MATH 416 Abstract Linear Algebra

Week 5 - Homework 4
Assigned: Fri. Sept. 26, 2025
Due: Fri. Oct. 3, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (6 points): Injectivity and surjectivity
Let V be a (finite-dim.) vector space over a field $\mathbb{F}$. Let $v_1, \ldots, v_m \in V$ and define the linear map

$$T\colon \mathbb{F}^m \to V, \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto \sum_{i=1}^{m} x_i v_i.$$

(i) Prove that T is injective if and only if $\{v_1, \ldots, v_m\}$ are linearly independent.

(ii) Prove that T is surjective if and only if $V = \operatorname{span}\{v_1, \ldots, v_m\}$.

Exercise 2 (4 points): Linear maps as matrices I
Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, and fix bases $\mathcal{B}_V = \{v_1, \ldots, v_n\}$ for V and $\mathcal{B}_W = \{w_1, \ldots, w_m\}$ for W. In the following, we abbreviate $\mathcal{M}(\cdot) = \mathcal{M}(\cdot)_{\mathcal{B}_V, \mathcal{B}_W}$.
Show that:

(i) $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ for $S, T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

(ii) $\mathcal{M}(aT) = a\mathcal{M}(T)$ for $a \in \mathbb{F}$ and $T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

Exercise 3 (4 points): Linear maps as matrices II
Consider the following linear map:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^4, \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$$

(i) Determine $\mathcal{M}(T)_{\mathcal{S}_3, \mathcal{S}_4}$, where $\mathcal{S}_3$ and $\mathcal{S}_4$ are the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

(ii) Let now

$$\mathcal{B}_V = \left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right) \right\} \qquad \mathcal{B}_W = \left\{ \left(\begin{smallmatrix} 1 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ -1 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right) \right\}$$

and determine $\mathcal{M}(T)_{\mathcal{B}_V,\mathcal{B}_W}$.

Remark: You do not need to show that $\mathcal{B}_V$ and $\mathcal{B}_W$ are indeed bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

(optional) Bonus Question (2 points): **Projection Operators**

Let $P \in \mathcal{L}(V)$ such that $P^2 = P$. Such operators are called *projection operators* or *projectors* and they play a fundamental role in linear algebra, representation theory, quantum mechanics, data science, and many more fields.

(i) Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.