ECE457A: Assignment 2 Solution

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2024-10-11

Question 1 Solution

1. Problem Formulation

State Representation: For this simulated annealing problem, the representation of a state is simply the current coordinate pair of x_1 and x_2 values. And thus, an arbitrary state can be represented as $S = (x_1, x_2)$

Initial State: The initial state of this problem was chosen to be a random coordinate, $(x_1, x_2) \; \forall \in [-100, 100]$.

Goal State: The goal state of this problem is to find the coordinate which minimizes the *Easom* function (within some degree of error). This goal state is $S = (\pi, \pi)$.

Actions: The actions available for the simulated annealing algorithm are as follows:

- change current state values of x_1 and x_2 using the neighbourhood function
- accept the new state if the new cost is less than the current cost
- accept or reject the new state if the new cost is less than or equal to the current cost based on the probability function $P = e^{\frac{-\Delta c}{t}}$

Cost: Due to the structure of this problem, I decided that the best cost function was to use a variation of the *Easom* function itself. Since the function is essentially constant for values surrounding the minimum at (π, π), it is important that the cost function is extremely sensitive to small changes so that it is much more likely to detect the steep function decrease. Therefore, the cost function chosen is the following:

$$c(x_1, x_2) = -cos(x_1)cos(x_2)(-(x_1 - \pi)^2 - (x - 2 - \pi)^2) \cdot 10^6$$

Neighbourhood Function: The neighbourhood function in determines how the current state may change from one iteration to another in simulated annealing. For the neighbourhood function for this problem, it was chosen to be an addition or subtraction of a random uniform distribution within the bounds $(-10, 10)$ from the current state. It was also important to saturate the new state values to the bounds given in the problem $[-100, 100]$.

$$x_{1_{i+1}} = min(max(x_{1_i} + rand.unif(-10, 10), -100), 100)$$

$$x_{2_{i+1}} = min(max(x_{2_i} + rand.unif(-10, 10), -100), 100)$$

2. Solution Description

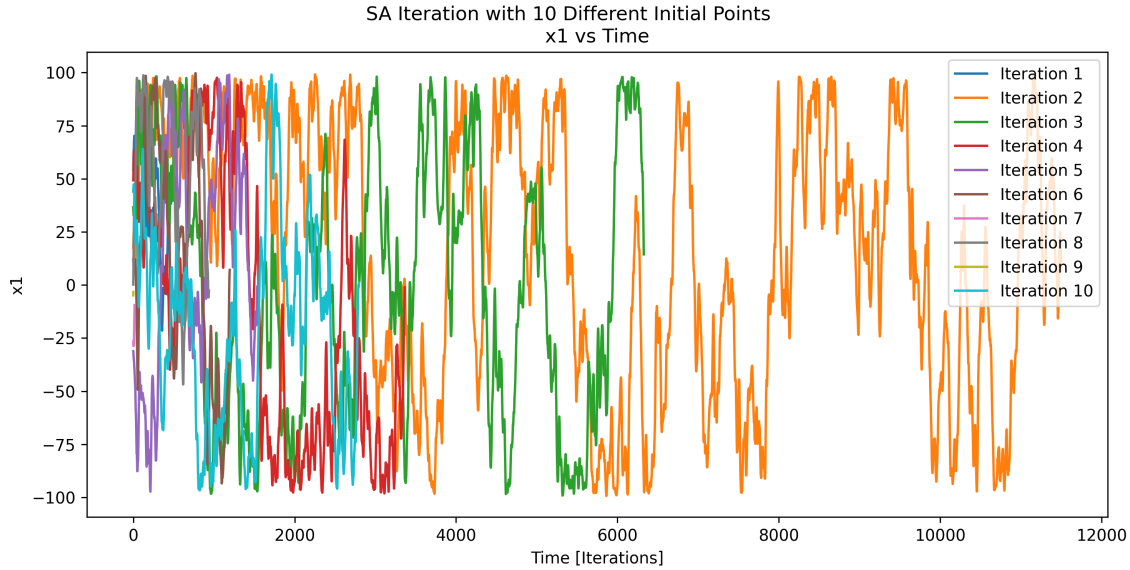
Pseudo Code from Simulated Annealing:

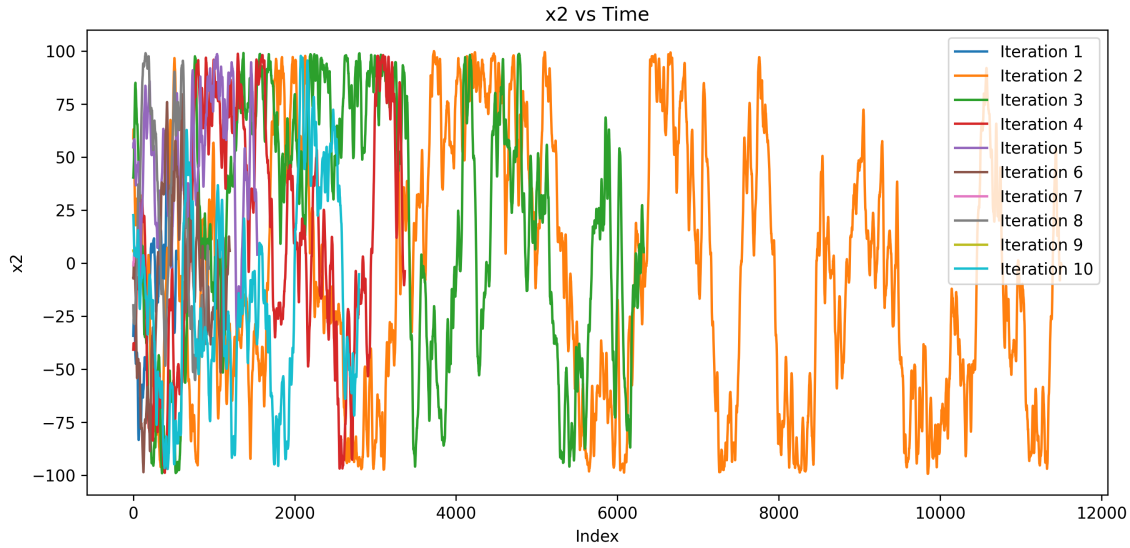
```
define simulated_annealing(temp_0, alpha, bounds, max_iterations):
  curr_x = random (x1, x2) values in bounds
  curr_cost = cost_function(curr_x)

  loop in max_iterations:
    new_x = neighbour_function(curr_x)
    new_cost = cost_function(curr_x)
    delta_cost = new_cost - curr_cost

    if delta_cost < 0:
      curr_x = new_x
      curr_cost = new_cost
    else:
      if random < probability_function(delta_cost, temp)
```

