**MATH 416 Abstract Linear Algebra**

Midterm 2 – Practice Exam 2

**Exam Instructions:** This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"Chance favors the prepared mind."*
>
> — Louis Pasteur

**Question 1** (10 points): **Null Spaces and Ranges**

For this entire problem, let $V, W$ be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$.

(i) (2 points) What is the definition of the range of $T$?

> **Solution.** The range of $T$ is the set of all vectors in $W$ that can be reached by acting $T$ on some vector in $V$. Formally,
>
> $$\operatorname{range} T = \{Tv : v \in V\}. \tag{1}$$

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. What is range $D$?

> **Solution.** All polynomials can be reached by applying the differentiation operator to a polynomial of a higher degree. Because $D$ is defined here from all polynomials to all polynomials, the range is simply $\mathcal{P}(\mathbb{R})$.

(iii) (6 points) Prove that range $T$ is a subspace of $W$.

> *Proof.* We must show that $0 \in \operatorname{range} T$ and that it is closed under scalar multiplication and addition. For the zero vector, note that $T0 = 0$, thus $0 \in \operatorname{range} T$. Now, let $u, v \in \operatorname{range} T$ and $a, b \in \mathbb{F}$. Then, there exists $u' \in V$ such that $u = Tu'$ and $v' \in V$ such that $v = Tv'$. It follows that
>
> $$\begin{aligned} au + bv &= aTu' + bTv', && (2) \\ &= T(au' + bv'), && (3) \end{aligned}$$
>
> and $au' + bv' \in V$ because it is a vector space (and thus closed under linear combinations of elements). Thus, $au + bv \in \operatorname{range} T$, as desired. Thus, range $T$ is a subspace of $W$.
>
> $\square$

**Question 2** (10 points): **Matrices, Invertibility, and Change of Basis**
Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T(x, y) = (3x + y,\, x + 2y).$$

Let the *standard basis* of $\mathbb{R}^2$ be

$$E = \{e_1 = (1, 0),\, e_2 = (0, 1)\},$$

and let

$$B = \{b_1 = (1, 1),\, b_2 = (1, -1)\}$$

be another basis of $\mathbb{R}^2$.

(i) (3 points) Find $\mathcal{M}(T)_E$, the matrix of $T$ in the standard basis $E$.

> **Solution.** In the standard basis we can usually just eye-ball the matrix; however, to be formal, we should consider the action of $T$ on the standard basis
>
> $$T(1, 0) = (3, 1), \tag{4}$$
> $$T(0, 1) = (1, 2). \tag{5}$$
>
> These become the columns of the matrix in the standard basis
>
> $$\mathcal{M}(T)_E = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}. \tag{6}$$

(ii) (4 points) Express each vector in $B$ in terms of the standard basis, and compute the *change-of-basis matrix*

$$P_{B \to E}$$

(from $B$-coordinates to $E$-coordinates). Then find its inverse $P_{E \to B}$.

> **Solution.** To change from $B$ to $E$, we need to consider acting $I$ on the elements of $B$ and then expressing them in terms of the standard basis vectors as follows
>
> $$I(1, 1) = (1, 1) = 1(1, 0) + 1(0, 1), \tag{7}$$
> $$I(1, -1) = (1, -1) = 1(1, 0) + (-1)(0, 1). \tag{8}$$
>
> Taking these coefficients as the columns of the change-of-basis matrix, we obtain
>
> $$P_{B \to E} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{9}$$

To find the inverse, we may work in the opposite direction. We have

$$I(1,0) = (1,0) = \frac{1}{2}(1,1) + \frac{1}{2}(1,-1), \tag{10}$$
$$I(0,1) = (0,1) = \frac{1}{2}(1,1) + \left(-\frac{1}{2}\right)(1,-1) \tag{11}$$

from which we may form

$$P_{E\to B} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}. \tag{12}$$

To check that we did not make any algebra mistakes, observe

$$P_{B\to E}P_{E\to B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{13}$$

(iii) (3 points) Compute the matrix of $T$ in the basis $B$, denoted $\mathcal{M}(T)_B$, using your result from (ii). In words, does the invertibility of $T$ depend on the choice of basis?

