Otsu's Binarization Algorithm: More Steps in Derivation 1

$w_0(t) = \sum_{i=0}^{t} \hat{h}(i)$, $\quad w_1(t) = \sum_{i=t+1}^{L-1} \hat{h}(i)$,

$\mu_0(t) = \frac{1}{w_0(t)} \sum_{i=0}^{t} i\hat{h}(i)$, $\quad \mu_1(t) = \frac{1}{w_1(t)} \sum_{i=t+1}^{L-1} i\hat{h}(i)$,

$v_0(t) = \frac{1}{w_0(t)} \sum_{i=0}^{t} \hat{h}(i)\left(i-\mu_0(t)\right)^2$, $\quad v_1(t) = \frac{1}{w_1(t)} \sum_{i=t+1}^{L-1} \hat{h}(i)\left(i-\mu_1(t)\right)^2$,

- For better performance, use recurrence relation on t
- For every t, $\mu = \sum_{i=0}^{L-1} i\hat{h}(i)$, $v = \sum_{i=0}^{L-1} (i-\mu)^2 \hat{h}(i)$
- Rewrite the equation of v:

$$
\begin{aligned}
v &= \sum_{i=0}^{t} (i-\mu_0(t)+\mu_0(t)-\mu)^2\, \hat{h}(i) + \sum_{i=t+1}^{L-1} (i-\mu_1(t)+\mu_1(t)-\mu)^2 \hat{h}(i) \\
&= \sum_{i=0}^{t} \left[\left(i-\mu_0(t)\right)^2 + \underbrace{2(i-\mu_0(t))(\mu_0(t)-\mu)}_{0} + (\mu_0(t)-\mu)^2\right] \hat{h}(i) + \\
&\quad \sum_{i=t+1}^{L-1} \left[\left(i-\mu_1(t)\right)^2 + \underbrace{2(i-\mu_1(t))(\mu_1(t)-\mu)}_{0} + (\mu_1(t)-\mu)^2\right] \hat{h}(i) \\
&= \sum_{i=0}^{t} \left[\left(i-\mu_0(t)\right)^2 + (\mu_0(t)-\mu)^2\right] \hat{h}(i) + \sum_{i=t+1}^{L-1} \left[\left(i-\mu_1(t)\right)^2 + (\mu_1(t)-\mu)^2\right] \hat{h}(i) \\
&= \sum_{i=0}^{t} \left(i-\mu_0(t)\right)^2 \hat{h}(i) + (\mu_0(t)-\mu)^2 \sum_{i=0}^{t} \hat{h}(i) + \sum_{i=t+1}^{L-1} \left(i-\mu_1(t)\right)^2 \hat{h}(i) + (\mu_1(t)-\mu)^2 \sum_{i=t+1}^{L-1} \hat{h}(i) \\
&\boxed{= \{(\mu_0(t)-\mu)^2 w_0(t) + (\mu_1(t)-\mu)^2 w_1(t)\} + \{w_0(t)v_0(t) + w_1(t)v_1(t)\}} \\
&= \{(\mu_0(t)-\mu)^2 w_0(t) + (\mu_1(t)-\mu)^2 (1-w_0(t))\} + v_{within}(t) \qquad (\because w_0(t)+w_1(t)=1\ ) \\
&\boxed{= w_0(t)(1-w_0(t))(\mu_0(t)-\mu_1(t))^2 + v_{within}(t)} \qquad (\because w_0(t)\mu_0(t)+w_1(t)\mu_1(t)=\mu\ ) \\
&= v_{between}(t)\ + v_{within}(t)
\end{aligned}
$$

Minimize $v_{within}(t)$ → Maximize $v_{between}(t)$, where $v_{between}(t) = w_0(t)\left(1-w_0(t)\right)\left(\mu_0(t)-\mu_1(t)\right)^2, t \in \{0,1,\dots,L-1\}$

Otsu's Binarization Algorithm: More Steps in Derivation 2

$$= \boxed{\{(\mu_0(t)-\mu)^2 w_0(t) + (\mu_1(t)-\mu)^2 w_1(t)\} + \{w_0(t)v_0(t) + w_1(t)v_1(t)\}}$$

$$= \{(\mu_0-\mu)^2 w_0 + (\mu_1-\mu)^2 w_1\} + v_{within}(t)$$

(Omit t from $\mu_0(t), \mu_1(t), w_0(t), w_1(t)$ for simplicity)

$$= \{(\mu_0^2 - 2\mu_0\mu + \mu^2)w_0 + (\mu_1^2 - 2\mu_1\mu + \mu^2)w_1\} + v_{within}(t)$$

$$= (\mu_0^2 w_0 + \mu_1^2 w_1) - 2\mu(\mu_0 w_0 + \mu_1 w_1) + \mu^2(w_0 + w_1) + v_{within}(t)$$

$$= (\mu_0^2 w_0 + \mu_1^2 w_1) - 2\mu^2 + \mu^2 + v_{within}(t)$$

$(\because w_0(t)\mu_0(t) + w_1(t)\mu_1(t) = \mu)$ $(\because w_0(t) + w_1(t) = 1)$

$$= (\mu_0^2 w_0 + \mu_1^2 w_1) - \mu^2 + v_{within}(t)$$

$(\because w_0(t)\mu_0(t) + w_1(t)\mu_1(t) = \mu)$ $(\because w_0(t) + w_1(t) = 1)$

$$= \{\mu_0^2 w_0(w_0 + w_1) + \mu_1^2 w_1(w_0 + w_1)\} - (w_0\mu_0 + w_1\mu_1)^2 + v_{within}(t)$$

$$= \{\cancel{\mu_0^2 w_0^2} + \cancel{\mu_1^2 w_1^2} + w_0 w_1(\mu_0^2 + \mu_1^2)\} - (\cancel{w_0^2\mu_0^2} + \cancel{w_1^2\mu_1^2} + 2w_0 w_1 \mu_0 \mu_1) + v_{within}(t)$$

$$= w_0 w_1(\mu_0^2 + \mu_1^2 - 2\mu_0\mu_1) + v_{within}(t)$$

$$= w_0 w_1(\mu_0 - \mu_1)^2 + v_{within}(t) = \boxed{w_0(1-w_0)(\mu_0-\mu_1)^2 + v_{within}(t)}$$

Minimize $v_{within}(t)$ → Maximize $v_{between}(t)$, where $v_{between}(t) = w_0(t)\big(1-w_0(t)\big)\big(\mu_0(t)-\mu_1(t)\big)^2, t \in \{0,1,\ldots,L-1\}$