Garson, James (2006). *Modal Logic for Philosophers*, ch. 5.

Exercise 5.1

Define validity in the case where <**W, R**> is dense, transitive, and serial. What system do you think is adequate with respect to this notion of satisfiability?

1. Let a C4-D4 model be any K-model <**W, R, a**> where **R** is dense, transitive, and serial. Then:
2. An argument H/C is C4-D4 valid iff it has no C4-D4 counterexample
3. Iff it is not the case that H/~C is C4-D4 satisfiable
4. If for no model <**W, R, a**> such that <**W, R**> is dense, transitive, and serial, H/~C is satisfiable
5. The system adequate for this notion of satisfiability is that which results from adding the (C4), (D), and (4) axioms to K

Exercise 5.2

Invent two more conditions that could plausibly hold given **R** is **earlier than**.

1. *Linearity*: if **wRv** and **wRu**, then either **vRu** or **uRv** or **v=u**
2. *Beginning time*: $\exists w \forall v \forall u(((v \in W \,\&\, v \neq w) \supset wRv) \,\&\, (((v \in W \,\&\, v \neq w) \supset uRv) \supset u = w))$

Exercise 5.3

Give the truth condition for D

$\mathbf{a}_w(\mathbf{D}A)=T$ iff for all **v** in **W**, if **wRv**, then $\mathbf{a}_v(A)=T$

Exercise 5.4

In a situation of eternal recurrence, there are numerically distinct times that are identical with respect to all propositions that hold at them. But in a possible worlds interpretation, one might think that something like the identity of indiscernibles holds, i.e. worlds with exactly the same propositions holding at them are identical.

Exercise 5.5

Show that ($\Box$5) is equivalent to ($\Box$) when **R** is universal

1. Let <**W, R**> be a universal frame, i.e. for all **w, v** in **W**, **wRv**. And let <**W, R, a**> be a model on that frame, **w** an arbitrarily chosen world in **W**, and A a formula evaluated on the model. Then $\mathbf{a}_w(\Box_5 A)=T$ iff $\mathbf{a}_w(\Box A)$.
2. First, presume $\mathbf{a}_w(\Box_5 A)=T$
3. Then for all **v** in **W**, $\mathbf{a}_v(A)=T$
4. So for all **v** in **W**, if **wRv**, then $\mathbf{a}_v(A)=T$
5. i.e. $\mathbf{a}_w(\Box A)=T$
6. Next, presume $\mathbf{a}_w(\Box A)=T$
7. Then for all **v** in **W**, if **wRv,** then $\mathbf{a}_v(A)=T$
8. And since **R** is universal, for all **v** in **W**, **wRv**.
9. Therefore, by *modus ponens*, for all **v** in **W**, $\mathbf{a}_v(A)=T$,
10. i.e. $\mathbf{a}_w(\Box_5 A)=T$

Exercise 5.6

Invent another reading of $\Box$ for which the accessibility relation is neither transitive nor symmetric

1. Let $\mathbf{a}_w(\Box A)=T$ iff for all worlds **v** such that v *resembles* **w**, $\mathbf{a}_v(A)=T$

2. Resemblance is not transitive, since a world **v** accessing **u** and accessed by **w** could resemble **w** in different respects than that in which it resembles **u**
3. And resemblance is not symmetric, since for any two things in a resemblance relation, one is usually given priority. For instance, a son may resemble his father, and a painting may resemble what it depicts, but not vice versa.

Exercise 5.7

Given that **wRv** iff **v** is in *M*, show that (O) is equivalent to (O**M**).

(O) $\mathbf{a}_w$(OA)=T iff for all **v**, if **wRv** then $\mathbf{a}_v$(A)=T

1. $\mathbf{a}_w$(OA)=T iff for all **v**, if **wRv** then $\mathbf{a}_v$(A)=T
2. But **wRv** iff **v is in** *M*.
3. So $\mathbf{a}_w$(OA)=T iff for all **v**, if **v** is in *M*, then $\mathbf{a}_v$(A)=T

Exercise 5.8

Show that any **R** such that **wRv** iff **v** is in *M* is serial, dense, transitive, and shift reflexive. Show that such an *R* need not be reflexive or symmetric.

