MATH 416 Abstract Linear Algebra

Exam 1 – Practice Exam

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using. *I would encourage you to complete the practice exam under exam conditions (50 min, no book or phone, and fear in your heart[1]).*

Question 1 (5 points): **Linear Systems and Gaussian Elimination**
Consider the system of equations.

$$\begin{cases} x + 2y - z = 1, \\ 2x + 4y - 2z = 2, \\ 3x + 6y - 3z = 3. \end{cases}$$

Solve the system, determine the dimension of the solution space, and describe the space geometrically.

[1] This is a joke. You are all going to do great!

Question 2 (5 points): **Complex Numbers are *Real*ly Cool (5 points)**

Let $z = a + bi$ and $w = c + di$ be complex numbers with $a, b, c, d \in \mathbb{R}$. Recall that the conjugate of z is $\overline{z} = a - bi$ and $|z| = \sqrt{a^2 + b^2}$. Further, recall Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$.

1. (1 point) Show that $\overline{z + w} = \overline{z} + \overline{w}$.
2. (1 point) Show that $\overline{zw} = \overline{z}\,\overline{w}$.
3. (2 points) Find two distinct square roots of i in standard form. *Note: you may use either standard or polar form to solve this, but your answer must be expressed in standard form.*
4. (1 point) Compute $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i)^8$.

Question 3 (10 points): **Vector Subspaces, Bases, and Dimension**
Recall that a polynomial is a function $p : \mathbb{R} \to \mathbb{R}$ of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m, \tag{1}$$

for all $x \in \mathbb{R}$. We denote the set of all such functions $\mathcal{P}(\mathbb{R})$. When we assume $m < \infty$, we denote the set $\mathcal{P}_m(x)$. In class, we claimed this is a vector space. In this problem, we develop that fact. Note that addition and scalar multiplication are defined as follows. For all $f, g \in \mathcal{P}(x)$ the sum $f + g \in \mathcal{P}(x)$ is the function defined by

$$(f+g)(x) = f(x) + g(x). \tag{2}$$

Similarly, for all $\lambda \in \mathbb{R}$ and all $f \in \mathcal{P}(x)$, the product $\lambda f \in \mathcal{P}(x)$ is the function defined by

$$(\lambda f)(x) = \lambda f(x) \tag{3}$$

(i) Consider the set $B = \{1, x, x^2, \ldots, x^m\}$. Show B is a basis for $\mathcal{P}_m(\mathbb{R})$. Use this to determine the dimension of $\mathcal{P}_m(\mathbb{R})$.

(ii) Now, let $U_i := \operatorname{span}\{x^i\} = \{cx^i : c \in \mathbb{R}\}$ for all $1 \le i \le m$ be subspaces of $\mathcal{P}(\mathbb{R})$. Show that

$$\mathcal{P}_m(\mathbb{R}) = U_0 \oplus U_1 \oplus \cdots \oplus U_m. \tag{4}$$

Show that this result can be used to conclude the value of $\dim \mathcal{P}_m(\mathbb{R})$. *Hint: recall (from HW3) that for subspaces $V_1, \cdots, V_m$ of V, we always have* $\dim(V_1 + \cdots + V_m) \le \dim V_1 + \cdots + \dim V_m$.

Question 4 (10 points): **The Vector Space of Linear Maps**

We saw in class that the set $\mathcal{L}(U, V)$ of all linear maps from U to V is, indeed, a vector space. In fact, it is one of the most important vector spaces we will study in this course. Recall also that $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials over $\mathbb{R}$.

(i) Define a function $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $(Tp)(x) = x^2 p(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. Show that this function is a linear map.

(ii) Let p' denote the derivative of p. Consider $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ defined as $Dp = p'$ for all $p \in \mathcal{P}(\mathbb{R})$. Using fundamental definitions from calculus, one can show that this is, indeed a linear map. Using this fact and (i), show that $DT \neq DT$. That is, these two maps do not commute.