Modal Logic for Philosophers, Ch. 1
Exercise 1.1
a) ~~p
~p → ⊥
(p → ⊥) → ⊥

b) ~p & ~q
~(~p → ~~q)
(~p → ~~q) → ⊥
((p → ⊥) → ~~q) → ⊥
((p → ⊥) → (~q → ⊥)) → ⊥
((p → ⊥) → ((q → ⊥) → ⊥)) → ⊥

c) p v (q & r)
~p → (q & r)
(p → ⊥) → (q & r)
(p → ⊥) → ~(q → ~r)
(p → ⊥) → ((q → ~r) → ⊥)
(p → ⊥) → ((q → (r → ⊥)) → ⊥)

d) ~(p v q)
(p v q) → ⊥
(~p → q) → ⊥
((p → ⊥) → q) → ⊥

e) ~(p ↔ q)
(p ↔ q) → ⊥
((p → q) & (q → p)) → ⊥
(~((p → q) → ~(q → p))) → ⊥
(~((p → q) → ((q → p) → ⊥))) → ⊥
(((p → q) → ((q → p) → ⊥)) → ⊥) → ⊥
Exercise 1. 2
a) p→q / (q→⊥)→(p→⊥)

1.	p → q		
2.		q → ⊥	[Hyp]
3.		p	[Hyp]
4.		q	[1, 3, MP]
5.		⊥	[2, 4, MP]
6.		p → ⊥	[3-5, CP]
7.	(q → ⊥) → (p → ⊥)		[2-6, CP]

b) p→q, p→(q→⊥) / q→⊥

1.	p → q		
2.	p → (q → ⊥)		
3.		p	[Hyp]
4.		q	[1, 3, MP]
5.		q → ⊥	[2, 3, MP]
6.		⊥	[4, 5, MP]
7.	p → ⊥		[3-6, CP]

c) / (p→q)→(~q→~p)

1.		p→q		[Hyp]
2.			~q	[Hyp]
3.			q→⊥	[2, Def~]
4.			p	[Hyp]
5.			q	[1, 4, MP]
6.			⊥	[3, 5, MP]
7.			p→⊥	[4-6, CP]
8.			~p	[7, Def~]
9.		~q→~p`		[2-8, CP]
10.	(p→q)→(~q→~p)			[1-9, CP]

Exercise 1.3

1.	A	[Hyp]
2.	...	
3.	B	
4.	~B	
5.	⊥	[3, 4, ⊥In]
6.	A→⊥	[1-5, CP]
7.	~A	[6, Def~]

Exercise 1.4
a) &Int

1.	A	
2.	B	
3.	~(A&B)	[Hyp]
4.	~~(A→~B)	[3, Def&]
5.	A→~B	[4, DN]
6.	~B	[1, 5, MP]
7.	B	[2, Reit]
8.	A&B	[3-7, ~Out]

b) &Out

1. A&B
2. ~(A→~B) [1, Def&]
3. B [1, &Out]
4. ~A [Hyp]
5. A [Hyp]
6. B [Hyp]
7. A [Reit, 5]
8. ~A [Reit, 4]
9. ~B [6-8, ~In]
10. A→~B [5-9, CP]
11. ~(A→~B) [2, Reit]
12. A [4-11, ~Out]

c) vIn (1)

1. A
2. ~A [Hyp]
3. ~B [Hyp]
4. A [1, Reit]
5. ~A [2, Reit]
6. B [3-5, ~Out]
7. ~A→B [2-6, CP]
8. AvB [7, Defv]

d) vIn (2)

1. B
2. ~A [Hyp]
3. B [1, Reit]
4. ~A→B [2-3, CP]
5. AvB [4, Defv]

e) ↔In

1. A→B
2. B→A
3. (A→B)&(B→A) [1,2, &In]
4. A↔B [3, Def ↔]

e) ↔Out (1)

1. A↔B
2. (A→B)&(B→A) [1, Def ↔]
3. A→B [2, &Out]

f) ↔ Out (2)

1. A↔B
2. (A→B)&(B→A) [1, Def ↔]
3. B→A [2, &Out]

Exercise 1.5

a) MT

1. A→B
2. ~B
3. A [Hyp]
4. B [1, 4, MP]
5. ~B [2, Reit]
6. ~A [3-5, ~In]

b) CN

1. A→B
2. ~B [Hyp]
3. ~A [1, 2, MT]
4. ~B→~A [2, 3, CP]

c) DM (1)

1. ~(AvB)
2. A [Hyp]
3. AvB [2, vIn]
4. ~(AvB) [1, Reit]
5. ~A [2-4, ~In]
6. B [Hyp]
7. AvB [vIn]
8. ~(AvB) [1, Reit]
9. ~B [6-8, ~In]
10. ~A&~B [5, 9, &In]

d) DM (2)

