Math 416: Abstract Linear Algebra

Date: Oct. 29, 2025

Lecture: 23

Announcements

- ☐ HW 7 due Friday @ 8pm
- ☐ Midterms will be graded by Friday
  - ↳ corrections due next Fri (Nov 7)

Last time

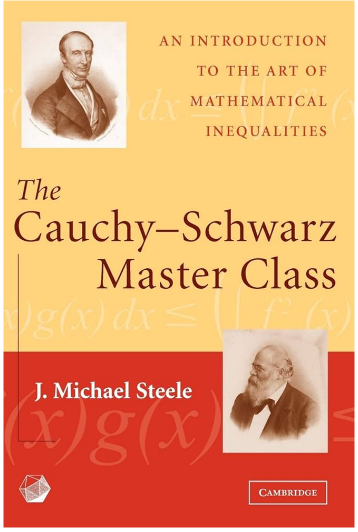
- ☐ Upper-triangular matrices (existence proof)
- ☐ Diagonalizable & commuting operators

This time

- ☐ Inner product, norms, & inequalities

Reading/watching

- ☐ §6A of Axler
- ☐ Cauchy-Schwarz Masterclass by Steele



Inner product spaces

Vector space

↳ set w/ addition & scalar multiplication that satisfies certain properties

Inner-product spaces

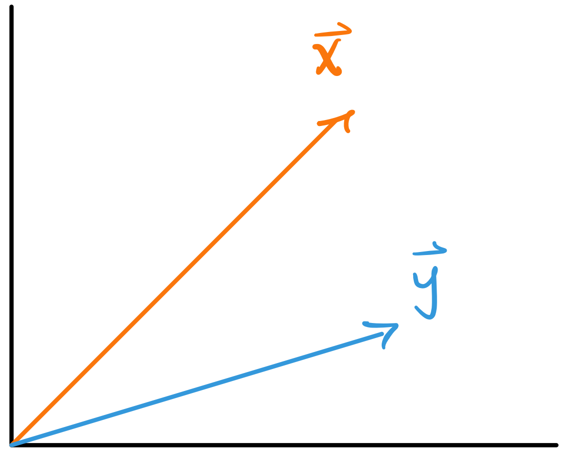
↳ vector spaces w/ an associated inner product

generalizes notion of dot product from $\mathbb{R}^n$.

↳ This will allow us to talk about lengths of vectors & angles between them in a well-defined way

Generalizing the dot product

Consider two vectors in $\mathbb{R}^2$



Their dot product is defined as $\vec{x}\cdot\vec{y} = x_1 y_1 + x_2 y_2$ & their lengths are

$$\|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}} = \underbrace{\sqrt{x_1^2 + x_2^2}}_{\text{pythagorean thm!}}$$

This general dot product satisfies:

- $\vec{x}\cdot\vec{x} \geq 0 \quad \forall\ \vec{x} \in \mathbb{R}^n$ (equal iff $\vec{x} = \vec{0}$)
- for $\vec{y}$ fixed, $\vec{x}\cdot\vec{y}$ is linear
- $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x} \quad \forall\ \vec{x},\vec{y} \in \mathbb{R}^n$

Now, we want to generalize this to both real & complex vector spaces.

Recall that the length of a complex # is $|z| = \sqrt{z\bar{z}}$. Motivated by this, we define the <u>norm</u> of a complex vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ as

$$\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

$$\iff \quad \|z\|^2 = z_1\bar{z}_1 + \cdots + z_n\bar{z}_n$$

want to view as the inner product w/ itself

We define the inner product on $\mathbb{C}^n$ as

$$w_1\bar{z}_1 + \cdots + w_n\bar{z}_n$$

note physicists define it $\bar{w}_1 z_1 + \cdots + \bar{w}_n z_n$

Inner products

As we have seen throughout the course, the structures we study hold for vectors that are far more general than vectors in $\mathbb{R}^n$, $\mathbb{C}^n$.

Motivated by them, though, we make the following formal def.

6.2 definition: *inner product*

An *inner product* on V is a function that takes each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbf{F}$ and has the following properties.

positivity
$\langle v, v\rangle \geq 0$ for all $v \in V$.

definiteness
$\langle v, v\rangle = 0$ if and only if $v = 0$.

additivity in first slot
$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ for all $u, v, w \in V$.

homogeneity in first slot
$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

conjugate symmetry
$\langle u, v\rangle = \overline{\langle v, u\rangle}$ for all $u, v \in V$.

6.4 definition: *inner product space*

An *inner product space* is a vector space V along with an inner product on V.

Examples

- Euclidean inner product

$$\langle (w_1, \dots, w_n), (z_1, \dots, z_n) \rangle = w_1 \overline{z_1} + \cdots + w_n \overline{z_n}$$

$$\forall \quad w, z \in \mathbb{C}^n$$

- Inner product on continuous, real-valued functions on $[a,b]$

$$\langle f, g \rangle = \int_a^b fg$$

- random variables

$$\langle X, Y \rangle = \mathbb{E}[XY]$$

Properties

6.6 *basic properties of an inner product*

(a) For each fixed $v \in V$, the function that takes $u \in V$ to $\langle u, v \rangle$ is a linear map from V to $\mathbf{F}$.