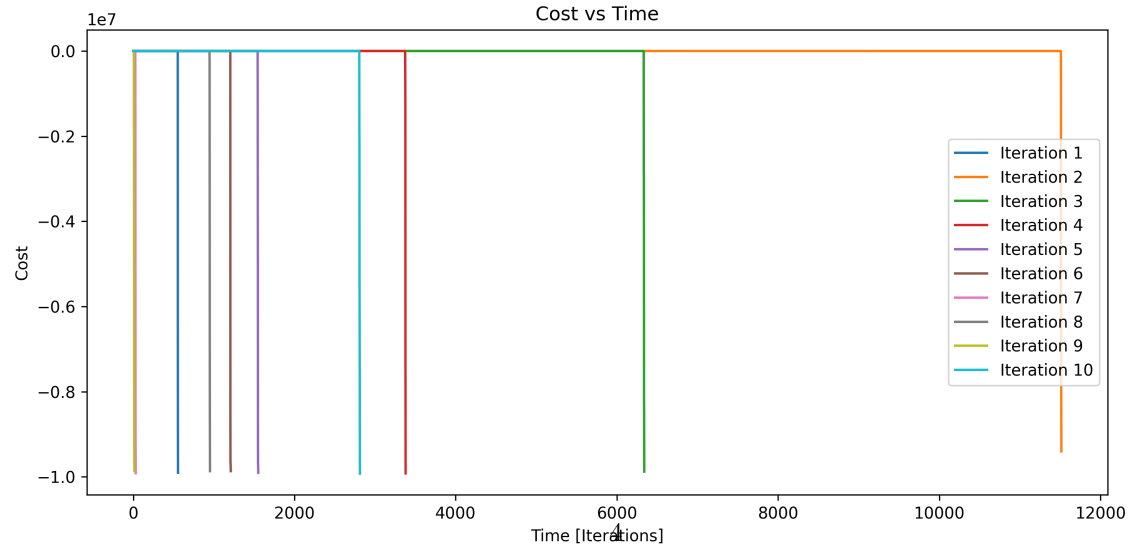
```
      curr_x = new_x
      curr_cost = new_cost
  temp = cool_function(temp, alpha)
return curr_x, curr_cost
```

3. Solution Profile for Different Variates

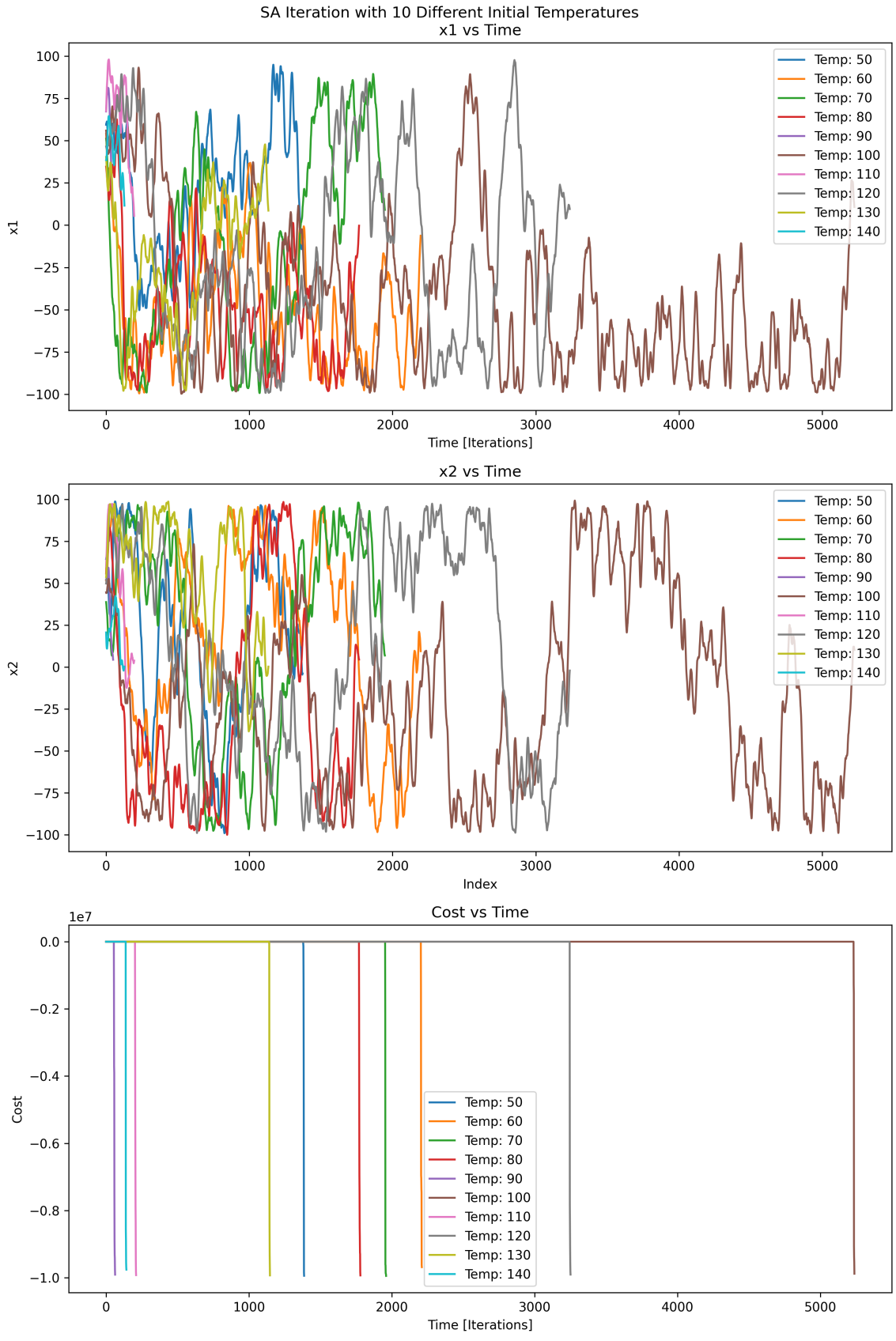
Experiment 1: 10 Different Initial Points



