Math 416: Abstract Linear Algebra

Date: Sept. 5, 2025

Lecture: 5

Announcements

- ☐ HW 1 was due before class
- ☐ HW 2 is now live. Due 9/12.
  - ↳ Get started early!
- ☐ Updated office hours:
  - Tuesdays 5 - 5:50 Davenport 212
  - Wednesdays 2-2:50 Davenport 132

Last time

- Gauss-Jordan elimination (RREF)
- Complex #s

This time

- More complex #s
- Vector spaces

Recommended reading / watching
- § 1A - 1B of Axler
- 3blue1brown videos (see canvas)

Complex #s (Recap)

Def. (complex #s)

$$\overset{\mathrm{Re}[z]}{a} + \overset{\mathrm{Im}[z]}{b}i$$

- $z = a + bi \quad , \quad a, b \in \mathbb{R}$

- set of all such #s is denoted $\mathbb{C} = \{a+bi : a, b \in \mathbb{R}\}$

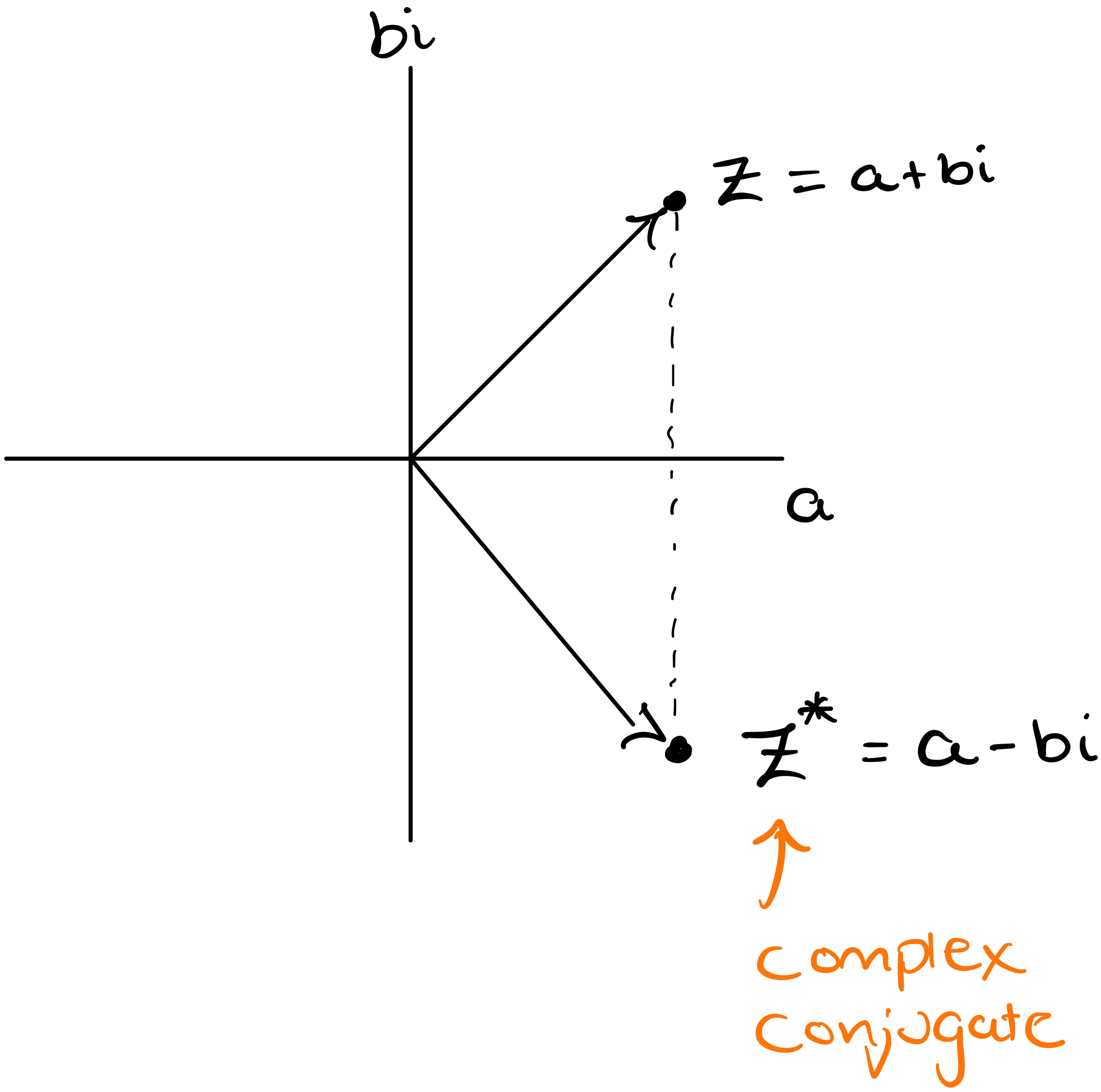
- addition:

$$(a+bi) + (c+di) = (a+b) + (c+d)i$$

- multiplication:

$$(a+bi)(c+di) = (ac-bd) + (ad+bc)i$$

Complex Plane



Challenge

Express $Re[z]$ & $Im[z]$ in terms of z & z^*.

$$Re[z] = \frac{z+z^*}{2}, \quad Im[z] = \frac{z-z^*}{2i}$$

Warm up

a) Show $zz^* \in \mathbb{R}$.

b) What is $\dfrac{a+bi}{c+di}$?

Soln.

a) $z = a+bi$, so $z^* = a-bi$

$zz^* = (a+bi)(a-bi) = a^2+b^2 \in \mathbb{R}$

because reals are closed under addition and mult.

b) This fact indicates how we can divide two complex #s

$$\frac{a+bi}{c+di} = \left(\frac{a+bi}{c+di}\right) \cdot 1$$

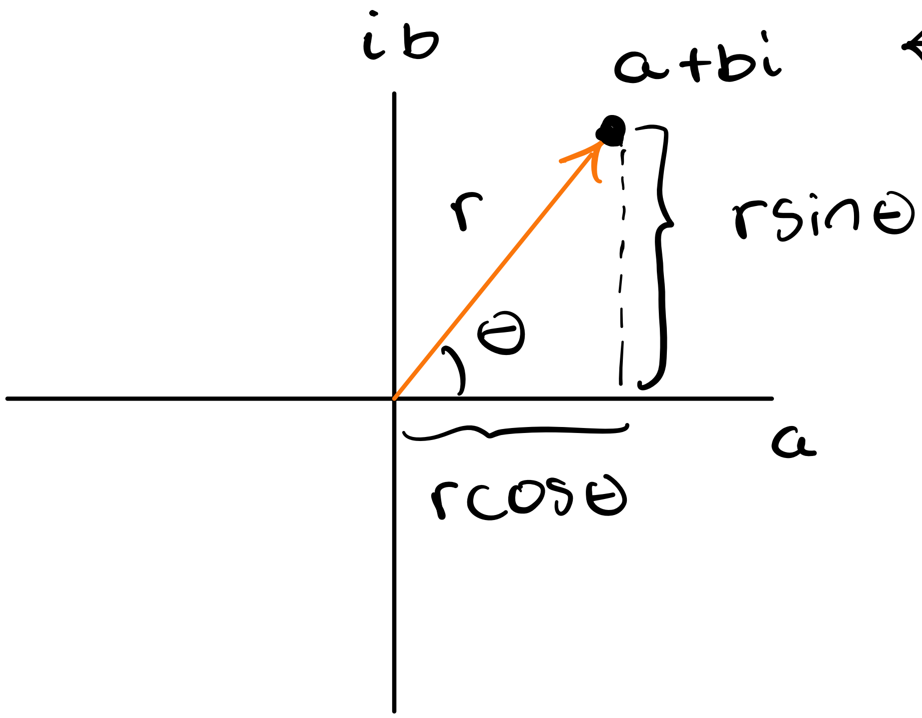
$$= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)$$

$$\frac{a+bi}{c+di} = \frac{(ac+bd)}{c^2+d^2} + \frac{(bc-ad)i}{c^2+d^2}$$

Polar form

Multiplication & division are very tedious in standard form.

Polar form makes these operations simple.



$$a+bi \iff \underbrace{r\cos\theta + r i \sin\theta}_{\text{polar form}}$$

Question:

How do we find r in terms of a, b?