1. ~(A&B)
2. ~~A [Hyp]
3. A [2, DN]
4. B [Hyp]
5. A&B [3, 4, &In]
6. ~(A&B) [1, Reit]
7. ~B [4-6, ~In]
8. ~~A→~B [2-7, CP]
9. ~Av~B [8, Defv]

e) → F (2)

1. ~(A→B)
2. B [Hyp]
3. A [Hyp]

4. B [2, Reit]
5. A→B [3-4, CP]
6. ~(A→B) [1, Reit]
7. ~B [2-6, ~In]

Exercise 1.6

a)

1. pvq
2. p→r
3. q→s
4. p [Hyp] q [Hyp]
5. r [MP] s [MP]
6. rvs [vIn] rvs [vIn]
7. rvs [vOut]

b)

1. pv(q&r)
2. p [Hyp] q&r [Hyp]
3. pvq [vIn] q [&Out]
4. pvr [vIn] r [&Out]
5. (pvq)&(qvr) [&In] pvq [vIn]
6. pvr [vIn]
7. [&In] (pvq)&(pvr)
8. (pvq)&(pvr) [vOut]

c)

1. ~pv~q
2. ~p ~q [Hyp]
3. p&q p&q [Hyp]
4. p q [&Out]
5. ~p ~q [Reit]
6. ~(p&q) ~(p&q) [~In]
7. ~(p&q) [vOut]

d)

1. pv(qvr)
2. p qvr [Hyp]
3. pvq q r
4. (pvq)vr pvq (pvq)vr [vIn]
5. (pvq)vr [vIn]
6. (pvq)vr [vOut]
7. (pvq)vr [vOut]

Exercise 1.7

a) □p / □(pvq)

1. □p
2. □
3. p [1, □Out]
4. pvq [3, vIn]
5. □(pvq) [2-4, □In]

b) □(p→q) / □p→□q

1. □(p→q)
2. □p [Hyp]
3. □
4. p [□Out]
5. p→q [□Out]
6. q [4, 5, MP]
7. □q [3-6, □In]
8. □p→□q [2-7, CP]

c) □(p&q) / □p&□q

1. □(p&q)
2. □
3. p&q [1, □Out]
4. p [3, &Out]
5. □p [2-4, □Out]
6. □
7. p&q [1, □Out]
8. q [7, &Out]
9. □q [6-8, □Out]
10. □p&□q [5, 9, &In]

d) □(pvq), □(p→r), □(q→r) / □r

1. □(pvq)
2. □(p→r)
3. □(q→r)
4. □
5. pvq [1, □Out]
6. p→r [2, □Out]
7. q→r [3, □Out]
8. p q [Hyp]
9. r r [6,8; 7,8; MP]
10. r [8-9, vOut]
11. □r [4-10, □In]

e) □pv□q / □(pvq)

1.	□pv□q			
2.	□p	□q		[Hyp]
3.		□	□	
4.		p	q	[□Out]
5.		pvq	pvq	[vIn]
6.	□(pvq)			[□In]

Exercise 1.8

a) / □(p&q)↔(□p&□q)

1.	□(p&q)		[Hyp]
2.		□	
3.		p&q	[1, □Out]
4.		p	[3, &Out]
5.	□p		[2-4, □In]
6.		□	
7.		p&q	[1, □Out]
8.		q	[7, &Out]
9.	□q		[6-8, □In]
10.	□p&□q		[5, 9, &In]
11.	□(p&q)→□p&□q		[1-10, CP]
12.	□p&□q		[Hyp]
13.	□p		[12, &Out]
14.	□q		[12, &Out]
15.		□	
16.		p	[13, □Out]
17.		q	[14, □Out]
18.		p&q	[16, 17, &In]
19.	□(p&q)		[15-18, □In]
20.	(□p&□q)→□(p&q)		[12-19, CP]
21.	□(p&q)↔(□p & □q)		[11, 20, ↔In]

b) / (□pv□q) → □(pvq)

1.	□pv□q			[Hyp]
2.	□p	□q		[Hyp]
3.		□	□	
4.		p	q	[□Out]
5.		pvq	pvq	[vIn]
6.	□(pvq)	□(pvq)		[□In]
7.	□(pvq)			[vIn]

c) / (◊pv◊q)↔ ◊(pvq)

