Jacob Archambault
Fitting, Melvin and Richard L. Mendelsohn (1998). *First-Order Modal Logic*, ch. 1.

Exercise 1.3.1.
Using the definition verify that, in fact, ((□P&◊Q)→◊(P&Q)) is a formula.

Rules
1. Every propositional letter is a formula
2. If X is a formula, so is ~X.
3. If X and Y are formulas, and ○ is a binary connective, (X○Y) is a formula.
4. If X is a formula, so are □X and ◊X

Proof
1. P is a formula [Rule 1]
2. So □P is a formula [1, Rule 4]
3. Q is a formula [Rule 1]
4. So ◊Q is a formula [3, Rule 4]
5. So (□P&◊Q) is a formula [2,4, Rule 3]
6. (P&Q) is a formula [1, 3, Rule 3]
7. So ◊(P&Q) is a formula [6, Rule 4]
8. So ((□P&◊Q)→◊(P&Q)) is a formula [5, 7, Rule 3]

Exercise 1.5.1
1. Show that we can define a truth-functional operator for which □P→P is a thesis but for which P→□P is not.
2. Show that if we also require that ~□P not be a thesis, no truth-functional operator will do.

The following table gives the only possible truth-functional interpretation of □ that specifies condition 1. But by doing so, it automatically fails condition 2.

P	□P	□P→P	P→□P	~□P
1	0	1	0	1
0	0	1	1	1

In order to make □P→P true when P is false, □P must be false as well. And in order to avoid making P→□P true when P is true, □P must be false when P is true. So if the desired theses are to be obtained, then □P must always be false. But if this is so, then ~□P must be a theorem. So, there is no truth functional operator that satisfies both the conditions of 1 and the condition of 2.

Exercise 1.6.1
Show that, at each possible world Γ of a modal model,
a) Γ||-(□X↔~◊~X) and
b) Γ||-(◊X↔~□~X).

1. To show that Γ||-(□X↔~◊~X):
2. First, we show that Γ||-(□X→~◊~X)
3. Second, we show that Γ||-(~◊~X→□X).
4. To show the first, it is sufficient to show for an arbitrary Γ that if Γ||-□X, then Γ||-~◊~X. This is shown by *reductio*
5. Presume Γ||-□X and Γ⊮~◊~X
6. Then for all Δ, if Γ*R*Δ, then Δ||-X [5, □]
7. And Γ||-◊~X [5, ~]
8. So there is some Δ such that Γ*R*Δ, and Δ||-~X [7, ◊]
9. So there is some Δ such that Γ*R*Δ and Δ⊮X [8, ~]
10. ⊥ [6, 9]
11. **Therefore, Γ||-(□X→~◊~X)** [3-10]
12. To show the second, we show that if Γ⊮□X, then Γ⊮~◊~X
13. Presume Γ⊮□X
14. Then there is some Δ such that Δ*R*Γ and Δ⊮X [13, □–]
15. So Δ||-~X [14, ~]
16. So, since Γ*R*Δ, Γ||-◊~X [15, ◊]
17. So, Γ||-~~◊~X [16, ~~]
18. So Γ⊮~◊~X [17, ~]
19. **Therefore, Γ||-(~◊~X→□X)** **[13-18]**
20. **Therefore, Γ||-(□X↔~◊~X)** **[11, 19]**

1. To show that for every Γ, Γ ϵ *g*, Γ||-(◊X↔~□~X):
2. First, we show that Γ||-◊X→~□~X
3. Second, we show that Γ||-~□~X→◊X

4. The first is proven by *reductio*
5. Presume $\Gamma \nVdash \Diamond X \rightarrow \sim\Box\sim X$
6. Then $\Gamma \Vdash \Diamond X$ and $\Gamma \nVdash \sim\Box\sim X$
7. So there is some $\Delta \Box g$ such that $\Gamma R \Delta$ and $\Delta \Vdash X$ [6, ◊]
8. And $\Gamma \Vdash \Box\sim X$
9. That is, for *all* Δ, if $\Gamma R \Delta$, then $\Delta \Vdash \sim X$ [8, □]
10. So for *all* Δ, if $\Gamma R \Delta$, then $\Delta \nVdash X$ [9, ~]
11. $\bot$ [7, 9]
12. **Therefore, $\Gamma \Vdash \Diamond X \rightarrow \sim\Box\sim X$**
13. To show that $\Gamma \Vdash \sim\Box\sim X \rightarrow \Diamond X$, we show that if $\Gamma \Vdash \sim\Box\sim X$, then $\Gamma \Vdash \Diamond X$:
14. Presume $\Gamma \Vdash \sim\Box\sim X$
15. Then $\Gamma \nVdash \Box\sim X$ [14, ~]
16. So it is not the case that for all Δ, if $\Gamma R \Delta$, then $\Delta \Vdash \sim X$
17. So, there is some Δ such that $\Gamma R \Delta$ and $\Delta \Vdash X$
18. Which is to say that $\Gamma \Vdash \Diamond X$ [17,◊]
19. **Therefore, $\Gamma \Vdash \sim\Box\sim X \rightarrow \Diamond X$**
20. **Therefore, $\Gamma \Vdash \Diamond X \leftrightarrow \sim\Box\sim X$**

Exercise 1.6.2

Show that if a world Γ of a model has no worlds accessible to it, then at Γ every formula is necessary, but none are possible.

1. Let Γ be a world such that there is no Δ for which $\Gamma R \Delta$, and let A be an arbitrary formula.
2. Then, $\Gamma \Vdash \Box A$ iff for all Δ, if $\Gamma R \Delta$, then $\Delta \Vdash A$
3. But since there is no world for which $\Gamma R \Delta$, the antecedent condition on "if $\Gamma R \Delta$, then $\Delta \Vdash A$" fails.
4. Therefore, by the truth table for the material conditional, "if $\Gamma R \Delta$, then $\Delta \Vdash A$" is true.
5. **And so $\Gamma \Vdash \Box A$**
6. Now, $\Gamma \Vdash \Diamond A$ iff there is some world Δ such that $\Gamma R \Delta$ and $\Delta \Vdash A$
7. But there is *no* world Δ such that $\Gamma R \Delta$.
8. **Therefore, $\Gamma \nVdash \Diamond A$**

