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The Left Distributive Law and the Freeness of an Algebra of Elementary Embeddings

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The left distributive law for a single binary operation is the law $a(bc)=(ab)(ac)$. It has been studied in universal algebra (Stein [S], Kepka and Nemec [KN], Kepka [K1, K2]; see also Jezek *et al.* [JKN] for a bibliography on the *two-sided* distributive law) and it has been studied more recently by set theorists because of its connection with elementary embeddings. For λ a limit ordinal let $\mathscr{E}_\lambda$ be the collection of all $j\colon V_\lambda \to V_\lambda$, j an elementary embedding of $(V_\lambda, \in)$ into itself, j not the identity. Then the existence of a λ such that $\mathscr{E}_\lambda \neq \varnothing$ is a large cardinal axiom (see Gaifman [G] and Kanamori *et al.* [KRS]). For $j \in \mathscr{E}_\lambda$ let $\kappa_0 = \mathrm{cr}(j)$, the critical point of j, and $\kappa_{n+1} = j(\kappa_n)$. Then λ must equal $\sup\{\kappa_n : n<\omega\}$ (Kunen [Ku]). Martin [M1], Woodin [M2], Martin and Steel [MS1, MS2], and Woodin [W], in deep work, showed that slight strengthenings of this axiom imply determinacy properties of the real line, and subsequently that considerable weakenings suffice, making the axiom unnecessary for those determinacy properties.

There is a natural operation $\cdot$ on $\mathscr{E}_\lambda$ (write uv for $u \cdot v$ in this and similar contexts below). For $j \in \mathscr{E}_\lambda$, j extends to a map $j\colon V_{\lambda+1} \to V_{\lambda+1}$ by defining, for $A \subseteq V_\lambda$, $j(A) = \bigcup_{\alpha<\lambda} j(A \cap V_\alpha)$. Then j may or may not be an elementary embedding of $(V_{\lambda+1}, \in)$ into itself, but at least j is elementary from $(V_\lambda, \in, A)$ into $(V_\lambda, \in, jA)$. In the special case that A, as a set of ordered pairs, is a $k \in \mathscr{E}_\lambda$, we have that $j(k) \in \mathscr{E}_\lambda$. Let $j \cdot k = j(k)$. Then the operation $\cdot$ on $\mathscr{E}_\lambda$ is nonassociative, noncommutative, and left distributive.

Another operation on $\mathscr{E}_\lambda$ is composition: if $k, l \in \mathscr{E}_\lambda$, then $k \circ l \in \mathscr{E}_\lambda$. Let Σ be the set of laws $a\circ(b\circ c) = (a\circ b)\circ c$, $(a\circ b)c = a(bc)$, $a(b\circ c) = ab\circ ac$, $a\circ b = ab\circ a$. Then $\mathscr{E}_\lambda$ satisfies Σ, and Σ implies the left distributive law $(a(bc) = (a\circ b)c = (ab\circ a)c = ab(ac))$.

For $j \in \mathscr{E}_\lambda$, let $\mathscr{A}_j$ be the closure of $\{j\}$ under $\cdot$. Let $\mathscr{P}_j$, the set of "polynomials in j," be the closure of $\{j\}$ under $\cdot$ and $\circ$. Results in which $\mathscr{A}_j$, $\mathscr{P}_j$ and their governing equations are involved appear in [M1, M2, L], Dougherty

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[Do], and Dehornoy [De1–4]. A natural question, noticed independently by a number of people, is whether $\mathscr{A}_j$ and $\mathscr{P}_j$ are the one-generated free algebras $\mathscr{A}$ and $\mathscr{P}$ subject to, respectively, the left distributive law and to Σ. In this paper a normal form theorem (Theorem 28(1)) for the free algebras is proved, from which the freeness of $\mathscr{A}_j$ and $\mathscr{P}_j$ can be derived.

$\mathscr{A}$ and $\mathscr{P}$ have the usual representations as sets of words under an equivalence relation. That is, let A (respectively P) be the set of terms in one letter x in the language of $\cdot$ (respectively, in the language of $\cdot$ and $\circ$). For example, $((xx)x)(x \circ x(xx)) \in P$. We first make the definitions for the case of P. For z, $w \in P$, z is a *component* of w if $z = w$ or $w = uv$ or $u \circ v$ with z a component of u or of v. Say $z \equiv w$ if there is a sequence $z = z_0, z_1, ..., z_n = w$ with z_{i+1} obtained from z_i by substituting, for some component of z_i, a word equivalent to it by one of the laws of Σ. Then $\mathscr{P} = P/\equiv$.

If $u_0, u_1, ..., u_n \in P$, then by convention $u_0 u_1 u_2 \cdots u_{n-1} u_n$ and $u_0 u_1 u_2 \cdots u_{n-1} \circ u_n$ are, respectively, $((((u_0 u_1) u_2) \cdots) u_{n-1}) u_n$ and $((((u_0 u_1) u_2) u_2) \cdots) u_{n-1}) \circ u_n$. Make this convention also for elementary embeddings. For u, $v \in P$, write $u < v$ if for some $v' \equiv v$, u is a proper component of v'. Write $u <_L v$ (u is a left component of v) if $v \equiv u a_0 a_1 \cdots a_n$ or $v \equiv u a_0 a_1 \cdots a_{n-1} \circ a_n$ for some $a_0, ..., a_n \in P$, $n \geqslant 0$. Then $<$, $<_L$ are preserved under $\equiv$ so they are relations on $\mathscr{P}$, and it is not hard to see that $<$, $<_L$ are transitive and that $x < w$, $x <_L w$ for all $w \in P$, $w \neq x$.

Make the same definitions and remarks for $\mathscr{A}$, where the left distributive law is used instead of Σ. The justification for using the same notation $\equiv$, $<$, $<_L$ for the analogous relations over $\mathscr{A}$ is Lemma 3: if u, $v \in A$ and $u \equiv v$ ($u < v$, $u <_L v$) in the sense of $\mathscr{P}$, then $u \equiv v$ ($u < v$, $u <_L v$) in the sense of $\mathscr{A}$. Also, it is easy to check from Lemma 3 that $<_L$ is irreflexive on $\mathscr{A}$ if and only if $<_L$ is irreflexive on $\mathscr{P}$ (if $w <_L w$ in $\mathscr{P}$, then wx is equivalent in $\mathscr{P}$ to some $a \in A$, and $a <_L a$ in $\mathscr{A}$). A statement about $\equiv$, $<$, $<_L$ will be, unless otherwise indicated, an assertion that both the $\mathscr{P}$ and $\mathscr{A}$ versions hold.

THEOREM A. *For no $k_0, k_1, ..., k_n \in \mathscr{E}_\lambda$ is $k_0 = k_0 k_1 \cdots k_n$ or $k_0 = k_0 k_1 \cdots k_{n-1} \circ k_n$.*

COROLLARY. *If for some λ, $\mathscr{E}_\lambda \neq \varnothing$, then $<_L$ is irreflexive.*

Proof. Suppose $w = w a_0 a_1 \cdots a_n$ or $w \equiv w a_0 a_1 \cdots a_{n-1} \circ a_n$. Applying the homomorphism of $\mathscr{P}$ into $\mathscr{P}_j$ induced by $[x] \to j$ yields embeddings contradicting Theorem A.

The irreflexivity of $<_L$ can be proved from the weaker assumption that for every n there is an n-huge cardinal; it is unknown whether it is provable from ZFC.

For $w \in P$ let $w^{(0)} = w$ and $w^{(i+1)} = w^{(i)}w^{(i)}$. Then $i < m$ implies $w^{(i)} <_{\mathrm{L}} w^{(m)}$, in fact by induction using the left distibutive law, $i \leqslant n$ implies $w^{(i)}w^{(n)} \equiv w^{(n+1)}$. The main point of the normal form theorem is:

THEOREM B. *Assume $<_{\mathrm{L}}$ is irreflexive. Then for each $p \in \mathscr{P}$, p is expressible uniquely in the form $a_0 a_1 \cdots a_n$ or $a_0 a_1 \cdots a_{n-1} \circ a_n$, where each $a_m \in \mathscr{P}$, a_0 is an $x^{(i)}$, $a_1 <_{\mathrm{L}} a_0$, and for $2 \leqslant m \leqslant n$, $a_m \leqslant_{\mathrm{L}} a_0 a_1 \cdots a_{m-2}$ (and in the case $p = a_0 a_1 \cdots a_{n-1} \circ a_n$, $a_n <_{\mathrm{L}} a_0 a_1 \cdots a_{n-2}$).*

The normal form theorem is the only result about $\mathscr{P}$ in this paper which does not have an analog for $\mathscr{A}$; the word $x^{(2)}x^{(1)}(x^{(2)}xx)$, for instance, is in A but its normal form equivalent $x^{(2)}(x^{(1)} \circ x)x$ is not in A. The set of words in normal form has a lexicographical ordering which coincides with $<_{\mathrm{L}}$, and one can obtain

COROLLARY. *Assume $<_{\mathrm{L}}$ is irreflexive. Then for $\mathscr{P}$ and $\mathscr{A}$*

(1) $< = <_{\mathrm{L}}$.
(2) *$<$ is a linear ordering.*
(3) *The word problems are decidable.*

COROLLARY. *For each $j \in \mathscr{E}_\lambda$, $\mathscr{A}_j \cong \mathscr{A}$ and $\mathscr{P}_j \cong \mathscr{P}$.*

Proof. If $j \in \mathscr{E}_\lambda$, then $<_{\mathrm{L}}$ is irreflexive by the corollary to Theorem A, so $<_{\mathrm{L}}$ is a linear ordering by the corollary to Theorem B. If $w \in P$, let $w[j]$ be the homomorphic image of w in $\mathscr{A}_j$ obtained from the map $[x] \to j$. If, say, $\mathscr{A}_j$ were not free, there would be w, $u \in A$, $w \not\equiv u$ but $w[j] = u[j]$. Since $<_{\mathrm{L}}$ is a linear ordering, we would have, say, $w <_{\mathrm{L}} u$, whence $u \equiv ww_0 \cdots w_n$; thus $u[j] = w[j]\ w_0[j] \cdots w_n[j] = u[j]\ w_0[j] \cdots w_n[j]$, contradicting Theorem A. Similarly $\mathscr{P}_j$ is free.

After I proved these results in spring 1989, P. Dehornoy told me what he had obtained by his different approach to these problems. He proves in ZFC that for any u, $v \in A$ at least one of $u <_{\mathrm{L}} v$, $u \equiv v$, $v <_{\mathrm{L}} u$ holds. His method, combined with Theorem A of this paper, affords an alternative proof of the linearity of $<_{\mathrm{L}}$ on $\mathscr{A}$ (and its corollary the decidability of the word problem for $\mathscr{A}$, and (with Theorem A) the freeness of $\mathscr{A}_j$, and the analogous results for $\mathscr{P}$ and $\mathscr{P}_j$ via Section 1). His result is shorter than Section 3 of this paper, at the expense of not getting normal forms. See [De 3, 4] for this interesting method.

The organization of the sections is preliminaries in Section 1, elementary embeddings and Theorem A in Section 2, and the normal form theorem in Section 3.

1

For $p \in P$ write $p = b_0 b_1 \cdots b_{n-1} * b_n$ to mean that either $p = b_0 b_1 \cdots b_{n-1} b_n$ or $p = b_0 b_1 \cdots b_{n-1} \circ b_n$.

LEMMA 1. (1) $a(b_0 b_1 \cdots b_{n-1} * b_n) \equiv (ab_0)(ab_1) \cdots (ab_{n-1}) * (ab_n)$.

(2) *$ac <_{\mathrm{L}} a \circ c$ and if $c <_{\mathrm{L}} b$, $a \circ c <_{\mathrm{L}} ab$.*

Proof. (1) is clear. For (2) $ac <_{\mathrm{L}} ac \circ a \equiv a \circ c$, and if $b \equiv cu_0 u_1 \cdots u_{n-1} * u_n$, then $ab \equiv (a \circ c)\, u_0 (au_1) \cdots (au_{n-1}) * (au_n) >_{\mathrm{L}} a \circ c$.

