Garson, James (2006). *Modal Logic for Philosophers*, ch. 3.

Exercise 3.1

a) If $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(A\to B)=T$, then $\mathbf{a}_w(B)=T$

1. Presume $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(A\to B)=T$.
2. Then $\mathbf{a}_w(A)=F$ or $\mathbf{a}_w(B)=T$
3. But $\mathbf{a}_w(A)=T$
4. So $\mathbf{a}_w(B)=T$
5. Therefore, if $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(A\to B)=T$, then $\mathbf{a}_w(B)=T$

<table>
<tr><td colspan="2">A
A→B</td></tr>
<tr><td>~A</td><td>B</td></tr>
</table>

b) If $\mathbf{a}_w(A\to B)=T$ and $\mathbf{a}_w(B)=F$, then $\mathbf{a}_w(A)=F$

1. Presume $\mathbf{a}_w(A\to B)=T$ and $\mathbf{a}_w(B)=F$.
2. Then $\mathbf{a}_w(A)=F$ or $\mathbf{a}_w(B)=T$
3. But $\mathbf{a}_w(B)=F$
4. So $\mathbf{a}_w(A)=F$
5. Therefore, if $\mathbf{a}_w(A\to B)=T$ and $\mathbf{a}_w(B)=F$, then $\mathbf{a}_w(A)=F$

<table>
<tr><td colspan="2">A→B
~B</td></tr>
<tr><td>~A</td><td>B</td></tr>
</table>

c) $\mathbf{a}_w(A\&B)=T$ iff $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$.
To show that if $\mathbf{a}_w(A\&B)=T$, then $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$

1. Presume $\mathbf{a}_w(A\&B)=T$
2. Then $\mathbf{a}_w({\sim}(A\to{\sim}B))=T$
3. So $\mathbf{a}_w(A\to{\sim}B)=F$
4. So $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w({\sim}B)=F$
5. That is, $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$
6. Therefore, if $\mathbf{a}_w(A\&B)=T$, then $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$

<table>
<tr><td>A&B</td></tr>
<tr><td>~(A→~B)
A
~~B
B</td></tr>
</table>

To show that if $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$, then $\mathbf{a}_w(A\&B)=T$:

1. Presume that $\mathbf{a}_w(A\&B)=F$
2. Then $\mathbf{a}_w({\sim}(A\to{\sim}B))=F$
3. So $\mathbf{a}_w(A\to{\sim}B)=T$
4. So $\mathbf{a}_w(A)=F$ or $\mathbf{a}_w({\sim}B)=T$
5. That is, $\mathbf{a}_w(A)=F$ or $\mathbf{a}_w(B)=F$
6. So if $\mathbf{a}_w(A\&B)=F$, then $\mathbf{a}_w(A)=F$ or $\mathbf{a}_w(B)=F$
7. So by contraposition, if neither $\mathbf{a}_w(A)=F$ nor $\mathbf{a}_w(B)=F$, then $\mathbf{a}_w(A\&B)\neq F$.
8. That is, if $\mathbf{a}_w(A)=T$ and $\mathbf{a}_w(B)=T$, then $\mathbf{a}_w(A\&B)=T$

<table>
<tr><td colspan="2">~(A&B)</td></tr>
<tr><td colspan="2">~~(A→~B)
A→~B</td></tr>
<tr><td>~A</td><td>~B</td></tr>
</table>

Exercise 3.2

Show that the following v diagram rules follow from the rules for ~ and →

<table>
<tr><td colspan="2">AvB</td></tr>
<tr><td>A</td><td>B</td></tr>
</table>

<table>
<tr><td>~(AvB)</td></tr>
<tr><td>~A
~B</td></tr>
</table>

For the first:

1. Let $\mathbf{a}_w(AvB)=T$

2. Then $\mathbf{a}_w(\sim A \to B)=T$
3. Then $\mathbf{a}_w(\sim A)=F$ or $\mathbf{a}_w(B)=T$
4. That is, $\mathbf{a}_w(A)=T$ or $\mathbf{a}_w(B)=T$

For the second:

1. Presume $\mathbf{a}_w(AvB)=F$
2. Then $\mathbf{a}_w(\sim A \to B)=F$
3. So $\mathbf{a}_w(\sim A)=T$ and $\mathbf{a}_w(\sim B)=T$
4. That is, $\mathbf{a}_w(A)=F$ and $\mathbf{a}_w(B)=F$

Exercise 3.3

a) for the following diagram rule for (◊F), draw diagrams for **before** and **after** the rule is applied.

