MATH 416 Abstract Linear Algebra

Week 9 - Homework 7
Assigned: Fri. Oct. 24, 2025
Due: Fri. Oct. 31, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (5 points): **Eigenspaces and Diagonalizable Operators**
Note that these two problems are both related to eigenspaces and diagonalizable operators, but are otherwise distinct.

(a) (1 point) Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$E(\lambda, T) = E(\lambda^{-1}, T^{-1}) \tag{1}$$

for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

> Recall that $E(\lambda, T) = \{v \in V : Tv = \lambda v\}$. Thus, for arbitrary $v \in V$, we have
>
> $$Tv = \lambda v \Leftrightarrow T^{-1}(Tv) = T^{-1}(\lambda v), \tag{2}$$
> $$\Leftrightarrow (T^{-1}T)v = \lambda(T^{-1}v), \tag{3}$$
> $$\Leftrightarrow \lambda^{-1}v = T^{-1}v. \tag{4}$$
>
> Thus, any $v \in E(\lambda, T)$ is also in $E(\lambda^{-1}, T^{-1})$ and vice versa. As such, these are the same set of vectors.

(b) (4 points) Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote distinct non-zero eigenvalues of T. Prove that

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \text{range } T. \tag{5}$$

> In class, we proved a less restrictive result that said for distinct (but not necessarily non-zero) eigenvalues $\lambda_1, \ldots, \lambda_m$
>
> $$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V. \tag{6}$$
>
> So, if none of T's eigenvalues are zero, we know there does not exist $v \in V$ such that $v \neq 0$ and $Tv = 0$. Thus, the null space is trivial which implies T is injective (and thus surjective).

A surjective operator satisfies range $T = V$ and we obtain the desired result. Now, suppose that T has a zero eigenvalue. Then, from the result in class we have

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) + \dim E(0, T) \leq \dim V, \tag{7}$$

$$\implies \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) + \dim \text{null}\, T \leq \dim V, \tag{8}$$

$$\implies \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V - \dim \text{null}\, T, \tag{9}$$

$$\implies \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim \text{range}\, T, \tag{10}$$

where the last line holds from the fundamental theorem of linear maps. Note that we also used the fact that $E(0, T)$ is just different notation for the null space. Remember that, in general, the eigenspace is defined as

$$E(\lambda, T) = \text{null}\, (T - \lambda I). \tag{11}$$

Thus, when $\lambda = 0$, this is exactly null T.

Exercise 2 (5 points): **Commuting Operators**

Suppose A is a diagonal matrix *with distinct entries on the diagonal* and B is a matrix of the same size as A. Show that $AB = BA$ if and only if B is a diagonal matrix.

Proof. ($\Rightarrow$) First, let us show that if $AB = BA$, then B is a diagonal matrix. Note from the definition of matrix multiplication, we have

$$(AB)_{jk} = \sum_l A_{jl}B_{lk} = A_{jj}B_{jk} \quad \text{and} \quad (BA)_{jk} = \sum_l B_{jl}A_{lk} = B_{jk}A_{kk} = A_{kk}B_{jk}. \qquad (12)$$

If $AB = BA$, then all components must be equal and we have

$$(AB)_{jk} = (BA)_{jk}, \qquad (13)$$
$$\implies A_{jj}B_{jk} = A_{kk}B_{jk}, \qquad (14)$$
$$\implies (A_{jj} - A_{kk})B_{jk} = 0. \qquad (15)$$

Since A is a matrix with distinct entries along the diagonal, when $j \neq k$, $(A_{jj} - A_{kk}) \neq 0 \implies B_{jk} = 0$. It follows that B_{jk} can only be non-zero when $j = k$, i.e. if B is diagonal.

($\Leftarrow$) If B is a diagonal matrix, then we have the product of two diagonal matrices whose commutativity follows from the definition of matrix multiplication and the commutativity of the underlying field. $\square$

Exercise 3 (5 points): **Inner Products and Norms**

(a) (3 points) Suppose u, v are non-zero vectors in $\mathbb{R}^2$. Prove that

$$\langle u, v \rangle = \|u\|\|v\| \cos \theta, \tag{16}$$

where θ is the angle between u, v when we think of u, v as arrows with bases at the origin. *Hint: Use the law of cosines on the triangle formed by u, v, and $u - v$.*

(b) (2 points) Once $n > 3$, we lose the ability to picture vectors in $\mathbb{R}^n$ geometrically. In light of part (a), how might we define the angle between vectors in $\mathbb{R}^n$? Use Cauchy-Schwarz to explain why this definition makes sense.