Say $p \in P$ is an *explicit composition* if $p = a \circ b$ for some a, b; p is a *composition* if $p \equiv a \circ b$ for some a, b. Let EC and C be the sets of explicit compositions and compositions, respectively. There are natural functions H and K which assign to each $w \in P$ an equivalent composition of members of $P - C$, respectively, an equivalent composition of members of A. Define

$$H(a_0(a_1(\cdots(a_{n-1}x)))) = (a_0(a_1(\cdots(a_{n-1}x))),$$

and

$$H(a_0(a_1(\cdots(a_{n-1}(a \circ b)))))$$
$$= H(a_0(a_1(\cdots(a_{n-1}a)))) \circ H(a_0(a_1(\cdots(a_{n-1}b)))).$$

Let $\hat{P}$ be the set of members of P of the form $(((w_0 \circ w_1) \circ w_2) \circ \cdots) \circ w_n$, each $w_i \in A$. For w, $u \in \hat{P}$, $w = ((w_0 \circ w_1) \circ \cdots) \circ w_n$, $u = ((u_0 \circ u_1) \circ \cdots) \circ u_m$ define $w \mathbin{\hat{\circ}} u = (((((w_0 \circ w_1) \circ \cdots) \circ w_n) \circ u_0) \circ \cdots) \circ u_m$, and $w \mathbin{\hat{\cdot}} u = (((w_0(w_1(\cdots(w_n u_0)))) \circ w_0(w_1(\cdots(w_n u_1)))) \circ \cdots) \circ w_0(w_1(\cdots(w_n u_m)))$, elements of $\hat{P}$ equivalent to $w \circ u$ and wu, respectively. Define $K\colon P \to \hat{P}$ by $K(x) = x$, $K(u \circ v) = K(u) \mathbin{\hat{\circ}} K(v)$, $K(uv) = K(u) \mathbin{\hat{\cdot}} K(v)$. (Simultaneously with the definition of K one verifies part (3) of the following lemma.)

LEMMA 2. (1) *$w \in C$ if and only if w is of the form $a_0(a_1(\cdots(a_{n-1}(b \circ c))))$; $w \notin C$ if and only if w is of the form $a_0(a_1(\cdots(a_{n-1}x)))$.*

(2) *For $w \in P$, $H(w) \equiv w$.*

(3) *For $w \in P$, $K(w) \equiv w$; if $w \in A$, then $K(w) = w$; if $w \notin C$, then $K(w) \in A$.*

Proof. For (1), each of those forms is preserved under substitution using Σ; (2) and (3) are clear by induction.

To prove that $\equiv$, $<$, $<_{\mathrm{L}}$ on $\mathscr{A}$ are the relations inherited from those on $\mathscr{P}$, let them for a moment be denoted by $\equiv'$, $<'$, $<'_{\mathrm{L}}$.

LEMMA 3. $\equiv' = \equiv \restriction A$, $<' = < \restriction A$, $<'_{\mathrm{L}} = <_{\mathrm{L}} \restriction A$.

Proof that $\equiv' = \equiv \restriction A$. For u, $v \in \hat{P}$, $u = u_0 \hat{\circ} \cdots \hat{\circ} u_n$, $v = v_0 \hat{\circ} \cdots \hat{\circ} v_m$, write $u \mathrel{\hat{\equiv}} v$ if $u_0(u_1(\cdots(u_n x))) \equiv' v_0(v_1(\cdots(v_m x)))$.

LEMMA 3.1. (1) *If* $a, b \in A$, *then* $a \mathrel{\hat{\equiv}} b \Leftrightarrow a \equiv' b$.

(2) *If* w, w', v, $v' \in \hat{P}$ *and* $w \mathrel{\hat{\equiv}} w'$, $v \mathrel{\hat{\equiv}} v'$, *then* $w \mathbin{\hat{\circ}} v \mathrel{\hat{\equiv}} w' \mathbin{\hat{\circ}} v'$ *and* $w \mathbin{\hat{\cdot}} v \mathrel{\hat{\equiv}} w' \mathbin{\hat{\cdot}} v'$.

(3) *For* a, b, $c \in \hat{P}$

(i) $a \mathbin{\hat{\circ}} (b \mathbin{\hat{\circ}} c) = (a \mathbin{\hat{\circ}} b) \mathbin{\hat{\circ}} c$.

(ii) $(a \mathbin{\hat{\circ}} b) \mathbin{\hat{\cdot}} c = a \mathbin{\hat{\cdot}} (b \mathbin{\hat{\cdot}} c)$

(iii) $a \mathbin{\hat{\cdot}} (b \mathbin{\hat{\circ}} c) \mathrel{\hat{\equiv}} (a \mathbin{\hat{\cdot}} b) \mathbin{\hat{\circ}} (a \mathbin{\hat{\cdot}} c)$

(iv) $a \mathbin{\hat{\circ}} b \mathrel{\hat{\equiv}} (a \mathbin{\hat{\cdot}} b) \mathbin{\hat{\circ}} a$.

Proof. Left to the reader. For the second part of (2), it is seen that if $u_0, u_1, ..., u_n, v_0, v_1, ..., v_r \in A$ and $u_0(u_1(\cdots(u_n x))) \equiv' v_0(v_1(\cdots(v_r x)))$, then for any $w \in \hat{P}$ $(w \mathbin{\hat{\cdot}} u_0)((w \mathbin{\hat{\cdot}} u_1)(\cdots((w \mathbin{\hat{\cdot}} u_n)x))) \equiv' (w \mathbin{\hat{\cdot}} v_1)(\cdots((w \mathbin{\hat{\cdot}} v_r)x)))$.

LEMMA 3.2. *For* w, $v \in P$, $w \equiv v$ *if and only if* $K(w) \mathrel{\hat{\equiv}} K(v)$.

Proof. ($\Rightarrow$) If u' comes from $u \in P$ by replacement of a component by a word equivalent to it by one of the laws of Σ, then $K(u')$ is similarly obtained from $K(u)$ by the corresponding law in Lemma 3.1(3). Then $K(u) \mathrel{\hat{\equiv}} K(u')$ follows by induction on the length of u, using Lemma 3.1(2).

($\Leftarrow$) Since $u \equiv K(u)$ for all $u \in P$, it suffices to show that if w_0, $w_1, ..., w_n$, v_0, $v_1, ..., v_r \in A$ and $((w_0 \circ w_1) \circ \cdots) \circ w_n \equiv' ((v_0 \circ v_1) \circ \cdots) \circ v_r$, then $((w_0 \circ w_1) \circ \cdots) \circ w_n \equiv ((v_0 \circ v_1) \circ \cdots) \circ v_r$ (which follows by induction on the derivation witnessing $w_0(w_1(\cdots(w_n x))) \equiv' v_0(v_1(\cdots(v_r x)))$).

Thus $\equiv' = \equiv \restriction A$, that is, if w, $u \in A$ and $w \equiv u$, then $K(w) \mathrel{\hat{\equiv}} K(u)$, but $K(w) = w$ and $K(u) = u$, whence $w \equiv' u$ by Lemma 3.1(1).

For $<'_{\mathrm{L}} = <_{\mathrm{L}} \restriction A$, if u, $v \in A$ and $u <_{\mathrm{L}} v$, then for some a_0, $a_1, ..., a_n$, $v \equiv u a_0 a_1 \cdots a_n \equiv K(u a_0 a_1 \cdots a_n) = u b_0 \cdots b_m \in A$ for some $b_0, ..., b_m$, from the definition of K and the fact that $v \notin C$. Then $v \equiv' u b_0 \cdots b_m$, as desired. The proof that $<' = < \restriction A$ is similar.

2

We assume familiarity with the basics of elementary embeddings. For $k \in \mathscr{E}_\lambda$, $\theta < \lambda$, let $\log_k(\theta)$ be the least δ with $k(\delta) \geqslant \theta$. Let $k \overset{*}{\cap} V_\theta$ code up k's action restricted to V_θ, i.e., $k \overset{*}{\cap} V_\theta = \{\langle x, y\rangle : x \in V_\theta,\ y \in V_\theta,\ y \in k(x)\}$. Let $k =^\theta l$ mean that $k \overset{*}{\cap} V_\theta = l \overset{*}{\cap} V_\theta$. The following may be easily checked.

LEMMA 4. *Let* $\mu = \log_k \theta$.

(i) $k =^{\theta} l \Rightarrow \mu = \log_l \theta$.

(ii) $k \upharpoonright V_{\mu+1} = l \upharpoonright V_{\mu+1} \Rightarrow k =^{\theta} l \Rightarrow k \upharpoonright V_{\mu} = l \upharpoonright V_{\mu}$.

(iii) $kl \mathbin{\overset{*}{\cap}} V_{\theta} = k(l \mathbin{\overset{*}{\cap}} V_{\mu}) \cap V_{\theta}$.

(iv) $k =^{\theta} k'$ *and* $l =^{\mu} l' \Rightarrow kl =^{\theta} k'l'$.

(v) $k =^{\theta} k'$ *and* $l =^{\theta} l' \Rightarrow k \circ l =^{\theta} k' \circ l'$.

LEMMA 5. *Suppose* $k, l_0, l_1, ..., l_n \in \mathscr{E}_{\lambda}$. *Let* $\operatorname{cr} k = \gamma$ *and* $\theta_n = \inf\{kl_0 \cdots l_i(\gamma) : i < n\}$ *(say* $\theta_n = \lambda$ *if* $n = 0$*). Then* $kl_0 \cdots l_n =^{\theta_n} k(l_0 \cdots l_n)$.

Proof. By induction on n; $n = 0$ is by definition, suppose it holds for $n-1$. Since $\operatorname{cr} k = \gamma$, $l_n =^{\gamma} kl_n$. Applying $kl_0 \cdots l_{n-1}$ to both sides, and using the induction hypothesis and Lemma 4(iv), $kl_0 \cdots l_{n-1} l_n =^{kl_0 \cdots l_{n-1}(\gamma)} kl_0 \cdots l_{n-1}(kl_n) =^{\theta_{n-1}} k(l_0 \cdots l_{n-1})(kl_n) = k(l_0 \cdots l_{n-1} l_n)$. Since $\theta_n = \inf\{\theta_{n-1}, kl_0 \cdots l_{n-1}(\gamma)\}$, we are done.

Note that $\theta_n = \inf\{k(l_0 \cdots l_i)(\gamma) : i < n\}$. Namely, suppose it is already true for $n-1$. If either of $kl_0 \cdots l_{n-1}(\gamma)$, $k(l_0 \cdots l_{n-1})(\gamma)$ is less than θ_{n-1}, then $kl_0 \cdots l_{n-1}(\gamma) = k(l_0 \cdots l_{n-1})(\gamma) = \theta_n$.

LEMMA 6. *Suppose* $k, l_0, l_1, ..., l_n \in \mathscr{E}_{\lambda}$, $\operatorname{cr} k = \gamma$, *and* $\operatorname{cr} kl_0$, $\operatorname{cr} kl_0 l_1, ...,$ $\operatorname{cr} kl_0 l_1 \cdots l_{n-1} < \gamma$. *Then* $\operatorname{cr} kl_0 l_1 \cdots l_n \neq \gamma$.

Proof. Since each $\operatorname{cr} kl_0 \cdots l_i < \gamma$ $(i < n)$, each $\theta_i > \gamma$ $(i \leqslant n)$. Given $\operatorname{cr} kl_0 l_1 \cdots l_n = kl_0 \cdots l_{n-1}$ $(\operatorname{cr} l_n)$. If this ordinal were γ, then $\operatorname{cr} l_n < \gamma$; then from $\theta_{n-1} > \gamma$, $\gamma = kl_0 \cdots l_{n-1}(\operatorname{cr} l_n) = k(l_0 \cdots l_{n-1})(k(\operatorname{cr} l_n)) \in \operatorname{range} k$, a contradiction.

THEOREM 7. (i) *If* $k_n \in \mathscr{E}_{\lambda}$ $(n < \omega)$, *then* $\{\operatorname{cr}(k_0 k_1 \cdots k_n) : n < \omega\}$ *is infinite.*

(ii) *For* $k_0 k_1 \cdots k_n \in \mathscr{E}_{\lambda}$ $(n > 0)$, $k_0 \neq k_0 k_1 \cdots k_n$ *and* $k_0 \neq k_0 k_1 \cdots k_{n-1} \circ k_n$.

Proof. (i) If $\{\operatorname{cr}(k_0 k_1 \cdots k_n) : n < \omega\}$ were finite, there would be $M < N$ with $\operatorname{cr}(k_0 k_1 \cdots k_M) = \operatorname{cr}(k_0 k_1 \cdots k_N) = \gamma$ and $\operatorname{cr}(k_0 k_1 \cdots k_i) < \gamma$ for all i, $M < i < N$, contradicting Lemma 6.

