Flow Field Calculation
Part III: SIMPLE Steps in Detail (Upwind Scheme)

1 SIMPLE Scheme Overview

The following are the SIMPLE scheme steps; details are given below.

Step A: Generate initial guesses for pressure and velocity: $p^{*,old}$, $u^{*,old}$, and $v^{*,old}$.
Step B: Solve the relaxed discretized x-momentum equation to obtain $u^{*,new}$.

$$\begin{aligned}\frac{a_{i,J}}{\alpha_u} u_{i,J}^{*,new} = a_{i+1,J} u_{i+1,J}^{*,new} + a_{i-1,J} u_{i-1,J}^{*,new} + a_{i,J+1} u_{i,J+1}^{*,new} \\ + a_{i,J-1} u_{i,J-1}^{*,new} + \left(p_{I-1,J}^{*,old} - p_{I,J}^{*,old}\right)\Delta y + b_{i,J} \\ + \left[\frac{(1-\alpha_u)}{\alpha_u} a_{i,J}\right] u_{i,J}^{*,old}\end{aligned} \tag{B.1a}$$

Step C: Solve the relaxed discretized y-momentum equation to obtain $v^{*,new}$.

$$\begin{aligned}\frac{a_{I,j}}{\alpha_v} v_{I,j}^{*,new} = a_{I+1,j} v_{I+1,j}^{*,new} + a_{I-1,j} v_{I-1,j}^{*,new} + a_{I,j+1} v_{I,j+1}^{*,new} \\ + a_{I,j-1} v_{I,j-1}^{*,new} + \left(p_{I,J-1}^{*,old} - p_{I,J}^{*,old}\right)\Delta x + b_{I,j} \\ + \left[\frac{(1-\alpha_v)}{\alpha_v} a_{I,j}\right] v_{I,j}^{*,old}\end{aligned} \tag{C.1a}$$

Step D: Solve the pressure correction equation to obtain p'.

$$\begin{aligned}a_{I,J} p'_{I,J} = a_{I+1,J} p'_{I+1,J} + a_{I-1,J} p'_{I-1,J} + a_{I,J+1} p'_{I,J+1} + a_{I,J-1} p'_{I,J-1} \\ + b'_{I,J}\end{aligned} \tag{D.1a}$$

Step E: Calculate the new pressure field.

$$p^{new} = p^{*,old} + \alpha_p p' \tag{E.1}$$

Step F: Find the new velocity fields using the pressure correction field.

$$u^{new} = u^{*,new} + d_{i,J}\left(p'_{I-1,J} - p'_{I,J}\right) \tag{F.1}$$

$$v^{new} = v^{*,new} + d_{I,j}\left(p'_{I,J-1} - p'_{I,J}\right) \tag{F.2}$$

Step G: Set $p^{*,old} = p^{new}$, $u^{*,old} = u^{new}$, $v^{*,old} = v^{new}$, and iterate until converged.

2 Step A: Initial Guesses

Generate initial guesses for pressure and velocity, here denoted as $p^{*,old}$, $u^{*,old}$, and $v^{*,old}$.

3 Step B: $u^{*,new}$

Solve the relaxed discretized x-momentum equation (Eq. B.1a) to obtain $u^{*,new}$. See Sec. 4 and 6 of the Part II notes, observing that $b_{i,J}$ appears here but not in the Part II notes because Part II only deals with interior nodes with no momentum source terms. Here, $b_{i,J}$ is included to allow for additional source terms and certain boundary conditions, the latter of which is discussed in the Part IV notes.

$$\begin{aligned}\frac{a_{i,J}}{\alpha_u}u_{i,J}^{*,new} = {} & a_{i+1,J}u_{i+1,J}^{*,new} + a_{i-1,J}u_{i-1,J}^{*,new} + a_{i,J+1}u_{i,J+1}^{*,new} \\ & + a_{i,J-1}u_{i,J-1}^{*,new} + \left(p_{I-1,J}^{*,old} - p_{I,J}^{*,old}\right)\Delta y + b_{i,J} \\ & + \left[\frac{(1-\alpha_u)}{\alpha_u}a_{i,J}\right]u_{i,J}^{*,old}\end{aligned} \tag{B.1a}$$

$$a_{i+1,J} = D_e\Delta y + \max(-F_e, 0)\,\Delta y \tag{B.1b}$$

$$a_{i-1,J} = D_w\Delta y + \max(F_w, 0)\,\Delta y \tag{B.1c}$$

$$a_{i,J+1} = D_n\Delta x + \max(-F_n, 0)\,\Delta x \tag{B.1d}$$

$$a_{i,J-1} = D_s\Delta x + \max(F_s, 0)\,\Delta x \tag{B.1e}$$

$$\begin{aligned}a_{i,J} = {} & a_{i+1,J} + a_{i-1,J} + a_{i,J+1} + a_{i,J-1} + (F_e - F_w)\Delta y \\ & + (F_n - F_s)\Delta x\end{aligned} \tag{B.1f}$$