1.	◊pv◊q				[Hyp]
2.		◊p	◊q		[Hyp]
3.		~□~p	~□~q		[Def◊]
4.			□~(pvq)	□~(pvq)	[Hyp]
5.			□	□	
6.			~(pvq)	~(pvq)	[□Out]
7.			~p&~q	~p&~q	[DM]
8.			~p	~q	[&Out]
9.			□~p	□~q	[□In]
10.			~□~p	~□~q	[Reit]
11.		~□~(pvq)	~□~(pvq)		[~In]
12.		◊(pvq)	◊(pvq)		[Def◊]
13.	◊(pvq)				
14.	(◊pv◊q)→◊(pvq)				[1-13, CP]
15.	◊(pvq)				[Hyp]
16.	~□~(pvq)				[Def◊]
17.		~(◊pv◊q)			[Hyp]
18.		~◊p&~◊q			[DM]
19.		~◊p			[&Out]
20.		~◊q			[&Out]
21.		~~□~p			[Def◊]
22.		~~□~q			[Def◊]
23.		□~p			[DN]
24.		□~q			[DN]
25.			□		
26.			pvq		[Hyp]
27.			~p→q		[Defv]
28.			~p		[□Out]
29.			q		[MP]
30.			~q		[□Out]
31.			~(pvq)		[~In]
32.		□~(pvq)			[□In]
33.		~□~(pvq)			[Reit]
34.	◊pv◊q				[~Out]
35.	◊(pvq)→(◊pv◊q)				[CP]
36.	(◊pv◊q)↔◊(pvq)				[↔In]

d) / ◊(p&q)→(◊p&◊q)

1.	◊(p&q)	[Hyp]
2.	~□~(p&q)	[Def◊]

3.		~(◊p&◊q)				[Hyp]
4.		~◊pv~◊q				[DM]
5.			~◊p		~◊q	[Hyp]
6.			~~□~p		~~□~q	[Def◊]
7.			□~p		□~q	[DN]
8.				□	□	
9.				~p	~q	[□Out]
10.				~pv~q	~pv~q	[vIn]
11.			□(~pv~q)		□(~pv~q)	[□In]
12.		□(~pv~q)				[vIn]
13.			□			
14.				p&q		[Hyp]
15.				p		[&Out]
16.				~~p		[~~In]
17.				q		[&Out]
18.				~pv~q		[□Out]
19.				~~p→~q		[Defv]
20.				q		[MP]
21.				~q		[Reit]
22.			~(p&q)			[~In]
23.		□~(p&q)				[□In]
24.		~□~(p&q)				[Reit]
25.	◊p&◊q					[~Out]
26.	◊(p&q)→(◊p&◊q)					[CP]

Exercise 1.9

1.	◊A			
2.		□		
3.			A	[Hyp]
4.			…	
5.			B	
6.		A→B		[3-5, CP]
7.	□(A →B)			[2-6, □In]
8.	~□~A			[1, Def◊]
9.		~◊B		[Hyp]
10.		~~□~B		[9, Def◊]
11.		□~B		[10, DN]
12.			□	
13.			A → B	[7, □Out]
14.			~B	[11, □Out]
15.			~A	[13, 14, MT]
16.		□~A		[12-15, □In]
17.		~□~A		[8, Reit]
18.	◊B			[9-17, ~Out]

Exercise 1.10

a) (◊pv◊q)↔◊(pvq)

1.		◊pv◊q			[Hyp]
2.			◊p	◊q	[Hyp]
3.			□, p	□, q	
4.			pvq	pvq	[3, vIn]
5.			◊(pvq)	◊(pvq)	[2, 4, ◊Out]
6.		◊(pvq)			[1, 2-5, vOut]
7.	(◊pv◊q)→◊(pvq)				[1-6, CP]
8.		◊(pvq)			[Hyp]
9.			~(◊pv◊q)		[Hyp]
10.			~◊p&~◊q		[9, DM]
11.			~◊p		[10, &Out]
12.			~◊q		[10, &Out]
13.			~~□~p		[11, Def◊]
14.			□~p		[13, DN]
15.				□, pvq	
16.				~p	[14, □Out]
17.				~p→q	[15, Defv]
18.				q	[16, 17, MP]
19.			◊q		[15, 18, ◊Out]
20.			~◊q		[12, Reit]
21.		◊pv◊q			[9-20, ~Out]
22.	◊(pvq)→(◊pv◊q)				[8-21, CP]
23.	(◊pv◊q)↔◊(pvq)				[7, 22, ↔In]

b.1) ~□A / ◊~A

1.	~□A		
2.		~◊~A	[Hyp]
3.		~~□~~A	[2, Def◊]
4.		□~~A	[3, DN]
5.		□	
6.		~~A	[4, □Out]
7.		A	[6, DN]
8.		□A	[5-7, □In]