(ii) If $k_0 = k_0 k_1 \cdots k_n$, then the sequence $\langle k_0, k_1, ..., k_n, k_1, ..., k_n, ... \rangle$ would contradict (i). If $n > 1$ and $k_0 = k_0 k_1 \cdots k_{n-1} \circ k_n$, then $k_0 k_1 = k_0 k_1 \cdots k_{n-1}(k_n k_1)$, a contradiction. And $k_0 \neq k_0 \circ k_1$ is clear (it also follows as above by setting $k_0 \circ k_1 = k_0 k_1 \circ k_0$).

To obtain the corollary that $<_{\mathrm{L}}$ is irreflexive from just the assumption that for every n there is an n-huge cardinal, assume that, say,

$w \equiv wa_0 a_1 \cdots a_m$ and canonically pick an n large enough such that if $l: V_\alpha \to V_\beta$ is an elementary embedding, $\operatorname{cr} l = \kappa$, and $l^n(\kappa) < \alpha$, then for every component u of $wa_0 a_1 \cdots a_m$, the object $u[l] \overset{*}{\cap} V_{l^n(\kappa)}$ has critical point less than $l^{n-1}(\kappa)$. Then working with these restrictions as in Theorem 7 yields a contradiction.

The results in this section were first noted around the time of [L] in trying to understand the proliferation of critical points of members of $\mathscr{A}_j$. See [L, Do] for further results about the structure of $\mathscr{E}_\lambda$ and its members.

3

We work in $\mathscr{P}$ throughout.

For $a, b \in P$, the *finite iterates* of $\langle a, b \rangle$ are the words a, ab, aba, $aba(ab)$, $aba(ab)(aba)$, ... If w_n is the nth member of this sequence ($w_1 = a$, $w_2 = ab$, $w_{m+2} = w_{m+1} w_m$), then $a \circ b \equiv w_{n+1} \circ w_n$ for all n, by iterating the law $u \circ v = uv \circ u$. So each finite iterate of $\langle a, b \rangle$ is $<_{\mathrm{L}} a \circ b$.

The next lemma defines simultaneously a set NF of words in "normal form" and a lexicographical linear ordering $<_{\mathrm{Lex}}$ on NF ($<_{\mathrm{Lex}}$ will shortly be seen to coincide with $<_{\mathrm{L}}$ on NF words, and ultimately with $<$). Write $w \leqslant_{\mathrm{Lex}} v$ for $w <_{\mathrm{Lex}} v$ or $w = v$. An *atomic* NF-word is an $x^{(i)}$ ($i < \omega$).

The first condition for a word v to be in normal form is that v is of the form $a_0 a_1 \cdots a_{n-1} * a_n$ where a_0 is an $x^{(i)}$ and if $n > 0$, $a_1 \neq a_0$. Note that not every $v \in P$ can be parsed this way (for example, words of the form $(a \circ b)c$ or $(a \circ b) \circ c$ cannot be). But such a parsing of v, if it exists, is unique. That is, let $v = c_n * d_n$, $c_n = c_{n-1} d_{n-1}$, $c_{n-1} = c_{n-2} d_{n-2}$, and so on to $c_0 = x$. Then for some unique i, $v = x^{(0)} x^{(0)} x^{(1)} \cdots x^{(i-1)} (= x^{(i)})$, or v is of the form $x^{(i)} * d_{i+1}$ or $x^{(i)} d_{i+1} \cdots d_{n-1} * d_n$, with $d_{i+1} \neq x^{(i)}$. Call $a_0 a_1 \cdots a_{n-1} * a_n$ as above the *normal form parsing* of v. If $x^{(i)} a_1 a_2 \cdots a_{n-1} * a_n$ is the NF parsing of v, define i to be the *rank* of v. In the sequel, a phrase such as "$v = b_0 b_1 \cdots b_{n-1} * b_n$ is in NF" will not presume that $b_0 b_1 \cdots b_{n-1} * b_n$ is the NF parsing of v, but it will be presumed that $\operatorname{rank} b_0 = \operatorname{rank} v$, i.e., the NF parsing of b_0 is $x^{(i)} a_0 a_1 \cdots a_{k-1}$ and the NF parsing of v is $x^{(i)} a_0 a_1 \cdots a_{k-1} b_1 \cdots b_{n-1} * b_n$. However, a phrase such as "$w = x^{(i)} a_1 \cdots a_{n-1} * a_n$ is in NF" will always presume $x^{(i)} a_1 \cdots a_{n-1} * a_n$ is the NF parsing of w.

LEMMA 8. *There is a unique set $NF \subset P$ and a linear order $<_{\mathrm{Lex}}$ on NF such that*

(i) *Each $w \in NF$ has a NF parsing $w = a_0 a_1 \cdots a_{n-1} * a_n$ as above with each $a_m \in NF$. In particular, each atomic NF-word $x^{(i)}$ is in NF; $x^{(i)} <_{\mathrm{Lex}} x^{(m)}$ iff $i < m$.*

(ii) $w \in NF \Leftrightarrow$ *letting* $w = a_0 a_1 \cdots a_{n-1} * a_n$ *be the normal form parsing of* w, *then each* $a_m \in NF$, a_0 *is an* $x^{(i)}$, $a_1 <_{\mathrm{Lex}} a_0$, *and for* $2 \leqslant m \leqslant n$, $a_m \leqslant_{\mathrm{Lex}} a_0 a_1 \cdots a_{m-2}$, *and in case* $* = \circ$ *and* $n > 1$, $a_n <_{\mathrm{Lex}} a_0 a_1 \cdots a_{n-2}$. *(So in the case* $n = 1$, $x^{(i)} a_1 \in NF$ *iff* $x^{(i)} \circ a_1 \in NF$ *iff* $a_1 \in NF$ *and* $a_1 <_{\mathrm{Lex}} x^{(i)}$.)

(iii) *If* $a \circ b \in NF$, *then the finite iterates of* $\langle a, b \rangle$ *are all in NF.*

(iv) *Define the* associated sequence *of an atomic NF-word* $x^{(i)}$ *to be* $\langle x^{(i)} \rangle$. *If* $w = x^{(i)} a_1 \cdots a_n \in NF$, *then the associated sequence of* w *is* $\langle x^{(i)}, a_1, a_2, ..., a_n \rangle$. *If* $w = x^{(i)} a_1 \cdots a_{n-1} \circ a_n \in NF$, *then the associated sequence of* w *is* $\langle x^{(i)}, a_1, a_2, ..., a_n, ..., a_k, ... \rangle_{k < \omega}$ *where for* $k > n$, $a_k = x^{(i)} a_1 \cdots a_{k-2}$. *That is,* $\langle a_k : n < k < \omega \rangle$ *is the sequence of finite iterates of* $\langle x^{(i)} a_1 \cdots a_{n-1}, a_n \rangle$. *Given* v, $u \in NF$, *let their associated sequences be* $\langle v_n : n < N \rangle$ *and* $\langle u_n : n < M \rangle$, *where* N, $M \leqslant \omega$. *Then* $v <_{\mathrm{Lex}} u$ *if and only if either* $\langle v_n : n < N \rangle$ *is a proper initial segment of* $\langle u_n : n < M \rangle$ *or* $v_n <_{\mathrm{Lex}} u_n$ *for* n *least with* $v_n \neq u_n$.

Remarks and Proof. Regarding (ii), the reason that $a_n <_{\mathrm{Lex}} x^{(i)} a_1 \cdots a_{n-2}$ is required for $w = x^{(i)} a_1 \cdots a_{n-1} \circ a_n$ to be in NF is that if $a_n = x^{(i)} a_1 \cdots a_{n-2}$, then $w \equiv x^{(i)} a_1 \cdots a_{n-2} \circ a_{n-1}$ (by the $a \circ b = ab \circ a$ law) and the expression $x^{(i)} a_1 \cdots a_{n-1} \circ a_n$ would not be appropriately minimal.

Call $M \subset P$ good if M is closed under components, M satisfies (i) and (iii), the relation $<_{\mathrm{Lex}}$ on M determined by (i) and (iv) linearly orders M, and M satisfies the $\Rightarrow$ direction of (i). Let N be a maximal good set. Since $\{x^{(i)} : i < \omega\}$ is good ((iii) is true vacuously), $N \neq \varnothing$. We want to show that N satisfies the $\Leftarrow$ direction of (ii).

Claim. *If* a, $b \in N$ *and either* $a = x^{(i)}$ *and* $b <_{\mathrm{Lex}} a$, *or* $a = x^{(i)} a_1$ *and* $b \leqslant_{\mathrm{Lex}} x^{(i)}$, *or* $a = x^{(i)} a_1 \cdots a_n$ *and* $b \leqslant_{\mathrm{Lex}} x^{(i)} a_1 \cdots a_{n-1}$, *then* $ab \in N$.

Proof. Let $N' = N \cup \{ab\}$. Then N' is closed under components, satisfies (i) and (iii) and the $\Rightarrow$ direction of (ii), and since the associated sequence of ab is a sequence of members of N, it may be compared uniquely, in the manner of (iv), with the associated sequence of any member of N', using the linearity of $<_{\mathrm{Lex}}$ on N. Thus $<_{\mathrm{Lex}}$ linearly orders N', so N' is good and $ab \in N$ by the maximality of N.

Claim. *If* a, $b \in N$ *and either* $a = x^{(i)}$ *and* $b <_{\mathrm{Lex}} a$, *or* $a = x^{(i)} a_1$ *and* $b <\mathrm{Lex}\ x^{(i)}$, *or* $a = x^{(i)} a_1 \cdots a_n$ *and* $b <_{\mathrm{Lex}} x^{(i)} a_1 \cdots a_{n-1}$, *then* $a \circ b \in N$.

Proof. By iterating the previous claim, every finite iterate of $\langle a, b \rangle$ is in N. Let $N' = N \cup \{a \circ b\}$. Again N' is closed under components, satisfies (i) and (iii) and the $\Rightarrow$ direction of (ii), and the presence of the finite iterates of $\langle a, b \rangle$ in N allows lexicographic comparison of $a \circ b$'s associated sequence with that of any other member of N'. Note that the assumption about b in the claim guarantees that $a \circ b$'s associated sequence is different from that of any other member of N'. So $<_{\text{Lex}}$ linearly orders N'; done as before.

Thus N is good and satisfies the $\Leftarrow$ direction of (ii); such an N must be unique, so $N = NF$ is as desired.

Remarks. Every $w \in NF$ is uniquely expressible in the form $a_0 \circ (a_1 \circ (\cdots \circ (a_{n-1} \circ a_n)))$, each $a_i \notin EC$, for some $n \geqslant 0$. By (ii), if $w = a_0 a_1 \cdots a_{n-1} * a_n \in NF$, then for each $m < n$, $a_0 a_1 \cdots a_m \in NF$ and $a_0 \cdots a_m <_{\text{Lex}} w$. By induction it is then seen that if $u \in NF$ and w is a component of u, then $w <_{\text{Lex}} u$. By (iv), if $a \circ b \in NF$, then $a \circ b$ is the $<_{\text{Lex}}$ sup of the finite iterates of $\langle a, b\rangle$. It will follow from Lemma 10 that each $w \in P$ is equivalent to at most one member of NF. The normal form theorem says that each $w \in P$ has a normal form equivalent.

LEMMA 9. *Let $\prec$ be the smallest transitive relation on NF such that*

(i) *u a proper component of w implies $u \prec w$.*

(ii) rank $u <$ rank w *implies $u \prec w$.*

(iii) *If u is a finite iterate of $\langle a, b\rangle$, then $u \prec (a \circ b)$.*

Then $\prec$ is well founded.

Proof. Need to show there is no sequence $\langle w_n : n < \omega\rangle$ of NF words with each $w_{n+1} \prec w_n$ by an application of (i), (ii), or (iii). Pick such a sequence with rank w_0 as small as possible. Then each w_n has the same rank as w_0, as applications of (i), (ii), (iii) do not raise the rank. So w_{n+1} comes from w_n by an application of (i) or (iii). Pick a w_n so that no proper component of w_n is a w_m for any $m > n$. Then $w_n = a \circ b$ and w_{n+1} is a finite iterate of $\langle a, b\rangle$. By the construction of finite iterates, a proper component of a finite iterate of $\langle a, b\rangle$ is either a finite iterate of $\langle a, b\rangle$ or a component of a or of b. So then $w_{n+2}, w_{n+3}, \ldots$ can only be obtained by applying (i), until a w_m is reached which is a component of a or of b. This contradicts the minimality of w_n.