(b) $\langle 0, v \rangle = 0$ for every $v \in V$.

(c) $\langle v, 0 \rangle = 0$ for every $v \in V$.

(d) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.

(e) $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$ for all $\lambda \in \mathbf{F}$ and all $u, v \in V$.

Proof. a, b, c are immediate from def.

d) $\langle u, v+w \rangle = \overline{\langle v+w, u \rangle}$, conjugate symmetry

$= \overline{\langle v,u \rangle + \langle w,u \rangle}$, linearity

$= \langle \overline{v,u} \rangle + \langle \overline{w,u} \rangle$

$= \langle u,v \rangle + \langle u,w \rangle$

e) same logic as d)

Norms

Every inner product induces a norm via the following def.

6.7 definition: *norm,* $\|v\|$

For $v \in V$, the *norm* of v, denoted by $\|v\|$, is defined by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

6.9 *basic properties of the norm*

Suppose $v \in V$.

(a) $\|v\| = 0$ if and only if $v = 0$.

(b) $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbf{F}$.

Orthogonality

6.10 definition: *orthogonal*

Two vectors $u, v \in V$ are called *orthogonal* if $\langle u, v\rangle = 0$.

6.11 *orthogonality and* 0

(a) 0 is orthogonal to every vector in V.

(b) 0 is the only vector in V that is orthogonal to itself.

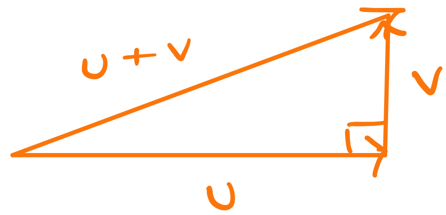
Inequalities on inner-product spaces

6.12 *Pythagorean theorem*

Suppose $u, v \in V$. If u and v are orthogonal, then

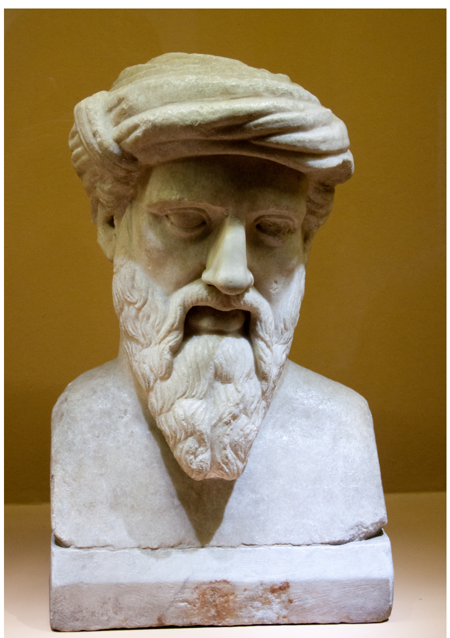
$$\|u + v\|^2 = \|u\|^2 + \|v\|^2.$$

Proof.

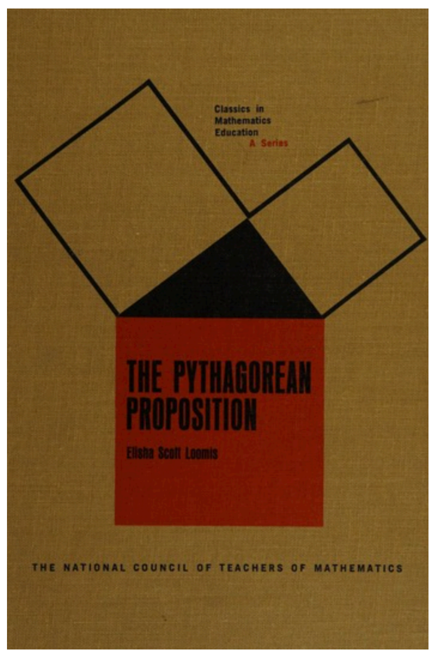


$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + \underbrace{\langle u,v\rangle}_{0} + \underbrace{\langle v,u\rangle}_{0} + \langle v,v\rangle \\ &= \|u\|^2 + \|v\|^2 \qquad \square\end{aligned}$$

Babylonians (~1800 BCE)



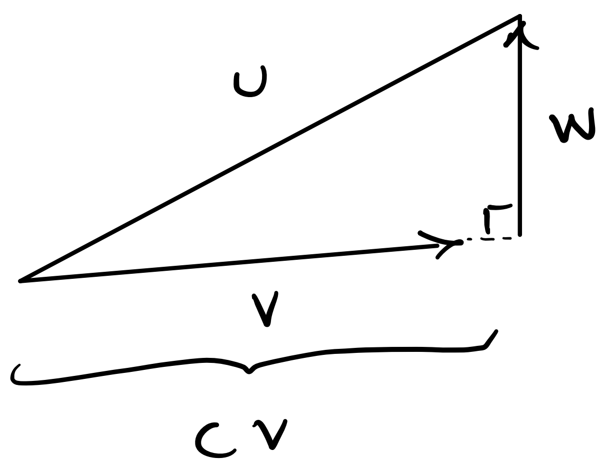
Pythagoras (~500 BCE)



