Math 416: Abstract Linear Algebra

Date: Oct. 13, 2025

Lecture: 18

Announcements

- ☐ HW6 is due Fri, Oct. 17 @ 8pm
- ☐ Grades will be updated today!
- ☐ Midterm 2 : Fri, Oct 24 @ 1pm

Last time

- ☐ basis change (wrap-up) & polynomials

This time

- ☐ Invariant subspaces, eigenvals/vecs

Reading/watching

- ☐ §5A of Axler
- ☐ 3blue1brown eigenvals/eigenvecs

Fundamental Thm of Algebra

Recall: a func. $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial of degree m if $\exists$ exist non-zero $a_1, \dots, a_m \in \mathbb{F}$ s.t.

$$p(z) = a_0 + a_1 z + \dots + a_m z^m$$

$\forall \; z \in \mathbb{F}$.

- A zero (or root) of $p \in \mathcal{P}(\mathbb{F})$ is a $\lambda \in \mathbb{F}$ s.t. $p(\lambda) = 0$.
- Let m be a pos. integer and $p \in \mathcal{P}_m(\mathbb{F})$. Then p has at most m zeros in $\mathbb{F}$.

↳ see Axler 4.8

The next result is essential in the proof of the existence of eigenvalues in the next chapter.

↳ though, its importance reaches far beyond lin alg.

Thm 4.12 (Fundamental thm of algebra)

Every nonconstant polynomial w/ complex coeff has a zero in $\mathbb{C}$.

Another important & equiv statement of the theorem is that non-constant $p \in \mathcal{P}_m(\mathbb{F})$ has a factorization of the form

$$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$. And crucially, this factorization is unique up to re-ordering of factors.

Ch. 5 Roadmap

- Ch 5 initiates the study of linear operators (i.e. maps from a space to itself).
- A fruitful technique will be studying how an operator acts on various subspaces
    - ↳ this will lead us naturally to invariant subspaces, eigenvals, & eigenvecs

Statue of Leonardo of Pisa (1170–1250, approximate dates), also known as Fibonacci. Exercise 21 in Section 5D shows how linear algebra can be used to find the explicit formula for the Fibonacci sequence shown on the front cover.

Invariant subspaces

Eigenvalues

Def. 5.1 (operator)

A linear map from a vector space to itself is called an operator

Recall from Ch.2, we had a HW problem showing that $\forall$ finite-dim. V, $\exists$ 1-dim subspaces s.t.

$$V = \bigoplus_{k=1}^{n} V_k$$

Our strategy for understanding $T \in \mathcal{L}(V)$ will be to study its behavior on each subspace V_k.

When restricting the domain to V_k, we denote the map T (restricted to V_k domain) as

$$T|_{V_k}$$

We will be most interested in subspaces that retain their elements under the action of T.

Def. 5.2 (invariant subspace)

Suppose $T \in \mathcal{L}(V)$. A subspace $U \subseteq V$ is called invariant under T if $\forall\, u \in U$, $Tu \in U$.

$T|_U$ is then an operator on U

Example

Differentiation : $T \in \mathcal{L}(P(\mathbb{R}))$
defined via $Tp = p'$.

Invariant subspace : $\mathcal{P}_m(\mathbb{R})$ for all finite m, b/c $p \in \mathcal{P}_m(\mathbb{R})$ has degree at most m, thus p' has degree at most $m-1$.

General examples

If $T \in \mathcal{L}(V)$, the following subspaces are all invariant under T.

- $\{0\}$
- V
- null T
- range T

* note, these may not all be distinct (i.e. range T can equal V)

Question

Must an operator $T \in \mathcal{L}(V)$ have invar. subspaces other than $\{0\}$ & V ?

↳ we will see that the answer is yes when $\dim_{\mathbb{C}} V > 1$ ($\dim_{\mathbb{R}} V > 2$)

Eigenspaces

Take any $v \in V$ w/ $v \neq 0$ and let

$$U = \{\lambda v : \lambda \in \mathbb{F}\} = \operatorname{span}(v).$$

- Every 1D subspace of V is of this form.
- If U is invariant under T, then

  eigenvectors

  $$Tv \in U \implies \exists\, \lambda \in \mathbb{F} \text{ s.t. } Tv = \lambda v$$

  eigenvalue
- "eigen-" is a German prefix meaning "own". – charaterizing an intrinsic property
- V has a one-dim subspace invariant under T iff T has an eigenvalue

Eigenvalues & eigenvectors

Prop 5.7 (Equiv. conditions to be an eigenvalue)

Suppose $\dim V < \infty$, $T \in \mathcal{L}(V)$, & $\lambda \in \mathbb{F}$. Then the following are all equivalent:

a) λ is an eigenvalue of T

b) $T - \lambda I$ is not injective

c) $T - \lambda I$ is not surjective

d) $T - \lambda I$ is not invertible

Proof. To see a) $\Leftrightarrow$ b), note

$$Tv = \lambda v \iff Tv - \lambda v = 0$$

$$(T - \lambda I)v = 0$$

$v \neq 0$ by def. of eigenval/vec

remaining equivalences follow from equiv. of injectivity, surjectivity, & invertibility of operators (Axler 3.65)