**Solution.** To obtain $\mathcal{M}(T)_B$, we recall that $\mathcal{M}(T)_B = P_{E\to B}\mathcal{M}(T)_E P_{B\to E}$. Thus, we obtain

$$\mathcal{M}(T)_B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 7/2 & 1/2 \\ 1/2 & 3/2. \end{pmatrix} \tag{14}$$

Invertibility can be determined without ever choosing a basis. Because the matrix of a product of maps is simply the product of the matrices (regardless of bases) we know that if $T$ is invertible, $\mathcal{M}(T)$ is invertible regardless of the chosen basis.
Above would be sufficient justification. However, we can go a step further and show that if $T$ is invertible, then there exists a $T^{-1}$ such that $TT^{-1} = I$. This implies

$$\mathcal{M}(I) = \mathcal{M}(TT^{-1}) = \mathcal{M}(T)\mathcal{M}(T^{-1}), \tag{15}$$

which, by the uniqueness of the inverse, implies $\mathcal{M}(T^{-1}) = \mathcal{M}(T)^{-1}$. This holds for all bases and we are done.

**Question 3** (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**

(i) (3 points) Suppose $T \in \mathcal{L}(V)$ and $V_1, \ldots, V_m$ are subspaces of $V$ invariant under $T$. Prove that $V_1 + \cdots + V_m$ is invariant under $T$.

*Proof.* Let $v_i \in V_i$ for all $i$. Then, any $v \in V_1 + \cdots + V_m$ may be written as

$$v = v_1 + \cdots + v_m \tag{16}$$

for some $v_i \in V_i$. Since each $V_i$ is a $T$-invariant subspace, we know that $Tv_i \in V_i$. Thus,

$$\begin{aligned} Tv &= T\left(v_1 + \cdots + v_m\right), & (17) \\ &= Tv_1 + \cdots Tv_m, & (18) \end{aligned}$$

but each $Tv_i \in V_i$, thus $Tv \in V_1 + \cdots V_m$ and we conclude that $V_1 + \cdots V_m$ is $T$-invariant, as desired. □

(ii) (3 points) Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of $T$.

**Solution.** From the definition of an eigenvalue, we have $T(x, y) = (\lambda x, \lambda y)$. Expanding the definition of the map, we obtain

$$(-3y, x) = (\lambda x, \lambda y). \tag{19}$$

This yields the simultaneous equations $-3y = \lambda x$ and $x = \lambda y$. Clearly, neither $x$ nor $y$ can be zero because it would imply the other component is zero. Plugging the second into the first, we see

$$-3y = \lambda^2 y \implies (\lambda^2 + 3)y = 0. \tag{20}$$

We see that the only way this is true is if $\lambda^2 + 3 = 0$, which has no solutions over the reals. Thus, as defined, the operator does not have any eigenvalues. If instead, it was defined over complex numbers, the eigenvalues would clearly be $\pm\sqrt{3}i$.

(iii) (4 points) Suppose $P \in \mathcal{L}(V)$ such that $P^2 = P$. Prove that if $\lambda$ is an eigenvalue of $P$, then $\lambda = 0$ or $\lambda = 1$.

*Proof.* If $\lambda$ is an eigenvalue of $P$, then for some non-zero $v \in V$, we have

$$Pv = \lambda v. \tag{21}$$

We may then write $P^2 v = P(\lambda v) = \lambda(Pv) = \lambda^2 v$. However, $P^2 = P$, thus we have

$$\lambda v = \lambda^2 v \implies (\lambda - \lambda^2) v = 0. \tag{22}$$

Finally, because $v \neq 0$ by the assumption that $\lambda$ is an eigenvalue, we may conclude $(\lambda - \lambda^2) = 0$ or, equivalently, $\lambda(1 - \lambda) = 0$. Thus, $\lambda = 0$ or $\lambda = 1$. $\square$