Conservative Dynamic Systems
Part 1. Aperiodic Pulsations in a Cubic Field

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MATH 263_VO1 Computing I

1. Mathematical Formulation

Consider a conservative system *f(x)*, which is governed by the Hamiltonian Ordinary Differential Equation (ODE).

$$\frac{\left(\frac{d}{dx} f(x)\right)^2}{2} + P_3(f(x)) = 0 \tag{1}$$

With the polynomial field $P_3(f(x))$ of the third order:

$$P_3(f(x)) = \frac{8 r^2 (f-p)^2 (f-q-p)}{q} \tag{2}$$

$$P_3(f(x)) = \frac{200 (f-3)^2 (f-6)}{3} \tag{3}$$

Computing *f(x)* experimentally by means of the Hamiltonian ODE:

$$\frac{\left(\frac{d}{dx} f(x)\right)^2}{2} + \frac{200 (f(x)-3)^2 (f(x)-6)}{3} = 0 \tag{4}$$

Setting the initial condition

$$f(0) = p + q \tag{5}$$

After specifications of parameters *p* and *q* the above initial condition becomes:

$$f(0) = 6 \tag{6}$$

$$P_3 = \frac{200 (f-3)^2 (f-6)}{3}$$

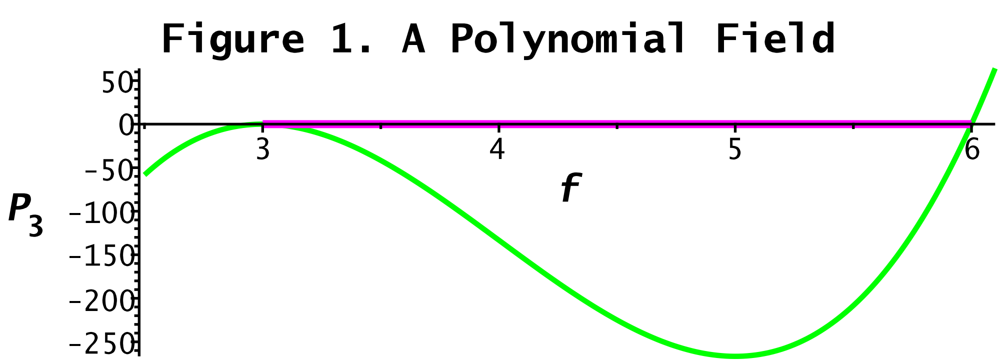


Figure 1 shows the polynomial field P_3 together with the total virtual energy E_t. The system above executes an aperiodic pulsation between roots $x = 3$ and $x = 6$, virtual kinetic energy $K_e = 1/2(df/dx)^2$ is balanced by the virtual potential energy $P_3(f(x))$; total virtual energy $E_t = 1/2(df/dx)^2 + P_3(f(x)) = 0$.

These two values have an <u>inverse relationship</u>, as the virtual kinetic energy increases, the virtual potential energy decreases and vice versa, when the virtual potential energy increases, the virtual kinetic energy decreases until the system velocity vanishes; $df/dx = 0$, at the roots $x = 3$ and $x = 6$ of the polynomial field. The aforementioned results in the conservative system being "locked" in the polynomial well.

In order to transform the polynomial field towards the canonical form $R_3 = 2h^2(h - 1)$ for the aperiodic pulsation $h = \operatorname{sech}(u)^2$ we execute a change of variable below, and use the package PDEtools to make a horizontal shift of P_3 .

$$\{x = y, f(x) = g(y) + 3\}$$
$$\{y = x, g(y) = f(x) - 3\}$$
$$\frac{\left(\frac{d}{dy} g(y)\right)^2}{2} + \frac{200\, g(y)^2 (g(y) - 3)}{3} = 0 \tag{7}$$

