

# CSE 311: Foundations of Computing

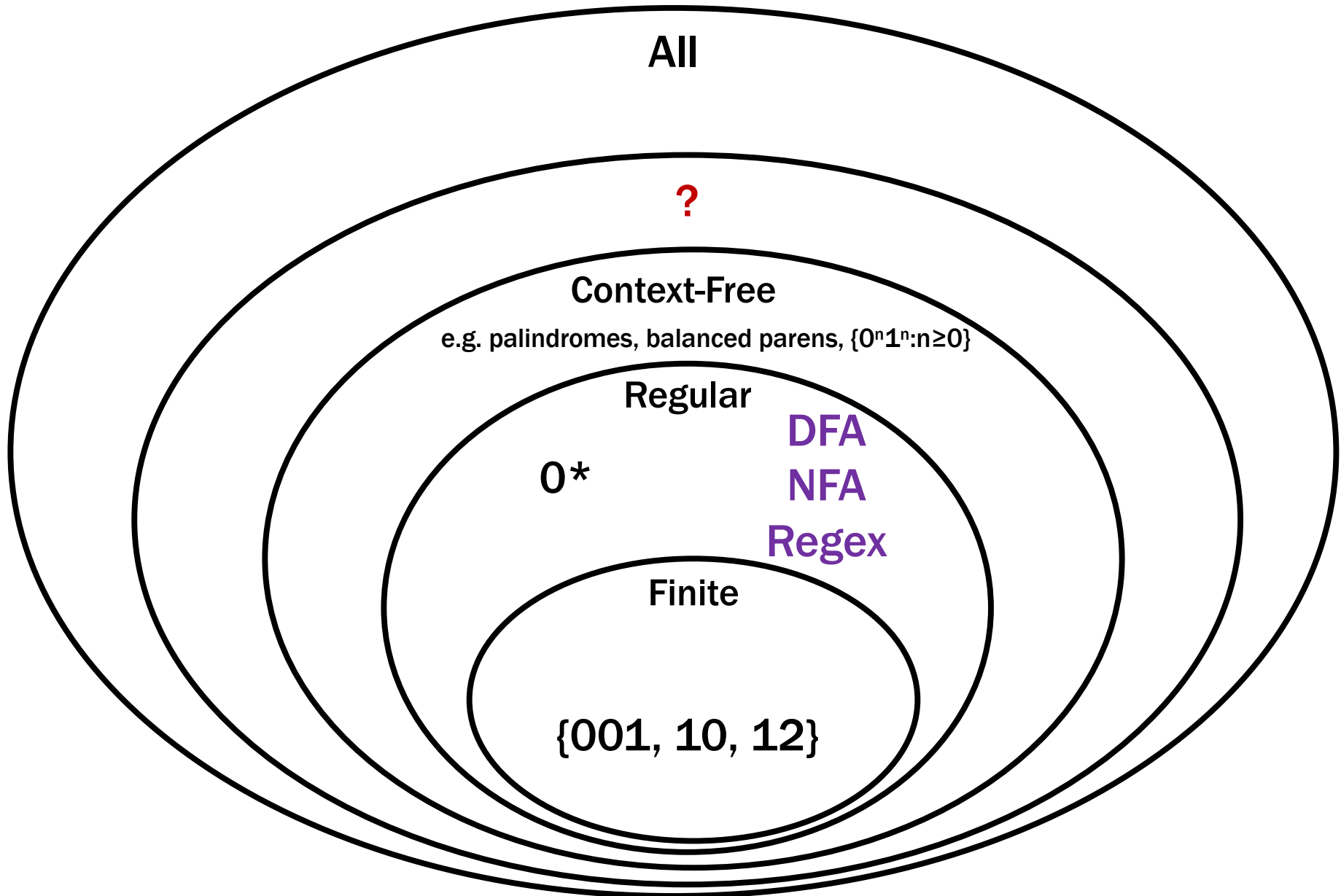
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## Lecture 26: Cardinality, Uncomputability



# Last time: Languages and Representations

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# General Computation

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# Computers from Thought

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Computers as we know them grew out of a desire to avoid bugs in mathematical reasoning.

Hilbert in a famous speech at the International Congress of Mathematicians in 1900 set out the goal to **mechanize all of mathematics**.

