# Math 416: Abstract Linear Algebra

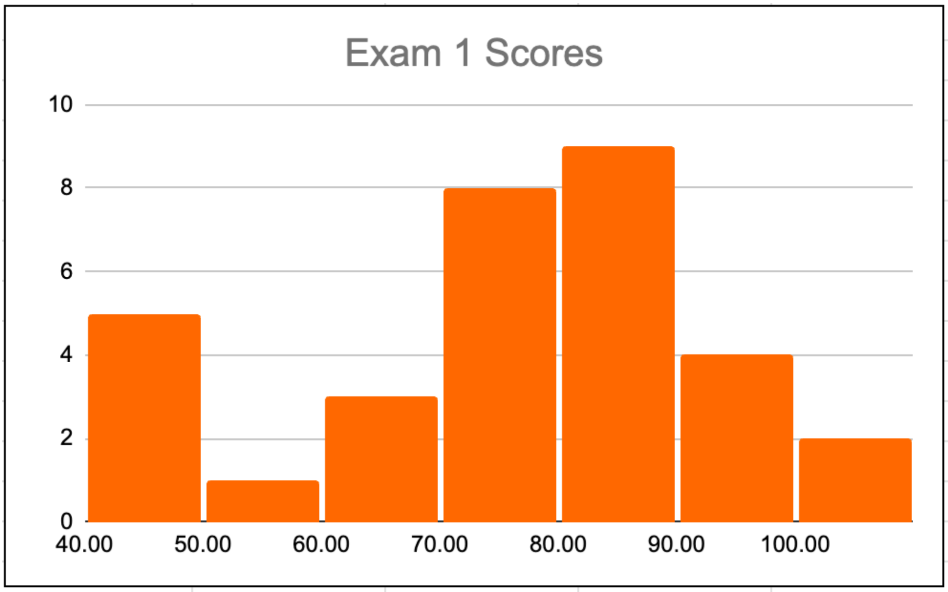
Date: Sept. 28, 2025

Lecture: 12

## Announcements

- ☐ HW4 is due Fri, Oct. 3 @ 8pm
- ☐ office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Davenport 132
- ☐ Exam #1 corrections
  - ↳ Monday 11:59pm ← Oct. 6
  - ↳ half credit for each correction

# Exam 1 Results




mean : 75%

median : 79%

Standard deviation : 18%

After corrections

mean : ~88% if everyone does them!

# This time

- [ ] Fundamental theorem of linear maps

## Recommended reading/watching

- [ ] §3B of Axler
- [ ] 3blue1brown column space/null space

# Next time

- [ ] Matrices

# Null space and injectivity

- For $T \in \mathcal{L}(V,W)$, the null space is the subset $\text{null}\, T = \{v \in V : Tv = 0\}$

  ↳ also called the kernel, $\ker T$

**Prop 3.13** ($\text{null}\, T \leq V$)

Suppose $T \in \mathcal{L}(V,W)$. Then $\text{null}\, T \leq V$.

"subspace of"

*Proof.* Check $0 \in \text{Null}\, T$ & closure under add. & mult. (pg. 60)

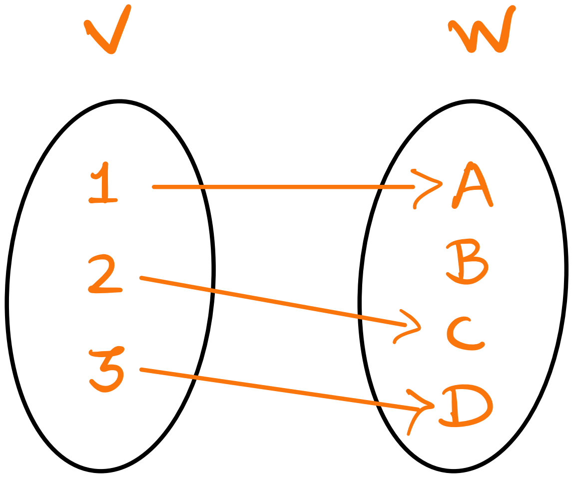
**Def.** Injective

A func $f: V \to W$ is injective (or one-to-one) if

$$Tu = Tv \Rightarrow u = v$$

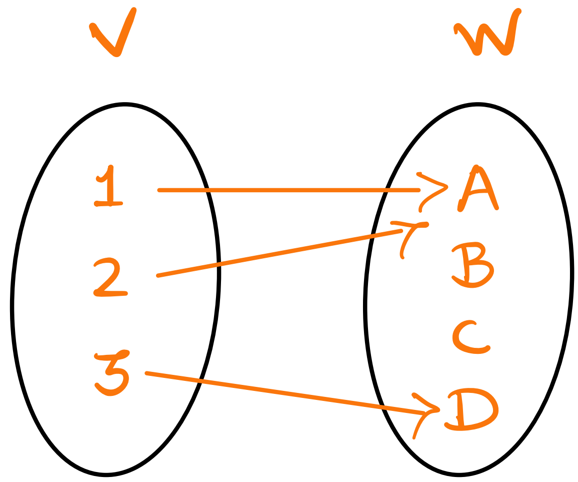
↳ equiv. $u \neq v \Rightarrow Tu \neq Tv$

# Injectivity




injective

e.g. $f(x) = x$




not injective

e.g. $f(x) = x^2$

**Prop. 3.15** (injective $\Leftrightarrow$ null $T = \{0\}$)

Let $T \in \mathcal{L}(V,W)$. $T$ is injective iff null $T = \{0\}$.

