Conservative Dynamic Systems
Part 2. Periodic Pulsations in a Cubic Field

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MATH 263_VO1 Computing I

1. Mathematical Formulation

Consider a conservative system $f(x)$, which is governed by the Hamiltonian Ordinary Differential Equation (ODE)

$$\frac{\left(\frac{d}{dx} f(x)\right)^2}{2} + P_3(f(x)) = 0 \tag{1}$$

with the polynomial field $P_3(f(x))$ of the third order

$$P_3(f(x)) = \frac{8 r^2 (u (f-p) + v q) (f-p) (f-q-p)}{q} \tag{2}$$

$$P_3(f(x)) = \frac{200 \left(\frac{81 f}{100} - \frac{93}{50}\right) (f-3) (f-6)}{3} \tag{3}$$

To compute experimentally $f(x)$ by means of the the Hamiltonian ODE,

$$\frac{\left(\frac{d}{dx} f(x)\right)^2}{2} + \frac{200 \left(\frac{81 f(x)}{100} - \frac{93}{50}\right) (f(x)-3) (f(x)-6)}{3} = 0 \tag{4}$$

we also need an initial condition

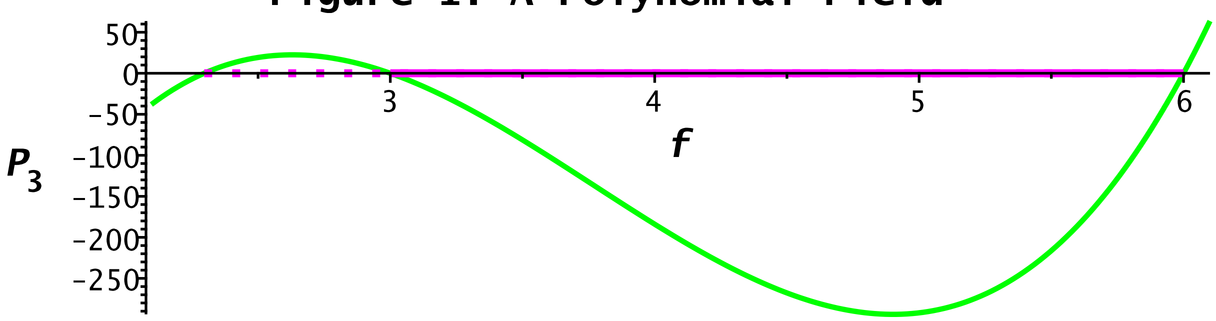
$$f(0) = p + q \tag{5}$$

which after specification of parameters p and q becomes

$$f(0) = 6 \tag{6}$$

$$P_3 = \frac{200 \left(\frac{81 f}{100} - \frac{93}{50}\right) (f-3) (f-6)}{3}$$

Figure 1. A Polynomial Field



The polynomial field U_3 together with the total virtual energy E_t are shown in **Figure 1** above. The system performs a periodic pulsation between roots $x = 3$ and $x = 6$ since a virtual kinetic energy K

$_e = 1/2$

$(df/dx)^2$ is balanced by the virtual potential energy $U_3(f(x))$; total virtual energy $E_t = 1/2(\mathrm{df/dx})^2 + U_3(f(x)) = 0$. These two values have an inverse relationship, as the virtual kinetic energy increases, the virtual potential energy decreases and vice versa, when the virtual potential energy increases, the virtual kinetic energy decreases until the system velocity vanishes; $df/dx = 0$, at the roots $x = 3$ and $x = 6$ of the polynomial field. The aforementioned results in the conservative system being "locked" in the polynomial well.

In order to transform the polynomial field towards the canonical form $R_3 = 2(\kappa h + \lambda^2)h(h - 1)$ for the periodic pulsations $h = cn(u, \kappa)^2$ we execute a change of variable below, and use the package PDEtools to make a horizontal shift of U_3.

$$Eqe_8 := \{y = x, g(y) = f(x) - 3\}$$

$$\{x = y, f(x) = g(y) + 3\}$$

$$\{y = x, g(y) = f(x) - 3\}$$

$$\frac{\left(\frac{d}{dy} g(y)\right)^2}{2} + \frac{200\left(\frac{81\, g(y)}{100} + \frac{57}{100}\right) g(y)\,(g(y) - 3)}{3} = 0 \tag{7}$$

