HW3

Monday, February 12, 2024 10:21 AM

ME EN 541 – HW 3
Handed out 6 Feb 2024
Due 12 Feb 2024

Problem	Points	Topic
1	15	Richardson extrapolation
2	20	Models with source terms
3	20	Models with source terms

1. Explore Richardson extrapolation on HW 2 Problem 5 as follows.
   a. Calculate the left-most ("base") temperature simulated using 5, 10, and 20 control volumes. Estimate the order of the simulation using these values.
   b. Predict the grid-independent base temperature using the estimated simulation order from part (a) and the base temperature values from the 10- and 20-control volume cases.
   c. Plot base temperature for 5, 10, 20, 40, …, 5120, 10240 control volumes. Show the predicted grid-independent solution from part b on this plot. Also generate a table that lists, for these control volumes, the number of control volumes, the simulated base temperature, and the percent difference between the simulated solutions and the grid-independent solution.

Problem Setup for Problems 2 and 3
Consider cooling of a circular fin by means of convective and radiative heat transfer along its length. For constant diameter, the steady state conservation equation is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{4}{D}\left[h(T-T_\infty)+\varepsilon\sigma\left(T^4-T_{surr}^4\right)\right]=0$$

where k is the thermal conductivity, D the diameter, T_∞ the ambient temperature, T_{surr} the temperature of the surroundings, ε the fin surface emissivity, and σ the Stefan-Boltzmann constant (5.67×10^{-8} W/m^{2}K^4).

Formulate this problem for solution via the control-volume method, and apply the two cases specified below. For both cases, the temperature at the base is fixed, the tip of the fin can be considered to be adiabatic, and the following parameters apply:

L = 2 cm, D = 3 mm
k = 401 W/mk (copper), h = 10 W/m^{2}K
T_B = 400 K, T_∞ = 273 K, T_{surr} = 273 K

Note the following in developing your model:

- An iterative method is required since the source term is nonlinear. **Use the TDMA method** to solve the linear system at each iteration.
- Iterate until the maximum temperature change at any node between iterations is < 0.0001.
- Appropriately linearize the source term.

2. Validate your model by predicting the temperature distribution $T(x)$ and the fin heat transfer q_f for the convection-only case ($\varepsilon = 0$). For this case there is an analytical solution:

$$\frac{T - T_\infty}{T_B - T_\infty} = \frac{\cosh(n(L-x))}{\cosh(nL)}$$

   where $n^2 = 4h/Dk$. Plot the maximum error vs. number of control volumes.

3. Predict and plot the steady-state temperature distribution for the combined convection and radiation case with $h = 10$ W/m^{2}K and $\varepsilon = 1$. Use an appropriate number of CVs as determined from the previous problem. Comparing this problem with the previous problem, comment on the relative importance of convective vs. radiative heat transfer on the tip temperature.

1. Explore Richardson extrapolation on HW 2 Problem 5 as follows.
   a. Calculate the left-most ("base") temperature simulated using 5, 10, and 20 control volumes. Estimate the order of the simulation using these values.
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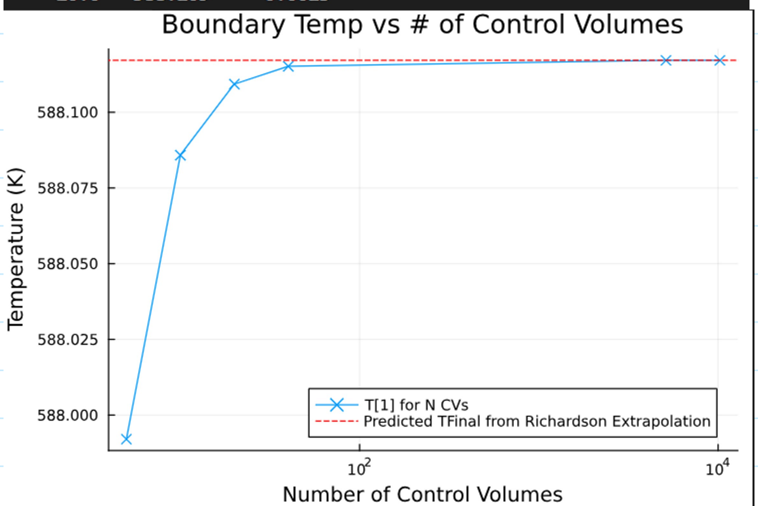
   c. Plot base temperature for 5, 10, 20, 40, …, 5120, 10240 control volumes. Show the predicted grid-independent solution from part b on this plot. Also generate a table that lists, for these control volumes, the number of control volumes, the simulated base temperature, and the percent difference between the simulated solutions and the grid-independent solution.

