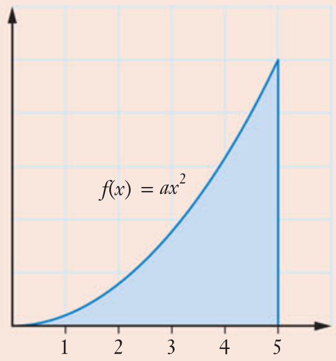
b Drawing the graph gives a parabola in the domain [0, 5].



A PDF has area 1.

So $\int_0^5 ax^2\,dx = 1$

$$\int_0^5 ax^2\,dx = a\int_0^5 x^2\,dx$$

$$= a\left[\frac{x^3}{3}\right]_0^5$$

$$= a\left[\frac{5^3}{3} - \frac{0^3}{3}\right]$$

$$= a\left[\frac{125}{3}\right]$$

$$= \frac{125a}{3}$$

So $\quad \dfrac{125a}{3} = 1$

$$a = \frac{3}{125}$$

Exercise 10.01 Probability density functions

1 State whether each random variable is discrete or continuous.

- a The size of T-shirts worn by people
- b The speed of cars as they pass a certain point
- c The volume of water in a dam
- d The number of seats on an aeroplane
- e The weight of babies born in August

2 Which of the following functions describe continuous probability distributions?

a $f(x) = 0.2$ in the domain [1, 6]

b $f(x) = \dfrac{x}{12}$ in the domain [0, 6]

c $f(x) = \begin{cases} \dfrac{x^3}{324} & \text{for } 0 \le x \le 3 \\ 0 & \text{for all other } x \end{cases}$

d $f(x) = \dfrac{x^2}{21}$ in the interval $1 \le x \le 4$

e $f(x) = \begin{cases} \dfrac{x}{8} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

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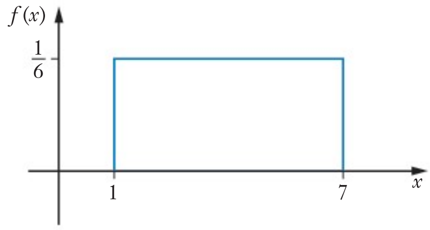
3 Which of the following graphs are of probability density functions?

a



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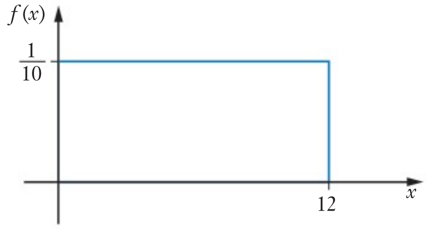
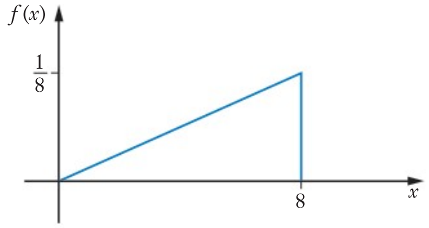
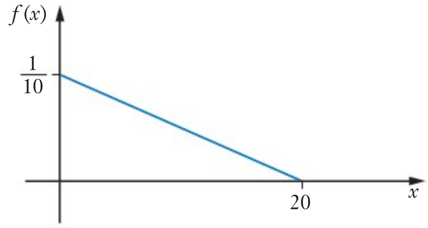


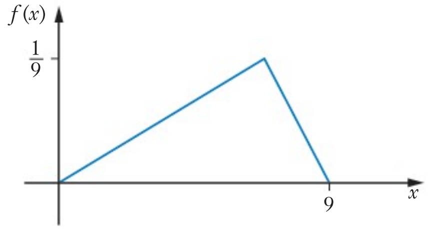
c



d



e



4 A probability density function has the equation $f(x) = \frac{x^4}{3355}$ over the domain [2, b]. Evaluate b.

5 Given the continuous probability distribution $f(x) = \begin{cases} kx^3 & \text{for } 0 \le x \le 5 \\ 0 & \text{for all other } x \end{cases}$

find the value of k.

6 A probability density function is given by $f(x) = ae^x$ over a certain domain. Find the exact value of a if the domain is:

a [1, 3] b [1, 7] c [0, 4]

7 A function is given by $f(x) = \frac{x^2}{72}$.
Over what domain starting at $x = 0$ is this a probability density function?

8 A PDF is given by $f(x) = \frac{2x^5}{87\ 381}$ over the interval $1 \le x \le b$. Find the value of b.

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10.02 Calculating probabilities

Since $P(X = x) = 0$ for continuous probability distributions, we can only find the probability of a range of values $P(a \le X \le b)$.

Also, since $P(X = a) = 0$ and $P(X = b) = 0$, it makes no difference whether we use $\le$ or $<$, $\ge$ or $>$.

$P(a < X < b) = P(a \le X \le b)$

WS Probability density functions

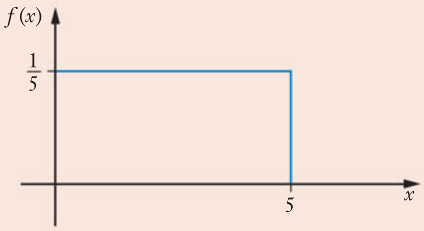
WS Uniform and triangular probability density functions

EXAMPLE 3

For the probability density function, find:

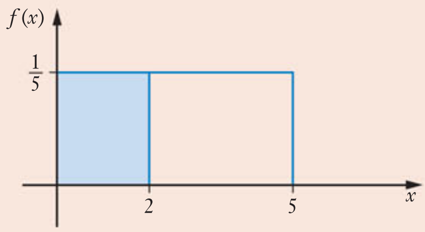
a $P(X \le 2)$

b $P(1 < X < 4)$



Solution

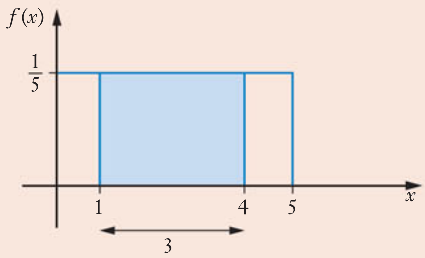
a $P(X \le 2)$ is the shaded area between $x = 0$ and $x = 2$.



$$P(X \le 2) = 2 \times \frac{1}{5}$$
$$= \frac{2}{5}$$

b $P(1 < X < 4)$ is the shaded area between $x = 1$ and $x = 4$.

Notice that $P(1 < X < 4) = P(1 \le X \le 4)$ since $P(X = 1) = P(X = 4) = 0$



$$P(1 < X < 4) = 3 \times \frac{1}{5}$$
$$= \frac{3}{5}$$

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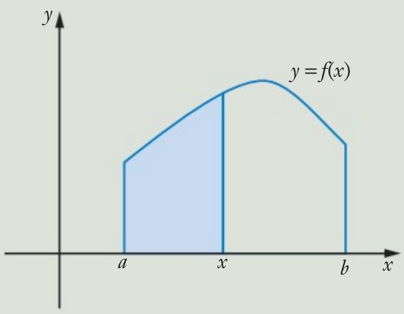
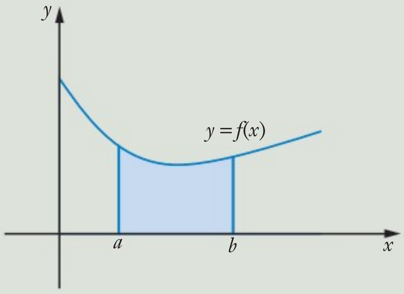
Probabilities in probability density functions

$P(X \le x) = \int_a^x f(x)\ dx$ where $y = f(x)$ is a PDF defined in the domain [a, b].



$P(a \le X \le b) = \int_a^b f(x)\,dx$ where $y = f(x)$ is a PDF and a and b are in the defined domain.



EXAMPLE 4

A function is given by $f(x) = \dfrac{3x^2}{117}$ defined in the domain [2, 5]. Find:

a $P(X \le 4)$

b $P(3 \le X \le 4)$

Solution

a $P(X \le 4) = \int_2^4 \dfrac{3x^2}{117}\,dx$ (since domain is [2, 5])

$$= \frac{1}{117}\int_2^4 3x^2\,dx$$

$$= \frac{1}{117}\left[x^3\right]_2^4$$

$$= \frac{1}{117}(4^3 - 2^3)$$

$$= \frac{56}{117}$$

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b $P(3 \le X \le 4) = \int_3^4 \frac{3x^2}{117}\,dx$

$= \frac{1}{117}\int_3^4 3x^2\,dx$

$= \frac{1}{117}\left[x^3\right]_3^4$

$= \frac{1}{117}(4^3 - 3^3)$

$= \frac{37}{117}$

Uniform distributions

In Year 11, Chapter 10, Discrete probability distributions, you learned that with a uniform probability distribution, every outcome has the same probability.

EXAMPLE 5

A continuous probability function $y = f(x)$ is uniform in the domain [5, 15].

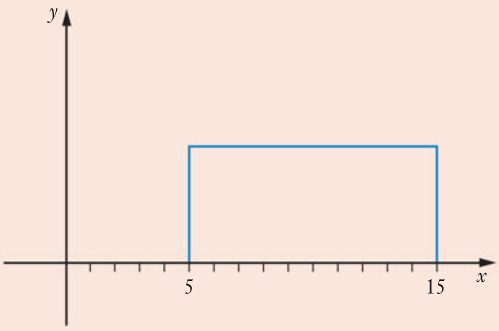
a Sketch the probability density function.

b Find:

i $P(X \ge 8)$ ii $P(7 \le X \le 10)$ iii $P(8 < X < 11)$

Solution

a A uniform distribution has all equal probabilities so will have the same height. This gives a rectangle.



PDF has area 1:

$A = bh$

$= (15 - 5) \times h$

$= 10h$

So $10h = 1$

$h = \frac{1}{10}$

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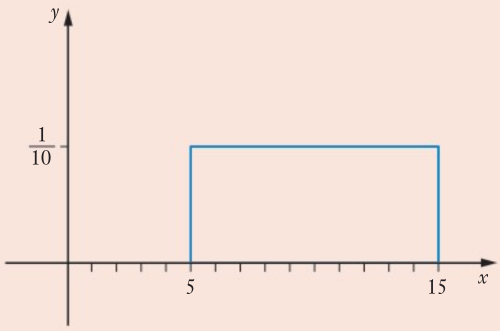
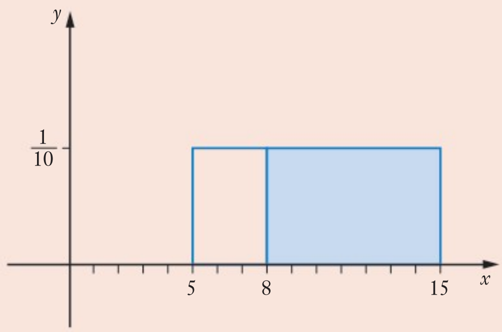


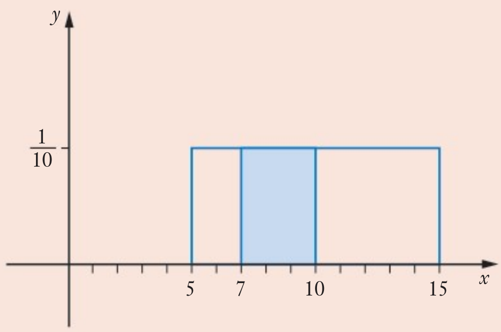
b i $P(X \geq 8) = (15 - 8) \times \dfrac{1}{10}$

$= \dfrac{7}{10}$



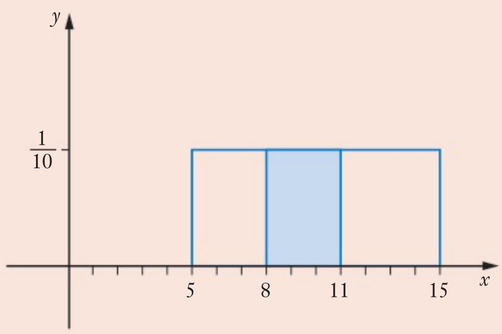
ii $P(7 \leq X \leq 10) = (10 - 7) \times \dfrac{1}{10}$

$= \dfrac{3}{10}$



iii $P(8 < X < 11) = (11 - 8) \times \dfrac{1}{10}$

$= \dfrac{3}{10}$



Notice in the example that intervals with the same width have the same probability.

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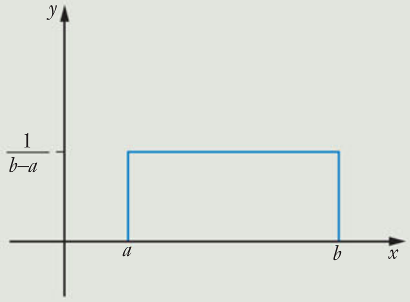
Uniform continuous probability distributions

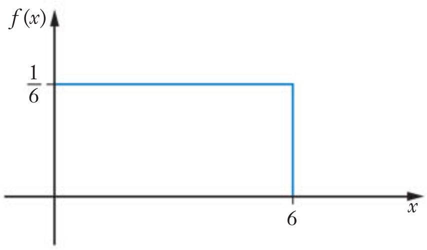
$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$$



Equal intervals along the x-axis will have the same probability.

Exercise 10.02 Calculating probabilities

1 For the continuous probability distribution graphed, find:



a $P(X \le 3)$

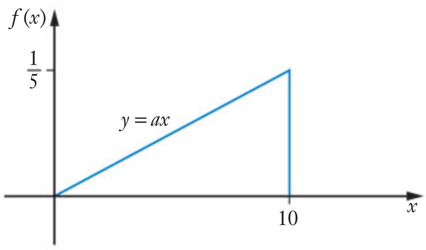
b $P(1 \le X \le 2)$

c $P(1 \le X \le 4)$

d $P(X < 4)$

e $P(X \ge 4)$

2 A probability density function is shown.



a Find the equation of the linear function $y = ax$.

b Find:

i $P(X < 9)$

ii $P(X \le 3)$

iii $P(4 \le X \le 7)$

iv $P(2 < X < 6)$

v $P(X > 5)$

3 The continuous probability distribution is defined by $f(x) = ax^2$ in the domain $[0, 5]$.

a Evaluate a.

b Find:

i $P(X \le 3)$ ii $P(1 < X < 4)$ iii $P(X > 2)$

iv $P(X < 1)$ v $P(3 \le X < 4)$

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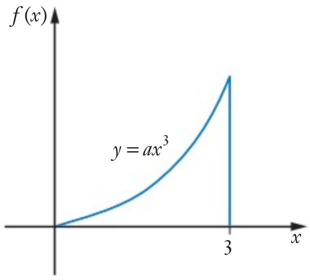
c

4 The continuous random variable X has the PDF shown.



a Evaluate a.

b Find:

i $P(1 \le X \le 3)$ ii $P(X < 2)$

iii $P(1 \le X \le 2)$ iv $P(X \le 1)$

5 A continuous probability function is given by $f(x) = ke^x$, defined on the domain $[1, 6]$.

a Find the exact value of k.

b Find each exact probability:

i $P(2 \le X \le 5)$ ii $P(X < 4)$ iii $P(X \ge 3)$

6 a Show that $y = \sin x$ is a probability density function in the domain $\left[0, \frac{\pi}{2}\right]$.

b Find each exact probability:

i $P\left(X \le \frac{\pi}{3}\right)$

ii $P\left(0 < X < \frac{\pi}{4}\right)$

iii $P\left(X > \frac{\pi}{6}\right)$

7 a Show that the uniform distribution $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{for all other } x \text{ values} \end{cases}$

is a probability density function.

b If $a = 3$ and $b = 7$, find:

i $P(X \le 6)$ ii $P(X \ge 5)$ iii $P(5 \le X \le 6)$

10.03 Cumulative distribution function

WS Cumulative distribution function

In the previous section you integrated the PDF each time to find $P(X \le x)$, a cumulative probability. The cumulative distribution function is a general formula for finding $P(X \le x)$ directly.

Cumulative distribution function (CDF)

The cumulative distribution function is given by $F(x) = \int_a^x f(x)\,dx$ where $y = f(x)$ is a PDF defined in the domain $[a, b]$.

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EXAMPLE 6

A continuous probability function is given by $f(x) = \frac{4x^3}{255}$ defined in the domain $[1, 4]$.

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \leq 3)$

ii $P(X < 1.6)$

Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \frac{4x^3}{255}$ is a PDF defined in the domain $[1, 4]$.

$$= \int_1^x \frac{4x^3}{255}\,dx$$

$$= \frac{1}{255}\int_1^x 4x^3\,dx$$

$$= \frac{1}{255}\left[x^4\right]_1^x$$

$$= \frac{1}{255}(x^4 - 1^4)$$

$$= \frac{1}{255}(x^4 - 1)$$

$$= \frac{x^4 - 1}{255}$$

b Using $F(x) = \frac{x^4 - 1}{255}$ to find $P(X \leq x)$:

i For $P(X \leq 3)$:

$$F(3) = \frac{3^4 - 1}{255}$$

$$= \frac{81 - 1}{255}$$

$$= \frac{80}{255}$$

$$= \frac{16}{51}$$

So $P(X \leq 3) = \frac{16}{51}$

ii For $P(X < 1.6)$:

$$F(1.6) = \frac{1.6^4 - 1}{255}$$

$$= 0.0218$$

So $P(X < 1.6) = 0.0218$

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We can use the cumulative distribution function to find probabilities such as $P(X \geq a)$ or $P(a \leq X \leq b)$.

EXAMPLE 7

A continuous probability function is given by $f(x) = \frac{3x^2}{335}$ defined in the domain [2, 7].

a Find the cumulative distribution function.

b Use the CDF to find:

i $P(X \geq 4)$ ii $P(3.5 \leq X \leq 6.2)$

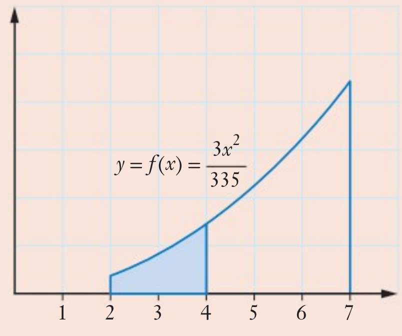
Solution

a $F(x) = \int_a^x f(x)\,dx$ where $f(x) = \frac{3x^2}{335}$ is a PDF defined in the domain [2, 7].

$$= \int_2^x \frac{3x^2}{335}\,dx$$
$$= \frac{1}{335}\int_2^x 3x^2\,dx$$
$$= \frac{1}{335}\left[x^3\right]_2^x$$
$$= \frac{1}{335}(x^3 - 2^3)$$
$$= \frac{1}{335}(x^3 - 8)$$
$$= \frac{x^3 - 8}{335}$$

b We use $F(x) = \frac{x^3 - 8}{335}$ to find $P(X \leq x)$.

i



To find $P(X \geq 4)$, first find $P(X \leq 4)$:

The shaded part of the PDF is $P(X \leq 4)$.

$$P(X \leq 4) = \frac{4^3 - 8}{335} = \frac{56}{335}$$

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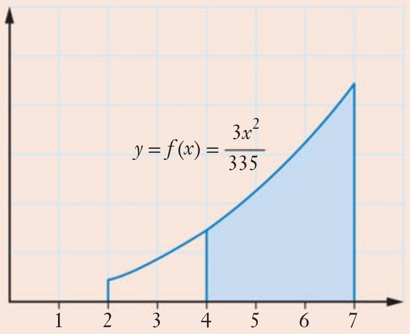


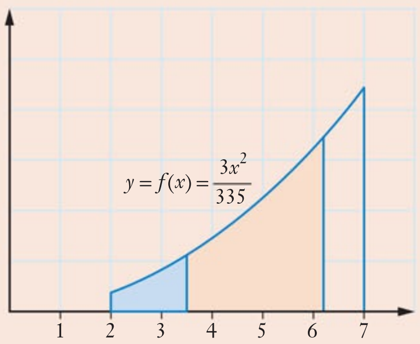
Area under the curve is 1:

$P(X \geq 4) = 1 - P(X \leq 4)$

$= 1 - \frac{56}{335}$

$= \frac{279}{335}$

ii



$P(X \leq 3.5)$ is the area to the left of $x = 3.5$

$P(X \leq 6.2)$ is the area to the left of $x = 6.2$

So $P(3.5 \leq X \leq 6.2) = P(X \leq 6.2) - P(X \leq 3.5)$

$= \frac{6.2^3 - 8}{335} - \frac{3.5^3 - 8}{335}$

$= 0.688 - 0.104$

$= 0.583$

Mode of a continuous probability distribution

We sometimes want to know what the highest probability is. This is the mode.

Mode

The mode is the maximum point of the probability density function.

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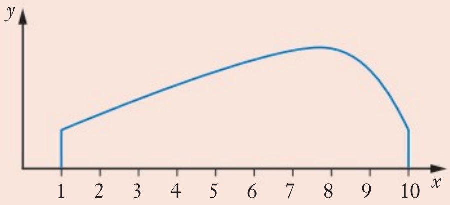
EXAMPLE 8

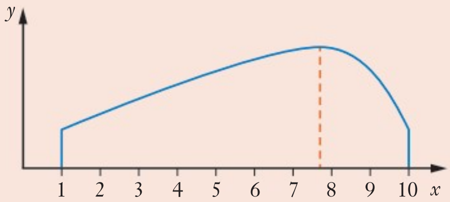
a Find the mode of the continuous probability distribution shown below.



b A continuous probability distribution is defined on the interval $1 \le x \le 5$ and has equation $f(x) = \frac{3x(6-x)}{92}$. Find the mode.

Solution

a



The highest point of the PDF is at $x = 7.7$

So the mode is 7.7.

b
$$f(x) = \frac{3x(6-x)}{92}$$
$$= \frac{3}{92}(6x - x^2)$$

The function is a parabola with $a < 0$ so will have a maximum turning point.
We use calculus to see if this point lies within the defined domain [1, 5].
(We could also use $x = -\frac{b}{2a}$ for the axis of symmetry of a parabola).

$$f'(x) = \frac{3}{92}(6-2x)$$

For stationary points:

$$f'(x) = 0$$
$$\frac{3}{92}(6-2x) = 0$$
$$6 - 2x = 0$$
$$6 = 2x$$
$$3 = x$$

$x = 3$ lies in the domain [1, 5].

$$f''(x) = \frac{3}{92}(-2)$$
$$= -\frac{3}{46}$$
$$< 0$$

Concave down so a maximum turning point.

So the mode is 3.

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Exercise 10.03 Cumulative distribution function

1 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \dfrac{x^2}{9}$ defined in the domain [0, 3]

b $f(x) = \dfrac{4x^3}{1296}$ defined in the domain [0, 6]

c $f(x) = \dfrac{e^x}{e^4 - 1}$ in the interval $0 \le x \le 4$

d $f(x) = \dfrac{4(x-2)^3}{625}$ in the domain [2, 7]

e $f(x) = \dfrac{3x(8-x)}{135}$ in the domain [2, 5]

2 a Find the cumulative distribution function for $f(x) = \begin{cases} \dfrac{5x^4}{7776} & \text{for } 1 \le x \le 6 \\ 0 & \text{for all other values} \end{cases}$.

b Find:

i $P(X \le 3)$ ii $P(X \le 2)$ iii $P(X < 5)$

iv $P(X > 4)$ v $P(2 \le X \le 4)$

3 A continuous probability distribution is given by $f(x) = \dfrac{4x^3}{2320}$ in the domain [3, 7].

a Find the cumulative distribution function.

b Find:

i $P(X \le 4)$ ii $P(X \le 6)$ iii $P(X \ge 5)$

iv $P(X > 4)$ v $P(4 \le X < 6)$

4 A continuous probability distribution is defined by $f(x) = \dfrac{2e^{2x}}{e^{10} - 1}$ in the domain [0, 5].

a Find the cumulative distribution function.

b Calculate each probability correct to 2 significant figures.

i $P(X \le 2)$ ii $P(X \le 4)$ iii $P(X > 3)$

iv $P(X \ge 2.8)$ v $P(2 \le X \le 4)$

5 a Evaluate a if $f(x) = ax^3$ is a continuous probability distribution defined in the domain [0, 9].

b Find the cumulative distribution function.

c Find:

i $P(X \le 5)$ ii $P(X \le 4)$ iii $P(X > 8)$

iv $P(X \ge 3)$ v $P(2 \le X \le 6)$

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6 a Find the exact value of a if $f(x) = \frac{a}{x}$ is a continuous probability distribution defined in the domain $[1, 6]$.

b Find the cumulative distribution function.

c Find to 2 decimal places:

i $P(X \le 3)$ ii $P(X \le 2)$ iii $P(X > 5)$

iv $P(X \ge 4)$ v $P(2 \le X \le 5)$

7 a Show that $y = \cos x$ is a probability density function in the domain $\left[\frac{3\pi}{2}, 2\pi\right]$.

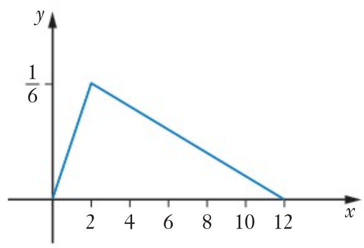
b Find the cumulative distribution function.

c Find each probability in exact form:

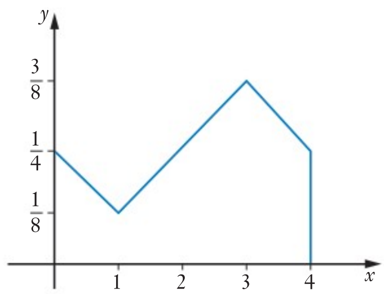
i $P\left(X \le \frac{5\pi}{3}\right)$ ii $P\left(X \ge \frac{7\pi}{4}\right)$ iii $P\left(\frac{5\pi}{3} \le X \le \frac{11\pi}{6}\right)$

8 Find the mode of each continuous probability distribution.

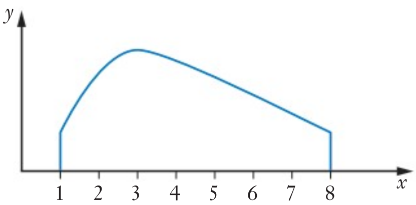
a



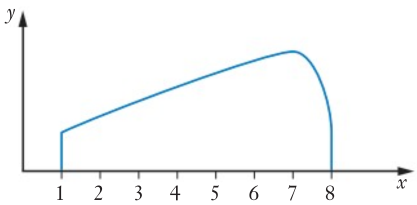
b



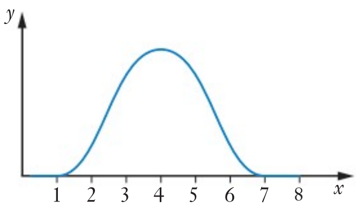
c



d



e



f $f(x) = -\frac{3(x^2 - 8x - 9)}{434}$ defined in the domain $[0, 7]$

g $f(x) = \frac{4e^{4x}}{e^8(e^{16} - 1)}$ defined in the interval $2 \le x \le 6$

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h $f(x) = -\frac{3(x^2 - 16x + 15)}{1100}$ defined in the domain [1, 11]

i $f(x) = \frac{2(2x^3 - 33x^2 + 168x + 3)}{2105}$ defined in the interval $0 \leq x \leq 5$

j $f(x) = \frac{3x^2}{342}$ defined in the interval $1 \leq x \leq 7$

9 a Find the mode of the function $f(x) = -\frac{3}{22}(x^2 - 6x + 5)$ defined on the domain [2, 4].

b Find the cumulative distribution function.

c Find $P(X \leq a)$ where a is the mode.

10 The times that athletes took to finish a race varied between 3 and 7 minutes and are represented by the continuous probability function $f(x) = \frac{1}{116}(x^3 - 9x^2 + 24x + 1)$ defined in the domain [3, 7].

a Find the cumulative distribution function.

b Find the probability that an athlete will finish this race:

i in less than 5 minutes

ii in 4 minutes or more

iii in between 4 and 5 minutes

c What is the most likely time in which an athlete would finish the race?

10.04 Quantiles

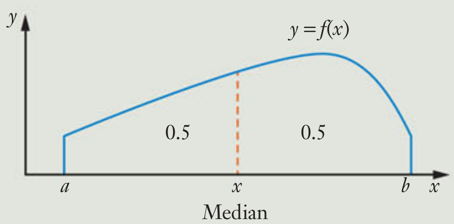
WS Quantiles

Median

For a continuous probability distribution, the median is the value of x that splits the distribution into halves. Because the PDF has an area of 1, the area on each side of the median is $\frac{1}{2}$.

Median

The median lies at the point x where $\int_a^x f(x)\,dx = 0.5$ given $y = f(x)$ is a PDF defined in the domain [a, b].



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The median of a continuous probability distribution

EXAMPLE 9

Find the median of the continuous probability distribution defined as $f(x) = \frac{x^2}{21}$ in the domain [1, 4].

Solution

For $f(x) = \frac{x^2}{21}$ defined in the domain [1, 4], first find the cumulative distribution function (CDF):

$$\int_1^x \frac{x^2}{21}\,dx = \frac{1}{21}\int_1^x x^2\,dx$$

$$= \frac{1}{21}\left[\frac{x^3}{3}\right]_1^x$$

$$= \frac{1}{21}\left(\frac{x^3}{3} - \frac{1^3}{3}\right)$$

$$= \frac{1}{21}\left(\frac{x^3 - 1}{3}\right)$$

$$= \frac{x^3 - 1}{63}$$

For the median:

$$\int_a^x f(x)\,dx = 0.5$$

$$\frac{x^3 - 1}{63} = 0.5$$

$$x^3 - 1 = 31.5$$

$$x^3 = 32.5$$

$$x = \sqrt[3]{32.5}$$

$$\approx 3.2$$

So the median is 3.2.

This means $P(X < 3.2) = 0.5$.

Quartiles, deciles and percentiles

You learned about quartiles, deciles and percentiles in Chapter 7, Statistics. They are values that separate a proportion of a set of data. For example, $Q_1 >$ bottom 25% of scores, $Q_3 >$ bottom 75% of scores, 2nd decile > bottom 20% of scores and 67th percentile > bottom 67% of scores.

EXAMPLE 10

A continuous probability distribution is defined as $f(x) = \frac{x^4}{11605}$ in the domain [4, 9].

Find, correct to 2 decimal places:

a the 1st quartile

b the 38th percentile

c the 7th decile

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Solution

First, find the CDF.

$$\int_4^x \frac{x^4}{11605}\,dx = \frac{1}{11605}\int_x^4 x^4\,dx$$

$$= \frac{1}{11605}\left[\frac{x^5}{5}\right]_4^x$$

$$= \frac{1}{11605}\left(\frac{x^5}{5} - \frac{4^5}{5}\right)$$

$$= \frac{1}{11605}\left(\frac{x^5 - 1024}{5}\right)$$

$$= \frac{x^5 - 1024}{58\,025}$$

a 1st quartile: 25%

$$\int_a^x f(x)\;dx = 0.25$$

$$\frac{a^5 - 1024}{58\,025} = 0.25$$

$$a^5 - 1024 = 14\,506.25$$

$$a^5 = 15\,530.25$$

$$a \approx 6.89$$

So the 1st quartile is 6.89.

This means $P(X < 6.89) = 0.25$.

b 38th percentile: 38%

$$\int_a^x f(x)\;dx = 0.38$$

$$\frac{a^5 - 1024}{58\,025} = 0.38$$

$$a^5 - 1024 = 22\,049.5$$

$$a^5 = 23\,073.5$$

$$a \approx 7.46$$

So the 38th percentile is 7.46.

This means $P(X < 7.46) = 0.38$.

c 7th decile: 70%

$$\int_a^x f(x)\;dx = 0.7$$

$$\frac{a^5 - 1024}{58\,025} = 0.7$$

$$a^5 - 1024 = 40\,617.5$$

$$a^5 = 41\,641.5$$

$$a \approx 8.39$$

So the 7th decile is 8.39.

This means $P(X < 8.39) = 0.7$.

Exercise 10.04 Quantiles

1 Find the median of each continuous random variable correct to 2 decimal places.

a $f(x) = \dfrac{3x^2}{511}$ defined on the interval $1 \le x \le 8$

b $f(x) = \dfrac{4x^3}{2401}$ defined in the domain $[0, 7]$

c $f(x) = \dfrac{5x^4}{16\,807}$ in the interval $0 \le x \le 7$

d $f(x) = \dfrac{3(x-3)^2}{16}$ in the domain $[1, 5]$

e $f(x) = \dfrac{(3x+1)^2}{244}$ in the interval $0 \le x \le 4$

f $f(x) = \dfrac{4x^3}{6560}$ defined in the domain $[1, 9]$

g $f(x) = \dfrac{3x^2}{1034}$ defined in the domain $[3, 11]$

h $f(x) = \dfrac{6x^5}{15\,625}$ in the interval $0 \le x \le 5$

i $f(x) = \dfrac{(2x-1)^4}{16105}$ in the domain $[1, 6]$

j $f(x) = \dfrac{x(x^2-3)^3}{3570}$ defined in the domain $[2, 4]$

2 For each continuous probability distribution, find:

i the 1st quartile   ii the 2nd decile   iii the 77th percentile

a $f(x) = \dfrac{3x^2}{973}$ defined in the domain $[3, 10]$

b $f(x) = \dfrac{x^3}{324}$ defined in the interval $0 \le x \le 6$

c $f(x) = \dfrac{5x^4}{3124}$ defined in the interval $1 \le x \le 5$

3 For the continuous probability distribution $f(x) = \dfrac{3x^2}{512}$ defined in the domain $[0, 8]$, find:

a the median

b the 35th percentile

4 For the continuous probability distribution $f(x) = \dfrac{x^2}{168}$ defined on the interval $2 \le x \le 8$, find:

a the median   b the 1st quartile   c the 3rd quartile

d the 67th percentile   e the 14th percentile   f the 8th decile

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5 For the continuous probability distribution defined as $f(x) = \frac{x^2}{576}$ on the interval $0 \le x \le 12$, find:

a the 20th percentile

b the median

c the 3rd quartile

6 For the continuous probability distribution $f(x) = \frac{x^3}{1020}$ defined in the interval $2 \le x \le 8$, find:

a the cumulative probability function

b $P(X \le 5)$

c $P(X > 4)$

d $P(3 \le X \le 7)$

e the median

f the 3rd quartile

g the 9th decile

h the 23rd percentile

10.05 Normal distribution

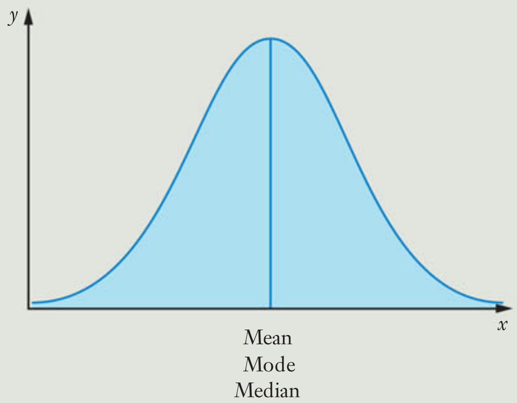
WS The normal distribution

The normal distribution is a special continuous probability distribution. Its probability density function is often called a bell curve because of its shape.

Normal distribution

The normal distribution is a symmetrical bell-shaped function.

The mean, mode and median are equal, at the centre of the probability density function.



There are many examples of data that are normally distributed, such as IQ, birth weights, ages, reaction times and exam results in a school.

We use the population mean μ and standard deviation σ for the normal distribution.

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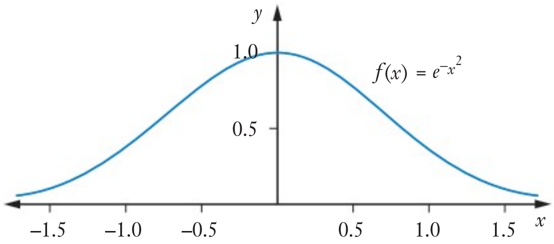
TECHNOLOGY

The normal distribution

A good estimate for the shape of the normal distribution is $f(x) = e^{-x^2}$.



Use your graphing techniques and technology to sketch this function.

Given a population that is normally distributed with mean $\mu = 10$ and standard deviation $\sigma = 2$, use technology to sketch the graph of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

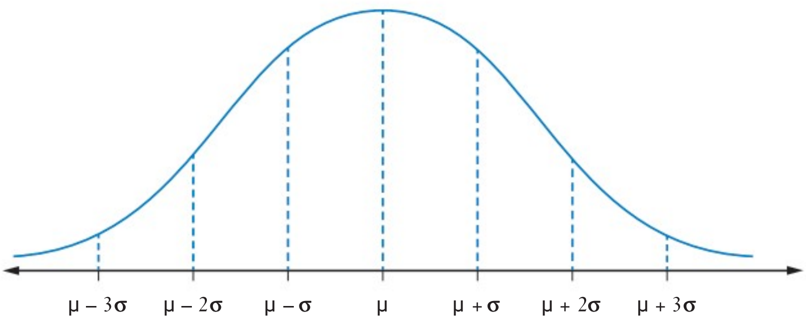
This is the actual equation of the normal distribution.

What do you notice about this graph? Use technology and the trapezoidal rule with many subintervals to approximate the area under the graph. What do you find?

Research the normal distribution and its features.

Graphing a normal distribution

In a normal distribution, most of the data lies within 3 standard deviations of the mean, with the mean in the centre.



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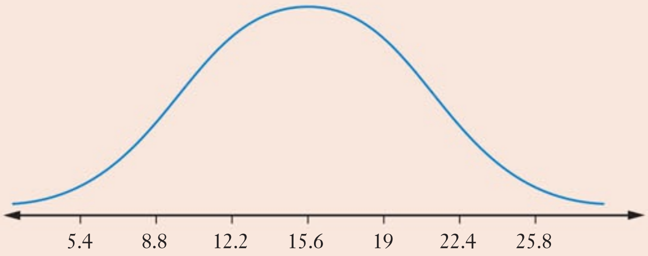
EXAMPLE 11

a A set of data is normally distributed with mean 8.3 and standard deviation 1.2. Sketch the probability distribution function.

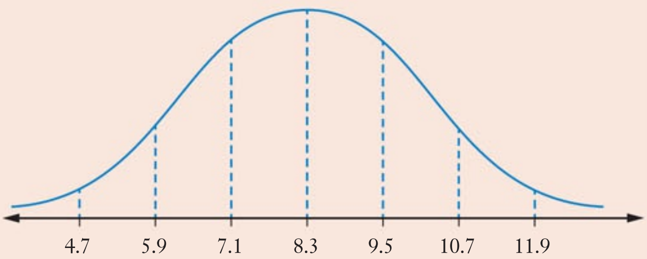
b A normal distribution has the probability density function below. Find its mean and standard deviation.



Solution

a

$$\begin{aligned} \mu + \sigma &= 8.3 + 1.2 \\ &= 9.5 \\ \mu + 2\sigma &= 8.3 + 2(1.2) \\ &= 10.7 \\ \mu + 3\sigma &= 8.3 + 3(1.2) \\ &= 11.9 \\ \mu - \sigma &= 8.3 - 1.2 \\ &= 7.1 \\ \mu - 2\sigma &= 8.3 - 2(1.2) \\ &= 5.9 \\ \mu - 3\sigma &= 8.3 - 3(1.2) \\ &= 4.7 \end{aligned}$$



b $\mu = 15.6$ (centre of the distribution)

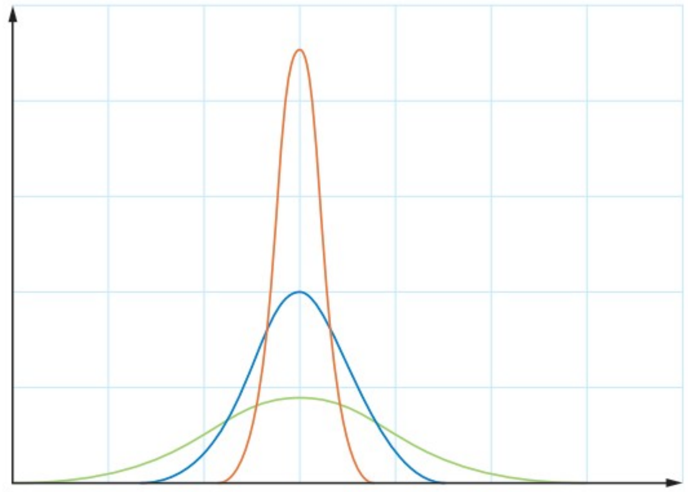
$\mu + \sigma = 19$

$15.6 + \sigma = 19$

$\sigma = 19 - 15.6$

$= 3.4$ (check that all the differences between all values on the graph are 3.4)

The normal distribution can have different shapes depending on the size of the standard deviation. In the diagram, the green curve shows the normal distribution with the highest standard deviation.



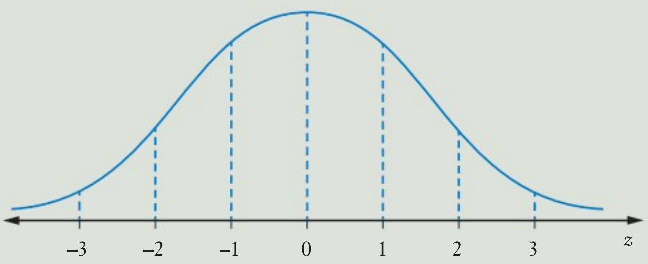
Standard normal distribution

It is difficult to find probabilities in the normal distribution by integration because the equation of the cumulative distribution function is complicated. To get around this, tables of probabilities have been developed from the standard normal distribution. This is a normal curve that has been transformed so that the mean is 0 and the standard deviation is 1.
The values of a standard normal distribution are called z rather than x, also known as z-scores or standardised scores.

Standard normal distribution

$\mu = 0$
$\sigma = 1$

Area under the curve is 1.



Probability tables for the standard normal distribution

WS
Normal distribution table

Probability tables for the standard normal distribution begin next page. A copy can also be downloaded from NelsonNet. Values represent the area to the left of (or less than) the z-score. Row labels show the z-score to one decimal place. Column labels show the second decimal place.

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z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
–3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
–3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
–3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
–3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
–3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
–2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
–2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
–2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
–2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
–2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
–2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
–2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
–2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
–2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
–2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
–1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
–1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
–1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
–1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
–1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
–1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
–1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
–1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
–1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
–1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
–0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
–0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
–0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
–0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
–0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
–0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
–0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
–0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
–0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
–0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

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EXAMPLE 12

For a standard normal distribution, use the table to find:

a $P(z \le 0.6)$

b $P(z \le -1.83)$

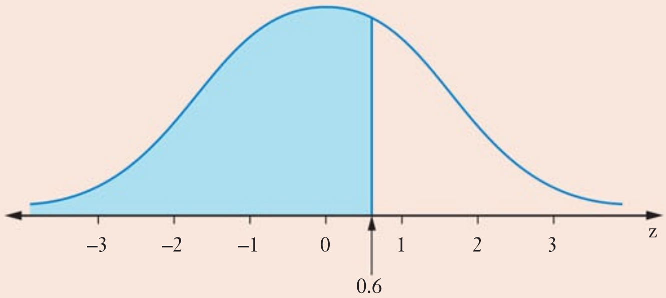
c $P(z < 2.34)$

d $P(z \ge -2.7)$

e $P(-0.3 \le z \le 1.4)$

Solution

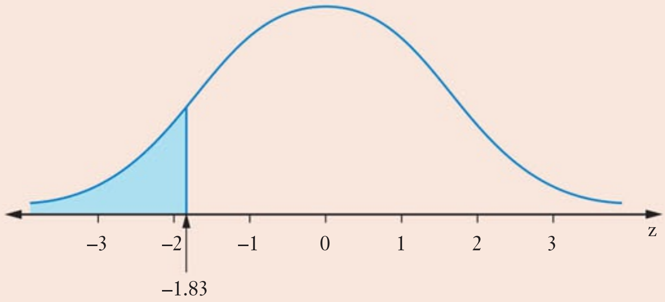
a The table gives the area under the PDF for the standard normal distribution.



Find 0.6 in the left column of the table.

$P(z \le 0.6) = 0.7257$

b



For -1.83, find -1.8 in the left column of the table and the entry under 0.03 in this row.

$P(z \le -1.83) = 0.0336$

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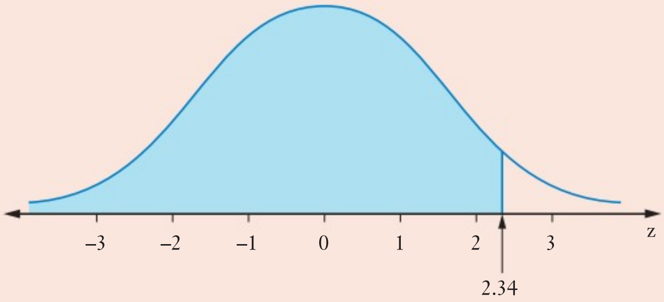
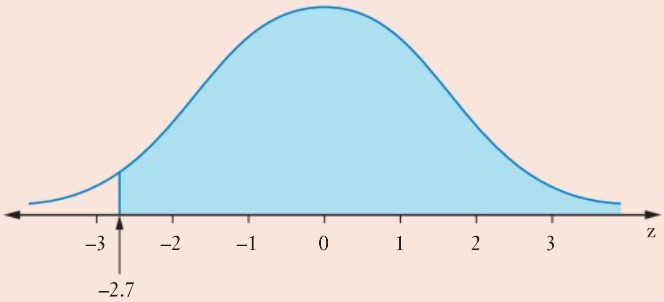
c



For 2.34, find 2.3 in the left column of the table and the entry under 0.04 in this row.

$P(z < 2.34) = 0.9904$

d



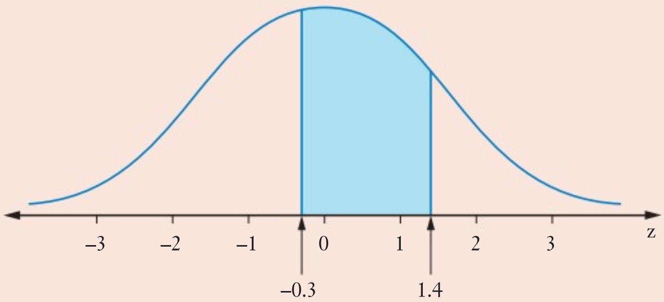
Find –2.7 in the left column of the table.

$P(z \leq -2.7) = 0.0035$

$P(z \geq -2.7) = 1 - 0.0035$

$= 0.9965$

e



$$\begin{aligned} P(-0.3 \leq z \leq 1.4) &= P(z \leq 1.4) - P(z \leq -0.3) \\ &= 0.9192 - 0.3821 \\ &= 0.5371 \end{aligned}$$

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Quartiles, deciles and percentiles

EXAMPLE 13

For a standard normal distribution, use the table on pages 441–442 to find:

a the median b the lower quartile c the upper quartile

d the 84th percentile e the 6th decile

Solution

a The median separates the bottom half of the scores.

We want to find the score with the cumulative probability of 0.5 in the table.

Median = 0 (this is the same value as the mean)

b The lower quartile separates the bottom 0.25 of the data.

In the table there are 2 values close to 0.25:

$P(z \leq -0.67) = 0.2514$ and $P(z \leq -0.68) = 0.2483$

0.2514 is closer to 0.25 so the lower quartile is approximately –0.67.

(Note: 0.2514 – 0.25 = 0.0014, and 0.25 – 0.2483 = 0.0017.)

c The upper quartile separates the bottom 0.75 of the data.

In the table there are 2 values close to 0.75:

$P(z \leq 0.67) = 0.7486$ and $P(z \leq 0.68) = 0.7517$

0.7486 is closer to 0.75 so the upper quartile is approximately 0.67.

Notice that the values for the lower and upper quartiles are ±0.67 because the normal distribution is symmetrical.

d The 84th percentile separates the bottom 0.84 of the data.

In the table there are 2 values close to 0.84:

$P(z \leq 0.99) = 0.8389$ and $P(z \leq 1) = 0.8413$

0.8389 is closer to 0.84 so the 84th percentile is approximately 0.99.

e The 6th decile separates the bottom 0.6 of the data.

In the table there are 2 values close to 0.6:

$P(z \leq 0.25) = 0.5987$ and $P(z \leq 0.26) = 0.6026$

0.5897 is closer to 0.6 so the 6th decile is approximately 0.25.

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EXAMPLE 14

Shade the area of the normal distribution where the values are:

a above the top 20% of data

b above the top 10% of data

Solution

a $P(z > a) = 20\%$, so $P(z \le a) = 80\%$ (0.8 or the 8th decile)

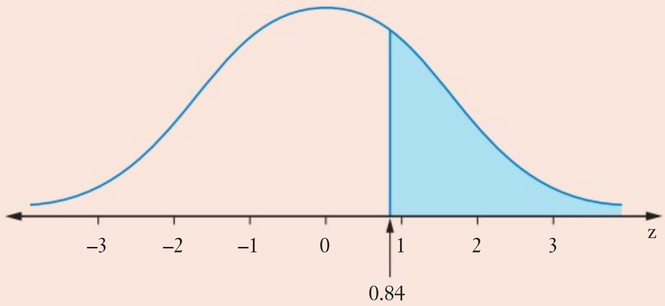
From the table:

$P(z \le 0.84) = 0.7995$ and $P(z \le 0.85) = 0.8023$

0.7995 is closer to 0.8 so the 8th decile is approximately 0.84

This means all values to the left of 0.84 lie below 80%.

So the top 20% lies to the right of 0.84.



b $P(z > a) = 10\%$, so $P(z \le a) = 90\%$ (0.9 or the 9th decile)

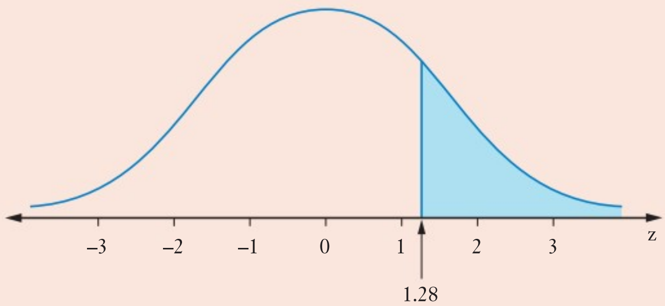
From the table:

$P(z \le 1.28) = 0.8997$ and $P(z \le 1.29) = 0.9015$

0.8997 is closer to 0.9 so the 9th decile is approximately 1.28.

This means all values to the left of 1.28 lie below 90%.

So the top 10% lies to the right of 1.28.



