MATH 416 Abstract Linear Algebra

Homework 3
Assigned: Fri. Sept. 12, 2025
Due: Fri. Sept. 19, 2025 (by 1pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (3 points): **Linear independence and span**

(i) (2 points) Let $z_1 = 1 + i$ and $z_2 = 1 - i$. First consider the complex numbers $\mathbb{C}$ as a vector space over the field $\mathbb{R}$, and show that $\{z_1, z_2\}$ is linearly independent over $\mathbb{R}$. Then consider $\mathbb{C}$ as a vector space over itself (i.e., $\mathbb{F} = \mathbb{C}$), and show that now $\{z_1, z_2\}$ is linearly dependent.

(ii) (1 point) Let $\{v_1, \dots, v_m\}$ be a set of linearly independent vectors in V, and let $w \in V$. Show that, if $\{v_1 + w, \dots, v_m + w\}$ are linearly dependent, then $w \in \langle v_1, \dots v_m \rangle$.

Exercise 2 (3 points): **Bases I**

(i) Let $\{u_1, u_2, u_3\}$ be the following vectors in $\mathbb{R}^2$:

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_3 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Show that $\{u_1, u_2, u_3\}$ is not a basis of $\mathbb{R}^2$, but $\{u_i, u_j\}$ is a basis for any $1 \le i < j \le 3$.

(ii) Prove that the following set of vectors $\{v_1, v_2, v_3\}$ forms a basis of $\mathbb{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(iii) Prove that the following set of vectors $\{w_1, w_2, w_3\}$ does not form a basis of $\mathbb{R}^3$:

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad w_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$$

Exercise 3 (3 points): **Bases II**
Let U be the subspace of $\mathbb{R}^5$ defined by[1]

$$U = \left\{ (x_1, x_2, x_3, x_4, x_5)^T \in \mathbb{R}^5 \colon x_1 = 3x_2 \text{ and } x_3 = 7x_4 \right\}$$

(i) Find a basis for U.

(ii) Extend the basis you found in (i) to a basis of $\mathbb{R}^5$.

(iii) Find a subspace $W \leq \mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Exercise 4 (3 points): **Dimension I**
Show that the subspaces of $\mathbb{R}^3$ are precisely $\{0\}$, all lines in $\mathbb{R}^3$ containing the origin, all planes in $\mathbb{R}^3$ containing the origin, and $\mathbb{R}^3$.

Exercise 5 (4 points): **Dimension II**
Suppose that $V_1, \ldots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \cdots + V_m$ is finite dimensional and

$$\dim (V_1 + \cdots + V_m) \leq \dim V_1 + \cdots + \dim V_m.$$

Exercise 6 (4 points): **Dimension III**
Suppose V is finite dimensional, with $\dim V = n \geq 1$. Prove that there exists one-dimensional subspaces $V_1, \ldots, V_m$ of V such that

$$V = V_1 \oplus \cdots \oplus V_m.$$

[1] Here, ^{T} denotes transposition, and $x = (x_1, x_2, x_3, x_4, x_5)^T = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$.