MATH 416 Abstract Linear Algebra

Week 5 - Homework 4
Assigned: Fri. Sept. 26, 2025
Due: Fri. Oct. 3, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (6 points): Injectivity and surjectivity
Let V be a (finite-dim.) vector space over a field $\mathbb{F}$. Let $v_1, \dots, v_m \in V$ and define the linear map

$$T \colon \mathbb{F}^m \to V, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \mapsto \sum_{i=1}^{m} x_i v_i.$$

(i) Prove that T is injective if and only if $\{v_1, \dots, v_m\}$ are linearly independent.

(ii) Prove that T is surjective if and only if $V = \operatorname{span}\{v_1, \dots, v_m\}$.

Solution. First, let us recall the following definitions. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{F}^m$. From (Axler 3.14), we know that a function T is injective if $T\boldsymbol{x} = T\boldsymbol{y}$ implies $\boldsymbol{x} = \boldsymbol{y}$. Moreover, recall from (Axler 2.15) that a list is linearly independent if the only choice of coefficients that makes

$$\sum_{i=1}^{m} x_i v_i = \mathbf{0}_V \tag{1}$$

is $x_i = 0$ for all i.

(i) *Proof.* ($\Rightarrow$) Let us first assume T is injective and show that this implies $\{v_1, \dots, v_m\}$ is linearly independent. For this first part of the proof, let us assume that Eq. (1) holds for some collection of coefficients that need not all be zero. By this assumption, we have

$$T\boldsymbol{x} = \sum_{i=1}^{m} x_i v_i = \mathbf{0}_V. \tag{2}$$

But of course we may also consider the action of T on $\mathbf{0} \in \mathbb{F}^m$. A linear map always sends the zero vector in the input space to the zero vector in the output space, thus we have $T\mathbf{0} = \mathbf{0}_V$. Equating these, we obtain $T\boldsymbol{x} = T\mathbf{0}$ which, because we have assumed T is injective, implies $\boldsymbol{x} = \mathbf{0}$. Thus, $x_1 = \cdots = x_m = 0$ which implies that $\{v_1, \dots, v_m\}$

is linearly independent.

($\Leftarrow$) Now let us assume the list is linearly independent and proceed to show that T must be injective. If the list is linearly independent, then the only way to satisfy

$$\sum_{i=1}^{m} x_i v_i = \mathbf{0}_V \tag{3}$$

is to take $x_i = 0$ for all i. Thus, the only $\boldsymbol{x} \in \mathbb{F}^m$ that is sent to the zero vector is $\mathbf{0} \in \mathbb{F}^m$, thus by (Axler 3.15), T is injective. □

(ii) For this problem, recall that a map $T : V \to W$ is surjective if its range is equal to W.

Proof. ($\Rightarrow$) Let us first assume that T is surjective and proceed to show that $V = \operatorname{span}\{v_1, \dots, v_m\}$. First, note that

$$\operatorname{span}\{v_1, \dots, v_m\} = \left\{ \sum_{i=1}^{m} x_i v_i : x_i \in \mathbb{F}\ \forall i \right\}. \tag{4}$$

Moreover, the range of T is

$$\operatorname{range} T = \{\sum_{i=1}^{m} x_i v_i : (x_1, \dots, x_m)^T \in \mathbb{F}^m\}. \tag{5}$$

Thus we see that $\operatorname{span}\{v_1, \dots, v_m\} = \operatorname{range} T$. If T is surjective, then $\operatorname{range} T = V$ which, in turn, implies that $V = \operatorname{span}\{v_1, \dots, v_m\}$ as desired.

($\Leftarrow$) This argument holds in reverse as well. If $V = \operatorname{span}\{v_1, \dots, v_m\}$ is spanned by these vectors and the range is, by definition, equal to the span, then $\operatorname{range} T = V$ and thus T is surjective.

□

Exercise 2 (4 points): Linear maps as matrices I

Let V, W be finite-dimensional vector spaces over a field $\mathbb{F}$, and fix bases $\mathcal{B}_V = \{v_1, \dots, v_n\}$ for V and $\mathcal{B}_W = \{w_1, \dots, w_m\}$ for W. In the following, we abbreviate $\mathcal{M}(\cdot) = \mathcal{M}(\cdot)_{\mathcal{B}_V, \mathcal{B}_W}$.

