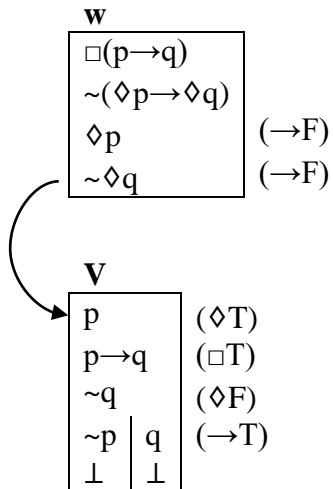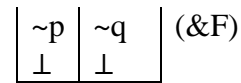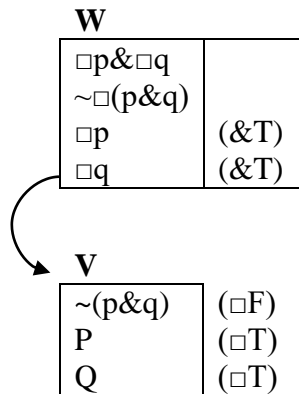Garson, James (2006). *Modal Logic for Philosophers*, ch. 4

Exercise 4.1 Use diagrams to show that the following arguments are K-valid

a) □(p→q) / ◊p→◊q
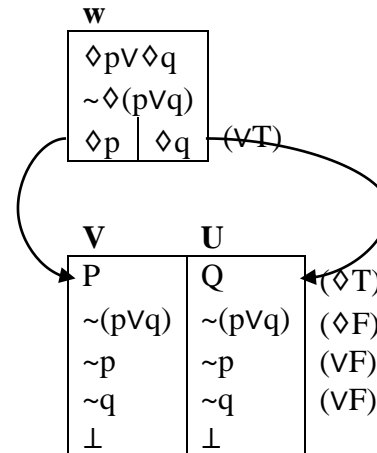
w

□(p→q)	
~(◊p→◊q)	
◊p	(→F)
~◊q	(→F)

V

p		(◊T)
p→q		(□T)
~q		(◊F)
~p	q	(→T)
⊥	⊥	

b) □p&□q / □(p&q)

W

□p&□q	
~□(p&q)	
□p	(&T)
□q	(&T)

V

~(p&q)	(□F)
P	(□T)
Q	(□T)

~p	~q	(&F)
⊥	⊥	

c) ◊p∨◊q / ◊(p∨q)

w

◊p∨◊q		
~◊(p∨q)		
◊p	◊q	(∨T)

V   **U**

P	Q	(◊T)
~(p∨q)	~(p∨q)	(◊F)
~p	~p	(∨F)
~q	~q	(∨F)
⊥	⊥	

Exercise 4.2

Show K-valid with trees.

a) □p, □(p→q) / ◇r→□q

V

□p	
□(p→q)	
~(◇r→□q)	
◇r	(→F)
~□q	(→F)

W

~q		(□F)
p		(□T)
p→q		(□T)
~p	q	(→T)
⊥	⊥	

b) ◇p&◇q / ◇(pvr)

V

◇p&◇q	
~◇(pvr)	
◇p	(&T)
◇q	(&T)

W

p	(◇T)
~(pvr)	(◇F)

~p	(VF)
~r	(VF)
⊥	

c) ◇~q→□□~~p / □□pV□q

V

◇~q→□□~~p		
~(□□pV□q)		
~□□p		(VF)
~□q		(VF)
~◇~q	□□~~p	(→T)

W   **U**

~q	(□F)	~□p	(□F)
~~q	(◇F)	□~~p	(□T)
⊥			

W'