Exercise 1.7.1

Show that $\nVdash (\Diamond P \& \Diamond Q) \rightarrow \Diamond(P \& Q)$

1. Let <g, R, ||-> be a model where g = {Γ, Δ, Ω}, $\Gamma R \Omega$ and $\Gamma R \Delta$, and $\Delta \Vdash P$, $\Omega \Vdash Q$. All other atomic sentences at all other worlds are false and no other worlds access any others..
2. Then, since $\Delta \Vdash P$ and $\Gamma R \Delta$, $\Gamma \Vdash \Diamond P$
3. Similarly, since $\Omega \Vdash Q$ and $\Gamma R \Omega$, $\Gamma \Vdash \Diamond Q$
4. So $\Gamma \Vdash \Diamond P \& \Diamond Q$
5. But, since there is no world Ψ such that $\Gamma R \Psi$ and $\Psi \Vdash P \& Q$, $\Gamma \nVdash \Diamond(P \& Q)$
6. So $\Gamma \Vdash \Diamond P \& \Diamond Q$ and $\Gamma \Vdash \Diamond(P \& Q)$
7. That is, $\Gamma \nVdash (\Diamond P \& \Diamond Q) \rightarrow \Diamond(P \& Q)$

Exercise 1.7.2

Show that $\Gamma \nVdash_k \Diamond P \rightarrow \Box\Diamond P$

1. Let <*g*, *R*, ||-> be a model where *g* = {Γ, Δ}, $\Gamma R \Delta$, and $\Delta \Vdash P$. No other relations hold on worlds, and for all remaining propositional parameters at all worlds are valuated as false.
2. Since $\Gamma R \Delta$ and $\Delta \Vdash P$, $\Gamma \Vdash \Diamond P$.
3. But since Δ accesses no worlds $\Delta \nVdash \Diamond P$
4. And since $\Gamma R \Delta$, there is some world that Γ accesses where $\Diamond P$ fails.
5. Therefore, $\Gamma \nVdash \Box\Diamond P$
6. So, since $\Gamma \Vdash \Diamond P$ and $\Gamma \nVdash \Box\Diamond P$, $\Gamma \nVdash \Diamond P \rightarrow \Box\Diamond P$

Exercise 1.7.3

Show that $\Gamma \nVdash_k (\Diamond\Box P \& \Diamond\Box Q) \rightarrow \Diamond\Box(P \& Q)$

1. Let <*g*, *R*, ||-> be a model where *g* = {Γ, Δ, Z, Θ, K}. $\Gamma R \Delta$, $\Gamma R Z$, $\Delta R \Theta$, $Z R K$, and no other relations hold on worlds. $\Theta \Vdash P$, $K \Vdash Q$, and all remaining propositions at all worlds are valuated as false.
2. So, since Δ only $R\Theta$ and $\Theta \Vdash P$, $\Delta \Vdash \Box P$
3. Similarly, since Z only RK and $K \Vdash Q$, $Z \Vdash \Box Q$
4. Since $\Gamma R \Delta$ and $\Delta \Vdash \Box P$, $\Gamma \Vdash \Diamond\Box P$
5. And since $Z \Vdash \Box Q$ and $\Gamma R Z$, $\Gamma \Vdash \Diamond\Box Q$
6. So $\Gamma \Vdash \Diamond\Box P \& \Diamond\Box Q$ [4, 5, &]
7. But for every world in *g*, $\nVdash P \& Q$
8. Since $\Delta R \Theta$ and $\Theta \nVdash P \& Q$, $\Delta \nVdash \Box(P \& Q)$

9. Similarly, since ZRK and $K\nVdash P\&Q$, $Z\nVdash\Box(P\&Q)$
10. So at every world W such that ΓRW, $W\nVdash\Box(P\&Q)$
11. So $\Gamma\nVdash\Diamond\Box(P\&Q)$
12. Therefore, $\Gamma\nVdash_k(\Diamond\Box P\&\Diamond\Box Q)\to\Diamond\Box(P\&Q)$ [6, 11]

Exercise 1.7.4

Show that the following are true at every possible world of every model:

1. $(\Box P\&\Box Q)\to\Box(P\&Q)$
2. $\Box(P\to Q)\to(\Box P\to\Box Q)$

To show that $\Vdash_k(\Box P\&\Box Q)\to\Box(P\&Q)$:

1. Let $\langle g, R, \Vdash\rangle$ be a model and Γ an arbitrary member of g. Then, if $\Gamma\Vdash(\Box P\&\Box Q)$, $\Gamma\Vdash\Box(P\&Q)$.
2. Presume $\Gamma\Vdash(\Box P\&\Box Q)$.
3. Then $\Gamma\Vdash\Box P$ and $\Gamma\Vdash\Box Q$
4. So for every world Δ in g, if $\Gamma R\Delta$, then $\Delta\Vdash P$ and $\Vdash Q$
5. So for every world Δ in g, if $\Gamma R\Delta$ then $\Delta\Vdash P\&Q$ [4, &]
6. Therefore, $\Gamma\Vdash\Box(P\&Q)$ [5, $\Box$]

To show that $\Vdash_k\Box(P\to Q)\to(\Box P\to\Box Q)$:

1. Let $\langle g, R, \Vdash\rangle$ be a model and Γ an arbitrary member of g. Then, if $\Gamma\Vdash\Box(P\to Q)$, $\Gamma\Vdash(\Box P\to\Box Q)$. By *reductio*:
2. Presume $\Gamma\Vdash\Box(P\to Q)$ and $\Gamma\nVdash(\Box P\to\Box Q)$
3. Then, for every world Δ in g, if $\Gamma R\Delta$, then $\Delta\Vdash P\to Q$
4. That is, either $\Delta\nVdash P$ or $\Delta\Vdash Q$
5. Now, since $\Gamma\nVdash(\Box P\to\Box Q)$, $\Gamma\Vdash\Box P$ and $\Gamma\nVdash\Box Q$
6. So for every world Δ in g, if $\Gamma R\Delta$, then $\Delta\Vdash P$
7. So for all Δ such that $\Gamma R\Delta$, since $\Delta\Vdash P$ and $\Delta\Vdash P\to Q$, $\Delta\Vdash Q$
8. That is, $\Gamma\Vdash\Box Q$
9. But $\Gamma\nVdash\Box Q$
10. $\bot$
11. Therefore, $\Gamma\Vdash\Box(P\to Q)\to(\Box P\to\Box Q)$

Exercise 1.7.5

Show that:

a) If R is reflexive, $\Box P\to P$ is true at every member of g.

1. Let $\langle g, R, \Vdash\rangle$ be a model where R is reflexive. Then if $\Gamma\Vdash\Box P$, $\Gamma\Vdash P$.
2. Let Γ be an arbitrary member of g such that $\Gamma\Vdash\Box P$.
3. Then for every world W in g, if ΓRW, then $W\Vdash P$
4. But since R is reflexive, $\Gamma R\Gamma$. Therefore, $\Gamma\Vdash P$
5. Therefore, $\Gamma\Vdash\Box P\to P$
6. Now, since Γ was arbitrary, this is true for any member of g.
7. Therefore, if R is reflexive, then $\Box P\to P$ is true at every member of g.

b) If R is symmetric, $P\to\Box\Diamond P$ is true at every member of g.