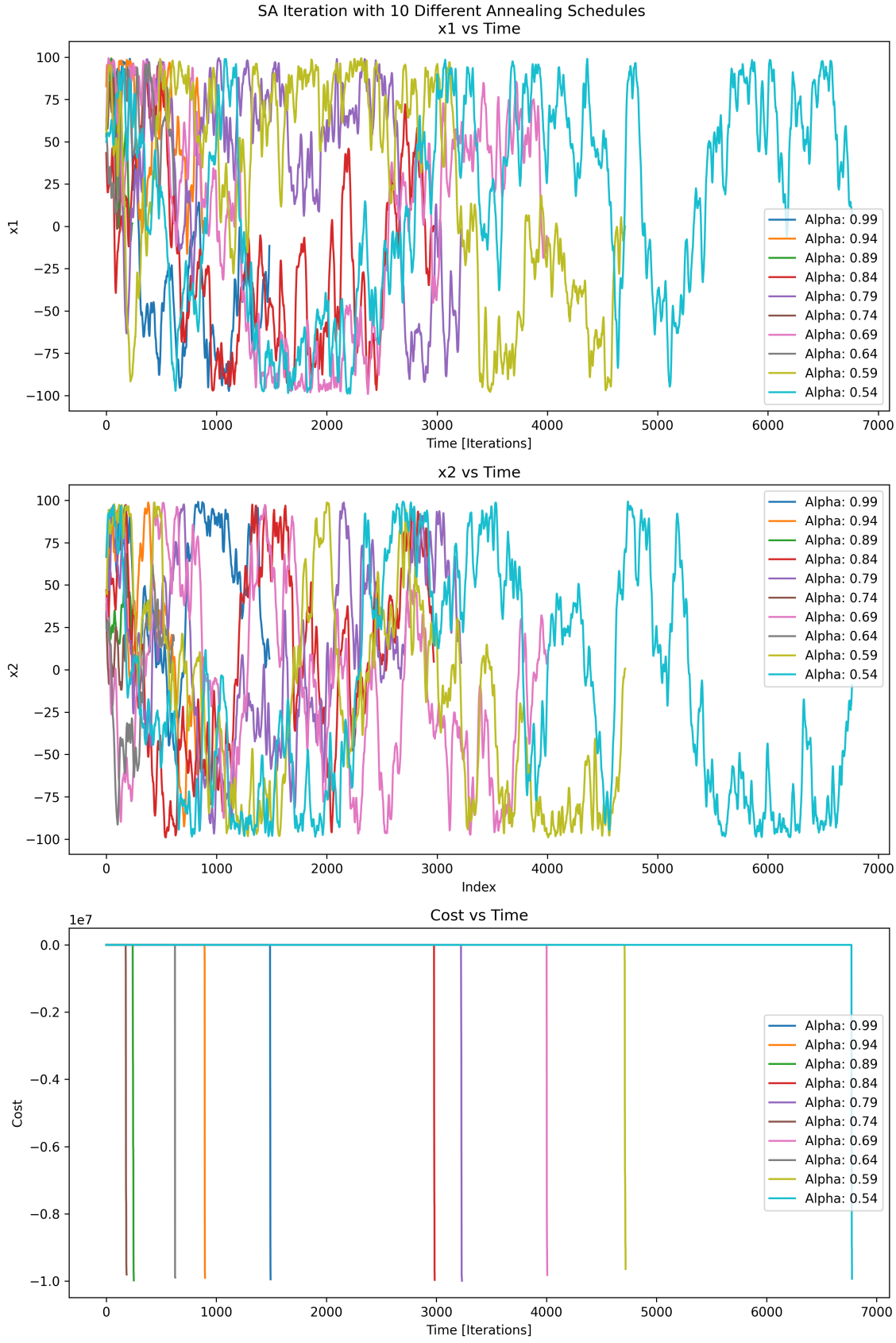




Experiment 2: 10 Different Initial Temperatures



Experiment 3: 10 Different Annealing Schedules



4. Observations

From experiment one, it can be seen that the algorithm is generally very good at quickly finding a solution to this optimization problem, regardless of the initial guess. This can be seen in the plot as 8/10 of the executions took less than 3500 iterations. The two outliers in the plot show that there is some range of initial guess that requires more iterations of the algorithm to reach the goal.

From experiment two, it appears that temperature values around 100 perform the best. These tests were performed with a constant alpha value, and thus it appears that temperature around 100 worked the best for this cooling.