(◊F) if $\mathbf{a}_w(\Diamond A)=F$ and $\mathbf{wRv}$, then $\mathbf{a}_v(A)=F$

Before

$\boxed{\sim\Diamond A}_w$

↓

______

$\square_v$

After

$\boxed{\sim\Diamond A}_w$

↓

______

$\boxed{\sim A}_v$ (◊F)

b) show that (◊F) follows from (□), (~), and (Def◊) using diagrams

$\sim\Diamond A^w$

$\sim\sim\square\sim A^w$
$\square\sim A^w$

↓

$\sim A^v$

Exercise 3.4

a) Show that $|=_K A$ iff for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$

1. First, we show that if $|=_K A$, then for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$.
2. Presume $|=_K A$
3. Then the argument /A is K-valid
4. That is, /A has no K-counterexample
5. So ~A is not K-satisfiable
6. That is, for a language containing ~A , there is no model **<W, R, a>** and a world **w** in **W** where $\mathbf{a}_w(\sim A)=T$
7. So for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(\sim A)=F$\
8. That is, for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$
9. Therefore, if $|=_K A$, then for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$.
10. Next, we show that if for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$, then $|=_K A$
11. Presume $|\neq_K A$
12. Then /A is not K-valid
13. Then /A has a counterexample
14. Then /~A is K-satisfiable
15. That is, there is a K-model **<W, R, a>** such that for some **w** in **W**, $\mathbf{a}_w(\sim A)=T$
16. So there is a K-model **<W, R, a>** such that for some **w** in **W**, $\mathbf{a}_w(A)=F$
17. That is, it is not the case that for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$.
18. Thus, if $|\neq_K A$, then it is not the case that for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$.
19. So by contraposition, if for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$, then $|=_K A$
20. Therefore, $|=_K A$ iff for all models **<W, R, a>** and all **w** in **W**, $\mathbf{a}_w(A)=T$

b) Show that $H|=_K A$ iff for all models <**W, R, a**> and all **w** in **W**, if $\mathbf{a}_w(H)=T$, then $\mathbf{a}_w(A)=T$

1. First, we show that if $H|=_K A$, then for all models **<W, R, a>**

and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(A)=T$.

2. Presume $H \models_K A$
3. Then the argument H/A is K-valid
4. That is, H/A has no K-counterexample
5. So the list H, ~A is not K-satisfiable
6. That is, for a language containing ~A , there is no model **<W, R, a>** and a world **w** in **W** such that $\mathbf{a}_{\mathbf{w}}(H)=T$ and $\mathbf{a}_{\mathbf{w}}(\sim A)=T$
7. So for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(\sim A)=F$
8. That is, for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(A)=T$
9. Therefore, if $H \models_K A$, then for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(A)=T$.
10. Next, we show that if for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(A)=T$, then $\models_K A$
11. Presume $H \not\models_K A$
12. Then H/A is not K-valid
13. Then H/A has a counterexample
14. Then the list H, ~A is K-satisfiable
15. That is, there is a K-model **<W, R, a>** such that for some **w** in **W**, $\mathbf{a}_{\mathbf{w}}(H)=T$ and $\mathbf{a}_{\mathbf{w}}(\sim A)=T$
16. So there is a K-model **<W, R, a>** such that for some **w** in **W**, $\mathbf{a}_{\mathbf{w}}(H)=T$ and $\mathbf{a}_{\mathbf{w}}(A)=F$
17. That is, it is not the case that for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$ then $\mathbf{a}_{\mathbf{w}}(A)=T$.
18. Thus, if $\not\models_K A$, then it is not the case that for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$ then $\mathbf{a}_{\mathbf{w}}(A)=T$.
19. So by contraposition, if for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$ then $\mathbf{a}_{\mathbf{w}}(A)=T$, then $\models_K A$
20. Therefore, $H \models_K A$ iff for all models **<W, R, a>** and all **w** in **W**, if $\mathbf{a}_{\mathbf{w}}(H)=T$, then $\mathbf{a}_{\mathbf{w}}(A)=T$

c) Create a diagram that shows that $H \not\models_K C$, that is, that the argument H/C has a K-counterexample

<table>
<tr><td colspan="2">H</td></tr>
<tr><td>C</td><td>~C</td></tr>
</table>