MATH 416 Abstract Linear Algebra

Week 7 - Homework 6
Assigned: Fri. Oct. 10, 2025
Due: Fri. Oct. 17, 2025 (by 8pm)

Reminder: I encourage you to work together and use resources as needed. Please remember to state who you collaborated with and what resources you used.

Exercise 1 (7 points): **Basis change matrices**
Let $V = \mathbb{R}^3$, and consider the standard basis $\mathcal{S} = \{e_1, e_2, e_3\}$ and the bases $\mathcal{B} = \{v_1, v_2, v_3\}$ and $\mathcal{B}' = \{w_1, w_2, w_3\}$ with

$$\begin{aligned} v_1 &= (1,1,1)^T & v_2 &= (1,-1,0)^T & v_3 &= (1,0,1)^T \\ w_1 &= (1,0,1)^T & w_2 &= (1,-1,1)^T & w_3 &= (1,1,0)^T. \end{aligned}$$

(i) (2 points) Compute $A = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$ and $B = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}}$ and verify that $B = A^{-1}$.

(ii) (2 points) Compute $C = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{S}}$ and $D = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$ and verify that $D = C^{-1}$.

(iii) (2 points) Compute $E = \mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$ and $F = \mathcal{M}(I_V)_{\mathcal{B}',\mathcal{B}}$ and verify that $F = E^{-1}$.

(iv) (1 point) What is the relationship between $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{B}'}$, $\mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}$, and $\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$?

Exercise 2 (3 points): **Linear maps as matrices**
Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map defined by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} 2x - 3y \\ x + y + z \\ 3y - z \end{pmatrix},$$

and let $\mathcal{S}, \mathcal{B}, \mathcal{B}'$ be the bases from Exercise 1.

(i) (2 points) Determine $\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}$ and $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'}$ using the definition of the matrix representation of a linear map.

(ii) (1 point) Verify that $\mathcal{M}(T)_{\mathcal{B},\mathcal{B}'} = \mathcal{M}(I_V)_{\mathcal{S},\mathcal{B}'}\mathcal{M}(T)_{\mathcal{S},\mathcal{S}}\mathcal{M}(I_V)_{\mathcal{B},\mathcal{S}}$.

Exercise 3 (5 points): **Reverse Triangle Inequality**

For this problem, let $w, z \in \mathbb{C}$. And recall that $\bar{z}$ denotes the complex conjugate of z.

(i) (1 point) Prove that $|\mathrm{Re}[z]| \leq |z|$ and $|\mathrm{Im}[z]| \leq |z|$.

(ii) (1 point) Prove that $|zw| = |z||w|$.

(iii) (3 points) Prove the reverse triangle inequality

$$||w| - |z|| \leq |w - z|, \tag{1}$$

for all $w, z \in \mathbb{C}$.

Hint: see page 121 of Axler or Lecture 17 notes for a proof of the standard triangle inequality.