Tutorial: Intro to Proofs

A supplement for Math 416 by **Jacob Beckey**

This handout introduces several canonical proof techniques. For each type of proof, you will see one example worked out in detail and one left open for you to attempt. I've included historical context to remind you that it took thousands of years for humans to develop these techniques; don't get discouraged if you don't master it in an afternoon! Linear algebra is a wonderful opportunity to learn proof techniques, and we will do so together throughout the semester.

Direct Proofs

Example 1: The Pythagorean Theorem

The Pythagorean theorem is one of the oldest known results in mathematics. Integer solutions to the equation date back to Mesopotamia nearly 4000 years ago; however, the first logical proof is attributed to the Pythagoreans.

Theorem 1 (Pythagorean Theorem)**.** *In a right triangle with legs a and b and hypotenuse c,*

$$a^2 + b^2 = c^2.$$

According to *The Pythagorean Proposition* by Elisha Scott Loomis [Loo40], there are at least 367 distinct proofs of Theorem 1. Amazingly, new proofs are still being discovered to this day. Very recently, two **high school students** Ne'Kiya Jackson and Calcea Johnson discovered a purely *trigonometric proof* of the Pythagorean Theorem [JJ24] – a feat that was deemed impossible in the early 1900s [Loo40].

Proof. We will prove this famous theorem directly by drawing the right picture. By observation, we see that

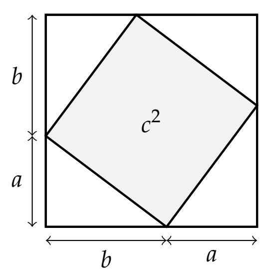

Figure 1: A helpful picture for directly proving the Pythagorean Theorem.

there are 4 triangles with areas of $ab/2$ that, together with one square with area c^2, make up the total square area of $(a+b)^2$. Thus, we may write

$$(a+b)^2 = 4 \cdot \frac{1}{2}ab + c^2, \tag{1}$$
$$a^2 + 2ab + b^2 = 2ab + c^2, \tag{2}$$
$$a^2 + b^2 = c^2. \tag{3}$$

□

This was an absolute triumph of early mathematics. Amazingly, it arises constantly in modern applications, for example, Einstein's famous mass-energy equivalence for particles of energy E, momentum p, and mass m, is formally equivalent to the Pythagorean theorem; $E^2 = (mc^2)^2 + (pc)^2$, which, for $p = 0$ (stationary particles) becomes the more well-known: $E = mc^2$. Equally fascinating is that the equation $a^n + b^n = c^n$ does not hold for $n > 2$! This took mathematicians 350 years to prove. Andrew Wiles' proof of *Fermat's last theorem* was over 100 pages [Wil95; Sin97]!

To Try: A Basic Result from Number Theory

Theorem 2 (Square of an odd number is odd)**.** *Show that if n is odd, then n^2 is odd.*

Proof by Contradiction

Example: There is no smallest positive rational number

Theorem 3. *There is no smallest positive rational number.*

Proof. Assume, for contradiction, that there exists a smallest positive rational number, call it $r > 0$. Now consider $r/2$. This is also a positive rational number, and

$$r/2 < r,$$

which contradicts our assumption that r was the smallest positive rational number. Therefore, no smallest positive rational number exists. $\square$

To Try: The Irrationality of $\sqrt{2}$

Legend says the Pythagoreans were shocked to discover $\sqrt{2}$ is not rational. According to legend, the discovery was so shocking that the Pythagoreans reportedly tried to keep it secret, because it contradicted their worldview.

Theorem 4. *Prove that $\sqrt{2}$ is irrational.*

Hint: Assume $\sqrt{2} = \frac{p}{q}$ in lowest terms (i.e. two integers p, q share no common factors).

