Garson, James (2006). *Modal Logic for Philosophers*. Ch. 2

Exercise 2.1

a) A→◊A

1.	□~A→~A				
2.		A			
3.		~~A			[~~In]
4.		~□~A			[1,3,MT]
5.		◊A			[4,Def◊]
6.	A→◊A				[2-5,CP]

b) □A→◊A

1.	□~A→~A				[M]
2.		□A			
3.		A			[2,M]
4.		~~A			[3,~~In]
5.		~□~A			[1,4,MT]
6.		◊A			[4,Def◊]
7.	□A→◊A				

c) Prove M in K + (A→◊A)

1.	~A → ◊~A				
2.		□A			
3.			◊~A		
4.			~□~~A		[3,Def◊]
5.				□, A	[2,□Out]
6.				~~A	[5,~~In]
7.			□~~A		[6,□In]
8.		~◊~A			[2-7,~In]
9.		~~A			[1,3,MT]
10.		A			[4,DN]
11.	□A→A				[2-10,CP]

Exercise 2.3

a) □□A↔□A in S4

1.		□□A			
2.		□A			[1,M]
3.	□□A→□A				[1-2,CP]
4.		□A			
5.		□□A			[4,(4)]
6.	□→□□A				[4-5,CP]
7.	□□A↔□A				[3, 6,↔In]

b) □□~A/□~~□~A in K

1.	□□~A				
2.		□, □~A			[□Out]
3.		~~□~A			[2,~~In]
4.	□~~□~A				[2-3,□In]

c) ◊◊A↔◊A in S4

1.		◊◊A			
2.		~□~~□~A			[1, Def◊]
3.			~◊A		
4.			~~□~A		[3, Def◊]
5.			□~A		[4, DN]
6.			□□~A		[5, (4)]
7.				□, □~A	[5, □Out]
8.				~~□~A	[7, ~~In]
9.			□~~□~A		[7-8, □In]
10.		◊A			[3-9, ~Out]
11.	◊◊A→◊A				[1-10, CP]
12.		◊A			
13.			~◊◊A		
14.			~~□~◊A		[13, Def◊]
15.			□~◊A		[14, DN]
16.			~◊A		[15, M]
17.		◊◊A			[16, ~Out]
18.	◊A→◊◊A				[12-17, CP]
19.	◊◊A↔◊A				[11, 18, ↔In]

d) □◊◊A↔□◊A in S4

1.		◊◊A			
2.		~□~~□~A			[1, Def◊]
3.			~◊A		
4.			~~□~A		[3, Def◊]
5.			□~A		[4, DN]
6.			□□~A		[5, (4)]
7.				□, □~A	[5, □Out]

8.				~~□~A	[7, ~~In]
9.			□~~□~A		[7-8, □In]
10.		◊A			[3-9, ~Out]
11.	◊◊A→◊A				[1-10, CP]
12.		◊A			
13.			~◊◊A		
14.			~~□~◊A		[13, Def◊]
15.			□~◊A		[14, DN]
16.			~◊A		[15, M]
17.		◊◊A			[16, ~Out]
18.	◊A→◊◊A				[12-17, CP]
19.	□(◊◊A→◊A)				[11, Nec]
20.	□(◊A→◊◊A)				[18, Nec]
21.	□◊◊A→□◊A				[19, Dist]
22.	□◊A→□◊◊A				[20, Dist]
23.	□◊◊A↔□◊A				[21, 22, ↔In]

e) □◊A↔◊A in S5

1.	□◊A→◊A	[M]
2.	◊A→□◊A	[(5)]
3.	□◊A↔◊A	[1, 2, ↔In]

f) (B) in S5

1.		A	
2.		~~A	[1, ~~In]
3.		□~A→~A	[M]
4.		~□~A	[2, 3, MT]
5.		◊A	[4, Def◊]
6.		□◊A	[5, (5)]
7.	A→□◊A		[1-6, CP]

g) □~□~~A / □~□A in K

1.	□~□~~A				
2.		□, ~□~~A			
3.			□A		
4.				□, A	[3, □Out]
5.				~~A	[4, ~~In]
6.			□~~A		[4, 5, □In]
7.		~□A			[3-6, ~In]
8.	□~□A				[7, □In]

