MATH 416 Abstract Linear Algebra

Exam 1 – September 24, 2025

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"Mathematicians are not people who find math easy, they're the people who enjoy that it's difficult."*
>
> — Matt Parker

Question 1 (5 points): **Linear Systems of Equations**

Linear algebra was invented[1] to facilitate the solution of linear systems of equations. If the field began there, it seems fitting that our first exam should as well!

(i) (2 points) A school cafeteria sells three types of sandwiches: turkey (\$2 each), ham (\$1 each), and veggie (\$1 each). On Monday total of 10 sandwiches were sold, the total revenue was \$14, and the number of turkey sandwiches sold was equal to the number of veggie sandwiches. Set up a system of equations to model the situation and solve for the quantity of each sandwich sold. *Hint: let x, y, z stand for turkey, ham, and veggie sandwhich quantities, respectively.*

> **Solution.** Translating this into a linear system, we get
>
> $$1x + 1y + 1z = 10, \tag{1}$$
> $$2x + 1y + 1z = 14, \tag{2}$$
> $$1x + 0y - 1z = 0, \tag{3}$$
>
> which we can convert to an augmented
>
> $$\begin{bmatrix} 1 & 1 & 1 & 10 \\ 2 & 1 & 1 & 14 \\ 1 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & -1 & -1 & -6 \\ 0 & -1 & -2 & -10 \end{bmatrix}$$
>
> $$\xrightarrow{R_3 \to R_3 - R_2, R_2 \to -R_2,\ R_3 \to -R_3} \begin{bmatrix} 1 & 1 & 1 & 10 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$
>
> Using back-substitution, we find $(x, y, z) = (4, 2, 4)$.

(ii) (3 points) Suppose a linear system of equations $A\mathbf{x} = \mathbf{b}$ has at least two distinct solutions. Prove that the system must in fact have infinitely many solutions.

> **Proof.** The $\mathbf{x}_1 \neq \mathbf{x}_1$ be two solutions to $A\mathbf{x} = b$. Then, for any $c \in \mathbb{R}$, we have
>
> $$A\left(c\mathbf{x}_1 + (1-c)\mathbf{x}_2\right) = cA\mathbf{x}_1 + (1-c)A\mathbf{x}_2 = cb + (1-c)b = b \tag{4}$$
>
> Because this holds for all real numbers, we have constructed an infinite number of distinct solutions from the assumption that we had at least two disinct solutions.

[1] or discovered, depending on your philosophy...

Bonus (1 point): The method we now call "Gaussian elimination" was not invented by Gauss. From where does this algorithm originate?

Answer. Ancient China ($\sim$ 10th-2nd centuries BCE).

Question 2 (5 points): **Complex Numbers are Essential (5 points)**

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. Recall that the conjugate of z is $\bar{z} = a - bi$ and $|z| = \sqrt{a^2 + b^2}$. Consider the following problems regarding complex numbers.

(i) (1 point) In terms of z and $\bar{z}$, write down a condition for a number to be real. Do the same for a purely imaginary number. Is there a number that is both real and purely imaginary?

Solution. A complex number is real if $z = \bar{z}$, it is imaginary if $z = -\bar{z}$, and the only number that satisfies both is $z = 0$.

(ii) (2 point) Write the real and imaginary parts of z in terms of z and $\bar{z}$.

Solution. We have that $a = (z + \bar{z})/2$ and $b = (z - \bar{z})/2i$.

(iii) (2 points) Show that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ for $z \neq 0$.

Solution. We can write

$$\begin{aligned} \frac{1}{z} &= \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi}, && (5) \\ &= \frac{a-bi}{a^2+b^2}, && (6) \\ &= \frac{\bar{z}}{|z|^2}. && (7) \end{aligned}$$

Question 3 (10 points): **Vector Subspaces, Bases, and Dimension**

Recall that a polynomial is a function $p : \mathbb{R} \to \mathbb{R}$ of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, for all $x \in \mathbb{R}$. Recall that addition and scalar multiplication of polynomials are defined as follows. For all $f, g \in \mathcal{P}(x)$ the sum $f + g \in \mathcal{P}(x)$ is the function defined by

$$(f+g)(x) = f(x) + g(x). \tag{8}$$

Similarly, for all $\lambda \in \mathbb{R}$ and all $f \in \mathcal{P}(x)$, the product $\lambda f \in \mathcal{P}(x)$ is the function defined by

$$(\lambda f)(x) = \lambda f(x). \tag{9}$$

Let $\mathcal{P}_4 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_0, \ldots, a_4 \in \mathbb{R}\}$ be the vector space of all real polynomials of degree at most 4. A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$.

(i) (3 points) Show that the set of all even polynomials in $\mathcal{P}_4$ forms a subspace of $\mathcal{P}_4$. (You may simply state that the odd case works similarly.)

Proof. Let $a, b \in \mathbb{R}$ and $g, h \in \mathcal{P}_4^e$, where we have added a subscript to denote the even functions within $\mathcal{P}_4$. To verify that the zero map is in $\mathcal{P}_4^e$, we may write

$$0(-x) = (g-g)(-x) = g(-x) - g(-x) = g(x) - g(x) = (g-g)(x) = 0(x) \tag{10}$$

Similarly for linear combinations, we have

$$\begin{aligned} (af + bg)(-x) &= (af)(-x) + (bg)(-x), && (11) \\ &= af(-x) + bg(-x), && (12) \\ &= af(x) + bg(x), && (13) \\ &= (af + bg)(x). && (14) \end{aligned}$$

These are the two requirements to form a subsapce, so we are done.

