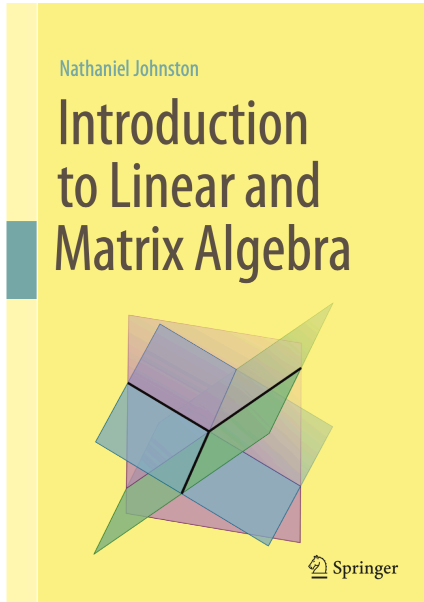
Math 416: Abstract Linear Algebra

Date: August 29, 2025

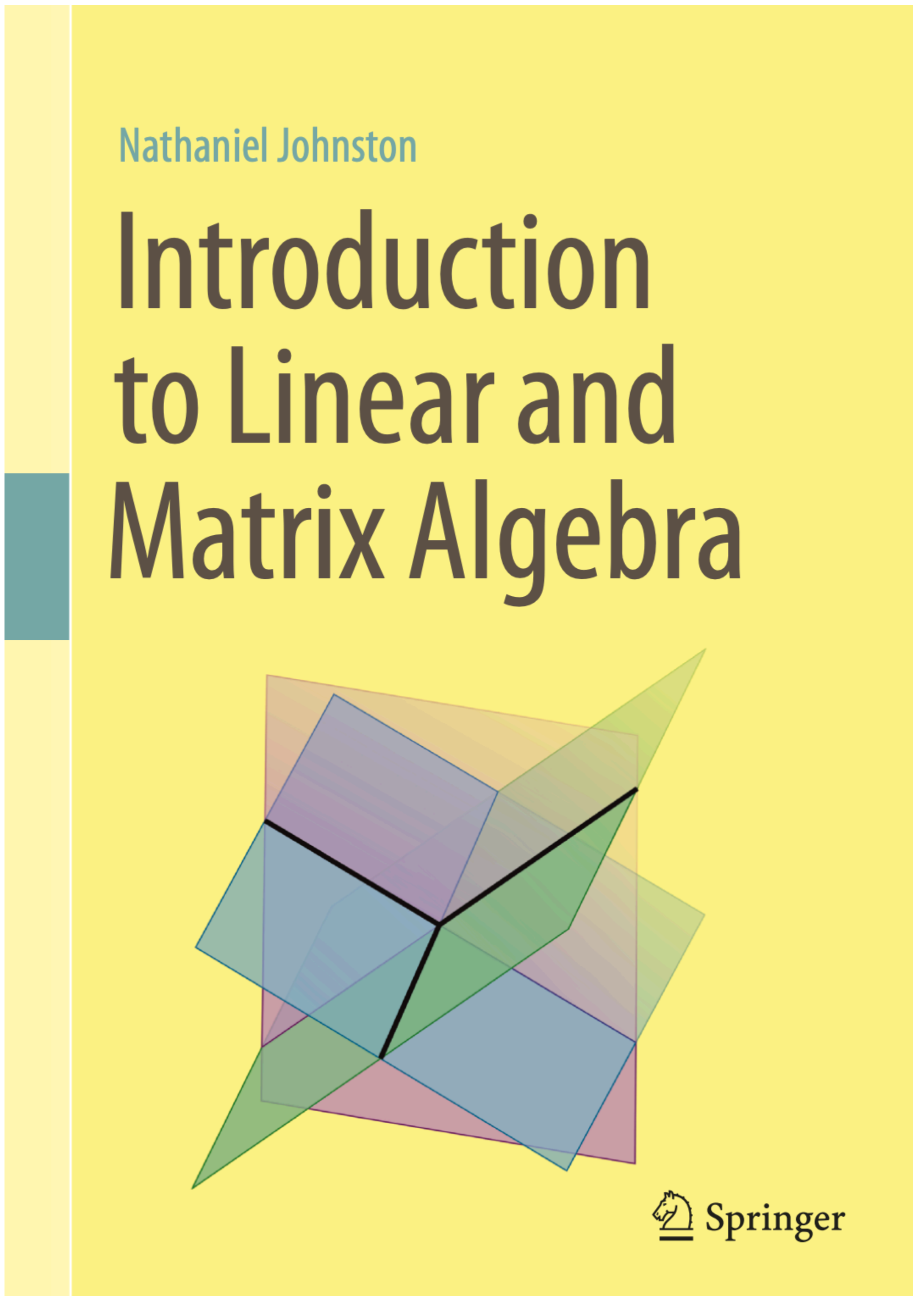
Lecture: 3



Announcements

- ☐ Pre-lecture quiz was due at start of class
- ☐ Intro to proofs tutorial is on Canvas (not graded)
- ☐ HW 1 is now live. Due 9/5
- ☐ Updated office hours:
  - Tuesdays 5-5:50 Davenport 212
  - Wednesdays 2-2:50 Davenport 132

Accessible using your U of I credentials



This & 3blue1brown will set you up for success on HW1.

Last time

Row picture

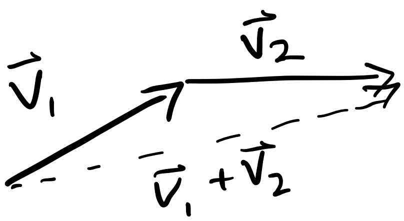
$$\begin{aligned} 2x + 4y &= 5 \\ -1x + 6y &= 2 \end{aligned}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Column picture

$$\underbrace{x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 4 \\ 6 \end{bmatrix}}_{\text{linear combination}} = \underbrace{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}_{\text{lies in span}\left\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \end{bmatrix}\right\}}$$

Vector addition



The Matrix equation

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\quad & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}\right\} \longrightarrow A\vec{x} = \vec{b}$$