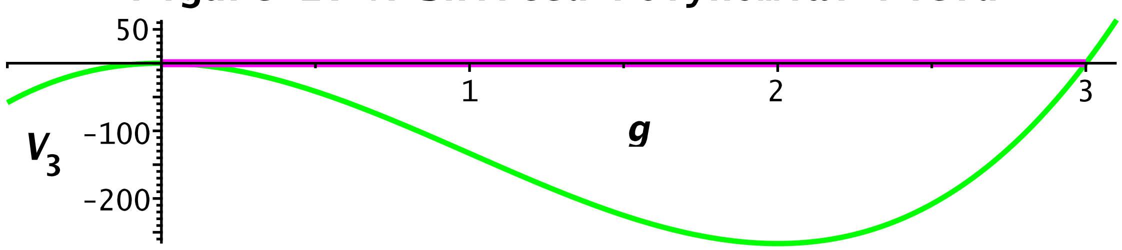
Initial condition for a conservative system $g(y)$ is converted to the following:

$$g(0) = 3 \tag{8}$$

As seen in **Figure 2** below**,** the center root of P_3 is moved to the origin by way of canonical transformation.

$$V_3 = \frac{200\, g^2 (g - 3)}{3}$$

Figure 2. A Shifted Polynomial Field



We then scale $g(y)$ by the positive root using another change of variables and simplification of the Hamiltonian ODE.

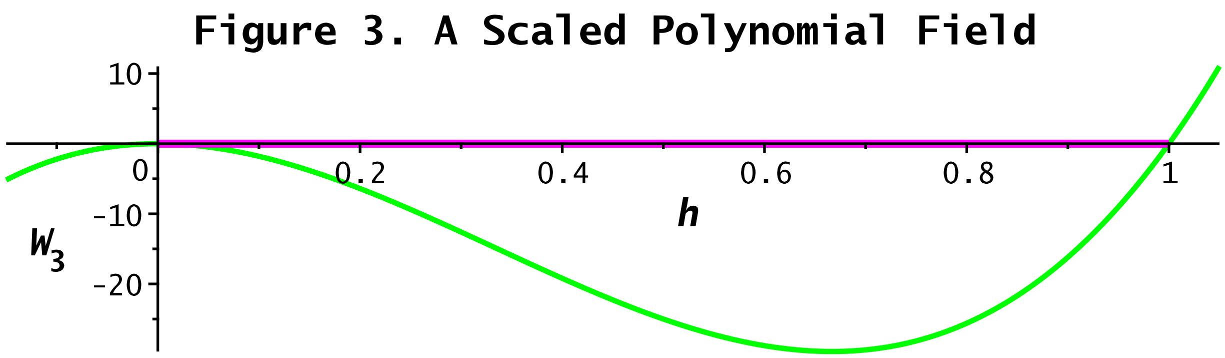
$$\{y = z, g(y) = 3\, h(z)\}$$
$$\left\{z = y, h(z) = \frac{g(y)}{3}\right\}$$
$$\frac{\left(\frac{d}{dz} h(z)\right)^2}{2} + \frac{200\, h(z)^2 (3\, h(z) - 3)}{3} = 0 \tag{9}$$

An initial condition for a conservative system $h(z)$ then becomes

$$h(0) = 1 \tag{10}$$

The second congruent transformation moves the positive root of V_3 to 1 (see Figure 3).

$$W_3 = \frac{200\, h^2 (3\, h - 3)}{3}$$



Thus, we have reduced the Hamiltonian ODE to the scaled form (9) for the conservative system $h(z)$ subjected to the initial condition (10) (the Cauchy condition). When the Hamiltonian problem for $h(z)$ is solved, a conservative solution for the original system $f(x)$ is provided by substitutions:

$$f(x) = 3\,h(z) + 3,\ z = x \tag{11}$$

2. An Exact Hyperbolic Solution of the Hamiltonian ODE

Primarily, we solve the Hamiltonian ODE using the classical method of undetermined coefficients. We are looking for a hyperbolic solution $h(z)$ of Equation (9) in the following form:

$$h(z) = A\,\mathrm{sech}(2\,a\,(z-c))^2 \tag{12}$$

Where A is an undetermined amplitude of an aperiodic pulsation, a is a positive undetermined parameter, which characterizes scaling of a virtual time z, and c is an undetermined coefficient, which controls a shift in z.