1. Let **<W, R>** be a frame, and *M* non-empty subset of **W**, and **R** a binary relation on **W** such that for all **w, v,** in **W**, **wRv** iff **v** is in *M*. We show for each condition on **R** that if **wRv** iff **v** is in *M*, then that condition holds on **R**. Then we show that **R** need not be reflexive or symmetric.
2. Presume that **wRv** iff **v** is in *M* for some arbitrarily chosen **w, v**
3. Then, since *M* is non-empty, there is some **v** such that **wRv**, i.e. **R** is serial.
4. And since **v** is in M, **vRv**. i.e., **R** is shift reflexive
5. Third, presume that there is some world **u** such that **vRu**.
6. Then **v** is in *M*.
7. So **wRu**, i.e., **R** is transitive
8. Now, since **R** is shift reflexive, there is at least one **u** – namely **v** – such that **wRu** and **uRv**. I.e., **R** is dense.
9. Next, we show that **R** need not be reflexive or symmetric.
10. Countermodel for reflexivity and symmetry: **W**={w, u}, *M*={u}. **R**={**<w, u>, <u, u>}**
11. Since only **u** is in *M*, and all worlds in **W Ru**, **wRv** iff **v** is in *M*.
12. But it is not the case that **wRw**. So **R** is not reflexive
13. Further, **wRu,** but it is not the case that **uRw**. So **R** is not symmetric.

Exercise 5.9

Show that given (O**M**), O is not world relative, that is, show that for any two **w, u** in **W**, $\mathbf{a}_w$(OA) is $\mathbf{a}_u$(OA).

1. Let **<W, R, a>** be a model with a non-empty subset *M* satisfying the conditions stated in Exercise 5.8. Then for any **w, u** in **W**, $\mathbf{a}_w$(OA) = $\mathbf{a}_u$(OA)
2. Presume $\mathbf{a}_w$(OA) = T
3. $\mathbf{a}_w$(OA) = T iff for all **v** in *M*, $\mathbf{a}_v$(A) = T
4. And $\mathbf{a}_u$(OA) = T iff for all **v** in *M*, $\mathbf{a}_v$(A) = T
5. So $\mathbf{a}_w$(OA) = T iff $\mathbf{a}_u$(OA) = T
6. Next, presume $\mathbf{a}_w$(OA) = F
7. $\mathbf{a}_w$(OA) = F iff for some **v** in *M*, $\mathbf{a}_v$(A)=F
8. $\mathbf{a}_u$(OA) = F iff for some **v** in *M*, $\mathbf{a}_v$(A)=F
9. So $\mathbf{a}_w$(OA) = F iff $\mathbf{a}_u$(OA) = F
10. Since **w, u** were arbitrarily chosen, it follows that for any **w, u** in **W**, $\mathbf{a}_w$(OA) = $\mathbf{a}_u$(OA)

Exercise 5.10

Show that (T) and (Tf) are equivalent when f is the function defined from a serial and unique **R** by f(**w**)=**v** iff **wRv**.

1. $\mathbf{A}_w$(TnA)=T iff for all **v**, if **wRv**, then $\mathbf{a}_v$(A)=T (T)
2. ...iff for all **v**, if f(**w**)=**v**, then $\mathbf{a}_v$(A)=T (Def**f**)
3. ...iff if f(**w**)=**v**, then $\mathbf{a}_v$(A)=T (Uniqueness)
4. ...iff $\mathbf{a}_{\mathbf{f(w)}}$(A)=T (Tf)

Exercise 5.11

Show that (TT) is valid in a semantics where a model <**W, F, a**> contains a set **F** of constant functions from **W** into **W**, where **a** assigns to each term n a member **a**(n) of **F**, and (Tn) is used as the truth clause for Tn

(Tn) $\mathbf{a}_w$ TnA=T iff $\mathbf{a}_{a(n)(w)}$(A)=T

1. TnA↔TmTnA fails...
2. ...iff for some model <**W, F, a**> and some world **w** in **W**, $\mathbf{a}_{(w)}$(TnA↔TmTnA)=F
5. ...iff $\mathbf{a}_{(w)}$(TnA)≠$\mathbf{a}_{(w)}$(TmTnA)
6. ...iff $\mathbf{a}_{a(n)(w)}$(A)≠$\mathbf{a}_{(w)}$(TmTnA)=
7. ...iff $\mathbf{a}_{a(n)(w)}$(A)≠$\mathbf{a}_{a(m)(w)}$(TnA)
8. ...iff $\mathbf{a}_{a(n)(w)}$(A)≠$\mathbf{a}_{a(n)(a(m)(w))}$(A)
9. But since **a**(**n**) is a constant function, **a**(**n**)(**w**)=**a**(**n**)((**a**(**m**)(**w**)).
10. So TnA↔TmTnA fails iff $\mathbf{a}_{a(n)(w)}$(A)≠$\mathbf{a}_{a(n)(w)}$(A).
11. Since the consequent is impossible, TnA↔TmTnA is valid.