h) ◊□A→A in B

1.	~A →□◊~A						[B]
2.		◊□A					[Hyp]
3.		~□~□A					[2, Def◊]
4.			~A				[Hyp]
5.			□◊~A				[1, 3, MP]
6.				□, ◊~A			[5, □Out]
7.				~□~~A			[6, Def◊]
8.					□A		[Hyp]
9.						□, A	[8, □Out]
10.						~~A	[9, ~~In]
11.					□~~A		[10, □In]
12.				~□A			[8-11, ~In]
13.			□~□A				[12, □In]
14.		A					[4-13, ~Out]
15.	◊□A→A						[2-14, CP]

i) ◊□A↔□A in S5

1.	□~□A→~□A					[M]
2.		□A				[Hyp]
3.		~~□A				[2, ~~In]
4.		~□~□A				[1, 3, MT]
5.		◊□A				[4, Def◊]
6.	□A→◊□A					[2-5, CP]
7.	◊~A→□◊~A					[(5)]
8.		◊□A				[Hyp]
9.			~□A			[Hyp]
10.			◊~A			[9, ~□]
11.			□◊~A			[7, 10, MP]
12.			□~□~~A			
13.				□A		
14.					□, A	[13, □Out]
15.					~~A	[14, ~~In]
16.				□~~A		[15, □In]
17.			◊□~~A			[13-16, ◊Out]
18.			~□~□~~A			[17, Def◊]

19.	□A	[9-18, ~Out]
20.	◊□A→□A	[8-19, CP]
21.	◊□A↔□A	[6, 20, ↔In]

Exercise 2.4

a) Prove (4) in S5

1.	□A	[Hyp]
2.	□~□A	[Hyp]
3.	~□A	[2, M]
4.	~□~□A	[2-3, ~In]
5.	◊□A	[4, Def◊]
6.	□◊□A	[5, (5)]
7.	□, ◊□A	[Hyp]
8.	□A	
9.	□, A	[8, □Out]
10.	~~A	[9, ~~In]
11.	□~~A	[10, □In]
12.	◊□~~A	[8-11, ◊Out]
13.	~□~□~~A	[12, Def◊]
14.	~□A	[Hyp]
15.	◊~A	[14, ~□]
16.	□◊~A	[15, (5)]
17.	□~□~~A	[16, Def◊]
18.	□A	[14-17, ~Out]
19.	□□A	[7-18, □In]
20.	□A→□□A	[1-19, CP]

b) Using the previous result, explain why S5 is equivalent to M+(4)+(5), and M+(4)+(B)+(5).

The equivalence holds because (4) and (B) do not add anything to S5 that could not have been proven without their aid: that is, the axioms (4) and (5) are conservative extensions of S5.

c) Prove S5 is equivalent to M+(4)+(B) by proving (5) in M+(4)+(B).

1.	◊A	[Hyp]
2.	□◊◊A	[1, B]
3.	□~□~~□~A	[2, Def◊]
4.	□, ~□~~□~A	[3, □Out]
5.	~◊A	[Hyp]
6.	□~A	[5, ~◊]
7.	□□~A	[6, (4)]
8.	□, □~A	[7, □Out]
9.	~~□~A	[8, ~~In]
10.	□~~□~A	[9, □In]
11.	◊A	[5-10,~Out]
12.	□◊A	[4-11, □In]
13.	◊A→□◊A	[1-12, CP]

