Math 416: Abstract Linear Algebra

Date: Oct. 15, 2025

Lecture: 19

Announcements

- ☐ HW6 is due Fri, Oct. 17 @ 8pm
- ☐ Grades are posted! Contact me w/ any issues
- ☐ Midterm 2 : $\boxed{\text{Fri, Oct 24 @ 1pm}}$

Last time

- ☐ Invariant subspaces, eigenvals/vecs

This time

- ☐ Eigenvals/vecs, polynomials of operators

Reading/watching

- ☐ §5A of Axler
- ☐ 3blue1brown eigenvals/eigenvecs

Eigenvalues & eigenvectors

Prop 5.7 (Equiv. conditions to be an eigenvalue)

Suppose $\dim V < \infty$, $T \in \mathcal{L}(V)$, & $\lambda \in \mathbb{F}$. Then the following are all equivalent:

a) λ is an eigenvalue of T

b) $T - \lambda I$ is not injective

c) $T - \lambda I$ is not surjective

d) $T - \lambda I$ is not invertible

Proof. To see a) $\Leftrightarrow$ b), note

$$Tv = \lambda v \iff Tv - \lambda v = 0$$
$$(T - \lambda I)v = 0$$

$v \neq 0$ by def. of eigenval/vec

remaining equivalences follow from equiv. of injectivity, surjectivity, & invertibility of operators (Axler 3.65)

Prop. 5.11 (linearly indep. eigenvecs)

Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly indep.

To first gain some intuition, let v_1, v_2 be eigenvectors corresponding to distinct $\lambda_1, \lambda_2 \in \mathbb{F}$. Then, we wish to show

$$a_1 v_1 + a_2 v_2 = 0 \implies a_1 = a_2 = 0$$

$$\implies a_1 \lambda_1 v_1 + a_2 \lambda_1 v_2 = 0$$

Consider the following

$$T(a_1 v_1 + a_2 v_2) = a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0$$

Subtracting, we obtain

b/c $\lambda_1 \neq \lambda_2 \therefore \lambda_1 - \lambda_2 \neq 0$

$$a_2 \lambda_1 v_2 - a_2 \lambda_2 v_2 = 0$$

$$(\lambda_1 - \lambda_2)(a_2 v_2) = 0 \implies a_2 = 0 \implies a_1 = 0$$

- This argument can be turned into an inductive proof to handle the general case.

- Challenge: think of a non-inductive proof!

- We also immediately have the following cor.

Cor. 5.12 (# of distict eigenvalues)

Suppose V is finite-dim. Then each $T \in \mathcal{L}(V)$ has at most $\dim V$ distinct eigenvals.

Proof. By above, distinct eigvals have eigvecs that are LI.

$$\text{Length of LI list} \leq \dim V$$

by Axler 2.22.

Polynomials of Operators

Lets define some notation that will allow us to talk about polynomials of operators

Let $T \in \mathcal{L}(V)$ & $m \in \mathbb{Z}^+$. Then

- $T^m \in \mathcal{L}(V)$ is defined to be $T^m := \underbrace{T \cdots T}_{m \text{ times}}$
- $T^0 := I_V \in \mathcal{L}(V)$
- If T is invertible, $T^{-m} := (T^{-1})^m$

From these, we can derive the following:

$$T^m T^n = (\underbrace{T \cdots T}_{m \text{ times}})(\underbrace{T \cdots T}_{n \text{ times}})$$

$$= (\underbrace{T \cdots T}_{m+n \text{ times}})$$

$$= T^{m+n}$$

$$(T^m)^n = (\underbrace{T^m \cdots T^m}_{n \text{ times}})$$

$$= T^{mn}$$

If T is invertible, $m, n \in \mathbb{Z}$. Otherwise $m, n \in \{0, 1, \ldots\}$

We may now define what it means to have a polynomial of an operator.

Suppose $T \in \mathcal{L}(V)$ & $p \in \mathcal{P}(\mathbb{F})$ w/

$$p(z) = a_0 z^0 + a_1 z^1 + \cdots + a_m z^m$$

$\forall z \in \mathbb{F}$. Then, $p(T) \in \mathcal{L}(V)$ is defined as

$$p(T) = a_0 I + a_1 T + \cdots + a_m T^m$$

Example

Suppose $D \in \mathcal{L}(P(\mathbb{R}))$ def. by $Dq = q'$ and let $p(x) = 7 - 3x + 5x^2$.

Let $q(x) = x^3$. What is $(p(D))q$?

$$p(D)q(x) = (7I - 3D + 5D^2)x^3$$

$$= 7x^3 - 3(3x^2) + 5(6x)$$

$$\boxed{p(D)q(x) = 7x^3 - 9x^2 + 30x}$$

Products of polynomials

If $p, q \in \mathcal{P}(\mathbb{F})$, then we define

$$(pq)(z) = p(z)q(z) \quad \forall\, z \in \mathbb{F}$$

From this def., we may derive the following properties:

a) $(pq)(T) = p(T)q(T)$

b) $p(T)q(T) = q(T)p(T)$

Informal proof: these obviously hold for products of polynomials. Let $z \mapsto T$ & the result still holds. (See Axler 5.17)

Finally, let us note that null $p(T)$ is invariant under T.

↳ If $u \in \text{null}\, p(T)$, $p(T)u = 0$. Thus

$$\underbrace{p(T)(Tu) = T(p(T)u)}_{\text{mult. property}} = T(0) = 0$$

$\Rightarrow Tu \in \text{null}\, p(T)$.

Similar for range $p(T)$