Exercise 5.12

Show that the semantics of Exercise 5.11 is equivalent to one where the truth clause reads: $\mathbf{a}_w$(TnA)=T iff $\mathbf{a}_{a(n)}$(A)=T, and a model is such that **a**(n) is not a function from **W** into **W**, but instead a member of **W**.

1. Let **M** = <**W, F, a**> be a model where **W** is a set of worlds, **F** a set of constant functions from **W** into **W**, and **a** an assignment of propositions at worlds to truth values, and function symbols to members of **F**. Then we define another model **M' =** <**W', a'**>, exactly like **M**, except for each function symbol **n** assigned to a member of **F** by **a**, **a'** assigns **n** directly to the output of **n**. We prove that $\mathbf{a}_w$(A)=T in **M** iff $\mathbf{a}_w$(A)=T in **M'**.
2. Since **M'** is exactly like **M** except with respect to **F**, all formulae not depending on F will have equivalent assignments. In particular, atomic and truth functional valuations will be identical. Hence, it is only necessary to show that valuations whose truth value depends on **F** in **M** remain the same in **M'**, i.e. those involving locative operators.
3. So let TnA be an arbitrary formula. Then $\mathbf{a}_w$(TnA)=T in **M** iff $\mathbf{a}_{a(n)(w)}$(A)=T
4. But since **n** is a constant function, there is some world **w'** such that for any world **w**, **n**(**w**)=**w'**.
5. But by construction, **a'**(**n**)=**w'** in **M'**
6. So $\mathbf{a}_w$(TnA)=T in **M** iff $\mathbf{a'}_{a'(n)}$(A)=T in **M'**
7. i.e. $\mathbf{a}_w$(TnA)=T in **M** iff **a'**(TnA)=T in **M'**

Exercise 5.13

Show that the above three formulas are provable from (↪) in M, assuming that ↪ is identified with >.

Transitivity: ((A>B)&(B>C)) > (A>C)

1. First, we translate the formula: (□(□(A→B)&□(B→C))→□(A→C)).
2. Next, we show that the formula is valid in M.
3. Assume it isn't.

4. Then for some model **M**=<**W, R, a**>, and some world **u** in **W**, $\mathbf{a}_u$(□(□(A→B)&□(B→C))→□(A→C))=0
5. i.e. for some world **w** in **W**, **uRw** and $\mathbf{a}_w$(□(A→B)&□(B→C))→□(A→C))=0
6. So $\mathbf{a}_w$(□(A→B)&□(B→C))=1, and $\mathbf{a}_w$□(A→C)=0.
7. So $\mathbf{a}_w$(□(A→B))=$\mathbf{a}_w$(□(B→C))=1, and $\mathbf{a}_w$□(A→C)=0.
8. So $\mathbf{a}_w$(□(A→B))=$\mathbf{a}_w$(□(B→C))=1, and for some **v** in **W**, **wRv** and $\mathbf{a}_v$(A→C)=0.
9. So $\mathbf{a}_w$(□(A→B))=$\mathbf{a}_w$(□(B→C))=1, and for some **v** in **W**, **wRv** and $\mathbf{a}_v$(A)=1 and $\mathbf{a}_v$(C)=0.
10. And for all **u** in **W**, if **wRu**, then $\mathbf{a}_u$(A→B)=$\mathbf{a}_u$(B→C)=1.
11. So in particular, $\mathbf{a}_v$(A→B)=$\mathbf{a}_v$(B→C)=1.
12. I. e. $\mathbf{a}_v$(A)=0 or $\mathbf{a}_v$(B)=1, and $\mathbf{a}_v$(B)=0 or $\mathbf{a}_v$(C)=1.
13. Since $\mathbf{a}_v$(A)=1, $\mathbf{a}_v$(B)=1.
14. But since $\mathbf{a}_v$(C)=0, $\mathbf{a}_v$(B)=0.
15. So $\mathbf{a}_v$(B)=0=1.
16. ⊥
17. Therefore, there is no such world **w** or model **M**, i.e. ((A>B)&(B>C)) > (A>C) is valid.

