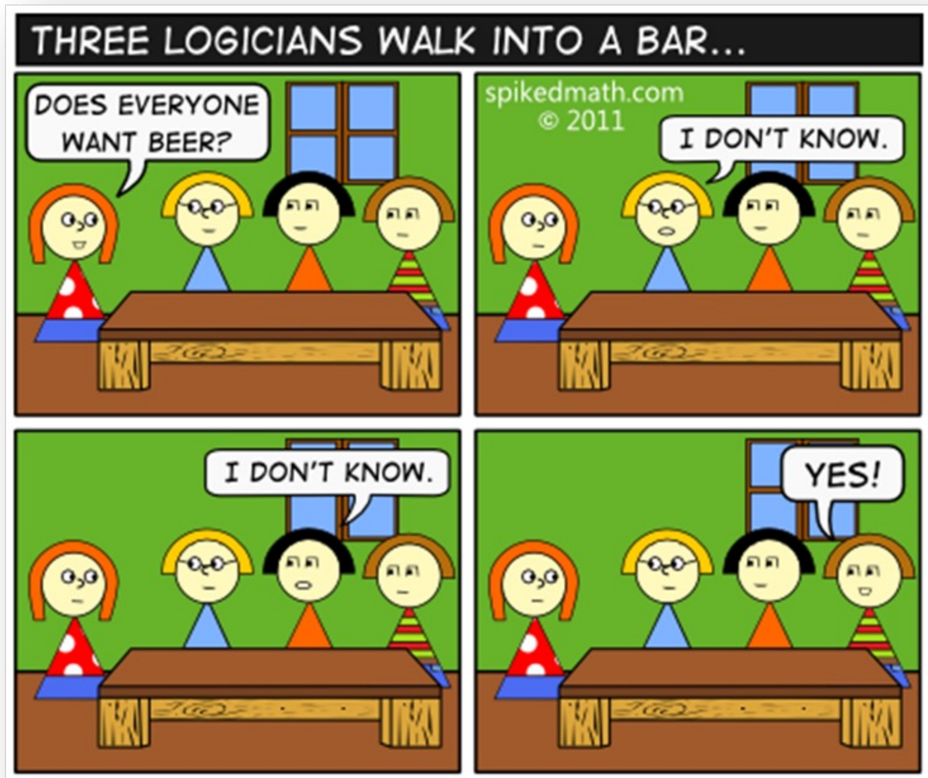
CSE 311: Foundations of Computing

Lecture 6: Predicate Logic



Last class: Canonical Forms

- Truth table is the **unique signature** of a 0/1 function

- The same truth table can have many gate realizations
  - We've seen this already
  - Depends on how good we are at Boolean simplification

- Canonical forms
  - Standard forms for a Boolean expression
  - We all produce the same expression

Sum-of-Products Canonical Form

- AKA **Disjunctive Normal Form (DNF)**
- AKA **Minterm Expansion**

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1. Read T rows off truth table
2. Convert to Boolean Algebra

001 → A'B'C
011 → A'BC
101 → AB'C
110 → ABC'
111 → ABC

3. Add the minterms together

F= A'B'C + A'BC + AB'C + ABC' + ABC

Sum-of-Products Canonical Form

Product term (or minterm)

- **ANDed product of literals – input combination for which output is true**
- **each variable appears exactly once, true or inverted (but not both)**

A	B	C	minterms
0	0	0	$A'B'C'$
0	0	1	$A'B'C$
0	1	0	$A'BC'$
0	1	1	$A'BC$
1	0	0	$AB'C'$
1	0	1	$AB'C$
1	1	0	ABC'
1	1	1	ABC

F in canonical form:

$$F(A, B, C) = A'B'C + A'BC + AB'C + ABC' + ABC$$

canonical form $\neq$ minimal form

$$\begin{aligned} F(A, B, C) &= A'B'C + A'BC + AB'C + ABC + ABC' \\ &= (A'B' + A'B + AB' + AB)C + ABC' \\ &= ((A' + A)(B' + B))C + ABC' \\ &= C + ABC' \\ &= ABC' + C \\ &= AB + C \end{aligned}$$

