FEM - calculator

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Here are some mathematical equations that are used when finite elements are to be calculated:

$$\dot{q_1} = \frac{k}{L}(T_1 - T_2) \tag{1}$$

$$\dot{q_2} = -\frac{k}{L}(T_1 - T_2) \tag{2}$$

$$\dot{\bar{q}}_2^e = \dot{\bar{k}}^e \cdot \dot{\bar{T}}^e \tag{3}$$

$$\bar{k}^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} = \frac{k}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4}$$

$$\dot{\bar{T}}^e = \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} \tag{5}$$

$$\dot{\bar{q}}^e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} \tag{6}$$

$$\bar{c^e} = \frac{L \cdot c \cdot \rho}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{7}$$

$$\bar{K} = \begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 \\ k_{12}^1 & (k_{22}^1 + k_{11}^2) & k_{12}^2 & 0 \\ 0 & k_{21}^2 & (k_{22}^2 + k_{11}^3) & k_{12}^3 \\ 0 & 0 & k_{21}^3 & k_{22}^3 \end{bmatrix} \tag{8}$$

$$\bar{C} \cdot \dot{\bar{T}} + \bar{K} \cdot \bar{T} = \bar{Q} \tag{9}$$

$$\dot{Q}_i = \epsilon\sigma(T_r^4 - T_{s,i}^4) + h(T_g - T_{s,i}) \tag{10}$$

$$\dot{\bar{T}} \cong \frac{\Delta\bar{T}}{\Delta t} = \frac{\bar{T}^{j+1} - \bar{T}^j}{\Delta t} \tag{11}$$

$$\bar{C}\left[\tfrac{\bar{T}^{j+1} - T^j}{\Delta t}\right] + \bar{K}\bar{T} = \bar{Q} \tag{12}$$

$$\bar{T}^{j+1} = \left(\tfrac{\bar{C}}{\Delta t} + \bar{K}\right)^{-1} \cdot \left(\bar{Q}^j + \bar{C}\bar{T}^j\right)\Delta t \tag{13}$$

Finite elements are calculated through dividing an element/wall into several small parts as shown in figure 1. Thereafter an iterative process is performed for every time-step until equilibrium is reached for respective time step. This specific Finite Element calculator is based on incoming heat flux and the ambient temperature on the none exposed side of the element.

Fire compartment	...	m - 1	m	m + 1	...	Ambient temperature

Whereas each segment between each node can be described as:

T_1 q_1	k, ρ, c L, A	T_2 q_2

Where L is the distance between the nodes, A is the section area. Furthermore k is the thermal conductivity of the material, c the specific heat capacity of relevant material and ρ the density.

Let's do a test calculation with concrete in order to understand what is happening during static boundary conditions to simulate a simpler case.

Concrete	k	c	ρ	A	L
	1.5 $\frac{W}{mK}$	900 $\frac{Ws}{kgK}$	2300 $\frac{kg}{m^3}$	1 m^2	0.05 m

T_1 is assumed to be 600°C, T_2, which is the ambient temperature, is commonly assumed to be 20°C which will be the case here as well. q_1 and q_2 will be dynamically calculated thus the expected temperature rise will otherwise be linear, that would be too much of a simplification.

The iterative process will start with calculating $\bar{k}^e$, equation (4), specifically in this case the thermal conductivity is assumed to be constant, otherwise the material would have initially have contracted the ambient temperature.

$$\bar{k}^e = \begin{bmatrix} k_{11}^e & k_{12}^e \\ k_{21}^e & k_{22}^e \end{bmatrix} = \frac{k}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1.5}{900}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

In this simple case $\bar{k}^e$ will be constant and does not have to be tampered with. $\bar{C}^e$ is another constant and will not be tampered with after this initial calculation:

$$\bar{c^e} = \frac{0.05 \cdot 900 \cdot 2300}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 51750\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The global heat balance equation will be slightly altered with to achieve an equation that can be used in one dimensional cases:

$$\bar{C}\cdot\dot{T}+\bar{K}\bar{T}=\bar{Q}\rightarrow \bar{c}^{e}\dot{\bar{T}}+\bar{k}^{e}\dot{\bar{T}}^{e}=\dot{\bar{q}}^{e} \tag{14}$$

Which if derived a bit further equates to:

$$\bar{c}^{e}\left[\frac{\bar{T}^{j+1}-T^{j}}{\Delta t}\right]+\bar{k}^{e}\dot{\bar{T}}^{e}=\dot{\bar{q}}^{e} \tag{15}$$

We also need an equation in order to calculate $\bar{T}^{j+1}$ which is as follows:

$$\bar{T}^{j+1}=\left(\frac{\bar{c}^{e}}{\Delta t}+\bar{k}^{e}\right)^{-1}\left(\dot{\bar{q}}^{e}+\bar{c}^{e}\dot{\bar{T}}^{e}\right) \tag{16}$$

We can now start the calculations, to start with equation (5) and (3) is combined to get a starting value of $\bar{T}$ as well as $\dot{\bar{q}}^{e}$. In the next stage $\bar{T}^{j+1}$ is calculated to receive a new $\dot{\bar{q}}^{e}$ which is then inserted in equation (16) again to get the next temperature.