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Exercise 10.05 Normal distribution

1 Draw a probability density function for the normal distribution with:

a mean 9 and standard deviation 2

b mean 8.6 and standard deviation 0.3

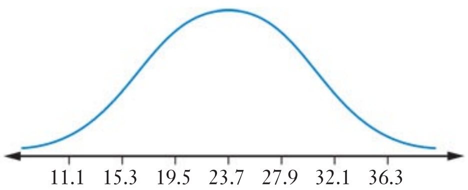
c mean 11.5 and standard deviation 1.4

d mean 27 and standard deviation 2.5

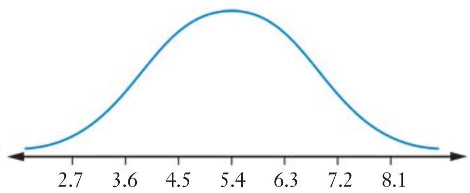
e mean 115.2 and standard deviation 3.2

2 What is the mean and standard deviation of each normal distribution?

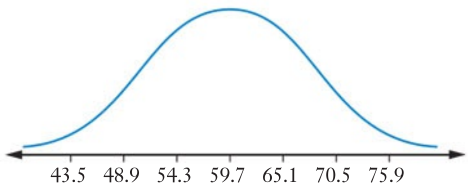
a



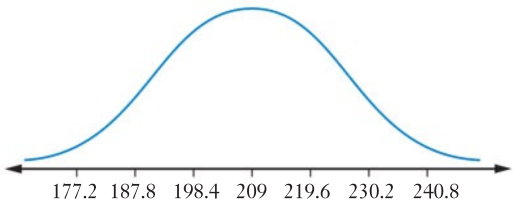
b



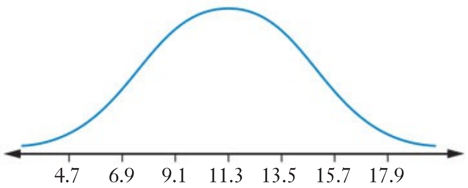
c



d



e



3 Draw the probability density function for a standard normal distribution.

4 Use the probability table for a standard normal distribution on pages 441–442 to find:

a $P(z \le 0)$ b $P(z \le 1)$

c $P(z \le 2)$ d $P(z \le 3)$

e $P(z \le -1)$ f $P(z \le -2)$

g $P(z \le -3)$ h $P(z \le 1.5)$

i $P(z < -2.67)$ **j** $P(z \le 3.09)$

5 Use the probability table to find:

a $P(z \ge -0.46)$ b $P(z > 2.11)$

c $P(z \ge -2.01)$ d $P(-2.4 \le z \le -1.76)$

e $P(-2.2 \le z \le 2.2)$ f $P(1.21 < z < 1.89)$

g $P(-1.45 \le z \le 3.1)$ h $P(-1 \le z \le 1)$

i $P(-2 \le z \le 2)$ **j** $P(-3 \le z \le 3)$

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6 Use the probability table to find:

a the 8th decile
b the 3rd quartile
c the 29th percentile
d the 2nd decile
e the 89th percentile
f the 12th percentile
g the 3rd decile
h the 1st quartile
i the 63rd percentile

7 Sketch a normal curve and draw on it the area where values lie in the:

a bottom 30%
b bottom 15%
c top 67%
d top 24%
e top 12%

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10.06 Empirical rule

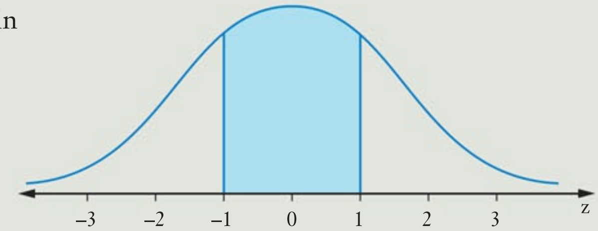
In Question 5 of the previous exercise you calculated $P(-1 \le z \le 1)$, $P(-2 \le z \le 2)$ and $P(-3 \le z \le 3)$, the probability that values in a normal distribution will fall within 1, 2 or 3 standard deviations of the mean respectively. These probabilities are part of the empirical rule.

WS The empirical rule

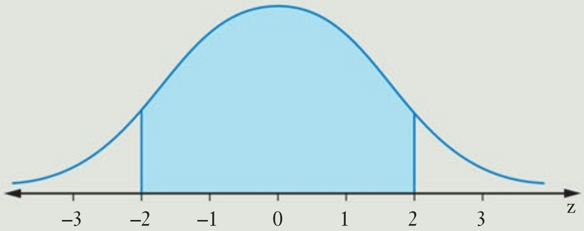
WS The normal curve

Empirical rule

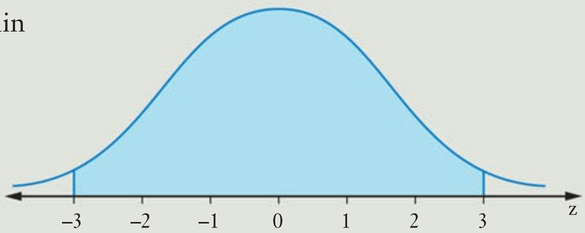
Approximately 68% of scores lie within 1 standard deviation of the mean, with z-scores between –1 and 1.



Approximately 95% of scores lie within 2 standard deviations of the mean, with z-scores between –2 and 2.



Approximately 99.7% of scores lie within 3 standard deviations of the mean, with z-scores between –3 and 3.



EXAMPLE 15

a A data set is normally distributed with mean 20 and standard deviation 3.

What percentage of scores lie between:

i 17 and 23? ii 14 and 26? iii 11 and 29?

b A normal distribution has $\mu = 65.2$ and $\sigma = 1.3$. What percentage of scores lie between:

i 61.3 and 65.2? ii 65.2 and 66.5? iii 62.6 and 69.1?

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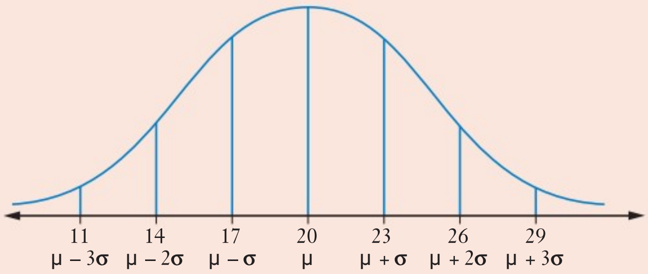
Solution

a We can draw the PDF for the normal curve.



i Scores between 17 and 23 are within 1 standard deviation of the mean.

So about 68% of scores lie between 17 and 23.

ii Scores between 14 and 26 are within 2 standard deviations of the mean.

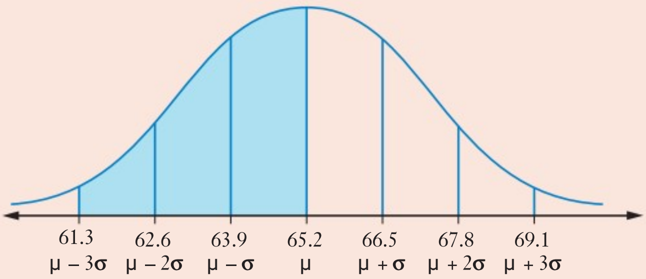
So about 95% of scores lie between 14 and 26.

iii Scores between 11 and 29 are within 3 standard deviations of the mean.

So about 99.7% of scores lie between 11 and 29.

b We can draw the PDF for the normal curve.

i



Scores between 61.3 and 69.1 are within 3 standard deviations of the mean (99.7% of data).

Scores between 61.3 and 65.2 are half this area.

$99.7\% \div 2 = 49.85\%$

So 49.85% of scores lie between 61.3 and 65.2.

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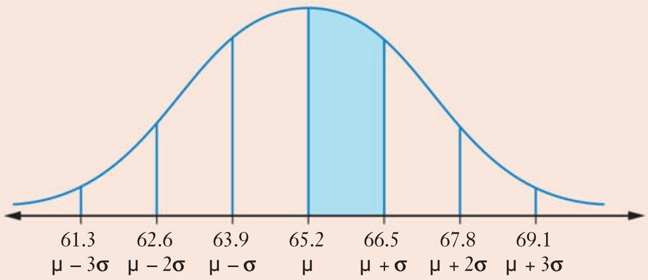
ii



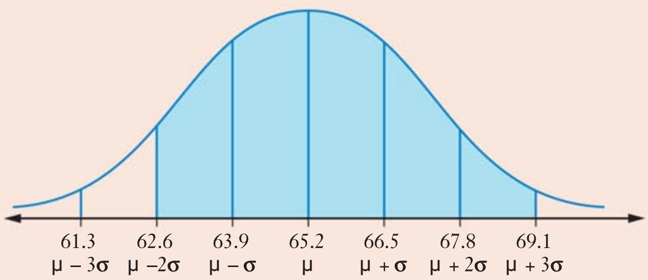
Scores between 63.9 and 66.5 are within 1 standard deviation of the mean (68% of data).

Scores between 65.2 and 66.5 are half this area.

$68\% \div 2 = 34\%$

So 34% of scores lie between 65.2 and 66.5.

iii



Scores between 62.6 and 65.2 are half the area within 2 standard deviations of the mean $\left(\frac{1}{2} \times 95\%\right)$.

Scores between 65.2 and 69.1 are half the area within 3 standard deviations of the mean $\left(\frac{1}{2} \times 99.7\%\right)$.

$$\begin{aligned}\text{Total area} &= \frac{1}{2} \times 95\% + \frac{1}{2} \times 99.7\% \\ &= 47.5\% + 49.85\% \\ &= 97.35\%\end{aligned}$$

So 97.35% of scores lie between 62.6 and 69.1.

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Exercise 10.06 Empirical rule

1 What is the approximate percentage of data in a normal distribution that lies within:

a 1 standard deviation of the mean?

b 2 standard deviations of the mean?

c 3 standard deviations of the mean?

2 A set of data has a mean of 15 and a standard deviation of 1.5. If the data set is normally distributed, find the percentage of data that lies between:

a 13.5 and 16.5 b 12 and 18 c 10.5 and 19.5

3 A set of data is normally distributed with a mean of 8.4 and a standard deviation of 0.9. Find the percentage of data that lies between:

a 7.5 and 9.3 b 6.6 and 10.2 c 5.7 and 11.1

4 A set of data is normally distributed with mean 18 and standard deviation 2. Find the percentage of data that lies between:

a 16 and 20 b 14 and 22 c 12 and 24

d 16 and 18 e 18 and 24 f 12 and 22

5 A normal distribution has a mean of 65 and standard deviation 4.

a Sketch the graph of its PDF.

b Find the percentage of data that lies between:

i 57 and 73 ii 61 and 65 iii 65 and 77

iv 57 and 69 v 61 and 73

6 A normal distribution has a mean of 9.7 and standard deviation 2.1. Find the percentage of data that lies between:

a 7.6 and 11.8 b 9.7 and 11.8 c 9.7 and 13.9

d 5.5 and 9.7 e 3.4 and 11.8

WS The standard normal curve

WS z-scores

WS Areas under the normal curve

z-scores

10.07 z-scores

We can transform any normal distribution into a standard normal distribution by using z-scores.

z-scores

To convert a raw score, x, into a z-score, use the formula:

$$z = \frac{x - \mu}{\sigma}$$

where μ is the mean and σ is the standard deviation of the distribution.

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EXAMPLE 16

A data set is normally distributed with mean 15 and standard deviation 2.

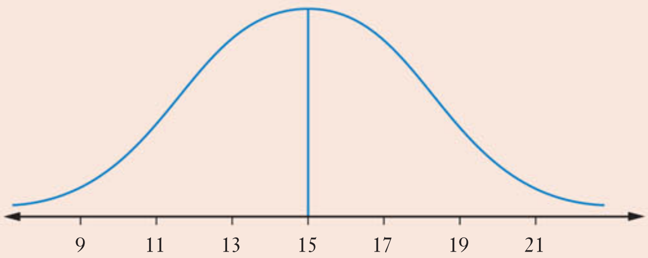
a Draw the probability density function for this distribution.

b Use z-scores to convert each x value to standardised scores:

i $x = 19$ ii $x = 15$ iii $x = 9$

Solution

a Drawing the PDF gives the curve below.



b $\mu = 15$ and $\sigma = 2$

i For $x = 19$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{19 - 15}{2}$$

$$= 2$$

This is because 19 is 2 standard deviations above the mean.

ii For $x = 15$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{15 - 15}{2}$$

$$= 0$$

This is because 15 is the mean.

iii For $x = 9$:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{9 - 15}{2}$$

$$= -3$$

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EXAMPLE 17

A data set is normally distributed with mean 23.8 and standard deviation 1.2.

a Find the z-score for each raw score and describe where it is on the standard normal distribution.

i 25.7 ii 20.6 iii 28.3

b Which of the scores from part a are very unlikely?

c Find the value of a raw score whose z-score is -1.82, correct to 2 decimal places.

Solution

a i For $x = 25.7$:

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{25.7-23.8}{1.2}$$
$$\approx 1.58$$

A score of 25.7 lies 1.58 standard deviations above (to the right of) the mean.

ii For $x = 20.6$:

$$z = \frac{x-\mu}{\sigma}$$
$$= \frac{20.6-23.8}{1.2}$$
$$\approx -2.67$$

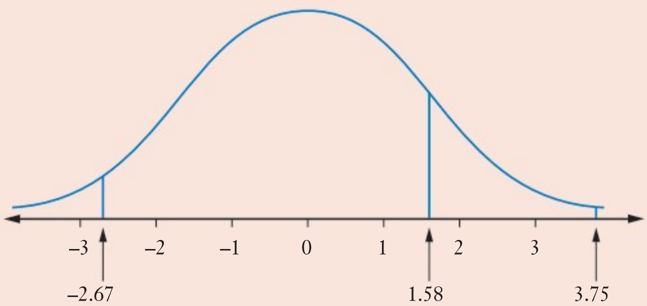
A score of 20.6 lies 2.67 standard deviations below (to the left of) the mean.

iii For $x = 28.3$:

$$z = \frac{28.3-23.8}{1.2}$$
$$\approx 3.75$$

A score of 28.3 lies 3.75 standard deviations above (to the right of) the mean.

b We can draw each score on the standard normal distribution.



99.7% of scores lie within 3 standard deviations of the mean.
So a z-score of 3.75 is very unlikely.

This means that a score of 28.3 is very unlikely.

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c

$$z = \frac{x - \mu}{\sigma}$$

$$-1.82 = \frac{x - 23.8}{1.2}$$

$$-2.184 = x - 23.8$$

$$21.616 = x$$

So the raw score is 21.62.

Exercise 10.07 z-scores

1 A data set is normally distributed with mean 18 and standard deviation 1.3.

a Find the z-score for each raw score.

i 18 ii 19.3 iii 20.6

iv 21.9 v 16.7 vi 15.4

vii 14.1

b Which raw score has a z-score of:

i 1.5? ii −2.1?

2 The length of fish caught in a fishing competition had a mean of 53.1 cm and standard deviation 8.7.

a If the lengths of fish almost certainly lie within 3 standard deviations of the mean, between which lengths would the fish almost certainly lie?

b Find the z-score for each length.

i 53.1 cm ii 61.8 cm iii 44.4 cm

iv 70.5 cm v 35.7 cm vi 79.2 cm

vii 27 cm viii 65 cm

3 A sample of overnight temperatures at Thredbo in June showed a mean temperature of 6.8°C and a standard deviation of 1.1.

a Almost all temperatures lie within 3 standard deviations of the mean. Within what range do almost all temperatures lie?

b Find the z-score for each temperature.

i 6.8°C ii 7.9°C iii 9°C

iv 10.1°C v 5.7°C vi 4.6°C

vii 3.5°C viii 6°C

4 A survey showed that the mean volume of juice in an orange is 66.4 mL with a standard deviation of 5.8.

a The volume of juice very probably lies within 2 standard deviations of the mean. Between which 2 volumes do they lie?

b Find the z-scores for each volume.

i 66.4 mL	ii 72.2 mL	iii 78 mL
iv 83.8 mL	v 60.6 mL	vi 54.8 mL
vii 49 mL	viii 90 mL	

c Find the volume that has a z-score of:

i 1.2	ii 2.9	iii −0.6	iv −2.3

5 a Find the z-score for each raw score below if the mean is 68 and the standard deviation is 4.5.

i 80	ii 53.2	iii 78.6
iv 62.1	v 90	vi 59.7
vii 82.7	viii 56.4	

b From your answers to part a, which scores are most unlikely?

c Which scores in part a lie within 2 standard deviations of the mean?

d Which scores in part a lie within 3 standard deviations of the mean?

6 The mean diameter of a batch of circular discs is 14.2 mm with standard deviation 1.4.

a What is the z-score for a disc with a diameter of:

i 16 mm?	ii 12 mm?

b Find the diameter of a disc with z-score:

i −2.1	ii 1.3	iii 3.2
iv −0.76	v 1.95	

7 A set of data that is normally distributed has a mean of 23 and standard deviation 2. Which raw score has a z-score of 2.5?

8 A data set that is normally distributed has a standard deviation of 4.5. A score of 39 has a z-score of 2.7. What is the mean?

9 Find the standard deviation of a normally distributed data set if the mean is 89 and a raw score of 59 has z-score of −0.6.

10 A set of data that is normally distributed has a mean of 53.4 and standard deviation of 5.6. Find the raw score that has a z-score of:

a 0	b −2	c 1
d 2.8	e −1.7	

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11 The standard deviation of a normal distribution is 3.3 and the z-score of 45 is –1. Calculate the mean.

12 The mean of a normally distributed data set is 16 and standard deviation is 1.9.

a Find the scores between which:

i 95% of data lies ii 68% of data lies iii 99.7% of data lies

b Calculate the z-score of:

i 20 ii 13.5

c Find the raw score that has a z-score of:

i –3 ii 1.1

13 A normal distribution has a mean of 104.7 and standard deviation 5.1.

a Find the scores that lie within 1 standard deviation of the mean.

b Calculate the z-score of:

i 80 ii 103

c Which score has a z–score of:

i 2? ii –1.3?

10.08 Applications of the normal distribution

The normal distribution is often used in quality control and predicting outcomes.

WS Applying the normal distribution

WS Normal distribution worded problems 1

WS Normal distribution worded problems 2

WS Continuous random variables assignment

WS Continuous random variables problems

EXAMPLE 18

a A company produces 1 kg packets of sugar. A quality control check found that the weight of the packets was normally distributed with a mean weight of 0.995 kg and standard deviation 0.03 kg. The company policy is to reject any packet with a weight outside 2 standard deviations from the mean.

i What is the smallest weight allowed by the company?

ii What is the largest weight allowed?

iii What percentage of packets will be rejected?

iv What percentage of large packets will be rejected?

b The mean shelf life of a spice is 13.4 weeks and the standard deviation is 1.8 weeks.

i Would a shelf life of 20 weeks be unusual? Why?

ii Find the z-score for a shelf life of 15.5 weeks.

iii What percentage of shelf lives would be expected to be between 13.4 and 15.5 weeks?

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Solution

a $\mu = 0.995$

i Smallest weight allowed:

$\mu - 2\sigma = 0.995 - 2 \times 0.03$

$= 0.935$

ii Largest weight allowed:

$\mu + 2\sigma = 0.995 + 2 \times 0.03$

$= 1.055$

iii About 95% of weights lie within 2 standard deviations of the mean, so 5% will lie outside this area.

The company rejects 5% of the packets of sugar.

iv Since the normal distribution is symmetrical, the 5% is made up of 2.5% of larger and 2.5% of smaller packets.

So the company rejects 2.5% of larger packets.

b i $\mu = 13.4$

$\mu - 3\sigma = 13.4 - 3 \times 1.8$

$= 8$

$\mu + 3\sigma = 13.4 + 3 \times 1.8$

$= 18.8$

So the shelf life of spices almost certainly lies between 8 and 18.8 weeks.

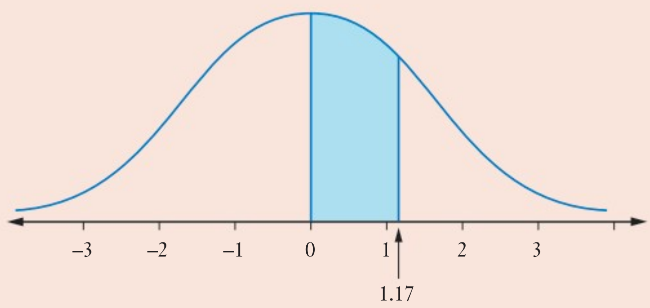
A shelf life of 20 weeks is outside this range, so it would be unusual.

ii $z = \dfrac{x - \mu}{\sigma}$

$= \dfrac{15.5 - 13.4}{1.8}$

$= 1.17$

iii



$\mu = 13.4$ so its z-score $= 0$

$P(13.4 \le X \le 15.5) = P(0 \le z \le 1.17)$

$= P(X \le 1.17) - P(X \le 0)$

$= 0.8790 - 0.5000$ (using the table on pages 441–442)

$= 0.379$

So 37.9% of shelf lives would be expected to be between 13.4 and 15.5 weeks.

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Using z-scores also allows us to compare 2 data sets.

EXAMPLE 19

In Year 7 at a school the mean weight of students was 59.4 kg and the standard deviation was 3.8 kg. In Year 8, the mean was 63.5 kg and the standard deviation was 1.7 kg.

John in Year 7, and Deng in Year 8, both weighed 68 kg. Which student was heavier in relation to his Year?

Solution

For John:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 59.4}{3.8}$$

$$\approx 2.263$$

For Deng:

$$z = \frac{x - \mu}{\sigma}$$

$$= \frac{68 - 63.5}{1.7}$$

$$\approx 2.647$$

A z-score of 2.647 is higher than a z-score of 2.263.

So Deng weighed more in relation to his Year group than John.

Exercise 10.08 Applications of the normal distribution

1 A machine at the mint produces coins with a mean diameter of 24 mm and a standard deviation of 0.2 mm.

- a What percentage of coins will have a diameter between:
  - i 23.4 mm and 24.6 mm?
  - ii 24 mm and 24.4 mm?
  - iii 23.8 mm and 24.2 mm?
  - iv 23.4 mm and 24 mm?
  - v 23.4 mm and 24.2 mm?
- b A coin selected at random has a diameter of 24.8 mm.
  - i What is its z-score?
  - ii Is this diameter unusual? What could you say about this?
- c
  - i Convert a diameter of 23.7 mm to a z-score.
  - ii Find the probability of producing a coin with a diameter of less than 23.7 mm.
  - iii Find the probability of producing a coin with a diameter between 23.7 mm and 23.9 mm.

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2 The maximum temperature in April is normally distributed with a mean of 25.3°C and a standard deviation of 3.4°C.

a What percentage of the time in April would you expect the temperature to be between:

i 21.9° and 28.7°? ii 18.5° and 32.1°? iii 15.1° and 35.5°?

iv 25.3° and 28.7°? v 21.9° and 32.1°?

b The temperature drops to 14°.

i What is its z-score?

ii Is this temperature unusual? Why?

c The temperature rises to 38°.

i What is its z-score?

ii Is this temperature unusual? Why?

3 A company surveyed the amount of time it took to assemble its product. The times were normally distributed with mean 8 minutes and standard deviation 1.7 minutes.

a What is the z-score for a time of 15 minutes? Would this be an acceptable amount of time to assemble the product? Why?

b Between what times would 95% of times lie?

c What percentage of times would lie between 4.6 and 13.1 minutes?

d i Find the z-score for a time of 7 minutes 30 seconds.

ii What percentage of times would lie between 7 minutes 30 seconds and 8 minutes?

e Find the probability of times lying between:

i 6 and 8 minutes ii 6 and 9.7 minutes iii 5 and 10 minutes

f What percentage of times would lie between:

i 3 and 12 minutes? ii 7 and 11 minutes? iii 3.5 and 11.9 minutes?

4 A certain brand of perfume is sold in 20 mL bottles. The volume of perfume in the bottles was tested in a quality control check. The mean was found to be 19.9 mL with a standard deviation of 0.4 mL.

a What percentage of bottles have a volume between:

i 19.1 mL and 20.7 mL?

ii 19.9 mL and 21.1 mL?

iii 20 mL and 21 mL?

b Between which two volumes do 99.7% of bottles lie?

c Comment on a bottle that has a volume of 23 mL.

d Find the probability that a bottle of perfume will have a volume between 18.9 and 19.3 mL.

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5 A farmer is only allowed to deliver standard size apples to the markets. The mean diameter must be 7.5 cm with a standard deviation of 0.3 cm. Only apples within 2 standard deviations of the mean are allowed.

a What percentage of apples are allowed?

b What is the largest diameter allowed?

c What is the smallest diameter allowed?

6 A certain brand of car battery has a mean life of 3.1 years with a standard deviation of 0.3 years.

a What is the minimum life you could reasonably expect from a battery of this type?

b What percentage of batteries would have a life between:

i 2.8 and 3.4 years?

ii 2.5 and 3.5 years?

c Would a life of over 4 years be unusual? Why?

7 A Jack Russell terrier has a mean height of 28 cm and a standard deviation of 0.833 cm.

a What range of heights (to 1 decimal place) would you expect for this breed of dog?

b What is the z-score of a height of 27.2 cm?

c What percentage of dogs would be between 27.2 cm and 30 cm tall?

d Would a dog 24 cm tall be typical of this breed? Why?

8 A factory produces 5 kg bags of bread mix. In a quality check, the mean weight of a bag of bread mix was found to be 4.95 kg, with a standard deviation of 0.15 kg. The factory rejects bags that weigh less than 4.65 kg or more than 5.25 kg.

a What percentage of bags does the factory reject?

b What percentage of bags does the factory reject because their weight is too small?

c The manager of the factory decided that too many bags are being rejected and that in future only those outside the normal range of 3 standard deviations would be rejected.

i What percentage will be rejected?

ii What weights will be rejected?

9 A company manufactures steel rods with a mean diameter of 10.6 cm and a standard deviation of 0.5 cm. The manufacturer rejects rods that are outside acceptable limits.

a If the company rejects 0.3% of rods, find:

i the percentage of rods it accepts

ii the largest diameter it will accept

iii the smallest diameter it will accept

b In one batch of rods, the diameters are 10.9 cm, 9.6 cm, 8.3 cm, 11.4 cm and 12.6 cm. Which ones will be rejected?

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10 A manufacturer of canned fruit guarantees that the minimum weight in each can is 250 g. A random check showed that the mean weight was 252.5 g with a standard deviation of 0.4 g. Comment on this guarantee. Is it realistic? Why?

11 A butcher shop advertises that it will give a free leg of lamb to any customer who can prove that a packet of mince with a mean weight of 1 kg weighs less than 980 g. If the mean weight is 1 kg with a standard deviation of 10 g, what percentage of customers should expect to receive a free leg of lamb?

12 The width of a type of door almost certainly lies within the range 814 mm to 826 mm (3 standard deviations of the mean).

a What is the standard deviation?

b What is the mean width of the door?

c Within what widths do 95% of these doors lie?

13 The mean time for a ferry to travel from one port to another is 3.1 hours. About 47.5% of the time the ferry takes between 3.1 hours and 3.5 hours.

a What is the standard deviation?

b What is the minimum time you would expect the ferry to take?

14 Xavier takes 78 minutes to drive to Epping and 65 minutes to drive to the city. The mean time to Epping is 75.3 minutes with a standard deviation of 2.6, while the mean time to the city is 62.7 minutes with a standard deviation of 1.7.

a Calculate the z-scores for Xavier's trips to Epping and the city.

b Which was the longer trip in comparison with the mean?

15 Kieran scored 78 in an exam where the mean was 69.5 and the standard deviation was 8.5. Cameron scored 71 in an exam where the mean was 61.2 and the standard deviation was 4.8.

a Find Kieran's z-score.

b Calculate Cameron's z-score.

c Which student scored higher in comparison with the other students in each exam?

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10. TEST YOURSELF

For Questions 1 to 3, choose the correct answer A, B, C or D.

WS
Formula sheet: Statistics and probability

Qz
Practice quiz

1 Which function does not describe a continuous probability distribution?

A $f(x) = \frac{x^2}{21}$ for the interval $1 \le x \le 4$

B $f(x) = \frac{e^x}{e^3 - 1}$ in the domain $[0, 3]$

C $f(x) = \frac{x^4}{625}$ in the domain $[1, 5]$

D $f(x) = \frac{4x^3}{625}$ in the interval $0 \le x \le 5$

2 The percentage of z-scores between -2 and 2 is:

A 95% B 68% C 47.5% D 49.85%

3 Which random variable is not continuous?

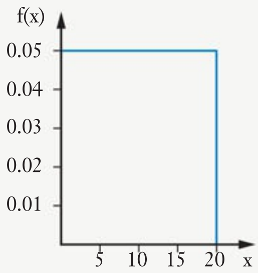
A The temperature of different freezers

B The mass of rocks found at the site of a volcano

C The length of the arms of people

D Shoe sizes of people

4 For the uniform continuous probability distribution shown, find:



a $P(X \le 15)$ b $P(X \le 8)$

c $P(7 \le X \le 18)$ d $P(4 < X < 13)$

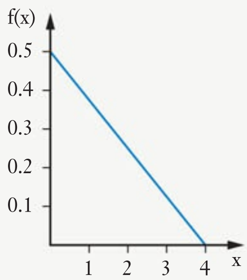
e $P(X \ge 6)$ f the median

g the 18th percentile h the 89th percentile

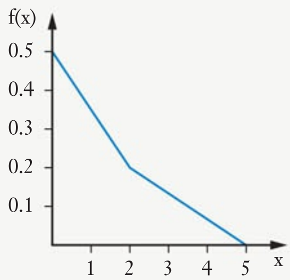
i the 6th decile j the 3rd quartile

5 State whether each graph represents a probability density function.

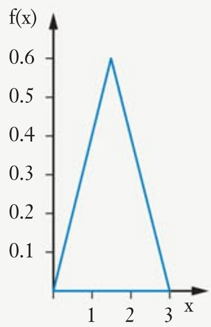
a



b



c



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6 Find the median of the continuous random variable $f(x) = \frac{3x^2}{124}$ defined on the interval $1 \le x \le 5$. Answer correct to 2 decimal places.

7 a Find the cumulative distribution function for $f(x) = \begin{cases} \frac{3x^2}{511} & \text{for } 1 \le x \le 8 \\ 0 & \text{for all other } x \end{cases}$

b Find:

i $P(X \le 3)$ ii $P(X \le 5)$ iii $P(X > 6)$

iv $P(X \ge 4)$ v $P(2 \le X \le 7)$

8 For each continuous probability distribution, find:

i the mode ii the median

a $f(x) = \frac{2x}{15}$ defined in the domain $[1, 4]$

b $f(x) = \frac{x^2}{243}$ defined in the interval $0 \le x \le 9$

9 The birth weights of babies born at St John's Hospital were measured and found to be normally distributed with mean 3.2 kg and standard deviation 0.31.

a Find the range of weights in which 99.7% of the weights of these babies would lie.

b Find the z-score (to 2 decimal places) for a weight of:

i 3.9 kg ii 3.5 kg

c Use the standard normal probability table on pages 441–442 to find the probability that a baby born at the hospital would have a birth weight between 3.5 kg and 3.9 kg.

10 A function is given by $f(x) = \frac{3x^5}{2048}$. Over what domain starting at $x = 0$ is this a probability density function?

11 Draw a probability density function for the normal distribution with:

a mean 15 and standard deviation 0.5

b mean 3.4 and standard deviation 0.2

12 A factory produces 2 kg bags of nails. In a quality check, the mean weight of a bag of nails was found to be 1.95 kg, with a standard deviation of 0.08 kg. The factory rejects bags that weigh less than 1.71 kg or more than 2.19 kg.

a What percentage of bags does the factory reject?

b After a complaint, the manager decided to reject bags that weigh less than 1.87 kg or more than 2.11 kg. What percentage will be rejected?

13 For the PDF $f(x) = \frac{3x^2}{316}$ defined on the interval $3 \le x \le 7$, find:

a the median b the 3rd quartile c the 4th decile

d the 63rd percentile e the 28th percentile

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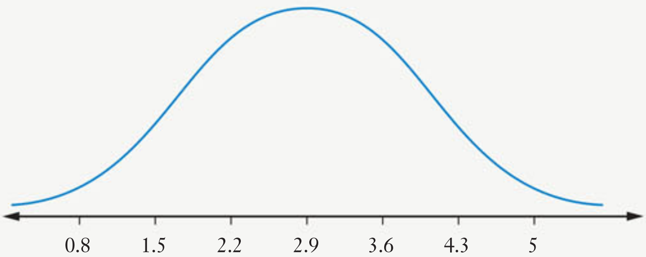
14 Find the mean and standard deviation of this normal distribution.



15 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \frac{x^4}{625}$ defined on the interval $0 \leq x \leq 5$

b $f(x) = \frac{x^6}{117\,649}$ defined on [0, 7]

c $f(x) = \frac{e^x}{e^6 - 1}$ in the interval $0 \leq x \leq 6$

d $f(x) = \frac{x}{40}$ for [1, 9]

16 Use the probability table for a standard normal distribution on pages 441–442 to find:

a $P(z \leq 0.54)$ b $P(z \leq 1.32)$ c $P(z \leq -3)$

d $P(z \leq -0.71)$ e $P(z \geq -1)$ f $P(z \geq 2.5)$

g $P(z \geq -1.08)$ h $P(-2.3 \leq z \leq -1.09)$ i $P(1.1 \leq z \leq 3.11)$

17 A set of data is normally distributed with mean 12.5 and standard deviation 1.5. Find the percentage of data that lies between:

a 9.5 and 15.5 b 12.5 and 14 c 11 and 17

d 10 and 15 e 12 and 13

18 A probability density function is given by $f(x) = ae^x$ over a certain domain. Find the exact value of a if the domain is:

a [0, 5] b [1, 4]

19 Shade on the standard normal distribution the area where values lie:

a in the bottom 20%

b in the bottom 32%

c in the top 15%

d in the top 30%

e between the bottom 10% and the top 40%

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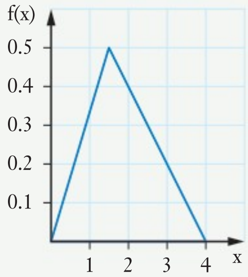
20 Find the mode of each continuous probability distribution.

a $f(x) = \dfrac{x}{30}$ defined in the domain [2, 8]

b



c $f(x) = \dfrac{4}{189}(x^3 - 9x^2 + 24x)$ defined in the domain [0, 3]

21 Klare took 9.3 s to finish a race where the mean was 8.3 s and the standard deviation was 1.2. Simon took 8.9 s to run in the next heat where the mean was 8.1 s and the standard deviation was 0.8.

a Find Klare's z-score.

b Calculate Simon's z-score.

c Which person had the better time in comparison with the other runners in their heat?

22 Show that $f(x) = \dfrac{x^3}{600}$ defined on the interval $1 \le x \le 7$ is a probability density function.

23 Circular tables are made with a mean radius of 1.1 m with standard deviation 0.02 m.

a Find the z-score for a table with a radius of:

i 1.15 m ii 1.07 m

b Find the radius of a table with z-score:

i 1 ii −2 iii 3.1 iv −0.63 v 1.27

24 A normal distribution has a standard deviation of 1.6. A score of 87.9 has a z-score of −1.3. What is the mean?

25 The standard deviation of a normal distribution is 1.9 and the z-score of 52.4 is 1.7. Calculate the mean.

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10. CHALLENGE EXERCISE

1 A probability density function is given by $f(x) = \dfrac{5x^4}{3124}$ over the interval $a \le x \le b$.

Find the value of a and b if the median of the PDF is 4.353 031.

2 A normal distribution has mean μ and standard deviation σ.
Evaluate μ and σ given that 95% of scores lie between 12.4 and 14.

3 Find the mode of the continuous probability distribution $f(x) = \dfrac{4}{249}(x+2)(x-4)^2$ defined in the domain [0, 3].

4 Find the cumulative distribution function for the continuous probability distribution

$f(x) = \dfrac{3x(x^2+1)^2}{62\,000}$ defined on the domain [3, 7].

5 For a normal distribution, $P(X \le 23.8) = 91.92\%$ and $P(X \le 17.15) = 30.85\%$.
Find the mean and standard deviation.

6 A good model for a normal distribution is the function $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Use technology to show that this is a probability density function given $\mu = 6$ and $\sigma = 1$.

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Practice set 4

In Questions 1 to 5, select the correct answer A, B, C or D.

1 An amount of \$6000 is invested at 3.5% p.a. with interest paid quarterly. Find the balance after 10 years.

A \$8501.45 B \$8463.59 C \$6546.16 D \$6899.96

2 Which function does not describe a continuous probability distribution?

A $f(x) = \dfrac{3x^2}{124}$ for the interval $1 \le x \le 5$

B $f(x) = \dfrac{8x(x^2+1)^3}{1000}$ in the domain $[0, 3]$

C $f(x) = 4x^3$ in the domain $[0, 1]$

D $f(x) = 2\cos 2x$ in the interval $0 \le x \le \dfrac{\pi}{4}$

3 The percentage of scores between z-scores of −3 and 3 is:

A 95% B 68% C 99.7% D 49.85%

4 The percentage of data in a normal distribution that lies within 2 standard deviations of the mean is:

A 68% B 95% C 99.7% D 34%

5 The median of a continuous probability distribution f(x) defined in the domain [a, b] is:

A x where $\int_a^b f(x)\,dx = x$

B x at the maximum value of $f(x)$

C x where $\int_a^x f(x)\,dx = 0.5$

D x where $\int_a^b 0.5\,dx = F(x)$

6 Find each integral.

a $\int e^{3x}\,dx$ b $\int (4x-3)\,dx$ c $\int \sec^2 4x\,dx$ d $\int \dfrac{dx}{x-3}$

7 Use the table of future values of an annuity on page 390 to answer each question.

a Mahmoud wants to deposit \$1000 at the end of each year for 20 years so he has a nest egg when he retires from work. If interest is 7% p.a., find how much Mahmoud will have.

b Georgina wants to save a certain amount at the end of each quarter for 3 years so that she will have \$10 000 for an overseas trip. If interest is 8% p.a. paid quarterly, how much will Georgina need to save each quarter?

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8 Alice promises her son a sum of money on his 18th birthday, made up of \$10 for his 1st year of life, \$15 for his 2nd year, \$20 for his 3rd year and so on, up to 18 years. How much will her son receive?

9 A plant grows so that it increases its height each month by 0.2 of its previous month's height. If it grows to 3 m, find its height in the first month.

10 a Find the cumulative distribution function for $f(x) = \frac{x^2}{114}$ defined for [1, 7].

b Find as a fraction:

i $P(X \le 5)$ ii $P(X \le 2)$ iii $P(X > 3)$

iv $P(X \ge 4)$ v $P(3 \le X \le 6)$

c Find correct to 2 decimal places:

i the median ii the 93rd percentile

11 A farmer places 20 bales of hay in a row in the shed. He then stacks 17 on top of these, then 14 in the next row up and so on, continuing with this pattern.

a How many bales of hay are in the top row?

b How many rows are there?

c How many bales of hay are stacked in the shed?

12 Ryanna has \$120 000 in a superannuation trust fund. She withdraws \$1600 each month as a pension.

a If the trust fund earns 6% p.a. paid monthly, find the amount left in the fund after:

i 1 month ii 2 months iii 3 months

b i Show that the amount left after n months is given by:

$120\,000(1.005)^n - 1600(1 + 1.005 + 1.005^2 + \ldots + 1.005^{n-1})$

ii How much will be in the trust fund after 20 months?

13 I put \$2000 in the bank, where it earns interest at the rate of 12% p.a., paid quarterly. How much will there be in my account after 3 years?

14 Find the median of the continuous random variable $f(x) = \frac{4x^3}{609}$ defined on the interval $2 \le x \le 5$.

15 Express $0.1\dot{7}$ as a fraction.

16 Find any stationary points and points of inflection on the graph of the function $y = x^3 - 6x^2 - 15x + 1$.

17 I borrow \$5000 at 18% interest p.a. compounded monthly and make equal monthly payments over 3 years, at the end of which the loan is fully paid out.
Find the amount of each monthly payment.

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18 The volume of blood in adult humans is normally distributed with mean 4.7 L and standard deviation 0.4.

a What would be the range of blood volumes for 95% of adults?

b What percentage of blood volumes would lie between 4.3 L and 5.1 L?

c Use the table of standard normal probabilities on pages 441–442 to find the z-score for a volume of:

i 6 L ii 5.2 L iii 3.9 L iv 4.1 L

d Find the probability for the blood volume X:

i $P(X \leq 6)$ ii $P(X \leq 3.9)$ iii $P(X \geq 4.1)$

iv $P(X \geq 5.2)$ v $P(3.9 \leq X \leq 5.2)$

e Find the percentage of blood volumes that lie between:

i 4 L and 5 L ii 4.2 L and 5.2 L iii 3.6 L and 5.8 L

19 Use the repayments table for reducing balance loans on $1000 on page 401 to find the monthly repayments for a loan of:

a $25 000 at 4% p.a. over 5 years

b $100 000 at 2.5% p.a. over 25 years

c $128 500 at 6% p.a. over 15 years

d $2400 at 7.5% p.a. over 10 years

20 Find the cumulative distribution function for each continuous probability distribution.

a $f(x) = \frac{3(x+2)^2}{335}$ defined in the domain [0, 5]

b $f(x) = \frac{x^3}{156}$ defined in the domain [1, 5]

c $f(x) = 2 \cos x$ in the interval $0 \leq x \leq \frac{\pi}{6}$

21 Find the mode of the continuous probability distribution $f(x) = \frac{4x - x^2}{9}$ defined in the domain [1, 4].

22 Use the probability tables for a standard normal distribution on pages 441–2 to find:

a $P(z \leq 1.35)$ b $P(z \geq -0.88)$ c $P(z \leq -1)$

d $P(z \geq 2.04)$ e $P(-3.12 \leq z \leq 2.81)$

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23 Li scored 72% in her first maths exam in which the class mean was 69% with standard deviation 0.8. She scored 65% in her second exam with class mean 55% and standard deviation 1.2. In which exam did Li do better in relation to her class?

24 I borrow $10 000 over 5 years at 1.85% monthly interest. How much do I need to pay each month?

25 A function is given by $f(x) = \frac{x^4}{1555}$. Over what domain starting at $x = 1$ is this a probability density function?

26 A data set is normally distributed with mean 12.5 and standard deviation 1.3. Find the raw score for each z-score.

a 0.4 b −1.5 c 2.96 d −3

27 Find the mode of the continuous probability distribution $f(x) = \frac{3x^2}{512}$ defined in the domain [0, 8].

28 A normal distribution has a standard deviation of 2.5. A score of 19 has a z-score of 0.2. What is the mean?

29 A factory produces 350 mL cans of soft drink. In a quality check, the mean volume of soft drink in cans was found to be 349.8 mL, with a standard deviation of 0.2 mL. The factory rejects cans that are outside 2 standard deviations of the mean.

a Find the percentage and volumes of cans that the factory rejects.

b The factory manager decides they are rejecting too many cans and that only cans whose volume is outside 3 standard deviations of the mean will be rejected. What percentage and volumes of cans will the factory reject?

30 A sum of $2500 is put into a bank account where it earns 2.4% p.a. Find the amount in the bank after 4 years if interest is paid:

a annually b quarterly c monthly

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ANSWERS

Answers are based on full calculator values and only rounded at the end, even when different parts of a question require rounding. This gives more accurate answers. Answers based on reading graphs may not be accurate.

Chapter 1

Exercise 1.01

1 a 14, 17, 20 b 23, 28, 33
c 44, 55, 66 d 85, 80, 75
e 1, –1, –3 f –15, –24, –33
g $2, 2\frac{1}{2}, 3$ h 3.1, 3.7, 4.3
i 32, –64, 128 j $\frac{27}{320}, \frac{81}{1280}, \frac{243}{5120}$

2 a 1456 b 63 c 78
d 126 e 91 f 441

3 $\frac{1}{16}, \frac{1}{32}, \frac{1}{64}$ 4 38, 51, 66, 83

5 21, 34, 55, 89, 144

6

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Exercise 1.02

1 a $y = 13$ b $x = -4$ c $x = 72$
d $b = 11$ e $x = 7$ f $x = 42$
g $k = 2$ h $x = 1$ i $t = -2$
j $t = 3$

2 a 46 b 78 c 94
d –6 e 67

3 a 590 b –850 c 414
d 1610 e –397

4 a –110 b 12.4 c –8.3
d 37 e $15\frac{4}{5}$

5 $T_n = 2n + 1$

6 a $T_n = 8n + 1$ b $T_n = 2n + 98$
c $T_n = 3n + 3$ d $T_n = 6n + 74$
e $T_n = 4n - 25$ f $T_n = 20 - 5n$
g $T_n = \frac{n+6}{8}$ h $T_n = -2n - 28$
i $T_n = 1.2n + 2$ j $T_n = \frac{3n-1}{4}$

7 28th term 8 54th term

9 30th term 10 15th term

11 Yes 12 No

13 Yes 14 $n = 13$

15 $n = 30, 31, 32, \ldots$ 16 –2

17 103 18 785

19 a $d = 8$ b 87

20 $d = 9$ 21 $a = 12, d = 7$

22 173 23 $a = 5$

24 280 25 1133

26 a $T_2 - T_1 = T_3 - T_2 = d = \log_5 x$
b $80 \log_5 x$ or $\log_5 x^{80}$ c 8.6

27 a $T_2 - T_1 = T_3 - T_2 = d = \sqrt{3}$
b $50\sqrt{3}$

28 26 29 122b 30 38th term

Exercise 1.03

1 a 375 b 555 c 480

2 a 2640 b 4365 c 240

3 a 2050 b –2575

4 a –4850 b 4225

5 a 28 875 b 3276 c –1419
d 6426 e 6604 f 598
g –2700 h 11 704 i –290
j 1284

6 21 7 8 8 11

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9 8 and 13 terms

10 $a = 14, d = 4$ 11 $a = -3, d = 5$

12 2025 13 3420 14 1010

15 a $(2x + 4) - (x + 1) = (3x + 7) - (2x + 4) = x + 3$

b $25(51x + 149)$

16 1290 17 16

18 $S_n = S_{n-1} + T_n$

So $S_n - S_{n-1} = T_n$

19 a 816 b 4234

Exercise 1.04

1 a No b Yes, $r = -\frac{3}{4}$ c Yes, $r = \frac{2}{7}$

d No e No f No

g Yes, $r = 0.3$ h Yes, $r = -\frac{3}{5}$ i No

j Yes, $r = -8$

2 a $x = 196$ b $y = -48$

c $a = \pm 12$ d $y = \frac{2}{3}$

e $x = 2$ f $p = \pm 10$

g $y = \pm 21$ h $m = \pm 6$

i $x = 4 \pm 3\sqrt{5}$ **j** $k = 1 \pm 3\sqrt{7}$

k $t = \pm\frac{1}{6}$ l $t = \pm\frac{2}{3}$

3 a $T_n = 5^{n-1}$ b $T_n = 1.02^{n-1}$

c $T_n = 9^{n-1}$ d $T_n = 2 \times 5^{n-1}$

e $T_n = 6 \times 3^{n-1}$ f $T_n = 8 \times 2^{n-1} = 2^{n+2}$

g $T_n = \frac{1}{4} \times 4^{n-1} = 4^{n-2}$ h $T_n = 1000(-0.1)^{n-1}$

i $T_n = -3(-3)^{n-1} = (-3)^n$ **j** $T_n = \frac{1}{3}\left(\frac{2}{5}\right)^{n-1}$

4 a 1944 b 9216 c -8192

d 3125 e $\frac{64}{729}$

5 a 256 b 26 244 c 1.369

d -768 e $\frac{3}{1024}$

6 a 234 375 b 268.8 c $-81\,920$

d $\frac{2187}{156\,250}$ e 27

7 a 3×2^{19} b 7^{19} c 1.04^{20}

d $\frac{1}{4}\left(\frac{1}{2}\right)^{19} = \frac{1}{2^{21}}$ e $\left(\frac{3}{4}\right)^{20}$

8 11^{49} 9 6th 10 5th

11 No 12 7th 13 11th

14 9th 15 $n = 5$ 16 $r = 3$

17 a $r = -6$ b -18

18 $a = \frac{1}{10}, r = \pm 2$

19 $n = 7$ 20 $208\frac{2}{7}$

Exercise 1.05

1 a 2 097 150 b 7 324 218

2 a 720 600 b 26 240

3 a 131 068 b 0.5

4 a 7812 b $35\frac{55}{64}$ c 8403

d 273 e 255

5 a 1792 b 3577

6 148.58 7 133.33 8 $n = 9$

9 10 terms 10 $a = 9$ 11 10 terms

12 a 2046 b 100 c 2146

Puzzles

1 Choice 1 gives \$465.00. Choice 2 gives \$10 737 418.23 so choice 2 is better.

2 382 apples

Exercise 1.06

1 a Yes, $13\frac{1}{2}$ b No

c Yes, $12\frac{4}{5}$ d No

e Yes, 3 f Yes, $\frac{25}{32}$

g No h Yes, $-1\frac{5}{22}$

i No **j** Yes, $1\frac{3}{7}$

2 a 80 b $426\frac{2}{3}$ c $66\frac{1}{3}$ d 12

e $\frac{7}{10}$ f 54 g $-10\frac{2}{7}$ h $\frac{9}{20}$

i 48 **j** $-\frac{16}{39}$

3 a 0.58 b 0.15 c 0.000080

d 0.016 e 0.89

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4 $a = 4$ 5 $r = \frac{2}{5}$ 6 $a = 5\frac{3}{5}$

7 $r = \frac{7}{8}$ 8 $r = -\frac{1}{4}$ 9 $r = -\frac{2}{3}$

10 $a = 3, r = \frac{2}{3}$ or $a = 6, r = \frac{1}{3}$

11 $a = 192, r = -\frac{1}{4}, S = 153\frac{3}{5}$

12 $a = 1, r = \frac{2}{3}, S = 3$ or $a = -1, r = -\frac{2}{3}, S = -\frac{3}{5}$

13 $a = 150, r = \frac{3}{5}, S = 375$

14 $a = \frac{2}{5}, r = \frac{2}{3}, S = 1\frac{1}{5}$

15 $a = 3, r = \frac{2}{5}$ or $a = 2, r = \frac{3}{5}$

16 $x = \frac{21}{32}$

17 a $|k| < 1$ b $-\frac{2}{5}$ c $k = \frac{3}{4}$

18 See worked solutions.

Test yourself 1

1 C 2 C 3 B

4 a $T_n = 4n + 5$ b $T_n = 14 - 7n$

c $T_n = 2 \times 3^{n-1}$ d $T_n = 200\left(\frac{1}{4}\right)^{n-1}$

e $T_n = (-2)^n$

5 a 2 b 1185 c 1183

d $T_{15} = S_{15} - S_{14}, S_{15} = S_{14} + T_{15}$

e $n = 16$

6 a i b ii c i d iii

e i f ii g ii h i

i i j i

7 $n = 108$

8 $a = -33, d = 13$

9 a 59 b 80 c 18th

10 a $x = 25$ b $x = \pm 15$

11 $x = 3$

12 a 136 b 44 c 6

13 $121\frac{1}{2}$

14 a $S_n = n(2n + 3)$ b $S_n = \frac{1.07^n - 1}{0.07}$

15 a $|x| < 1$ b $2\frac{1}{2}$ c $x = \frac{1}{3}$

16 $d = 5$

17 $x = -\frac{2}{17}, 2$

18 1300

29 a 735 b 4315

20 $n = 20$

21 $n = 11$

Challenge exercise 1

1 a 8.1 b 19th

2 a $\frac{\pi}{4}$ b $\frac{9\pi}{4}$ c $\frac{33\pi}{4}$

3 a 2 097 170 b −698 775

4 6th 5 17 823

6 $n \geq 5$

7 a $a = 7, r = -2$ b −56

8 a $\operatorname{cosec}^2 x$

b $r = \cos^2 x$

$-1 < \cos x < 1$ where $\cos^2 x \neq 0, 1$

So $0 < \cos^2 x < 1$

Since $|r| < 1$, the series has a limiting sum.

