Jacob Archambault
Fitting, Melvin, and Richard L. Mendelsohn (1998). *First-Order Modal Logic*. ch. 4

Exercise 4.1.1
Verify that the following are formulas, and determine which are the free variable occurrences. Also, determine which of the following are sentences.

a) $((\forall x)\Diamond P(x, y) \rightarrow (\exists y)\Box Q(x, y))$

1. $Q(x, y)$ [atomic: x, y free]
2. $\Box Q(x, y)$ [1, rule 4: x, y free]
3. $(\exists y)\Box Q(x, y))$ [2, rule 5: x free]
4. $P(x, y)$ [atomic: x, y free]
5. $\Diamond P(x, y)$ [4, rule 4: x, y free]
6. $(\forall x)\Diamond P(x, y)$ [5, rule 5: y free]
7. $((\forall x)\Diamond P(x, y) \rightarrow (\exists y)\Box Q(x, y))$ [3, 6, r. 3: 1st y, 2nd x free.]

b) $(\exists x)(\Box P(x) \rightarrow (\forall x)\Box P(x))$

1. $P(x)$ [atomic: x free]
2. $\Box P(x)$ [1, rule 4: x free]
3. $(\forall x)\Box P(x)$ [2, rule 5: sentence]
4. $\Box P(x) \rightarrow (\forall x)\Box P(x)$ [2, 3, rule 3: 1st x free]
5. $(\exists x)(\Box P(x) \rightarrow (\forall x)\Box P(x))$ [4, rule 5: sentence]

c) $(\forall x)((\exists y)R(x, y) \rightarrow R(y, x))$

1. $R(y, x)$ [atomic: y, x free]
2. $R(x, y)$ [atomic: x, y free]
3. $(\exists y)R(x, y)$ [2, rule 5: x free]
4. $(\exists y)R(x, y) \rightarrow R(y, x)$ [1, 3, rule 3: 2nd x, y free]
5. $(\forall x)((\exists y)R(x, y) \rightarrow R(y, x))$ [4, rule 5: 2nd y free]

Exercise 4.6.1

1. $[(\exists x)\Diamond P(x) \;\&\; \Box(\forall x)(P(x) \rightarrow Q(x))] \rightarrow (\exists x)\Diamond Q(x)$ – Valid
2. $(\forall x)\Box P(x) \rightarrow \Box(\forall x)P(x)$ – Valid
3. $\Box(\forall x)P(x) \rightarrow (\forall x)\Box P(x)$ – Valid
4. $(\exists x)\Box P(x) \rightarrow \Box(\exists x)P(x)$ – Valid
5. $\Box(\exists x)P(x) \rightarrow (\exists x)\Box P(x)$ – Invalid
6. $(\exists x)\Diamond[\Box P(x) \rightarrow (\forall x)\Box P(x)]$ –Invalid
7. $(\exists x)\Diamond[P(x) \rightarrow (\forall x)\Box P(x)]$ – Invalid
8. $(\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$ – valid

Exercise 4.6.2
Give a proof of the proposition $(\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$

1. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ be a constant domain model, and Γ an arbitrary member of $\mathcal{G}$.Then $\mathcal{M}, \Gamma \Vdash (\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$
2. To show this, we show that if $\mathcal{M}, \Gamma \Vdash (\exists x)(\forall y)\Box R(x, y)$, then $\mathcal{M}, \Gamma \Vdash (\forall y)\Box(\exists x)R(x, y)$.
3. Assume $\mathcal{M}, \Gamma \Vdash_v (\exists x)(\forall y)\Box R(x, y)$
4. So for some x-variant v' of v, $\mathcal{M}, \Gamma \Vdash_{v'} (\forall y)\Box R(x, y)$
5. So for every y-variant v'' of v', $\mathcal{M}, \Gamma \Vdash_{v''} \Box R(x, y)$
6. So for all Δ, if $\Gamma R \Delta$, then $\mathcal{M}, \Delta \Vdash_{v''} R(x, y)$
7. i.e. $\langle v''(x), v''(y)\rangle \in \mathcal{I}(R, \Delta)$
8. And since v'' doesn't differ from v' on x, $v''(x) = v'(x)$, and $\langle v'(x), v''(y)\rangle \in \mathcal{I}(R, \Delta)$
9. Since v'' is *any* y-variant of v' let v'' be such that $v''(y) = v(y)$.
10. So for some x-variant v' of v, $\mathcal{M}, \Delta \Vdash_{v'} R(x, y)$
11. Therefore, $\mathcal{M}, \Delta \Vdash_v (\exists x)R(x, y)$
12. And since Δ is any world accessible from Γ, it follows that $\mathcal{M}, \Gamma \Vdash_v \Box(\exists x)R(x, y)$
13. Since v was chosen arbitrarily, it follows that for any y-variant of v', $\mathcal{M}, \Gamma \Vdash_{v'} \Box(\exists x)R(x, y)$
14. Therefore, $\mathcal{M}, \Gamma \Vdash_{v'} (\forall y)\Box(\exists x)R(x, y)$
15. And since v doesn't differ from v' with respect to any free variables of the above formula (there are none), it follows that $\mathcal{M}, \Gamma \Vdash_v (\forall y)\Box(\exists x)R(x, y)$
16. Therefore, $\mathcal{M}, \Gamma \Vdash_v (\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$