Soln. If we naively apply pythagorean thm, we obtain

$$r^2 = (a)^2 + (bi)^2$$
$$= a^2 - b^2$$

or

$$r^2 = (r\cos\theta)^2 + (r \cdot i \cdot \sin\theta)^2$$
$$= r^2(\cos^2\theta - \sin^2\theta)$$

For complex #s, the length is actually $r = \sqrt{ZZ^*}$

$$ZZ^* = (a+bi)(a-bi)$$
$$= a^2 + b^2$$

and

$$ZZ^* = (r\cos\theta + ir\sin\theta)(r\cos\theta - ir\sin\theta)$$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2 \checkmark$$

Euler's formula

In HW2, ex 1, you will prove one of the most beautiful formulas in math:

$$\underbrace{e^{i\theta} = \cos\theta + i\sin\theta}_{\text{Euler's formula}}$$

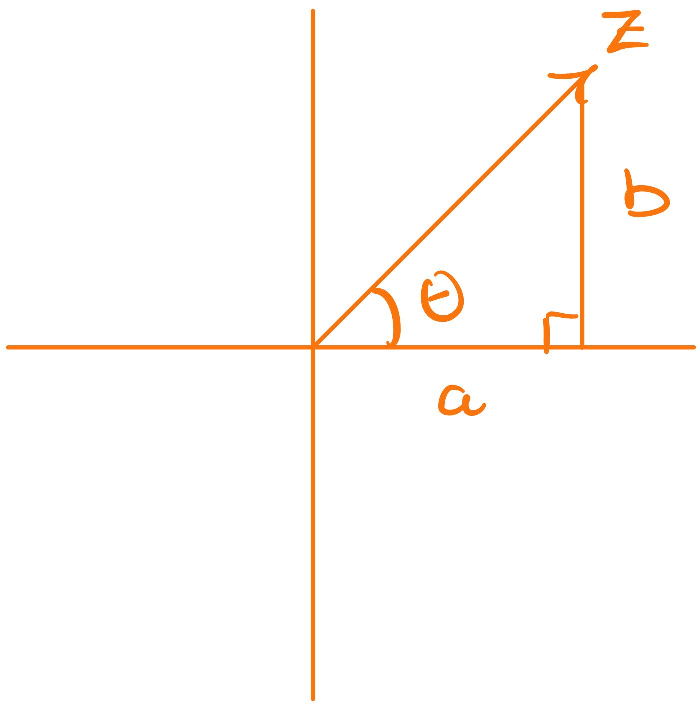
Thus, all complex #s can be written as

$$z = a+bi \iff z = re^{i\theta}$$

Question

what is θ in terms of a & b?

Soln. Draw a picture!



$$\tan\theta = \frac{b}{a}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Multiplication is very easy in this form:

$$z_1 = r_1 e^{i\theta_1}, \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$z z^* = re^{i\theta} re^{-i\theta} = r^2 e^0 = r^2$$

Complex #s Summary

- Add / subtract $\longrightarrow$ standard form

$$(a+bi) \pm (c+di) = (a \pm b) + (c \pm d)i$$

- Mult / divide $\longrightarrow$ exponential form

$$z_1 = r_1 e^{i\theta_1} \quad , \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

- You will prove Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\theta = \pi \quad : \quad \boxed{e^{i\pi} + 1 = 0}$$

Vector spaces

1.19 definition: *addition, scalar multiplication*

- An *addition* on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$.
- A *scalar multiplication* on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Now we are ready to give the formal definition of a vector space.

1.20 definition: *vector space*

A *vector space* is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

commutativity
$u + v = v + u$ for all $u, v \in V$.

associativity
$(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and for all $a, b \in \mathbf{F}$.

additive identity
There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$.

additive inverse
For every $v \in V$, there exists $w \in V$ such that $v + w = 0$.

multiplicative identity
$1v = v$ for all $v \in V$.

distributive properties
$a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$.

The following geometric language sometimes aids our intuition.

1.21 definition: *vector, point*

Elements of a vector space are called *vectors* or *points*.

Lemma. A vector space has a unique additive identity

Proof. Suppose 0 & $0'$ are both additive identities in V.

$$0' = 0' + 0, \quad 0 \text{ is an add iden.}$$

$$= 0 + 0', \quad \text{commutativity}$$

$$= 0, \quad 0' \text{ is an add iden}$$

Thus $0' = 0$ & the identity is unique in a vector space.

Try the following for more practice:

- Additive inverse is unique
- $0v = 0 \quad \forall \; v \in V$
- $a0 = 0 \quad \forall \; a \in \mathbb{F}$
- $(-1)v = -v \quad \forall \; v \in V$

See Axler §1.B for proofs after you try them!