MATH 416 Abstract Linear Algebra

Midterm 3 – November 19, 2025

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

"An expert is a [person] who has made all the mistakes, which can be made, in a very narrow field."

—Niels Bohr

Question 1 (10 points): **Inner Product Spaces**

Let U be a *subset* of V. Recall that the orthogonal complement $U^{\perp}$ is defined as the set of all vectors in V that are orthogonal to every vector in U. Moreover, recall that P_U denotes the orthogonal projector of V onto U.

(a) (4 points) Prove that if U is a subset of V, then $U^{\perp}$ is a subspace of V.

(b) (6 points) Prove that $P_U \in \mathcal{L}(V)$ (i.e. that it is a linear operator on V). *Hint: You may use the fact that $V = U \oplus U^{\perp}$, which implies all $v \in V$ take the form $v = u + w$, where $u \in U$ and $w \in U^{\perp}$.*

Bonus (1 point): Prove $\|P_U v\| \leq \|v\|$ for all $v \in V$.

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(a) (5 points) Prove that the eigenvalues of a self-adjoint operator are real.

(b) (5 points) Prove that if all of the eigenvalues of a normal operator on a complex vector space are real, then the operator is self-adjoint. *Hint: Recall that, with respect to an orthonormal basis,* $\mathcal{M}(T)^* = \mathcal{M}(T^*)$.

Bonus (1 point): What is another term for self-adjoint?

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(a) (4 points) Suppose that $T \in \mathcal{L}(V)$. Show that if T is self-adjoint and all of its eigenvalues are non-negative, then T is a positive operator. *Hint: use the spectral theorem!*

(b) (6 points) Show that the product of two unitary operators on V is a unitary operator on V and that the inverse of a unitary operator is also a unitary operator. *Hint: It may be useful to recall that all unitaries satisfy* $U^* = U^{-1}$.

Bonus (1 point): What does part (b) imply about the set of unitary operators along with the binary operation of operator composition?

(Optional) Bonus Challenge Problem (1 points)

Let H be an operator on a finite dimensional vector space and suppose $U = e^{iH}$, what condition on H guarantees U is unitary.

Final Bonus Opportunity (1 point)

It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?