~p	(□F)
~~p	(□T)
⊥	
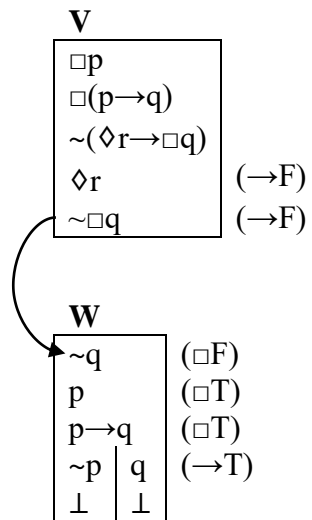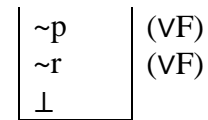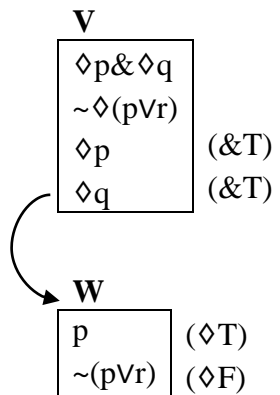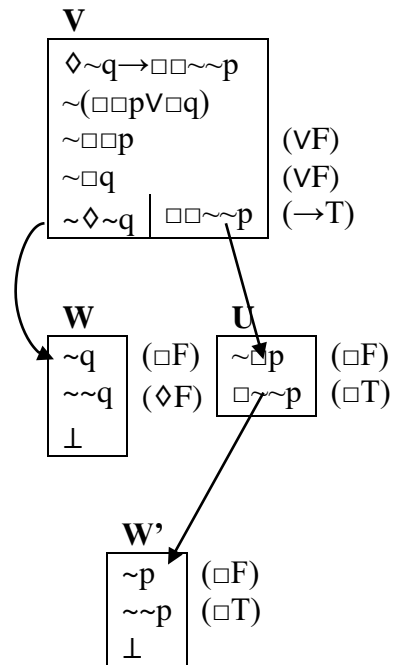
Exercise 4.3

Add formulas to the diagram we just completed to verify that it counts as a counterexample to p→□q / □(p→q).
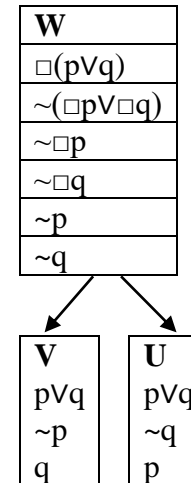
```
┌─────────────┐
│ W           │
│ p→□q        │
│ ~□(p→q)     │
│             │
│ ~p          │
│ ~q          │
└─────────────┘
       │
       ▼
┌─────────────┐
│ V           │
│ ~(p→q)      │
│ p           │
│ ~q          │
└─────────────┘
```
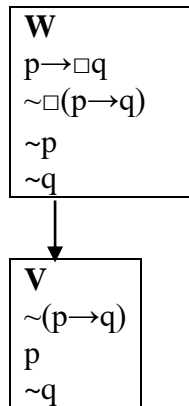
Exercise 4.4

Verify that the last diagram is a counterexample to □(pVq) / □pV□q.

```
┌─────────────┐
│ W           │
├─────────────┤
│ □(pVq)      │
├─────────────┤
│ ~(□pV□q)    │
├─────────────┤
│ ~□p         │
├─────────────┤
│ ~□q         │
├─────────────┤
│ ~p          │
├─────────────┤
│ ~q          │
└─────────────┘
       │
   ┌───┴───┐
   ▼       ▼
┌─────┐ ┌─────┐
│ V   │ │ U   │
│ pVq │ │ pVq │
│ ~p  │ │ ~q  │
│ q   │ │ p   │
└─────┘ └─────┘
```
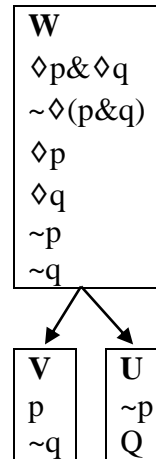
Exercise 4.5

Construct counterexamples to the following arguments using the tree method. Verify that the diagrams you create are counterexamples.
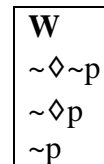
a) □p→□q / □(p→q)

W
□p→□q
~□(p→q)
~□p
~p
~q

V
~(p→q)
p
~q

U
~p
~q

b) ◊p&◊q / ◊(p&q)

