Quantum Channels

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January 27, 2021

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Chapter 0

Prerequisites

0.1 Hilbert Spaces and Linear Operators

Throughout this course, $\mathcal{H}$ denotes a finite-dimensional Hilbert space (complex vector space with an associated inner product). Using Dirac's "bra-ket" notation we denote elements of the Hilbert space (called kets) as

$$|\psi\rangle \in \mathcal{H}. \tag{1}$$

The elements of the dual Hilbert space are called bras and are denoted

$$\langle\psi| \in \mathcal{H}^*, \tag{2}$$

where $\langle\psi| = (|\psi\rangle)^\dagger$. Here, $X^\dagger := \bar{X}^T$ denotes the Hermitian adjoint (also called the conjugate transpose). We denote

$$B(\mathcal{H}_1, \mathcal{H}_2) := \{\text{linear maps from } \mathcal{H}_1 \text{ to } \mathcal{H}_2\} \tag{3}$$

and the set of all linear maps to and from the same space will be denoted $B(\mathcal{H}) = B(\mathcal{H}, \mathcal{H})$. An operator $X \in B(\mathcal{H})$ is *normal* if $XX^T = X^T X$. Every normal operator has a *spectral decomposition*. That is, there exists a unitary U and a diagonal matrix D whose entries are the eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{C}$ of X such that

$$X = UDU^\dagger. \tag{4}$$

In other words,

$$X = \sum_{i=1}^{d} \lambda_i |\psi_i\rangle \langle\psi_i| \tag{5}$$

where $X|\psi_i\rangle = \lambda_i |\psi_i\rangle$ and $U = (|\psi_i\rangle, \ldots, |\psi_d\rangle)$. If X is Hermitian, $X = X^\dagger$, then $\lambda_i \in \mathbb{R}$. An operator X is positive semi-definite (PSD) if

$$\langle\varphi| X |\varphi\rangle \geq 0 \qquad \forall |\varphi\rangle \in \mathcal{H}. \tag{6}$$

As a consequence, $X \geq 0$ and $\lambda_i \geq 0$. It holds that PSD $\implies$ Hermitian $\implies$ normal. Unless otherwise stated, we will always assume we are working in an orthonormal basis.

0.2 Quantum States

A quantum state ρ in a Hilbert space $\mathcal{H}$ is a PSD linear operator with

$$\rho \in B(\mathcal{H}), \quad \rho \geq 0, \quad \mathrm{tr}\rho = 1. \tag{7}$$

This means that the state has eigenvalues $\{\lambda_i\}_{i=1}^d$ satisfying $\lambda_i \geq 0$ and $\sum_{i=1}^d \lambda_i = 1$. Thus, $\{\lambda_i\}_{i=1}^d$ forms a probability distribution.

A *pure quantum state* ψ is a quantum state with rank 1. We can find $|\psi\rangle \in \mathcal{H}$ such that $\psi = |\psi\rangle\langle\psi|$. In this case, ψ is called a *projector*. A *mixed state* is a quantum state with rank > 1. Mixed states are convex combinations of pure states. That is, for every quantum state ρ with $r = \mathrm{rank}(\rho)$ there are pure states $|\psi_i\rangle_{i=1}^k \quad (k \geq r)$ and a probability distribution $\{p_i\}_{i=1}^k$ such that

$$\rho = \sum_{i=1}^k p_i |\psi_i\rangle\langle\psi_i|. \tag{8}$$

The spectral decomposition of ρ is a special case of this property.

0.3 Composite systems, partial trace, entanglement

Let A and B be two quantum systems with Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The *joint system* AB is described by the Hilbert space $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$. We denote quantum states of the joint system as $\rho_{AB} \in \mathcal{H}_{AB}$. The marginal of the bipartite state, denoted ρ_A, is uniquely defined as the operator satisfying

$$\rho_A := \mathrm{tr}_B \rho_{AB}, \tag{9}$$

which is defined via $\mathrm{tr}(\rho_{AB}(X_A \otimes \mathbb{I}_B)) = \mathrm{tr}\rho_A X_A \quad \forall X_A \in B(\mathcal{H}_A)$. For a Hilbert space with $|B| := \dim\mathcal{H}_B$, the explicit form of the partial trace is

$$\mathrm{tr}_B \rho_{AB} = \sum_{i=1}^{|B|} (\mathbb{I}_A \otimes \langle i|_B) \rho_{AB} (\mathbb{I}_A \otimes |i\rangle_B), \tag{10}$$

for some basis $\{|i\rangle_B\}_{i=1}^{|B|}$ of $\mathcal{H}_B$.

