MATH 416 Abstract Linear Algebra

Midterm 3 – Practice Exam 2

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"Sometimes the questions are complicated and the answers are simple."*
>
> —Dr. Seuss

Question 1 (10 points): **Inner Product Spaces**

(i) (5 points) Suppose $u, v \in V$. Then,

$$\|u + v\| \leq \|u\| + \|v\|. \tag{1}$$

This inequality is an equality if and only if one of u, v is a non-negative real multiple of the other.

(ii) (5 points) Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that

$$U \text{ is invariant under } T \iff P_U T P_U = T P_U. \tag{2}$$

Question 2 (10 points): **Self-adjoint, Normal Operators, and the Spectral Theorem**

(i) (5 points) Recall that we say two operators A and B commute if $AB = BA$. Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then T is normal if and only if there exist commuting self-adjoint operators A and B such that $T = A + iB$.

(ii) (5 points) Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that

$$T^2 - 5T + 6I = 0. \tag{3}$$

Question 3 (10 points): **Positive Operators, Isometries, and Unitary Operators**

(i) (5 points) Suppose T is a positive operator on V and $v \in V$ is such that $\langle Tv, v \rangle = 0$. Then $Tv = 0$.

(ii) (5 points) Suppose $\mathbb{F} = \mathbb{C}$ and $A, B \in \mathcal{L}(V)$ are self-adjoint. Show that $A + iB$ is unitary if and only if $AB = BA$ and $A^2 + B^2 = I$.