MATH 416 Abstract Linear Algebra

Midterm 2 – October 24, 2025

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"The art of doing mathematics is finding that special case that contains all the germs of generality."*
>
> — David Hilbert

Question 1 (10 points): **Null Spaces and Ranges**
For this problem, let V, W be finite dimensional vector spaces and assume $T \in \mathcal{L}(V, W)$. Recall, also, that the set of all polynomials is denoted $\mathcal{P}(\mathbb{R})$.

(i) (2 points) Formally state the definition of an injective map. Do the same for a surjective map.

Solution. A map is injective if for all $u, v \in V$, $Tu = Tv \implies u = v$ and surjective if range $T = W$.

Bonus (1 point): What is another name for injective? What about surjective?

Solution. *One-to-one* and *onto* are alternatives for injective and surjective, respectively.

(ii) (2 points) Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation map defined as $Dp = p'$. Determine whether D is injective, surjective, both, or neither.

Solution. Recall that we showed a map is injective if and only if its null space contains only the zero vector. In this case, null D also contains all constant polynomials, thus D is not injective. It is, however, surjective. In calculus terms, this is just saying that an anti-derivative $p \in \mathcal{P}(\mathbb{R})$ exists for all $q \in \mathcal{P}(\mathbb{R})$ (i.e. $Dp = q$ or, equivalently, $p' = q$).

(iii) (6 points) Recall that the fundamental theorem of linear maps relates the input space dimension to the size of the null space and range of any map as

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T.$$

Use this to prove that if $\dim V > \dim W$, there are no injective linear maps from V to W.

Proof. T is injective if and only if null $T = \{0\}$. This is equivalent to saying $\dim \operatorname{null} T = 0$. Thus, let us show that if $\dim V > \dim W$, then $\dim \operatorname{null} T > 0$. Using the fundamental theorem, we may write

$$\begin{aligned} \dim \operatorname{null} T &= \dim V - \dim \operatorname{range} T, && (1) \\ &\geq \dim V - \dim W, && (2) \\ &> 0, && (3) \end{aligned}$$

by assumption. Note that the first inequality follows from the fact that the range is a subspace of the codomain (i.e. range $T \leq W$). □

Question 2 (10 points): **Invertibility and Isomorphisms**
Recall that a linear map $T \in \mathcal{L}(V, W)$ is an *isomorphism* if it is linear and bijective (i.e. injective and surjective). When this is the case, we say V and W are *isomorphic*.

(i) (4 points) Prove that if V and W are isomorphic, then $\dim V = \dim W$.

Proof. If V and W are isomorphic, then there exists an invertible linear map $T \in \mathcal{L}(V, W)$. Thus, $\dim \operatorname{null} T = 0$ (injectivity) and $\operatorname{range} T = W$ (surjectivity). From the fundamental theorem of linear maps we then have that

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T \implies \dim V = 0 + \dim W = \dim W, \tag{4}$$

as desired. □

(ii) (6 points) Let $V = \mathcal{P}_1(\mathbb{R})$ (the space of polynomials of degree at most 1), $W = \mathbb{R}^2$, and define

$$T(a + bx) = (a, b).$$

Show that T is an isomorphism. *Hint: Let $p = a_p + b_p x$ and $q = a_q + b_q x$ be two arbitrary polynomials in V.*

Solution. We must show that T is linear and injective. Because the input and output spaces have the same dimension, injectivity is equivalent to surjectivity and, by extension, invertibility. To see linearity, let $p = a_p + b_p x$ and $q = a_q + b_q x$ be two polynomials in V and let $\alpha, \beta \in \mathbb{R}$. Then,

$$\begin{aligned}
T(\alpha p + \beta q) &= T(\alpha(a_p + b_p x) + \beta(a_q + b_q x)), && (5)\\
&= T((\alpha a_p + \beta a_q) + (\alpha b_p + \beta b_q)x), && (6)\\
&= (\alpha a_p + \beta a_q, \alpha b_p + \beta b_q), && (7)\\
&= (\alpha a_p, \alpha b_p) + (\beta a_q, \beta b_q), && (8)\\
&= \alpha(a_p, b_p) + \beta(a_q, b_q), && (9)\\
&= \alpha T(a_p + b_p x) + \beta T(a_q + b_q x), && (10)\\
&= \alpha T(p) + \beta T(q), && (11)
\end{aligned}$$

as desired. To show that it is injective, let T act on an arbitrary $p = a_p + b_p x \in V$ and set the result equal to the zero vector. We have

$$T(a_p + b_p x) = (a_p, b_p) = (0, 0), \tag{12}$$

which is obviously only true if the input is the zero polynomial. Thus, T is injective. It follows, then, that T is invertible. An invertible linear map is an isomorphism, so we are done.

