MATH 416 Abstract Linear Algebra

Week 11 - Homework 9
Assigned: Fri. Nov. 7, 2025
Due: Fri. Nov. 14, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (5 points): **Minimization via Orthogonal Projection**

Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and $\int_0^1 |2 + 3x - p(x)|^2 dx$ is as small as possible.

Solution. Define an inner product on $\mathcal{P}_3(\mathbb{R})$ by

$$\langle f, g \rangle = \int_0^1 fg.$$

Let $q(x) = 2 + 3x$, and let

$$U = \{\, p \in \mathcal{P}_3(\mathbb{R}) : p(0) = 0,\ p'(0) = 0 \,\}.$$

With this notation, our problem is to find the closest point $p \in U$ to q. To do this, first we find an orthonormal basis of U. A polynomial p satisfying $p(0) = 0$ and $p'(0) = 0$ has constant term 0 and first-degree term also equal to 0. Thus a basis of U is

$$(x^2,\, x^3).$$

Apply the Gram–Schmidt procedure to this basis, getting

$$e_1 = \sqrt{5}\, x^2, \qquad e_2 = \sqrt{7}\,(-5x^2 + 6x^3).$$

Thus with e_1, e_2 as above, e_1, e_2 is an orthonormal basis of U. By 6.61 and 6.57(i), the closest point $p \in U$ to q is given by the formula

$$p = \langle q, e_1 \rangle e_1 \ +\ \langle q, e_2 \rangle e_2.$$

A short computation now shows that

$$p(x) = 24x^2 \ -\ \frac{203}{10} x^3.$$

Exercise 2 (5 points): **Adjoints and Self-Adjoint Operators**

(a) (3 points) Suppose V is finite dimensional and φ is a linear functional on V (i.e. $\varphi \in \mathcal{L}(V, \mathbb{F})$). Then, there is a unique vector $v \in V$ such that

$$\varphi(u) = \langle u, v \rangle, \tag{1}$$

for every $u \in V$.

Proof. We will first prove the existence of a $v \in V$ such that $\varphi(u) = \langle u, v \rangle$ for every $u \in V$. Let $e_1, \ldots, e_n$ be an orthonormal basis of V. Then, we may expand any $u \in V$ in this basis to write

$$\begin{aligned}
\varphi(u) &= \varphi(\langle u, e_1 \rangle e_1 + \cdots + \langle u, e_n \rangle e_n), && \text{Axler 6.30(a)} \quad (2)\\
&= \langle u, e_1 \rangle \varphi(e_1) + \cdots + \langle u, e_n \rangle \varphi(e_n), && \text{Linearity of } \varphi(\cdot) \quad (3)\\
&= \langle u, \overline{\varphi(e_1)} e_1 + \cdots \overline{\varphi(e_n)} e_n \rangle. && (4)
\end{aligned}$$

Thus, by taking $v := \overline{\varphi(e_1)} e_1 + \cdots \overline{\varphi(e_n)} e_n$, we have $\varphi(u) = \langle u, v \rangle$, as desired. To show that this is unique, suppose for the sake of contradiction that there exist unequal $v_1, v_2 \in V$ such that $\varphi(u) = \langle u, v_1 \rangle = \langle u, v_2 \rangle$. Then, $0 = \langle u, v_1 \rangle - \langle u, v_2 \rangle = \langle u, v_1 - v_2 \rangle$, for all $u \in V$. Taking $u = v_1 - v_2$ implies $\langle u, u \rangle = 0$, which further implies $u = 0$ (by Axler 6.11). Thus, $v_1 = v_2$ and we are done. □

(b) (2 points) Use (a) to argue why the definition of the adjoint makes sense.

Solution. First, let us recall the definition of the adjoint. Let $T \in \mathcal{L}(V, W)$. The adjoint of T is the function $T^* : W \to V$ such that $\langle Tv, w \rangle = \langle v, T^* w \rangle$ for every $v \in V$ and every $w \in W$. Then, if we fix $w \in W$, the linear functional

$$v \mapsto \langle Tv, w \rangle \tag{5}$$

on V maps any vector $v \in V$ to $\langle Tv, w \rangle$. But Riesz representation theorem tells us that there exists a unique vector in V such that the functional is obtained by taking the inner product with this vector. We denote this unique vector as $T^* w$.

Hint: The result in part (a) is called the Riesz representation theorem and you may find it useful to peruse Axler 6B to learn more!

Exercise 3 (5 points): **Spectral Theorem**
Consider the self-adjoint matrix

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}.$$

(a) (2 points) Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

Proof. ($\Rightarrow$) If a normal operator T is self-adjoint, then all its eigenvalues are real. Suppose $v \neq 0$ is an eigenvector of a self-adjoint T with eigenvalue λ. Then

$$\begin{aligned} 0 &= \langle (T - T^*)v, v\rangle, && (6)\\ &= \langle Tv, v\rangle - \langle T^*v, v\rangle, && (7)\\ &= \langle Tv, v\rangle - \langle v, Tv\rangle, && (8)\\ &= \lambda\|v\|^2 - \overline{\lambda}\|v\|^2, && (9)\\ \implies \lambda &= \overline{\lambda}, && (10) \end{aligned}$$

thus $\lambda \in \mathbb{R}$.
($\Leftarrow$) If $\lambda_i \in \mathbb{R}$ for all i, then $\lambda_i v_i = \overline{\lambda_i} v_i \implies Tv_i = T^* v_i$. For normal operators, though, the eigenvectors form a basis, thus by showing this holds on an arbitrary basis element, we have shown it holds on the whole space.