LEMMA 10. *If $<_{\text{L}}$ is irreflexive, then for $u, v \in NF$, $u <_{\text{Lex}} v \Leftrightarrow u <_{\text{L}} v$.*

Proof. ($\Rightarrow$) (Irreflexivity not used) By induction on $\max\{\|u\|, \|v\|\}$, where for $w \in NF$, $\|w\|$ is the ordinal rank of w under the well founded partial ordering $\prec$.

If $v = a \circ b$ and $u <_{\text{Lex}} v$, then since v is the $<_{\text{Lex}}$ sup of the finite iterates of $\langle a, b\rangle$, there is such a finite iterate v' with $u <_{\text{Lex}} v'$. Then $u <_{\text{L}} v'$ by the induction hypothesis, and $v' <_{\text{L}} v$, so $u <_{\text{L}} v$.

So we may assume v's associated sequence is $b_0, \ldots, b_n$, some $n \geqslant 0$. Let the associated sequence of u be $a_0 a_1, \ldots, a_m \cdots$ ($m < k$, where $1 \leqslant k \leqslant \omega$). By the induction hypothesis and the transitivity of $<_{\mathrm{L}}$ we may assume that n is least with $b_n >_{\mathrm{Lex}} a_n$. Thus, either $v = x^{(i)}$ and $u = x^{(i')}d_1 \cdots d_{n-1} * d_n$, some $i' < i$, $n \geqslant 0$, or $v = x^{(i)}b_1 b_2 \cdots b_n$ and u's associated sequence is $x^{(i)}b_1 \cdots b_{n-1} a_n a_{n+1} \cdots a_m \cdots (m < k$, where $n+1 \leqslant k \leqslant \omega)$ with $a_n <_{\mathrm{Lex}} b_n$. In the latter case, if u is of the form $a \circ b$, then a cannot be of the form $x^{(i)}b_1 \cdots b_r$ for some $r < n-1$, since then a_n would by definition be $x^{(i)}b_1 \cdots b_{n-2}$, and $a_n <_{\mathrm{Lex}} b_n <_{\mathrm{Lex}} x^{(i)}b_1 \cdots b_{n-2}$ by the definition of normal form. Treat these cases in a unified way: in the case $v = x^{(i)}$, let $c = x^{(i')}$ and recall $c(x^{(i-1)}) \equiv x^{(i)}$; in the case $v = x^{(i)}b_1 \cdots b_n$ $(n \geqslant 1)$, let $c = x^{(i)}b_1 \cdots b_{n-1}$.

By the induction hypothesis, if u' and u'' are proper components of u and $u' <_{\mathrm{Lex}} u''$, then $u' <_{\mathrm{L}} u''$. In either case then, there are d, d', $d_1 \cdots d_n$ $(n \geqslant 0)$ such that

$$v \equiv cd$$

$$u \equiv cd'd_1 d_2 \cdots d_{n-1} * d_n,$$

where $d >_{\mathrm{L}} d'$, $d_1 \leqslant_{\mathrm{L}} c$, $d_2 \leqslant_{\mathrm{L}} cd'$, ..., $d_m \leqslant_{\mathrm{L}} cd' \cdots d_{m-2}$ $(3 \leqslant m \leqslant n)$.

Let $c_1 = c$, $c_0 = cd'$, $c_m = cd'd_1 \cdots d_m$ $(m \leqslant n)$. Claim that for each i, $v \geqslant_{\mathrm{L}} c_{i+1} \circ c_i$, namely we will find u_i with

$$v \geqslant_{\mathrm{L}} c_{i+1}(c_i u_i).$$

For $i = -1$, let $d \geqslant_{\mathrm{L}} d'u_0$. Then from Lemma 1, $v \equiv cd \geqslant_{\mathrm{L}} c(d'u_0) \equiv (cd')(cu_0) = c_0(c_{-1}d_0)$. Suppose the claim is true for i. Then $v \geqslant_{\mathrm{L}} c_{i+1}(c_i u_i) \equiv c_{i+1}c_i(c_{i+1}u_i)$. We are given $d_{i+2} \leqslant_{\mathrm{L}} c_i$. In the case $d_{i+2} = c_i$, let $u_{i+1} = u_i : v \geqslant_{\mathrm{L}} c_{i+1}c_i(c_{i+1}u_i) = c_{i+2}(c_{i+1}u_{i+1})$. In the case $d_{i+2} <_{\mathrm{L}} c_i$ write $c_i \geqslant_{\mathrm{L}} d_{i+2}u_{i+1}$; then $v \geqslant_{\mathrm{L}} c_{i+1}c_i(c_{i+1}u_i) \geqslant_{\mathrm{L}} c_{i+1}c_i \geqslant_{\mathrm{L}} c_{i+1}(d_{i+2}u_{i+1}) = c_{i+1}d_{i+2}(c_{i+1}u_{i+1}) = c_{i+2}(c_{i+1}u_{i+1})$.

To conclude $v >_{\mathrm{L}} u$, assume without loss of generality that $*$ is $\circ$. By case $i = n-1$ of the claim, $v \geqslant_{\mathrm{L}} cd'd_1 \cdots d_{n-1}d_n(cd'd_1 \cdots d_{n-1}u_{n-1}) \equiv cd'd_1 \cdots d_{n-1}(d_n u_{n-1}) \equiv (cd'd_1 \cdots d_{n-1} \circ d_n)\, u_{n-1} \equiv uu_{n-1}$.

For the $(\Leftarrow)$ direction, if $u \nless_{\mathrm{Lex}} v$, then $u \geqslant_{\mathrm{Lex}} v$ since $<_{\mathrm{Lex}}$ is a linear order; thus $u \geqslant_{\mathrm{L}} v$, so $u \nless_{\mathrm{L}} v$ must hold by the irreflexivity of $<_{\mathrm{L}}$.

For $w \in P$ let $|w|$ be the (necessarily unique) $u \in NF$ such that $w \equiv u$, if such a u exists. The notation $|w|$ and the assumption that $<_{\mathrm{L}}$ is interchangeable with $<_{\mathrm{Lex}}$, which occasionally appear below, use the axiom that for some λ, $\mathscr{E}_\lambda \neq \varnothing$ (just its corollary that $<_{\mathrm{Lex}}$ is irreflexive) for justification. So make that assumption from here through the end of the paper.

Regarding the proof that $|w|$ exists for all $w \in P$: since $x \in NF$, it suffices to show that if p, $q \in NF$, then $|pq|$, $|p \circ q|$ exist. In certain cases this will involve putting q in "p-normal form" (similar to normal form with the role of x replaced by p), then computing pq and $p \circ q$ in p-normal form, then putting the result back into normal form. The result, that words can be moved from one type of normal form to another, entails showing (Theorem 16) that if u, v are in p-normal form and u is reasonably larger than v ("$u \sqsupset_p v$") then uv and $u \circ v$ can be put into p-normal form directly without the device of changing from p to other bases.

Let $B = NF - C$. B is the set of *bases*—words p for which the notion of p-normal form will be defined.

LEMMA 11. (1) *The following are equivalent*:

(i) $p \in B$.

(ii) $p \in NF$ *has normal form representation* $x^{(i)}$ *or* $x^{(i)}a_0 a_1 \cdots a_n$ *where* $a_n \in B$.

(iii) $p \in NF$ *is of the form* $u_1(u_2(u_3(\cdots(u_n x))))$ *for some* $u_1 \cdots u_n$, $n \geqslant 0$.

(iv) *For some* $w \in A$, $p = |w|$.

(2) *If* $q \in NF$, *then there are* $p_0, p_1, \ldots, p_n \in B$ (*some* $n \geqslant 0$) *with* $q \equiv p_0 \circ p_1 \circ \cdots \circ p_n$.

Proof. (1) Use Lemma 10. For (2), let $p_0 \circ p_1 \circ \cdots \circ p_n$ be the $H(q)$ of Lemma 2 (note that if $q = a_1(a_2(\cdots a_n(b \circ c))) \in NF$, then, since $c <_{\mathrm{L}} b <_{\mathrm{L}} b \circ c$, $q_0 = a_1(a_2(\cdots(a_n b)))$ and $q_1 = a_1(a_2(\cdots(a_n c)))$ are in NF).

For $w \in P$ recall $w^{(0)} = w$, $w^{(i+1)} = w^{(i)} w^{(i)}$, and that if $i \leqslant m$, then $w^{(i)} w^{(m)} \equiv w^{(m+1)}$. For $p \in B$ the notion of p-normal form is defined similarly to that of normal form. A word in normal form is built up from the atomic symbols $x^{(i)}$ $(i < \omega)$; the similar atomic symbols for p-normal form will be $p^{(i)}$ $(i < \omega)$ and q $(q \in NF,\ q <_{\mathrm{L}} p)$. We state as a lemma the existence of a definition of, and properties of, p-normal form.

LEMMA 12. *For* $p \in B$ *there is a unique subset* p–NF *of* P *and a linear ordering*, $<^p_{\mathrm{Lex}}$ *of* p–NF *such that*

(i) $w \in p$–NF *if and only if either*

(a) $w \in NF$ *and* $w <_{\mathrm{L}} p$, *or*

(b) $w = p^{(i)}$ *for some* i, *or*

(c) $w = a_0 a_1 \cdots a_{n-1} * a_n$, *where* $a_0 = p^{(i)}$, *each* $a_m \in p$–NF, $a_1 <^p_{\mathrm{Lex}} a_0$, *and for* $2 \leqslant m \leqslant n$, $a_m \leqslant^p_{\mathrm{Lex}} a_0 a_1 \cdots a_{m-2}$. *Additionally if* $* = \circ$ *and* $n \geqslant 2$, *then* $a_n <^p_{\mathrm{Lex}} a_0 a_1 \cdots a_{n-2}$.

(ii) *Let an* atomic *p–NF word be a word of the form* $p^{(i)}$ $(i < \omega)$ *or a* $w \in NF$ *with* $w <^{\mathrm{L}} p$. *For* v, u *atomic p–NF words, let* $v <^p_{\mathrm{Lex}} u \Leftrightarrow v <_{\mathrm{L}} u$. *Let the p-associated sequence of an atomic p–NF word w be* $\langle w \rangle$. *With this as atomic step, the definition of p-associated sequence and the inductive definition of* $<^p_{\mathrm{Lex}}$ *on p–NF are the analogs of those definitions for NF.*

(iii) *p–NF is closed under components and finite iterates, and $u \circ v \in p$–NF is the $<^p_{\mathrm{Lex}}$ sup of the finite iterates of $\langle u, v\rangle$.*

(iv) *Letting* rank $p^{(i)}a_1 \cdots a_{n-1} * a_n = i$, rank $q = -1$ $(q \in NF$, $q <_{\mathrm{L}} p)$, *then $p^{(i)}$ is the $<^p_{\mathrm{Lex}}$ least word of rank i, and x is the $<^p_{\mathrm{Lex}}$ least word of rank -1.*

(v) *There is a well founded partial ordering $\prec_p$ of p–NF satisfying the conditions of Lemma 9, which agrees with $\prec$ on $\{w \in NF : w <_{\mathrm{L}} p\}$.*

(vi) *For $u, v \in p$–NF, $u <^p_{\mathrm{Lex}} v \Leftrightarrow u <_{\mathrm{L}} v$.*

(vii) *Each $w \in P$ is equivalent to at most one $u \in p$–NF.*

Proof. Similar to the proofs for NF.

So $NF = x$–NF. If $p \in B$, $a \in NF$, and $a <_{\mathrm{L}} p$, then pa, $p \circ a \in p$–NF though they need not be in NF. Again, drop the $<^p_{\mathrm{Lex}}$ notation in favor of $<_{\mathrm{L}}$. Let $|w|_p$ be the $u \in p$–NF with $u \equiv w$, if there is one ($|w| = |w|_x$).

For $w, v \in P$ let $v \ll_p w$ mean that $w \in p$–NF, $w \notin EC$, and either $w = p^{(i)}$ and $v <_{\mathrm{L}} w$, or $w = p^{(i)}a_0a_1 \cdots a_n$ (some $n \geqslant 0$) and $v \leqslant_{\mathrm{L}} p^{(i)}a_0 \cdots a_{n-1}$. Note that v need not be in p–NF; when $v \in p$–NF, $v \ll_p w$ means that $w \geqslant_{\mathrm{L}} p$ and $wv \in p$–NF. If $v' \leqslant_{\mathrm{L}} v \ll_p w$, then $v' \ll_p w$.