(a) *Proof.* Form a triangle with sides u, v, and $u - v$. The law of cosines yields

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos \theta. \tag{17}$$

From the definition of the norm on $\mathbb{R}^n$, we have

$$\|u - v\|^2 = \langle u - v, u - v \rangle, \tag{18}$$
$$= \langle u, u \rangle - \langle u, v \rangle - \langle v, u \rangle + \langle v, v \rangle, \tag{19}$$
$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\langle u, v \rangle. \tag{20}$$

Equating these and canceling like terms yields the desired result. $\square$

(b) From Cauchy-Schwarz, we know that

$$|\langle u, v \rangle| \le \|u\|\|v\| \implies \frac{|\langle u, v \rangle|}{\|u\|\|v\|} \le 1. \tag{21}$$

With this in mind, it makes sense to define the angle between two vectors in $\mathbb{R}^n$, in general, as

$$\theta := \arccos \left(\frac{|\langle u, v \rangle|}{\|u\|\|v\|} \right), \tag{22}$$

which is well-defined because $\arccos x$ is well-defined when $|x| \le 1$.

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(Optional) Bonus Problem (3 points): *Fibonacci Sequence via Linear Algebra*

This problem is not strictly useful, but it is fun and cool, so I encourage you to try it anyway! The Fibonacci sequence arises in many unexpected places throughout mathematics, physics, and nature generally. In this problem, you will use your newly acquired linear algebra skills to derive a closed-form formula for the n-th term in the sequence.

The *Fibonacci Sequence* is defined recursively via the following equations

$$F_0 = 0, \tag{23}$$
$$F_1 = 1, \tag{24}$$
$$F_n = F_{n-2} + F_{n-1}, \quad \forall\, n \geq 2. \tag{25}$$

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(x,y) = (y, x+y)$.

(a) (1 point) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each non-negative integer n.

> *Proof.* We can prove this formula inductively. For $n = 0$, we have $T^0(0,1) = (0,1)$ and $T^1(0,1) = (1,1)$, as desired. Now, assume this holds for $n-1$. To show this implies it holds for n, observe
>
> $$\begin{aligned} T^n(0,1) &= T(T^{n-1}(0,1)), & \text{definition of operator powers} & \tag{26}\\ &= T(F_{n-1}, F_n), & \text{induction hypothesis} & \tag{27}\\ &= (F_n, F_{n-1} + F_n), & \text{base case} & \tag{28}\\ &= (F_n, F_{n+1}), & \text{definition of Fibonacci} & \tag{29} \end{aligned}$$
>
> as desired. $\square$

(b) (1 point) Find the eigenvalues and corresponding eigenvectors of T.

> **Solution.** The eigenvalue equation $T(x,y) = \lambda(x,y)$ implies
>
> $$\lambda^2 - \lambda - 1 = 0. \tag{30}$$
>
> The quadratic formula yields
>
> $$\lambda_\pm = \frac{1 \pm \sqrt{5}}{2}. \tag{31}$$
>
> Using the eigenvector equation, we have the two equations $(y, x+y) = (\lambda x, \lambda y)$.

Using these two equations, one can find the eigenvectors

$$v_{\pm} = \left(1, \frac{1 + \sqrt{5}}{2} \right) \tag{32}$$

(c) (1 point) Use your solution in (b) to compute $T^n(0, 1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]. \tag{33}$$

Solution. The set $\{v_+, v_-\}$ forms an eigenbasis for the operators. Note, also, that $(0, 1) = \frac{1}{\sqrt{5}}(v_+ - v_-)$. It follows that

$$T^n(0, 1) = T^n \frac{1}{\sqrt{5}}(v_+ - v_-), \tag{34}$$

$$= \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n v_+ - \left(\frac{1 - \sqrt{5}}{2} \right)^n v_- \right), \tag{35}$$

Noting from (a) that the first entry of $T^n(0, 1)$ is F_n, we obtain

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]. \tag{36}$$

This is known as *Binet's formula*, named after French mathematician Jacques Philippe Marie Binet, though it was already known by Abraham de Moivre and Daniel Bernoulli (see wikipedia).