## 1. Theory of Probability Exam 1

**Question 1** (1.1)**.** Let $\{f_n\}$ be a sequence of measurable functions on the measure space $(S, \mathcal{S}, \mu)$ where $\mu$ is finite. Consider the following statements: (1) Every subsequence $\{f_{n_k}\}$ has a further subsequence which converges a.e. to 0, (2) $f_n \to 0$, in measure. Show $1 \implies 2$ $(2 \implies 1)$.

*Proof.* (a)
Suppose $f_n \not\to 0$ in measure. This means that, for some choice of $\epsilon, \delta > 0$, $\mu(|f_n - f| \geq \epsilon) > \delta$ for infinitely many $n$. Construct a sequence $\{n_k\}$ of values of $n$ for which the above inequality holds. Consider $\{f_{n_k}\}$. No subsequences of this sequence converge in measure, so no subsequences of this sequence converge a.e. (since a.e. convergence implies convergence in measure). □

**Question 2** (1.2)**.** Let $f \in \mathcal{L}^0_+(S, \mathcal{S}, \mu)$, where $\mu$ is a finite measure, be such that $\lim sup_{t\to\infty} t^{p_0} \mu(f > t) < \infty$, where $p_0 > 1$. Show that $f \in \mathcal{L}^p(S, \mathcal{S}, \mu)$ for all $p \in [1, p_0)$.

*Proof.* From our assumption, there exists some $C$, $N$, such that $\mu(f > t)$
$leqCt^{-p_0}$, for all $t > N$.
$\int_S f^p d$
$mu = \int_0^{inftyμ(f>t)t^p dt \leq \mu(S)^2 N^p + int_N^\infty \mu(f>t)t^p dt \leq C_1 + C\int_N^\infty t^{p-p_0} dt}$. This is certainly finite whenever $p - p_0 > 1$.*** □