Show that:

(i) $\mathcal{M}(S + T) = \mathcal{M}(S) + \mathcal{M}(T)$ for $S, T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

(ii) $\mathcal{M}(aT) = a\mathcal{M}(T)$ for $a \in \mathbb{F}$ and $T \in \mathcal{L}_{\mathbb{F}}(V, W)$.

Solution. Recall that for $T \in \mathcal{L}_{\mathbb{F}}(V,W)$ the matrix $\mathcal{M}(T)$ is determined by the images of the basis vectors $v_1,\ldots,v_n$: the j-th column of $\mathcal{M}(T)$ is the coordinate column $[T(v_j)]_{\mathcal{B}_W} \in \mathbb{F}^m$.

(i) Let $S,T \in \mathcal{L}_{\mathbb{F}}(V,W)$. For each basis vector v_j we have

$$(S+T)(v_j) \;=\; S(v_j)+T(v_j).$$

Taking coordinates with respect to $\mathcal{B}_W$ gives

$$[(S+T)(v_j)]_{\mathcal{B}_W} \;=\; [S(v_j)]_{\mathcal{B}_W} + [T(v_j)]_{\mathcal{B}_W}.$$

Thus the j-th column of $\mathcal{M}(S+T)$ equals the sum of the j-th columns of $\mathcal{M}(S)$ and $\mathcal{M}(T)$. Since this holds for every $j=1,\ldots,n$, we conclude

$$\mathcal{M}(S+T) \;=\; \mathcal{M}(S)+\mathcal{M}(T).$$

(ii) Let $a \in \mathbb{F}$ and $T \in \mathcal{L}_{\mathbb{F}}(V,W)$. For each j,

$$(aT)(v_j) \;=\; a\,T(v_j).$$

Taking coordinates yields

$$[(aT)(v_j)]_{\mathcal{B}_W} \;=\; a\,[T(v_j)]_{\mathcal{B}_W}.$$

Hence every column of $\mathcal{M}(aT)$ is a times the corresponding column of $\mathcal{M}(T)$, so

$$\mathcal{M}(aT) \;=\; a\,\mathcal{M}(T).$$

These two properties show that the map $\mathcal{M} : \mathcal{L}_{\mathbb{F}}(V,W) \to \mathbb{F}^{m\times n}$ sending a linear map to its matrix is itself linear (once bases are fixed).

Exercise 3 (4 points): Linear maps as matrices II
Consider the following linear map:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^4, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 + x_3 \\ x_1 - x_2 - x_3 \\ x_1 + x_2 \\ x_2 - x_3 \end{pmatrix}$$

(i) Determine $\mathcal{M}(T)_{\mathcal{S}_3,\mathcal{S}_4}$, where $\mathcal{S}_3$ and $\mathcal{S}_4$ are the standard bases in $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

Solution.
The input is a vector in $\mathbb{R}$, which can be represented as a matrix in $\mathbb{R}^{3,1}$. Thus, to produce a 4-by-1 matrix, our matrix representation of T must be 4-by-3. By (Axler 3.31), the matrix representing T is obtained by letting T act on the basis vectors of the input space. The transformed vectors become the column of T. For example, the first basis vector is

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1-0+0\\1-0-0\\1+0\\0+0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix}. \tag{6}$$

The same process can be carried out for the other two basis vectors. These columns become the columns of $\mathcal{M}(T)_{\mathcal{S}_3,\mathcal{S}_4}$. We thus have

$$\mathcal{M}(T)_{\mathcal{S}_3,\mathcal{S}_4} = \begin{pmatrix}1 & -1 & 1\\1 & -1 & -1\\1 & 1 & 0\\0 & 1 & -1\end{pmatrix}. \tag{7}$$

We can then verify that

$$\begin{pmatrix}1 & -1 & 1\\1 & -1 & -1\\1 & 1 & 0\\0 & 1 & -1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}x_1-x_2+x_3\\x_1-x_2-x_3\\x_1+x_2\\x_2-x_3\end{pmatrix}. \tag{8}$$

(ii) Let now

$$\mathcal{B}_V = \left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \qquad \mathcal{B}_W = \left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}$$

and determine $\mathcal{M}(T)_{\mathcal{B}_V,\mathcal{B}_W}$.