Chapter 2

Exercise 2.01

1 a Vertical translation 3 units up

b Vertical translation 7 units down

c Vertical translation 1 unit down

d Vertical translation 5 units up

2 a Vertical translation 1 unit up

b Vertical translation 4 units down

c Vertical translation 8 units up

3 Vertical translation 9 units up

4 a $y = x^2 - 3$ b $f(x) = 2^x + 8$

c $y = |x| + 1$ d $y = x^3 - 4$

e $f(x) = \log x + 3$ f $y = \frac{2}{x} - 7$

5 a Vertical translation 1 unit down

b Vertical translation 6 units up

6 a i $y = 2x^3 - 2$ ii $y = 2x^3 + 6$

b i $y = |x| - 3$ ii $y = |x| - 6$

c i $y = e^x + 1$ ii $y = e^x + 5$

d i $f(x) = \log_e x + 10$ ii $f(x) = \log_e x - 8$

7 a $(1, -1)$ b $(1, -9)$ c $(1, -3 + m)$

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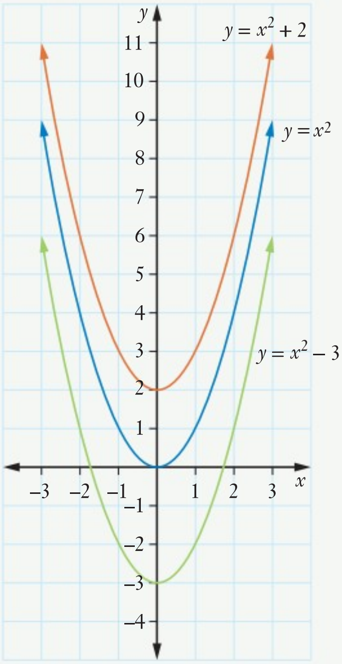
8 a (−1, 1) b (−1, 5)

9 a



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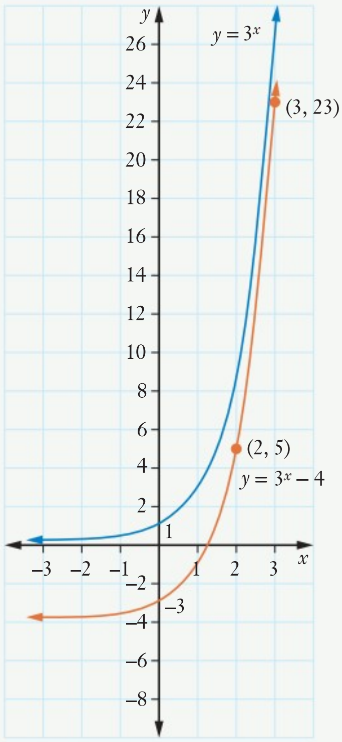
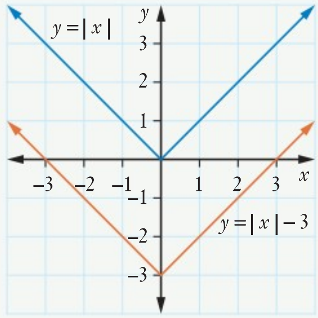
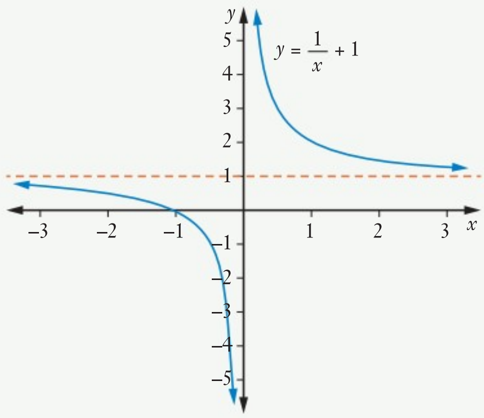


c

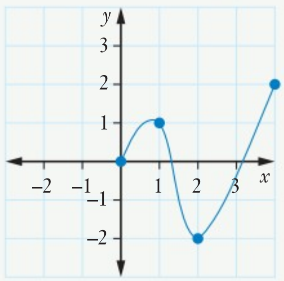


10 a Vertical translation 1 unit up

b



11 a



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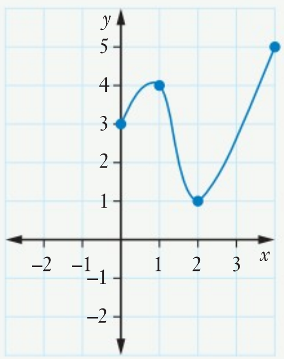

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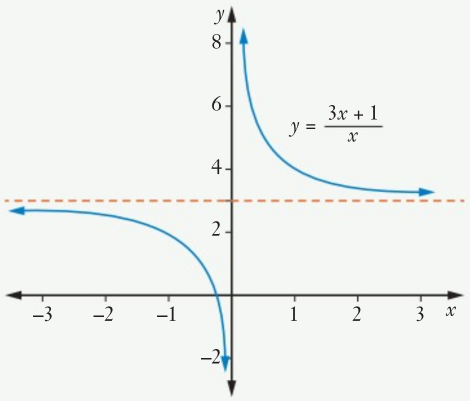
12 b



Exercise 2.02

1 a Horizontal translation 4 units to the right

b Horizontal translation 2 units to the left

2 a Horizontal translation 5 units to the right

b Horizontal translation 3 units to the left

3 a $y = (x + 3)^2$ b $f(x) = 2^{x-8}$

c $y = |x + 1|$ d $y = (x - 4)^3$

e $f(x) = \log(x + 3)$

4 Horizontal translation 3 units to the right

5 a Horizontal translation 2 units to the left

b Horizontal translation 5 units to the right

6 a i $y = -(x + 4)^2$ ii $y = -(x - 8)^2$

b i $y = |x - 3|$ ii $y = |x + 4|$

c i $y = e^{x+6}$ ii $y = e^{x-5}$

d i $f(x) = \log_2(x - 5)$ ii $f(x) = \log_2 x$

7 a $(5, -3)$

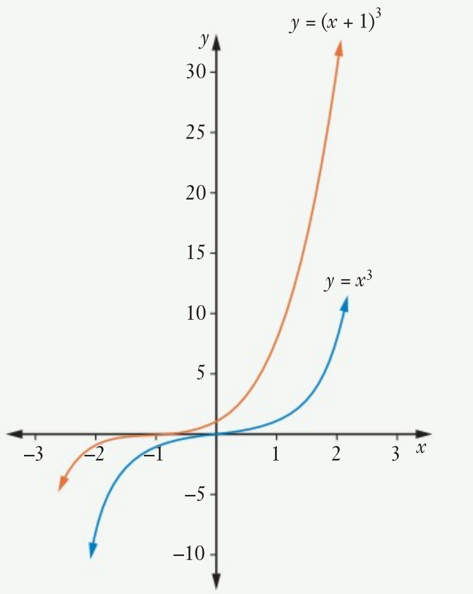
b $(-8, -3)$

c $(1 + t, -3)$

8 a $(3, 2)$

b $(-9, 2)$

9 a



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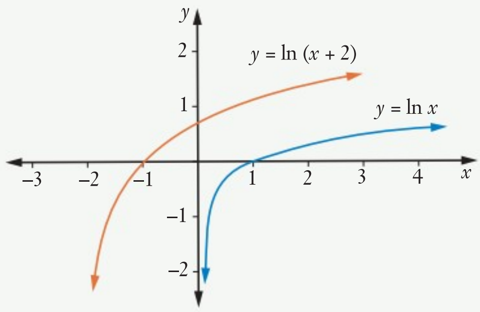
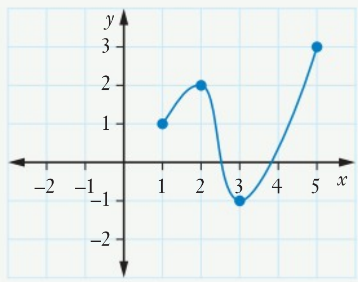


10 a



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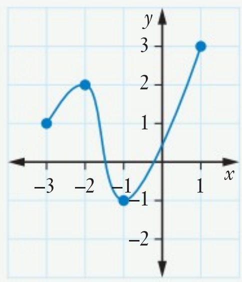

11 a $f(x) = x^5 - 5$ b $f(x) = (x - 3)^5$

c $f(x) = x^5 + 2$ d $f(x) = (x + 7)^5$

12 $(7, -2)$

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Exercise 2.03

1 a i Vertical dilation scale factor 6 (stretched)
ii Vertical dilation scale factor $\frac{1}{2}$ (compressed)
iii Vertical dilation scale factor -1 (reflection in x-axis)
b i Vertical dilation scale factor 2 (stretched)
ii Vertical dilation scale factor $\frac{1}{6}$ (compressed)
iii Vertical dilation scale factor -1 (reflection in x-axis)
c i Vertical dilation scale factor 4 (stretched)
ii Vertical dilation scale factor $\frac{1}{7}$ (compressed)
iii Vertical dilation scale factor $\frac{4}{3}$ (stretched)
d i Vertical dilation scale factor 9 (stretched)
ii Vertical dilation scale factor $\frac{1}{3}$ (compressed)
iii Vertical dilation scale factor $\frac{3}{8}$ (compressed)
e i Vertical dilation scale factor 5 (stretched)
ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)
iii Vertical dilation scale factor -1 (reflection in x-axis)
f i Vertical dilation scale factor 9 (stretched)
ii Vertical dilation scale factor -1 (reflection in x-axis)
iii Vertical dilation scale factor $\frac{2}{5}$ (compressed)

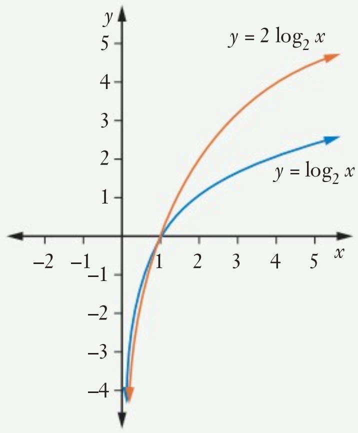
2 a $y = 6x^2$; domain $(-\infty, \infty)$; range $[0, \infty)$
b $y = \frac{\ln x}{4}$; domain $(0, \infty)$; range $(-\infty, \infty)$
c $f(x) = -|x|$; domain $(-\infty, \infty)$; range $(-\infty, 0]$
d $f(x) = 4e^x$; domain $(-\infty, \infty)$; range $(0, \infty)$
e $y = \frac{7}{x}$; domain $(-\infty, 0) \cup (0, \infty)$; range $(-\infty, 0) \cup (0, \infty)$

3 a $y = 5 \cdot 3^x$ b $f(x) = \frac{x^2}{3}$ c $y = -x^3$
d $y = \frac{1}{2x}$ e $y = \frac{2|x|}{3}$

4 a (3, 24) b (3, – 6) c (3, 72) d (3, 5)

5 a (4, 4) b (4, 6) c (4, 36) d (4, 16)
e (4, – 12)

6 a



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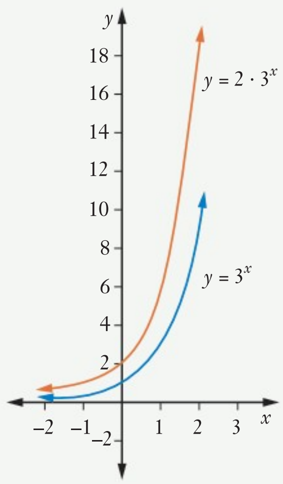

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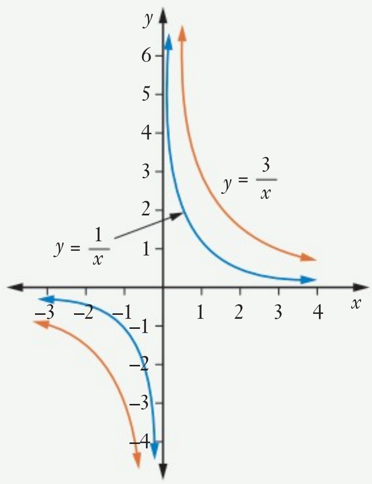

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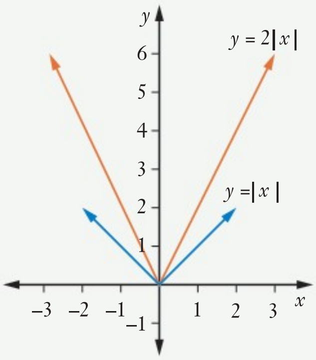
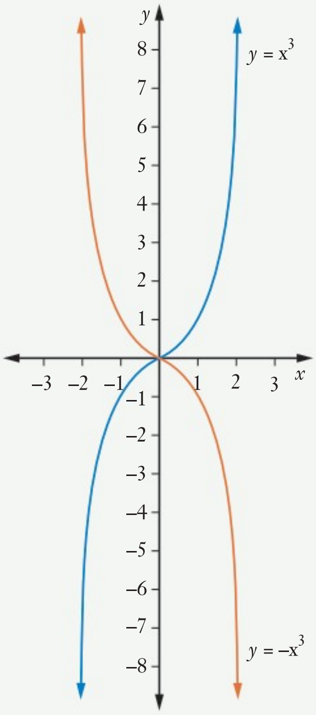
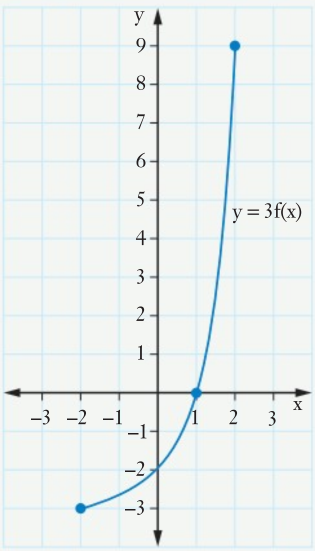
d



e



7 a



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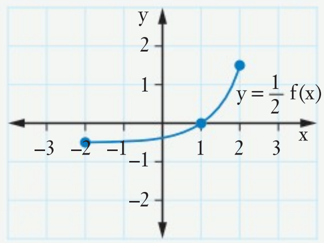

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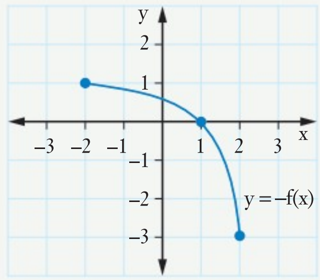

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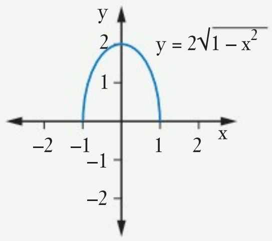

Exercise 2.04

1 a Horizontal dilation scale factor $\frac{1}{8}$ (compressed)

b Horizontal dilation scale factor 5 (stretched)

c Horizontal dilation scale factor $\frac{7}{3}$ (stretched)

d Horizontal dilation scale factor -1(reflection in y-axis). Note that this is the same graph as $f(x) = x^4$.

2 a i Horizontal dilation scale factor $\frac{1}{2}$ (compressed)

ii Horizontal dilation scale factor $\frac{1}{5}$ (compressed)

iii Horizontal dilation scale factor 3 (stretched)

b i Vertical dilation scale factor 4 (stretched)

ii Horizontal dilation scale factor 2 (stretched)

iii Horizontal dilation scale factor -1 (reflection in y-axis)

c i Horizontal dilation scale factor $\frac{1}{7}$ (compressed)

ii Vertical dilation scale factor $\frac{1}{8}$ (compressed)

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iii Horizontal dilation scale factor $\frac{4}{3}$ (stretched)

d i Horizontal dilation scale factor $\frac{1}{5}$ (compressed) (or vertical dilation scale factor 5, stretched)

ii Horizontal dilation scale factor 2 (stretched) (or vertical dilation scale factor $\frac{1}{2}$, compressed)

iii Horizontal dilation scale factor $\frac{5}{3}$ (stretched) (or vertical dilation scale factor $\frac{3}{5}$, compressed)

e i Horizontal dilation scale factor $\frac{1}{3}$ (compressed)

ii Vertical dilation scale factor -1 (reflection in x-axis)

iii Horizontal dilation scale factor 2 (stretched)

f i Vertical dilation scale factor 8 (stretched)

ii Horizontal dilation scale factor -1 (reflection in y-axis)

iii Horizontal dilation scale factor 7 (stretched)

3 a $f(x) = |5x|$; domain $(-\infty, \infty)$; range $[0, \infty)$

b $y = \left(\frac{x}{3}\right)^2$; domain $(-\infty, \infty)$; range $[0, \infty)$

c $y = (-x)^3$; domain $(-\infty, \infty)$; range $(-\infty, \infty)$

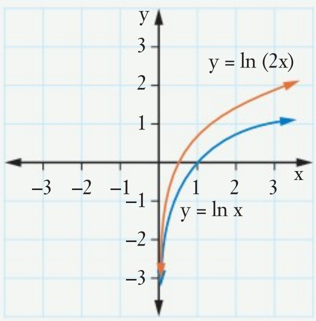
d $y = \frac{e^x}{9}$; domain $(-\infty, \infty)$; range $(0, \infty)$

e $y = -\log_4 x$; domain $(0, \infty)$; range $(-\infty, \infty)$

4 a $(-1, 7)$ b $(2, 7)$ c $(-6, 7)$

5 a $(-72, 1)$ b $(-48, 1)$ c $(-6, 1)$

6 a



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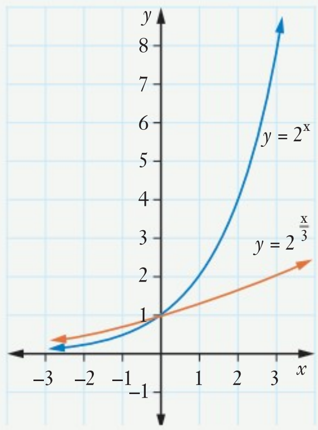

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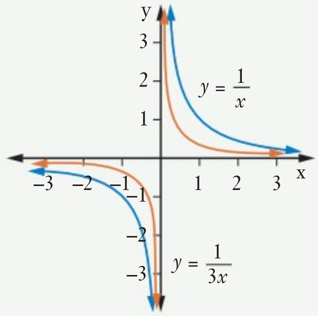

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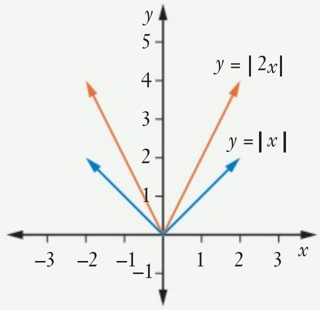
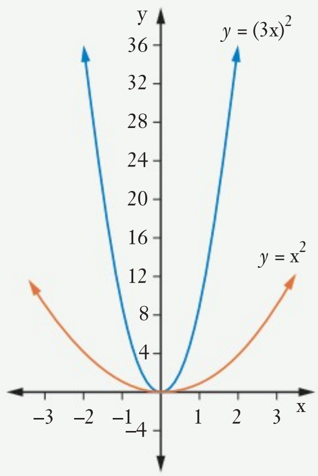


e



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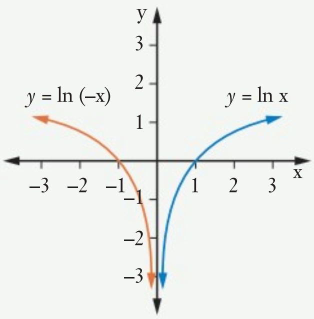

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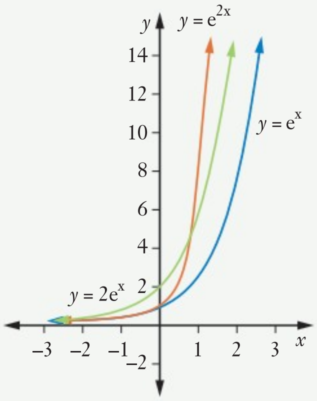
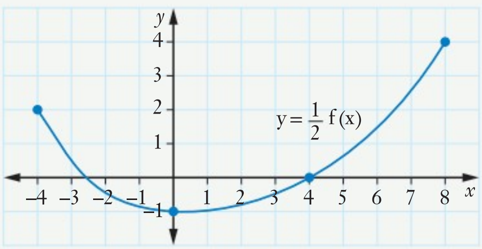


8 A reflection in y-axis transforms $y = f(x)$ into $y = f(-x)$.

a Since $y = x^2$ is an even function, $f(x) = f(-x)$, so a reflection in the y-axis doesn't change the function.

b Since $y = |x|$ is an even function, $f(x) = f(-x)$ so a reflection in the y-axis doesn't change the function.

9 a



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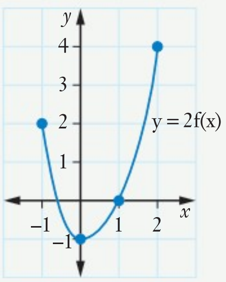

Exercise 2.05

1 a $(5, -11)$ b $(-1, -2)$
c $(9, 3)$ d $(-2, -17)$

2 a $f(x) = -4x^5$ b $f(x) = -\frac{1}{243}x^5$

3 a $y = (x+4)^3 - 3$ b $f(x) = |x-1| + 9$
c $f(x) = 3x - 6$ d $y = -e^x + 2$
e $y = (2x)^3 - 5$ f $f(x) = \frac{6}{x}$
g $f(x) = 3\sqrt{-2x}$ h $y = \ln\frac{x}{3} + 2$
i $f(x) = 3\log_2 4x$ j $y = \left(\frac{x}{2}\right)^2 - 3$

4 a Horizontal translation 1 unit to the right, vertical translation 7 units up

b Vertical dilation scale factor 4, vertical translation 1 unit down

c Vertical dilation scale factor 5, reflection in x-axis, vertical translation 3 units down

d Horizontal translation 7 units to the left, vertical dilation scale factor 2

e Rewrite as $y = 6[2(x-2)]^3 + 5$. Horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 2 units to the right, vertical dilation scale factor 6, vertical translation 5 units up

f Rewrite as $y = 2[3(x+3)]^3 - 10$. Horizontal dilation scale factor $\frac{1}{3}$, horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 10 units down

5 a Horizontal translation 3 units to the left, vertical dilation scale factor 2, vertical translation 1 unit down

b Horizontal dilation scale factor $\frac{1}{3}$, reflection in x-axis, vertical translation 9 units up

c Horizontal dilation scale factor $\frac{1}{5}$, vertical dilation scale factor 2, vertical translation 3 units down

d Horizontal translation 7 units to the right, vertical dilation scale factor 4, vertical translation 1 unit up

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e Reflection in y-axis, horizontal dilation scale factor $\frac{1}{2}$, horizontal translation 1 unit to the left, vertical translation 1 unit down

f Horizontal dilation scale factor $\frac{1}{2}$, reflection in x-axis, vertical translation 8 units up

6 a $(9, -31)$ b $(4, 5)$
c $(5, -25)$ d $(-8, -67)$
e Change to $y = -2f[2(x - 2)] - 3$: $(6, 21)$

7 a $(x + 3, y - 6)$ b $(-x, y + 6)$
c $(x - 5, 2y)$ d $(3x, y + 5)$
e $(5x - 30, -8y - 1)$

8 a $y = f(x + 1) - 2$ b $y = f(x - 5) + 3$
c $y = -f(x - 4)$ d $y = f(-x) + 2$
e $y = -f\left(\frac{x}{4}\right)$ f $y = 2f(x) - 2$

9 a $f(x) = -\frac{9}{x} + 3$ b $y = 5(x + 2)^2 - 6$
c $f(x) = 8 \ln\left[\frac{1}{2}(x) - 5\right] - 3$
d $y = 9\sqrt{-(x + 4)} + 4$ e $f(x) = -|6x| + 7$
f $y = [4(x + 4)]^3 = 64(x + 4)^3$
g $y = 6(2^{x-2} + 5)$

10 a Domain $(-\infty, \infty)$, range $[5, \infty)$
b Domain $(-\infty, \infty)$, range $[-2, \infty)$
c Domain $(-\infty, 2) \cup (2, \infty)$, range $(-\infty, 1) \cup (1, \infty)$
d Domain $(-\infty, \infty)$, range $(2, \infty)$
e Domain $(2, \infty)$, range $(-\infty, \infty)$

11 a $y = (x + 1)^2 - 8$
b Horizontal translation 1 unit to the left, vertical translation 8 units down

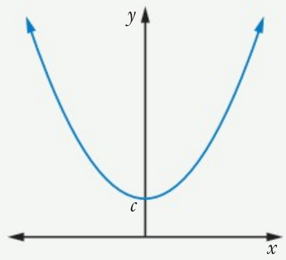
12 Horizontal translation 5 units to the right, vertical translation 28 units down

13 a $(2x + 3, 2y + 5)$ b $\left(\frac{x}{3} - 6, -y - 2\right)$

14 a Circle $(x - 3)^2 + (y - 4)^2 = 9$ or $x^2 - 6x + y^2 - 8x + 16 = 0$
b Translated 2 units to the right, 3 units down

Exercise 2.06

1 a i



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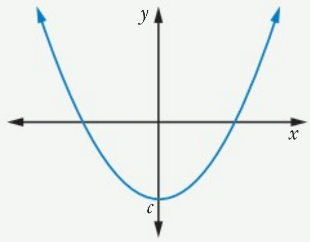
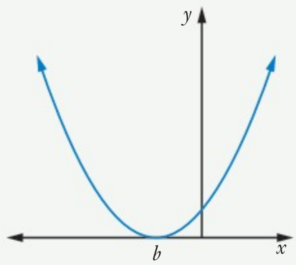


b i



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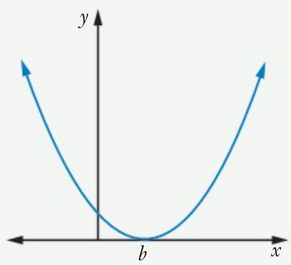
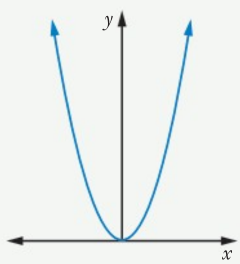


c i



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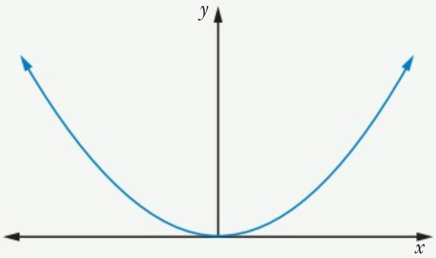

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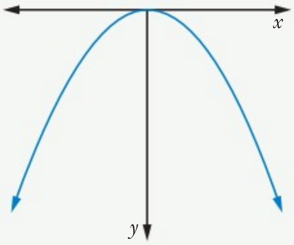
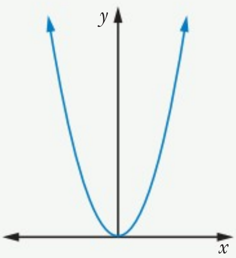


d i, iii



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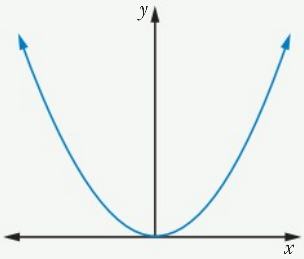
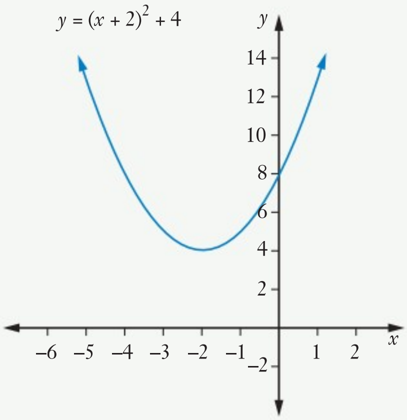


2 a



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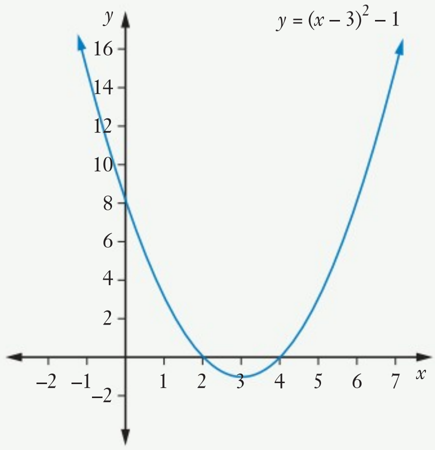

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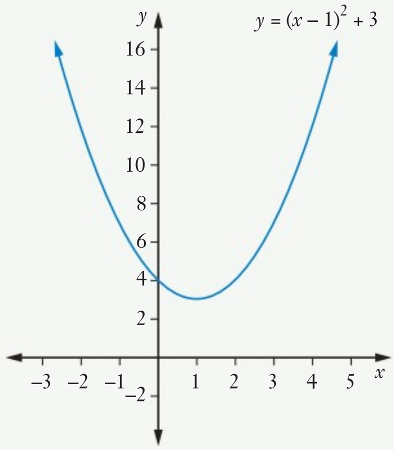

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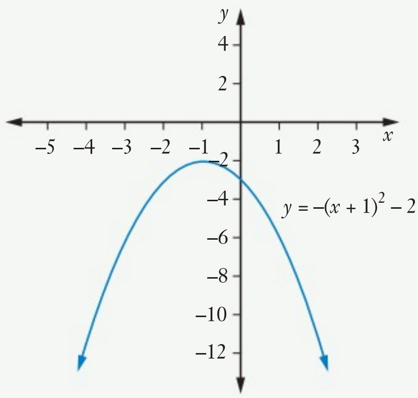

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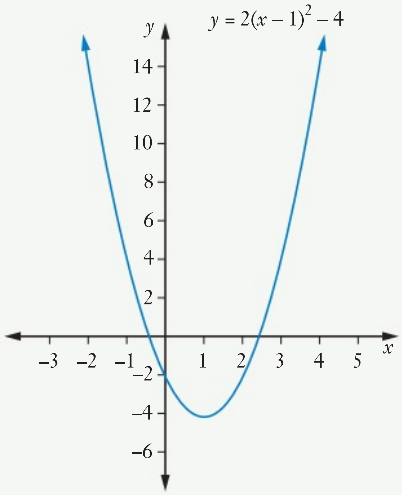
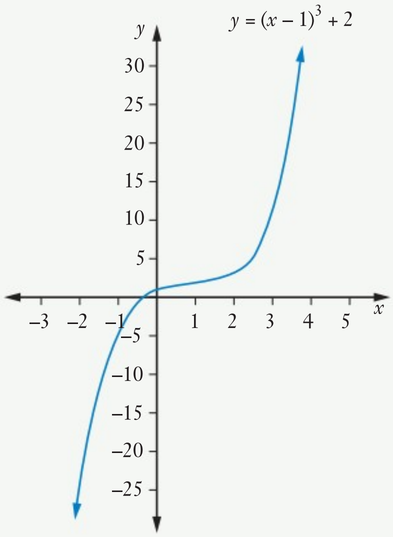
e



3 a



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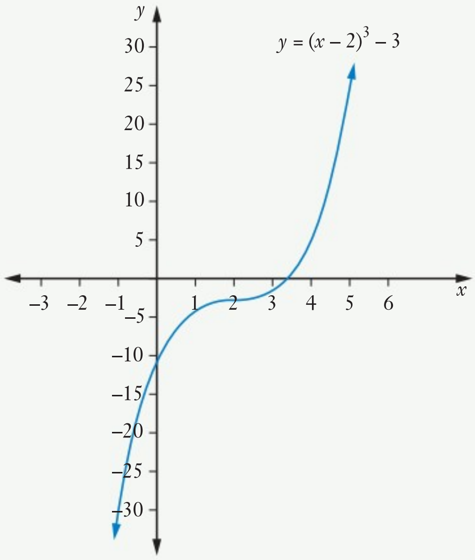

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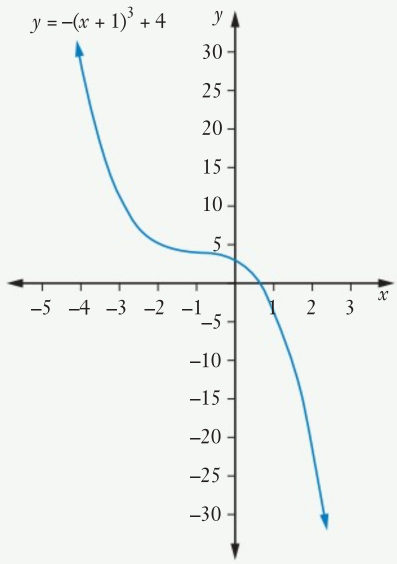

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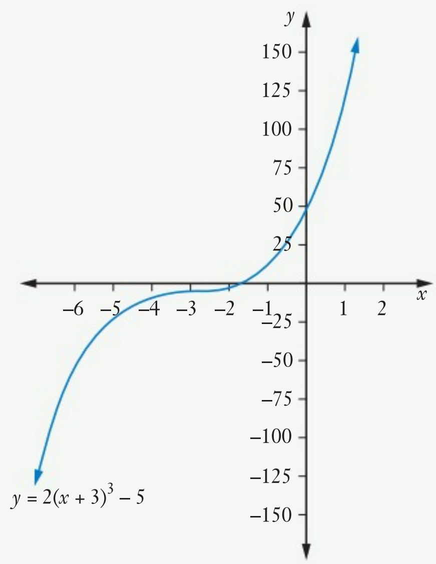

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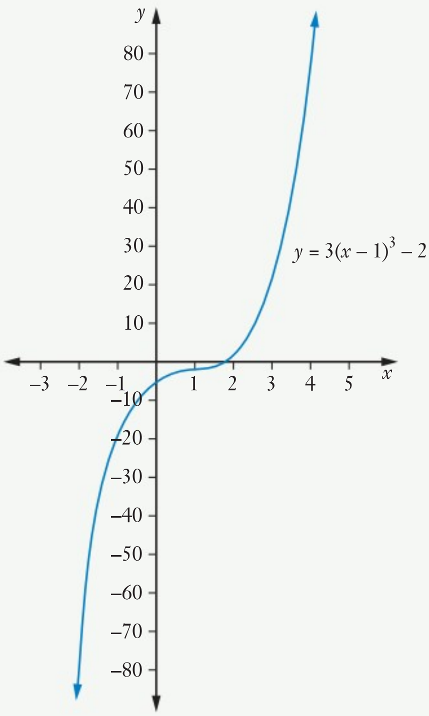
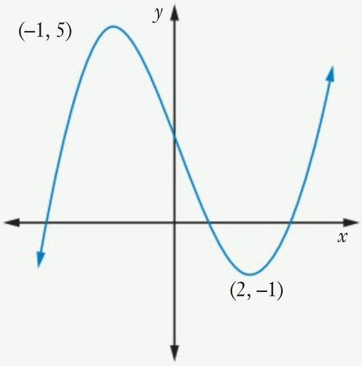
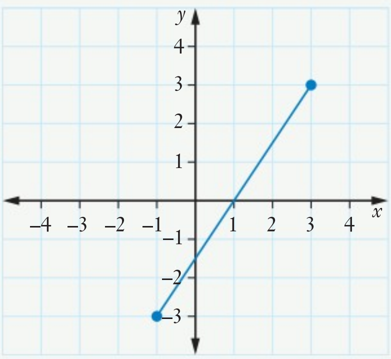


4 a $(2, -1)$ and $(-1, 5)$

b



5 a i



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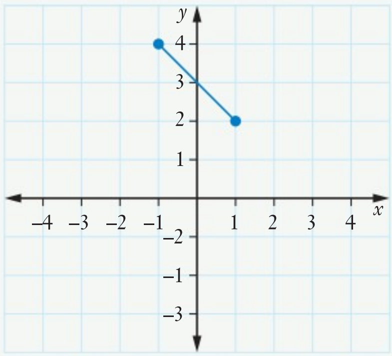
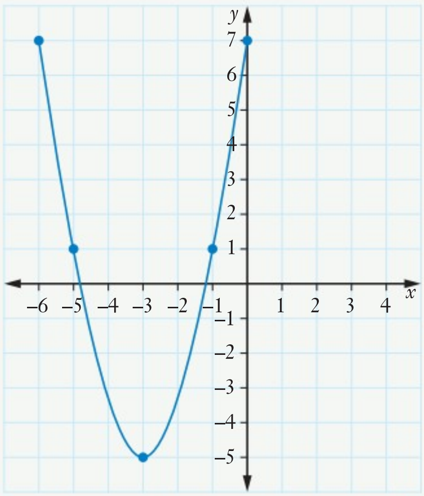


b i



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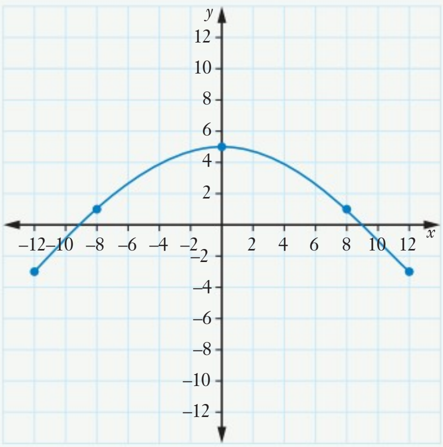

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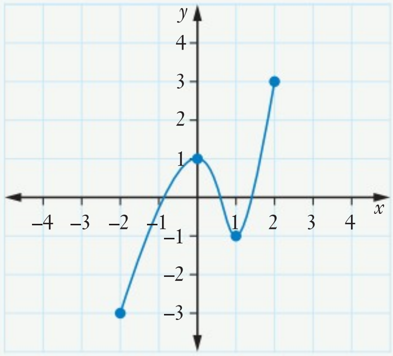
c i



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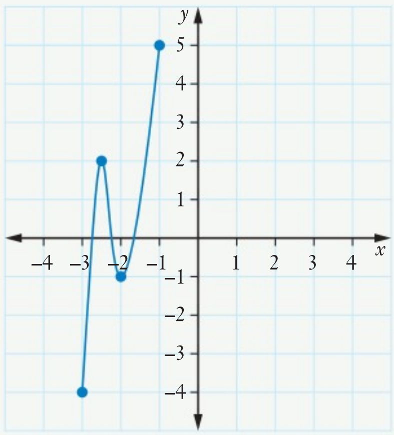

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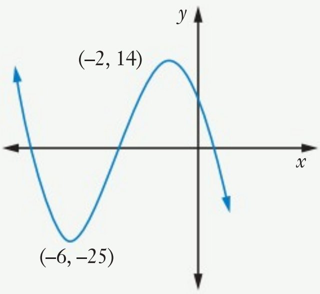

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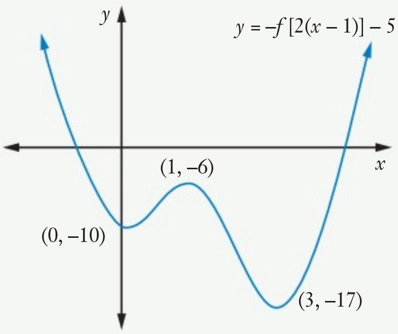
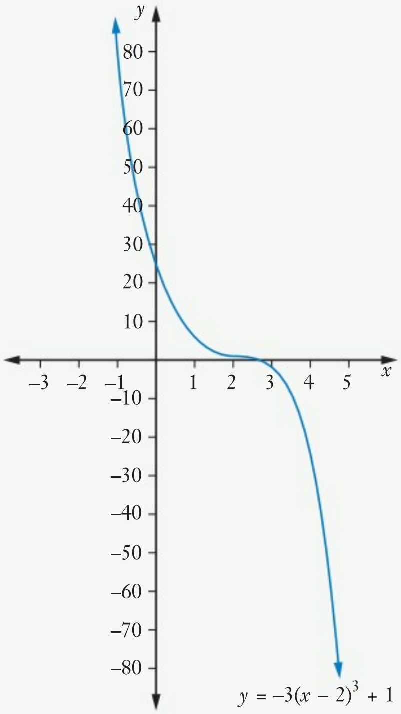


8 a



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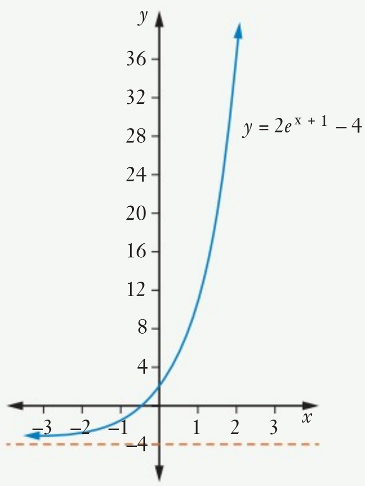

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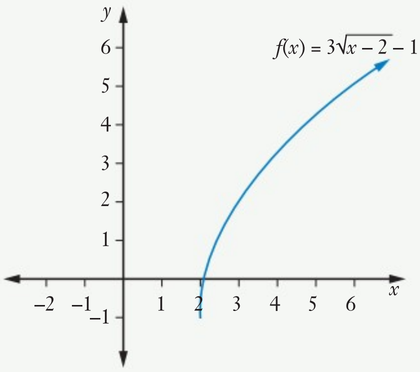

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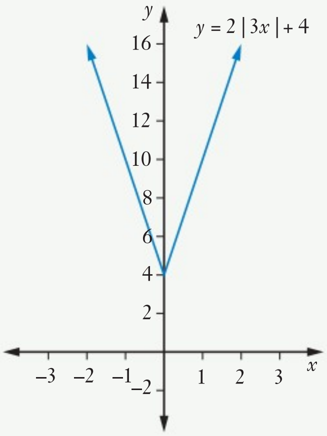

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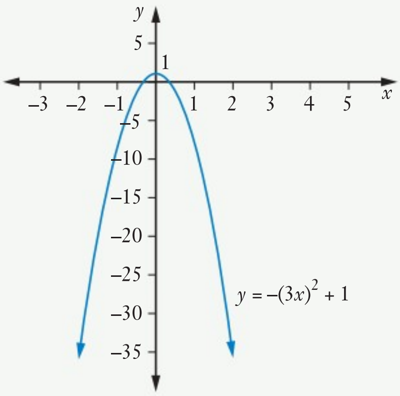
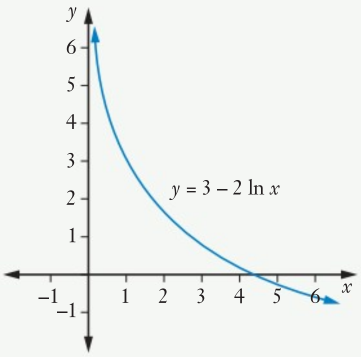


9 a



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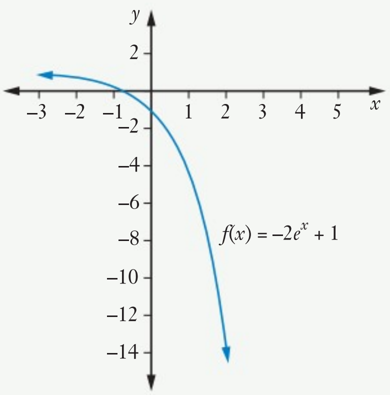

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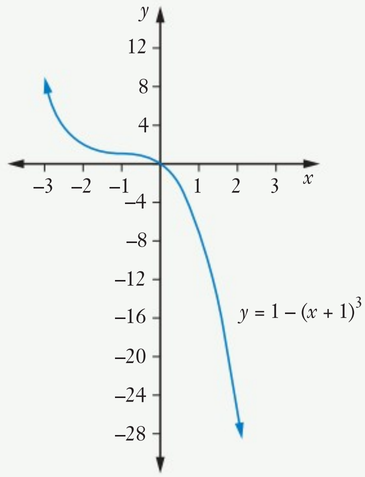

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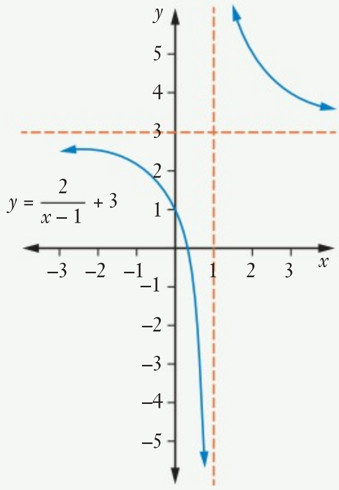

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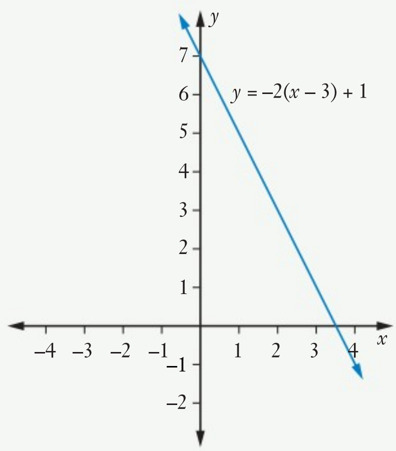

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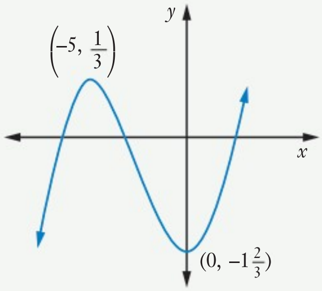
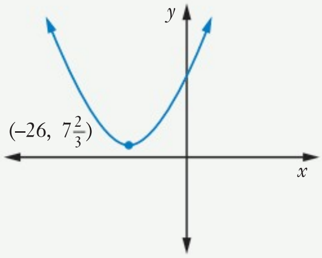
10 a $(-5, \frac{1}{3})$

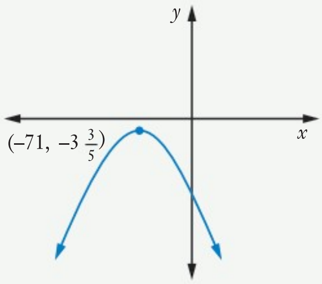
b



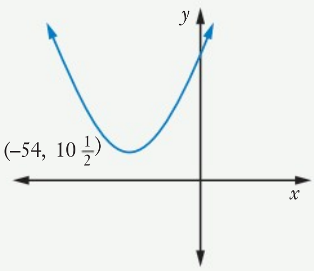
11 a $(-26, 7\frac{2}{3})$



b $(-71, -3.6)$



c $(-54, 10\frac{1}{2})$



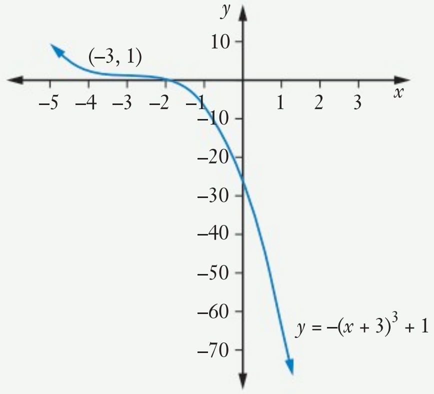
Exercise 2.07

1 a 2 b 0 c 1 d 3
e 0 f 1 g 2 h 0
i 1 j 0

2 a i $x = -2, 0$ ii $x = 0.6, -2.6$
iii $x = -2.2, 0.2$ b $x = 0.2, -2.2$

3 a $x = 1.4$ b $x = 1.9$ c $x = 0.2$
d $x > 2.2$ e $x \leq 3.1$

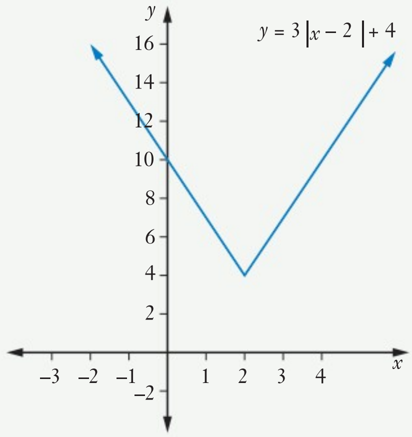
4 a



b i $x = -2$ ii $x = -0.8$ iii $x = -0.3$

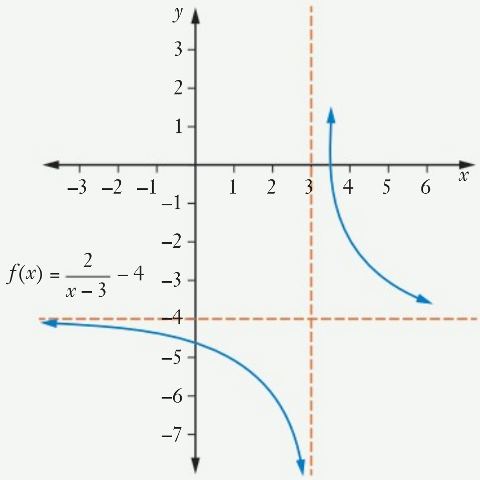
c $x = -2$

5 a



b None c $x = 0, 4$

6 a



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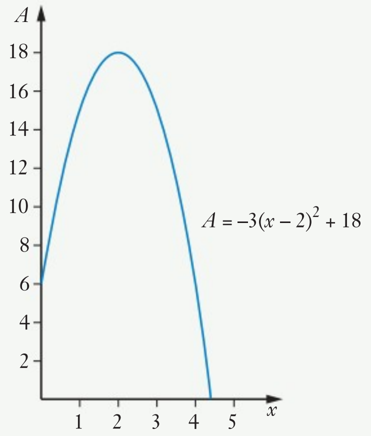
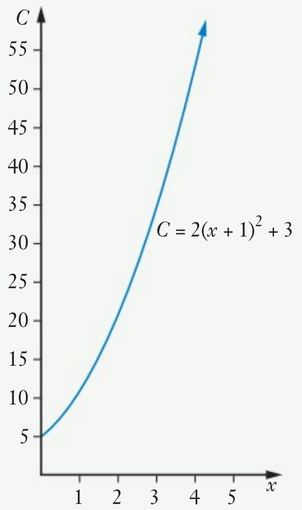
b $x = 1$ c $x = 4$

7 a



b $x = 0.4, 3.6$

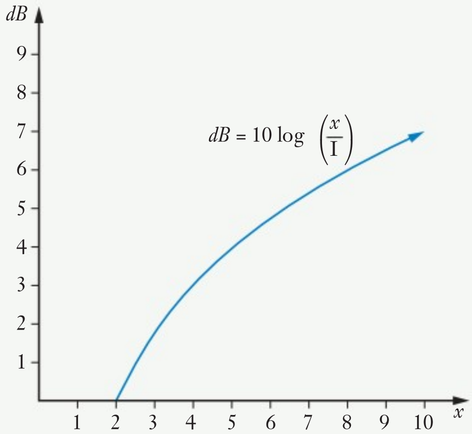
8 a



b $5000

c The costs are $20 000 when 2 products are made.

9 a



b i $x = 6$ ii $x = 3$

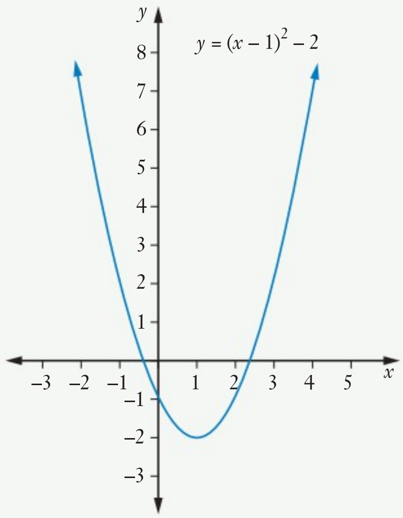
10 a i $t = 3.5$. After 3.5 minutes the temperature is 50°C.

ii $t = 8$. After 8 minutes the temperature is 30°C.

b i $t = 0.74$ ii $t = 11.85$

c 24°C (room temperature)

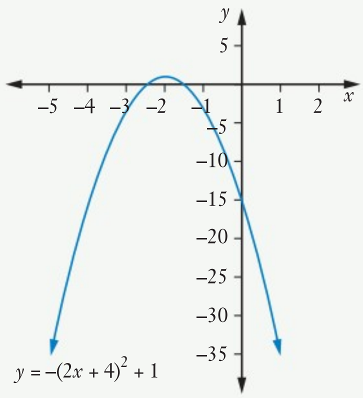
11 a



b i $x = 3, -1$ ii $x \le -1, x \ge 3$

iii $-1 < x < 3$

12 a



b i $x = -3, -1$ ii $-3 < x < -1$

iii $x \le -3, x \ge -1$

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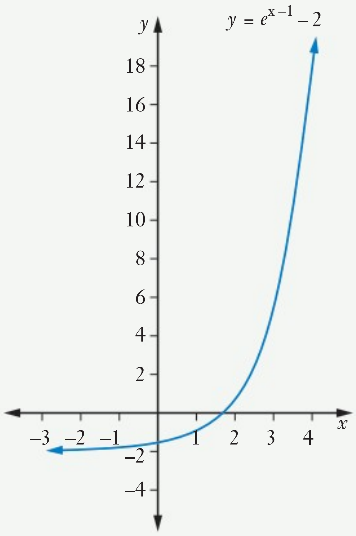
Test yourself 2

1 B　　2 D　　3 C

4 a



b $x = 3.1$　　c $x = 4.09$

5 a (6, 107)　　b (6, 40)

c (–24, 177)　　d (17, –75)

e (16, –31)

6 a i $y = x^3 + 3$　　ii $y = (x + 7)^3$

b i $y = 3|x|$　　ii $y = \left|\frac{x}{2}\right|$

c $f(x) = 5\ln(-x)$　　d $f(x) = -\frac{1}{x-4}$

e $f(x) = 9 \cdot 3^{3x-2} - 6$

7 a c is a vertical translation c units up if $c > 0$ or down if $c < 0$.

b is a horizontal translation b units to the right if $b < 0$ and to the left if $b > 0$.

k is a vertical dilation with scale factor k, stretched if $k > 1$ and compressed if $0 < k < 1$.

a is a horizontal dilation with scale factor $\frac{1}{a}$, compressed if $a > 1$ and stretched if $0 < a < 1$.

b i Reflection in the x-axis

ii Reflection in the y-axis

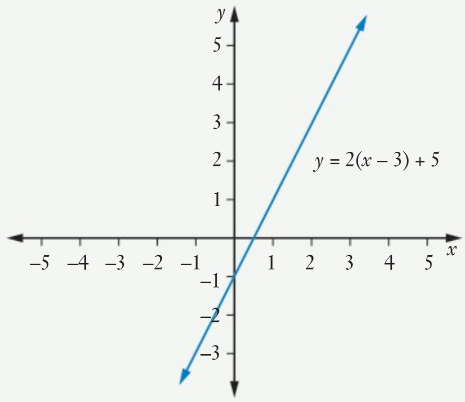
8 $f(x) = -3x^2 + 1$

$f(-x) = -3(-x)^2 + 1$

$= -3x^2 + 1$

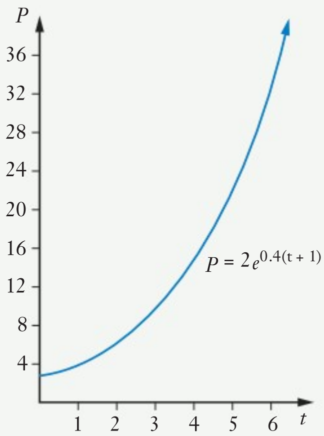
$= f(x)$ so even

9 a



b i $x \leq 4$　　ii $x > 5$

10 a



b $t = 1.3$. It takes 1.3 years for the population to reach 50 000.

