MATH 416 Abstract Linear Algebra

Homework 3
Assigned: Fri. Sept. 12, 2025
Due: Fri. Sept. 19, 2025 (by 1pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (3 points): **Linear independence and span**

(i) (2 points) Let $z_1 = 1 + i$ and $z_2 = 1 - i$. First consider the complex numbers $\mathbb{C}$ as a vector space over the field $\mathbb{R}$, and show that $\{z_1, z_2\}$ is linearly independent over $\mathbb{R}$. Then consider $\mathbb{C}$ as a vector space over itself (i.e., $\mathbb{F} = \mathbb{C}$), and show that now $\{z_1, z_2\}$ is linearly dependent.

(ii) (1 point) Let $\{v_1, \ldots, v_m\}$ be a set of linearly independent vectors in V, and let $w \in V$. Show that, if $\{v_1 + w, \ldots, v_m + w\}$ are linearly dependent, then $w \in \langle v_1, \ldots v_m \rangle$.

Proof. (i) To show linear independence over $\mathbb{R}$ we must show that the only way to write 0 in terms of z_1, z_2 is to choose real numbers $a = b = 0$. We write

$$0 = az_1 + bz_2, \tag{1}$$
$$0 + 0i = a + ai + b - bi, \tag{2}$$
$$0 + 0i = a + b + (a - b)i. \tag{3}$$

Equating real and imaginary parts, this forces $a = -b$ and $a = b$, which has only one solution in the reals: $a = b = 0$.

If instead $a = b + di$ and $b = e + fi$ for real numbers b, d, e, f, then we have

$$0 = (b + di)(1 + i) + (e + fi)(1 - i), \tag{4}$$
$$0 + 0i = (b + bi + di - d) + (e - ei + fi + f), \tag{5}$$
$$0 + 0i = (b - d + e + f) + (b + d - e + f)i, \tag{6}$$

which forces $0 = (b - d + e + f)$ and $0 = (b + d - e + f)$. This implies

$$b - d + e + f = b + d - e + f \implies d = e. \tag{7}$$

Thus, any such scalars in the field will allow us to write 0 in terms of z_1, z_2, implying they are linearly dependent.

(ii) Suppose $\{v_1 + w, \ldots, v_m + w\}$ is linearly dependent. Then there exist scalars c_j, not all zero, such that

$$\sum_{j=1}^{m} c_j(v_j + w) = 0$$
$$\sum_{j=1}^{m} c_j v_j + \Big(\sum_{j=1}^{m} c_j\Big) w = 0.$$

If $\sum c_j = 0$, then $\sum c_j v_j = 0$, contradicting independence of $\{v_j\}$. Hence $\sum c_j \neq 0$, and

$$w = -\frac{1}{\sum c_j} \sum_{j=1}^{m} c_j v_j \in \langle v_1, \ldots, v_m \rangle.$$

$\square$

Exercise 2 (3 points): **Bases I**

(i) Let $\{u_1, u_2, u_3\}$ be the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Show that $\{u_1, u_2, u_3\}$ is not a basis of $\mathbb{R}^2$, but $\{u_i, u_j\}$ is a basis for any $1 \leq i < j \leq 3$.

(ii) Prove that the following set of vectors $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(iii) Prove that the following set of vectors $\{w_1, w_2, w_3\}$ does not form a basis of $\mathbb{R}^3$:

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

(i) To see that this set is linearly dependent, simply note that $u_3 = u_2 + u_1$. We can verify pairwise independence in the following manner. Consider writing $0 = au_1 + bu_2$ for

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \tag{8}$$

which implies $a = b = 0$. The same thing can be checked for the remaining combinations.

(ii) As above, we can check linear independence by showing the only way to make $0 = a_1v_1 + a_2v_2 + a_3v_3$ is by choosing $a_1 = a_2 = a_3 = 0$.

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_2 \end{pmatrix}. \tag{9}$$

Clearly, the last entry forces $a_2 = 0$ which, in turn, forces the other two coefficients to be zero. By (Axler 2.38) we know that a linearly independent list of the right length is a basis and we are done.

(iii) Observe that $w_3 = 2w_2 + 2w_1$. Constructing any such example suffices to show the list is linearly dependent.

Exercise 3 (3 points): **Bases II**
Let U be the subspace of $\mathbb{R}^5$ defined by[1]

$$U = \left\{ (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 \colon x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}$$

(i) Find a basis for U.

(ii) Extend the basis you found in (i) to a basis of $\mathbb{R}^5$.

(iii) Find a subspace $W \leq \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Solution.

(i) First, note that we can write this subspace as

$$U = \{(3x_2, x_2, 7x_4, x_4, x_5)^T \in \mathbb{R}^5 : x_2, x_4, x_5 \in \mathbb{R}\}. \tag{10}$$

Then, we can simply read off the basis by writing the general vector in the column picture. Let $u \in U$, then u is of the form

$$u = x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{pmatrix} + + x_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{11}$$

Thus, the set $\{(3,1,0,0,0)^T, (0,0,7,1,0)^T, (0,0,0,0,1)^T\}$ is a basis for U. This is *a* basis of U, but is not unique.

(ii) We need to choose two vectors that do not lie in the span of the three vectors given above. In this case, we can do this easily by inspection. Consider adding $(1,0,0,0,0)^T$ and $(0,0,1,0,0)^T$ to the list. By inspection, we see that there is no way to write either of these in terms of the other three. Again, this is not unique.