Exercise 2.5

a) □□□A↔□A in S4

1.	□□□A	[Hyp]
2.	□□A	[1, M]
3.	□A	[2, M]
4.	□□□A→□A	[1-3, CP]
5.	□A	[Hyp]
6.	□□A	[5, (4)]
7.	□□□A	[6, (4)]
8.	□A→□□□A	[5-7, CP]
9.	□□□A↔□A	[4,8,↔In]

b) ◊◊◊A↔◊A in S4

1.	◊◊◊A	[Hyp]
2.	~◊◊A	[Hyp]
3.	□~◊A	[2,~◊]
4.	□□~◊A	[3,(4)]
5.	□, □~◊A	[4,□Out]
6.	~~□~◊A	[5,~~In]
7.	□~~□~◊A	[6,□In]
8.	□~◊◊A	[7,Def◊]
9.	~□~◊◊A	[1, Def◊]
10.	◊◊A	[2-9,~In]
11.	~◊A	[Hyp]
12.	□~A	[11, ~◊]
13.	□□~A	[12,(4)]
14.	□, □~A`	[13, □Out]
15.	~~□~A	[14, ~~In]

16.			□~~□~A		[15, □In]
17.			□~◊A		[16, Def◊]
18.			~□~◊A		[10, Def◊]
19.		◊A			[11-18,~Out]
20.	◊◊◊A→◊A				[1-19, CP]
21.		◊A			
22.		~□~A			[21, Def◊]
23.		□□~A→□~A			[M]
24.		□□□~A→□□~A			[M]
25.		~□□~A			[22,23,MT]
26.		~□□□~A			[24,25,MT]
27.		◊~□□~A			[26,~□]
28.			□, ~□□~A		
29.			◊~□~A		[28,~□]
30.			◊◊A		[29,Def◊]
31.		◊◊◊A			[28-30,◊Out]
32.	◊A→◊◊◊A				[21-31,CP]
33.	◊◊◊A↔◊A				[20,32,↔In]

c) □□□□A↔□A in S4

1.		□□□□A			[Hyp]
2.		□□□A			[1,M]
3.		□□A			[2,M]
4.		□A			[3,M]
5.	□□□□A→□A				[1-4,CP]
6.		□A			[Hyp]
7.		□□A			[6,(4)]
8.		□□□A			[7,(4)]
9.		□□□□A			[8,(4)]
10.	□A→□□□□A				[6-9,CP]
11.	□□□□A↔□A				[5,10,↔In]

Exercise 2.6

a) ◊□◊A↔◊A in S5

1.		◊□◊A			
2.			□,□◊A		
3.			◊A		[2, M]
4.		◊◊A			[2-3, ◊Out]
5.			~◊A		[Hyp]
6.			□~A		[5, ~◊]
7.			□□~A		[6,(4)]
8.				□, □~A	[7,□Out]
9.				~~□A	[8,~~In]
10.			□~~□~A		[9,□In]
11.			~□~~□~A		[4,Def◊]
12.		◊A			[5-11,~Out]
13.	◊□◊A→◊A				[1-12,CP]
14.		◊A			[Hyp]
15.			□, A		
16.			□◊A		[15,(B)]
17.		◊□◊A			[16,◊Out]
18.	◊□◊A→◊A				[14-17,CP]
19.	◊□◊A↔◊A				[13,18,↔In]

b) □◊□A↔□A in S5

1.		□◊□A			
2.			□, ◊□A		[1,□Out]
3.				□A	
4.				□, A	[3,□Out]
5.				~~A	[4,~~In]
6.				□~~A	[5,□In]
7.			◊□~~A		[3-6,◊Out]
8.				~A	[Hyp]
9.				□◊~A	[3,(B)]
10.				□~□~~A	[5,Def◊]
11.				~□~□~~A	[7,Def◊]
12.			A		[8-11,~Out]
13.		□A			[12,□In]
14.	□◊□A→□A				[1-13,CP]
15.		□A			[Hyp]
16.		□◊□A			[15,(B)]
17.	□A→□◊□A				[15-16,CP]
18.	□◊□A↔□A				[14,17,↔In]

Exercise 2.7

a) Prove □(A→◊A) in M

1.	A	[Hyp]
2.	~◊A	[Hyp]
3.	□~A	[2,~◊]
4.	~A	[3,M]
5.	◊A	[2-4,~Out]
6.	A→◊A	[1-5,CP]
7.	□(A→◊A)	[6, Nec]

Exercise 2.8

Find the duals of the following sentences:

a) □A→□□A = ◊◊A→◊A

b) (□A&□B)↔□(A&B) = (◊Av◊B)↔◊(AvB)

c) ◊A→□◊A = ◊□A→□A

d) □(AvB)→(□Av□B) = (◊A&◊B)→◊(A&B)

e) Vx□Ax↔□VxAx = Ǝx◊Ax↔◊ƎxAx

f) □(□A→A) has no dual.