In agreement with the Hamiltonian ODE, the conservative system is located at a stagnation point at the initial moment $z = 0$ since substitution of the initial value of $h(z)$, $h(0) = 1$, in (9) produces

$$\left(\frac{d}{dz}\,h(z)\right)\Bigg|_{\{z=0\}} = 0 \tag{13}$$

The first derivative of $h(z)$ then becomes

$$\frac{d}{dz}\,h(z) = -4\,A\,a\tanh(2\,a\,(z-c))\,\mathrm{sech}(2\,a\,(z-c))^2 \tag{14}$$

Since $\mathrm{sech}(u)^2 > 0$ for all parameters and values of z and $\tanh(0) = 0$, the initial condition for the first derivative (13) may be satisfied for all A and a if and only if $c = 0$

$$\tanh(2\,a\,(z-c))\Big|_{z=0} = -\tanh(2\,a\,c)$$

$$-\tanh(2\,a\,c) = 0$$

$$2\,a\,c = 0$$

$$c = 0 \tag{15}$$

For $c = 0$, evaluation of $h(z)$ at the initial moment $z = 0$ gives

$$h(z) = A\,\mathrm{sech}(2\,a\,z)^2$$

$$h(0) = A \tag{16}$$

Therefore, the initial condition $h(0) = 1$, is satisfied for $A = 1$. So, a reduced hyperbolic solution, and its first derivative, which satisfy initial conditions (10) and (13) become

$$h(z) = \mathrm{sech}(2\,a\,z)^2$$

$$\frac{d}{dz}\,h(z) = -4\,a\tanh(2\,a\,z)\,\mathrm{sech}(2\,a\,z)^2 \tag{17}$$

To find the last undetermined parameter a, we compute the first derivative squared

$$\left(\frac{d}{dz}\,h(z)\right)^2 = 16\,a^2\tanh(2\,a\,z)^2\,\mathrm{sech}(2\,a\,z)^4 \tag{18}$$

and simplify it by the hyperbolic Pythagorean identity

$$\begin{gathered}\mathrm{sech}(u)^2+\tanh(u)^2=1\\ \tanh(u)^2=1-\mathrm{sech}(u)^2\end{gathered} \tag{19}$$

as follows:

$$\tanh(2\,a\,z)^2=1-\mathrm{sech}(2\,a\,z)^2 \tag{20}$$

Substitution of (18), (17), and (20) in the Hamiltonian ODE (9) yields

$$8\,a^2\left(1-\mathrm{sech}(2\,a\,z)^2\right)\mathrm{sech}(2\,a\,z)^4+\frac{200\,\mathrm{sech}(2\,a\,z)^4\left(3\,\mathrm{sech}(2\,a\,z)^2-3\right)}{3}=0 \tag{21}$$

Thus, the Hamiltonian ODE is reduced to a Hamiltonian Algebraic Equation (AE). Collecting like terms in sech $(2\,a\,z)^2$ gives

$$\begin{gathered}u=\mathrm{sech}(2\,a\,z)^2\\ 8\,a^2\,(1-u)\,u^2+\frac{200\,u^2\,(3\,u-3)}{3}=0\\ -8\,(a-5)\,(a+5)\,(u-1)\,u^2=0\\ -8\,(a-5)\,(a+5)\left(\mathrm{sech}(2\,a\,z)^2-1\right)\mathrm{sech}(2\,a\,z)^4=0\end{gathered} \tag{22}$$

For the Hamiltonian AE to be satisfied for all z and positive a, a coefficient of Equation (22) must vanish

$$-8\,(a-5)\,(a+5)=0 \tag{23}$$

Solving Equation (23) for positive a yields

$$a=5 \tag{24}$$

Finally, we substitute a in Equation (17) to find an exact hyperbolic solution

$$h(z)=\mathrm{sech}(10\,z)^2 \tag{25}$$

Using backward substitutions (11) yields an exact hyperbolic solution in the initial variables x and $f(x)$in the following form

