Math 416: Abstract Linear Algebra

Date: Sept. 8, 2025

Lecture: 6

Announcements

- ☐ HW2 is now live. Due 9/12.
  - ↳ Get started early!
- ☐ Updated office hours:
  - Tuesdays 5 - 5:50 Davenport 212
  - Wednesdays 2-2:50 Davenport 132

Last time

- ☐ More complex #s
- ☐ Vector spaces

This time

- ☐ Examples of vector spaces
- ☐ Subspaces and direct sums

Recommended reading/watching

- ☐ § 1B - 1C of Axler

Vector spaces

1.19 definition: *addition, scalar multiplication*

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Now we are ready to give the formal definition of a vector space.

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

Group axioms!

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity
$1v = v$ for all $v \in V$.

Correction: last time I said $1 \in V$... should be $1 \in \mathbb{F}$

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

The following geometric language sometimes aids our intuition.

1.21 definition: *vector, point*

Elements of a vector space are called *vectors* or *points*.

Vector space "=" abelian group under addition + scalar mult. by a field

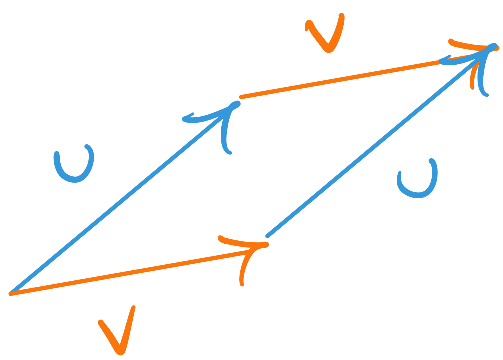
~ Vector space axioms ~

Visualized

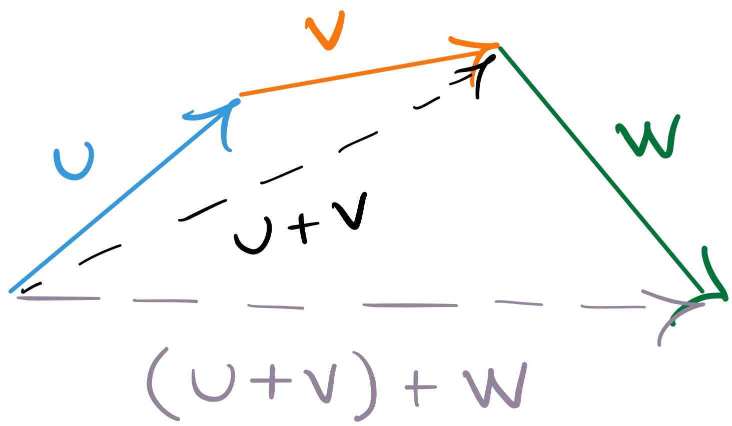
In $\mathbb{R}^2$, we can visualize the axioms

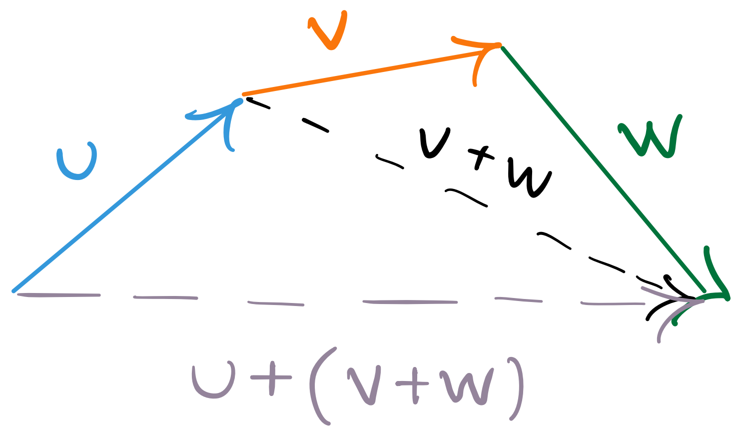
Commutativity

$u + v = v + u$

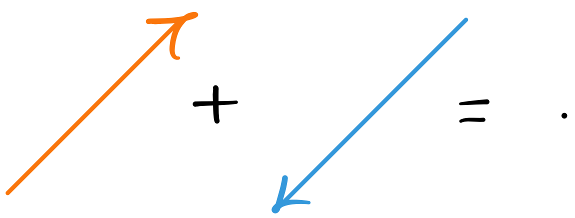


Associativity





Additive inverse



and so on...

Lemma. 1.26 A vector space has a unique additive identity

Proof. Suppose 0 & $0'$ are both additive identities in V.

$$0' = 0' + 0, \quad 0 \text{ is an add iden.}$$
$$= 0 + 0', \quad \text{commutativity}$$
$$= 0, \quad 0' \text{ is an add iden}$$

Thus $0' = 0$ & the identity is unique in a vector space.

Lemma 1.27 (unique additive inverse)

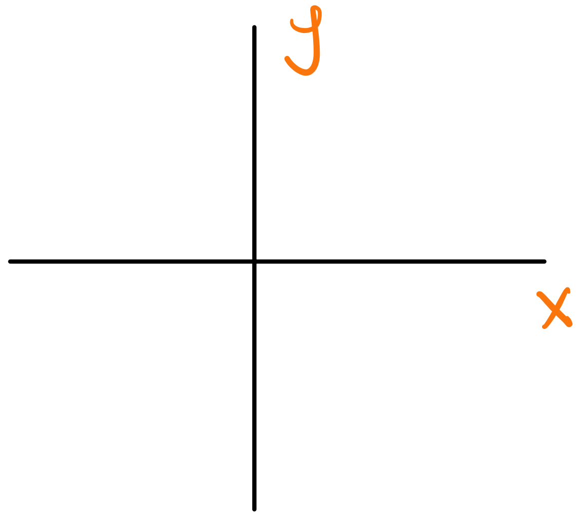
Every element in a vector space has a unique additive inverse.

