Quantum Channels

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Chapter 0

Prerequisites

0.1 Hilbert Spaces and Linear Operators

Throughout this course, $\mathcal{H}$ denotes a finite-dimensional Hilbert space (complex vector space with an associated inner product). Using Dirac's "bra-ket" notation we denote elements of the Hilbert space (called kets) as

$$|\psi\rangle \in \mathcal{H}. \tag{1}$$

The elements of the dual Hilbert space are called bras and are denoted

$$\langle\psi| \in \mathcal{H}^*, \tag{2}$$

where $\langle\psi| = (|\psi\rangle)^\dagger$. Here, $X^\dagger := \bar{X}^T$ denotes the Hermitian adjoint (also called the conjugate transpose). We denote

$$B(\mathcal{H}_1, \mathcal{H}_2) := \{\text{linear maps from } \mathcal{H}_1 \text{ to } \mathcal{H}_2\} \tag{3}$$

and the set of all linear maps to and from the same space will be denoted $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$. An operator $X \in B(\mathcal{H})$ is *normal* if $XX^T = X^T X$. Every normal operator has a *spectral decomposition*. That is, there exists a unitary U and a diagonal matrix D whose entries are the eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ of X such that

$$X = UDU^\dagger. \tag{4}$$

In other words,

$$X = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle \langle\psi_i| \tag{5}$$

where $X|\psi_i\rangle = \lambda_i |\psi_i\rangle$ and $U = (|\psi_i\rangle, \ldots, |\psi_d\rangle)$. If X is Hermitian, $X = X^\dagger$, then $\lambda_i \in \mathbb{R}$. An operator X is positive semi-definite (PSD) if

$$\langle\varphi| X |\varphi\rangle \geq 0 \qquad \forall |\varphi\rangle \in \mathcal{H}. \tag{6}$$

As a consequence, $X \geq 0$ and $\lambda_i \geq 0$. It holds that PSD $\implies$ Hermitian $\implies$ normal. Unless otherwise stated, we will always assume we are working in an orthonormal basis.

0.2 Quantum States

A quantum state ρ in a Hilbert space $\mathcal{H}$ is a PSD linear operator with

$$\rho \in B(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}\rho = 1. \tag{7}$$

This means that the state has eigenvalues $\{\lambda_i\}_{i=1}^d$ satisfying $\lambda_i \geq 0$ and $\sum_{i=1}^d \lambda_i = 1$. Thus, $\{\lambda_i\}_{i=1}^d$ forms a probability distribution.

A *pure quantum state* ψ is a quantum state with rank 1. We can find $|\psi\rangle \in \mathcal{H}$ such that $\psi = |\psi\rangle\langle\psi|$. In this case, ψ is called a *projector*. A *mixed state* is a quantum state with rank > 1. Mixed states are convex combinations of pure states. That is, for every quantum state ρ with $r = \mathrm{rank}(\rho)$ there are pure states $|\psi_i\rangle_{i=1}^k \quad (k \geq r)$ and a probability distribution $\{p_i\}_{i=1}^k$ such that

$$\rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|. \tag{8}$$

The spectral decomposition of ρ is a special case of this property.

0.3 Composite systems, partial trace, entanglement

Let A and B be two quantum systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The *joint system AB* is described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. We denote quantum states of the joint system as $\rho_{AB} \in \mathcal{H}_{AB}$. The marginal of the bipartite state, denoted ρ_A, is uniquely defined as the operator satisfying

$$\rho_A := \mathrm{tr}_B \rho_{AB}, \tag{9}$$

which is defined via $\mathrm{tr}(\rho_{AB}(X_A \otimes \mathbb{I}_B)) = \mathrm{tr}\rho_A X_A \quad \forall X_A \in B(\mathcal{H}_A)$. For a Hilbert space with $|B| := \dim \mathcal{H}_B$, the explicit form of the partial trace is

$$\mathrm{tr}_B \rho_{AB} = \sum_{i=1}^{|B|} (\mathbb{I}_A \otimes \langle i|_B) \rho_{AB} (\mathbb{I}_A \otimes |i\rangle_B), \tag{10}$$

for some basis $\{|i\rangle_B\}_{i=1}^{|B|}$ of $\mathcal{H}_B$.

A *product state* on AB is a state of the form $\rho_A \otimes \sigma_B$. The state is called *separable* if it lies in the convex hull of product states:

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \sigma_B^i \tag{11}$$

for some states $\{\rho_A^i\}_i$ and $\{\sigma_B^i\}_i$ and probability distribution $\{p_i\}_i$. A state is called *entangled*, if it is not separable. An entangled state of particular interest is the *maximally entangled state*. Let $d = \dim \mathcal{H}$, $\{|i\rangle\}_{i=1}^d$ be a basis for $\mathcal{H}$. A maximally entangled state is expressed as

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \quad \in \mathcal{H} \otimes \mathcal{H} \tag{12}$$

0.4 Measurements

The most general measurement is given by a *positive operator-valued measure* (POVM) $E = \{E_i\}_i$ where $E_i \geq 0 \quad \forall i$ and $\sum_i E_i = \mathbb{I}$. Then, for a quantum system $\mathcal{H}$ in state ρ, the probability of obtaining measurement outcome i is given by $p_i = \mathrm{tr}[\rho E_i]$. So, we have

$$\sum_i p_i = \sum_i \mathrm{tr}[\rho E_i] = \mathrm{tr}\left[\rho \sum_i E_i\right] = \mathrm{tr}[\rho \mathbb{I}] = \mathrm{tr}\rho = 1, \tag{13}$$

for all normalized quantum states. A *projective measurement* $\Pi = \{\Pi_i\}$ is a POVM with the added property of orthogonality, which for projectors means

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i. \tag{14}$$

Any basis $\{|e_i\rangle\}_{i=1}^{\dim \mathcal{H}}$ gives rise to a projective measurement $\Pi = \{|e_i\rangle \langle e_i|\}_{i=1}^{\dim \mathcal{H}}$.