$$f(x)=3\,\mathrm{sech}(10\,x)^2+3 \tag{26}$$

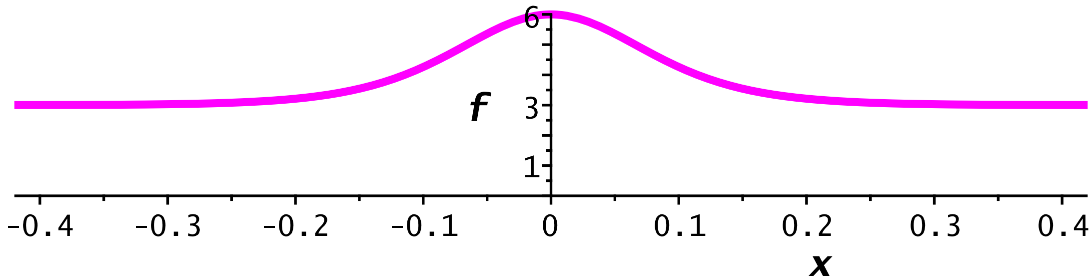
To verify the exact experimental solution, we substitute solution (26) in the initial Hamiltonian ODE (4) and reduce it to identity.

$$0=0 \tag{27}$$

The initial condition in variables x and $f(x)$ is fulfilled as well since

$$f(0)=6 \tag{28}$$

Figure 4. The aperiodic pulsation



The aperiodic solution (26) of the Hamiltonian ODE (4) with the virtual potential (3) (see Figure 1) represents an elevation pulsation, which is shown in Figure 4. The pulsatory solution (26) subjected to the initial condition (6) is computed on the interval [0,+∞) and afterwards continued symmetrically in Figure 4 for all times on the interval (-∞, +∞), whereas any moment along the trajectory of the conservative system may be treated as an initial one.
We now construct a static visualization of the computed solution a three-dimensional (3-D) space [x, f ,P_3] called the 3-D Hamiltonian map, where P_3 is a virtual potential of the Hamiltonian ODE (4).

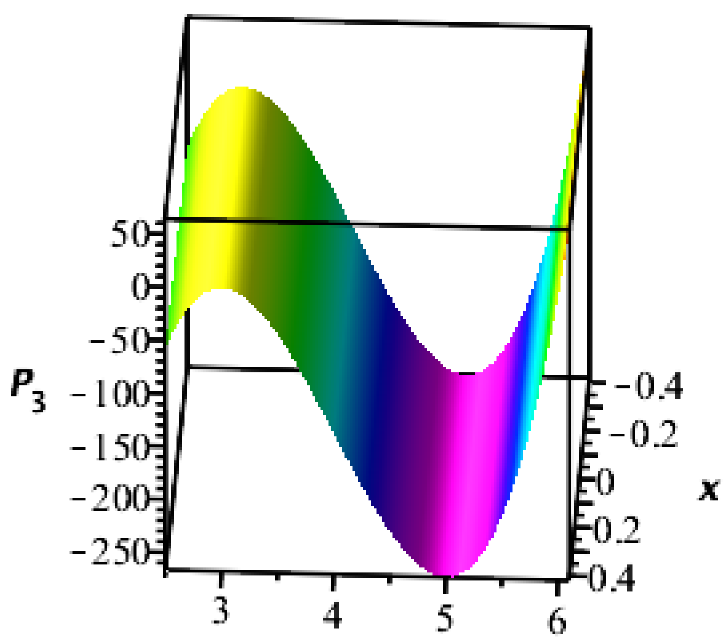
$$P_3 = \frac{200\,(f-3)^2\,(f-6)}{3}$$

$$f2d := 3\,\mathrm{sech}(10\,x)^2 + 3$$

$$f3d := 1800\,\mathrm{sech}(10\,x)^4\,(\mathrm{sech}(10\,x) - 1)\,(\mathrm{sech}(10\,x) + 1) \qquad \textbf{(29)}$$

Compute then 50 frames of the static 3-D Hamiltonian map, where the argument x is parametrized by a virtual time t.

