MATH 416 Abstract Linear Algebra

Week 6 - Homework 5

Assigned: Fri. Oct. 3, 2025

Due: Fri. Oct. 10, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (10 points): **Rank of a Matrix**

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(i) (3 points). Compute the column rank and the row rank of A by finding maximal linearly independent sets of columns and of rows.

Solution. Write the columns of A as

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Observe that

$$c_3 = c_1 + c_2,$$

so c_3 is a linear combination of c_1 and c_2. Thus the set $\{c_1, c_2, c_3\}$ has at most two independent vectors, and $\{c_1, c_2\}$ is linearly independent because

$$\alpha c_1 + \beta c_2 = 0 \implies \alpha = \beta = 0.$$

Hence a maximal linearly independent set of columns is $\{c_1, c_2\}$ and the *column rank* is 2. The rows of A are

$$r_1 = (1, 0, 1), \quad r_2 = (0, 1, 1), \quad r_3 = (1, 1, 2),$$

and we similarly have $r_3 = r_1 + r_2$. The rows r_1, r_2 are linearly independent (their first two coordinates show this), so a maximal linearly independent set of rows is $\{r_1, r_2\}$ and the *row rank* is 2. In particular $\text{rank}(A) = 2$.

(ii) (3 points). Prove the following proposition (the *column–row factorization*): If the column rank of a matrix $A \in M_{m\times n}(\mathbb{F})$ is r, then there exist matrices

$$C \in M_{m\times r}(\mathbb{F}), \qquad R \in M_{r\times n}(\mathbb{F})$$

such that

$$A = CR.$$

Hint: let the columns of C be a maximal linearly independent set of columns of A, and argue that every column of A is a linear combination of these.

Proof. Write $A = [a_1 \ \cdots \ a_n]$ where each $a_j \in \mathbb{F}^m$ is a column of A. Because the column rank of A is r, there is a maximal linearly independent subset of columns

$$\{c_1, \ldots, c_r\} \subset \{a_1, \ldots, a_n\}.$$

Form the matrix $C \in M_{m\times r}(\mathbb{F})$ whose columns are these vectors:

$$C = [\, c_1 \ \cdots \ c_r \,].$$

Then, every column a_j of A can be written as a linear combination of the c_i's:

$$a_j = \sum_{i=1}^{r} x_{ij}\, c_i \qquad (x_{ij} \in \mathbb{F}).$$

For each $j = 1, \ldots, n$ collect the coefficients $x_{1j}, \ldots, x_{rj}$ into the column vector $x_j \in \mathbb{F}^r$. Now form the matrix $R \in M_{r\times n}(\mathbb{F})$ whose j-th column is x_j:

$$R = [\, x_1 \ \cdots \ x_n \,].$$

With these definitions, the j-th column of the product CR is

$$Cx_j = \sum_{i=1}^{r} x_{ij}\, c_i = a_j,$$

so every column of CR equals the corresponding column of A. Hence $A = CR$, as required. $\square$

(iii) (1 point). Find C and R such that their product yields the A from part (i).

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

The first two columns of A are linearly independent, and the third column is their sum. Hence the column rank is $r = 2$.
We take

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then

$$CR = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} = A.$$

Thus the column–row factorization (CR decomposition) of A is

$$\boxed{A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{R}.}$$

(iv) (3 points). Use the factorization in (ii) to prove that the column rank of any matrix equals its row rank. *Hint: Take transposes and interpret column rank of A^T as row rank of A.*

Proof. Let $A \in M_{m\times n}(\mathbb{F})$ and suppose its column rank is r. By the column–row factorization from part (ii), there exist matrices

$$C \in M_{m\times r}(\mathbb{F}) \quad \text{and} \quad R \in M_{r\times n}(\mathbb{F})$$

such that

$$A = CR.$$

Now take transposes:

$$A^T = R^T C^T.$$

Here $A^T \in M_{n\times m}(\mathbb{F})$, and by the same factorization argument, the column space of A^T is contained in the span of the r columns of R^T, so the column rank of A^T is at

most r. Thus
$$\text{column rank}(A^T) \le r = \text{column rank}(A).$$
Applying the same reasoning to A^T (which yields $A = (A^T)^T$), we obtain the reverse inequality
$$\text{column rank}(A) \le \text{column rank}(A^T).$$
Therefore,
$$\text{column rank}(A) = \text{column rank}(A^T).$$
Finally, note that the column rank of A^T is by definition the row rank of A, since the rows of A are the columns of A^T. Hence
$$\boxed{\text{column rank}(A) = \text{row rank}(A)}.$$
$\square$

Exercise 2 (10 points): **Invertibility, Isomorphisms, and Basis Change**

(i) (2 points). Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. Let $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ be invertible. Then $T^{-1} \in \mathcal{L}(V, U)$ and $S^{-1} \in \mathcal{L}(W, V)$ exist. Consider the composition
$$T^{-1}S^{-1} : W \xrightarrow{S^{-1}} V \xrightarrow{T^{-1}} U.$$
We claim that $T^{-1}S^{-1}$ is the inverse of ST. First, compute
$$(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = SI_VS^{-1} = SS^{-1} = I_W.$$
Similarly,
$$(T^{-1}S^{-1})(ST) = T^{-1}(S^{-1}S)T = T^{-1}I_VT = T^{-1}T = I_U.$$
Hence $T^{-1}S^{-1}$ is both a left- and right-inverse of ST, and therefore
$$(ST)^{-1} = T^{-1}S^{-1}.$$
In particular, ST is invertible. $\square$

(ii) (4 points). Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \Leftrightarrow S, T \text{ are both invertible.}$$

Proof. First, if S and T are invertible then, as in the previous exercise,

$$(ST)^{-1} = T^{-1}S^{-1},$$

so ST is invertible. Conversely, assume ST is invertible. Then, ST is injective and surjective. We now show T and S are invertible.

