MATH 416 Abstract Linear Algebra

Final Exam – Dec. 17, 2025

Exam Instructions: This is a **closed-book** exam and you have **3 hours** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"The beauty of mathematics only shows itself to more patient followers."*
>
> —Maryam Mirzakhani

Question 1 (10 points): **The Vector Space of Linear Maps**

We saw in class that the set $\mathcal{L}(U, V)$ of all linear maps from U to V is, indeed, a vector space. Recall also that $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials over $\mathbb{R}$.

(i) (5 points) Suppose $m, b \in \mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ is a linear map if and only if $b = 0$. *Hint: remember that a linear map takes 0 to 0, that is $T(0) = 0$.*

> **Proof.** First, suppose that $f(x) = mx + b$ is a linear map. Consider $f(0)$:
>
> $$f(0) = f(x + (-x)), \tag{1}$$
> $$= f(x) + f(-x), \quad \text{Linearity assumption} \tag{2}$$
> $$= (mx + b) + (m(-x) + b), \tag{3}$$
> $$= 2b. \tag{4}$$
>
> A linear map must map zero to zero, thus we conclude that $b = 0$. Now, assume that $b = 0$ so that $f(x) = mx$. This is clearly linear
>
> $$f(ax + cy) = m(ax + cy) = a(mx) + c(my) = af(x) + bf(y), \tag{5}$$
>
> and we are done.

(ii) (1 point) Define a function $T : \mathcal{P}_m(\mathbb{R}) \to \underline{\quad}$ by $(Tp)(x) = xp(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. What space does T map into?

> **Answer.** Because we are multiplying by x we will increase the maximum degree of the polynomial by 1, yielding $\mathcal{P}_{m+1}(\mathbb{R})$.

(iii) (4 points) Consider $S \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ defined as $(Sp)(x) = p(x + a)$ for all $p \in \mathcal{P}(\mathbb{R})$. With T defined as in (ii), show that $ST \neq TS$.

> **Proof.** First applying T and then S, we obtain
>
> $$((ST)p)(x) = S((Tp)(x)), \tag{6}$$
> $$= S(xp(x)), \tag{7}$$
> $$= (x + a)p(x + a). \tag{8}$$

However, in the opposite direction, we see

$$((TS)p)(x) = T((Sp)(x)), \tag{9}$$
$$= T(p(x+a)), \tag{10}$$
$$= xp(x+a), \tag{11}$$

thus we see the two maps are unequal (i.e. they do not commute).

Question 2 (10 points): **Null Spaces and Ranges**

For this entire problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the range of T?

> **Solution.** The range of T is the set of all vectors in W that can be reached by acting T on some vector in V. Formally,
>
> $$\text{range } T = \{Tv : v \in V\}. \tag{12}$$

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is range D?

> **Solution.** All polynomials can be reached by applying the differentiation operator to a polynomial of a higher degree. Because D is defined here from all polynomials to all polynomials, the range is simply $\mathcal{P}(\mathbb{R})$.

(iii) (6 points) Prove that range T is a subspace of W.

> *Proof.* We must show that $0 \in$ range T and that it is closed under scalar multiplication and addition. For the zero vector, note that $T0 = 0$, thus $0 \in$ range T. Now, let $u, v \in$ range T and $a, b \in \mathbb{F}$. Then, there exists $u' \in V$ such that $u = Tu'$ and $v' \in V$ such that $v = Tv'$. It follows that
>
> $$au + bv = aTu' + bTv', \tag{13}$$
> $$= T(au' + bv'), \tag{14}$$
>
> and $au' + bv' \in V$ because it is a vector space (and thus closed under linear combinations of elements). Thus, $au + bv \in$ range T, as desired. Thus, range T is a subspace of W. $\qquad\square$

Question 3 (10 points): **Inner Product Spaces and Positive Operators**

(i) (5 points) Suppose $u, v \in V$. Then,

$$\|u + v\| \leq \|u\| + \|v\|. \tag{15}$$

This inequality is an equality if and only if one of u, v is a non-negative real multiple of the other.

> *Proof.* Expanding the norm and using conjugate symmetry gives
>
> $$\|u + v\|^2 = \|u\|^2 + 2\operatorname{Re}\langle u, v \rangle + \|v\|^2.$$
>
> By Cauchy–Schwarz,
> $$\operatorname{Re}\langle u, v \rangle \leq |\langle u, v \rangle| \leq \|u\|\,\|v\|,$$
>
> so
> $$\|u + v\|^2 \leq (\|u\| + \|v\|)^2,$$
>
> and taking square roots yields the inequality. Equality holds iff both inequalities above are equalities, which occurs exactly when $|\langle u, v \rangle| = \|u\|\,\|v\|$ and $\langle u, v \rangle \geq 0$, i.e. when u and v are linearly dependent and point in the same direction. Thus one is a non-negative real multiple of the other. $\square$

(ii) (5 points) Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

> *Proof.* Because T is positive, it is self-adjoint. The hypothesis $\langle Tv, v \rangle = 0$ and the Cauchy–Schwarz inequality give
>
> $$0 = |\langle Tv, v \rangle| \leq \|Tv\|\,\|v\|.$$
>
> If $v = 0$, the conclusion is trivial. If $v \neq 0$, then $\|v\| > 0$, so the inequality forces $\|Tv\| = 0$. Hence $Tv = 0$. See page 254 of Axler for another proof technique. $\square$

Question 4 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(a) (5 points) Prove that the eigenvalues of a self-adjoint operator are real.