11 $f(x) = (x + 4)^4$

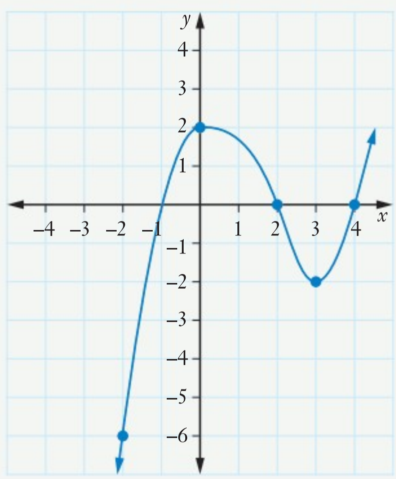
12 (3, – 11)

13 a $x = 2.88, 1.12$　　b $x < 1.12, x > 2.88$

c $1.12 \leq x \leq 2.88$

14 a $(x + 3, -7y - 4)$　　b $x = -6, y = -1$

15 a



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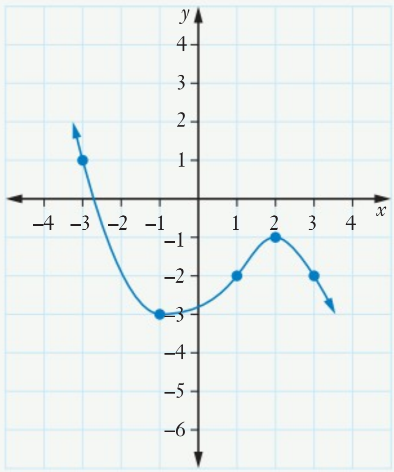
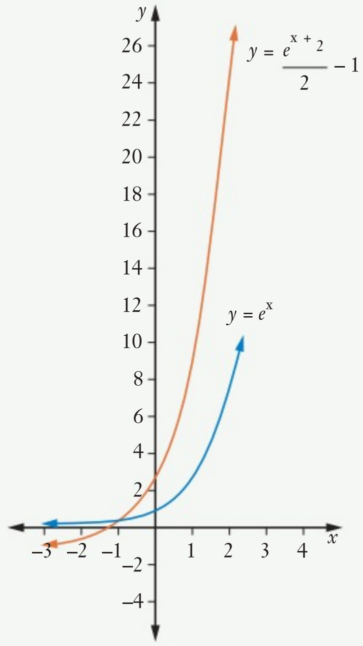
b

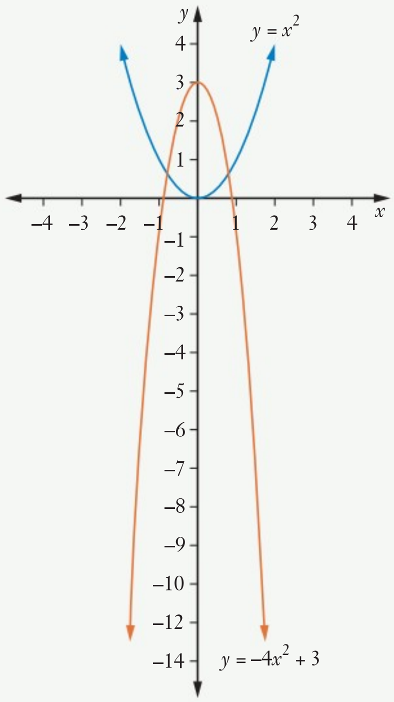


c



16 a $-3 \le x \le 1$ b $x < -3,\ x > 1$

17 a



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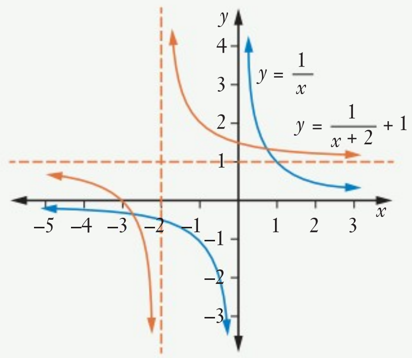

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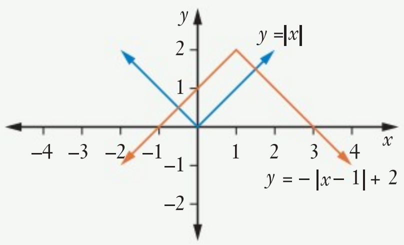

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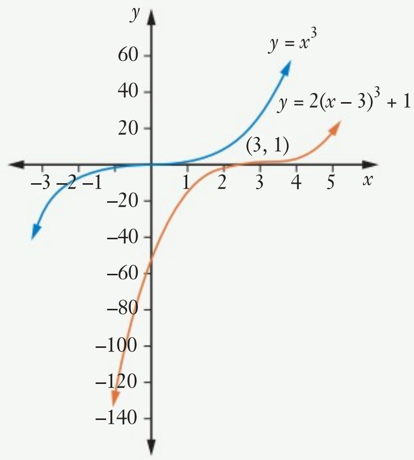

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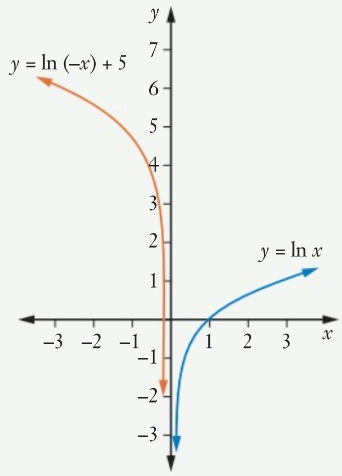
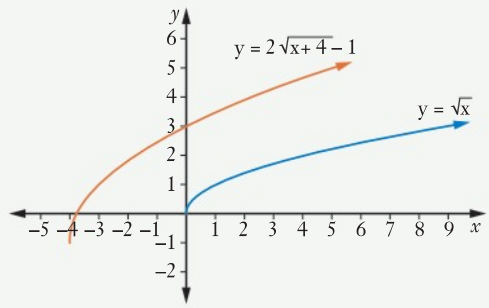
f

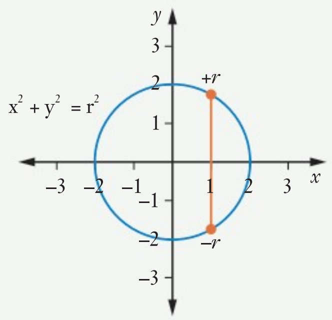


g



18 a 2 b 1 c 0 d 1
e 1 f 0 g 1 h 0
i 4 j 3

19 a The circle is not a function since a vertical line cuts the graph in more than one point.

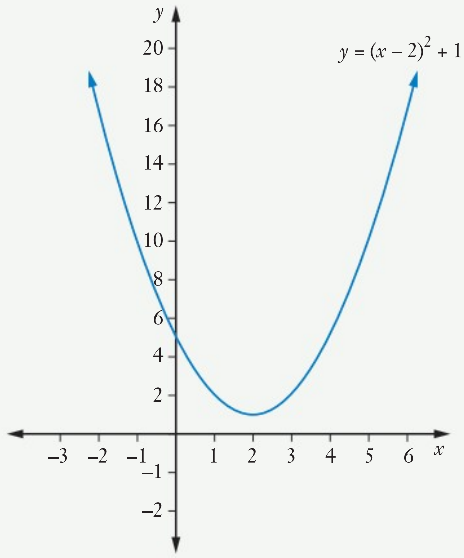


b $y=\sqrt{r^2-x^2}$ and $y=-\sqrt{r^2-x^2}$

c An ellipse (oval)

20 $(32,-1)$

21 a



b i $x=-1, 5$ ii $x<-1, x>5$

iii $-1 \le x \le 5$

22 a $(x-1, 3y-5)$ b $\left(\frac{1}{2}[x+6], -2y+4\right)$

c $(-x, 5y-3)$ d $\left(-\frac{1}{3}[x+3], -3y-1\right)$

23 a Stretched b Compressed
c Stretched d Compressed
e Stretched

24 a Domain $(-\infty, \infty)$; range $[-10, \infty)$
b Domain $(-\infty, \infty)$; range $(-\infty, 2]$
c Domain $(-\infty, 3) \cup (3, \infty)$; range $(-\infty, -5) \cup (-5, \infty)$

Challenge exercise 2

1 a $h=-2t^2+4t+1$ b 2.2 seconds

c $h=-2(t-1)^2+3$. Horizontal translation 1 unit to the right, reflection in the x-axis, vertical dilation scale factor 2, vertical translation 3 units up.

2 a i $(1,-8)$ ii $(2, 0)$ iii $(3,-2)$

b $x-2y-17=0$

c Horizontal dilation with scale factor 2 and horizontal translation 17 units to the right, OR: vertical dilation with scale factor $\frac{1}{2}$ and vertical translation $\frac{17}{2}$ units down

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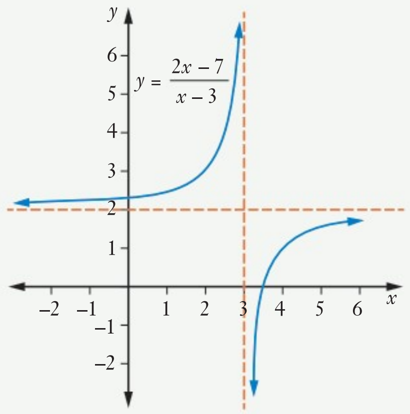
3 b



c i $x < 3, x \geq 3.5$ ii $x > 3$

4 a A horizontal dilation with scale factor $\frac{1}{a} = 2$:

$a = \frac{1}{2}$

$y = \frac{1}{(ax)} = \frac{1}{\frac{1}{2}x} = \frac{2}{x}$

A vertical dilation with scale factor $k = 2$:

$y = \left(\frac{1}{x}\right) = 2\left(\frac{1}{x}\right) = \frac{2}{x}$

So these transformations have the same effect on $y = \frac{1}{x}$.

b No. Horizontal dilation gives $y = \frac{4}{x}$, vertical dilation gives $y = \frac{2}{x}$.

5 a $x = -\frac{b}{2a}$

b Horizontal translation

c i $x = -1$ ii $x = 3$

iii $x = -b$ iv $x = -\frac{b}{a}$

6 a $y = -3 \sin \frac{x}{2} - 1$

b Amplitude 3, period 4π, centre -1

7 $x^2 - 6x + y^2 + 8 = 0$

8 Reflection in y-axis, horizontal dilation scale factor $\frac{1}{3}$, horizontal translation 2 units to the left, vertical dilation scale factor 3, vertical translation 5 units down

9 $y = -x^3 + 3x$

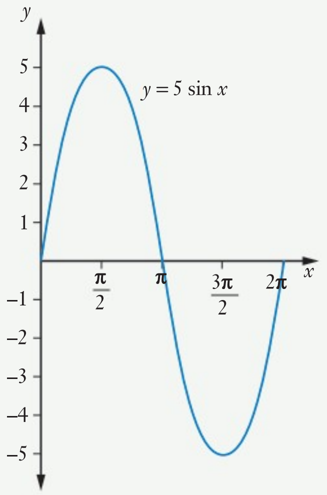
Chapter 3

Exercise 3.01

1 a Phase shift b Amplitude

c Period d Centre

2 a



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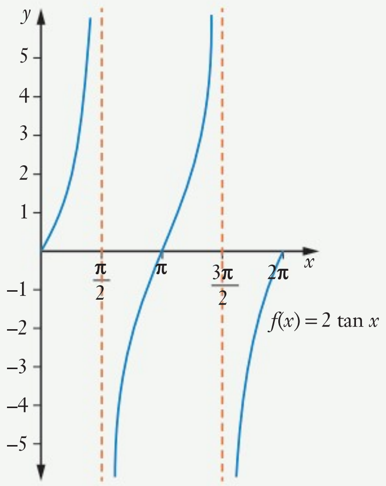

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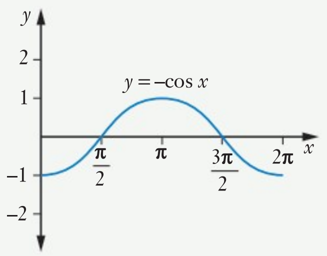

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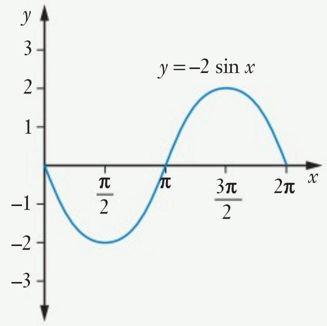
d



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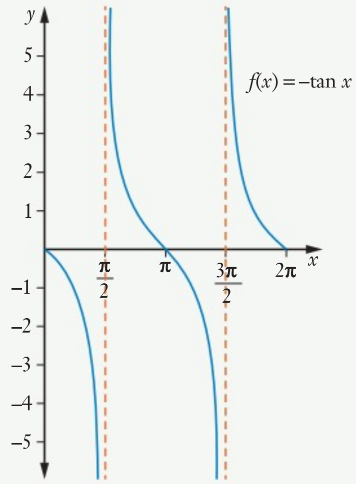
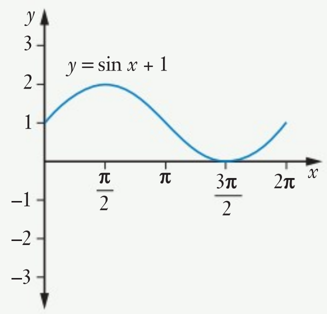


3 a



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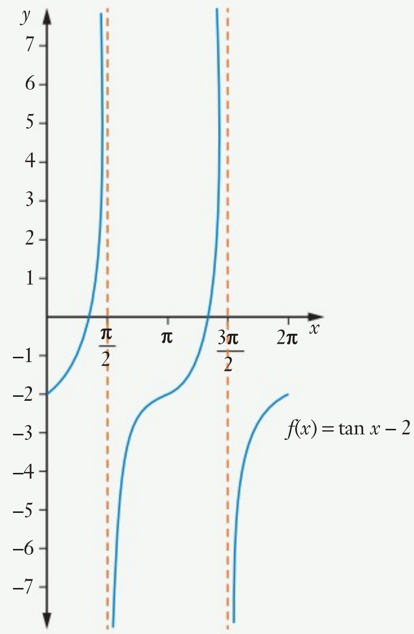

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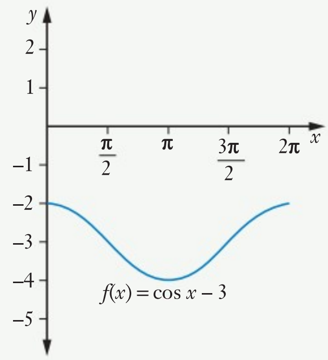

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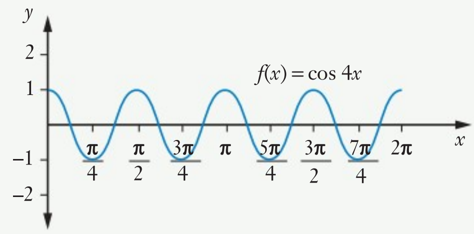

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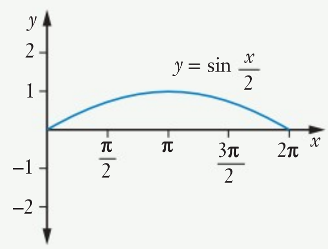

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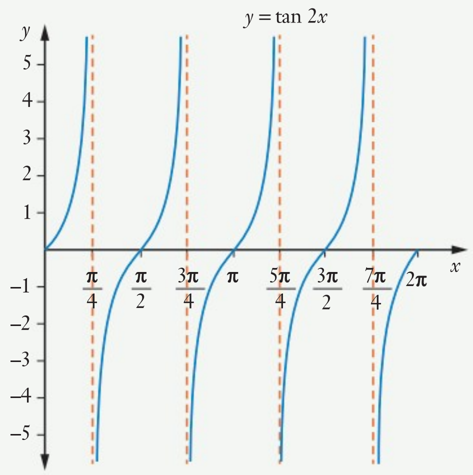

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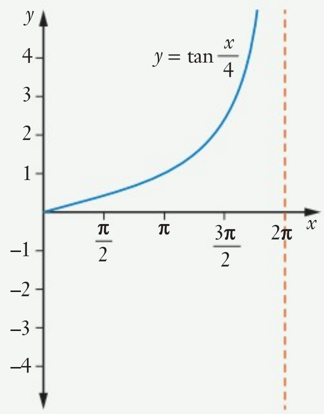
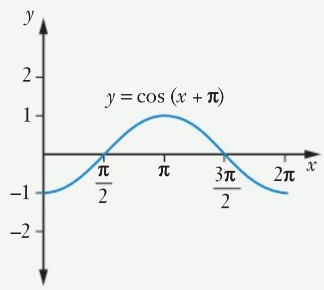


5 a



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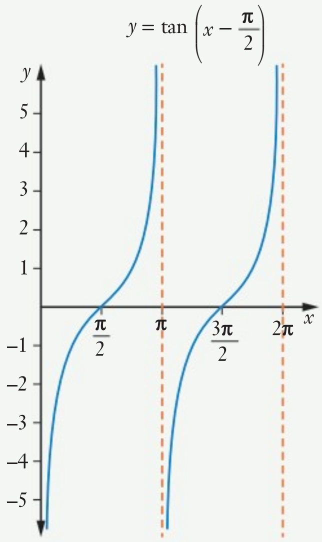

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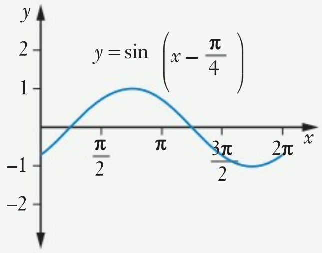


6 a $y = 9 \sin x$ b $y = -\sin x$

c $y = \sin x - 4$ d $y = \sin 2x$

e $y = \sin (x - \pi)$

7 a $y = 4 \cos x$ b $y = \cos\left(x + \frac{\pi}{3}\right)$

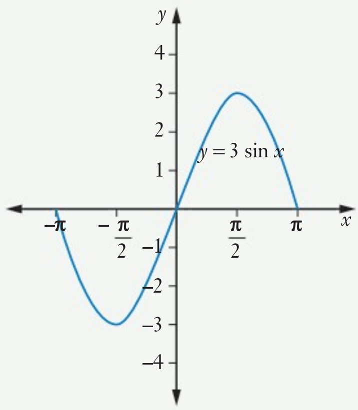
c $y = \cos x + 8$ d $y = \cos 4x$

e $y = 7 \cos x$

8 a $y = \tan \frac{x}{2}$ b $y = \tan\left(x - \frac{\pi}{6}\right)$

c $y = \tan (-x)$

9 a



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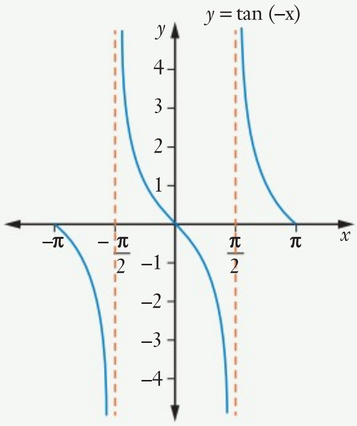

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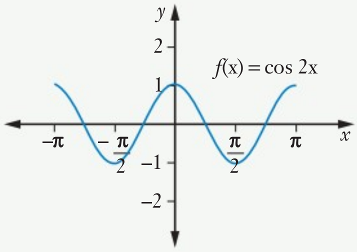

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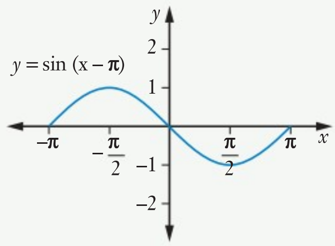

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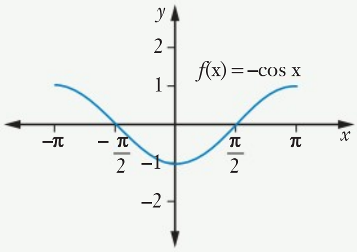
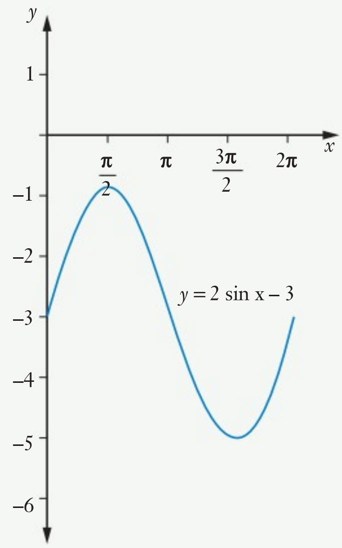


Exercise 3.02

1 a



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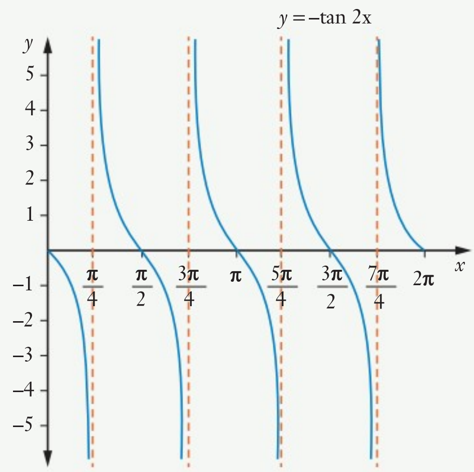

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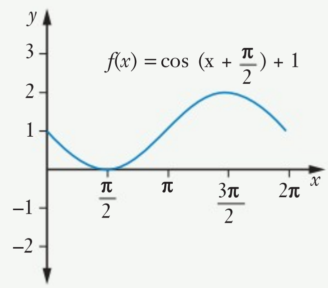
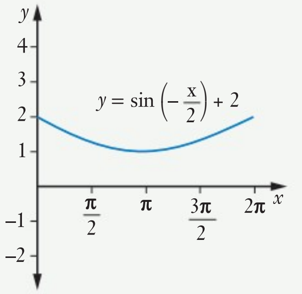
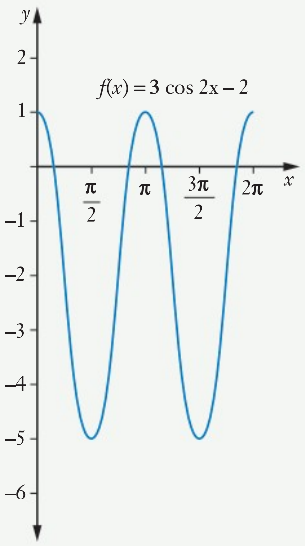


d



e



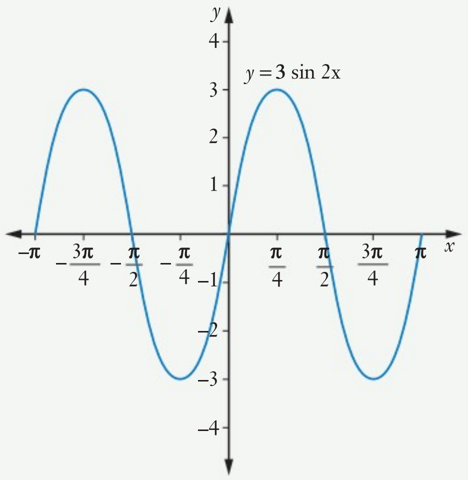
2 a $y = 5\sin[3(x + 5)] - 6$

b Amplitude 5, period $\frac{2\pi}{3}$, centre −6, phase shift 5 units to the left

3 a $y = 4\cos\left[6\left(x - \frac{\pi}{3}\right)\right] + 2$

b $y = -\cos(x + \pi) - 5$

4 a



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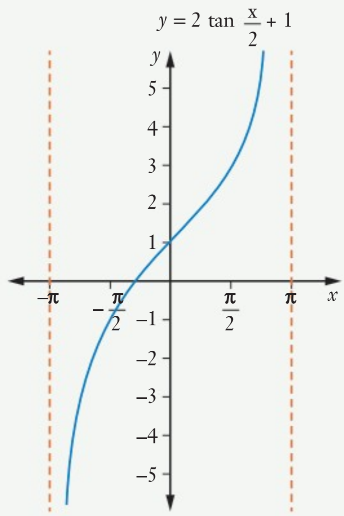

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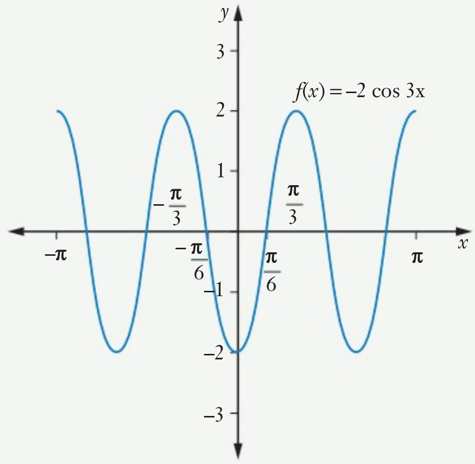
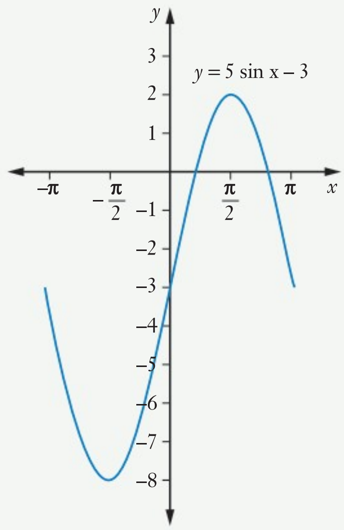


d



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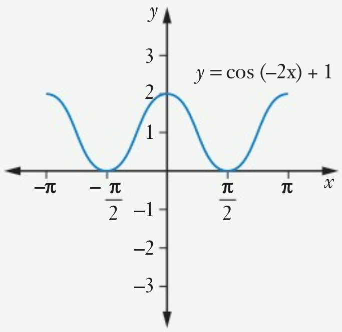
e



5 a No amplitude, period $\frac{\pi}{4}$, centre -5

b Amplitude 8, period 2π, phase shift π units to the left, centre -3

c Amplitude 5, period π, phase shift 3 units to the right, centre 1

6 a $y = 7\sin[2(x-1)] - 3$

b $y = -\cos 5x + 2$

c $y = -\tan\left[\frac{1}{2}(x+2)\right]$

d $y = 4\sin\left[-\frac{1}{3}(x+5)\right] + 2$

7 No amplitude, period $\frac{2\pi}{a}$, phase shift b units to the right when $b < 0$, to the left when $b > 0$, centre c

8 $y = \tan(4x - 3)$

9 a 15 m

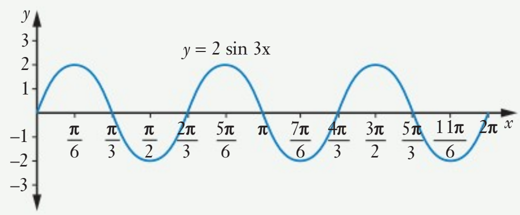
b Amplitude 10, period 12

c $D = 10\cos\frac{\pi t}{6} + 15$

10 $B = 20\sin\frac{\pi t}{30} + 100$

Exercise 3.03

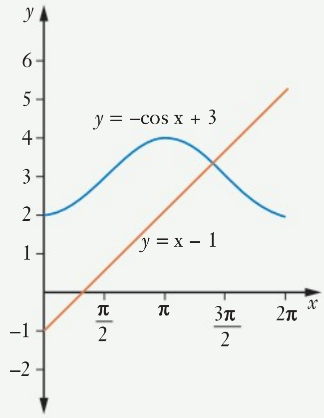
1



a 6

b $x = 0.2, 0.9, 2.3, 3, 4.3, 5.1$

2 a



b i $x = 4.4$ ii $x = 0, 2\pi$

3 a $x = 15°, 75°, 195°, 255°$

b $x = 45°, 105°, 165°, 225°, 285°, 345°$

c $x = 240°, 300°$ d $x = 105°, 285°$

e $x = 120°, 300°$

4 a $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

b $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

c $x = 0, \frac{2\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$

d $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

e $x = \pi$

5 a $x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$

b $x = -\frac{\pi}{2}, 0$ c $x = -\frac{\pi}{4}, \frac{3\pi}{4}$

d $x = -\pi, 0, \pi$

e $x = -\pi, -\frac{3\pi}{4}, -\frac{\pi}{2}, -\frac{\pi}{4}, 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

6 a $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

b $x = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}, \frac{19\pi}{9}$

7 a Amplitude 15, period 12, centre 20

b $t = 0, 12, 24, \ldots$ months. Average temperature of 35° occurs in January of each year.

8 a $y = 7\sin\frac{\pi t}{5} + 13$

b $t = 2.5, 12.5, 22.5, 32.5$ seconds

c 7.5 seconds

d $0, 5, 10, 15, 20, 25, \ldots$ seconds

9 a Amplitude 1, period $\frac{1}{440}$

b i $x = 0.0002, 0.0009, 0.0025, 0.0032, 0.0045, 0.0055, 0.007, 0.0075, 0.0093$

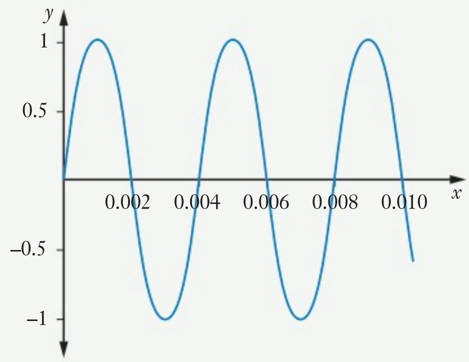
ii $x = 0, 0.001, 0.0021, 0.00035, 0.0045, 0.0056, 0.0068, 0.008, 0.009$

c i $x = 0.00019, 0.00095, 0.0025, 0.0032, 0.0047, 0.0055, 0.0070, 0.0078, 0.0093$

ii $x = 0, 0.0011, 0.0023, 0.0034, 0.0045, 0.0057, 0.0068, 0.0080, 0.0091$

d $y = 3 \sin (880\pi x)$

e



f 440 Hz

10 a Physical 23 days, emotional 28 days, intellectual 33 days

b 7 days, 21 days c 6 days, 19 days, 32 days

d around 7 days

11 a 15 is the centre of motion (equilibrium of spring)

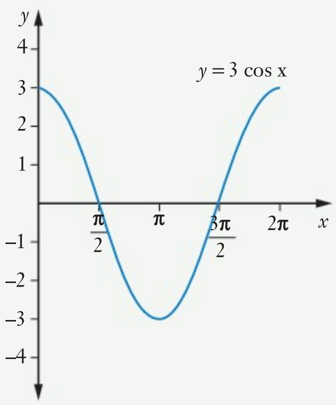
b 27 cm maximum, 3 cm minimum

c 3 cm d $\pi, 3\pi, 5\pi, \ldots$ seconds

Test yourself 3

1 D 2 A 3 B

4 a



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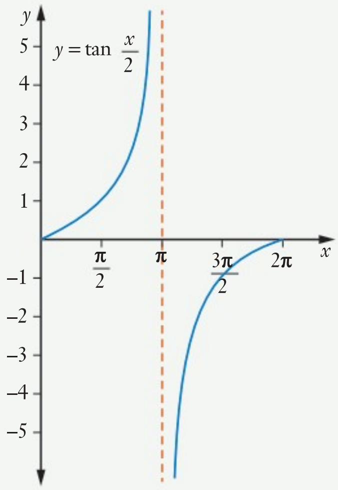

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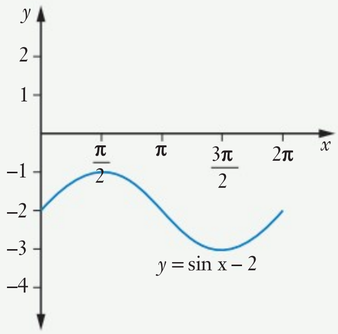
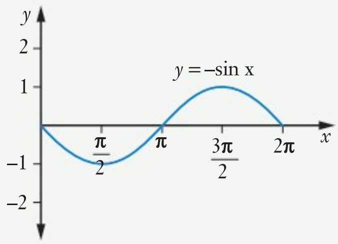


d



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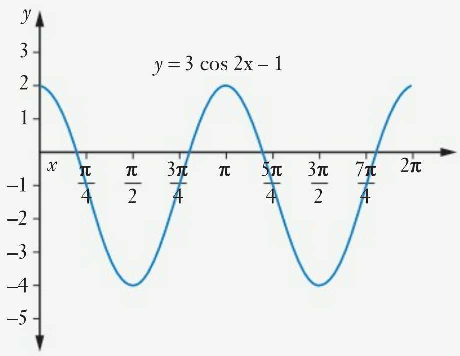

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5 a Maximum 13 m; t = 0, 3, 6, 9, … h; minimum 7 m; t = 1.5, 4.5, 7.5 …h

b t = 0.6, 2.4, 3.6, 5.4, 6.6, 8.4, …h; times when water level in the lock is 11 m

6 a $2\operatorname{cosec}^2 x$ b sec A

c 1 d sin x

7 a $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

b $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$

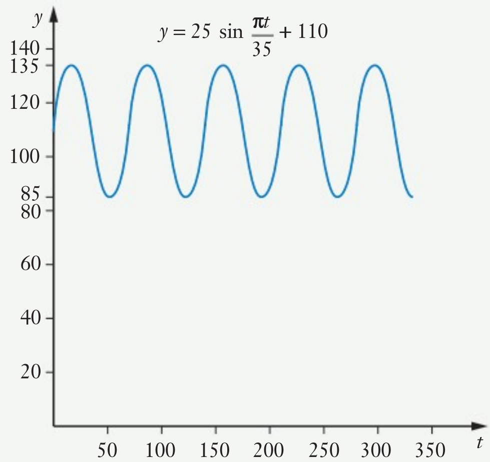
c $x = \frac{3\pi}{2}$

d $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$

e x = 1.15, 1.99, 4.29, 5.13

8 a $y = 25\sin\frac{\pi t}{35} + 110$

b



9 a $x = \pm\frac{\pi}{6}, \pm\frac{5\pi}{6}$ b $x = \frac{\pi}{12}, -\frac{11\pi}{12}$

c $x = \pm\frac{\pi}{4}$

10 a $h = 15\sin\frac{2\pi t}{13} + 65$

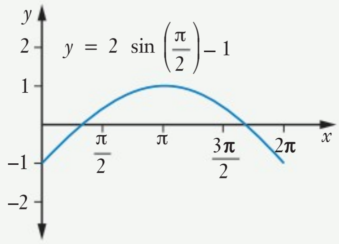
b 0, 6.5, 13, 19.5, … hours

11 a $y = \cos 2x$ b $y = 5\cos x$

c $y = -\cos x$ d $y = \cos\left(x - \frac{\pi}{6}\right)$

e $y = \cos x + 4$

12 a



b x = 1, 5.2 c $x = \frac{\pi}{3}, \frac{5\pi}{3}$

13 a Amplitude 2, period $\frac{2\pi}{3}$, centre −1

b Amplitude 1, period 4π, centre 0, phase shift 2π units to the left

c No amplitude, reflection in x-axis, period $\frac{\pi}{5}$, centre 0, phase shift $\frac{\pi}{20}$ to the right

14 a x = 0°, 180°, 360° b x = 155°, 245°

c x = 10°54′, 59°6′, 190°54′, 239°6′

15 a x = −112.5°, −22.5°, 67.5°, 157.5°

b x = 90°, −30°, −150°

Challenge exercise 3

1 a Amplitude 2, period π, phase shift $\frac{\pi}{4}$ units to the right

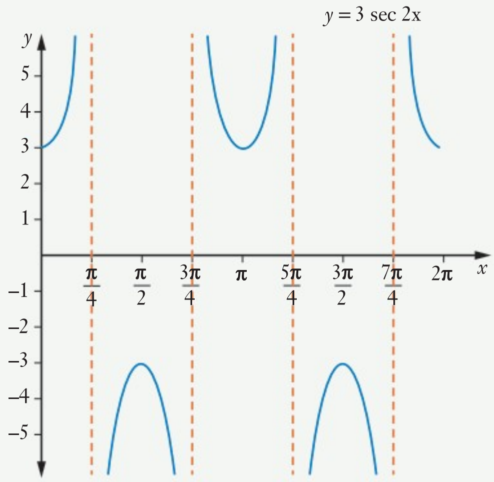
b $x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

2 a $y = 8\cos x + 4$

b $y = 2\sin\left[8\left(x - \frac{\pi}{3}\right)\right] + 3$

c $y = \tan\left[\frac{1}{2}\left(x + \frac{\pi}{2}\right)\right]$

3



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4 $x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$

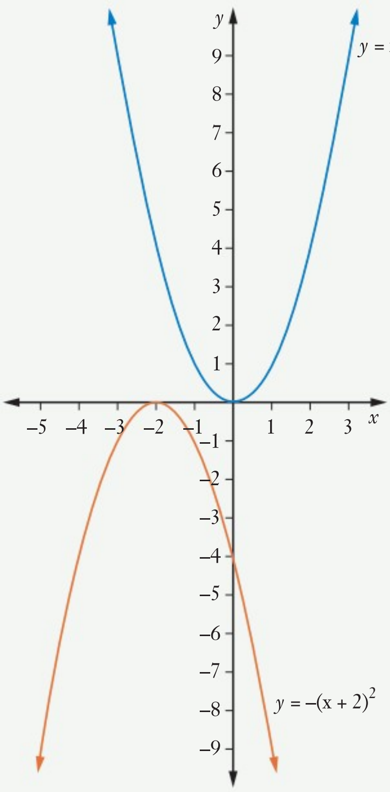
5 a $y = 4\cos\left(\frac{1}{3}\left[x - \frac{\pi}{6}\right]\right) - 20$

b Amplitude 4, period 6π, centre -20, phase shift $\frac{\pi}{6}$ to the right

Practice set 1

1 B	2 C	3 C	4 D
5 D	6 C	7 D	

8 a



b Reflection in x-axis, horizontal translation 2 units to the left

9 a $x = 67° 30', 157° 30', 247° 30', 337° 30'$

b $x = 20°, 100°, 140°, 220°, 260°, 340°$

c $x = 150°, 210°$

d $x = 60°, 240°$

e $x = 0°, 90°, 180°, 270°, 360°$

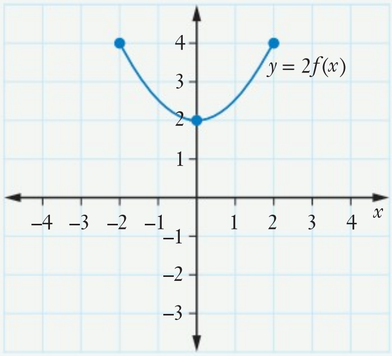
10 a Amplitude 4, period $\frac{2\pi}{5}$, centre 0

b Amplitude 2, reflection in x-axis, centre 1, phase shift $\frac{\pi}{6}$ units to the right

c Period 4π, phase shift 8 units to the left

11 $\frac{3}{4^{10}}$

12 a



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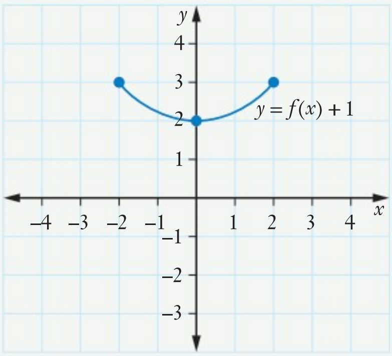

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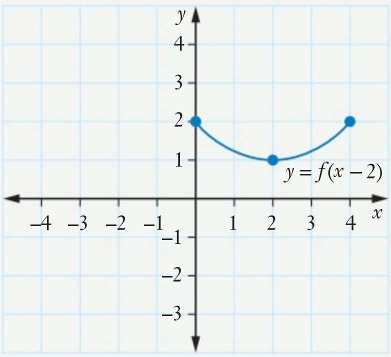

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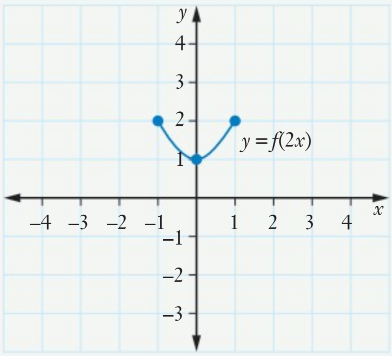


13 $\pm 104, 52, \pm 26, 13, \ldots$

14 44th term

15 Vertical dilation, scale factor 4 (stretch), horizontal translation 1 unit to the right, vertical translation 3 units down.

16 a $x = 30°, 41°49', 138°11', 150°$

b $x = 45°, 105°, 225°, 285°$

17 a Centre 10; max distance 16 cm, 4 cm

b 1 second

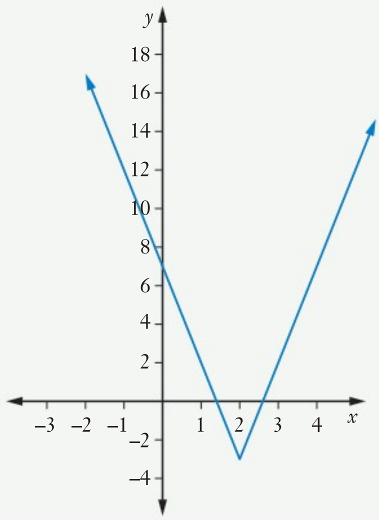
18 $y = 4\sqrt{3(x+7)} - 1$

19 a 199 b 5050

20 a $T_1 = 4, T_2 = 11, T_3 = 18, T_{12} = 81$

b 1410 c 29th term

21 a



b i $x = 1, 3$ ii $x = 0, 4$ iii $x = 2$

c Domain $(-\infty, \infty)$, range $[-3, \infty)$

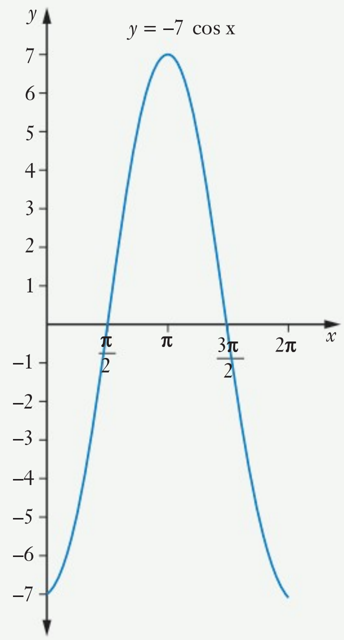
22 a $-\frac{1}{2}$ b $-\frac{\sqrt{3}}{2}$ c 1

d $-\frac{1}{\sqrt{2}}$ e $-\frac{1}{\sqrt{3}}$

23 $T_n = 11n - 26$

24 $x = 40°$

25 a



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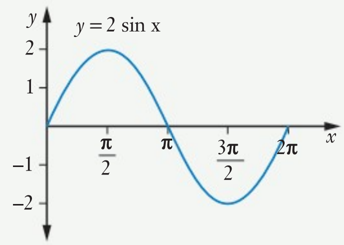

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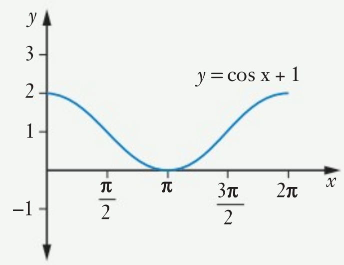

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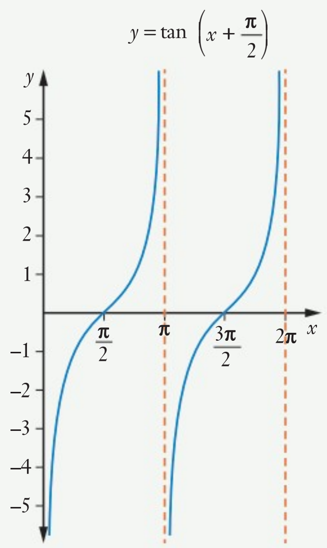

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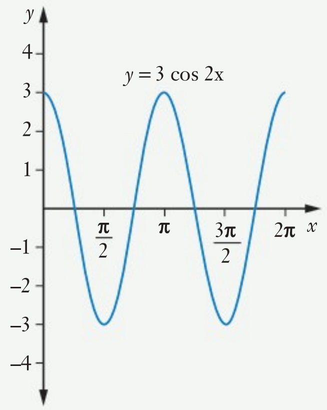

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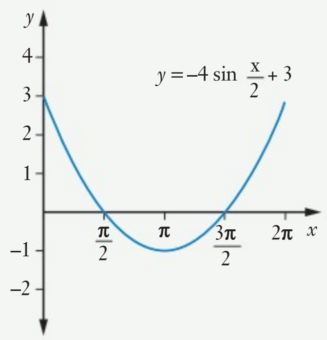


26 Proofs (see worked solutions)

27 a $y = 2\sin x$ b $y = \sin\frac{x}{2}$

c $y = \sin x - 3$ d $y = -5\sin x$

e $y = \sin\left(x + \frac{\pi}{2}\right)$

f $y = 5\sin\left[\frac{1}{3}(x - \pi)\right] + 1$

28 $x = \frac{5}{6}$

29 a $x = 0.9, 5.38$ b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

c $x = \frac{\pi}{6}, \frac{5\pi}{6}$ d $x = \frac{\pi}{2}, \frac{3\pi}{2}$

e $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

f $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

30 $n = 4$ 31 $\frac{1}{10}$

32 a Centre 18, amplitude 20, so maximum temperature is 38°C, minimum −2°C. Period is 12, so it is an annual cycle.

b July c January

d 8°C e April and October

33 $-1 \le x \le 3$

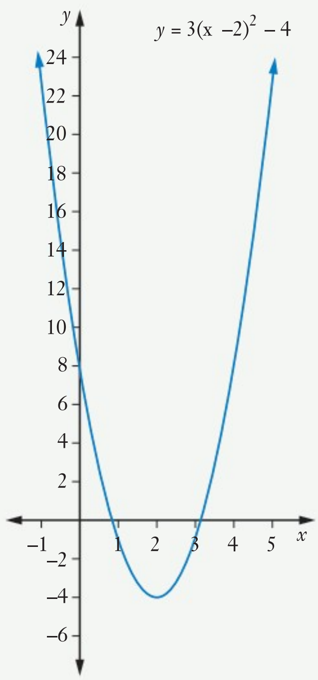
34 $f(-x) = 3(-x)^2 - 2$
$= 3x^2 - 2$
$= f(x)$ so even

35 a $\log 3, \log 3^2, \log 3^3, \ldots = \log 3, 2\log 3, 3\log 3, \ldots$

Arithmetic series, since $2\log 3 - \log 3 = 3\log 3 - 2\log 3 = \log 3$

b $210 \log 3$

36 a



b i $x \le 0, x \ge 4$ ii $0 < x < 4$

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Chapter 4

Exercise 4.01

1 a $12x^3 - 6x^2 + 7$ b 2
  c $12x - 3$

2 $20x^4 + 18x$ 3 $6\pi t^2 - 6t$ 4 $f'(-2) = 101$

5 a $-5x^{-6}$ b $\frac{2}{3}x^{-\frac{1}{3}}$ c $-\frac{2}{x^3}$
  d $\frac{1}{4\sqrt[4]{x^3}}$ e $\frac{20}{x^5}$

6 $\frac{1}{12}$

7 a $21(3x-1)^6$ b $3(2x-1)(x^2-x+2)^2$
  c $\frac{7}{2\sqrt{7x-2}}$ d $-\frac{3}{(3x-2)^2}$
  e $\frac{2x}{3\sqrt[3]{(x^2-3)^2}}$

8 a $3x^2 + 8x$ b $24x + 4$ c $12x^2 + 4$
  d $4(x^2 - 1)(5x^2 + 3x - 1)$ e $\frac{x^2(7x+6)}{\sqrt{x+1}}$

9 a $-\frac{13}{(x-5)^2}$ b $\frac{x^2(8x-21)}{(4x-7)^2}$
  c $\frac{2(x^2-3x-3)}{(2x-3)^2}$ d $\frac{-6x+23}{(2x+9)^3}$
  e $\frac{3x-7}{\sqrt{(2x-1)^3}}$

10 a -6 b 3

11 a $\frac{1}{14}$ b $\frac{1}{5}$

12 a $7x - y - 24 = 0$ b $51x - y - 72 = 0$

13 a $x - 4y - 3 = 0$ b $x + 3y - 6 = 0$

14 $x = 4$ 15 $(2, 6), (-2, -10)$

16 $3x + y + 8 = 0$ 17 $(4, 2)$

18 a $17x - y - 3 = 0$ b $x + 17y + 51 = 0$

19 a $6t$ b $-\frac{2}{(t-3)^2}$ c $\frac{2}{3\sqrt[3]{(2x+3)^2}}$

20 a i 10 kg s^{-1} ii 17 kg s^{-1}
  b i 13 kg s^{-1} ii 123 kg s^{-1}

21 -2.18 Pa/m^3

22 a i 2 m ii 1.5 m b 2 s
  c i 2 m s^{-1} ii 0 m s^{-1} iii -4 m s^{-1}

Exercise 4.02

1 a $7e^{7x}$ b $-e^{-x}$ c $6e^{6x-2}$
  d $2xe^{x^2+1}$ e $(3x^2+5)e^{x^3+5x+7}$
  f $5e^{5x}$ g $-2e^{-2x}$ h $10e^{10x}$
  i $2e^{2x} + 1$ **j** $2x + 2 - e^{1-x}$
  k $5(1 + 4e^{4x})(x + e^{4x})^4$ l $e^{2x}(2x + 1)$
  m $\frac{e^{3x}(3x-2)}{x^3}$ n $x^2e^{5x}(5x + 3)$
  o $\frac{4e^{2x+1}(x+2)}{(2x+5)^2}$

2 $3e$

3 a $3^x\ln 3$ b $10^x\ln 10$ c $3(2^{3x-4})\ln 2$

4 5 5 $x + y - 1 = 0$

6 a $3e^3$ b $-\frac{1}{3e^3}$

7 a $y = 2ex - e$ b $x + 2ey - 2e^2 - 1 = 0$

8 $x\ln 4 - y + 4 = 0$

9 a i 29 627 ii 35 826
  b i 1044 people/year ii 1240 people/year
  c i 1126 people/year ii 1361 people/year

10 a 55 042 cm min^{-1}
  b i 142.8 cm min^{-1} ii 1087 cm min^{-1}
  iii 177 722 205.4 cm min_{-1}

11 a 20 g b 7 g
  c -0.091 g/year
  d i -0.147 g/year ii -0.051g/year
  iii -0.0063g/year

12 a 66 079.4 cm b 132 158.8 cm s^{-1}

Exercise 4.03

1 a $1 + \frac{1}{x}$ b $-\frac{1}{x}$ c $\frac{3}{3x+1}$
  d $\frac{2x}{x^2-4}$ e $\frac{15x^2+3}{5x^3+3x-9}$
  f $\frac{10x^2+2x+5}{5x+1}$ g $6x + 5 + \frac{1}{x}$
  h $\frac{8}{8x-9}$ i $\frac{6x+5}{(x+2)(3x-1)}$
  j $\frac{-30}{(4x+1)(2x-7)}$ k $\frac{5}{x}(1 + \ln x)^4$
  l $9\left(\frac{1}{x}-1\right)(\ln x - x)^8$ m $\frac{4}{x}(\ln x)^3$

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n $6\left(2x+\frac{1}{x}\right)(x^2+\ln x)^5$

o $1+\ln x$ p $\frac{1-\ln x}{x^2}$

q $\frac{2x\ln x+2x+1}{x}$

r $3x^2\ln(x+1)+\frac{x^3}{x+1}$ s $\frac{1}{x\ln x}$

t $\frac{x-2-x\ln x}{x(x-2)^2}$ u $\frac{e^{2x}(2x\ln x-1)}{x(\ln x)^2}$

v $e^x\left(\frac{1}{x}+\ln x\right)$ w $\frac{10\ln x}{x}$

2 $f'(1)=-\frac{1}{2}$ 3 $\frac{1}{x\ln 10}$

4 $x-2y-2+2\ln 2=0$ 5 $x-y-2=0$

6 $-\frac{2}{5}$ 7 $5x+y-\ln 5-25=0$

8 $5x-19y+19\ln 19-15=0$

9 $\frac{2}{(2x+5)\ln 3}$

10 $(2\ln 2)x+y-1-4\ln 2=0$

11 a 20 000
   b i 10.6 years ii 43.6 years
   c $P=20\,000e^{0.021t}$ d 452 kangaroos/year
   e i 447 kangaroos/year
      ii 466 kangaroos/year
      iii 518 kangaroos/year

Exercise 4.04

1 a $4\cos 4x$ b $-3\sin 3x$
  c $5\sec^2 5x$ d $3\sec^2(3x+1)$
  e $\sin(-x)$ f $3\cos x$
  g $-20\sin(5x-3)$ h $-6x^2\sin(x^3)$
  i $14x\sec^2(x^2+5)$ j $3\cos 3x-8\sin 8x$
  k $\sec^2(\pi+x)+2x$ l $x\sec^2 x+\tan x$
  m $3\sin 2x\sec^2 3x+2\tan 3x\cos 2x$
  n $\frac{x\cos x-\sin x}{2x^2}$
  o $\frac{3\sin 5x-5(3x+4)\cos 5x}{\sin^2 5x}$
  p $9(2+7\sec^2 7x)(2x+\tan 7x)^8$
  q $2\sin x\cos x=\sin 2x$ r $-45\sin 5x\cos^2 5x$

  s $e^x+2\sin 2x$ t $-\frac{1}{x}\cos(1-\ln x)$
  u $(e^x+1)\cos(e^x+x)$ v $\frac{\cos x}{\sin x}=\cot x$
  w $e^{3x}(3\cos 2x-2\sin 2x)$
  x $\frac{e^{2x}(2\tan 7x-7\sec^2 7x)}{\tan^2 7x}$

2 12 3 $6\sqrt{3}x-12y+6-\pi\sqrt{3}=0$

4 $-\frac{\sin x}{\cos x}=-\tan x$ 5 $-\frac{2}{3\sqrt{3}}=-\frac{2\sqrt{3}}{9}$

6 $\sec^2 x\,e^{\tan x}$

7 $8x+24\sqrt{2}y-72-\pi=0$

8 Proof (see worked solutions).

9 a $\frac{\pi}{180}\sec^2 x°$ b $-\frac{\pi}{60}\sin x°$ c $\frac{\pi}{900}\cos x°$

10 $\sin^3 x(4\cos^2 x-\sin^2 x)$

11 a 750 b 525 c 975
   d 2.6, 6.4, 11.6, 15.4 … days
   e i −136 fish/day ii 155 fish/day
     iii −101 fish/day iv 0 fish/day
   f 0.23, 4.27, 9.23, 13.27, … days

12 a i 9 m ii 13 m
   b 0.2, 5.8, 12.2, 17.8, … h
   c i 0 mh^{-1} ii 3.6 mh^{-1} iii 4.2 mh^{-1}
   d 1.5, 10.5, 13.5, 22.5, … h

Exercise 4.05

1 $7x^6-10x^4+4x^3-1$; $42x^5-40x^3+12x^2$; $210x^4-120x^2+24x$; $840x^3-240x+24$

2 $72x^7$

3 $f'(x)=10x^4-3x^2$, $f''(x)=40x^3-6x$

4 $f'(1)=11$, $f''(-2)=168$

5 $7x^6-12x^5+16x^3$; $42x^5-60x^4+48x^2$; $210x^4-240x^3+96x$

6 $\frac{dy}{dx}=4x-3$, $\frac{d^2y}{dx^2}=4$

7 $f'(-1)=-16$, $f''(2)=40$ 8 $-4x^{-5}$; $20x^{-6}$

9 $-\frac{1}{32}$ 10 26

11 $x=\frac{7}{18}$ 12 $x>\frac{1}{3}$

13 $20(4x-3)^4$; $320(4x-3)^3$

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14 $f'(x) = -\frac{1}{2\sqrt{2-x}}$; $f''(x) = -\frac{1}{4\sqrt{(2-x)^3}}$

15 $f'(x) = -\frac{16}{(3x-1)^2}$; $f''(x) = \frac{96}{(3x-1)^3}$

16 $\frac{dv^2}{dt^2} = 24t + 16$

17 $b = \frac{2}{3}$

18 196

19 $b = -2.7$

20 $\frac{dy}{dx} = 4e^{4x} - 4e^{-4x}$

$\frac{d^2y}{dx^2} = 16e^{4x} + 16e^{-4x} = 16y$

21, 22 Proofs (see worked solutions).