Exercise 4.7.1
Which of the sentences of Exercise 4.6.1 are valid in all *varying domain* models and which are not?

1. $[(\exists x)\Diamond P(x) \;\&\; \Box(\forall x)(P(x) \rightarrow Q(x))] \rightarrow (\exists x)\Diamond Q(x)$ – Invalid

2. $(\forall x)\Box P(x) \rightarrow \Box(\forall x)P(x)$ – Invalid
3. $\Box(\forall x)P(x)\rightarrow (\forall x)\Box P(x)$ – Invalid
4. $(\exists x)\Box P(x) \rightarrow \Box(\exists x)P(x)$ – Invalid
5. $\Box(\exists x)P(x) \rightarrow (\exists x)\Box P(x)$ – Invalid
6. $(\exists x)\Diamond[\Box P(x) \rightarrow (\forall x)\Box P(x)]$ –Invalid
7. $(\exists x)\Diamond[P(x) \rightarrow (\forall x)\Box P(x)]$ – Invalid
8. $(\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$ – Invalid

Exercise 4.8.1

Let Φ be the formula $(\exists x)(\Box P(x) \rightarrow \Box(\forall x)P(x)$? What is $\Phi^{\mathcal{E}}$ (the existence relativization of Φ)?

Answer: $(\exists x)(E(x)\ \&\ (\Box P(x)\rightarrow\Box(\forall x)(E(x)\rightarrow P(x)))$

Exercise 4.8.2

Prove the following:

Let Φ be a sentence not containing the symbol $\mathcal{E}$. Then Φ is valid in every varying domain model only if its *existence relativization*, denoted $\Phi^{\mathcal{E}}$, is valid in every constant domain model, where $\Phi^{\mathcal{E}}$ is defined recursively as follows:

1. If A is atomic, $A^{\mathcal{E}} = A$
2. $(\neg X)^{\mathcal{E}} = \neg(X^{\mathcal{E}})$.
3. For a binary connective $\circ$, $(X \circ Y)^{\mathcal{E}} = (X^{\mathcal{E}} \circ Y^{\mathcal{E}})$.
4. $(\Box X)^{\mathcal{E}} = (\Box X^{\mathcal{E}})$.
5. $(\Diamond X)^{\mathcal{E}} = (\Diamond X^{\mathcal{E}})$.
6. $((\forall x)\Phi)^{\mathcal{E}} = (\forall x)(\mathcal{E}(x) \supset \Phi^{\mathcal{E}})$
7. $((\exists x)\Phi)^{\mathcal{E}} = (\exists x)(\mathcal{E}(x)\ \&\ \Phi^{\mathcal{E}})$

