Math 416: Abstract Linear Algebra

Date: Sept. 15, 2025

Lecture: 9

Announcements

- ☐ HW3 is now live. Due 9/19
- ☐ Updated office hours:
  - Tuesdays 5 - 5:50 Davenport 336
  - Wednesdays 2-2:50 Daveport 132
- ☐ Exam #1 : Wed. 9/24
  - ↳ Fair game:
    - basic matrix LA
    - Sec. 1A - 3B of axler is fair game

Last time

- ☐ tutorial: linear indep., bases

This time

- ☐ Bases & dimension

Recommended reading/watching

- ☐ §2B & §2C of Axler
- ☐ 3blue1brown: linear combos, span, & bases

Next time

- ☐ Linear maps (§3A of Axler)

Reminder: Special Colloquim

↳ 180 Bevier Hall

↳ Tomorrow 4-5pm

James Maynard, Oxford

2022 Fields medal winner

- Famous for significant progress on the twin prime conjecture

- This talk will be on another famous open problem: the Riemann hypothesis

- See canvas for videos on related topics!

Def. 2.26 Basis

A basis of V is a list of vectors that is:

- linearly independent
- spans V

A basis consists of:

- a minimal # of vectors that span the space
- max # of LI vectors

Ex. Canonical basis

$$\{e_1, \dots, e_n\}$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$\forall \; v \in \mathbb{F}^n, \; \exists \; x_i \in \mathbb{F} \quad \text{s.t.} \quad v = \sum_{i=1}^{n} x_i e_i$$

(non)-Examples

- Let $v_1, v_2, v_3 \in \mathbb{R}^2$ be

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

↳ Immediately clear it is not a basis b/c we have 3 vecs in $\mathbb{R}^2$

↳ Are all pairs a basis for $\mathbb{R}^2$?

yes, see HW 3, ex 2

Prop 2.28 Criterion for basis

A set $\{v_1, \ldots, v_m\}$ of vectors $v_i \in V$ is a basis iff every $w \in V$ can be expressed

$$w = \sum_i a_i v_i \quad \text{for } \underline{\text{unique}} \text{ scalars } a_i \in \mathbb{F}$$

Proof. ($\Rightarrow$) Suppose $\{v_1, \ldots, v_m\}$ is a basis of V.

As a basis, $\{v_1, \ldots, v_m\}$ spans V, thus $\exists$ $a_i \in \mathbb{F}$ s.t.

$$v = \sum_i a_i v_i .$$

Suppose $\exists$ $c_i \in \mathbb{F}$ s.t.

$$v = \sum_i c_i v_i \qquad (*)$$

$$0 = v - v = \sum_i (a_i - c_i) v_i$$

$\Rightarrow$ $(a_i - c_i) = 0$ $\forall i$ b/c $\{v_1, \ldots, v_m\}$ is linearly idep.

$\Rightarrow$ $a_i = c_i$ $\forall i$

($\Leftarrow$) Suppose every $v \in V$ can be written uniquely as

$$v = \sum_i a_i v_i$$

This implies the list spans V.

To show LI, suppose $\exists\, a_i \in \mathbb{F}$ s.t.

$$0 = a_1 v_1 + \ldots + a_n v_n.$$

The uniqueness of this decomp implies $a_1 = \ldots = a_n = 0$. Thus, the set is LI and hence a basis.

$\square$

Basis facts

- every spanning list contains a basis
- every finite-dim vector space has a basis
- every LI list extends to a basis

Prop. 2.33 Suppose V is finite-dim and $U \leq V$. Then $\exists$ $W \leq V$ s.t. $V = U \oplus W$

Proof Sketch. (see page 42)

§2C Dimension

How should we define the dimension of a vector space?

Intuitively, it feels like we should define the dim. as the # of vectors in a basis.

But what if different bases have a different lengths?

- recall:

length of LI list $\leq$ length spanning list (*)

- suppose V is finite dim & let B_1 & B_2 be two bases for V. Then B_1 is LI in V & B_2 spans V. By (*) $|B_1| \leq |B_2|$. But the arg. works in reverse, so $|B_2| \leq |B_1|$. $\therefore |B_1| = |B_2|$

Def. 2.35 dimension

- dimension of finite-dim vector space is the length of any basis of V
- we denote this $\underline{\dim V}$

<u>Examples</u>

- $\dim \mathbb{F}^n = n$
- Let $\mathcal{P}_m(\mathbb{F})$ be the set of all polynomials w/ coeff. in $\mathbb{F}$ and degree at most m

  $\hookrightarrow \dim \mathcal{P}_m(\mathbb{F}) = m+1$

  $\hookrightarrow \underbrace{x^0, x^1, \ldots, x^m}_{m+1}$

- $U = \{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$

Properties of bases

Suppose V is finite-dim. Then, the following hold:

- $U \subseteq V \implies \dim U \leq \dim V$
- Every LI list in V of length V is a basis of V

  ↳ same for spanning list
- If $U \subseteq V$ & $\dim U = \dim V$, then $U = V$.
- If $V_1, V_2 \subseteq V$, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

↳ See proof on pg. 47