1. Let $\langle g, R, \Vdash\rangle$ be a model where R is symmetric, and Γ an arbitrary member of g. Then $\Gamma\Vdash P\to\Box\Diamond P$
2. We prove this by *reductio*. Presume $\Gamma\nVdash P\to\Box\Diamond P$
3. Then $\Gamma\Vdash P$ and $\Gamma\nVdash\Box\Diamond P$
4. So there is some Δ such that $\Gamma R\Delta$ and $\Delta\nVdash\Diamond P$
5. That is, for all W such that ΔRW, $W\nVdash P$
6. But since $\Gamma R\Delta$ and R is symmetric, $\Delta R\Gamma$.
7. Therefore $\Gamma\nVdash P$ [by 5, 6]
8. But $\Gamma\Vdash P$ [3]
9. $\bot$
10. Therefore, $\Gamma\Vdash P\to\Box\Diamond P$.
11. And since Γ was arbitrary, if R is symmetric, then $P\to\Box\Diamond P$ is true at every member of g.

c) If R is both symmetric and transitive, $\Diamond P\to\Box\Diamond P$ is true at every member of g.

1. Let $\langle g, R, \Vdash\rangle$ be a model where R is symmetric and transitive, and Γ an arbitrary member of g. Then, $\Gamma\Vdash\Diamond P\to\Box\Diamond P$
2. By *reductio*. Presume $\Gamma\nVdash\Diamond P\to\Box\Diamond P$.
3. Then $\Gamma\Vdash\Diamond P$ and $\Gamma\nVdash\Box\Diamond P$
4. Then for some Δ, $\Gamma R\Delta$ and $\Delta\Vdash P$
5. And for some Ω, $\Gamma R\Omega$ and $\Omega\nVdash\Diamond P$
6. That is, for all W, if ΩRW, $\nVdash P$
7. But since $\Gamma R\Omega$ and R is symmetric, $\Omega R\Gamma$.
8. And since R is transitive, and $\Gamma R\Delta$ and $\Omega R\Gamma$, $\Omega R\Delta$
9. Therefore, $\Delta\nVdash P$

10. But $\Delta \Vdash P$
11. $\bot$
12. Therefore, if R is symmetric and transitive, then for any arbitrary member Γ of g, $\Gamma \Vdash \Diamond P \to \Box\Diamond P$

Exercise 1.8.1.
Show that $\Box P \to \Diamond P$ is valid in all serial models.

1. Let $\langle g, R, \Vdash\rangle$ be a model where R is serial and Γ is a member of g where $\Gamma \Vdash \Box P$. Then $\Gamma \Vdash \Diamond P$.
2. $\Gamma \Vdash \Box P$ [Hyp]
3. So for all Δ in g, if $\Gamma R \Delta$, then $\Delta \Vdash P$
4. But since R is serial, there is some Δ such that $\Gamma R \Delta$.
5. So for some Δ in g, $\Gamma R \Delta$ and $\Delta \Vdash P$ [3, 4]
6. So $\Gamma \Vdash \Diamond P$ [4]
7. Therefore, $\Gamma \Vdash \Box P \to \Diamond P$

Exercise 1.8.2.
Prove that a frame $\langle g, R, \rangle$ is transitive iff every formula of the form $\Box P \to \Box\Box P$ is valid in it

1. Let $\langle g, R, \Vdash\rangle$ be a model where R is transitive. Then for any Γ in g, $\Gamma \Vdash \Box P \to \Box\Box P$
2. By *reductio*:
3. Presume $\Gamma \nVdash \Box P \to \Box\Box P$
4. Then $\Gamma \Vdash \Box P$ and $\Gamma \nVdash \Box\Box P$
5. So for some Δ, $\Gamma R \Delta$ and $\Delta \nVdash \Box P$
6. So for there is some Ω such that $\Delta R \Omega$ and $\Omega \nVdash P$
7. But since R is transitive, $\Gamma R \Delta$ and $\Delta R \Omega$, $\Gamma R \Omega$.
8. And since $\Gamma \Vdash \Box P$ and $\Gamma R \Omega$, $\Omega \Vdash P$
9. $\bot$
10. **Therefore, if $\langle g, R\rangle$ is a transitive frame on a model $\langle g, R, \Vdash\rangle$, then for all Γ in g, $\Gamma \Vdash \Box P \to \Box\Box P$**
11. Next, we prove that if a frame $\langle g, R\rangle$ is non-transitive, then $\Box P \to \Box\Box P$ is not valid on it.
12. To do this, we specify a non-transitive model for which $\Box P \to \Box\Box P$ does not hold.
13. Let Γ, Δ, and Ω be the only members of g. $\Gamma R \Delta$ and $\Delta R \Omega$. No other relations hold on worlds. $\Delta \Vdash P$, and all other atomic formulae are set to false at all worlds.
14. So, since Δ is the only world that Γ accesses and $\Delta \Vdash P$, $\Gamma \Vdash \Box P$
15. But since $\Delta R \Omega$ and $\Omega \nVdash P$, $\Delta \nVdash \Box P$
16. And since $\Gamma R \Delta$ and $\Delta \nVdash \Box P$, $\Gamma \nVdash \Box\Box P$
17. **Therefore, if $\langle g, R\rangle$ is not transitive, then $\Box P \to \Box\Box P$ is not valid on that frame.**
18. **Therefore, a frame $\langle g, R, \rangle$ is transitive iff every formula of the form $\Box P \to \Box\Box P$ is valid in it**

Exercise 1.8.3
There is a modal logic, sometimes called **S4.3** (due to Dummett and Lemon) which is characterized semantically by the class of frames $\langle g, R\rangle$ for which R is reflexive, transitive, and *linear*, that is, for all Γ, Δ in g, either $\Gamma R \Delta$ or $\Delta R \Gamma$.

Show that $\Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$ is valid in all S4.3 frames

1. Let $\langle g, R\rangle$ be a reflexive, transitive, and linear frame on a model $\langle g, R, \Vdash\rangle$, and let Γ be an arbitrary world in g. We show that $\Gamma \Vdash \Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$.
2. By *reductio*:
3. Presume $\Gamma \nVdash \Box(\Box P \to \Box Q) \vee \Box(\Box Q \to \Box P)$
4. Then $\Gamma \nVdash \Box(\Box P \to \Box Q)$ and $\Gamma \nVdash \Box(\Box Q \to \Box P)$
5. So for some world Δ in g, $\Gamma R \Delta$ and $\Delta \nVdash (\Box P \to \Box Q)$
6. And for some world Ω in g, $\Gamma R \Omega$ and $\Omega \nVdash (\Box Q \to \Box P)$
7. So $\Delta \Vdash \Box P$ and $\Delta \nVdash \Box Q$
8. So for some Z in g, $\Delta R Z$ and $Z \nVdash Q$
9. Similarly, $\Omega \Vdash \Box Q$ and $\Omega \nVdash \Box P$
10. So for some Θ in g, $\Omega R \Theta$ and $\Theta \nVdash P$
11. Since $\langle g, R\rangle$ is linear, either $\Theta R Z$ or $Z R \Theta$
12. Presume $\Theta R Z$
13. Then since $\langle g, R\rangle$ is transitive and $\Omega R \Theta$, $\Omega R Z$
14. Since $\Omega \Vdash \Box Q$ and $\Omega R Z$, $Z \Vdash Q$
15. But $Z \nVdash Q$
16. $\bot$
17. Therefore, $\sim(\Theta R Z)$