1. We prove the contrapositive, by induction on the length of Φ.
2. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I}\rangle$ be a constant domain model, and $\Phi^{\mathcal{E}}$ an invalid formula on $\mathcal{M}$, perhaps making use of the existence predicate. Then we construct a varying-domain model $\mathcal{M}' = \langle \mathcal{G}', \mathcal{R}', \mathcal{D}', \mathcal{I}'\rangle$ as follows: $\mathcal{G}'$ and $\mathcal{R}'$ are exactly the same as $\mathcal{G}$, and $\mathcal{R}$ in $\mathcal{M}$; $\mathcal{D}'$ is a domain function on members of $\mathcal{G}'$, such that for each member Γ of $\mathcal{G}'$, $\mathcal{D}'(\Gamma) = \mathcal{I}(\mathcal{E}, \Gamma)$; and $\mathcal{I}'$ is exactly like $\mathcal{I}$ except perhaps with respect to $\mathcal{E}$. We show that for any formula X not containing $\mathcal{E}$, for any world $\Gamma \in \mathcal{G}$, and for any evaluation v. $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \Leftrightarrow \mathcal{M}', \Gamma \Vdash_v X$

For the base case, if $\Phi^{\mathcal{E}}$ is an atomic formula X:

3. Then, the equivalence holds, since *ex hypothesi* X does not contain $\mathcal{E}$, and the existence relativization of an atomic formula is that formula itself.

Now for the various inductive cases:

If $\Phi^{\mathcal{E}}$ is a negation of the form $\neg X$:

4. $\mathcal{M}, \Gamma \Vdash_v \neg X^{\mathcal{E}}$ iff $\mathcal{M}, \Gamma, \nVdash X^{\mathcal{E}}$.
5. ...iff, $\mathcal{M}', \Gamma \nVdash_v X$, by the induction hypothesis (IH hereafter)
6. ...iff $\mathcal{M}', \Gamma \Vdash_v \neg X$.

If $\Phi^{\mathcal{E}}$ is a conjunction:

7. $\mathcal{M}, \Gamma \Vdash_v (X^{\mathcal{E}}\ \&\ Y^{\mathcal{E}})$ iff $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}}$ and $\mathcal{M}, \Gamma \Vdash_v Y^{\mathcal{E}}$.
8. ...iff, respectively, $\mathcal{M}'\ \Gamma \Vdash_v X$, and $\mathcal{M}', \Gamma \Vdash_v Y$ [IH]
9. ...iff $\mathcal{M}', \Gamma \Vdash_v X\ \&\ Y$.

If $\Phi^{\mathcal{E}}$ is a disjunction:

10. $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \vee Y^{\mathcal{E}}$ iff $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}}$ or $\mathcal{M}, \Gamma \Vdash_v Y^{\mathcal{E}}$.
11. ...iff, respectively, $\mathcal{M}', \Gamma \Vdash_v X$ or $\mathcal{M}', \Gamma \Vdash_v Y$. [IH]
12. ...iff $\mathcal{M}', \Gamma \Vdash_v X \vee Y$.

If $\Phi^{\mathcal{E}}$ is a material implication:

13. $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \supset Y^{\mathcal{E}}$. iff $\mathcal{M}, \Gamma \nVdash_v X^{\mathcal{E}}$ or $\mathcal{M}, \Gamma \Vdash_v Y^{\mathcal{E}}$.
14. ...iff, respectively, $\mathcal{M}', \Gamma \nVdash_v X$ or $\mathcal{M}', \Gamma \Vdash_v Y$. [IH]
15. ...iff $\mathcal{M}'\ \Gamma \Vdash_v X \supset Y$.

If $\Phi^{\mathcal{E}}$ is a biconditional:

16. $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \equiv Y^{\mathcal{E}}$. iff $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}}$ and $\mathcal{M}, \Gamma \Vdash_v Y^{\mathcal{E}}$; or $\mathcal{M}, \Gamma \nVdash_v Y^{\mathcal{E}}$ and $\mathcal{M}, \Gamma, \nVdash_v X^{\mathcal{E}}$.
17. The first iff $\mathcal{M}', \Gamma \Vdash_v X$ and $\mathcal{M}', \Gamma \Vdash_v Y$, [IH], i.e. only if $\mathcal{M}', \Gamma \Vdash_v X \equiv Y$
18. The second iff $\mathcal{M}', \Gamma \nVdash_v X$ and $\mathcal{M}', \Gamma \nVdash_v Y$, [IH], and so only if $\mathcal{M}', \Gamma \Vdash_v X \equiv Y$.
19. And $\mathcal{M}', \Gamma \Vdash_v X \equiv Y$ iff ($\mathcal{M}', \Gamma \Vdash_v X$ and $\mathcal{M}', \Gamma \Vdash_v Y$; or $\mathcal{M}', \Gamma \nVdash_v X$ and $\mathcal{M}', \Gamma \nVdash_v Y$)
20. So $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \equiv Y^{\mathcal{E}}$. iff ($\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}}$ and $\mathcal{M}, \Gamma \Vdash_v Y^{\mathcal{E}}$; or $\mathcal{M}, \Gamma \nVdash_v Y^{\mathcal{E}}$ and $\mathcal{M}, \Gamma, \nVdash_v X^{\mathcal{E}}$) iff ($\mathcal{M}', \Gamma \Vdash_v X$ and $\mathcal{M}', \Gamma \Vdash_v Y$; or $\mathcal{M}', \Gamma \nVdash_v X$ and $\mathcal{M}', \Gamma \nVdash_v Y$) iff $\mathcal{M}', \Gamma \Vdash_v X \equiv Y$.