$$D_e = \frac{\mu}{x_{i+1} - x_i} \tag{B.1g}$$

$$D_w = \frac{\mu}{x_i - x_{i-1}} \tag{B.1h}$$

$$D_n = \frac{\mu}{y_{J+1} - y_J} \tag{B.1i}$$

$$D_s = \frac{\mu}{y_J - y_{J-1}} \tag{B.1j}$$

$$F_e = \frac{\rho}{2}\left(u_{i+1,J}^{*,old} + u_{i,J}^{*,old}\right) \tag{B.1k}$$

$$F_w = \frac{\rho}{2}\left(u_{i,J}^{*,old} + u_{i-1,J}^{*,old}\right) \tag{B.1l}$$

$$F_n = \frac{\rho}{2}\left(v_{I,j+1}^{*,old} + v_{I-1,j+1}^{*,old}\right) \tag{B.1m}$$

$$F_s = \frac{\rho}{2}\left(v_{I,j}^{*,old} + v_{I-1,j}^{*,old}\right) \tag{B.1n}$$

where $\Delta x = (x_I - x_{I-1})$, $\Delta y = \left(y_{j+1} - y_j\right)$.

Notes:

a) The objective of this step is to find the new values of u^* throughout the computation domain. We distinguish these as $u^{*,new}$.

b) The a coefficients (B.1b through B.1f) form a matrix, A_u. These coefficients are generated using the already-known $u^{*,old}$ and $v^{*,old}$ values (i.e., either initial guess in the first iteration, or subsequently, values from the previous iteration). They are not changed when solving for $u^{*,new}$ at this step.

c) Everything in the last three terms of Eq. (B.1a) (the pressure, $b_{i,J}$, and $u_{i,J}^{*,old}$ terms) is known. These terms can thus be combined to form a single term, B_u, forming a matrix system to be solved: $[A_u]\{u^{*,new}\} = \{B_u\}$.

d) When creating $[A_u]$ don't forget to divide $a_{i,J}$ terms by α_u (LHS of Eq. B.1a).

e) These equations incorporate the upwind scheme for convection.

4 Step C: $v^{*,new}$

Solve the relaxed discretized y-momentum equation to obtain $v^{*,new}$ (see Sec. 5 and 6 of Part II notes):

$$\begin{aligned}\frac{a_{I,j}}{\alpha_v} v_{I,j}^{*,new} &= a_{I+1,j} v_{I+1,j}^{*,new} + a_{I-1,j} v_{I-1,j}^{*,new} + a_{I,j+1} v_{I,j+1}^{*,new} \\ &\quad + a_{I,j-1} v_{I,j-1}^{*,new} + \left(p_{I,J-1}^{*,old} - p_{I,J}^{*,old}\right)\Delta x + b_{I,j} \\ &\quad + \left[\frac{(1-\alpha_v)}{\alpha_v} a_{I,j}\right] v_{I,j}^{*,old}\end{aligned} \tag{C.1a}$$

$$a_{I+1,j} = D_e \Delta y + \max(-F_e, 0)\,\Delta y \tag{C.1b}$$

$$a_{I-1,j} = D_w \Delta y + \max(F_w, 0)\,\Delta y \tag{C.1c}$$

$$a_{I,j+1} = D_n \Delta x + \max(-F_n, 0)\,\Delta x \tag{C.1d}$$

$$a_{I,j-1} = D_s \Delta x + \max(F_s, 0)\,\Delta x \tag{C.1e}$$

$$\begin{aligned}a_{I,j} &= a_{I+1,j} + a_{I-1,j} + a_{I,j+1} + a_{I,j-1} + (F_e - F_w)\Delta y \\ &\quad + (F_n - F_s)\Delta x\end{aligned} \tag{C.1f}$$

$$D_e = \frac{\mu}{x_{I+1} - x_I} \tag{C.1g}$$

$$D_w = \frac{\mu}{x_I - x_{I-1}} \tag{C.1h}$$

$$D_n = \frac{\mu}{y_{j+1} - y_j} \tag{C.1i}$$

$$D_s = \frac{\mu}{y_j - y_{j-1}} \tag{C.1j}$$

$$F_e = \frac{\rho}{2}\left(u_{i+1,J}^{*,new} + u_{i+1,J-1}^{*,new}\right) \tag{C.1k}$$

$$F_w = \frac{\rho}{2}\left(u_{i,J}^{*,new} + u_{i,J-1}^{*,new}\right) \tag{C.1l}$$

$$F_n = \frac{\rho}{2}\left(v_{I,j}^{*,old} + v_{I,j+1}^{*,old}\right) \tag{C.1m}$$

$$F_s = \frac{\rho}{2}\left(v_{I,j-1}^{*,old} + v_{I,j}^{*,old}\right) \tag{C.1n}$$

where $\Delta x = (x_{i+1} - x_i)$, $\Delta y = \left(y_J - y_{J-1}\right)$.

Notes:

a) The a coefficients used here to obtain $v^{*,new}$ are generated using the $u^{*,new}$ values obtained in Step B and the $v^{*,old}$ values from Step A.

b) Similar to Step B, the following is solved: $[A_v]\{v^{*,new}\} = \{B_v\}$.