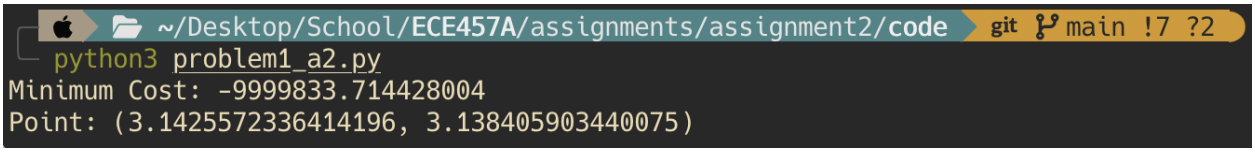
From experiment three, it can be seen that the lower values of alpha do not perform as well in finding a solution and take more iterations. With an alpha value too small, this decreases the temperature to rapidly until it approaches zero and causes the probability function to be undefined.

5. Performance of SA

Through the experimentation shown above, the best solution that was output from my program was a minimum value of approximately $x_1 = 3.142557$ and $x_2 = 3.138406$, which a cost of $c(x_1, x_2) = -9999833.714$. For this iteration, the initial points were chosen randomly, with the remaining settings of the `sim_annealing()` function as follows:

- $t_0 = 100$
- $\alpha = 0.99$

The output of the program with these settings during this iteration can be seen in the figure below.



These settings results in good solution due to a good cooling function and initial temperature. The geometric cooling function used for this algorithm results in the values being more likely to be accepted near the beginning of the execution and not accepted later in the execution. This allows the simulated annealing algorithm to be able so efficiently search the state space until it detects the minimum.

The time complexity of this program and can be described as the following:

- `easom()` performs one simple calculation and thus the space and time complexity are both $O(1)$
- `cost()` performs one calculation with x and therefore, has time and space complexity of $O(1)$
- `cool()` simply updates the temperature value and so time and space complexity are both $O(1)$
- `sim_annealing()` this algorithm contains one `for` loop which is dependent on the number of iterations and with each iteration, the cost and coordinates are stored in lists. Therefore, the time complexity is $O(i)$ and the space complexity is $O(2i) \implies O(i)$, where i is the number of iterations.

6. Code

```
from math import cos, exp, pi
from random import uniform, random
import matplotlib.pyplot as plt
import numpy as np


def easom(x1, x2):
    return -cos(x1)*cos(x2)*exp(-(x1 - pi)**2 - (x2 - pi)**2)

def cost(x1, x2, t):
    return easom(x1, x2) * 1E7

def cool(temp, alpha):
    return temp ** alpha

def sim_annealing(t_0, alpha, bounds, max_iterations, x = None):
    # Initial random guess within bounds
    if x is None:
        x1 = (random() - 0.5) * bounds[0][1]
        x2 = (random() - 0.5) * bounds[1][1]
    else:
        x1, x2 = x

    current_x = list()
    current_x.append((x1, x2))
    current_cost = list()
    current_cost.append(cost(x1, x2, t = "easom"))

    temp = t_0
    for i in range(max_iterations):
        # new x's in neighbourhood and ensure in bounds
        neighbourhood_factor = 10
        new_x1 = min(max(current_x[-1][0] + uniform(-1, 1) * neighbourhood_factor, bounds[0][0]), bounds
        new_x2 = min(max(current_x[-1][1] + uniform(-1, 1) * neighbourhood_factor, bounds[1][0]), bounds

        new_x = (new_x1, new_x2)
        new_cost = cost(new_x1, new_x2, t = "easom")

        delta_cost = new_cost - current_cost[-1]

        if temp < 1E-5:
            return current_x, current_cost

        # accept new x
        if delta_cost < 0:
            current_x.append(new_x)
            current_cost.append(new_cost)
        else:
            if random() < exp(-delta_cost/temp):
                current_x.append(new_x)
                current_cost.append(new_cost)
```

```
        temp = cool(temp, alpha)

    return current_x, current_cost

x, c = sim_annealing(t_0 = 100, alpha = 0.99, bounds = [(-100, 100), (-100, 100)], max_iterations = 200

print(f"Minimum Cost: {c[-1]}\nPoint: {x[-1]}")

# Moving average function for smoothing
def moving_average(data, window_size=10):
    return np.convolve(data, np.ones(window_size)/window_size, mode='valid')

'''
## Experiments
exp1_x = list()
exp1_cost = list()
for i in range(10):
    x, c = sim_annealing(t_0 = 100, alpha = 0.99 - (0.05*i), bounds = [(-100, 100), (-100, 100)], max_i
    exp1_x.append(x)
    exp1_cost.append(c)

window_size = 15
fig, (ax1, ax2, ax3) = plt.subplots(3, 1, figsize=(10, 15))

# Plot x1 vs index
for i, x_values in enumerate(exp1_x):
    x1_values = [x[0] for x in x_values]
    smoothed_x1 = moving_average(x1_values, window_size)
    ax1.plot(range(len(smoothed_x1)), smoothed_x1, label=f'Alpha: {round(0.99 - (0.05*i), 2)}')
ax1.set_title('x1 vs Time')
ax1.set_xlabel('Time [Iterations]')
ax1.set_ylabel('x1')
ax1.legend()

# Plot x2 vs index
for i, x_values in enumerate(exp1_x):
    x2_values = [x[1] for x in x_values]
    smoothed_x2 = moving_average(x2_values, window_size)
    ax2.plot(range(len(smoothed_x2)), smoothed_x2, label=f'Alpha: {round(0.99 - (0.05*i), 2)}')
ax2.set_title('x2 vs Time')
ax2.set_xlabel('Index')
ax2.set_ylabel('x2')
ax2.legend()

# Plot cost vs index
for i, cost_values in enumerate(exp1_cost):
    ax3.plot(range(len(cost_values)), cost_values, label=f'Alpha: {round(0.99 - (0.05*i), 2)}')
ax3.set_title('Cost vs Time')
ax3.set_xlabel('Time [Iterations]')
ax3.set_ylabel('Cost')
ax3.legend()

# Display the plots
fig.suptitle("SA Iteration with 10 Different Annealing Schedules")
```

```
plt.tight_layout()
plt.savefig('experiment3_plot.png', dpi=300)
plt.show()
'''
```

Question 2 Solution

1. Problem Description and Strategy

Problem Formulation

State Representation: For the Conga game, the state representation is the board describing how many stones are in each tile on the board and to which player the stones belong. This is represented in the program by two 2-dimensional 4x4 matrices where the player matrix contains either `"B"` for black stones, `"W"` for white pieces, and `None` for empty squares. For the stones matrix, each index contains the number of square on the corresponding tile, regardless of the colour.

