Math 416: Abstract Linear Algebra

Date: Sept. 19, 2025

Lecture: 11

Announcements

- ☐ HW3 is due before class
  - ↳ 6pm moving forward
- ☐ office hours:
  - Tuesdays 5 - 5:50 Davenport 336
  - Wednesdays 2-2:50 Davenport 132
- ☐ Exam #1 : Wed. 9/24
  - ↳ Fair game:
    - basic matrix LA
    - Sec. 1A - 3A of axler is fair game

Survey Take-aways

- HW takes ~4hrs (standard dev. 3hrs)
- Clickers?
- Harder worked examples in class so homework is more manageable

Last time

- ☐ Linear maps

This time

- ☐ Vector space of linear maps

Recommended reading/watching

- ☐ §3A of Axler
- ☐ 3blue1brown : linear transformations

Next time

- ☐ Review session

Chapter 3

Linear Maps

So far our attention has focused on vector spaces. No one gets excited about vector spaces. The interesting part of linear algebra is the subject to which we now turn—linear maps.

Def. 3.1 (linear map)

A linear map from V to W is a function $T : V \to W$ satisfying

additivity.

$$T(u+v) = Tu + Tv, \ \forall\, u, v \in V$$

homogeneity.

$$T(\lambda v) = \lambda(Tv) \ \forall\, \lambda \in \mathbb{F} \ \& \text{ all } v \in V$$

Lemma 3.4 (linear map lemma)

Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ s.t.

$$Tv_k = w_k \qquad \forall\, k \in [n]$$

$\{1, 2, \dots, n\}$

Proof. See lec. 10 notes or (Axler 3.4)

Take-aways

- Existence: we can find a linear map that takes on whatever values we wish on the basis
- Uniqueness: linear map is completely determined by the values it takes on a basis

Algebraic operations on $\mathcal{L}(V,W)$

- $S,T \in \mathcal{L}(V,W)$ & $\lambda \in \mathbb{F}$.

$$\boxed{(S+T)(v) = Sv + Tv \quad \& \quad (\lambda T)(v) = \lambda(Tv)}$$

- $\mathcal{L}(V,W)$ is a vector space w/ the above defn. of addition and scalar mult.

Additive identity: 0 map defined as $0v = 0 \quad \forall v$

Mult. identity: $I v = v \quad \forall v$

Commutativity (under addition): Let $T, S \in \mathcal{L}(V,W)$.

$$\forall v \in V \quad (S+T)(v) = S(v) + T(v)$$
$$= T(v) + S(v)$$
$$= (T+S)(v)$$

where we have used that $T(v), S(v) \in W$ and W is a vector space. Similar for the other required properties.

- product of linear maps

let $T \in \mathcal{L}(U,V)$, $S \in \mathcal{L}(V,W)$.

then $ST \in \mathcal{L}(U,W)$ is defined by

$$\underbrace{(ST)(u) = S(Tu) \quad \forall \ u \in U.}_{\text{just standard composition of functions}}$$

$$h(x) := g(f(x)) \iff (g \circ f)(x)$$

Properties of products of lin. maps

- Associativity : $(T_1 T_2) T_3 = T_1 (T_2 T_3)$

- Identity : $\underset{\substack{\uparrow \\ I \in \mathcal{L}(V)}}{TI} = \underset{\substack{\uparrow \\ I \in \mathcal{L}(W)}}{IT} = T \quad \forall \ T \in \mathcal{L}(V,W)$

- Distributive : $(S_1 + S_2) T = S_1 T + S_2 T$

$$S(T_1 + T_2) = ST_1 + ST_2$$

$$T, T_1, T_2 \in \mathcal{L}(U, V)$$

$$S, S_1, S_2 \in \mathcal{L}(V, W)$$

(non)- Commuting maps

Let $T, S \in \mathcal{L}(V)$. If $TS = ST$, then we say T & S commute. Unfortunately, linear operators do <u>not</u> always commute.

ex. Consider $T_a : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ defined $\forall\, a \in \mathbb{R}$ as $(T_a f)(x) = f(x+a)$

Let $f, g \in \mathcal{P}(\mathbb{R})$. Then

$$\begin{aligned}
\big(T_a(f+g)\big)(x) &= (f+g)(x+a) && \mathcal{P}(\mathbb{R}) \text{ is a vec space} \\
&= f(x+a) + g(x+a) && \text{def of } T_a \\
&= (T_a f)(x) + (T_a g)(x) && \mathcal{L}(\mathcal{P}(\mathbb{R})) \text{ is a vec} \\
&= [T_a f + T_a g](x)
\end{aligned}$$

$$T_a(\lambda f)(x) = (\lambda f)(x+a)$$

$$= \lambda f(x+a)$$

$$= \lambda(T_a f)(x)$$

$\mathcal{P}(\mathbb{R})$ is a vec. space

Claim. E_a & E_b commute $\forall\ a,b \in \mathbb{R}$.

Proof. For all $f \in \mathcal{P}(\mathbb{R})$, we have

$$(E_a E_b)f(x) = E_a(E_b f)(x)$$

$$= E_a(f(x+b))$$

$$= f(x+b+a)$$

$$= f((x+a)+b)$$

$$= E_b(f(x+a))$$

$$= E_b(E_a f)(x)$$

$$= (E_b E_a) f(x)$$

$$\Rightarrow \quad E_a E_b = E_b E_a.$$

Let $M_x : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ be defined as

$$(M_x f)(x) = x f(x).$$

Claim. E_a & M_x do not commute.

Proof. Let $f \in \mathcal{P}(\mathbb{R})$.

$$\Big(\big(E_a M_x\big) f\Big)(x) = \Big(E_a\,(M_x f)\Big)(x)$$

$$= E_a\big(x f(x)\big)$$

$$= (x+a) f(x+a)$$

$$= M_{x+a}\big(f(x+a)\big)$$

$$= M_{x+a}\big(E_a f\big)(x)$$

$$= \underbrace{(M_{x+a} E_a)}_{\neq M_x E_a} f(x)$$

unless $a = 0$.