(iii) Inspired by our choice of basis above, we consider

$$W = \{(x_1, 0, x_3, 0, 0) : x_1, x_3 \in \mathbb{R}\}. \tag{12}$$

[1] Here, ^{T} denotes transposition, and $x = (x_1, x_2, x_3, x_4, x_5)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.

Via Lemma in Axler, we know that $U+W$ is a direct sum decomposition of V iff $U \cap W = \{0\}$. This can be verified simply by setting

$$(3x_2, x_2, 7x_4, x_4, x_5)^T = (x_1, 0, x_3, 0, 0)^T. \tag{13}$$

It is then clear that the only vector satisfying this equality is the zero vector, and we are done.

Exercise 4 (3 points): **Dimension I**
Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, all lines in $\mathbb{R}^3$ containing the origin, all planes in $\mathbb{R}^3$ containing the origin, and $\mathbb{R}^3$.

Proof. We know that $\dim \mathbb{R}^3$ has dimension 3, thus by (Axler 2.37) any subspace of $\mathbb{R}^3$ must have dimension 0,1,2, or 3. We have

0. There is only one vector space with dimension 0: the trivial vector space $\{0\}$.
1. If $\dim U = 1$, we know by (Axler 2.31) there exists a basis and it must contain one vector. Such a vector space must go through the origin, thus it can only be a line through the origin.
2. If $\dim U = 2$, we know that there exists a basis with two linearly independent vectors u_1, u_2. Thus, $U = \text{span} u_1, u_2$ which is a plane through the origin.
3. If $\dim U = 3$, $U = V$ by (Axler 2.39).

□

Exercise 5 (4 points): **Dimension II**
Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite dimensional and

$$\dim (V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

Let us first prove this by induction on m. Crucial will be the use of (Axler 2.43):

$$\dim (V_1 + V_2) = \dim V_1 + \dim V_2 - \dim (V_1 \cap V_2) \tag{14}$$

for subspaces $V_1, V_2 \leq V$.

Proof. Consider the base case $m = 1$, we have $\dim V_1 = \dim V_1$. Our induction hypothesis is that, for some $m > 1$,

$$\dim\left(V_1 + \cdots + V_{m-1}\right) \leq \dim V_1 + \cdots + \dim V_{m-1}. \tag{15}$$

We must then show that the case with m subspaces holds. To make this rigorous, recall from Axler 1.40 that the sum of subspaces is itself a subspace. Then, we have

$$\begin{aligned}
&\dim\left(V_1 + \cdots + V_{m-1} + V_m\right) = \dim\left(\left(V_1 + \cdots + V_{m-1}\right) + V_m\right), \\
&\leq \dim\left(V_1 + \cdots + V_{m-1}\right) + \dim V_m - \dim\left(\left(V_1 + \cdots + V_{m-1}\right)\right) \cap V_m), \\
&\leq \dim\left(V_1 + \cdots + V_{m-1}\right) + \dim V_m, \\
&\leq \dim V_1 + \cdots + \dim V_{m-1} + \dim V_m,
\end{aligned}$$

where the first inequality is by 2.43 in Axler; the second is that the the dimension of the intersection is non-negative, so dropping it cannot decrease the sum; and the final inequality follows by the induction hypothesis. This completes the proof. □

For fun (we are having fun, right?) let us also prove this directly.

Proof. For each $i \in [m]$, let B_i be a basis for V_i, which we know exists because V_i is finite dimensional (Axler 2.31). Then, consider the union of these bases

$$S := B_1 \cup \cdots \cup B_m. \tag{16}$$

This set of vectors spans $V_1 + \cdots + V_m$ because every vector in this sum can be expressed as a linear combination of vectors from each of the subspaces (not necessarily uniquely, thus it need not be a basis). Because each B_i is finite, their union is also finite, thus $V_1 + \cdots + V_m$ is finite dimensional. We know that the length of basis is at most the length of a spanning set, thus

$$\begin{aligned}
\dim\left(V_1 + \cdots + V_m\right) &\leq |S|, && (17) \\
&= |B_1 \cup \cdots \cup B_m|, && (18) \\
&\leq \sum_{i=1}^{m} |B_i|, && (19) \\
&= \dim V_1 + \cdots + \dim V_m, && (20)
\end{aligned}$$

as desired. Note that the second inequality holds because the size of a union of finite sets is at most the sum of their sizes. □

Remark. In probability theory, this second inequality is called the *union bound* or *Boole's inequality* after the famous Irish mathematician George Boole. He was largely self-taught and yet developed a great deal of original mathematics including Boolean algebra, which now underlies all of our modern technology.

Exercise 6 (4 points): **Dimension III**

Suppose V is finite dimensional, with $\dim V = n \geq 1$. Prove that there exists one-dimensional subspaces $V_1, \ldots, V_m$ of V such that

$$V = V_1 \oplus \cdots \oplus V_m.$$

Proof. Let $v_1, \ldots, v_m$ be a basis for V. Then, to construct such one-dimensional subspaces, just take $V_k = \operatorname{span}(v_k)$ for all $k \in [m]$. Because $v_1, \ldots, v_m$ is a basis for V, every $v \in V$ can be expressed *uniquely* as

$$v = \sum_{k=1}^{m} a_k v_k, \tag{21}$$

where each $v_i \in V_k$. By the definition of a direct sum (Axler 1.41), this implies that $V = V_1 \oplus \cdots \oplus V_m$, as desired. $\square$