In the 1930s, work of Gödel and Turing showed that Hilbert's program is **impossible**.

Gödel's Incompleteness Theorem

Undecidability of the Halting Problem

Both of these employ an idea we will see called **diagonalization**.

The ideas are simple but so revolutionary that their inventor Georg Cantor was initially shunned by the mathematical leaders of the time:

Poincaré referred to them as a “**grave disease infecting mathematics**.”

Kronecker fought to keep Cantor's papers out of his journals.



Full employment for mathematicians  
and computer scientists!!

# Cardinality

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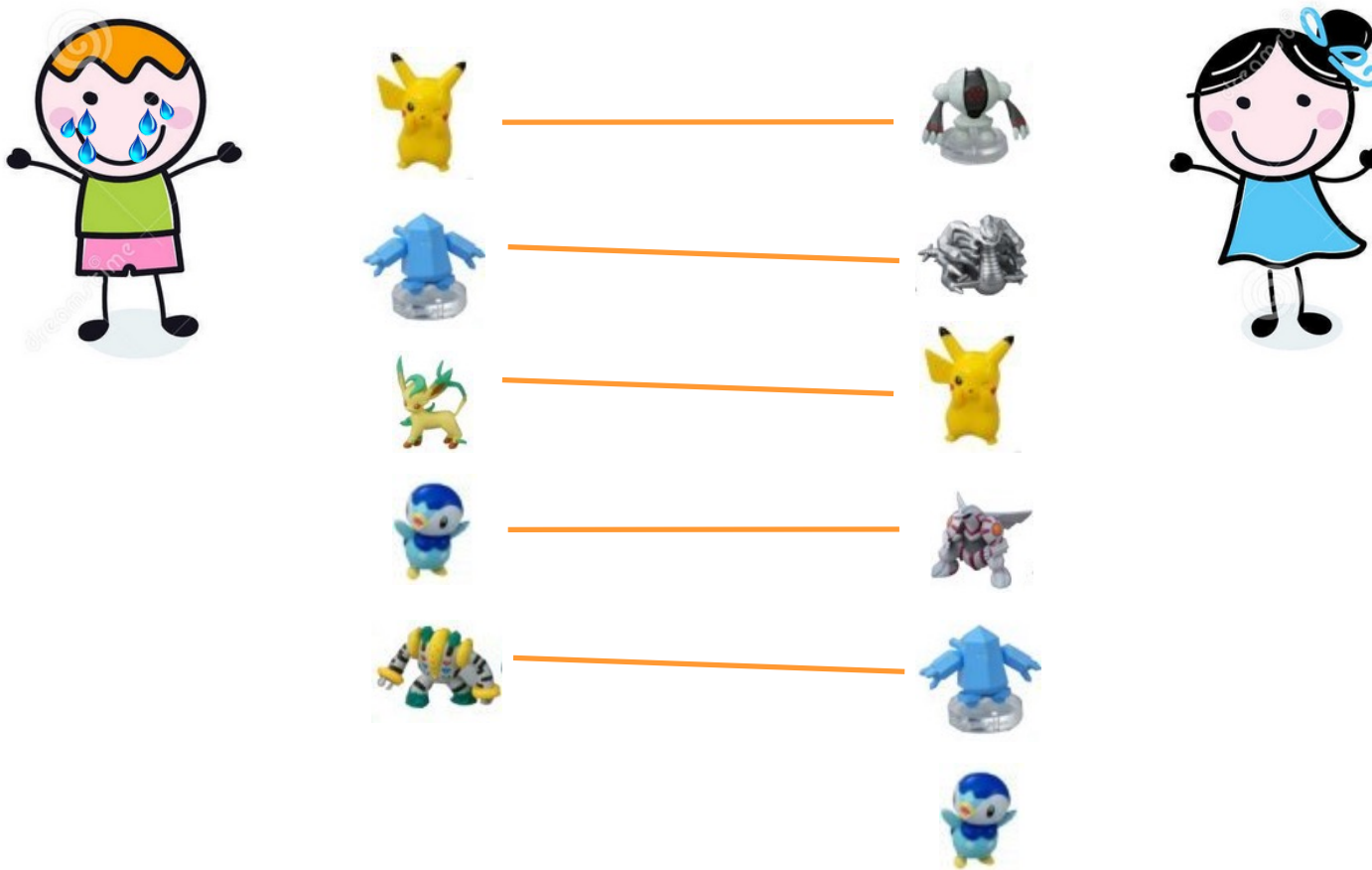
What does it mean that two sets have the same size?