Figure 5. The animated 3-D Hamiltonian Map



Hint: use options FPS =1-3 (Frames Per Second), loop, and forward to show the math movie in detail.
In Figure 5, we display the animated 3-D Hamiltonian map of the aperiodic pulsation of elevation for

the conservative system (26) in a cubic field (3). The yellow curve shows the 3-D trajectory of the conservative system along the virtual potential, while its 2-D projection on the plane of variables [*x*, *f* (*x)*], which is displayed by the magenta curve, coincides with that in Figure 4. The conservative systems starts at the virtual time $x \Rightarrow -\infty$ on the top of the potential barrier located at the virtual coordinate $f = 3$, where its virtual velocity $df/dx \Rightarrow 0$. The dynamic system then hits at the virtual moment $x = 0$ he potential wall located at the virtual coordinate $f = 8$, where again its virtual velocity $df/dx = 0$. The pulsatory system finishes at the virtual time $x \Rightarrow +\infty$ on the top of the potential barrier located at the virtual coordinate $f = 3$, where its virtual velocity vanishes once more, $df/dx \Rightarrow 0$

3. Solving by the Expansion in Invariant Hyperbolic Structures

Secondly, we develop an application of the method of Decomposition in Invariant Structures (DIS) to solve the problem for the Newtonian ODE that is obtained by differentiation of the Hamiltonian ODE (14)

$$\left(\frac{d}{dz}\,h(z)\right)\left(600\,h(z)^2+\frac{d^2}{dz^2}\,h(z)-400\,h(z)\right)=0 \tag{30}$$

since a coefficient of the first derivative dh/dZ must vanish for Equation (30) to be satisfied for all virtual times z

In agreement with Figure 3, the forcing field

$$\frac{d}{dz}\,W_3(z)=0 \tag{31}$$

has extrema at $h = 0$ (a local maximum) and $h = \frac{2}{3}$ (a local minimum). Compared with the Hamiltonian ODE, the Newtonian ODE (31) is more robust since the order of algebraic nonlinearity is two, while the order of algebraic nonlinearity of the Hamiltonian ODE (9) is three. Since Equation (30) is ODE of the second order, the initial condition (10) should be complemented by a second condition, which is a condition on vanishing at infinity (an aperiodicity condition),

$$h(\infty)=0 \tag{32}$$

Substitution of the condition on vanishing at infinity in the Hamiltonian ODE (9) shows that the first derivative of h (z) also vanishes at infinity because

$$\left.\left(\frac{d}{dz}\,h(z)\right)\right|_{\{z=\infty\}}=0 \tag{33}$$

To solve the Newtonian ODE (31) subjected to conditions (10) and (32)-(33), we construct an invariant hyperbolic structure as the following expansion in even powers of tanh=(v z) = th with undetermined coefficients v and c_{2m}

$$h(z)=1+c_2\,th^2+c_4\,th^4+c_6\,th^6+c_8\,th^8+c_{10}\,th^{10}+c_{12}\,th^{12} \tag{34}$$

where m = 1, 2, ... M

Hint: display this and all following equations of the computational order M_c for debugging and then hide them to simplify the interactive worksheet. This and following equations of order M_s are used for tracing the algorithm and presenting the computed results. Primarily, debug equations of order M_s, then copy them, paste, replace s with c, and debug equations of order M_c.

We immediately verify that the initial condition (10) is satisfied identically as tanh(0) = 0

$$h(0)=1 \tag{35}$$

Similarly, the condition on vanishing at infinity expressed through undetermined coefficients c_{2m} becomes

$$0 = 1 + c_2 + c_4 + c_6 + c_8 + c_{10} + c_{12} \quad \textbf{(36)}$$

since tanh(∞) = 1

The second-order derivative of $h(z)$ is invariant since

$$\frac{d^2}{dz^2}\,h(z) = \left(2\,c_2 + \left(-8\,c_2 + 12\,c_4\right)\,th^2 + \left(6\,c_2 - 32\,c_4 + 30\,c_6\right)\,th^4 + \left(20\,c_4 - 72\,c_6 + 56\,c_8\right)\,th^6 + \left(42\,c_6 - 128\,c_8 + 90\,c_{10}\right)\,th^8 + \left(72\,c_8 - 200\,c_{10} + 132\,c_{12}\right)\,th^{10} + \left(110\,c_{10} - 288\,c_{12}\right)\,th^{12} + 156\,c_{12}\,th^{14}\right)\,\nu^2 \quad \textbf{(37)}$$

