NMM 2276B Final Examination

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Prof. Najeh Zarir

April 10, 2022 10:00 AM– 01:00 PM; 180 minutes

Print Name: ______________ Student Number: ______________

Signature: ______________ UWO email/ user ID: ______________

Exam seating: Room ______________ Row: ______________ Seat: ______________

BEFORE STARTING THE EXAMINATION, READ THE FOLLOWING:

This examination has 8 long solution questions on pages 3-10. For each question, show your work in full detail. Unjustified answers will receive little or no credit.

In addition to the question pages, the exam has:

- Cover page: Please fill your information on the cover page before the start of the exam. Make sure you enter your name, student number, and email ID. In addition, enter the exam room code, seating row and number. These numbers must be identical to the numbers you received in PostEM on the owl site. You also need to sign the cover page in the provided space.
- Below these instructions on the cover page, there is an empty box which you can use flag potential issues with the exam.
- Page number 8 has been left blank intentionally. Use it for extra work if you wish.
- The last page (No. 13) contains the formula sheet. You are free to detach the sheet but you must write your name on it and submit it with your exam.

Please write your answers in the space provided. Extra blank pages are provided as part of the exam. Please write your name and student ID on each page that is not stapled with the exam booklet and submit everything together.
This is a closed-book closed-notes exam. No books, notes or electronic devices allowed. Having a mobile phone, even turned off, on your person during the exam will be considered an attempt to cheat and will result in an academic offence investigation.

This space is for you to flag potential issues with any questions.

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Problem 1: Gradient, Divergence, and Curl.

Part 1a. Find a vector that gives the direction in which the given function increases most rapidly at the indicated point. Find the maximum rate of change.

$$F(x,y) = xy \cdot e^{(x-y)} \text{ at } (5,5)$$

Part 1b. Calculate $\nabla \bullet (\nabla \times \mathbf{F})$ for

$$\mathbf{F}(x,y,z) = x^2 y\,\hat{\mathbf{i}} + xy^2\,\hat{\mathbf{j}} + 2xyz\,\hat{\mathbf{k}}$$

Part 1c. Let $\mathbf{a}$ be a constant vector and $\mathbf{r} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$. Verify that:

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

Problem 2: Line Integration.

Part 2a. Show that the given line integral is independent of path.

$$\int_{(-1,1)}^{(0,0)} \mathbf{F} \bullet \mathrm{d}\mathbf{r}$$

$$\text{for } \mathbf{F} = (5x + 4y)\,\hat{\mathbf{i}} + (4x - 8y^3)\,\hat{\mathbf{j}}$$

Part 2b. Evaluate the integral by finding a potential function evaluated at the integration limits.

Part 2c. Evaluate the integral explicitly using any path between the endpoints.

Problem 3: Green's Theorem.

Use Green's Theorem to calculate the work done by the given force **F** around the closed curve shown below.

$$\mathbf{F}(x,y) = -xy^2\,\hat{\mathbf{i}} + x^2y\,\hat{\mathbf{j}}$$

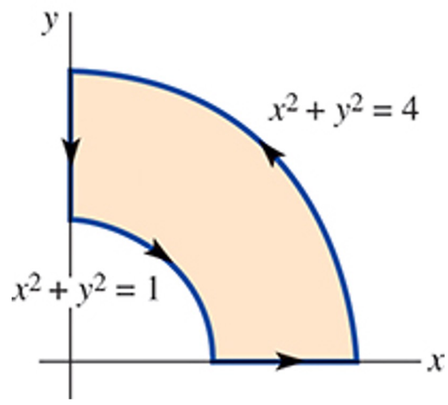

Figure 1: Integration path for Problem 3

Problem 4: Stoke's Theorem.

Verify Stoke's Theorem for the following vector field, curve, and surface.

$$\mathbf{F}(x, y, z) = xy\,\hat{\mathbf{i}} + 2yz\,\hat{\mathbf{j}} + xz\,\hat{\mathbf{k}}$$

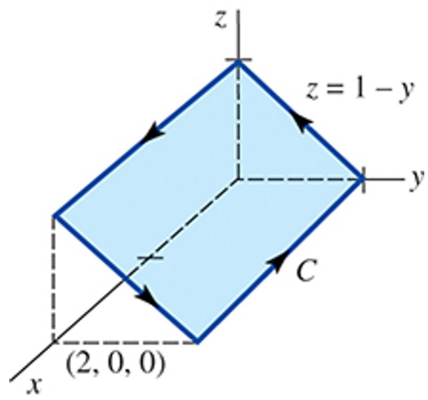

Figure 2: Curve and Surface for Problem 4

Problem 5: Divergence Theorem.

