MATH 416 Abstract Linear Algebra

Week 15 - Homework 10

Assigned: Wed. Dec. 3, 2025

Due: Wed. Dec. 10, 2025 (by 11:59pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (6 points): **Trace**

(i) Prove that the trace is linear (i.e $\text{tr}\,[A+B] = \text{tr}\,[A] + \text{tr}\,[B]$ and $\text{tr}\,[zA] = z\,\text{tr}\,[A]$).

> *Proof.* To prove linearity, it suffices to show that $\text{tr}(\alpha A + \beta B) = \alpha\,\text{tr}(A) + \beta\,\text{tr}(B)$ for any scalars α, β and $n \times n$ matrices A, B. By the definitions of scalar multiplication and matrix addition, the diagonal entry $(\alpha A + \beta B)_{ii}$ is equal to $\alpha A_{ii} + \beta B_{ii}$. Using the definition of the trace:
>
> $$\begin{aligned} \text{tr}(\alpha A + \beta B) &= \sum_{i=1}^{n} (\alpha A + \beta B)_{ii} \\ &= \sum_{i=1}^{n} (\alpha A_{ii} + \beta B_{ii}) \\ &= \alpha \sum_{i=1}^{n} A_{ii} + \beta \sum_{i=1}^{n} B_{ii} \\ &= \alpha\,\text{tr}(A) + \beta\,\text{tr}(B). \end{aligned}$$
>
> Thus, the trace is a linear map. □

(ii) Show that the space of traceless matrices, $\{A \in M_n(\mathbb{F}) \colon \text{tr}(A) = 0\}$, is a subspace of $M_n(\mathbb{F})$. What is its dimension?

> *Proof.* Let $W = \{A \in M_n(\mathbb{F}) : \text{tr}(A) = 0\}$ be the set of traceless matrices. First, the zero matrix 0_n is in W since $\text{tr}(0_n) = 0$. Next, we show closure under linear combinations. Let $A, B \in W$ and let α, β be scalars. By the linearity of the trace:
>
> $$\text{tr}(\alpha A + \beta B) = \alpha\,\text{tr}(A) + \beta\,\text{tr}(B) = \alpha(0) + \beta(0) = 0.$$
>
> Thus, $\alpha A + \beta B \in W$, so W is a subspace of $M_n(\mathbb{F})$. The space $M_n(\mathbb{F})$ has dimension n^2. The condition $\text{tr}(A) = 0$ imposes exactly one linear constraint on the n^2 entries (specifically, the sum of the n diagonal entries must be zero). Subtracting this single

constraint from the total dimension gives:

$$\dim(W) = n^2 - 1.$$

$\square$

(iii) Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Prove that $\operatorname{tr} T^* = \overline{\operatorname{tr} T}$.
Hint: use Axler 8.55 in the fourth edition.

Proof. Let $e_1, \ldots, e_n$ be an orthonormal basis of V. By Axler 8.55 (4th ed.), the trace of an operator on an inner product space is given by:

$$\operatorname{tr}(T) = \sum_{j=1}^{n} \langle Te_j, e_j \rangle.$$

Applying this formula to the adjoint operator T^*, we have:

$$\begin{aligned}
\operatorname{tr}(T^*) &= \sum_{j=1}^{n} \langle T^* e_j, e_j \rangle \\
&= \sum_{j=1}^{n} \langle e_j, Te_j \rangle && \text{(definition of adjoint)} \\
&= \sum_{j=1}^{n} \overline{\langle Te_j, e_j \rangle} && \text{(conjugate symmetry of inner product)} \\
&= \overline{\sum_{j=1}^{n} \langle Te_j, e_j \rangle} && \text{(additivity of complex conjugation)} \\
&= \overline{\operatorname{tr}(T)}.
\end{aligned}$$

$\square$

Exercise 2 (9 points): **Determinant**

(i) Prove that $\det(AB) = \det(A)\det(B)$ for all 2-by-2 matrices A, B.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. First, we calculate the product of the individual determinants:

$$\begin{aligned}\det(A)\det(B) &= (ad-bc)(eh-fg) \\ &= adeh - adfg - bceh + bcfg. \quad (*)\end{aligned}$$

Next, we compute the matrix product AB:

$$AB = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

Now, we compute the determinant of this product matrix:

$$\begin{aligned}\det(AB) &= (ae+bg)(cf+dh) - (af+bh)(ce+dg) \\ &= (aecf + aedh + bgcf + bgdh) - (afce + afdg + bhce + bhdg).\end{aligned}$$

Since scalar multiplication is commutative (e.g., $aecf = afce$), the first and last terms inside the parentheses cancel out:

$$\begin{aligned}\det(AB) &= aedh + bgcf - afdg - bceh. \\ &= adeh - adfg - bceh + bcfg.\end{aligned}$$

Comparing this result with $(*)$, we see that $\det(AB) = \det(A)\det(B)$. □

(ii) Consider the matrix

$$A = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix}. \tag{1}$$

Solve the equation $\det(A - \lambda I) = 0$ to find the eigenvalues of A.

Solution.
First, we set up the characteristic equation $\det(A - \lambda I) = 0$:

$$A - \lambda I = \begin{pmatrix} 6 & -1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 6-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix}.$$

Now, we compute the determinant of this matrix:

$$\begin{aligned}\det(A - \lambda I) &= (6-\lambda)(3-\lambda) - (-1)(2) \\ &= (18 - 6\lambda - 3\lambda + \lambda^2) + 2 \\ &= \lambda^2 - 9\lambda + 20.\end{aligned}$$

To find the eigenvalues, we set this characteristic polynomial equal to zero and solve for λ:

$$\lambda^2 - 9\lambda + 20 = 0$$
$$(\lambda - 4)(\lambda - 5) = 0.$$

Thus, the eigenvalues are $\lambda = 4$ and $\lambda = 5$.

(iii) Let λ_1, λ_2 be two eigenvalues of a matrix A. Define the mean and the product of the eigenvalues of a 2-by-2 matrix as

$$m = \frac{1}{2}\operatorname{tr}[A] = \frac{1}{2}(\lambda_1 + \lambda_2), \tag{2}$$
$$p = \det(A) = \lambda_1\lambda_2. \tag{3}$$

Prove that the eigenvalues of A are given as

$$\lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}. \tag{4}$$

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The eigenvalues are found by solving the characteristic equation $\det(A - \lambda I) = 0$. First, we compute the determinant explicitly:

$$\begin{aligned}0 = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ &= \lambda^2 - (a+d)\lambda + (ad - bc)\end{aligned}$$

We identify the terms in the coefficients: $a + d = \operatorname{tr}(A)$, $ad - bc = \det(A)$.
Using the definitions given in the problem statement ($2m = \operatorname{tr}(A)$ and $p = \det(A)$),

we substitute these into the polynomial:

$$\lambda^2 - 2m\lambda + p = 0.$$

Now we solve for λ using the quadratic formula:

$$\begin{aligned}
\lambda &= \frac{-(-2m) \pm \sqrt{(-2m)^2 - 4(1)(p)}}{2} \\
&= \frac{2m \pm \sqrt{4m^2 - 4p}}{2} \\
&= \frac{2m \pm 2\sqrt{m^2 - p}}{2} \\
&= m \pm \sqrt{m^2 - p}.
\end{aligned}$$

□