MATH 416 Abstract Linear Algebra

Midterm 2 – Practice Exam 2

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

"Chance favors the prepared mind."

— Louis Pasteur

Question 1 (10 points): **Null Spaces and Ranges**

For this entire problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the range of T?

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is range D?

(iii) (6 points) Prove that range T is a subspace of W.

Question 2 (10 points): **Matrices, Invertibility, and Change of Basis**
Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T(x, y) = (3x + y,\, x + 2y).$$

Let the *standard basis* of $\mathbb{R}^2$ be

$$E = \{e_1 = (1, 0),\, e_2 = (0, 1)\},$$

and let

$$B = \{b_1 = (1, 1),\, b_2 = (1, -1)\}$$

be another basis of $\mathbb{R}^2$.

(i) (3 points) Find $\mathcal{M}(T)_E$, the matrix of T in the standard basis E.

(ii) (4 points) Express each vector in B in terms of the standard basis, and compute the *change-of-basis matrix*

$$P_{B \to E}$$

(from B-coordinates to E-coordinates). Then find its inverse $P_{E \to B}$.

(iii) (3 points) Compute the matrix of T in the basis B, denoted $\mathcal{M}(T)_B$, using your result from (ii). In words, does the invertibility of T depend on the choice of basis?

Question 3 (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**

(i) (3 points) Suppose $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

(ii) (3 points) Suppose $\mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

(iii) (4 points) Suppose $P \in \mathcal{L}(V)$ such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.