Summary: Using the same code and setup from Last homework, use Richardson Extrapolation to estimate the order of the simulation and estimate the grid-independent base temperature
Assumptions: Steady state, the harmonic mean accurately describes what is happening, and the problem is fully described as explained last homework K

→ 588.11712 K (I rounded the output)

Part A-C

```
The order of the simulation is 2.0.
For Ts = [587.992047966878, 588.0858129520578, 588.1092942637172]
The grid converged value is 588.12.
7×3 Matrix{Any}:
     "N"      "T[1]"     "Percent Difference"
    5.0   587.992    -0.0213
   10.0   588.086    -0.0053
   20.0   588.109    -0.0013
```



Reflection: It is amazing how accurate the Richardson Extrapolation estimation is. Note that I plotted with an log x axis, as without it the first 4 points are very close. My code from last week didn't converge, however I fixed some things, and even though it doesn't converge well still, the temperatures I have checked do all match to several decimal places

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Note the following in developing your model:

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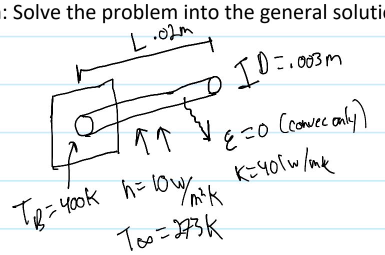
$$\frac{T-T_\infty}{T_B-T_\infty}=\frac{\cosh(n(L-x))}{\cosh(nL)}$$

where $n^2 = 4h/Dk$. Plot the maximum error vs. number of control volumes.

Summary: Given a circular cooling fin with convective and radiative heat transfer, solve iteratively using the TDMA method and compare to the analytical solution.
Assumptions: steady state, adiabatic tip, fixed base temperature, homogeneous material
Approach: Solve the problem into the general solution, modify into TDMA, code up and compare to analytical

Sketch:

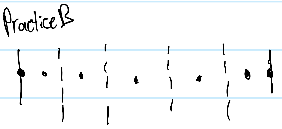


$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)-\frac{4}{D}\left[\underbrace{h(T-T_\infty)}_{\text{convection}}+\underbrace{\varepsilon\sigma\left(T^4-T_{surr}^4\right)}_{\text{radiation}}\right]=0$$

$$\sigma = 5.67\times10^{-8}\ \frac{W}{m^2K}$$

$$S(T) = -\frac{4}{D}\left[h(T-T_\infty)+\varepsilon\sigma\left(T^4-T_{surr}^4\right)\right]$$

CV approach Practice B



"Non-lazy" faster iterative method $\rightarrow S(T)=S_c+S_pT_p$ (linearization)

we could say

$$S = S^* + (T_p-T_p^*)\left(\frac{dS}{dT}\right)^*$$

Note: T^* = most recently available value, not last iterations
first iteration T^* = guess

$$S_c + S_pT_p = S^* + (T_p-T_p^*)\left(\frac{dS}{dT}\right)^* \rightarrow S^* - T_p^*\left(\frac{dS}{dT}\right)^* + \left(\frac{dS}{dT}\right)^* T_p = S_c + S_pT_p$$

$$S(T) = -\frac{4h}{D}(T-T_\infty) - \frac{4\varepsilon\sigma}{D}\left(T^4-T_{surr}^4\right)$$

$$S_c = S^* - T_p^*\left(\frac{dS}{dT}\right)^*$$

$$S_p = \left(\frac{dS}{dT}\right)^*$$

$$\frac{dS}{dT} = -\frac{4h}{D} - \frac{16\varepsilon\sigma}{D}T^3$$

$$\frac{dS}{dT} = -\frac{4h}{D} - \frac{16\varepsilon\sigma}{D}T^3 \qquad S_p = \left(\frac{dS}{dT}\right)^*$$

