MATH 416 Abstract Linear Algebra

Midterm 3 – Practice Exam 1 – Solution

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."*
>
> —John von Neumann

Question 1 (10 points): **Inner Product Spaces**

(i) (5 points) Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is injective.

Proof. Suppose for the sake of contradiction that $T - \sqrt{2}I$ is not injective. Then, there exists a non-zero eigenvector $v \in V$ such that $Tv = \sqrt{2}v$. It follows that $\|Tv\| = \sqrt{2}\|v\|$, which implies

$$\sqrt{2}\|v\| = \|Tv\| \leq \|v\|, \tag{1}$$

but this implies $v = 0$. This implies that v is not an eigenvector, contradicting our assumption. Thus $T - \sqrt{2}I$ is injective. □

(ii) (5 points) Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Prove that

$$\|v - P_U v\| \leq \|v - u\|, \tag{2}$$

where P_U is the orthogonal projection onto U.

Proof. We may write

$$\begin{aligned} \|v - P_U v\|^2 &\leq \|v - P_U v\|^2 + \|P_U v - u\|^2, && (3)\\ &= \|(v - P_U v) + (P_U v - u)\|^2, && (4)\\ &= \|v - u\|^2, && (5) \end{aligned}$$

and taking the square root of both sides yields the desired result. Note that, in the second line we have used Pythagorean theorem because $P_U v - u \in U$ and $v - P_U v \in U^{\perp}$. □

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(i) (5 points) Suppose $T \in \mathcal{L}(V, W)$. Prove that range $T^* = (\text{null}\, T)^{\perp}$.

Proof. It seems somewhat easier to reason about the null space here, so let us take the orthogonal complement of both sides of our desired equivalence and instead prove that null $T = (\text{range}\, T^*)^{\perp}$. Letting $w \in W$, we may then write the following equivalences:

$$
\begin{aligned}
w \in \text{null}\, T &\iff Tw = 0, && (6)\\
&\iff \langle Tw, v\rangle = 0 \quad \text{for all } v \in V, && (7)\\
&\iff \langle w, T^* v\rangle = 0 \quad \text{for all } v \in V, && (8)\\
&\iff w \in (\text{range}\, T^*)^{\perp}. && (9)
\end{aligned}
$$

Again taking the orthogonal complement of both sides gives the desired equivalence. $\square$

(ii) (5 points) Suppose $\mathbb{F} = \mathbb{C}$. Suppose $T \in \mathcal{L}(V)$ is normal and only has one eigenvalue. Prove that T is a scalar multiple of the identity operator.

Proof. Let λ be the eigenvalue of T. If T is normal, then there exists a spectral decomposition of T in terms of an orthonormal basis consisting of eigenvectors of T. Then, any $v \in V$ can be decomposed in terms of this basis as $v = \sum_i a_i v_i$, which allows us to write

$$Tv = \sum_i a_i T v_i = \lambda \sum_i a_i v_i = \lambda v = (\lambda I)v. \tag{10}$$

But v was arbitrary, thus $T = \lambda I$, as desired. $\square$

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(i) (5 points) Suppose that $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

Proof. Let $w \in W$. Then, we may write

$$\langle S^*TSw, w\rangle = \langle T(Sw), Sw\rangle \geq 0, \tag{11}$$

because $Sw \in V$ and T is a positive operator on V. $\square$

(ii) (5 points) Prove that all eigenvalues of a unitary operator have absolute value 1. Using this, prove that if S is a unitary operator, then there is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.

Proof. Suppose U is a unitary with eigenvalue λ. Then, $\|v\| = \|Uv\| = |\lambda|\|v\|$, which implies $|\lambda\| = 1$ because $\|v\| \neq 0$. Next, we want to apply the spectral theorem. Note that all unitaries satisfy $U^*U = UU^* = I$, thus they are normal and the spectral theorem applies. But we just showed that all eigenvalues of a unitary have magnitude 1, so we are done. $\square$