21. So $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \equiv Y^{\mathcal{E}}$ iff $\mathcal{M}', \Gamma \Vdash_v X \equiv Y$.

If $\Phi^{\mathcal{E}}$ is a boxed formula:

22. $\mathcal{M}, \Gamma, \Vdash_v \Box X^{\mathcal{E}}$ iff for all Δ in $\mathcal{G}$, if $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Delta \Vdash_v X^{\mathcal{E}}$.
23. ...iff $\mathcal{M}', \Delta \Vdash_v X$. (by IH, equivalence of $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{R}$, $\mathcal{R}'$)
24. ...iff $\mathcal{M}', \Gamma \Vdash_v \Box X$.

If $\Phi^{\mathcal{E}}$ is a formula of the form $\Diamond X^{\mathcal{E}}$:

25. $\mathcal{M}, \Gamma \Vdash_v \Diamond X^{\mathcal{E}}$ iff for some Δ in $\mathcal{G}$, $\Gamma\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \Vdash_v X^{\mathcal{E}}$.
26. ...iff $\mathcal{M}', \Delta \Vdash_v X$. (by IH, equivalence of $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{R}$, $\mathcal{R}'$)
27. ...iff $\mathcal{M}', \Gamma \Vdash_v \Diamond X$.

If $\Phi^{\mathcal{E}}$ is a universally quantified sentence:

28. $\mathcal{M}, \Gamma \Vdash_v (\forall x)(\mathcal{E}(x) \supset X^{\mathcal{E}})$ iff for every x-variant v' of v, $\mathcal{M}, \Gamma \Vdash_{v'} \mathcal{E}(x) \supset X^{\mathcal{E}}$.
29. ...iff, for every x-variant v' of v, $\mathcal{M}, \Gamma \nVdash_{v'} \mathcal{E}(x)$ or $\mathcal{M}, \Gamma \Vdash_{v'} X^{\mathcal{E}}$
30. The first iff $v'(x) \notin \mathcal{I}(\mathcal{E}, \Gamma)$.
31. ...iff $v'(x) \notin \mathcal{D}'(\Gamma)$. (by construction of $\mathcal{D}'$)
32. The second iff $\mathcal{M}', \Gamma \Vdash_{v'} X$ (by IH)
33. So $\mathcal{M}, \Gamma \Vdash_v (\forall x)(\mathcal{E}(x) \supset X^{\mathcal{E}})$ iff for every x-variant v' of v, $v'(x) \notin \mathcal{D}'(\Gamma)$ or $\mathcal{M}', \Gamma \Vdash_{v'} X$
34. That is, iff for every x-variant v' s.t. $v'(x) \in \mathcal{D}'(\Gamma)$, $\Gamma \Vdash_{v'} X$
35. i.e. iff $\mathcal{M}', \Gamma \Vdash_v (\forall x)X$

If $\Phi^{\mathcal{E}}$ is an existentially quantified sentence:

36. $\mathcal{M}, \Gamma \Vdash_v (\exists x)(\mathcal{E}(x) \,\&\, X^{\mathcal{E}})$ iff for some x-variant v' of v, $\mathcal{M}, \Gamma \Vdash_{v'} \mathcal{E}(x) \,\&\, X^{\mathcal{E}}$
37. ...iff $\mathcal{M}, \Gamma \Vdash_{v'} \mathcal{E}(x)$ and $\mathcal{M}, \Gamma \Vdash_{v'} X^{\mathcal{E}}$.
38. ...iff $v'(x) \in \mathcal{I}(\mathcal{E}, \Gamma)$ and $\mathcal{M}', \Gamma \Vdash_{v'} X$ (the latter by IH)
39. ...iff $v'(x) \in \mathcal{D}'(\Gamma)$ and $\mathcal{M}', \Gamma \Vdash_{v'} X$ (by construction of $\mathcal{D}'$)
40. ...iff for some x-variant v' of v at Γ, $\mathcal{M}', \Gamma \Vdash_{v'} X$
41. ...iff $\mathcal{M}', \Gamma \Vdash (\exists x)X$.

Summing up:

42. So if $\Phi^{\mathcal{E}}$ is atomic, for any world $\Gamma \in \mathcal{G}$, and for any evaluation v. $\mathcal{M}, \Gamma \Vdash_v \Phi^{\mathcal{E}} \Leftrightarrow \mathcal{M}', \Gamma \Vdash_v \Phi$
43. And if $\Phi^{\mathcal{E}}$ is any possible syntactic extension of a sentence for which the above equivalence holds, then it is still the case that $\mathcal{M}, \Gamma \Vdash_v \Phi^{\mathcal{E}} \Leftrightarrow \mathcal{M}', \Gamma \Vdash_v \Phi$
44. And so for any formula X not containing $\mathcal{E}$, for any world $\Gamma \in \mathcal{G}$, and for any evaluation v, $\mathcal{M}, \Gamma \Vdash_v X^{\mathcal{E}} \Leftrightarrow \mathcal{M}', \Gamma \Vdash_v X$
45. Therefore, if $\Phi^{\mathcal{E}}$ is invalid on some constant domain model, then there is a corresponding varying domain model on which Φ is invalid.
46. And so if Φ is valid on every varying domain model, then $\Phi^{\mathcal{E}}$ is valid on every constant domain model.

Exercise 4.9.1

Which of the sentences of Exercise 4.6.1 are valid in all *monotonic domain* models and which are not? Similarly for *anti-monotonic domain* models.

a) monotonic domain models

1. $[(\exists x)\Diamond P(x) \,\&\, \Box(\forall x)(P(x) \rightarrow Q(x))] \rightarrow (\exists x)\Diamond Q(x)$ – Valid
2. $(\forall x)\Box P(x) \rightarrow \Box(\forall x)P(x)$ – Invalid
3. $\Box(\forall x)P(x) \rightarrow (\forall x)\Box P(x)$ – Valid
4. $(\exists x)\Box P(x) \rightarrow \Box(\exists x)P(x)$ – Valid
5. $\Box(\exists x)P(x) \rightarrow (\exists x)\Box P(x)$ – Invalid
6. $(\exists x)\Diamond[\Box P(x) \rightarrow (\forall x)\Box P(x)]$ – Invalid
7. $(\exists x)\Diamond[P(x) \rightarrow (\forall x)\Box P(x)]$ – Invalid
8. $(\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$ – Valid

b) anti-monotonic domain models

1. $[(\exists x)\Diamond P(x) \,\&\, \Box(\forall x)(P(x) \rightarrow Q(x))] \rightarrow (\exists x)\Diamond Q(x)$ – Invalid
2. $(\forall x)\Box P(x) \rightarrow \Box(\forall x)P(x)$ – Valid
3. $\Box(\forall x)P(x) \rightarrow (\forall x)\Box P(x)$ – Invalid
4. $(\exists x)\Box P(x) \rightarrow \Box(\exists x)P(x)$ – Invalid
5. $\Box(\exists x)P(x) \rightarrow (\exists x)\Box P(x)$ – Invalid
6. $(\exists x)\Diamond[\Box P(x) \rightarrow (\forall x)\Box P(x)]$ –Invalid
7. $(\exists x)\Diamond[P(x) \rightarrow (\forall x)\Box P(x)]$ – Invalid
8. $(\exists x)(\forall y)\Box R(x, y) \rightarrow (\forall y)\Box(\exists x)R(x, y)$ – Invalid

Exercise 4.9.2

Show that a varying domain augmented frame is anti-monotonic if and only if every model based on it is one in which the Barcan formula is valid.