g) □A→◊A = □A→◊A

h) A→□◊A = ◊□A→A

Exercise 2.9

a) □A→A = A→◊A

b) □A→□□A = ◊◊A→◊A

c) ◊A→□◊A = ◊□A→□A

Exercise 2.10

a) |-$_{M}$ A→◊A

1.	A	[Hyp]
2.	□~A	[Hyp]
3.	~A	[2,M]
4.	~□~A	[2-3,~In]
5.	◊A	

b) |-$_{4}$ ◊◊A→◊A

1.	◊◊A	[Hyp]
2.	□~A	[Hyp]
3.	□□~A	[2,(4)]
4.	□, □~A	[3,□Out]
5.	~~□~A	[4,~~In]
6.	□~~□~A	[5,□In]
7.	~□~~□~A	[1,Def◊]
8.	~□~A	[2-7,~In]
9.	◊A	[8,Def◊]
10.	◊◊A→◊A	[1-9,CP]

c) |-$_{B}$ ◊□A→A

1.	◊□A	[Hyp]
2.	□, □A	
3.	□, A	[2,□Out]
4.	~~A	[3,~~In]
5.	□~~A	[4,□In]
6.	◊□~~A	[2-5,◊Out]
7.	~A	[Hyp]
8.	□◊~A	[7,(B)]
9.	□~□~~A	[8,Def◊]
10.	~□~□~~A	[6,Def◊]
11.	A	[7-10,~Out]
12.	◊□A→A	[1-11,CP]

d) |-$_{5}$ ◊□A→□A

1.	◊□A	[Hyp]
2.	□, □A	
3.	□, A	[2,□Out]
4.	~~A	[3,~~In]
5.	□~~A	[4,□In]
6.	◊□~~A	[2-5,◊Out]
7.	~□A	[Hyp]
8.	◊~A	[7,~□]
9.	□◊~A	[8,(5)]
10.	□~□~~A	[9,Def◊]
11.	~□~□~~A	[6,Def◊]
12.	□A	[7-11,~Out]
13.	◊□A→□A	[1-12,CP]

Exercise 2.11

|-$_{D+OO}$OA→OPA

1.	**O**A	
2.	**OO**A	[1, **OO**]
3.	**O, O**A	[2,**O**Out]
4.	**P**A	[3,**D**]

5.		**OP**A			[4,**O**In]
6.	**O**A→**OP**A				[1-5,CP]

Exercise 2.12

a) **PG**A→A

1.		**PG**A			[Hyp]
2.			**H, G**A		
3.				**G**, A	[2,**G**Out]
4.				~~A	[3,~~In]
5.			**G**~~A		[4,**G**In]
6.		**PG**~~A			[2-5,**P**In]
7.			~A		[Hyp]
8.			**HF**~A		[7, **HF**]
9.			**H**~**G**~~A		[8, Def**F**]
10.			~**H**~**G**~~A		[6, Def**P**]
11.		A			[7-10, ~Out]
12.	**PG**A→A				[1-11, CP]

b) **FH**A→A

1.		**FH**A			[Hyp]
2.			**G, H**A		
3.				**H,** A	[2, **H**Out]
4.				~~A	[3, ~~In]
5.			**H**~~A		[4, **H**In]
6.		**FH**~~A			[2-5, **F**In]
7.			~A		[Hyp]
8.			**GP**~A		[7, **GP**]
9.			**G~H**~~A		[8, Def**P**]
10.			~**G~H~~**A		[6, Def**F**]
11.		A			[7-10, ~Out]
12.	**FH**A→A				[1-11,CP]

Exercise 2.13

The *dual* of a sentence of the form A→B is B*→A*, where A* is the result of replacing each occurrence in A of &, v, V, E, **F**, **G, H**, and **P** respectively by v, &, E, V, **G**, **F**, **P**, and **H**.

