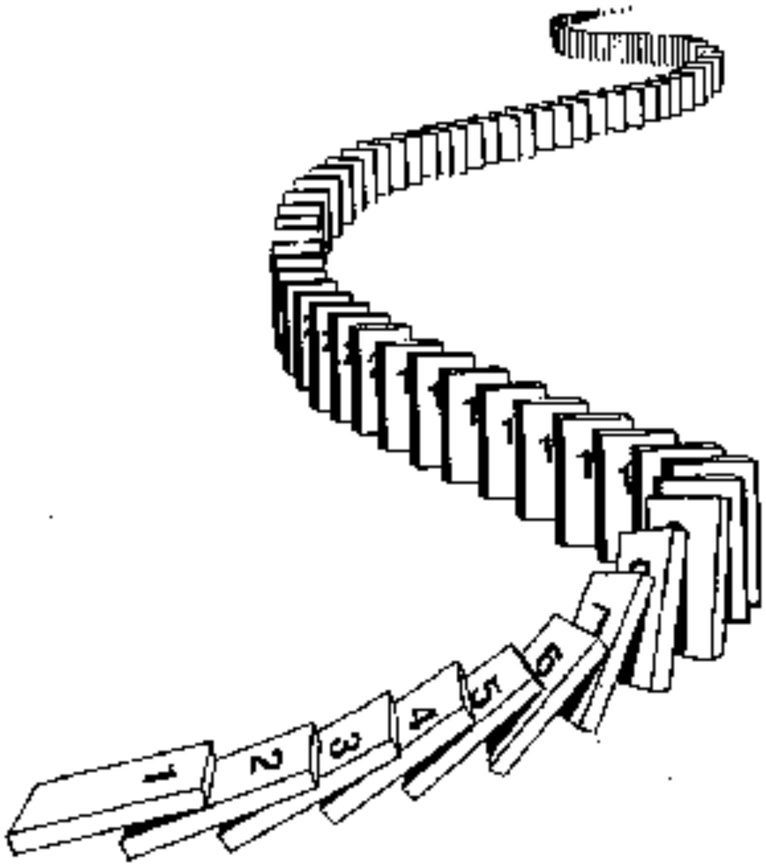
CSE 311: Foundations of Computing

Lecture 15: Induction



More Logic

Induction

Mathematical Induction

Method for proving statements about all natural numbers

- **A new logical inference rule!**
  - It only applies over the natural numbers
  - The idea is to **use** the special structure of the naturals to prove things more easily

- **Particularly useful for reasoning about programs!**

```
for (int i=0; i < n; n++) { … }
```

  - Show P(i) holds after i times through the loop

Prove $\forall a, b, m > 0\ \forall k \in \mathbb{N}\ ((a \equiv_m b) \rightarrow (a^k \equiv_m b^k))$

Let $a, b, m > 0$ be arbitrary. Let $k \in \mathbb{N}$ be arbitrary.

Suppose that $a \equiv_m b$.

We know $((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$ by multiplying congruences. So, applying this repeatedly, we have:

$$((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$$
$$\left((a^2 \equiv_m b^2) \wedge (a \equiv_m b)\right) \rightarrow (a^3 \equiv_m b^3)$$

…

$$\left((a^{k-1} \equiv_m b^{k-1}) \wedge (a \equiv_m b)\right) \rightarrow (a^k \equiv_m b^k)$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there such a property of the natural numbers!

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(3)?

Induction Is A Rule of Inference

Domain: Natural Numbers

$$\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \\ \hline \therefore \forall n\ \ P(n) \end{array}$$

How do the givens prove P(5)?

$$P(0) \xrightarrow{P(0)\to P(1)} P(1) \xrightarrow{P(1)\to P(2)} P(2) \xrightarrow{P(2)\to P(3)} P(3) \xrightarrow{P(3)\to P(4)} P(4) \xrightarrow{P(4)\to P(5)} P(5)$$

First, we have P(0).

Since P(n) → P(n+1) for all n, we have P(0) → P(1).

Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.

Since P(n) → P(n+1) for all n, we have P(1) → P(2).

Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)

4. $\forall$k (P(k) $\rightarrow$ P(k+1))
5. $\forall$n P(n) Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)
2. Let k be an arbitrary integer ≥ 0

3. $P(k) \rightarrow P(k+1)$
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Using The Induction Rule In A Formal Proof

$$\frac{\begin{array}{c} P(0) \\ \forall k\ (P(k) \longrightarrow P(k+1)) \end{array}}{\therefore \forall n\ \ P(n)}$$

1. P(0)	
2. Let k be an arbitrary integer ≥ 0	
3.1. P(k)	Assumption
3.2. ...	
3.3. P(k+1)	
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3
5. $\forall n\ P(n)$	Induction: 1, 4

Translating to an English Proof

$$P(0)$$
$$\forall k\ (P(k) \longrightarrow P(k+1))$$
$$\therefore \forall n\ \ P(n)$$

1. Prove P(0) **Base Case**

2. Let k be an arbitrary integer ≥ 0 **Inductive Hypothesis**
    3.1. Suppose that P(k) is true
    3.2. ...
    3.3. Prove P(k+1) is true **Inductive Step**

3. $P(k) \rightarrow P(k+1)$ Direct Proof Rule
4. $\forall k\ (P(k) \rightarrow P(k+1))$ Intro $\forall$: 2, 3
5. $\forall n\ P(n)$ Induction: 1, 4

Conclusion

Translating to an English Proof

1. Prove P(0)		**Base Case**
2. Let k be an arbitrary integer ≥ 0		**Inductive Hypothesis**
3.1. Assume that P(k) is true		
3.2. ...		**Inductive Step**
3.3. Prove P(k+1) is true		
3. $P(k) \rightarrow P(k+1)$	Direct Proof Rule	
4. $\forall k\, (P(k) \rightarrow P(k+1))$	Intro $\forall$: 2, 3	
5. $\forall n\, P(n)$	Induction: 1, 4	
		Conclusion

Induction English Proof Template

[...Define P(n)...]