23 $n = -15$

24 $y = 2\cos 5x$; $\frac{dy}{dx} = -10\sin 5x$

$\frac{d^2y}{dx^2} = -50\cos 5x = -25y$

25 $f(x) = -2\sin x$; $f'(x) = -2\cos x$

$f''(x) = 2\sin x = -f(x)$

26 $y = 2\sin 3x - 5\cos 3x$; $\frac{dy}{dx} = 6\cos 3x + 15\sin 3x$

$\frac{d^2y}{dx^2} = -18\sin 3x + 45\cos 3x = -9y$

27 $a = -7$, $b = -24$

28 $f''(2) = \frac{3}{4\sqrt{2}} = \frac{3\sqrt{2}}{8}$

29 a 8 m b 38 m c 31 m s^{-1}

d 26 m s^{-2}

30 a 4 cm

b Maximum 20 cm, minimum 4 cm

c i 0 cm s^{-1} ii 25.1 cm s^{-1}

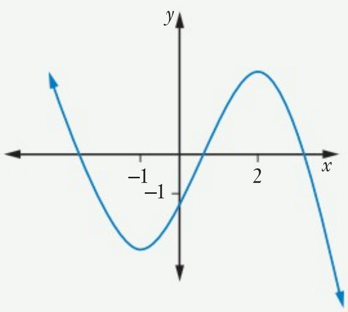
d i $-8\pi^2$ or -79 cm s^{-2} ii $8\pi^2$ or 79 cm s^{-2}

iii 0 cm s^{-2}

e $\frac{d^2h}{dt^2} = -\pi^2(h - 12)$

Exercise 4.06

1 a



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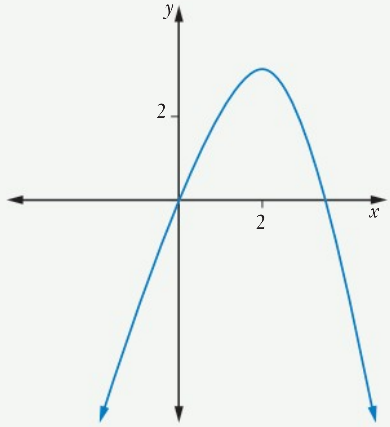

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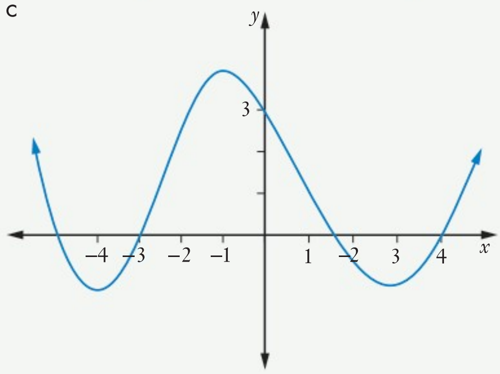

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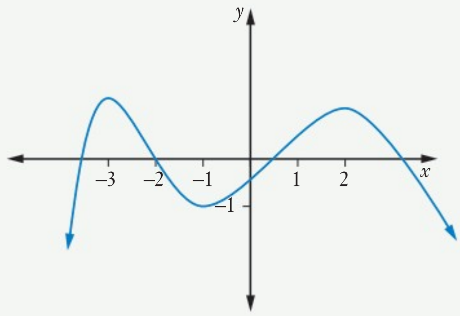

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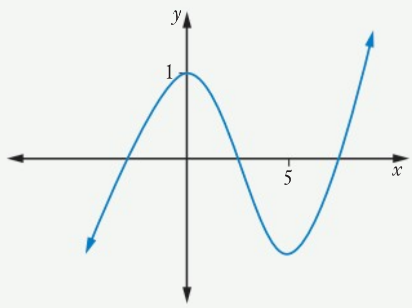

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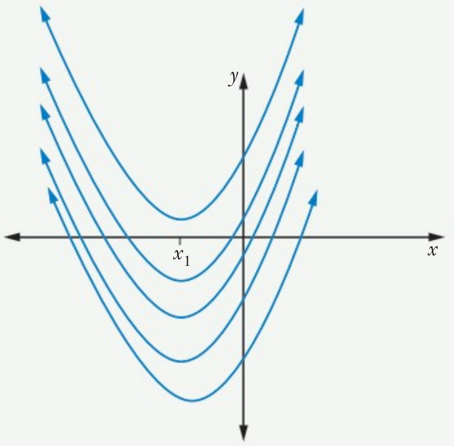
2 a



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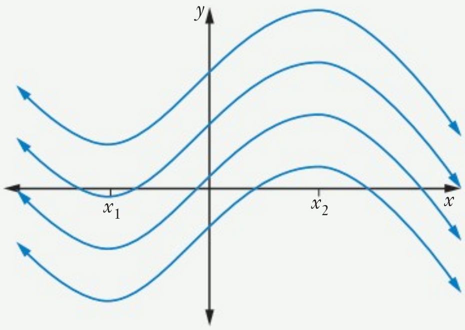

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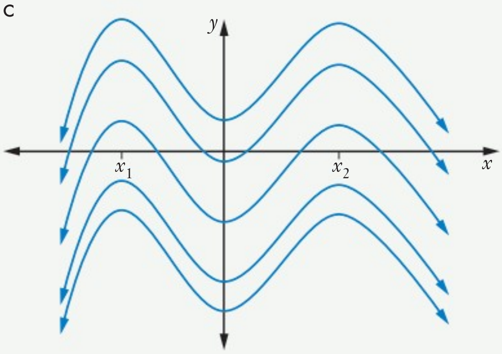

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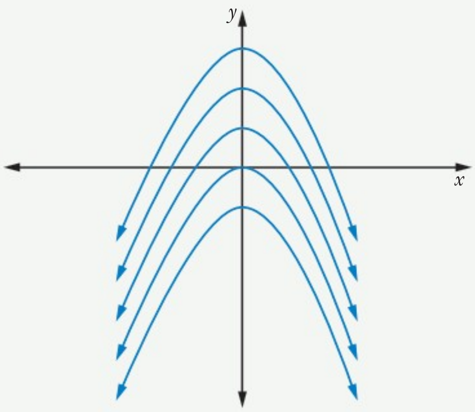

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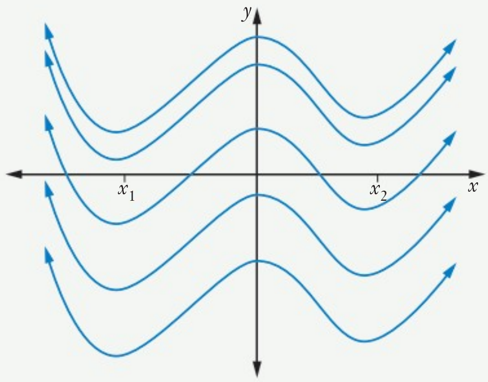

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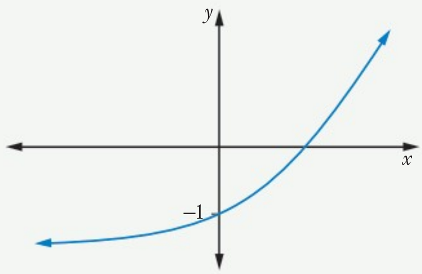

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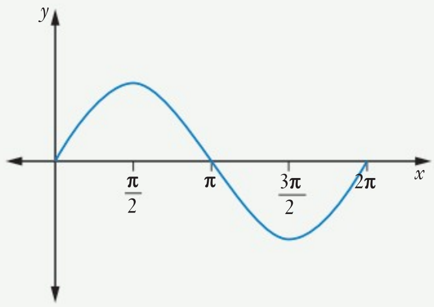

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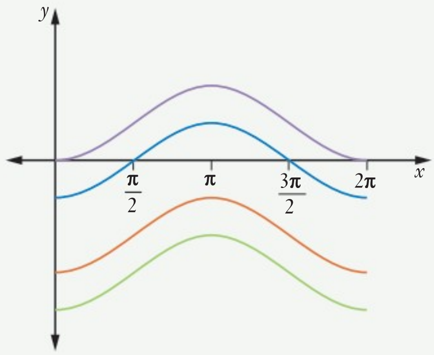

Exercise 4.07

1 a $x^2 - 3x + C$

b $\frac{x^3}{3} + 4x^2 + x + C$

c $\frac{x^6}{6} - x^4 + C$

d $\frac{(x-1)^3}{3} + C$

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e $6x + C$

f $\frac{(3x+2)^6}{18} + C$

g $\frac{4(2x-7)^5}{5} + C$

2 a $f(x) = 2x^3 - \frac{x^2}{2} + C$

b $f(x) = \frac{x^5}{5} - x^3 + 7x + C$

c $f(x) = \frac{x^2}{2} - 2x + C$

d $f(x) = \frac{x^3}{3} - x^2 - 3x + C$ e $f(x) = \frac{2x^{\frac{3}{2}}}{3} + C$

3 a $y = x^5 - 9x + C$ b $y = -\frac{x^{-3}}{3} + 2x^{-1} + C$

c $y = \frac{x^4}{20} - \frac{x^3}{3} + C$ d $y = -\frac{2}{x} + C$

e $y = \frac{x^4}{4} - \frac{x^2}{3} + x + C$

4 a $\frac{2\sqrt{x^3}}{3} + C$ b $-\frac{x^{-2}}{2} + C$

c $-\frac{1}{7x^7} + C$ d $2x^{\frac{1}{2}} + 6x^{\frac{1}{3}} + C$

e $-\frac{x^{-6}}{6} + 2x^{-1} + C$

5 a $\frac{(x^2+5)^5}{5} + C$ b $\frac{(x^3-1)^{10}}{10} + C$

c $\frac{(2x^2+3)^4}{2} + C$ d $\frac{3(x^5+1)}{7} + C$

e $\frac{(x^3-4)^8}{16} + C$ f $\frac{(2x^6-7)^9}{108} + C$

g $\frac{(x^2-x+3)^5}{5} + C$ h $\frac{(x^3+2x^2-7x)^{11}}{11} + C$

i $\frac{(x^2-6x-1)^6}{12} + C$

6 $y = \frac{x^4}{4} - x^3 + 5x - \frac{1}{4}$

7 $f(x) = 2x^2 - 7x + 11$ 8 $f(1) = 8$

9 $y = 2x - 3x^2 + 19$ 10 $x = 16\frac{1}{3}$

11 $y = 4x^2 - 8x + 7$ 12 $y = 2x^3 + 3x^2 + x - 2$

13 $f(x) = x^3 - x^2 - x + 5$ 14 $f(2) = 20.5$

15 $y = \frac{4x^3}{3} - 15x - 14\frac{1}{3}$

16 $y = \frac{x^3}{3} - 2x^2 + 3x - 4\frac{2}{3}$

17 $f(x) = x^4 - x^3 + 2x^2 + 4x - 2$

18 $y = 3x^2 + 8x + 8$

19 $f(-2) = 77$ 20 $x = 3t^2 - 5t - 2$

21 $x = 2t^4 - 2t^3 + 3t^2 - 8t - 3$

Exercise 4.08

1 a $-\cos x + C$ b $\tan x + C$ c $\sin x + C$

d $\frac{1}{7}\tan 7x + C$ e $-\frac{1}{2}\cos(2x - \pi) + C$

2 a $e^x + C$ b $\frac{1}{6}e^{6x} + C$ c $\ln|x| + C$

d $\ln|3x - 1| + C$ e $\frac{1}{2}\ln|x^2 + 5| + C$

3 a $e^x + 5x + C$ b $\sin x + 2x^2 + C$

c $\frac{x^2}{2} + \ln|x| + C$

d $2x^4 - x^3 + 3x^2 - 3x + \ln|x| + C$

e $-\frac{1}{5}\cos 5x - \frac{1}{9}\tan 9x + C$

4 $y = \sin x - 5$ 5 $f(x) = 5\ln|x| + 3$

6 $y = 2\sin 2x + \sqrt{3}$

7 $f(x) = 3e^{3x} - 8e^6x + 14e^6$

8 a $P = 25\,000e^{0.054t} + 10\,000$ b $52\,900$

9 $x = e^{3t} + 4$

10 a $\frac{dx}{dt} = 3\cos 3t$

b -0.3 cm c $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \ldots$ s

Test yourself 4

1 D 2 B 3 A 4 C

5 a $5e^{5x}$ b $-2e^{1-x}$ c $\frac{1}{x}$

d $\frac{4}{4x+5}$ e $e^x(x+1)$ f $\frac{1 - \ln x}{x^2}$

g $10e^x(e^x + 1)^9$

6 a $-\sin x$ b $2\cos x$ c $\sec^2 x$

d $x\cos x + \sin x$ e $\frac{x\sec^2 x - \tan x}{x^2}$

f $-3\sin 3x$ g $5\sec^2 5x$

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7 $3x - y + 3 = 0$

8 $12x + 4\sqrt{2}y - 4 - 3\pi = 0$

9 $\dfrac{d^2x}{dt^2} = -4\cos 2t$

10 a $\dfrac{1}{\sqrt{1-x^2}}$ b $\dfrac{3}{1+9x^2}$

c $-\dfrac{10}{\sqrt{1-25x^2}}$

11 $-\dfrac{e^2}{e^2+1}$

12 a $2x^5 - x^4 + 3x^2 - 3x + C$

b $\dfrac{1}{5}e^{5x} + C$

c $\dfrac{1}{9}\tan 9x + C$

d $\ln|x+5| + C$

e $\dfrac{1}{2}\sin 2x + C$

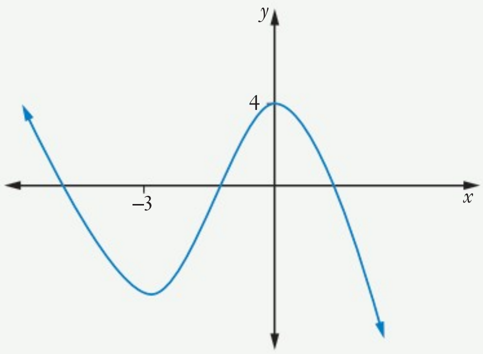
f $-4\cos\dfrac{x}{4} + C$

13 $-3\sqrt{3}$

14 $y = 2x^3 + 6x^2 - 5x - 33$

15 $\dfrac{1}{5\sqrt[5]{(x-3)^4}}$

16



17 $2x + y - \ln 2 - 4 = 0$

18 $4x + 8y - 8 - \pi = 0$

19 a $\dfrac{1}{\sqrt{1-x^2}}$ b $-\dfrac{1}{\sqrt{25-x^2}}$ c $\dfrac{1}{1+x^2}$

d $\dfrac{4}{\sqrt{1-16x^2}}$ e $\dfrac{2}{4+x^2}$

20 a $40x(5x^2+7)^3$

b $4(16x-3)(2x-3)^6$

c $\dfrac{23}{(3x+4)^2}$

d $2x^2e^x(x+3)$

e $\dfrac{3(x+1)\sec^2 3x - \tan 3x}{(x+1)^2}$

21 a 0

b $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}; \dfrac{d}{dx}\left(\dfrac{\pi}{2}\right) = 0$

22 $f(x) = \dfrac{5x^3}{2} + 6x^2 - 49x + 59$

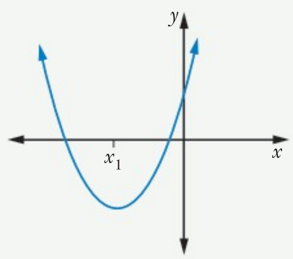
23 a -8 b 26 c -90

24 a $f^{-1}(x) = x^2 - 1$ b $P = (3, 8)$

c $6x - y - 10 = 0$

25 $\dfrac{x}{1+x^2} + \tan^{-1}x$

26 a



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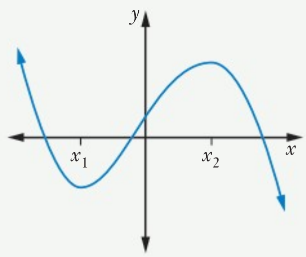

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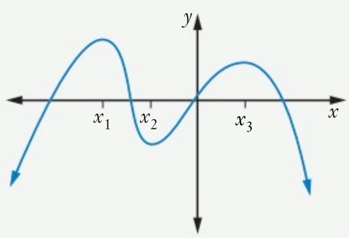

27 $f(x) = 2x^3 - 3x^2 - 31x + 68$

28 $4x - 6\sqrt{3}y + \sqrt{3}\pi - 6 = 0$

29 a $\dfrac{(3x^4-5)^7}{84} + C$ b $\dfrac{3(x^2+1)^{10}}{20} + C$

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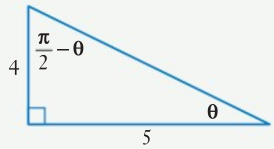
Challenge exercise 4

1 $2e$

2 a



b 0

c $\theta + \frac{\pi}{2} - \theta$

3 $\frac{20x^2 - 120x - 1}{(4x^2 + 1)^4}$; $\frac{-24(20x^3 - 140x^2 - 3x + 5)}{(4x^2 + 1)^5}$

4 a $e^{x^2} + C$ b $-\frac{1}{3}\cos(x^3) + C$

5 $(2x\cos 2x + \sin 2x)e^{x\sin 2x}$

6 $y = \frac{x^3}{3} - x^2 - 15x - 1$

7 a $\frac{2x}{\sqrt{1 - x^4}}$ b $\frac{e^x}{1 + e^{2x}}$

c $\frac{\cos x - \sin x}{\sin x + \cos x}$

8 $25\frac{5}{6}$

9 $\frac{1 + \ln x - x\ln x}{e^x}$

10 a $\theta = \tan^{-1}(4t)$ b $3° 32'$

11 a $x = y + e^y$ b $P = (1 + e, 1)$

c $x - (1 + e)y = 0$

12 a $\frac{\sec^2 x}{\tan x}$ or $\frac{1}{\sin x \cos x}$ b $-\ln|\cos x| + C$

13 a $-\frac{\cos\left(x^3 - \pi\right)}{3} + C$ b $\frac{e^{x^2}}{2} + C$

Chapter 5

Exercise 5.01

1 $x < 2$

2 $x < \frac{1}{4}$

3 $(-\infty, 0)$

4 a $x < 1.5$ b $x > 1.5$ c $x = 1.5$

5 $f'(x) = -2 < 0$ for all x

6 $y' = 3x^2 > 0$ for all $x \neq 0$

7 $(0, 0)$

8 $x = -3, 2$

9 a $(1, -4)$ b $(0, 9)$

c $(1, 1)$ and $(2, 0)$ d $(0, 1)$, $(1, 0)$ and $(-1, 0)$

10 $(2, 0)$

11 $x = 2, 5$

12 $p = -12$

13 $a = 1\frac{1}{2}$, $b = -6$

14 a $\frac{dy}{dx} = 3x^2 - 6x + 27$

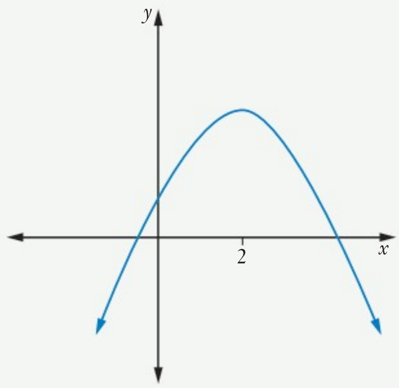
b The quadratic function has $a > 0$

$b^2 - 4ac = -288 < 0$

So $3x^2 - 6x + 27 > 0$ for all x.

The function is monotonic increasing for all x.

15



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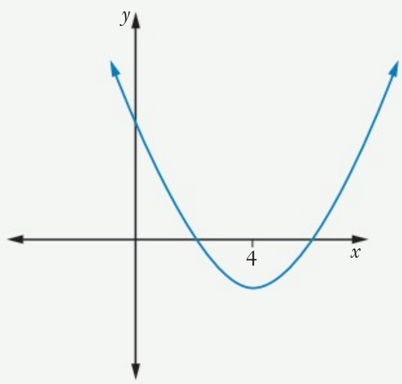

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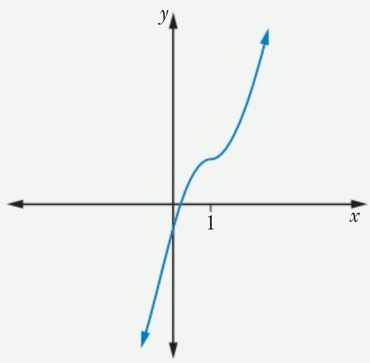

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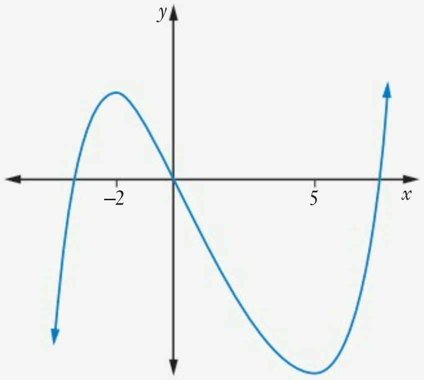

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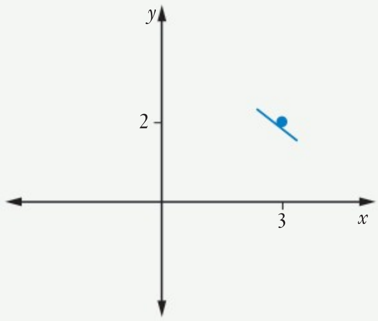

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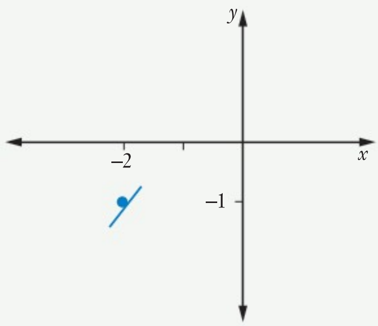


21 (2, 0) and $\left(\frac{2}{3}, 3\frac{13}{81}\right)$

22 $\frac{3x+2}{2\sqrt{x+1}}$; $\left(-\frac{2}{3}, -\frac{2\sqrt{3}}{9}\right)$

23 $a = -1.75$

24 $y' = \frac{1}{2\sqrt{x}} \neq 0$

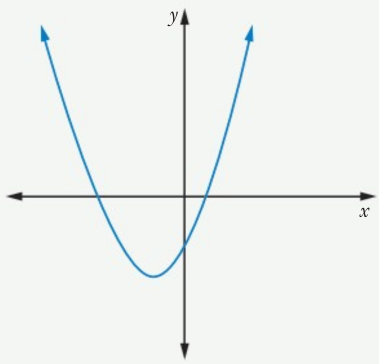
25 $y' = -\frac{3}{x^4} \neq 0$

Exercise 5.02

1 (0, –1); show $y' < 0$ on LHS, $y' > 0$ on RHS
2 (0, 0), minimum
3 (–2, 11); show $f'(x) > 0$ on LHS and $f'(x) < 0$ on RHS
4 (–1, –2), minimum
5 (4, 0) minimum
6 (0, 5) maximum, (4, –27) minimum
7 (0, 5) maximum, (2, 1) minimum
8 (0, –3) maximum, (1, –4) minimum, (–1, –4) minimum
9 (1, 0) minimum, (–1, 4) maximum
10 $m = -3$
11 $x = -3$ minimum
12 $x = 0$ minimum, $x = -1$ maximum
13 a $\frac{dp}{dx} = 2 - \frac{50}{x^2}$
b (–5, –20) maximum, (5, 20) minimum
14 $\left(1, \frac{1}{2}\right)$ minimum
15 (2.06, 54.94) maximum, (–2.06, –54.94) minimum
16 (4.37, 54.92) minimum, (–4.37, –54.92) maximum
17 a $\frac{3600-2x^2}{\sqrt{3600-x^2}}$
b (42.4, 1800) maximum, (– 42.4, –1800) minimum

Exercise 5.03

1 $x > -\frac{1}{3}$
2 $x < 3$
3 $y'' = -8 < 0$
4 $y'' = 2 > 0$
5 $\left(-\infty, 2\frac{1}{3}\right)$
6 (1, 9)
7 (1, –17) and (–1, –41)
8 (0, –2): $y'' < 0$ on LHS, $y'' > 0$ on RHS
9 a No
b Yes – point of inflection at (0, 0)
c Yes – point of inflection at (0, 0)
d Yes – point of inflection at (0, 0)
e No
10



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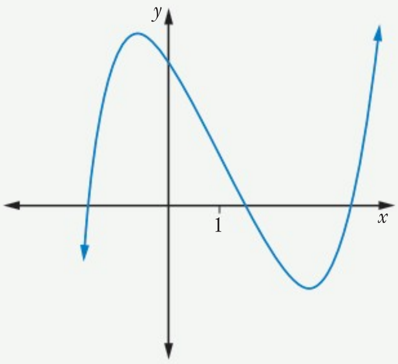


12 None: (2, 31) is not a point of inflection since concavity does not change.

13 Show that $\frac{12}{x^4} > 0$ for all $x \neq 0$.

14 a (0, 7), (1, 0) and (–1, 14)

b (0, 7)

15 a $12x^2 + 24 \neq 0$ and there are no points of inflection.

b The curve is always concave upwards.

16 $a = 2$

17 $p = 4$

18 $a = 3, b = -3$

19 a (0, –8), (2, 2)

b $\frac{dy}{dx} = 6x^5 - 15x^4 + 21$

At (0, –8):

$\frac{dy}{dx} \neq 0$

At (2, 2):

$\frac{dy}{dx} \neq 0$

Exercise 5.04

1 a $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$ b $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} < 0$

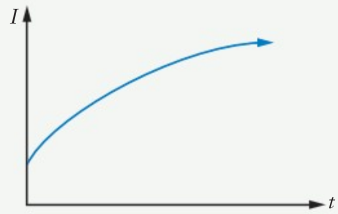
c $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} < 0$ d $\frac{dy}{dx} < 0, \frac{d^2y}{dx^2} > 0$

e $\frac{dy}{dx} > 0, \frac{d^2y}{dx^2} > 0$

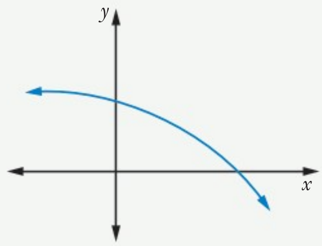
2 a $\frac{dP}{dt} > 0, \frac{d^2P}{dt^2} < 0$

b No, the rate is decreasing.

3



4 a



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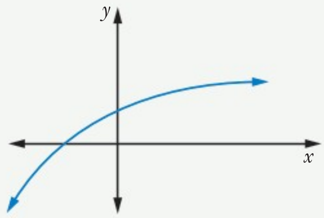

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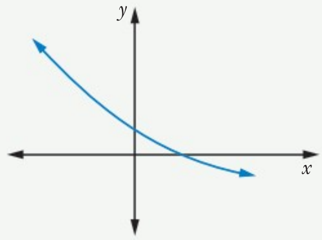

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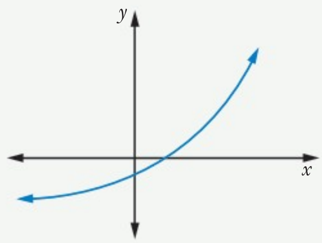

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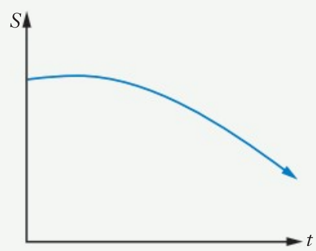

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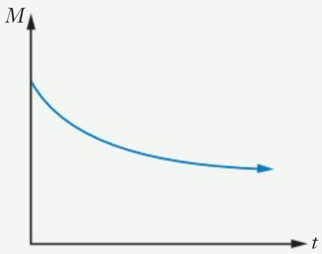
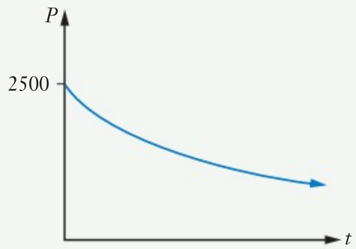
6 

7 $\frac{dM}{dt} < 0, \frac{d^2M}{dt^2} > 0$

8 a The number of fish is decreasing.

b The population rate is increasing.

c 

9 The level of education is increasing, but the rate of increase is slowing down.

10 The population is decreasing, and the rate of change in population is decreasing.

Exercise 5.05

1 (1, 0), minimum

2 (0, 1), minimum (flat)

3 (2, –5); $y'' = 6 > 0$ so minimum

4 (0.5, 0.25); $y'' = -2 < 0$ so maximum

5 (0, –5); $f''(x) < 0$ on LHS, $f''(x) > 0$ on RHS

6 Yes – point of inflection at (0, 3)

7 (–2, –78) minimum, (–3, –77) maximum

8 (0, 1) maximum, (–1, –4) minimum, (2, –31) minimum

9 (0, 1) maximum, (0.5, 0) minimum, (–0.5, 0) minimum

10 a (4, 176) maximum, (5, 175) minimum

b (4.5, 175.5)

11 (3.67, 0.38), maximum

12 (0, –1) minimum, (–2, 15) maximum, (–4, –1) minimum

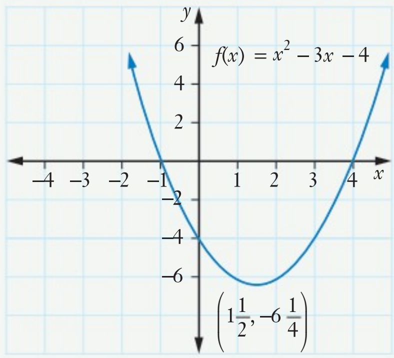
13 a $a = 4$

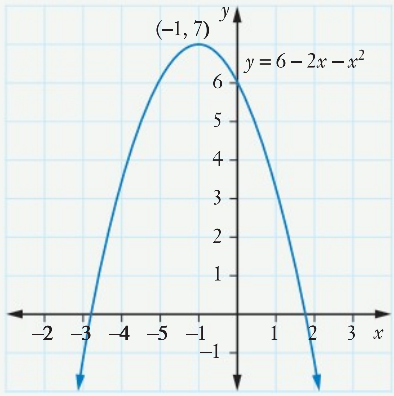
b $\left(\frac{1}{2}, 0\right)$, minimum

14 $m = -4$

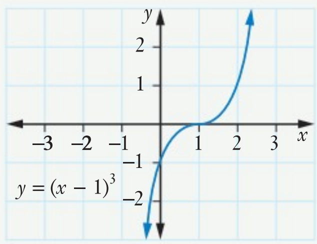
15 $a = 3, b = -9$

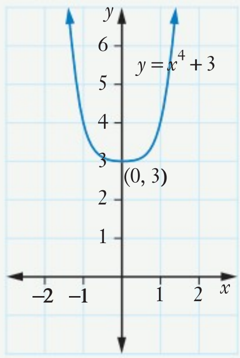
Exercise 5.06

1 

2 

3 (1, 0) point of inflection



4 

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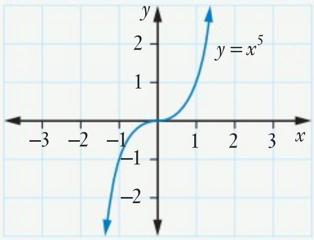
5 (0, 0)



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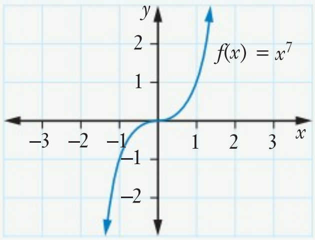
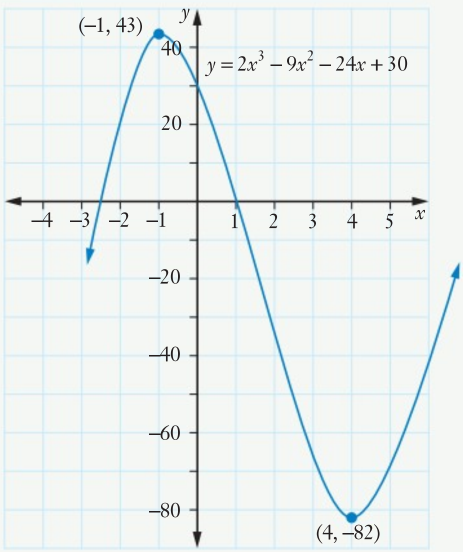


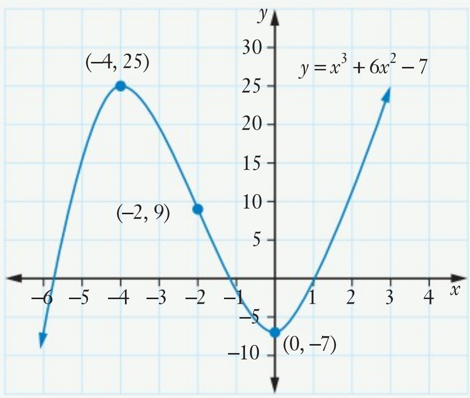
7 (–1, 43) maximum, (4, –82) minimum



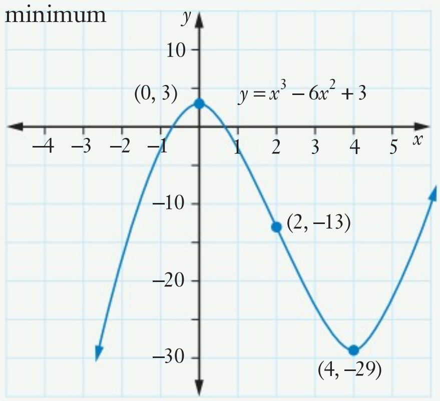
8 a (0, –7) minimum, (–4, 25) maximum

b (–2, 9)

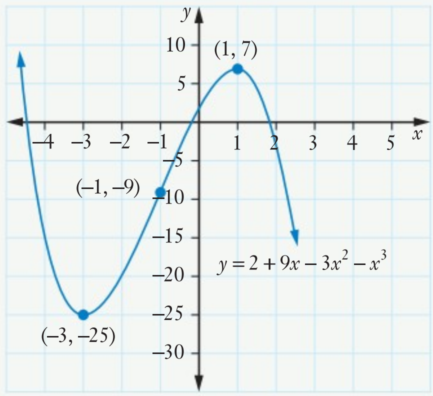
c



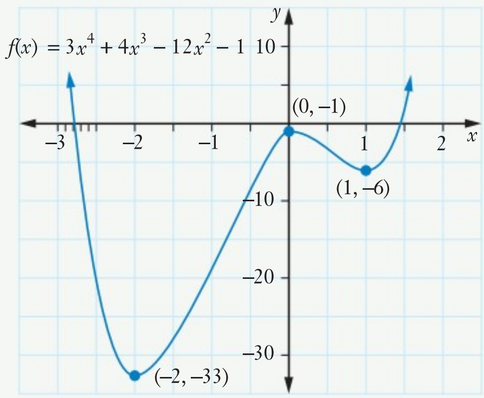
9 (0, 3) maximum, (2, –13) point of inflection, (4, –29) minimum



10 (–3, –25) minimum, (–1, –9) point of inflection, (1, 7) maximum



11 (–2, –33) minimum, (0, –1) maximum, (1, –6) minimum



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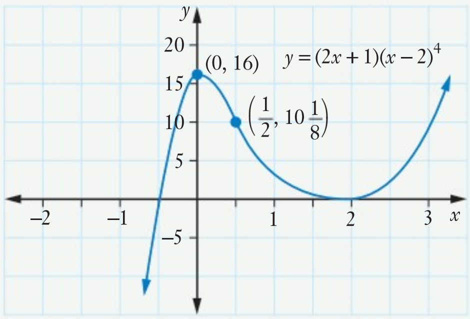
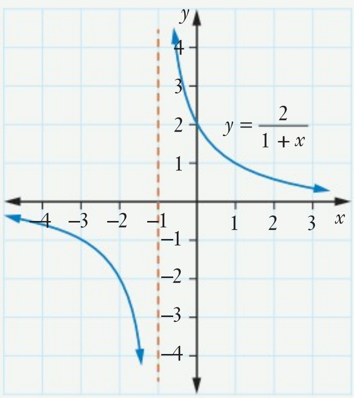
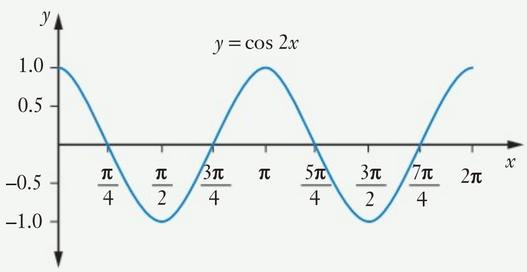
12 (0, 16) maximum, $\left(\frac{1}{2}, 10\frac{1}{8}\right)$ point of inflection, (2, 0) minimum



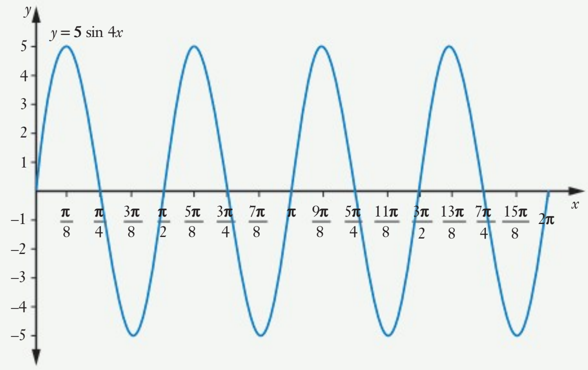
13 $\frac{dy}{dx} = -\frac{2}{(1+x)^2} \neq 0$ for any x



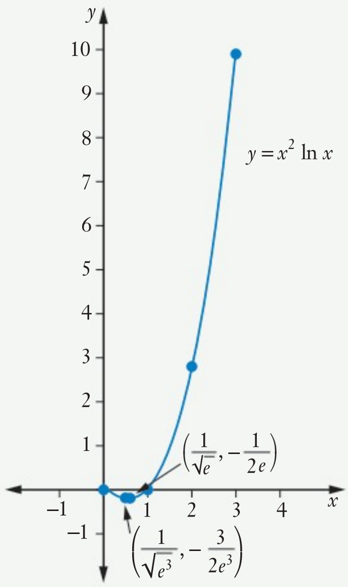
14 a



b



15 a



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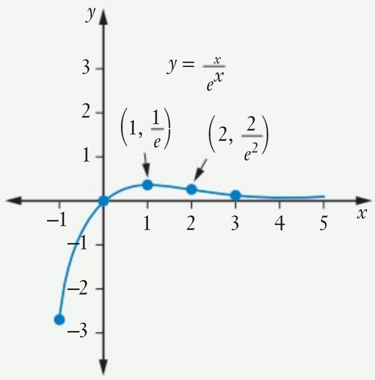

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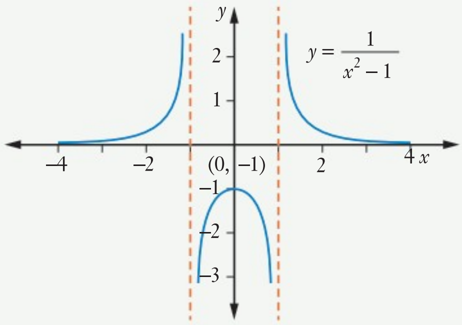
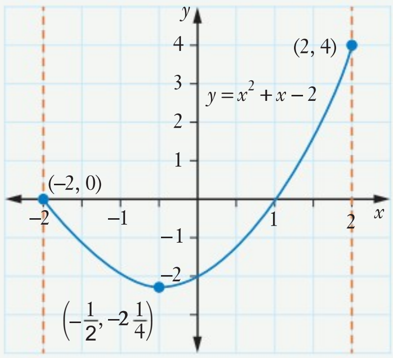
c

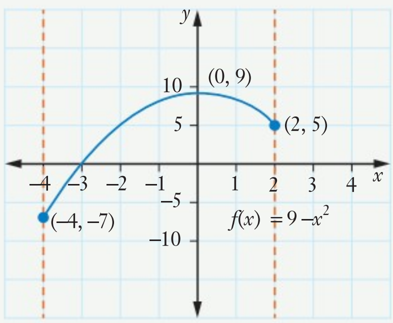


Exercise 5.07

1 Maximum value is 4.

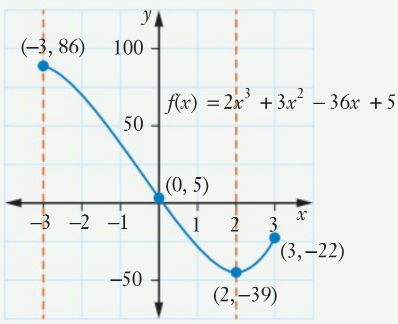


2 Maximum value is 9, minimum value is −7.



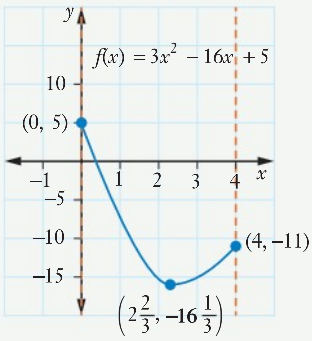
3 Maximum value is 25.

4 Maximum value is 86, minimum value is −39.



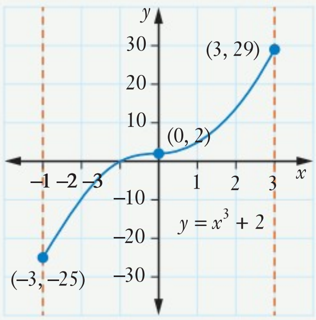
5 Maximum value is −2.

6 Maximum value is 5, minimum value is $-16\frac{1}{3}$.

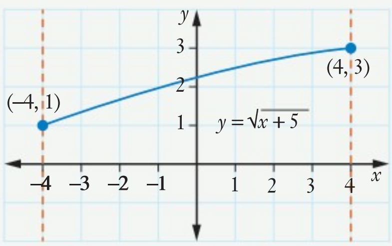


7 Global maximum 29, local maximum −3, global minimum −35, local minimum −35, −8

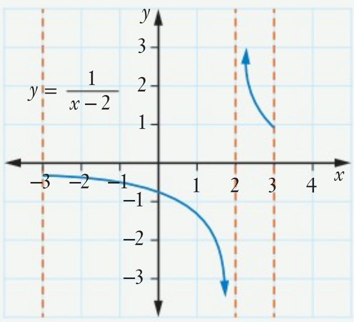
8 Minimum −25, maximum 29



9 Maximum 3, minimum 1



10 Maximum ∞, minimum −∞



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Investigation

The disc has radius $\frac{30}{7}$ cm. (This result uses Stewart's theorem – research this.)

Exercise 5.08

See worked solutions for full proofs.

1 $\frac{50}{x} = y$

$P = 2x + 2y$

2 $y = 60 - x$

$A = xy$

3 $\frac{20}{x} = y$

$S = x + y$

4 $\frac{400}{\pi r^2} = h$

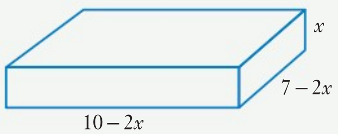
$S = 2\pi r^2 + 2\pi r h$

5 a $x + y = 30$

b $A = \left(\frac{1}{4}x\right)^2 + \left(\frac{1}{4}y\right)^2$

6 a $x^2 + y^2 = 280^2 = 78\,400$

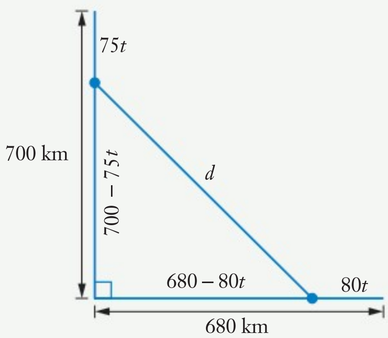
b $A = xy$

7 

$V = x(10 - 2x)(7 - 2x)$

8 Profit per person = Cost – Expenses

$= (900 - 100x) - (200 + 400x)$

9 

$d^2 = (700 - 75t)^2 + (680 - 80t)^2$

10 Distance AB: $d = \sqrt{x^2 + 0.5^2}$

$t = \frac{\sqrt{x^2 + 0.25}}{5}$

Distance BC: $d = 7 - x$

$t = \frac{7 - x}{4}$

Exercise 5.09

See worked solutions for full proofs.

1 2 s, 16 m

2 7.5 km

3 a $y = 30 - x$

b Maximum area is 225 m^2

4 a $\frac{4000}{x} = y$

$P = 2x + 2y$

b 63.2 m by 63.2 m

c \$12 322.88

5 4 m by 4 m

6 14 and 14

7 –2.5 and 2.5

8 $x = 1.25$ m, $y = 1.25$ m

9 a $V = x(30 - 2x)(80 - 2x)$

b $x = 6\frac{2}{3}$ cm

c 7407.4 cm^3

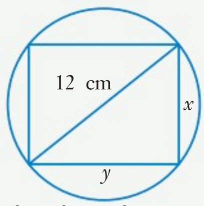
10 a $\frac{54}{r^2} = h$

$S = 2\pi r(r + h)$

b Radius is 3 m.

