MATH 416 Abstract Linear Algebra

Midterm 3 – Practice Exam 2 – Solution

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"Sometimes the questions are complicated and the answers are simple."*
>
> —Dr. Seuss

Question 1 (10 points): **Inner Product Spaces**

(i) (5 points) Suppose $u, v \in V$. Then,

$$\|u + v\| \leq \|u\| + \|v\|. \tag{1}$$

This inequality is an equality if and only if one of u, v is a non-negative real multiple of the other.

> *Proof.* See page 190 of Axler. $\square$

(ii) (5 points) Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that

$$U \text{ is invariant under } T \iff P_U T P_U = T P_U. \tag{2}$$

> *Proof.* ($\Rightarrow$) Suppose U is invariant under T. Let $v \in V$. Write $v = u + w$ with $u \in U$ and $w \in U^{\perp}$, so that $P_U v = u$. Because U is T-invariant, we have $Tu \in U$, hence
>
> $$T P_U v = Tu \in U.$$
>
> Applying P_U to this gives
>
> $$P_U T P_U v = P_U(T P_U v) = P_U(Tu) = Tu = T P_U v.$$
>
> Since this holds for every $v \in V$, we conclude that
>
> $$P_U T P_U = T P_U.$$
>
> ($\Leftarrow$) Now suppose $P_U T P_U = T P_U$. Let $u \in U$. Then $P_U u = u$, and applying the assumed identity to u gives
>
> $$P_U T u = P_U T P_U u = T P_U u = Tu.$$
>
> Thus $Tu = P_U T u$, which means $Tu \in U$, because the range of P_U is U. Hence U is invariant under T. $\square$

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(i) (5 points) Recall that we say two operators A and B commute if $AB = BA$. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.

Proof. ($\Longrightarrow$) Suppose T is normal, so $TT^* = T^*T$. Define

$$A = \frac{T + T^*}{2} \quad \text{and} \quad B = \frac{T - T^*}{2i}.$$

Then

$$A^* = \left(\frac{T + T^*}{2}\right)^* = \frac{T^* + T}{2} = A \quad \text{and} \quad B^* = \left(\frac{T - T^*}{2i}\right)^* = \frac{T^* - T}{-2i} = \frac{T - T^*}{2i} = B,$$

so A and B are self-adjoint. A short computation shows

$$A + iB = \frac{T + T^*}{2} + i\frac{T - T^*}{2i} = \frac{T + T^*}{2} + \frac{T - T^*}{2} = T,$$

hence $T = A + iB$.
Because T is normal, we have

$$TT^* = T^*T.$$

Using $T = A + iB$ and $T^* = A - iB$, this becomes

$$(A + iB)(A - iB) = (A - iB)(A + iB).$$

Expanding both sides, we obtain

$$A^2 - iAB + iBA + B^2 \;=\; A^2 + iAB - iBA + B^2.$$

Cancelling A^2 and B^2 from both sides gives

$$-iAB + iBA = iAB - iBA.$$

Rearranging,

$$2i(BA - AB) = 0,$$

so $BA - AB = 0$, which means $AB = BA$. Thus we have written T as

$$T = A + iB$$

with A and B self-adjoint and commuting.

($\Longleftarrow$) Conversely, suppose A and B are self-adjoint operators that commute and $T = A + iB$. Then

$$A^* = A, \quad B^* = B,$$

so

$$T^* = (A + iB)^* = A - iB.$$

We now compute

$$TT^* = (A + iB)(A - iB) = A^2 - iAB + iBA + B^2,$$

and

$$T^*T = (A - iB)(A + iB) = A^2 + iAB - iBA + B^2.$$

Because A and B commute, we have $AB = BA$, and hence

$$TT^* = A^2 + B^2 = T^*T.$$

Thus T is normal. Therefore, T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$. $\square$

(ii) (5 points) Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that

$$T^2 - 5T + 6I = 0. \tag{3}$$

Proof. If v is an eigenvector of T with eigenvalue 2, then

$$(T^2 - 5T + 6I)v = \big((T - 3I)(T - 2I)\big)v = (T - 3I)\big((T - 2I)v\big) = (T - 3I)(0) = 0.$$

Similarly, if v is an eigenvector of T with eigenvalue 3, then

$$(T^2 - 5T + 6I)v = \big((T - 2I)(T - 3I)\big)v = (T - 2I)\big((T - 3I)v\big) = (T - 2I)(0) = 0.$$

By the complex spectral theorem or real spectral theorem (whichever is appropriate), there is an orthonormal basis of the domain of T consisting of eigenvectors of T. The equations above show that $T^2 - 5T + 6I$ applied to any such basis vector equals 0. Thus

$$T^2 - 5T + 6I = 0.$$

$\square$

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(i) (5 points) Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v\rangle = 0$. Then $Tv = 0$.

Proof. Because T is positive, it is self-adjoint. The hypothesis $\langle Tv, v\rangle = 0$ and the Cauchy–Schwarz inequality give

$$0 = |\langle Tv, v\rangle| \leq \|Tv\|\,\|v\|.$$

If $v = 0$, the conclusion is trivial. If $v \neq 0$, then $\|v\| > 0$, so the inequality forces $\|Tv\| = 0$. Hence $Tv = 0$. See page 254 of Axler for another proof technique. □

(ii) (5 points) Suppose $\mathbb{F} = \mathbb{C}$ and $A, B \in \mathcal{L}(V)$ are self-adjoint. Show that $A + iB$ is unitary if and only if $AB = BA$ and $A^2 + B^2 = I$.

Proof. Let $T = A + iB$. Because A and B are self-adjoint, we have

$$T^* = (A + iB)^* = A - iB.$$

($\Longrightarrow$) Suppose T is unitary. Then $T^*T = TT^* = I$. Compute

$$T^*T = (A - iB)(A + iB) = A^2 + B^2 + i(AB - BA),$$

and

$$TT^* = (A + iB)(A - iB) = A^2 + B^2 - i(AB - BA).$$

Since $T^*T = I$ and $TT^* = I$, we get

$$A^2 + B^2 + i(AB - BA) = I \quad \text{and} \quad A^2 + B^2 - i(AB - BA) = I.$$

Subtracting these two equations gives $2i(AB - BA) = 0$, hence $AB = BA$. Substituting back, we obtain $A^2 + B^2 = I$.
($\Longleftarrow$) Conversely, suppose $AB = BA$ and $A^2 + B^2 = I$. Then

$$T^*T = A^2 + B^2 + i(AB - BA) = A^2 + B^2 = I,$$

and similarly

$$TT^* = A^2 + B^2 - i(AB - BA) = A^2 + B^2 = I.$$

Thus $T^*T = TT^* = I$, so T is unitary. Hence $A + iB$ is unitary if and only if $AB = BA$ and $A^2 + B^2 = I$. $\square$