Contraposition: (A>~B) > (B>~A)

1. Assume (A>~B) >(B>~A), i.e. □(□(A→~B) → □(B→~A)) isn't valid.
2. Then for some model **M**=<**W**, **R**, **a**> and some **w** in **W**, $\mathbf{a}_w$(□(□(A→~B) → □(B→~A)))=0.
3. So for some world **v** in **W**, **wRv** and $\mathbf{a}_v$(□(A→~B) → □(B→~A))=0.
4. So $\mathbf{a}_v$(□(A→~B))=1 and $\mathbf{a}_v$(□(B→~A))=0.
5. So for some **u** in **W**, **vRu** and $\mathbf{a}_u$(B→~A)=0.
6. i.e. $\mathbf{a}_u$(B)=1 and $\mathbf{a}_u$(~A)=0
7. i.e. $\mathbf{a}_u$(B)=$\mathbf{a}_u$(A)=1.
8. And for all **w'**in **W**, if **vRw'**, then $\mathbf{a}_{w'}$(A→~B)=1.
9. So in particular, $\mathbf{a}_u$(A→~B)=1.
10. i.e. $\mathbf{a}_u$(A)=0 or $\mathbf{a}_u$(~B)=1.
11. i.e. a_u(A)=0 or a_u(B)=0.
12. Since $\mathbf{a}_u$(A)=1, $\mathbf{a}_u$(B)=0.
13. But $\mathbf{a}_u$(B)=1.
14. ⊥
15. Therefore, (A>~B)>(B>~A) is valid in **M**

Strengthening Antecedents: (A>B) > ((A&C)>B)

1. Assume(A>B) > ((A&C)>B), i.e. □(□(A→B) → □((A&C)→B) isn't valid.
2. Then for some model <**W, R, a**> and world **w** in **W**, $\mathbf{a}_w$(□(□(A→B) → □((A&C)→B)))=0.
3. So for some world **v** in **W**, **wRv** and $\mathbf{a}_v$(□(A→B) → □((A&C)→B))=0
4. So $\mathbf{a}_v$(□(A→B))=1 and $\mathbf{a}_v$(□((A&C)→B))=0.
5. So $\mathbf{a}_v$(□(A→B))=1 and for some **u** in **W**, **vRu** and $\mathbf{a}_u$((A&C)→B)=0.
6. So $\mathbf{a}_u$((A&C)=1, i.e. a_u(A)=$\mathbf{a}_u$(C)=1, and $\mathbf{a}_u$(B)=0.
7. And for all **v'** in **W**, if **vRv'**, then $\mathbf{a}_{v'}$(A→B)=1.
8. So in particular, $\mathbf{a}_u$(A→B)=1.
9. So $\mathbf{a}_u$(A)=0 or $\mathbf{a}_u$(B)=1.
10. But $\mathbf{a}_u$(A)=1, so $\mathbf{a}_u$(B)=1.
11. But $\mathbf{a}_u$(B)=0.
12. ⊥
13. Therefore, (A>B) > ((A&C)>B) is valid in **M**.

Exercise 5.14

If I flip the switch, the light will turn on. So if I flip the switch and the circuit is shorted, the light will turn on.

Exercise 5.15

We have not discussed two conditions that appear on this list: the euclidean condition and convergence. Draw diagrams for these conditions. Then consider which of them holds when **R** is **earlier than** for various structures of time.

*Euclidean condition**:** wRv&wRu →vRu*

$$\begin{tikzcd} w \arrow[r] \arrow[d] & v \arrow[dl] \\ u & \end{tikzcd}$$

Convergence: wRv&wRu→∃x(vRx&uRx)

$$\begin{tikzcd} w \arrow[r] \arrow[d] & v \arrow[d] \\ u \arrow[r] & x \end{tikzcd}$$

If **R** is **earlier than**, then it is transitive, and arguably connected and dense.