> *Proof.* Let λ be an eigenvalue of T with eigenvector v. Then
>
> $$\lambda\|v\|^2 = \langle Tv, v\rangle = \langle v, T^*v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\|v\|^2. \tag{16}$$
>
> Subtracting the first and last expressions give $(\lambda - \overline{\lambda})\|v\|^2 = 0$. Because $v \neq 0$, we conclude that $\lambda = \overline{\lambda}$. That is $\lambda \in \mathbb{R}$. $\qquad\square$

(b) (5 points) Suppose that $T \in \mathcal{L}(V)$. Show that if T is self-adjoint and all of its eigenvalues are non-negative, then T is a positive operator. *Hint: use the spectral theorem!*

> *Proof.* Because T is self-adjoint, the spectral theorem tells us that any vector $v \in V$ may be expressed as
>
> $$v = \sum_i a_i v_i, \tag{17}$$
>
> where v_i are eigenvectors of T, with eigenvalues λ_i all non-negative. Then, we may right
>
> $$\langle Tv, v\rangle = \langle T\sum_i a_i v_i, \sum_j a_j v_j\rangle, \tag{18}$$
>
> $$= \langle \sum_i a_i Tv_i, \sum_j a_j v_j\rangle, \tag{19}$$
>
> $$= \langle \sum_i a_i \lambda_i v_i, \sum_j a_j v_j\rangle, \tag{20}$$
>
> $$= \sum_i \sum_j \lambda_i a_i \overline{a_j}\langle v_i, v_j\rangle, \tag{21}$$
>
> $$= \sum_i \lambda_i |a_i|^2, \tag{22}$$
>
> $$\geq 0, \tag{23}$$
>
> because $\lambda_i \geq 0$ for all i. $\qquad\square$

Question 5 (10 points): **Determinant and Trace**

(a) (2 points) For an invertible matrix A, prove that $\det\left(A^{-1}\right) = (\det A)^{-1}$.

> *Proof.* If A is invertible, there exists A^{-1} such that $AA^{-1} = I$. Thus, we may write
>
> $$\det(I) = \det\left(AA^{-1}\right), \qquad (24)$$
> $$= \det(A)\det\left(A^{-1}\right). \qquad (25)$$
>
> Noting that $\det(I) = 1$, we obtain the desired result. $\qquad\square$

(b) (2 points) Determine whether the following matrix is invertible

$$X = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 1 & 5 \end{pmatrix}. \qquad (26)$$

> **Solution.** The matrix is invertible if and only if the determinant is non-zero, thus we compute
>
> $$\det(X) = 2(4\cdot 5 - 7\cdot 1) - 3(3\cdot 5 - 7\cdot 4) + 5(3\cdot 1 - 4\cdot 4), \qquad (27)$$
> $$= 26 + 39 - 65, \qquad (28)$$
> $$= 0, \qquad (29)$$
>
> therefore X is not invertible.

(c) (2 point) What is the sum of the eigenvalues of the above matrix? *Hint: do not try to actually compute each eigenvalue.*

> **Solution.** Recall that the trace is basis independent. Moreover, we can always go to a basis that makes X upper-triangular (i.e. it has the eigenvalues along the diagonal). Thus, the trace of a matrix is always the sum of the eigenvalues. In this case, that is $2 + 4 + 5 = 11$.

(d) (4 points) Suppose that A, B, C are 3-by-3 matrices with $\det(A) = 2$, $\det(B) = 3$, and $\det(C) = 5$. Compute each of the following determinants:

(a) $\det(AB)$

(b) $\det\left(2A^{-3}B^{-2}(CB)^4\right)$

> **Solution.**
>
> (a) $\det(AB) = \det(A)\det(B) = 2 \cdot 3 = 6$
>
> (b) We can successively apply this property to the composition of these maps.
>
> $$\det\left(2A^{-3}B^{-2}(CB)^4\right) = \det(2I) \cdot \det(A)^{-3} \cdot \det(B)^{-2} \cdot \det(C)^4 \det(B)^4, \quad (30)$$
>
> $$= 2^3 \cdot \det(A)^{-3} \cdot \det(B)^2 \cdot \det(C)^4, \quad (31)$$
>
> $$= 8 \cdot \frac{1}{8} \cdot 9 \cdot 625, \quad (32)$$
>
> $$= 5625 \quad (33)$$