MATH 416 Abstract Linear Algebra

Midterm 3 – November 19, 2025

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"An expert is a [person] who has made all the mistakes, which can be made, in a very narrow field."*
>
> —Niels Bohr

Question 1 (10 points): **Inner Product Spaces**
Let U be a *subset* of V. Recall that the orthogonal complement $U^\perp$ is defined as the set of all vectors in V that are orthogonal to every vector in U. Moreover, recall that P_U denotes the orthogonal projector of V onto U.

(a) (4 points) Prove that if U is a subset of V, then $U^\perp$ is a subspace of V.

Proof. To verify that a *subset* of V is actually a *subspace* of V, we need to check that the zero vector is contained and that it is closed under addition and scalar multiplication. Clearly, for any element of $u \in U$, $\langle 0, u\rangle = 0$, thus $0 \in U^\perp$. To prove closure, suppose $a, b \in \mathbb{F}$ and let $w_1, w_2 \in U^\perp$. Then, for an arbitrary element $u \in U$, we have

$$\begin{aligned}
\langle aw_1 + bw_2, u\rangle &= a\langle w_1, u\rangle + b\langle w_2, u\rangle, && \text{linearity of first slot} && (1)\\
&= a0 + b0, && w_1, w_2 \in U^\perp, && (2)\\
&= 0, && && (3)
\end{aligned}$$

thus $aw_1 + bw_2 \in U^\perp$. Thus, $U^\perp$ is a subspace of V. $\square$

(b) (6 points) Prove that $P_U \in \mathcal{L}(V)$ (i.e. that P_U is a linear operator on V). *Hint: You may use the fact that $V = U \oplus U^\perp$, which implies all $v \in V$ take the form $v = u + w$, where $u \in U$ and $w \in U^\perp$.*

Proof. To show that something is a linear operator, we must show that $P_U(av_1 + bv_2) = aP_Uv_1 + bP_Uv_2$. Using the hint, we recall that an arbitrary element of v may be written uniquely as $v = u + w$, where $u \in U$ and $w \in U^\perp$. Letting $u_1, u_2 \in U$ and $w_1, w_2 \in U^\perp$, we may write

$$\begin{aligned}
P_U(av_1 + bv_2) &= P_U(au_1 + aw_1 + bu_2 + bw_2), && (4)\\
&= P_U(\underbrace{au_1 + bu_2}_{\in U} + \underbrace{aw_1 + bw_2}_{\in U^\perp}), && (5)\\
&= au_1 + bu_2, && (6)\\
P_U(av_1 + bv_2) &= aPv_1 + bPv_2. && (7)
\end{aligned}$$

$\square$

Bonus (1 point): Prove $\|P_Uv\| \leq \|v\|$ for all $v \in V$.

Proof. Let $v \in V$ with unique decomposition $v = u + w$. Note, further that $P_Uv = u$. Then, by the Pythagorean theorem, we may write

$$\|v\|^2 = \|u\|^2 + \|w\|^2 \geq \|u\|^2 = \|P_U v\|^2. \tag{8}$$

Taking the square root yield the desired inequality. $\square$

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(a) (5 points) Prove that the eigenvalues of a self-adjoint operator are real.

Proof. Let λ be an eigenvalue of T with eigenvector v. Then

$$\lambda\|v\|^2 = \langle Tv, v\rangle = \langle v, T^*v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \overline{\lambda}\|v\|^2. \tag{9}$$

Subtracting the first and last expressions give $(\lambda - \overline{\lambda})\|v\|^2 = 0$. Because $v \neq 0$, we conclude that $\lambda = \overline{\lambda}$. That is $\lambda \in \mathbb{R}$. □

(b) (5 points) Prove that if all of the eigenvalues of a normal operator on a complex vector space are real, then the operator is self-adjoint. *Hint: Recall that, with respect to an orthonormal basis,* $\mathcal{M}(T)^* = \mathcal{M}(T^*)$.