18. Therefore, by disjunctive syllogism, $Z R \Theta$
19. Since $<g, R>$ is transitive and $\Delta R Z$, $\Delta R \Theta$
20. And since $\Delta R \Theta$ and $\Delta \Vdash \Box P$, $\Theta \Vdash P$
21. But $\Theta \nVdash P$
22. $\bot$
23. Therefore, $\Gamma \Vdash \Box(\Box P \rightarrow \Box Q) \vee \Box(\Box Q \rightarrow \Box P)$
24. Therefore, $\Box(\Box P \rightarrow \Box Q) \vee \Box(\Box Q \rightarrow \Box P)$ is valid in all **S4.3** frames

Show that $P \rightarrow \Box\Diamond P$ can be falsified in some **S4.3** frame

1. Let $<g, R>$ be a reflexive, transitive, linear frame where $g=\{\Gamma, \Delta\}$ and $\Gamma R \Delta$. $\Gamma \Vdash P$ and $\Delta \nVdash P$
2. Since $\Delta R \Delta$ and R*'s* nothing else in g, $\Delta \nVdash \Diamond P$
3. And since $\Gamma R \Delta$, $\Gamma \nVdash \Box\Diamond P$
4. So since $\Gamma \Vdash P$ and $\Gamma \nVdash \Box\Diamond P$, $\Gamma \nVdash P \rightarrow \Box\Diamond P$

Show that $\Diamond\Box(P \rightarrow Q) \rightarrow (\Diamond\Box P \rightarrow \Diamond\Box Q)$ is valid in all **S4.3** frames.

1. Let $<g, R, \Vdash>$ be a model on an **S4.3** frame, and Γ an arbitrary member of g. Assume $\Gamma \nVdash \Diamond\Box(P \rightarrow Q) \rightarrow (\Diamond\Box P \rightarrow \Diamond\Box Q)$
2. Then $\Gamma \Vdash \Diamond\Box(P \rightarrow Q)$ and $\Gamma \nVdash \Diamond\Box P \rightarrow \Diamond\Box Q$
3. So for some Δ in g, $\Gamma R \Delta$ and $\Delta \Vdash \Box(P \rightarrow Q)$
4. And $\Gamma \Vdash \Diamond\Box P$ and $\Gamma \nVdash \Diamond\Box Q$
5. So for some Ω in g, $\Omega \Vdash \Box P$
6. So for some Θ in g, $\Gamma R \Theta$ and $\Theta \nVdash Q$
7. Since R is linear, either $\Delta R \Omega$ or $\Omega R \Delta$. We show that neither of these can be the case.
8. First, Presume $\Delta R \Omega$
9. Then, since R is transitive, $\Omega \nVdash \Box Q$
10. So for some Θ in g, $\Omega R \Theta$ and $\Theta \nVdash Q$
11. Then, since $\Delta \Vdash \Box(P \rightarrow Q)$, and R is transitive, $\Delta R \Theta$
12. So $\Theta \Vdash P \rightarrow Q$
13. Therefore, either $\Theta \nVdash P$ or $\Theta \Vdash Q$
14. But $\Theta \nVdash Q$
15. So $\Theta \nVdash P$
16. But since $\Omega R \Theta$ and $\Omega \Vdash \Box P$, $\Theta \Vdash P$
17. $\bot$
18. Next, presume $\Omega R \Delta$
19. Then, since R is transitive and $\Gamma \nVdash \Diamond\Box Q$, $\Delta \nVdash \Box Q$
20. So for some Θ in g, $\Delta R \Theta$ and $\Theta \nVdash Q$
21. Then, since $\Delta \Vdash \Box(P \rightarrow Q)$, $\Theta \Vdash P \rightarrow Q$
22. And since $\Omega \Vdash \Box P$, and R is transitive, $\Theta \Vdash P$
23. Therefore by modus ponens, $\Theta \Vdash Q$
24. $\bot$
25. Therefore, $\Diamond\Box(P \rightarrow Q) \rightarrow (\Diamond\Box P \rightarrow \Diamond\Box Q)$ is valid in all **S4.3** frames.

Exercise 1.8.4

This exercise has many parts, but each one is not difficult. Show the inclusions between logics that are given in the diagram above are the only ones that hold.

That **K** is a sublogic of **D**:

1. Every formula X that is **K**-valid is valid on every frame with no conditions. Hence, since adding conditions to a frame cannot decrease the number of valid formulas on that frame, X remains valid on a frame that is serial. Hence, If $\Vdash_{\boldsymbol{K}} X$ then $\Vdash_{\boldsymbol{D}} X$.
2. On the other hand, not every formula of **D** is a formula of **K**. Counterexample: $\Vdash_{\boldsymbol{D}} \Box A \rightarrow \Diamond A$, but $\nVdash_{\boldsymbol{K}} \Box A \rightarrow \Diamond A$. The first half of this was shown in Exercise 1.8.1. The second, given that **K** has models where some worlds access none, is a consequence of the result of exercise 1.6.2.

That **K** is a sublogic of **K4**

1. The proof that every formula proven in **K** is also proven in **K4** is identical to that given for **D**, with seriality substituted for transitivity.
2. The proof that there is some formula proven in **K4** not proven in **K** is given because $\Vdash_{\boldsymbol{K4}} \Box A \rightarrow \Box\Box A$, but $\nVdash_{\boldsymbol{K}} \Box A \rightarrow \Box\Box A$. Given that **K** has non-transitive frames, this result is a consequence of the result of exercise 1.8.2.

That **D** is a sublogic of **T**

1. A frame is reflexive iff every world in it accesses itself. But

such a frame is also one in which every world accesses something. Therefore, every reflexive frame is serial. Therefore, every reflexive frame proves whatever a serial frame proves, so if $\Vdash_{\boldsymbol{D}} A$, then $\Vdash_{\boldsymbol{T}} A$.