~ 370 distinct proofs

Orthogonal decomposition

It will often be useful to write a vector u as a scalar multiple of v plus a vector w that is orthogonal to v



From this picture we see

$u = cv + w$ & of course $u = u + cv - cv$

$u = cv + (u - cv)$

Thus $w = (u - cv)$. Now, we want to find c s.t. $\langle w, v \rangle = 0$. We have

$0 = \langle u - cv, v \rangle \Rightarrow \langle u, v \rangle = c \langle v, v \rangle \Rightarrow c = \dfrac{\langle u, v \rangle}{\|v\|^2}$

$$u = \frac{\langle u, v \rangle}{\|v\|^2} v + \left(u - \frac{\langle u, v \rangle}{\|v\|^2} v \right)$$

6.13 *an orthogonal decomposition*

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u, v\rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u, v\rangle}{\|v\|^2}v$. Then

$$u = cv + w \quad \text{and} \quad \langle w, v\rangle = 0.$$

This allows us to prove one of the most important inequalities in all of math

6.14 *Cauchy–Schwarz inequality*

Suppose $u, v \in V$. Then

$$|\langle u, v\rangle| \leq \|u\|\,\|v\|.$$

This inequality is an equality if and only if one of u, v is a scalar multiple of the other.

Proof. If $v=0$, holds trivially. If $v \neq 0$, we may consider

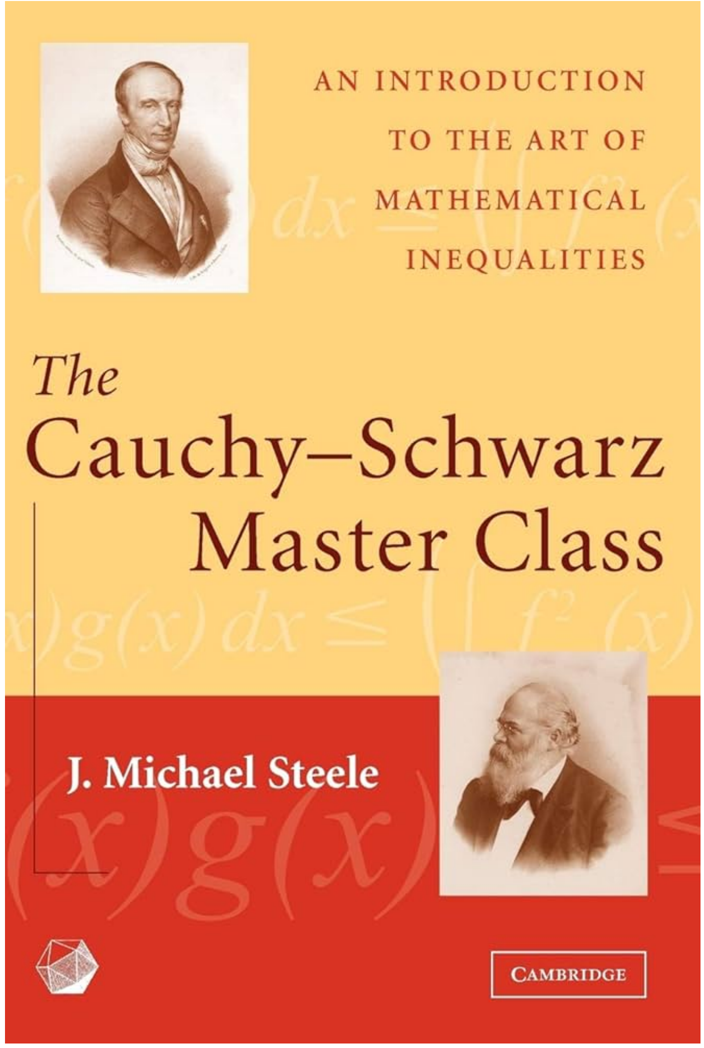
$$u = \frac{\langle u,v\rangle}{\|v\|^2} v + w$$

$$\|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2 \quad , \quad \text{pythagoras}$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^2} + \underbrace{\|w\|^2}_{\geq 0}$$

$$\|u\|^2 \geq \frac{|\langle u,v\rangle|^2}{\|v\|^2} \qquad \square$$

It is so important that whole books have been written about it



6.17 *triangle inequality*

Suppose $u, v \in V$. Then

$$\|u + v\| \leq \|u\| + \|v\|.$$

This inequality is an equality if and only if one of u, v is a nonnegative real multiple of the other.

6.21 *parallelogram equality*

Suppose $u, v \in V$. Then

$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$