# Cardinality

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What does it mean that two sets have the same size?



# 1-1 and onto

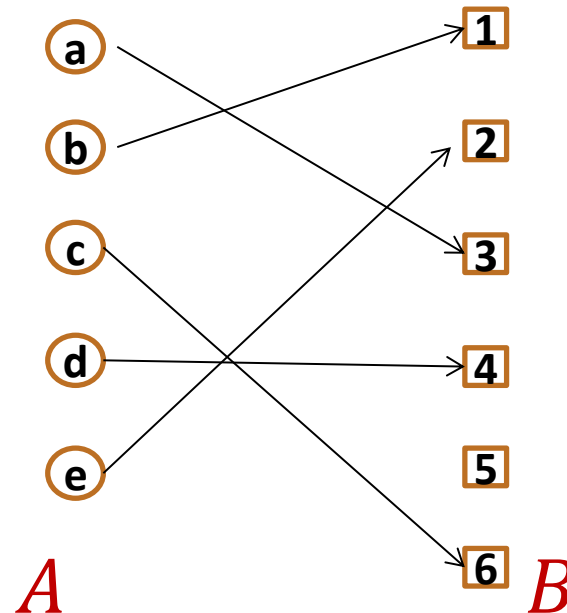
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A function  $f : A \rightarrow B$  is **one-to-one** (1-1) if every output corresponds to at most one input;

i.e.  $f(x) = f(x') \Rightarrow x = x'$  for all  $x, x' \in A$ .

A function  $f : A \rightarrow B$  is **onto** if every output gets hit;

i.e. for every  $y \in B$ , there exists  $x \in A$  such that  $f(x) = y$ .

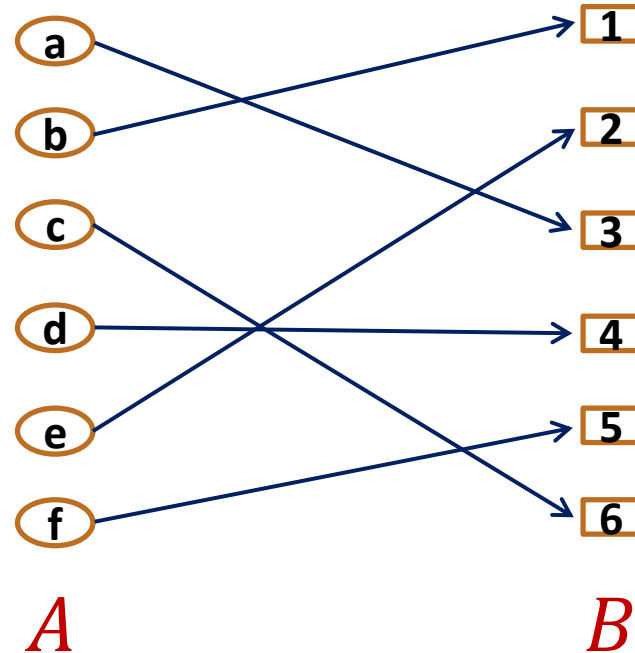


1-1 but not onto

# Cardinality

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**Definition:** Two sets  $A$  and  $B$  have the same **cardinality** if there is a one-to-one correspondence between the elements of  $A$  and those of  $B$ . More precisely, if there is a **1-1 and onto** function  $f : A \rightarrow B$ .



1-1 proves  $\leq$   
onto proves  $\geq$

The definition also makes sense for infinite sets!



# Cardinality

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Do the natural numbers and the even natural numbers have the same cardinality?

Yes!

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ...

0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 ...

What's the map  $f : \mathbb{N} \rightarrow 2\mathbb{N}$  ?  $f(n) = 2n$

# Countable sets

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**Definition:** A set is **countable** iff it has the same cardinality as some subset of  $\mathbb{N}$ .