11 a $S = 2\pi r^2 + \frac{17200}{r}$

b 2324 m^2

12 a 

$x^2 + y^2 = 12^2$

$A = xy = x\sqrt{144 - x^2}$

b 72 cm^2

13 a $\frac{400}{x} = y$

$A = (x - 10)(y - 10)$

b 100 cm^2

14 1.12 m^2

15 a 7.5 m by 7.5 m

b 2.4 m

16 160.17 cm^2

17 1.68 m, 1.32 m

18 a $d^2 = (200 - 80t)^2 + (120 - 60t)^2$

b Minimum distance 24 km

19 a $d = (x^2 - 2x + 5) - (4x - x^2)$

$= 2x^2 - 6x + 5$

b 0.5

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20 a $s = \frac{d}{t}$

So $t = \frac{d}{s}$

$= \frac{1500}{s}$

Cost of trip taking t hours:

$C = (s^2 + 9000)t$

$= (s^2 + 9000)\frac{1500}{s}$

$= 1500\left(s + \frac{9000}{s}\right)$

b 95 km h^{-1} c \$2846

Test yourself 5

1 A 2 C 3 D 4 C

5 (–3, –11) maximum, (–1, –15) minimum

6 $x > 1\frac{1}{6}$

7 50 m

8 $x > -1$

9 $\left(\frac{1}{2}, -1\right)$

10 a $\frac{375}{\pi r^2} = h$

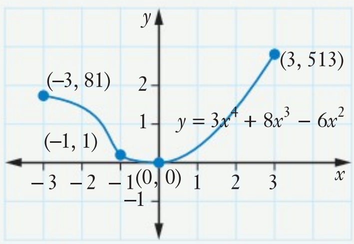
$S = 2\pi r^2 + 2\pi r h$

b 3.9 cm

11 a (0, 0) and (–1, 1)

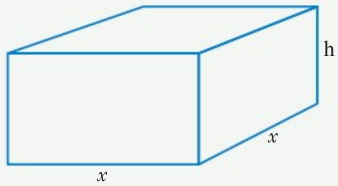
b (0, 0) minimum, (–1, 1) horizontal point of inflection

c



d Maximum value 513, minimum value 0

12 a



$\frac{125 - x^2}{2x} = h$

$V = x^2 h$

b 6.45 cm by 6.45 cm by 6.45 cm

13 150 products

14 a $y = \sqrt{25 - x^2}$

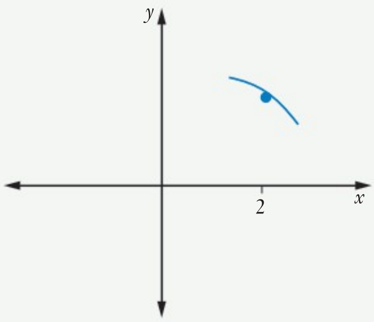
$A = \frac{1}{2}xy$

b 6.25 m^2

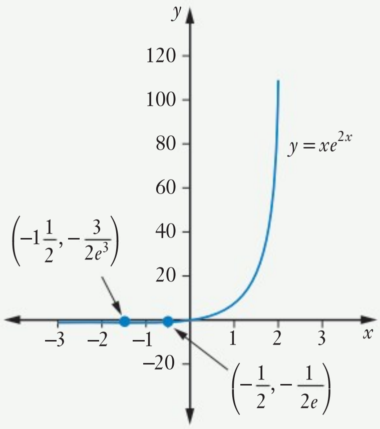
15 (0, 1) and (3, –74)

16 179

17



18



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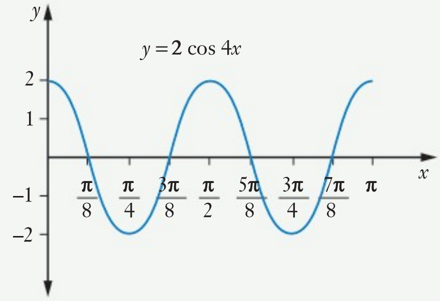
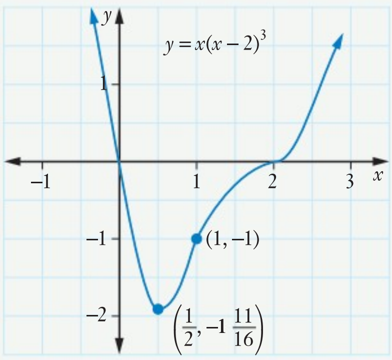


Challenge exercise 5

1



2 16 m^2

3 27; –20.25

4 $f'(0.6) = f''(0.6) = 0$ and concavity changes

5 Proof. See Worked Solutions. $r = s = 12.5$

6 $y = x^2 + 2x + 3$

7 a $y' = 0$ at (0, 0)

b $y'' > 0$ on LHS and RHS

c $y'' < 0$ on LHS, $y'' > 0$ on RHS

8 Minimum –1, maximum $-\dfrac{1}{5}$

9 87 km h^{-1}

Chapter 6

Exercise 6.01

1 a 4.125 units^2 b 6.625 units^2

2 a 1.17 units^2 b 1.67 units^2

c 1.27 units^2 d 1.52 units^2

3 a 6.5 units^2 b 5 units^2 c 132 units^2

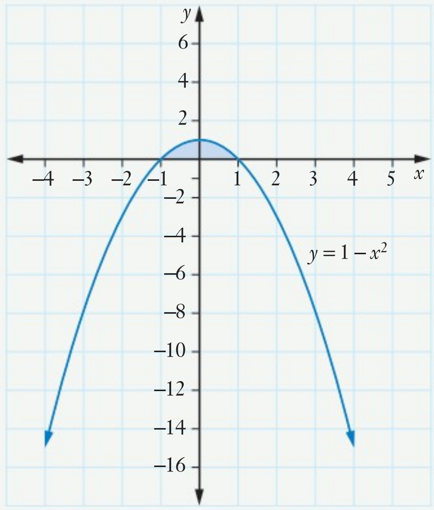
d $\dfrac{\pi\left(1+\sqrt{2}\right)}{8\sqrt{2}} = \dfrac{\pi}{16}\left(2+\sqrt{2}\right)\text{units}^2$ e 6.5 units^2

4 a 28 units^2 b 156 units^2 c 140 units^2

5 a $3\frac{3}{7}\text{ units}^2$ b 5.5 units^2 c $\dfrac{\pi}{4}\text{ units}^2$

d $\dfrac{3}{2}(e + e^4)$ e 23 units^2

6 a



b 1 unit^2

7 a 4.41 units^2 b 4.91 units^2

c 4.5 units^2 d 4.5 units^2

8 $\dfrac{25\pi}{2}\text{ units}^2$

9 a $\dfrac{9\pi}{2}\text{ units}^2$

b i 2.5 units^2 ii 8.1 units^2

10 a 22.4 units^2 b 3.3 units^2 c 1 unit^2

11 a i 6 units^2 ii 14 units^2

b i $\dfrac{\pi\sqrt{2}}{4}\text{ units}^2$ ii $\dfrac{\pi\left(\sqrt{2}+2\right)}{4}\text{units}^2$

12 a 60.625 units^2 b 73.125 units^2

Exercise 6.02

1 a 2.5 b 10

c 2.4 d 0.225

2 a 28 b 22

3 a 0.39 b 0.41

4 a 1.08 b 0.75 c 0.65

d 0.94 e 0.92

5 a 75.1 b 16.5 c 650.2

6 a 28.9 m^2 b 39.25 m^2

c 7.45 km^2 d 492.25 m^2

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Exercise 6.03

1 a 8 b 10 c 217
d −1 e 10 f 54
g $3\frac{1}{3}$ h 16 i 50

2 a $\frac{2}{3}$ b $21\frac{1}{4}$ c 0
d $4\frac{2}{3}$ e $1\frac{1}{4}$ f $4\frac{1}{3}$
g 0 h $2\frac{1}{3}$ i 0
j $6\frac{2}{9}$ k 100 l 54
m $15\frac{5}{6}$ n $22\frac{2}{3}$ o 0.0126

3 a 70 m b 38 km c 258 cm
d $72\frac{2}{3}$ m e 142 cm

4 a 750 L b 51 000 L c 810 750

Exercise 6.04

1 a $\frac{x^3}{3}+C$ b $\frac{x^6}{2}+C$ c $\frac{2x^5}{5}+C$
d $\frac{m^2}{2}+m+C$ e $\frac{t^3}{3}-7t+C$ f $\frac{h^8}{8}+5h+C$
g $\frac{y^2}{2}-3y+C$ h x^2+4x+C i $\frac{b^3}{3}+\frac{b^2}{2}+C$

2 a $\frac{x^3}{3}+x^2+5x+C$
b $x^4-x^3+4x^2-x+C$
c $x^6+\frac{x^5}{5}+\frac{x^4}{2}+C$
d $\frac{x^8}{8}-\frac{3x^7}{7}-9x+C$
e $\frac{x^4}{2}+\frac{x^3}{3}-\frac{x^2}{2}-2x+C$
f $\frac{x^6}{6}+\frac{x^4}{4}+4x+C$
g $\frac{4x^3}{3}-\frac{5x^2}{2}-8x+C$
h $\frac{3x^5}{5}-\frac{x^4}{2}+\frac{x^2}{2}+C$
i $\frac{3x^4}{2}+\frac{5x^3}{3}-4x+C$
j $-x^{-3}-\frac{x^{-2}}{2}-2x^{-1}+C$

3 a $-\frac{1}{7x^7}+C$ b $\frac{3x^{\frac{4}{3}}}{4}+C$
c $\frac{x^4}{4}-x^3+x^2+C$ d $x-2x^2+\frac{4x^3}{3}+C$
e $\frac{x^3}{3}+\frac{3x^2}{2}-10x+C$ f $-\frac{3}{x}+C$
g $-\frac{1}{2x^2}+C$
h $-\frac{4}{x}-x+\frac{3}{2x^2}-\frac{7}{4x^4}+C$
i $\frac{y^3}{3}+\frac{y^{-6}}{6}+5y+C$
j $\frac{t^4}{4}-\frac{t^3}{3}-2t^2+4t+C$
k $\frac{2\sqrt{x^3}}{3}+C$ l $-\frac{1}{2t^4}+C$ m $\frac{3\sqrt[3]{x^4}}{4}+C$
n $\frac{2\sqrt{x^5}}{5}+C$ o $\frac{2\sqrt{x^3}}{3}+x+C$

4 900° 5 $y=x^3-x^2+x+6$
6 −11 079.25 cm s^{-1} 7 21 000 L

Exercise 6.05

1 a $\frac{(3x-4)^3}{9}+C$ b $\frac{(x+1)^5}{5}+C$
c $\frac{(5x-1)^{10}}{50}+C$ d $\frac{(3y-2)^8}{24}+C$
e $\frac{(4+3x)^5}{15}+C$ f $\frac{(7x+8)^{13}}{91}+C$
g $-\frac{(1-x)^7}{7}+C$ h $\frac{\sqrt{(2x-5)^3}}{3}+C$
i $-\frac{2(3x+1)^{-3}}{9}+C$ j $-3(x+7)^{-1}+C$
k $-\frac{1}{16(4x-5)^2}+C$ l $\frac{3\sqrt[3]{(4x+3)^4}}{16}+C$

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m $-2(2-x)^{\frac{1}{2}}+C$ n $\dfrac{2\sqrt{(t+3)^5}}{5}+C$

o $\dfrac{2\sqrt{(5x+2)^7}}{35}+C$

2 a 288.2 b $-1\frac{1}{4}$ c $-\frac{1}{8}$

d $60\frac{2}{3}$ e $\frac{1}{6}$ f $\frac{1}{7}$

g $4\frac{2}{3}$ h $1\frac{1}{5}$ i $\frac{3}{5}$

3 a $\frac{1}{3}(x^4+5)^3+C$ b $\frac{1}{6}(x^2-3)^6+C$

c $\frac{1}{4}(x^3+1)^4+C$ d $\frac{1}{5}(x^2+3x-2)^5+C$

e $\frac{1}{42}(3x^2-7)^7+C$ f $-\frac{1}{45}(4-5x^3)^3+C$

g $\frac{1}{15}(2x^6-3)^5+C$ h $\frac{3}{80}(5x^2+3)^8+C$

i $\frac{1}{12}(x^2+4x)^6+C$ **j** $\frac{1}{12}(3x^3-6x-2)^4+C$

4 a $108\frac{2}{3}$ b $-\frac{1}{18}$

c 66 812.75 d $10\ 159\frac{1}{32}$

e 564 537.6 f $-1236\frac{2}{3}$

g $-19\ 839\frac{3}{14}$ h $96\frac{4}{9}$

i −474 618 565.3

5 $y=\frac{1}{15}((x^3-2)^5+61)$

6 a $x=\frac{1}{10}[(x^2-3)^5-1]$ b 777.5 m

Exercise 6.06

1 a $\frac{1}{4}e^{4x}+C$ b $-e^{-x}+C$ c $\frac{1}{5}e^{5x}+C$

d $-\frac{1}{2}e^{-2x}+C$ e $\frac{1}{4}e^{4x+1}+C$ f $-\frac{3}{5}e^{5x}+C$

g $\frac{1}{2}e^{2t}+C$ h $\frac{1}{7}e^{7x}-2x+C$

i $e^{x-3}+\frac{x^2}{2}+C$

2 a $\frac{1}{5}(e^5-1)$ b $e^{-2}-1=\frac{1}{e^2}-1$

c $\frac{2e^7}{3}(e^9-1)$ d $19-\frac{1}{2}e^4(e^2-1)$

e $\frac{1}{2}e^4+1\frac{1}{2}$ f $e^2-e-1\frac{1}{2}$

g $\frac{1}{2}e^6+e^{-3}-1\frac{1}{2}$

3 a 0.32 b 268.29

c 37 855.68 d 346.85

e 755.19

4 a $\frac{1}{\ln 5}5^x+C$ b $\frac{1}{3\ln 7}7^{3x}+C$

c $\frac{1}{2\ln 3}3^{2x-1}+C$

5 a $x(2+x)e^x$ b x^2e^x+C

6 $f(x)=\frac{1}{6}(e^{2x^3}-1)$ 7 $2e^3+5$ m

Exercise 6.07

1 a $\ln|2x+5|+C$ b $\ln|2x^2+1|+C$

c $\ln|x^5-2|+C$

d $\frac{1}{2}\ln|x|+C$ or $\frac{1}{2}\ln|2x|+C$

e $2\ln|x|+C$ f $\frac{5}{3}\ln|x|+C$

g $\ln|x^2-3x|+C$ h $\frac{1}{2}\ln|x^2+2|+C$

i $\frac{3}{2}\ln|x^2+7|+C$ **j** $\frac{1}{2}\ln|x^2+2x-5|+C$

2 a $\ln|4x-1|+C$ b $\ln|x+3|+C$

c $\frac{1}{6}\ln|2x^3-7|+C$ d $\frac{1}{12}\ln|2x^6+5|+C$

e $\frac{1}{2}\ln|x^2+6x+2|+C$

3 a 0.5 b 0.7 c 1.6

d 3.1 e 0.5

4 a RHS = LHS

b $\ln|x+3|+2\ln|x-3|+C$

5 a RHS = LHS

b $x-5\ln|x-1|+C$

6 $f(x)=\frac{1}{9}\ln\left|\frac{3x^3-1}{2}\right|$ 7 8.95 m

8 4

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Exercise 6.08

1 a $\sin x + C$ b $-\cos x + C$
c $\tan x + C$ d $-\frac{45}{\pi}\cos x° + C$
e $-\frac{1}{3}\cos 3x + C$ f $\frac{1}{7}\cos 7x + C$
g $\frac{1}{5}\tan 5x + C$ h $\sin(x+1) + C$
i $-\frac{1}{2}\cos(2x-3) + C$ **j** $\frac{1}{2}\sin(2x-1) + C$
k $\cos(\pi - x) + C = -\cos x + C$
l $\sin(x + \pi) + C = -\sin x + C$
m $\frac{2}{7}\tan 7x + C$ n $-8\cos\frac{x}{2} + C$
o $9\tan\frac{x}{3} + C$

2 a 1 b $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$
c $\frac{2}{\sqrt{2}} = \sqrt{2}$ d $-\frac{1}{3}$
e $\frac{1}{\pi}$ f $\frac{1}{2}$
g $\frac{3}{4}$ h $-\frac{1}{5}$

3 a $\sin\left(x + \frac{\pi}{3}\right) + C$ or $\frac{\sin x + \sqrt{3}\cos x}{2} + C$
b $\cos(\pi - x) + C = -\cos x + C$

4 $y = \frac{1}{4}\sin 4x + \frac{\pi}{4}$

5 a $x = 18\sin\frac{2\pi t}{3} + 2$ cm
b i $9\sqrt{3} + 2$ cm ii $-9\sqrt{3} + 2$ cm

6 a $d = -24\cos\frac{\pi t}{6} + 26$ b 14 m
c 50 m; 2 m; 26 m d 12 h

Exercise 6.09

1 $1\frac{1}{3}$ units^2 2 36 units^2 3 4.5 units^2
4 $10\frac{2}{3}$ units^2 5 $\frac{1}{6}$ units^2 6 14.3 units^2
7 4units^2 8 0.4 units^2 9 8 units^2
10 24.25 units^2 11 $e^2(e^2 - 1)$ units^2
12 $\frac{1}{4}(e - e^{-3})$ units^2 13 2.86 units^2
14 29.5 units^2 15 4 units^2
16 $\frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$ units^2 17 0.86 units^2
18 2 units^2 19 $9\frac{1}{3}$ units^2
20 $11\frac{2}{3}$ units^2 21 $\frac{1}{6}$ units^2
22 $\frac{2}{3}$ units^2 23 $\frac{1}{3}$ units^2
24 $\ln 3 - \ln 2 = \ln 1.5$ units^2
25 $\ln 2$ units^2
26 0.61 units^2 27 $5\frac{1}{3}$ units^2
28 18 units^2 29 3.98 units^2
30 $\frac{a^4}{2}$ units^2

Exercise 6.10

1 $21\frac{1}{3}$ units^2 2 20 units^2
3 $4\frac{2}{3}$ units^2 4 1.5 units^2
5 $1\frac{1}{4}$ units^2 6 $2\frac{1}{3}$ units^2
7 $10\frac{2}{3}$ units^2 8 $\frac{1}{6}$ units^2
9 $3\frac{7}{9}$ units^2 10 2 units^2
11 $11\frac{1}{4}$ units^2 12 60 units^2
13 4.5 units^2 14 $1\frac{1}{3}$ units^2
15 1.9 units^2 16 47.2 units^2

Exercise 6.11

1 $1\frac{1}{3}$ units^2 2 $1\frac{1}{3}$ units^2
3 $\frac{1}{6}$ units^2 4 $10\frac{2}{3}$ units^2
5 $20\frac{5}{6}$ units^2 6 8units^2
7 $\frac{2}{3}$ units^2 8 $166\frac{2}{3}$ units^2
9 $\frac{5}{12}$ units^2 10 $\frac{2}{3}$ units^2
11 $\frac{1}{12}$ units^2 12 $\frac{1}{3}$ units^2
13 36 units^2 14 $2\frac{2}{3}$ units^2

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15 $(\pi - 2)$ units^2

16 $\frac{1}{2} + \ln 2$ units^2

17 $2\sqrt{2}$ units^2

18 $\frac{1}{2}(e^4 - 5)$ units^2

19 $\sqrt{3} - \frac{\pi}{3} = \frac{3\sqrt{3} - \pi}{3}$ units^2

24 $85\frac{1}{3}$ units^2

25 a $T = 40e^{-0.4t} + 175$

b i 180° ii 175°

26 a $\frac{1}{\sqrt{2}}$ b $\sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Test yourself 6

1 D 2 B 3 A 4 D

5 a 0.535 b 0.5

6 a $\frac{3x^2}{2} + x + C$ b $\frac{5x^2}{2} - x + C$

c $\frac{2\sqrt{x^3}}{3} + C$ d $\frac{(2x+5)^8}{16} + C$

e $\frac{(3x^4 - 2)^5}{60} + C$

7 $\frac{3^x}{\ln 3} + C$

8 a 14 units^2 b 27 units^2 c 28 units^2

9 a 2 b 0 c $2\frac{1}{5}$

d $6\frac{7}{9}$ e $228\frac{3}{8}$

10 $e(e^2 - 1)$ units^2

11 3 units^2

12 $-\frac{180}{\pi}\cos x° + C$

13 $2\frac{2}{3}$ units^2

14 a $\frac{1}{4}e^{4x} + C$ b $\frac{1}{2}\ln|x^2 - 9| + C$

c $-e^{-x} + C$ d $\ln|x + 4| + C$

e $\frac{(x^2 - 6x + 1)^9}{18} + C$

15 $4\frac{1}{2}$

16 $\frac{9\pi}{4}$ units^2

17 $\frac{3}{4}$ units^2

18 $\frac{(7x+3)^{12}}{84} + C$

19 3 units^2

20 $\frac{1}{2}e^4(e^6 - 1)$ units^2

21 5.36 units^2

22 36

23 a $-\frac{1}{2}\cos 2x + C$ b $3\sin x + C$

c $\frac{1}{5}\tan 5x + C$ d $x - \cos x + C$

27 a $\frac{(2x-1)^5}{2} + C$ b $\frac{x^6}{8} + C$

28 $\frac{1}{\sqrt{2}}$ units^2

29 a $\frac{1}{18}(x^3 - 2)^6 + C$ b $\frac{1}{50}(5x^2 + 2)^5 + C$

c $\frac{5}{24}(2x^4 - 1)^3 + C$ d $\frac{1}{8}(x^2 + 4x - 3)^4 + C$

30 2 units^2

31 a $\sqrt{2}$ units^2

b 1 unit^2

c $1 + \sqrt{2}$ units^2

32 a $f(x) = \frac{1}{20}[3(2x^2 - 1)^5 + 57]$

b $f(x) = \frac{1}{2}\tan 2x$

c $f(x) = \frac{1}{5}e^{5x}$

d $f(x) = \frac{1}{16}[(x^4 - 15)^4 - 1]$

e $f(x) = \frac{3}{4}\ln(x^4 + 1) + 2$

33 a $x = \frac{2\sqrt{t^3 + 9}}{3} - 4$

b 3.7 m

c 7.6 s

Challenge exercise 6

1 a Show $f(-x) = -f(x)$

b 0

c 12 units^2

2 a RHS = LHS b $\ln\sqrt{3} = \frac{\ln 3}{2}$

3 9 units^2

4 $\frac{3 - \sqrt{3}}{6}$

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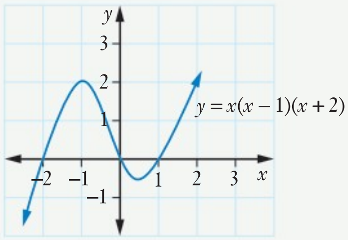
5 $\frac{1}{8}$

6 7.3 units^2

7 a



b $3\frac{1}{12}$ units^2

8 11.68 units^2

9 a $\frac{3(x+2)}{2\sqrt{x+3}}$

b $\frac{2x\sqrt{x+3}}{3} + C$

10 a $x(1 + 2 \ln x)$

b $18 \ln 3$

11 $\frac{5}{12}$ units^2

12 a 6879.7

b 1375.93

c 1527.62

13 a 6879.7

b 1375.9 units^2

c 1527.6 units^2

Practice set 2

1 C	2 D	3 B
4 B	5 A	6 C

7 $x < \frac{1}{2}$

8 $x^3 - x^2 + x + C$

9 24

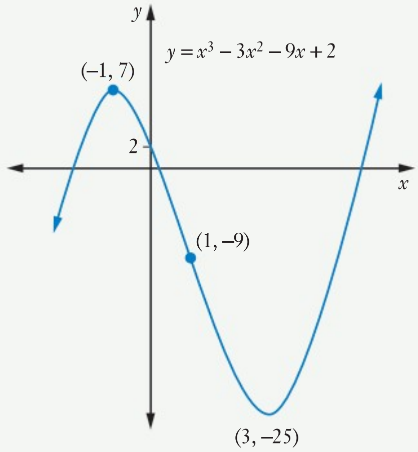
10 a 8 b $\frac{2\pi}{3}$ c 5

11 8 m

12 $\frac{d^2y}{dx^2} = 7(-7 \sin 7x)$

13 $\frac{x^9}{3} + 2x^2 + C$

14



15 $1\frac{1}{3}$ units^2

16 $\frac{3}{4} \ln |2x^2 - 5| + C$

17 $f'(3) = 20$; $f''(-2) = -16$

18 68

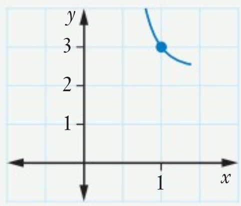
19 $-\frac{7}{9}, \infty$

20 3

21 $\frac{1}{4}(\tan x + 1)^4 + C$

22 a $f(1) = 3, f'(1) = -2, f''(1) = 18$

b Curve is decreasing and concave upwards at (1, 3).



23 a $P = 8x + 4y = 4$

$4y = 4 - 8x$

$y = 1 - 2x$

$A = 3x^2 + y^2$

b Rectangle $\frac{2}{7}$ m $\times$ $\frac{6}{7}$ m, square with sides $\frac{3}{7}$ m

24 $y = 3\left[\frac{1}{2}(x + 4)\right]^2 + 5$

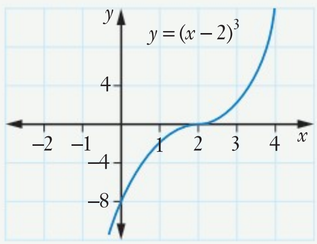
25 $f(-1) = 0$

26 12

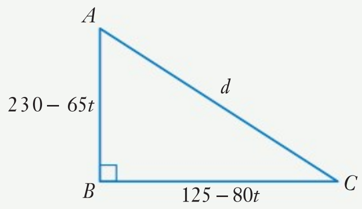
27 0.944

28 a (2, 0); point of inflection

b



29 a



$d^2 = (230 - 65t)^2 + (125 - 80t)^2$

b 2.3 h c 99.8 km

30 a $-\frac{25}{128}$ b $66\frac{77}{128}$ c $66\frac{2}{3}$

31 450 cm^2

32 $\frac{(3x+5)^8}{24} + C$

33 1.099

34 (0, 3) maximum, (1, 2) minimum, (−1, 2) minimum

35 $2\frac{8}{15}$

36 $10\frac{2}{3}$ units^2

37 $f(2) = -16$

38 9 units^2

39 $3x^2 + 2e^{2x}$

40 a 7900 L b 400 min (6 h 40 min)

41 $\frac{3e}{2}(e^9 - 1)$

42 953 m

43 $f(x) = x^3 - 4x^2 - 3x + 20$

44 $\frac{12}{4x+3}$

45 $\frac{1}{3}\ln(3x^2 + 3x - 2) + C$

46 a $\frac{1-2x}{e^{2x}}$ b $\frac{1}{x \ln 3}$

47 $x - y + 2 = 0$

48 $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$; minimum

49 $y = 2\cos 3x + 3$

50 1

51 a $\cot x$ b $5e^{5x}\sec^2(e^5 + 1)$

52 0.393

53 a 6 units^2 b 14 units^2

54 a $e^x(\sin x + \cos x)$ b $3\tan^2 x \sec^2 x$

c $-6\sin\left(3x - \frac{\pi}{2}\right)$

55 $12x - 2y - 2 - 3\pi = 0$

56 a $3e^x \sin^2(e^x)\cos(e^x)$ b $\frac{\sec^2(\ln x + 1)}{x}$

57 $(5 - e)$ units^2

58 a $\frac{1}{3}e^{3x} + C$ b $\frac{1}{\pi}\tan \pi x + C$

c $\frac{1}{2}\ln x + C$ d $5\sin\left(\frac{x}{5}\right) + C$

e $-\frac{1}{8}\cos 8x + C$

59 $3x - 2\ln x - \frac{5}{x} + C$

60 $\frac{1}{5}e^{5x} + \frac{1}{\pi}\cos \pi x + C$

61 $(e^2 - 1)$ units^2

62 −1

Chapter 7

Note: Answers obtained from reading graphs are approximate.

Exercise 7.01

1 a N b N c N
d C e N f N
g N h C i N
j C k N l C
m N n N o N
p N q N r N
s C t N

2 a O b C c N
d N e C f D
g D h N i C
j N k D l N
m N

3 a e.g. types of sport, hair colour
b e.g. heights, test scores, prices
c e.g. rankings, test scores, clothing sizes
d e.g. karate belt colour, size of take-away coffee cups, Olympic medals
e e.g. race times, length, rainfall
f e.g. types of trees, housing, cars

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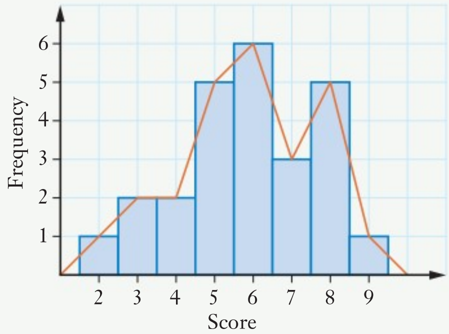
Exercise 7.02

1 a i

Score	Frequency
2	1
3	2
4	2
5	5
6	6
7	3
8	5
9	1

ii



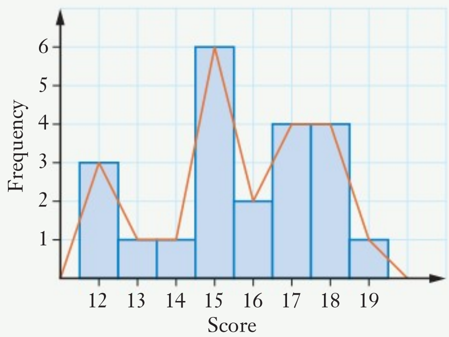
iii Highest 9, lowest 2

iv 6

b i

Pizzas	Frequency
12	3
13	1
14	1
15	6
16	2
17	4
18	4
19	1

ii

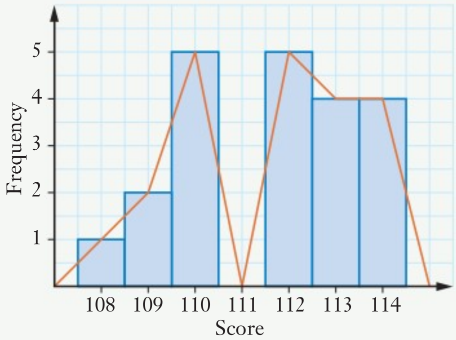


iii Highest 19, lowest 12 iv 15

c i

Gym attendance	Frequency
108	1
109	2
110	5
111	0
112	5
113	4
114	4

ii



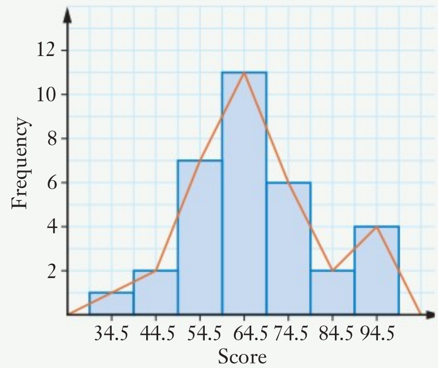
iii Highest 114, lowest 108

iv 110 and 112

d i

Results	Class centre	Frequency
30–39	34.5	1
40–49	44.5	2
50–59	54.5	7
60–69	64.5	11
70–79	74.5	6
80–89	84.5	2
90–99	94.5	4

ii



iii Highest 94.5, lowest 34.5 iv 64.5

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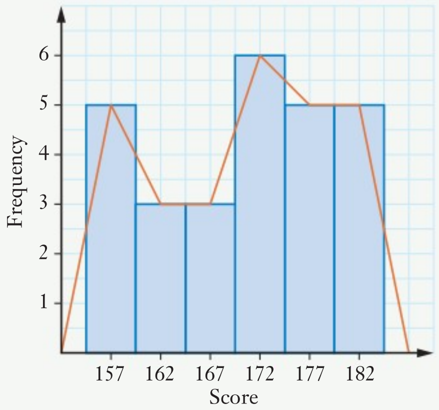
e i

Height (cm)	Class centre	Frequency
155–159	157	5
160–164	162	3
165–169	167	3
170–174	172	6
175–179	177	5
180–184	182	5

ii



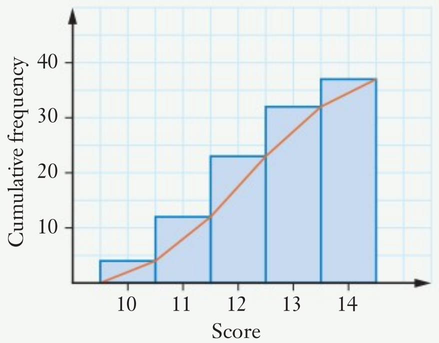
iii Highest 182, lowest 157

iv 172

2 a i

Number of cars	Frequency	Cumulative frequency
10	4	4
11	8	12
12	11	23
13	9	32
14	5	37

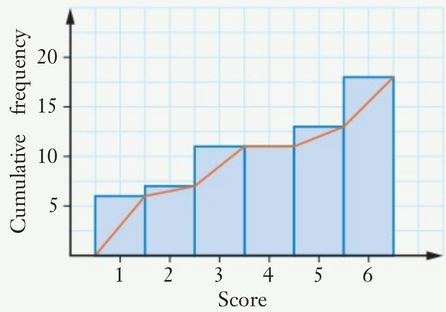
ii



b i

Score	Frequency	Cumulative frequency
1	7	7
2	1	8
3	3	11
4	0	11
5	2	13
6	5	18

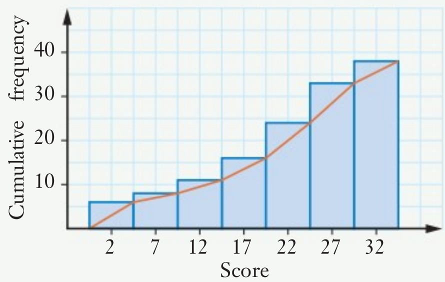
ii



c i

Sales	Class centre	Frequency	Cumulative frequency
0–4	2	6	6
5–9	7	2	8
10–14	12	3	11
15–19	17	5	16
20–24	22	8	24
25–29	27	9	33
30–34	32	5	38

ii



d i

Scores	Class centre	Frequency	Cumulative frequency
0–19	9.5	3	3
20–39	29.5	2	5
40–59	49.5	7	12
60–79	69.5	6	18
80–99	89.5	1	19

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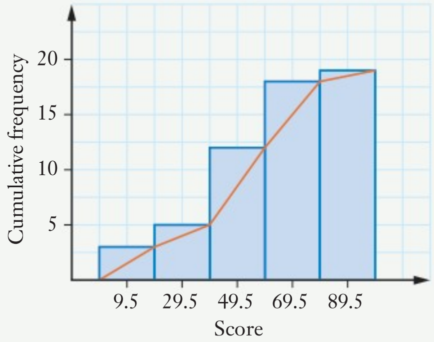
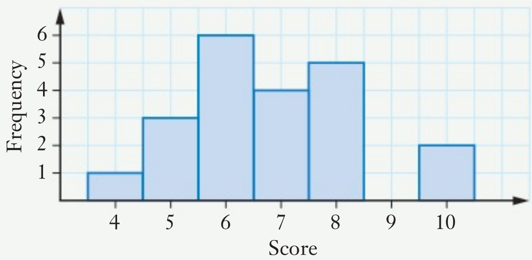
ii



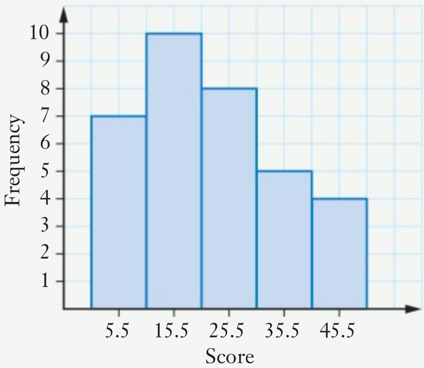
3 a



b 11 times

c 6 rescues

4 a



b 38.2%

5 a

	Play soccer	Do not play soccer
Play tennis	12	35
Do not play tennis	27	28

b i 11.8% ii 27.5%

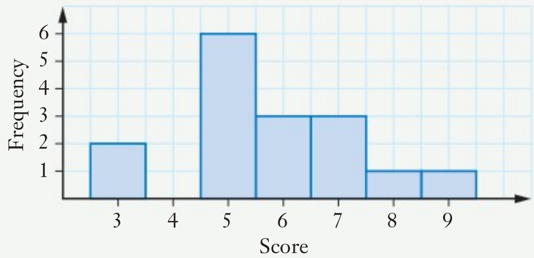
c 36.5%

d $\frac{4}{13}$

e 74.5%

6 a 16 b 5

c

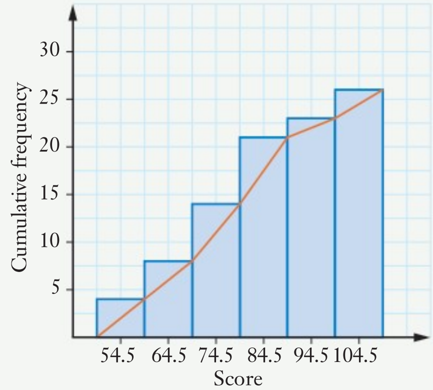


d 87.5% e $\frac{1}{8}$

7 a

Weight (kg)	Class centre	Frequency	Cumulative frequency
50–59	54.5	4	4
60–69	64.5	4	8
70–79	74.5	6	14
80–89	84.5	7	21
90–99	94.5	2	23
100–109	104.5	3	26

b



c i 12 ii 4

d 50% e $\frac{7}{13}$

8 a 25%

b

Sport	Frequency
Tennis	90
Soccer	120
Athletics	60
Cricket	180
Basketball	150
Volleyball	120

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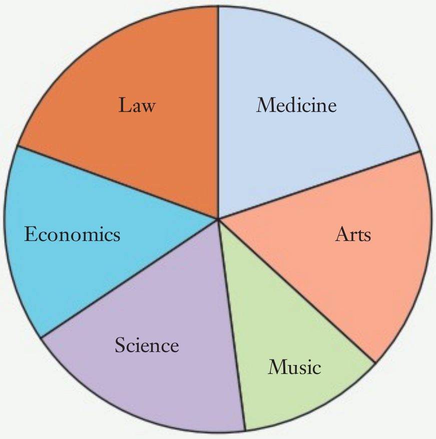
9 a 19.3% b 31%

c



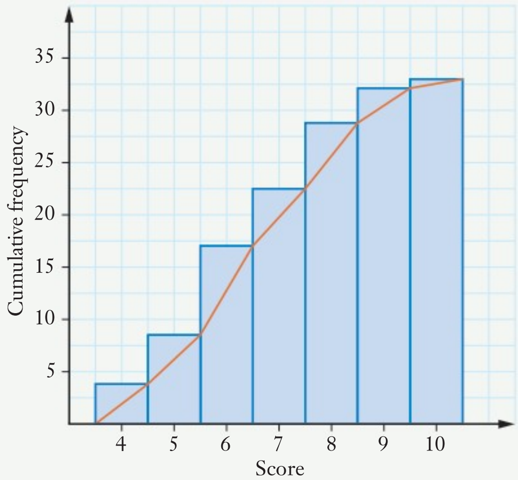
10 a 87 b 55.9%

c 6.5% d 76

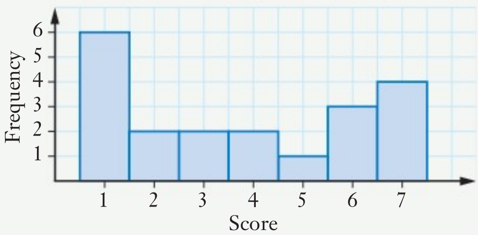
11 a 430 b 48.6% c 50.2%

d 221 e $\frac{112}{221}$

12



13 a

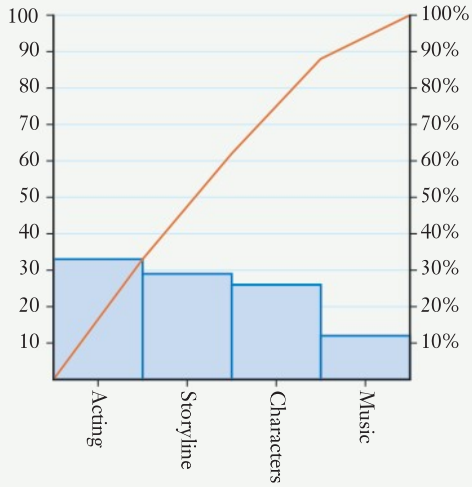


b

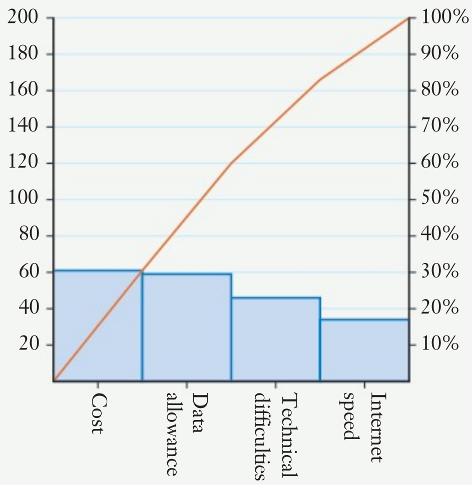
Junk mail items	Frequency
1	6
2	2
3	2
4	2
5	1
6	3
7	4

14 Stem-and-leaf plots list individual scores so retain all details. Not easy to draw, and can be long. A grouped frequency distribution table groups scores so individual data is lost. Easy to draw and compact.

15 a



b



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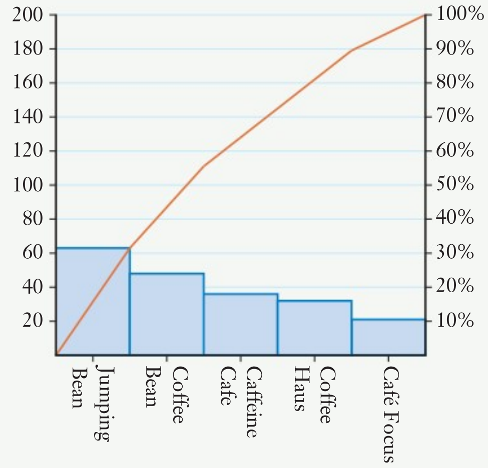
c



Exercise 7.03

1 a i 5 ii 5 iii 5

b i 5 ii 4 iii 5

c i 16 ii 18 iii 17

d i 5.7 ii 4 iii 5.5

e i 1.51 ii 1.49 iii 1.49

2 a Brown b Tabby

3 a i 6 ii 6 iii 6

b i 52.4 ii 52 iii 51

c i 17.56 ii 18 iii 20

d i 103.3 ii 104 iii 104

4 a 3 b 1.5

c 3.5 d 3

5 a i 9.24 ii 8–10

b i 17 ii 20–24

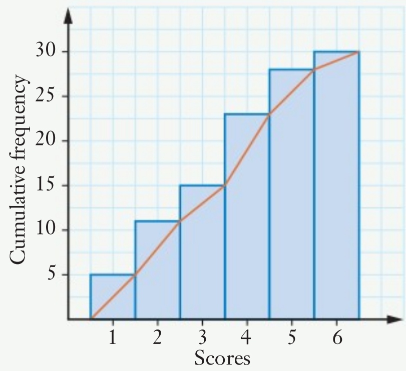
c i 68.7 ii 70–84

d i 39 ii 20–24, 50–54 (bimodal)

6 a i

Athletes	Frequency	Cumulative frequency
1	5	5
2	6	11
3	4	15
4	8	23
5	5	28
6	2	30

ii

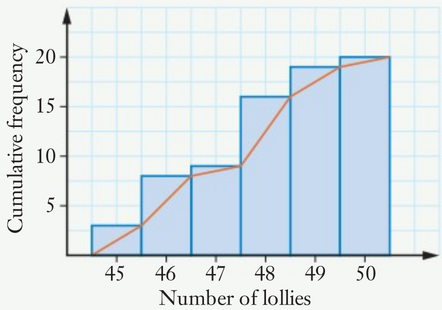


iii 3.5

b i

Lollies	Frequency	Cumulative frequency
45	3	3
46	5	8
47	1	9
48	7	16
49	3	19
50	1	20

ii



iii 48

c i

Time (h)	Class centre	Frequency	Cumulative frequency
1–5	3	7	7
6–10	8	5	12
11–15	13	3	15
16–20	18	6	21
21–25	23	7	28
26–30	28	2	30

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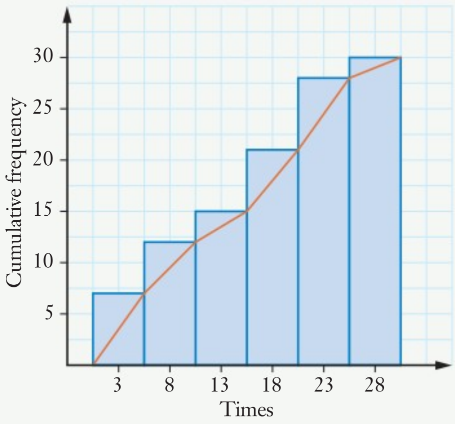
ii

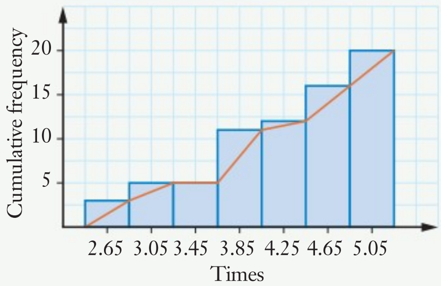


iii 15.5

d i

Time (min)	Class centre	Frequency	Cumulative frequency
2.5–2.8	2.65	3	3
2.9–3.2	3.05	2	5
3.3–3.6	3.45	0	5
3.7–4.0	3.85	6	11
4.1–4.4	4.25	1	12
4.5–4.8	4.65	4	16
4.9–5.2	5.05	4	20

ii



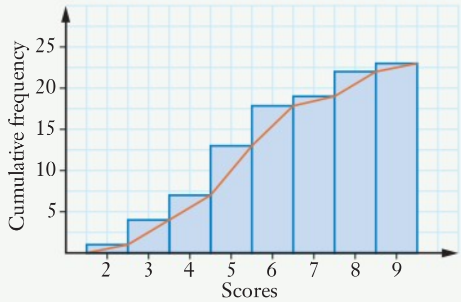
iii 4.0

7 a i

Score	Frequency	Cumulative frequency
2	1	1
3	3	4
4	3	7
5	6	13
6	5	18
7	1	19
8	3	22
9	1	23

ii 5.35 iii 5

iv



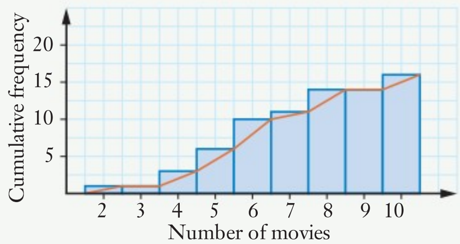
v 5.5

b i

Number of movies	Frequency	Cumulative frequency
2	1	1
3	0	1
4	2	3
5	3	6
6	4	10
7	1	11
8	3	14
9	0	14
10	2	16

ii 6.25 iii 6

iv



v 6.5

c i

Ages	Class interval	Frequency	Cumulative frequency
20–29	24.5	6	6
30–39	34.5	10	16
40–49	44.5	5	21
50–59	54.5	11	32
60–69	64.5	7	39
70–79	74.5	3	42
80–89	84.5	6	48

ii 54.5 iii 50–59

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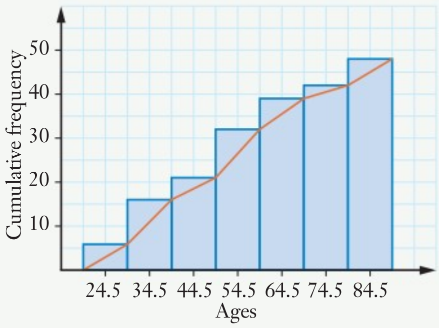
iv



v 54.5

8 a i 32

ii Mean = 62.4, no mode, median = 64

iii Mean = 66.25, no mode, median = 66

b i 1

ii Mean = 6.6, mode = 6, median = 7

iii Mean = 6.8, mode = 6, median = 7

c i 97

ii Mean = 47.1, mode = 50, median = 46.5

iii Mean = 43.2, mode = 50, median = 43

d i 3

ii Mean = 6.1, mode = 5, median = 6

iii Mean = 6.3, mode = 5, median = 6

9 Outlier 1. It changes the mean.

10 a 17. One student had a very low score on the test.

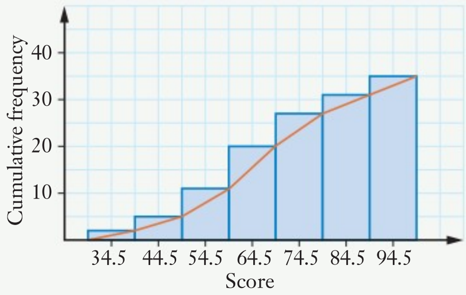
b

Class	Class centre	Frequency	Cumulative frequency
10–19	14.5	1	1
20–29	24.5	0	1
30–39	34.5	2	3
40–49	44.5	3	6
50–59	54.5	6	12
60–69	64.5	9	21
70–79	74.5	7	28
80–89	84.5	4	32
90–99	94.5	4	36

c i Mean = 65.6, modal class = 60–69

ii Mean = 67.1, modal class = 60–69

d



Median = 66

Exercise 7.04

1 a i 2 ii 3 iii 4.5

b i 11 ii 12 iii 13

c i 7 ii 8 iii 9

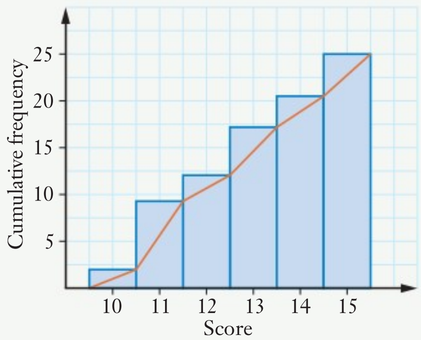
d i 2 ii 3 iii 5

2 $Q_1 = 2$, $Q_3 = 4.5$

3 a 23rd percentile = 24, 55th percentile = 25.5, 91st percentile = 28

b 2nd decile = 24, 8th decile = 27

4 a



b i 11 ii 14 iii 12

iv 13.5 v 10.5

5 a 42 b 37 c 38

d 43 e 37–47

6 a 100 b 20% c 12

d 14 e 69th

7 a 12.5%

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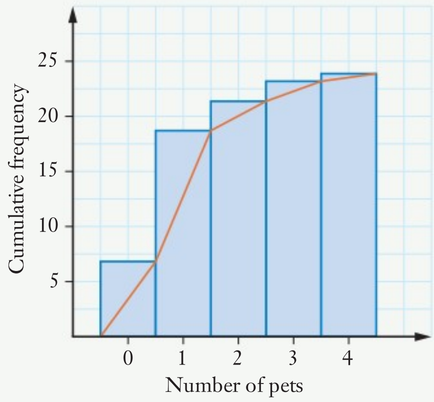
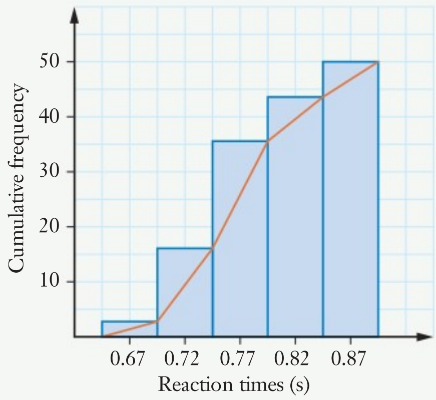
b



c i 1 ii 0 iii 1.5

8 a 0.774 b 38%

c



d i 0.74 ii 0.77 iii 0.72–0.82

9 a i 51 ii 38 iii 79

b i 11 ii 6 iii 14.5

c i 132.9 ii 115.3 iii 154.6

d i 15.5 ii 14.5 iii 17.5

Exercise 7.05

1 a 16 b 71

c 39 d 5

2 a i 6.5 ii 5 iii 4

b i 2 ii 5 iii 1

c i 40 ii 100 iii 55

d i 2 ii 6 iii 1

3 a i 7 ii 5 iii 2

b i 5 ii 5 iii 2

c i 17 ii 9 iii 4

d i 14 ii 19 iii 5

e i 6 ii 7 iii 2

4 a 6.76 b 7 c 7

d 4 e 1.5

5 a No outlier b 19 c 1

Exercise 7.06

1 a i 5.4 ii 2.1

b i 52.5 ii 14.6

c i 123.3 ii 16.1

d i 6.3 ii 1.8

e i 17.6 ii 1.96

2 a i 7.2 ii 52.1

b i 1.96 ii 3.8

c i 13.5 ii 183.2

d i 14 ii 196.6

e i 2.3 ii 5.2

3 a i 8.4 ii 2.4 iii 5.8

b i 4.4 ii 1.7 iii 2.9

c i 34.1 ii 1.5 iii 2.39

d i 51.2 ii 14.9 iii 222

4 a 4 b 1.5 c 1.14

5 a i 73.4 ii 16.3

b Yes, 20–29 i 74.8 ii 14.4

6 a Yes, 53.

