# Math 416: Abstract Linear Algebra

Date: Oct. 20, 2025

Lecture: 21

## Announcements

- ☐ practice exams available
- ☐ review on wednesday (bring questions!)
- ☐ Midterm 2 : $\boxed{\text{Fri, Oct 24 @ 1pm}}$

## Last time

- ☐ Existence of eigenvals & upper-triangular matrices

## This time

- ☐ Upper-triangular matrices

## Reading/watching

- ☐ §5C of Axler

We are approximately half way through the course

### The author's top ten

Listed below are the author's ten favorite results in the book, in order of their appearance in the book. Students who leave your course with a good understanding of these crucial results will have an excellent foundation in linear algebra.

- any two bases of a vector space have the same length (2.34) ✓
- fundamental theorem of linear maps (3.21) ✓
- existence of eigenvalues if $\mathbf{F} = \mathbf{C}$ (5.19) ✓
- upper-triangular form always exists if $\mathbf{F} = \mathbf{C}$ (5.47) today!
- Cauchy–Schwarz inequality (6.14)
- Gram–Schmidt procedure (6.32)
- spectral theorem (7.29 and 7.31)
- singular value decomposition (7.70)
- generalized eigenspace decomposition theorem when $\mathbf{F} = \mathbf{C}$ (8.22)
- dimension of alternating $n$-linear forms on $V$ is 1 if $\dim V = n$ (9.37)

# Warm-up

Define $T \in \mathcal{L}(\mathbb{F}^3)$ by

$$T(x,y,z) = (2x+y, 5y+3z, 8z).$$

What are the eigenvalues of $T$?
↳ and eigenvectors!

Soln. Start w/ eigenvalue def.

$$T(x,y,z) = \lambda(x,y,z)$$

$$(2x+y, 5y+3z, 8z) = (\lambda x, \lambda y, \lambda z)$$

$8z = \lambda z$ : If $z \neq 0$, $\boxed{\lambda = 8}$.

To find eigenvector, plug $\lambda = 8$ into other eqs.

$5y + 3z = 8z \Rightarrow 5y = 5z \Rightarrow y = z$

and $2x + y = 8x \Rightarrow y = 6x \Longrightarrow 6x = y$

Thus, eigenvectors are scalar multiples of $(1,6,6)$

Check! $T(1,6,6) = (8,48,48) = 8(1,6,6)$ ✓

If $z=0$, $5y+3z=\lambda y \Rightarrow 5y=\lambda y$

$5y=\lambda y$: If $y\neq 0$, $\boxed{\lambda=5}$

$2x+y=5x \Rightarrow y=3x$ & $z=0$

So scalar multiples of $(1,3,0)$ are eigenvectors w/ eigenval $\lambda=5$

Check! $T(1,3,0)=(5,15,0)=5(1,3,0)$ ✓

If $y=0$, $2x+y=\lambda x \Rightarrow 2x=\lambda x$

$2x=\lambda y$: $x$ cannot be zero at this point or we would have $(x,y,z)=(0,0,0)$.

Thus, $x\neq 0$ & $\boxed{\lambda=2}$

and the eigenspace is simply spanned by $(1,0,0)$.

Check! $T(1,0,0)=(2,0,0)=2(1,0,0)$ ✓

# Upper-triangular matrices

**Example.** The map above is upper triangular

$$T(x,y,z) = (2x+y,\ 5y+3z,\ 8z).$$

$$\Big\downarrow \text{ standard basis}$$

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

A matrix $A \in M_n(\mathbb{F})$ is called upper-triangular if $A_{ij} = 0$ whenever $i > j$. That is,

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$*$ denotes potentially non-zero entries we do not care about

When can we achieve this form?

Prop. 5.39 (conditions for upper-triangular)

Let $T \in \mathcal{L}(V)$ & $v_1, \ldots, v_n$ be a basis of $V$.

Then, the following are equivalent.

a) $\mathcal{M}(T)$ w.r.t $v_1, \ldots, v_n$ is upper-tri

b) $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under $T$ $\forall$ $j = 1, \ldots, n$

c) $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ $\forall$ $j = 1, \ldots, n$

Proof. We will show $a \Leftrightarrow c$ & $b \Leftrightarrow c$.

