MATH 416 Abstract Linear Algebra

Exam 1 – Practice Exam

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using. Organize your solutions logically, simplify final answers when possible, and state any assumptions you make. *I would encourage you to complete the practice exam under exam conditions (50 min, no book or phone, fear in your heart[1]).*

Question 1 (5 points): **Linear Systems and Gaussian Elimination**
Consider the system of equations.

$$\begin{cases} x + 2y - z = 1, \\ 2x + 4y - 2z = 2, \\ 3x + 6y - 3z = 3. \end{cases}$$

Solve the system, determine the dimension of the solution space, and describe the space geometrically.

Solution. Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ 3 & 6 & -3 & 3 \end{array}\right].$$

Perform row operations to reduce to echelon form:
Eliminate below the first pivot:

$$R_2 \leftarrow R_2 - 2R_1, \quad R_3 \leftarrow R_3 - 3R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

[1] This is a joke. You are all going to do great!

Now the system reduces to a single independent equation:

$$x + 2y - z = 1.$$

Let $y = s$ and $z = t$ be free parameters. Then

$$x = 1 - 2s + t.$$

General solution:

$$(x, y, z) = (1 - 2s + t,\ s,\ t), \quad s, t \in \mathbb{R}.$$

This describes a plane in $\mathbb{R}^3$ passing through the point $(1, 0, 0)$ with direction vectors $(-2, 1, 0)$ and $(1, 0, 1)$.

Question 2 (5 points): **Complex Numbers are *Real*ly Cool (5 points)**

Let $z = a + bi$ and $w = c + di$ be complex numbers with $a, b, c, d \in \mathbb{R}$. Recall that the conjugate of z is $\overline{z} = a - bi$ and $|z| = \sqrt{a^2 + b^2}$. Further, recall Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.

1. (1 point) Show that $\overline{z + w} = \overline{z} + \overline{w}$.
2. (1 point) Show that $\overline{zw} = \overline{z}\,\overline{w}$.
3. (2 points) Find two distinct square roots of i in standard form. *Note: you may use either standard or polar form to solve this, but your answer must be expressed in standard form.*
4. (1 point) Compute $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^8$.

Solutions.

1. $\overline{z + w} = \overline{a + c + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id) = \overline{z} + \overline{w}$.
2. $\overline{zw} = \overline{(a + bi)(c + di)} = \overline{(ac - bd) + i(ad + bc)} = (ac - bd) - i(ad + bc) = (a - bi)(c - di) = \overline{z}\,\overline{w}$.
3. This is easiest to compute in polar form. We seek z such that $z^2 = i$. Write

$$i = e^{i(\pi/2 + 2\pi k)}, \quad k \in \mathbb{Z} \implies z = e^{i(\pi/4 + \pi k)}, \quad k = 0, 1. \tag{1}$$

We can use Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ to convert to standard form:

$$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \quad \text{and} \quad z = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i.$$

We can also compute this in standard form, but it is more involved. Let $z = x + yi$ satisfy $z^2 = i$. Then $(x+yi)^2 = x^2 - y^2 + 2xyi = 0 + 1 \cdot i$, so

$$x^2 - y^2 = 0, \quad 2xy = 1.$$

From $x^2 = y^2$, we have $y = \pm x$. If $y = x$, then $2x^2 = 1 \implies x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$. If $y = -x$, then $-2x^2 = 1 \implies x^2 = -\frac{1}{2}$, impossible.

So the two distinct roots are

$$z_1 = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \quad z_2 = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

(the negative is the other square root, since $(-z_1)^2 = z_1^2 = i$).

4. Let $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$. Using $\cos\theta = \frac{\sqrt{2}}{2}$, $\sin\theta = \frac{\sqrt{2}}{2}$, we see $z = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$. Then using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, we obtain

$$z^8 = (e^{i\pi/4})^8 = e^{i2\pi} = 1.$$

Question 3 (10 points): **Vector Subspaces, Bases, and Dimension**
Recall that a polynomial is a function $p : \mathbb{R} \to \mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m, \tag{2}$$

for all $x \in \mathbb{R}$. We denote the set of all such functions $\mathcal{P}(\mathbb{R})$. When we assume $m < \infty$, we denote the set $\mathcal{P}_m(x)$. In class, we claimed this is a vector space. In this problem, we develop that fact. Note that addition and scalar multiplication are defined as follows. For all $f, g \in \mathcal{P}(x)$ the sum $f + g \in \mathcal{P}(x)$ is the function defined by

$$(f+g)(x) = f(x) + g(x). \tag{3}$$

Similarly, for all $\lambda \in \mathbb{R}$ and all $f \in \mathcal{P}(x)$, the product $\lambda f \in \mathcal{P}(x)$ is the function defined by

$$(\lambda f)(x) = \lambda f(x) \tag{4}$$

(i) Consider the set $B = \{1, x, x^2, \ldots, x^m\}$. Show B is a basis for $\mathcal{P}_m(\mathbb{R})$. Use this to determine the dimension of $\mathcal{P}_m(\mathbb{R})$.

