Math 416: Abstract Linear Algebra

Date: Oct. 10, 2025

Lecture: 17

Announcements

- HW5 is due Fri, Oct. 10 @ 8pm
- office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Davenport 132

Last time

- Linear maps as matrix mult. & basis change

This time

- basis change (wrap-up) & polynomials

Reading

- Ch. 4 of Axler

Isomorphic vector spaces

- An <u>isomorphism</u> is an invertible linear map
- Two vec. spaces V, W are isomorphic if $\exists$ an isomorphism between them. We denote this $V \cong W$

Prop. 3.70 (dim. shows whether vector spaces are isomorphic)

V & W are isomorphic $\Leftrightarrow$ $\dim V = \dim W$

proof. $(\Rightarrow)$ If $V \cong W$, $\exists$ an isomorphism $T: V \to W$. Invertibility implies $\operatorname{null} T = \{0\}$ & $\operatorname{range} T = W$. FTLM implies

$$\dim V = 0 + \dim W \quad \checkmark$$

$(\Leftarrow)$ Suppose $\dim V = \dim W$.

Let $v_1, \ldots, v_n$ & $w_1, \ldots, w_n$ be bases of V & W, resp. Define

$$T(c_1 v_1 + \cdots + c_n v_n) = c_1 w_1 + \cdots + c_n w_n$$

- T is well defined b/c $v_1, \ldots, v_n$ is a basis
- T is surjective b/c $w_1, \ldots, w_n$ spans W
- T is inj. ($\text{null}\, T = \{0\}$) b/c $w_1, \ldots, w_n$ is LI.
- T is inj. & surj., thus invertible.
  - $\hookrightarrow \Rightarrow V \cong W$.

$\square$

Remarks

- These results imply every finite-dim vec. space V is isomorphic to $\mathbb{F}^n$ w/ $n = \dim V$.
  - $\hookrightarrow$ e.g. $\mathcal{P}_m(\mathbb{F}) \cong \mathbb{F}^{m+1}$
- $\mathcal{L}(V, W) \cong \mathbb{F}^{m,n}$ (prop. 3.71)
  - $\hookrightarrow \dim \mathcal{L}(V,W) = (\dim V)(\dim W)$ (prop 3.72)

Change of basis

Is I always $\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$?

Consider choosing two different bases for the input & output spaces.

Ex. $B_1 = \{(4,2), (5,3)\}$ & $B_2 = \{(1,0), (0,1)\}$ are bases of $\mathbb{F}^2$.

$$I(4,2) = (4,2) = 4(1,0) + 2(0,1)$$

$$I(5,3) = (5,3) = 5(1,0) + 3(0,1)$$

Thus $M_{B_1, B_2}(I) = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}$

What about the other way?

$$B_1 \leftrightarrow B_2$$

$$I(1,0) = (1,0) = \frac{3}{2}(4,2) + (-1)(5,3)$$

$$I(0,1) = (0,1) = -\frac{5}{2}(4,2) + 2(5,3)$$

So, $M_{B_2,B_1}(I) = \begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix}$

Now, how do these relate?

$$M_{B_1,B_2}(I)\, M_{B_2,B_1}(I) = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 3/2 & -5/2 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By (Axler 3.82) this is always the case! Using these ideas, one can prove the following change of basis formula (from $B_1 \to B_2$)

$$M_{B_1}[T] = M_{B_2,B_1}[I]\, M_{B_2}[T]\, M_{B_1,B_2}[I]$$

$$A \quad = \quad C^{-1} \quad B \quad C$$

Polynomials

- We need a few results from this chapter that will be used in later chapters
- That said, the results here are not technically linear algebra!

Warm-up

Consider $\mathbb{C}$ as a vector space. What is its dimension?

answer below

Answer. The question is somewhat ill-posed unless we specify the field over which the space is defined!

↳ $\mathbb{C}$ as a vector space over $\mathbb{C}$ has dim $= 1$

↳ $\mathbb{C}$ as a vector space over $\mathbb{R}$ has dim $= 2$

Thus, we say $\mathbb{C} \cong \mathbb{R}^2$ when $\mathbb{C}$ is thought of as a vector space <u>over $\mathbb{R}$</u>.

Take-away: the underlying field matters when discussing dimension.

A few facts about complex #s

The following facts will be useful in our course & many future math courses!

Let $z \in \mathbb{C}$. Then $z = a + bi$ for $a, b \in \mathbb{R}$. Recall also that $|z| = \sqrt{z\bar{z}}$. We then have the following facts:

complex conjugate

- $|\mathrm{Re}\, z| \leq |z|$ & $|\mathrm{Im}\, z| \leq z$

- $|z_1 z_2| = |z_1|\,|z_2| \quad \forall\ z_1, z_2 \in \mathbb{C}$

- most importantly,

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

one of the most frequently used ineq. in all of math!

The first two follow from the def of complex #s. Let's prove the triangle inequality.

Claim. $|z+w| \leq |z| + |w| \quad \forall\ z,w \in \mathbb{C}$

Proof.
$$
\begin{aligned}
|w+z|^2 &= (w+z)(\overline{w+z}) \\
&= (w+z)(\bar{w}+\bar{z}) \\
&= w\bar{w} + w\bar{z} + \bar{z}w + z\bar{z} \\
&= |w|^2 + 2\mathrm{Re}(w\bar{z}) + |z|^2 \\
&= |w|^2 + |z|^2 + 2\mathrm{Re}(w\bar{z}) \\
&\leq |w|^2 + |z|^2 + 2|w\bar{z}| \\
&= |w|^2 + |z|^2 + 2|w||\bar{z}| \\
&= |w|^2 + |z|^2 + 2|w||z| \\
&= (|w|+|z|)^2
\end{aligned}
$$

note: you will prove the reverse triangle ineq. in HW6! yay!!!

$$\Rightarrow \quad |w+z| \leq |w| + |z| \qquad \square$$

Fundamental Thm of Algebra

Recall: a func. $p : \mathbb{F} \to \mathbb{F}$ is called a polynomial of degree m if $\exists$ exist non-zero $a_1, \dots, a_m \in \mathbb{F}$ s.t.

$$p(z) = a_0 + a_1 z + \dots + a_m z^m$$

$\forall$ $z \in \mathbb{F}$.

- A zero (or root) of $p \in \mathcal{P}(\mathbb{F})$ is a $\lambda \in \mathbb{F}$ s.t. $p(\lambda) = 0$.

- Let m be a pos. integer and $p \in \mathcal{P}_m(\mathbb{F})$. Then p has at most m zeros in $\mathbb{F}$.

↳ see Axler 4.8

The next result is essential in the proof of the existence of eigenvalues in the next chapter.

↳ though, its importance reaches far beyond lin alg.

Thm 4.12 (Fundamental thm of algebra)

Every nonconstant polynomial w/ complex coeff has a zero in $\mathbb{C}$.

Another important & equiv statement of the theorem is that non-constant $p \in \mathcal{P}_m(\mathbb{F})$ has a factorization of the form

$$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$. And crucially, this factorization is <u>unique</u> up to re-ordering of factors.