Equivalent: A set  **$S$**  is countable iff there is an *onto* function  **$g$**  :  $\mathbb{N} \rightarrow S$

Equivalent: A set  **$S$**  is countable iff we can order the elements  
 **$S = \{x_1, x_2, x_3, \dots\}$**

**The set  $\mathbb{Z}$  of all integers**

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# The set $\mathbb{Z}$ of all integers

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0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 ...

0 1 -1 2 -2 3 -3 4 -4 5 -5 6 -6 7 -7 ...

# The set $\mathbb{Q}$ of rational numbers

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We can't do the same thing we did for the integers.

Between any two rational numbers there are an infinite number of others.

## The set of positive rational numbers

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$1/1$	$1/2$	$1/3$	$1/4$	$1/5$	$1/6$	$1/7$	$1/8$	...
$2/1$	$2/2$	$2/3$	$2/4$	$2/5$	$2/6$	$2/7$	$2/8$	...
$3/1$	$3/2$	$3/3$	$3/4$	$3/5$	$3/6$	$3/7$	$3/8$	...
$4/1$	$4/2$	$4/3$	$4/4$	$4/5$	$4/6$	$4/7$	$4/8$	...
$5/1$	$5/2$	$5/3$	$5/4$	$5/5$	$5/6$	$5/7$	...	
$6/1$	$6/2$	$6/3$	$6/4$	$6/5$	$6/6$	...		
$7/1$	$7/2$	$7/3$	$7/4$	$7/5$	....			
...	...	...	...	...				

# The set of positive rational numbers

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The set of all positive rational numbers **is countable**.

$$\mathbb{Q}^+ = \{1/1, 2/1, 1/2, 3/1, 2/2, 1/3, 4/1, 2/3, 3/2, 1/4, 5/1, 4/2, 3/3, 2/4, 1/5, \dots\}$$

List elements in order of numerator+denominator, breaking ties according to denominator.

Only  $k$  numbers have total of sum of  $k + 1$ , so every positive rational number comes up some point.

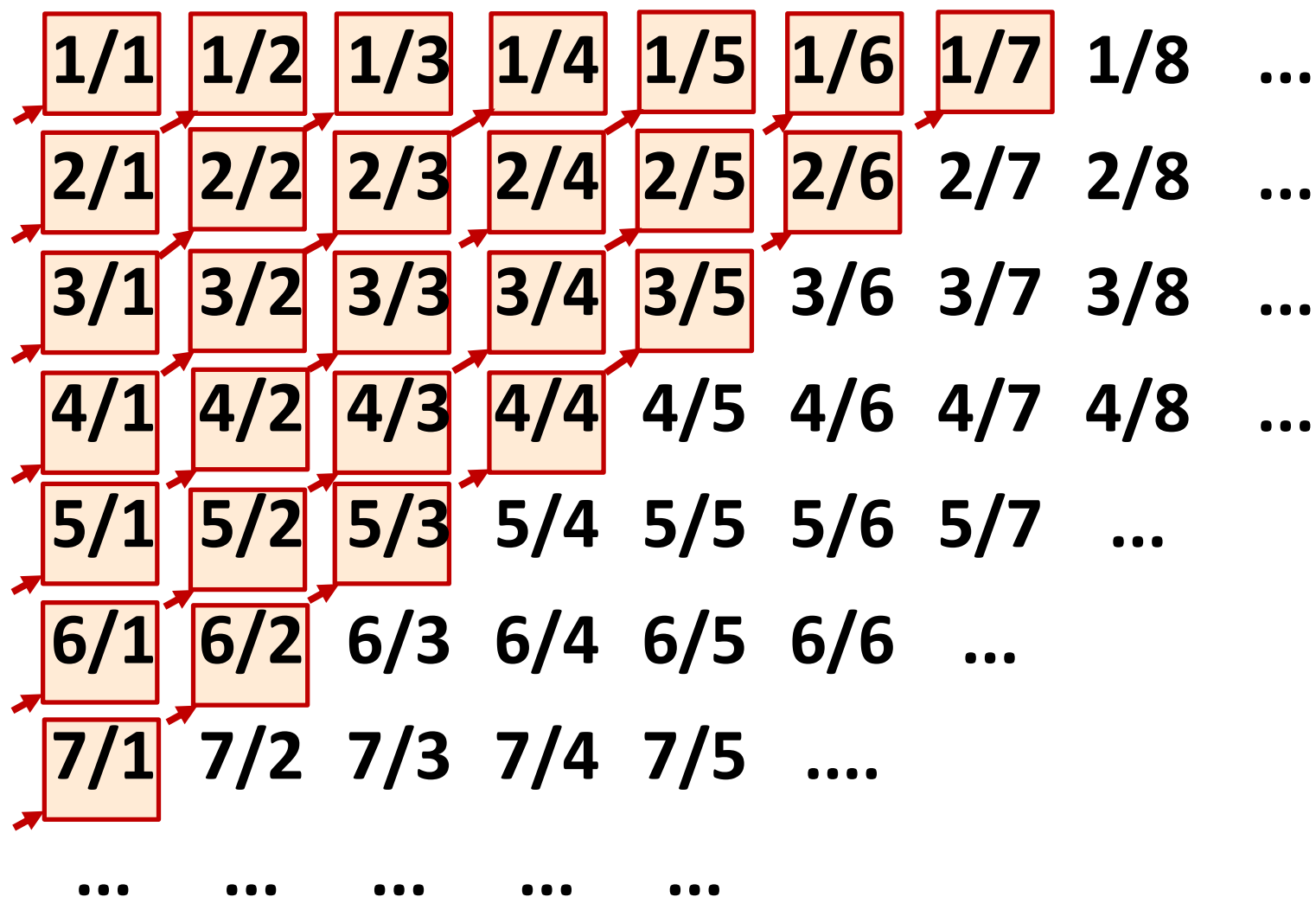
The technique is called “**dovetailing**.”

