MATH 416 Abstract Linear Algebra

Homework 1
Assigned: Fri. August 29, 2025
Due: Fri. Sept. 5, 2025 (by 1pm)

Exercise 1 (10 points): In class, we saw one way of taking products between two vectors. In this problem, you will practice your proof writing skills to prove a few fundamental results about the dot product. First, let us formally define the dot product between two real vectors.

Definition 1. For vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, the **dot product** is defined as

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i.$$

We also define the **norm** (or length) of a vector as

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Now, let's warm up with some elementary results.

1a. (1 point) Compute $(1, 2, 3) \cdot (4, 5, 6)$.

Solution. Using Def. 1, we have

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \times 4 + 2 \times 5 + 3 \times 6 = 4 + 10 + 18 = 32. \tag{1}$$

1b. (1 point) Show that $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (commutativity).

Proof. As with many basic proofs in linear algebra, one can prove this directly using properties of the underlying field (in this case just the real numbers, $\mathbb{R}$). We may write

$$\begin{aligned}
\mathbf{x} \cdot \mathbf{y} &= \sum_{i=1}^{n} x_i y_i, && \text{Def. 1,} && (2) \\
&= \sum_{i=1}^{n} y_i x_i, && \text{Commutativity of reals,} && (3) \\
&= \mathbf{y} \cdot \mathbf{x}, && \text{Def. 1,} && (4)
\end{aligned}$$

and we are done. □

1c. (1 point) Show that $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$ (distributivity).

Proof. As above, one can prove this directly using properties of the reals. We have

$$\begin{aligned} \mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) &= \sum_{i=1}^{n} x_i(y_i + z_i), \quad \text{Def. 1} && (5) \\ &= \sum_{i=1}^{n} x_i y_i + x_i z_i, \quad \text{Distributivity of reals.} && (6) \\ &= \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}, \quad \text{Def. 1} && (7) \end{aligned}$$

and we are done. □

1d. (1 point) Show that $\mathbf{x} \cdot \mathbf{x} \geq 0$ and equals 0 if and only if $\mathbf{x} = \mathbf{0}$.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. By definition, the dot product of $\mathbf{x}$ with itself is

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2.$$

Step 1: Show $\mathbf{x} \cdot \mathbf{x} \geq 0$

Each term $x_i^2 \geq 0$ since the square of a real number is nonnegative. Therefore,

$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0.$$

Step 2: Show $\mathbf{x} \cdot \mathbf{x} = 0 \iff \mathbf{x} = \mathbf{0}$.

To show that P is true "if and only if" Q is true (in symbols $P \iff Q$), we must first assume P and show it implies Q. Then, we must assume Q and show that it implies P.

- ($\Rightarrow$) Suppose $\mathbf{x} \cdot \mathbf{x} = 0$. Then

  $$x_1^2 + x_2^2 + \cdots + x_n^2 = 0.$$

  Since each $x_i^2 \geq 0$, the sum of nonnegative numbers is zero if and only if each term is zero. Thus,

  $$x_1 = x_2 = \cdots = x_n = 0 \implies \mathbf{x} = \mathbf{0}.$$

- ($\Leftarrow$) Conversely, if $\mathbf{x} = \mathbf{0}$, then clearly

$$\mathbf{x} \cdot \mathbf{x} = 0^2 + 0^2 + \cdots + 0^2 = 0.$$

Step 3: Conclusion

We have show that for any $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x} \cdot \mathbf{x} \geq 0, \quad \text{and } \mathbf{x} \cdot \mathbf{x} = 0 \iff \mathbf{x} = \mathbf{0}.$$

□

With these results in mind, we now turn to the proof of one of the most ubiquitous inequalities in mathematics, the *Cauchy-Schwarz Inequality*.

Theorem 2 (Cauchy–Schwarz Inequality)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$*,*

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|.$$

1e. (6 points) Prove the Cauchy-Schwarz inequality.

