Tutorial: Linear Independence, Dependence, and Bases

A supplement for Math 416 by **Jacob Beckey**

Goal: By the end of this activity, you will be able to define linear independence and dependence, explain how spanning and independence interact, and describe what a basis is.

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1. Warming Up with Span

Consider the three vectors in $\mathbb{R}^2$:

$$v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (1, 1, 0).$$

1. What subspace of $\mathbb{R}^3$ is spanned by $\{v_1, v_2, v_3\}$?

2. Do we need all three vectors to span this subspace? Why or why not?

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2. Linear Dependence

A set of vectors is called *linearly independent* if ________________________________. Otherwise, it is called *linearly dependent*.

1. Show that $v_3 = (1, 1, 0)$ can be written as a linear combination of v_1 and v_2. Can any vector in this list be written as a combination of the other two?

2. Give another example (your own!) of a linearly dependent set of vectors in $\mathbb{R}^3$. Explain why it is dependent.

3. Linear Independence

A list of vectors $v_1, \ldots, v_m$ is called *linearly independent* if the only way to write

$$a_1 v_1 + \cdots + a_m v_m = 0$$

is with $a_1 = \cdots = a_m = 0$.

1. Check whether the following set is independent in $\mathbb{R}^3$:

$$\{(1,0,0), (0,1,0), (0,0,1)\}.$$

2. Check whether the following set is independent in $\mathbb{R}^3$:

$$\{(1,0,0), (0,1,0), (1,1,0)\}.$$

4. Comparing Spanning and Independence

Suppose U is a subspace of $\mathbb{R}^n$.

1. If you have a spanning set for U, can you always remove some vectors until you get a linearly independent spanning set?

2. If you have an independent set in U, can you always add more vectors from U until it spans U?

3. What does this suggest about the relationship between the length of a linearly independent list and the length of a spanning list in the same subspace?

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5. Basis

Definition: A *basis* for a subspace U is a list of vectors that ___________ and also ___________.

1. Which of the following sets form a basis for $\mathbb{R}^2$? Justify.
   (a) $\{(1,0),(0,1)\}$

   (b) $\{(1,0),(1,1)\}$

   (c) $\{(1,0),(0,0)\}$

   (d) $\{(1,0),(0,1),(1,1)\}$

2. In your own words: why is having a basis more useful than just having any spanning set?

Preview: Next time we will prove that every subspace has a basis, and the number of vectors in any basis is called the ***dimension****.*

Challenge Problems

1. Prove that any set of vectors that contains the zero vector is linearly dependent.

2. Suppose a set $S = \{v_1, \ldots, v_m\}$ is linearly dependent. Prove that at least one v_j is a linear combination of the others.

3. Prove that if a spanning set for a subspace U is minimal (no proper subset spans U), then it must be linearly independent.

4. Prove that if a set of vectors in a subspace U is linearly independent and maximal (you cannot add more vectors from U and keep independence), then it must span U.

5. Let U be a subspace of $\mathbb{R}^n$. Suppose S is a spanning set for U with m vectors, and I is an independent set in U with k vectors. Prove that $k \leq m$. (Hint: combine Problems 3 and 4.)

6. Prove that if $\{v_1, \ldots, v_m\}$ is linearly independent, then every $u \in \operatorname{span}(v_1, \ldots, v_m)$ has a unique representation as a linear combination of $v_1, \ldots, v_m$.