LEMMA 13. *Suppose $p \in B$.*

(i) *If $v' \in p$–NF, $v' <_{\mathrm{L}} v \ll_p w$, then $w \circ v' \in p$–NF.*

(ii) *If $v \in p$–NF, $v \ll_p w$, then $|w \circ v|_p$ exists.*

(iii) *If $w = a_0(a_1(a_2(\cdots(a_{n-1}(a_n u))))) \in p$–$NF$, then $w' = a_0 \circ (a_1 \circ (a_2 \circ \cdots \circ (a_{n-1} \circ a_n))) \in p$–$NF$.*

Proof. (i) is immediate.

(ii) If $w = p^{(i)}$, or $w = p^{(i)}a_0a_1 \cdots a_n$ and $v <_{\mathrm{L}} p^{(i)}a_0 \cdots a_{n-1}$, then $|w \circ v|_p = w \circ v$. Suppose that $w = p^{(i)}a_0a_1 \cdots a_n$ and $v = p^{(i)}a_1a_2 \cdots a_{n-1}$. Then let m $(0 \leqslant m < n)$ be greatest such that $a_{m+1} <_{\mathrm{L}} p^{(i)}a_0a_1 \cdots a_{m-1}$. If no such m exists, $|w \circ v|_p = p^{(i)} \circ a_0$; if m exists, $|w \circ v|_p = p^{(i)}a_0 \cdots a_m \circ a_{m+1}$.

(iii) This is not used below. We have $a_0 \notin EC$, and $a_1(a_2 \cdots (a_{n-1}(a_n u))) \in p$–$NF$. Since $w \equiv w'u$, $w' <_{\mathrm{L}} w$. If $p >_{\mathrm{L}} x$ and $w <_{\mathrm{L}} p$, we are done: $w' \in NF$, $w' <_{\mathrm{L}} w <_{\mathrm{L}} p$, so $w' \in p$–NF. So assume $w \geqslant_{\mathrm{L}} p$, with $p >_{\mathrm{L}} x$. By induction we may assume that $a_1 \circ (a_2 \circ (\cdots (a_{n-1} \circ a_n)) \in p$–$NF$. Then $a_0 \gg_p a_1(a_2(\cdots(a_{n-1}(a_n u)))) >_{\mathrm{L}} a_1 \circ (a_2 \circ (\cdots \circ (a_{n-1} \circ a_n)))$. Thus $a_0 \circ (a_1 \circ (a_2 \circ \cdots \circ (a_{n-1} \circ a_n))) \in p$–$NF$ by (i).

For $p \in B$, a, $b \in p$–NF, define $a \sqsupset_p b$ (a strengthening of $a >_{\mathrm{L}} b$) by $\prec_p$-induction on a (write $\sqsupset_x = \sqsupset$).

(1) If $a <_{\mathrm{L}} p$, then $a \sqsupset_p b$ iff $b \in NF$, $a \sqsupset b$, $a \circ b <_{\mathrm{L}} p$.

(2) If $a = p^{(n)}$, then $a \sqsupset_p b$ iff $a >_{\mathrm{L}} b$.

(3) If $a = p^{(i)} a_0 a_1 \cdots a_n (n \geqslant 0)$, then $a \sqsupset_p b$ iff either $a \gg_p b$ or there is a $b_n \in p\text{-}NF$ such that b's p-associated sequence begins with $\langle p^{(i)}, a_0, ..., a_{n-1}, b_n \rangle$, where $a_n \sqsupset_p b_n$, $a_n \circ b_n \ll_p p^{(i)} a_0 a_1 \cdots a_{n-1}$.

(4) If for $n \geqslant 0$, $a = p^{(i)} a_0 a_1 \cdots a_{n-1} \circ a_n$, then $a \sqsupset_p b$ if $a_n \sqsupset_p b$ and $a_n \circ b \ll_p p^{(i)} a_0 a_1 \cdots a_{n-1}$.

The words $a \circ b$, $a_n \circ b_n$, $a_n \circ b$ of (1), (3), and (4) are not assumed in the definition to have $p\text{-}NF$ equivalents, though Theorem 16 says they will have them. Another way to put the condition in (3) is that either $b \leqslant_{\mathrm{L}} p^{(i)} a_0 \cdots a_{n-1}$ or $b = p^{(i)} a_0 \cdots a_{n-1} \circ b_n$, or $b = p^{(i)} a_0 \cdots a_{n-1} b_n e_1 \cdots e_{m-1} * e_m$, with b_n as in (3).

For $p >_{\mathrm{L}} x$, the definition of $\sqsupset_p$ involves the prior definition of $\sqsupset_x$. Proofs below, by induction on $<_p$ about the $\sqsupset_p$ relation for all $p \in B$, could be done first for $p = x$, then for all $p >_{\mathrm{L}} x$. Because the $p = x$ case will be the same proof but simpler (there are no words $w <_{\mathrm{L}} x$ to worry about), the proofs below will be phrased for all p simultaneously.

Lemma 14. *If $p \in B$, a, b, $c \in p\text{-}NF$, then*

(i) $a \gg_p b \Rightarrow a \sqsupset_p b$.

(ii) $a \sqsupset_p b \Rightarrow a >_{\mathrm{L}} b$.

(iii) $u \circ v \in p\text{-}NF \Rightarrow u \sqsupset_p v$.

(iv) *If $a \sqsupset_p b$ and $c \in p\text{-}NF$, $c \leqslant_{\mathrm{L}} b$, then $a \sqsupset_p c$.*

Proof. (i) is by definition.

(ii) By induction on a. Case (1) reduces to the $p = x$ case. Case (2) and the first part of Case (3) are immediate. In the second part of Case (3), the induction hypothesis yields $a_n >_{\mathrm{L}} b_n$, whence by Lemma 12(vi), $a >_{\mathrm{L}} b$. In Case (4), $a_n >_{\mathrm{L}} b$ by the induction hypothesis, and $a >_{\mathrm{L}} p^{(i)} a_0 \cdots a_{n-1} >_{\mathrm{L}} a_n$.

(iii) If $u \circ v \geqslant_{\mathrm{L}} p$, then u is of the form $p^{(i)} a_1 \cdots a_n$ and even $uv \in p\text{-}NF$ would imply $u \sqsupset_p v$. If $u \circ v <_{\mathrm{L}} p$, then $u \circ v \in NF$, whence $u \gg_x v$, so $u \sqsupset v$, and thus $u \sqsupset_p v$.

(iv) By induction on a. Case (1) reduces to the $p = x$ case and the fact that $c <_{\mathrm{L}} b$ implies $a \circ c <_{\mathrm{L}} a \circ b$ (Lemma 1). Case (2) is immediate, and Cases (3) and (4) are immediate by the induction hypothesis.

To help move words from one base to another, a slightly different type of normal form ("(p, r)-normal form" $(p, r \in B)$) will be considered. Lemma 15 notes a situation common to both normal forms. An abstract treatment could put these types of normal forms under one rubric, but we treat them separately here. Call a word $a_0a_1 \cdots a_n$ *prenormal* if for $2 \leqslant i \leqslant n$, $a_i \leqslant_L a_0a_1 \cdots a_{i-2}$; let $a_0a_1 \cdots a_{n-1} \circ a_n$ be pre-normal if in addition, $a_n <_L a_0a_1 \cdots a_{n-2}$ in the case $n \geqslant 2$. (This definition depends on the sequence $\langle a_0, a_1, ..., a_n \rangle$, not just the word $a_0a_1 \cdots a_{n-1} * a_n$.)

LEMMA 15. *Let $w = ab$. Let v be prenormal, in one of the forms $a \circ c$, $acu_1 \cdots u_{n-1} * u_n$. Then the following give prenormal equivalents to wv and $w \circ v$:*

(i) $ab(a \circ c) \equiv a(b \circ c)a \circ a(bc)$.

(ii) $ab \circ (a \circ c) \equiv a \circ (b \circ c)$.

(iii) $ab(acu_1 \cdots u_n) \equiv a(b \circ c)\, u_1(abu_2) \cdots (abu_n)$.

(iv) $ab \circ (acu_1 \cdots u_n) \equiv a(b \circ c)\, u_1(abu_2) \cdots (abu_n) \circ ab$.

(v) $ab(acu_1 \cdots u_{n-1} \circ u_n) \equiv a(b \circ c)\, u_1(abu_2) \cdots (abu_{n-1}) \circ abu_n$.

(vi) $ab \circ (acu_1 \cdots u_{n-1} \circ u_n) \equiv a(b \circ c)\, u_1(abu_0) \cdots (abu_{n-1}) \circ (ab \circ u_n)$.

Proof. (i) $ab(a \circ c) \equiv ab(aca \circ ac) \equiv ab(aca) \circ ab(ac) \equiv a(b \circ c)a \circ a(bc)$. Prenormal since $a(bc) <_L a(b \circ c)$.

(ii) $ab \circ (a \circ c) \equiv (ab \circ a) \circ c \equiv (a \circ b) \circ c \equiv a \circ (b \circ c)$, vacuously prenormal.

(iii)–(vi) The equivalences are easily checked. The prenormality of the forms follows from the fact that $ab <_L a(b \circ c)$, that $u_1 \leqslant_L a$, that $u_i \leqslant_L acu_0 \cdots u_{i-2}$ implies $abu_i \leqslant_L a(b \circ c)\; u_0(abu_1) \cdots (abu_{i-2})$, and that $u_n <_L acu_0 \cdots u_{n-2}$ implies $abu_n <_L ab \circ u_n <_L a(b \circ c)\; u_0(abu_1) \cdots (abu_{n-2})$ (Lemma 1).

For example then, if ab and $acu_1 \cdots u_{n-1}u_n$ are in p–NF, $|b \circ c|_p$ exists, $b \circ c \ll_p a$, and $|abu_2|_p$, $|abu_3|p, ..., |abu_{n-1}|_p$, $|ab \circ u_n|_p$ exist, then by (vi), $|ab \circ (acu_1 \cdots u_{n-1} \circ u_n)|_p$ exists and equals $a\, |b \circ c|_p\, u_1\, |abu_2|_p \cdots |abu_{n-1}|_p \circ |ab \circ u_n|_p$.

THEOREM 16. *If $a \sqsupset_p b$, then $|ab|_p$, $|a \circ b|_p$ exist and $|ab|_p \sqsupset_p a$.*

Proof. By induction on $\prec_p$, first with respect to a, then with respect to b.

Case 1. $a <_L p$. Then $a \sqsupset b$, so $|ab|$, $|a \circ b|$ exist by the case $p = x$ of the theorem. And since $ab <_L a \circ b <_L p$, $|ab| = |ab|_p$ and $|a \circ b| = |a \circ b|_p$. To see $|ab|_p \sqsupset_p a$, use that $|ab|_p = |ab|$, that $|ab| \sqsupset_p a$ by the $p = x$ case, and that $|ab| \circ a \equiv ab \circ a \equiv a \circ b <_L p$.

Case 2. $a = p^{(i)}$. Then $a >_L b$, whence ab, $a \circ b$ are in p–NF, and so is aba, giving $ab \sqsupset_p a$.

Case 3. $a = p^{(i)}a_0 \cdots a_{n-1}a_n$ $(n \geqslant 0)$. Let $p^{(i)}a_0 \cdots a_{n-1} = r$, $a_n = s$, so $a = rs$.

Case 3.1. $b \leqslant_{\mathrm{L}} r$. Then ab, $aba \in p\text{–}NF$, $ab \sqsupset_p a$. Also, since $b \ll_p a$, $|a \circ b|_p$ exists by Lemma 13(ii).

If $b \nleqslant_{\mathrm{L}} r$, then there is a $t \in p\text{–}NF$ with $s \sqsupset_p t$ and $s \circ t \ll_p r$, and either $b = rt$, $b = r \circ t$, or for some $m \geqslant 0$, $b = rtv_0 \cdots v_m$ or $rtv_0 \cdots v_{m-1} \circ v_m$. Namely, t is the "b_n" of the definition of $\sqsupset_p$. Note that by the induction hypothesis on a, $|st|_p$ and $|s \circ t|_p$ exist and $|st|_p \sqsupset_p s$. Also, since $a \sqsupset_p b >_{\mathrm{L}} v_i$, $a \sqsupset_p v_i$ by Lemma 15(iv); thus by the induction hypothesis on b, $|av_i|_p$ exists.

Case 3.2. $b = rt$. Then $ab \equiv r(st)$, $a \circ b \equiv r(s \circ t)$, and $st <_{\mathrm{L}} s \circ t \ll_p r$. So $|ab|_p = r(|st|_p)$ and $|a \circ b|_p = r(|s \circ t|_p)$.