2. On the other hand, $\Vdash_{\boldsymbol{T}} \Box A \rightarrow A$, but $\nVdash_{\boldsymbol{D}} \Box A \rightarrow A$. The first of these is a consequence of 1.7.5a. The second is shown below.
3. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a serial model where $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{(\Gamma, \Delta), (\Delta, \Gamma)\}$, $\Delta \Vdash_{\boldsymbol{D}} P$, and all other atomic formulae at worlds are set to false.
4. Then, since $\Gamma \mathcal{R} \Delta$ and only Δ, $\Gamma \Vdash_{\boldsymbol{D}} \Box A$.
5. But $\Gamma \nVdash_{\boldsymbol{D}} A$
6. Therefore, $\Gamma \nVdash_{\boldsymbol{D}} \Box A \rightarrow A$

That **K** is a sublogic of **T**

1. This follows since **K** is a sublogic of **D**, which is a sublogic of **T**, by transitivity of the sublogic relation.

That **T** is not a sublogic of **K**

1. This follows since **T** is not a sublogic of **D**. Therefore, it is not a sublogic of what is strictly weaker than **D**, e.g. **K**

That **T** is a sublogic of **B**

1. A **B** frame is one which is reflexive and symmetric. Therefore, it is reflexive. So whatever a reflexive frame proves, **B** proves. Therefore, if $\Vdash_{\boldsymbol{T}} A$, $\Vdash_{\boldsymbol{B}} A$.
2. But B proves formulae of the form $A \rightarrow \Box\Diamond B$, which are falsifiable on some T frames. The first half of this is given from exercise 1.7.5b. The latter is given below, where we prove that **B** is not a sublogic of **S4**.

That **K, D** are sublogics of **B**.

1. These results follow from **T**'s being a sublogic of **D** along with the transitivity of the sublogic relation.

That **T** is a sublogic of **S4**.

1. **S4** is characterized as by the collection of models on frames that are reflexive and transitive. Therefore, whatever holds on a frame that is merely reflexive also holds on an **S4** frame. So in particular, if $\Vdash_{\boldsymbol{T}} A$, then $\Vdash_{\boldsymbol{S4}} A$.

That **K, D** are sublogics of **S4**.

1. This follows from **T**'s being a sublogic of **S4**, **K** and **D** being sublogics of **T**, and the transitivity of the sublogic relation.

That **K4** is a sublogic of **S4**.

1. S4 is characterized as the collection of models on frames that are reflexive and transitive. Therefore, whatever is a formally valid consequence on the class of transitive models alone is also a consequence of **S4**. And so, since **K4** is defined as the class of models on transitive frames, if $\Vdash_{\boldsymbol{K4}} A$, then $\Vdash_{\boldsymbol{S4}} A$.

That **B** is not a sublogic of **S4**

1. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a model on a reflexive, transitive frame $\langle \mathcal{G}, \mathcal{R} \rangle$, where $\mathcal{G} = \{\Gamma, \Delta\}$. $\mathcal{R} = \{(\Gamma, \Gamma), (\Gamma, \Delta), (\Delta, \Delta)\}$. $\Gamma \Vdash A$, and all other atomic sentences at worlds are set to false.
2. $\Delta \nVdash A$
3. So $\Delta \nVdash \Diamond A$
4. Therefore, since $\Gamma \mathcal{R} \Delta$, $\Gamma \nVdash \Box\Diamond A$
5. Therefore, since $\Gamma \Vdash A$, $\Gamma \nVdash A \rightarrow \Box\Diamond A$.

That **B** is not a sublogic of **K**, **D**, **K4**, or **T**

1. This follows because **B** is not a sublogic of **S4**, all of the above *are* sublogics of **S4**, and what is not a sublogic of a stronger logic is also not a sublogic of a strictly weaker one.

That **B** is a sublogic of **S5**.

1. **B** is characterized by the class of frames that are reflexive and symmetric, while **S5** is characterized by the class of frames that are reflexive, symmetric, and transitive. Therefore, since whatever is proven over the class of reflexive, symmetric frames remains proven over the set of these frames plus transitivity, if $\Vdash_{\boldsymbol{B}} A$ then $\Vdash_{\boldsymbol{S5}} A$.

That **S4** is a sublogic of **S5**

1. **S4** is the class of models on frames that are reflexive and transitive, while **S5** is characterized by the subset of these frames that are also symmetric. Therefore, whatever is proven in a reflexive, transitive frame remains proven in one which is also symmetric. Therefore, if $\Vdash_{\boldsymbol{S4}} A$ then $\Vdash_{\boldsymbol{S5}} A$.

That **K, D, T,** and **K4** are sublogics of **S5.**

1. This follows since **S4** is a sublogic of S5, and since the above

are all sublogics of **S4**

That neither **S5, S4, T, B,** nor **D** is a sublogic of **K4**

1. Every serial frame proves schemata of the form $\Box A \rightarrow \Diamond A$, and all of the above are serial (or reflexive, and therefore serial). We show that this schema fails on some **K4** frame.
2. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a model on a transitive frame $\langle \mathcal{G}, \mathcal{R} \rangle$ such that $\mathcal{G} = \{\Gamma\}$, and $\mathcal{R}$ relates nothing to anything else.
3. Since $\mathcal{R}$ fails to relate Γ to anything, it is vacuously true that if it relates Γ to a Δ, and Δ to Ω, it relates Γ to Ω.
4. Similarly, $\mathcal{M}, \Gamma \Vdash_{K4} \Box A$ holds vacuously
5. But since $\mathcal{R}$ relates Γ to nothing, $\mathcal{M}, \Gamma \nVdash_{K4} \Diamond A$
6. Therefore, $\mathcal{M}, \Gamma \nVdash_{K4} \Box A \rightarrow \Diamond A$.

That neither **K4** nor **S4** nor **S5** is a sublogic of **K**, **D, T,** or **B**

1. Since **K4**, **S4**, and **S5** are transitive, $\Vdash_{K4, S4, S5} \Box A \rightarrow \Box\Box A$. But since **D**, **T**, and **B** have non-transitive frames, formulae of this type are not valid consequences in these logics.

