Math 416: Abstract Linear Algebra

Date: Oct. 27, 2025

Lecture: 22

Announcements

- ☐ HW 7 due Friday @ 8pm
- ☐ Midterms will be graded by Friday
  - ↳ corrections due next Fri (Nov 7)

Last time

- ☐ upper-triangular matrices

This time

- ☐ Upper-triangular matrices (existence proof)
- ☐ Diagonalizable & commuting operators

Reading/watching

- ☐ §5C-E of Axler 4th ed.

Recall from lecture 21 we proved the following prop. relating invariant subspaces & upper-tri. matrices

5.39 *conditions for upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent.

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

We know eigenvalues are related to invariant subspaces & we have now related upper-tri matrices to invariant subspaces...

So, if eigenvalues always exist, does a basis making an operator upper-triangular always exist?

5.47 *if $\mathbf{F} = \mathbf{C}$, then every operator on V has an upper-triangular matrix*

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Note: I prefer the proof in the third edition of Axler b/c it is slightly more self contained.

Proof. We will use induction on $\dim V$.

Base case ($\dim V = 1$): Trivially true.

Induction Hypothesis: Suppose $\dim V > 1$ & result holds for all complex VS w/ dimension strictly less than $\dim V$.

In any such space, we know an eigenvalue of T will exist. Let λ be said eigenval & let

$$U = \operatorname{range}(T - \lambda I)$$

B/c λ is an eigenval $\Rightarrow$ $T - \lambda I$ is not surj. Thus $\dim U < \dim V$.

Moreover, U is invariant under T. To see this, let $u \in U$. Then

$$Tu = Tu - \lambda u + \lambda u$$

$$= \underbrace{(T - \lambda I)u}_{\in U} + \underbrace{\lambda u}_{\in U}$$

$\therefore$ $Tu \in U$, so U is invar. under T.

This establishes that $T|_U$ is an operator on U. By ind. hyp., $\exists$ basis $u_1, \ldots, u_m$ of U s.t. $T|_U$ is upper-tri.

Then, using Prop 5.39, $\forall\ j \in \{1, \ldots, m\}$

$$Tu_j = (T|_U)(u_j) \in \operatorname{span}(u_1, \ldots, u_j)$$

Extend $u_1, \ldots, u_m$ to a basis $u_1, \ldots, u_m, v_1, \ldots, v_n$ of V. For each $k \in \{1, \ldots, n\}$, we have

$$Tv_k = Tv_k - \lambda v_k + \lambda v_k$$
$$= \underbrace{(T - \lambda I)v_k}_{\in U} + \lambda v_k \implies Tv_k \in \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_n)$$

Thus, again by 5.39, we have that T has an upper-triang. w.r.t $u_1, \ldots, u_m, v_1, \ldots, v_n$. $\square$

If we can find this basis, which is not always easy, we can simply read off the eigenvals.

Example. From before,

$$T(x,y,z) = (2x+y,\ 5y+3z,\ 8z).$$

$$\Big\downarrow \text{ standard basis}$$

$$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 3 \\ 0 & 0 & 8 \end{pmatrix}$$

Thus, eigenvalues are simply 2, 5, 8.

See Axler 5.41 for proof.

5.41 *determination of eigenvalues from upper-triangular matrix*

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

Diagonalizable Operators

Upper-triangular matrices exist for all operators on complex vector spaces.

- pro: we can read off eigenvalues!
- cons: - eigenvectors require more work to be determined
  - raising operator to higher powers is still tedious

We'd like to find a basis for V s.t. $T \in \mathcal{L}(V)$ has matrix

$$\mathcal{M}(T) = \begin{pmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix}$$

Such matrices are called diagonal & the corresponding operator is called diagonalizable.

Eigenspaces

When can we diagonalize $T \in \mathcal{L}(V)$?

The following def. will allow us to succinctly state the conditions

5.52 definition: *eigenspace*, $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The *eigenspace* of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by

$$E(\lambda, T) = \text{null}(T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

↳ $E(\lambda, T)$ is a subspace of V

↳ λ is an eigenval of T iff $E(\lambda, T) \neq \{0\}$

Example

Let $T \in \mathcal{L}(V)$ & $\{v_1, v_2, v_3\}$ be a basis of V s.t.

$$M(T) = \begin{pmatrix} 8 & & 0 \\ & 5 & \\ 0 & & 5 \end{pmatrix}$$

Then, $E(8, T) = \text{span}(v_1)$

& $E(5, T) = \text{span}(v_2, v_3)$

With the above def in mind, we have the following proposition.

5.54 *sum of eigenspaces is a direct sum*

Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) \leq \dim V.$$

Proof. To show that $\sum_{i=1}^{m} E(\lambda_i, T)$ is a direct sum, suppose

$$\sum_{i=1}^{m} v_i = 0,$$

where $v_i \in E(\lambda_i, T) \ \forall i \in [1, m]$.