To check $|ab|_p = r(|st|_p) \sqsupset_p rs = a$, we need that $|st|_p \sqsupset_p s$ and $|st|_p \circ s \ll_p r$. These are given above, using that $|st|_p \circ s \equiv s \circ t$.

Case 3.3. $b = r \circ t$. Then by Lemma 15,

$$ab = rs(r \circ t) \equiv r(s \circ t)r \circ r(st),$$

$$a \circ b = rs \circ (r \circ t) \equiv r \circ (s \circ t).$$

In the ab case, since $st <_{\mathrm{L}} s \circ t \ll_p r$, $r(|s \circ t|_p)r$ and $r(|st|_p)$ are in $p\text{–}NF$. Thus $ab \equiv r(|s \circ t|_p)r \circ r(|st|_p) \in p\text{–}NF$.

In the $a \circ b$ case, $|s \circ t|_p \ll_p r$ and $|r \circ (s \circ t)|_p$ exists by Lemma 13(ii).

To see that $|ab|_p \sqsupset_p a$, we need from the form of $|ab|_p$ that $r(|st|_p) \sqsupset_p rs$ and $r(|st|_p) \circ rs \ll_p r(|s \circ t|_p)r$. For the former, we have $|st|_p \sqsupset_p s$ and $st \circ s \equiv s \circ t \ll_p r$. For the latter, $r(st) \circ rs \equiv r(s \circ t) \ll_p r(|s \circ t|_p)r$.

Case 3.4. $b = rtv_0 \cdots v_m$, some $m \geqslant 0$. Then $ab \equiv r(s \circ t)\, v_0(av_1) \cdots (av_m)$ is prenormal, and $a \circ b \equiv r(s \circ t)\, v_0(av_1) \cdots (av_m) \circ a$ is prenormal. We have that $|s \circ t|_p$ and the $|av_i|_p$'s exist, and $s \circ t \ll_p r$. Thus $ab \equiv r(|s \circ t|_p)\, v_0\, |av_1|_p \cdots |av_m|_p \in p\text{–}NF$, and $a \circ b \equiv r(|s \circ t|_p)\, v_0\, |a_1|_p \cdots |av_m|_p \circ a \in p\text{–}NF$.

We also have $|ab|_p a \in p\text{–}NF$, so $|ab|_p \sqsupset_p a$.

Case 3.5. $b = rtv_0 \cdots v_{m-1} \circ v_m$. For notational convenience assume $m \geqslant 2$; the cases $m = 0$, 1 are similar. As before $ab = rs(b) \equiv r(s \circ t)\, v_0(av_1) \cdots (av_{m-1}) \circ (av_m)$ in prenormal form, and $a \circ b \equiv r(s \circ t)\, v_0(av_1) \cdots (av_{m-1}) \circ (a \circ v_n)$, in prenormal form. Thus using the induction hypothesis on a, $ab \equiv r(|s \circ t|_p)\, v_0(|av_1|_p) \cdots (|av_{m-1}|) \circ (|av_m|_p) \in p\text{–}NF$, $a \circ b \equiv r(|s \circ t|p)\, v_0 \cdots |av_{m-1}| \circ |av_m| \in p\text{–}NF$.

To check $|ab|_p \sqsupset_p a$, we need, by the form of $|ab|p$, that $|av_m|p \sqsupset a$ and that $av_m \circ a \ll_p r(|s \circ t|_p)\, v_0(|av_1|_p) \cdots (|av_{m-1}|_p)$. We have $|av_m|_p \sqsupset a$ by the induction hypothesis on b. For the latter property, by Lemma 1

$av_m \circ a \ \equiv\ a \circ v_m <_{\mathrm{L}} a(rtv_0 \cdots v_{m-2}) \ \equiv\ r(|s \circ t|_p)\ v_0(|av_1|_p) \cdots (|av_{m-2}|_p) \ll_p r(|s \circ t|_p)\, v_0(|av_1|_p) \cdots (|av_{m-1}|_p)$.

Case 4. $a = u \circ v$ ($u \geqslant_{\mathrm{L}} p$). Thus $v \sqsupset_p b$ and $v \circ b \ll_{\geqslant} u$. By the induction hypothesis for a, $|vb|_p$ and $|v \circ b|_p$ exist. Since $vb <_{\mathrm{L}} v \circ b$, $vb \ll_p u$; thus $|ab|_p = u(|vb|_p)$. To check $|ab|_p \sqsupset a$ we need $|ab|_p \sqsupset_p v$ (given by the induction hypothesis for a) and $vb \circ v \ll_p u$ (given). To check that $|a \circ b|_p$ exists, write $a \circ b \equiv u \circ (v \circ b)$; since $|v \circ b|_p \ll_p u$, $|a \circ b|_p$ exists by Lemma 13(ii).

This completes the proof of Theorem 16.

The following fact about *NF* is not needed in the *p–NF* version, but that version also holds (by occasionally checking the $u \circ v <_p p$ clause).

LEMMA 17. (1) *Suppose* $\tilde{a} = a_1(a_2(\cdots(a_n(u \circ v)))) \in NF$. *Let* $\tilde{u} = a_1(a_2(\cdots(a_n u)))$, $\tilde{v} = a_1(a_2(\cdots(a_n v)))$. *Then*

(i) $\tilde{u} \sqsupset \tilde{v}$.

(ii) *If* $\tilde{a} \sqsupset w$, *then* $\tilde{v} \sqsupset w$ *and* $\tilde{u} \sqsupset |\tilde{v} \circ w|$.

(2) *If* $a \sqsupset w$, *then* $aw \equiv b_0(b_1(\cdots(b_n w)))$, *with each* $b_i \in B$, *so that* $b_n \sqsupset w$ *and* $b_i \sqsupset |b_{i+1}(b_{i+2}(\cdots(b_n w)))|$ *for* $0 \leqslant i < n$, *and if* $n > 0$, *then each length* $b_i <$ *length* a.

Proof. (1) We have that $\tilde{u}$ and $\tilde{v}$ are in *NF*, and in (ii), given that $\tilde{v} \sqsupset w$, $|\tilde{v} \circ w|$ exists by Theorem 16. Prove (i) and (ii) by induction on n. If $n = 0$, $\tilde{a} = u \circ v$ and (i), (ii) follow by definition and the fact that $\gg$ implies $\sqsupset$. Suppose (i) and (ii) are true for $n-1$, so $\tilde{u}_1 = a_2(a_3(\cdots(a_n u))) \sqsupset a_2(a_3(\cdots(a_n v))) = \tilde{v}_1$, and letting $\tilde{a}_1 = a_2(a_3(\cdots(a_n(u \circ v))))$, $\tilde{u}_1 \circ \tilde{v}_1 \equiv \tilde{a}_1 \ll \tilde{a}$. Thus (i) holds for n.

For (ii), consider the case $w \leqslant_{\mathrm{L}} a_1$. Then $\tilde{v} \sqsupset w$, and $\tilde{v} \circ w = a_1 \tilde{v}_1 \circ w$, so $|\tilde{v} \circ w|$'s associated sequence is an initial segment of $\langle x^{(i)}, d_1, \ldots, d_k, \tilde{v}_1, w \rangle$, where $a_1 = x^{(i)} d_1 \cdots d_k$. Thus $\tilde{u} = x^{(i)} d_1 \cdots d_k \tilde{u}_1 \sqsupset |\tilde{v} \circ w|$.

If $w >_{\mathrm{L}} a_1$ but $w \sqsupset \tilde{a}$, then $w = a_1 \circ t$ or $a_1 t e_1 \cdots e_{m-1} * e_m$, where $t \sqsubset \tilde{a}_1$, $\tilde{a}_1 \circ t \ll a_1$. By the induction hypothesis (ii) applied to $\tilde{a}_1 \sqsupset t$, $\tilde{v}_1 \sqsupset t$, and $\tilde{u}_1 \sqsupset |\tilde{v}_1 \circ t|$. To conclude that $\tilde{v} \sqsupset w$ we need $\tilde{v}_1 \circ t \ll a_1$; this holds since $\tilde{v}_1 \circ t <_{\mathrm{L}} \tilde{u}_1$ and $\tilde{u}_1 \ll a_1$.

To show $\tilde{u} \sqsupset |\tilde{v} \circ w|$, recall $\tilde{u}_1 \sqsupset |\tilde{v}_1 \circ t|$ from above and, since $\tilde{a} \sqsupset w$, $a_1(|\tilde{u}_1 \circ \tilde{v}_1|) \sqsupset a_1 \circ t$, whence $\tilde{u}_1 \circ v_{1} \circ t \ll a_1$.

Case 1. $w = a_1 \circ t$. Then $\tilde{v} \circ w = a_1 \tilde{v}_1 \circ (a_1 \circ t) \equiv a_1 \circ (\tilde{v}_1 \circ t)$ by Lemma 15(ii). Since $\tilde{v}_1 \circ t \ll a_1$, either $a_1 \circ |\tilde{v}_1 \circ t|$ is in *NF* or for some initial segment de of a_1, $|a_1 \circ (\tilde{v}_1 \circ t)| = d \circ e$. In the latter case clearly $\tilde{u} \sqsupset |\tilde{v} \circ w|$, so assume $|\tilde{v} \circ w| = a_1 \circ |\tilde{v}_1 \circ t|$. Then $\tilde{u} \sqsupset |\tilde{v}_1 \circ t|$ follows from the facts preceding Case 1.

Case 2. $w = a_1 te_1 \cdots e_{m-1} * e_m$. Assume without loss of generality that $* = 0$. Then $\tilde{v} \circ w = a_1 \tilde{v}_1 \circ (a_1 te_1 \cdots e_{m-1} \circ e_m) \equiv (a_1(\tilde{v}_1 \circ t)\, e_1\, |\tilde{v}e_2| \cdots |\tilde{v}e_{m-1}| \circ |\tilde{v}e_m|) \circ \tilde{v} = a_1(\tilde{v}_1 \circ t)\;\; e_1\, |\tilde{v}e_2| \cdots |\tilde{v}e_{m-1}| \circ |\tilde{v} \circ e_m|$. (The last term exists since $v \sqsupset w >_{\mathrm{L}} e_m$.) This expression is in *NF*, and the facts preceding Case 1 again yield $\tilde{u} \sqsupset |\tilde{v} \circ w|$.

This finishes the proof of (1).

(2) By induction on a. If $a \in B$, then done. Otherwise $a = a_1(a_2(\cdots(a_n(u \circ v))))$. Then by (i), $a_1(a_2(\cdots(a_n v))) \sqsupset w$ and $a_1(a_2(\cdots(a_n u))) \sqsupset |a_1(a_2(\cdots(a_n v)))w|$. Apply the induction hypothesis first to $a_1(a_2(\cdots(a_n v)))$, then to $a_1(a_2(\cdots(a_n u)))$.

The notion of (p, r)-normal form will be defined via the next lemma. For $p, r \in B$ an *atomic* (p, r)*–NF* word is one of the form pq ($q \in r$–*NF*) or a $w \in NF$ with $w \leqslant_{\mathrm{L}} p$. Note that the atomic (p, r)–*NF* words are linearly ordered by $<_{\mathrm{L}}$ (by Lemma 1, the linear ordering of r–*NF* under $<_{\mathrm{L}}$, and irreflexivity of $<_{\mathrm{L}}$).