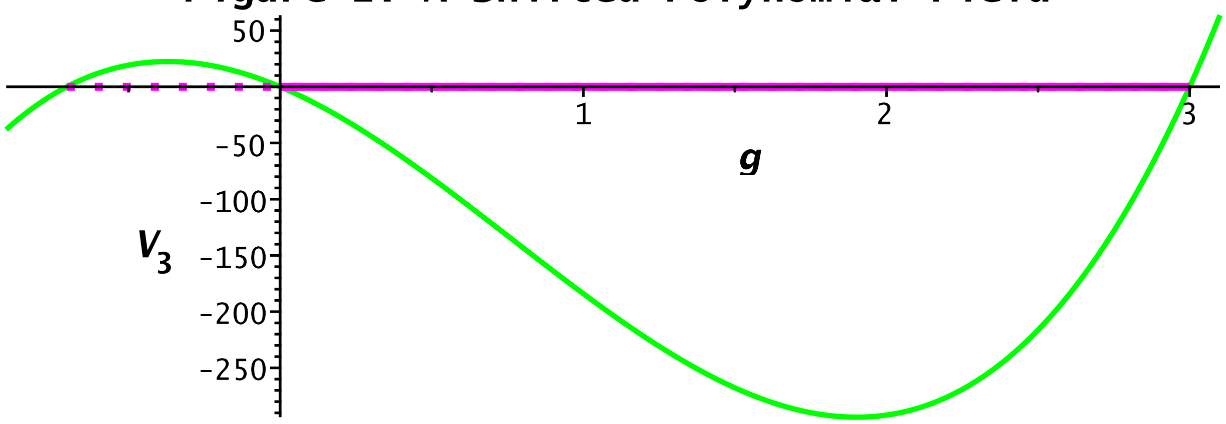
An initial condition for a conservative system $g(y)$ is converted to the following:

$$g(0) = 3 \tag{8}$$

As it is shown in Figure 2, the canonical transformation moves the center root of P_3 to the origin.

$$V_3 = \frac{200\left(\frac{81\, g}{100} + \frac{57}{100}\right) g\,(g - 3)}{3}$$

Figure 2. A Shifted Polynomial Field



We then scale $g(y)$ by the positive root using another change of variables and simplification of the Hamiltonian ODE.

$$\{y = z, g(y) = 3\, h(z)\}$$

$$\left\{z = y, h(z) = \frac{g(y)}{3}\right\}$$

$$\frac{\left(\frac{d}{dz} h(z)\right)^2}{2} + 2\,(81\, h(z) + 19)\, h(z)\,(h(z) - 1) = 0 \tag{9}$$

We then define an elliptic modulus κ and a complementary modulus λ, which are related by the Pythagorean identity,

$$\kappa^2 = \frac{81}{100}, \lambda^2 = \frac{19}{100}$$

$$\kappa^2 + \lambda^2 = 1 \tag{10}$$

Expressing the Hamiltonian ODE for $h(z)$ with the help of κ^2 and λ^2 then becomes

$$\frac{\left(\frac{d}{dz} h(z)\right)^2}{2} + 200\left(h(z)\,\kappa^2 + \lambda^2\right) h(z)\,(h(z) - 1) = 0 \tag{11}$$

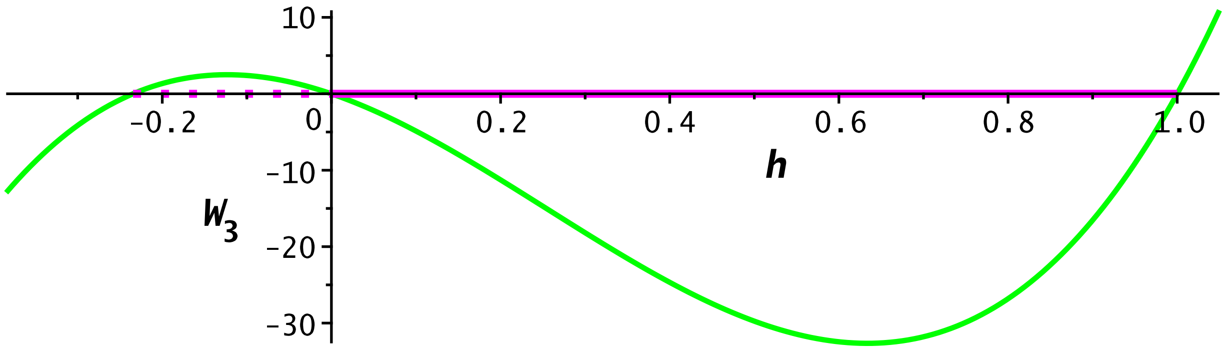
An initial condition for a conservative system $h(z)$ then becomes