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

4
Multiply the maxterms together

F =

1
Read F rows off truth table

2
Negate all bits

3
Convert to Boolean Algebra

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

F

Product-of-Sums Canonical Form

- AKA **Conjunctive Normal Form (CNF)**
- AKA **Maxterm Expansion**

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

1 Read F rows off truth table	2 Negate all bits	3 Convert to Boolean Algebra
000	111	A + B + C
010	101	A + B′ + C
100	011	A′ + B + C

$\rightarrow$ F

4 Multiply the maxterms together

F = (A + B + C)(A + B′ + C)(A′ + B + C)

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a **minterm** expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Product-of-Sums: Why does this procedure work?

Useful Facts:

- We know (F')' = F
- We know how to get a **minterm** expansion for F'

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$F' = A'B'C' + A'BC' + AB'C'$$

Taking the complement of both sides...

$$(F')' = (A'B'C' + A'BC' + AB'C')'$$

And using DeMorgan/Comp....

$$F = (A'B'C')' \ (A'BC')' \ (AB'C')'$$

$$F = (A + B + C)(A + B' + C)(A' + B + C)$$

Product-of-Sums Canonical Form

Sum term (or maxterm)

- ORed sum of literals – input combination for which output is false
- each variable appears exactly once, true or inverted (but not both)

A	B	C	maxterms
0	0	0	$A+B+C$
0	0	1	$A+B+C'$
0	1	0	$A+B'+C$
0	1	1	$A+B'+C'$
1	0	0	$A'+B+C$
1	0	1	$A'+B+C'$
1	1	0	$A'+B'+C$
1	1	1	$A'+B'+C'$

F in canonical form:

$$F(A, B, C) = (A + B + C)\,(A + B' + C)\,(A' + B + C)$$

canonical form ≠ minimal form

$$\begin{aligned} F(A, B, C) &= (A + B + C)\,(A + B' + C)\,(A' + B + C) \\ &= (A + B + C)\,(A + B' + C) \\ &\quad\ (A + B + C)\,(A' + B + C) \\ &= (A + C)\,(B + C) \end{aligned}$$

Predicate Logic

Predicate Logic

- Propositional Logic
  - Allows us to analyze complex propositions in terms of their simpler constituent parts (a.k.a. atomic propositions) joined by connectives

- Predicate Logic
  - Lets us analyze them at a deeper level by expressing how those propositions depend on the objects they are talking about

"All positive integers x, y, and z satisfy $x^3 + y^3 \neq z^3$."

Predicate Logic

Adds two key notions to propositional logic

- Predicates

- Quantifiers

Predicates

Predicate

– **A function that returns a truth value, e.g.,**

Cat(x) ::= “x is a cat”

Prime(x) ::= “x is prime”

HasTaken(x, y) ::= “student x has taken course y”

LessThan(x, y) ::= “$x < y$”

Sum(x, y, z) ::= “$x + y = z$”

GreaterThan5(x) ::= “$x > 5$”

HasNChars(s, n) ::= “string s has length n”

Predicates can have varying numbers of arguments and input types.

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This non-empty set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or …

(2) "x is prime", "$x = 0$", "$x < 0$", "x is a power of two"

"numbers" or "integers" or "integers greater than 5" or …

(3) "student x has taken course y" "x is a pre-req for z"

"students and courses" or "university entities" or …

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x **in the domain for which** $P(x)$ **is true**

read as "there exists x, P of x"

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)	**T**	**e.g. 2, 4, 6, ...**
$\forall x$ Odd(x)	**F**	**e.g. 2, 4, 6, ...**
$\forall x$ (Even(x) $\vee$ Odd(x))	**T**	**every integer is either even or odd**
$\exists x$ (Even(x) $\wedge$ Odd(x))	**F**	**no integer is both even and odd**
$\forall x$ Greater(x+1, x)	**T**	**adding 1 makes a bigger number**
$\exists x$ (Even(x) $\wedge$ Prime(x))	**T**	**Even(2) is true and Prime(2) is true**

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y$ Greater(y, x)

For every positive integer x, there is a positive integer y, such that y > x.