Exercise 2.14

a) Tn(AvB)↔(TnAvTnB)

1.		Tn(AvB)					[Hyp]
2.			~(TnAvTnB)				[Hyp]
3.			~TnA&~TnB				[2, DM]
4.			~TnA				[3, &Out]
5.			Tn~A				[4, T~]
6.				Tn, AvB			[1, TnOut]
7.				~A→B			[6, Defv]
8.				~A			[5, TnOut]
9.				B			[8, 9, MP]
10.			TnB				[9, TnIn]
11.			~TnB				[3, &Out]
12.		TnAvTnB					[2-11, ~Out]
13.	Tn(AvB)→(TnAvTnB)						[1-12, CP]
14.		TnAvTnB					
15.			TnA		TnB		[Hyp]
16.				Tn, A		Tn, B	[TnOut]
17.				AvB		AvB	[vIn]
18.			Tn(AvB)			Tn(AvB)	[TnIn]
19.		Tn(AvB)					[14, 15-18, vOut]
20.	(TnAvTnB)→Tn(AvB)						[14-19, CP]
21.	Tn(AvB)↔(TnAvTnB)						[13, 20, ↔In]

b) Tn(A&B)↔(TnA&TnB)

1.		Tn(A&B)		[Hyp]
2.			Tn, A&B	[1, TnOut]
3.			A	[2, &Out]
4.		TnA		[3, TnIn]
5.			Tn, A&B	[1, TnOut]
6.			B	[5, &Out]
7.		TnB		[6, TnIn]
8.		TnA&TnB		[4, 7, &In]
9.	Tn(A&B)→(TnA&TnB)			[1-8, CP]
10.		TnA&TnB		[Hyp]
11.		TnA		[10, &Out]
12.		TnB		[10, &Out]
13.			Tn, A	[11, TnOut]

14. B [12, TnOut]
15. A&B [13, 14, &In]
16. Tn(A&B) [15, TnIn]
17. (TnA&TnB)→Tn(A&B) [10-16, CP]
18. Tn(A&B)↔(TnA&TnB) [9, 17, ↔In]

c) Tn(A→B)↔(TnA→TnB)

1. Tn(A→B) [Hyp]
2. TnA [Hyp]
3. Tn, A→B [1, TnOut]
4. A [2, TnOut]
5. B [3, 4, MP]
6. TnB [5, TnIn]
7. TnA→TnB [2-6, CP]
8. Tn(A→B)→(TnA→TnB) [1-7, CP]
9. TnA→TnB [Hyp]
10. ~Tn(A→B) [Hyp]
11. Tn~(A→B) [10, T~]
12. Tn, ~(A→B) [11, TnOut]
13. A [12, →F]
14. TnA [13, TnIn]
15. TnB [9, 14, MP]
16. Tn, B
17. A [Hyp]
18. B [16, Reit]
19. A→B [17-18, CP]
20. Tn(A→B) [19, TnIn]
21. Tn(A→B) [10-20, ~Out]
22. (TnA→TnB)→Tn(A→B) [9-21, CP]
23. Tn(A→B)↔(TnA→TnB) [8, 22, ↔In]

Exercise 2.15

a) Show that TmTnA↔TnA is provable in T plus (TT)

1. TmTnA [Hyp]
2. ~TnA [Hyp]
3. Tn~A [2, T~]
4. TmTn~A [3, (TT)]
5. Tm, Tn~A [4, TmOut]
6. ~TnA [5, T~]
7. Tm~TnA [6, TmIn]
8. ~TmTnA [7, T~]
9. TnA [2-8, ~Out]
10. TmTnA→TnA [1-9, CP]
11. TnA [Hyp]
12. TmTnA [11, (TT)]
13. TnA→TmTnA [11-12, CP]
14. TmTnA↔TnA [10, 13, ↔In]

Exercise 2.17

a) Prove ~□⊥→~□~□⊥ in GL

1. □(□⊥→⊥)→□⊥ [(GL)]
2. □~□⊥→□⊥ [1, Def~]
3. ~□⊥→~□~□⊥ [2, CN]