0.5 Entropies

The *Shannon entropy* $H(p)$ of a probability distribution $p = \{p_i, \dots, p_d\}$ is defined as $H(p) = -\sum_{i=1}^d p_i \log p_i$, where the logarithm is base 2 unless otherwise specified. Note that when the logarithm is base 2, the entropy has units of *bits*. The *von Neumann entropy* $S(\rho)$ of a quantum state ρ is defined as

$$S(\rho) = -\mathrm{tr}\left[\rho \log \rho\right] = H(\{\lambda_i, \dots, \lambda_d\}), \tag{15}$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ is a spectral decomposition of ρ and where the logarithm of an operator is obtained by first diagonalizing the matrix representing the operator and then taking the logarithm of the diagonal elements. That is,

$$\log \rho = \sum_{i:\lambda_i > 0} \log(\lambda_i) |\psi_i\rangle \langle \psi_i| \,. \tag{16}$$

Chapter 1

Representations

1.1 Unitary evolution of (closed) quantum systems

1.1.1 Quantum mechanics

We start by stating some broadly-accepted facts or axioms about quantum mechanics.

- we represent quantum systems using (in our case) finite dimensional Hilbert spaces, denoted $\mathcal{H}$
- Observables: measurable quantities are Hermitian operators $A \in B(\mathcal{H})$, $A = A^\dagger$, $A = \sum_i a_i \left|a_i\right\rangle \left\langle a_i\right|$ where $\langle a_i|a_j\rangle = \delta_{ij}$ and $a_i \in \mathbb{R}$.
- eigenvalues $a_i \in \mathbb{R}$ are the possible measurement outcomes of the observable A.
- quantum states on $\mathcal{H}$: $\rho \in B(\mathcal{H}), \rho \geq 0, \mathrm{tr}\rho = 1, \equiv \sum_i \lambda_i = 1, \rho = \sum_i \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \rho \left|\psi_i\right\rangle = \lambda_i \left|\psi_i\right\rangle$
- pure quantum states have rank 1: i.e. $\exists \left|\psi\right\rangle \in \mathcal{H}$ s.t. $\rho = \left|\psi\right\rangle \left\langle\psi\right|$
- given a state ρ and an observable A on the Hilbert space, $\mathcal{H}$, the probability of a specific outcome is given by $p_i = \mathrm{tr}(\rho \left|a_i\right\rangle \left\langle a_i\right|) = \langle a_i|\rho|a_i\rangle$
- expected outcomes of measurement: $\langle A\rangle_\rho = \sum_i p_i a_i = \mathrm{tr}(\rho A)$

1.1.2 Evolution of quantum systems

There are essentially three *pictures* of quantum evolution:

1. Evolving states (Schrodinger picture)
2. Evolving observables (Heisenberg picture)
3. Evolving both (interaction picture)

We primarily focus on the Schrodinger picture in this course. Now, what do we require of a formalism that describes quantum evolution?

1. Evolution operation must be linear
2. Transformation $T : B(\mathcal{H}) \to B(\mathcal{H})$ should preserve the structure of the Hilbert space. Mathematically, this means

$$\mathrm{tr}(\psi\phi) = \mathrm{tr}(T(\psi)T(\phi)) \quad \forall \psi, \phi \in B(\mathcal{H}) \tag{1.1}$$

Theorem 1. *Wigner's Theorem: The above requirements imply that $T(X)$ must be either unitary or anti-unitary. That is,*

$$T(X) = UXU^\dagger \quad \textit{or} \quad UX^TU^\dagger \tag{1.2}$$

for some unitary U.

1.2 Open systems and noisy evolution

Because interaction with the environment (some external, inaccessible system) is unavoidable, the closed system assumption is not realistic. Even if the environment is inaccessible, for all cases we are interested in we can first write a new system as the system we are interested in plus the environment we cannot access. That is, SE= system + environment and

$$X_{SE} \mapsto UX_{SE}U^\dagger, \quad \text{where } U \in \mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E). \tag{1.3}$$

Note that $\mathcal{U}$ denotes the unitary group. We can then trace out the environment to recover the evolved system of interest via

$$X_S = \mathrm{Tr}_E(UX_{SE}U^\dagger) \tag{1.4}$$

This partial trace over the environment corresponds to a *noisy and irreversible* evolution of S. Recall that we are focusing on the Schrodinger picture of quantum mechanics. This means we evolve quantum states with maps $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. Maps that evolve quantum states must satifsy the following requirements.

1. Linearity
2. Map states to states
   (a) T should preserve trace: $\mathrm{tr}(T(X)) = \mathrm{tr}X$
   (b) $(X \geq 0 \implies T(X) \geq 0) \Leftrightarrow T \geq 0$ (short-hand notation)
   (c) Complete positivity: $T \otimes \mathbb{I}_n \geq 0, \quad \forall n \in \mathbb{N}$

These requirements lead us to our first definition.

Definition 1.2.1. *A quantum channel is a linear, completely positive (CP), trace-preserving (TP) map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$.*

Given a map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$, the *adjoint map* is $T^\dagger : B(\mathcal{H}_2) \to B(\mathcal{H}_1)$ defined via

$$\langle T^\dagger(X), Y\rangle = \langle X, T(Y)\rangle \tag{1.5}$$

for all $X \in B(\mathcal{H}_2)$ and $Y \in B(\mathcal{H}_1)$. Our map, $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is:

- CP iff $T^\dagger$ is CP.