5 Step D: p'

Solve the pressure correction equation to obtain p' (see Sec. 9 of Part I notes):

$$a_{I,J}p'_{I,J} = a_{I+1,J}p'_{I+1,J} + a_{I-1,J}p'_{I-1,J} + a_{I,J+1}p'_{I,J+1} + a_{I,J-1}p'_{I,J-1} + b'_{I,J} \tag{D.1a}$$

$$a_{I,J} = a_{I+1,J} + a_{I-1,J} + a_{I,J+1} + a_{I,J-1} \tag{D.1b}$$

$$a_{I+1,J} = (\rho d \Delta y)_{i+1,J} \tag{D.1c}$$

$$a_{I-1,J} = (\rho d \Delta y)_{i,J} \tag{D.1d}$$

$$a_{I,J+1} = (\rho d \Delta x)_{I,j+1} \tag{D.1e}$$

$$a_{I,J-1} = (\rho d \Delta x)_{I,j} \tag{D.1f}$$

$$b'_{I,J} = -(\rho u^{*,new}\Delta y)_{i+1,J} + (\rho u^{*,new}\Delta y)_{i,J} - (\rho v^{*,new}\Delta x)_{I,j+1} + (\rho v^{*,new}\Delta x)_{I,j} \tag{D.1g}$$

$$d_{i+1,J} = \frac{\Delta y \alpha_u}{a_{i+1,J}} \tag{D.1h}$$

$$d_{i,J} = \frac{\Delta y \alpha_u}{a_{i,J}} \tag{D.1i}$$

$$d_{I,j+1} = \frac{\Delta x \alpha_v}{a_{I,j+1}} \tag{D.1j}$$

$$d_{I,j} = \frac{\Delta x \alpha_v}{a_{I,j}} \tag{D.1k}$$

where $\Delta x = (x_{i+1} - x_i)$, $\Delta y = \left(y_{j+1} - y_j\right)$.

Notes:

a) $a_{i,J}, a_{i+1,J}, a_{I,j}$, and $a_{I,j+1}$ are primary coefficients for their respective nodal locations. For example, $a_{i,J}$ is the primary coefficient (Eq. B.1f) used when solving $u^{*,new}$ in Step B. Refer to Fig. 9.1 of Part I notes. These coefficients are calculated prior to solving for $u^{*,new}$ and $v^{*,new}$ in Steps B and C, respectively, and are <u>not</u> recalculated in this step prior to solving for p'.

b) Relaxation has been incorporated at this step in D.1h through D.1k.

c) The parentheses on the RHS of Eqns. (D.1c) through (D.1f) each contain three separate variables: ρ, d, and Δy. For example, the RHS of D.1c could alternatively be written as $\rho_{i+1,J}d_{i+1,J}\Delta y_{i+1,J}$.

d) The following is solved: $\left[A_{p'}\right]\{p'\} = \left\{B_{p'}\right\}$.

6 Step E: $\boldsymbol{p^{new}}$

Calculate the new pressure field (see Sec. 10 of Part I notes):

$$p^{new} = p^{*,old} + \alpha_p p' \tag{E.1}$$

Notes:

a) p' is the pressure correction field obtained in Step D.

b) In the first iteration, $p^{*,old}$ is the initial guess. In subsequent iterations, it is the value obtained in the previous iteration.

c) The relaxation factor α_p is used to only add a fraction of the pressure correction p' to $p^{*,old}$, thereby stabilizing the solver.

7 Step F: $\boldsymbol{u^{new}, v^{new}}$

Find the new velocity fields using the pressure correction field (see Eqns. 8.4, 8.5, 8.12, and 8.13 of Part I notes):

$$u^{new} = u^{*,new} + d_{i,J}\left(p'_{I-1,J} - p'_{I,J}\right) \tag{F.1}$$

$$v^{new} = v^{*,new} + d_{I,j}\left(p'_{I,J-1} - p'_{I,J}\right) \tag{F.2}$$

Notes:

1) $u^{*,new}$ is the output of Step B, $v^{*,new}$ is the output of Step C, and p' is the output of Step D.

2) Relaxation does not explicitly appear in Eqns. (F.1) and (F.2), but it has been included via Eqns. (D.1h) through (D.1k) when solving for the pressure correction terms as well as via Eqns. (B.1a) and (C.1a) when solving for $u^{*,new}$ and $v^{*,new}$.

8 Step G: Iterate

Set $p^{*,old} = p^{new}$, $u^{*,old} = u^{new}$, and $v^{*,old} = v^{new}$ and iterate steps B through G until convergence is reached.

9 Implementation Note

When generating the code for the SIMPLE solver, be careful with the order of the subscripted indices in these notes and the indexing convention of the code. For example, with $a_{i,J}$ in these notes, i refers to x-position and J refers to y-position. However, in MATLAB and many other programs, a matrix entry such as `a(m,n)` is indexed as `a(row,column)`. If we follow the intuitive interpretation that `row` denotes y-position and `column` denotes x-position, one can see that the order of the MATLAB indices is opposite that of the subscripts in these notes.