9. ~□A [1, Reit]
10. ◊~A [2-8, ~Out]

b.2) ~◊A / □~A
1. ~◊A
2. ~~□~A [Def◊]
3. □~A [2, DN]

c.1) □~□p /□◊~p
1. □~□p
2. □
3. ~□p [1, □Out]
4. ◊~p [3, ~□]
5. □◊~p [2-4, □In]

c.2) ◊~◊p / ◊□~p
1. ◊~◊p
2. □, ~◊p
3. □~p [2, ~◊]
4. ◊□~p [3, ◊Out]

Exercise 1.11

a) &Out

L |- A&B

L |- A, L |- B

b) IP

L, ~A |- ⊥

L |- A

c) ◊Out

L, ◊A, □, A |- B

L, ◊A |- ◊B

Exercise 1.12

c) □(p&q) / □p&□q
1. □(p&q) |-□(p&q) [Hyp]
2. □(p&q), □ |- (p&q) [□Out]
3. □(p&q), □ |- p [&Out]
4. □(p&q), □ |- q [&Out]
5. □(p&q) |- □p [□In]
6. □(p&q), |- □q [□In]
7. □(p&q) |- □p&□q [&In]

d)□(pvq), □(p→r), □(q→r) / □r
1. □(pvq), □(p→r), □(q→r) |- □(pvq), □(p→r), □(q→r) [Hyp]
2. □(pvq), □(p→r), □(q→r), □ |- pvq [□Out]
3. □(pvq), □(p→r), □(q→r), □ |- p→r [□Out]
4. □(pvq), □(p→r), □(q→r), □ |- q→r [□Out]
5. □(pvq), □(p→r), □(q→r), □, p |- r [MP]
6. □(pvq), □(p→r), □(q→r), □, q |- r [MP]
7. □(pvq), □(p→r), □(q→r), □ |- r [vOut]
8. □(pvq), □(p→r), □(q→r), |- □r [□Out]

Exercise 1.13

a) □p |- □(pvq)
1. p |- p [Hyp]
2. p |- pvq [vIn]
3. □p |- □(pvq) [GN]

b) □p, □(p→q) |- □q
1. p, p→q |- p [Hyp]
2. p, p→q |- p→q [Hyp]
3. p, p→q |- q [MP]
4. □p, □(p→q) |- □q [GN]

c) □(pvq), □(p→r), □(q→r) |- □r
1. pvq, p→r, q→r |- pvq [Hyp]
2. pvq, p→r, q→r |- p→r [Hyp]
3. pvq, p→r, q→r |- q→r [Hyp]
4. pvq, p→r, q→r, p |- p [Hyp]
5. pvq, p→r, q→r, p |- r [MP]
6. pvq, p→r, q→r, q |- q [Hyp]
7. pvq, p→r, q→r, q |- r [MP]
8. pvq, p→r, q→r |- r [vOut]
9. □(pvq), □(p→r), □(q→r) |- □r [GN]

d) □~p, □(pvq) |- □q
1. ~p, pvq |- ~p [Hyp]
2. ~p, pvq |- pvq [Hyp]
3. ~p, pvq |- ~p→q [Defv]

4. ~p, pvq |- q [MP]
5. □~p, □(pvq) |- □q [GN]

e) □(p→q), □~q |- □~p

1. p→q, ~q |- p→q [Hyp]
2. p→q, ~q |- ~q [Hyp]
3. p→q, ~q |- ~p [MT]
4. □(p→q), □~q |- □~p [GN]

Exercise 1.14

B_1, B_2, B_3 |- A [Given]
B_1, B_2 |- B_3→A [CP]
B_1 |- B_2→(B_3→A) [CP]
|- B_1→(B_2→(B_3→A)) [CP]
|- □(B_1→(B_2→(B_3→A))) [Nec]
|- (□(B_1→(B_2→(B_3→A))))→(□B_1→□(B_2→(B_3→A))) [Dist]
|- □B_1→□(B_2→(B_3→A)) [MP]
□B_1 |- □B_1 [Hyp]
□B_1 |- □(B_2→(B_3→A)) [MP]
□B_1 |- (□(B_2→(B_3→A)))→(□B_2→□(B_3→A)) [Dist]
□B_1 |- □B_2→□(B_3→A) [MP]
□B_1, □B_2 |- □B_2 [Hyp]
□B_1, □B_2 |- □(B_3→A) [MP]
□B_1, □B_2 |-□(B_3→A)→(□B_3→□A) [Dist]
□B_1, □B_2 |- □B_3→□A [MP]
□B_1, □B_2, □B_3 |- □B_3 [Hyp]
□B_1, □B_2, □B_3 |- □A [MP]