The second power of $h(z)$ is also invariant because

$$h(z)^2 = 1 + 2\,c_2\,th^2 + \left(c_2^2 + 2\,c_4\right)\,th^4 + \left(2\,c_2\,c_4 + 2\,c_6\right)\,th^6 + \left(2\,c_2\,c_6 + c_4^2 + 2\,c_8\right)\,th^8 + \left(2\,c_2\,c_8 + 2\,c_4\,c_6 + 2\,c_{10}\right)\,th^{10} + \left(2\,c_2\,c_{10} + 2\,c_4\,c_8 + c_6^2 + 2\,c_{12}\right)\,th^{12} + \left(2\,c_2\,c_{12} + 2\,c_4\,c_{10} + 2\,c_6\,c_8\right)\,th^{14} + \left(2\,c_4\,c_{12} + 2\,c_6\,c_{10} + c_8^2\right)\,th^{16} + \left(2\,c_6\,c_{12} + 2\,c_8\,c_{10}\right)\,th^{18} + \left(2\,c_8\,c_{12} + c_{10}^2\right)\,th^{20} + 2\,c_{10}\,c_{12}\,th^{22} + c_{12}^2\,th^{24} \quad \textbf{(38)}$$

Substitution of the second derivative (37) of $h(z)$, the second (38) and first (34) powers of $h(z)$ in the Newtonian ODE (30) and collecting powers of th gives

$$2\,c_2\,\nu^2 + 200 + \left(800\,c_2 + \left(-8\,c_2 + 12\,c_4\right)\,\nu^2\right)\,th^2 + \left(600\,c_2^2 + 800\,c_4 + \left(6\,c_2 - 32\,c_4 + 30\,c_6\right)\,\nu^2\right)\,th^4 + \left(1200\,c_2\,c_4 + 800\,c_6 + \left(20\,c_4 - 72\,c_6 + 56\,c_8\right)\,\nu^2\right)\,th^6 + \left(1200\,c_2\,c_6 + 600\,c_4^2 + 800\,c_8 + \left(42\,c_6 - 128\,c_8 + 90\,c_{10}\right)\,\nu^2\right)\,th^8 + \left(1200\,c_2\,c_8 + 1200\,c_4\,c_6 + 800\,c_{10} + \left(72\,c_8 - 200\,c_{10} + 132\,c_{12}\right)\,\nu^2\right)\,th^{10} + \left(1200\,c_2\,c_{10} + 1200\,c_4\,c_8 + 600\,c_6^2 + 800\,c_{12} + \left(110\,c_{10} - 288\,c_{12}\right)\,\nu^2\right)\,th^{12} + \left(156\,c_{12}\,\nu^2 + 1200\,c_2\,c_{12} + 1200\,c_4\,c_{10} + 1200\,c_6\,c_8\right)\,th^{14} + \left(1200\,c_4\,c_{12} + 1200\,c_6\,c_{10} + 600\,c_8^2\right)\,th^{16} + \left(1200\,c_6\,c_{12} + 1200\,c_8\,c_{10}\right)\,th^{18} + \left(1200\,c_8\,c_{12} + 600\,c_{10}^2\right)\,th^{20} + 1200\,c_{10}\,c_{12}\,th^{22} + 600\,c_{12}^2\,th^{24} = 0 \quad \textbf{(39)}$$

Thus, the Newtonian ODE is reduced to a Newtonian AE. For the Newtonian AE (39) to be satisfied for all virtual times, coefficients of th^{2m} should vanish. Vanishing coefficients of th^{2m} for $m = 0, 1, \ldots M-1$ we derive the following system of recurrent relation to compute v and $c_{2,\ldots}\,c_{2m}, \ldots c_{2M}$.

$$\left[2\,c_2\,\nu^2 + 200 = 0,\ 800\,c_2 + \left(-8\,c_2 + 12\,c_4\right)\,\nu^2 = 0,\ 600\,c_2^2 + 800\,c_4 + \left(6\,c_2 - 32\,c_4 + 30\,c_6\right)\,\nu^2 = 0,\ 1200\,c_2\,c_4 + 800\,c_6 + \left(20\,c_4 - 72\,c_6 + 56\,c_8\right)\,\nu^2 = 0,\ 1200\,c_2\,c_6 + 600\,c_4^2 + 800\,c_8 + \left(42\,c_6 - 128\,c_8 + 90\,c_{10}\right)\,\nu^2 = 0,\ 1200\,c_2\,c_8 + 1200\,c_4\,c_6 + 800\,c_{10} + \left(72\,c_8 - 200\,c_{10} + 132\,c_{12}\right)\,\nu^2 = 0\right] \quad \textbf{(40)}$$