W
◊p&◊q
~◊(p&q)
◊p
◊q
~p
~q

V
p
~q

U
~p
Q

c) ~◊~p / ◊p
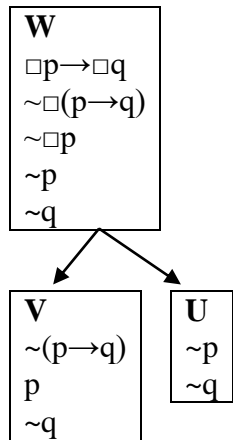
W
~◊~p
~◊p
~p

Exercise 4.6

Verify that the counterexample diagram just given is indeed a K-counterexample to □p, □~q→□~p / □q

```
┌─────────────┐
│ W           │
│ ~p          │
│ ~q          │
│ □p          │
│ □~q→□~p     │
│ ~□q         │
│ ~□~q        │
└─────────────┘
      ╱    ╲
     ↙      ↘
┌──────┐  ┌──────┐
│ V    │  │ U    │
│ ~~q  │  │ ~q   │
│ q    │  │ p    │
│ p    │  └──────┘
└──────┘
```
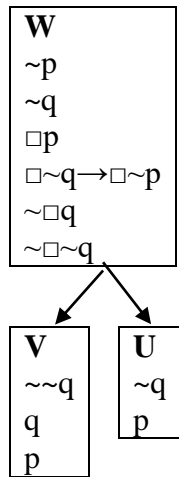
Exercise 4.7

Suppose we pruned the left-hand branch in the above tree. What would the counterexample diagram look like?

```
┌─────────────┐
│ W           │
│ □(pVq)      │
│ pVq         │
│ ~(□pV□q)    │
│ ~□p         │
│ ~□q         │
│ q           │
│ ~p          │
└─────────────┘
      ╱    ╲
     ↙      ↘
┌──────┐  ┌──────┐
│ V'   │  │ U'   │
│ pVq  │  │ pVq  │
│ p    │  │ q    │
│ ~q   │  │ ~p   │
└──────┘  └──────┘
```
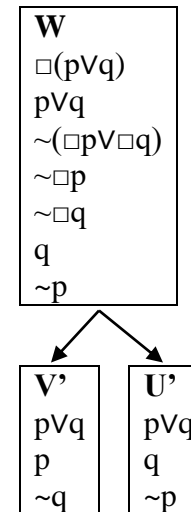
Exercise 4.8

Use the tree method to determine whether the following arguments are K-valid or K-invalid. If an argument is K-invalid create counterexample diagrams, and verify that each diagram is a counterexample. For e) and f) you may use (↔T) and (↔F). Or if you like, you could convert the ↔ into → and & and work the trees from there.

a) ◊(p∨q) / ◊p∨◊q

```
┌─────────────┐
│ W           │
│ ◊(p∨q)      │
│ ~(◊p∨◊q)    │
│ ~◊p         │
│ ~◊q         │
└─────────────┘
        │
        ↓
┌─────────────┐
│ V           │
│ p∨q         │
│ P    │ q    │
│ ~p   │ ~p   │
│ ⊥    │ ~q   │
│      │ ⊥    │
└─────────────┘
```

b) / □(~□p→(◊~p&◊q))

```
┌───────────────────────┐
│ W                     │
│ ~□(~□p→(◊~p&◊q))      │
│ ~p                    │
│ ~q                    │
└───────────────────────┘
        │
        ↓
┌───────────────────────┐
│ V                     │
│ ~(~□p→(◊~p&◊q))       │
│ ~□p                   │
│ ~(◊~p&◊q)             │
│ ~◊q                   │
│ ~p                    │
│ ~q                    │
└───────────────────────┘
        │
        ↓
┌───────┐
│ U     │
│ ~p    │
│ ~q    │
└───────┘
```

c) ◊p / □(◊p∨◊q)

```
W
◊p
~□(◊p∨◊q)
~p
~q
```
↓
```
V
P
~q
```

d) □q, ◊p∨◊q / ◊(p&q)

```
W
□q
◊p∨◊q
~◊(p&q)
◊q
~p
~q
```
↓
```
V
q
~(p&q)
~p
```

e) □(p↔q) / □p↔□q

```
W
□(p↔q)
~(□p↔□q)
□p      ~□p
~□q      □q
```

V			U		
~q			~p		
P			q		
p↔q			p↔q		
p	~p		p	~p	
q	~q		q	~q	
⊥	⊥		⊥	⊥	

f) □p↔□q / □(p↔q)

```
W
□p↔□q
~□(p↔q)
~□p
~□q
~p
~q
```

```
V
~(p↔q)
p
~q
```
```
U
~p
~q
```

g) □(q→p)V□(~p→q), ~□(~pVq) / □(~p→~q)

```
W
□(q→p)V□(~p→q)
~□(~pVq)
~□(~p→~q)
□(~p→q)
~p
~q
```

```
V
~(~pVq)
~p→q
p
~q
```

```
U
~(~p→~q)
~p→q
~p
q
```

h) □p / ◇p

```
W
□p
p
```