$Q_3 + 1.5 \times IQR = 30.5 + 15.75 = 46.25$

$Q_1 - 1.5\ IQR = 20 - 15.75 = 4.25$

So 53 is outside 4.25–46.25.

b i 8.4 ii 5.3

7 a 9.2 b 3.5 c 12.6

Exercise 7.07

1 a Positively skewed, unimodal

b Negatively skewed, unimodal

c Positively skewed, multimodal

d Symmetrical, unimodal

e Positively skewed, unimodal

f Bimodal

g Negatively skewed, unimodal

h Bimodal

i Positively skewed, unimodal

j Multimodal

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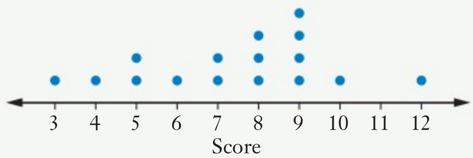
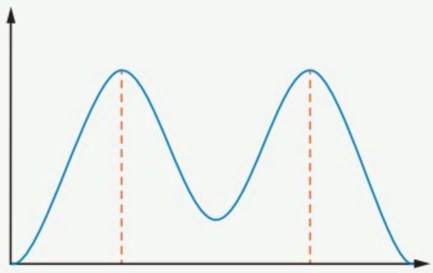
2 a



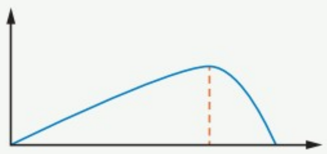
b Positively skewed

3 a Positively skewed, unimodal
b Symmetrical, unimodal
c Bimodal
d Negatively skewed, unimodal
e Symmetrical, unimodal

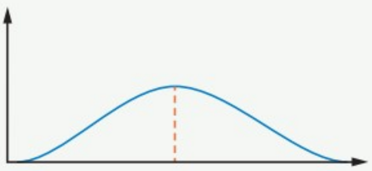
4 a



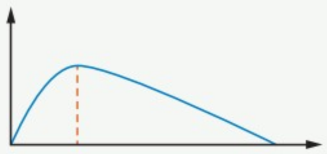
b



c



d



5 Set 1: bimodal, Set 2: positively skewed, unimodal

6 a Plot 1: positively skewed,
Plot 2: negatively skewed
b 5 c 1

7 a

Score	Class centre	Frequency
145–149	147	3
150–154	152	3
155–159	157	3
160–164	162	6
165–169	167	3
170–174	172	1
175–179	177	6
180–184	182	4
185–189	187	1

b Bimodal

8

9 a i 7 ii 7 iii 7
b Symmetrical

10 Class discussion

Exercise 7.08

1 a i 7 ii 4
b i 2 ii 2
c i 3 ii 2
d Class discussion

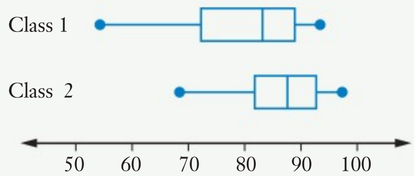
2 a

Sample 1		Sample 2
9 8 7	14	
9 8 7 5 3 0	15	1 7 7 9
9 7 6 4 2 0	16	2 4 5 6 8
7 6 2 1 0	17	2 3 3 6 8 9
	18	0 1 1 1 2

b Sample 1: 161, Sample 2: 172.5
c Sample 1: 30, Sample 2: 31
d Class discussion

3 a i Bank 1: 5.08, Bank 2: 2.77, Bank 3: 11.95
ii Bank 1: 4, Bank 2: 1, Bank 3: 13
b Bank 1: 4.1, Bank 2: 2.4, Bank 3: 4.4
c Class discussion

4 a



b i Class 1: 82.5, Class 2: 88

ii Class 1: 17, Class 2: 10.5

iii Class 1: 79.4, Class 2: 85.8

iv Class 1: 38, Class 2: 28

5 a

Camera 1		Camera 2
9 9 9 6 5	6	
7 6 5 4 4 3 3 2	7	
9 5 4 3 0	8	
0	9	
3	10	
	11	3 6 8 9 9
	12	0 0 2 2 3 4 5
	13	0 0 1 5 5 8
	14	0 2

b Camera 1: 77.3kmh^{-1}; Camera 2: 126.1kmh^{-1}

c Camera 1: 60 kmh^{-1}; Camera 2: 110 kmh^{-1}

6 a 17.5 b 17 c 10

d 2.67

7 a i 5.57 ii 5.86

b i 3.16 ii 1.25

c The average test score is similar in both tests, but test 1 has a wider spread of scores.

8 a Parents: 2, Children: 4

b i Parents: 4, Children: 3

ii Parents: 5, Children: 6 c 6

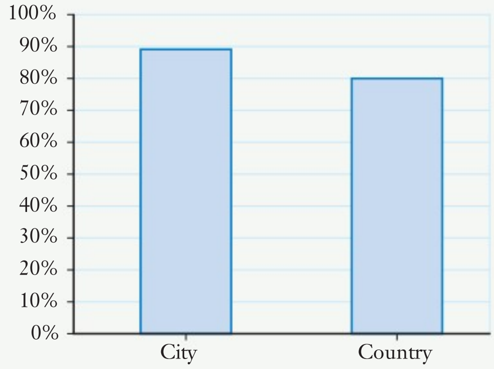
9 a Science: 70, English: 63.5

b Science: 69.8, English: 68.8

c Science: 12.8, English: 13.4 d 10

10 a The graph makes it look as if city access is about 3 times as high as country access.

b



11

Test yourself 7

1 B 2 C 3 D

4 D 5 D 6 A

7 Mode 8, median 7, range 5

8 a 5–9 b 10.75 c 9.5

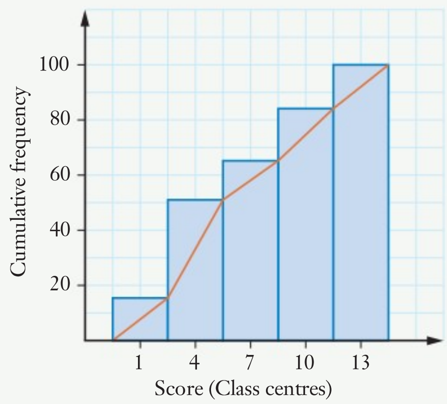
d i 6.6 ii 43.97

9 Mean 55.7, standard deviation 23.5

10 a 10 b 11

c 14 d 6

11 a



b 5.5

c i 3 ii 10 iii 13

iv 4 v 12

12 a 1.5 b 1 c 5 d 2

13 Mode

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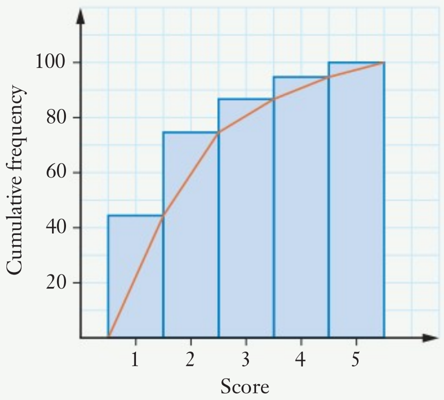
14 a



b i 2 ii 1.5

15 a 7.6 b 1.35 c 6

16 Mean 44.8, mode 49, median 45

17 a 3 b 3.5

18 a i 17 ii 14.42 iii 14, 18

iv 15

b 2

c i 8 ii 15.1 iii 14, 18

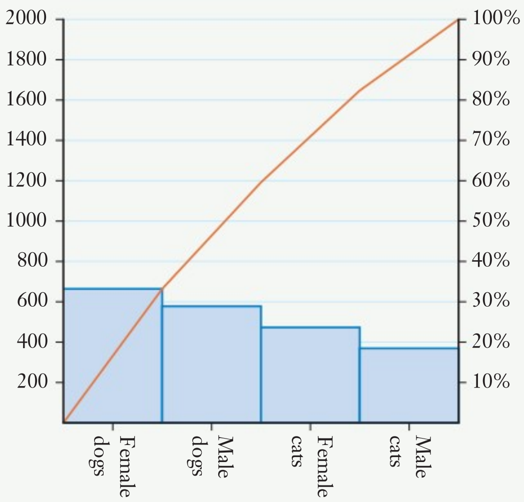
iv 15

d Affects range and mean.

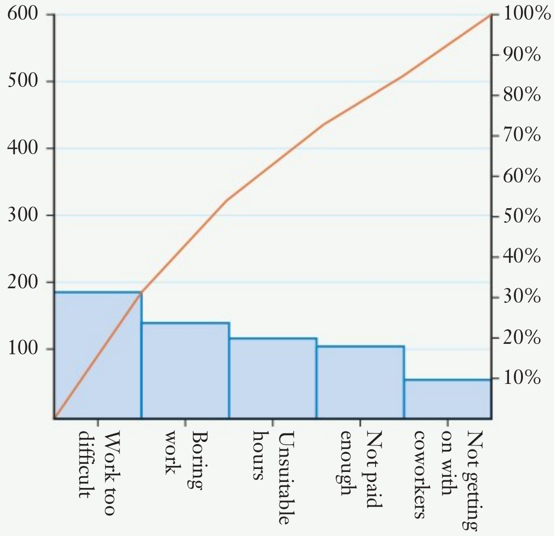
19 a i 54.6% ii 22.7% iii 27.7%

iv 59.6%

b



20



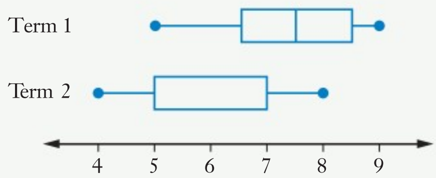
21 a North = 47.5, West = 43

b Mean 46.5, standard deviation 7.8

c Mean 43.4, standard deviation 10.7

d Slightly more mushrooms found in the North region on average, spread higher in West region so results more variable.

22 a



b Term 1: 7.5; Term 2: 5

c Term 1: 2; Term 2: 2

d Term 1: mean 7.3, standard deviation 1.4; Term 2: mean 5.6, standard deviation 1.2

e Students did better on average in Term 1. The 2 assessments had a similar spread.

Challenge exercise 7

1 a Bimodal. Female heights may have their own mode and male heights have their own (higher) mode.

b Mean 168.3, variance 92.2

2 4

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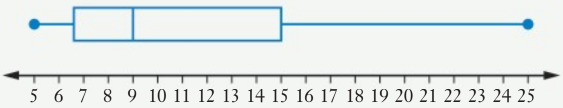
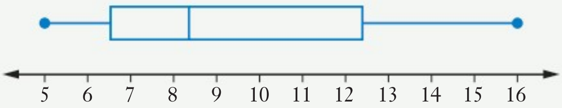
3 a



b Positively skewed. There is an outlier at 25.



Without the outlier it is still slightly positively skewed, but it is more symmetrical.

4 a i 15 ii 5.66

b

Score x	$x-\bar{x}$	$(x-\bar{x})^2$
7	−8	64
11	−4	16
15	0	0
19	4	16
23	8	64
	$\Sigma(x-\bar{x})=0$	$\Sigma(x-\bar{x})^2=160$

c 5.66

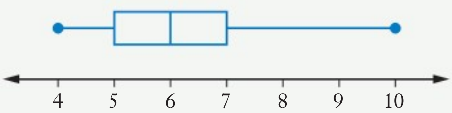
5 25.375

6 a Set B has a higher average or centre, and more consistent results (less spread out than set A).

b Set B has a higher average and is less consistent (more spread out).

7 a Mean 6.04, standard deviation 1.41

b



c Positively skewed

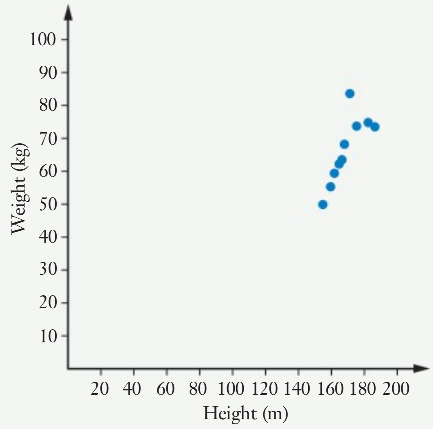
8 a Positively skewed

b Negatively skewed

Chapter 8

Exercise 8.01

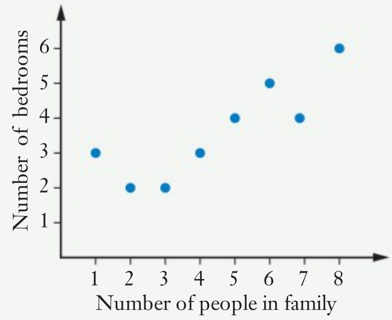
1 a



b The shape seems to be linear in a positive direction.

c As height increases, weight increases.

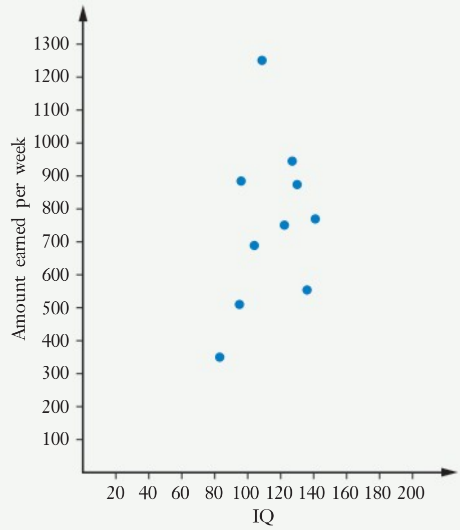
2 a



b The scatterplot is non-linear, or linear (positive) with outliers.

c With some exceptions, the more family members, the more bedrooms in the home.

3 a



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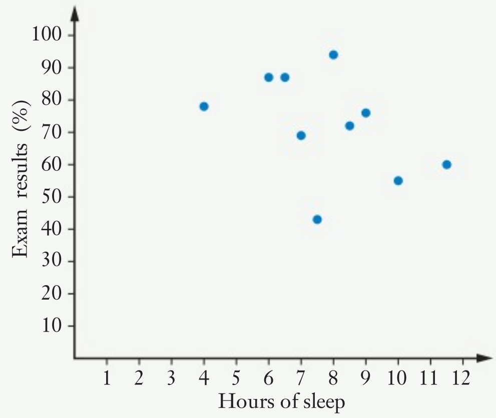
b The scatterplot has no pattern, or is linear (positive) with outliers.

c With some exceptions, the higher the IQ, the more the person earns.

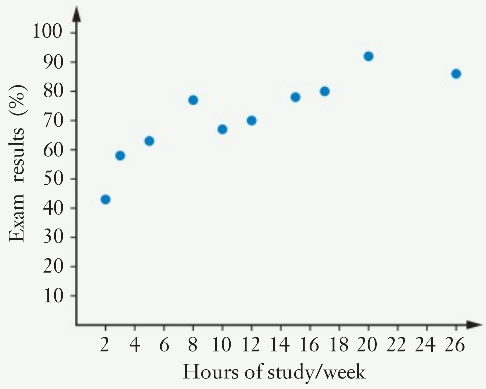
4 a



b The scatterplot has no pattern.

c There is little or no correlation found between amount of sleep and exam results.

5 a



b The shape is linear in a positive direction.

c As the number of hours of study increases, exam results increase.

6 a Non-linear

b Linear in negative direction

c No pattern

d Linear in positive direction

e Linear in negative direction with outliers

f Non-linear

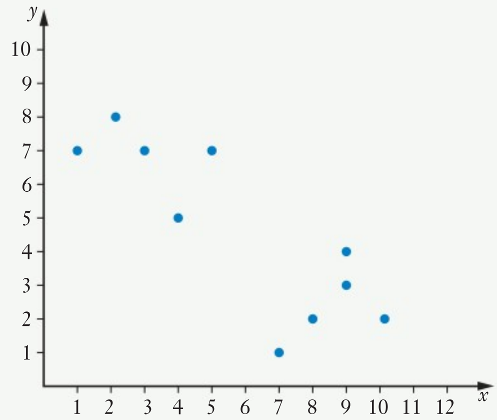
g No pattern

h Linear in positive direction

i Non-linear

j Non-linear

7 a



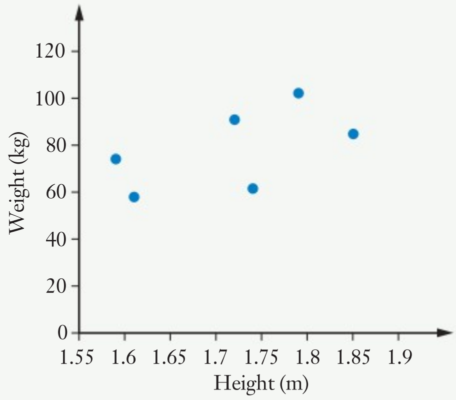
b Linear in a negative directon

8 a B	b C	c C	d D
e A	f C	g B	h A
i C	j D		

Exercise 8.02

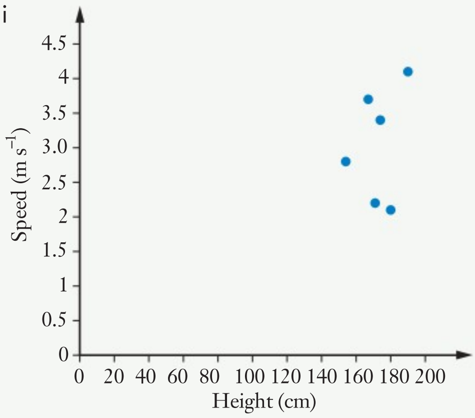
1 a G	b B	c F	d A
e C	f D		

2 a i



ii 0.571

b i



ii 0.284

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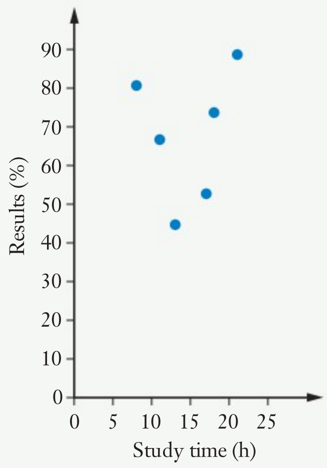
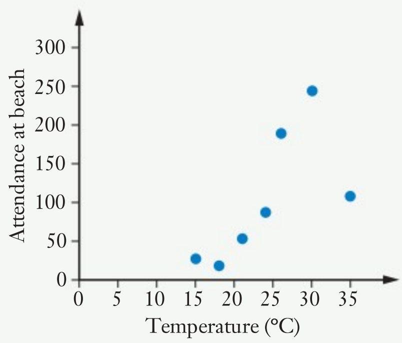
c i



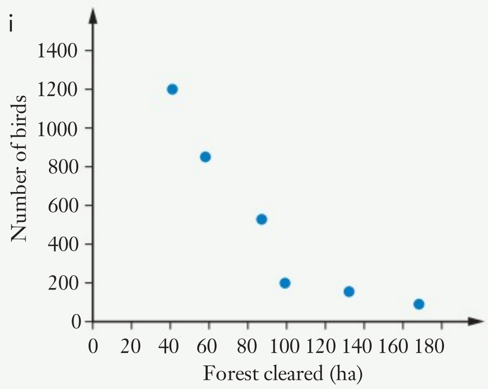
ii 0.181

d i



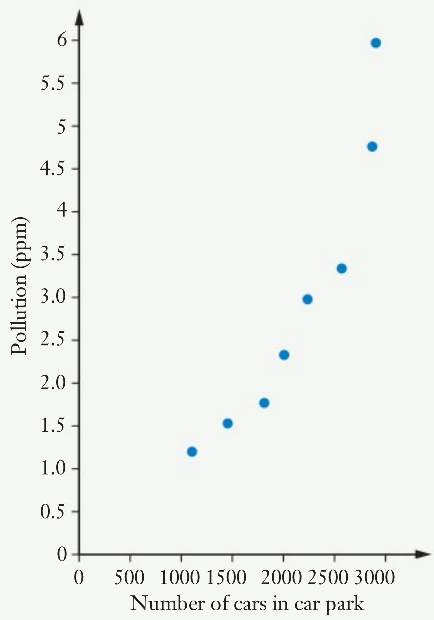
ii 0.685

e i



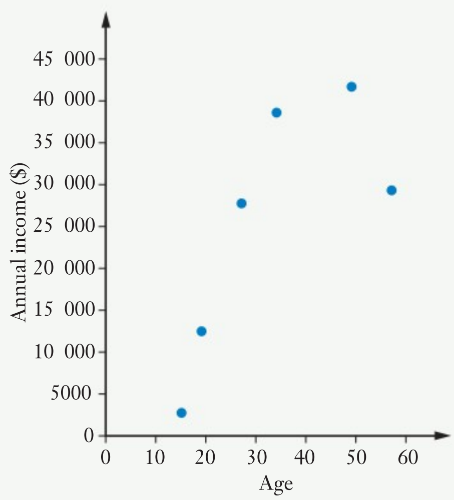
ii −0.908

f i



ii 0.930

g i



ii 0.749

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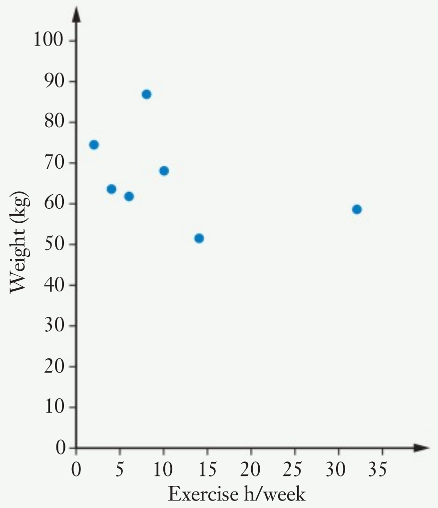
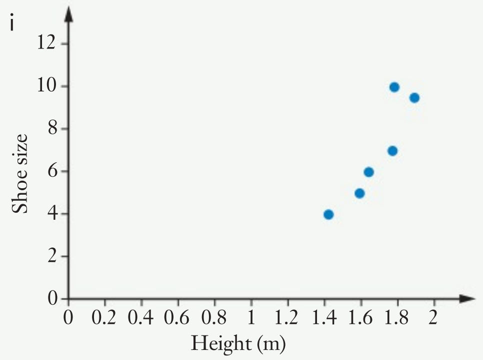
h i



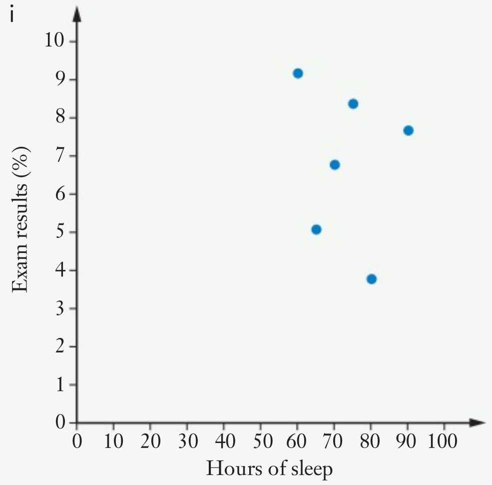
ii −0.430

i i



ii 0.905

j i



ii −0.182

3 d, e, f, i (Class discussion)

4 a 0.40 b −0.39
c 0.99 d −0.60

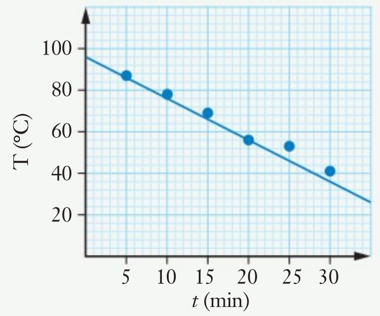
5 a, c and f (Class discussion)

Exercise 8.03

All answers are approximate.

1 a $y = 0.6x + 0.7$ b $N = -2t + 23$
c $V = 5.6x + 32$ d $P = 1.4t + 6.3$
e $A = -8x + 98$ f $x = 5.2t + 480$

2 a

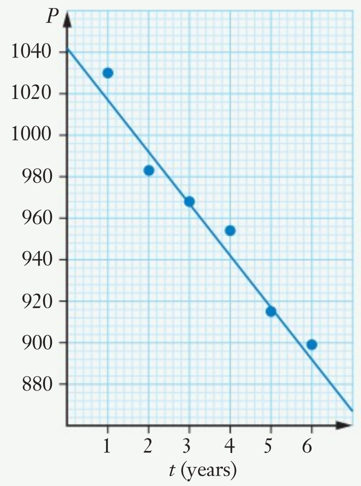


b $T = -2t + 95$

c i 61°C ii 25°C

d Good model when interpolating, but not for extrapolating because it can only cool to room temperature (23°C) but not below this.

3 a



b $P = -25t + 1045$

c 870

d 42 years

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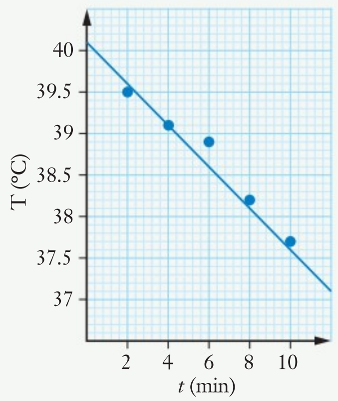
4 a

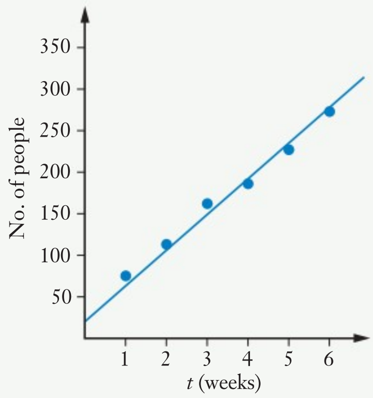


b $T = -0.24t + 40.1$

c $36.5°$

d No (class discussion)

5 a



b $N = 37.5t + 42.5$

c 418

d No, because the number of people cannot keep increasing forever.

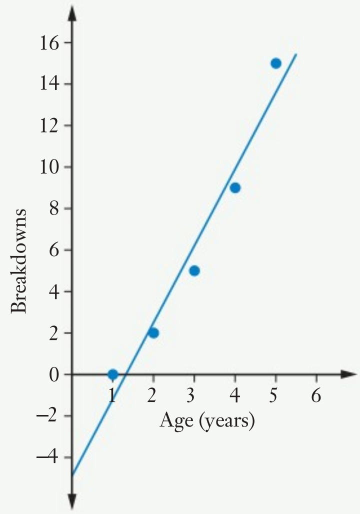
Exercise 8.04

1 a i $r = 0.9923$ ii $m = 2.43$

b i $r = 0.976$ ii $m = 3.89$

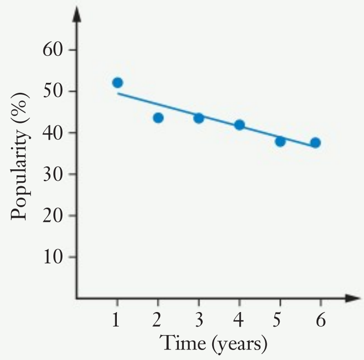
2 a i $y = 3.7x - 4.9$

ii



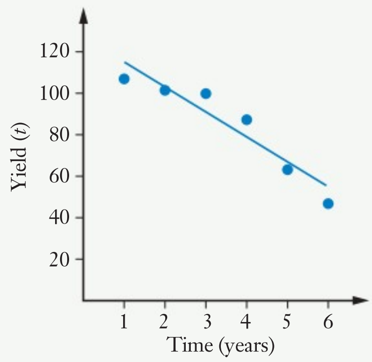
b i $y = -2.65x + 52.18$

ii



c i $y = -12.24x + 127.43$

ii



d i $y = 69.88x + 355.43$

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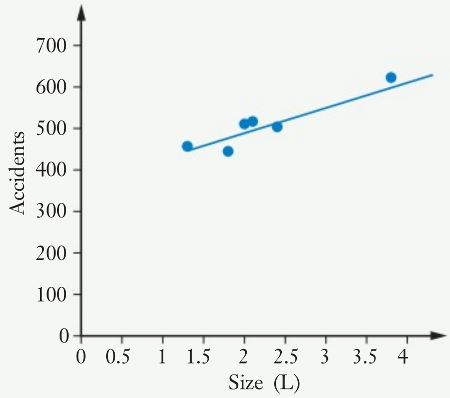
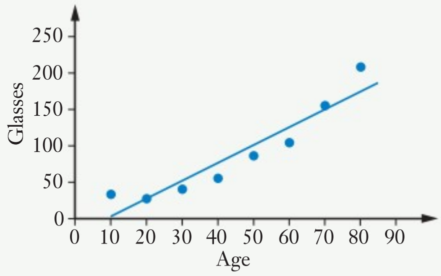
ii



e i $y = 2.49x - 22.36$

ii



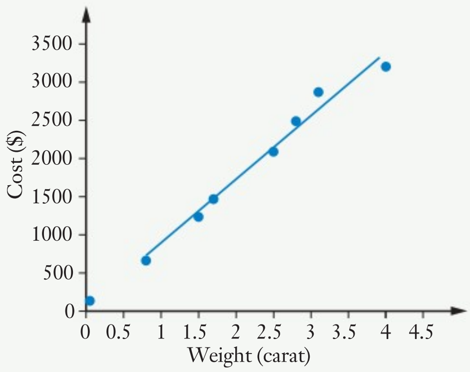
3 a $r = -0.954$, which shows a high negative correlation. This means that the regression line has a negative gradient and that the mass decreases as time increases.

b $m = -0.614t + 23.56$

c i 12.5 kg ii -13.3kg

d The answer to c ii shows that extrapolation is not useful in this situation. The ice has melted to 0 volume before an hour has passed and the equation is no longer relevant.

4 a



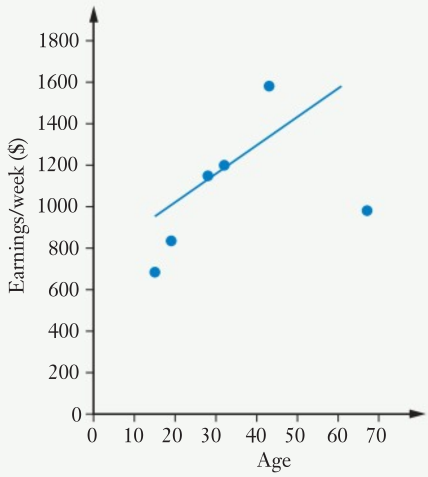
b 0.993. This is a high positive correlation.

c $y = 831.7x + 69.2$

d $1732.59

e 11.94 carats

5 a



b 0.395

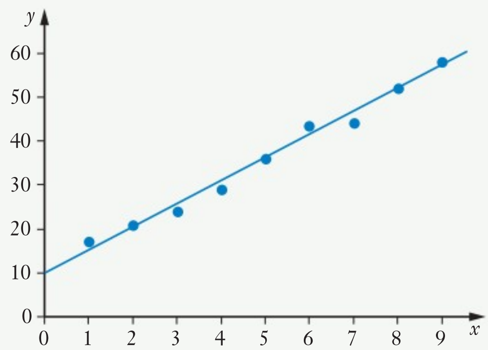
c $y = 6.6x + 853.5$ d $1181.92

e No. For example. using the equation to find the age of a person earning $1800 would give a 145-year-old!

Test yourself 8

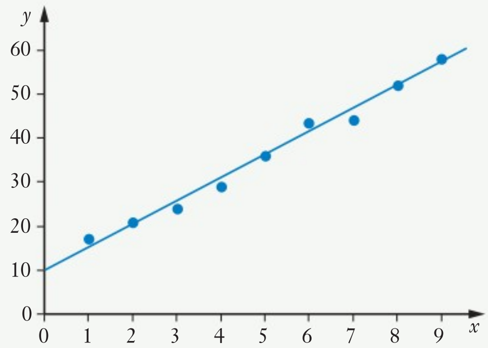
1 C 2 B 3 D 4 A 5 C

6 a



$y = 5x + 10$

b



$y = 5.1833x + 10.083$

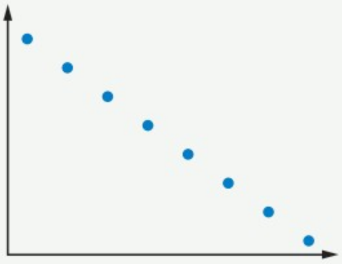
7 a C b A c D d C
e B f A g C h D

8 a 0.84 b $y = 0.758x + 10.88$
c 56 d 78

9 c, d are causal (Class discussion)

Challenge exercise 8

1 $r = 0.8$, $\bar{y} = 6.4$

2 a


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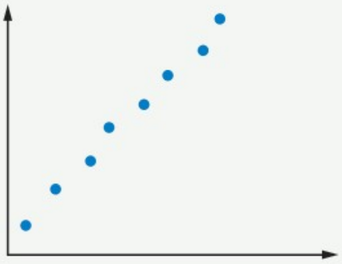

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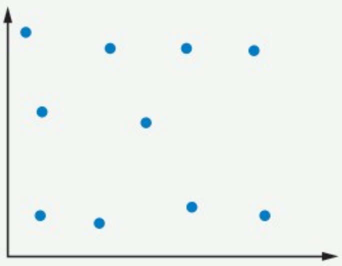
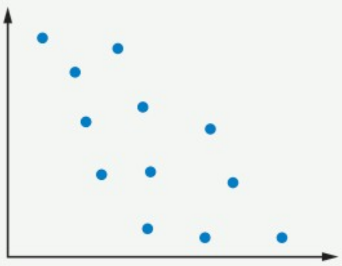
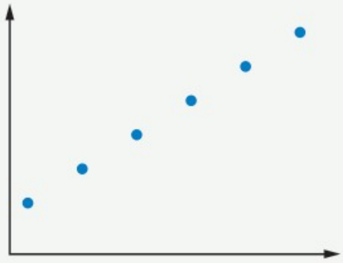
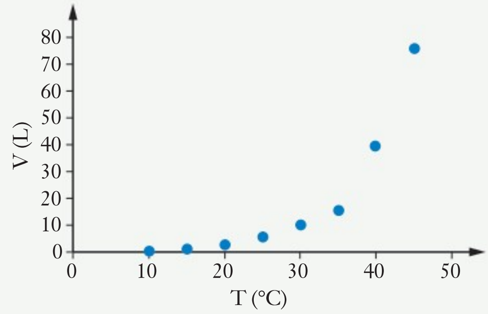
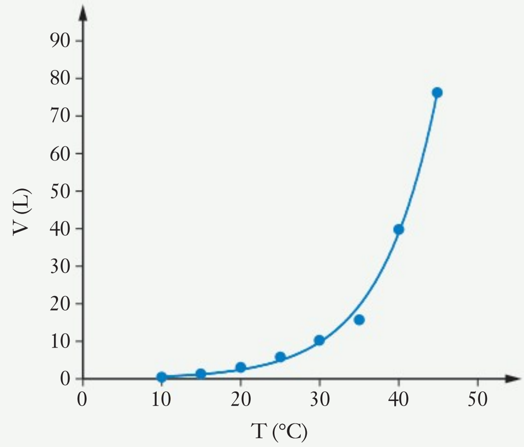


d


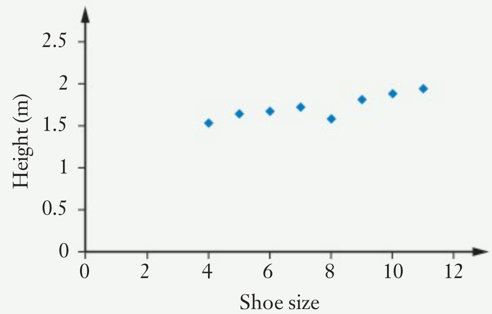
e


3 a


b The correlation is non-linear, so a least squares regression line would not be a good approximation

c


$V = 0.1563e^{0.1379T}$

4 a


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b 0.876

c $y = 0.052x + 1.34$

d i 0.992 ii $y = 0.054x + 1.35$

e No. Shoe sizes cannot keep increasing

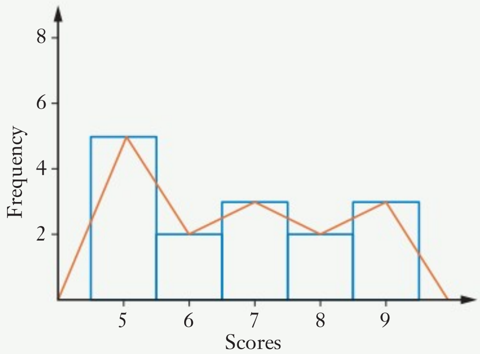
Practice set 3

1 A 2 A 3 D 4 D 5 B

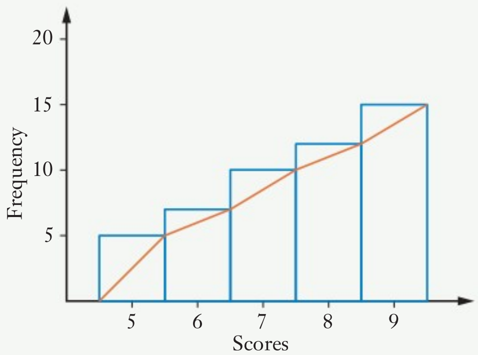
6 a

Score	Frequency
5	5
6	2
7	3
8	2
9	3

b



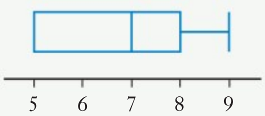
c



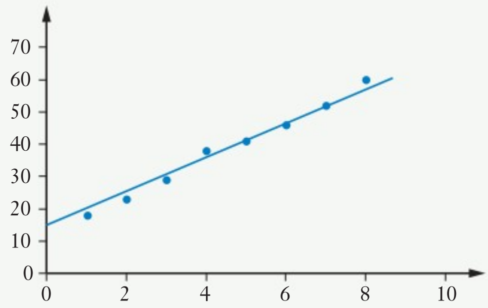
d 7

e 3

f

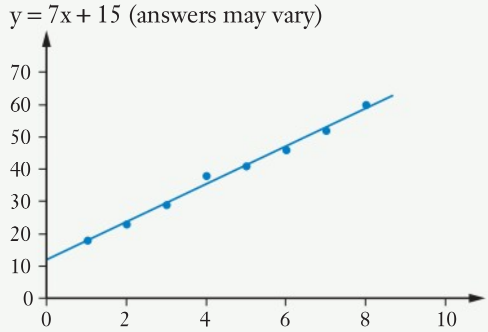


7 a



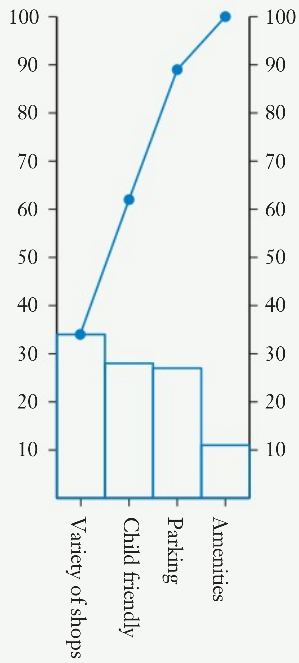
$y = 7x + 15$ (answers may vary)

b



$y = 5.869x + 11.964$

8



9 Mean 6.375, standard deviation 1.87, variance 3.48

10 a Mean 8.04, standard deviation 1.54

b 3 is not between 4 and 12, so 3 is an outlier.

c Mean 8.26, standard deviation 1.15

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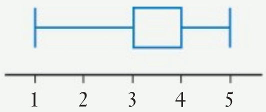
11 Mean 6.43, mode 6, median 6, range 5

12 a 52

b Mean 3.58, standard deviation 1.01

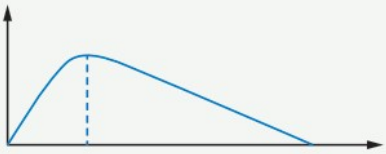
c 4

d

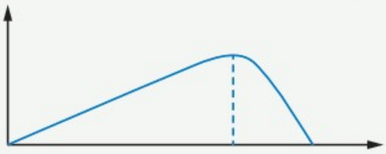


13 –0.82

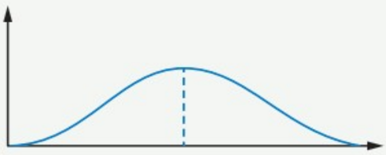
14 a



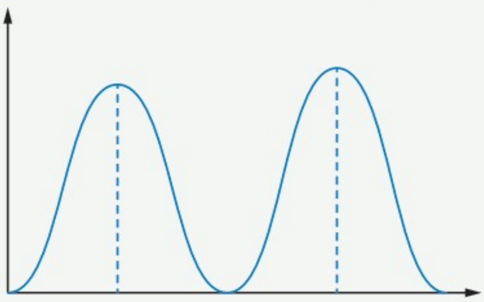
b



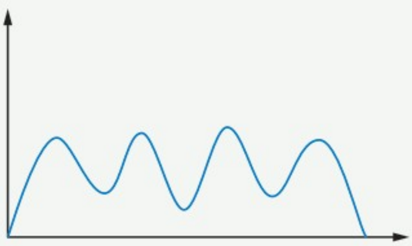
c



d



e



15 a categorical nominal

b quantitative continuous

c categorical ordinal

d quantitative discrete

e quantitative discrete

16 a C b D c B d A

e D f C

17 a 3.5 b 2.1 c 4.5 d 2.5

18 $(4, -143)$ min, $(-2, 73)$ max, $(1, -35)$ point of inflection

19 a $x = \frac{\pi}{6}, \frac{5\pi}{6}$

b $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

c $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

20 a 399 b 20 100

Chapter 9

Exercise 9.01

1 a 210 b 13th c 57

2 a 39 b 29th c 32

3 a $3n + 3$

b $S_n = \frac{1}{2}n[2 \times 6 + (n - 1) \times 3]$

4 a 0.01 m b 91.5 m

5 a 7 b 10.5 m

6 a \$2050 b \$2100 c \$2150

d \$2500 e \$3500

7 a 1.25 m b 8

8 a 4.5–5 kg b 18th

9 a 49 b 4mm

10 a $3k$ m b $3k(k + 1)$ m c 9

Exercise 9.02

1 a i 93% ii 86.49% iii 80.44%

b 33.67% c 19 weeks

2 a 96.04% b 34 days c 114 days

3 a i \$23 200 ii \$26 912

iii \$31 217.92

b \$102 345.29 c 6.2 years

4 a 77.4% b 13.5 years c 31.4 years

5 a $\frac{4}{9}$ b $\frac{7}{9}$ c $1\frac{2}{9}$ d $\frac{25}{99}$

e $2\frac{9}{11}$ f $\frac{7}{30}$ g $1\frac{43}{90}$ h $1\frac{7}{450}$

i $\frac{131}{990}$ **j** $2\frac{361}{999}$

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6 0.625 m
7 15 m
8 20 cm
9 3 m
10 a 4.84 m
b After 3 years
11 a 74.7 cm
b 75 cm
12 300 cm
13 3.5 m
14 32 m
15 a 1, 8, 64, 512, …
b 16 777 216 people
c 19 173 961 people

Exercise 9.03

1 a $6895.85 b $6999.79 c $7633.37
d $6857.36 e $7207.26
2 a $2852.92 b $12 720.32 c $4038.13
d $5955.08 e $87 362
3 $893 262
4 $3552.86
5 $21 173.72
6 $1069.23
7 $4902.96
8 a $1125.68 b $2209.20 c $14 930
d $96 630 e $305 900 f $902.22
g $1300.26 h $3090.90 i $1061.50
j $3866.10
9 a $8705.49 b $4971.66 c $4631.99
d $9705.91 e $8227.07

Exercise 9.04

1 a $740.12 b $14 753.64
c $17 271.40 d $9385.69
e $5298.19
2 a $2007.34 b $2015.87 c $2020.28
3 a $4930.86 b $4941.03
4 a $408.24 b $410.51
5 a $971.40 b $972.12
6 a $1733.99 b $1742.21 c $1747.83
7 a $3097.06 b $13.47
8 a $22 800.81 b $945.92
9 $691.41
10 $1776.58
11 $14 549.76
12 $1 301 694.62
13 a $4113.51 b $555.32 c $9872.43
d $238.17 e $10 530.59
14 $4543.28
15 4 years
16 8 years
17 a $x = 7$ b $x = 5$ c $x = 8$
d $x = 6.5$ e $x = 8.5$
18 $7.68
19 Kate $224.37
20 Account A, $844.94
21 a i $39 400.53 ii $41 812. 16
b 27th year
22 a i $128 547 ii $175 196.37
iii $230 700.16
b 28th year
23 a i 4 years ii 17 years
b i 3.1% ii 1.8%
24 a 1.9990
b $A = 1.08^9$
$= 1.9990$ to 4 decimal places
25 FV interest factor is 1.5209.
$A = 1.15^3$
$= 1.5209$ to 4 decimal places

Exercise 9.05

1 a $10 125 b $3745.92 c $2720.63
d $20 410 e $6194.05
2 a $49 759.29 b $5524.36
c $94 334.25 d $2074.02
e $274 948.05 f $3981.08
g $20 103.20 h $15 559.48
i $327 214.50 j $1 474 272
3 a $10 789.40 b $22 839.56
c $69 720.25 d $9511.37
e $11 513.83
4 $477 076.25
5 $2074.70
6 a $1469.39 b $2811.42 c $1266.10
d $614.91 e $30 242.55
7 $296.59
8 a i $47 000 ii $43 880 iii $40 635.20
b $43 756.80 c $41 833.42
9 a $125 750 b $126 507.50
c $127 272.58 e $125 376.88
d $125 757.53
10 Graph A:
a 18 months b $10 400
c 36 months (3 years)
Graph B:
a 27 months (2 years 3 months)
b $12 200
c 46 months (3 years 10 months)

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11 a 18 b 21

12 a 9% b 3%

Exercise 9.06

1 $40 728.17 2 $3383.22

3 $65 903.97 4 $2846.82

5 $6181.13 6 $4646.71

7 $21 426.03

8 a $26 361.59 b $46 551.94

9 $45 599.17

10 a $7922.80 b $1584.56

11 $500 for 30 years ($11 640.86 better off)

12 Yes, $259.80 over 13 $818.72

14 a $37.38

b Proof (see worked solutions).

Exercise 9.07

1 a $19 234.54 b $3177.64

c $99 990.13 d $1798.81

e $45 659.71

2 a i $5978.52 ii $978.52

iii 6.5%

b i $24 225.60 ii $8325.60

iii 10.5%

c i $159 785.28 ii $79 785.28

iii 8.3%

d i $272 127 ii $37 127

iii 0.63%

e i $1710.48 ii $362.48

iii 13.4%

3 a $154.64 b $304.20 c $854.64

d $2605.80 e $2549.04 f $686.37

g $568.89 h $3021.75 i $3922

j $5680

4 a $3946.90 b $947 256

c $266 756 d 1.96%

5 a 5 years b 10 years c 20 years

d 30 years e 10 years f 25 years

g 10 years h 15 years i 10 years

j 20 years

6 a 6% b 4.5% c 2%

d 8% e 5.5% f 7%

g 2.5% h 3% i 7.5%

j 4%

Exercise 9.08

1 $1047.62 2 $394.46 3 $139.15

4 a $966.45 b $1265.79

5 $2519.59

6 a $868.46 b $55 907.60

7 a $77.81 b $2645.42

8 $78 700

9 a Get Rich $949.61, Capital Bank $491.27

b $33 427.80 more through Capital Bank

10 a $875.60 b $43 778.80

11 $61 292.20

12 NSW Bank totals $5791.25, Sydney Bank totals $5557.88 so Sydney Bank is better.

13 a $249.69 b $13 485.12

14 a $13 251.13 b $374.07

c $20 199.78

15 a $1835.68 b $9178.41

16 a $10.36

b Proof (see worked solutions).