Initial State: The initial state is the board configuration at the beginning of the game. This means that the player board has a `"B"` in the position $(1, 4)$ and a `"W"` in the $(4, 1)$ position. For the stones board, there are 10 stones in both the $(1, 4)$ and $(4, 1)$ positions.

$$\begin{bmatrix} B & - & - & - \\ - & - & - & - \\ - & - & - & - \\ - & - & - & W \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix}$$

Goal State: The goal state for this problem is any configuration in which a player does not have any legal moves. One such configuration is as follows:

$$\begin{bmatrix} B & W & - & - \\ W & W & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \begin{bmatrix} 10 & 3 & 0 & 0 \\ 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Actions: An action is defined as performing any legal move in Conga. This is defined in the assignment documentation, however, it involves removing all stones from a tile and moving in one of the 8 possible directions. Travel as far as possible in that direction and stop until you reach the edge of the board or a tile with enemy stones on it. Place 1 stone in each tile along that path, with all remaining stones being added to the last tile. Player turns alternate every action.

Cost: The cost defined for this problem in the number of legal moves available to white subtracted by the number of legal moves available for black. The means that positive cost values favour white and negative cost values favour black.

```
Cost = (# legal white moves) - (# legal black moves)
```

Pseudo Code

```
def minmax:
  if depth is 0 or game over:
    return evaluation

  if maximizing:
    max_eval = -inf
    moves = get moves
    for move in moves:
      make move
      eval = minmax(depth - 1, minimizing)
      alpha = max(alpha, eval)
      if beta <= alpha
        prune branch
```

```
    return max eval

  if minimizing:
    max_eval = inf
    moves = get moves
    for move in moves:
      make move
      eval = minmax(depth - 1, maximizing)
      beta = min(beta, eval)
      if beta <= alpha
        prune branch
    return max eval
```

2. Observations of Algorithm vs. Random Agent

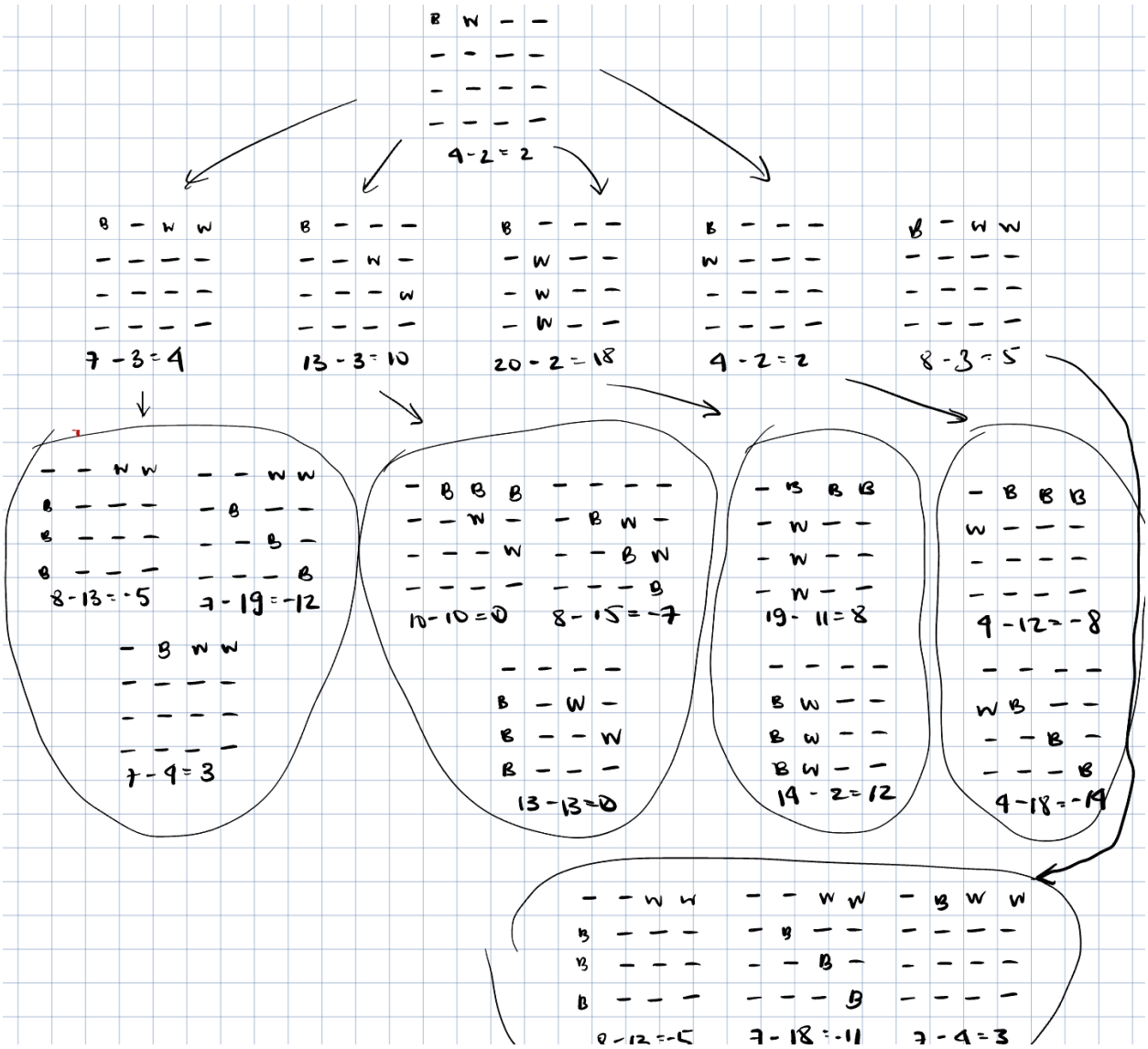
Against a random opponent, the minmax/alpha beta pruning opponent performed very well against the random opponent. Out of 100 games, 77 of the games were won by the AI, 2 were lost, and 21 of the game reached the maximum number of iterations set by the program.

