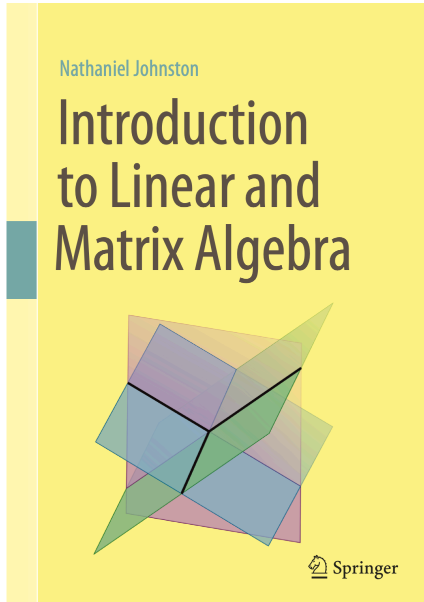
Math 416: Abstract Linear Algebra

Date: Sept. 3, 2025

Lecture: 4



Announcements

- ☐ HW 1 is now live. Due 9/5

- ☐ Updated office hours:
  - Tuesdays 5 - 5:50 Davenport 212
  - Wednesdays 2-2:50 Davenport 132

today after class!

Last time

- ☐ Trichotomy of linear systems
- ☐ Row echelon form
- ☐ motivated Gaussian elimination

This time

- ☐ Clarification: Trichotomy for linear systems
- ☐ Gaussian elimination example
- ☐ reduced row echelon form (RREF)

- ☐ Start Complex #s
  - ↳ § 1A of Axler
  - ↳ § "Imaginary #s are Real" series on Youtube

Trichotomy of linear systems (our first proof)

Theorem. (trichotomy of linear systems)

Every system of linear equations has either zero, one, or infinitely many solutions.

Proof. We will prove this by contradiction.

Logically, we either have

- 0 soln, 1 soln, infinitely many soln.

OR

- m soln., where $m \in \mathbb{N}$ s.t. $1 < m < \infty$

Assume for the sake of contradiction that $\exists$ finitely many soln. $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_m$. Without loss of generality (w.l.o.g), consider $\vec{x}_1$ & $\vec{x}_2$. Let $\vec{x}' := c\vec{x}_1 + (1-c)\vec{x}_2$, $c \in \mathbb{R}$

$$\begin{aligned} A\vec{x}' = A(c\vec{x}_1 + (1-c)\vec{x}_2) &= cA\vec{x}_1 + (1-c)A\vec{x}_2 \\ &= c\vec{b} + (1-c)\vec{b} \\ &= \vec{b} \end{aligned}$$

"therefore"

$\therefore$ $\exists$ infinitely many soln. a contradiction! We conclude that there can only be 0, 1, or ∞ soln. $\square$

Augmented Matrix

This was a lot of tedious writing.

To ease this process, we often use an augmented matrix

$$A\vec{x} = \vec{b} \longrightarrow [A \mid b]$$

$$\begin{aligned} x + 3y - 2z &= 5 \\ x + 5y - 8z &= 9 \\ 2x + 4y + 5z &= 12 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right]$$

Elementary row operations

you will prove (HW 1, ex 2) that the following operations are allowed

1. multiplication. $R_i \to cR_i \quad \forall c \in \mathbb{R}$
2. permutation. $i \leftrightarrow j$
3. addition $R_i + R_j \to R_i$

Gaussian Elimination

- Algorithizes the above process

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 1 & 5 & -8 & 9 \\ 2 & 4 & 5 & 12 \end{array}\right] \xrightarrow{\text{Gaussian elim.}} \left[\begin{array}{ccc|c} 1 & 3 & -2 & 5 \\ 0 & 2 & -6 & 4 \\ 0 & 0 & 3 & 6 \end{array}\right]$$

↳ alg. always terminates

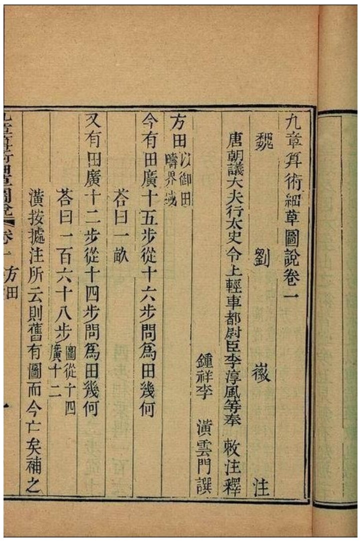
↳ n eqs. & n unknowns requires $O(n^3)$ operations (see additional resources for proof)