A *product state* on AB is a state of the form $\rho_A \otimes \sigma_B$. The state is called *separable* if it lies in the convex hull of product states:

$$\rho_{AB} = \sum_i p_i \rho_A^i \otimes \sigma_B^i \tag{11}$$

for some states $\{\rho_A^i\}_i$ and $\{\sigma_B^i\}_i$ and probability distribution $\{p_i\}_i$. A state is called *entangled*, if it is not separable. An entangled state of particular interest is the *maximally entangled state*. Let $d = \dim\mathcal{H}$, $\{|i\rangle\}_{i=1}^d$ be a basis for $\mathcal{H}$. A maximally entangled state is expressed as

$$|\phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle \otimes |i\rangle \quad \in \mathcal{H} \otimes \mathcal{H} \tag{12}$$

0.4 Measurements

The most general measurement is given by a *positive operator-valued measure* (POVM) $E = \{E_i\}_i$ where $E_i \geq 0 \quad \forall i$ and $\sum_i E_i = \mathbb{I}$. Then, for a quantum system $\mathcal{H}$ in state ρ, the probability of obtaining measurement outcome i is given by $p_i = \mathrm{tr}[\rho E_i]$. So, we have

$$\sum_i p_i = \sum_i \mathrm{tr}[\rho E_i] = \mathrm{tr}\left[\rho \sum_i E_i\right] = \mathrm{tr}[\rho \mathbb{I}] = \mathrm{tr}\rho = 1, \tag{13}$$

for all normalized quantum states. A *projective measurement* $\Pi = \{\Pi_i\}$ is a POVM with the added property of orthogonality, which for projectors means

$$\Pi_i \Pi_j = \delta_{ij} \Pi_i. \tag{14}$$

Any basis $\{|e_i\rangle\}_{i=1}^{\dim \mathcal{H}}$ gives rise to a projective measurement $\Pi = \{|e_i\rangle \langle e_i|\}_{i=1}^{\dim \mathcal{H}}$.

0.5 Entropies

The *Shannon entropy* $H(p)$ of a probability distribution $p = \{p_i, \ldots, p_d\}$ is defined as $H(p) = -\sum_{i=1}^{d} p_i \log p_i$, where the logarithm is base 2 unless otherwise specified. Note that when the logarithm is base 2, the entropy has units of *bits*. The *von Neumann entropy* $S(\rho)$ of a quantum state ρ is defined as

$$S(\rho) = -\mathrm{tr}\left[\rho \log \rho\right] = H(\{\lambda_i, \ldots, \lambda_d\}), \tag{15}$$

where $\rho = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i|$ is a spectral decomposition of ρ and where the logarithm of an operator is obtained by first diagonalizing the matrix representing the operator and then taking the logarithm of the diagonal elements. That is,

$$\log \rho = \sum_{i:\lambda_i > 0} \log(\lambda_i) |\psi_i\rangle \langle \psi_i| \,. \tag{16}$$

Chapter 1

Representations

1.1 Unitary evolution of (closed) quantum systems

1.1.1 Quantum mechanics

We start by stating some broadly-accepted facts or axioms about quantum mechanics.

- we represent quantum systems using (in our case) finite dimensional Hilbert spaces, denoted $\mathcal{H}$
- Observables: measurable quantities are Hermitian operators $A \in B(\mathcal{H})$, $A = A^\dagger$, $A = \sum_i a_i \left|a_i\right\rangle \left\langle a_i\right|$ where $\langle a_i|a_j\rangle = \delta_{ij}$ and $a_i \in \mathbb{R}$.
- eigenvalues $a_i \in \mathbb{R}$ are the possible measurement outcomes of the observable A.
- quantum states on $\mathcal{H}$: $\rho \in B(\mathcal{H}), \rho \geq 0, \mathrm{tr}\rho = 1, \equiv \sum_i \lambda_i = 1, \rho = \sum_i \lambda_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \rho\left|\psi_i\right\rangle = \lambda_i \left|\psi_i\right\rangle$
- pure quantum states have rank 1: i.e. $\exists \left|\psi\right\rangle \in \mathcal{H}$ s.t. $\rho = \left|\psi\right\rangle\left\langle\psi\right|$
- given a state ρ and an observable A on the Hilbert space, $\mathcal{H}$, the probability of a specific outcome is given by $p_i = \mathrm{tr}(\rho \left|a_i\right\rangle \left\langle a_i\right|) = \langle a_i|\rho|a_i\rangle$
- expected outcomes of measurement: $\left\langle A\right\rangle_\rho = \sum_i p_i a_i = \mathrm{tr}(\rho A)$

1.1.2 Evolution of quantum systems

There are essentially three *pictures* of quantum evolution:

1. Evolving states (Schrodinger picture)
2. Evolving observables (Heisenberg picture)
3. Evolving both (interaction picture)

We primarily focus on the Schrodinger picture in this course. Now, what do we require of a formalism that describes quantum evolution?