That **S5** is not a sublogic of ***S4***

1. **S5** is symmetric, and therefore validates formulae of the form $A \rightarrow \Box\Diamond A$ as was shown in exercise 1.7.5b. To show that **S5** is strictly stronger than **S4**, we show that this formula is invalid on some **S4** frame.
2. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a reflexive, transitive model, where $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{(\Gamma, \Gamma), (\Gamma, \Delta), (\Delta, \Delta)\}$, $\Gamma \Vdash A$, and all other atomic propositions are set to false.
3. So $\mathcal{M}, \Gamma \Vdash_{S4} A$
4. But $\mathcal{M}, \Delta \nVdash_{S4} A$
5. Therefore, $\mathcal{M}, \Delta \nVdash_{S4} \Diamond A$
6. Therefore, $\mathcal{M}, \Gamma \nVdash_{S4} \Box\Diamond A$
7. Therefore, $\mathcal{M}, \Gamma \nVdash_{S4} A \rightarrow \Box\Diamond A$

Exercise 1.9.1

Show that $\emptyset \vDash_K \{\Box P \rightarrow P\} \Rightarrow \Box\Box P \rightarrow P$ does not hold, but $\{\Box P \rightarrow P\} \vDash_K \emptyset \Rightarrow \Box\Box P \rightarrow P$ does.

a) $\emptyset \nvDash_K \{\Box P \rightarrow P\} \Rightarrow \Box\Box P \rightarrow P$

1. To show that the entailment does not hold, we specify a countermodel.
2. We let $\langle \mathcal{G}, \mathcal{R} \rangle$ be a **K** frame, and $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ a model on the frame s. t. $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{(\Gamma, \Delta)\}$. All atomic formulae at worlds are interpreted as false.
3. $\Delta \nVdash P$
4. And since Δ doesn't relate to anything, $\Delta \Vdash \Box P$
5. So $\Delta \nVdash \Box P \rightarrow P$
6. So since $\Gamma\mathcal{R}\Delta$, $\Gamma \nVdash \Box\Box P \rightarrow P$
7. Since $\Gamma\mathcal{R}\Delta$ and $\Delta \nVdash P$, $\Gamma \nVdash \Box P$
8. Therefore, $\Gamma \Vdash \Box P \rightarrow P$
9. So $\Gamma \Vdash \Box P \rightarrow P$ and $\Gamma \nVdash \Box\Box P \rightarrow P$
10. Therefore, $\emptyset \nvDash_K \{\Box P \rightarrow P\} \Rightarrow \Box\Box P \rightarrow P$

b) $\{\Box P \rightarrow P\} \vDash_K \emptyset \Rightarrow \Box\Box P \rightarrow P$

1. We let $\mathfrak{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ be the collection of **K** frames, with $\{\Box P \rightarrow P\}$ as a global assumption. We show that $\Box P \rightarrow P \vDash_K \emptyset \Rightarrow \Box\Box P \rightarrow P$
2. Let Γ be an arbitrary member of $\mathcal{G}$ s. t. $\mathcal{M}, \Gamma \Vdash \Box\Box P$
3. So for every member Δ of $\mathcal{G}$ s. t. $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash \Box P$
4. And since $\Box P \rightarrow P$ is a global assumption, $\mathcal{M}, \Delta \Vdash P$
5. Therefore, for every member Δ of $\mathcal{G}$, s. t. $\Gamma\mathcal{R}\Delta$, $\Delta \Vdash P$
6. Therefore, $\Gamma \Vdash \Box P$
7. And, since $\{\Box P \rightarrow P\}$ is a global assumption, $\Gamma \Vdash P$.
8. Therefore, if $\Gamma \Vdash \Box\Box P$, then $\Gamma \Vdash P$
9. Therefore, $\Gamma \Vdash \Box\Box P \rightarrow P$
10. And since Γ was chosen arbitrarily, $\Box P \rightarrow P \vDash_K \emptyset \Rightarrow \Box\Box P \rightarrow P$

Exercise 1.9.2

Prove proposition 1.9.4 (monotonicity):

Suppose $S \vDash_L U \rightarrow X$, $S \subseteq S'$, and $U \subseteq U'$. Then $S' \vDash_L U' \rightarrow X$.

1. Let **L** be a collection of frames, S a set of formulae valid on every frame in **L**, U a collection of formulae, and X a single formula. Suppose $S \vDash_L U \rightarrow X$, $S \subseteq S'$, and $U \subseteq U'$
2. Next, assume $S' \vDash_L U' \rightarrow X$ fails.
3. So for every frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ in **L** where S is valid; for every

point Γ in $\mathcal{G}$ where all members of U are true, X is true as well.

4. And there is a frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$ in **L** such that: for every model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ based on $\mathcal{F}$, and every world Γ in $\mathcal{G}$, $\mathcal{M}$, $\Gamma \Vdash S'$; and there is some world Δ in $\mathcal{G}$ such that $\mathcal{M}, \Delta \Vdash U'$ and $\mathcal{M}, \Delta \nVdash X$
5. But since $S \subseteq S'$, for every world Γ in $\mathcal{G}$, $\mathcal{M}, \Gamma \Vdash S$
6. That is, S is valid on $\mathcal{F}$'
7. And since $\mathcal{M}, \Delta \Vdash U'$ and $U \subseteq U'$, $\mathcal{M}, \Delta \Vdash U$
8. So U is true at Δ
9. Therefore, Δ is an element in **L** where U is true on a frame where every member of S is valid.
10. Therefore, contrary to our hypothesis, $\Delta \Vdash X$
11. Therefore, by *reductio*, $S' \vDash_L U' \rightarrow X$.
12. Therefore, if $S \vDash_L U \rightarrow X$, $S \subseteq S'$, and $U \subseteq U'$, then $S' \vDash_L U' \rightarrow X$.

Exercise 1.9.3

Let $\Box^n Y$ denote $\Box\Box\ldots\Box Y$, where we have written a string of n occurrences of $\Box$. Use the various facts given above concerning modal consequence and show the following version of the deduction theorem: $S \cup \{Y\} \vDash_L U \rightarrow X$ if and only if, for some n, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$.

1. We prove the contrapositive in each direction.
2. First, assume $S \cup \{Y\} \vDash_L U \rightarrow X$ fails
3. Then for some frame $\langle \mathcal{G}, \mathcal{R} \rangle$ in **L**, for some model $\langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ based on this frame in which all members of $S \cup \{Y\}$ are valid; and for some world $\Gamma \in \mathcal{G}$ at which all members of U are true, $\Gamma \nVdash X$
4. So for some frame $\langle \mathcal{G}, \mathcal{R} \rangle$ in **L**, for some model $\langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ based on this frame in which all members of $S \cup \{Y\}$ are true in every world in $\mathcal{G}$; and for some world $\Gamma \in \mathcal{G}$ at which all members of U are true, $\Gamma \nVdash X$
5. And, since all the members of $S \cup \{Y\}$ are true in every world in $\mathcal{G}$, all the members of $\{Y\}$ are as well
6. And so, in particular, $\Gamma \Vdash Y$
7. And, since $\Gamma \nVdash X$, $\Gamma \nVdash Y \supset X$
8. And so for some n – namely, where $n = 0$ – it is not the case that $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$.
9. **Therefore, $S \cup \{Y\} \vDash_L U \rightarrow X$ if for some n, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$**
10. Next, assume that for no n does it hold that, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$
11. So for every value of n, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$
12. That is, for every value of n, there is a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ in **L**, and a model $\langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ based on this frame in which all members of S are valid; and a world $\Gamma \in \mathcal{G}$ at which all members of U are true, yet $\Gamma \nVdash (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$
13. So $\Gamma \Vdash (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y)$ and $\Gamma \nVdash X$
14. So $\Gamma \Vdash \Box^0 Y$, and $\Gamma \Vdash \Box^1 Y$, ..., and $\Gamma \Vdash \Box^n Y$ and $\Gamma \nVdash X$
15. So for every value of n, there is a frame $\langle \mathcal{G}, \mathcal{R} \rangle$ in **L**, and a model $\langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ based on this frame in which all the members of S are valid, and a world $\Gamma \in \mathcal{G}$ where all the members of $U \cup \{\Box^0 Y, \Box^1 Y, \ldots, \Box^n Y\}$ are true, and yet $\Gamma \Vdash \cancel{X}$
16. So by local deduction, for every value of n $S \vDash_L U \cup \{\Box^0 Y, \Box^1 Y, \ldots, \Box^n Y\} \rightarrow X$ fails
17. And so, generalizing on n, $S \vDash_L U \cup \{\Box Y, \Box\Box Y, \Box\Box\Box Y, \ldots\} \rightarrow X$ fails
18. So by the global deduction theorem, $S \cup \{Y\} \vDash_L U \rightarrow X$ fails
19. **So $S \cup \{Y\} \vDash_L U \rightarrow X$ only if, for some n, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$.**
20. **So $S \cup \{Y\} \vDash_L U \rightarrow X$ if and only if, for some n, $S \vDash_L U \rightarrow (\Box^0 Y \,\&\, \Box^1 Y \,\&\, \ldots \,\&\, \Box^n Y) \supset X$.**