Remark: You do not need to show that $\mathcal{B}_V$ and $\mathcal{B}_W$ are indeed bases for $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

Solution. To determine the form of the map in non-standard bases is a bit more tedious. We still consider the action of T on the input basis vectors, but then we must express the result as a linear combination of the vectors in the second basis. Consider the first element of $\mathcal{B}_V$. We have

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1-0+0\\1-0-0\\1+0\\0+0\end{pmatrix} = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} = 1w_1 + (-1)w_2 + 0w_3 + 1w_4, \tag{9}$$

where we have let w_i denote the i-th entry of $\mathcal{B}_W$. The coefficients in this expansion then become the first column of $\mathcal{M}(T)_{\mathcal{B}_V,\mathcal{B}_W}$. For the other two basis vectors, we have

$$T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1-1+0\\1-1-0\\1+1\\1-0\end{pmatrix} = \begin{pmatrix}0\\0\\2\\1\end{pmatrix} = 0w_1 + 0w_2 + 1w_3 + 1w_4, \tag{10}$$

and finally

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1-1+1\\1-1-1\\1+1\\1-1\end{pmatrix} = \begin{pmatrix}1\\-1\\2\\0\end{pmatrix} = 1w_1 + 1w_2 + 0w_3 + 0w_4. \tag{11}$$

Putting these coefficients into the corresponding column of the matrix, we obtain

$$\mathcal{M}(T)_{\mathcal{B}_V,\mathcal{B}_W} = \begin{pmatrix}1 & 0 & 1\\-1 & 0 & 1\\0 & 1 & 0\\1 & 1 & 0\end{pmatrix}. \tag{12}$$

(optional) Bonus Question (2 points): **Projection Operators**
Let $P \in \mathcal{L}(V)$ such that $P^2 = P$. Such operators are called *projection operators* or *projectors* and they play a fundamental role in linear algebra, representation theory, quantum mechanics, data science, and many more fields. The following problem gives a hint as to why they are so useful.

(i) Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

Solution. From (Axler 3.13), we know $\operatorname{null} P \leq V$. Similarly, From (Axler 3.18), we know $\operatorname{range} P \leq V$. Thus, both the null space and the range are subspaces of V. To show that

$V = \operatorname{null} P + \operatorname{range} P$, we must show that an arbitrary $v \in V$ can be written as a sum of an element from the null space and an element from the range. To see this, we write

$$\begin{aligned} v &= v + 0, && (13)\\ &= v + (-Pv + Pv), && (14)\\ &= (v - Pv) + Pv, && (15) \end{aligned}$$

where we claim that the first term is in the null space and the second is in the range. We have $P(v - Pv) = Pv - P^2v = Pv - Pv = 0$, thus $(v - Pv) \in \operatorname{null} P$. Similarly, $P(Pv) = P^2v = Pv$, thus $Pv \in \operatorname{range} P$. It follows that $v \in \operatorname{null} P + \operatorname{range} P$. Because v was arbitrary, $V = \operatorname{null} P + \operatorname{range} P$.

To show that this sum is direct, it suffices (by Axler 1.46) to show that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. To do so, suppose $u \in \operatorname{null} P \cap \operatorname{range} P$. Because u is in the null space, $Pu = 0$. Because it is in the range, we know there exists a $w \in V$ such that $u = Pw$. Applying P to this second equation, we see $Pu = P^2w = Pw$. However $Pu = 0$, thus $Pw = 0$. But, $u = Pw$, thus $u = 0$. Because u was an arbitrary element of $\operatorname{null} P \cap \operatorname{range} P$, we may conclude that $\operatorname{null} P \cap \operatorname{range} P = \{0\}$. Thus by (Axler 1.46) we have

$$V = \operatorname{null} P \oplus \operatorname{range} P, \tag{16}$$

as desired.