Math 416: Abstract Linear Algebra

Date: Oct. 6, 2025

Lecture: 15

Announcements

- ☐ HW5 is due Fri, Oct. 10 @ 8pm

- ☐ office hours:
  - Tuesdays 5 - 5:50 Davenport 336
  - Wednesdays 2-2:50 Daveport 132

Last time

- ☐ Invertibility

This time

- ☐ Invertibility (cont) & isomorphic vector spaces

Reading

- ☐ §3D of Axler

Prop. 3.63 invertibility $\iff$ injective & surjective

Proof. ($\Rightarrow$) Suppose T is invertible.

To see injectivity, let $u, v \in V$ & $Tu = Tv$.

Then,
$$\begin{aligned} u &= I_V u \\ &= (T^{-1}T)u \\ &= T^{-1}(Tu) \\ &= T^{-1}(Tv) \\ &= (T^{-1}T)v \\ &= v \end{aligned}$$

For surjectivity, let $w \in W$. Then $w = T(T^{-1}w)$, so $w \in \operatorname{range} T$. But w was arbitrary, so $W = \operatorname{range} T$ & we are done.

↓ cont.

($\Leftarrow$) Suppose T is injective & surjective.

- $\forall$ $w \in W$, let $S(w)$ be the unique elem. of V s.t. $T(S(w)) = w$ (existence & uniqueness follows from inj. & surjectivity)

- $T \circ S(w) = w \quad \forall\, w \in W \implies T \circ S = I_W$

- next, let $v \in V$. Then

$$\begin{aligned} T((S \circ T)v) &= (T \circ S)(Tv) \\ &= I_W(Tv) \\ &= Tv \end{aligned}$$

T is inject. $\implies (S \circ T)v = v \quad \therefore S \circ T = I_V$

- Finally, we must show S is <u>linear</u>

  $\hookrightarrow$ $w_1, w_2 \in W$: $T(S(w_1) + S(w_2))$

  $= T(S(w_1)) + T(S(w_2)) = w_1 + w_2$

  $\hookrightarrow$ by def. of S, $S(w_1 + w_2) = S(w_1) + S(w_2)$

- same for homogeneity.

$\square$

Questions

Determine if the following maps are injective and/or surjective.

1. $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$, $T_1(x,y) = (x,y,0)$

   ↳ injective, not surjective

2. $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$, $T_2(x,y,z) = (x+y, y+z)$

   ↳ not injective, but surjective

3. $T_3 : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, $T_3\, p(x) = x^2 p(x)$

   ↳ injective : $x^2 p(x) = 0 \Rightarrow p(x) = 0$

   ↳ not surjective b/c $\nexists$ $p(x)$ s.t. $x^2 p(x) = 1$, for ex.

Fortunately, in finite-dim, when $\dim V = \dim W$, things are much simpler...

Prop 3.65 (injectivity $\Leftrightarrow$ surjectivity when $\dim V = \dim W < \infty$)

Suppose $\dim V = \dim W < \infty$ and $T \in \mathcal{L}(V,W)$. Then,

T is invertible $\Leftrightarrow$ T is inject. $\Leftrightarrow$ T is surj.

Proof. Fund. thm. of linear maps:

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \quad (1)$$

If T is inject., $\dim \operatorname{null} T = 0$. Then, (1)

$$\Rightarrow \dim \operatorname{range} T = \underbrace{\dim V = \dim W}_{\text{assumption}}$$

via Axler 2.39

$$\Rightarrow T \text{ surjective}$$

If T is surjective, $\dim \operatorname{range} T = \dim W$

$$(1) \Rightarrow \dim \operatorname{null} T = \dim V - \dim \operatorname{range} T$$
$$= \dim V - \dim W$$
$$= 0$$

$\therefore$ T is injective. Thus, if T is either inj. or surj., it is both. If it is both, it is invertible by (Axler 3.63). $\square$

Isomorphic vector spaces

- An <u>isomorphism</u> is an investible linear map
- Two vec. spaces V, W are isomorphic if $\exists$ an isomorphism between them. We denote this $V \cong W$

Remarks

- These results imply every finite-dim vec. space V is isomorphic to $\mathbb{F}^n$ w/ $n = \dim V$.

  $\hookrightarrow$ e.g. $\mathcal{P}_m(\mathbb{F}) \cong \mathbb{F}^{m+1}$

- $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$ (prop. 3.71)

  $\hookrightarrow$ $\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$ (prop 3.72)