Math 416: Abstract Linear Algebra

Date: Oct. 1, 2025

Lecture: 13

Announcements

- ☐ HW4 is due Fri, Oct. 3 @ 8pm
- ☐ office hours:
  - Tuesdays 5-5:50 Davenport 336
  - Wednesdays 2-2:50 Davenport 132
- ☐ Exam #1 corrections
  - ↳ Monday 11:59pm ← Oct. 6
  - ↳ half credit for each correction
- ☐ Math Talk: "Prime Numbers" 3301 CIF Thursday 5pm
  - ↳ Free pizza! & free knowledge!

Last time

- ☐ Fundamental theorem of linear maps

This time

- ☐ Matrices

Recommended reading / watching

- ☐ §3B-C of Axler
- ☐ 3blue1brown nonsquare matrices

Next time

- ☐ Invertibility

Fundamental Theorem of Linear Maps

Theorem 3.21 (Fundamental thm of lin maps)

Suppose V is finite-dim. & $T \in \mathcal{L}(V,W)$.

Then range T is finite-dim. &

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

Proof sketch.

- $\dim V < \infty \implies \dim \operatorname{null} T < \infty$
- let $\{u_1, \dots, u_m\}$ be a basis of null T ($\implies \dim \operatorname{null} T = m$)
- extend to basis of V by adding $v_1, \dots, v_n$ ($\implies \dim V = m+n$)
- Show $\{Tv_1, \dots, Tv_n\}$ is a basis of range T ($\implies \dim \operatorname{range} T = n$)

* See lec 12 or Axler pg. 62-63 for full proof.

"Rank-nullity theorem"

In matrix linear alg. students may be asked to memorize the following formula

$$n = (\#\text{pivots}) + (\#\text{ free variables})$$

We can see this as a special case of the above theorem, when $T: \mathbb{R}^n \to \mathbb{R}^m$

$$\dim(\mathbb{R}^n) = \dim(\text{column space}) + \dim(\text{null space})$$

$$n = (\#\text{pivots}) + (\#\text{ free variables})$$

Corollary 3.22

Suppose V & W are finite-dim vector spaces s.t. $\dim V > \dim W$. Then no lin map from V to W is injective.

recall: (Axler 3.15) T injective $\iff$ $\operatorname{null} T = \{0\}$

Proof. Let $T \in \mathcal{L}(V, W)$. Then

$$\begin{aligned}
\dim \operatorname{null} T &= \dim V - \operatorname{range} T \\
&\geq \dim V - \dim W \quad , \quad \operatorname{range} T \subseteq W \\
&> 0 \quad , \quad \dim V > \dim W
\end{aligned}$$

$\Rightarrow$ T is not injective by (Axler 3.15).

Corollary 3.24 If $\dim V < \dim W$ $\nexists$ $T \in \mathcal{L}(V, W)$ s.t. T is surjective.

Proof.

$$\begin{aligned}
\dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\
&\leq \dim V \\
&< \dim W
\end{aligned}$$

$\square$

Matrices

An m-by-n matrix A is a rect. array w/ m rows & n columns

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \cdots & A_{mn} \end{pmatrix}$$

and A_{jk} represents an arbitrary element.

Def. matrix of a linear map, $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V, W)$ w $\{v_1, \ldots, v_n\}$ a basis of V & $\{w_1, \ldots, w_m\}$ a basis of W. Then, the matrix of T w.r.t these bases is defined as

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

when bases are not clear from context, we write $\mathcal{M}(T)_{B_V, B_W}$

$$\mathcal{M}(T) = \begin{array}{c|c} & \begin{matrix} v_1 & \cdots & v_k & \cdots & v_n \end{matrix} \\ \begin{matrix} w_1 \\ \vdots \\ w_m \end{matrix} & \begin{pmatrix} & & A_{1k} & & \\ & & \vdots & & \\ & & A_{mk} & & \end{pmatrix} \end{array}$$

k-th column consists of scalars needed to write Tv_k as linear combos of $w_1, \ldots w_m$:

$$Tv_k = \sum_{j=1}^{m} A_{jk} w_j$$

Examples

Let $T \in \mathcal{L}(\mathbb{F}^2, \mathbb{F}^3)$ be defined as

$$T(x,y) = (x+3y,\ 2x+5y,\ 7x+9y)$$

What is $\mathcal{M}(T)$ in the standard basis?

Soln. $\underbrace{T(1,0)}_{e_1} = (1+3\cdot 0, 2\cdot 1+5\cdot 0, 7\cdot 1+9\cdot 0)$

$= (1, 2, 7)$

$\underbrace{T(0,1)}_{e_2} = (0+3\cdot 1, 2\cdot 0+5\cdot 1, 7\cdot 0+9\cdot 1)$

$= (3, 5, 9)$

$$\Rightarrow \quad M(T) = \begin{pmatrix} T(e_1) & T(e_2) \\ & \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix}$$

Next time, we will formally define matrix mult. and can verify

$$\begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 7 & 9 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+3y \\ 2x+5y \\ 7x+9y \end{pmatrix}$$ as desired

Suppose instead we want the matrix w.r.t $\mathcal{B}_{\mathbb{F}^2} = \{(1,1), (1,0)\}$

$$\mathcal{B}_{\mathbb{F}^3} = \{\underbrace{(1,0,0)}_{v_1}, \underbrace{(0,1,0)}_{v_2}, \underbrace{(1,1,1)}_{v_3}\}$$

Soln.

$$T(1,1) = (4,7,16) = a v_1 + b v_2 + c v_3$$

$$= -12 v_1 + (-9) v_2 + 16 v_3$$

$$T(1,0) = (1,2,7) = a' v_1 + b' v_2 + c' v_3$$

$$= -6 v_1 + (-5) v_2 + 7 v_3$$

$$\mathcal{M}(T)_{\mathcal{B}_{F^2}, \mathcal{B}_{F^3}} = \begin{pmatrix} -12 & -6 \\ -9 & -5 \\ 16 & 7 \end{pmatrix}$$

Addition & scalar mult.

Let A & C be $m \times n$ matrices

- $(A + C)_{jk} = A_{jk} + C_{jk}$

  ↪ implies $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$

  (see HW 4)

- $(\lambda A)_{jk} = \lambda A_{jk}$

  ↪ implies $\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$

  (see HW 4)

- $\mathbb{F}^{m,n} = \{ m \times n \text{ matrices w/ entries in } \mathbb{F} \}$

- $\mathbb{F}^{m,n}$ forms a vector space!