Proof by Contrapositive

In logic, every conditional statement "If P, then Q" has a *contrapositive*: "If not Q, then not P". A key fact is that a statement and its contrapositive are **logically equivalent**: proving one proves the other. Contraposition is often useful in proofs because the direct approach ("Assume P, show Q") can be tricky, while reasoning from the negation of the conclusion can simplify the argument. For example, instead of trying to directly show

$$\text{If } n^2 \text{ is even, then } n \text{ is even,}$$

it is easier to prove the contrapositive:

$$\text{If } n \text{ is odd, then } n^2 \text{ is odd.}$$

By showing this simpler statement, we automatically establish the truth of the original claim.

Example: Products of Consecutive Integers

Theorem 5. *If the sum of two integers $a + b$ is odd, then a and b cannot both be even or odd (i.e. the sum must contain one of each).*

Proof by Contrapositive. Instead of proving the statement directly, we prove the contrapositive:

Contrapositive: If both a and b are even, or both are odd, then $a + b$ is even. Assume a and b are both even. Then we can write

$$a = 2k, \quad b = 2\ell$$

for some integers k and ℓ. Then their sum is

$$a + b = 2k + 2\ell = 2(k + \ell),$$

which is divisible by 2, i.e., even. Next, assume a and b are both odd. Then we can write

$$a = 2k + 1, \quad b = 2\ell + 1$$

for some integers k and ℓ. Then

$$a + b = (2k + 1) + (2\ell + 1) = 2(k + \ell + 1),$$

which is also divisible by 2, i.e., even. Since the contrapositive is true, the original statement follows: if $a + b$ is odd, then one of a or b is odd and the other is even. □

To Try: Odd Products

Theorem 6. *If the product of two integers ab is odd, then both a and b are odd.*

Proof by Induction

Example: Gauss and the Sum of Integers

Carl Friedrich Gauss, as a child, discovered a shortcut for adding $\sum_{i=1}^{n} i := 1 + 2 + \cdots + n$.

Theorem 7 (Gauss' Trick)**.** *Prove by induction that*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \tag{4}$$

Proof. We will proceed by induction.

- **Base case:** $n = 1$:

$$\sum_{i=1}^{1} = 1 = \frac{1(1+1)}{2} \tag{5}$$

- **Inductive hypothesis:** Assume the claim holds for $n = m - 1$. That is,

$$\sum_{i=1}^{m-1} i = \frac{(m-1) \cdot m}{2}. \tag{6}$$

- **Induction:** We must now show this holds for $n = m$. We have

$$\begin{aligned}
\sum_{i=1}^{m} i &= \sum_{i=1}^{m-1} i + m, && (7)\\
&= \frac{(m-1) \cdot m}{2} + m, \quad \text{inductive hypothesis} && (8)\\
&= \frac{m^2}{2} - \frac{m}{2} + m, && (9)\\
&= \frac{m^2 - m + 2m}{2}, && (10)\\
&= \frac{m(m+1)}{2}. && (11)
\end{aligned}$$

□

Challenge: can you prove this proposition without using induction? A young Carl Friedrich Gauss is said to have been given this problem as a punishment; however, he saw a beautiful trick and finished the problem in minutes [Col06].

To Try: Sum of Odd Numbers

Blaise Pascal cites Maurolycus' *Arithmetica*, published in 1575, as one of the first examples of mathematical induction. While it is debated who deserves credit for the first ever proof by induction, reproducing Maurolycus' proof makes for a great exercise.

Theorem 8. *Thus sum of the first n odd numbers is n^2.*

If you would like to take a deeper dive into methods of proof, I recommend *Proofs: A Long-Form Mathematics Textbook* by Jay Cummings [Proofs21]. There are many great texts out there for an introduction to abstract math, this one is accessible and affordable!

References

[Col06] Steven Colyer. "Gauss's Day of Reckoning". In: *American Scientist* 94.3 (2006). Accessed: 2025-08-26, p. 200. URL: https://www.americanscientist.org/article/gausss-day-of-reckoning.

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