Math 416

Lecture 2
Wed. 8/27/25

Announcements

- Proof tutorial on canvas
- Office hours today
  - ↪ 2 - 2:50pm (132 Davenport)
  - ↪ proofs!
- Today's plan
  - ↪ simultaneous eqs.
  - ↪ matrices, vectors
  - ↪ matrix operations

Warm-up

Solve for x & y.

$$2x - y = 0 \qquad (1)$$

$$-x + 2y = 3 \qquad (2)$$

plug into
Eq. (2)

$$y = 2x \longrightarrow -x + 2(2x) = 3$$

$$3x = 3$$

back
Substitute info
Eq. (1)

$$x = 1$$

$$y = 2 \cdot 1 = 2$$

Solution: $\boxed{x = 1,\ y = 2}$

Matrix Equations

$$2x - y = 0 \qquad (1)$$

$$-x + 2y = 3 \qquad (2)$$

Column picture

vectors

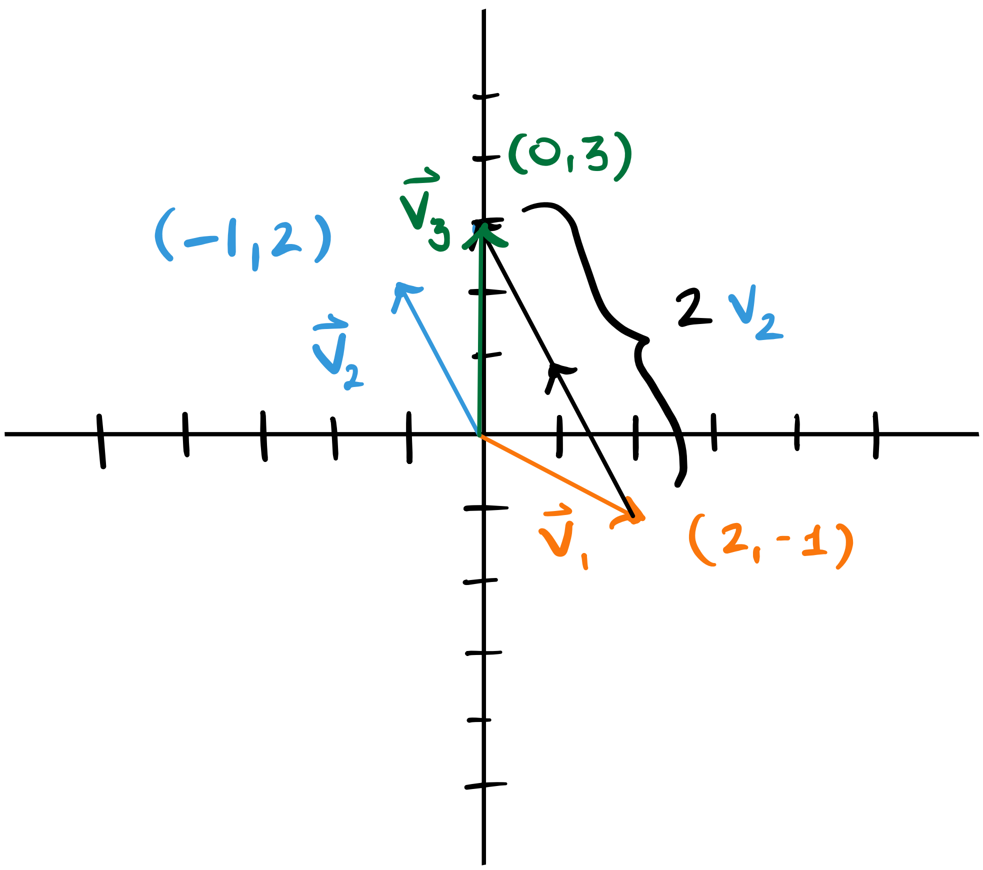
$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x \, \vec{v}_1 + y \, \vec{v}_2 = \vec{v}_3$$

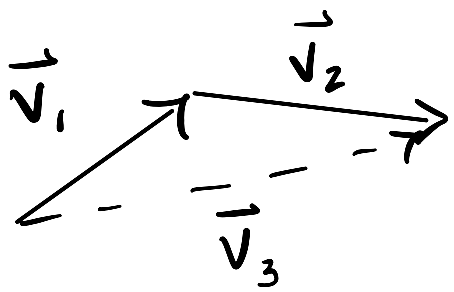
- In words, $\vec{v}_3$ is a <u>linear combination</u> of $\vec{v}_1$ & $\vec{v}_2$.
- Equivalently, we say $\vec{v}_3$ lies in the <u>span</u> of $\{\vec{v}_1, \vec{v}_2\}$

Column picture visualized

$$2x - y = 0$$
$$-x + 2y = 3$$
$$\longrightarrow \quad x\underbrace{\begin{bmatrix}2\\-1\end{bmatrix}}_{\vec{v}_1} + y\underbrace{\begin{bmatrix}-1\\2\end{bmatrix}}_{\vec{v}_2} = \underbrace{\begin{bmatrix}0\\3\end{bmatrix}}_{\vec{v}_3}$$



vector addition



$$\longleftrightarrow \quad \begin{bmatrix}a\\b\end{bmatrix} + \begin{bmatrix}c\\d\end{bmatrix} = \begin{bmatrix}a+c\\b+d\end{bmatrix}$$

Clicker Question

Geometrically, what is the solution to these equations?

