MATH 416 Abstract Linear Algebra

Exam 1 – September 24, 2025

Exam Instructions: This is a **closed-book** exam and you have **50 minutes** to complete it. Show all work clearly; **partial credit** will be awarded for reasoning that demonstrates useful thinking even if the final answer is incorrect. When proving statements, always start from the **basic definitions** and clearly indicate on each line which definitions, properties, or theorems you are using.

> *"Mathematicians are not people who find math easy, they're the people who enjoy that it's difficult."*
>
> — Matt Parker

Question 1 (5 points): **Linear Systems of Equations**

Linear algebra was invented[1] to facilitate the solution of linear systems of equations. If the field began there, it seems fitting that our first exam should as well!

(i) (2 points) A school cafeteria sells three types of sandwiches: turkey (\$2 each), ham (\$1 each), and veggie (\$1 each). On Monday total of 10 sandwiches were sold, the total revenue was \$14, and the number of turkey sandwiches sold was equal to the number of veggie sandwiches. Set up a system of equations to model the situation and solve for the quantity of each sandwich sold. *Hint: let x, y, z stand for turkey, ham, and veggie sandwhich quantities, respectively.*

(ii) (3 points) Suppose a linear system of equations $A\mathbf{x} = \mathbf{b}$ has at least two distinct solutions. Prove that the system must in fact have infinitely many solutions.

Bonus (1 point): The method we now call "Gaussian elimination" was not invented by Gauss. From where does this algorithm originate?

[1] or discovered, depending on your philosophy...

Question 2 (5 points): **Complex Numbers are Essential (5 points)**

Let $z = a + bi$ be a complex number with $a, b \in \mathbb{R}$. Recall that the conjugate of z is $\bar{z} = a - bi$ and $|z| = \sqrt{a^2 + b^2}$. Consider the following problems regarding complex numbers.

(i) (1 point) In terms of z and $\bar{z}$, write down a condition for a number to be real. Do the same for a purely imaginary number. Is there a number that is both real and purely imaginary?

(ii) (2 point) Write the real and imaginary parts of z in terms of z and $\bar{z}$.

(iii) (2 points) Show that $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ for $z \neq 0$.

Question 3 (10 points): **Vector Subspaces, Bases, and Dimension**
Recall that a polynomial is a function $p : \mathbb{R} \to \mathbb{R}$ of the form $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, for all $x \in \mathbb{R}$. Recall that addition and scalar multiplication of polynomials are defined as follows. For all $f, g \in \mathcal{P}(x)$ the sum $f + g \in \mathcal{P}(x)$ is the function defined by

$$(f + g)(x) = f(x) + g(x). \tag{1}$$

Similarly, for all $\lambda \in \mathbb{R}$ and all $f \in \mathcal{P}(x)$, the product $\lambda f \in \mathcal{P}(x)$ is the function defined by

$$(\lambda f)(x) = \lambda f(x). \tag{2}$$

Let $\mathcal{P}_4 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_0, \ldots, a_4 \in \mathbb{R}\}$ be the vector space of all real polynomials of degree at most 4. A polynomial $p(x)$ is called **even** if $p(-x) = p(x)$ and **odd** if $p(-x) = -p(x)$.

(i) (3 points) Show that the set of all even polynomials, denoted $\mathcal{P}_4^e$, forms a subspace of $\mathcal{P}_4$. (You may simply state that the odd case works similarly.)

(ii) (3 points) Write down a basis for $\mathcal{P}_4^e$ and state its dimension. Do the same for the subspace of odd polynomials in $\mathcal{P}_4^o$ and state its dimension.

(iii) (4 points) Show that every polynomial in $\mathcal{P}_4$ can be uniquely written as the sum of an even polynomial and an odd polynomial; that is, show that $\mathcal{P}_4 = \mathcal{P}_4^e \oplus \mathcal{P}_4^o$.

Bonus (1 point): Chapter 2 in Axler has the same title as the first modern linear algebra textbook. What was the title of the textbook? Who was the author?

Question 4 (10 points): **The Vector Space of Linear Maps**

We saw in class that the set $\mathcal{L}(U, V)$ of all linear maps from U to V is, indeed, a vector space. In fact, it is one of the most important vector spaces we will study in this course. Recall also that $\mathcal{P}(\mathbb{R})$ denotes the set of all polynomials over $\mathbb{R}$.

(i) (5 points) Suppose $m, b \in \mathbb{R}$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = mx + b$ is a linear map if and only if $b = 0$. *Hint: remember that a linear map takes 0 to 0, that is* $T(0) = 0$.

(ii) (1 point) Define a function $T : \mathcal{P}_m(\mathbb{R}) \to \underline{\quad}$ by $(Tp)(x) = xp(x)$ for all $p(x) \in \mathcal{P}(\mathbb{R})$. What space does T map into?

(iii) (4 points) Consider $S \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ defined as $(Sp)(x) = p(x + a)$ for all $p \in \mathcal{P}(\mathbb{R})$. With T defined as in (ii), show that $ST \neq TS$.

Bonus Challenge Problem (2 points)
The following problem is designed to demonstrate that neither homogeneity nor additivity alone suffice to imply that a function is a linear map.

(i) (1 point) Give an example of a function from $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\varphi(av) = a\varphi(v) \tag{3}$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but φ is not linear.

(ii) (1 point) Give an example of a function $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z), \tag{4}$$

for all $w, z \in \mathbb{C}$ but φ is not linear. *Note: here $\mathbb{C}$ can be thought of as a complex vector space.*

Final Bonus Opportunity (1 point)
It is always discouraging to study broadly only to find a certain topic you focused on was not included on the exam. If this happened to you, take the space below to explain the topic to me in simple terms. Why is this topic important for linear algebra?