$$h(0) = 1 \tag{12}$$

The second congruent transformation moves the positive root of V_3 to 1 (see Figure 3)

$$W_3 = 200\left(h\,\kappa^2 + \lambda^2\right) h\,(h - 1)$$

Figure 3. A Scaled Polynomial Field



Thus, we have reduced the Hamiltonian ODE to the scaled form (11) for the conservative system $h(z)$ subjected to the initial condition (12) (the Cauchy condition). When the Hamiltonian problem for $h(z)$ is solved, a conservative solution for the original system $f(x)$ is provided by substitutions

$$f(x) = 3\,h(z) + 3, z = x \tag{13}$$

2. An Exact Elliptic Solution

Define notations of the Jacobi elliptic functions *sn, cn, dn,* and the complete elliptic integral of the first kind *Ke.*

$$sn, cn, dn, Kc \tag{14}$$

Primarily, we solve the Hamiltonian ODE using the classical method of undetermined coefficients. We are looking for an elliptic solution $h(z)$ of (11) in the following form:

$$h(z) = A\,cn(2\,a\,(z - c), \kappa)^2 \tag{15}$$

where A is an undetermined amplitude of a periodic pulsation, a is a positive undetermined parameter, which characterizes scaling of a virtual time z, and c is an undetermined coefficient, which controls a shift in z.

In agreement with the Hamiltonian ODE, the conservative system is located at a stagnation point at the initial moment $z = 0$ since substitution of the initial value of $h(z)$, $h(0) = 1$, in (11) yields

$$\left(\frac{d}{dz}\, h(z)\right)\bigg|_{\{z=0\}} = 0 \tag{16}$$

The first derivative of $h(z)$ becomes

$$\frac{d}{dz}\, h(z) = -4\, A\, a\, sn(2\, a\, (z-c), \kappa)\, dn(2\, a\, (z-c), \kappa)\, cn(2\, a\, (z-c), \kappa) \tag{17}$$

Since $dn(u, \kappa) > 0$ for all parameters and values of z, $cn(0, \kappa) = 1$ and $sn(0, \kappa) = 0$ the initial condition for the first derivative (16) is satisfied for all A and a if $c = 0$ as

$$sn(2\, a\, (z-c), \kappa)\Big|_{z=0} = -sn(2\, a\, c, \kappa)$$

$$-sn(2\, a\, c, \kappa) = 0$$

$$2\, a\, c = 0$$

$$c = 0 \tag{18}$$

For $c = 0$, evaluation of $h(z)$ at the initial moment $z = 0$ gives

$$h(z) = A\, cn(2\, a\, z, \kappa)^2$$

$$h(0) = A \tag{19}$$

Therefore, the initial condition (12), $h(0) = 1$, is satisfied for $A = 1$. So, a reduced elliptic solution $h(z)$, and its first derivative, which satisfy initial conditions (12) and (16), becomes

$$h(z) = cn(2\, a\, z, \kappa)^2$$

$$\frac{d}{dz}\, h(z) = -4\, a\, sn(2\, a\, z, \kappa)\, dn(2\, a\, z, \kappa)\, cn(2\, a\, z, \kappa) \tag{20}$$