More generally:

- Put all elements into *finite* groups
- Order the groups
- List elements in order by group (arbitrary order within each group)

# The set of positive rational numbers

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**The set  $\mathbb{Q}$  of rational numbers**

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**Claim:  $\Sigma^*$  is countable for every finite  $\Sigma$**

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**Dictionary/Alphabetical/Lexicographical order is bad**

- Never get past the A's
- A, AA, AAA, AAAA, AAAAA, AAAAAA, ....

**Claim:  $\Sigma^*$  is countable for every finite  $\Sigma$**

---

Dictionary/Alphabetical/Lexicographical order is bad

- Never get past the A's
- A, AA, AAA, AAAA, AAAAA, AAAAAA, ....

Instead, use same “dovetailing” idea, except that we group based on length: only  $|\Sigma|^k$  strings of length  $k$ .

e.g.  $\{0,1\}^*$  is countable:

$\{\epsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 111, \dots\}$

# **The set of all Java programs is countable**

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**Java programs are just strings in  $\Sigma^*$  where  $\Sigma$  is the alphabet of ASCII characters.**

**Since  $\Sigma^*$  is countable, so is the set of all Java programs.**

**More generally, any subset of a countable set is countable: it has same cardinality as an (even smaller) subset of  $\mathbb{N}$**

**OK OK... Is Everything Countable ?!!**

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# Are the real numbers countable?

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**Theorem [Cantor]:**

The set of real numbers between 0 and 1 is **not** countable.

Proof will be by contradiction.

Uses a new method called diagonalization.

# Real numbers between 0 and 1: $[0,1)$

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Every number between 0 and 1 has an infinite decimal expansion:

$$1/2 = 0.500000000000000000000000000000...$$

$$1/3 = 0.333333333333333333333333333333...$$

$$1/7 = 0.14285714285714285714285714285...$$

$$\pi-3 = 0.14159265358979323846264...$$

$$1/5 = 0.199999999999999999999999999999...$$

$$= 0.200000000000000000000000000000...$$

Representation is unique except for the cases that the decimal expansion ends in all 0's or all 9's. We will never use the all 9's representation.

# Proof that $[0,1)$ is not countable

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Suppose, for a contradiction, that there is a list of them:

$r_1$     0.50000000...

$r_2$     0.33333333...

$r_3$     0.14285714...

$r_4$     0.14159265...

$r_5$     0.12122122...

$r_6$     0.25000000...

$r_7$     0.71828182...

$r_8$     0.61803394...

...    ...