3. Program Output

Here is an example of the program output, showing the last state of a game where black loses.

```
------BLACK-------

 -  -  -  W     0  0  0  1
 -  -  W  W     0  0  2  3
 W  -  W  -     1  0  1  0
 -  -  W  B     0  0  2  10

------WHITE-------

 -  -  -  W     0  0  0  1
 -  -  -  W     0  0  0  3
 W  -  W  W     1  0  1  2
 -  -  W  B     0  0  2  10
```

Here is the drawing work showing the branching posibilities of moves with white as the maximizing player.



4. Discussion on Time and Memory Complexity

The time complexity of this program with only the minmax algorithm is $O(b^d)$ and with alpha-beta pruning is $O(b^{d/2})$, where b is the branching factor and d is the depth of the search.

For the space complexity, the algorithm must store the game state for every possible move at every depth, and thus the space complexity for minmax and alpha-beta pruning is $O(d)$.

Question 3 Solution

1. Optimization Formulation

State Representation: The state representation of this optimization problem is a list containing the 20 departments involved in the problem in the order they place in the locations (going down the rows from left to right). For example,

$$S = 15, 5, 3, 7, 12, 11, 20, 2, 19, 1, 17, 16, 4, 14, 6, 13, 9, 8, 11, 18$$

Initial State: The initial state for this optimization problem is any random configuration of department.

Goal State: The goal state is a configuration of department placements which produces a minimum cost of 2570 in the case of this implementation.

Actions: The actions that can be performed in this optimization involve rearranging which department are place in each location. This can involve swapping two departments, swapping multiple departments, or completely rearranging the locations of departments.

Cost: The cost is a function of the distance of any two locations and the flow of one department with another.