Test yourself 9

1 C 2 B 3 A

4 $21 980.94 5 $1672.74

6 a Each slat rises 3 mm so the bottom one rises up 30 × 3 mm or 90 mm.

b 87 mm

c 90, 87, 84, … is an arithmetic sequence with $a = 90$, $d = -3$.

d 42 mm e 1395 mm

7 a $121 320 b $58 820 c 18.8%

8 a $723.22 b $9995 c $2925.27

d $16 785.90 e $11 345.01

9 a $24 050 b $220 250

10 a $184 867.25 b $182 829.42

c $180 786.49

11 $1285.19

12 a 10 stacks b 110 boxes

13 a $\frac{4}{9}$ b $\frac{13}{18}$ c $1\frac{19}{33}$

14 $3400.01

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15 a $59 000 b $15 988.89
16 a $2385.04 b $2392.03
17 a $1432.86 b $343 886.91
18 a $2651.56 b $9872.43
19 a 39 words/min b 15 weeks
20 4.8 m
21 a i $33 000 ii $23 000
b i 14 years ii 28 years
c 32 years

Challenge exercise 9

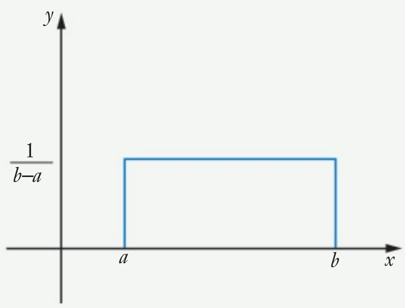
1 $1799.79
2 a $40 b $2880
3 $8522.53
4 a i 115.8° ii 51.4°
b i 6.6 min ii 10.9 min
5 a $\$1000(1.06)^{25}$ b $\$1000(1.06)^{24}$
c $\$1000(1.06)^{23}$ d $1000(1.06)
e $S_{25} = \dfrac{1000 \times 1.06 \times (1.06^{25} - 1)}{1.06 - 1}$
f $58 156.38
6 a $10 100 b $11 268.25
c $4212.41 d $2637.22
7 $466 563.74

Chapter 10

Exercise 10.01

1 a Discrete b Continuous
c Continuous d Discrete
e Continuous
2 a, d 3 a, d
4 b= 7 5 $k = \dfrac{4}{625}$
6 a $a = \dfrac{1}{e(e^2 - 1)}$
b $a = \dfrac{1}{e(e^6 - 1)}$ c $a = \dfrac{1}{e^4 - 1}$
7 [0, 6] 8 b= 8

Exercise 10.02

1 a $\frac{1}{2}$ b $\frac{1}{6}$ c $\frac{1}{2}$
d $\frac{2}{3}$ e $\frac{1}{3}$
2 a $y = \dfrac{x}{50}$
b i $\frac{81}{100}$ ii $\frac{9}{100}$ iii $\frac{33}{100}$
iv $\frac{8}{25}$ v $\frac{3}{4}$
3 a $a = \dfrac{3}{125}$
b i $\frac{27}{125}$ ii $\frac{63}{125}$ iii $\frac{117}{125}$
iv $\frac{1}{125}$ v $\frac{37}{125}$
4 a $a = \dfrac{4}{81}$
b i $\frac{80}{81}$ ii $\frac{16}{81}$ iii $\frac{5}{27}$
iv $\frac{1}{81}$
5 a $k = \dfrac{1}{e(e^5 - 1)}$
b i $\dfrac{e(e^3 - 1)}{e^5 - 1}$ ii $\dfrac{e^3 - 1}{e^5 - 1}$ iii $\dfrac{e^2(e^3 - 1)}{e^5 - 1}$
6 a
$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin x \, dx &= [-\cos x]_0^{\frac{\pi}{2}} \\ &= -\cos\frac{\pi}{2} - (-\cos 0) \\ &= -0 + 1 \\ &= 1\end{aligned}$$
So y = sin x is a PDF in the domain $\left[0, \frac{\pi}{2}\right]$
b i $\frac{1}{2}$ ii $1 - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2} - 1}{\sqrt{2}}$ iii $\dfrac{\sqrt{3}}{2}$
7 a


$A = bh$
$= (b - a) \times \dfrac{1}{b - a}$
= 1 so PDF

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b i $\frac{3}{4}$ ii $\frac{1}{2}$ iii $\frac{1}{4}$

Exercise 10.03

1 a $F(x)=\frac{x^3}{27}$ b $F(x)=\frac{x^4}{1296}$

c $F(x)=\frac{e^x-1}{e^4-1}$ d $F(x)=\frac{(x-2)^4}{625}$

e $F(x)=\frac{12x^2-x^3-40}{135}$

2 a $F(x)=\frac{x^5-1}{7776}$

b i $\frac{121}{3338}$ ii $\frac{31}{7776}$ iii $\frac{781}{1944}$

iv $\frac{2251}{2592}$ v $\frac{31}{243}$

3 a $F(x)=\frac{x^4-81}{2320}$

b i $\frac{35}{464}$ ii $\frac{243}{464}$ iii $\frac{111}{145}$

iv $\frac{429}{464}$ v $\frac{13}{29}$

4 a $F(x)=\frac{e^{2x}-1}{e^{10}-1}$

b i 0.0024 ii 0.14 iii 0.98

iv 0.99 v 0.13

5 a $a=\frac{4}{6561}$ b $F(x)=\frac{x^4}{6561}$

c i $\frac{625}{6561}$ ii $\frac{256}{6561}$ iii $\frac{2465}{6561}$

iv $\frac{80}{81}$ v $\frac{1280}{6561}$

6 a $a=\frac{1}{\ln 6}$ b $F(x)=\frac{\ln x}{\ln 6}$

c i 0.61 ii 0.39 iii 0.10

iv 0.23 v 0.51

7 a Show that $\int_{\frac{3\pi}{2}}^{2\pi}\cos x\,dx=1$

b $F(x)=\sin x+1$

c i $\frac{2-\sqrt{3}}{2}$ ii $\frac{1}{\sqrt{2}}$

iii $\frac{\sqrt{3}-1}{2}$

8 a 2 b 3 c 3 d 7

e 4 f 4 g 6 h 8

i 4 **j** 7

9 a 3

b $F(x)=-\frac{1}{22}(x^3-9x^2+15x-2)$

c $\frac{1}{2}$

10 a $F(x)=\frac{1}{464}(x^4-12x^3+48x^2+4x-201)$

b i $\frac{9}{29}$ ii $\frac{393}{464}$ iii $\frac{73}{464}$

c 7 minutes

Exercise 10.04

1 a 6.35 b 5.89 c 6.09 d 3

e 3.11 f 7.57 g 8.79 h 4.45

i 5.29 **j** 3.73

2 a i 6.47 ii 6.05 iii 9.19

b i 4.24 ii 4.01 iii 5.62

c i 3.79 ii 3.62 iii 4.75

3 a 6.35 b 5.64

4 a 6.38 b 5.12 c 7.28 d 7.02

e 4.28 f 7.44

5 a 7.02 b 9.52 c 10.90

6 a $F(x)=\frac{x^4-16}{4080}$ b $\frac{203}{1360}$

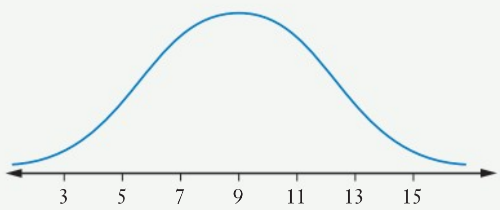
c $\frac{16}{17}$ d $\frac{29}{51}$

e 6.73 f 7.45

g 7.79 h 5.56

Exercise 10.05

1 a



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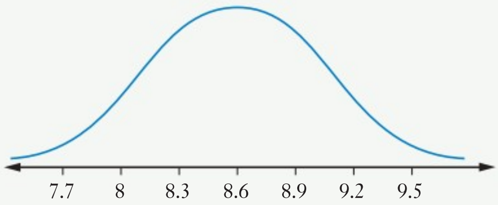
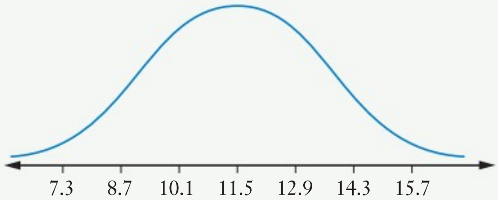
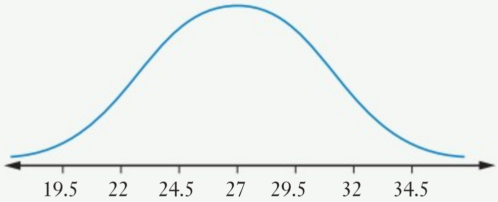
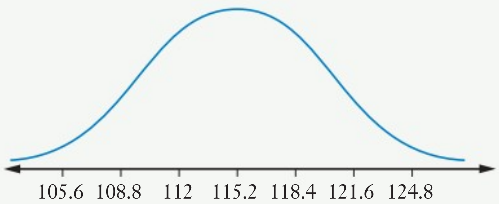
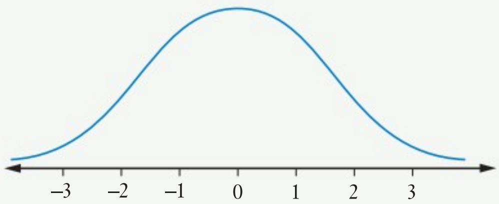
b 

c 

d 

e 

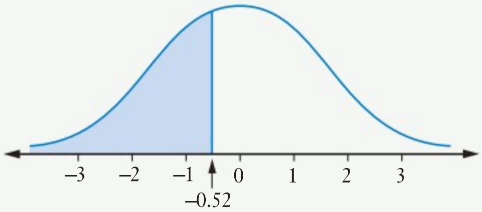
2 a $\mu = 23.7, \sigma = 4.2$ b $\mu = 5.4, \sigma = 0.9$
c $\mu = 59.7, \sigma = 5.4$ d $\mu = 209, \sigma = 10.6$
e $\mu = 11.3, \sigma = 2.2$

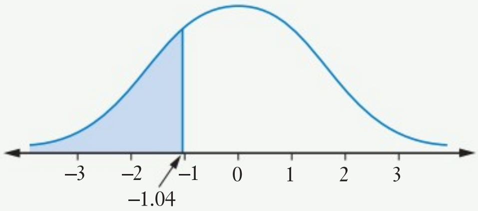
3 

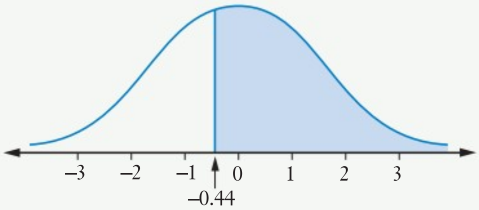
4 a 0.5 b 0.8413 c 0.9772
d 0.9987 e 0.1587 f 0.0228
g 0.0013 h 0.9332 i 0.0038
j 0.999

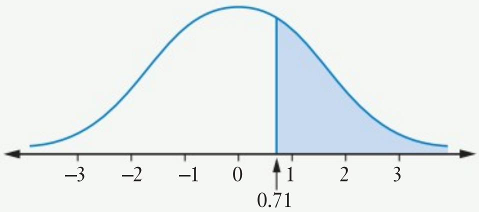
5 a 0.6772 b 0.0174 c 0.9778
d 0.031 e 0.9722 f 0.0837
g 0.9255 h 0.6826 i 0.9544
j 0.9974

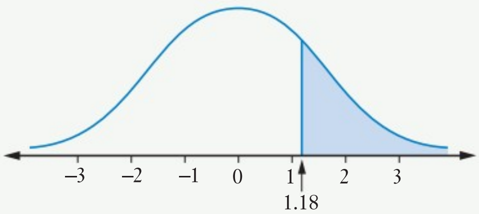
6 a 0.84 b 0.67 c –0.53
d –0.84 e 1.23
f –1.17 or – 1.18
g –0.52 h –0.67 i 0.33

7 a 

b 

c 

d 

e 

Exercise 10.06

1 a 68% b 95% c 99.7%
2 a 68% b 95% c 99.7%
3 a 68% b 95% c 99.7%
4 a 68% b 95% c 99.7%
d 34% e 49.85% f 97.35%

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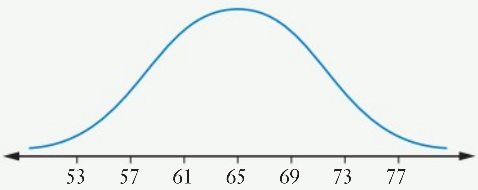
5 a



b i 95% ii 34% iii 49.85%
iv 81.5% v 81.5%

6 a 68% b 34% c 47.5%
d 47.5% e 83.85%

Exercise 10.07

1 a i 0 ii 1 iii 2
iv 3 v –1 vi –2
vii –3
b i 19.95 ii 15.27

2 a 27 cm to 79.2 cm
b i 0 ii 1 iii –1
iv 2 v –2 vi 3
vii –3 viii 1.4

3 a 3.5° to 10.1°
b i 0 ii 1 iii 2
iv 3 v –1 vi –2
vii –3 viii –0.7

4 a 54.8 mL to 78 mL
b i 0 ii 1 iii 2
iv 3 v –1 vi –2
vii –3 viii 4.1
c i 73.36 mL ii 83.22 mL
iii 62.92 mL iv 53.06 mL

5 a i 2.7 ii –3.3 iii 2.4
iv –1.3 v 4.9 vi –1.8
vii 3.3 viii –2.6
b 53.2, 90 and 82.7 c 62.1 and 59.7
d 80, 78.6, 62.1, 59.7 and 56.4

6 a i 1.29 ii –1.57
b i 11.26 mm ii 16.02 mm
iii 18.68 mm iv 13.136 mm
v 16.93 mm

7 28 8 26.85 9 50

10 a 53.4 b 42.2 c 59
d 69.08 e 43.88

11 48.3

12 a i 12.2 to 19.8 ii 14.1 to 17.9
iii 10.3 to 21.7
b i 2.1 ii –1.3
c i 10.3 ii 18.09

13 a 99.6 to 109.8
b i –4.8 ii –0.3
c i 114.9 ii 98.07

Exercise 10.08

1 a i 99.7% ii 47.5% iii 68%
iv 49.85% v 83.85%
b i 4 ii Yes: outside normal range
c i –1.5 ii 0.0668 iii 0.2417

2 a i 68% ii 95% iii 99.7%
iv 34% v 81.5%
b i –3.3 ii Yes: outside normal range
c i 3.7 ii Yes: outside normal range

3 a 4.1 – no: outside normal range
b 4.6 min to 11.4 min
c 97.35%
d i –0.29 ii 11.41%
e i 0.381 ii 0.721 iii 0.8418
f i 98.9% ii 68.32% iii 98.5%

4 a i 95% ii 49.85% iii 39.83%
b 18.7 mL to 21.1 mL
c Unusual – outside range
d 0.0606

5 a 95% b 8.1 cm c 6.9 cm

6 a 2.2 years
b i 68% ii 88.54%
c Yes – outside normal range

7 a 25.5 cm to 30.5 cm
b –0.96 c 82.33%
d No – outside normal range

8 a 5% b 2.5% c i 0.3%
ii greater than 5.4 kg, less than 4.5 kg

9 a i 99.7% ii 12.1 cm iii 9.1 cm
b 8.3 cm, 12.6 cm

10 Yes: Minimum weight within normal limits is 251.3 g.

11 2.5%

12 a 2 mm b 820 mm
c 816 mm to 824 mm

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13 a 0.2 h b 2.5 h

14 a Epping 1.04, city 1.35 b City

15 a 1 b 2.04 c Cameron

Test yourself 10

1 C 2 A 3 D

4 a 0.75 b 0.4 c 0.55
d 0.45 e 0.7 f 10
g 3.6 h 17.8 i 12
j 15

5 a Yes b Yes c No

6 3.98

7 a $F(x) = \frac{x^3 - 1}{511}$

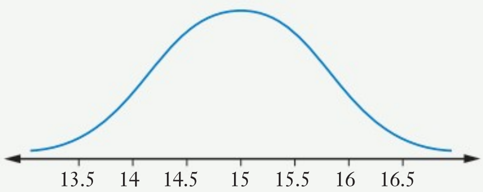
b i $\frac{26}{511}$ ii $\frac{124}{511}$ iii $\frac{296}{511}$
iv $\frac{64}{73}$ v $\frac{335}{511}$

8 a i 4 i i 2.9
b i 9 i i 7.1

9 a 2.27 kg to 4.13 kg
b i 2.26 ii 0.97
c 0.1541

10 [0, 4]

11 a



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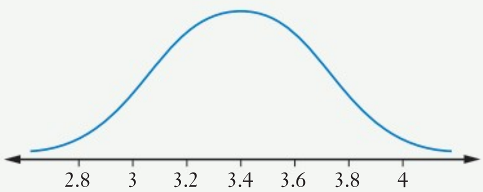


12 a 0.3% b 18.5%

13 a 5.7 b 6.4 c 5.35
d 6.09 e 4.87

14 Mean 2.9, standard deviation 0.7

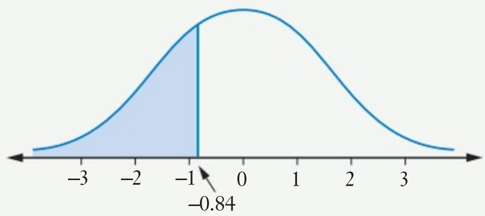
15 a $F(x) = \frac{x^5}{3125}$ b $F(x) = \frac{x^7}{823543}$
c $F(x) = \frac{e^x - 1}{e^6 - 1}$ d $F(x) = \frac{x^2 - 1}{80}$

16 a 0.7054 b 0.9066 c 0.0013
d 0.2389 e 0.8413 f 0.0062
g 0.8599 h 0.1272 i 0.1348

17 a 95% b 34% c 83.85%
d 90.5% e 25.86%

18 a $a = \frac{1}{e^5 - 1}$ b $a = \frac{1}{e(e^3 - 1)}$

19 a



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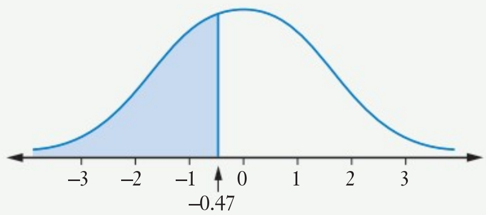
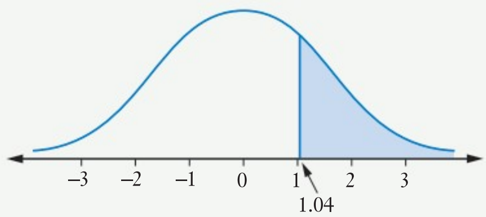
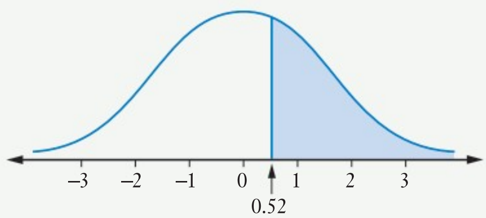


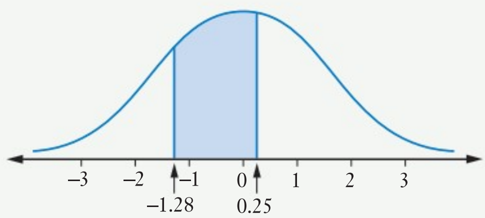
c



d



e



20 a 8 b 1.5 c 2

21 a 0.83 b 1 c Klare

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22 $F(x) = \frac{1}{600}\left[\frac{x^4}{4}\right]_1^7 = 1$ so PDF

23 a i 2.5 ii −1.5

b i 1.12 m ii 1.06 m

iii 1.16 m iv 1.09 m

v 1.13 m

24 89.98

25 49.17

Challenge exercise 10

1 $a = 1, b = 5$

2 $\mu = 13.2, \sigma = 0.4$

3 0

4 $F(x) = \frac{(x^2+1)^3 - 1000}{124000}$

5 Mean 18.9, standard deviation 3.5

6 Show area is 1.

Practice set 4

1 A 2 B 3 C 4 B 5 C

6 a $\frac{1}{3}e^{3x} + C$ b $2x^2 - 3x + C$

c $\frac{1}{4}\tan 4x + C$ d $\ln|x-3| + C$

7 a \$40 995.50 b \$745.60

8 \$945 9 2.4 m

10 a $F(x) = \frac{x^3 - 1}{342}$

b i $\frac{62}{171}$ ii $\frac{7}{342}$ iii $\frac{158}{171}$

iv $\frac{31}{38}$ v $\frac{21}{38}$

c i 5.56 ii 6.83

11 a 2 b 7 c 77

12 a i \$119 000

ii \$117 995

iii \$116 984.98

b i See worked solutions.

ii \$99 020.88

13 \$2851.52

14 4.23

15 $\frac{8}{45}$

16 (−1, 9) maximum, (5, − 99)minimum, (2, − 45) point of inflection

17 \$180.76

18 a 3.9 L to 5.5 L b 68%

c i 3.25 ii 1.25

iii −2 iv −1.5

d i 0.9994 ii 0.0228

iii 0.9332 iv 0.1056

v 0.8716

e i 73.33%

ii 78.88%

iii 99.4%

19 a \$460.50 b \$449

c \$1084.54 d \$28.49

20 a $F(x) = \frac{(x+2)^3 - 8}{335}$

b $F(x) = \frac{x^4 - 1}{624}$

c $F(x) = 2\sin x$

21 2

22 a 0.9115 b 0.8106

c 0.1587 d 0.0207

e 0.9966

23 2nd exam

24 \$277.33

25 [1, 6]

26 a 13.02 b 10.55

c 16.348 d 8.6

27 8

28 18.5

29 a 5%, reject cans below 349.4 and above 350.2 mL

b 0.3%, reject cans below 349.2 and above 350.4 mL

30 a \$2748.78 b \$2751.11 c \$2751.63

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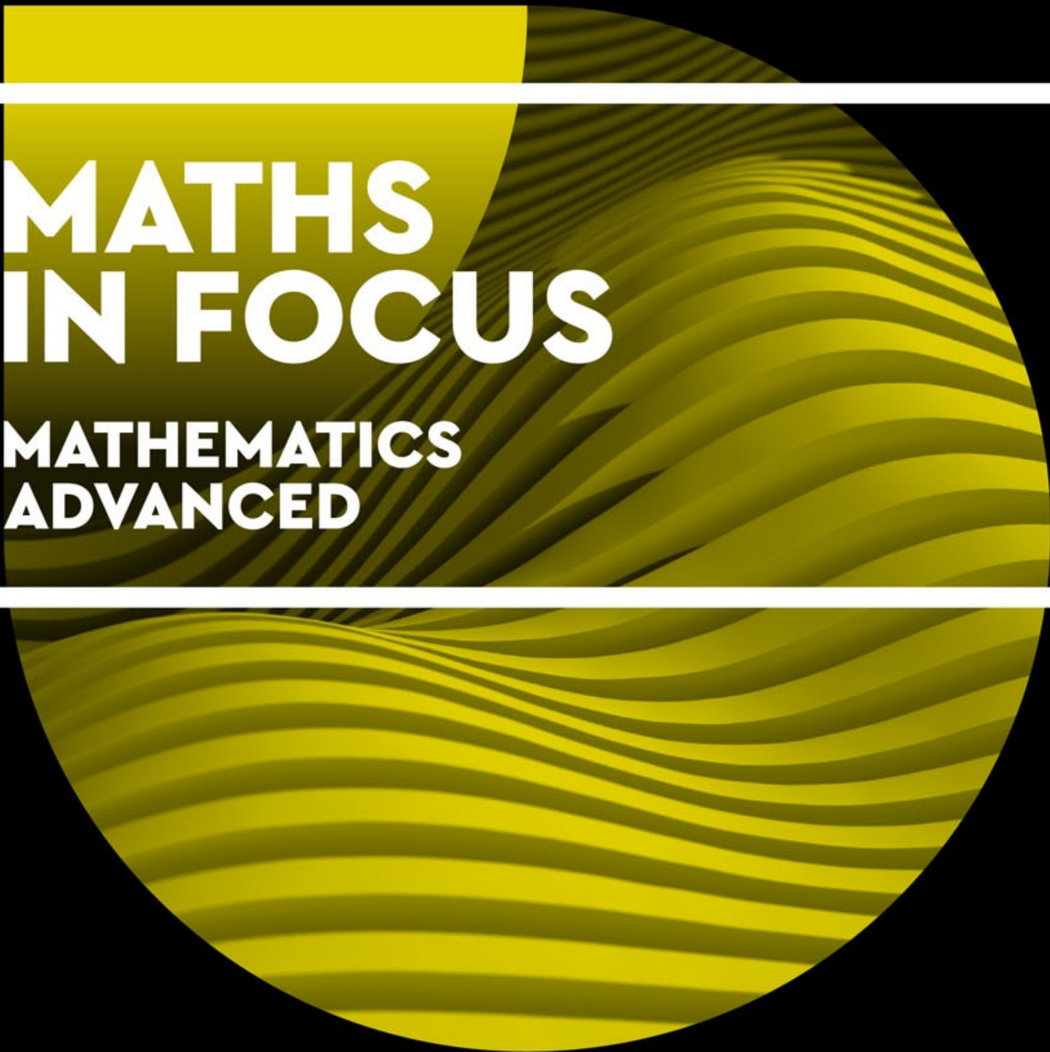
MATHEMATICS ADVANCED



YEAR 12

Margaret Grove

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PREFACE

Maths in Focus 12 Mathematics Advanced has been rewritten for the new Mathematics Advanced syllabus (2017). In this 3rd edition of the book, teachers will find those familiar features that have made Maths in Focus a leading senior mathematics series, such as clear and abundant worked examples in plain English, comprehensive sets of graded exercises, chapter Test Yourself and Challenge exercises, Investigations, and practice sets of mixed revision and exam-style questions.

The Mathematics Advanced course is designed for students who intend to study at university in a field that requires mathematics, especially calculus and statistics. This book covers the content of the Year 12 Mathematics Advanced course. The theory follows a logical order, although some topics may be learned in any order. We have endeavoured to produce a practical text that captures the spirit of the course, providing relevant and meaningful applications of mathematics.

The NelsonNet student and teacher websites contain additional resources such as worksheets, video tutorials and topic tests. We wish all teachers and students using this book every success in embracing the new senior mathematics course.

ABOUT THE AUTHOR

Margaret Grove has spent over 30 years teaching HSC Mathematics, most recently at Bankstown TAFE College. She has written numerous senior mathematics texts and study guides over the past 25 years, including the bestselling *Maths in Focus* series for Mathematics and Mathematics Extension 1.

Margaret thanks her family, especially her husband Geoff, for their support in writing this book.

CONTRIBUTING AUTHORS

Gaspare Carrozza and Haroon Ha from Homebush Boys High School wrote many of the NelsonNet worksheets.

Scott Smith created the video tutorials.

Tania Eastcott and Elizabeth Nabhan wrote the topic tests.

Roger Walter wrote the ExamView questions.

Shane Scott, Brandon Pettis and George Dimitriadis wrote the worked solutions to all exercise sets.

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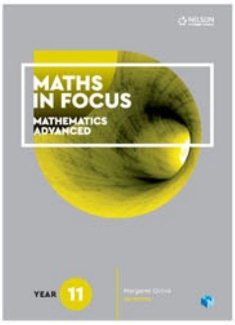

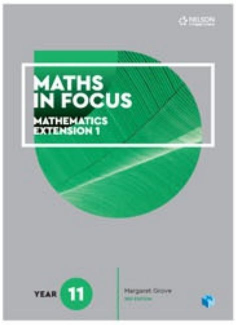

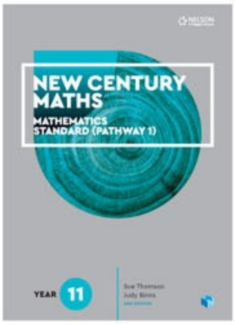

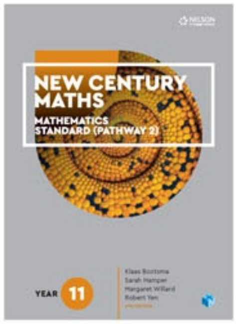

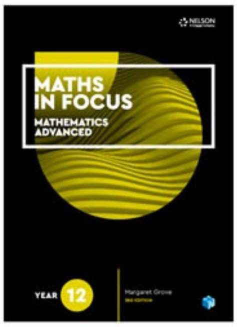

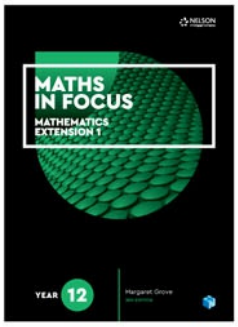

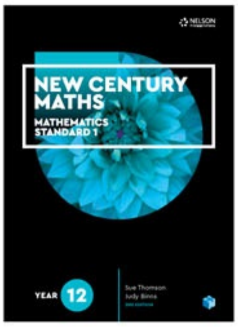

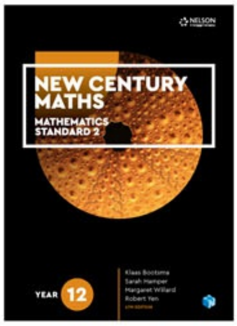

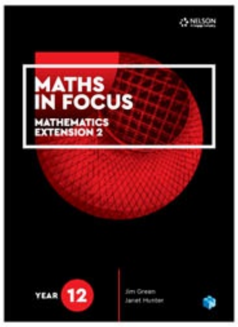

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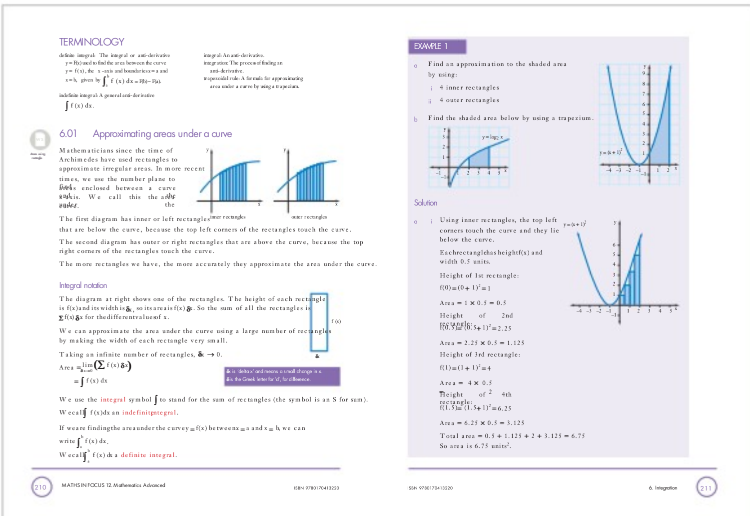
ABOUT THIS BOOK

AT THE BEGINNING OF EACH CHAPTER

- Each chapter begins on a double-page spread showing the Chapter contents and a list of chapter outcomes

- Terminology is a chapter glossary that previews the key words and phrases from within the chapter



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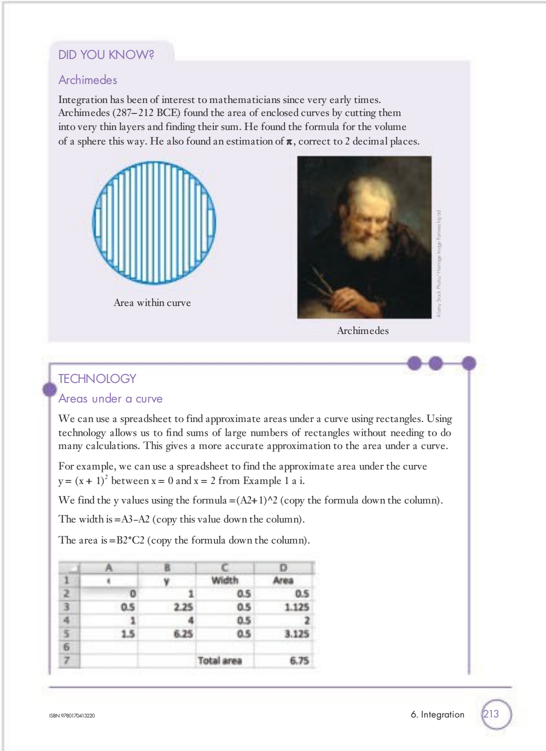
IN EACH CHAPTER

- Important facts and formulas are highlighted in a shaded box.
- Important words and phrases are printed in red and listed in the Terminology chapter glossary.
- Graded exercises include exam-style problems and realistic applications.
- Worked solutions to all exercise questions are provided on the NelsonNet teacher website.
- Investigations explore the syllabus in more detail, providing ideas for modelling activities and assessment tasks.
- Did you know? contains interesting facts and applications of the mathematics learned in the chapter.



DID YOU KNOW?

Archimedes

Integration has been of interest to mathematicians since very early times. Archimedes (287–212 BCE) found the area of enclosed curves by cutting them into very thin layers and finding their sum. He found the formula for the volume of a sphere this way. He also found an estimation of π, correct to 2 decimal places.

Area within curve

Archimedes

TECHNOLOGY

Areas under a curve

We can use a spreadsheet to find approximate areas under a curve using rectangles. Using technology allows us to find sums of large numbers of rectangles without needing to do many calculations. This gives a more accurate approximation to the area under a curve.

For example, we can use a spreadsheet to find the approximate area under the curve $y = (x + 1)^2$ between $x = 0$ and $x = 2$ from Example 1 a i.

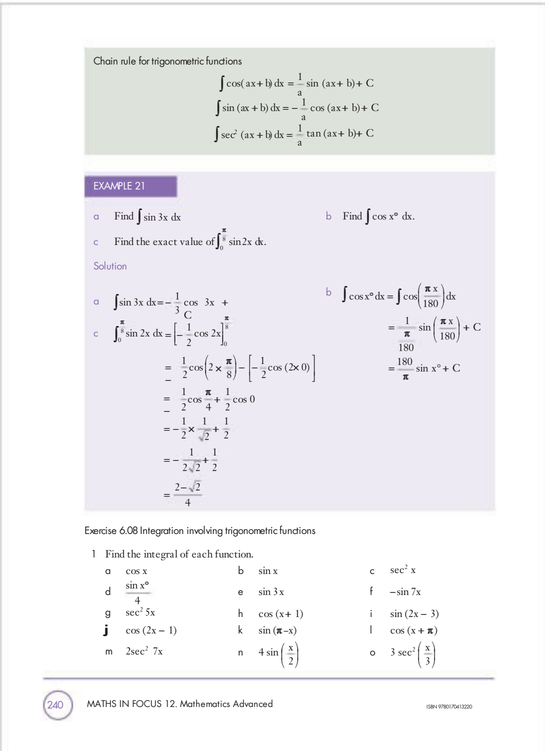
We find the y values using the formula =(A2+1)^2 (copy the formula down the column).

The width is =A3–A2 (copy this value down the column).

The area is =B2*C2 (copy the formula down the column).

	A	B	C	D
1	x	y	Width	Area
2	0	1	0.5	0.5
3	0.5	2.25	0.5	1.125
4	1	4	0.5	2
5	1.5	6.25	0.5	3.125
6				
7			Total area	6.75

ISBN 9780170413220 6. Integration 213



Chain rule for trigonometric functions

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$
$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$
$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

EXAMPLE 21

a Find $\int \sin 3x\,dx$ b Find $\int \cos x^\circ\,dx$.

c Find the exact value of $\int_0^{\frac{\pi}{8}} \sin 2x\,dx$.

Solution

a $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x + C$

b $\int \cos x^\circ\,dx = \int \cos\left(\frac{\pi x}{180}\right)dx = \frac{1}{\frac{\pi}{180}}\sin\left(\frac{\pi x}{180}\right) + C = \frac{180}{\pi}\sin x^\circ + C$

c $\int_0^{\frac{\pi}{8}} \sin 2x\,dx = \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{8}} = -\frac{1}{2}\cos\left(2\times\frac{\pi}{8}\right) - \left[-\frac{1}{2}\cos(2\times 0)\right] = -\frac{1}{2}\cos\frac{\pi}{4} + \frac{1}{2}\cos 0 = -\frac{1}{2}\times\frac{1}{\sqrt{2}} + \frac{1}{2} = -\frac{1}{2\sqrt{2}} + \frac{1}{2} = \frac{2-\sqrt{2}}{4}$

Exercise 6.08 Integration involving trigonometric functions

1 Find the integral of each function.

a $\cos x$ b $\sin x$ c $\sec^2 x$

d $\frac{\sin x^\circ}{4}$ e $\sin 3x$ f $-\sin 7x$

g $\sec^2 5x$ h $\cos(x+1)$ i $\sin(2x-3)$

j $\cos(2x-1)$ k $\sin(\pi - x)$ l $\cos(x+\pi)$

m $2\sec^2 7x$ n $4\sin\left(\frac{x}{2}\right)$ o $3\sec^2\left(\frac{x}{3}\right)$

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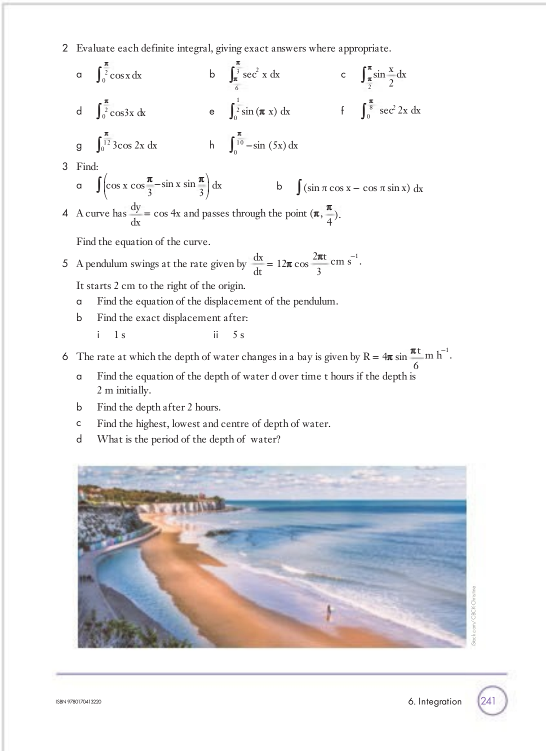

2 Evaluate each definite integral, giving exact answers where appropriate.

a $\int_0^{\frac{\pi}{2}} \cos x\,dx$ b $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x\,dx$ c $\int_{\frac{\pi}{2}}^{\pi} \sin\frac{x}{2}\,dx$

d $\int_0^{\frac{\pi}{2}} \cos 3x\,dx$ e $\int_0^{\frac{1}{2}} \sin(\pi x)\,dx$ f $\int_0^{\frac{\pi}{8}} \sec^2 2x\,dx$

g $\int_0^{\frac{\pi}{12}} 3\cos 2x\,dx$ h $\int_0^{\frac{\pi}{10}} -\sin(5x)\,dx$

3 Find:

a $\int\left(\cos x\cos\frac{\pi}{3} - \sin x\sin\frac{\pi}{3}\right)dx$ b $\int(\sin\pi\cos x - \cos\pi\sin x)\,dx$

4 A curve has $\frac{dy}{dx} = \cos 4x$ and passes through the point $\left(\pi, \frac{\pi}{4}\right)$.

Find the equation of the curve.

5 A pendulum swings at the rate given by $\frac{dx}{dt} = 12\pi\cos\frac{2\pi t}{3}$ cm s^{-1}.

It starts 2 cm to the right of the origin.

a Find the equation of the displacement of the pendulum.

b Find the exact displacement after:

i 1 s ii 5 s

6 The rate at which the depth of water changes in a bay is given by $R = 4\pi\sin\frac{\pi t}{6}$ m h^{-1}.

a Find the equation of the depth of water d over time t hours if the depth is 2 m initially.

b Find the depth after 2 hours.

c Find the highest, lowest and centre of depth of water.

d What is the period of the depth of water?

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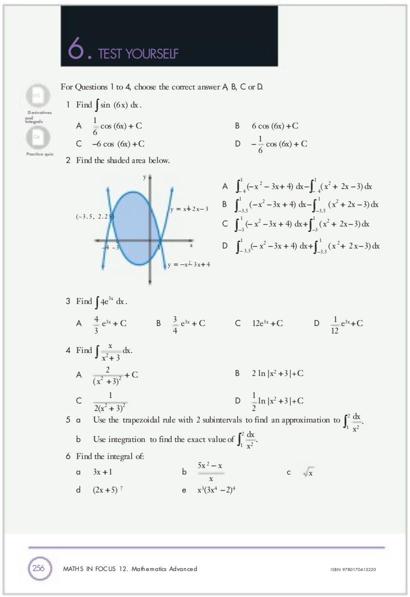
AT THE END OF EACH CHAPTER

- Test Yourself contains chapter revision exercises.
- If you have trouble completing the Test Yourself exercises, you need to go back and revise the chapter before trying the exercises again.
- Challenge Exercise contains chapter extension questions. Attempt these only after you are confident with the Test Yourself exercises, because these are more difficult and are designed for students who understand the topic really well.
- Practice sets (after several chapters) provide a comprehensive variety of mixed exam-style questions from various chapters, including short-answer, free-response and multiple-choice questions.



6. TEST YOURSELF

For Questions 1 to 4, choose the correct answer A, B, C or D.

1 Find $\int \sin(6x)\,dx$.

A $\frac{1}{6}\cos(6x)+C$ B $6\cos(6x)+C$

C $-6\cos(6x)+C$ D $-\frac{1}{6}\cos(6x)+C$

2 Find the shaded area below.

A $\int_{-4}^{1}(-x^2-3x+4)\,dx-\int_{-4}^{1}(x^2+2x-3)\,dx$

B $\int_{-3.5}^{1}(-x^2-3x+4)\,dx-\int_{-3.5}^{1}(x^2+2x-3)\,dx$

C $\int_{-3}^{1}(-x^2-3x+4)\,dx+\int_{-3}^{1}(x^2+2x-3)\,dx$

D $\int_{-3.5}^{1}(-x^2-3x+4)\,dx+\int_{-3.5}^{1}(x^2+2x-3)\,dx$

3 Find $\int 4e^{3x}\,dx$.

A $\frac{4}{3}e^{3x}+C$ B $\frac{3}{4}e^{3x}+C$ C $12e^{3x}+C$ D $\frac{1}{12}e^{3x}+C$

4 Find $\int \frac{x}{x^2+3}\,dx$.

A $\frac{2}{(x^2+3)^2}+C$ B $2\ln|x^2+3|+C$

C $\frac{1}{2(x^2+3)^2}$ D $\frac{1}{2}\ln|x^2+3|+C$

5 a Use the trapezoidal rule with 2 subintervals to find an approximation to $\int_1^2 \frac{dx}{x^2}$.

b Use integration to find the exact value of $\int_1^2 \frac{dx}{x^2}$.

6 Find the integral of:

a $3x+1$ b $\frac{5x^2-x}{x}$ c $\sqrt{x}$

d $(2x+5)^7$ e $x^3(3x^4-2)^4$

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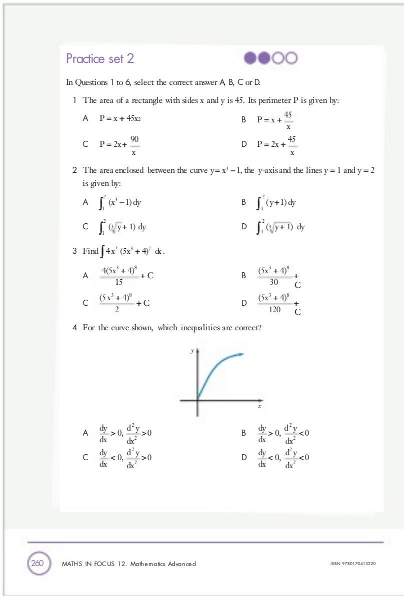
AT THE END OF THE BOOK

- Answers and Index (worked solutions on the teacher website).



Practice set 2

In Questions 1 to 6, select the correct answer A, B, C or D.

1 The area of a rectangle with sides x and y is 45. Its perimeter P is given by:

A $P=x+45x^2$ B $P=x+\frac{45}{x}$

C $P=2x+\frac{90}{x}$ D $P=2x+\frac{45}{x}$

2 The area enclosed between the curve $y=x^3-1$, the y-axis and the lines $y=1$ and $y=2$ is given by:

A $\int_1^2 (x^3-1)\,dy$ B $\int_1^2 (y+1)\,dy$

C $\int_1^2 (\sqrt[3]{y}+1)\,dy$ D $\int_1^2 (\sqrt[3]{y+1})\,dy$

3 Find $\int 4x^2(5x^3+4)^7\,dx$.

A $\frac{4(5x^3+4)^8}{15}+C$ B $\frac{(5x^3+4)^8}{30}+C$

C $\frac{(5x^3+4)^8}{2}+C$ D $\frac{(5x^3+4)^8}{120}+C$

4 For the curve shown, which inequalities are correct?

A $\frac{dy}{dx}>0, \frac{d^2y}{dx^2}>0$ B $\frac{dy}{dx}>0, \frac{d^2y}{dx^2}<0$

C $\frac{dy}{dx}<0, \frac{d^2y}{dx^2}>0$ D $\frac{dy}{dx}<0, \frac{d^2y}{dx^2}<0$

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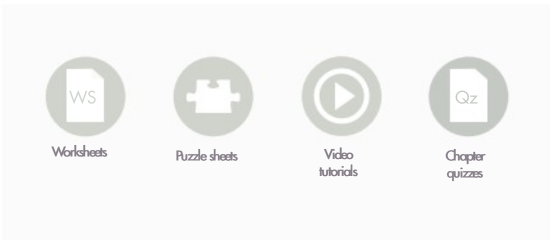
NELSONNET STUDENT WEBSITE

Margin icons link to print (PDF) and multimedia resources found on the NelsonNet student website, www.nelsonnet.com.au. These include:



- Worksheets and puzzle sheets that are write-in enabled PDFs
- Video tutorials: worked examples explained by ‘flipped classroom’ teachers
- ExamView quizzes: interactive and self-marking

NELSONNET TEACHER WEBSITE

The NelsonNet teacher website, also at www.nelsonnet.com.au, contains:

- A teaching program, in Microsoft Word and PDF formats
- Topic tests, in Microsoft Word and PDF formats
- Worked solutions to each exercise set
- Chapter PDFs of the textbook
- ExamView exam-writing software and questionbanks
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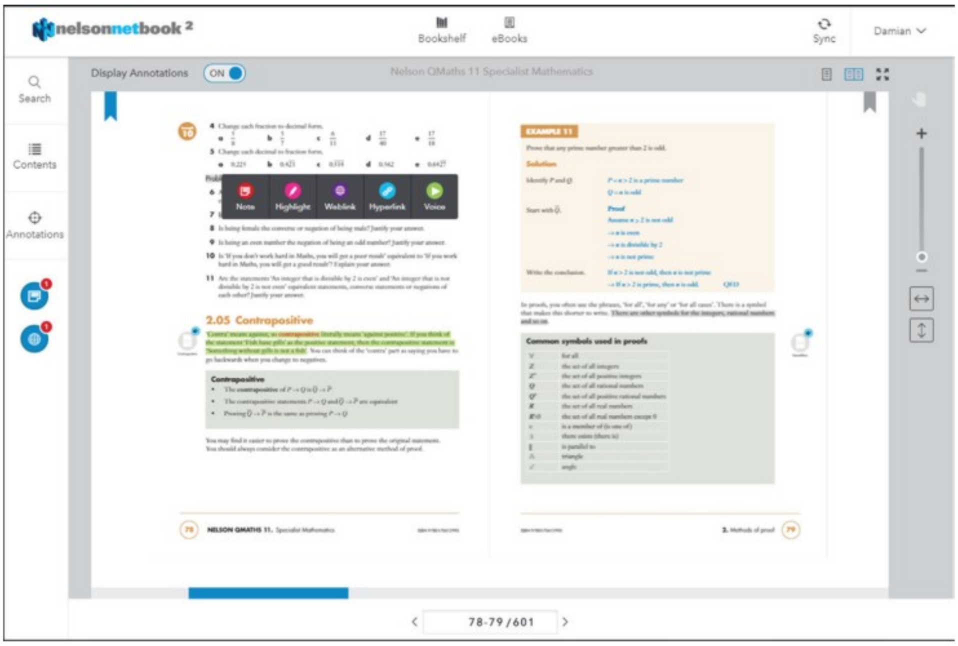
NELSONNETBOOK

NelsonNetBook is the web-based interactive version of this book found on NelsonNet.

- To each page of NelsonNetBook you can add notes, voice and sound bites, highlighting, weblinks and bookmarks
- Zoom and Search functions
- Chapters can be customised for different groups of students.



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STUDY SKILLS

The Year 11 course introduces the basics of topics such as calculus that are then applied in the Year 12 course. You will struggle in the HSC if you don't set yourself up to revise the Year 11 topics as you learn new Year 12 topics. Your teachers will be able to help you build up and manage good study habits. Here are a few hints to get you started. There is no right or wrong way to learn. Different styles of learning suit different people. There is also no magical number of hours a week that you should study, as this will be different for every student. But just listening in class and taking notes is not enough, especially when learning material that is totally new.

If a skill is not practised within the first 24 hours, up to 50% can be forgotten. If it is not practised within 72 hours, up to 85–90% can be forgotten! So it is really important that, whatever your study timetable, new work must be looked at soon after it is presented to you.

With a continual succession of new work to learn and retain, this is a challenge. But the good news is that you don't have to study for hours on end!

IN THE CLASSROOM

In order to remember, first you need to focus on what is being said and done.

According to an ancient proverb:

I hear and I forget
I see and I remember
I do and I understand.

If you chat to friends and just take notes without really paying attention, you aren't giving yourself a chance to remember anything and will have to study harder at home.

If you are unsure of something that the teacher has said, the chances are that others are also not sure. Asking questions and clarifying things will ultimately help you gain better results, especially in a subject like mathematics where much of the knowledge and skills depends on being able to understand the basics.

Learning is all about knowing what you know and what you don't know. Many students feel like they don't know anything, but it's surprising just how much they know already. Picking up the main concepts in class and not worrying too much about other less important parts can really help. The teacher can guide you on this.

Here are some pointers to get the best out of classroom learning:

- Take control and be responsible for your own learning
- Clear your head of other issues in the classroom
- Active, not passive, learning is more memorable
- Ask questions if you don't understand something

- Listen for cues from the teacher
- Look out for what are the main concepts.

Note-taking varies from class to class, but here are some general guidelines:

- Write legibly
- Use different colours to highlight important points or formulas
- Make notes in textbooks (using pencil if you don't own the textbook)
- Use highlighter pens to point out important points
- Summarise the main points
- If notes are scribbled, rewrite them at home.

AT HOME

You are responsible for your own learning and nobody else can tell you how best to study. Some people need more revision time than others, some study better in the mornings while others do better at night, and some can work at home while others prefer a library.

- Revise both new and older topics regularly
- Have a realistic timetable and be flexible
- Summarise the main points
- Revise when you are fresh and energetic
- Divide study time into smaller rather than longer chunks
- Study in a quiet environment
- Have a balanced life and don't forget to have fun!

If you are given exercises out of a textbook to do for homework, consider asking the teacher if you can leave some of them till later and use these for revision. It is not necessary to do every exercise at one sitting, and you learn better if you can spread these over time.

People use different learning styles to help them study. The more variety the better, and you will find some that help you more than others. Some people (around 35%) learn best visually, some (25%) learn best by hearing and others (40%) learn by doing.

- Summarise on cue cards or in a small notebook
- Use colourful posters
- Use mind maps and diagrams
- Discuss work with a group of friends
- Read notes out aloud
- Make up songs and rhymes
- Do exercises regularly
- Role-play teaching someone else

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ASSESSMENT TASKS AND EXAMS

You will cope better in exams if you have practised doing sample exams under exam conditions. Regular revision will give you confidence, and if you feel well prepared this will help get rid of nerves in the exam. You will also cope better if you have had a reasonable night's sleep before the exam.

One of the biggest problems students have with exams is in timing. Make sure you don't spend too much time on questions you're unsure about, but work through and find questions you can do first.

Divide the time up into smaller chunks for each question and allow some extra time to go back to questions you couldn't do or finish. For example, in a 3-hour exam with 50 questions, allow around 3 minutes for each question. This will give an extra half hour at the end to tidy up and finish off questions. Alternatively, in a 3-hour exam with questions worth a total of 100 marks, allow around 1.5 minutes per mark.

- Read through and ensure you know how many questions there are
- Divide your time between questions with extra time at the end
- Don't spend too much time on one question
- Read each question carefully, underlining key words
- Show all working out, including diagrams and formulas
- Cross out mistakes with a single line so it can still be read
- Write legibly

AND FINALLY…

Study involves knowing what you don't know, and putting in a lot of time into concentrating on these areas. This is a positive way to learn. Rather than just saying 'I can't do this', say instead 'I can't do this yet', and use your teachers, friends, textbooks and other ways of finding out.

With the parts of the course that you do know, make sure you can remember these easily under exam pressure by putting in lots of practice.

Remember to look at new work

today, tomorrow, in a week, in a month.

Some people hardly ever find time to study while others give up their outside lives to devote their time to study. The ideal situation is to balance study with other aspects of your life, including going out with friends, working and keeping up with sport and other activities that you enjoy.

Good luck wit*h* your studies!

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MATHEMATICAL VERBS

A glossary of 'doing words' commonly found in mathematics problems

analyse: study in detail the parts of a situation

apply: use knowledge or a procedure in a given situation

classify, identify: state the type, name or feature of an item or situation

comment: express an observation or opinion about a result

compare: show how two or more things are similar or different

construct: draw an accurate diagram

describe: state the features of a situation

estimate: make an educated guess for a number, measurement or solution, to find roughly or approximately

evaluate, calculate: find the value of a numerical expression, for example, 3×8^2 or $4x + 1$ when $x = 5$

expand: remove brackets in an algebraic expression, for example, expanding $3(2y + 1)$ gives $6y + 3$

explain: describe why or how

factorise: opposite to expand, to insert brackets by taking out a common factor, for example, factorising $6y + 3$ gives $3(2y + 1)$

give reasons: show the rules or thinking used when solving a problem. See also justify

increase: make larger

interpret: find meaning in a mathematical result

justify: give reasons or evidence to support your argument or conclusion. See also give reasons

rationalise: make rational, remove surds

show that, prove: (in questions where the answer is given) use calculation, procedure or reasoning to prove that an answer or result is true

simplify: give a result in its most basic, shortest, neatest form, for example, simplifying a ratio or algebraic expression

sketch: draw a rough diagram that shows the general shape or ideas, less accurate than construct

solve: find the value(s) of an unknown pronumeral in an equation or inequality

substitute: replace a variable by a number and evaluate

verify: check that a solution or result is correct, usually by substituting back into the equation or referring back to the problem

write, state: give the answer, formula or result without showing any working or explanation (This usually means that the answer can be found mentally, or in one step)

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