- TP iff $T^\dagger$ is unital: $T^\dagger(\mathbb{I}_2) = \mathbb{I}_1$.

$$\langle T^\dagger(\mathbb{I}_2), Y\rangle = \langle \mathbb{I}_2, T(Y)\rangle = \mathrm{tr}(T(Y)) = \mathrm{tr}Y = \langle \mathbb{I}_1, Y\rangle \quad \forall \quad Y \in B(\mathcal{H}_1) \tag{1.6}$$

This chain of equalities implies that $T^\dagger(\mathbb{I}_2) = \mathbb{I}_1$. Note that unital quantum channels are both TP and unital.

1.3 Choi–Jamiołkowski isomorphism

We now turn to a very useful tool for studying quantum channels.

Definition 1.3.1. *Let $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ be a linear map. The Choi operator $\tau \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined as*

$$\tau := (\mathbb{I}_1 \otimes T)(\gamma) \tag{1.7}$$

where $\gamma := |\gamma\rangle\langle\gamma|$ and $|\gamma\rangle = \sum_{i=1}^{\dim \mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$. Note that $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_1}$ is an orthonormal basis for $\mathcal{H}_1$.

The explicit form of this operator is then

$$\tau = \sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|) \tag{1.8}$$

Example. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$, the we can express the Choi operator with the block matrix given by

$$\tau = \begin{pmatrix} T(|0\rangle\langle 0|) & T(|1\rangle\langle 0|) \\ T(|0\rangle\langle 1|) & T(|1\rangle\langle 1|) \end{pmatrix} \tag{1.9}$$

where the elements of this matrix are themselves operators $T(|i\rangle\langle j|)$.

Proposition 1.3.1. *Let $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. The map $T \mapsto \tau = (\mathbb{I}_1 \otimes T)(\gamma)$ is a bijection between $\{T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)\}$ and $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, with the inverse mapping $\tau \mapsto T(X) := tr_1(\tau(X^T \otimes \mathbb{I})))$.*

Proof. Let $\tau = (\mathbb{I} \otimes T)(\gamma)$, where as usual $\gamma = |\gamma\rangle\langle\gamma|$. Tracing over the first system, we have

$$\begin{aligned}
\mathrm{tr}_1(\tau(X^T \otimes \mathbb{I})) &= \mathrm{tr}_1\left((\sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)(X^T \otimes \mathbb{I})) \right) && (1.10)\\
&= \sum_{i,j} \mathrm{tr}(|i\rangle\langle j| X^T) T(|i\rangle\langle j|) && (1.11)\\
&= \sum_{i,j} \left(\sum_i \langle i| (|i\rangle\langle j| X^T) |i\rangle T(|i\rangle\langle j|) \right) && (1.12)\\
&= \sum_{i,j} \langle j| X^T |i\rangle T(|i\rangle\langle j|) && (1.13)\\
&= \sum_{i,j} x_{ij} T(|i\rangle\langle j|), \quad \langle j| X^T |i\rangle = \langle i| X |j\rangle := x_{ji} && (1.14)\\
&= T(\sum_{i,j} x_{ij} |i\rangle\langle j|), \quad \text{linearity of } T && (1.15)\\
&= T(X) && (1.16)
\end{aligned}$$

It remains to be shown that $T \mapsto (\mathbb{I} \otimes T)(\gamma)$ is surjective. We note that there exists $|\psi_i\rangle, |\phi_i\rangle$ such that $\tau = \sum_i |\psi_i\rangle\langle\phi_i| \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ where $|\psi_i\rangle \neq |\phi_i\rangle$.

Claim: For every vector $|\psi\rangle \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \exists V \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$|\psi\rangle = (\mathbb{I}_1 \otimes V)|\gamma\rangle. \tag{1.17}$$

To see this, let $|\psi\rangle = \sum_{i,j} p_{ij} |i\rangle \otimes |e_j\rangle$ where $\{|i\rangle\}$ is γ's basis and $\{e_j\}$ is an arbitrary basis. Then we can construct V as

$$V = \sum_{i,j} p_{ij} |e_j\rangle\langle i|. \tag{1.18}$$

Recall $\tau = \sum_i |\psi_i\rangle\langle\phi_i| \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We next claim that there exists L_i, K_i such that

$$\begin{aligned}
|\psi_i\rangle &= (\mathbb{I} \otimes K_i)|\gamma\rangle, && (1.19)\\
|\phi_i\rangle &= (\mathbb{I} \otimes L_i)|\gamma\rangle. && (1.20)\\
& && (1.21)
\end{aligned}$$

This implies

$$\begin{aligned}
\tau &= \sum_i |\psi_i\rangle\langle\phi_i| && (1.22)\\
&= \sum_i (\mathbb{I} \otimes K_i)|\gamma\rangle\langle\gamma| (\mathbb{I} \otimes L_i)^\dagger && (1.23)\\
&= (\mathbb{I} \otimes T)(\gamma), && (1.24)
\end{aligned}$$

where we have identified

$$T(X) = \sum_i K_i X L_i \tag{1.25}$$

as the linear map we sought. This completes the proof of the proposition. □

1.3.1 Recap

- Quantum systems are modeled by Hilbert spaces
- Quantum state: $\rho \in B(\mathcal{H}), \rho \geq 0, \mathrm{tr}\rho = 1$. The state has an eigendecomposition given by

$$\rho = \sum_i \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \qquad \text{where } \rho\left|\psi_i\right\rangle = \lambda_i \left|\psi_i\right\rangle, \lambda_i \geq 0, \langle\psi_i|\psi_j\rangle = \delta_{ij} \tag{1.26}$$

