# MATH 416 Abstract Linear Algebra

Week 10 - Homework 8
**Assigned:** Fri. Oct. 31, 2025
**Due:** Fri. Nov. 7, 2025 (by 8pm)

**Reminder:** I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

**Exercise 1** (5 points): Orthogonal Bases
Suppose $e_1, \ldots, e_n$ is an orthonormal basis of $V$.

(a) (3 points) Prove that if $v_1, \ldots, v_n$ are vectors in $V$ such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}} \tag{1}$$

for each $k$, then $v_1, \ldots, v_n$ is a basis of $V$.

(b) Show that there exist $v_1, \ldots, v_n \in V$ such that

$$\|e_k - v_k\| \leq \frac{1}{\sqrt{n}} \tag{2}$$

for each $k$, but $v_1, \ldots, v_n$ is *not* linearly independent.

*Note: The first part of this exercise shows that if we perturb an orthonormal basis an appropriate amount, we still have a basis. The second part shows that we can't increase the* $1/\sqrt{n}$.

**Exercise 2** (5 points): Inner products and orthogonal complements

(i) (3 points) Let $U \leq \mathbb{R}^4$ be the subspace spanned by the vectors $v_1 = (1, 2, 3, -4)^T$ and $v_2 = (-5, 4, 3, 2)^T$. Find orthonormal bases for $U$ and its orthogonal complement $U^\perp$ for the standard inner product $\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$.

(ii) (2 points) Consider the following inner product on $\mathbb{R}^3$: $\langle x, y \rangle_{\text{alt}} := 2x_1 y_1 + x_2 y_2 + 2x_3 y_3$. Compute $\{v\}^\perp$ for the vector $v = (1, -2, 1)^T \in \mathbb{R}^3$.

*Warning: If you use the Gram-Schmidt (GS) procedure for this example, then you need to use the inner product $\langle x, y \rangle_{\text{alt}}$ and the associated norm $\|x\|_{\text{alt}} := \sqrt{\langle x, x \rangle_{\text{alt}}}$ in the GS-formulas.*

**Exercise 3** (5 points): Orthogonal projections

(i) (2 points) Suppose $u, v \in V$. Prove that $\langle u, v \rangle = 0 \Leftrightarrow \|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.

(ii) (1 point) Let $U \leq V$ be a subspace of a finite-dimensional inner product space $V$. Show that $P_{U^\perp} = I_V - P_U$.

(iii) (2 points) Suppose $V$ is finite-dimensional and $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and $\|Pv\| \leq \|v\|$ for every $v \in V$. Prove that there exists a subspace $U$ of $V$ such that $P = P_U$.