1. Evolution operation must be linear
2. Transformation $T : B(\mathcal{H}) \to B(\mathcal{H})$ should preserve the structure of the Hilbert space. Mathematically, this means

$$\operatorname{tr}(\psi\phi) = \operatorname{tr}(T(\psi)T(\phi)) \quad \forall \psi, \phi \in B(\mathcal{H}) \tag{1.1}$$

Theorem 1. *Wigner's Theorem: The above requirements imply that $T(X)$ must be either unitary or anti-unitary. That is,*

$$T(X) = UXU^\dagger \quad \text{or} \quad UX^TU^\dagger \tag{1.2}$$

for some unitary U.

1.2 Open systems and noisy evolution

Because interaction with the environment (some external, inaccessible system) is unavoidable, the closed system assumption is not realistic. Even if the environment is inaccessible, for all cases we are interested in we can first write a new system as the system we are interested in plus the environment we cannot access. That is, SE= system + environment and

$$X_{SE} \mapsto UX_{SE}U^\dagger, \quad \text{where } U \in \mathcal{U}(\mathcal{H}_S \otimes \mathcal{H}_E). \tag{1.3}$$

Note that $\mathcal{U}$ denotes the unitary group. We can then trace out the environment to recover the evolved system of interest via

$$X_S = \operatorname{Tr}_E(UX_{SE}U^\dagger) \tag{1.4}$$

This partial trace over the environment corresponds to a *noisy and irreversible* evolution of S. Recall that we are focusing on the Schrodinger picture of quantum mechanics. This means we evolve quantum states with maps $T : B(\mathcal{H}_1) \to \mathcal{H}_2$. Maps that evolve quantum states must satifsy the following requirements.

1. Linearity
2. Map states to states
   (a) T should preserve trace: $\operatorname{tr}(T(X)) = \operatorname{tr}X$
   (b) $(X \geq 0 \implies T(X) \geq 0) \Leftrightarrow T \geq 0$ (short-hand notation)
   (c) Complete positivity: $T \otimes \mathbb{I}_n \geq 0, \quad \forall n \in \mathbb{N}$

These requirements lead us to our first definition.

Definition 1.2.1. *A quantum channel is a linear, completely positive (CP), trace-preserving (TP) map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$.*

Given a map $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$, the *adjoint map* is $T^\dagger : B(\mathcal{H}_2) \to B(\mathcal{H}_1)$ defined via

$$\langle T^\dagger(X), Y\rangle = \langle X, T(Y)\rangle \tag{1.5}$$

for all $X \in B(\mathcal{H}_2)$ and $Y \in B(\mathcal{H}_1)$. Our map, $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ is:

- CP iff $T^\dagger$ is CP.

- TP iff $T^\dagger$ is unital: $T^\dagger(\mathbb{I}_2) = \mathbb{I}_1$.

$$\langle T^\dagger(\mathbb{I}_2), Y\rangle = \langle \mathbb{I}_2, T(Y)\rangle = \mathrm{tr}(T(Y)) = \mathrm{tr}Y = \langle \mathbb{I}_1, Y\rangle \quad \forall \quad Y \in B(\mathcal{H}_1) \tag{1.6}$$

This chain of equalities implies that $T^\dagger(\mathbb{I}_2) = \mathbb{I}_1$. Note that unital quantum channels are both TP and unital.

1.3 Choi–Jamiołkowski isomorphism

We now turn to a very useful tool for studying quantum channels.

Definition 1.3.1. *Let $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$ be a linear map. The Choi operator $\tau \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined as*

$$\tau := (\mathbb{I}_1 \otimes T)(\gamma) \tag{1.7}$$

where $\gamma := |\gamma\rangle\langle\gamma|$ and $|\gamma\rangle = \sum_{i=1}^{dim\mathcal{H}_1} |i\rangle \otimes |i\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$. Note that $\{|i\rangle\}_{i=1}^{dim\mathcal{H}_1}$ is an orthonormal basis for $\mathcal{H}_1$.

