MATH 416 Abstract Linear Algebra

Final Exam – Dec. 17, 2025

Exam Instructions: This is a **closed-book** exam and you have **3 hours** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"The beauty of mathematics only shows itself to more patient followers."*
>
> —Maryam Mirzakhani

Question 1 (10 points): **The Vector Space of Linear Maps**
We saw in class that the set $\mathcal{L}(U, V)$ of all linear maps from U to V is, indeed, a vector space. Recall also that $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials over $\mathbb{R}$.

(i) (5 points) Suppose $m, b \in \mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ is a linear map if and only if $b = 0$. *Hint: remember that a linear map takes 0 to 0, that is* $T(0) = 0$.

(ii) (1 point) Define a function $T : \mathcal{P}_m(\mathbb{R}) \to \_\_\_$ by $(Tp)(x) = xp(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. What space does T map into?

(iii) (4 points) Consider $S \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ defined as $(Sp)(x) = p(x + a)$ for all $p \in \mathcal{P}(\mathbb{R})$. With T defined as in (ii), show that $ST \neq TS$.

Question 2 (10 points): **Null Spaces and Ranges**

For this entire problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the range of T?

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is range D?

(iii) (6 points) Prove that range T is a subspace of W.

Question 3 (10 points): **Inner Product Spaces and Positive Operators**

(i) (5 points) Suppose $u, v \in V$. Then,

$$\|u + v\| \leq \|u\| + \|v\|. \tag{1}$$

This inequality is an equality if and only if one of u, v is a non-negative real multiple of the other.

(ii) (5 points) Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

Question 4 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(a) (5 points) Prove that the eigenvalues of a self-adjoint operator are real.

(b) (5 points) Suppose that $T \in \mathcal{L}(V)$. Show that if T is self-adjoint and all of its eigenvalues are non-negative, then T is a positive operator. *Hint: use the spectral theorem!*

Question 5 (10 points): **Determinant and Trace**

(a) (2 points) For an invertible matrix A, prove that $\det\left(A^{-1}\right) = (\det A)^{-1}$.

(b) (2 points) Determine whether the following matrix is invertible

$$X = \begin{pmatrix} 2 & 3 & 5 \\ 3 & 4 & 7 \\ 4 & 1 & 5 \end{pmatrix}. \tag{2}$$

(c) (2 point) What is the sum of the eigenvalues of the above matrix? *Hint: do not try to actually compute each eigenvalue.*

(d) (4 points) Suppose that A, B, C are 3-by-3 matrices with $\det(A) = 2$, $\det(B) = 3$, and $\det(C) = 5$. Compute each of the following determinants:

  (a) $\det(AB)$

  (b) $\det\left(2A^{-3}B^{-2}(CB)^4\right)$