Question 3 (10 points): **Invariant Subspaces, Eigenvalues, and Eigenvectors**
Let $T \in \mathcal{L}(\mathbb{R}^2)$ be defined by

$$T(x, y) = (3x + y,\, 2y).$$

Recall that a number $\lambda \in \mathbb{F}$ (in this problem $\mathbb{F} = \mathbb{R}$ is called an *eigenvalue* of T if there exists a non-zero $v \in V$ such that $Tv = \lambda v$.

(i) (2 points) Find *two distinct eigenvalues* of T and a corresponding eigenvector for each one.

Solution. Quickest way to find the eigenvalues is to note that, in the standard basis, T is upper triangular

$$\mathcal{M}(T) = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}, \tag{13}$$

thus the eigenvalues are $\lambda = 3$ with corresponding eigenvector $(1, 0)$ and $\lambda = 2$. To find the eigenvector corresponding to $\lambda = 2$, we do need to use the definition of an eigenvalue to obtain

$$3x + y = \lambda x, \tag{14}$$
$$2y = \lambda y. \tag{15}$$

Plugging in $\lambda = 2$ to the first equation, we find that $y = -x$. Thus, an eigenvector is $(1, -1)$.

(ii) (2 points) Using your answers from (i), *verify directly* that your two eigenvectors are linearly independent. (4 points) Then, prove in general that if v_1, v_2 are eigenvectors of a linear operator corresponding to *distinct eigenvalues*, they must be linearly independent.

Solution. To be linearly independent, we must have that the only solution to

$$a(1, 0) + b(1, -1) = (0, 0) \tag{16}$$

is $a = b = 0$. The above implied $a = -b$ and $-b = 0$. Thus $a = b = 0$ and the two vectors are linearly independent. To see this for two general, but distinct eigenvectors, let $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$. We wish to show that $av_1 + bv_2 = 0$ only when $a = b = 0$. To show this, let us multiply through by λ_1 to obtain $a\lambda_1 v_1 + b\lambda_1 v_2 = 0$. Consider instead acting T on both sides to obtain

$$T(av_1 + bv_2) = a\lambda_1 v_1 + b\lambda_2 v_2 = 0. \tag{17}$$

Equating these, we obtain

$$a\lambda_1 v_1 + b\lambda_1 v_2 = a\lambda_1 v_1 + b\lambda_2 v_2 \implies b(\lambda_1 - \lambda_2)v_2 = 0. \tag{18}$$

We know $v_2 \neq 0$ because it is an eigenvector and $\lambda_1 - \lambda_2 \neq 0$ by assumption. Thus, $b = 0$ which, in turn, forces $a = 0$ and we are done.

(Optional) Bonus Challenge Problem (3 points)
Generalize the your proof from the last problem. That is, prove that if we have m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with corresponding eigenvectors $v_1, \ldots, v_m$, then the list of eigenvectors is linearly independent.

Proof. We prove by induction on m that if $v_1, \ldots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_m$, then $v_1, \ldots, v_m$ are linearly independent. We proved the base case ($m = 2$) just above.

Inductive Step. Assume the statement holds for some $m = k \geq 2$. Let $v_1, \ldots, v_{k+1}$ be eigenvectors of T with distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$. Suppose

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k + a_{k+1} v_{k+1} = 0.$$

Applying T to both sides gives

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \cdots + a_k \lambda_k v_k + a_{k+1} \lambda_{k+1} v_{k+1} = 0.$$

Subtracting λ_{k+1} times the original equation yields

$$\sum_{i=1}^{k} a_i (\lambda_i - \lambda_{k+1})\, v_i = 0.$$

Because the eigenvalues are distinct, each $\lambda_i - \lambda_{k+1} \neq 0$ for $1 \leq i \leq k$. Thus, the scalars $a_i(\lambda_i - \lambda_{k+1})$ must all be zero. By the inductive hypothesis (applied to $v_1, \ldots, v_k$), the list $v_1, \ldots, v_k$ is linearly independent, so

$$a_1 = a_2 = \cdots = a_k = 0.$$

Returning to the original equation, we now have $a_{k+1} v_{k+1} = 0$. Since $v_{k+1} \neq 0$, it follows that $a_{k+1} = 0$. Therefore, all coefficients are zero, and $v_1, \ldots, v_{k+1}$ are linearly independent. Thus, the result holds for all $m \geq 2$. $\square$

Final Bonus Opportunity (1 point)

It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?