Math 416: Abstract Linear Algebra

Date: Oct. 8, 2025

Lecture: 16

Announcements

- HW5 is due Fri, Oct. 10 @ 8pm
- office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Daveport 132

Last time

- Invertibility & isomorphic vector spaces

This time

- Linear maps as matrix mult. & basis chage

Reading

- §3D of Axler

Isomorphic vector spaces

- An <u>isomorphism</u> is an invertible linear map
- Two vec. spaces V, W are isomorphic if $\exists$ an isomorphism between them. We denote this $V \cong W$

Prop. 3.70 (dim. shows whether vector spaces are isomorphic)

V & W are isomorphic $\iff \dim V = \dim W$

proof. ($\Rightarrow$) If $V \cong W$, $\exists$ an isomorphism $T: V \to W$. Invertibility implies $\operatorname{null} T = \{0\}$ & $\operatorname{range} T = W$.
FTLM implies

$$\dim V = 0 + \dim W \quad \checkmark$$

($\Leftarrow$) Suppose $\dim V = \dim W$.
Let $v_1, \ldots, v_n$ & $w_1, \ldots, w_n$ be bases of V & W, resp. Define

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

- T is well defined b/c $v_1, \dots, v_n$ is a basis
- T is surjective b/c $w_1, \dots, w_n$ spans W
- T is inj. ($\text{null}\, T = \{0\}$) b/c $w_1, \dots, w_n$ is LI.
- T is inj. & surj., thus invertible.
  - $\hookrightarrow \Rightarrow V \cong W.$

$\square$

Remarks

- These results imply every finite-dim vec. space V is isomorphic to $\mathbb{F}^n$ w/ $n = \dim V$.
  - $\hookrightarrow$ e.g. $\mathcal{P}_m(\mathbb{F}) \cong \mathbb{F}^{m+1}$
- $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$ (prop. 3.71)
  - $\hookrightarrow \dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ (prop 3.72)

Linear maps thought of as matrix mult.

Let $v_1, \dots, v_n$ be a basis for V.

We define the matrix of v w.r.t this basis as

$$M(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$\forall \; v = b_1 v_1 + \dots + b_n v_n$

Ex. What is the matrix of

$$2 - 7x + 5x^3 + x^4$$

w.r.t the standard basis of $\mathcal{P}_4(\mathbb{R})$?

Soln. Standard basis is $\{1, x^1, x^2, x^3, x^4\}$

So we have

$$\begin{bmatrix} 2 \\ -7 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

With this def, we can think of linear maps as matrix mult. : $M(Tv) = M(T)M(v)$.

Invertible Matrices

Note that we will denote the identity matrix

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

We will use I to represent both the operator and the matrix.

- We say $A \in \mathbb{F}^{m,n}$ is invertible if $\exists$ B s.t. $AB = BA = I$.

  ↳ B is unique, so we denote it A^{-1}

Exercise

- Show $(A^{-1})^{-1} = A$ $\forall$ invertible A.

$AA^{-1} = A^{-1}A = I$, thus by uniqueness of the inverse, A must be the inverse of A^{-1}.

- Suppose A & C are invertible. Show AC is invertible & $(AC)^{-1} = C^{-1}A^{-1}$.

$$\begin{aligned} (AC)(C^{-1}A^{-1}) &= A(CC^{-1})A^{-1} \\ &= AIA^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

$$\begin{aligned} &(C^{-1}A^{-1})(AC) \\ &= C^{-1}(A^{-1}A)C \\ &= C^{-1}IC \\ &= I \end{aligned}$$

Thus by uniqueness of inverses,

$$(AC)^{-1} = C^{-1}A^{-1}$$

Change of basis

Is I always $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$?

Consider choosing two different bases for the input & output spaces.

Ex. $B_1 = \{(4,2), (5,3)\}$ & $B_2 = \{(1,0), (0,1)\}$ are bases of $\mathbb{F}^2$.

$I(4,2) = (4,2) = 4(1,0) + 2(0,1)$

$I(5,3) = (5,3) = 5(1,0) + 3(0,1)$

Thus $M_{B_1, B_2}(I) = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

What about the other way?

$B_1 \leftrightarrow B_2$

$I(1,0) = (1,0) = \frac{3}{2}(4,2) + (-1)(5,3)$

$I(0,1) = (0,1) = -\frac{5}{2}(4,2) + 2(5,3)$

So, $$M_{B_2,B_1}(I) = \begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix}$$

Now, how do these relate?

$$M_{B_1,B_2}(I)\, M_{B_2,B_1}(I) = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By (Axler 3.82) this is always the case! Using these ideas, one can prove the following change of basis formula (from $B_1 \Rightarrow B_2$)

$$M_{B_1}[T] = M_{B_2,B_1}[I]\, M_{B_2}[T]\, M_{B_1,B_2}[I]$$

$$A \quad = \quad C^{-1} \quad B \quad C$$