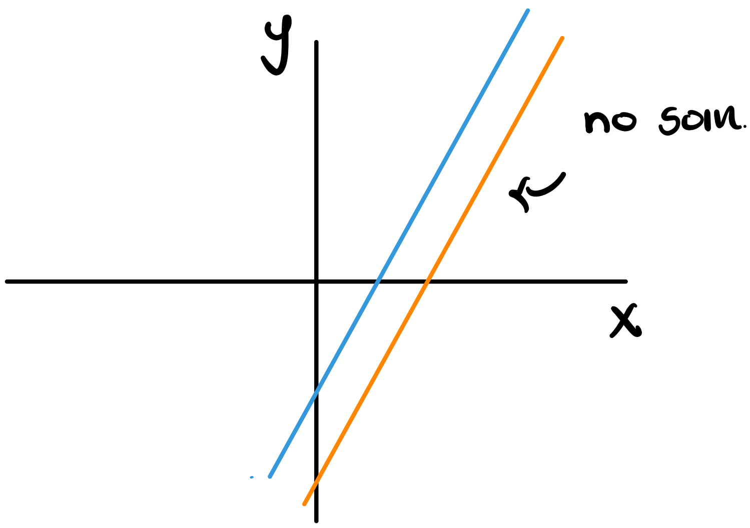
This time

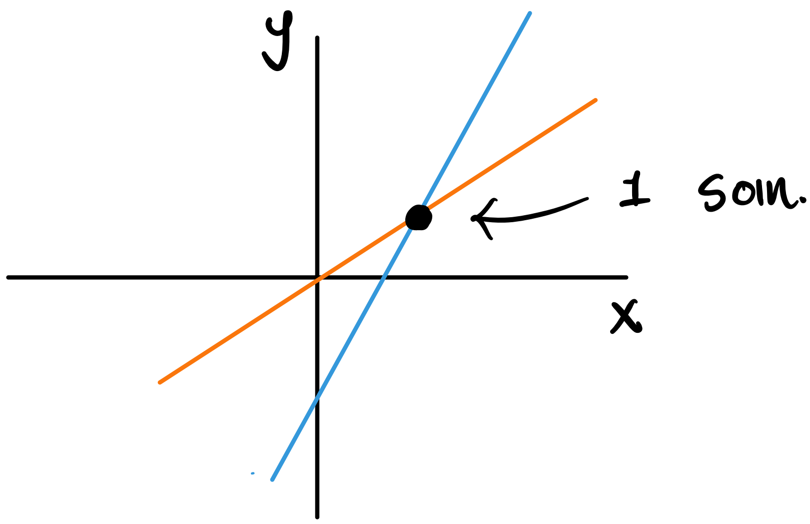
- ☐ Trichotomy for linear systems
- ☐ Row echelon form
- ☐ Gaussian elimination

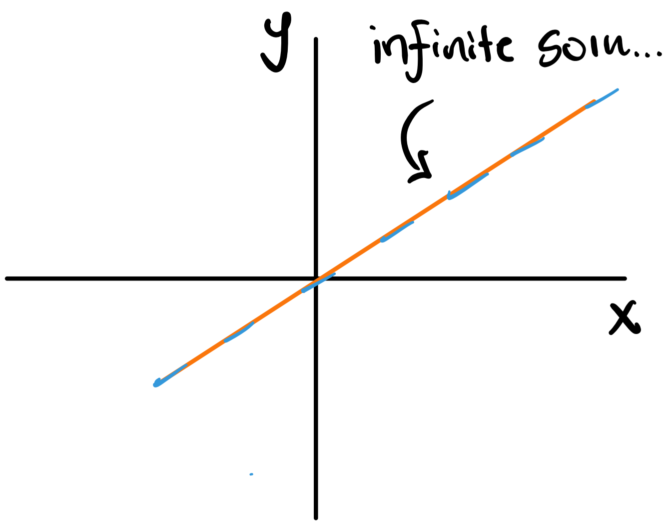
Trichotomy of linear systems

Simple question: can a linear system ever have exactly 2 solutions?

↳ Check smallest case!







Intuitively, we see that lines can only intersect zero, one, or infinitely many times!

Trichotomy of linear systems (our first proof)

Theorem. (trichotomy of linear systems)

Every system of linear equations has either zero, one, or infinitely many solutions.

Proof. We will prove this by contradiction.

Logically, we either have

- 0 soln, 1 soln, infinitely many soln,

OR

- m soln., where $m \in \mathbb{N}$ s.t. $1 < m < \infty$

Assume for the sake of contradiction that $\exists$ finitely many soln. $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$. Without loss of generality (w.l.o.g), consider $\vec{x}_1$ & $\vec{x}_2$. Let $\vec{x}' := c\vec{x}_1 + (1-c)\vec{x}_2$, $c \in \mathbb{R}$

$$\begin{aligned} A\vec{x}' = A(c\vec{x}_1 + (1-c)\vec{x}_2) &= cA\vec{x}_1 + (1-c)A\vec{x}_2 \\ &= c\vec{b} + (1-c)\vec{b} \\ &= \vec{b} \end{aligned}$$

"therefore"

$\therefore$ $\exists$ infinitely many soln. a contradiction! We conclude that there can only be 0, 1, or ∞ soln. $\square$

Row Echelon Form

Clicker Question

What is the solution (if one exists) of

$$\begin{aligned} x + 3y - 2z &= 5 \\ 2y - 6z &= 4 \\ 3z &= 6 \end{aligned}$$

a) There are no solutions!

$\boxed{\text{b) } (-15, 8, 2)}$

c) $(15, -8, 2)$

d) There are infinitely many solutions!

Row Echelon Form

Why was that so easy?