LEMMA 18. *For $p, r \in B$ there is a unique subset (p, r)–NF of P and a linear ordering $<_{\mathrm{Lex}}$ of (p, r)–NF such that*

(i) *$w \in (p, r)$–NF if and only if either*

(a) *$w \in NF$ and $w \leqslant_{\mathrm{L}} p$, or*

(b) *$w = pq$ for some $q \in r$–NF, or*

(c) *$w = a_0 a_1 \cdots a_{n-1} * a_n$, where each $a_m \in (p, r)$–NF, $a_0 = pq$ for some $q \in r$–NF, $a_1 \leqslant^{p,r}_{\mathrm{Lex}} p$ (i.e., $a_1 \in NF$ and $a_1 \leqslant_{\mathrm{L}} p$), and for $2 \leqslant m \leqslant n$, $a_m \leqslant^{p,r}_{\mathrm{Lex}} a_0 a_1 \cdots a_{m-2}$. Additionally, if $* = \circ$ and $n \geqslant 2$, then $a_n <^{p,r}_{\mathrm{Lex}} a_0 a_1 \cdots a_{n-2}$.*

(ii) *For v, w atomic (p, r)–NF words, let $v <^{p,r}_{\mathrm{Lex}} w \Leftrightarrow v <_{\mathrm{L}} w$. With this as atomic step, the definition of (p, r)-associated sequences and the inductive definition of $<^{p,r}_{\mathrm{Lex}}$ on (p, r)–NF are the analogs of those definitions for NF.*

(iii) *(p, r)–NF is closed under finite iterates, and $u \circ v \in (p, r)$–NF is the $<^{p,r}_{\mathrm{Lex}}$ sup of the finite iterates of $\langle u, v \rangle$ ((p, r)–NF is not literally closed under components, but it is if we exclude components of the atomic (p, r)–NF words pq).*

(iv) *Letting* $\mathrm{rank}(pqa_0 \cdots a_{n-1} * a_n) = 1 + (r$*–NF rank of* $q)$, *and* $\mathrm{rank}\, w = -1$ *$(w \in NF,\ w \leqslant_{\mathrm{L}} p)$, there is a well founded partial ordering $\prec_{p,r}$ of (p, r)–NF satisfying the conditions of Lemma 9, which agrees with $\prec$ on $\{w \in NF : w <_{\mathrm{L}} p\}$, such that if $q \prec_r q'$, then $pqa_0 \cdots a_{n-1} * a_n \prec_{p,r} pq'b_0 \cdots b_{i-1} * b_i$.*

(v) *For $u, v \in (p, r)$–NF, $u <^{p,r}_{\mathrm{Lex}} v \Leftrightarrow u <_{\mathrm{L}} v$.*

(vi) *Each $w \in P$ is equivalent to at most one $u \in (p, r)$–NF.*

Proof. Similar to the proof for *NF*.

Again, write $<_{\mathrm{L}}$ for $<_{\mathrm{Lex}}^{p,r}$, and $|w|_{p,r}$ for the $u \equiv w$, $u \in (p, r)$–*NF* if there is one.

For $w, v \in P$ let $w \ll_{p,r} v$ mean that $v \in (p, r)$–*NF*, with either $v = pq$ and $w \leqslant_{\mathrm{L}} p$ or $v = pqa_0 \cdots a_n$ (some $n \geqslant 0$) and $w \leqslant_{\mathrm{L}} pqa_0 \cdots a_{n-1}$.

So if $w \in (p, r)$–*NF*, $w \ll_{p,r} v$, *then* $vw \in (p, r)$–*NF*, $|v \circ w|_{p,r}$ exists as in Lemma 13(ii), and $w' \ll_{p,r}$ for all $w' \in (p, r)$–*NF* with $w' <_{\mathrm{L}} w$.

For $a, b \in (p, r)$–*NF*, define $a \sqsupset_{p,r} b$ by $\prec_{p,r}$-induction on a.

(1) For $a <_{\mathrm{L}} p$, $a \sqsupset_{p,r} b$ iff $a \sqsupset_p b$ (iff $a \sqsupset b$ and $|a \circ b| <_{\mathrm{L}} p$).

(2) For $a = p$, $a \sqsupset_{p,r} b$ iff $a \sqsupset_p b$ (iff $b <_{\mathrm{L}} p$).

(3) For $a = pq$, $a \sqsupset_{p,r} b$ iff $b \leqslant_{\mathrm{L}} p$ or for some $n \geqslant 0$ $b = p(q')\, b_1 \cdots b_{n-1} * b_n$ with $q' \sqsubset_r q$.

(4) For $a = pqa_0 a_1 \cdots a_{n-1} a_n$, $a \sqsupset_{p,r} b$ iff $b \ll_{p,r} a$ or there is a $b_n \in (p, r)$–*NF* such that b's (p, r)-associated sequence begins with $\langle pq, a_0, a_1, ..., a_{n-1}, b_n \rangle$, with $b_n \sqsubset_{p,r} a_n$ and $a_n \circ b_n \ll_{p,r} pqa_0 a_1 \cdots a_{n-1}$.

(5) For $a = pqa_0 a_1 \cdots a_{n-1} \circ a_n$, $a \sqsupset_{p,r} b$ iff $a_n \sqsupset_{p,r} b$ and $a_n \circ b \ll_{p,r} pqa_0 a_1 \cdots a_{n-1}$.

Lemma 19. *The* (p, r)–*NF analogs of the statements of Lemma* 14 *hold.*

Theorem 20. *Suppose* $p, r \in B$, *and for every* $w \in NF$, $|w|_r$ *exists. Then for* $a, b \in (p, r)$–*NF with* $a \sqsupset_{p,r} b$, $|ab|_{p,r}$ *and* $|a \circ b|_{p,r}$ *exist, and* $|ab|_{p,r} \sqsupset_{p,r} a$.

Proof. By $<_{p,r}$-induction on a, then on b.

Case 1. $a <_{\mathrm{L}} p$. Then $a \sqsupset b$ and $a \circ b <_{\mathrm{L}} p$. By Theorem 16, $|ab|$ and $|a \circ b|$ exist, and $|ab| \sqsupset a$. We are done since $|ab| <_{\mathrm{L}} |a \circ b| <_{\mathrm{L}} p$.

Case 2. $a = p$. Then $a >_{\mathrm{L}} b$, and $b \in NF$, so by the assumption of the theorem, $|b|_r$ exists. Then $|ab|_{p,r} = p\,|b|_r$, $a \sqsupset_{p,r} b$, and $|a \circ b|_{p,r} = p\,|b|_r \circ p$.

Case 3. $a = pq$.

Case 3.1. $b \leqslant_{\mathrm{L}} p$. Then $a \sqsupset_{p,r} b$, and the (p, r)-normal forms of ab, $a \circ b$ are pqb, $pq \circ b$.

Case 3.2. $b = pq' b_1 b_2 \cdots b_{n-1} * b_n$, some $n \geqslant 0$, with $q' \sqsubset r\, q$. Then $|qq'|_r$ and $|q \circ q'|_r$ exist by Theorem 16, and $|ab_i|_{p,r}$ and $|a \circ b_i|_{p,r}$ exist by the induction hypothesis, $1 \leqslant i \leqslant n$, and $b_1 \leqslant_{\mathrm{L}} p$ by the definition of (p, r)–*NF*.

Case 3.2.1. $n = 0$, i.e., $b = pq'$. Then $|ab|_{p,r} = p\,|qq'|_r$, $|ab|_{p,r} \sqsupset_{p,r} a$, and $|a \circ b|_{p,r} = p\,|q \circ q'|_r$.

For $n \geqslant 1$, without loss of generality assume $* = \circ$.

Case 3.2.2. $n = 1$. Then $b = pq' \circ b_1 \equiv pq'b_1 \circ pq'$. So $ab \equiv a(pq'b_1 \circ pq') \equiv p(q \circ q')\, b_1 \circ p(qq') \equiv p\, |q \circ q'|_r\, b_1 \circ p\, |qq'|_r$, which is in (p, r)–NF since $b_1 \leqslant_{\mathrm{L}} p$ and $qq' <_{\mathrm{L}} q \circ q'$. And $a \circ b \equiv ab \circ a \equiv p\, |q \circ q'|_r\, b_1 \circ (p\, |qq'|_r \circ pq) \equiv p\, |q \circ q'|_r\, b_1 \circ p\, |q \circ q'|_r \equiv p\, |q \circ q'|_r \circ b_1 \in (p, r)$–$NF$. To check $|ab|_{p,r} \sqsupset_{p,r} a$, we need $p\, |qq'|_r \sqsupset_{p,r} pq$ (which holds since Theorem 16 gives $|qq'|_r \sqsupset_r q$), and $p\, |qq'|_r \circ pq \equiv p(q \circ q') \ll_{p,r} p\, |q \circ q'|_r\, b_1$ (which is clear).

Case 3.2.3. $n > 1$. So $b = pq'b_1 \cdots b_{n-1} \circ b_n$. Then $ab \equiv p\, |q \circ q'|_r\, b_1\, |ab_2|_{p,r} \cdots |ab_{n-1}|_{p,r} \circ |ab_n|_{p,r}$, which is prenormal as in Lemma 15 and thus in (p, r)–NF. And $a \circ b \equiv p\, |q \circ q'|_r\, b_1\, |ab_2|_{p,r} \cdots |ab_{n-1}|_{p,r} \circ |a \circ b_n|_{p,r}$ is similarly in (p, r)–NF by Lemma 15. For $ab \sqsupset_{p,r} a$ we need $|ab_n|_{p,r} \sqsupset a$ (true by the induction hypothesis) and $ab_n \circ a \equiv a \circ b_n \ll_{p,r} p\, |q \circ q'|_r\, b_1\, |ab_2|_{p,r} \cdots |ab_{n-1}|_{p,r}$ (given just above).

Case 4. $a = pqa_0a_1 \cdots a_m$. Then if $b \ll_p a$, then $|ab|_{p,r} = ab$, $|ab|_{p,r} \sqsupset_{p,r} a$, and $|a \circ b|_{p,r}$ exists as in Lemma 13(ii).

So, letting $v = pqa_0 \cdots a_{m-1}$, assume that $b = vb_m \cdots b_{k-1} * b_k$ $(k \geqslant m)$, where $a_m \sqsupset_{p,r} b_m$ and $a_m \circ b_m \ll_{p,r} v$. We have that $|a_m b_m|_{p,r}$ and $|a_m \circ b_m|_{p,r}$ exist by the induction hypothesis on a (so that $v\, |a_m b_m|_{p,r}$ and $v\, |a_m \circ b_m|_{p,r} \in (p, r)$–$NF$), and that $|ab_i|$ exists for all i by the induction hypothesis on b.

The argument now is just as in Case 3, with va_m (respectively $vb_m \cdots * b_k$) playing the role of Case 3's pq (respectively $pq'b_1 \cdots * b_n$).

Case 5. $a = pqa_0 \cdots a_{m-1} \circ a_m$ (some $m \geqslant 0$) where $a_m \sqsupset_{p,r} b$ and $a_m \circ b \ll_{p,r} pqa_0 \cdots a_{m-1}$. Then $|ab|_{p,r} = pqa_0 \cdots a_{m-1}\, |a_m b|_{p,r}$, and $|a \circ b|_{p,r}$ exists by Lemma 13(ii). For $|ab|_{p,r} \sqsupset_{p,r} a$, we need $|a_m b| \sqsupset_{p,r} a_m$ (true by $a_m \sqsupset_{p,r} b$ and the induction hypothesis) and $a_m b \circ a_m \ll_{p,r} pqa_0 \cdots a_{m-1}$ (given).

Suppose d and e are bases for normal forms, i.e., d and e are either of the form p $(p \in B)$ or (p, r) $(p, r \in B)$. Then write $d \to e$ to mean that for every $w \in d$–NF, $|w|_e$ exists. Let $d \leftrightarrow e$ iff $d \to e$ and $e \to d$. A hypothesis of Theorem 20 was $x \to r$.

LEMMA 21. (i) $x \leftrightarrow x^{(i)}$.

(ii) *If* $p \in B$, $p \leftrightarrow x$, *then* $p \leftrightarrow |p^{(1)}|$.

(iii) *If* $p \in B$, $a \in p$–NF, *then* $|pa|_p$ *exists and* $|p \circ a|_p$ *exists.*

Proof. (i) In fact $NF = x^{(i)}$–NF. By induction on length w $(w \in P)$, using $x^{(i+n)} = (x^{(i)})^{(n)}$, it is seen that $w \in NF$ iff $w \in x^{(i)}$–NF.

(ii) Note that $|p^{(1)}|$ exists since $p^{(1)} \in p$–NF and $p \to x$. Let $F(p^{(i)}) = |p^{(1)}|^{(i-1)}$ for $i \geqslant 1$ and for $w \in p$–NF with $w < p^{(1)}$ let $F(w) = |w|$. As in (i),

F extends to a one–one F: p–$NF \to |p^{(1)}|$–NF with $w \equiv F(w)$. F is onto since $x \to p$, i.e., every $u \in NF$ with $u <_{\mathrm{L}} p^{(1)}$ is in range F.

(iii) By induction on a. If $a <_{\mathrm{L}} p$, then $pa \in p$–NF. If $a = p^{(i)} a_1 a_2 \cdots a_{n-1} * a_n$, then $pa \equiv p^{(i+1)} |pa_1|_p \; |pa_2|_p \cdots |pa_{n-1}|_p * |pa_n|_p$ by the induction hypothesis and Lemma 1, and again by Lemma 1 this word is in p–NF.

For $p \circ a$, if $p >_{\mathrm{L}} a$, then $|p \circ a|_p = p \circ a$. If $p \leqslant_{\mathrm{L}} a$ assume without loss of generality that $a = p^{(i)} a_1 \cdots a_{n-1} \circ a_n$. Then $|p \circ a|_p = p^{(i+1)} |pa_1|_p \cdots |pa_{n-1}|_p \circ |p \circ a_n|_p$.

