Math 416: Abstract Linear Algebra

Date: Sept. 17, 2025

Lecture: 10

Announcements

- ☐ HW3 is now live. Due 9/19
- ☐ Updated office hours:
  - Tuesdays 5 - 5:50 Davenport 336
  - Wednesdays 2-2:50 Daveport 132
- ☐ Exam #1 : Wed. 9/24

  ↳ Fair game:
  - basic matrix LA
  - Sec. 1A - 3B of axler is fair game

Last time

- Bases & dimension

This time

- Linear maps

Recommended reading/watching

- §3A of Axler
- 3blue1brown : linear transformations

Next time

- The FUNdamental Theorem of linear maps

Def. 2.26 Basis

A basis of V is a list of vectors that is:

- linearly independent
- spans V

A basis consists of:

- a minimal # of vectors that span the space
- max # of LI vectors

Basis facts

- every spanning list contains a basis
- every finite-dim vector space has a basis
- every LI list extends to a basis

Def. 2.35 dimension

- dimension of finite-dim vector space is the length of any basis of V
- we denote this $\underline{\dim V}$

Examples

- $\dim \mathbb{F}^n = n$
- Let $\mathcal{P}_m(\mathbb{F})$ be the set of all polynomials w/ coeff. in $\mathbb{F}$ and degree at most m

  $\hookrightarrow \dim \mathcal{P}_m(\mathbb{F}) = m+1$

  $\hookrightarrow \underbrace{x^0, x^1, \ldots, x^m}_{m+1}$

- $U = \{(x,y,z) \in \mathbb{F}^3 : x+y+z=0\}$

Facts about dimension

Suppose V is finite-dim. Then, the following hold:

- $U \leq V \implies \dim U \leq \dim V$
- Every LI list in V of length $\dim V$ is a basis of V
  - ↳ same for spanning list
- If $U \leq V$ & $\dim U = \dim V$, then $U = V$.
- If $V_1, V_2 \leq V$, then

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

↳ See proof on pg. 47

Chapter 3

Linear Maps

So far our attention has focused on vector spaces. No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn—linear maps.

Def. 3.1 (linear map)

A linear map from V to W is a function $T: V \to W$ satisfying

additivity.

$$T(u+v) = Tu + Tv, \quad \forall\, u,v \in V$$

homogeneity.

$$T(\lambda v) = \lambda(Tv) \quad \forall\, \lambda \in \mathbb{F} \text{ \& all } v \in V$$

Notation

- $\mathcal{L}(V,W) = \{\text{all lin. maps from } V \text{ to } W\}$
- $\mathcal{L}(V) := \mathcal{L}(V,V)$

Examples of linear maps

- $0 \in \mathcal{L}(V,W)$ is defined $\underset{\text{func.}}{0}\, v = \underset{\text{vec}}{0}$
- $I \in \mathcal{L}(V)$ is defined by $Iv = v \quad \forall\, v \in V$

  ↳ $\mathbb{1}$, 1, I all used as well

- Differentiation

  $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ s.t. $Dp = p'$

  ↳ in more standard calc notation

  $$\frac{d}{dx}\left[A f(x) + B g(x)\right] = A f'(x) + B g'(x)$$

  $$\forall\ A, B \in \mathbb{R}\ \&\ f(x), g(x) \in \mathcal{P}(\mathbb{R}_x)$$

- Multiplication by x^2

  Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ be defined as

  $$(Tp)(x) = x^2 p(x) \quad \forall\ x \in \mathbb{R}$$

- $\mathbb{R}^3$ to $\mathbb{R}^2$

  Define $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ by

  $$T(x_1, x_2, x_3) = (2x_1 - x_2 + 3x_3,\ 7x_1 + 5x_2 - 6x_3)$$

- $\mathbb{F}^n$ to $\mathbb{F}^m$. Let $A_{j,k} \in \mathbb{F}$ $\forall$ $j = 1, \ldots, m$, $k = 1, \ldots, n$