I will provide a somewhat standard proof of the inequality. There are many proofs and all that I have seen have some elements that are not terribly obvious on first pass.

Proof Using Quadratic Formula. If $\mathbf{y} = \mathbf{0}$, the inequality is trivial, so we will assume $\mathbf{y} \neq \mathbf{0}$. Consider the function

$$f(t) = \|\mathbf{x} - t\mathbf{y}\|^2 = (\mathbf{x} - t\mathbf{y}) \cdot (\mathbf{x} - t\mathbf{y}) \geq 0 \quad \text{for all } t \in \mathbb{R}. \tag{8}$$

Expanding the dot product:

$$\begin{aligned} f(t) &= (\mathbf{x} - t\mathbf{y}) \cdot (\mathbf{x} - t\mathbf{y}), \\ &= \mathbf{x} \cdot \mathbf{x} - 2t(\mathbf{x} \cdot \mathbf{y}) + t^2(\mathbf{y} \cdot \mathbf{y}) \\ &= \|\mathbf{x}\|^2 - 2t(\mathbf{x} \cdot \mathbf{y}) + t^2\|\mathbf{y}\|^2. \end{aligned}$$

This is a quadratic in t and is always nonnegative. To map it to the standard formula, let $a = \|\mathbf{y}\|^2$, $b = -2(\mathbf{x} \cdot \mathbf{y})$, $c = \|\mathbf{x}\|^2$. The, this is just the standard form of a quadratic $at^2 + bt + c$. Let's define the discriminant as $\Delta := b^2 - 4ac$. (One can verify that) we can always express a quadratic as

$$f(t) = a\left(t + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}. \tag{9}$$

Thus, because $a = \|\mathbf{y}\|^2 \geq 0$ and $f(t) \geq 0$, we must have $\Delta \leq 0$. It follows that

$$\begin{aligned}
&(-2\mathbf{x}\cdot\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0 && (10)\\
\implies &\quad 4(\mathbf{x}\cdot\mathbf{y})^2 - 4\|\mathbf{x}\|^2\|\mathbf{y}\|^2 \leq 0 && (11)\\
\implies &(\mathbf{x}\cdot\mathbf{y})^2 \leq \|\mathbf{x}\|^2\|\mathbf{y}\|^2 && (12)\\
\implies &|\mathbf{x}\cdot\mathbf{y}| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|, && (13)
\end{aligned}$$

as desired. □

Here, we assumed we were dealing with real vectors. This inequality can be extended easily for complex vectors and it is used all the time to derive really important results in quantum mechanics, quantum information, and many other mathematical fields. There is a beautiful book on the art of mathematical inequalities and their proofs called *The Cauchy-Schwarz Masterclass* [Ste04] that I *highly* recommend.

Exercise 2 (5 points): Consider the following system of linear equations:

$$\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1\\
&\vdots\\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m,
\end{aligned}$$

where $a_{ij}, b_i \in \mathbb{R}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Show that the set of solutions of this system does not change under the following operations:

1. **Multiplication.** Multiply both sides of an equation by $\lambda \neq 0$.
2. **Permutation.** Swapping any two rows.
3. **Addition.** Add one row to another one.

Hint: Assume there exists a solution $\mathbf{x} = (x_1, \ldots, x_n)^T$ of the original system of linear equations, and show that it is also a solution of the system transformed via the operations in 1, 2, and 3

Proof. Write the ith equation of the system as

$$E_i(\mathbf{x}) = a_{i1}x_1 + \cdots + a_{in}x_n = b_i.$$

1. **Multiplication.** Replacing $E_i(\mathbf{x}) = b_i$ by $\lambda E_i(\mathbf{x}) = \lambda b_i$ with $\lambda \neq 0$ does not change the solution set: any $\mathbf{x}$ satisfying $E_i(\mathbf{x}) = b_i$ also satisfies the scaled version, and conversely division by λ recovers the original. The operation is reversible by multiplying instead by $1/\lambda$.

