# MATH 416 Abstract Linear Algebra

Midterm 2 – October 24, 2025

**Exam Instructions:** This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"The art of doing mathematics is finding that special case that contains all the germs of generality."*
>
> — David Hilbert

**Question 1** (10 points): **Null Spaces and Ranges**

For this problem, let $V, W$ be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$. Recall, also, that the set of all polynomials is denoted $\mathcal{P}(\mathbb{R})$.

(i) (2 points) Formally state the definition of an injective map. Do the same for a surjective map.

**Bonus** (1 point): What is another name for injective? What about surjective?

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. Determine whether $D$ is injective, surjective, both, or neither.

(iii) (6 points) Recall that the fundamental theorem of linear maps relates the input space dimension to the size of the null space and range of any map as

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Use this to prove that if $\dim V > \dim W$, there are no injective linear maps from $V$ to $W$.

**Question 2** (10 points): **Invertibility and Isomorphisms**

Recall that a linear map $T \in \mathcal{L}(V, W)$ is an *isomorphism* if it is linear and bijective (i.e. injective and surjective). When this is the case, we say $V$ and $W$ are *isomorphic*.

(i) (4 points) Prove that if $V$ and $W$ are isomorphic, then $\dim V = \dim W$.

(ii) (6 points) Let $V = \mathcal{P}_1(\mathbb{R})$ (the space of polynomials of degree at most 1), $W = \mathbb{R}^2$, and define

$$T(a + bx) = (a, b).$$

Show that $T$ is an isomorphism. *Hint: Let $p = a_p + b_p x$ and $q = a_q + b_q x$ be two arbitrary polynomials in $V$.*

**Question 3** (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**
Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T(x, y) = (3x + y,\, 2y).$$

Recall that a number $\lambda \in \mathbb{F}$ (in this problem $\mathbb{F} = \mathbb{R}$ is called an *eigenvalue* of $T$ if there exists a non-zero $v \in V$ such that $Tv = \lambda v$.

(i) (4 points) Find *two distinct eigenvalues* of $T$ and a corresponding eigenvector for each one.

(ii) (2 points) Using your answers from (i), *verify directly* that your two eigenvectors are linearly independent. (4 points) Then, prove in general that if $v_1, v_2$ are eigenvectors of a linear operator corresponding to *distinct eigenvalues*, they must be linearly independent.

**(Optional) Bonus Challenge Problem** (3 points)

Generalize the your proof from the last problem. That is, prove that if we have $m$ distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with corresponding eigenvectors $v_1, \ldots, v_m$, then the list of eigenvectors is linearly independent.

**Final Bonus Opportunity** (1 point)

It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?