$$\left(\frac{dS}{dT}\right)^* = -\frac{4h}{D} - \frac{16\varepsilon\sigma}{D}\left(T_p^*\right)^3$$

$$S^* = S(T_p^*) \qquad S_C = \cancel{-\frac{4h}{D}(T_p^*)} + \frac{4h}{D}T_\infty \underbrace{- \frac{4\varepsilon\sigma}{D}T^{*4} + \frac{4\varepsilon\sigma}{D}T_{surr}^4 \cancel{- T_p^*\left(-\frac{4h}{D}\right)} - T_p^*\left(-\frac{16\varepsilon\sigma}{D}T_p^{*3}\right)}_{\frac{12\varepsilon\sigma}{D}\left(T_p^*\right)^4}$$

$$\boxed{S_C = \frac{4}{D}hT_\infty + \frac{4\varepsilon\sigma}{D}T_{surr}^4 + \frac{12\varepsilon\sigma}{D}\left(T_p^*\right)^4}$$

$$\boxed{S_p = -\frac{4h}{D} - \frac{16\varepsilon\sigma}{D}\left(T_p^*\right)^3} \qquad S(T) = S_C + S_pT_p$$

General Form

$$a_pT_p = a_ET_E + a_WT_W + b$$

general equation $\int_w^e \frac{d}{dx}\left(k\frac{dT}{dx}\right) + \int_w^e S\,dx = 0$ → $(S_C + S_pT_p)\Delta x_p$

$$k\frac{dT}{dx}\Big|_e - k\frac{dT}{dx}\Big|_w + S_C\Delta x_p + S_pT_p\Delta x_p$$

$$\frac{k_E(T_E - T_P)}{\delta x_e} - \frac{k_W(T_P - T_W)}{\delta x_W} + S_C\Delta x_p + S_pT_p\Delta x_p$$

$$\frac{k_ET_E}{\delta x_e} + \frac{k_WT_W}{\delta x_W} + S_C\Delta x_p = T_p\left(-S_p\Delta x_p + \frac{k_E}{\delta x_e} + \frac{k_W}{\delta x_W}\right)$$

$$\boxed{a_E = \frac{k_E}{\delta x_e} \quad a_W = \frac{k_W}{\delta x_W} \quad b = S_C\Delta x_p \quad a_p = a_E + a_W - S_p\Delta x_p}$$

General (Nodes 3-5) (with SC4)

Node 1

$T_p = T_B$

so $a_E = 0$

$a_W = 0$

$a_p = a_E + a_W + 1$

$b = T_{Base}$

$\Delta x_p = 0$ $S_p \to 0$ $S_c \to 0$

Node 2

$a_E = \frac{k_E}{\delta x}$

$a_W = \frac{2k_W}{\delta x}$

$a_p = a_E + a_W - S_p\Delta x_p$ ← is this $\frac{dx}{2}$?

$b = S_C\Delta x_p$

Node 6

$a_E = \frac{2k_E}{\delta x}$

$a_W = \frac{k_W}{\delta x}$

a_p = standard

b = standard

Node 7

adiabatic tip

So $k\frac{dT}{dx} = 0$ $T_p = T_W$

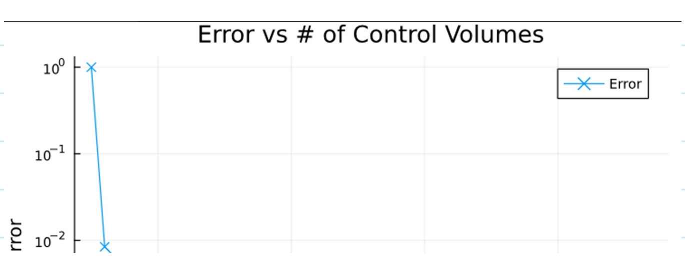
$a_W = 1$

$a_p = 1$

$\Delta x_p = 0$ so $S_p \to 0$ $S_c \to 0$

Convert to TDMA $\overset{a_p}{a_i}\phi_i = \overset{a_E}{b_i}\phi_{i+1} + \overset{a_W}{c_i}\phi_{i-1} + \overset{b}{d_i}$