$$cost = (flow) \cdot (rectilineardistance)$$

2. Performance Evaluation

Experiment 1: Different Initial Solutions

```
Iteration 0:
Initial config: [3, 4, 6, 9, 5, 7, 14, 0, 2, 16, 13, 15, 10, 18, 19, 12, 17, 1, 11, 8]
Best configuration found: [17, 1, 13, 2, 8, 18, 14, 11, 9, 15, 3, 19, 7, 10, 12, 16, 6, 4, 0, 5]
Cost: 2574.0

Iteration 1:
Initial config: [11, 3, 16, 8, 7, 6, 2, 9, 1, 12, 13, 4, 18, 0, 5, 10, 19, 17, 14, 15]
Best configuration found: [16, 4, 6, 0, 5, 18, 14, 19, 7, 12, 3, 1, 11, 10, 15, 17, 13, 9, 2, 8]
Cost: 2570.0

Iteration 2:
Initial config: [19, 5, 14, 0, 11, 13, 9, 8, 7, 10, 12, 15, 16, 3, 6, 1, 18, 17, 2, 4]
Best configuration found: [8, 2, 9, 13, 17, 15, 10, 11, 1, 3, 12, 7, 19, 14, 18, 5, 0, 6, 4, 16]
Cost: 2570.0

Iteration 3:
Initial config: [9, 2, 14, 19, 0, 10, 17, 6, 5, 4, 15, 11, 18, 1, 8, 16, 13, 7, 12, 3]
Best configuration found: [5, 0, 4, 6, 16, 12, 10, 7, 19, 3, 15, 9, 11, 14, 18, 8, 2, 13, 1, 17]
Cost: 2574.0

Iteration 4:
Initial config: [0, 18, 5, 3, 6, 2, 19, 14, 13, 11, 8, 9, 15, 10, 17, 4, 7, 12, 16, 1]
Best configuration found: [16, 4, 6, 0, 5, 18, 14, 19, 7, 12, 3, 1, 11, 10, 15, 17, 13, 9, 2, 8]
Cost: 2570.0

Iteration 5:
Initial config: [8, 13, 14, 2, 9, 0, 18, 15, 17, 7, 5, 3, 10, 4, 6, 11, 19, 12, 1, 16]
Best configuration found: [8, 2, 9, 13, 17, 15, 10, 11, 1, 3, 12, 7, 19, 14, 18, 5, 0, 6, 4, 16]
Cost: 2570.0
```

```
Iteration 6:
Initial config: [14, 4, 7, 11, 17, 0, 5, 19, 1, 16, 12, 6, 9, 18, 8, 15, 3, 2, 13, 10]
Best configuration found: [16, 6, 4, 0, 5, 3, 19, 7, 10, 12, 18, 14, 11, 9, 15, 17, 1, 13, 2, 8]
Cost: 2574.0

Iteration 7:
Initial config: [0, 7, 1, 6, 14, 5, 10, 12, 19, 17, 11, 9, 8, 4, 2, 18, 13, 16, 15, 3]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 8:
Initial config: [6, 9, 14, 1, 19, 2, 18, 17, 0, 5, 15, 7, 8, 16, 3, 4, 11, 10, 13, 12]
Best configuration found: [17, 13, 9, 2, 8, 3, 1, 11, 10, 15, 18, 14, 19, 7, 12, 16, 4, 6, 0, 5]
Cost: 2570.0

Iteration 9:
Initial config: [1, 3, 9, 15, 13, 11, 6, 19, 18, 14, 10, 12, 4, 5, 0, 8, 16, 17, 7, 2]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 10:
Initial config: [12, 13, 8, 19, 0, 18, 5, 6, 4, 2, 11, 14, 16, 3, 7, 1, 17, 10, 15, 9]
Best configuration found: [17, 13, 9, 2, 8, 3, 1, 11, 10, 15, 18, 14, 19, 7, 12, 16, 4, 6, 0, 5]
Cost: 2570.0

Iteration 11:
Initial config: [18, 7, 9, 16, 3, 10, 8, 5, 2, 0, 19, 15, 6, 13, 4, 1, 17, 14, 11, 12]
Best configuration found: [2, 17, 9, 0, 5, 13, 1, 11, 6, 4, 8, 14, 7, 19, 12, 15, 18, 10, 3, 16]
Cost: 2588.0

Iteration 12:
Initial config: [10, 11, 6, 12, 2, 5, 4, 14, 7, 13, 19, 8, 16, 0, 9, 15, 3, 17, 18, 1]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 13:
Initial config: [10, 19, 3, 0, 6, 1, 2, 14, 18, 9, 15, 13, 11, 17, 5, 7, 8, 16, 12, 4]
Best configuration found: [16, 4, 6, 0, 5, 18, 14, 19, 7, 12, 3, 1, 11, 10, 15, 17, 13, 9, 2, 8]
Cost: 2570.0

Iteration 14:
Initial config: [8, 10, 18, 7, 11, 15, 1, 0, 12, 19, 4, 6, 5, 17, 2, 14, 9, 13, 16, 3]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 15:
Initial config: [11, 6, 15, 5, 13, 3, 0, 7, 10, 18, 16, 17, 1, 14, 4, 12, 19, 2, 9, 8]
Best configuration found: [5, 0, 9, 17, 2, 4, 6, 11, 1, 13, 12, 19, 7, 14, 8, 16, 3, 10, 18, 15]
Cost: 2588.0

Iteration 16:
Initial config: [8, 17, 19, 10, 12, 2, 4, 14, 3, 5, 7, 6, 13, 16, 18, 9, 15, 11, 1, 0]
Best configuration found: [5, 0, 6, 19, 16, 12, 4, 10, 7, 3, 15, 9, 11, 14, 18, 8, 2, 13, 1, 17]
```

```
Cost: 2580.0

Iteration 17:
Initial config: [11, 6, 10, 14, 13, 12, 19, 3, 15, 2, 4, 1, 9, 8, 7, 16, 5, 18, 0, 17]
Best configuration found: [8, 2, 9, 13, 17, 15, 10, 11, 1, 3, 12, 7, 19, 14, 18, 5, 0, 6, 4, 16]
Cost: 2570.0

Iteration 18:
Initial config: [10, 2, 15, 8, 13, 18, 5, 7, 17, 6, 1, 4, 9, 0, 14, 11, 19, 12, 3, 16]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 19:
Initial config: [5, 8, 18, 16, 13, 2, 10, 14, 7, 12, 3, 1, 0, 4, 15, 6, 19, 11, 9, 17]
Best configuration found: [16, 6, 4, 0, 5, 3, 19, 7, 10, 12, 18, 14, 11, 9, 15, 17, 1, 13, 2, 8]
Cost: 2574.0
```

Experiment 2: Different Tabu Sizes

Tabu List Size = 2

```
Iteration 0:
Initial config: [6, 5, 14, 10, 17, 2, 15, 9, 7, 8, 11, 12, 16, 0, 4, 13, 1, 18, 3, 19]
Best configuration found: [8, 2, 13, 1, 17, 15, 9, 11, 14, 18, 12, 10, 7, 19, 3, 5, 0, 4, 6, 16]
Cost: 2574.0

Iteration 1:
Initial config: [18, 3, 2, 7, 9, 0, 10, 17, 6, 19, 8, 4, 12, 11, 16, 5, 14, 1, 13, 15]
Best configuration found: [5, 0, 6, 19, 16, 12, 4, 10, 7, 3, 15, 9, 11, 14, 18, 8, 2, 13, 1, 17]
Cost: 2580.0
```

Tabu List Size = 8

```
Iteration 0:
Initial config: [3, 2, 17, 0, 10, 11, 4, 1, 6, 14, 19, 12, 8, 13, 5, 9, 15, 18, 16, 7]
Best configuration found: [15, 0, 10, 7, 16, 12, 4, 6, 19, 3, 5, 9, 11, 14, 18, 8, 2, 13, 1, 17]
Cost: 2574.0

Iteration 1:
Initial config: [9, 7, 11, 19, 0, 1, 5, 16, 15, 13, 8, 10, 12, 14, 4, 17, 2, 18, 6, 3]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0
```

From the experiment results, it appears that the test with the larger tabu list size performed better than the test with the smaller one. Not only did the larger tabu size find better solutions, it was also able to find an optimal solution in one of the two executions.