B/c each v_i is an eigenvector corresponding to distinct eigenvalue, $\{v_i\}_{i=1}^{m}$ is a LI list. Thus $v_i = 0 \ \forall i$ & so $\sum_{i=1}^{m} (\lambda_i, T)$ is a direct sum (by Axler 1.45). Moreover

$$\dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T) = \dim\left[E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\right]$$

$$\xrightarrow{\text{Axler 2.37}} \leq \dim V$$

$\square$

Note: in the above inequality, we simply used that the dimension of a subspace is less than or equal to the dim of the full space.

If $m = \dim V$ (i.e. we have a # of distinct eigenvalues equal to the dimension of the space) then each eigenspace is 1D and we have m of them thus

$$\sum_{k=1}^{m} \dim E(\lambda_k, T) = \dim V.$$

However, if we have multiplicities, we will "miss" some dimensions. In other words, eigenvectors corresponding to the same eigenvalue may be linearly dependent.

Lets do a concrete example

Example

Consider $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t.

$\mathcal{M}(T) = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ w.r.t. standard basis.

Let v_1, v_2 be two eigenvectors of T w/ matrices $v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $v_2 = \begin{pmatrix} c \\ d \end{pmatrix}$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2\begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = 2\begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{pmatrix} 2a+b \\ 0a+2b \end{pmatrix} = \begin{pmatrix} 2a \\ 2b \end{pmatrix} \qquad \begin{pmatrix} 2c+d \\ 2d \end{pmatrix} = \begin{pmatrix} 2c \\ 2d \end{pmatrix}$$

$$\Rightarrow b = 0 \qquad \Rightarrow d = 0$$

$$\Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \Rightarrow v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus $E(2, T) = \operatorname{span}(v_1, v_2) = \operatorname{span}(v_1)$

$\therefore \dim E(2, T) = 1 < 2$.

Conditions for diagonalizability

5.55 *conditions equivalent to diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Proof. (a $\Leftrightarrow$ b)

$T \in \mathcal{L}(V)$ has a matrix w.r.t basis $\{v_k\}_{k=1}^n$ iff $Tv_k = \lambda_k v_k$ $\forall$ k essentially by def.

$$\begin{array}{c|c} & \begin{matrix} v_1 & \cdots & v_n \end{matrix} \\ \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} & \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \end{array} \iff Tv_k = \lambda_k v_k$$

(b $\Rightarrow$ c) If b) holds, V has a basis consisting of eigenvecs of T. Thus, all $v \in V$ may be expressed

$$v = \sum_{k=1}^{m} \underbrace{a_k v_k}_{\in E(\lambda_k, T)}$$

Thus, $V = \sum_{k=1}^{m} E(\lambda_k, T)$ &

by Axler 5.54,

$$V = \bigoplus_{k=1}^{m} E(\lambda_k, T),$$

so c) holds.

(c ⇒ d) Follows from Axler 3.94

$\sum_{k=1}^{m} V_k$ is a direct sum iff

$$\dim\left(\sum_{k=1}^{m} V_k\right) = \sum_{k=1}^{m} \dim V_k$$

(d ⇒ b) Want to show

$\dim V = \sum_{k=1}^{m} \dim E(\lambda_k, T)$ ⇒ V has a basis consisting of eigvecs of T.

Choose basis of each $E(\lambda_k, T)$ & consider list $v_1, \ldots, v_n$ ($n = \dim V$). To see that this list is LI (& thus a basis) suppose $\sum_{k=1}^{n} a_k v_k = 0$. Partition n into m bins and let u_k be the sum of all $a_j v_j$ s.t. $v_j \in E(\lambda_k, T)$ $\forall$ $k = 1, \ldots, m$.

Thus $u_k \in E(\lambda_k, T) \Rightarrow u_1 + \cdots + u_m = 0$.

But $u_1, \ldots, u_m$ are eigenvecs corresponding to distinct eigenvals, thus $u_1, \ldots, u_m$ must be LI $\Rightarrow u_k = 0 \ \forall \ k$. But each u_k is a sum of $a_j v_j$ terms where v_j was a basis of $E(\lambda_k, T)$.

Thus $a_j = 0 \ \forall j$ & $v_1, \ldots, v_n$ is LI as desired.

$\square$

5.58 *enough eigenvalues implies diagonalizability*

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.

Proof. Eigenvecs corresponding to distinct eigenvalues are LI.

An LI list of length $= \dim V$ is a basis of V. Thus, if we have $\dim V$ LI eigenvecs, T is diag. w.r.t to this basis $\square$

Commuting Operators

5.71 definition: *commute*

- Two operators S and T on the same vector space *commute* if $ST = TS$.
- Two square matrices A and B of the same size *commute* if $AB = BA$.

5.74 *commuting operators correspond to commuting matrices*

Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute if and only if $\mathcal{M}\big(S, (v_1, \ldots, v_n)\big)$ and $\mathcal{M}\big(T, (v_1, \ldots, v_n)\big)$ commute.

5.75 *eigenspace is invariant under commuting operator*

Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbf{F}$. Then $E(\lambda, S)$ is invariant under T.

5.76 *simultaneous diagonalizability $\Longleftrightarrow$ commutativity*

Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.

5.78 *common eigenvector for commuting operators*

Every pair of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.

5.80 *commuting operators are simultaneously upper triangularizable*

Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.