↳ The system was already in **triangular form** so we could easily solve by **back-substitution**

Group work

Put the following system into triangular form

$$x + 3y - 2z = 5 \qquad (1)$$

$$x + 5y - 8z = 9 \qquad (2)$$

$$2x + 4y + 5z = 12 \qquad (3)$$

A solution

$$\begin{array}{r} (2) \\ -(1) \\ \hline (2') \end{array} \qquad \begin{array}{r} x+5y-8z=9 \\ -\;\; x-3y+2z=-5 \\ \hline 2y-6z=4 \end{array} \quad (2')$$

$$\begin{array}{r} (3) \\ -2(1) \\ \hline (3') \end{array} \qquad \begin{array}{r} 2x+4y+5z=12 \\ -\;\; 2x-6y+4z=-10 \\ \hline -2y+9z=2 \end{array} \quad (3')$$

new system

$$\begin{array}{rl} x+3y-2z=5 & (1) \\ 2y-6z=4 & (2') \\ -2y+9z=2 & (3') \end{array}$$

$$\begin{array}{r} (2') \\ +(3') \\ \hline (3'') \end{array} \qquad \begin{array}{r} 2y-6z=4 \\ -2y+9z=2 \\ \hline 3z=6 \end{array}$$

Row-echelon form

$$\begin{array}{r} x+3y-2z=5 \\ 2y-6z=4 \\ 3z=6 \end{array}$$

Augmented Matrix

This was a lot of tedious writing. To ease this process, we often use an augmented matrix

$$A\vec{x} = \vec{b} \longrightarrow [A \mid b]$$

$$\begin{aligned} x + 3y - 2z &= 5 \\ x + 5y - 8z &= 9 \\ 2x + 4y + 5z &= 12 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right]$$

Elementary row operations

you will prove (HW 1, ex 2) that the following operations are allowed

1. multiplication. $R_i \to cR_i \quad \forall c \in \mathbb{R}$
2. permutation. $i \leftrightarrow j$
3. addition $R_i + R_j \to R_i$

Gaussian Elimination

- Algorithizes the above process

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right] \xrightarrow{\text{Gaussian elim.}} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

↳ alg. always terminates

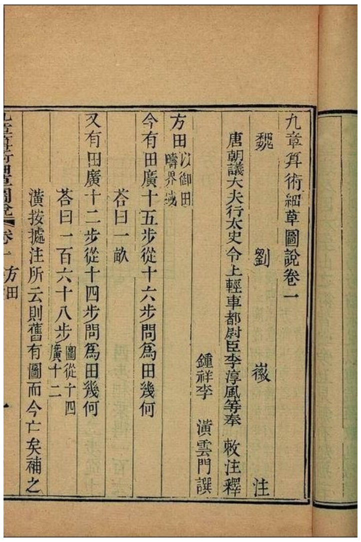
↳ n eqs. & n unknowns requires $O(n^3)$ operations (see additional resources for proof)

↳ we will practice in HW 1, ex 3

↳ See §2.1.3 of Johnston's textbook for details

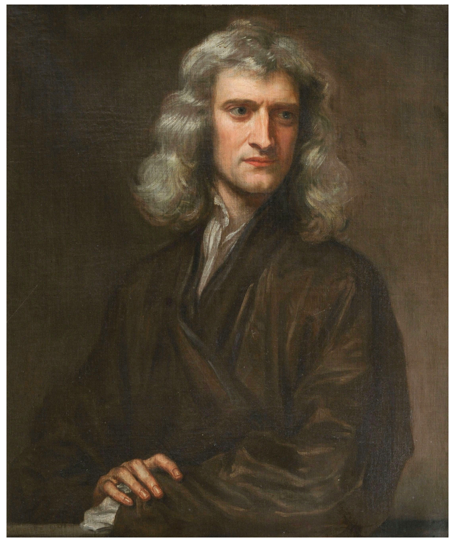
History of Gaussian Eimination

~ 200 BCE
9 chapters on mathematical art (chinese scholars)



九章算術細草圖說卷一
魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 敕注釋
鍾祥李 潢雲門譔
方田以御田疇界域
今有田廣十五步從十六步問為田幾何
荅曰一畝
又有田廣十二步從十四步問為田幾何
荅曰一百六十八步圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

1670 Newton
rediscovers method



1810
Gauss devises
clean notation
for this method

19th century: Harvard
"computers" adopt
notation of Gauss

See "How Ordinary Elimination became
Gaussian Elimination" by Grcar (on canvas)