HW2

Tuesday, January 30, 2024 11:19 AM

JACOB CHILD

ME EN 541 – HW 2
Handed out 25 January 2024
Due 1 February 2024

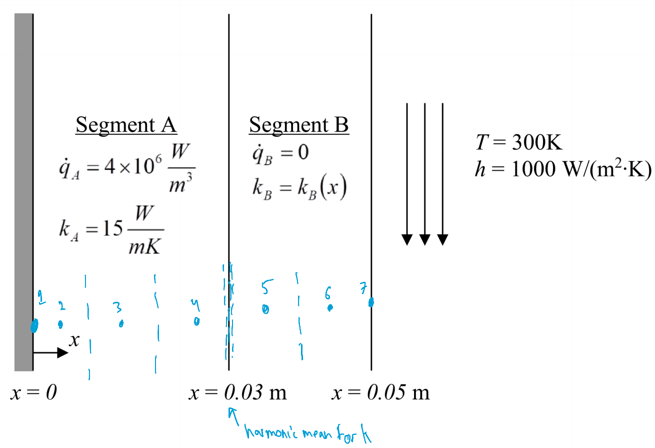
Problem	Points	Topic
1	5	Control volume discretization, Practice A
2	5	Control volume discretization, Practice B
3	20	Control volume by hand, constant *k*
4	20	Control volume code, constant *k*
5	30	Control volume code, variable *k*

1. Derive the discretized form of the convective heat transfer boundary condition for the left boundary of a 1-D steady-state conduction problem using the control volume approach and grid deployment Practice A. Make the standard assumption of a linearized source term, and state your final results in terms of the general FV discretized equation (i.e., explicitly state the coefficients a_w, a_e, a_P, and b).

2. Repeat the previous problem for the right boundary and using grid deployment Practice B.

Problems 3-5:
Consider one-dimensional conduction in a plane composite wall, as in the following figure. The left side is perfectly insulated. Inside segment A the thermal conductivity is constant and uniform heat generation exists. Inside segment B the thermal conductivity may be a function of *x*, but no generation occurs. The right surface is in contact with a moving fluid. Write a program to calculate the steady-state temperature distribution using the CV method, accounting for non-uniform thermal conductivity and heat generation. Note the following where applicable:

a) Use grid deployment Practice B and equal-sized interior volumes. It is important to have a control surface at the interface between wall segments A and B.
b) Properly evaluate the interface thermal conductivity using the harmonic mean.
c) Solve the linear system using any subroutine of your choosing.
d) Submit a copy of your source code with your HW.



3. Assume the thermal conductivity of segment B is constant, $k_B = 60$ W/(m·K).
   a) Beginning with the governing equation, develop, by hand, the system of equations that corresponds to this problem using the control volume approach with 5 equally-spaced control volumes (i.e., there will be 7 nodes, including the left and right boundaries). Remember to use Practice B. Express your answer in the form **AT** = **b**, where **A** is the matrix of a coefficients, **T** is the temperature vector, and **b** is the vector of b values at each node.
   b) Solve, by any method, the system of equations. Report T for each node.

4. Assume the thermal conductivity of segment B is constant, $k_B = 60$ W/(m·K).
   a) Write a code to solve for the temperature at each note using 5 equally-spaced control volumes. Refer to the previous hand-worked problem to verify that your code is set up correctly. Plot the temperature profile and report the temperatures at both boundaries.
   b) Repeat using 20 equally-spaced control volumes. Is this result what you expected?

5. Assume the thermal conductivity in segment B varies as $k_B(x) = 137e^{25x-2}$ W/(m·K).
   a) Modify your code to find the temperature for this scenario. Plot the temperature profile and report the temperatures at both boundaries using 5 equal-spaced control volumes.
   b) Repeat using 20 equally-spaced control volumes.
   c) Plot the predicted maximum temperature in the wall (in Kelvin) as a function of the number of control volumes (i.e., perform a grid-refinement study). What is the "grid-converged" maximum temperature to 3 decimal places (i.e., 0.001), and how many control volumes does it take to obtain this result?

