MATH 416 Abstract Linear Algebra

Midterm 3 – Practice Exam 1

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."

—John von Neumann

Question 1 (10 points): **Inner Product Spaces**

(i) (5 points) Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is injective.

(ii) (5 points) Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$. Prove that

$$\|v - P_U v\| \leq \|v - u\|, \tag{1}$$

where P_U is the orthogonal projection onto U.

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(i) (5 points) Suppose $T \in \mathcal{L}(V, W)$. Prove that range $T^* = (\text{null } T)^\perp$.

(ii) (5 points) Suppose $\mathbb{F} = \mathbb{C}$. Suppose $T \in \mathcal{L}(V)$ is normal and only has one eigenvalue. Prove that T is a scalar multiple of the identity operator.

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(i) (5 points) Suppose that $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

(ii) (5 points) Prove that all eigenvalues of a unitary operator have absolute value 1. Using this, prove that if S is a unitary operator, then there is an orthonormal basis of V consisting of eigenvectors of S whose corresponding eigenvalues all have absolute value 1.