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ \ldots\ A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

↪ It turns out every map from $\mathbb{F}^n \to \mathbb{F}^m$ is of this form (see HW 4)

Lemma 3.4 (linear map lemma)

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ s.t.

$$Tv_k = w_k \qquad \forall\, k \in [n]$$

$\{1, 2, \dots, n\}$

Proof. Existence. $\{v_k\}_{k=1}^{n}$ is a basis, so $\forall\, v \in V$, $\exists\, c_k \in \mathbb{F}$ s.t.

$$v = \sum_k c_k v_k .$$

$\{v_k\}$ is basis, so this is unique, thus

(*) $$T(c_1 v_1 + \dots + c_n v_n) = c_1 w_1 + \dots + c_n w_n$$

defines a func from V to W.

For each k, let $c_k = 1$ and $c_j = 0 \;\forall\, j \neq k$. This yields

$$Tv_k = w_k$$

Thus, we have a function from V to W.

Consider $u, v \in V$. Let $a_k, c_k, \alpha, \beta \in \mathbb{F}$.

$$u = a_1 v_1 + \cdots + a_n v_n$$

$$v = c_1 v_1 + \cdots + c_n v_n$$

$$T(\alpha u + \beta v)$$

$$= T\left((\alpha a_1 + \beta c_1) v_1 + \cdots + (\alpha a_n + \beta c_n) v_n \right)$$

$$\overset{(*)}{=} (\alpha a_1 + \beta c_1) w_1 + \cdots + (\alpha a_n + \beta c_n) w_n$$

$$= \alpha (a_1 w_1 + \cdots + a_n w_n) + \beta (c_1 w_1 + \cdots + c_n w_n)$$

$$= \alpha T u + \beta T v,$$

thus T is a linear map from V to W.

↓ cont.

To prove uniqueness, suppose $S \in \mathcal{L}(V,W)$ s.t. $Sv_k = w_k \quad \forall\ k \in [n]$. Let $c_1, \dots c_k \in \mathbb{F}$. For contradiction, assume $S(v) \neq T(v)$.

$$\begin{aligned} S(v) &= S\left(\sum_k c_k v_k\right) \\ &= \sum_k c_k S(v_k) \\ &= \sum_k c_k w_k \qquad \text{, assumption} \\ &= T(v) \qquad , \end{aligned}$$

which is a contradiction. $\square$

Take-aways

- Existence: we can find a linear map that takes on whatever values we wish on the basis
- Uniqueness: linear map is completely determined by the values it takes on a basis

Algebraic operations on $\mathcal{L}(V,W)$

- $S, T \in \mathcal{L}(V,W)$ & $\lambda \in \mathbb{F}$.

$$(S+T)(v) = Sv + Tv \quad \& \quad (\lambda T)(v) = \lambda(Tv)$$

- $\mathcal{L}(V,W)$ is a vector space w/ the above defn. of addition and scalar mult.

- product of linear maps

let $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$.
then $ST \in \mathcal{L}(U,W)$ is defined by

$$\underbrace{(ST)(u) = S(Tu)}_{\text{just standard composition of functions}} \quad \forall \ u \in U.$$

$$h(x) := g(f(x)) \iff (g \circ f)(x)$$

Properties of products of lin. maps

- Associativity : $(T_1 T_2) T_3 = T_1 (T_2 T_3)$
- Identity : $\underset{\substack{\uparrow \\ I \in \mathcal{L}(V)}}{TI} = \underset{\substack{\uparrow \\ I \in \mathcal{L}(W)}}{IT} = T \quad \forall\ T \in \mathcal{L}(V, W)$
- Distributive : $(S_1 + S_2) T = S_1 T + S_2 T$

$$S(T_1 + T_2) = ST_1 + ST_2$$

$$T, T_1, T_2 \in \mathcal{L}(U, V)$$

$$S, S_1, S_2 \in \mathcal{L}(V, W)$$