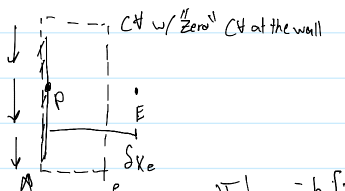
1. Derive the discretized form of the convective heat transfer boundary condition for the left boundary of a 1-D steady-state conduction problem using the control volume approach and grid deployment Practice A. Make the standard assumption of a linearized source term, and state your final results in terms of the general FV discretized equation (i.e., explicitly state the coefficients a_W, a_E, a_P, and b).

Summary: Find the general FV discretized eqn

Assumptions: Convective BC, Steady state, linearized source term

Practice A:

Nodes First



$$\int_0^e \frac{d}{dx}\left(k\frac{dT}{dx}\right) + \int_0^e S\,dx = 0\,dx$$

Convective BC $= -k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$

$$k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \int S\,dx$$

$\rightarrow \bar{S}\Delta x_P$

$$\text{Convective BC} = -k\frac{\partial T}{\partial x}\Big|_{x=0} = h\left[T_\infty - T(0,t)\right]$$

$$k_e\frac{T_E - T_P}{\delta x_e} + h\left(T_\infty - T_0\right) + \left(S_C + S_P T_P\right)\Delta x_P = 0$$

Generalized form $a_P T_P = a_W T_W + a_E T_E + b$

$$\frac{k_e T_E}{\delta x_e} - \frac{k_e T_P}{\delta x_e} + h\left(T_\infty - T_0\right) + S_e \Delta x_P + S_P T_P \Delta x_P = 0$$

$$\rightarrow \left(\frac{k_e}{\delta x_e} - S_P \Delta x_P\right) T_P = h\left(T_\infty - T_0\right) + \frac{k_e T_E}{\delta x_e} + S_e \Delta x_P$$

$a_P = \frac{k_e}{\delta x_e} - S_P \Delta x_P$ $\qquad$ $b = h(T_\infty - T_0) + S_e \Delta x_P$

$a_W = 0$ $\qquad$ $\boxed{a_P T_P = a_E T_E + b}$

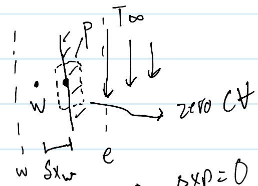
$a_E = \frac{k_e}{\delta x_e}$

2. Repeat the previous problem for the right boundary and using grid deployment Practice B.

Summary: Find the genera FV discretized eqn

Assumptions: Convective BC, steady state, linearized source term

Assuming also convection: RHB convec = $-k\frac{dT}{dx} = h(T_B - T_\infty)$



$$\int_e^w d\left(k\frac{dT}{dx}\right) + \int_e^w S\,dx = 0$$

$\Delta x_P = 0$ because of 0 CV

is this direction right?

nope, go e→w

$$k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \cancel{(S_C + S_P T_P)\Delta x_P}^{\,0} = 0$$

$$-h\left(T_P - T_\infty\right) - k_w\frac{T_P - T_W}{\delta x} = 0 \qquad a_P T_P = a_W T_W + a_E T_E + b$$

$k_w = k_P$ @ zero width CV

$$-hT_P + hT_\infty - \frac{k_w T_P}{\delta x_w} + \frac{k_w T_W}{\delta x_w}$$

$$\left(\frac{k_w}{\delta x_w} + h\right) T_P = hT_\infty + \frac{k_w}{\delta x_w} T_W$$

$a_P = \frac{k_w}{\delta x} + h$ $\qquad$ $a_E = 0$

$b = hT_\infty$

$a_W = \frac{k_w}{\delta x_w}$

$\boxed{a_P T_P = a_W T_W + b}$

Problems 3-5:
Consider one-dimensional conduction in a plane composite wall, as in the following figure. The left side is perfectly insulated. Inside segment A the thermal conductivity is constant and uniform heat generation exists. Inside segment B the thermal conductivity may be a function of x, but no generation occurs. The right surface is in contact with a moving fluid. Write a program to calculate the steady-state temperature distribution using the CV method, accounting for non-uniform thermal conductivity and heat generation. Note the following where applicable:

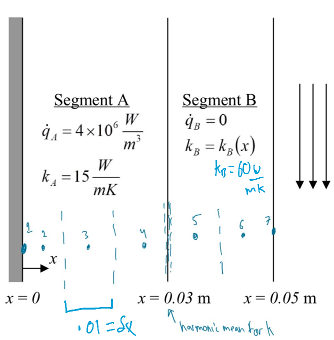
a) Use grid deployment Practice B and equal-sized interior volumes. It is important to have a control surface at the interface between wall segments A and B.
b) Properly evaluate the interface thermal conductivity using the harmonic mean.
c) Solve the linear system using any subroutine of your choosing.
d) Submit a copy of your source code with your HW.

Summary: with the given conditions solve for T at each node.

Assumptions: Steady state, the harmonic mean describes k @ $x = .03m$, insulated on the left $\frac{dT}{dx}\Big|_{x=0} = 0$

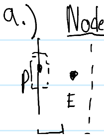
Approach: Start at Node 1 w/ BC and work my way right

remember segment A has $\dot{q}$ term



3. Assume the thermal conductivity of segment B is constant, $k_B = 60$ W/(m·K).
   a) Beginning with the governing equation, develop, by hand, the system of equations that corresponds to this problem using the control volume approach with 5 equally-spaced control volumes (i.e., there will be 7 nodes, including the left and right boundaries). Remember to use Practice B. Express your answer in the form **AT** = **b**, where **A** is the matrix of a coefficients, **T** is the temperature vector, and **b** is the vector of b values at each node.
   b) Solve, by any method, the system of equations. Report T for each node.

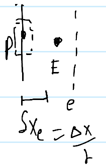
a.) Node 1 zero CV



$$\int_w^e d\left(\frac{k dT}{dx}\right) + \int_w^e S\,dx \rightarrow \frac{k dT}{dx}\Big|_e - \frac{k dT}{dx}\Big|_w + \cancel{S\Delta x_P}^{\,0\ \text{zero CV}} = 0 \qquad a_P T_P = a_W T_W + a_E T_E + b$$

generalized

$k_P = k_A = k_1$ $\qquad$ $\delta x_e = \frac{\Delta x}{2}$



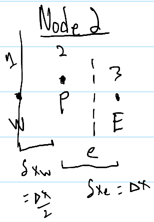
$$\int_w^e \frac{d}{dx}\left(k\frac{dT}{dx}\right) + \int_w^e S\,dx \rightarrow k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \cancel{S\Delta x_p} = 0 \qquad a_pT_p = a_wT_w + a_ET_E + b$$

$$k_p\frac{T_E - T_p}{\delta x_e} - \underset{\uparrow BC}{0} = 0 \qquad \text{generalized } \frac{k_pT_E}{\delta x_e} = \frac{k_pT_p}{\delta x_e}$$

$$k_p = k_A = k_1 \qquad a_p = \frac{k_1}{\delta x_e} \qquad a_E = \frac{k_1}{\delta x_e} \qquad \delta x_e = \frac{\Delta x}{2}$$

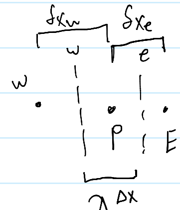
$$b = 0 \quad a_w = 0$$

Node 2



$$\int_w^e \left(k\frac{dT}{dx}\right) + \int_w^e S\,dx \rightarrow k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \underbrace{(S_c + S_pT_p)\Delta x_p}_{\dot q_A \Delta x}$$

$$k_e\frac{T_e - T_p}{\delta x_e} - k_w\frac{T_p - T_w}{\delta x/2} + \dot q_A\Delta x \rightarrow \left(\frac{k_e}{\delta x_e} + \frac{2k_w}{\delta x}\right)T_p = \frac{k_eT_e}{\delta x_e} + \frac{2k_wT_w}{\delta x} + \dot q_A\Delta x$$



$$a_p = \frac{k_e}{\delta x_e} + 2\frac{k_w}{\delta x}$$

$$a_E = \frac{k_e}{\delta x_e} \qquad b = \dot q_A\Delta x$$

$$a_W = \frac{2k_w}{\delta x}$$

Interior Nodes, 3-5

Generalized $k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \dot q_A\Delta x = 0$

$$k_e\frac{T_E - T_p}{\delta x} - k_w\frac{T_p - T_w}{\delta x} + \dot q_A\Delta x = 0 \rightarrow \left(\frac{k_e}{\delta x} + \frac{k_w}{\delta x}\right)T_p = \frac{k_eT_E}{\delta x} + \frac{k_wT_W}{\delta x} + \dot q_A\Delta x = 0$$