We will show that $P(n)$ is true for every $n \in \mathbb{N}$ by Induction.

Base Case: *[...proof of $P(0)$ here...]*

Induction Hypothesis:

Suppose that $P(k)$ is true for an arbitrary $k \in \mathbb{N}$.

Induction Step:

[...proof of $P(k+1)$ here...]

*The proof of $P(k+1)$ **must** invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Proof:

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 \qquad = \quad 1$
- $1 + 2 \qquad = \quad 3$
- $1 + 2 + 4 \qquad = \quad 7$
- $1 + 2 + 4 + 8 \qquad = \quad 15$
- $1 + 2 + 4 + 8 + 16 \qquad = \quad 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ **but that would literally take forever.**

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let $P(n)$ be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show $P(n)$ is true for all natural numbers by induction.
2. Base Case ($n=0$): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so $P(0)$ is true.
3. Induction Hypothesis: Suppose that $P(k)$ is true for some arbitrary integer $k \geq 0$, i.e., that $\boxed{2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1.}$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   Goal: Show P(k+1)**, i.e. show** $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

$$2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1 \quad \textbf{by IH}$$

Adding 2^{k+1} **to both sides, we get:**

$$2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$$

Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$**.**

So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$**, which is exactly** P(k+1)**.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** $P(n)$ **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** $P(n)$ **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** $P(0)$ **is true.**
3. **Induction Hypothesis: Suppose that** $P(k)$ **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$**.**
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly $P(k+1)$**.**

Alternative way of writing the inductive step

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   We can calculate

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0+2^1+ \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} \qquad \textbf{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$, **by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** “0 + 1 + 2 + … + n = n(n+1)/2”. **We will show** P(n) **is true for all natural numbers by induction.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0. **I.e., suppose** 1 + 2 + …+ k = k(k+1)/2

Prove $1 + 2 + 3 + \dots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** 1 + 2 + …+ k+ (k+1) = (k+1)(k+2)/2

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

1. **Let** P(n) **be** "0 + 1 + 2 + … + n = n(n+1)/2". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): 0 = 0(0+1)/2. **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** 1 + 2 + …+ k = k(k+1)/2
4. **Induction Step:**

$$\begin{aligned}1 + 2 + \ldots + k + (k+1) &= (1 + 2 + \ldots + k) + (k+1)\\ &= k(k+1)/2 + (k+1) \quad \textbf{by IH}\\ &= (k+1)(k/2 + 1)\\ &= (k+1)(k+2)/2\end{aligned}$$

   So, we have shown 1 + 2 + … + k + (k+1) = (k+1)(k+2)/2**,**

   which is exactly P(k+1)**.**

5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**

Another example of a pattern

- $2^0 - 1 = 1 - 1 = 0 = 3 \cdot 0$
- $2^2 - 1 = 4 - 1 = 3 = 3 \cdot 1$
- $2^4 - 1 = 16 - 1 = 15 = 3 \cdot 5$
- $2^6 - 1 = 64 - 1 = 63 = 3 \cdot 21$
- $2^8 - 1 = 256 - 1 = 255 = 3 \cdot 85$
- $\ldots$

Prove: $3 \mid (2^{2n}-1)$ **for all** $n \geq 0$

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** “3 | (2^{2n} – 1)”. **We will show** P(n) i**s true for all natural numbers by induction.**

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) i**s true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true**

Prove: $3 \mid (2^{2n}-1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$. **I.e., suppose that** $3 \mid (2^{2k} - 1)$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0} - 1 = 1 - 1 = 0 = 3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**. I.e., suppose that** $3 \mid (2^{2k} - 1)$
4. **Induction Step:**

   Goal: Show P(k+1)**, i.e. show** $3 \mid (2^{2(k+1)} - 1)$

Prove: $3 \mid (2^{2n} - 1)$ for all $n \geq 0$

1. **Let** P(n) **be** "$3 \mid (2^{2n} - 1)$". **We will show** P(n) **is true for all natural numbers by induction.**
2. **Base Case** (n=0): $2^{2\cdot 0}-1=1-1=0=3\cdot 0$ **Therefore** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**. I.e., suppose that** $3 \mid (2^{2k} - 1)$
4. **Induction Step:**

   By IH, $3 \mid (2^{2k} - 1)$ **so** $2^{2k} - 1 = 3j$ **for some integer** j

   So $2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4(2^{2k}) - 1 = 4(3j+1) - 1$

   $= 12j+3 = 3(4j+1)$

   Therefore $3 \mid (2^{2(k+1)} - 1)$ **which is exactly** P(k+1)**.**
5. **Thus** P(n) **is true for all** $n \in \mathbb{N}$**, by induction.**