Lemma 22. *For $p, r, r' \in B$, $r \to r' \Rightarrow (p, r) \to (p, r')$.*

Proof. By induction on $u \in (p, r)$–NF, show $|u|_{p, r'}$ exists. Let $H(w) = w$ ($w \in NF$, $w \leqslant_{\mathrm{L}} p$), and $H(pq) = p\, |q|_{r'}$ ($q \in r$–NF). Then H extends to H defined on (p, r)–NF such that $H(u) = |u|_{p, r'}$.

Lemma 23. *Suppose $p, r \in B$, $p \geqslant_{\mathrm{L}} r$, $|pr|$ exists, and $p \to x$. Then for any $q \in r$–NF, $|pq|_{|pr|}$ exists and $|pq|_{|pr|} \sqsupset_{|pr|} p$.*

Proof. Construct $|pq|_{|pr|}$ by induction on $q \in r$–NF.

If $q <_{\mathrm{L}} r$, then $q \in NF$ and, since $p \geqslant_{\mathrm{L}} r$, $pq \in p$–NF. So $|pq|$ exists by $p \to x$, and $|pq| <_{\mathrm{L}} |pr|$; thus $|pq|_{|pr|} = |pq|$. To check $|pq|_{|pr|} \sqsupset_{|pr|} p$, we have $|pq| \sqsupset p$ by Theorem 16, and $|pq| \circ p \equiv p \circ q <_{\mathrm{L}} pr$ by Lemma 1.

So assume $q = r^{(i)} a_0 \cdots a_{n-1} \circ a_n$. Then as each $|pa_m|_{|pr|}$ exists by the induction hypothesis, $pq \equiv |pr|^{(i)} \, |pa_0|_{|pr|} \cdots |pa_{n-1}|_{|pr|} * |pa_n|_{|pr|}$. To check $|pq|_{|pr|} \sqsupset_{|pr|} p$. In the case $* = \cdot$, then $|pq|_{|pr|} \geqslant_{|pr|} p$, whence $|pq|_{|pr|} \sqsupset_{|pr|} p$. So assume $* = \circ$, whence $|pq|_{|pr|} \equiv |pr|^{(i)} \, |pa_0|_{|pr|} \cdots |pa_{n-1}|_{|pr|} \circ |pa_n|_{|pr|}$. Then $|pa_n|_{|pr|} \sqsupset_{|pr|} p$ by the induction hypothesis, and $|p \circ a_n|_{|pr|}$ exists. And $|pa_n \circ p|_{|pr|} = |p \circ a_n|_{|pr|} <_{\mathrm{L}} |pr|^{(i)} \, |pa_0|_{|pr|} \cdots |pa_{n-2}|_{|pr|}$ by Lemma 1 and $a_n <_{\mathrm{L}} r^{(i)} a_0 \cdots a_{n-2}$. So $|pq|_{|pr|} \sqsupset_{|pr|} p$.

Lemma 24. *Assume $p, r \in B$, $p \geqslant_{\mathrm{L}} r$, $|pr|$ exists, and $p \to x$. Then $(p, r) \to |pr|$.*

Proof. By induction on $w \in (p, r)$–NF, show $|w|_{|pr|}$ exists.

Case 1. $w \leqslant_{\mathrm{L}} p$. Then $w \in NF$ and $w <_{\mathrm{L}} |pr|$ so $|w|_{|pr|} = w$.

Case 2. $w = pq$. This is Lemma 23.

Case 3. $w = pqa_0 a_1 \cdots a_{n-1} * a_n$. Then $a_0 \leqslant_{\mathrm{L}} p$, so by Lemma 23 (and the fact that $u \sqsupset_b v \geqslant_{\mathrm{L}} v'$ ($v' \in b$–NF) implies $u \sqsupset_b v'$) $|pq|_{|pr|} \sqsupset_{|p, r|} |a_0|_{|pr|}$. Now iterated Theorem 16 $n+1$ times to get the existence of $|w|_{|pr|}$.

LEMMA 25. (i) *If* $p \in B$ *and* $x \to p$, *then* $p \to (p, p)$.

(ii) *If* $p, r \in B$, $|pr|$ *exists*, $x \to p$, *and* $x \to (p, r)$, *then* $|pr| \to (p, r)$.

Proof. (i) By induction on $w \in p\text{–}NF$. If $w \leqslant_{\mathrm{L}} p$, then $|w|_{p,p} = w$. If $w = pa_0$ (respectively $p \circ a_0$), then $|w|_{p,p} = pa_0$ (respectively $pa_0 \circ p$). If $w = pa_0 a_1 \cdots a_{n-1} * a_n$ $(n > 0)$, then $pa_0 \in (p, p)\text{–}NF$ and, since $a_1 \leqslant_{\mathrm{L}} p$, $pa_0 \sqsupset_{p,p} |a_1|_{p,p}$. Then iterate Theorem 20 to get $|w|_{p,p}$. Suppose $w = p^{(i)} a_1 a_2 \cdots a_{n-1} * a_n$ for some $i > 0$, $n \geqslant 0$. Then $|p^{(i)}|_{p,p} = pp^{(i-1)}$. And since $a_1 <_{\mathrm{L}} p^{(i)}$, $|a_1|_{p,p}$ is either $\leqslant_{\mathrm{L}} p$ or of the form $pqb_1 \cdots b_{n-1} * b_n$, where $q \in p\text{–}NF$ must have rank $\leqslant i-2$, whence $q \sqsupset_p p^{(i-1)}$. In either case, $|p^{(i)}|_{p,p} \sqsupset_{p,p} |a_1|_{p,p}$ and again Theorem 20 applies.

(ii) By induction on $w \in |pr|\text{–}NF$. If $w <_{\mathrm{L}} |pr|$, then $w \in NF$ and by $x \to (p, r)$, $|w|_{p,r}$ exists. If $w = |pr|\, a_1 \cdots a_{n-1} * a_n$, then $a_1 <_{\mathrm{L}} pr$, and so either $|a_1|_{p,r} \leqslant_{\mathrm{L}} p$ or $|a_1|_{p,r} = pqb_1 b_2 \cdots b_{n-1} * b_n$, where $q <_{\mathrm{L}} r$, and thus $r \sqsupset_r q$. Thus $pr \sqsupset_{p,r} |a_1|_{p,r}$, and Theorem 20 applies. If $w = |pr|^{(i)} a_1 \cdots a_{n-1} * a_n$ for some $i > 0$, then again either $|a_1|_{p,r} \leqslant_{\mathrm{L}} p$ or $|a_1|_{p,r} = pqb_1 b_2 \cdots b_{n-1} * b_n$ where the r–NF rank of q is less than i, so Theorem 20 applies.

To do induction on B avoiding the problem that an initial segment of a member of B need not be in B, define B^* to be the smallest subset of P such that

(i) $x^{(i)} \in B^*$, for all i.

(ii) If $u, v \in B^*$ and $|u| \sqsupset |v|$, then $uv \in B^*$.

So the members of B^* are in general not in normal form.

LEMMA 26. $|u|$ *exists for each* $u \in B^*$ *and* $\{|u| : u \in B^*\} = B$.

Proof. That $|u|$ exists $(u \in B^*)$ and $\{|u| : u \in B^*\} \subseteq B$ follows by induction on u, using Theorem 16 and Lemma 2. Suppose $b \in B$ is of minimal length such that $b \notin \{|u| : u \in B^*\}$. Then b is not an $x^{(i)}$, so $b = x^{(i)} a_0 \cdots a_{n-1} a_n$, where $a_n \in B$. Let $w = x^{(i)} a_0 \cdots a_{n-1}$; then by Lemma 17, $b = wa_n \equiv b_0(b_1(\cdots(b_k a_n)))$, each $b_i \in B$, length $b_i \leqslant$ length $w <$ length b, and $b_k \sqsupset a_n$, $b_i \sqsupset |b_{i+1}(b_{i+2}(\cdots(b_k a_n)))|$ for $i < n$. By hypothesis a_n and each b_i are in $\{|u| : u \in B^*\}$; thus so is b.

LEMMA 27. *If* $b, c \in B$, *then* $b \leftrightarrow c$.

Proof. By Lemma 26 it suffices to show $|u| \leftrightarrow |v|$ for all $u, v \in B^*$. By Lemma 21(i), $x^{(i)} \leftrightarrow x^{(m)}$ for all i, m. Let $D \subset B^*$ be maximal such that $\{x^{(i)} : i < \omega\} \subset D$ and such that $|u| \leftrightarrow |v|$ $(u, v \in D)$. It suffices to show $u, v \in D$ and $|u| \sqsupset |v|$ implies $uv \in D$. To show $uv \in D$ it suffices to show $|u| \leftrightarrow |uv|$.

(a) $|u| \leftrightarrow (|u|, |u|)$.

(i) $|u| \rightarrow (|u|, |u|)$ follows from Lemma 25(i), since $x \rightarrow |u|$.

(ii) $(|u|, |u|) \rightarrow |u^{(1)}|$ by Lemma 24 ($|u^{(1)}|$ exists since $|u| \rightarrow x$), and $|u^{(1)}| \rightarrow u$ by Lemma 21.

(b) $(|u|, |u|) \leftrightarrow (|u|, |v|)$. This follows from Lemma 22 and $|u| \leftrightarrow |v|$.

(c) $(|u|, |v|) \leftrightarrow |uv|$.

(i) $(|u|, |v|) \rightarrow |uv|$ is Lemma 24.

(ii) Given $x \rightarrow |u|$, and, by (a) and (b), $x \rightarrow (|u|, |v|)$, Lemma 25(ii) gives $|uv| \rightarrow (|u|, |v|)$.

THEOREM 28. *Assume for some λ, $\mathscr{E}_\lambda \neq \varnothing$.*

(1) *$|w|_b$ exists for all $w \in P$ and $b \in B$.*

(2) *$<_{\mathrm{L}} = <$, and is a linear order of $\mathscr{P}$, $\mathscr{A}$.*

(3) *The word problems for $\mathscr{P}$, $\mathscr{A}$ are decidable.*

(4) *For each $j \in \mathscr{E}_\lambda$, $\mathscr{P}_j \cong \mathscr{P}$, $\mathscr{A}_j \cong \mathscr{A}$.*

Proof. (1) By Lemma 27, it suffices to prove it for $b = x$. It suffices to show that if $u, v \in NF$, then $|uv|$, $|u \circ v|$ exist. Suppose $u \in B$. Then by Lemma 27, $|v|_u$ exists. By Lemma 21(iii), $|uv|_u$ and $|u \circ v|_u$ exist. By Lemma 27, $|uv|$ and $|u \circ v|$ exist. By Lemma 11(2), an arbitrary $u \in NF$ is equivalent to a word of the form $p_0 \circ p_1 \circ \cdots \circ p_n$, each $p_i \in B$, and repeating the above procedure for each p_i yields $|uv|$ and $|u \circ v|$.

(2) Claim that for $w, u \in P$, $u <_{\mathrm{L}} wu$. Assume $w \in B$; then $|u|_w$ exists by (1). Then rank $|wu|_w =$ rank $|u|_w + 1$, so $|u|_w <_{\mathrm{L}} |wu|_w$. For arbitrary w, w is equivalent by Lemma 11 to a word of the form $p_0 \circ p_1 \circ \cdots \circ p_n$, $p_i \in B$; iterate this procedure to get $u <_{\mathrm{L}} p_n(u) <_{\mathrm{L}} p_{n-1}(p_n u) <_{\mathrm{L}} \cdots <_{\mathrm{L}} p_0(p_1(p_1(\cdots(p_n u))) \equiv wu$.

Thus if u is a proper component of w, then $u <_{\mathrm{L}} w$, whence $u < w \Rightarrow u <_{\mathrm{L}} w$. Since $u <_{\mathrm{L}} w \Rightarrow u < w$, $<$ equals $<_{\mathrm{L}}$ on $\mathscr{P}$. By Lemma 3 the same is true for $\mathscr{A}$. A derivation of $<_{\mathrm{L}} = <$ from the linearity of $<_{\mathrm{L}}$ has been given by R. McKenzie.

Finally, for (3), the existence of an algorithm deciding the equivalence of two words in P is a consequence of the linearity of $<$ (or of the linearity of $<_{\mathrm{L}}$, or of the existence of unique normal forms) and the derivation of (4) from the linearity of $<_{\mathrm{L}}$ and Theorem 7 was given in the introduction.

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