$$2x - y = 0 \qquad (1)$$

$$-x + 2y = 3 \qquad (2)$$

A) a number

B) a point

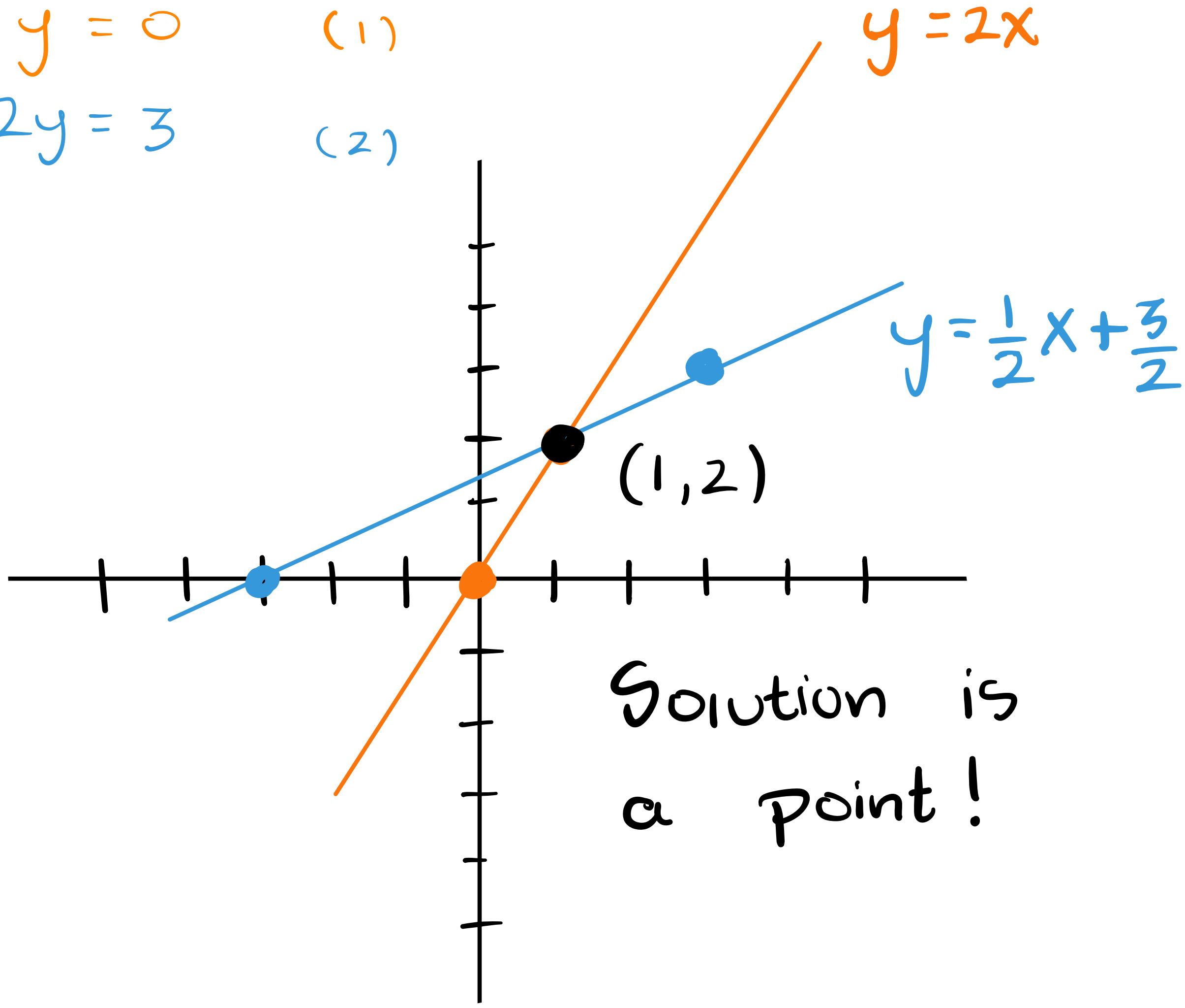
C) a line

D) a plane

Row picture visualized

$2x - y = 0 \quad (1)$

$-x + 2y = 3 \quad (2)$



Solution is a point!

Row picture

$$\underbrace{\overbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}^{\text{Matrix}}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{\vec{b}}$$

Matrix multiplication

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \xrightarrow{?} \begin{aligned} 2x - y &= 0 \quad (1) \\ -x + 2y &= 3 \quad (2) \end{aligned}$$

$$2x - y = 0$$
$$-1x + 2y = 3$$

Of course we will soon deal with larger matrices. So, we need some more notation...

Some matrix notation

$$\left.\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ \vec{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}\right\} 2 \times 2$$

In general, we have

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix}$$

Question

Can you write down an explict expression for

$A\vec{x} = ?$

Answer

Always start small!

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 \quad a_{22}x_2 \end{bmatrix}}_{\vec{b}}$$

In general, for $n \times n$ matrix times $n \times 1$ column vector, we have

$$b_i = \sum_j a_{ij} x_j$$

note: this is just the dot product between the i-th row of A & $\vec{x}$

Question

How many simultaneous equations do you think you could solve in a day?

answer: $\sim 7-8$

Gaussian elimination (next time)

- requires $O(n^3)$ operations
  - ↳ 8 eqs $\rightarrow$ $\sim \ll 1$ ms
  - ↳ 10,000 eqs $\rightarrow$ $\sim$ 10s
  - ↳ > 10,000 eqs $\rightarrow$ $\sim$ min - hours