Exercise 1.10.1

What is the quantifier translation of **PP***P*?

$(\exists t'')(t' < t \,\&\, (\exists t')(t' < t'' \,\&\,$ John wins the election at $t')$

Exercise 1.11.1

Prove each of (1.13), (1.14), and (1.15), using Hintikka's model sets.

(1.13) $\mathcal{K}_a\mathcal{K}_b P \supset \mathcal{K}_a P$

1	$\mathcal{K}_a\mathcal{K}_b P \in \mu$	Assumption
2	$\neg\mathcal{K}_a P \in \mu$	Assumption
3	$\mathcal{P}_a \neg P \in \mu$	(C.¬ K), 2
4	$\neg P \in \mu^*$	(C.P*), 3
5	$\mathcal{K}_a\mathcal{K}_b \mathrm{P} \in \mu^*$	(C.KK*), 1
6	$\mathcal{K}_b P \in \mu^*$	(C.K), 5
7	$P \in \mu^*$	(C.K), 6
	X	4, 7

(1.14) $\mathcal{K}_a P \supset \mathcal{P}_a P$

1	$\mathcal{K}_a P \in \mu$	Assumption
2	$\neg\mathcal{P}_a P \in \mu$	Assumption
3	$\mathcal{K}_a \neg P \in \mu$	(C.¬P), 2
4	$P \in \mu$	(C.K), 1
5	$\neg P \in \mu$ (C.K), 3	
	X	4, 5

(1.15) $\mathcal{K}_a P \equiv \mathcal{K}_a\mathcal{K}_a P$

Since no rule is mentioned for ≡, we break the formula down into its two conditionals,

(1.15a) $\mathcal{K}_a P \supset \mathcal{K}_a\mathcal{K}_a P$

and

(1.15b) $\mathcal{K}_a\mathcal{K}_a P \supset \mathcal{K}_a P$

(1.15a) $\mathcal{K}_a P \supset \mathcal{K}_a\mathcal{K}_a P$

1	$\mathcal{K}_a P \in \mu$	Assumption
2	$\neg\mathcal{K}_a\mathcal{K}_a P \in \mu$	Assumption
3	$\mathcal{P}_a \neg\mathcal{K}_a P \in \mu$	(C.¬K), 2
4	$\neg\mathcal{K}_a P \in \mu^*$	(C.P*), 3
5	$\mathcal{P}_a \neg P \in \mu^*$	(C.¬K), 4
6	$\neg P \in \mu^{**}$	(C.P*), 5
7	$\mathcal{K}_a P \in \mu^*$	(C.KK*), 1
8	$\mathcal{K}_a P \in \mu^{**}$	(C.KK*), 7
9	$P \in \mu^{**}$	(C.K), 8
	X	6, 9

(1.15b) $\mathcal{K}_a\mathcal{K}_a P \supset \mathcal{K}_a P$

1	$\mathcal{K}_a\mathcal{K}_a P \in \mu$	Assumption
2	$\neg\mathcal{K}_a P \in \mu$	Assumption
3	$\mathcal{K}_a P \in \mu$	(C.K), 1
	X	2, 3

Exercise 1.12.1

Can a square of opposition be constructed when ◊ is interpreted to mean "it is contingent that," where *contingent* means true but not necessary?

Yes. The construction is as follows:

$\Box P$	CONTRARIES	$\Diamond P \stackrel{\text{def}}{=} P \,\&\, \neg\Box P$
S		S
U		U
B		B
A		A
L		L
T		T
E		E
R		R
N		N
S		S
$\neg\Diamond P$	SUBCONTRARIES	$\neg\Box P$

That opposite corners are contradictories follows straightforwardly from the definition of a negation.

Next, since $\Diamond P$ implies $\neg\Box P$ by definition, it cannot be true alongside $\Box P$. But they can both be false – e,g. when $\neg P$ alone holds. This makes them contraries.

And since $\Diamond P$ is defined as $P \,\&\, \neg\Box P$, $\Diamond P$ implies $\neg\Box P$ by &

elimination. And so $\neg\Box P$ is subalternated to $\Diamond P$.

Also, since $\Diamond P$ is defined as P & $\neg\Box P$, its negation will be $\neg(P$ & $\neg\Box P)$. And, since, by DeMorgan's Law and double negation elimination, this is equivalent to $\neg P$ v $\Box P$, it will be implied by $\Box P$ by disjunction introduction. Therefore, $\neg\Diamond P$ is subalternated to $\Box P$.

Lastly, that the bottom corners are subcontraries follows indirectly from the above contrariety results: for, if both were false, then their contradictories would both be true, which cannot be the case given that they are contraries.

Exercise 1.12.2.

1 Verify that $\Box P \supset P$ is not valid in **D**

1. Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \Vdash \rangle$ be a countermodel, where $\mathcal{W} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{\langle\Gamma, \Delta\rangle, \langle\Delta, \Gamma\rangle\}$, $\Gamma \nVdash P$ and $\Delta \Vdash P$
2. Since $\Gamma\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \Vdash P$, P holds at all worlds accessible from Γ.
3. So $\mathcal{M}, \Gamma \Vdash \Box P$
4. But $\mathcal{M}, \Gamma \nVdash P$
5. Therefore, $\mathcal{M}, \Gamma \nVdash \Box P \supset P$.