$\exists y\ \forall x$ Greater(y, x)

There is a positive integer y such that, for every pos. int. x, we have y > x.

$\forall x\ \exists y$ (Greater(y, x) $\wedge$ Prime(y))

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x$ (Prime(x) $\rightarrow$ (Equal(x, 2) $\vee$ Odd(x)))

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\ \exists y$ (Sum(x, 2, y) $\wedge$ Prime(x) $\wedge$ Prime(y))

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x \, \exists y \, \text{Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that y > x.

$\exists y \, \forall x \, \text{Greater}(y, x)$

There is a positive integer y such that, for every pos. int. x, we have y > x.

$\forall x \, \exists y \, (\text{Greater}(y, x) \land \text{Prime}(y))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer, there is some larger positive integer.

$\exists y\ \forall x\ \text{Greater}(y, x)$

There is a positive integer that is larger than every other positive integer.

$\forall x\ \exists y\ (\text{Greater}(y, x) \wedge \text{Prime}(y))$

For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

"All red cats like tofu"

$$\forall x\ ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

"Some red cats don't like tofu"

$$\exists y\ ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg\text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= "x is a cat"
Red(x) ::= "x is red"
LikesTofu(x) ::= "x likes tofu"

When putting two predicates together like this, we use an "and".

"All Red cats like tofu"

When restricting to a smaller domain in a "for all" we use **implication.**

"Some red cats don't like tofu"

When restricting to a smaller domain in an "exists" we use **and.**

"Some" means "there exists".

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "$x > y$"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "$x = y$"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "$x + y = z$"

Translate the following statements to English

$\forall x\,(\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\,\exists y\,(\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist positive integers x and y such that x + 2 = y and x and y are prime.

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) ::= "x is even"	Greater(x, y) ::= "x > y"
Odd(x) ::= "x is odd"	Equal(x, y) ::= "x = y"
Prime(x) ::= "x is prime"	Sum(x, y, z) ::= "x + y = z"

Translate the following statements to English

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

$\exists x\, \exists y\, (\text{Sum}(x, 2, y) \wedge \text{Prime}(x) \wedge \text{Prime}(y))$

There exist prime numbers that differ by two.

Spot the domain restriction patterns

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) ::= “x is a cat”
Red(x) ::= “x is red”
LikesTofu(x) ::= “x likes tofu”

“All Red cats like tofu”

“Red cats like tofu”

When there’s no leading quantification, it means “for all”.

“Some red cats don’t like tofu”

“A red cat doesn’t like tofu”

“A” means “there exists”.

Statements with Quantifiers (Natural Translations)

Translations often (not always) sound more <u>natural</u> if we

1. Notice "domain restriction" patterns

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

2. Avoid introducing *unnecessary* variable names

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer, there is some larger positive integer.

3. Can sometimes drop "all" or "there is"

$\neg\ \exists x\ (\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$

No even prime is greater than 2.

More English Ambiguity

Implicit quantifiers in English are often **confusing**

<u>Three people</u> that are all friends can form a raiding party $\forall$

<u>Three people</u> I know are all friends with Mark Zuckerberg $\exists$

Formal logic removes this ambiguity

- quantifiers can always be specified
- unquantified variables that are not known constants (e.g, π) are **implicitly** $\forall$–quantified

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= “x is a purple fruit”

(*) $\forall$x PurpleFruit(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”

(b) “there exists a non-purple fruit”

(c) “all fruits are not purple”

Try your intuition! Which one seems right?