(a) $\Leftrightarrow$ c)) Recall def of $A = \mathcal{M}(T)$:

$T(v_j) = \sum_{i=1}^{n} A_{ij} v_i$. Then, we have

$$\begin{aligned} A \text{ upper-tri} &\Leftrightarrow A_{ij} = 0 \quad \forall\, i > j \\ &\Leftrightarrow T(v_j) = \sum_{i=1}^{j} A_{ij} v_i \\ &\Leftrightarrow T(v_j) \in \operatorname{span}(v_1, \ldots, v_j) \end{aligned}$$

(b) $\Rightarrow$ c)) If $\operatorname{span}(v_1, \ldots, v_j)$ is invar. under $T$, then $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ trivially.

(c) ⇒ b)) Fix $j \in \{1, \dots, n\}$. By c), we may write

$$T v_1 \in \operatorname{span}(v_1) \subset \operatorname{span}(v_1, \dots, v_j)$$
$$T v_2 \in \operatorname{span}(v_1, v_2) \subset \operatorname{span}(v_1, \dots, v_j)$$
$$\vdots$$
$$T v_j \in \operatorname{span}(v_1, \dots, v_j)$$

Thus $\forall j$, $v \in \operatorname{span}(v_1, \dots, v_j)$ is of the form

$$v = \sum_{i=1}^{j} \lambda_i v_i$$

$$\Rightarrow \quad T v = \sum_{i=1}^{j} \lambda_i \underbrace{T v_i}_{\in \operatorname{span}(v_1, \dots, v_j)}$$

$\Rightarrow$ $\operatorname{span}(v_1, \dots, v_j)$ is invariant under $T$. $\square$

Great! We have conditions for upper-triangularity ... but will all $T \in \mathcal{L}(V)$ admit this form?

## Prop 5.47 (Every $T \in \mathcal{L}(V_{\mathbb{C}})$ has upper-triang.)

Suppose $V$ is a finite-dim complex vector space. Then $T$ has an upper-triang. matrix w.r.t some basis.

Note: I prefer the proof in the third edition of Axler b/c it is slightly more self contained.

**Proof.** We will use induction on $\dim V$.

Base case ($\dim V = 1$): Trivially true.

Induction Hypothesis: Suppose $\dim V > 1$ & result holds for all complex VS w/ dimension strictly less than $\dim V$.

In any such space, we know an eigenvalue of $T$ will exist. Let $\lambda$ be said eigenval & let

$$U = \text{range}\,(T - \lambda I)$$

B/c $\lambda$ is an eigenval $\Rightarrow$ $T - \lambda I$ is not surj. thus $\dim U < \dim V$.

Moreover, $U$ is invariant under $T$. To see this, let $u \in U$. Then

$$Tu = Tu - \lambda u + \lambda u$$

$$= \underbrace{(T - \lambda I)u}_{\in U} + \underbrace{\lambda u}_{\in U}$$

$\therefore$ $Tu \in U$, so $U$ is invar. under $T$.

This establishes that $T|_U$ is an operator on $U$. By ind. hyp., $\exists$ basis $u_1, \dots, u_m$ of $U$ s.t. $T|_U$ is upper-tri.

Then, using Prop 5.39, $\forall\ j \in \{1, \dots, m\}$

$$Tu_j = (T|_U)(u_j) \in \operatorname{span}(u_1, \dots, u_j)$$

Extend $u_1, \dots, u_m$ to a basis $u_1, \dots, u_m, v_1, \dots, v_n$ of $V$. For each $k \in \{1, \dots, m, m+1, \dots, m+n\}$, we have

$$Tv_k = Tv_k - \lambda v_k + \lambda v_k$$
$$= \underbrace{(T - \lambda I)v_k + \lambda v_k}_{\in \operatorname{span}(u_1, \dots, u_m) \subset \operatorname{span}(u_1, \dots, u_m, v_1, \dots, v_k)}$$

Thus, again by 5.39, we have that $T$ has an upper-triang. w.r.t $u_1, \dots, u_m, v_1, \dots, v_n$. $\square$

If we can find this basis, which is not always easy, we can simply read off the eigenvals.

**Example.** From before,

$$T(x,y,z) = (2x+y, 5y+3z, 8z).$$

$$\Big\downarrow \text{ standard basis}$$

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

Thus, eigenvalues are simply 2, 5, 8.

See Axler 5.41 for proof.

**Note:** An upper diag. matrix of $T \in \mathcal{L}(V)$ w.r.t basis $v_1, \dots, v_n$ of $V$ will have eigenvals along diag. & $v_1$ will be an eigenvec. However, $v_2, \dots, v_n$ need not be eigenvectors! We will see the case when they <u>are</u> after the exam!