Math 416: Abstract Linear Algebra

Date: Oct. 31, 2025

Lecture: 24

Announcements

- ☐ HW7 due Friday @ 9pm
- ☐ Exam 2 mean: 81%! Corrections due <u>11/7</u>
- ☐ no office hours today or tuesday
  - ↳ email w/ any questions

Last time

- ☐ Diagonalizability (wrap up)
- ☐ Inner product, norms, & inequalities

This time

- ☐ orthonormal bases

Reading/watching

- ☐ §6B of Axler

Pre-lecture quiz

Suppose a, b, c, d are positive #s.

Prove $(a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$.

What is the equality condition?

Solution. Consider $u, v \in \mathbb{R}$ w/

$$u = \left(\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}\right), \quad v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$$

C-S inequality implies

$$|\langle u, v \rangle|^2 \leq \|u\|^2 \|v\|^2 \quad \text{w/ equality}$$

when $u = tv$ for $t \in \mathbb{R}$. Thus, we need only compute $|\langle u, v \rangle|^2$

$$|\langle u, v \rangle|^2 = \langle u, v \rangle^2 = \left(\sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + \sqrt{c} \cdot \frac{1}{\sqrt{c}} + \sqrt{d} \cdot \frac{1}{\sqrt{d}}\right)^2$$

$$= 16$$

$$u = tv \implies t = a = b = c = d.$$

Angle between vectors

What is the angle between the following vectors?

$$u = (1, 2, 3, 4, 5) \quad \& \quad v = (-9, 4, -1, 1, 11)$$

Soln. $\langle u, v \rangle = 55$

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{55}$$

$$\|v\| = \sqrt{\langle v, v \rangle} = 2\sqrt{55}$$

$$\theta = \arccos\left(\frac{\langle u, v \rangle}{\|u\| \|v\|}\right)$$

$$= \arccos\left(\frac{1}{2}\right)$$

$$\theta = 60°$$

Easy, challenging, & open problems

- Easy: how many mutually orthogonal vectors can you find in $\mathbb{R}^n$?

- Challenging: Suppose we just want all pairs of vectors to be "almost orthogonal." How long a list of such vectors can we construct?
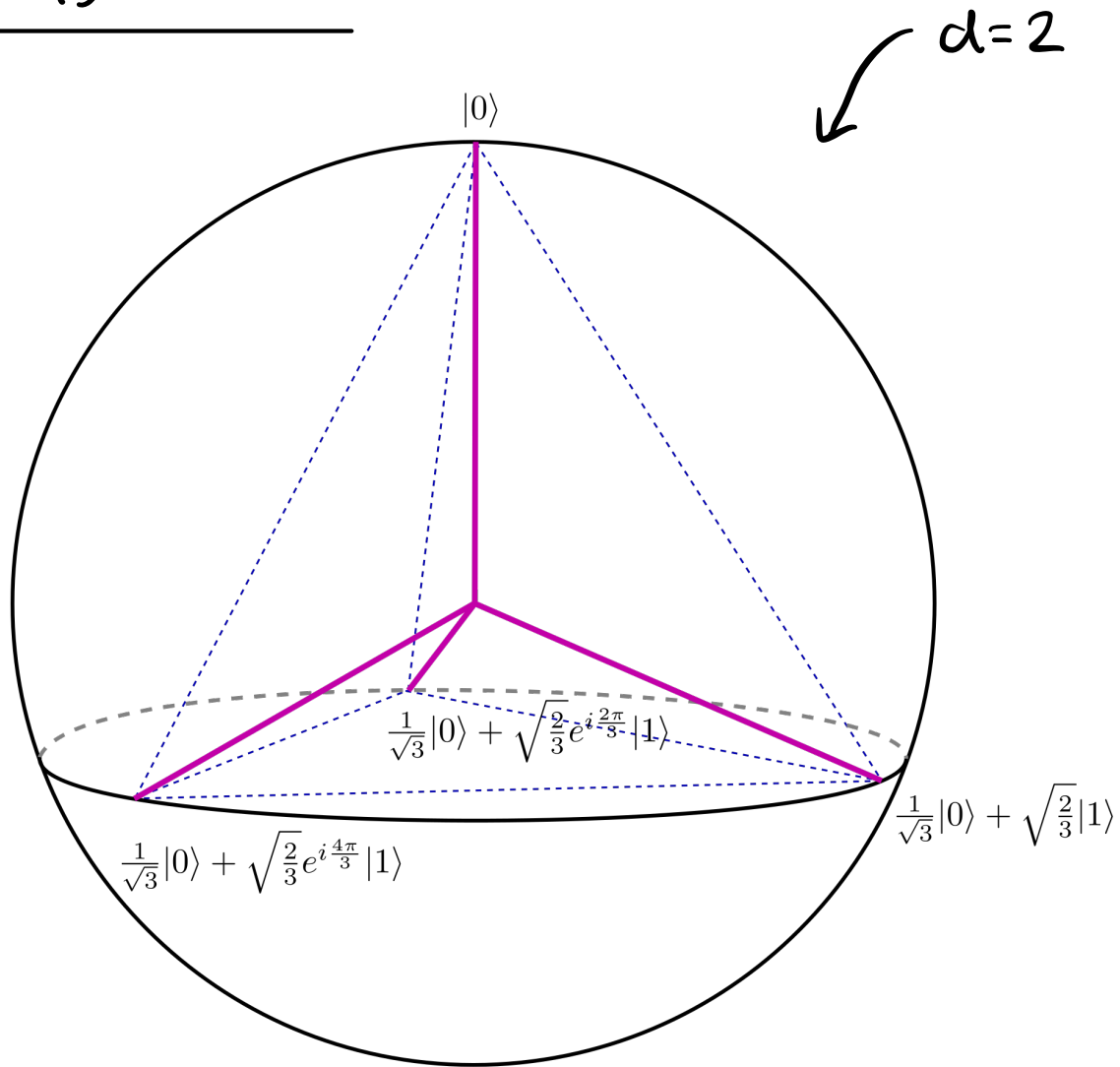
Theorem 2.1.1 (Existence of large ϵ-orthonormal sets). Let $\epsilon \in (0,1)$ be fixed. Then there exists a set of $k = \exp\{O(N\epsilon^2)\}$ unit vectors in $\mathbb{R}^N$ such that