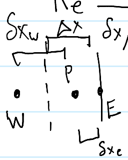
$$a_p = \frac{k_e}{\delta x} + \frac{k_w}{\delta x} \qquad a_W = \frac{k_w}{\delta x} \qquad a_E = \frac{k_e}{\delta x} \qquad b = \dot q_A\Delta x$$

* in material B $\dot q = 0$ @ node 5

Node 6

$$k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + \cancel{\dot q_B}^{\,0}\Delta x$$

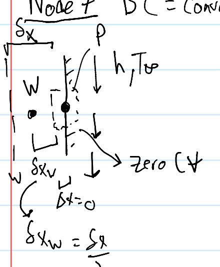
* use the harmonic mean every where for k_e & k_w remember $\delta x_e^- \neq \delta x_e$ in our case $\delta x_e^- = \frac{\delta x_e}{2}$



$$k_e\frac{T_e - T_p}{\delta x/2} - k_w\frac{T_w - T_p}{\delta x} = 0 \rightarrow \left(\frac{2k_e}{\delta x} + \frac{k_w}{\delta x}\right)T_p = \frac{2k_eT_E}{\delta x} + \frac{k_wT_w}{\delta x}$$

$$a_p = \frac{2k_e}{\delta x} + \frac{k_w}{\delta x} \qquad a_E = \frac{2k_e}{\delta x} \qquad a_W = \frac{k_w}{\delta x} \qquad b = 0$$

Node 7 BC = Convection



$$k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + 0 = 0$$

RHB Convec: $-k\frac{dT}{dx} = h(T_p - T_\infty)$

$$-h(T_p - T_\infty) - k_w\frac{T_p - T_w}{\delta x/2} = 0 \rightarrow \left(\frac{2k_w}{\delta x} + h\right)T_p = \frac{2k_wT_w}{\delta x} - hT_\infty$$

$$-hT_p + hT_\infty - \frac{k_wT_p}{\delta x/2} + \frac{k_wT_w}{\delta x/2}$$

$$a_p = \frac{2k_w}{\delta x} + h \qquad a_w = \frac{2k_w}{\delta x} \qquad b = hT_\infty$$

$$\begin{array}{c} 1\\2\\3\\4\\5\\6 \end{array}\begin{bmatrix} a_{p1} & -a_{E1} & 0 & 0 & 0 & 0 & 0 \\ -a_{w2} & a_{p2} & -a_{E2} & 0 & & & \\ 0 & -a_{w3} & a_{p3} & -a_{E3} & 0 & & \\ & 0 & -a_{w4} & a_{p4} & -a_{E4} & 0 & \\ & & 0 & -a_{w5} & a_{p5} & -a_{E5} & 0 \\ & & & 0 & -a_{w6} & a_{p6} & -a_{E6} \end{bmatrix}\begin{bmatrix} T_1\\T_2\\T_3\\T_4\\T_5\\T_6 \end{bmatrix} = \begin{bmatrix} b_1\\b_2\\b_3\\b_4\\b_5\\b_6 \end{bmatrix}$$

$$k_e = \frac{\delta x_e k_E k_p}{k_E\delta x_e^- + k_p\delta x_e^+}$$

$$k_w = \frac{\delta x_w k_W k_P}{\ }$$

$$
\begin{array}{c}
5 \\ 6 \\ 7
\end{array}
\begin{bmatrix}
0 & -q_{w5} & q_{P5} & -q_{E5} & 0 \\
 & 0 & -q_{w6} & q_{P6} & -q_{E6} \\
 & & 0 & -q_{w7} & q_{P7}
\end{bmatrix}
\begin{bmatrix} T_5 \\ T_6 \\ T_7 \end{bmatrix}
\begin{bmatrix} b_5 \\ b_6 \\ b_7 \end{bmatrix}
$$