# Proof that $[0,1)$ is not countable

---

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4	5	6	7	8	9	...
$r_1$	0.	5	0	0	0	0	0	0	0	...	...
$r_2$	0.	3	3	3	3	3	3	3	3	...	...
$r_3$	0.	1	4	2	8	5	7	1	4	...	...
$r_4$	0.	1	4	1	5	9	2	6	5	...	...
$r_5$	0.	1	2	1	2	2	1	2	2	...	...
$r_6$	0.	2	5	0	0	0	0	0	0	...	...
$r_7$	0.	7	1	8	2	8	1	8	2	...	...
$r_8$	0.	6	1	8	0	3	3	9	4	...	...
...	....	...	....	....	...	...	...	...	...	...	...

# Proof that $[0,1)$ is not countable

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Suppose, for a contradiction, that there is a list of them:

		1	2	3	4	5	6	7	8	9	...
$r_1$	0.	5	0	0	0	0	0	0	0	...	...
$r_2$	0.	3	3	3	3	3	3	3	3	...	...
$r_3$	0.	1	4	2	8	5	7	1	4	...	...
$r_4$	0.	1	4	1	5	9	2	6	5	...	...
$r_5$	0.	1	2	1	2	2	1	2	2	...	...
$r_6$	0.	2	5	0	0	0	0	0	0	...	...
$r_7$	0.	7	1	8	2	8	1	8	2	...	...
$r_8$	0.	6	1	8	0	3	3	9	4	...	...
...	....	...	....	....	...	...	...	...	...	...	...

# Proof that $[0,1)$ is not countable

---

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4	<div>Flipping rule: Only if the other driver deserves it.</div>					
$r_1$	0.	5	0	0	0						
$r_2$	0.	3	3	3	3	3	3	3	3	...	...
$r_3$	0.	1	4	2	8	5	7	1	4	...	...
$r_4$	0.	1	4	1	5	9	2	6	5	...	...
$r_5$	0.	1	2	1	2	2	1	2	2	...	...
$r_6$	0.	2	5	0	0	0	0	0	0	...	...
$r_7$	0.	7	1	8	2	8	1	8	2	...	...
$r_8$	0.	6	1	8	0	3	3	9	4	...	...
...	....	...	....	....	...	...	...	...	...	...	...

# Proof that $[0,1)$ is not countable

---

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4						
$r_1$	0.	5 <sup>1</sup>	0	0	0						
$r_2$	0.	3	3 <sup>5</sup>	3	3						
$r_3$	0.	1	4	2 <sup>5</sup>	8	5	7	1	4	...	...
$r_4$	0.	1	4	1	5 <sup>1</sup>	9	2	6	5	...	...
$r_5$	0.	1	2	1	2	2 <sup>5</sup>	1	2	2	...	...
$r_6$	0.	2	5	0	0	0	0 <sup>5</sup>	0	0	...	...
$r_7$	0.	7	1	8	2	8	1	8 <sup>5</sup>	2	...	...
$r_8$	0.	6	1	8	0	3	3	9	4 <sup>5</sup>	...	...
...	....	...	....	....	...	...	...	...	...	...	...

**Flipping rule:**

If digit is 5, make it 1.

If digit is not 5, make it 5.

# Proof that $[0,1)$ is not countable

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4						
$r_1$	0.	5 <sup>1</sup>	0	0	0						
$r_2$	0.	3	3 <sup>5</sup>	3	3						
$r_3$	0.	1	4	2 <sup>5</sup>	8	5	7	1	4	...	...
$r_4$	0.	1	4	1	5 <sup>1</sup>	9	2	6	5	...	...
$r_5$	0.	1	2	1	2	2 <sup>5</sup>	1	2	2	...	...
$r_6$	0.	2	5	0	0	0	0 <sup>5</sup>	0	0	...	...
$r_7$	0.	7	1	8	2	8	1	8 <sup>5</sup>	2	...	...

**Flipping rule:**

If digit is 5, make it 1.

If digit is not 5, make it 5.