Proof. ($\Rightarrow$) If $T$ inject., then null $T = \{0\}$

$\hookrightarrow$ show $\{0\} \subseteq$ null $T$ &

null $T \subseteq \{0\}$

($\Leftarrow$) If null $T = \{0\}$, then $T$ is inject.

$\hookrightarrow$ $u, v \in V$ s.t. $Tu = Tv$. $0 = Tu - Tv = T(u-v)$

$\Rightarrow u - v \in$ null $T$ $\Rightarrow u - v = 0$ or $u = v$.

# range and surjectivity

**Def. 3.16 (range)** or "image", im $T$

$T \in \mathcal{L}(V,W)$. range of $T$ is a subset of $W$ def. as

$$\text{range } T = \{Tv : v \in V\}$$

**Prop. 3.18** (range $T \leq W$)

Proof. Let $T \in \mathcal{L}(V,W)$. $T(0) = 0$ (Axler 3.10), so $0 \in$ range $T$. If $w_1, w_2 \in$ range $T$, $\exists$ $v_1, v_2 \in V$ s.t. $w_1 = Tv_1$ & $w_2 = Tv_2$.

Closure under add :

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2$$

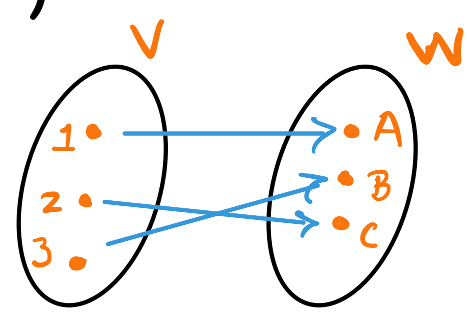
$$\therefore \; w_1 + w_2 \in \text{range } T.$$

Similar for Scalar mult. $\square$

**Def. 3.19 (surjective)** or "onto"

A func $f: V \to W$ is surjective if

$$\text{range } f = W$$



# Fundamental Theorem of Linear Maps

**Theorem 3.21** (Fundamental thm of lin maps)

Suppose $V$ is finite-dim. & $T \in \mathcal{L}(V,W)$.

Then range $T$ is finite-dim. &

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

**Proof.**
- Recall that $\operatorname{null} T \leq V$ (Axler 3.13)
- It follows from (Axler 2.37) that $\dim \operatorname{null} T \leq \dim V$, thus $\operatorname{null} T$ is finite-dim.
- Every finite-dim vec space has a basis by (Axler 2.31).

Let $\underbrace{u_1, \ldots, u_m}_{\text{LI in whole space } V}$ be a basis of $\operatorname{null} T$ ($\Rightarrow \dim \operatorname{null} T = m$).

$\therefore$ we can extend to a basis of $V$ (Axler 2.32) : $\underbrace{u_1, \ldots, u_m, v_1, \ldots, v_n}_{\Rightarrow \dim V = m+n}$

Need to show : 1. range $T$ is finite-dim. & 2. $\dim \operatorname{range} T = n$.

Plan: show $Tv_1, \ldots, Tv_n$ is a basis of range $T$.

Let $v \in V$. $u_1, \ldots, u_m, v_1, \ldots, v_n$ spans $V$, thus

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

$$\Rightarrow \quad Tv = a_1 \underbrace{Tu_1}_{0} + \cdots + a_m \underbrace{Tu_m}_{0} + b_1 Tv_1 + \cdots + b_n Tv_n$$

$$= b_1 Tv_1 + \cdots + b_n Tv_n \qquad (*)$$

(*) implies $Tv_1, \dots, Tv_n$ spans range $T$ which shows range $T$ is finite-dim.

It remains to show $Tv_1, \dots, Tv_n$ is LI. Let $c_i \in \mathbb{F}$ $\forall\, i \in [n]$.

$$c_1 Tv_1 + \dots + c_n Tv_n = 0$$

$$T(\underbrace{c_1 v_1 + \dots + c_n v_n}_{\in \text{Null}\, T}) = 0$$

$\Rightarrow$

but $u_1, \dots, u_m$ spans null $T$! Thus, there must exist $d_i \in \mathbb{F}$ s.t.

$$c_1 v_1 + \dots + c_n v_n = d_1 u_1 + \dots + d_m u_m$$

But $u_1, \dots, u_m, v_1, \dots v_n$ is a basis for $V$!

In particular, $u_1, \dots, u_m, v_1, \dots v_n$ is LI $\therefore$

$c$'s & $d$'s are both all zero. Thus

$$c_1 Tv_1 + \dots + c_n Tv_n = 0$$

only has the trivial soln: $c_i = 0$ $\forall\, i \in [n]$

implying $Tv_1, \dots, Tv_n$ is LI and hence a basis for range $T$.

$\square$

Special case: "Rank-nullity theorem"

in matrix linear eq., we see

$$n = (\#\text{pivots}) + (\#\text{ free variables})$$

$$\dim(\mathbb{R}^n) = \dim(\text{column space}) + \dim(\text{null space})$$

which is a special case of our more general result.

## Corollary 3.22

Suppose $V$ & $W$ are finite-dim vector spaces s.t. $\dim V > \dim W$. Then no lin map from $V$ to $W$ is injective.

recall: (Axler 3.15) $T$ injective $\iff$ $\operatorname{null} T = \{0\}$

**Proof.** Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \operatorname{range} T \\ &\geq \dim V - \dim W, \quad \operatorname{range} T \subseteq W \\ &> 0, \quad \dim V > \dim W \end{aligned}$$

$\implies$ $T$ is not injective by (Axler 3.15).

## Corollary 3.24

If $\dim V < \dim W$ $\nexists$ $T \in \mathcal{L}(V, W)$ s.t. $T$ is surjective.