MATH 416 Abstract Linear Algebra

Week 7 - Homework 6
Assigned: Fri. Oct. 10, 2025
Due: Fri. Oct. 17, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (7 points): **Basis change matrices**
Let $V = \mathbb{R}^3$, and consider the standard basis $\mathcal{S} = \{e_1, e_2, e_3\}$ and the bases $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{B}' = \{w_1, w_2, w_3\}$ with

$$\begin{aligned} v_1 &= (1,1,1)^T & v_2 &= (1,-1,0)^T & v_3 &= (1,0,1)^T \\ w_1 &= (1,0,1)^T & w_2 &= (1,-1,1)^T & w_3 &= (1,1,0)^T. \end{aligned}$$

(i) (2 points) Compute $A = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$ and $B = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}}$ and verify $B = A^{-1}$.

Solution. A's columns are simply v_1, v_2, v_3 in the standard basis

$$A = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \tag{1}$$

To find the form of B, we need to express the standard basis in terms of $\mathcal{B}$. Formally, we have

$$I_V e_1 = e_1 = (1,0,0) = 1v_1 + 1v_2 + (-1)v_3, \tag{2}$$
$$I_V e_2 = e_2 = (0,1,0) = 1v_1 + 0v_2 + (-1)v_3, \tag{3}$$
$$I_V e_3 = e_3 = (0,0,1) = (-1)v_1 + (-1)v_2 + 2v_3, \tag{4}$$

and taking those coefficients to form the columns of B, we have

$$B = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 2. \end{pmatrix} \tag{5}$$

By standard matrix multiplication, we may verify that $AB = BA = I$, thus $B = A^{-1}$ by the uniqueness of the inverse.

(ii) (2 points) Compute $C = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{S}}$ and $D = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$ and verify $D = C^{-1}$.

The basis $\mathcal{B}' = \{w_1, w_2, w_3\}$ gives columns

$$C = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{S}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Playing the same game as above, we find

$$D = \begin{pmatrix} -1 & 1 & 2 \\ 1 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix},$$

and again $CD = DC = I_V$ implying that $D = C^{-1}$ by uniqueness of the inverse.

(iii) (2 points) Compute $E = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$ and $F = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{B}}$ and verify $F = E^{-1}$.

$$I_V v_1 = I_V(1,1,1) = 2w_1 + (-1)w_2 + 0w_3, \tag{6}$$
$$I_V v_2 = I_V(1,-1,0) = -2w_1 + 2w_2 + 1w_3, \tag{7}$$
$$I_V v_3 = I_V(1,0,1) = 1w_1 + 0w_2 + 0w_3, \tag{8}$$

which yields the matrix

$$E = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'} = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{9}$$

Reversing the roles of the two bases, we find

$$F = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{B}} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 1 & 2 & -2 \end{pmatrix}, \tag{10}$$

which we can again multiply to verify that we have obtained the inverses in an appropriate manner.

(iv) (1 point) What is the relationship between $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$, $\mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$, and $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$?

The matrices compose in the obvious way: first convert $\mathcal{B}$-coordinates to standard coordinates, then convert standard coordinates to $\mathcal{B}'$-coordinates. Thus

$$\mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'} = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}\ \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}.$$

Equivalently, writing $A = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$ and $C = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{S}}$ (so $\mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'} = C^{-1}$), we have

$$E = C^{-1}A.$$

Exercise 2 (3 points): **Linear maps as matrices**
Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x - 3y \\ x + y + z \\ 3y - z \end{pmatrix},$$

and let $\mathcal{S}, \mathcal{B}, \mathcal{B}'$ be the bases from Exercise 1.

(i) (2 points) Determine $\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}$ and $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'}$ using the definition of the matrix representation of a linear map.