2. **Permutation.** Swapping two equations merely changes their order. A vector $\mathbf{x}$ satisfies all equations if and only if it satisfies them in any order. The operation is reversible by swapping back.

3. **Addition.** Replacing $E_j(\mathbf{x}) = b_j$ by $E_j(\mathbf{x}) + E_i(\mathbf{x}) = b_j + b_i$ also preserves the solution set: if $\mathbf{x}$ solves the original system, then $E_i(\mathbf{x}) = b_i$ and $E_j(\mathbf{x}) = b_j$, hence the new equation holds. Conversely, if $\mathbf{x}$ satisfies the new equation together with $E_i(\mathbf{x}) = b_i$ (unchanged), then subtracting yields $E_j(\mathbf{x}) = b_j$. The operation is reversible by replacing row j with row $j - E_i$.

   In each case, the key point is *reversibility*: the transformed system can be converted back to the original by another allowed operation. Thus the two systems are logically equivalent, and the solution set is unchanged.

Remark. The argument above relies on the fact that we are working over a *field* (such as $\mathbb{R}$ or $\mathbb{C}$), so that every nonzero scalar λ has a multiplicative inverse. Over more general rings (e.g. $\mathbb{Z}$), not all nonzero elements are invertible, so one cannot freely perform operation (1) in general. We will see more about fields in Homework 2.

□

Exercise 3 (10 points): Determine if the following systems of linear equations have solutions, and if yes find them. In one of the systems below there are multiple solutions. What is the dimensionality of the solution space? (1 point)

1. (3 points) System of linear equations in $\mathbb{R}^3$

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned}$$

Solution. Consider the system:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4, \\ x_1 + 3x_2 + 3x_3 &= 5, \\ 2x_1 + 6x_2 + 5x_3 &= 6. \end{aligned}$$

We write its augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{array}\right].$$

Step 1: Eliminate x_1 from rows 2 and 3

$$\begin{aligned} R_2 \to R_2 - R_1 : &\qquad [1\,3\,3\,|5] - [1\,2\,2\,|4] = [0\,1\,1\,|1], \\ R_3 \to R_3 - 2R_1 : &\qquad [2\,6\,5\,|6] - 2[1\,2\,2\,|4] = [0\,2\,1\,|-2]. \end{aligned}$$

The matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right].$$

Step 2: Eliminate x_2 from row 3

$$R_3 \to R_3 - 2R_2 : [0\,2\,1\,|-2] - 2[0\,1\,1\,|1] = [0\,0\,-1\,|-4].$$

The matrix is now in upper-triangular form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{array}\right].$$

Step 3: Back-substitution

From the last row:

$$-1 \cdot x_3 = -4 \implies x_3 = 4.$$

From the second row:

$$x_2 + x_3 = 1 \implies x_2 + 4 = 1 \implies x_2 = -3.$$

From the first row:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 = 4 &\implies x_1 + 2(-3) + 2(4) = 4 && (14)\\ &\implies x_1 + (-6+8) = 4 && (15)\\ &\implies x_1 + 2 = 4 && (16)\\ &\implies x_1 = 2. && (17) \end{aligned}$$

Solution

The unique solution of the system is

$$\boxed{(x_1, x_2, x_3) = (2, -3, 4).}$$

2. (3 points) System of linear equations in $\mathbb{R}^4$

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7\\ x_1 + x_2 + x_3 - x_4 &= 3\\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

Solution. Consider the system:

$$\begin{aligned} x_1 + 2x_2 + 0 \cdot x_3 + x_4 &= 7,\\ x_1 + x_2 + x_3 - x_4 &= 3,\\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1. \end{aligned}$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7\\ 1 & 1 & 1 & -1 & 3\\ 3 & 1 & 5 & -7 & 1 \end{array}\right]$$