To find the last undetermined parameter a, we compute the first derivative squared

$$\left(\frac{d}{dz}\, h(z)\right)^2 = 16\, a^2\, sn(2\, a\, z, \kappa)^2\, dn(2\, a\, z, \kappa)^2\, cn(2\, a\, z, \kappa)^2 \tag{21}$$

and simplify it by the elliptic Pythagorean identities

$$cn(u, \kappa)^2 + sn(u, \kappa)^2 = 1,\; sn(u, \kappa)^2 = 1 - cn(u, \kappa)^2$$

$$dn(u, \kappa)^2 - \kappa^2\, cn(u, \kappa)^2 = \lambda^2,\; dn(u, \kappa)^2 = \lambda^2 + \kappa^2\, cn(u, \kappa)^2 \tag{22}$$

as follows

$$sn(2\, a\, z, \kappa)^2 = 1 - cn(2\, a\, z, \kappa)^2$$

$$dn(2\, a\, z, \kappa)^2 = \lambda^2 + \kappa^2\, cn(2\, a\, z, \kappa)^2 \tag{23}$$

Substitution of (21), (20), and (23), in the Hamiltonian ODE (11) yields

$$8\, a^2\, \left(1 - cn(2\, a\, z, \kappa)^2\right) \left(\lambda^2 + \kappa^2\, cn(2\, a\, z, \kappa)^2\right) cn(2\, a\, z, \kappa)^2 + 200\, \left(\lambda^2 + \kappa^2\, cn(2\, a\, z, \kappa)^2\right) cn(2\, a\, z, \kappa)^2\, \left(cn(2\, a\, z, \kappa)^2 - 1\right) = 0 \tag{24}$$

Thus, the Hamiltonian ODE is reduced to a Hamiltonian Algebraic Equation (AE). Collecting like terms in $cn(2\, a\, z, \kappa)^2$ gives

$$u = cn(2\, a\, z, \kappa)^2$$

$$8\, a^2\, (1-u)\, \left(\kappa^2\, u + \lambda^2\right) u + 200\, \left(\kappa^2\, u + \lambda^2\right) u\, (u-1) = 0$$

$$-8\, \left(\kappa^2\, u + \lambda^2\right) u\, (a-5)\, (a+5)\, (u-1) = 0$$

$$-8\left(\lambda^2+\kappa^2\,cn(2\,a\,z,\kappa)^2\right)cn(2\,a\,z,\kappa)^2\,(a-5)\,(a+5)\left(cn(2\,a\,z,\kappa)^2-1\right)=0 \tag{25}$$

For the Hamiltonian AE to be satisfied for all z, κ, λ, and positive a, a coefficient of Equation (25) must vanish

$$-8\,(a-5)\,(a+5)=0 \tag{26}$$

Solving Equation (26) for a positive a yields

$$a=5 \tag{27}$$

Finally, we substitute a in Equation (20) to find an exact elliptic solution

$$h(z)=cn(10\,z,\kappa)^2 \tag{28}$$

Using backward substitutions (13) yields an exact elliptic solution in the initial variables x and $f(x)$ in the following forms

$$f(x)=3\,cn(10\,x,\kappa)^2+3 \tag{29}$$

To verify the exact experimental solution, we substitute (29) in the initial Hamiltonian ODE (4) and reduce it to identity