Proof. Let T be a normal operator on a finite-dimensional complex inner product space. By the spectral theorem, there is an orthonormal basis with respect to which $\mathcal{M}(T)$ is diagonal, with the eigenvalues of T on the diagonal. By assumption, all eigenvalues of T are real, so $\mathcal{M}(T)$ is a diagonal matrix with real entries on the diagonal. Hence taking the conjugate transpose leaves it unchanged:

$$\mathcal{M}(T)^* = \mathcal{M}(T).$$

With respect to this orthonormal basis, we also have

$$\mathcal{M}(T)^* = \mathcal{M}(T^*).$$

Therefore,

$$\mathcal{M}(T) = \mathcal{M}(T^*).$$

Since the two operators have the same matrix with respect to the same basis, it follows that $T = T^*$, so T is self-adjoint. □

Bonus (1 point): What is another term for self-adjoint?

Proof. Hermitian, which is in reference to Charles Hermite (1822-1901). □

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(a) (4 points) Suppose that $T \in \mathcal{L}(V)$. Show that if T is self-adjoint and all of its eigenvalues are non-negative, then T is a positive operator. *Hint: use the spectral theorem!*

Proof. Because T is self-adjoint, the spectral theorem tells us that any vector $v \in V$ may be expressed as

$$v = \sum_i a_i v_i, \tag{10}$$

where v_i are eigenvectors of T, with eigenvalues λ_i all non-negative. Then, we may right

$$\begin{aligned}
\langle Tv, v\rangle &= \langle T \sum_i a_i v_i, \sum_j a_j v_j\rangle, &(11)\\
&= \langle \sum_i a_i T v_i, \sum_j a_j v_j\rangle, &(12)\\
&= \langle \sum_i a_i \lambda_i v_i, \sum_j a_j v_j\rangle, &(13)\\
&= \sum_i \sum_j \lambda_i a_i \overline{a_j} \langle v_i, v_j\rangle, &(14)\\
&= \sum_i \lambda_i |a_i|^2, &(15)\\
&\geq 0, &(16)
\end{aligned}$$

because $\lambda_i \geq 0$ for all i. □

(b) (6 points) Show that the product of two unitary operators on V is a unitary operator on V and that the inverse of a unitary operator is also a unitary operator. *Hint: It may be useful to recall that all unitaries satisfy $U^* = U^{-1}$.*

Proof. Let U_1 and U_2 be two unitary operators. Then,

$$(U_1U_2)^*(U_1U_2) = (U_2^*U_1^*)(U_1U_2) = U_2^* I U_2 = I, \tag{17}$$
$$(U_1U_2)(U_1U_2)^* = (U_1U_2)(U_2^*U_1^*) = U_1^* I U_1 = I, \tag{18}$$

Thus, the product of two unitaries are unitary. Finally, recall that $U^* = U^{-1}$. We may write

$$(U^{-1})(U^{-1})^* = (U^{-1})(U^*)^* = U^{-1}U = I, \tag{19}$$

$$(U^{-1})^*(U^{-1}) = (U^*)^*(U^{-1}) = UU^{-1} = I. \tag{20}$$

□

Bonus (1 point): What does part (b) imply about the set of unitary operators along with the binary operation of operator composition?

It implies that the set of all unitaries with the binary operation of operator composition form a group. The unitary group plays a very important role in theoretical physics and math.

(Optional) Bonus Challenge Problem (1 points)
Let H be an operator on a finite dimensional vector space and suppose $U = e^{iH}$, what condition on H guarantees U is unitary.

Proof. $H = H*$ guarantees that U is unitary. Although we won't formally prove this, the idea is that

$$U^*U = e^{-iH^*}e^{iH} = e^{-i(H^*-H)}, \tag{21}$$

which equals the identity operator only when $H^* - H = 0$, or, equivalently, when $H = H^*$. $\square$

Final Bonus Opportunity (1 point)
It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?