$c_1 = 0$ & $b_N = 0$ for boundaries



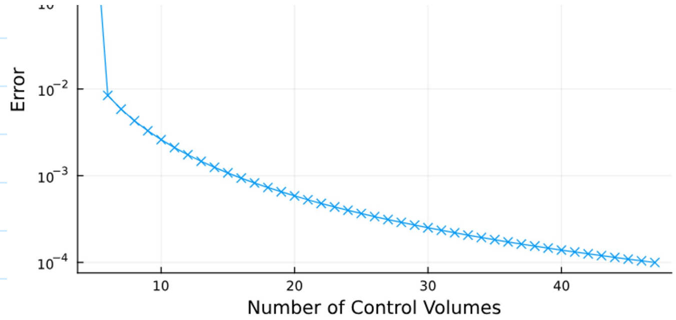
$$\phi_i = P_i\phi_{i+1} + Q_i$$

$$P_i = \frac{b_i}{a_i - c_iP_{i-1}}$$

$$Q_i = \frac{d_i + c_iQ_{i-1}}{a_i - c_iP_{i-1}}$$

↓

P_N



P_N

Q_N

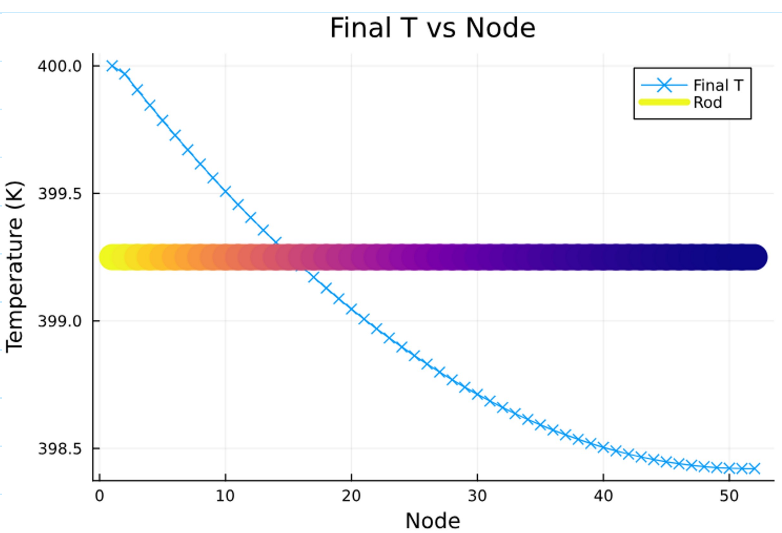
then $\phi_N = Q_N$

$Q_{N-1} = P_{N-1}\phi_N + Q_{N-1}$

Reflection: I iterated until the difference with analytical was less than 0.001. It seems to work with less control volumes than the finite difference method, although that could be because of something else I don't understand. Error also drops really fast once you start iterating because of Tpstar etc. It runs decently fast on my computer too. I got that my error was less than .0001 at 46 Control Volumes, so I will do problem 3 with 50 CVs

3. Predict and plot the steady-state temperature distribution for the combined convection and radiation case with $h = 10$ W/m^{2}K and $\varepsilon = 1$. Use an appropriate number of CVs as determined from the previous problem. Comparing this problem with the previous problem, comment on the relative importance of convective vs. radiative heat transfer on the tip temperature.

Summary: Using the code that I have from above change the needed values and CVs = 50, find the steady state temperature distribution
Approach: As I no longer have an analytical solution to compare to I will need to iterate several times and set an appropriate stopping criteria (max difference is < .0001
Assumptions: Same as above
Sketch: Same as above




```
julia> Ttip2 - Ttip3
0.7390889876621713
```

Reflection: There was only a .74 degree difference (399 vs 398 for radiation) between my tip temperatures once I added radiation. This shows, that in this case, convection dominates much more than radiation does. My solution only iterated a few times, so it is possible that I implemented something wrong. I included a visual representation of the rod for fun.