If diagonal element is  $0.x_{11}x_{22}x_{33}x_{44}x_{55}\dots$  then let's call the flipped number  $0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55}\dots$

**It cannot appear anywhere on the list!**

# Proof that $[0,1)$ is not countable

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4
$r_1$	0.	5 <sup>1</sup>	0	0	0
$r_2$	0.	3	3 <sup>5</sup>	3	3
$r_3$	0.	1	4	2 <sup>5</sup>	8
$r_4$	0.	1	4	1	5 <sup>1</sup>

**Flipping rule:**

If digit is 5, make it 1.

If digit is not 5, make it 5.

For every  $n \geq 1$ :

$r_n \neq 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \dots$

because the numbers differ on the  $n$ -th digit!

5	7	1	4	...	...
9	2	6	5	...	...
2 <sup>5</sup>	1	2	2	...	...
0	0 <sup>5</sup>	0	0	...	...
8	1	8 <sup>5</sup>	2	...	...

If diagonal element is  $0.x_{11}x_{22}x_{33}x_{44}x_{55} \dots$  then let's call the flipped number  $0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \dots$

**It cannot appear anywhere on the list!**

# Proof that $[0,1)$ is not countable

Suppose, for a contradiction, that there is a list of them:

		1	2	3	4
$r_1$	0.	5 <sup>1</sup>	0	0	0
$r_2$	0.	3	3 <sup>5</sup>	3	3
$r_3$	0.	1	4	2 <sup>5</sup>	8
$r_4$	0.	1	4	1	5 <sup>1</sup>

## Flipping rule:

If digit is 5, make it 1.

If digit is not 5, make it 5.

For every  $n \geq 1$ :

$$r_n \neq 0.\hat{x}_{11}\hat{x}_{22}\hat{x}_{33}\hat{x}_{44}\hat{x}_{55} \dots$$

because the numbers differ on the  $n$ -th digit!

5	7	1	4	...	...
9	2	6	5	...	...
2 <sup>5</sup>	1	2	2	...	...
0	0 <sup>5</sup>	0	0	...	...
8	1	8 <sup>5</sup>	2	...	...

So the list is incomplete, which is a contradiction.

Thus the real numbers between 0 and 1 are **not countable**: “uncountable”

**The set of all functions  $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$  is uncountable**



**The set of all functions  $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$  is uncountable**

Supposed listing of all the functions:

	1	2	3	4	5	6	7	8	9	...
f <sub>1</sub>	5	0	0	0	0	0	0	0	...	...
f <sub>2</sub>	3	3	3	3	3	3	3	3	...	...
f <sub>3</sub>	1	4	2	8	5	7	1	4	...	...
f <sub>4</sub>	1	4	1	5	9	2	6	5	...	...
f <sub>5</sub>	1	2	1	2	2	1	2	2	...	...
f <sub>6</sub>	2	5	0	0	0	0	0	0	...	...
f <sub>7</sub>	7	1	8	2	8	1	8	2	...	...
f <sub>8</sub>	6	1	8	0	3	3	9	4	...	...
...	...	....	....	...	...	...	...	...	...	

# The set of all functions $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

[illegible]

# The set of all functions $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$ is uncountable

Supposed listing of all the functions:

	1	2	3	4	<div>Flipping rule:</div> <div>If <math>f_n(n) = 5</math>, set <math>D(n) = 1</math></div> <div>If <math>f_n(n) \neq 5</math>, set <math>D(n) = 5</math></div>					
$f_1$	5 <sup>1</sup>	0	0	0						
$f_2$	3	3 <sup>5</sup>	3	3						
$f_3$	1	4	2 <sup>5</sup>	8						
$f_4$	1	4	1	5 <sup>1</sup>	9	2	6	5	...	...
$f_5$	1	2	1	2	2 <sup>5</sup>	1	2	2	...	...
$f_6$	2	5	0	0	0	0 <sup>5</sup>	0	0	...	...
$f_7$	7	1	8	2	8	1	8 <sup>5</sup>	2	...	...