Proof. We must show the set spans the space and is linearly independent. Clearly any $p(x) \in \mathcal{P}_m(\mathbb{R})$ can be written as a linear combination of elements of B. Thus, B *spans* $\mathcal{P}_m(\mathbb{R})$. To show linear independence we must argue that the zero polynomial $0 \in \mathcal{P}_m(\mathbb{R})$ can only be written as a linear combination of elements in B by choosing all coefficients to be zero. Suppose for contradiction there existed a set of coefficients (not all zero) such that

$$0 = a_0 \cdot 1 + a_1 \cdot x + a_2 \cdot x^2 \cdots a_m \cdot x^m. \tag{5}$$

But we also know that $0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \cdots 0 \cdot x^m$, where the zero on the left is in $\mathcal{P}_m(\mathbb{R})$ and the all of the zeroes on the right are in $\mathbb{R}$. Equating these two expressions forces all coefficients to be zero, and we have reached a contradiction. Thus, B is a basis of $\mathcal{P}_m(\mathbb{R})$. Further, $|B| = m + 1$, so we say $\dim \mathcal{P}_m(\mathbb{R}) = m + 1$. □

(ii) Now, let $U_i := \operatorname{span}\{x^i\} = \{cx^i : c \in \mathbb{R}\}$ for all $1 \leq i \leq m$ be subspaces of $\mathcal{P}(\mathbb{R})$. Show that

$$\mathcal{P}(\mathbb{R}) = U_0 \oplus U_1 \oplus \cdots \oplus U_m. \tag{6}$$

Show that this result can be used to conclude the value of $\dim \mathcal{P}_m(\mathbb{R})$. *Hint: recall (from HW3) that for subspaces $V_1, \cdots, V_m$ of V, we always have* $\dim(V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m$.

Proof. By definition of $\mathcal{P}(\mathbb{R})$, any $p \in \mathcal{P}(\mathbb{R})$ can be expressed as $p(x) = a_0 x_0 + a_1 x_1 + \cdots + a_m x_m$. By definition of the U_i's it is clear that we have expressed an arbitrary element of $\mathcal{P}(\mathbb{R})$ in terms of elements in $u_i \in U_i$ for all i. Thus, we conclude that

$$\mathcal{P}(\mathbb{R}) = U_0 + U_1 + \cdots + U_m. \tag{7}$$

To show that this is unique, recall from (i) that B was a basis, thus the decomposition is unique. □

Question 4 (10 points): **The Vector Space of Linear Maps**

We saw in class that the set $\mathcal{L}(U, V)$ of all linear maps from U to V is, indeed, a vector space. In fact, it is one of the most important vector spaces we will study in this course. Recall also that the set of all polynomials

(i) Define a function $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. Show that this function is a linear map.

Proof. A linear map is a function that also satisfies additivity and homogeneity. Let $a, b \in \mathbb{R}$ and $f, g \in \mathcal{P}(\mathbb{R})$. If T is linear, we must show that $(T(af + bg))(x) = a(Tf)(x) + b(Tg)(x)$. We may write

$$\begin{aligned}
(T(af+bg))(x) &= x^2(af+bg)(x), && \text{Definition of } T \\
&= x^2\left(af(x)+bg(x)\right), && \text{Definition of add. and mult. over } \mathcal{P}(\mathbb{R}) \\
&= ax^2 f(x) + bx^2 g(x), && \text{Distrib. and commutativity of reals} \\
&= a(Tf)(x) + b(Tg)(x) && \text{Definition of } T
\end{aligned}$$

□

(ii) Let p' denote the derivative of p. Consider $D \in \mathcal{P}(\mathbb{R})$ defined as $Dp = p'$ for all $p \in \mathcal{P}(\mathbb{R})$. Using fundamental definitions from calculus, one can show that this is, indeed a linear map. Using this fact and (i), show that $DT \neq DT$. That is, these two maps do not commute.

Solution. We can check this non-commutativity explicitly. We have

$$\begin{aligned}
((DT)p)(x) &= D((Tp)(x)), && (8) \\
&= Dx^2p(x). && (9) \\
&= 2xp(x) + x^2p'(x) && (10)
\end{aligned}$$

, while in the other direction we have

$$\begin{aligned}
((TD)p)(x) &= T((Dp)(x)), && (11) \\
&= Tp'(x), && (12) \\
&= x^2p'(x). && (13)
\end{aligned}$$

Clearly these are unequal on an arbitrary input, thus we say $TD \neq DT$, or the maps do not commute.