Use the Divergence Theorem to find the total outward flux of the following vector field through the given closed surface defining region D.

$$\mathbf{F}(x, y, z) = 15x^2y\,\hat{\mathbf{i}} + x^2z\,\hat{\mathbf{j}} + y^4\,\hat{\mathbf{k}}$$

D the region bounded by $x + y = 2, z = x + y, z = 3, y = 0$

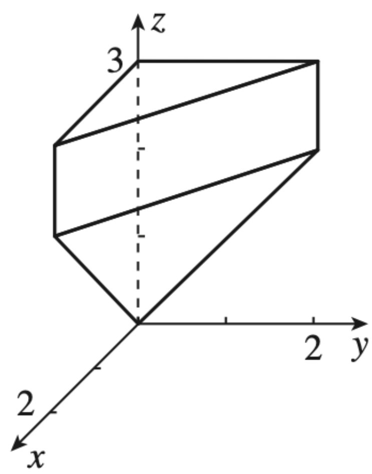

Figure 3: Surface and Volume for Problem 5

Problem 6: Fourier Series and Integrals.

1. Expand the function $f(x) = 2 + x^2, -\pi \leq x \leq \pi$ in an appropriate cosine or sine series.

2. Use an appropriate sine or cosine integral to represent the function

$$f(x) = \begin{cases} 2x, & |x| < 3\pi \\ 0, & |x| > 3\pi \end{cases}$$

Problem 7: Fourier Transforms.

Use the definition of the cosine Fourier transform pair

$$\mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\cos(\alpha x)dx = F(\alpha)$$
$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\cos(\alpha x)d\alpha = f(x)$$

to:

1. find $f(x)$ if

$$F(\alpha) = \begin{cases} 1-\alpha, & 0 \le \alpha \le 1 \\ 0, & \alpha > 1 \end{cases}$$

2. Show that your result in part (a) can be used to show that:

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$$

.

[Hint: Evaluate $F(\alpha)$ for some appropriate α. You may need to use $\cos(2\theta) = \cos^2\theta - \sin^2\theta$]

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Formula Sheet
Trigonometric identities:

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Fourier Series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x + b_n \sin \frac{n\pi}{p} x \right)$
$\mathbf{a}_0 = \frac{1}{p} \int_{-p}^{p} f(x)\, dx$
$\mathbf{a}_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos \frac{n\pi}{p} x\, dx$
$\mathbf{b}_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin \frac{n\pi}{p} x\, dx$

Fourier Cosine and Sine Series

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x$
$\mathbf{a}_0 = \frac{2}{p} \int_0^p f(x)\ dx$
$\mathbf{a}_n = \frac{2}{p} \int_0^p f(x) \cos \frac{x\pi}{p}\ x\, dx$
$\mathbf{f}(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x,$
$\mathbf{b}_n = \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x\, dx$

Fourier Integral

$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos(\alpha x) + B(\alpha) \sin(\alpha x)]\, d\alpha$
$\mathbf{A}(\alpha) = \int_{-\infty}^{+\infty} f(x) \cos(\alpha x)\, dx$
$\mathbf{B}(\alpha) = \int_{-\infty}^{+\infty} f(x) \sin(\alpha x)\, dx$

For even functions: $f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \cos(\alpha x)\, d\alpha$
For odd functions: $f(x) = \frac{2}{\pi} \int_0^{\infty} A(\alpha) \sin(\alpha x)\, d\alpha$
Where $A(\alpha) = \int_0^{+\infty} f(x) \cos(\alpha x)\, dx$ **and** $B(\alpha) = \int_0^{+\infty} f(x) \sin(\alpha x)\, dx$**.**

Fundamental Theorem for Line Integrals (with special requirements for F):

$$\int_C \mathbf{F} \bullet d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r} = \phi(B) - \phi(A)$$

Green's Theorem in the Plane:

$$\oint_C P \cdot dx + Q \cdot dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Stoke's Theorem:

$$\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, dS$$

Divergence Theorem:

$$\iint_S (\mathbf{F} \bullet \mathbf{n})\ dS = \iiint_D \nabla \bullet \mathbf{F}\, dV$$

Surface Integral detail (example for single case, other cases are similar):

$$\iint_S G(x,y,z)\, dS = \iint_R G(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$