- Schrodinger picture: evolution of quantum states
- Closed systems: evolution given by unitary maps (Wigner's theorem)
- Open systems: unitary evolution on system + environment. This induces noisy evolution on the system
- Quantum channel: $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$
  - linear
  - trace-preserving (TP): $\mathrm{tr}(X(T)) = \mathrm{tr}X \quad \forall X \in B(\mathcal{H}_1)$
  - completely positive (CP): $T \otimes \mathbb{I} \geq 0 \quad \forall n \in \mathbb{N}$
- Choi operator: $\tau \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\tau := (\mathbb{I} \otimes T)(\gamma)$ where $\left|\gamma\right\rangle = \sum_i \left|i\right\rangle \otimes \left|i\right\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_1$
- Choi–Jamiołkowski isomorphism: $T \mapsto \tau = (\mathbb{I}\otimes T)(\gamma)$ is a bijection $\{T : B(\mathcal{H}_1) \to B(\mathcal{H}_2) \text{ linear}\} \Leftrightarrow B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with inverse mapping given by

$$\tau \mapsto \left[T : X \mapsto \mathrm{tr}_1(\tau(X^T \otimes \mathbb{I}))\right]. \tag{1.27}$$

- Steering inequality: $\forall \left|\psi\right\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \quad \exists \quad K \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$\left|\psi\right\rangle = (\mathbb{I} \otimes K)\left|\gamma\right\rangle \tag{1.28}$$

Proposition 1.3.2. *Let T be a linear map that acts as $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. Let τ denote the associated Choi operator that is defined as $\tau = (\mathbb{I}\otimes T)(\gamma)$, with $\left|\gamma\right\rangle = \sum_i \left|i\right\rangle \otimes \left|i\right\rangle$. Then the following hold:*

1. *$T(X)^\dagger = T(X^\dagger)$ iff $\tau = \tau^\dagger$.*
2. *T is CP iff $\tau \geq 0$.*
3. *T is TP iff $\mathrm{tr}_2\tau = \mathbb{I}_1$.*
4. *T is unital iff $\mathrm{tr}_1\tau = \mathbb{I}_2$.*

Recall an operator S is unital if $S(\mathbb{I}_1) = \mathbb{I}_2$.

Proof. 1. ($\Rightarrow$) First we prove if $T(X)^\dagger = T(X^\dagger)$, then $\tau = \tau^\dagger$.

$$\tau^\dagger = \left(\sum_{i,j} |i\rangle \langle j| \otimes T(|i\rangle \langle j|)\right)^\dagger \tag{1.29}$$
$$= \sum_{i,j} |j\rangle \langle i| \otimes T(|i\rangle \langle j|)^\dagger \tag{1.30}$$
$$= \sum_{i,j} |j\rangle \langle i| \otimes T(|i\rangle \langle j|^\dagger) \tag{1.31}$$
$$= \sum_{i,j} |j\rangle \langle i| \otimes T(|j\rangle \langle i|) \tag{1.32}$$
$$= \tau \tag{1.33}$$

($\Leftarrow$) The other way, if $\tau = \tau^\dagger$, $T(X)^\dagger = T(X^\dagger)$. We know that $\tau = \sum_{i,j} |i\rangle \langle j| \otimes T(|i\rangle \langle j|)$ so $\tau = \tau^\dagger$ allows us to write

$$\left(\sum_{ij} |i\rangle \langle j| \otimes T(|i\rangle \langle j|)\right) = \left(\sum_{ij} |i\rangle \langle j| \otimes T(|i\rangle \langle j|)\right)^\dagger \tag{1.34}$$
$$= \left(\sum_{ij} |j\rangle \langle i| \otimes T(|i\rangle \langle j|)^\dagger\right) \tag{1.35}$$

This is a matrix equality, so each element must be the same. Sandwiching $\langle j| \otimes \mathbb{I}_2 \cdot |i\rangle \otimes \mathbb{I}_2$ around both sides of the expression above yields

$$T(|i\rangle \langle j|)^\dagger = T(|j\rangle \langle i|) \quad \forall i, j, \tag{1.36}$$

which implies

$$T(X)^\dagger = T(X^\dagger) \quad \forall X \in B(\mathcal{H}_1), \tag{1.37}$$

because all bounded operators can be expanded in the form $X = \sum_{i,j} x_{ij} |i\rangle \langle j|$ and because T is linear.

2. ($\Rightarrow$) First we prove that if T is CP, then $\tau \geq 0$. $\tau = (\mathbb{I} \otimes T)(\gamma), \gamma \geq 0 \implies \tau \geq 0$.

   ($\Leftarrow$) Going the other way, we want to prove that if $\tau \geq 0$, then T is CP. To do so, we need to show

$$(\mathbb{I}_n \otimes T)(\rho) \geq 0 \quad \forall \rho \in B(\mathbb{C}^n \otimes \mathcal{H}_2), \rho \geq 0. \tag{1.38}$$

3. ($\Rightarrow$) If T is TP, then $\mathrm{tr}_2\tau = \mathbb{I}_1$.

$$\mathrm{tr}_2\tau = \mathrm{tr}_2\left(\sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)\right) \tag{1.39}$$
$$= \sum_{i,j} |i\rangle\langle j| \,\mathrm{tr}(T(|i\rangle\langle j|)) \tag{1.40}$$
$$= \sum_{i,j} |i\rangle\langle j| \,T(\mathrm{tr}(|i\rangle\langle j|)) \tag{1.41}$$
$$= \sum_{i,j} |i\rangle\langle j| \,\delta_{ij} \tag{1.42}$$
$$= \sum_{i} |i\rangle\langle i|_1 \tag{1.43}$$
$$= \mathbb{I}_1 \tag{1.44}$$

($\Leftarrow$) If $\mathrm{tr}_2\tau = \mathbb{I}_2$, then $\mathrm{tr}T(X) = \mathrm{tr}X$.

$$\mathrm{tr}T(X) = \mathrm{tr}\left[\mathrm{tr}_1(\tau(X^T \otimes \mathbb{I}))\right] \tag{1.45}$$
$$= \mathrm{tr}(\tau(X^T \otimes \mathbb{I})) \qquad \mathrm{tr}(\mathrm{tr}_i(\cdot)) = \mathrm{tr}(\cdot) \tag{1.46}$$
$$= \mathrm{tr}\left(\mathrm{tr}_2(\tau)X^T\right) \tag{1.47}$$
$$= \mathrm{tr}X^T \tag{1.48}$$
$$= \mathrm{tr}X \tag{1.49}$$

