MATH 416 Abstract Linear Algebra

Midterm 2 – Practice Exam 1

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"There are two ways to do great mathematics. The first way is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent."*
>
> — Raoul Bott

Question 1 (10 points): **Null Spaces and Ranges**

For this entire problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the null space of T?

Solution. The null space is the set of all vectors that get mapped to the zero vector under a transformation. Formally, let $T \in \mathcal{L}(V, W)$. Then,

$$T = \{v \in V : Tv = 0\}. \tag{1}$$

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is null D?

Solution. The set of all vectors sent to zero under the differentiation map is the set of all constant polynomials.

(iii) (6 points) Prove that null T is a subspace of V.

Proof. To show that a vector space is a subspace, we must show that it contains the zero vector and is closed under addition and scalar multiplication. Linear maps send the zero vector to the zero vector, i.e. $T(0) = 0$. Thus, $0 \in \operatorname{null} T$. To show closure, suppose $u, v \in \operatorname{null} T$ and $a, b \in \mathbb{F}$. Then,

$$T(au + bv) = a(Tu) + b(Tv) = a \cdot 0 + b \cdot 0 = 0, \tag{2}$$

as desired. □

Question 2 (10 points): **Matrices, Invertibility, and Isomorphisms**
Throughout this problem, let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined via

$$T(x, y, z) = (-y, x, 4z).$$

If you have seen determinants before, please note they may not be used in your reasoning below.

(i) (3 points) Write down $\mathcal{M}(T)$ in the standard basis and describe, in words, what this transformation does to a vector in $\mathbb{R}^3$.

Solution. In the standard basis, we can determine the matrix by inspection. We have

$$\mathcal{M}(T) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}. \tag{3}$$

To understand what this transformation does, consider the action on each of the standard basis vectors. For the first two, we see

$$T(1,0,0) = (0,1,0), \tag{4}$$
$$T(0,1,0) = (-1,0,0), \tag{5}$$

so, geometrically, these vectors are rotated 90 degrees about the z axis. The last basis vector is mapped as

$$T(0,0,1) = (0,0,4), \tag{6}$$

which is a stretch of a factor of 4. Thus, an arbitrary vector in $\mathbb{R}^3$ is rotated about the z-axis by 90 degrees and stretched by a factor of 4.

(ii) (4 points) What two properties of T could be used to determine if T is invertible? Use either one of them to argue that T is invertible and then determine T^{-1} by reversing the logic you described in part (i).

Solution. In finite dimensions, we generally have to check that T is both surjective and injective. Because $T \in \mathbb{R}^3$, we know that the input and output space dimensions are the same, thus injectivity is equivalent to surjectivity. Therefore, we should just choose the easiest property to check. Usually, but not always, this is injectivity. For a map to be injective, the null space must contain only the zero vector. To see if this is the case, we consider an arbitrary vector $(a,b,c) \in \operatorname{null} T$. We may write

$$T(a,b,c) = (-b,a,4c) = (0,0,0). \tag{7}$$

Thus, $a = b = c = 0$, and so $\operatorname{null} T = \{0\}$. This implies that T is injective (and thus surjective and invertible). The easiest way to invert the map is to just reverse the logic described in i). That is, T^{-1} should be a rotation by -90 degrees about the z axis and a squash by a factor of 4. Mathematically,

$$T^{-1}(x,y,z) = (y,-x,\frac{1}{4}z). \tag{8}$$

(iii) (3 points) Using explicit matrix multiplication, show that $\mathcal{M}(T)\mathcal{M}(T^{-1}) = I$.

Solution. In the standard basis, we have

$$\mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{4}. \end{pmatrix} \tag{9}$$

To check explicitly that this is the inverse, we may multiply to find

$$\mathcal{M}(T)\mathcal{M}(T)^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

To double check, we can work at the level of the maps

$$\begin{aligned} (TT^{-1})(x,y,z) &= T(T^{-1}(x,y,z)), && (11) \\ &= T(y,-x,\frac{1}{4}z), && (12) \\ &= (x,y,z), && (13) \end{aligned}$$

which implies $TT^{-1} = I$ as desired. Note that, in principle, we would have to check the other order of the multiplication but on the exam you can just say the other direction works the same.

Question 3 (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**
Let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined by

$$T(x, y, z) = (2x + y,\, 2y + z,\, 2z).$$

(i) (2 points) A subspace $U \leq \mathbb{R}^3$ is *invariant under* T if ________________. Complete the definition, and give one simple example of a T-invariant subspace.

> **Solution.** A subspace $U \leq \mathbb{R}^3$ is invariant under T if for all $u \in U$, $Tu \in U$. Some T-invariant subspaces that exist for all T are null T, range T, $\{0\}$, V.

(ii) (5 points) Recall that a vector $v \neq 0$ is called an *eigenvector* of T if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$ (in this case $\mathbb{F} = \mathbb{R}$). Find all such vectors and their corresponding eigenvalues.

> **Solution.** To find the eigenvalues, we write
>
> $$(2x + y, 2y + z, 2z) = (\lambda x, \lambda y, \lambda z), \tag{14}$$
>
> which implies
>
> $$\begin{aligned} 2x + y &= \lambda x, &(15)\\ 2y + z &= \lambda y, &(16)\\ 2z &= \lambda z. &(17) \end{aligned}$$
>
> For the final equation, if $z \neq 0$, then $\lambda = 2$. If $z = 0$ and $y \neq 0$, we find $\lambda = 2$. Finally, if $z = y = 0$ but $x \neq 0$, we find $\lambda = 2$. Thus, we are in a "fully degenerate" case in which all three eigenvalues are 2. A more formal way of stating this is that we have one eigenvalue equal to 2, with *multiplicity* 3. Plugging $\lambda = 2$ back into the first equation above, we find that $y = 0$. This, in turn, implies $z = 0$ via the second equation. The eigenvector may not be zero, thus, it must be of the form $x(1, 0, 0)$, with $x \neq 0$.

(iii) (3 points) Let V be a finite-dimensional vector space over $\mathbb{F}$. Prove that $U \leq V$ is a one-dimensional subspace if and only if there exists a non-zero $v \in V$ such that

$$U = \{\lambda v : \lambda \in \mathbb{F}\}.$$

Proof. ($\Rightarrow$): If U is a one-dimensional subspace, it is spanned by a single vector, $v \in U$. In other words,

$$U = \text{span}\,(v) = \{\lambda v : \lambda \in \mathbb{F}\}. \tag{18}$$

($\Leftarrow$): If there exists a non-zero $v \in V$ such that $U = \{\lambda v : \lambda \in \mathbb{F}\}$, then, an arbitrary element of u lies in the span of v. Trivially, this list is linearly independent (there is only one element). A linearly independent list that spans a subspace is a basis. The length of that basis is the dimensions, thus U is a one-dimensional subspace. $\square$