MATH 416 Abstract Linear Algebra

Week 6 - Homework 5

Assigned: Fri. Oct. 3, 2025

Due: Fri. Oct. 10, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (10 points): **Rank of a Matrix**

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

(i) (3 points). Compute the column rank and the row rank of A by finding maximal linearly independent sets of columns and of rows.

(ii) (3 points). Prove the following proposition (the *column–row factorization*): If the column rank of a matrix $A \in M_{m\times n}(\mathbb{F})$ is r, then there exist matrices

$$C \in M_{m\times r}(\mathbb{F}), \qquad R \in M_{r\times n}(\mathbb{F})$$

such that

$$A = CR.$$

Hint: let the columns of C be a maximal linearly independent set of columns of A, and argue that every column of A is a linear combination of these.

(iii) (1 point). Find C and R such that their product yields the A from part (i).

(iv) (3 points). Use the factorization in (ii) to prove that the column rank of any matrix equals its row rank. *Hint: Take transposes and interpret column rank of A^T as row rank of A.*

Exercise 2 (10 points): **Invertibility, Isomorphisms, and Basis Change**

(i) (2 points). Suppose $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$ are both invertible. Prove that $ST \in \mathcal{L}(U,W)$ is invertible and $(ST)^{-1} = T^{-1}S^{-1}$.

(ii) (4 points). Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that

$$ST \text{ is invertible} \Leftrightarrow S, T \text{ are both invertible.}$$

(iii) (4 points). Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity.

> *Hints*. The $(\Leftarrow)$ direction shouldn't pose too many issues; however, I didn't like Axler's proof for the $(\Rightarrow)$ direction because he invokes a fact from chapter 5... In Wednesday's class, I intended to get to the following result. Let A, B be two matrices corresponding to $T \in \mathcal{L}(V)$. Moreover, let C be the identity matrix with input basis the same as A and output the same as B. Then, by Axler 3.84, we have
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> $$A = C^{-1}BC. \tag{1}$$
>
> This transformation enacts a basis transformation. Thus, if we assume T is the same matrix before and after such a transformation, we have
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> $$A = C^{-1}AC \tag{2}$$
>
> for all invertible C. This is equivalent to $CA = AC$. Now, it can be shown (and you can assume for this problem) that if A commutes with all invertible matrices, it commutes with all matrices (i.e. $AM = MA$ for all M). All that remains to complete this direction of the proof is showing that the only matrices that commute with all matrices are scalar multiples of the identity. Give this a go and don't worry about the grade, this problem was harder than I realized when I assigned it!

(optional) Bonus Question (2 points): **Rank-1 Decomposition**

Prove that every matrix $A \in M_{m\times n}(\mathbb{F})$ of rank r can be written as a sum of r matrices of rank 1. That is, show that there exist column vectors $u_1, \ldots, u_r \in \mathbb{F}^{m\times 1}$ and row vectors $v_1^T, \ldots, v_r^T \in \mathbb{F}^{1\times n}$ such that

$$A = u_1 v_1^T + u_2 v_2^T + \cdots + u_r v_r^T.$$

Motivation. This statement shows that we can build up matrices in terms of "rank-1 outer products" (i.e. a column times a row vector) building blocks. Later, we will see that the so-called *Singular Value Decomposition (SVD)* is a refinement of this idea: it expresses any matrix as a sum of rank-1 outer products, but with the additional structure that the vectors form orthonormal bases and the coefficients are nonnegative singular values. The SVD has far-reaching consequences for data science, machine learning, entanglement theory, and many more fields.