Random Variate Generation

CMS 380 Simulation and Stochastic Modeling

The Inverse CDF Method

Suppose that you have implemented a good quality PRNG and can use it to generate those uniformly distributed random values in $[0, 1)$. What do you do when you want, say, an exponentially distributed random variable in a simulation?

The answer: you need to *transform* a uniformly distributed random number (which you can get from your PRNG) into an exponentially distributed random number. This is the problem of *random variate generation*.

Recall that the CDF of the exponential distribution is

$$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$$

$F_X(x)$ is a probability, so it must be between 0 and 1.

Here's the strategy:

- Let u be uniform random value from the PRNG.
- Set $F_X(x) = u$ and solve for x.
- The resulting formula will give you a way to generate random values drawn from the distribution with F_X as its CDF.

Applying this strategy to the exponential CDF yields:

$$\begin{aligned} 1 - e^{-\lambda x} &= u \\ e^{-\lambda x} &= 1 - u \end{aligned}$$

Take the natural logarithm of both sides to undo the exponentiation:

$$-\lambda x = \ln(1 - u)$$

The final formula is

$$x = -\frac{\ln(1 - u)}{\lambda}$$

This can be simplified a little bit: if u is uniformly distributed in $[0, 1)$, then so is $1 - u$, so it's sufficient to use

$$x = -\frac{\ln u}{\lambda}$$

Variates calculated by this method will be exponentially distributed with parameter λ. Here's some Python code.

```
from math import log
from random import random

def randExp(mu):
    """Generate an exponential random variate with parameter mu.

        Input:  mu  the parameter of the distribution
        Output: a random variate x ~ exp(mu)

        Tip: use mu as the parameter because lambda is a Python
             keyword (it's used to create anonymous functions)
    """

    return -log(random()) / mu
```

That's it. Python's `log` method calculates the natural logarithm if you use it with only one input argument (it can take the base of the log as an optional second argument).

The Rayleigh distribution *is a continuous distribution that has applications to physics and aerodynamics. It has one parameter,* σ*, which is called the scale. It's CDF is*

$$F_X(x) = 1 - e^{\frac{-x^2}{2\sigma^2}}$$

Use the Inverse CDF method to derive a function for generating Rayleigh distributed random variates.

Set $F_X(x) = u$ and solve for x:

$$\begin{aligned}
1 - e^{\frac{-x^2}{2\sigma^2}} &= u \\
e^{\frac{-x^2}{2\sigma^2}} &= 1 - u \\
\frac{-x^2}{2\sigma^2} &= \ln(1 - u) \\
x^2 &= -2\sigma^2 \ln(1 - u) \\
x &= \sqrt{-2\sigma^2 \ln(1 - u)}
\end{aligned}$$

The last equation can be simplified a little bit by again replacing $1 - u$ with an equivalent u and bringing σ outside the square root:

$$x = \sigma\sqrt{-2\ln u}$$

Note that it isn't a problem to have -2 under the square root. In fact, because $u < 1$, $\ln u$ must be negative, so multiplying by -2 makes the entire term under the radical positive.