↳ we will practice in HW 1, ex 3

↳ See §2.1.3 of Johnston's textbook for details

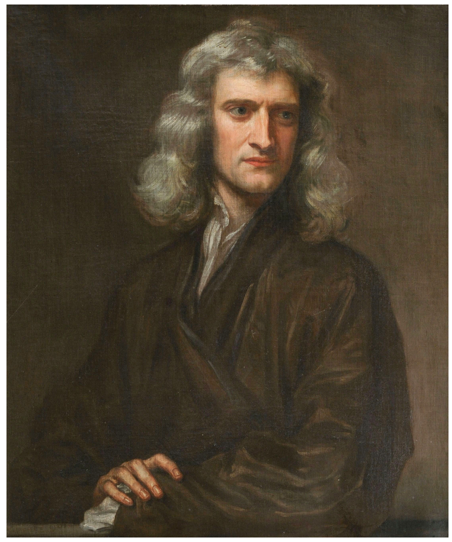
History of Gaussian Elimination

~ 200 BCE
9 chapters on mathematical art (chinese scholars)



九章算術細草圖說卷一
魏 劉 徽 注
唐朝議大夫行太史令上輕車都尉臣李淳風等奉 敕注釋
鍾祥李 潢雲門譔
方田 以御田疇界域
今有田廣十五步從十六步問爲田幾何
荅曰一畝
又有田廣十二步從十四步問爲田幾何
荅曰一百六十八步 圖從十四廣十二
潢按據注所云則舊有圖而今亡矣補之

1670 Newton
rediscovers method



1810
Gauss devises
clean notation
for this method

19th century: Harvard
"computers" adopt
notation of Gauss

See "How Ordinary Elimination became Gaussian Elimination" by Grcar (on canvas)

Gaussian Elimination

We will do the tricky case together and I refer you to § 2.1.3 of Johnston for the other cases.

Ex. $$\begin{aligned} x+y+z&=2 \\ 2x+2y+2z&=4 \\ x-y+z&=0 \end{aligned} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

Step 1. Position a leading entry

need nonzero entry in top left

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 1 & -1 & 1 & 0 \end{array}\right]$$

Step 2. Zero out leading entry's column

$$\left.\begin{aligned} R_2 &\to R_2 - 2R_3 \\ R_3 &\to R_3 - R_1 \end{aligned}\right\} \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

Step 3. Repeat until we cannot.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & -2 & 0 & -2 \end{array}\right] \xrightarrow{R_2 \to R_2 + 2R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 \end{array}\right]$$

$R_2 \leftrightarrow R_3$

Reduced echelon form!

$$\underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{is this unique?}}$$

No, in the last step, we could have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & -2 & 0 & -2 \end{array}\right] \xrightarrow{R_3 \to R_3 + \frac{1}{2}R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{Reduced echelon!}}$$

Reduced row echelon form (RREF)

- Reduce more, from right-to-left
  - ↪ refered to as Gauss-Jordan elim

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2 \to \frac{1}{4}R_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{\text{RREF is unique!}}$$

our solution is thus

$$\left.\begin{aligned} x + y + z &= 2 \\ y &= 1 \end{aligned}\right\} \quad \begin{aligned} x + z &= 1 \\ y &= 1 \end{aligned}$$

free variable

leading variables →

$$x = 1 - z$$
$$y = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-z \\ 1 \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Recap

Pg. 91 of Johnston

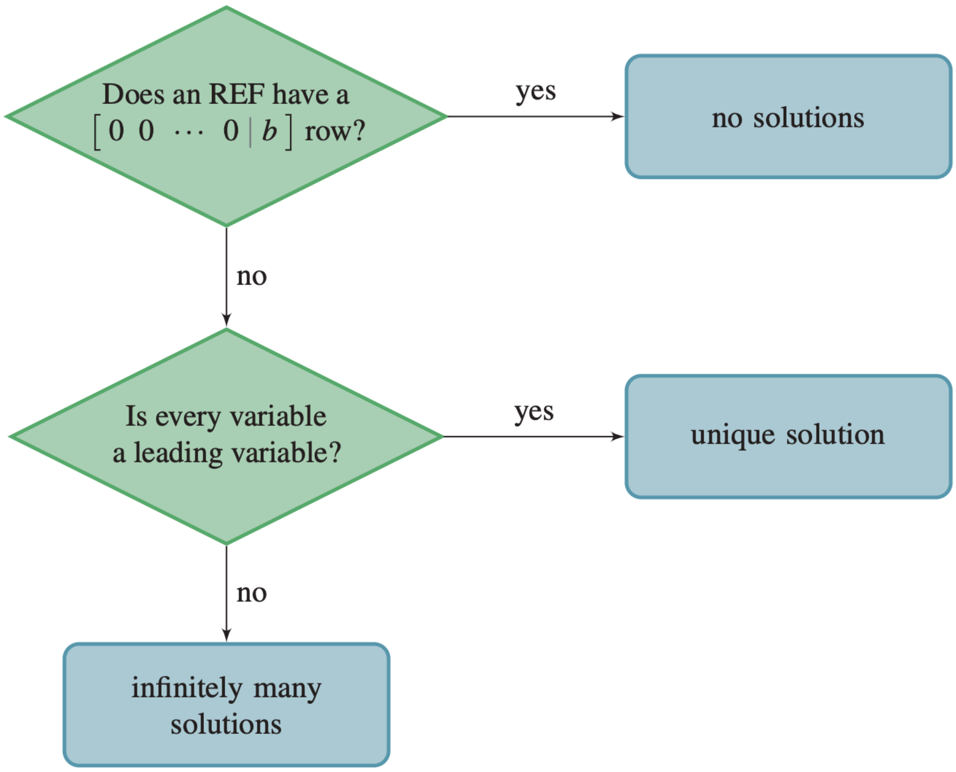


Figure 2.6: A flowchart that describes how to use a row echelon form of an augmented matrix to determine how many solutions the associated linear system has.