All remaining terms of the Newtonian AE (39), which are not vanished by the system of recurrent relations (40) constitute a remainder of the hyperbolic polynomial approximation of $h(z)$ in th^{2m}

$$Ra = \left(1200\,c_2\,c_{10} + 1200\,c_4\,c_8 + 600\,c_6^2 + 800\,c_{12} + \left(110\,c_{10} - 288\,c_{12}\right)\,\nu^2\right)\,th^{12} + \left(156\,c_{12}\,\nu^2 \quad \textbf{(41)}$$

$$+ 1200\, c_2\, c_{12} + 1200\, c_4\, c_{10} + 1200\, c_6\, c_8 \big)\, th^{14} + \big(1200\, c_4\, c_{12} + 1200\, c_6\, c_{10} + 600\, c_8^2 \big)\, th^{16} + \big(1200\, c_6\, c_{12} + 1200\, c_8\, c_{10} \big)\, th^{18} + \big(1200\, c_8\, c_{12} + 600\, c_{10}^2 \big)\, th^{20} + 1200\, c_{10}\, c_{12}\, th^{22} + 600\, c_{12}^2\, th^{24}$$

As we shall see later, the remainder may become smaller any prescribed tolerance ε f the order of approximation $2M$ is large enough because of smallness of the structural coefficients c_{2m}

We then express recurrent solutions of the algebraic system (40) for structural coefficients c_{2m} through $N = \frac{1}{\nu}$ as follows:

$$\Big[c_2 = -100\, N^2,\ c_4 = \frac{20000}{3} N^4 - \frac{200}{3} N^2,\ c_6 = -\frac{3400000}{9} N^6 + \frac{80000}{9} N^4 - \frac{460}{9} N^2,\ c_8 = \frac{1240000000}{63} N^8 - \frac{6800000}{9} N^6 + \frac{88000}{9} N^4 - \frac{880}{21} N^2,\ c_{10} = -\frac{552800000000}{567} N^{10} + \frac{9920000000}{189} N^8 - \frac{29240000}{27} N^6 + \frac{5744000}{567} N^4 - \frac{2252}{63} N^2,\ c_{12} = \frac{873760000000000}{18711} N^{12} - \frac{5528000000000}{1701} N^{10} + \frac{52576000000}{567} N^8 - \frac{2312000000}{1701} N^6 + \frac{17405600}{1701} N^4 - \frac{65080}{2079} N^2 \Big] \tag{42}$$

To satisfy the condition on vanishing at infinity, we substitute the recurrent solution for $c_{2,\ \ldots}\ c_{2m},\ \ldots\ c_{2M}$ through N, collect the like terms, and compute an algebraic equation for N of order $2M$

$$0 = \frac{873760000000000}{18711} N^{12} - \frac{7186400000000}{1701} N^{10} + \frac{93496000000}{567} N^8 - \frac{6081920000}{1701} N^6 + \frac{77729600}{1701} N^4 - \frac{679276}{2079} N^2 + 1 \tag{43}$$

Computation of numerical solutions by procedure fsolve with the system accuracy provided by the global parameter Digits yields

$$[-0.14599877982356,\ -0.10000000000000,\ 0.10000000000000,\ 0.14599877982356]$$
$$[-0.17399997362755,\ -0.10000000000000,\ 0.10000000000000,\ 0.17399997362755] \tag{44}$$

Selecting a smallest positive root, we obtain the same solutions for in orders of approximation M_s and M_c

$$\nu = 10.000000000000$$
$$\nu = 10.000000000000 \tag{45}$$

which coincides with the exact solution

$$\nu = 10 \tag{46}$$

which was computed in the previous section.