Proof. Suppose $\exists$ $v, w \in V$ s.t. $u + v = 0$ & $u + w = 0$. Then, we may write

$$
\begin{aligned}
v &= v + 0 && \text{, def. add ident.} \\
&= v + (u + w) && \text{, assumption} \\
&= (v + u) + w && \text{, associativity} \\
&= (u + v) + w && \text{, commutativity} \\
&= 0 + w && \text{, assumption} \\
&= w && \text{, def. add ident.}
\end{aligned}
$$

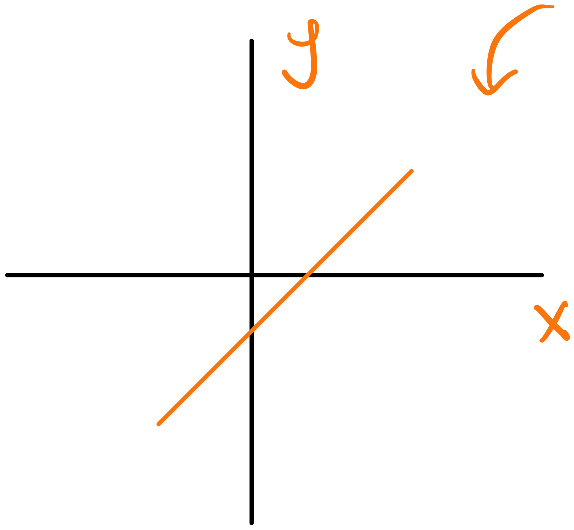
§1.C Subspaces

Consider $\mathbb{R}^2$.



A <u>subset</u> of $\mathbb{R}^2$ is a set of vectors U s.t. every element of U is also an element of $\mathbb{R}^2$.

<u>Example</u>



$U = \{(x, y) \in \mathbb{R}^2 \mid y = x - 1\}$

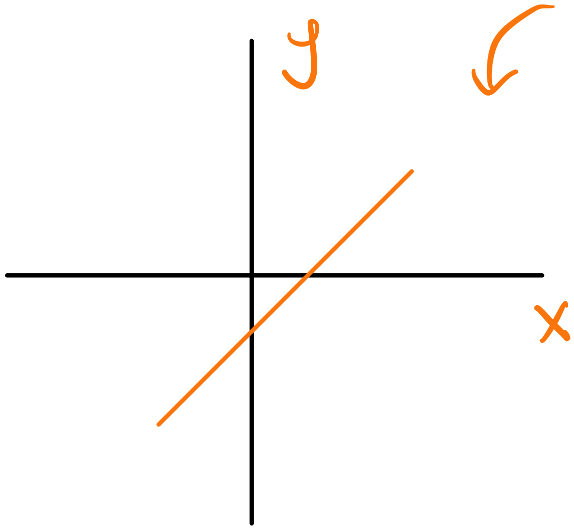
such that

Def. 1.33 (subspace) A subset U of V is called a subspace of V if U is also a vector space w/ the same additive identity, addition, & scalar mult as on V

Example (revisited)

Is U a subspace of V?

$U = \{(x,y) \in \mathbb{R} \mid y = x - 1\}$



Consider scalar mult. Let $x = 1, y = 2$

$$2(x,y) = 2(1,2)$$
$$= (2,4)$$

$$4 \neq 2 - 1 = 1$$

Moreover, we need the additive identity!

$\therefore$ U is <u>not</u> a subspace of V!

Lemma. 1.34

(Conditions for a subspace)

A subset U of V is a subspace iff U satisfies

1. additive identity

$$0 \in U$$

2. closed under addition

$$u, w \in U \Rightarrow u + w \in U$$

3. closed under scalar mult.

$$a \in \mathbb{F} \;\&\; v \in U \Rightarrow av \in U$$

Questions

1) What are the largest & smallest subspaces of a vector space V?

Answer: $\{0\}$ is the smallest

V is the largest

2) Let $b \in \mathbb{F}$. When is

$$\{(x_1, x_2, x_3) \in \mathbb{F}^3 \mid x_3 = 2x_1 + b\}$$

a subspace?

Answer: only when $b=0$.

Sums of subspaces

Def. 1.36 (Sums of subspaces)

Suppose $V_1, \dots, V_m$ are subspaces of V. Then

$$V_1 + \cdots + V_m$$

$$= \{v_1 + \cdots + v_m : v_1 \in V_1, \dots, v_m \in V_m\}$$

In words: $V_1 + \dots + V_m$ is the set of all possible sums of elements of $V_1 \dots, V_m$.

This is anaragous to unions of subsets in set theory!

Example

Let $V = \mathbb{R}^3$

x-y plane

$U = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$

$W = \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$

y-z plane

$V = U + W$ however, the sum will <u>not</u> be unique...

Consider $v = (1,1,1) \in \mathbb{R}^3$.

$$v = \underbrace{(1,1,0)}_{\in U} + \underbrace{(0,0,1)}_{\in W}$$

$$v = (1,0,0) + (0,1,1)$$