$$|\langle x_i | x_j \rangle| \leq \epsilon \quad \forall\, i \neq j. \tag{2.21}$$

- unsolved open problem: Let $d \geq 2$. Find d^2 vecs $v_1, \ldots, v_{d^2} \in \mathbb{C}^d$ s.t.

$$\forall\, i \neq j \quad |\langle v_i, v_j \rangle|^2 = \frac{1}{d+1} \quad \text{and} \quad \sum_{i=1}^{d^2} v_i v_i^* = d\, I_d$$

$*$ means conjugate transpose

SIC-POVM problem / Zauner's conjecture

What is known?

$d=2$

$|0\rangle$

$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\frac{2\pi}{3}}|1\rangle$

$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$

$\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\frac{4\pi}{3}}|1\rangle$

Shown analytically to exist in

$$d = 2\text{--}24, 28, 30, 31, 35, 37, 39, 43, 48, 124.$$

and several others. No known counter-examples ...

Back to our course...

Orthonormal bases

Consider $V = \mathbb{R}^2$ w/ the standard euclidean inner product (dot product).

Let $V_1 = (1,1)$ & $V_2 = (1,0)$.

Convert this to an orthonormal basis.

Soln. V_2 is normalized already but not orthogonal to $V_1 \ldots$

Step 1. Normalize U_1

$$e_1 = \frac{V_1}{\|V_1\|} = \frac{(1,1)}{\sqrt{2}} = \frac{1}{\sqrt{2}}(1,1)$$

Step 2. From V_2, subtract off component along V_1.

$$v_2' = v_2 - \langle v_2, e_1 \rangle e_1 = (1,0) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(1,1)\right)$$

$$= \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

Step 3. Normalize V_2'

$$e_2 := \frac{v_2'}{\|v_2'\|} = \frac{\left(\tfrac{1}{2}, -\tfrac{1}{2}\right)}{1/\sqrt{2}} = \frac{1}{\sqrt{2}}(1,-1).$$

This process naturally generalizes

6.32 Gram–Schmidt procedure

Suppose $v_1, \ldots, v_m$ is a linearly independent list of vectors in V. Let $f_1 = v_1$. For $k = 2, \ldots, m$, define f_k inductively by

$$f_k = v_k - \frac{\langle v_k, f_1 \rangle}{\|f_1\|^2} f_1 - \cdots - \frac{\langle v_k, f_{k-1} \rangle}{\|f_{k-1}\|^2} f_{k-1}.$$

For each $k = 1, \ldots, m$, let $e_k = \dfrac{f_k}{\|f_k\|}$. Then $e_1, \ldots, e_m$ is an orthonormal list of vectors in V such that

$$\operatorname{span}(v_1, \ldots, v_k) = \operatorname{span}(e_1, \ldots, e_k)$$

for each $k = 1, \ldots, m$.

More practice

6.34   example: *an orthonormal basis of* $\mathcal{P}_2(\mathbf{R})$

Suppose we make $\mathcal{P}_2(\mathbf{R})$ into an inner product space using the inner product given by

$$\langle p, q \rangle = \int_{-1}^{1} pq$$

for all $p, q \in \mathcal{P}_2(\mathbf{R})$. We know that $1, x, x^2$ is a basis of $\mathcal{P}_2(\mathbf{R})$, but it is not an orthonormal basis. We will find an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$ by applying the Gram–Schmidt procedure with $v_1 = 1$, $v_2 = x$, and $v_3 = x^2$.

See Axler 4e pg. 202!