Negations of Quantifiers

Predicate Definitions
PurpleFruit(x) ::= "x is a purple fruit"

(*) $\forall$x PurpleFruit(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"

(b) "there exists a non-purple fruit"

(c) "all fruits are not purple"

Domain of Discourse
{plum, apple}

(*) PurpleFruit(plum) $\wedge$ PurpleFruit(apple)

(a) PurpleFruit(plum) $\vee$ PurpleFruit(apple)

(b) $\neg$ PurpleFruit(plum) $\vee$ $\neg$ PurpleFruit(apple)

(c) $\neg$ PurpleFruit(plum) $\wedge$ $\neg$ PurpleFruit(apple)

De Morgan's Laws for Quantifiers

$$\boxed{\begin{array}{c} \neg\forall x\ P(x) \equiv \exists x\ \neg\ P(x) \\ \neg\ \exists x\ P(x) \equiv \forall x\ \neg\ P(x) \end{array}}$$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

"There is no integer larger than every other integer"

$$\neg \exists x\, \forall y\, (x \geq y)$$
$$\equiv \forall x\, \neg \forall y\, (x \geq y)$$
$$\equiv \forall x\, \exists y\, \neg (x \geq y)$$
$$\equiv \forall x\, \exists y\, (y > x)$$

"For every integer, there is a larger integer"

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

These are **equivalent** but not **equal**

They have different English translations, e.g.:

There is no unicorn $\neg \exists x\, \text{Unicorn}(x)$

Every animal is not a unicorn $\forall x\, \neg \text{Unicorn}(x)$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"No even prime is greater than 2"

$$\neg \exists x\, (\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x \neg(\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x\, (\neg(\text{Even}(x) \wedge \text{Prime}(x)) \vee \neg\text{Greater}(x, 2))$$
$$\equiv \forall x\, ((\text{Even}(x) \wedge \text{Prime}(x)) \rightarrow \neg\text{Greater}(x, 2))$$
$$\equiv \forall x\, ((\text{Even}(x) \wedge \text{Prime}(x)) \rightarrow \text{LessEq}(x, 2))$$

"Every even prime is less than or equal to 2."

De Morgan's Laws for Quantifiers

We just saw that

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$

Can similarly show that

$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)

Scope of Quantifiers

$\exists x\ (P(x) \land Q(x))$ **vs.** $\exists x\ P(x) \land \exists x\ Q(x)$

Scope of Quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $\exists x\, P(x) \wedge \exists x\, Q(x)$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Variables with the same name do not necessarily refer to the same object.

Scope of Quantifiers

Domain of Discourse
{1, 2, 3, 4}

Example: NotLargest(x) ::= $\exists$ y Greater (y, x)

$\equiv$ $\exists$ z Greater (z, x)

truth value:

doesn't depend on y **or** z **"bound variables"**

does depend on x **"free variable"**

quantifiers only act on free variables of the formula they quantify

$$\exists\, y\ (P(x,y) \rightarrow \forall\, x\ Q(y, x))$$

Scope of Quantifiers

Domain of Discourse
{1, 2, 3, 4}

Example: NotLargest(x) ::= $\exists$ y Greater (y, x)

$\equiv$ $\exists$ z Greater (z, x)

truth value:

doesn't depend on y or z "bound variables"

does depend on x "free variable"

quantifiers only act on free variables of the formula they quantify

$$\forall x\, (\exists y\, (P(x,y) \rightarrow \forall x\, Q(y, x)))$$

Quantifier "Style"

$$\forall x\,(\exists y\,(P(x,y) \rightarrow \forall x\, Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

$$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \wedge P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

“Every number has a number greater than or equal to it.”

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)))$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) ::= “x ≥ y”

“There is a number greater than or equal to all numbers.”

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)))$$

“Every number has a number greater than or equal to it.”

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)))$$

y

x	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

Quantification with Two Variables

expression	when true	when false
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.