Midterm2 Problem 11

Saturday, April 1, 2023 10:56 PM

11. Consider the open-loop system

$$Y(s) = \frac{(s+p)}{(s+a)(s+p)}U(s) = \frac{s+p}{s^2+(a+p)s+ap}U(s),$$

where it is clear that there is an exact pole-zero cancellation at $s = p$.

(a) Derive the equivalent state-space system in control canonic form. Are the resulting state-space equations in control canonic form controllable? Are the resulting state-space equations in control canonic form observable?

(b) Derive the equivalent state-space system in observer canonic form. Are the resulting state-space equations in observer canonic form observerable? Are the resulting state-space equations in observer canonic form controllable?

(c) Explain the answers that you got in parts (a) and (b). Can you design an observer based control system for this problem?

$$\frac{s+p}{s^2+(a+p)s+ap}$$

$m = 1$

$b_0 = p$

$n = 2$

$a_1 = (a+p)$

$a_0 = ap$

Consider the general SISO monic transfer function model given by

$$Y(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s), \tag{11.10}$$

where $m < n$. The block diagram of the system is shown in Fig. 11-1. The first

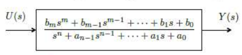

Figure 11-1: General transfer function for a SISO system.

step in deriving the control canonic state space equations is to artificially decompose the transfer function as shown in Fig. 11-2, where the numerator polynomial and the denominator polynomial appear in separate cascaded blocks, and where $Z(s)$ is an artificial intermediate variable. In the s-domain, the equations corre-

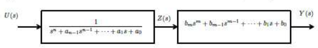

Figure 11-2: General transfer function for a SISO system decomposed into state dynamics and output dynamics.

sponding to Fig. 11-2 can be written as

$$Z(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}U(s)$$
$$Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0)Z(s).$$

Taking the inverse Laplace transform of these equations gives

$$\frac{d^n z}{dt^n} = u(t) - a_{n-1}\frac{d^{n-1}z}{dt^{n-1}} - \cdots - a_1\dot{z} - a_0 z(t) \tag{11.11}$$
$$y(t) = b_m\frac{d^m z}{dt^m} + b_{m-1}\frac{d^{m-1}z}{dt^{m-1}} + \cdots + b_1\dot{z} + b_0 z(t). \tag{11.12}$$

The next step in the derivation of the state space equations is to draw the analog computer implementation of Equations (11.11) and (11.12), which is shown in Fig. 11-3. Labeling the output of the integrators as the state variables $x_1, \ldots, x_n$,

$$Z(s) = \frac{1}{s^2+(a+p)s+ap}U(s)$$

$$Y(s) = (s+p)Z(s)$$

$$\frac{d^2z}{dt^2} = u(t) - (a+p)\dot{z} - ap\,z(t)$$

$$y(t) = \dot{z} + p\,z(t)$$

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \vdots\\ \dot{x}_n\end{pmatrix} = \begin{pmatrix}-a_{n-1} & -a_{n-2} & \ldots & -a_1 & -a_0\\ 1 & 0 & \ldots & 0 & 0\\ 0 & 1 & \ldots & 0 & 0\\ \vdots & & \ddots & & \vdots\\ 0 & 0 & \ldots & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} + \begin{pmatrix}1\\ 0\\ \vdots\\ 0\end{pmatrix}u \tag{11.13}$$

$$y = \begin{pmatrix}0 & \ldots & 0 & b_m & \ldots & b_1 & b_0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} + 0\cdot u.$$

$$\dot{x} = \begin{pmatrix}-(a+p) & -ap\\ 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}1\\ 0\end{pmatrix}u$$

$$Y = \begin{pmatrix}1 & p\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + (0)u$$

As an example, suppose that the transfer function model of the system is given by

$$Y(s) = \frac{9s+20}{s^3+6s^2-11s+8}U(s),$$

then the control canonic realization of the system is given by

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix}-6 & 11 & -8\\ 1 & 0 & 0\\ 0 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} + \begin{pmatrix}1\\ 0\\ 0\end{pmatrix}u \tag{11.18}$$
$$y = \begin{pmatrix}0 & 9 & 20\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}.$$

$$A = \begin{bmatrix}-(a+p) & -ap\\ 1 & 0\end{bmatrix} \qquad B = \begin{pmatrix}1\\ 0\end{pmatrix}$$

$$C = \begin{pmatrix}1 & p\end{pmatrix}$$

$$C_{AB} = \begin{bmatrix}1 & -(a+p)\\ 0 & 1\end{bmatrix}$$

$$C_{AB} = [B, AB, \ldots, A^{n-1}B]$$

$$\text{rank}(C) = n$$

A. Controllable = Yes } rank = 2 & n = 2

Observable = Yes, Augmenting the matrix shouldn't cause issues in this case

B.

We define the observer canonic realization as

$$\dot{x}_o = A_o x_o + B_o u$$
$$y = C_o x_o,$$

where

$$A_o \triangleq \begin{pmatrix}-a_{n-1} & 1 & \ldots & 0 & 0\\ -a_{n-2} & 0 & \ldots & 0 & 0\\ \vdots & & \ddots & & \vdots\\ -a_1 & 0 & \ldots & 0 & 1\\ -a_0 & 0 & \ldots & 0 & 0\end{pmatrix} \tag{13.8}$$

$$B_o \triangleq \begin{pmatrix}0\\ \vdots\\ 0\\ b_m\\ \vdots\\ b_1\end{pmatrix} \tag{13.9}$$

$$A_o = \begin{bmatrix}-(a+p) & 1\\ -ap & 0\end{bmatrix}$$

$$B_o = \begin{bmatrix}1\\ p\end{bmatrix}$$

$$B_o \triangleq \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b_m \\ \vdots \\ b_1 \\ b_0 \end{pmatrix} \tag{13.9}$$

$$C_o \triangleq \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{13.10}$$

Comparing the state space equations in observer canonic form in Equations (13.8)–(13.10) with the state space equations for control canonic form in Equations (11.14)-(11.16), we see that

$$A_o = A_c^\top$$
$$B_o = C_c^\top$$
$$C_o = B_c^\top.$$

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$$B_o = \begin{bmatrix} 1 \\ p \end{bmatrix}$$

$$C_o = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$O_c = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -(a+p) & 0 \end{bmatrix}$$

Observable = No? The rank is the number of linearly independent columns, and 0 0 is linearly dependent I think.
Controllable = Yes, I think so

C. In a we found the relevant matrices and found that it was controllable and observable, in b we found the relevant matrices and found that it was not observable, but was controllable. Because of that, I think we would not be able to design an observable controller