1. First we show that if a frame is anti-monotonic, then every model based on it is one in which the Barcan formula is valid
2. Then we show that if the Barcan formula is valid on every model of a frame, then that frame is anti-monotonic.
3. To show the first, we assume the contrary
4. So assume $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ is an anti-monotonic augmented frame on which the Barcan formula is invalid
5. Then for some model $\mathcal{M}$ on $\mathcal{F}$, and some element Γ in $\mathcal{G}$, $\mathcal{M}$, $\Gamma \nVdash_v (\forall x)\Box\Phi \supset \Box(\forall x)\Phi$
6. So $\mathcal{M}, \Gamma \Vdash_v (\forall x)\Box\Phi$ and $\mathcal{M}, \Gamma \nVdash_v \Box(\forall x)\Phi$
7. So for some world Δ, $\Gamma\mathcal{R}\Delta$ and $\mathcal{M}, \Delta \nVdash_v (\forall x)\Phi$
8. And so for some x-variant v' of v at Δ, $\mathcal{M}, \Delta \nVdash_{v'} \Phi$
9. Also, for every x-variant v'' of v at Γ, $\mathcal{M}, \Gamma \Vdash_{v'} \Box\Phi$
10. So on every x-variant v'' of v at Γ, and every world Z in $\mathcal{G}$, if $\Gamma\mathcal{R}Z$, then $\mathcal{M}$, Z $\Vdash_{v'} \Phi$
11. And since $\Gamma\mathcal{R}\Delta$, for every x-variant v'' of v at Γ, $\mathcal{M}, \Delta \Vdash_{v'} \Phi$
12. Now, since $\mathcal{F}$ is anti-monotonic, for any worlds w, w' in $\mathcal{G}$, if $w\mathcal{R}w'$, then $\mathcal{D}(w') \subseteq \mathcal{D}(w)$.
13. And in particular, $\mathcal{D}(\Delta) \subseteq \mathcal{D}(\Gamma)$
14. So every x-variant of v at Δ is an x-variant of v at Γ.
15. And in particular, $v'(x) \in \Gamma$
16. So contrary to 11, for some x-variant v'' of v at Γ, $\mathcal{M}, \Delta \nVdash_{v''} \Phi$
17. And so there is no model on $\mathcal{F}$ on which the Barcan formula is invalid
18. i.e. on every model on $\mathcal{F}$, the Barcan formula is valid.
19. **And so, since $\mathcal{F}$ was arbitrarily chosen, the Barcan formula is valid on every anti-monotonic frame.**
20. To show the second part of our proof, we show that if a frame *isn't* anti-monotonic, there is a model on that frame on which the Barcan formula fails.
21. Let $\mathcal{F} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D} \rangle$ be an arbitrary frame that is not anti-monotonic.
22. Then for some elements Γ, Δ in $\mathcal{G}$, and some object d in $\mathcal{D}(\mathcal{F})$, $\Gamma\mathcal{R}\Delta$, $d \in \mathcal{D}(\Gamma)$ and $d \notin \mathcal{D}(\Delta)$. We show that this suffices for some instance of the Barcan formula to fail at Γ for some model on $\mathcal{F}$.
23. First, we let Φ be some contingent formula, and construct a model $\mathcal{M}$ so that $\mathcal{M}, \Gamma \Vdash_v (\forall x)\Box\Phi$, i.e. for every x-variant w of v at Γ, and every world Z in $\mathcal{G}$ s.t. $\Gamma\mathcal{R}Z$, $\mathcal{M}$, Z $\Vdash_w \Phi$, and so $\mathcal{M}, \Delta \Vdash_w \Phi$.
24. To ensure that $\mathcal{M}, \Gamma \nVdash_v \Box(\forall x)\Phi$, we let v' be the x-variant at Δ assigning d to x and we stipulate that $\mathcal{M}, \Delta \nVdash_{v'} \Phi$
25. Therefore, $\mathcal{M}, \Delta \nVdash_{v'} (\forall x)\Phi$
26. So, since $\Gamma\mathcal{R}\Delta$, $\mathcal{M}, \Gamma \nVdash_v \Box(\forall x)\Phi$
27. Therefore, if a frame $\mathcal{F}$ isn't anti-monotonic, then one can construct a model $\mathcal{M}$ at a world Γ on $\mathcal{F}$ s. t. $\mathcal{M}, \Gamma \Vdash_v (\forall x)\Box\Phi$ and $\mathcal{M}, \Gamma \nVdash \Box(\forall x)\Phi$, i.e. such that $\mathcal{M}, \Gamma \nVdash_v (\forall x)\Box\Phi \supset \Box(\forall x)\Phi$.
28. i.e. if the Barcan formula *is* valid on every model of a frame, that frame must be anti-monotonic.
29. And so the Barcan formula is valid on a frame iff that frame is anti-monotonic.

