Math 416: Abstract Linear Algebra

Date: Sept. 10, 2025

Lecture: 7

Announcements

- ☐ HW2 is now live. Due 9/12.
- ☐ Updated office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Daveport 132

Last time

- Subspaces and direct sums

This time

- direct sum proofs
- Span

Next time

- tutorial: linear indep., bases, dimension

Recommended reading/watching

- §1C & §2A of Axler

Sums of subspaces

Def. 1.36 (Sums of subspaces)

Suppose $V_1, \dots, V_m$ are subspaces of V. Then

$$V_1 + \cdots + V_m$$

$$= \{v_1 + \cdots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}$$

In words: $V_1 + \dots + V_m$ is the set of all possible sums of elements of $V_1, \dots, V_m$.

This is anaragous to unions of subsets in set theory!

Example

Let $V = \mathbb{R}^3$

x-y plane

$U = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$

$W = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$

y-z plane

$V = U + W$ however, the sum will <u>not</u> be unique...

Consider $v = (1,1,1) \in \mathbb{R}^3$.

$$v = \underbrace{(1,1,0)}_{\in U} + \underbrace{(0,0,1)}_{\in W}$$

$$v = (1,0,0) + (0,1,1)$$

Direct sum

We often want a unique way of decomposing vectors.

A direct sum does exactly that!

Def. 1.41 (direct sum)

Let $V_k \leq V \quad \forall k$.

$\sum_{k=1}^{m} V_k = V_1 + \cdots + V_m$ is called a direct sum if each element of $V_1 + \cdots + V_m$ can only be written one way as $v_1 + \cdots + v_m$ w/ each $v_k \in V_k$.

Question from last time

I wrote $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$

which doesn't make sense notationally...

Analagous to the fact that a set unioned w/ itself is itself.

First, $\mathbb{R} + \mathbb{R} = \mathbb{R}$.

The x-axis is a subspace of $\mathbb{R}^2$

$$\mathbb{R}_x = \{(x,y) \in \mathbb{R}^2 : y = 0\}$$

Same for y

$$\mathbb{R}_y = \{(x,y) \in \mathbb{R}^2 : x = 0\}$$

Then, I claim

$$\mathbb{R}^2 = \mathbb{R}_x \oplus \mathbb{R}_y$$

The following lemma makes checking if a sum is direct simple.

Lemma. 1.46 (direct sum of 2 subspaces)

Let $U, W \leq V$. Then

$U + W$ is a direct sum $\iff U \cap W = \{0\}$

See page 23 for proof.

Claim. $\mathbb{R}^2 = \mathbb{R}_x \oplus \mathbb{R}_y$.

Proof. First, we must show that any $v \in \mathbb{R}^2$ may be expressed

$$v = \underbrace{v_x}_{\in \mathbb{R}_x} + \underbrace{v_y}_{\in \mathbb{R}_y}$$

An arbitrary element of $\mathbb{R}^2$ is

$$v = \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{x \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\in \mathbb{R}_x} + \underbrace{y \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\in \mathbb{R}_y} \checkmark$$

To show uniqueness, we will use Lemma 1.46. That is, we will check $\mathbb{R}_x \cap \mathbb{R}_y = \{0\}$. Suppose $v \in \mathbb{R}_x \cap \mathbb{R}_y$. Then $v \in \mathbb{R}_x$ & $v \in \mathbb{R}_y$

$$\Downarrow \qquad\qquad \Downarrow$$

$$v = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$\Longrightarrow \quad \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$\Longrightarrow \quad x = 0 \quad \& \quad y = 0$$

$\therefore$ The zero vector is the only vector in both $\mathbb{R}_x$ & $\mathbb{R}_y$.

Question

Suppose we have 3 subspaces

$$V_1, V_2, V_3 \leq \mathbb{F}^3.$$

Does $V_1 \cap V_2 = V_2 \cap V_3 = V_1 \cap V_3 = \{0\}$ impy $\mathbb{F}^3 = V_1 \oplus V_2 \oplus V_3$?

Answer. Unfortunately, no. Consider

$$V_1 = \{(x,y,0) \in \mathbb{F}^3\}$$

$$V_2 = \{(0,0,z) \in \mathbb{F}^3\}$$

$$V_3 = \{(0,y,y) \in \mathbb{F}^3\}$$

Then $0 \in \mathbb{F}^3$ can be written

$$\begin{array}{ccccccc} (0,0,0) & = & \underset{\in V_1}{(0,1,0)} & + & \underset{\in V_2}{(0,0,1)} & + & \underset{\in V_3}{(0,-1,-1)} \\ & & \updownarrow & & \updownarrow & & \updownarrow \\ (0,0,0) & = & (0,0,0) & + & (0,0,0) & + & (0,0,0) \end{array}$$

One can prove that $V_1 + \dots + V_m$ is a direct sum iff the only way to write

$$0 = v_1 + \dots + v_m \quad , \; v_k \in V_k$$

is taking $v_k = 0 \;\; \forall \; k$.

See proof on page 23 of Axler.

Ch 2: Finite-dimensional Vector Spaces

This course focuses finite-dim vector spaces. The infinite-dim case is the subject of functional analysis (e.g. Math 541).

Roadmap:

today

- §2A Span & $\underbrace{\text{linear indep.}}_{}$
- §2B Bases
- §2C Dimension

Friday

§ 2A Span & linear independence

Linear combos & span

- $a_1 v_1 + \cdots + a_m v_m$, $a_i \in \mathbb{F}$ $\forall i$

  is a <u>linear combination</u>

- the set of all such combos is called the span of $v_1, \ldots, v_m$

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_i \in \mathbb{F} \ \forall i\}$$

↪ If $V = \operatorname{span}(v_1, \ldots, v_m)$ we say $v_1, \ldots, v_m$ <u>spans</u> V.

- A vector space is <u>finite-dim</u> is $\exists$ a list of vectors that spans the space.