Step 1: T is injective. If $v \in \ker T$ then $Tv = 0$, hence

$$(ST)v = S(Tv) = S0 = 0,$$

so $v \in \ker(ST)$. Because ST is injective, $\ker(ST) = \{0\}$, therefore $\ker T = \{0\}$ and T is injective. In a finite-dimensional vector space an injective linear map is also surjective, so T is invertible.

Step 2: S is surjective. Since ST is surjective, for every $w \in V$ there exists $v \in V$ with

$$(ST)v = w,$$

so $w = S(Tv) \in \operatorname{range}(S)$. Hence $\operatorname{range}(S) = V$, i.e. S is surjective. Again using finite-dimensionality, a surjective linear map is injective, so S is invertible. Therefore S and T are both invertible. This completes the proof of the equivalence. $\square$

(iii) (4 points). Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity.

($\Rightarrow$) First, we assume that if T has the same matrix with respect to every basis of V and proceed to show that T must be a scalar multiple of the identity.
Let A, B be two matrices corresponding to $T \in \mathcal{L}(V)$. Moreover, let C be the identity matrix with input basis the same as A and output the same as B. Then, by Axler 3.84, we have that a basis transformation is enacted via the so-called *similarity transform*

$$A = C^{-1}BC. \tag{1}$$

Thus, if we assume T has the same matrix before and after such a transformation, we may write

$$A = C^{-1}AC \tag{2}$$

for invertible C. This is equivalent to $CA = AC$. Now, it can be shown (and you can assume for this problem) that if A commutes with all invertible matrices, it commutes with all matrices (i.e. $AM = MA$ for all M).

To proceed, denote the components of A as $A = (a_{pq})$ and let E_{ij} denote the matrix unit with a 1 in position (i, j) and zeros elsewhere. Since the matrices E_{ij} form a basis of $M_n(\mathbb{F})$, it suffices to assume that $AE_{ij} = E_{ij}A$ for all i, j and deduce that $A = \lambda I$. Compute the (k, ℓ)-entry of both sides of $AE_{ij} = E_{ij}A$. We have

$$(AE_{ij})_{k\ell} = a_{ki}\,\delta_{j\ell} \qquad \text{and} \qquad (E_{ij}A)_{k\ell} = \delta_{ki}\,a_{j\ell},$$

Where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. By assumption, these matrices commute, so we may equate them to obtain

$$a_{ki}\,\delta_{j\ell} = \delta_{ki}\,a_{j\ell} \qquad \text{for all } k, \ell.$$

We may interpret this equation as follows:

- If $k \neq i$ and $\ell = j$, then the left side is a_{ki} and the right side is 0, so $a_{ki} = 0$. Hence all off-diagonal entries of A vanish, and A is diagonal.
- If $k = i$ and $\ell = j$, then the equation reads $a_{ii} = a_{jj}$. Therefore, all diagonal entries are equal; denote this common value by λ.

Thus $A = \lambda I$, as desired. We will revisit this problem after (or during) Chapter 5. At that point, I hope it will make more sense!
($\Leftarrow$) Fortunately, the reverse direction is much easier. If we assume $A = \lambda I$, for $\lambda \in \mathbb{F}$, then

$$\begin{aligned} C^{-1}AC &= C^{-1}(\lambda I)C, && (3)\\ &= \lambda C^{-1}C, && (4)\\ &= \lambda I, && (5)\\ &= A, && (6) \end{aligned}$$

which tells us the somewhat obvious fact that if A is a scalar multiple of the identity then it commutes with all scalars.

(optional) Bonus Question (2 points): **Rank-1 Decomposition**

Prove that every matrix $A \in M_{m\times n}(\mathbb{F})$ of rank r can be written as a sum of r matrices of rank 1. That is, show that there exist column vectors $u_1, \dots, u_r \in \mathbb{F}^{m\times 1}$ and row vectors $v_1^T, \dots, v_r^T \in \mathbb{F}^{1\times n}$ such that

$$A = u_1 v_1^T + u_2 v_2^T + \cdots + u_r v_r^T.$$

Motivation. This statement shows that we can build up matrices in terms of "rank-1 outer products" (i.e. a column times a row vector) building blocks. Later, we will see that the so-called *Singular Value Decomposition (SVD)* is a refinement of this idea: it expresses any matrix as a sum of rank-1 outer products, but with the additional structure that the vectors form orthonormal bases and the coefficients are nonnegative singular values. The SVD has far-reaching consequences for data science, machine learning, entanglement theory, and many more fields.

Proof. Let $A \in M_{m\times n}(\mathbb{F})$ have rank r. By the column–row factorization (CR decomposition) there exist

$$C \in M_{m\times r}(\mathbb{F}), \qquad R \in M_{r\times n}(\mathbb{F})$$

such that $A = CR$. Write the columns of C and the rows of R as

$$C = [\, u_1 \ \cdots \ u_r \,], \qquad R = \begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix},$$

where each $u_i \in \mathbb{F}^{m\times 1}$ and each $v_i^T \in \mathbb{F}^{1\times n}$.

Now multiply C and R. Using the usual column–row expansion of a product we get

$$A = CR = \sum_{i=1}^{r} u_i v_i^T.$$

Each summand $u_i v_i^T$ is an outer product (a rank–at-most–one matrix). Moreover, because the r columns of C may be chosen to be a maximal independent set of columns of A, each u_i is nonzero, and likewise the corresponding v_i^T are nonzero; therefore each $u_i v_i^T$ has rank exactly 1. Hence A is written as a sum of r rank-1 matrices, as required. □