Exercise 4.9.3

Show the converse Barcan formula need not be valid in a model whose frame is anti-monotonic.

1. To show this, we specify a countermodel
2. Let $\mathcal{M} = \langle \mathcal{G}, \mathcal{R}, \mathcal{D}, \mathcal{I} \rangle$ where $\mathcal{G} = \{\Gamma, \Delta\}$, $\mathcal{R} = \{\langle \Gamma, \Delta \rangle\}$, $\mathcal{D}(\Gamma) = \{a, b\}$, $\mathcal{D}(\Delta) = \{b\}$. $\mathcal{I}(F, \Gamma) = \mathcal{I}(F, \Delta) = \{b\}$. $v(x) = a$. We show that $\mathcal{M}, \Gamma \Vdash \Box(\forall x)F(x) \supset (\forall x)\Box F(x)$.
3. Since $\mathcal{I}(F, \Delta) = \mathcal{D}(\Delta)$, it follows that $\mathcal{M}, \Delta \Vdash_{v'} F(x)$ for every x-variant v' of v at Δ.
4. And so $\mathcal{M}, \Delta \Vdash_v (\forall x)F(x)$
5. And since $\mathcal{R} = \{\langle \Gamma, \Delta \rangle\}$ and $\mathcal{M}, \Delta \Vdash_v (\forall x)F(x)$, it follows that $\mathcal{M}, \Gamma \Vdash_v \Box(\forall x)F(x)$
6. But since $a \notin \mathcal{I}(F, \Delta)$, it follows that $\mathcal{M}, \Delta \nVdash_v F(x)$.
7. And since $\Gamma\mathcal{R}\Delta$, it follows that $\mathcal{M}, \Gamma \nVdash_v \Box F(x)$
8. And since $a \in \mathcal{D}(\Gamma)$, for some x-variant of v at Γ – namely, v - $\mathcal{M}, \Gamma \nVdash_v \Box F(x)$.
9. Therefore, $\mathcal{M}, \Gamma \nVdash_v (\forall x)\Box F(x)$.

10. And since $\mathcal{M}, \Gamma \Vdash_v \Box(\forall x)F(x)$ and $\mathcal{M}, \Gamma \not\Vdash_v (\forall x)\Box F(x)$, it follows that $\mathcal{M}, \Gamma \not\Vdash_v \Box(\forall x)F(x) \supset (\forall x)\Box F(x)$.

Exercise 4.9.4.
Here are four formula schemes:

1. $(\exists x)\Box P(x) \supset \Box(\exists x)P(x)$
2. $\Box(\exists x)P(x) \supset (\exists x)\Box P(x)$
3. $(\forall x)\Diamond P(x) \supset \Diamond(\forall x)P(x)$
4. $\Diamond(\forall x)P(x) \supset (\forall x)\Diamond P(x)$

Just as we gave two versions of the Barcan formula, and observed they came in pairs, the same is true for the schemes above. Determine which pairs of schemes constitute equivalent assumptions.
Answer:
$(\exists x)\Box P(x) \supset \Box(\exists x)P(x) \equiv \Diamond(\forall x)P(x) \supset (\forall x)\Diamond P(x)$
And
$\Box(\exists x)P(x) \supset (\exists x)\Box P(x) \equiv (\forall x)\Diamond P(x) \supset \Diamond(\forall x)P(x)$

Exercise 4.9.5.
For the formula schemes in Exercise 4.9.4, determine the status of their validity assuming: constant domains; varying domains, monotonic domains, anti-monotonic domains.

Formulas 1 and 4 are valid on constant and monotonic domains, and invalid otherwise.

Formulas 2 and 3 are invalid on all of the above domains.