MATH 416 Abstract Linear Algebra

Week 15 - Homework 10
Assigned: Wed. Dec. 3, 2025
Due: Wed. Dec. 10, 2025 (by 11:59pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (6 points): **Trace**

(i) Prove that the trace is linear (i.e $\operatorname{tr}[A+B] = \operatorname{tr}[A] + \operatorname{tr}[B]$ and $\operatorname{tr}[zA] = z \operatorname{tr}[A]$).

(ii) Show that the space of traceless matrices, $\{A \in M_n(\mathbb{F})\colon\ \operatorname{tr}(A) = 0\}$, is a subspace of $M_n(\mathbb{F})$. What is its dimension?

(iii) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Prove that $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$.
Hint: use Axler 8.55 in the fourth edition.

Exercise 2 (9 points): **Determinant**

(i) Prove that $\det(AB) = \det(A)\det(B)$ for all 2-by-2 matrices A, B.

(ii) Consider the matrix

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}. \tag{1}$$

Solve the equation $\det(A - \lambda I) = 0$ to find the eigenvalues of A.

(iii) Let λ_1, λ_2 be two eigenvalues of a matrix A. Define the mean and the product of the eigenvalues of a 2-by-2 matrix as

$$m = \frac{1}{2}\operatorname{tr}[A] = \frac{1}{2}(\lambda_1 + \lambda_2), \tag{2}$$

$$p = \det(A) = \lambda_1 \lambda_2. \tag{3}$$

Prove that the eigenvalues of A are given as

$$\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}. \tag{4}$$