$$k_w = \frac{\delta x_w k_W k_P}{k_P \delta x_w^- + k_W \delta x_w^+}$$

Harmonic means for nodes 4&5

```
julia> A
7×7 Matrix{Float64}:
     1.0     -1.0      0.0      0.0      0.0       0.0       0.0
 -3000.0   4500.0  -1500.0      0.0      0.0       0.0       0.0
     0.0  -1500.0   3000.0  -1500.0      0.0       0.0       0.0
     0.0      0.0  -1500.0   3900.0  -2400.0       0.0       0.0
     0.0      0.0      0.0  -2400.0   8400.0   -6000.0       0.0
     0.0      0.0      0.0      0.0  -6000.0   18000.0  -12000.0
     0.0      0.0      0.0      0.0      0.0  -12000.0   13000.0
```

Reflection: The setup is quite tedious, but begins to be intuitive as you go along. As long as delta x is the same for east and west nodes it makes things quite simple, and because the interior nodes repeat the pattern it could be quite simple to scale with a computer. Because there is so much algebra though, there is a chance I made a sign error or something.

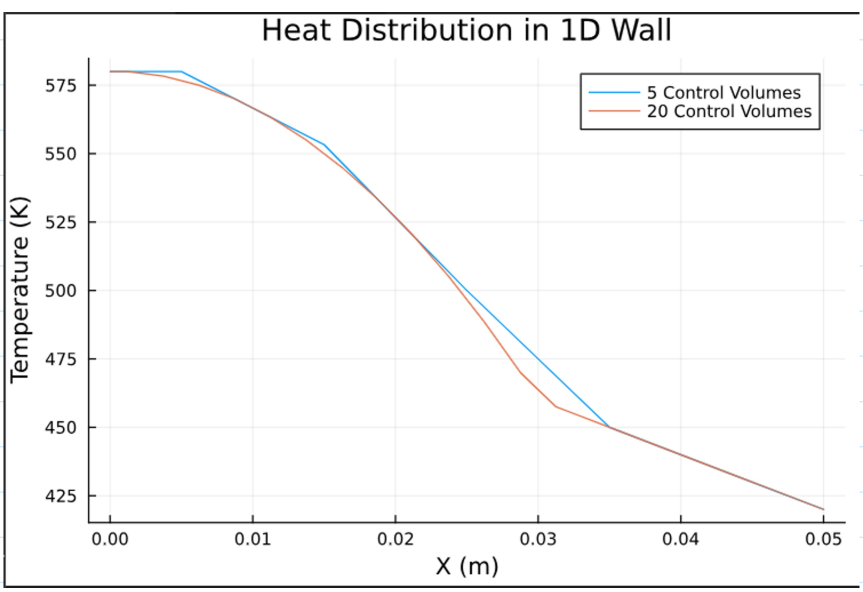
```
julia> b
7-element Vector{Float64}:
      0.0
  40000.0
  40000.0
  40000.0
      0.0
      0.0
 300000.0
```

```
julia> T
7-element Vector{Float64}:
 580.0000000000065
 580.0000000000058
 553.3333333333377
 500.0000000000029
 450.000000000002
 430.00000000000165
 420.0000000000015

julia> #In Kelvin
```

4. Assume the thermal conductivity of segment B is constant, $k_B = 60$ W/(m·K).
   a) Write a code to solve for the temperature at each note using 5 equally-spaced control volumes. Refer to the previous hand-worked problem to verify that your code is set up correctly. Plot the temperature profile and report the temperatures at both boundaries.
   b) Repeat using 20 equally-spaced control volumes. Is this result what you expected?

Summary: Using the equations and things derived above code a solver, then compare the temperature profile output with my hand calculations, then use more control volumes and compare with part a, discuss the results.
Assumptions: 1D, steady state, The harmonic mean is an accurate K representation.
Approach/Pseudocode: Create functions for every step along the way, given a number of control surfaces (N), call the Practice B function with a start and end range and have it return the locations of the nodes and surfaces



Reflection: Both of the profiles start at 580K and end on the right at 420K, I did not expect those to stay the same. The 5 CV estimation performed better than I expected, although the real struggle comes at the interface, where the 20CV method performs more realistically.