Substitution of ν in the structural coefficients c_{2m} returns that all coefficients vanish, besides c_2, with the system accuracy

$$[c_2 = -1.0000000000000,\ c_4 = 0.,\ c_6 = 0.,\ c_8 = 0.,\ c_{10} = -2.\ 10^{-14},\ c_{12} = 3.\ 10^{-14}]$$
$$\big[c_2 = -1.0000000000000,\ c_4 = 0.,\ c_6 = 0.,\ c_8 = 0.,\ c_{10} = -2.\ 10^{-14},\ c_{12} = 3.\ 10^{-14},\ c_{14} = -3.\ 10^{-14},\ c_{16} = 5.\ 10^{-14},\ c_{18} = 4.\ 10^{-14},\ c_{20} = 0.,\ c_{22} = -6.\ 10^{-14},\ c_{24} = -8.\ 10^{-14},\ c_{26} \tag{47}$$

$= 1.4\ 10^{-13}, c_{28} = -1.0\ 10^{-13}, c_{30} = 2.\ 10^{-14}, c_{32} = -3.\ 10^{-14}, c_{34} = -4.\ 10^{-14}, c_{36}$
$= -3.3\ 10^{-13}, c_{38} = -6.1\ 10^{-13}, c_{40} = -5.8\ 10^{-13}, c_{42} = -3.5\ 10^{-13}, c_{44} = -9.\ 10^{-14}, c_{46}$
$= -1.3\ 10^{-13}, c_{48} = 8.0\ 10^{-13}, c_{50} = 1.7\ 10^{-13}, c_{52} = -5.4\ 10^{-13}]$

and the remainder of the hyperbolic polynomial approximation is negligible with the system accuracy as

$$Ra = -1.0360000000000\ 10^{-9}\ th^{12} + 4.3200000000000\ 10^{-10}\ th^{14} + 2.400\ 10^{-25}\ th^{20} - 7.200\ 10^{-25}\ th^{22} + 5.400\ 10^{-25}\ th^{24}$$

$$\begin{aligned} Ra = {} & 1.749600\ 10^{-22}\ th^{104} - 1.101600\ 10^{-22}\ th^{102} - 5.010600\ 10^{-22}\ th^{100} \\ & + 2.474400\ 10^{-22}\ th^{98} + 4.158000\ 10^{-22}\ th^{96} + 8.36400\ 10^{-23}\ th^{94} + 2.281800\ 10^{-22}\ th^{92} \\ & - 4.50000\ 10^{-23}\ th^{90} - 4.079400\ 10^{-22}\ th^{88} - 4.987200\ 10^{-22}\ th^{86} - 7.42200\ 10^{-23}\ th^{84} \\ & + 3.034800\ 10^{-22}\ th^{82} + 5.400000\ 10^{-22}\ th^{80} + 4.802400\ 10^{-22}\ th^{78} \\ & + 4.542600\ 10^{-22}\ th^{76} + 4.524000\ 10^{-22}\ th^{74} + 7.0200\ 10^{-24}\ th^{72} - 2.02800\ 10^{-23}\ th^{70} \\ & + 2.7600\ 10^{-24}\ th^{68} + 7.74000\ 10^{-23}\ th^{66} + 3.37800\ 10^{-23}\ th^{64} + 2.71200\ 10^{-23}\ th^{62} \\ & + 7.99200\ 10^{-23}\ th^{60} - 4.93200\ 10^{-23}\ th^{58} - 3.64800\ 10^{-23}\ th^{56} \\ & - 1.4817600000000\ 10^{-7}\ th^{54} + 3.3474600000000\ 10^{-7}\ th^{52} \end{aligned} \tag{48}$$

So, all structural coefficients vanish with the system accuracy besides c = -1. This result is also in complete agreement with the exact solution since using the hyperbolic Pythagorean identity yields

$$h(z) = \operatorname{sech}(10\ z)^2$$

$$\operatorname{sech}(14\ z)^2 = 1 - \tanh(14\ z)^2 \tag{49}$$

Therefore, the presented computation shows a fast convergence of the DIS method in the considered case of invariant hyperbolic structure in $\tanh(v\ z)^{2m}$.