MATH 416 Abstract Linear Algebra

Week 11 - Homework 9

Assigned: Fri. Nov. 7, 2025

Due: Fri. Nov. 14, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (5 points): **Minimization via Orthogonal Projection**

Find $p \in \mathcal{P}_3(\mathbb{R})$ such that $p(0) = 0$, $p'(0) = 0$, and $\int_0^1 |2 + 3x - p(x)|^2 dx$ is as small as possible.

Exercise 2 (5 points): **Adjoints and Self-Adjoint Operators**

(a) (3 points) Suppose V is finite dimensional and φ is a linear functional on V (i.e. $\varphi \in \mathcal{L}(V, \mathbb{F})$). Then, there is a unique vector $v \in V$ such that

$$\varphi(u) = \langle u, v \rangle, \tag{1}$$

for every $u \in V$.

(b) (2 points) Use (a) to argue why the definition of the adjoint makes sense.

Hint: The result in part (a) is called the Riesz representation theorem and you may find it useful to peruse Axler 6B to learn more!

Exercise 3 (5 points): **Spectral Theorem**
Consider the self-adjoint matrix

$$A = \begin{pmatrix} 2 & 1-i \\ 1+i & 3 \end{pmatrix}.$$

(a) (2 points) Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

(b) (2 points) Find the eigenvalues of A and an orthonormal basis $\mathcal{B}$ for $\mathbb{C}^2$ consisting of eigenvectors.

(c) (1 point) Let $U = \mathcal{M}(I)_{\mathcal{B},\mathcal{S}}$, and compute U^*AU. What do you find?

(Optional) Bonus Question (3 points): *Self-adjoint maps and Pauli matrices*
Some of the most important objects in theoretical physics are the Pauli matrices $I, X, Y, Z \in M_2(\mathbb{C})$, defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let the *real* vector space of all self-adjoint complex (2×2)-matrices be defined $\mathcal{H}_2 = \{A \in M_2(\mathbb{C})\colon A^* = A\}$. Moreover, let us define an inner product on this space as

$$\langle A, B \rangle = \mathrm{tr}\,[AB], \tag{2}$$

where the *trace* of a matrix is defined as $\mathrm{tr}\,[A] = \sum_{i=1}^{n} A_{ii}$ (i.e. the sum of the diagonal terms).

(a) (1 point) Show that $\{I, X, Y, Z\}$ is a linearly independent list with respect to this inner product.

(b) (1 point) Formally prove that the $\dim_{\mathbb{R}} \mathcal{H}_2 = 4$.

(c) (1 points) What are the eigenvalues of these matrices?