□

(b) (2 points) Find the eigenvalues of A and an orthonormal basis $\mathcal{B}$ for $\mathbb{C}^2$ consisting of eigenvectors.

Solution. Let

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}, \qquad v = \begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

Then the eigenvalue equation $Av = \lambda v$ gives the two equations

$$2x + (1-i)y = \lambda x, \qquad (1+i)x + 3y = \lambda y.$$

We note that if either x or y is zero, then so is the other parameter, so we may assume neither is zero. It will thus be convenient to define $t = y/x$. Our two equations become

$$2 + (1-i)t = \lambda, \qquad (1+i) + 3t = \lambda t.$$

Equating these expressions for λ and simplifying,

$$(1-i)t^2 - t - (1+i) = 0.$$

One checks directly that the solutions are

$$t_1 = -\tfrac{1}{2} - \tfrac{i}{2}, \qquad t_2 = 1+i.$$

For t_1,

$$\lambda_1 = 2 + (1-i)t_1 = 1, \qquad v_1 = \begin{pmatrix} 2 \\ -(1+i) \end{pmatrix}.$$

For t_2,

$$\lambda_2 = 2 + (1-i)t_2 = 4, \qquad v_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

Since A is self-adjoint, $v_1 \perp v_2$. Normalizing gives the orthonormal eigenbasis

$$u_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -(1+i) \end{pmatrix}, \qquad u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1+i \end{pmatrix}.$$

Thus the eigenvalues are $\boxed{1, 4}$ and

$$\mathcal{B} = \{u_1, u_2\}$$

is an orthonormal basis of eigenvectors.

(c) (1 point) Let $U = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}$, and compute U^*AU. What do you find?

Solution. Let $\mathcal{B} = \{u_1, u_2\}$ be the orthonormal basis of eigenvectors of A with

$$Au_1 = 1 \cdot u_1, \qquad Au_2 = 4 \cdot u_2,$$

and let $U = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}$ be the change-of-basis matrix whose columns are u_1, u_2. Then U is unitary and

$$(U^*AU)_{ij} = \langle Au_j, u_i \rangle = \lambda_j \langle u_j, u_i \rangle = \lambda_j \delta_{ij},$$

so

$$U^*AU = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}.$$

(Optional) Bonus Question (3 points): *Self-adjoint maps and Pauli matrices*

Some of the most important objects in theoretical physics are the Pauli matrices $I, X, Y, Z \in M_2(\mathbb{C})$, defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let the *real* vector space of all self-adjoint complex (2×2)-matrices be defined $\mathcal{H}_2 = \{A \in M_2(\mathbb{C})\colon A^* = A\}$. Moreover, let us define an inner product on this space as

$$\langle A, B \rangle = \operatorname{tr}[AB], \tag{11}$$

where the *trace* of a matrix is defined as $\operatorname{tr}[A] = \sum_{i=1}^{n} A_{ii}$ (i.e. the sum of the diagonal terms).

(a) (1 point) Show that $\{I, X, Y, Z\}$ is an orthogonal list with respect to this inner product.

> **Solution.** This can be done by brute force because there are only four matrices; however, if you do all possible pairings you might notice the following facts. If either entry in the inner product is the identity, then the inner product becomes the trace of a Pauli matrix, which is zero by inspection (the diagonals of all non-identity Paulis sum to zero). Further, one might notice that $\langle A, B \rangle = \langle B, A \rangle$. This follows immediately from a fact called the *cyclicity of trace*, which can be expressed as $\operatorname{tr} AB = \operatorname{tr} BA$ for all A, B. Thus, we only need to check 3 inner products, all of which are easily shown to be zero ($\operatorname{tr} XY = \operatorname{tr} XZ = \operatorname{tr} YZ = 0$). Thus, this list of matrices is orthogonal with respect to this inner product.

(b) (1 point) Formally prove that the $\dim_{\mathbb{R}} \mathcal{H}_2 = 4$.

> **Solution.** In general, 2-by-2 matrices with complex entries require 8 real parameters to specify. However, the condition $A = A^*$ actually removes half of these free parameters. To see this, note that $A = A^* \implies a_{ii} = \overline{a_{ii}}$, thus $a_{11}, a_{22} \in \mathbb{R}$ (removing two real degrees of freedom). The off diagonal terms must satisfy $a_{ji} = \overline{a_{ij}}$, thus in our case, once we one of these off-diagonal terms, the other is fixed (removing two real degrees of freedom). Thus, $\dim_{\mathbb{R}} \mathcal{H}_2 = 4$, as desired.

(c) (1 points) What are the eigenvalues of these matrices?

> **Solution.** One can easily show that all eigenvalues of the Pauli matrices are ± 1.