Experiment 3: Dynamic Tabu Size

```
Iteration 0:
Initial config: [19, 7, 11, 2, 9, 5, 16, 18, 8, 13, 4, 15, 0, 10, 3, 17, 1, 14, 12, 6]
Best configuration found: [16, 3, 10, 18, 15, 12, 19, 7, 14, 8, 4, 6, 11, 1, 13, 5, 0, 9, 17, 2]
Cost: 2588.0

Iteration 1:
Initial config: [7, 6, 16, 11, 14, 0, 13, 18, 3, 17, 1, 2, 15, 8, 5, 4, 12, 9, 10, 19]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
```

```
Cost: 2570.0

Iteration 2:
Initial config: [10, 15, 0, 12, 4, 6, 5, 9, 13, 18, 8, 19, 7, 16, 11, 1, 14, 2, 17, 3]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 3:
Initial config: [6, 11, 10, 2, 15, 18, 4, 0, 5, 19, 16, 7, 8, 3, 13, 9, 1, 14, 12, 17]
Best configuration found: [8, 2, 9, 13, 17, 15, 10, 11, 1, 3, 12, 7, 19, 14, 18, 5, 0, 6, 4, 16]
Cost: 2570.0

Iteration 4:
Initial config: [6, 16, 8, 18, 4, 17, 1, 3, 7, 0, 14, 2, 11, 19, 9, 13, 10, 15, 5, 12]
Best configuration found: [16, 4, 6, 0, 5, 18, 14, 19, 7, 12, 3, 1, 11, 10, 15, 17, 13, 9, 2, 8]
Cost: 2570.0
```

For this experiment, I chose a dynamic tabu list size in the range [1, 10] and chose to update the size every 2000 iterations. The results were very positive, with 4/5 executions finding an optimal solution to the problem.

Experiment 4: Adding Aspiration Criteria

```
Iteration 0:
Initial config: [7, 3, 4, 15, 10, 11, 2, 16, 14, 18, 13, 9, 19, 8, 0, 17, 5, 6, 1, 12]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 1:
Initial config: [4, 7, 8, 2, 17, 10, 1, 14, 6, 9, 19, 11, 15, 3, 13, 12, 16, 18, 5, 0]
Best configuration found: [17, 13, 9, 2, 8, 3, 1, 11, 10, 15, 18, 14, 19, 7, 12, 16, 4, 6, 0, 5]
Cost: 2570.0

Iteration 2:
Initial config: [18, 3, 0, 11, 7, 15, 8, 19, 6, 10, 5, 1, 9, 14, 16, 12, 13, 2, 4, 17]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 3:
Initial config: [10, 15, 9, 2, 0, 3, 18, 13, 14, 12, 16, 6, 17, 19, 4, 7, 5, 11, 8, 1]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 4:
Initial config: [1, 12, 19, 14, 16, 15, 5, 18, 11, 0, 8, 6, 4, 9, 17, 3, 13, 2, 7, 10]
Best configuration found: [17, 13, 9, 2, 8, 3, 1, 11, 10, 15, 18, 14, 19, 7, 12, 16, 4, 6, 0, 5]
Cost: 2570.0
```

The aspiration criteria added in this experiment involves not adding moves to the tabu list that result in a configuration which has a cost equal to the current best cost. This experiment performed very well, with 5/5 of the tests finding an optimal solution to the problem.

Experiment 5: Frequency-Based Tabu List

```
Iteration 0:
Initial config: [19, 17, 7, 9, 5, 2, 18, 0, 14, 10, 16, 8, 13, 6, 1, 3, 15, 11, 12, 4]
```

```
Best configuration found: [17, 13, 9, 2, 8, 3, 1, 11, 10, 15, 18, 14, 19, 7, 12, 16, 4, 6, 0, 5]
Cost: 2570.0

Iteration 1:
Initial config: [5, 6, 10, 0, 1, 2, 4, 13, 7, 8, 3, 18, 9, 11, 16, 12, 19, 15, 17, 14]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 2:
Initial config: [8, 19, 9, 2, 14, 13, 0, 1, 4, 17, 18, 3, 7, 10, 11, 15, 6, 16, 12, 5]
Best configuration found: [16, 4, 6, 0, 5, 18, 14, 19, 7, 12, 3, 1, 11, 10, 15, 17, 13, 9, 2, 8]
Cost: 2570.0

Iteration 3:
Initial config: [4, 13, 14, 18, 12, 11, 5, 16, 19, 6, 8, 17, 3, 0, 15, 9, 2, 7, 10, 1]
Best configuration found: [5, 0, 6, 4, 16, 12, 7, 19, 14, 18, 15, 10, 11, 1, 3, 8, 2, 9, 13, 17]
Cost: 2570.0

Iteration 4:
Initial config: [18, 12, 19, 1, 2, 17, 7, 16, 9, 0, 8, 15, 6, 14, 11, 5, 3, 10, 4, 13]
Best configuration found: [16, 6, 4, 0, 5, 3, 19, 7, 10, 12, 18, 14, 11, 9, 15, 17, 1, 13, 2, 8]
Cost: 2574.0
```

For this experiment, I added a frequency based tabu list where if the best configuration is the best for more than 250 iterations, then the configuration is added to the tabu list. This hopes to avoid the algorithm getting stuck in local minima. This method performed quite well with 4/5 test runs achieving an optimal solution.

Time and Memory Complexity

For Tabu Search, the algorithm's time complexity is dominated by the neighbour generation function with is a function of $O(n^2)$. With each neighbour, the cost is calculated which introduces another complexity of $O(n^2)$, leading to an overall time complexity of $O(n^4)$. However, this is also affected by how the generation of the neighbours in performed. If only a constant number of neighbours is generated for each configuration, this bring the time complexity down to $O(n^2)$, since the functions are executed a constant number of times, leaving only the cost contributing to this complexity.

For the space complexity, the main aspect adding to the complexity is storing the generated neighbours of a configuration. the number of generated neighbours is n^2, with each neighbour containing n configurations, leading to a space complexity of $O(n^3)$. The tabu list is also a factor to consider, however, since this list is either a fixed value or dynamic with some complexity $O(n)$, this is dominated by the neighbour generation. Therefore, the overall space complexity is $O(n^3)$. This complexity is also dependent on the number of neighbours generated, similar to the time complexity.