For all  $n$ , we have  $D(n) \neq f_n(n)$ . Therefore  $D \neq f_n$  for any  $n$  and the list is incomplete!  $\Rightarrow \{f \mid f: \mathbb{N} \rightarrow \{0, 1, \dots, 9\}\}$  is **not** countable

# Uncomputable functions

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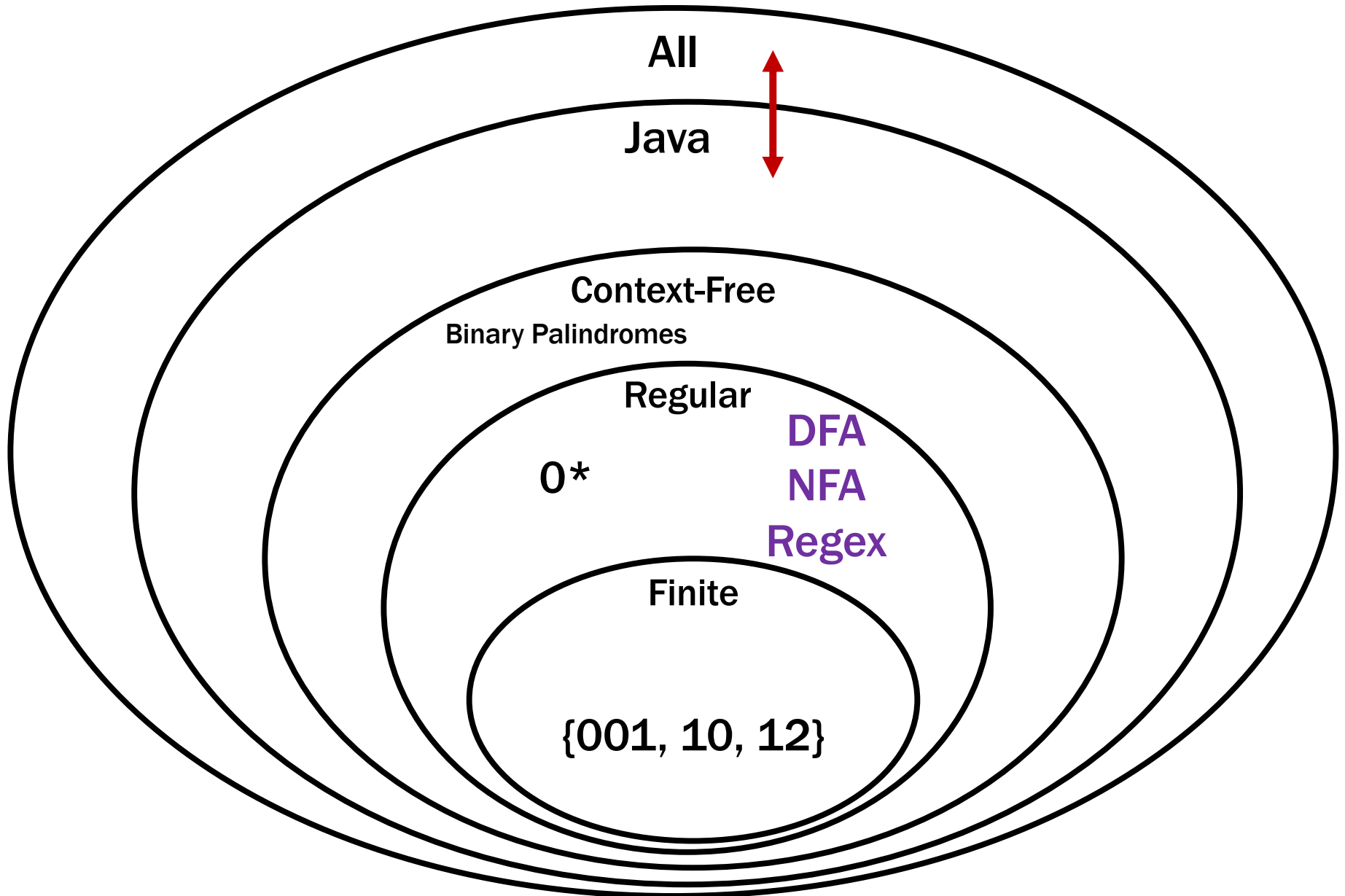
We have seen that:

- The set of all (Java) programs is countable
- The set of all functions  $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$  is not countable

So: There must be some function  $f : \mathbb{N} \rightarrow \{0, \dots, 9\}$  that is not computable by any program!

# Recall our language picture

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# Uncomputable functions

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Interesting... maybe.

Can we come up with an explicit function that is uncomputable?