The explicit form of this operator is then

$$\tau = \sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|) \tag{1.8}$$

Example. If $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}^2$, the we can express the Choi operator with the block matrix given by

$$\tau = \begin{pmatrix} T(|0\rangle\langle 0|) & T(|1\rangle\langle 0|) \\ T(|0\rangle\langle 1|) & T(|1\rangle\langle 1|) \end{pmatrix} \tag{1.9}$$

where the elements of this matrix are themselves operators $T(|i\rangle\langle j|)$.

Proposition 1.3.1. *Let $T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)$. The map $T \mapsto \tau = (\mathbb{I}_1 \otimes T)(\gamma)$ is a bijection between $\{T : B(\mathcal{H}_1) \to B(\mathcal{H}_2)\}$ and $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, with the inverse mapping $\tau \mapsto T(X) := tr_1(\tau(X^T \otimes \mathbb{I}))$.*

Proof. Let $\tau = (\mathbb{I} \otimes T)(\gamma)$, where as usual $\gamma = |\gamma\rangle\langle\gamma|$. Tracing over the first system, we have

$$\begin{aligned}
\mathrm{tr}_1(\tau(X^T \otimes \mathbb{I})) &= \mathrm{tr}\left((\sum_{i,j} |i\rangle\langle j| \otimes T(|i\rangle\langle j|)(X^T \otimes \mathbb{I})) \right) && (1.10)\\
&= \sum_{i,j} \mathrm{tr}(|i\rangle\langle j| X^T) T(|i\rangle\langle j|) && (1.11)\\
&= \sum_{i,j} \left(\sum_i \langle i| (|i\rangle\langle j| X^T) |i\rangle T(|i\rangle\langle j|) \right) && (1.12)\\
&= \sum_{i,j} \langle j| X^T |i\rangle T(|i\rangle\langle j|) && (1.13)\\
&= \sum_{i,j} x_{ij} T(|i\rangle\langle j|), \quad \langle j| X^T |i\rangle = \langle i| X |j\rangle := x_{ji} && (1.14)\\
&= T(\sum_{i,j} x_{ij} |i\rangle\langle j|), \quad \text{linearity of } T && (1.15)\\
&= T(X) && (1.16)
\end{aligned}$$

It remains to be shown that $T \mapsto (\mathbb{I} \otimes T)(\gamma)$ is surjective. We note that there exists $|\psi_i\rangle, |\phi_i\rangle$ such that $\tau = \sum_i |\psi_i\rangle\langle\phi_i| \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ where $|\psi_i\rangle \neq |\phi_i\rangle$.

Claim: For every vector $|\psi\rangle \in B(\mathcal{H}_1 \otimes \mathcal{H}_2) \exists V \in B(\mathcal{H}_1, \mathcal{H}_2)$ such that

$$|\psi\rangle = (\mathbb{I}_1 \otimes V)|\gamma\rangle. \tag{1.17}$$

To see this, let $|\psi\rangle = \sum_{i,j} p_{ij} |i\rangle \otimes |e_j\rangle$ where $\{|i\rangle\}$ is γ's basis and $\{e_j\}$ is an arbitrary basis. Then we can construct V as

$$V = \sum_{i,j} p_{ij} |e_j\rangle\langle i|. \tag{1.18}$$

Recall $\tau = \sum_i |\psi_i\rangle\langle\phi_i| \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We next claim that there exists L_i, K_i such that

$$\begin{aligned}
|\psi_i\rangle &= (\mathbb{I} \otimes K_i)|\gamma\rangle, && (1.19)\\
|\phi_i\rangle &= (\mathbb{I} \otimes L_i)|\gamma\rangle. && (1.20)\\
& && (1.21)
\end{aligned}$$

This implies

$$\begin{aligned}
\tau &= \sum_i |\psi_i\rangle\langle\phi_i| && (1.22)\\
&= \sum_i (\mathbb{I} \otimes K_i)|\gamma\rangle\langle\gamma| (\mathbb{I} \otimes L_i)^\dagger && (1.23)\\
&= (\mathbb{I} \otimes T)(\gamma), && (1.24)
\end{aligned}$$

where we have identified

$$T(X) = \sum_i K_i X L_i \tag{1.25}$$

as the linear map we sought. This completes the proof of the proposition. □

Bibliography