Step 2: Eliminate x_1 from rows 2 and 3

$$\begin{aligned} R_2 \to R_2 - R_1: &\qquad [1\;1\;1\;-1\;|3] - [1\;2\;0\;1\;|7] = [0\;-1\;1\;-2\;|-4],\\ R_3 \to R_3 - 3R_1: &\qquad [3\;1\;5\;-7\;|1] - 3[1\;2\;0\;1\;|7] = [0\;-5\;5\;-10\;|-20]. \end{aligned}$$

The matrix becomes:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & -1 & 1 & -2 & -4 \\ 0 & -5 & 5 & -10 & -20 \end{array}\right]$$

Step 3: Simplify row 2 and eliminate x_2 from row 3

$$R_2 \to -R_2 = [0\,1\,\,-1\,2\,|4],$$
$$R_3 \to R_3 - 5R_2 = [0\,\,-5\,5\,\,-10\,|-20] - 5[0\,1\,\,-1\,2\,|4] = [0\,0\,0\,0\,|0].$$

Now the matrix is:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 7 \\ 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 4: Express leading variables in terms of free variables

Let $x_3 = t$ and $x_4 = s$ be free parameters ($t, s \in \mathbb{R}$).
From the second row:

$$x_2 - x_3 + 2x_4 = 4 \implies x_2 = 4 + t - 2s.$$

From the first row:

$$\begin{aligned} x_1 + 2x_2 + x_4 = 7 &\implies x_1 + 2(4 + t - 2s) + s = 7 && (18)\\ &\implies x_1 + 8 + 2t - 4s + s = 7 && (19)\\ &\implies x_1 + 2t - 3s = -1 && (20)\\ &\implies x_1 = -1 - 2t + 3s. && (21) \end{aligned}$$

Step 5: General solution

$$\boxed{(x_1, x_2, x_3, x_4) = (-1 - 2t + 3s,\ 4 + t - 2s,\ t,\ s), \quad t, s \in \mathbb{R}.}$$

This solution set has two free parameters, so the dimension of the solution space is 2.

3. (3 points) System of linear equations in $\mathbb{R}^4$

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Step 1: Write the augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{array}\right]$$

Step 2: Eliminate x_1 from rows 2 and 3

$$\begin{aligned} R_2 \to R_2 + \frac{3}{2}R_1 &: [-3\ 4\ -5\ -6\ |3] + \frac{3}{2}[2\ 1\ 7\ -7\ |2] = [0\ 5.5\ 5.5\ -16.5\ |6], \\ R_3 \to R_3 - \frac{1}{2}R_1 &: [1\ 1\ 4\ -5\ |2] - \frac{1}{2}[2\ 1\ 7\ -7\ |2] = [0\ 0.5\ 0.5\ -1.5\ |1]. \end{aligned}$$

Multiply rows 2 and 3 by 2 to avoid fractions:

$$\begin{aligned} R_2 &\to 2R_2 = [0\ 11\ 11\ -33\ |12], \\ R_3 &\to 2R_3 = [0\ 1\ 1\ -3\ |2]. \end{aligned}$$

Now the matrix is:

$$\left[\begin{array}{cccc|c} 2 & 1 & 7 & -7 & 2 \\ 0 & 11 & 11 & -33 & 12 \\ 0 & 1 & 1 & -3 & 2 \end{array}\right]$$

Step 3: Eliminate x_2 from row 2 using row 3

$$R_2 \to R_2 - 11R_3 = [0\ 11\ 11\ -33\ |12] - 11[0\ 1\ 1\ -3\ |2] = [0\ 0\ 0\ 0\ |-10].$$

Step 4: Check consistency

The last row now reads: $0x_1 + 0x_2 + 0x_3 + 0x_4 = -10$, which is impossible.

Conclusion

Since the augmented matrix produces a row with all zeros in the coefficients and a nonzero entry in the augmented column the system is *inconsistent*.

$$\boxed{\text{The system has no solution.}}$$

References

[Ste04] J. Michael Steele. *The Cauchy–Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities*. Cambridge, UK: Cambridge University Press, 2004. ISBN: 052154677X. DOI: 10.1017/CBO9780511817106.