MATH 416 Abstract Linear Algebra

Midterm 2 – Practice Exam 1

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"There are two ways to do great mathematics. The first way is to be smarter than everybody else. The second way is to be stupider than everybody else – but persistent."*
>
> — Raoul Bott

Question 1 (10 points): **Null Spaces and Ranges**

For this entire problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the null space of T?

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is null D?

(iii) (6 points) Prove that null T is a subspace of V.

Question 2 (10 points): **Matrices, Invertibility, and Isomorphisms**
Throughout this problem, let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined via

$$T(x, y, z) = (-y, x, 4z).$$

If you have seen determinants before, please note they may not be used in your reasoning below.

(i) (3 points) Write down $\mathcal{M}(T)$ in the standard basis and describe, in words, what this transformation does to a vector in $\mathbb{R}^3$.

(ii) (4 points) What two properties of T could be used to determine if T is invertible? Use either one of them to argue that T is invertible and then determine T^{-1} by reversing the logic you described in part (i).

(iii) (3 points) Using explicit matrix multiplication, show that $\mathcal{M}(T)\mathcal{M}(T^{-1}) = I$.

Question 3 (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**
Let $T \in \mathcal{L}(\mathbb{R}^3)$ be defined by

$$T(x, y, z) = (2x + y,\, 2y + z,\, 2z).$$

(i) (2 points) A subspace $U \subseteq \mathbb{R}^3$ is *invariant under* T if ________________. Complete the definition, and give one simple example of a T-invariant subspace.

(ii) (5 points) Recall that a vector $v \neq 0$ is called an *eigenvector* of T if $Tv = \lambda v$ for some $\lambda \in \mathbb{F}$ (in this case $\mathbb{F} = \mathbb{R}$). Find all such vectors and their corresponding eigenvalues.

(iii) (3 points) Let V be a finite-dimensional vector space over $\mathbb{F}$. Prove that $U \leq V$ is a one-dimensional subspace if and only if there exists a non-zero $v \in V$ such that

$$U = \{\lambda v : \lambda \in \mathbb{F}\}.$$