Math 416: Abstract Linear Algebra

Date: Oct. 3, 2025

Lecture: 14

Announcements

- ☐ HW4 is due Fri, Oct. 3 @ 8pm

- ☐ office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Davenport 132

- ☐ Exam #1 corrections
  - ↳ Monday 11:59pm ← Oct. 6
  - ↳ half credit for each correction

Last time

- Matrices

This time

- Matrix multiplication & invertibility

Recommended reading/watching

- §3C-D of Axler

Next time

- Isomorphic vector spaces
  & change of basis

Addition & scalar mult.

Let A & C be $m \times n$ matrices

- $(A + C)_{jk} = A_{jk} + C_{jk}$

  ↪ implies $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

  (see HW 4)

- $(\lambda A)_{jk} = \lambda A_{jk}$

  ↪ implies $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

  (see HW 4)

- $\mathbb{F}^{m,n} = \{ m \times n \text{ matrices w/ entries in } \mathbb{F} \}$

- $\mathbb{F}^{m,n}$ forms a vector space!

Matrix multiplication

How should we define matrix multiplication?

↪ to ensure $\mathcal{M}(ST) = \mathcal{M}(ST)$

Def. 3.41 matrix mult.

A : $m \times n$, B : $n \times p$ } AB is $m \times p$ w/ j,k-th entry given by

$$(AB)_{j,k} = \sum_{r=1}^{n} A_{j,r} B_{r,k}$$

$$(AB)_{1,1} = A_{1,1}B_{1,1} + A_{1,2}B_{2,1}$$

Ex. $A_{1,\cdot}$ $B_{\cdot,1}$

$$\underbrace{\begin{pmatrix} 4 & 3 \\ 5 & 2 \\ 6 & 1 \end{pmatrix}}_{3 \times 2} \underbrace{\begin{pmatrix} 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \end{pmatrix}}_{2 \times 4} = \underbrace{\begin{pmatrix} 33 & & & \\ & & & \\ & & & \end{pmatrix}}_{3 \times 4}$$

See Axler 3C for more details...

Remark. (Matrix Algebras)

- when dealing w/ $n \times n$ matrices, we have the vector space structure, but now we also have a method of multiplying two elements...

↪ Thus, the set of all $n \times n$ matrices w/ complex coeff. $M_n(\mathbb{C})$ forms a matrix algebra over $\mathbb{C}$

Transpose and rank

- column rank: dim span{columns of A}
- row rank: dim span{rows of A}
- A^T : transpose of A, $(A^T)_{jk} = A_{jk}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

- HW 5: you will show column rank = row rank

Invertible linear maps

Def. 3.59 (invertible)

- $T \in \mathcal{L}(V,W)$ is <u>invertible</u> if $\exists$ $S \in \mathcal{L}(W,V)$ s.t. $ST = I_V$ and $TS = I_W$
- we call S an <u>inverse</u> of T.

Prop. 3.60 (unique inverse)

An invertible lin. map T has a unique inverse, which justifies the notation T^{-1}!

Proof. Suppose $T \in \mathcal{L}(V,W)$ is invertible and $S_1, S_2 \in \mathcal{L}(W,V)$ are inverses of T. Then

$$S_1 = S_1 I_W = S_1(TS_2) = (S_1 T) S_2 = I_V S_2 = S_2$$

thus $S_1 = S_2$.

Prop. 3.63 invertibility $\Longleftrightarrow$ injective & surjective

Proof. ($\Rightarrow$) Suppose T is invertible.

To see injectivity, let $u, v \in V$ & $Tu = Tv$.

Then,
$$\begin{aligned} u &= I_V u \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= v \end{aligned}$$

For surjectivity, let $w \in W$. Then $w = T(T^{-1}w)$, so $w \in \operatorname{range} T$. But w was arbitrary, so $W = \operatorname{range} T$ & we are done.

$\downarrow$ cont.

($\Leftarrow$) Suppose T is injective & surjective.

- $\forall\, w \in W$, let $S(w)$ be the unique elem. of V s.t. $T(S(w)) = w$ (existence & uniqueness follows from inj. & surjectivity)

- $T \circ S\,(w) = w \quad \forall\, w \in W \quad \Rightarrow T \circ S = I_W$

- next, let $v \in V$. Then

$$
\begin{aligned}
T((S \circ T)v) &= (T \circ S)(Tv) \\
&= I_W(Tv) \\
&= Tv
\end{aligned}
$$

T is inject. $\Rightarrow (S \circ T)v = v \quad \therefore S \circ T = I_V$

- Finally, we must show S is <u>linear</u>
  - $\hookrightarrow w_1, w_2 \in W$: $T(S(w_1) + S(w_2))$
    $= T(S(w_1)) + T(S(w_2)) = w_1 + w_2$
  - $\hookrightarrow$ by def. of S, $S(w_1 + w_2) = S(w_1) + S(w_2)$
- same for homogeneity.

$\square$