- Gaussian elim $\longrightarrow$ Row echelon form (not unique)
- Gauss-Jordan elim $\longrightarrow$ Reduced row echelon form (unique!)

We will now begin our formal treatment of LA.

Complex #s

Def. (complex #s)

Re[z] Im[z]

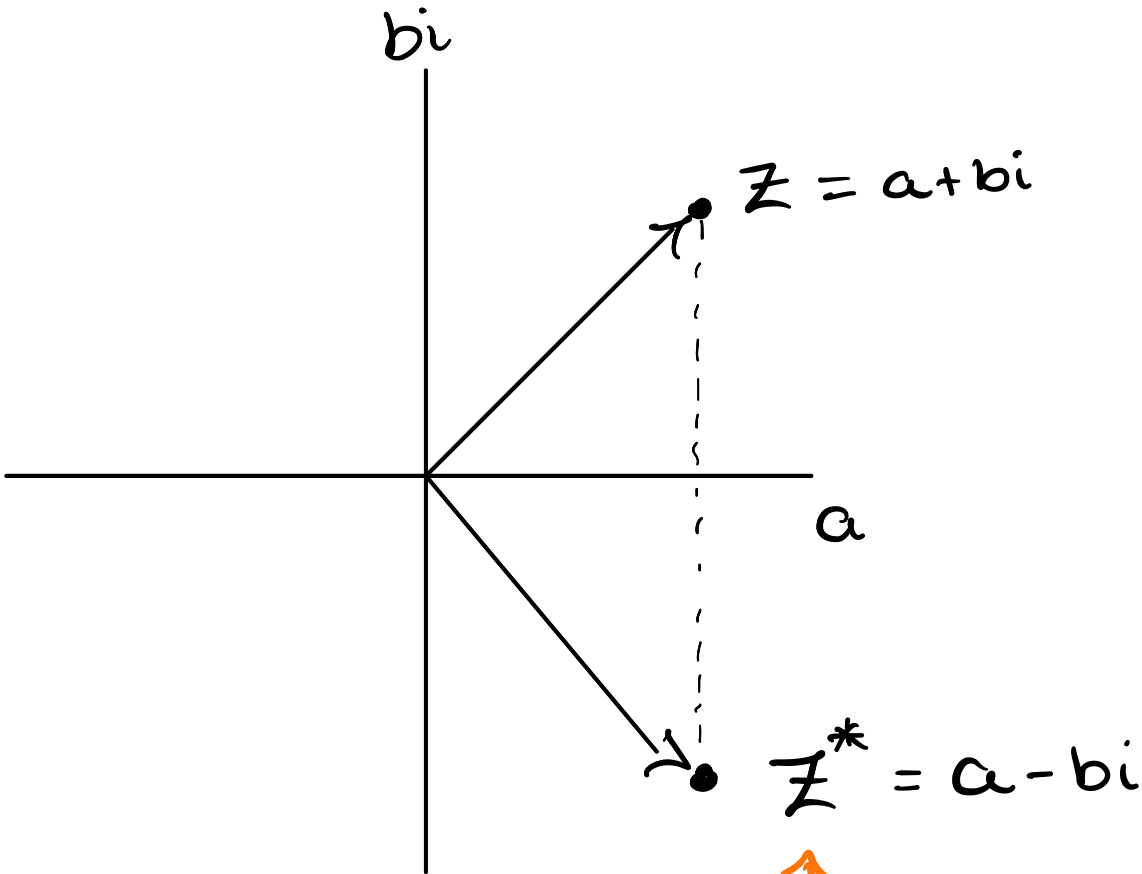
- $z = a + bi$, $a, b \in \mathbb{R}$
- set of all such #s is denoted $\mathbb{C} = \{a+bi : a,b \in \mathbb{R}\}$
- addition:

  $$(a+bi)+(c+di) = (a+b) + (c+d)i$$

- multiplication:

  $$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Complex Plane



complex conjugate

Challenge

Express $\mathrm{Re}[z]$ & $\mathrm{Im}[z]$ in terms of z & z^*.

$$\mathrm{Re}[z] = \frac{z+z^*}{2}, \quad \mathrm{Im}[z] = \frac{z-z^*}{2i}$$