First, let us compute the action on the standard basis vectors. We have

$$T(1,0,0) = (2,1,0), \tag{11}$$
$$T(0,1,0) = (-3,1,3), \tag{12}$$
$$T(0,0,1) = (0,1,-1), \tag{13}$$

which become the columns of the matrix

$$\mathcal{M}(T)_{\mathcal{S},\mathcal{S}} = \begin{pmatrix} 2 & -3 & 0 \\ 1 & 1 & 1 \\ 0 & 3 & -1 \end{pmatrix} \tag{14}$$

For the non-standard bases, we (annoyingly) have to solve the system of 3 equations with 3 unknowns. Doing so, we obtain

$$T(1,1,1) = (-1,3,2) = 8w_1 + (-6)w_2 + (-3)w_3, \tag{15}$$
$$T(1,-1,0) = (5,0,-3) = -11w_1 + 8w_2 + 8w_3, \tag{16}$$
$$T(1,0,1) = (2,2,2) = (-2)w_1 + 1w_2 + 3w_3, \tag{17}$$

which yields the matrix

$$\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'} = \begin{pmatrix} 8 & -11 & -2 \\ -6 & 8 & 1 \\ -3 & 8 & 3 \end{pmatrix} \tag{18}$$

(ii) (1 point) Verify that $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'} = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$.

This is done by explicit matrix multiplication (i.e. by Mathematica)!

Exercise 3 (5 points): **Reverse Triangle Inequality**

For this problem, let $w, z \in \mathbb{C}$. And recall that $\bar{z}$ denotes the complex conjugate of z.

(i) (1 point) Prove that $|\mathrm{Re}[z]| \leq |z|$ and $|\mathrm{Im}[z]| \leq |z|$.

Proof. Let $z = a + bi$, where $a, b \in \mathbb{R}$. Then, $\mathrm{Re}[z] = a$ and $|z| = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$. We may then write

$$|z| = \sqrt{a^2 + b^2} \geq \sqrt{a^2} = |\mathrm{Re}[z]|, \tag{19}$$

as desired. Equality is reached when $b = 0$ (i.e. when z is real). Similar argument holds for the imaginary part. Geometrically, this is simply saying that the hypotenuse of a right triangle is always at least as long as either of the other sides. □

(ii) (1 point) Prove that $|zw| = |z||w|$.

Proof. This fact follows from more basic facts about complex numbers. We have

$$\begin{aligned} |zw| &= \sqrt{(zw)(\overline{zw})}, && (20) \\ &= \sqrt{(zw)(\bar{z}\bar{w})}, \quad \text{multiplicativity of complex conjugate} && (21) \\ &= \sqrt{(z\bar{z})(w\bar{w})}, \quad \text{commutativity of complex numbers} && (22) \\ &= \sqrt{z\bar{z}}\sqrt{w\bar{w}}, && (23) \\ &= |z||w|, && (24) \end{aligned}$$

as desired. □

(iii) (3 points) Prove the reverse triangle inequality

$$||w| - |z|| \leq |w - z|, \tag{25}$$

for all $w, z \in \mathbb{C}$.

Hint: see page 121 of Axler or Lecture 17 notes for a proof of the standard triangle inequality.

Proof. To prove the reverse triangle inequality, one typically uses the standard triangle inequality and the classic trick of adding zero in a clever way. Observe

$$\begin{aligned} |z| &= |(z - w) + w|, && (26) \\ &\leq |z - w| + |w|, \quad \text{triangle} && (27) \\ &= |w - z| + |w|, && (28) \end{aligned}$$

which implies that $|w - z| \geq |z| - |w|$. Similarly, we may write

$$\begin{aligned} |w| &= |(w - z) + z|, && (29) \\ &\leq |w - z| + |z|, \quad \text{triangle} && (30) \end{aligned}$$

to obtain $|w - z| \geq |w| - |z|$, or $-|w - z| \leq |z| - |w|$. Together, these imply

$$\begin{aligned} -|w - z| \leq |z| - |w| &\leq |w - z|, && (31) \\ \implies \quad ||w| - |z|| &\leq |w - z|, && (32) \end{aligned}$$

as desired. $\square$