2 Verify that $\Box P \supset \Diamond P$ is valid in **D**

1. Assume the contrary
2. Then for some model $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \Vdash \rangle$ and some world Γ in $\mathcal{W}$, $\mathcal{M}, \Gamma \Vdash \Box P$ and $\mathcal{M}, \Gamma \nVdash \Diamond P$
3. So, for all worlds Δ s. t. $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash P$
4. And since $\mathcal{R}$ is serial, there *is* some such world Δ such that $\Gamma\mathcal{R}\Delta$.
5. So, given that $\mathcal{M}, \Delta \Vdash P$, it follows, contrary to our hypothesis, that $\mathcal{M}, \Gamma \Vdash \Diamond P$.
6. And so for no world Γ on any model $\mathcal{M}$ on a **D** frame does it hold that $\mathcal{M}, \Gamma \Vdash \Box P$ and $\mathcal{M}, \Gamma \nVdash \Diamond P$ – i. e., on no **D** model does it hold that $\mathcal{M}, \Gamma \nVdash \Box P \supset \Diamond P$.
7. So $\vDash_{\mathbf{D}} \Box P \supset \Diamond P$.

3 Verify that $\Box P \supset \Diamond P$ is not valid in **K**

1. Let $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \Vdash \rangle$ be a countermodel, where $\mathcal{W} = \{\Gamma\}$, $\mathcal{R}$ is empty, and $\Vdash$ evaluates all atomic formulae as false at Γ.
2. Then, since $\mathcal{R}$ relates Γ to nothing, $\mathcal{M}, \Gamma \Vdash \Box P$ holds vacuously.
3. But, since $\mathcal{R}$ relates Γ to nothing, it also does not relate it to a world where P holds. And so $\mathcal{M}, \Gamma \nVdash \Diamond P$.
4. Therefore, $\mathcal{M}, \Gamma \nVdash \Box P \supset \Diamond P$.

Exercise 1.12.3

Show that $\Diamond P \supset \Box P$ is not valid in **K**

1. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a **K**-model where $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{\langle\Gamma, \Delta\rangle, \langle\Gamma, \Gamma\rangle\}$, $\Gamma \Vdash P$ and $\Delta \nVdash P$
2. Since $\Gamma\mathcal{R}\Gamma$ and $\mathcal{M}, \Gamma \Vdash P$, $\mathcal{M}, \Gamma \Vdash \Diamond P$
3. But since $\Gamma\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \nVdash P$, $\mathcal{M}, \Gamma \nVdash \Box P$
4. So $\mathcal{M}, \Gamma \nVdash \Diamond P \supset \Box P$
5. So $\Diamond P \supset \Box P$ fails on some **K**-model, and thus doesn't hold on every **K**-model
6. So $\Diamond P \supset \Box P$ isn't valid in **K**

Exercise 1.12.4

We showed that modality collapses in **T** if $\Box P$ v $\Box\neg P$. Show that the argument must break down for **K**.

1. We let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ be a **K**-model where $\mathcal{G} = \{\Gamma\}$, $\mathcal{R}$ is empty, and $\Gamma \Vdash P$
2. Since $\mathcal{R}$ is empty, it holds vacuously that $\mathcal{M}, \Gamma \Vdash \Box P$ and $\mathcal{M}, \Gamma \Vdash \Box\neg P$.
3. Therefore, it also holds that $\mathcal{M}, \Gamma \Vdash \Box P$ v $\Box\neg P$
4. But since $\mathcal{R}$ is empty, $\mathcal{M}, \Gamma \nVdash \Diamond P$ and $\mathcal{M}, \Gamma \nVdash \Diamond\neg P$
5. And so $\mathcal{M}, \Gamma \nVdash \Box P \supset \Diamond P$, and $\mathcal{M}, \Gamma \nVdash \Box\neg P \supset \Diamond\neg P$
6. That is, necessities do not entail, and so are distinct from, their corresponding possibilities.

Exercise 1.12.5

Show that, in **K**, a formula P is valid if and only if $\Box P$ is valid.

1. First we show that $\vdash P$ if $\vdash \Box P$

2. Then we show that $\vdash P$ only if $\vdash \Box P$
3. To show the first, assume the contrary.
4. Then for every **K**-frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, for every **K**-model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ on $\mathcal{F}$, and every world Γ in that model, $\mathcal{M}, \Gamma \Vdash \Box P$.
5. And yet for some frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, for some **K**-model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ on $\mathcal{F}$, for some world Z on $\mathcal{M}$, $\mathcal{M}$, Z $\nVdash P$
6. So for every world Δ in $\mathcal{M}$ s. t .$\Gamma\mathcal{R}\Delta$, $\Delta \Vdash P$.
7. But since Δ is any arbitrary world in $\mathcal{G}$, $\mathcal{M}, \Delta \Vdash \Box P$.
8. And so for any world Δ in $\mathcal{G}$ on $\mathcal{M}$, if $\mathcal{M}, \Delta \Vdash P$, then $\mathcal{M}, \Delta \Vdash \Box P$.
9. Now we assumed that for some world Z in $\mathcal{M}$, $\mathcal{M}$, Z $\nVdash P$.
10. Then if there is any world Δ in $\mathcal{M}$ s. t. $\Delta\mathcal{R}$Z, $\mathcal{M}, \Delta \nVdash \Box P$, and so $\Box P$ is invalid, contrary to hypothesis.
11. And if not, we can construct a model $\mathcal{M}$' on a frame $\mathcal{F}$' in all other respects like $\mathcal{M}$ on $\mathcal{F}$, except that there is some world Δ in $\mathcal{M}$' s. t. $\Delta\mathcal{R}$Z. And so $\mathcal{M}$', $\Delta \nVdash \Box P$
12. And so, contrary to our hypothesis, there is some frame and model on which $\Box P$ is invalid.
13. **And so, if $\vdash_K \Box P$, then $\vdash_K P$**
14. Next, assume that $\vdash_K P$ and $\nvdash_K \Box P$
15. Then for every **K**-frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, for every **K**-model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ on $\mathcal{F}$, and every world Γ in that model, $\mathcal{M}, \Gamma \Vdash P$.
16. And yet for some frame $\mathcal{F} = \langle \mathcal{G}, \mathcal{R} \rangle$, for some **K**-model $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \Vdash \rangle$ on $\mathcal{F}$, for some world Z on $\mathcal{M}$, $\mathcal{M}$, Z $\nVdash \Box P$
17. Then for some world Δ in $\mathcal{G}$, Z$\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \nVdash P$
18. But since P holds at every world, $\mathcal{M}, \Delta \Vdash P$
19. Therefore, if $\vdash_K P$, then $\vdash_K \Box P$
20. Therefore, $\vdash_K P$ iff $\vdash_K \Box P$