MATH 416 Abstract Linear Algebra

Week 10 - Homework 8
Assigned: Fri. Oct. 31, 2025
Due: Fri. Nov. 7, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (5 points): Orthogonal Bases
Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V.

(a) (3 points) Prove that if $v_1, \ldots, v_n$ are vectors in V such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}} \tag{1}$$

for each k, then $v_1, \ldots, v_n$ is a basis of V.

Proof. We know that $\dim V = n$ because $e_1, \ldots, e_n$ is a basis. Thus, we only need to show that $v_1, \ldots, v_n$ is linearly independent to conclude that it is a basis. Suppose $\sum_{k=1}^n a_k v_k = 0$. Then, consider

$$\begin{aligned}
\|\sum_{k=1}^n a_k e_k\| &= \|\sum_{k=1}^n a_k e_k - a_k v_k\|, && (2)\\
&\le \sum_{k=1}^n |a_k| \|e_k - v_k\|, & \text{triangle inequality} \quad & (3)\\
&\le \frac{1}{\sqrt{n}} \sum_{k=1}^n |a_k|, & \text{by assumption} \quad & (4)\\
&\le \frac{1}{\sqrt{n}} \left(\sum_{k=1}^n |a_k|^2\right)^{1/2} \left(\sum_{k=1}^n 1^2\right)^{1/2} & \text{Cauchy-Schwarz} \quad & (5)\\
&= \left(\sum_{k=1}^n |a_k|^2\right)^{1/2}, && (6)\\
&= \|\sum_{k=1}^n a_k e_k\|. && (7)
\end{aligned}$$

The only way these inequalities can lead to an equality is if they were all saturated. In particular, the second and third line become

$$\sum_{k=1}^{n} |a_k| \|e_k - v_k\| = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} |a_k|, \tag{8}$$

which would imply $\|e_k - v_k\| = \frac{1}{\sqrt{n}}$ for all k if all $a_k \neq 0$. However, we assumed that $\|e_k - v_k\| < \frac{1}{\sqrt{n}}$, thus, $a_k = 0$ for all k. We may then conclude that $v_1, \ldots, v_n$ is a linearly independent list of the right length (and thus a basis). $\square$

This problem shows how powerful Cauchy-Schwarz is. The trick was to apply it to $\sum_{k=1}^{n} |a_k|$ by viewing this sum as an inner product between $u = (|a_1|, \ldots, |a_n|)$ and $w = (1, \ldots, 1)$.

(b) Show that there exist $v_1, \ldots, v_n \in V$ such that

$$\|e_k - v_k\| \leq \frac{1}{\sqrt{n}} \tag{9}$$

for each k, but $v_1, \ldots, v_n$ is *not* linearly independent.

Note: The first part of this exercise shows that if we perturb an orthonormal basis an appropriate amount, we still have a basis. The second part shows that we can't increase the $1/\sqrt{n}$.

In part a), we used the strict inequality condition to force all of the coefficients in the decomposition to be zero. If, instead, we have an inequality, then this line breaks down.

Proof. To show formally that there exists $v_1, \ldots, v_n$ satisfying the above inequality that is linearly *dependent*, consider (for each $k = 1, \ldots, n$) vectors of the form

$$v_k = e_k - \frac{1}{n} \sum_{j=1}^{n} e_j. \tag{10}$$

Then, we have

$$\|v_k - e_k\| = \frac{1}{n} \sqrt{\langle \sum_{j=1}^{n} e_j, \sum_{l=1}^{n} e_l \rangle}, \tag{11}$$

$$= \frac{1}{\sqrt{n}}. \tag{12}$$

However, note that

$$\sum_{k=1}^{n} v_k = \sum_{k=1}^{n} \left(e_k - \frac{1}{n} \sum_{j=1}^{n} e_j \right), \tag{13}$$

$$= \sum_{k=1}^{n} e_k - \sum_{j=1}^{n} e_j, \tag{14}$$

$$= 0. \tag{15}$$

Thus, $v_1, \ldots, v_n$ is linearly dependent. □

Exercise 2 (5 points): Inner products and orthogonal complements

(i) (3 points) Let $U \leq \mathbb{R}^4$ be the subspace spanned by the vectors $v_1 = (1,2,3,-4)^T$ and $v_2 = (-5,4,3,2)^T$. Find orthonormal bases for U and its orthogonal complement $U^\perp$ for the standard inner product $\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$.

Solution. Let $v_1 = (1,2,3,-4)^T$ and $v_2 = (-5,4,3,2)^T$. We apply Gram–Schmidt with the standard inner product.

First,

$$\|v_1\|^2 = 1+4+9+16 = 30 \quad \Rightarrow \quad u_1 = \frac{1}{\sqrt{30}}(1,2,3,-4)^T.$$

Next,

$$\langle v_2, u_1 \rangle = \frac{v_2 \cdot v_1}{\sqrt{30}} = \frac{4}{\sqrt{30}},$$

so

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1 = v_2 - \frac{4}{30}v_1 = \frac{1}{15}(-77,56,39,38)^T,$$

and

$$\|w_2\|^2 = \frac{802}{15} \quad \Rightarrow \quad u_2 = \frac{1}{\sqrt{12030}}(-77,56,39,38)^T.$$

Thus, an orthonormal basis for U is

$$\boxed{\left\{ \frac{1}{\sqrt{30}}(1,2,3,-4)^T,\ \frac{1}{\sqrt{12030}}(-77,56,39,38)^T \right\}}.$$

To find $U^\perp$, solve $x \cdot v_1 = x \cdot v_2 = 0$. One finds two independent solutions, for example

$$a = (-3,-9,7,0)^T, \qquad b = (10,9,0,7)^T.$$

Normalizing a gives

$$w_1 = \frac{1}{\sqrt{139}}(-3,-9,7,0)^T.$$

Orthogonalizing and normalizing b yields

$$w_2 = \frac{1}{\sqrt{2{,}731{,}211}}(1057,252,777,973)^T.$$

Thus, an orthonormal basis for $U^\perp$ is

$$\boxed{\left\{ \frac{1}{\sqrt{139}}(-3,-9,7,0)^T,\ \frac{1}{\sqrt{2{,}731{,}211}}(1057,252,777,973)^T \right\}}.$$

(ii) (2 points) Consider the following inner product on $\mathbb{R}^3$: $\langle x, y\rangle_{\text{alt}} := 2x_1y_1 + x_2y_2 + 2x_3y_3$. Compute $\{v\}^\perp$ for the vector $v = (1, -2, 1)^T \in \mathbb{R}^3$.

