Math 416: Abstract Linear Algebra

Date: Oct. 17, 2025

Lecture: 20

We're half way there! woah!

Announcements

- ☐ HW6 is due Fri, Oct. 17 @ 8pm
- ☐ practice exams available today @ 5pm
- ☐ Midterm 2 : $\boxed{\text{Fri, Oct 24 @ 1pm}}$

Last time

- ☐ Eigenvals/vecs, polynomials of operators

This time

- ☐ Existence of eigenvals & upper-triangular matrices

Reading/watching

- ☐ §5B/C of Axler & Down w/ determinants!
- ☐ 3blue1brown eigenvals

We are approximately half way through the course

The author's top ten

Listed below are the author's ten favorite results in the book, in order of their appearance in the book. Students who leave your course with a good understanding of these crucial results will have an excellent foundation in linear algebra.

- any two bases of a vector space have the same length (2.34) ✓
- fundamental theorem of linear maps (3.21) ✓
- existence of eigenvalues if $\mathbf{F} = \mathbf{C}$ (5.19) today!
- upper-triangular form always exists if $\mathbf{F} = \mathbf{C}$ (5.47) today & monday!
- Cauchy–Schwarz inequality (6.14)
- Gram–Schmidt procedure (6.32)
- spectral theorem (7.29 and 7.31)
- singular value decomposition (7.70)
- generalized eigenspace decomposition theorem when $\mathbf{F} = \mathbf{C}$ (8.22)
- dimension of alternating n-linear forms on V is 1 if $\dim V = n$ (9.37)

Before proving eigenvalues exist, we need to recall two earlier results!

Thm 4.13 (fund. thm of algebra, second version)

If $p \in \mathcal{P}(\mathbb{C})$ is a non-constant polynomial, then p has a unique factorization of the form

$$p(z) = \overset{\neq 0}{c}(z-\lambda_1)\cdots(z-\lambda_m)$$

where $c, \underbrace{\lambda_1, \ldots, \lambda_m}_{\text{zeros of } p} \in \mathbb{C}$

Coeff. of z^m in p

Prop 5.7 (Equiv. conditions to be an eigenvalue)

Suppose $\dim V < \infty$, $T \in \mathcal{L}(V)$, & $\lambda \in \mathbb{F}$. Then the following are all equivalent:

a) λ is an eigenvalue of T

b) $T - \lambda I$ is not injective

c) $T - \lambda I$ is not surjective

d) $T - \lambda I$ is not invertible

Thm 5.19 (Existence of eigenvalues)

Every operator on a finite-dim nonzero complex vector space has an eigenvalue.

Proof. Suppose V is a complex vector space w/ $\dim V = n > 0$ & $T \in \mathcal{L}(V)$. Choose $v \in V$ s.t. $v \neq 0$.

Then $\{v, Tv, T^2v, \ldots, T^nv\}$ cannot be LI b/c $\dim V = n$ & this has $n+1$ vecs.

$\Rightarrow \exists\ a_0, \ldots, a_n$ not all zero s.t.

$$(*) \qquad 0 = a_0 v + a_1 Tv + \ldots + a_n T^n v$$

Moreover, note $a_1, \ldots, a_n$ cannot all be zero b/c that would imply $0 = a_0 v \Rightarrow a_0 = 0$ due to the assumption that $v \neq 0$.

fund. thm of alg.

Now, consider the polynomial

$$a_0 + a_1 z + \cdots + a_n z^n = c\,(z - \lambda_1) \cdots (z - \lambda_m)$$

$c, \lambda_i \in \mathbb{C}\ \forall i$ $\qquad m \leq n$

We may the write $(*)$ as

$$\begin{aligned} 0 &= a_0 v + a_1 Tv + \cdots + a_n T^n v \\ &= (a_0 + a_1 T + \cdots + a_n T^n)\, v \\ &= c\,(T - \lambda_1 I) \cdots (T - \lambda_m I)\, v \end{aligned}$$

What does this tell us? Well, $c \neq 0$ otherwise the polynomial would be zero (contradicting the assumption) and $v \neq 0$ also by assumption...

Thus, there must exist a $j \in \{1, \ldots, m\}$ s.t.

$$(T - \lambda_j I)v = 0$$

$\Rightarrow$ $T - \lambda_j I$ is not injective

$\Rightarrow$ λ_j is an eigenvalue of T

$\hookrightarrow$ by Axler 5.7

$\square$

Note: this proof is actually from version 3 of LADR by Axler. See also his article "down with determinants"

Upper-triangular matrices

Operators correspond to square matrices

$$A := \mathcal{M}(T) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

$\forall \quad T \in \mathcal{L}(V)$ w/ $\dim V = n$.

Example. Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(x,y,z) = (2x+y, 5y+3z, 8z).$$

What is the matrix of T w.r.t. the standard basis?

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

In general matrices will consist of n^2 #s. A central goal of LA is to find bases s.t. $\mathcal{M}(T)$ has many zeros....

We know already that we can find a basis s.t. $\mathcal{M}(T)$ looks like

$$\begin{pmatrix} \lambda & & * \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix}$$

when $T \in \mathcal{L}(V)$ for complex V.

"proof!" Let λ be an eigenvalue of T (always exists!) and let v be the corresponding eigenvector. Extend v to a basis. $\mathcal{M}(T)$ w.r.t this basis has the above form.

We can often do better!

A matrix $A \in M_n(\mathbb{F})$ is called upper-triangular if $A_{ij} = 0$ whenever $i > j$. That is,

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$*$ denotes potentially non-zero entries we do not care about