4. ($\Rightarrow$) T is unital if $\mathrm{tr}_1\tau = \mathbb{I}_2$.

$$\mathrm{tr}_1\tau = \sum_{i,j} \mathrm{tr}(|i\rangle\langle j|_1) T(|i\rangle\langle j|_1) \tag{1.50}$$
$$= \sum_{i} T(|i\rangle\langle i|_1) \tag{1.51}$$
$$= T(\mathbb{I}_1) \tag{1.52}$$
$$= \mathbb{I}_2 \tag{1.53}$$

($\Leftarrow$) If $\mathrm{tr}_1\tau = \mathbb{I}_2$, then T is unital.

$$T(\mathbb{I}_1) = \mathrm{tr}_1\left(\tau(\mathbb{I}_1^T \otimes \mathbb{I}_2)\right) \tag{1.54}$$
$$= \mathrm{tr}_1\tau \tag{1.55}$$
$$= \mathbb{I}_2 \tag{1.56}$$

□

1.3.2 Examples of CP maps

1. unitary maps are CP: $(\mathbb{I} \otimes U)|\gamma\rangle\langle\gamma|(\mathbb{I} \otimes U)^\dagger \geq 0$
2. isometries are CP: $V : \mathcal{H}_1 \to \mathcal{H}_2 : V^\dagger V = \mathbb{I}_1 \Leftrightarrow \langle\varphi| V^\dagger V |\psi\rangle = \langle\varphi|\psi\rangle \quad \forall\varphi,\psi$, and where $\dim\mathcal{H}_1 \leq \dim\mathcal{H}_2$.

3. trace: $(\mathbb{I}_1 \otimes \mathrm{tr})(\gamma) = \sum_{i,j} |i\rangle \langle j| \,\mathrm{tr}(|i\rangle \langle j|) = \mathbb{I}_1 \geq 0$. Note that this also implies that the patial trace, $\mathrm{tr}_2 : B(\mathcal{H}_1 \otimes \mathcal{H}_2) \to B(\mathcal{H}_1)$ is CPTP. We will later show that all channels can be expressed as unitary evolution on system + environment followed by a partial trace over the environment.
4. $X \mapsto \sum_i K_i X K_i^\dagger$ are CP (more on this later).

We have seen examples of CP maps. What about a map that is not CP? The transposition map, $V : X \mapsto X^T$ with respect to a fixed basis, is positive but not *completely positive.*

Consider the following operator:

$$\mathbb{F} = (\mathbb{I} \otimes V)(\gamma) \tag{1.57}$$
$$= \sum_{i,j} |i\rangle \langle j| \otimes V(|i\rangle \langle j|) \tag{1.58}$$
$$= \sum_{i,j} |i\rangle \langle j| \otimes |j\rangle \langle i| \,. \tag{1.59}$$

We call this the swap operator because $|\psi_i\rangle, |\psi_2\rangle \in \mathcal{H}_1 : \mathbb{F}(|\psi_1\rangle \otimes |\psi_2\rangle) = |\psi_1\rangle \otimes |\psi_1\rangle$.

1.4 Kraus representation and the isometric picture

We saw before that $X \mapsto \sum_i K_i X K_i^\dagger$ are CP. The converse is also true.

Proposition 1.4.1.

1. *A map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is CP iff $\exists \{K_i\}_i$ with $K_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T(X) = \sum_i K_i X K_i^\dagger$. We call the K_i's the Kraus operators of T.*
2. *The Kraus rank, $r(T)$, the minimal number of Kraus operators needed to represent T, is equal to the rank of the Choi operator, $\tau = (\mathbb{I} \otimes T)(\gamma)$. It is always true that*
$$r(T) \leq \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2 \tag{1.60}$$
3. *There exists a Kraus representation with $r = \mathrm{rank}(\tau)$ operators such that $\langle K_i, K_j \rangle = tr(K_i^\dagger K_j) = c_i \delta_{i,j}$ for some constant c_i that results from using an unnormalized γ state.*
4. *T is TP iff $\sum_i K_i^\dagger K_i = \mathbb{I}_1$, T is unital iff $\sum_i K_i K_i^\dagger = \mathbb{I}_2$*
5. *Any two Kraus representations $\{K_i\}_i$ and $\{L_i\}_i$ of a channel T are related by a unitary U via*
$$K_i = \sum_{i,j} U_{ij} L_j. \tag{1.61}$$
This means that $T(X) = \sum_i K_i X K_i^\dagger = \sum_j L_j X L_j^\dagger$.

Proof.

1. A map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is CP iff $\exists \{K_i\}_i$ with $K_i \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that $T(X) = \sum_i K_i X K_i^\dagger$. We call the K_i's the Kraus operators of T.

   ($\leftarrow$) is obvious. **Why?**

($\Rightarrow$) Going the other way, we know T is CP $\Leftrightarrow \tau \geq 0$, by Prop 4.

Claim: $X \geq 0 \Leftrightarrow \exists\{|\psi_i\rangle\}_{i=1}^r$ with $r \geq \operatorname{rank} X$ such that $X = \sum_i |\psi_i\rangle\langle\psi_i|$ with $\langle\psi_i|\psi_j\rangle \neq \delta_{ij}$ in general.

Proof. ($\Leftarrow$) $X = \sum_i |\psi_i\rangle\langle\psi_i|$ for some $\{|\psi_i\rangle\}_i$.