(ii) (3 points) Write down a basis for the subspace of even polynomials in $\mathcal{P}_4$ and state its dimension. Do the same for the subspace of odd polynomials in $\mathcal{P}_4$ and state its dimension.

Solution. A basis for $\mathcal{P}_4^e$ is simply $\{x^0, x^2, x^4\}$, making it's dimension 3. Similarly, for $\mathcal{P}_4^o$, a basis is given by $\{x^1, x^3\}$ yielding dimension 2.

(iii) (4 points) Show that every polynomial in $\mathcal{P}_4$ can be uniquely written as the sum of an even polynomial and an odd polynomial; that is, show that $\mathcal{P}_4 = \text{Even} \oplus \text{Odd}$.

Proof. The standard basis for $\mathcal{P}_4$ is $\{x^i\}_{i=1}^4$, thus every element of the full space can be written *uniquely* as

$$p(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4, \tag{15}$$

$$= \underbrace{\left(a_0x^0 + a_2x_2 + a_4x^4\right)}_{\in\mathcal{P}_4^e} + \underbrace{\left(a_1x^1 + a_3x^3\right)}_{\in\mathcal{P}_4^o}. \tag{16}$$

We see that we can write an arbitrary element of $p(x)$ as a sum of an element from $\mathcal{P}_4^e$ and an element from $\mathcal{P}_4^o$, thus $\mathcal{P}_4 = \mathcal{P}_4^e + \mathcal{P}_4^o$. Because we expressed this sum in terms of a basis of the full space, we know the sum is unique, thus $\mathcal{P}_4 = \mathcal{P}_4^e \oplus \mathcal{P}_4^o$.

Bonus (1 point): Chapter 2 in Axler has the same title as the first modern linear algebra textbook. What was the title of the textbook? Who was the author?

Answer. The textbook was *Finite-dimensional Vector Spaces* by Paul Halmos.

Question 4 (10 points): **The Vector Space of Linear Maps**

We saw in class that the set $\mathcal{L}(U,V)$ of all linear maps from U to V is, indeed, a vector space. In fact, it is one of the most important vector spaces we will study in this course. Recall also that $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials over $\mathbb{R}$.

(i) (5 points) Suppose $m, b \in \mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ is a linear map if and only if $b = 0$. *Hint: remember that a linear map takes 0 to 0, that is* $T(0) = 0$.

Proof. First, suppose that $f(x) = mx + b$ is a linear map. Consider $f(0)$:

$$\begin{aligned} f(0) &= f(x + (-x)), && (17) \\ &= f(x) + f(-x), \quad \text{Linearity assumption} && (18) \\ &= (mx + b) + (m(-x) + b), && (19) \\ &= 2b. && (20) \end{aligned}$$

A linear map must map zero to zero, thus we conclude that $b = 0$. Now, assume that $b = 0$ so that $f(x) = mx$. This is clearly linear

$$f(ax + cy) = m(ax + cy) = a(mx) + c(my) = af(x) + bf(y), \tag{21}$$

and we are done.

(ii) (1 point) Define a function $T : \mathcal{P}_m(\mathbb{R}) \to \_\_$ by $(Tp)(x) = xp(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. What space does T map into?

Answer. Because we are multiplying by x we will increase the maximum degree of the polynomial by 1, yielding $\mathcal{P}_{m+1}(\mathbb{R})$.

(iii) (4 points) Consider $S \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ defined as $(Sp)(x) = p(x + a)$ for all $p \in \mathcal{P}(\mathbb{R})$. With T defined as in (ii), show that $ST \neq TS$.

Proof. First applying T and then S, we obtain

$$\begin{aligned} ((ST)p)(x) &= S((Tp)(x)), && (22) \\ &= S(xp(x)), && (23) \\ &= (x + a)p(x + a). && (24) \end{aligned}$$

However, in the opposite direction, we see

$$
\begin{aligned}
((TS)p)(x) &= T((Sp)(x)), && (25)\\
&= T(p(x+a)), && (26)\\
&= xp(x+a), && (27)
\end{aligned}
$$

thus we see the two maps are unequal (i.e. they do not commute).

Bonus Challenge Problem (2 points)
The following problem is designed to demonstrate that neither homogeneity nor additivity alone suffice to imply that a function is a linear map.

(i) (1 point) Give an example of a function from $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v) \tag{28}$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

Solution. An example of a function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ with such a property would be

$$\varphi(x, y) = \frac{x^2}{y}. \tag{29}$$

This is clearly not a linear function and yet

$$\varphi(ax, ay) = \frac{a^2x^2}{ay} = a \cdot \frac{x^2}{y} = a\varphi(x, y). \tag{30}$$

(ii) (1 point) Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z), \tag{31}$$

for all $w, z \in \mathbb{C}$ but φ is not linear. *Note: here $\mathbb{C}$ can be thought of as a complex vector space.*

Solution. An example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ with such a property would be

$$\varphi(z) = \bar{z}. \tag{32}$$

We showed in the practice exam that $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$. However, if we are viewing this as a complex vector space, then we need to be able to multiply by complex coefficients. Consider $\varphi(iz) = \overline{iz} = -i\overline{z} \neq i\overline{z}$ as would be required by homogoneity.

Final Bonus Opportunity (1 point)
It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?