5. Assume the thermal conductivity in segment B varies as $k_B(x) = 137e^{25x-2}$ W/(m·K).
   a) Modify your code to find the temperature for this scenario. Plot the temperature profile and report the temperatures at both boundaries using 5 equal-spaced control volumes.
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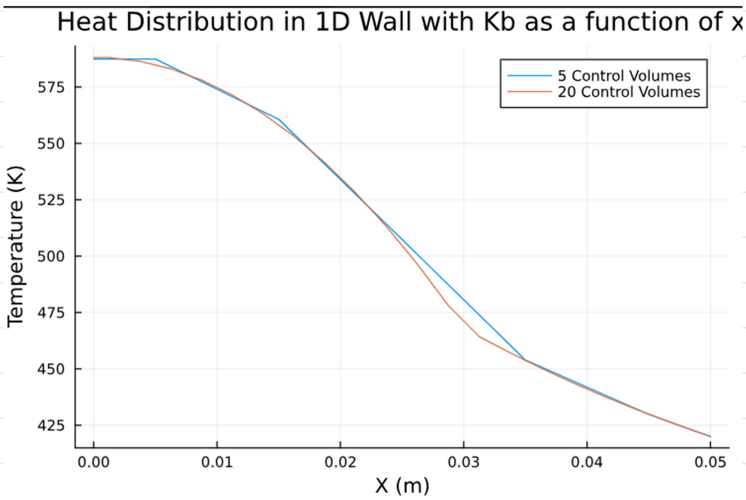
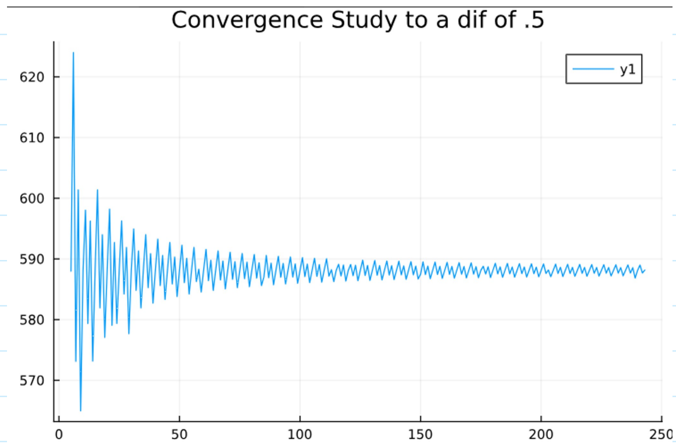
   c) Plot the predicted maximum temperature in the wall (in Kelvin) as a function of the number of control volumes (i.e., perform a grid-refinement study). What is the "grid-converged" maximum temperature to 3 decimal places (i.e., 0.001), and how many control volumes does it take to obtain this result?

Summary: Use the same code from the problem above, just modify kb to be a function of x. Use a function with a while loop to do a convergence study.





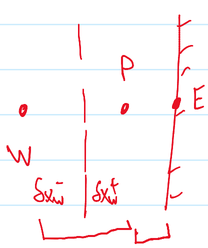
Reflection: The fact that my system is not converging and that it oscillates a little bit is concerning, I think it is a bug with my Harmonic mean values, but I can't figure it out and spent quite a lot of time on it. It is interesting to see that the 20 Control volume study does make a difference in the overall shape, and especially at the interface.

Debug N = 5 K_B Changing

5000.8 16423 11422

11422 12422

Node 6



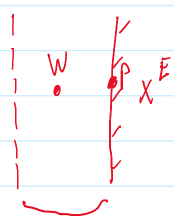
δx_w δx_E = $\frac{\delta x}{2}$

$\frac{\delta x\, k_e k_p}{\delta x^+}$

$\delta x_e^- = \frac{\delta x}{2}$

$\delta x_e^+ = 0$

Node 7



δx_w?

$= \delta x_w +$

$\delta x_w^- = 0$?