We know, $X \geq 0 \Leftrightarrow \langle\varphi|X|\varphi\rangle \geq 0 \quad \forall\,|\varphi\rangle$. This allows us to write

$$\langle\varphi|\left(\sum_i |\psi_i\rangle\langle\psi_i|\right)|\varphi\rangle = \sum_i |\langle\varphi|\psi_i\rangle|^2 \geq 0 \tag{1.62}$$

($\Rightarrow$) $X \geq 0 : X = \sum_i \lambda_i |X_i\rangle\langle X_i|$ where $X|X_i\rangle = \lambda_i|X_i\rangle$ and $\langle X_i|X_j\rangle = \delta_{ij}$. Let $|\psi_i\rangle := \sqrt{\lambda_i}\,|X_i\rangle \implies X = \sum_i |\psi_i\rangle\langle\psi_i|$. □

Returning to the proof of the first item of the proposition, we have that $\tau \geq 0 \Rightarrow \exists\{|\psi_i\rangle\}$ such that $\tau = \sum_i |\psi_i\rangle\langle\psi_i|$. Further, for all i there exists K_i such that $|\psi_i\rangle = (\mathbb{I}_1 \otimes K_i)|\gamma\rangle$. The we can write

$$\tau = \sum_i |\psi_i\rangle\langle\psi_i| \tag{1.63}$$

$$= \sum_i (\mathbb{I}_1 \otimes K_i)|\gamma\rangle\langle\gamma|(\mathbb{I}_1 \otimes K_i)^\dagger \tag{1.64}$$

by the C-J isomorphism we conclude that $T(X) = \sum_i K_i X K_i^\dagger$.

2. Clear from proof of 1.) $r = \operatorname{rank}(\tau) \implies$ at least r pure states in the decomposition of τ. This implies that there are at least r Kraus operators.

3. There exists a Kraus representation with $r = \operatorname{rank}(\tau)$ operators such that $\langle K_i, K_j\rangle = \operatorname{tr}(K_i^\dagger K_j) = c_i\delta_{i,j}$ for some constant c_i that results from using an unnormalized γ state.

   Let $\tau = \lambda_i |X_i\rangle\langle X_i|$ be the spectral decomposition of τ with $|\varphi_i\rangle = \sqrt{\lambda_i}\,|X_i\rangle$. We have

$$\tau = \sum_i |\varphi_i\rangle\langle\varphi_i| \tag{1.65}$$

$$= \sum_i (\mathbb{I} \otimes L_i)|\gamma\rangle\langle\gamma|(\mathbb{I} \otimes L_i)^\dagger \tag{1.66}$$

$$\Rightarrow \quad T = \sum_{i=1}^{r(\tau)} L_i \cdot L_i^\dagger \tag{1.67}$$

   We also need to show $\langle L_i, L_j\rangle = c\delta_{ij}$. We know

$$|\varphi_i\rangle = (\mathbb{I}_1 \otimes L_i)|\gamma\rangle \tag{1.68}$$

so, we can write

$$
\begin{aligned}
c\delta_{ij} &= \langle\varphi_i|\varphi_j\rangle && (1.69)\\
&= \langle\gamma|\left(\mathbb{I}_1 \otimes L_i^\dagger\right)(\mathbb{I}\otimes L_i)\,|\gamma\rangle && (1.70)\\
&= \sum_{k,l} \langle k|l\rangle \langle k|\, L_i^\dagger L_j\, |l\rangle && (1.71)\\
&= \sum_{k} \langle k|\, L_i^\dagger L_j\, |k\rangle && (1.72)\\
&= \mathrm{tr}\left(L_i^\dagger L_j\right) && (1.73)\\
&= \langle L_i, L_j\rangle && (1.74)
\end{aligned}
$$

4. T is TP iff $\sum_i K_i^\dagger K_i = \mathbb{I}_1$, T is unital iff $\sum_i K_i K_i^\dagger = \mathbb{I}_2$.

$$
\begin{aligned}
\mathrm{tr}(T(X)) &= \sum_i \mathrm{tr}\left(K_i X K_i^\dagger\right) && (1.75)\\
&= \sum_i \mathrm{tr}\left(K_i^\dagger K_i X\right) \quad \text{cyclicity of trace} && (1.76)\\
&= \mathrm{tr}\left(\sum_i K_i^\dagger K_i X\right) \quad \text{linearity of trace} && (1.77)\\
&= \mathrm{tr}X \quad \forall X && (1.78)\\
\Leftrightarrow \sum_i K_i^\dagger K_i &= \mathbb{I}_1 && (1.79)
\end{aligned}
$$

The other way is even easier.

$$
\begin{aligned}
T(\mathbb{I}_1) &= \sum_i K_i \mathbb{I} K_i^\dagger && (1.80)\\
&= \sum_i K_i K_i^\dagger && (1.81)\\
&= \mathbb{I}_2. && (1.82)
\end{aligned}
$$

5. Any two Kraus representations $\{K_i\}_i$ and $\{L_i\}_i$ of a channel T are related by a unitary U via

$$K_i = \sum_{i,j} U_{ij} L_j. \tag{1.83}$$

This means that $T(X) = \sum_i K_i X K_i^\dagger = \sum_j L_j X L_j^\dagger$.

Claim: $\sum_i |\psi_i\rangle\langle\psi_i| = \sum_j |\varphi_j\rangle\langle\varphi_j|$ iff there exists a unitary, U, with $|\psi_i\rangle = \sum_j U_{ij}|\varphi_j\rangle$.