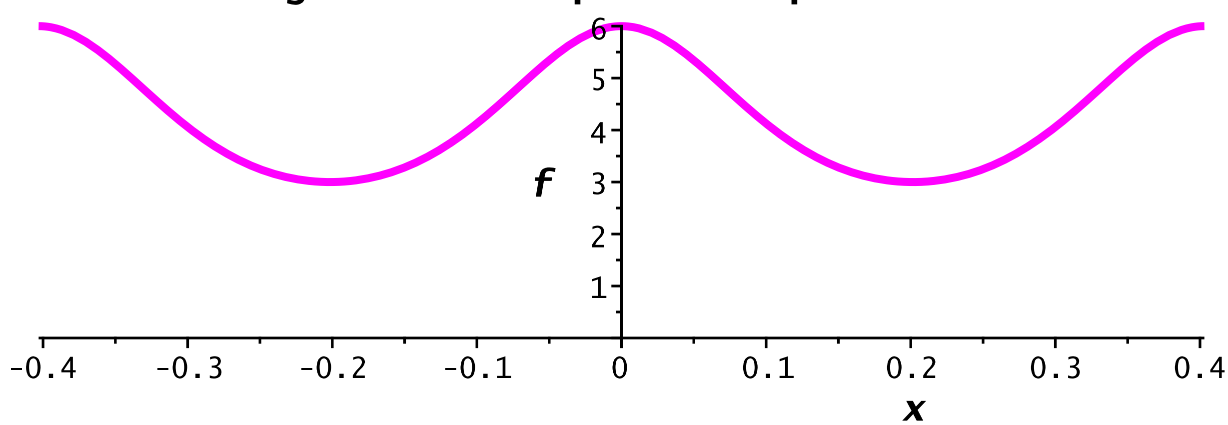
$$0=0 \tag{30}$$

The initial condition in variables x and $f(x)$ is fulfilled as well since

$$f(0)=6 \tag{31}$$

$$Puls := 3\,cn(10\,x,0.8100000000)^2+3$$

Figure 4. The periodic pulsation



The periodic solution (29) of the Hamiltonian ODE (4) with the virtual potential (3) (see Figure 1) represents an elevation pulsation, which is shown in Figure 4. The pulsatory solution (29) subjected to the initial condition (6) is computed on the interval [0, +∞) and afterwards continued symmetrically in Figure 4 for all times on the interval (-∞, +∞), whereas any moment along the trajectory of the conservative system may be treated as an initial one.
We now construct a static visualization of the computed solution a three-dimensional (3-D) space [x, f, P_3] called the 3-D Hamiltonian map, where P_3 is a virtual potential of the Hamiltonian ODE (4).

$$P_3=\frac{200\left(\frac{81\,f}{100}-\frac{93}{50}\right)(f-3)\,(f-6)}{3}$$

$$f2d := 3\,cn(10\,x,0.8100000000)^2+3$$

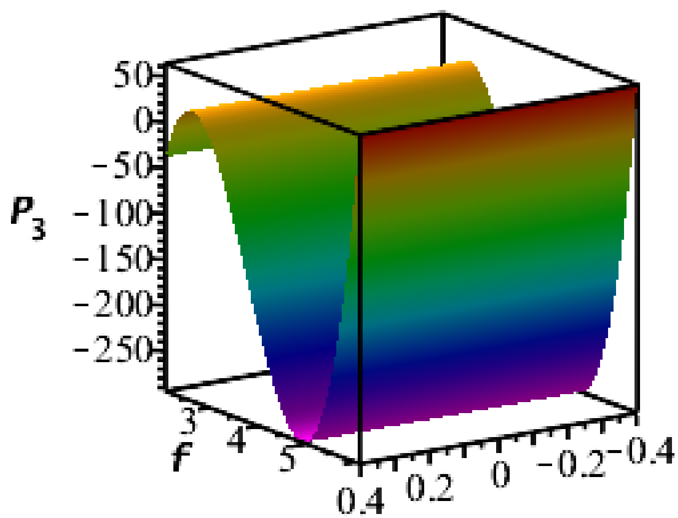
$$f3d := 1458.000000\left(cn(10.\,x,0.8100000000)^2+0.2345679012\right)cn(10.\,x, \tag{32}$$

$0.8100000000)^2\ (cn(10.\,x, 0.8100000000) + 1.000000000)\ (cn(10.\,x, 0.8100000000) - 1.)$

We then compute 50 frames of the static 3-D Hamiltonian map, where the argument x is parametrized by a virtual time t.

$$CPU\_time := 39.344 \quad (33)$$

Figure 5. The animated 3-D Hamiltonian Map



Use options FPS =1-3 (Frames Per Second), loop, and forward to show the math movie in detail.

In Figure 5, we display the animated 3-D Hamiltonian map of the periodic elevation pulsation for the conservative system (29) in a cubic field (3). The yellow curve shows the 3-D trajectory of the conservative system along the virtual potential, while its 2-D projection on the plane of variables $[x, f(x)]$ displayed by the magenta curve coincides with that in Figure 4. The conservative systems hits at the virtual time $x = 0$ the potential wall located at the virtual coordinate $f = 8$, where its virtual velocity df/dx = 0. The dynamic system then hits at the virtual moment $x = Kc(\kappa) / 2a$ the potential wall located at the virtual coordinate $f = 3$, where its virtual velocity again vanishes, df/dx = 0. The system finishes it period at the virtual time $x = Kc(\kappa) / a$ when it is again reflected by the potential barrier located at the virtual coordinate $f = 8$, where its virtual velocity vanishes once more, df/dx = 0.