Solution. For $\langle x, y\rangle_{\text{alt}} = 2x_1y_1 + x_2y_2 + 2x_3y_3$ and $v = (1, -2, 1)^T$,

$$\langle x, v\rangle_{\text{alt}} = 2x_1 - 2x_2 + 2x_3 = 0 \quad \Longleftrightarrow \quad x_1 - x_2 + x_3 = 0.$$

Hence

$$\boxed{\{v\}^\perp = \operatorname{span}\{(1, 1, 0)^T,\ (-1, 0, 1)^T\}.}$$

An $\langle\cdot,\cdot\rangle_{\text{alt}}$-orthonormal basis may be obtained by Gram–Schmidt:

$$e_1 = \tfrac{1}{\sqrt{3}}(1, 1, 0)^T, \qquad e_2 = \tfrac{1}{2\sqrt{6}}(-1, 2, 3)^T.$$

Warning: If you use the Gram-Schmidt (GS) procedure for this example, then you need to use the inner product $\langle x, y\rangle_{\text{alt}}$ and the associated norm $\|x\|_{\text{alt}} := \sqrt{\langle x, x\rangle_{\text{alt}}}$ in the GS-formulas.

Exercise 3 (5 points): Orthogonal projections

(i) (2 points) Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0 \Leftrightarrow \|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.

Proof. ($\Rightarrow$) Suppose $\langle u, v\rangle = 0$. Then, by Pythagorean theorem, we have

$$\begin{aligned} \|u + av\|^2 &= \|u\|^2 + \|av\|^2, &(16)\\ &\geq \|u\|^2, &(17)\\ \implies \|u + av\| &\geq \|u\|, &(18) \end{aligned}$$

as desired.

($\Leftarrow$) Assume $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$. Squaring gives

$$\|u + av\|^2 - \|u\|^2 = 2\Re(\bar{a}\langle u, v\rangle) + |a|^2\|v\|^2 \geq 0 \quad \text{for all } a.$$

If $v = 0$ we are done. Otherwise set

$$a = -\frac{\langle u, v\rangle}{\|v\|^2}.$$

Then

$$0 \leq \|u + av\|^2 - \|u\|^2 = 2\Re\left(-\frac{\overline{\langle u, v\rangle}\langle u, v\rangle}{\|v\|^2}\right) + \frac{|\langle u, v\rangle|^2}{\|v\|^2} = -\frac{|\langle u, v\rangle|^2}{\|v\|^2}.$$

Hence $|\langle u, v\rangle|^2 \leq 0$, so $\langle u, v\rangle = 0$.
Therefore $\langle u, v\rangle = 0 \iff \|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$. □

(ii) (1 point) Let $U \leq V$ be a subspace of a finite-dimensional inner product space V. Show that $P_{U^\perp} = I_V - P_U$.

Proof. Because V is finite-dimensional and $U \leq V$ is a subspace, every $v \in V$ has a unique orthogonal decomposition

$$v = u + w, \qquad u \in U,\ w \in U^\perp.$$

By definition of orthogonal projection,

$$P_U v = u \quad \text{and} \quad P_{U^\perp} v = w.$$

Hence

$$(I_V - P_U)v = v - P_U v = (u + w) - u = w = P_{U^\perp} v$$

for all $v \in V$. Therefore $P_{U^\perp} = I_V - P_U$. $\square$

(iii) (2 points) Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $\|Pv\| \leq \|v\|$ for every $v \in V$. Prove that there exists a subspace U of V such that $P = P_U$.

Proof. Let $U := \operatorname{range}(P)$. We will show $\operatorname{range}(P) \perp \operatorname{null}(P)$, hence $V = U \oplus U^\perp$ and $P = P_U$. To show that $\operatorname{range}(P) \perp \operatorname{null} P$, first take $u \in \operatorname{range} P$ and $w \in \operatorname{null} P$. Then $u = Pv$ for some v, and $Pw = 0$. For any $a \in \mathbb{F}$,

$$\|u\| = \|P(u + aw)\| \leq \|u + aw\| \qquad (\text{by } \|Pz\| \leq \|z\|).$$

By part (a), $\langle u, w \rangle = 0 \iff \|u\| \leq \|u + aw\| \ \forall a$, and we may conclude $\langle u, w \rangle = 0$. Thus $\operatorname{range} P \perp \operatorname{null} P$. Now for any $v \in V$,

$$v = Pv + (v - Pv),$$

with $Pv \in U$ and $P(v - Pv) = Pv - P^2 v = 0$, so $v - Pv \in \operatorname{null} P \subset U^\perp$. Hence $V = U \oplus U^\perp$. Finally, P restricted to U acts as the identity on U (i.e. if $u \in U$, then $u = Pw$ for some w and $Pu = P^2 w = u$) and P vanishes on $U^\perp$ (since $U^\perp \subset \operatorname{null} P$). Therefore P is the orthogonal projection onto U, i.e.

$$P = P_U.$$

$\square$