Proof. $|\Psi\rangle = \sum_i |\psi_i\rangle \otimes |i\rangle$, where $\{|i\rangle\}$ is a reference systems' orthonormal basis. Also, $|\Phi\rangle = \sum_j |\varphi_j\rangle \otimes |j\rangle$. Then, there always exists an isometry V such that $(\mathbb{I}\otimes V)\,|\Psi\rangle = |\Phi\rangle$. □

The above result is useful for our proof because the isometry V above can always be extended to a unitary.

$$\begin{aligned}
|\varphi_i\rangle &= (\mathbb{I} \otimes \langle i|)\,|\Phi\rangle && (1.84)\\
&= (\mathbb{I} \otimes \langle i|)\,(\mathbb{I} \otimes V)\,|\Psi\rangle && (1.85)\\
&= (\mathbb{I} \otimes \langle i|)\,(\mathbb{I} \otimes V) \sum_j |\psi_j\rangle \otimes |j\rangle && (1.86)\\
&= \sum_j |\psi_j\rangle\,\langle i|V|j| && (1.87)\\
&= \sum_j V_{ij}\,|\psi_j\rangle && (1.88)
\end{aligned}$$

□

We now want to relate the Kraus representation to the isometric picture. Recall that an isometry is a map

$$V : \mathcal{H}_1 \to \mathcal{H}_2 : \langle\psi|\varphi\rangle = \langle\psi|V^\dagger V|\varphi\rangle \quad \forall\,|\psi\rangle\,,|\varphi\rangle \in \mathcal{H}_1 \tag{1.89}$$

where $\dim \mathcal{H}_2 \geq \mathcal{H}_1$. Note that if the dimensions are equal, V is unitary. We often consider a $\mathcal{H}_2 = \mathcal{H}_B \otimes \mathcal{H}_E$.

Proposition 1.4.2. *1. $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is CP iff $\exists\, V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathbb{C}^r$ where $r \geq r(T)$ and $T(X) = tr_E V X V^\dagger$ with environment equal to $\mathbb{C}^r$.*

2. T is TP iff $V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathbb{C}^r$ is an isometry.

Proof. 1. ($\Leftarrow$) is done.
($\Rightarrow$) T is CP: $\exists\{K_i\}_{i=1}^n$ such that $T(X) = \sum_i K_i X K_i^\dagger, r \geq r(T)$. Choose an orthonormal basis $\{|i\rangle\}_{i=1}^r$ in $\mathbb{C}^r : V = \sum_{i=1}^r K_i \otimes |i\rangle$. We then have

$$\mathrm{tr}_E V X V^\dagger = \sum_{i,j} \mathrm{tr}_1\left((K_i \otimes |i\rangle) X (K_j \otimes |j\rangle)^\dagger\right) \tag{1.90}$$

2. T is TP iff $V^\dagger V = \mathbb{I}$.
($\Rightarrow$)

$$\begin{aligned}
\mathrm{tr}X &= \mathrm{tr}T(X) && (1.91)\\
&= \mathrm{tr}\left(\mathrm{tr}_E V X V^\dagger\right) && (1.92)\\
&= \mathrm{tr}(V X V^\dagger) && (1.93)\\
&= \mathrm{tr}X \quad \forall X \Leftrightarrow V^\dagger V = \mathbb{I} && (1.94)
\end{aligned}$$

□

- The isometry V in proposition above is called the Stinespring isometry or the Stinespring dilation of the channel T.

- Given Stinespring isometry V, a Kraus representation $\{K_i\}_i$ is obtained via $K_i = (\mathbb{I} \otimes \langle i|)V$. This is evident if we recall

$$\mathrm{tr}_E Y = \sum_i (\mathbb{I} \otimes \langle i|) Y (\mathbb{I} \otimes |i\rangle) \tag{1.95}$$

1.5 Unitary picture and open system dynamics

For quantum channel $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$:

$$\exists \text{ Stinespring isometry } V : \mathcal{H}_1 \to \mathcal{H}_2 \otimes \mathbb{C}^r \text{ such that} \tag{1.96}$$

$$T(X) = \mathrm{tr}_E V X V^\dagger \tag{1.97}$$

where $r = \dim \mathcal{H}_1 \dim \mathcal{H}_2$. Complete V to a unitary acting on $\mathcal{H}_1 = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$ such that $V = U(\mathbb{I}_{d_1} \otimes |\varphi\rangle)$ where $|\varphi\rangle$ is some fixed vector in $\mathbb{C}^{d_2} \otimes \mathbb{C}^{d_2}$. What we then have is

$$T(X) = \mathrm{tr}_E V X V^\dagger \tag{1.98}$$

$$= \mathrm{tr}_{E'} U (X \otimes \varphi) U^\dagger \tag{1.99}$$

where the environment E' is $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. This all allows us to consider noisy quantum evolution as unitary evolution on a larger closed system followed by a tracing out of the environment.

1.6 Linear representation

Let T be a linear map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. A key result from linear algebra is that **a linear map on a finite-dimensional Hilbert space can always be represented with a matrix with respect to a fixed basis.**

- If we are in a bounded Hilbert space $B(\mathcal{H})$, we have an associated inner product (called the Hilbert-Schmidt inner product) which is given as

$$\langle X, Y \rangle = \mathrm{tr}(X^\dagger Y). \tag{1.100}$$

- We can think of operators $X \in B(\mathcal{H})$ as vectors, and maps $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ as matrices.
- Let $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}}$ be an orthonormal basis for $\mathcal{H}$. Then $\{|i\rangle \langle j|\}_{i,j=1}^{\dim \mathcal{H}}$ is a basis for $B(\mathcal{H})$. Note that $\dim \mathcal{H} = d \implies \dim B(\mathcal{H}) = d^2$.
- Define a linear mapping

$$\mathrm{vec} : B(\mathcal{H}) \to \mathcal{H} \otimes \mathcal{H}. \tag{1.101}$$

  So,

$$|i\rangle \langle j| \mapsto |i\rangle \otimes |j\rangle + \text{ linear extension} \tag{1.102}$$

- $\{|i\rangle \otimes |j\rangle\}_{i,j=1}^{d}$ is a basis for $\mathcal{H} \otimes \mathcal{H} \implies$ vec is an isomorphism, $B(\mathcal{H}) \cong \mathcal{H} \otimes \mathcal{H}$

Properties of vec:

- $\text{vec}(|\psi\rangle\langle\psi|) = |\psi\rangle \otimes |\varphi^*\rangle$ where $*$ denotes the complex conjugate in the basis $\{|i\rangle\}_i$
- $\langle X, Y\rangle_{B(\mathcal{H})} = \langle \text{vec}(X), \text{vec}(Y)\rangle_{\mathcal{H}\otimes\mathcal{H}}$
- $\text{vec}(AXB) = (A \otimes B^T)\text{vec(X)}$

Let $\mathcal{N} : B(\mathcal{H} \to B(\mathcal{H})$ be a linear map. Then via the vec map, this will correspond to some operator in $N \in B(\mathcal{H}\otimes\mathcal{H})$. In particular, there exist A_i, B_i such that $N = \sum_i A_i \otimes B_i$. Then, for $X \in B(\mathcal{H})$ we would have

$$\mathcal{N}(X) = \sum_i A_i X B_i^T, \tag{1.103}$$

by using the identity $\text{vec}(AXB^T) = (A \otimes B)\text{vec}(X)$. When considering quantum channels, we have $\mathcal{N}(X) = \sum_i K_i X K_i^\dagger$ which via vec will be $N = \sum_i K_i \otimes K_i^*$. This is called the *transfer matrix* of $\mathcal{N}$. The unitality condition $\sum_i K_i K_i^\dagger = \mathbb{I}_2$ under vec becomes

$$N|\gamma\rangle = (\sum_i K_i^\dagger \otimes K_i^T)|\gamma\rangle = |\gamma\rangle = \text{vec}(\mathbb{I}) \tag{1.104}$$

The trace-preserving condition $\sum_i K_i^\dagger K_i = \mathbb{I}_2$ under vec becomes

$$N^\dagger|\gamma\rangle = (\sum_i K_i^\dagger \otimes K_i^T)|\gamma\rangle = |\gamma\rangle. \tag{1.105}$$

Note that if $T = \sum_i A_i \otimes B_i^T$, then $T^\dagger = \sum_i B_i^T \otimes A_i$. The linear representation is useful, because it translates channel composition into matrix multiplication.

Proposition 1.6.1. *Let $\mathcal{M} : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$, $\mathcal{N} : B(\mathcal{H}_2) \to B(\mathcal{H}_3)$ be linear maps with transfer matrices M, N, respectively. Then, $N \cdot M$ is the transfer matrix of $\mathcal{N} \circ \mathcal{M} : B(\mathcal{H}_1) \to B(\mathcal{H}_3)$.*

Proof.

$$\mathcal{M} = \sum_i A_i X B_i^T \quad A_i, B_i \in B(\mathcal{H}_1, \mathcal{H}_2) \tag{1.106}$$

$$\mathcal{N} = \sum_i C_i X D_i^T \quad C_i, D_i \in B(\mathcal{H}_2, \mathcal{H}_3) \tag{1.107}$$

The transfer matrices are

$$M = \sum_i A_i \otimes B_i, \tag{1.108}$$

$$N = \sum_i C_i \otimes D_i. \tag{1.109}$$

We then have

$$\mathcal{N} \circ \mathcal{M} = \sum_{i,j} C_i A_j X B_j^T D_i^T \tag{1.110}$$

$$N \cdot M = \sum_{i,j} C_i A_j \otimes D_i B_j \tag{1.111}$$

□

Proposition 1.6.2. *Let $\mathcal{N} : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ be a linear map with transfer matrix N and $\tilde{\tau} = (\mathcal{N} \otimes \mathbb{I}_2)(\gamma)$. Then, $\tilde{\tau} = N^\Gamma$ where Γ is an involution ($\Gamma^2 = \mathbb{I}$) defined by*

$$\langle i,j|X^\Gamma|k,l\rangle = \langle i,k|X|j,l\rangle \quad |i,j\rangle := |i\rangle \otimes |j\rangle \tag{1.112}$$

Proof. Without loss of generality, $\mathcal{N} = XY^\Gamma$, where $X, Y \in B(\mathcal{H}_1, \mathcal{H}_2)$. Let $\{|i\rangle\}, \{|\alpha\rangle\}$ denote a basis on $\mathcal{H}_1, \mathcal{H}_2$, respectively. Then we have

$$X = \sum_{i,\alpha} x_{\alpha,i} |\alpha\rangle \langle i| , \tag{1.113}$$

$$Y = \sum_{j,\beta} y_{\beta,j} |\beta\rangle \langle j| , \tag{1.114}$$

which allows us to write

$$N = X \otimes Y \sum_{i,j,\alpha,\beta} x_{\alpha,i} y_{\beta,j} |\alpha\rangle \langle i| \otimes |\beta\rangle \langle j| , \tag{1.115}$$

then

$$\begin{aligned}
\tilde{\tau} &= (\mathcal{N} \otimes \mathbb{I})(\gamma), & (1.116)\\
&= \sum_{k,l} X |k\rangle \langle l| Y^T \otimes |k\rangle \langle l| , & (1.117)\\
&= \sum_{k,l,i,j,\alpha,\beta} x_{\alpha,i} y_{\beta,j} |\alpha\rangle \langle i|k\rangle \langle l|j\rangle \langle \beta| \otimes |k\rangle \langle l| , & (1.118)\\
&= \sum_{i,j,\alpha,\beta} x_{\alpha,i} y_{\beta,j} |\alpha\rangle \langle \beta| \otimes |i\rangle \langle j| , & (1.119)
\end{aligned}$$

so we conclude

$$\langle \alpha, \beta| N |i,j\rangle = \langle \alpha, i|\tilde{\tau}|\beta, j\rangle \tag{1.120}$$

□

Note that $\tilde{\tau} \in B(\mathcal{H}_2 \otimes \mathcal{H}_1, \mathcal{H}_1 \otimes \mathcal{H}_2)$ while $N \in B(\mathcal{H}_1 \otimes \mathcal{H}_1, \mathcal{H}_2 \otimes \mathcal{H}_2)$.

Bibliography